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Liao Lingmin

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UNIVERSITÉ DE PICARDIE JULES VERNE  
FACULTÉ DE MATHÉMATIQUES ET D'INFORMATIQUE  
Laboratoire Amiénois de Mathématiques Fondamentale et  
Appliquée UMR CNRS 6140

**THÈSE**

En vue de l'obtention du

GRADE DE DOCTEUR DE L'UNIVERSITÉ DE PICARDIE  
JULES VERNE

Discipline : Mathématiques

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**ÉTUDES SUR LA RÉCURRENCE DE CERTAINS  
SYSTÈMES DYNAMIQUES TOPOLOGIQUES ET  
ARITHMÉTIQUES**

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**Lingmin LIAO**

Jury:

|                              |  |             |
|------------------------------|--|-------------|
| <b>M. BUGEAUD Yann</b>       | Prof. Université Louis Pasteur, Strasbourg | Rapporteur  |
| <b>M. CHABERT Jean-Luc</b>   | Prof. Université de Picardie, Amiens       | Examinateur |
| <b>M. DURAND Fabien</b>      | Prof. Université de Picardie, Amiens       | Examinateur |
| <b>M. FAN Ai-Hua</b>         | Prof. Université de Picardie, Amiens       | Directeur   |
| <b>M. HOST Bernard</b>       | Prof. Université de Marne-la-Vallée, Paris | Examinateur |
| <b>M. KAHANE Jean-Pierre</b> | Prof. Université Paris Sud, Orsay          | Président   |
| <b>M. SCHMELING Jörg</b>     | Prof. Lund University, Lund                | Rapporteur  |



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- (3) Generic points in systems of specification and Banach valued Birkhoff ergodic average, (avec A.-H. Fan et J. Peyrière), Discrete Contin. Dyn. Syst., 21 (2008) 1103-1128.
- (4) On Khintchine exponents and Lyapunov exponents of continued fractions, (avec A.-H. Fan, B.-W. Wang et J. Wu), Ergod. Th. Dynam. Sys., à paraître.
- (5) Dimension of some non-normal continued fraction sets, (avec J.-H. Ma et B.-W. Wang), Math. Proc. Camb. Phil. Soc., à paraître.
- (6) On the frequency of partial quotients of regular continued fractions, (avec A.-H. Fan et J.-H. Ma), prépublication.
- (7) Besicovitch-Eggleston sets in the countable symbolic space, (avec A.-H. Fan, J.-H. Ma et B.-W. Wang), prépublication.

**N.B. :** L'article (7) n'est pas joint dans la thèse.



## Résumé

Cette thèse est consacrée à l'étude de quelques aspects de la récurrence de trois classes de systèmes dynamiques : systèmes dynamiques  $p$ -adiques polynomiaux, systèmes topologiques ayant la propriété de spécification et système de Gauss associé aux fractions continues.

Dans une première partie, on étudie d'abord les polynômes à coefficients dans  $\mathbb{Z}_p$  d'ordre supérieur à 2, comme des systèmes dynamiques sur  $\mathbb{Z}_p$ . Nous prouvons que pour un tel système,  $\mathbb{Z}_p$  est composé des composants minimaux et de leurs bassins d'attraction. Pour tout polynôme quadratique sur  $\mathbb{Z}_2$ , nous exhibons tous ses composants minimaux. On étudie également les polynômes localement dilatants et transitifs. Nous montrons que la restriction d'un tel polynôme sur son ensemble de Julia est conjugué à un sous-shift de type fini.

Dans une deuxième partie, nous prouvons que pour un système dynamique compact ayant la propriété de spécification, l'entropie topologique de l'ensemble des points génériques d'une mesure invariante est égale à l'entropie de la mesure. En corollaire, nous établissons un principe variationnel pour le spectre d'entropie topologique des moyennes de Birkhoff à valeurs dans un espace de Banach.

La dernière partie est consacrée à l'étude des fractions continues. Nous travaillons en s'appuyant sur la théorie de l'opérateur de Ruelle, les spectres multifractals complets de l'exposant de Khintchine et de l'exposant de Lyapunov, qui ne sont ni concaves ni convexes. Notre résultat sur le spectre de Lyapunov complète celui de Pollicott et Weiss. Nous avons aussi bien étudié les fractions continues extrêmement non-normales et la fréquence des quotients partiels. Notre travail sur la fréquence complète complète celui de Billingsley et Henningsen.

**Mots clés:** systèmes dynamiques  $p$ -adiques, systèmes dynamiques topologiques, fractions continues, minimalité, points génériques, analyse multifractale, exposants de Khintchine et de Lyapunov



## Abstract

In this thesis, we study some aspects of recurrence in three classes of dynamical systems:  $p$ -adic polynomial dynamical systems, systems satisfying the specification property and Gauss dynamics associated to continued fractions.

In the first part, we begin with the study of polynomials with coefficients in  $\mathbb{Z}_p$  with order larger than 2 as dynamical systems on  $\mathbb{Z}_p$ . We prove that for such a system,  $\mathbb{Z}_p$  is decomposed into minimal components and their attracting basins. For any quadratic polynomial on  $\mathbb{Z}_2$ , we exhibit all its minimal components. We then study transitive locally expanding polynomials. We prove that such a polynomial, restricted to its Julia subset, is conjugate to a subshift of finite type.

In the second part, we prove that for a dynamical system with specification property, the topological entropy of the set of generic points of any invariant measure is equal to the entropy of the measure. Consequently, we obtain a variational principle for the topological entropy spectrum of Banach valued Birkhoff ergodic averages.

The last part is devoted to continued fractions. By applying the Ruelle operator theory, we obtain the multifractal spectra of the Khintchine exponent and Lyapunov exponent, which are neither concave nor convex. Our result on Lyapunov spectrum completes that of Pollicott and Weiss. We have also well studied the extremely non-normal continued fractions and the frequency of partial quotients. Our work on the frequency completes that of Billingsley and Henningsen.

**Key words:**  $p$ -adic dynamical systems, topological dynamical systems, continued fractions, minimality, generic points, multifractal analysis, Khintchine exponent, Lyapunov exponent.



# Contents

|  |            |
|--|------------|
| <b>Résumé</b>  | <b>v</b>   |
| <b>Abstract</b>  | <b>vii</b> |
| <b>Contents</b>  | <b>ix</b>  |
| <b>Chapter 1 Introduction</b>  | <b>1</b>   |
| 1.1 An overview of the thesis . . . . .                                      | 1          |
| 1.2 $p$ -adic dynamical systems . . . . .                                    | 6          |
| 1.2.1 $p$ -adic numbers and $p$ -adic dynamical systems . . . . .            | 6          |
| 1.2.2 Minimality of polynomial dynamical systems on $\mathbb{Z}_p$ . . . . . | 7          |
| 1.2.3 $p$ -adic repellers in $\mathbb{Q}_p$ . . . . .                        | 11         |
| 1.3 Entropy and Hausdorff dimension in dynamical systems . . . . .           | 12         |
| 1.3.1 Birkhoff averages and multifractal analysis . . . . .                  | 12         |
| 1.3.2 Level sets of Birkhoff averages in saturated systems . . . . .         | 14         |
| 1.3.3 Khintchine and Lyapunov Spectra in continued fractions . . . . .       | 16         |
| 1.3.4 Non-normal continued fractions . . . . .                               | 21         |
| 1.3.5 The Frequency of partial quotients in continued fractions . . . . .    | 23         |
| <b>Chapter 2 Preliminaries</b>   | <b>27</b>  |
| 2.1 Basic definitions and properties of dynamical systems . . . . .          | 27         |
| 2.1.1 Topological dynamical systems . . . . .                                | 27         |
| 2.1.2 Measure-preserving dynamical systems and ergodic theory . . . . .      | 29         |
| 2.1.3 Topological and measurable dynamical systems . . . . .                 | 31         |
| 2.2 $p$ -adic numbers, $p$ -adic analysis and $p$ -adic dynamics . . . . .   | 32         |
| 2.3 Entropy and Hausdorff dimension . . . . .                                | 35         |
| 2.3.1 Topological entropy and measure-theoretic entropy . . . . .            | 35         |

|   |   |           |
|---|---|-----------|
| 2.3.2   | Carathéodory structure and Hausdorff dimension . . . . .                  | 38        |
| 2.3.3   | Bowen's entropy and Bowen Lemma . . . . .                                 | 39        |
| 2.4   | Basic properties of continued fractions . . . . .                         | 40        |
| <b>Chapter 3 Minimality of the Polynomials on <math>\mathbb{Z}_p</math></b>     |   | <b>43</b> |
| 3.1   | Introduction . . . . .  | 43        |
| 3.2   | Induced dynamics on $\mathbb{Z}/p^n\mathbb{Z}_p$ ( $p \geq 3$ ) . . . . . | 46        |
| 3.3   | Induced dynamics on $\mathbb{Z}/p^n\mathbb{Z}_p$ ( $p = 2$ ) . . . . .    | 52        |
| 3.4   | Minimal decomposition . . . . .   | 56        |
| 3.5   | Conjugacy classes of Minimal subsystems . . . . .                         | 60        |
| 3.6   | 2-adic Quadratic Polynomials . . . . .                                    | 64        |
| <b>Chapter 4 The <math>p</math>-adic Repellers in <math>\mathbb{Q}_p</math></b> |   | <b>77</b> |
| 4.1   | Statement of results . . . . .  | 77        |
| 4.2   | Proofs . . . . .  | 79        |
| 4.3   | Examples. . . . .   | 83        |
| <b>Chapter 5 Level Sets of Birkhoff Averages in Saturated Systems</b>           |   | <b>85</b> |
| 5.1   | Introduction and statement of the results . . . . .                       | 85        |
| 5.2   | Preliminary . . . . .   | 90        |
| 5.3   | Systems with specification property are saturated . . . . .               | 91        |
| 5.3.1   | Dynamical Moran sets and their entropies . . . . .                        | 92        |
| 5.3.2   | Box-counting of $G_\mu$ . . . . .   | 95        |
| 5.3.3   | Saturatedness of systems with specification . . . . .                     | 99        |
| 5.4   | Variational principle . . . . .   | 106       |
| 5.4.1   | Proofs of Theorems 5.2 and 5.3 . . . . .                                  | 106       |
| 5.4.2   | $\ell^\infty(\mathbb{Z})$ -valued ergodic average . . . . .               | 108       |
| 5.5   | An example: recurrence in an infinite number of cylinders . . . . .       | 110       |
| 5.5.1   | Symbolic space . . . . .  | 110       |
| 5.5.2   | Example: Frequency of dyadic digital blocks . . . . .                     | 112       |

|                  |  |            |
|------------------|--|------------|
| <b>Chapter 6</b> | <b>Khinchine and Lyapunov Spectra</b>  | <b>119</b> |
| 6.1              | Statement of the results   | 119        |
| 6.2              | Preliminary  | 123        |
| 6.2.1            | Jacobian estimate and the level sets of Lyapunov exponents                         | 124        |
| 6.2.2            | Exponents $\gamma(x)$ and $\lambda(x)$   | 126        |
| 6.2.3            | Pointwise dimension  | 128        |
| 6.3              | Fast growth rate: proof of Theorem 6.1   | 129        |
| 6.3.1            | Lower bound  | 129        |
| 6.3.2            | Upper bound  | 133        |
| 6.4              | Ruelle operator theory   | 136        |
| 6.4.1            | Conformal infinite iterated function systems                                       | 136        |
| 6.4.2            | Continued fraction dynamical system  | 140        |
| 6.4.3            | Further study on $P(t, q)$   | 144        |
| 6.4.4            | Properties of $(t(\xi), q(\xi))$   | 145        |
| 6.5              | Khinchine spectrum   | 150        |
| 6.5.1            | Proof of Theorem 1.2 (1) and (2)   | 150        |
| 6.5.2            | Proof of Theorem 1.2 (3) and (4)   | 153        |
| 6.6              | Lyapunov spectrum  | 155        |
| <b>Chapter 7</b> | <b>Non-normal Continued Fractions</b>  | <b>163</b> |
| 7.1              | Statement of the results   | 163        |
| 7.2              | Some combinatorics of words and the relation between $\mathbb{E}$ and $\mathbb{F}$ | 166        |
| 7.3              | A “good seed” for $\mathbb{E}$   | 168        |
| 7.4              | Upper bound estimate   | 171        |
| 7.5              | Lower bound estimate   | 172        |
| <b>Chapter 8</b> | <b>The Frequency of Partial Quotients of Continued Fractions</b>                   | <b>175</b> |
| 8.1              | Introduction   | 175        |
| 8.2              | Preliminary  | 177        |
| 8.3              | Upper bound  | 179        |

|       |                                    |            |
|-------|------------------------------------|------------|
| 8.3.1 | Some Lemmas . . . . .              | 179        |
| 8.3.2 | Proof of the upper bound . . . . . | 181        |
| 8.4   | Lower bound . . . . .              | 185        |
|       | <b>Bibliography</b>                | <b>189</b> |

# Chapter 1

## Introduction

### 1.1 An overview of the thesis

Let  $X$  be an abstract space,  $G$  be a semigroup and  $T_t : X \rightarrow X$  ( $t \in G$ ) be a family of transformations on  $X$  satisfying  $T_{s+t} = T_s \circ T_t$ . Then  $\Gamma := \{T_t\}_{t \in G}$  defines a semigroup action of  $G$  on  $X$ . We call  $(X, \Gamma)$  a dynamical system. In this thesis, we study some discrete dynamical systems, where the semigroup  $G$  is generated by a single transformation  $T : X \rightarrow X$ . In this case  $(X, \Gamma)$  will also be denoted by  $(X, T)$ .

There are two ways to study dynamical systems: topological way and measure-theoretic way, which correspond respectively to two theories: Theory of Topological Dynamical Systems and Ergodic Theory.

In a topological dynamical system, the state space  $X$  is usually supposed to be a compact space ( in the present thesis we will also study some noncompact topological spaces) and  $T$  a homeomorphism or surjective continuous transformation (in the present thesis, we will also consider some piecewise continuous transformations). The study of topological dynamical systems concerns properties like periodicity, recurrence, wandering, transitivity, minimality, mixing property, chaotic property and so on. In the ergodic theory, the state space  $X$  is supposed to admit a measurable structure and  $T$  is a measure-preserving transformation. There are many concepts in the ergodic theory, which are similar to those in topological dynamical systems. Two of major results in the ergodic theory are Birkhoff ergodic Theorem ([18], 1931) and Von Neumann's mean ergodic theorem ([138], 1932).

There are also two ways to classify dynamical systems through topological conjugacy or measure-theoretic conjugacy, which are related to two important conjugate invariants: topological entropy and measure-theoretic entropy. The measure-theoretic entropy was introduced into the ergodic theory by Kolmogorov

([91], 1958) and the topological entropy was introduced by Adler, Konheim and McAndrew ([1], 1965). Ornstein ([115], 1970) showed that measure-theoretic entropy characterizes Bernoulli systems. A shortcoming of the entropy is that it can not distinguish systems with zero entropy.

In this thesis we will focus on the study of the minimality and the chaotic property of  $p$ -adic dynamical systems, on the study of the topological entropies of sets of generic points in topological dynamical systems satisfying specification and on the study of recurrence of Gauss dynamics associated to continued fractions.

The thesis is divided into three parts.

In the first part, we study iterations of a polynomial of degree larger than 2 with coefficients in the field  $\mathbb{Q}_p$  of  $p$ -adic numbers ( $p$  being a prime).

If the coefficients of the polynomial  $f$  are in the ring  $\mathbb{Z}_p$  of  $p$ -adic integers, i.e.  $f \in \mathbb{Z}_p[x]$ , then  $f : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$  is a 1-Lipschitz map and  $(\mathbb{Z}_p, f)$  becomes a compact topological dynamical system with zero entropy. We will study the minimality of the whole system  $(\mathbb{Z}_p, f)$  and the minimal subsystems of  $(\mathbb{Z}_p, f)$  when the whole system  $(\mathbb{Z}_p, f)$  is not minimal. Remark that the minimality of  $f$  on a compact and open invariant subset of  $\mathbb{Z}_p$  is equivalent to the ergodicity, uniquely ergodicity and strictly ergodicity, with the normalized Haar measure as the invariant measure (see Proposition 2.24).

One of our main theorems (Theorem 3.18) asserts that if  $(\mathbb{Z}_p, f)$  is a polynomial dynamical system with  $f \in \mathbb{Z}_p[x]$  and  $\deg f \geq 2$ , then the space  $\mathbb{Z}_p$  is decomposed into three parts:

$$\mathbb{Z}_p = A \sqcup B \sqcup C,$$

where  $A$  is the finite set (maybe empty) consisting of all periodic points of  $f$ ,  $B = \bigsqcup_i B_i$  is the union of all (at most countably many) clopen invariant sets such that each subsystem  $f : B_i \rightarrow B_i$  is minimal, and each point in  $C$  (if any) lies in the attracting basin of a periodic orbit or of a minimal component  $B_i$ .

According to the above result, the minimality of the whole system  $(\mathbb{Z}_p, f)$  means that there is only one minimal component in  $B$  and  $A = C = \emptyset$ . Thus we face two problems: one is determine the minimality of the system  $(\mathbb{Z}_p, f)$ , the other is to find the exact decomposition, especially all the components of  $B$ , when  $(\mathbb{Z}_p, f)$  is not minimal.

People have mainly worked on the first problem, which we will discuss in Subsection 1.2.2. The second problem for the affine transformation has been solved by Fan, Li, Yao and Zhou ([50]). This decomposition problem seems much more difficult for higher order polynomials. In this thesis, we can solve the problem for the system  $(\mathbb{Z}_2, f)$  where  $f \in \mathbb{Z}_2[x]$  is a quadratic polynomial (Theorem 3.23-3.30).

The other class of  $p$ -adic polynomial dynamical systems we would like to study consists of transitive locally expanding polynomial dynamical systems in the field  $\mathbb{Q}_p$ . Let  $f \in \mathbb{Q}_p[x]$ . Consider  $f : X \rightarrow \mathbb{Q}_p$  where  $X \subset \mathbb{Q}_p$  is a compact open set of  $\mathbb{Q}_p$ . We define the so-called Julia set  $J_f = \bigcap_{n=0}^{\infty} f^{-n}(X)$ . It is clear that  $f^{-1}(J_f) = J_f$  and then  $f(J_f) \subset J_f$ . We are interested in the dynamical system  $(J_f, f)$ . If  $f$  is transitive and locally expanding, the system  $(J_f, f)$  is proved to be conjugate to some subshift of finite type (Theorem 4.1). Actually the same result holds for some more general  $p$ -adic dynamical systems in  $\mathbb{Q}_p$  called  $p$ -adic repellers in  $\mathbb{Q}_p$ , which are not necessarily polynomial (Chapter 4).

In the rest parts of this thesis, we are concerned with interesting invariant subsets in a dynamical system. These subsets are related to Birkhoff averages of real or Banach valued functions. Their sizes are described either by Bowen's generalized topological entropy or by Hausdorff dimension.

Let  $(X, T)$  be a dynamical system. In the cases we study,  $X$  is usually a metric space and even compact, and  $T$  is continuous or piecewise continuous. Let  $\Phi : X \rightarrow \mathbb{R}$  be a real valued function. The Birkhoff averages of  $\Phi$  are defined by

$$A_n \Phi(x) := \frac{1}{n} \sum_{j=0}^{n-1} \Phi(T^j x) \quad (n \geq 1).$$

We are interested in the level sets of the limit of Birkhoff averages of  $\Phi$ :

$$X_\Phi(\alpha) := \left\{ x : \lim_{n \rightarrow \infty} A_n \Phi(x) = \alpha \right\}, \quad \alpha \in \mathbb{R} \cup \{-\infty, \infty\}.$$

We would like to use topological entropy and/or Hausdorff dimension to measure the sizes of  $X_\Phi(\alpha)$ .

We are also interested in the set of divergent points

$$D := \{x : \text{the limit of } A_n \Phi(x) \text{ does not exist}\},$$

and subsets of  $D$  defined by different asymptotic behaviors of  $A_n \Phi(x)$ .

In the case  $X_\Phi(\infty) \neq \emptyset$ , we consider finer structures of the set  $X_\Phi(\infty)$ . So, to this end, we consider the level sets of fast Birkhoff averages of polynomial speed:

$$X_\Phi(\alpha, \beta) := \left\{ x : \lim_{n \rightarrow \infty} \frac{1}{n^\beta} \sum_{j=0}^{n-1} \Phi(T^j x) = \alpha \right\} \quad (\beta > 1),$$

or of exponential speed:

$$X_\Phi(\alpha, a) := \left\{ x : \lim_{n \rightarrow \infty} \frac{1}{a^n} \sum_{j=0}^{n-1} \Phi(T^j x) = \alpha \right\} \quad (a > 1).$$

The level sets of Birkhoff averages of a Banach-valued function  $\Phi$  are also studied (Theorem 5.2 and Theorem 5.3).

In the second part of the thesis (Chapter 5), we consider a topological dynamical system  $(X, T)$ , where  $X$  is a compact metric space and  $T : X \rightarrow X$  is a continuous transformation. We suppose that  $(X, T)$  satisfies specification property (see the definition in Subsection 1.3.2). Let  $\mathbb{B}$  be a real Banach space and  $\mathbb{B}^*$  its dual space. Let  $\Phi : X \rightarrow \mathbb{B}^*$  be a  $\mathbb{B}^*$ -valued continuous function. Given  $\alpha \in \mathbb{B}^*$ , we study the level sets  $X_\Phi(\alpha)$  of the Birkhoff averages of  $\Phi$ , where the limit,  $\lim_{n \rightarrow \infty} A_n \Phi(x)$ , is in the sense of the weak star topology  $\sigma(\mathbb{B}^*, \mathbb{B})$ . The topological entropies of the level sets  $X_\Phi(\alpha)$  for  $\alpha \in \mathbb{B}^*$  are determined by a variational principle (Theorem 5.2 and Theorem 5.3). More significant is that a system satisfying specification property is saturated in the sense that the topological entropy of the set of generic points with respect to an invariant measure is equal to the measure-theoretic entropy of the measure (Theorem 5.1). This is a converse theorem under specification of a Theorem due to Bowen [24]. We illustrate our results by applying them to study the frequencies of blocks in the dyadic development of real numbers.

The third part of the thesis (the last three chapters) is devoted to continued fractions. We consider the Gauss transformation  $T$  on  $[0, 1)$  (for the definition see Subsection 1.3.3). Define  $a_1(x) = \lfloor 1/x \rfloor$ , then  $a_n(x) = a_1(T^{n-1}(x))$  for  $n \geq 2$ . The  $a_n(x)$ 's are the partial quotients of continued fraction expansion of  $x = [a_1(x), a_2(x), \dots] \in \mathbb{Q}^c \cap [0, 1]$ .

In Chapter 6, the problem that we study corresponds to  $\Phi(x) = \log a_1(x)$  or  $\log |T'(x)|$ . We call  $\gamma(x) := \lim_{n \rightarrow \infty} A_n \log a_1(x)$  the Khintchine exponent of  $x$ , if

the limit exists. We call  $\lambda(x) := \lim_{n \rightarrow \infty} A_n \log |T^n|(x)$  the Lyapunov exponent of  $x$ , if the limit exists. The Hausdorff dimensions of the level sets of Khintchine exponents and Lyapunov exponents, which are called Khintchine spectrum and Lyapunov spectrum respectively, are obtained (Theorem 6.2 and Theorem 6.3). In both cases of Khintchine exponents and Lyapunov exponents, we have  $X_\Phi(\infty) \neq \emptyset$ . Thus we are led to study the level sets of fast Khintchine exponents and Lyapunov exponents (Theorem 6.1).

In Chapter 7, we consider the frequencies of blocks of partial quotients (also called words) in continued fractions. Let  $(W_i)_{i \geq 1}$ ,  $W_i \in \mathbb{N}^*$  be a sequence of all words of finite length (in some fixed order), where  $\mathbb{N}^*$  stands for the set of all finite words in  $\mathbb{N}^{\mathbb{N}}$ . Let  $\Phi(x) = (\mathbb{1}_{W_i}(a_1(x) \cdots a_{|W_i|}(x)))_{i \geq 1}$ , where  $\mathbb{1}_{W_i}$  stands for the indicator function of the word  $W_i$ . We are interested in the set of points such that  $A_n \Phi(x)$  extremely oscillate. Denote by  $\text{acc}(A_n \Phi(x))$  the set of accumulation points of  $A_n \Phi(x)$  in the weak star topology of  $\ell^\infty$ . We study the set

$$\mathbb{F} := \left\{ x \in \mathbb{Q}^c \cap [0, 1] : \text{acc}(A_n \Phi(x)) = \bigcup_{y \in \mathbb{Q}^c \cap [0, 1]} \text{acc}(A_n \Phi(y)) \right\}.$$

We prove that  $\dim_H(\mathbb{F}) = 1/2$  (Theorem 7.2). We also prove that the Hausdorff dimension of the set of extreme non-normal continued fractions defined by Olsen ([113]), which is a subset of  $\mathbb{F}$ , is also  $1/2$  (Theorem 7.4). The Hausdorff dimension of this set was previously conjectured to be 0.

In Chapter 8, we take  $\Phi = (\mathbb{1}_{[1]}, \mathbb{1}_{[2]}, \dots)$ , where  $\mathbb{1}_{[j]}$  stands for the indicator function of the cylinder  $[j]$  (i.e., the fundamental interval  $[1/(j+1), 1/j)$ ). Here we are interested in the frequencies of partial quotients. Let  $\vec{p} = (p_1, p_2, \dots)$  be a probability vector. We study the level set  $X_\Phi(\vec{p})$ , which is the set of points for which the frequency of the digit  $j$  in the continued fraction expansion of  $x = [a_1, a_2, \dots]$  is equal to  $p_j$  for each  $j \geq 1$ . We compute the exact Hausdorff dimension of  $X_\Phi(\vec{p})$  (Theorem 8.1). Partial results were obtained by Billingsley and Henningsen. The results as well as the methods are different from those in the study of the Khintchine and Lyapunov exponents.

In the following we will detail the above presentation.

## 1.2 $p$ -adic dynamical systems

Before we present our results on  $p$ -adic dynamical systems in Subsections 1.2.2 and 1.2.3, we start with some history remarks on  $p$ -adic numbers and  $p$ -adic dynamical systems.

### 1.2.1 $p$ -adic numbers and $p$ -adic dynamical systems

The  $p$ -adic numbers were introduced by Hensel in 1897. Simply speaking, the field of  $p$ -adic numbers  $\mathbb{Q}_p$  is the topological completion of the field  $\mathbb{Q}$  with respect to some non-Archimedean absolute value ( $p$ -adic absolute value  $|\cdot|_p$ ). The ring of  $p$ -adic integers  $\mathbb{Z}_p$  is the unique maximal ideal in  $\mathbb{Q}_p$ . In 1918, Ostrowski ([117]) proved that every non-trivial absolute value on  $\mathbb{Q}$  is either equivalent to the classical Archimedean absolute value or to one of the  $p$ -adic absolute value. We can consider the algebraic extensions of  $\mathbb{Q}_p$ . The absolute value can also be uniquely extended. However, any extension of finite order of  $\mathbb{Q}_p$  is not algebraically closed. Thus the algebraic closure  $\mathbb{Q}_p^a$  of  $\mathbb{Q}_p$  is an infinite extension. But  $\mathbb{Q}_p^a$  is not topologically complete. Fortunately, the topological completion of  $\mathbb{Q}_p^a$  is algebraically closed. This ultimate field, denoted by  $\mathbb{C}_p$ , is called the field of complex  $p$ -adic numbers.

In 1987, Volovich ([139]) applied the  $p$ -adic numbers to establish the  $p$ -adic string theory. The  $p$ -adic physical investigations then stimulated people's high interests in  $p$ -adic dynamical systems. Earlier work of Oselies and Zieschang ([116]) appearing in 1975, concerned with the continuous automorphisms of the ring of  $p$ -adic integers  $\mathbb{Z}_p$  which are the multiplication transformations  $ax$  with  $a$  a unit in  $\mathbb{Z}_p$ . It seems to be the first study of  $p$ -adic dynamical systems. With the aim of studying the distribution of Fibonacci numbers, Coelho and Parry ([34]) also studied these multiplications. The power transformations  $x^n$  were studied by Gundlach, Khrennikov and Lindahl ([63]). The first work about the complex  $p$ -adic dynamical systems might be due to Herman and Yoccoz ([69]) in 1983. In 1994, Lubin ([99]) studied the iteration of analytic  $p$ -adic maps.

In the following, we will discuss two classes of transformations, 1-Lipschitz transformations and expansive transformations.

### 1.2.2 Minimality of polynomial dynamical systems on $\mathbb{Z}_p$

Let  $f \in \mathbb{Z}_p[x]$  be a polynomial with coefficients in  $\mathbb{Z}_p$ . It is a 1-Lipschitz transformation from  $\mathbb{Z}_p$  to  $\mathbb{Z}_p$ . We consider the compact topological dynamical system  $(\mathbb{Z}_p, f)$ .

As we have mentioned, we will prove that

**Theorem 1.1** (Theorem 3.18). *Let  $f \in \mathbb{Z}_p[x]$  with  $\deg f \geq 2$ . We have the following decomposition*

$$\mathbb{Z}_p = A \bigsqcup B \bigsqcup C$$

where  $A$  is the finite set consisting of all periodic points of  $f$ ,  $B = \bigsqcup_i B_i$  is the union of all (at most countably many) clopen invariant sets such that each subsystem  $f : B_i \rightarrow B_i$  is minimal, and each point in  $C$  lies in the attracting basin of a periodic orbit or of a minimal subsystem.

Two problems arise:

*Problem 1.* Under what conditions, the decomposition of  $(\mathbb{Z}_p, f)$  consists of only one piece of component in  $B$  and  $A = C = \emptyset$  i.e., the whole system  $(\mathbb{Z}_p, f)$  is minimal ?

*Problem 2.* If  $(\mathbb{Z}_p, f)$  is not minimal, how to find the exact and complete decomposition?

Concerning the first problem, a general criterion of minimality was given by Larin ([95]). He showed that for  $p \geq 5$ , the system  $(\mathbb{Z}_p, f)$  is minimal if and only if  $f$  is minimal modulo  $p^2$ ; and for  $p = 2$  or  $3$ , the system  $(\mathbb{Z}_p, f)$  is minimal if and only if  $f$  is minimal modulo  $p^3$ .

In a recent work, Chabert, Fan and Fares ([33]) obtained a criterion of minimality for 1-Lipschitz maps on Legendre sets in any discrete valuation domain.

There exist practical criteria of minimality by using the coefficients of the polynomial. Larin ([95]) showed that when  $p = 2$ , the system  $(\mathbb{Z}_2, f)$  where  $f(x) = \sum a_k x^k \in \mathbb{Z}_2[X]$  is minimal if and only if

$$\begin{aligned} a_0 &\equiv 1 \pmod{2}, \\ a_1 &\equiv 1 \pmod{2}, \\ 2a_2 &\equiv a_3 + a_5 + \cdots \pmod{4}, \\ a_2 + a_1 - 1 &\equiv a_4 + a_6 + \cdots \pmod{4}. \end{aligned}$$

For general polynomials in the case of  $p \geq 3$ , people don't know any necessary and sufficient condition for minimality by using the coefficients.

However, for special classes of polynomials, there are complete solutions. For example, it is proved (Larin [95], Fan, Li, Yao and Zhou [50]) that an affine map  $f(x) = ax + b$  with  $a, b \in \mathbb{Z}_p$ , is minimal if and only if

- (1)  $a \equiv 1 \pmod{p}$  when  $p \geq 3$  or  $a \equiv 1 \pmod{p^2}$  when  $p = 2$ ,
- (2)  $b \not\equiv 0 \pmod{p}$ .

For a quadratic polynomial  $f(x) = ax^2 + bx + c$  with  $a, b, c \in \mathbb{Z}_p$ , an exercise in Knuth ([89], p.32, Exercise 8), together with Larin's general criterion cited above or Theorem 2.25, implies that the system  $(\mathbb{Z}_p, f)$  is minimal if and only if

- (1)  $a \equiv 0 \pmod{p}, b \equiv 1 \pmod{p}, c \not\equiv 0 \pmod{p}$ , if  $p \geq 5$ ,
- (2)  $a \equiv 0 \pmod{9}, b \equiv 1 \pmod{3}, c \not\equiv 0 \pmod{3}$  or  
 $ac \equiv 6 \pmod{9}, b \equiv 1 \pmod{3}, c \not\equiv 0 \pmod{3}$ , if  $p \geq 3$ .
- (3)  $a \equiv 0 \pmod{2}, a + b \equiv 1 \pmod{4}$  and  $c \not\equiv 0 \pmod{2}$ , if  $p = 2$ .

It is well-known that any polynomial minimal system  $(\mathbb{Z}_p, f)$  is topologically conjugate to the adding machine  $(\mathbb{Z}_p, x + 1)$ , and the conjugacy is an isometry. See, for example, [33], where Chabert, Fan and Fares obtained a more general result on the minimality of a 1-Lipschitz map restricted on an invariant clopen set in any discrete valuation domain).

Coelho and Parry ([34]) studied the minimality of the multiplication  $f(x) := ax$  ( $a \in \mathbb{Z}_p$ ) on the group of units  $\mathbb{U} := \{x \in \mathbb{Z}_p : |x|_p = 1\}$  for  $p \geq 3$  instead of on the whole ring  $\mathbb{Z}_p$ . Here and in the follows  $|\cdot|_p$  is the  $p$ -adic absolute value. They showed that for  $a \in \mathbb{U}$ ,  $(\mathbb{U}, f|_{\mathbb{U}})$  is minimal if and only if  $a \pmod{p^2}$  generates the multiplicative group  $(\mathbb{Z}/p^2\mathbb{Z})^\times$ . For  $p = 2$ , Fan, Li, Yao and Zhou ([50]) showed that the system  $(\mathbb{U}_2, f|_{\mathbb{U}_2})$  where  $\mathbb{U}_2 := 1 + 4\mathbb{Z}_2$ , is minimal if and only if  $\pm a \pmod{8}$  generates the multiplicative group  $(\mathbb{Z}/8\mathbb{Z})^\times$ .

Gundlach, Khrennikov and Lindahl ([63]) studied the minimality of  $f(x) = x^n$  ( $n \geq 1$ ) on each ball  $1 + p^k\mathbb{Z}_p, k \geq 1$ , a subgroup of  $\mathbb{U}$ . After the work [63], Anashin ([5]) showed that for  $p \geq 3$ , and  $f(x) = x^n + p^{k+1}u(x)$  with  $k \geq 2$  and  $u \in \mathbb{Z}_p[X]$ , the transformation  $f$  restricted on  $1 + p^k\mathbb{Z}_p$  is minimal if and only if  $n$  is a generator of  $(\mathbb{Z}/p^2\mathbb{Z})^\times$ . Fan, Li, Yao and Zhou ([50]) showed that the system

$(\mathbb{U}_1, f|_{\mathbb{U}_1})$  where  $f(x) = ax^n$  ( $a \in \mathbb{Z}_p, n \geq 1$ ) and  $\mathbb{U}_1 = 1 + p\mathbb{Z}_p$  is topologically conjugate to the system  $(\mathbb{Z}_p, g)$  where  $g(x) = nx + \text{Log}(a)/\text{Log}(1+p)$ , and the logarithmic function is defined as the classical one:

$$\text{Log}(1+x) := \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}, \quad (|x|_p < 1).$$

Now let us look at the second problem to which the thesis will bring some contribution. Fan, Li, Yao and Zhou ([50]) gave an explicit and complete picture of the minimal decomposition of  $\mathbb{Z}_p$  ( $p \geq 2$ ) for affine transformations. This was one of our motivations to study polynomial of higher degree. We recall here some part of their results in the case  $p = 2$ . Let  $f(x) = ax$  ( $a \in \mathbb{Z}_2$ ). Denote  $\mathbb{U}_n = 1 + 2^{n+1}\mathbb{Z}_2$  ( $n \geq 0$ ). Then we have  $\mathbb{Z}_p = A \sqcup B \sqcup C$  where  $A = \{0\}$  consists of the unique fixed point,  $C$  is empty and  $B = \bigcup_{n \geq 0} 2^n \mathbb{U}$ . Furthermore, each  $(2^n \mathbb{U}, f|_{2^n \mathbb{U}})$  ( $n \geq 1$ ) is conjugate to  $(\mathbb{U}, f|_{\mathbb{U}})$ . About the subsystem  $(\mathbb{U}, f|_{\mathbb{U}})$ , we have

- (1) if  $a \in \mathbb{U}_2$ , then  $\mathbb{U}$  is decomposed into  $|a-1|_2^{-1}/2$  minimal components,
- (2) if  $a \in \mathbb{U} \setminus \mathbb{U}_2$ , then  $\mathbb{U}$  is decomposed into  $|a+1|_2^{-1}/2$  minimal components.

It seems much more difficult to study the decomposition for polynomials of higher order, even for quadratic polynomials. However, we can exploit the ideas from Desjardins and Zieve ([40]), and study the minimal decomposition of general polynomials (Theorem 1.1).

In Chapter 3, we will give a complete picture of the minimal decomposition of any 2-adic quadratic polynomial dynamical system. As we will see, we can reduce the problem to the quadratic polynomials of the following form

$$x^2 - \lambda, \quad x^2 + bx, \quad x^2 + x - d,$$

where  $\lambda \in \mathbb{Z}_2$ ,  $b \equiv 1 \pmod{2}$  and  $\sqrt{d} \notin \mathbb{Z}_2$ . The obtained results are stated in Theorems 3.22-3.30. We state here only one of them.

**Theorem 1.2** (Theorem 3.23). *Any 2-adic quadratic polynomial dynamical system which has only one fixed point in  $\mathbb{Z}_2$  is topologically conjugate to the transformation  $x^2 + x$  on  $\mathbb{Z}_2$ . For the dynamical system  $(\mathbb{Z}_2, x^2 + x)$ , we have*

- (1) the set  $1 + 2\mathbb{Z}_2$  is mapped into  $2\mathbb{Z}_2$ .

(2) the set  $2\mathbb{Z}_2$  is decomposed into

$$2\mathbb{Z}_2 = \{0\} \sqcup \left( \bigsqcup_{n \geq 2} 2^{n-1} + 2^n \mathbb{Z}_2 \right),$$

and for each  $n \geq 2$ ,  $2^{n-1} + 2^n \mathbb{Z}_2$  consists of  $2^{n-2}$  pieces of minimal components:

$$2^{n-1} + t2^n + 2^{2n-2} \mathbb{Z}_2, \quad t = 0, \dots, 2^{n-2} - 1.$$

In this case,  $A = \{0\}$ ,  $C = 1 + 2\mathbb{Z}_2$  and  $B = 2\mathbb{Z}_2$  consists of infinite minimal components  $2^{n-1} + t2^n + 2^{2n-2} \mathbb{Z}_2$  ( $n \geq 2, 0 \leq t < 2^{n-2}$ ).

In [33], Chabert, Fan and Fares recently proved that minimal sets of any 1-Lipschitz map are Legendre sets. We will prove that minimal sets of a polynomial are special Legendre sets. A set  $E \subset \mathbb{Z}_p$  is a *Legendre set* if for any  $s \geq 1$  and any  $x \in E/p^s \mathbb{Z}_p$ , the number

$$q_s := \text{Card} \{y \in E/p^{s+1} \mathbb{Z}_p : y \equiv x \pmod{p^s}\}$$

is independent of  $x \in E/p^s \mathbb{Z}_p$ . Let

$$p_s := q_1 q_2 \cdots q_s \quad (\forall s \geq 1).$$

It is clear that  $p_s = \text{Card } E/p^s \mathbb{Z}_p$ . We call  $(p_s)_{s \geq 1}$  the *structure sequence* of  $E$ . Consider the inverse limit

$$\mathbb{Z}_{(p_s)} := \varprojlim \mathbb{Z}/p_s \mathbb{Z}.$$

This is a profinite group, usually called an *odometer*, and the map  $\tau : x \mapsto x + 1$  is called the *adding machine* on  $\mathbb{Z}_{(p_s)}$ .

Chabert, Fan and Fares proved the following theorem.

**Theorem 1.3** ([33]). *Let  $E$  be a clopen set in  $\mathbb{Z}_p$  and  $f : E \rightarrow E$  be a 1-Lipschitz map. If the dynamical system  $(E, f)$  is minimal, then  $f$  is an isometry,  $E$  is a Legendre set and the system  $(E, f)$  is conjugate to the adding machine on  $(\mathbb{Z}_{(p_s)}, \tau)$  where  $(p_s)$  is the structure sequence of  $E$ . On the other hand, on any Legendre set there exists at least one minimal map.*

We improve the above result in the case of polynomials by giving more information on the structure sequences of polynomial minimal components.

**Theorem 1.4** (Theorem 3.21). *Let  $f \in \mathbb{Z}_p[x]$  with  $\deg f \geq 2$ . If  $E$  is a minimal clopen invariant set of  $f$ , then  $f : E \rightarrow E$  is conjugate to the adding machine on an odometer  $\mathbb{Z}_{(p_s)}$ , where*

$$(p_s) = (k, kd, kdp, kdp^2, \dots)$$

with some  $k \in \mathbb{N}, k \leq p$  and  $d|(p-1)$ .

In [57], Fan and Wang asked for the form of Legendre sets on which there is a minimal polynomial of given degree. The above theorem, which gives all the possible forms of Legendre sets on which there are minimal polynomials with coefficients in  $\mathbb{Z}_p$ , partially answers Fan and Wang's question.

### 1.2.3 $p$ -adic repellers in $\mathbb{Q}_p$

Now let us consider a class of expansive transformations on the field  $\mathbb{Q}_p$ . They are quite different from the polynomial dynamical systems on  $\mathbb{Z}_p$  that we have just studied. They exhibit chaotic behaviors.

Let  $f : X \rightarrow \mathbb{Q}_p$  be a map from a compact open set  $X$  of  $\mathbb{Q}_p$  into  $\mathbb{Q}_p$ . We assume that (i)  $f^{-1}(X) \subset X$ ; (ii)  $X = \bigsqcup_{i \in I} B_{p^{-\tau}}(c_i)$  can be written as a finite disjoint union of balls of centers  $c_i$  and of the same radius  $p^{-\tau}$  (with some  $\tau \in \mathbb{Z}$ ) such that for each  $i \in I$  there is an integer  $\tau_i \in \mathbb{Z}$  satisfying

$$|f(x) - f(y)|_p = p^{\tau_i} |x - y|_p \quad (\forall x, y \in B_{p^{-\tau}}(c_i)). \quad (1.1)$$

For such a map  $f$ , define its Julia set by  $J_f = \bigcap_{n=0}^{\infty} f^{-n}(X)$ . It is clear that  $f^{-1}(J_f) = J_f$  and then  $f(J_f) \subset J_f$ . We are interested in the dynamical system  $(J_f, f)$ .

The triple  $(X, J_f, f)$  is called a  $p$ -adic weak repeller if all  $\tau_i$  in (1.1) are nonnegative, but at least one is positive. We call it a  $p$ -adic repeller if all  $\tau_i$  in (1.1) are positive. For convenience, we will write  $\|f\| = p^{\tau_i}$  for any map having the property (1.1), which could be called the expanding ratio (resp. contractive ratio) of  $f$  on the ball  $B_{p^{-\tau}}(c_i)$  when  $\tau_i \geq 0$  (resp.  $\tau_i \leq 0$ ).

We have the following theorem.

**Theorem 1.5** (Theorem 4.1). *Any transitive  $p$ -adic weak repeller  $(X, J_f, f)$  is isometrically conjugate to some subshift of finite type endowed with a suitable metric.*

For the definitions of transitivity of the  $p$ -adic weak repeller and of the sub-shift, see Chapter 4.

Many polynomials in  $\mathbb{Q}_p[X]$  belong to the class of  $p$ -adic weak repellers. One class is of the form  $f(x) = p^{-m}P(x) \in \mathbb{Q}_p[x]$  with  $m > 0$  and  $P \in \mathbb{Z}_p[x]$  satisfying  $|P(x)|_p \geq |x|_p$  for all  $x \notin \mathbb{Z}_p$  and  $f'(x) \neq 0$  for all  $x \in X$  (by Theorem 4.2). Some concrete examples are fully examined at the end of Chapter 4.

In the literature, Thiran, Versteegen and Weyers [137] and Dremov, Shabat and Vytnova [41] studied the chaotic behavior of  $p$ -adic quadratic polynomial dynamical systems. Woodcock and Smart [147] proved that the so-called  $p$ -adic logistic map

$$\frac{x^p - x}{p} : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$$

is topologically conjugate to the full shift on the symbolic system with  $p$  symbols.

### 1.3 Entropy and Hausdorff dimension in dynamical systems

In the other parts of this thesis, we are concerned with following subjects, each of which corresponds to one chapter:

- (1) Entropies of level sets of Birkhoff averages for continuous functions in a compact topological dynamical system with specification property;
- (2) Hausdorff dimensions spectrum (from the multifractal point of view) of Khintchine exponents and Lyapunov exponents in continued fractions;
- (3) Sizes of some sets of non-normal continued fractions;
- (4) Frequency of partial quotients (described by Besicovitch-Eggleston sets) in continued fractions.

These studies are tightly related to Multifractal Analysis. So, we begin with some historic remarks on the study of Birkhoff averages and multifractal analysis.

#### 1.3.1 Birkhoff averages and multifractal analysis

Let  $(X, T)$  be a dynamical system with  $X$  a metric space and  $T$  a piecewise continuous transformation. Let  $\Phi$  be a continuous function taking values in  $\mathbb{R}$  or

in some real Banach space. As mentioned in Section 1.1, we are interested in level sets  $X_\Phi(\alpha)$  of Birkhoff averages of  $\Phi$ , subsets of the set  $D$  of divergent points, and level sets of fast Birkhoff averages,  $X_\Phi(\alpha, \beta)$  and  $X_\Phi(\alpha, a)$  if  $X_\Phi(\infty) \neq \emptyset$ . We will adopt two concepts, the entropy and Hausdorff dimension to measure the sizes of these sets.

As we know, entropies describe the complexity of dynamical systems. There are two kinds of entropies: measure-theoretic entropy and topological entropy. In 1973, Bowen ([24]) generalized the concept of topological entropy to any subset of the state space.

Denote respectively by  $h_{\text{top}}(A)$  and  $\dim_H(A)$  the topological entropy and Hausdorff dimension of a set  $A$ . The functions  $f(\alpha) := h_{\text{top}}(X_\Phi(\alpha))$  and  $g(\alpha) := \dim_H(X_\Phi(\alpha))$  are called multifractal spectra of  $\Phi$ .

In many well known cases, the spectra  $f(\alpha)$  and  $g(\alpha)$  obey the following formula (called variational principle)

$$h_{\text{top}}(X_\Phi(\alpha)) = \sup \left\{ h_\mu : \mu \text{ invariant and } \int \Phi d\mu = \alpha \right\}.$$

$$\dim_H(X_\Phi(\alpha)) = \sup \left\{ \frac{h_\mu}{\int \log |T'(x)| d\mu} : \mu \text{ invariant and } \int \Phi d\mu = \alpha \right\}.$$

In the literature, Fan and Feng ([48, 49]) proved a variational principle when  $(X, T)$  is a mixing subshift of finite type and  $\Phi$  is a continuous  $\mathbb{R}^d$ -valued function valued. Barreira, Saussol and Schmeling ([9]) obtained a variational principle for Hölder functions. Barreira and Saussol ([8]) established a variational principle in the case where  $T$  is a transformation with upper semi-continuous entropy, and  $\Phi$  is a function with a unique equilibrium. The case where  $T$  is  $\mathcal{C}^{1+\delta}$ -conformal mixing transformation and  $\Phi$  is continuous was studied by Feng, Lau and Wu ([58]). Takens and Verbitskiy ([135]) generalized the variational principle to systems satisfying specification.

Concerning the subsets of the set  $D$  of divergent points, Barreira and Schmeling ([7]) showed that for mixing subshifts of finite type and for a Hölder continuous  $\Phi$  which is not cohomologous to zero, the set  $D$  admits full entropy and Hausdorff dimension. Fan, Feng and Wu ([49]) proved that on a mixing subshift of finite type, if the Birkhoff averages of the continuous function  $\Phi$  do not have the same

limit for all points, then the set  $D$  has full Hausdorff dimension. Olsen([114]) considered the set of points whose accumulation points of Birkhoff average are prescribed. He obtained a variational principle for its Hausdorff dimension on a subshift space. Pfister and Sullivan ([123], [124]) studied the Billingsley dimension and topological entropy for the sets of Olsen type in both shift space and  $\beta$ -shift space.

It seems that level sets of fast Birkhoff averages  $X_\Phi(\alpha, \beta)$  and  $X_\Phi(\alpha, a)$  have not been examined before. Because in many classical cases, the set  $X_\Phi(\infty)$  is empty (for example, the Birkhoff averages of a continuous function  $\Phi$  in a subshift space). However, there are some other kinds of fast Birkhoff averages studied by Fan and Schmeling ([56]) in finite symbolic dynamical systems. They considered the points converging with different velocities, for example,

$$X_\Phi(\alpha, a, \beta) := \left\{ x : A_n \Phi(x) - \int \Phi d\mu \sim an^{\beta-1} \right\} \quad a \in \mathbb{R}, 0 \leq \beta < 1,$$

where  $u_n \sim v_n$  means  $u_n/v_n \rightarrow 1$ . They found that  $X_\Phi(\alpha, a, \beta)$  admits the same Hausdorff dimension as  $X_\Phi(\alpha)$ .

### 1.3.2 Level sets of Birkhoff averages in saturated systems

In this subsection we would like to state the main results of the second part of the thesis.

Let  $T : X \rightarrow X$  be a continuous transformation on a compact metric space  $X$  with metric  $d$ . Let  $\mathbb{B}$  be a real Banach space and  $\mathbb{B}^*$  its dual space. We consider  $\mathbb{B}^*$  as a locally convex topological space with the weak star topology  $\sigma(\mathbb{B}^*, \mathbb{B})$ . For any  $\mathbb{B}^*$ -valued continuous function  $\Phi : X \rightarrow \mathbb{B}^*$ , we consider the level sets  $X_\Phi(\alpha)$ ,  $\alpha \in \mathbb{B}^*$  of the Birkhoff average  $A_n \Phi(x)$ .

Denote by  $\mathcal{M}_{\text{inv}}$  the set of all  $T$ -invariant probability Borel measures on  $X$ . For  $\mu \in \mathcal{M}_{\text{inv}}$ , the set  $G_\mu$  of  $\mu$ -generic points is defined by

$$G_\mu := \left\{ x \in X : \frac{1}{n} \sum_{j=0}^{n-1} \delta_{T^j x} \xrightarrow{w^*} \mu \right\},$$

where  $\xrightarrow{w^*}$  stands for the weak star convergence of the measures.

A dynamical system  $(X, T)$  is said to be saturated if for any  $\mu \in \mathcal{M}_{\text{inv}}$ , we have  $h_{\text{top}}(G_\mu) = h_\mu$ .

Bowen ([24]) proved that on any compact dynamical system, we have  $h_{\text{top}}(G_\mu) \leq h_\mu$  for any  $\mu \in \mathcal{M}_{\text{inv}}$ . So, saturatedness means that  $G_\mu$  is of optimal topological entropy.

A dynamical system  $(X, T)$  is said to satisfy the *specification property* if for any  $\epsilon > 0$  there exists an integer  $m(\epsilon) \geq 1$  having the property that for any integer  $k \geq 2$ , for any  $k$  points  $x_1, \dots, x_k$  in  $X$ , and for any integers

$$a_1 \leq b_1 < a_2 \leq b_2 < \dots < a_k \leq b_k$$

with  $a_i - b_{i-1} \geq m(\epsilon)$  ( $\forall 2 \leq i \leq k$ ), there exists a point  $y \in X$  such that

$$d(T^{a_i+n}y, T^n x_i) < \epsilon \quad (\forall 0 \leq n \leq b_i - a_i, \quad \forall 1 \leq i \leq k).$$

The specification property was introduced by Bowen ([24]) who required that  $y$  is periodic. But the present day tradition doesn't require this. The specification property implies the topological mixing. Blokh ([19]) proved that these two properties are equivalent for continuous interval transformations. Mixing subshifts of finite type satisfy the specification property. In general, a subshift satisfies the specification if for any admissible words  $u$  and  $v$  there exists a word  $w$  with  $|w| \leq k$  (some constant  $k$ ) such that  $uwv$  is admissible. For  $\beta$ -shifts defined by  $T_\beta x = \beta x \pmod{1}$ , there is only a countable number of  $\beta$ 's such that the  $\beta$  shifts admit Markov partition (i.e. subshifts of finite type), but an uncountable number of  $\beta$ 's such that the  $\beta$ -shifts satisfy the specification property ([132]). For basic properties of systems satisfying specification property, we recommend the readers the book of Denker, Grillenberger and Sigmund ([39]).

We obtain the following theorem.

**Theorem 1.6** (Theorem 5.1). *A system satisfying specification property is saturated.*

Let  $\mathcal{M}_\Phi(\alpha) = \{\mu \in \mathcal{M}_{\text{inv}} : \int \Phi d\mu = \alpha\}$  where  $\int \Phi d\mu$  denotes the vector-valued integral in Pettis' sense. We also prove the following theorem.

**Theorem 1.7** (Theorem 5.2). *Suppose that the dynamical system  $(X, T)$  is saturated. Then*

- (a) if  $\mathcal{M}_\Phi(\alpha) = \emptyset$ , we have  $X_\Phi(\alpha) = \emptyset$ ;  
(b) if  $\mathcal{M}_\Phi(\alpha) \neq \emptyset$ , we have

$$h_{\text{top}}(X_\Phi(\alpha)) = \sup_{\mu \in \mathcal{M}_\Phi(\alpha)} h_\mu.$$

As application, we study the frequencies of blocks in the base-2 expansion of a real number. Except for a countable set, each  $t \in [0, 1]$  has a unique base-2 expansion,  $t = \sum_{n=1}^{\infty} \frac{t_n}{2^n}$  ( $t_n \in \{0, 1\}$ ). Let  $k \geq 1$ . Write  $0^k$  for the block of  $k$  consecutive zeroes and define the  $0^n$ -frequency of  $t$  as the limit (if exists)

$$f(t, k) = \lim_{n \rightarrow \infty} \frac{1}{n} \#\{1 \leq j \leq n : t_j t_{j+1} \cdots t_{j+k-1} = 0^k\}.$$

Let  $(a_1, a_2, \dots)$  be a sequence of non-negative numbers. We denote by  $S(a_1, a_2, \dots)$  the set of all numbers  $t \in [0, 1]$  such that  $f(t, k) = a_k$  for all  $k \geq 1$ . We prove

**Theorem 1.8** (Theorem 5.4). *The set  $S(a_1, a_2, \dots)$  is non-empty if and only if*

$$1 = a_0 \geq a_1 \geq a_2 \geq \dots; \quad a_i - 2a_{i+1} + a_{i+2} \geq 0 \quad (i \geq 0). \quad (1.2)$$

Under the above condition (1.2), we have

$$h_{\text{top}}(S(a_1, a_2, \dots)) = -h(1 - a_1) + \sum_{j=0}^{\infty} h(a_j - 2a_{j+1} + a_{j+2}) \quad (1.3)$$

where  $h(x) = -x \log x$ .

The rest of this introduction is devoted to present our works on continued fractions.

### 1.3.3 Khintchine and Lyapunov Spectra in continued fractions

It is known that the continued fraction of a real number can be generated by the Gauss transformation  $T : [0, 1) \rightarrow [0, 1)$  defined by

$$T(0) := 0, \quad T(x) := \frac{1}{x} \pmod{1}, \quad \text{for } x \in (0, 1). \quad (1.4)$$

Every irrational number  $x$  in  $[0, 1)$  is uniquely expanded as an infinite expansion of the form

$$x = \frac{1}{a_1(x) + \frac{1}{a_2(x) + \cdots + \frac{1}{a_n(x) + T^n(x)}}} = \frac{1}{a_1(x) + \frac{1}{a_2(x) + \frac{1}{a_3(x) + \cdots}}} \quad (1.5)$$

where  $a_1(x) = \lfloor 1/x \rfloor$  and  $a_n(x) = a_1(T^{n-1}(x))$  for  $n \geq 2$  are partial quotients of  $x$  ( $\lfloor x \rfloor$  denoting the integral part of  $x$ ). For simplicity, we will denote the second term in (1.5) by  $[a_1, a_2, \dots, a_n + T^n(x)]$  and the third term by  $[a_1, a_2, a_3, \dots]$ .

For any  $x \in [0, 1)$  with its continued fraction (1.5), we define its Khintchine exponent  $\gamma(x)$  and Lyapunov exponent  $\lambda(x)$  respectively by

$$\begin{aligned}\gamma(x) &:= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \log a_j(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log a_1(T^j(x)), \\ \lambda(x) &:= \lim_{n \rightarrow \infty} \frac{1}{n} \log \left| (T^n)'(x) \right| = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \left| T'(T^j(x)) \right|,\end{aligned}$$

if the limits exist. The Khintchine exponent of  $x$  stands for the average (geometric) growth rate of the partial quotients  $a_n(x)$ , and the Lyapunov exponent, which is extensively studied from dynamical system point of view, stands for the expanding rate of  $T$ . They are both Birkhoff averages.

The fact that the sets  $\{x : \gamma(x) = \infty\}$  and  $\{x : \lambda(x) = \infty\}$  are nonempty inspires our study of fast Birkhoff averages. Let  $\varphi : \mathbb{N} \rightarrow \mathbb{R}_+$ . Assume that  $\lim_{n \rightarrow \infty} \frac{\varphi(n)}{n} = \infty$ . The fast Khintchine exponent and fast Lyapunov exponent of  $x \in [0, 1]$ , relative to  $\varphi$ , are respectively defined by

$$\begin{aligned}\gamma^\varphi(x) &:= \lim_{n \rightarrow \infty} \frac{1}{\varphi(n)} \sum_{j=1}^n \log a_j(x) = \lim_{n \rightarrow \infty} \frac{1}{\varphi(n)} \sum_{j=0}^{n-1} \log a_1(T^j(x)), \\ \lambda^\varphi(x) &:= \lim_{n \rightarrow \infty} \frac{1}{\varphi(n)} \log \left| (T^n)'(x) \right| = \lim_{n \rightarrow \infty} \frac{1}{\varphi(n)} \sum_{j=0}^{n-1} \log \left| T'(T^j(x)) \right|.\end{aligned}$$

For real numbers  $\xi, \beta \geq 0$ , we are interested in the level sets of Khintchine exponents and Lyapunov exponents:

$$E_\xi := \{x \in [0, 1) : \gamma(x) = \xi\}, \quad F_\beta := \{x \in [0, 1) : \lambda(x) = \beta\},$$

and the level sets of fast Khintchine exponents and fast Lyapunov exponents:

$$E_\xi(\varphi) := \{x \in [0, 1) : \gamma^\varphi(x) = \xi\}, \quad F_\beta(\varphi) := \{x \in [0, 1) : \lambda^\varphi(x) = \beta\}.$$

The Khintchine spectrum and the Lyapunov spectrum are the dimensional functions:

$$t(\xi) := \dim E_\xi \quad \tilde{t}(\beta) := \dim F_\beta.$$

Functions

$$t^\varphi(\xi) := \dim E_\xi(\varphi) \quad \tilde{t}^\varphi(\beta) := \dim F_\beta(\varphi)$$

are called the fast Khintchine spectrum and the fast Lyapunov spectrum relative to  $\varphi$ .

It should be noticed (see Billingsley [16] and Walters [140]) that the transformation  $T$  is measure-preserving and ergodic with respect to the Gauss measure  $\mu_G$  defined by

$$d\mu_G = \frac{dx}{(1+x)\log 2}.$$

An application of Birkhoff ergodic theorem yields that for Lebesgue almost all  $x \in [0, 1)$ ,

$$\begin{aligned} \gamma(x) = \xi_0 &= \int \log a_1(x) d\mu_G = 2.6854\dots \\ \lambda(x) = \lambda_0 &= \int \log |T'(x)| d\mu_G = \frac{\pi^2}{6 \log 2} = 2.37314\dots \end{aligned}$$

Here  $\xi_0$  is called the Khintchine constant and  $\lambda_0$  the Lyapunov constant. Both constants are relative to the Gauss measure.

We can determine the multifractal spectra  $t(\cdot), \tilde{t}(\cdot), t^\varphi(\cdot), \tilde{t}^\varphi(\cdot)$ .

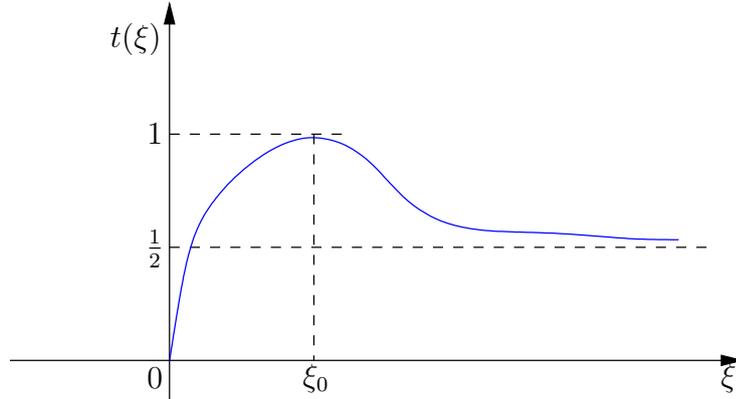
For the fast spectra  $t^\varphi(\cdot), \tilde{t}^\varphi(\cdot)$ , we prove

**Theorem 1.9** (Theorem 6.1). *If  $(\varphi(n+1) - \varphi(n)) \uparrow \infty$  and  $\lim_{n \rightarrow \infty} \frac{\varphi(n+1)}{\varphi(n)} := b \geq 1$ , then  $E_\xi(\varphi) = F_{2\xi}(\varphi)$  and  $\dim E_\xi(\varphi) = 1/(b+1)$  for all  $\xi \geq 0$ .*

For the Khintchine spectrum  $t(\cdot)$ , we have

**Theorem 1.10** (Theorem 6.2). *Let  $\xi_0 = \int \log a_1(x) d\mu_G(x)$ . For  $\xi \geq 0$ , the Hausdorff dimension  $t(\xi)$  of the set  $E_\xi$  has the following properties*

- 1)  $t(\xi_0) = 1, t(+\infty) = 1/2$ .
- 2)  $t'(\xi) < 0$  for all  $\xi > \xi_0, t'(\xi_0) = 0$  and  $t'(\xi) > 0$  for all  $\xi < \xi_0$ .
- 3)  $t'(0-) = +\infty, t'(+\infty) = 0$ .
- 4)  $t''(\xi_0) < 0$ , but  $t''(\xi_1) > 0$  for some  $\xi_1 > \xi_0$ . So  $t(\xi)$  is neither convex nor concave.

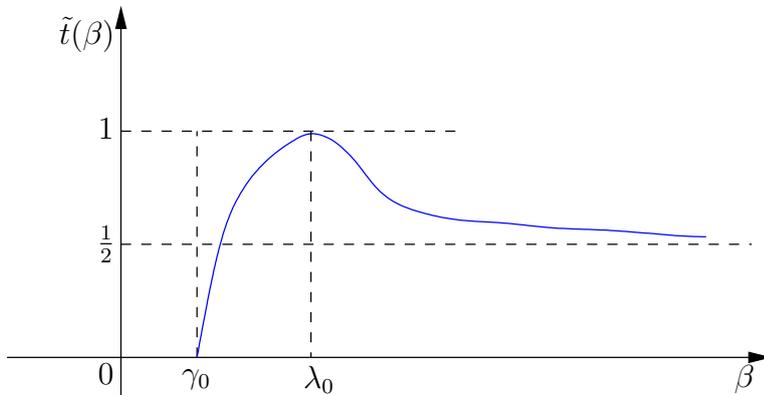


Khintchine spectrum

For Lyapunov spectrum  $\tilde{t}(\cdot)$ , we have

**Theorem 1.11** (Theorem 6.3). *Let  $\lambda_0 = \int \log |T'(x)| d\mu_G(x)$  and  $\gamma_0 = 2 \log \frac{1+\sqrt{5}}{2}$ . For any  $\beta \in [\gamma_0, \infty)$ , the Hausdorff dimension  $\tilde{t}(\beta)$  of the set  $F_\beta$  has the following properties*

- 1)  $\tilde{t}(\lambda_0) = 1$ ,  $\tilde{t}(+\infty) = 1/2$ .
- 2)  $\tilde{t}'(\beta) < 0$  for all  $\beta > \lambda_0$ ;  $\tilde{t}'(\lambda_0) = 0$  and  $\tilde{t}'(\beta) > 0$  for all  $\beta < \lambda_0$ .
- 3)  $\tilde{t}'(\gamma_0-) = +\infty$ ,  $\tilde{t}'(+\infty) = 0$ .
- 4)  $\tilde{t}''(\lambda_0) < 0$ , but  $\tilde{t}''(\beta_1) > 0$  for some  $\beta_1 > \lambda_0$ , i.e.,  $\tilde{t}(\beta)$  is neither convex nor concave.



Lyapunov spectrum

It should be noticed that the non-convexity shows a new phenomenon for the multifractal analysis in our settings.

For a better understanding of our motivation, we would like to review some classical works.

It was known to E. Borel ([20]) in 1909 that for Lebesgue almost all  $x \in [0, 1]$ , there exists a subsequence  $\{a_{n_r}(x)\}$  of  $\{a_n(x)\}$  such that  $a_{n_r}(x) \rightarrow \infty$ . A more explicit result due to Borel-Bernstein (see [14, 21, 22]) is the 0-1 law which hints that for almost all  $x \in [0, 1]$ ,  $a_n(x) > \varphi(n)$  holds for infinitely many  $n$ 's or finitely many  $n$ 's according as  $\sum_{n \geq 1} \frac{1}{\varphi(n)}$  diverges or converges. Then it arose a natural question to quantify the exceptional sets in terms of Hausdorff dimension. The first published work on this aspect was due to Jarnik ([73] in 1928) who was concerned with the set  $E$  of continued fractions with bounded partial quotients and with the sets  $E_2, E_3, \dots$ , where  $E_\alpha$  is the set of continued fractions whose partial quotients do not exceed  $\alpha$ . He successfully got that the set  $E$  is of full Hausdorff dimension, but he didn't find the exact dimensions of  $E_2, E_3, \dots$ . Later, lots of works were done to estimate  $\dim E_2$ , including those of Good ([62]), Bumby ([29]), Hensley ([67], [68]), Jenkinson and Pollicott ([74]), Mauldin and Urbański ([105]) and references therein. Up to now, the optimal approximation on  $\dim E_2$  is given by Jenkinson ([75]) in 2004:

$$\dim_H E_2 = 0.531280506277205141624468647368471785493059109018398779 \dots$$

which is accurate up to 54 decimal places. Other works which concern the Hausdorff dimensions of exceptional sets in continued fractions are Hirst [70, 71], Cusick [37], Moorthy [110], Luczak [101], Kesseböhmer and Zhu [79], Kesseböhmer and Stratmann [80], and Wang and Wu [144].

M. Pollicott and H. Weiss ([128]) initially studied the level set  $F_\beta$  of the Lyapunov exponents and obtained some partial results about the Lyapunov spectrum  $\tilde{t}(\cdot)$ . This is one of motivations for us to find a complete solution.

We briefly present our methods in finishing this subsection. The main tool of our study on the Khintchine spectrum and Lyapunov spectrum is the Ruelle-Perron-Frobenius operator with potential function

$$\Phi_{t,q}(x) = -t \log |T'(x)| + q \log a_1(x), \quad \Psi_t(x) = -t \log |T'(x)|$$

where  $(t, q)$  are suitable parameters. The usual way, as Pollicott and Weiss did ([128]), to obtain the spectrum through Ruelle theory (the study about the Ruelle-Perron-Frobenius operators and the thermodynamical formalism) fixes  $q$  and finds  $T(q)$  as the solution of  $P(T(q), q) = 0$ . (Here  $P(t, q)$  is the pressure corresponding to the potential function of two parameters). The information contained in the curve  $T(q)$  can only give some partial results ([128]). In the thesis, we look for multifractal information from the whole two-dimensional surface defined by the pressure  $P(t, q)$  rather than the single curve  $T(q)$ . This leads us to obtain complete graphs of the Khintchine spectrum and the Lyapunov spectrum.

There exist several works on pressure functions associated to different potentials in the Gauss dynamics. For a detailed study on pressure function associated to one potential function, we refer to the works of Mayer ([107], [108], [109]), and for pressure functions associated to two potential functions, we refer to the works of Pollicott and Weiss ([128]), Walters ([141], [142]) and Hanus, Mauldin and Urbański ([64]). We will use the theory developed in [64].

### 1.3.4 Non-normal continued fractions

We study two subsets of non-normal continued fractions. One is the set of continued fractions with maximal frequency oscillation, and the other is the set of extremely non-normal continued fractions which was previously defined by Olsen in [113].

Write the continued fraction expansion of  $x \in [0, 1] \cap \mathbb{Q}^c$  as  $x = [a_1(x), a_2(x), \dots]$ . Let  $\mathbb{N}^*$  denote the set of finite words over  $\mathbb{N}$ , which will be called digit strings alternatively. For any  $k$ -digit string  $W = (w_1, \dots, w_k) \in \mathbb{N}^*$ , we write

$$f(W, x, n) := \#\{1 \leq j \leq n - k + 1 : a_j(x) = w_1, \dots, a_{j+k-1}(x) = w_k\}$$

for the number of occurrences of  $W$  among the first  $n$  digits of  $x$ . We are concerned with certain sets of continued fractions which are determined by the limit behaviors of the sequence of all asymptotic frequencies, namely,  $\{\frac{1}{n}f(W, x, n) : n \geq 1, W \in \mathbb{N}^*\}$ .

We denote by  $\text{Freq}(W, x)$  the collection of all accumulation points of the sequence  $\{\frac{1}{n}f(W, x, n) : n \geq 1\}$ , and

$$\text{Freq}(W) = \bigcup_{x \in [0, 1] \cap \mathbb{Q}^c} \text{Freq}(W, x).$$

We are interested in the set

$$\mathbb{F} = \{x \in [0, 1) : \text{Freq}(W, x) = \text{Freq}(W) \text{ for all } W \in \mathbb{N}^*\}$$

of continued fractions with maximal frequency oscillation. This name is justified by the fact that, for any digit string  $W \in \mathbb{N}^*$ , the asymptotic frequencies of  $W$  in the continued fraction expansion of  $x \in \mathbb{F}$  oscillates to the maximal possible extent.

The set  $\mathbb{F}$  is a set of totally non-normal continued fractions. To be more precise, let  $\mu$  be any Borel probability measure which is ergodic with respect to the Gauss transformation (see [72]), then the Birkhoff ergodic theorem implies that for  $\mu$ -almost all  $x \in [0, 1)$

$$\lim_{n \rightarrow \infty} \frac{1}{n} f(W, x, n) = \mu \{y \in [0, 1) : a_j(y) = w_j, 1 \leq j \leq k\}$$

for every  $W = (w_1, \dots, w_k) \in \mathbb{N}^*$ . Let us say that  $x$  is a  $\mu$ -normal continued fraction if the last relation holds. It is then clear that  $\mathbb{F}$  is a set of non-normal continued fractions which has  $\mu$ -measure null. Therefore, the set  $\mathbb{F}$  is small in the measure-theoretic sense. However, it is large from the topological viewpoint. Indeed, one can show that  $\mathbb{F}$  is the intersection of a countable number of residual sets, hence the set  $\mathbb{F}$  is residual in  $[0, 1)$ , namely, its complementary set is of the first category. As a consequence, the packing dimension of  $\mathbb{F}$  is 1 (Theorem 7.1). However, it is interesting to note that the Hausdorff dimension of  $\mathbb{F}$  is of intermediate size.

**Theorem 1.12** (Theorem 7.2). *The set  $\mathbb{F}$  of continued fractions with maximal frequency oscillation is of Hausdorff dimension one-half.*

The other interesting set we will consider is the set  $\mathbb{E}$  of extremely non-normal continued fractions introduced by Olsen ([113]) in 2003. Let us first recall the definition. For each  $k \geq 1$ , define the simplex of probability vectors with index set  $\mathbb{N}^k$  by

$$\Delta_k := \left\{ (p(W))_{W \in \mathbb{N}^k} : p(W) \geq 0, \sum_W p(W) = 1 \right\}.$$

We will denote the elements of  $\Delta_k$  by vectors, and equip  $\Delta_k$  with the 1-norm

$$\|\vec{p} - \vec{q}\|_1 = \sum_{W \in \mathbb{N}^k} |p(W) - q(W)|.$$

Define the sub-simplex of shift invariant probability vectors in  $\Delta_k$  by

$$\mathbf{S}_k := \left\{ \vec{p} \in \Delta_k : \sum_i p(iV) = \sum_i p(Vi) \text{ for all } V \in \mathbb{N}^{k-1} \right\}.$$

Denote the vector of  $n$ -th asymptotic frequencies of the  $k$ -words occurring in  $x$  by

$$\Pi_k(x, n) := \left( \frac{1}{n} f(W, x, n) \right)_{W \in \mathbb{N}^k}.$$

It is clear that  $\Pi_k(x, n) \in \Delta_k$ . Let  $\mathbf{A}_k(x)$  denote the set of accumulation points of the sequence  $\{\Pi_k(x, n)\}_{n=1}^\infty$  with respect to the 1-norm  $\|\cdot\|_1$ . It is known that  $\mathbf{A}_k(x) \subset \mathbf{S}_k$  (see [113]). Then we define

$$\mathbb{E}_k = \{x \in [0, 1) \cap \mathbb{Q}^c : \mathbf{A}_k(x) = \mathbf{S}_k\}$$

and  $\mathbb{E} = \bigcap_{k=1}^\infty \mathbb{E}_k$ , which is called the set of extremely non-normal continued fractions.

The set  $\mathbb{E}$  is closely related to the set  $\mathbb{F}$  of continued fractions with maximal frequency oscillation. Indeed, we have  $\mathbb{E} \subset \mathbb{F}$  (Theorem 7.3).

In [113], Olsen proved that the set  $\mathbb{E}$  is residual in  $[0, 1)$  and has packing dimension 1. (Therefore the assertion that  $\mathbb{F}$  is residual and of packing dimension 1, comes directly from the fact that  $\mathbb{E} \subset \mathbb{F}$ .) As for the Hausdorff dimension, based on the analogous results in the decimal expansion case, it was conjectured in [113] that  $\dim(\mathbb{E}) = 0$ . However, we prove that

**Theorem 1.13** (Theorem 7.4). *The Hausdorff dimension of the extreme non-normal continued fraction set  $\mathbb{E}$  is one-half.*

### 1.3.5 The Frequency of partial quotients in continued fractions

We study the Besicovitch-Eggleston sets defined by the points with given frequencies of partial quotients of continued fractions.

The continued fraction of an irrational  $x$  is still denoted by  $x = [a_1, a_2, a_3, \dots]$ . For  $x \in [0, 1) \cap \mathbb{Q}^c$ , define the *frequency* of the digit  $j \in \mathbb{N}$  in the continued fraction expansion of  $x$  by

$$\tau_j(x) := \lim_{n \rightarrow \infty} \frac{\tau_j(x, n)}{n},$$

if the limit exists, where  $\tau_j(x, n) := \text{Card}\{k : a_k(x) = j, 1 \leq k \leq n\}$ .

A probability vector  $\vec{p} = (p_1, p_2, \dots)$ , i.e.,  $p_j \geq 0$  for all  $j \geq 1$  and  $\sum_{j=1}^{\infty} p_j = 1$ , will be referred to as a *frequency vector*. For such a given vector  $\vec{p}$ , we define the *Besicovitch-Eggleston set*

$$\mathcal{E}_{\vec{p}} := \{x \in [0, 1) \cap \mathbb{Q}^c : \tau_j(x) = p_j \ \forall j \geq 1\}.$$

Recall that Gauss transformation  $T$  admits an ergodic invariant measure  $\mu_G$ ,

$$d\mu_G = \frac{1}{(1+x)\log 2} dx.$$

By Birkhoff's ergodic theorem, for Lebesgue almost all points ( $\mu_G$ -a.e.)

$$\tau_j(x) = \int_{1/(j+1)}^{1/j} \frac{1}{(1+x)\log 2} dx = \frac{1}{\log 2} \log \frac{(i+1)^2}{i(i+1)}, \quad \forall j \geq 1.$$

Thus if we take

$$p_j = \frac{1}{\log 2} \log \frac{(i+1)^2}{i(i+1)} \quad \forall j \geq 1, \tag{1.6}$$

then  $\mathcal{E}_{\vec{p}}$  admits Lebesgue measure 1.

For any  $a_1, a_2, \dots, a_n \in \mathbb{N}$ , let

$$I(a_1, a_2, \dots, a_n) := \{x \in [0, 1) : a_1(x) = a_1, a_2(x) = a_2, \dots, a_n(x) = a_n\}$$

and call it a *rank  $n$  fundamental interval*. Denote by  $\mathcal{M}_{\text{erg}}$  the set of  $T$ -invariant ergodic measures. The entropy of  $\mu$  will be denoted by  $h_\mu$ . Let  $\mathcal{N}$  be a subset of  $\mathcal{M}_{\text{erg}}$  such that for any  $\mu \in \mathcal{N}$ , one has  $\mu(I(j)) = p_j$  for all  $j \geq 1$  and  $-2 \int \log x d\mu < \infty$ . Denote the Hausdorff dimension by  $\dim_H$ . With the convention  $\sup \emptyset = 0$ , we have the following theorem.

**Theorem 1.14** (Theorem 8.1). *For a given frequency vector  $\vec{p}$ , we have*

$$\dim_H(\mathcal{E}_{\vec{p}}) = \max \left\{ \frac{1}{2}, \sup_{\mu \in \mathcal{N}} \frac{h_\mu}{-2 \int \log x d\mu} \right\}. \tag{1.7}$$

The study of the Besicovitch-Eggleston type sets dates from the work of Besicovitch ([15]) in 1935 and Eggleston ([43]) in 1949, in which they studied the  $m$ -adic expansion case. For the Besicovitch-Eggleston sets in continued fractions,

Kinney and Pitcher ([88]) obtained a lower bound in 1966. They showed that if  $\sum_{j=1}^{\infty} p_j \log j < \infty$ , then

$$\dim_H \mathcal{E}_{\vec{p}} \geq \frac{\sum_{j=1}^{\infty} p_j \log p_j}{2 \int \log x d\mu_{\vec{p}}}, \quad (1.8)$$

where  $\mu_{\vec{p}}$  is the Bernoulli on  $[0, 1]$  such that  $\mu(I(j)) = p_j$  for all  $j \geq 1$ . This lower bound is in fact the Hausdorff dimension of the Bernoulli measure  $\mu_{\vec{p}}$ . However, by the result of Kifer, Peres and Weiss ([87]) in 2001, this lower bound is not always the best one. It is proved that there exists a universal constant  $\epsilon_0 > 10^{-7}$ , such that

$$\dim_H \mu_{\vec{p}} \leq 1 - \epsilon_0, \quad \text{for any Bernoulli measure } \mu_{\vec{p}}. \quad (1.9)$$

Take  $\vec{p}_0 = (p_j^0)_{j \geq 1}$  as (1.6), for example, then  $\dim_H(\mathcal{E}_{\vec{p}}) = 1$ . But by (1.9), any Bernoulli measure can obtain the Hausdorff dimension 1. Thus Kinney and Pitcher's result (1.8) can not serve as an exact lower bound for dimension of  $\mathcal{E}_{\vec{p}_0}$ .

In 1975, Billingsley and Henningsen ([17]) searched the lower bound by considering all ergodic measures and proved that if  $\sum_{j=1}^{\infty} p_j \log j < \infty$ , then

$$\dim_H(\mathcal{E}_{\vec{p}}) \geq \sup_{\mu \in \mathcal{N}} \frac{h_{\mu}}{-2 \int \log x d\mu}.$$

It is essentially a half part of the lower bound of Theorem 1.14, except that in Theorem 1.14, we do not assume  $\sum_{j=1}^{\infty} p_j \log j < \infty$ . At the end of Section 8.4, we will show that if

$$\sum_{j=1}^{\infty} p_j \log j = \infty, \quad (1.10)$$

then

$$\dim_H(\mathcal{E}_{\vec{p}}) = \frac{1}{2}.$$

In fact, the condition (1.10) implies  $\mathcal{N} = \emptyset$ , thus by the convention of  $\sup \emptyset = 0$ , the right side of (1.7) gives the value  $1/2$ .

The other half part of the lower bound of Theorem 1.14,  $\dim_H(\mathcal{E}_{\vec{p}}) \geq 1/2$ , is a corollary of Proposition 1.7 in [97]. However, we will give a direct proof in Chapter 8 (Section 8.4). The key point is that there are infinite possible choices of the digits which allow us to construct a big enough Cantor like subset of  $\mathcal{E}_{\vec{p}}$ .

To give an exact upper bound of the Hausdorff dimension of any Besicovitch-Eggleston set in continued fractions. We will use the techniques of [97] and [17] in estimating the lengths of fundamental intervals, to distinguish the effect of both two parts of  $1/2$  and the well-known variational principle in Theorem 1.14. We briefly explain how we made it. Denote the right side of (1.7) by  $s$ . We prove the upper bound of Theorem 1.14 by the definition of Hausdorff dimension. By using the rank- $n$  fundamental intervals as a cover of  $\mathcal{E}$ , for any  $\gamma > s$ , we will need to estimate the following

$$\sum_{x_1 \cdots x_n \in A} |I(x_1, \cdots, x_n)|^\gamma, \quad (1.11)$$

where  $x_1 \cdots x_n \in A$  signifies that the digits  $x_1, \dots, x_n$  satisfies some frequency conditions. We fix a large enough  $N$  and use Lemma 2.38 to pick up the digits which is larger than  $N$ . Suppose there are  $\tilde{n}$  digits which is smaller than  $N$  among  $x_1, \dots, x_n$ . We have (1.11) is less than

$$\left( \sum_{j=N+1}^{\infty} \frac{8}{(j+1)^{2\gamma}} \right)^{n-\tilde{n}} \sum_{x_1 \cdots x_{\tilde{n}} \in \tilde{A}} |I(x_1, \cdots, x_{\tilde{n}})|^\gamma,$$

where  $\tilde{A}$  signifies the set of words which is in  $A$  but with digits less than  $N$ . Then the first term is bounded by a constant, since  $\gamma > s \geq 1/2$ . The second term can be treated the similar way as that in Billingsley and Henningsen [17], since the digits  $x_1, \dots, x_{\tilde{n}} \in \tilde{A}$  contains only digits less than  $N$ . Since

$$\gamma > s \geq \sup_{\mu \in \mathcal{N}} \frac{h_\mu}{-2 \int \log x d\mu},$$

the second term is bounded from infinity. Combining these two terms, we have for any  $\gamma > s$ , the summation of (1.11) is strictly less than infinity, which implies the Hausdorff dimension of  $\mathcal{E}_{\bar{p}}$  less than  $s$ .

To end this subsection, we would like to remark that the methods to calculate the Hausdorff dimensions of Besicovitch-Eggleston sets are different from those to obtain the Khintchine and Lyapunov spectra. In the study of Khintchine and Lyapunov spectra, we find suitable measures which are supported on the level sets of Birkhoff averages though the theory of Ruelle operator. While to calculate the Hausdorff dimension of Besicovitch-Eggleston sets, we on one hand, find some Cantor like subset bounding the dimension from below; on the other hand, estimate the Hausdorff measure directly which induces the upper bounds.

## Chapter 2

### Preliminaries

#### 2.1 Basic definitions and properties of dynamical systems

An introduction to the study of dynamical systems can be found in the books: Denker, Grillenberger and Sigmund [39], Walters [140], Katok and Hasselblatt [78], Pollicott and Yuri [127], Kurta [94].

A terminology: we say the transformation satisfies some property if the dynamical system satisfies that one.

##### 2.1.1 Topological dynamical systems

A topological dynamical system is a pair  $(X, T)$  where  $X$  is a compact metric space with a metric  $d$ , and  $T : X \rightarrow X$  is a continuous transformation.

A point  $x \in X$  is called a fixed point if  $T(x) = x$ . A fixed point  $x \in X$  is said to be stable if for any  $\epsilon > 0$ , there exists  $\delta > 0$ , such that for any  $y \in X$  satisfying  $d(y, x) < \delta$ , one has  $d(T^n y, x) < \epsilon$  for all  $n \geq 0$ . A fixed point  $x \in X$  is said to be attracting if it is stable and moreover there exists  $\delta > 0$  such that for any  $y \in X$  satisfying  $d(y, x) < \delta$ , we have  $\lim_{n \rightarrow \infty} d(T^n y, x) = 0$ . A fixed point which is not stable will be called unstable.

A point  $x \in X$  is periodic if there exists a positive integer  $p$  such that  $T^p x = x$ . The least  $p$  will be called the period of  $x$ . A point  $x \in X$  is said to be eventually periodic if there exists some nonnegative integer  $m$  such that  $T^m x$  is periodic. A point is said to be preperiodic if it is eventually periodic but not periodic. A periodic point with period  $p$  is called stable or attracting if it is stable or attracting as a fixed point of  $T^p$ .

If  $T$  has derivative at the fixed point  $x$ , then we also say  $x$  is attracting if  $|T'(x)| < 1$ , repelling if  $|T'(x)| > 1$  and indifferent if  $|T'(x)| = 1$ . For the definition of the periodic point of period  $p$ , we say it is attracting, repelling and indifferent if  $T^p$  is so. It is easy to see repelling fixed points are unstable.

Two topological dynamical systems  $(X, T)$  and  $(Y, S)$  are said to be topologically conjugate if there exists a homeomorphism  $\phi : X \rightarrow Y$  such that  $\phi \circ T = S \circ \phi$ . The homeomorphism  $\phi$  is called a conjugacy. If  $\phi$  is not a homeomorphism but is continuous and surjective then  $(Y, S)$  is called a factor of  $(X, T)$  and  $\phi$  is called a factor map.

A topological dynamical system  $(X, T)$  is said to be transitive if there exists at least one point  $x \in X$  such that the orbit set  $O_T(x) := \{T^n x : n \in \mathbb{N}\}$  is dense. We call  $(X, T)$  a minimal system if for all points  $x \in X$ ,  $O_T(x)$  is dense. The system  $(X, T)$  is said to be chaotic if it is transitive and the set of periodic points is dense in  $X$  and  $X$  is infinite. The system  $(X, T)$  is said to be mixing if for any nonempty open sets  $U, V \subset X$ , there exists a positive integer  $N$  such that for all  $n \geq N$ ,  $T^n(U) \cap V \neq \emptyset$ .

A subset  $E \subset X$  is called  $T$ -invariant if  $TE = E$ . If  $E$  is closed and  $T$ -invariant, then  $(E, T|_E)$  becomes a subsystem. A closed and  $T$ -invariant subset  $E$  is called a minimal set with respect to  $T$  if  $(E, T|_E)$  is minimal.

**Proposition 2.1.** *For any topological dynamical system  $(X, T)$ , there exists a minimal set.*

**Proposition 2.2.** *If  $(X, T)$  is transitive or minimal, then for any  $f \in C(X)$  (the set of all complex valued continuous functions) satisfying  $f \circ T = f$ , we have  $f$  is a constant.*

A point  $x \in X$  is called an equicontinuous point if for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that for any  $y \in X$  satisfying  $d(x, y) < \delta$  and for all  $n \geq 0$ , one has  $d(T^n y, T^n x) < \epsilon$ . A topological dynamical system  $(X, T)$  is said to be equicontinuous if all the points in  $X$  are equicontinuous. It is said to be almost equicontinuous if the set of the equicontinuous points is a residual set. In particular, if  $T$  is an isometry then  $T$  is equicontinuous.

A system  $(X, T)$  is said to be sensitive if there exists an  $\epsilon > 0$ , for any  $x \in X$  and any  $\delta > 0$ , there exists  $y \in X$  satisfying  $d(y, x) < \delta$ , and an integer  $n \geq 0$  such that  $d(T^n y, T^n x) \geq \epsilon$ .

A system  $(X, T)$  is said to be expansive (positively expansive) if there exists an  $\epsilon > 0$  such that for any  $x \neq y \in X$ , there exists an integer  $n \geq 0$  such that  $d(T^n x, T^n y) \geq \epsilon$ .

**Theorem 2.3** ([2]). *A transitive system is either sensitive or almost equicontinuous.*

**Theorem 2.4** ([60],[2]). *A transitive equicontinuous system is minimal and a minimal almost equicontinuous system is equicontinuous.*

**Theorem 2.5** (see [94], p. 147). *A minimal equicontinuous system on a totally disconnected infinite space is topologically conjugate to an adding machine.*

**Theorem 2.6** ([60]). *A chaotic system is sensitive.*

**Proposition 2.7.** *A mixing dynamical system with at least two points is sensitive.*

**Proposition 2.8.** *If  $X$  is perfect and  $(X, T)$  is positively expansive then  $(X, T)$  is sensitive.*

### 2.1.2 Measure-preserving dynamical systems and ergodic theory

Let  $(X, \mathcal{B}, \mu)$  be a probability space where  $\mathcal{B}$  is a  $\sigma$ -algebra and  $\mu$  is a probability measure. A transformation  $T : X \rightarrow X$  is said to be measurable if  $T^{-1}(B) \in \mathcal{B}$ , and measure-preserving if  $T$  is measurable and  $\mu(T^{-1}B) = \mu(B)$  for all  $B \in \mathcal{B}$ . If  $T$  is measure-preserving, then  $(X, \mathcal{B}, \mu, T)$  will be called a measure-preserving dynamical system. If  $T$  is invertible and  $T^{-1}$  is also measure-preserving, then  $(X, \mathcal{B}, \mu, T)$  will be called a measure-theoretic dynamical system.  $T$  will be called an isomorphism from  $(X, \mathcal{B}, \mu)$  to itself.

Two measure-preserving systems  $(X, \mathcal{B}, \mu, T)$  and  $(Y, \mathcal{C}, \nu, S)$  are measure-theoretic conjugate if there exist  $X' \in \mathcal{B}, Y' \in \mathcal{C}$  such that  $\mu(X \setminus X') = \nu(Y \setminus Y') = 0$  and there exists an isomorphism  $\phi : X' \rightarrow Y'$  such that  $\nu = \mu \circ \phi^{-1}$  and  $\phi \circ T = S \circ \phi$   $\mu$ -a.e.. If  $\phi$  is not isomorphic but only measure-preserving and surjective, then  $(Y, \mathcal{C}, \nu, S)$  will be called a factor of  $(X, \mathcal{B}, \mu, T)$  and  $\phi$  will be called a factor map.

**Theorem 2.9** (Poincaré's Recurrence Theorem). *Let  $(X, \mathcal{B}, \mu, T)$  be a measure-preserving system. Let  $E \in \mathcal{B}$  satisfy  $\mu(E) > 0$ . Then there exists  $F \subset E$  with  $\mu(F) = \mu(E)$  such that for every point  $x \in F$ , there is a sequence of positive integers  $n_1 < n_2 < n_3 < \dots$  with  $T^{n_i}(x) \in E$  for each  $i$ .*

The measure-preserving system  $(X, \mathcal{B}, \mu, T)$  is said to be ergodic if for any  $B \in \mathcal{B}$  satisfying  $T^{-1}B = B$ , one has  $\mu(B) = 0$  or  $\mu(B) = 1$ . We also say  $T$  or  $\mu$  is ergodic in this case.

**Proposition 2.10.** *The following are equivalent:*

- (1)  $T$  is ergodic,
- (2) for any  $B \in \mathcal{B}$  satisfying  $\mu(T^{-1}B \triangle B) = 0$ , one has  $\mu(B) = 0$  or  $\mu(B) = 1$ ,
- (3) whenever  $f$  is measurable and  $f \circ T = f$  a.e., then  $f$  is constant a.e.,
- (4) whenever  $f \in L^2(\mu)$  and  $f \circ T = f$  a.e., then  $f$  is constant a.e..

**Theorem 2.11** (Birkhoff Ergodic Theorem). *Suppose  $(X, \mathcal{B}, \mu, T)$  is a measure preserving system and  $f \in L^1(\mu)$ . Then there exists a function  $f^* \in L^1(\mu)$  such that  $f^* \circ T = f^*$ ,  $\int f^* d\mu = \int f d\mu$  and*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = f^*(x), \quad \mu - a.e..$$

*If  $T$  is ergodic then  $f^*$  is the constant  $\int f d\mu$ ,  $\mu - a.e..$*

**Theorem 2.12** (von Neumann mean Ergodic Theorem). *Let  $(X, \mathcal{B}, \mu, T)$  be a measure-preserving system and  $f \in L^p(\mu)$ . There exists a function  $f^* \in L^p(\mu)$  such that  $f^* \circ T = f^*$ , and*

$$\lim_{n \rightarrow \infty} \left\| \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) - f^*(x) \right\|_p = 0.$$

A measure-preserving system  $(X, \mathcal{B}, \mu, T)$  is said to be weak-mixing if for any  $A, B \in \mathcal{B}$ , one has

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} | \mu(T^{-j}A \cap B) - \mu(A)\mu(B) | = 0.$$

It is said to be strong mixing if for any  $A, B \in \mathcal{B}$ , one has

$$\lim_{n \rightarrow \infty} | \mu(T^{-j}A \cap B) - \mu(A)\mu(B) | = 0.$$

### 2.1.3 Topological and measurable dynamical systems

The topological dynamical systems and the measure preserving dynamical systems are always mixed together. Let  $(X, T)$  be a topological dynamical system with  $X$  a compact metric space and  $T$  a continuous transformation. Let  $\mathcal{B}$  be the Borel  $\sigma$ -algebra. Denote  $\mathcal{M}$  denote the set of all Borel probability measures on  $X$  and  $\mathcal{M}_{\text{inv}}$  the set of all  $T$ -invariant measures in  $\mathcal{M}$ . The system  $(X, T)$  always possesses some  $T$ -invariant Borel probability  $\mu$ .

**Theorem 2.13** (Kryloff-Bogoliouboff ([93],1937)). *The set  $\mathcal{M}_{\text{inv}}$  is nonempty.*

We remark that Oxtoby and Ulam ([118],1939) gave a sufficient and necessary condition under which  $\mathcal{M}_{\text{inv}}$  is nonempty for general  $T$  on complete separable metric space.

For  $\mathcal{M}_{\text{inv}}$  we have

**Proposition 2.14.** *The set  $\mathcal{M}_{\text{inv}}$  is a convex subset of  $\mathcal{M}$ , closed in the weak topology. If  $T$  is not identity, then  $\mathcal{M}_{\text{inv}}$  is nowhere dense in  $\mathcal{M}$ .*

If  $\mathcal{M}_{\text{inv}}$  is a singleton, then the system  $(X, T)$  is called uniquely ergodic.

Denote the set of all complex valued functions by  $C(X)$ .

**Proposition 2.15.** *For the topological dynamical system  $(X, T)$ , the following are equivalent:*

- (1)  $T$  is uniquely ergodic,
- (2) for all  $f \in C(X)$ ,  $(1/n) \sum_{j=0}^{n-1} f(T^j x)$  converges uniformly to a constant,
- (3) for all  $f \in C(X)$ ,  $(1/n) \sum_{j=0}^{n-1} f(T^j x)$  converges pointwise to a constant.
- (4) there exists  $\mu \in \mathcal{M}_{\text{inv}}$  such that for all  $x \in X$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = \int f d\mu.$$

A system is called strictly ergodic if it is uniquely ergodic and minimal.

**Proposition 2.16.** *Let  $(X, T)$  be a topological dynamical system with  $T$  a homeomorphism. Suppose  $(X, T)$  is uniquely ergodic with  $\mathcal{M}_{\text{inv}} = \{\mu\}$ . Then  $T$  is minimal if and only if  $\mu(U) > 0$  for all nonempty open set  $U$ .*

**Proposition 2.17.** *Let  $(X, T)$  be a topological dynamical system with  $X$  a compact metric space and  $T$  being equicontinuous. Then the following are equivalent:*

- (1)  $T$  is minimal,
- (2)  $T$  is uniquely ergodic,
- (3)  $T$  is strictly ergodic,
- (4)  $T$  is ergodic for any  $\mu \in \mathcal{M}_{\text{inv}}$  with  $\mu(U) > 0$  for any open set  $U$ ,
- (5)  $T$  is ergodic for some  $\mu \in \mathcal{M}_{\text{inv}}$  with  $\mu(U) > 0$  for any open set  $U$ .

**Corollary 2.18.** *Let  $(X, T)$  be a rotation dynamical system on the compact metric group  $X$ . Then  $T$  is uniquely ergodic if and only if  $T$  is minimal. The Haar measure is the only invariant measure. In this case, the system is strictly ergodic.*

## 2.2 $p$ -adic numbers, $p$ -adic analysis and $p$ -adic dynamics

For an introduction to  $p$ -adic numbers and  $p$ -adic analysis, we recommend the book of Schikhoff ([131]). For  $p$ -adic dynamics, see the book of Khrennikov and Nilsson ([86]).

Let  $K$  be a field. An absolute value on  $K$  is a function  $|\cdot| : K \rightarrow [0, \infty)$  such that

- (1)  $|x| \geq 0$  for all  $x \in K$ , and  $|x| = 0$  if and only if  $x = 0$ ,
- (2)  $|xy| = |x||y|$  for all  $x, y \in K$ ,
- (3)  $|x + y| \leq |x| + |y|$  for all  $x, y \in K$ .

If we have  $|x + y| \leq \max\{|x|, |y|\}$  for all  $x, y \in K$  replacing the third one above, the absolute value will be called non-Archimedean, and the field  $K$  will be called a non-Archimedean field. The image under the function  $|\cdot|$  of the multiplicative group  $K^\times$  of  $K$  is a subgroup of  $(\mathbb{R}_+, \times)$ . If this subgroup is discrete we will say the absolute value is discrete.

Let  $p \geq 2$  be a prime number. For all  $n \in \mathbb{Z} \setminus \{0\}$ , we denote by  $v_p(n)$  the greatest integer  $k \geq 0$  such that  $p^k$  divides  $n$  in  $\mathbb{Z}$ . By convention, we put  $v_p(0) = +\infty$ . For all  $x = m/n \in \mathbb{Q}$  with  $m, n \in \mathbb{Z}$  and  $n \neq 0$ , define

$$|x|_p := p^{-v_p(m/n)} := p^{-(v_p(m) - v_p(n))}.$$

Then  $|\cdot|_p$  is a discrete non-Archimedean absolute value on  $\mathbb{Q}$ , called  $p$ -adic absolute value over  $\mathbb{Q}$ , while  $v_p$  is called the  $p$ -adic valuation over  $\mathbb{Q}$ .

The field  $\mathbb{Q}_p$  of  $p$ -adic numbers is the topological completion of  $\mathbb{Q}$  with respect to the metric topology induced by  $|\cdot|_p$ . The extension of  $|\cdot|_p$  (resp.  $v_p$ ) will still be denoted by the same notation.

It is known that  $|\cdot|_p$  is ultrametric over  $\mathbb{Q}_p$ : for all  $x, y \in \mathbb{Q}_p$  such that  $|x|_p \neq |y|_p$ , we have

$$|x + y|_p = \max \left\{ |x|_p, |y|_p \right\}.$$

A sequence  $(x_n)_{n \geq 0}$  in  $\mathbb{Q}_p$  is convergent if and only if

$$\lim_{n \rightarrow +\infty} v_p(x_n - x_{n-1}) = +\infty.$$

Any  $x \in \mathbb{Q}_p$ , can be written as

$$x = \sum_{n \geq v_p(x)} x_n p^n \quad (0 \leq x_n < p).$$

The ring  $\mathbb{Z}_p$  of  $p$ -adic integers is the local ring of  $\mathbb{Q}_p$ . It is the unit ball of  $\mathbb{Q}_p$ :  $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ . It is also the closure of  $\mathbb{N}$  with respect to the topology induced by  $|\cdot|_p$ . Furthermore,  $\mathbb{Q}_p$  is locally compact and  $\mathbb{Z}_p$  is compact.

Denote by  $\mathbb{Z}_p[x]$  be the set of all polynomials with coefficients in  $\mathbb{Z}_p$ . We have the following.

**Theorem 2.19** (Hensel's lemma). *Let  $f \in \mathbb{Z}_p[x]$ . Assume that there exist  $\alpha_0 \in \mathbb{Z}_p$  and  $\gamma \in \mathbb{N}$  such that*

$$\begin{aligned} f(\alpha_0) &\equiv 0 \pmod{p^{2\gamma+1}}, \\ f'(\alpha_0) &\equiv 0 \pmod{p^\gamma}, \\ f'(\alpha_0) &\not\equiv 0 \pmod{p^{\gamma+1}}. \end{aligned}$$

*Then there exists  $\alpha \in \mathbb{Z}_p$  such that  $f(\alpha) = 0$  and  $\alpha \equiv \alpha_0 \pmod{p^{\gamma+1}}$*

The following results concern about squares in  $\mathbb{Q}_p$ .

**Lemma 2.20** (Euler's criterion). *Let  $p \geq 3$ . We have  $u \in (\mathbb{Z}/p\mathbb{Z})^*$  is a square if and only if  $u^{\frac{p-1}{2}} \equiv 1 \pmod{p}$*

**Proposition 2.21.** *Suppose  $p \geq 3$ . Let  $x = p^{v_p(x)}u \in \mathbb{Q}_p \setminus \{0\}$ . Then  $x$  is a square in  $\mathbb{Q}_p$  if and only if  $v_p(x)$  is even and the image  $\bar{u}$  of  $u$  in  $(\mathbb{Z}_p/p\mathbb{Z}_p)^* = (\mathbb{Z}/p\mathbb{Z})^*$ , is a square.*

**Proposition 2.22.** *Suppose  $p = 2$ . Let  $x = p^{v_p(x)}u \in \mathbb{Q}_p \setminus \{0\}$ . Then  $x$  is a square in  $\mathbb{Q}_p$  if and only if  $v_p(x)$  is even and  $u \equiv 1 \pmod{8}$ .*

Let  $X \subset \mathbb{Q}_p$ . The set of all continuous functions  $f : X \rightarrow \mathbb{Q}_p$  is denoted by  $C(X)$  and the set of bounded continuous functions is denoted by  $C_b(X)$ . If  $X$  is compact, then  $C(X) = C_b(X)$ . Let  $B(X, \mathbb{Q}_p)$  be the set of all bounded functions from  $X$  to  $\mathbb{Q}_p$ . Define a norm on  $B(X, \mathbb{Q}_p)$  by

$$\|f\|_\infty = \sup_{x \in X} |f(x)|_p.$$

Then  $C_b(X)$  is a closed subspace of the non-Archimedean Banach space  $B(X, \mathbb{Q}_p)$ . A function  $f : X \rightarrow \mathbb{Q}_p$  is said to be locally constant if for any point  $x \in X$ , there exists a neighborhood  $U$  such that  $f$  is constant on  $U \cap X$ . Locally constant functions are continuous functions and dense in the Banach space  $C_b(X)$ .

Let  $U \subset \mathbb{Q}_p$  be an open set. The function  $f : U \rightarrow \mathbb{Q}_p$  is differentiable at  $x \in U$  if the following limit exists:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

In this case,  $f'(x)$  is called the derivative of  $f$  at  $x$ . If  $f$  is differentiable at every point in  $U$ , we say  $f$  is differentiable on  $U$ .

We remark that the classical mean-value theorem in  $p$ -adic analysis is invalid, and there exists some function with derivative equals zero everywhere but is not constant.

A function  $f : X \rightarrow \mathbb{Q}_p$  is continuously differentiable at a point  $a \in X$  if the following limit exists

$$\lim_{\Delta \ni (x,y) \rightarrow (0,0)} \frac{f(x) - f(y)}{x - y},$$

where  $\Delta = \{(x, y) : x, y \in X, x \neq y\}$ . Obviously, continuously differentiability implies differentiability.

**Proposition 2.23.** *If  $f : X \rightarrow \mathbb{Q}_p$  is continuously differentiable at  $a \in X$ , then there exists a neighborhood  $U$  of  $a$  such that*

$$|f(x) - f(y)|_p = |f'(a)|_p |x - y|_p \quad x, y \in X \cap U.$$

A function  $f : X \rightarrow \mathbb{Q}_p$  is 1-Lipschitz if

$$|f(x) - f(y)|_p \leq |x - y|_p \quad \forall x, y \in X.$$

Any polynomial  $f \in \mathbb{Z}_p[x]$  is 1-Lipschitz, and any 1-Lipschitz function is equicontinuous. Thus by Proposition 2.17, we have

**Proposition 2.24.** *The minimality of  $f \in \mathbb{Z}_p[x]$  on a closed subset of  $\mathbb{Z}_p$  is equivalent to its ergodicity, uniquely ergodicity and strictly ergodicity respect to the normalized Haar measure.*

For the ergodicity, we have the following theorem.

**Theorem 2.25** ([5], [33]). *Let  $X \subset \mathbb{Z}_p$  be a compact set, and  $f \in \mathbb{Z}_p[x]$ .  $f$  is measure-preserving (or, accordingly, ergodic) if and only if it is bijective (accordingly, transitive) on  $X/p^n\mathbb{Z}_p$  for all  $n \geq 1$ .*

## 2.3 Entropy and Hausdorff dimension

We recommend the book of Walters ([140]) and Pollicott and Yuri ([127]) for the definitions and properties of the classical topological entropy and measure-theoretic entropy, the books of Falconer ([44], [45]), Mattila ([102]), and Pesin ([119]) for the basic knowledge of Carathéodory structure and Hausdorff dimension, and the original paper of Bowen ([24]) for the definition of the generalized version of topological entropy and its properties.

### 2.3.1 Topological entropy and measure-theoretic entropy

Let  $(X, T)$  be a topological dynamical system with  $X$  a compact metric space and  $T$  a continuous transformation. Let  $\alpha = \{A_i\}, \beta = \{B_j\}$  be two (finite) covers of  $X$ . The refinement of  $\alpha$  and  $\beta$  is defined by

$$\alpha \vee \beta := \{A_i \cap B_j : A_i \cap B_j \neq \emptyset\}.$$

For finite cover  $\alpha = \{A_1, \dots, A_n\}$ , define the topological entropy of  $\alpha$  by  $H(\alpha) := \log N(\alpha)$ , where  $N(\alpha)$  is the smallest number of sets of subcovers of  $\alpha$ . The topological entropy of  $T$  relative to a cover  $\alpha$  is defined by

$$h(T, \alpha) := \limsup_{n \rightarrow \infty} \frac{1}{n} H(\bigvee_{i=0}^{n-1} T^{-i} \alpha).$$

The topological entropy of  $T$  is defined by

$$h_{\text{top}}(T) := \sup\{h(T, \alpha) : \alpha \text{ is a finite cover of } X\}.$$

**Proposition 2.26.** *Any expansive homeomorphism has finite topological entropy.*

**Proposition 2.27.** *The homeomorphisms of the unit circle and  $[0, 1]$  are of topological entropy zero.*

Let  $(X, \mathcal{B}, \mu, T)$  be a measure-preserving dynamical system. We say  $\alpha = \{A_i\}_{i \in I}$  is a countable measurable partition of the probability space  $(X, \mathcal{B}, \mu)$  if

- (1)  $X = \cup_i A_i$  up to a set of zero  $\mu$ -measure, and
- (2)  $A_i \cap A_j = \emptyset (i \neq j)$  up to a set of zero  $\mu$ -measure.

Define the information function  $I(\alpha) : X \rightarrow \mathbb{R}$  by

$$I(\alpha)(x) := - \sum_i \log \mu(A_i) \mathbb{1}_{A_i}(x),$$

where  $\mathbb{1}_{A_i}$  is the indicator function of the set  $A_i$ . The entropy of the partition  $\alpha$  is defined by

$$H(\alpha) := \int I(\alpha) d\mu = - \sum_{A \in \alpha} \mu(A) \log \mu(A).$$

Given a sub- $\sigma$ -algebra  $\mathcal{A} \subset \mathcal{B}$ , the conditional information function  $I(\alpha|\mathcal{A}) : X \rightarrow \mathbb{R}$  is defined by

$$I(\alpha|\mathcal{A})(x) := - \sum_i \log \mu(A_i|\mathcal{A})(x) \mathbb{1}_{A_i}(x),$$

where  $\mu(A_i|\mathcal{A})(x) = \mathbb{E}(\mathbb{1}_{A_i}(x)|\mathcal{A})(x)$  is the conditional measure. The conditional entropy of  $\alpha$  with respect to  $\mathcal{A}$  is defined by

$$H(\alpha|\mathcal{A}) := \int I(\alpha|\mathcal{A}) d\mu.$$

For partition  $\alpha = \{A_i\}$ , we write

$$\bigvee_{i=0}^{n-1} T^{-i} \alpha := \{A_{j_0} \cap T^{-1} A_{j_1} \cap \cdots \cap T^{-(n-1)} A_{j_n} : A_{j_i} \in \alpha\}$$

The entropy of partition  $\alpha$  is defined by

$$h(T, \alpha) := \lim_{n \rightarrow \infty} \frac{H_n(\alpha)}{n} = \inf_{n \geq 1} \frac{H_n(\alpha)}{n},$$

where  $H_n(\alpha) := H(\bigvee_{i=0}^{n-1} T^{-i} \alpha)$  can be proved to satisfy the subadditive property.

Given two partitions  $\alpha$  and  $\beta$ , the notation  $H(\alpha|\beta)$  means  $H(\alpha|\hat{\beta})$ , where  $\hat{\beta}$  is the  $\sigma$ -algebra generated by  $\beta$ .

**Proposition 2.28.** *We have an alternative definition of the entropy of  $\alpha$ :*

$$h(T, \alpha) = \lim_{n \rightarrow \infty} H(\alpha | \bigvee_{i=0}^{n-1} T^{-i} \alpha).$$

The measure-theoretic entropy of  $T$  is defined by

$$h_\mu(T) := \sup_{\{\alpha: H(\alpha) < \infty\}} h(T, \alpha).$$

**Proposition 2.29.** *Any rotation of a compact metric abelian group is of measure-theoretic entropy zero.*

Let  $(X, T)$  be a topological dynamical system with  $X$  a compact metric space and  $T$  a continuous transformation. The entropy map is defined by  $\mu \rightarrow h_\mu(T)$  from  $\mathcal{M}_{\text{inv}}$  to  $[0, \infty]$ .

**Proposition 2.30.** *The entropy map is affine, i.e.,*

$$h_{p\mu + (1-p)\nu}(T) = ph_\mu(T) + (1-p)h_\nu(T), \quad \mu, \nu \in \mathcal{M}_{\text{inv}}.$$

**Proposition 2.31.** *If  $T$  is expansive, then the entropy map is upper semi-continuous, i.e., for  $\mu \in \mathcal{M}_{\text{inv}}$ , for any  $\epsilon > 0$  there exists a neighborhood  $U$  of  $\mu$  such that for any  $\nu \in \mathcal{M}_{\text{inv}}$ , one has  $h_\nu(T) < h_\mu(T) + \epsilon$ .*

**Theorem 2.32.** *We have the following variational principle:*

$$h_{\text{top}}(T) = \sup\{h_\mu : \mu \in \mathcal{M}_{\text{inv}}\}.$$

### 2.3.2 Carathéodory structure and Hausdorff dimension

Let  $X$  be a metric space with metric  $d$ ,  $\mathcal{F}$  a collection of subsets of  $X$  and  $\eta$  a non-negative function on  $\mathcal{F}$ . Assume that

- (1) for any  $\delta > 0$ , there are  $E_1, E_2, \dots \in \mathcal{F}$  such that  $X = \bigcup_{i=1}^{\infty} E_i$  and  $\sup\{\text{diam}(E_i)\} \leq \delta$  (here and the follows, “diam” stands for the diameter of the sets),
- (2) for any  $\delta > 0$ , there is an  $\epsilon > 0$  such that  $\eta(E) \leq \delta$  for any  $E \in \mathcal{F}$  with  $\text{diam}(E) \leq \epsilon$ .

Then  $\tau = (\mathcal{F}, d, \eta)$  will be called a Carathéodory structure. For  $Z \subset X$  define

$$\psi_{\delta}(Z) := \inf \left\{ \sum_{i=1}^{\infty} \eta(E_i) : Z \subset \bigcup_i E_i, 0 < \text{diam}(E_i) < \delta \right\}.$$

We see that  $\psi_{\delta}$  is monotonic, thus we define the Carathéodory outer measure  $\psi = \psi(\tau) = \psi(\mathcal{F}, d, \eta)$  by

$$\psi(Z) := \lim_{\delta \rightarrow 0} \psi_{\delta}(Z) = \sup_{\delta > 0} \psi_{\delta}(Z).$$

We know that (see [102], p. 55.)

**Proposition 2.33.** *The function  $\psi$  is a Borel measure and if the members of  $\mathcal{F}$  are Borel sets, then  $\psi$  is Borel regular.*

Suppose  $X$  is separable. Take  $\mathcal{F}$  to be the collection of all subsets of  $X$ . For  $s \geq 0$ , let  $\eta(E) = \text{diam}^s(E)$  and assume  $0^0 = 1$  and  $\text{diam}(\emptyset) = 0$  for convenience. Then the Carathéodory outer measure becomes the classical Hausdorff outer measure. Denote by  $\mathcal{H}^s$  the Hausdorff outer measure. An important fact is that it will define the same Hausdorff outer measure  $\mathcal{H}^s$  if we replace  $\mathcal{F}$  by the collection of all the open sets or the collection of all the closed sets.

The Hausdorff dimension of a subset  $Z$  is then defined by

$$\dim Z := \inf\{s : \mathcal{H}^s(Z) < \infty\} = \sup\{s : \mathcal{H}^s(Z) = \infty\}.$$

We have

**Proposition 2.34.** *The Hausdorff dimension of  $Z$  will be the same if we replace  $\mathcal{F}$  by the family of the open balls or the family of close balls.*

### 2.3.3 Bowen's entropy and Bowen Lemma

We still consider the topological dynamical system  $(X, T)$  where  $X$  is a compact metric space with metric  $d$  and that  $T : X \rightarrow X$  is a continuous transformation.

For any integer  $n \geq 1$ , define the Bowen metric  $d_n$  on  $X$  by

$$d_n(x, y) = \max_{0 \leq j < n} d(T^j x, T^j y).$$

For any  $\epsilon > 0$ , denote by  $B_n(x, \epsilon)$  the open  $d_n$ -ball centered at  $x$  of radius  $\epsilon$ .

We follow Bowen's definition of entropy ([25]). Let  $Z \subset X$  be a subset of  $X$ . Let  $\epsilon > 0$ . We say a collection (at most countable)  $R = \{B_{n_i}(x_i, \epsilon)\}$  covers  $Z$  if  $Z \subset \bigcup_i B_{n_i}(x_i, \epsilon)$ . For such a collection  $R$ , put  $n(R) = \min_i n_i$ . Let  $s \geq 0$ . Define

$$H_n^s(Z, \epsilon) = \inf_R \sum_i \exp(-sn_i),$$

where the infimum is taken over all covers  $R$  of  $Z$  with  $n(R) \geq n$ . The quantity  $H_n^s(Z, \epsilon)$  is non-decreasing as a function of  $n$ , so the following limit exists

$$H^s(Z, \epsilon) = \lim_{n \rightarrow \infty} H_n^s(Z, \epsilon).$$

(We remind the readers that the function  $H^s(\cdot, \epsilon)$  is a Carathéodory outer measure with  $\mathcal{F}$  being the family of  $\epsilon$ -Bowen balls  $\{B_n(x, \epsilon)\}$ , the metric being  $d_n$  and the function  $\eta$  defined by  $\eta(B_n(x, \epsilon)) = e^{-n}$ .)

For the quantity  $H^s(Z, \epsilon)$  considered as a function of  $s$ , there exists a critical value, which we denote by  $h_{\text{top}}(Z, \epsilon)$ , such that

$$H^s(Z, \epsilon) = \begin{cases} +\infty, & s < h_{\text{top}}(Z, \epsilon) \\ 0, & s > h_{\text{top}}(Z, \epsilon). \end{cases}$$

One can prove that the following limit exists

$$h_{\text{top}}(Z) = \lim_{\epsilon \rightarrow 0} h_{\text{top}}(Z, \epsilon).$$

The quantity  $h_{\text{top}}(Z)$  is called the *topological entropy* of  $Z$ .

For  $x \in X$ , we denote by  $V(x)$  the set of all weak limits of the sequence of probability measures  $n^{-1} \sum_{j=0}^{n-1} \delta_{T^j x}$ . It is clear that  $V(x) \neq \emptyset$  and  $V(x) \subset \mathcal{M}_{\text{inv}}$  for any  $x$ . The following Bowen lemma is important in estimating the entropies.

**Lemma 2.35** (Bowen [24]). *For  $t \geq 0$ , consider the set*

$$B^{(t)} = \{x \in X : \exists \mu \in V(x) \text{ satisfying } h_\mu \leq t\}.$$

*Then  $h_{\text{top}}(B^{(t)}) \leq t$ .*

Let  $\mu \in \mathcal{M}_{\text{inv}}$  be an invariant measure. A point  $x \in X$  such that  $V(x) = \{\mu\}$  is said to be generic for  $\mu$ . Recall the definition of  $G_\mu$  (Subsection 1.3.2), we know that  $G_\mu$  is the set of all generic points for  $\mu$ . Bowen proved that  $h_{\text{top}}(G_\mu) \leq h_\mu$  for any invariant measure. This assertion can be deduced by using Lemma 2.35. In fact, the reason is that  $x \in G_\mu$  implies  $\mu \in V(x)$ . Bowen also proved that the inequality becomes equality when  $\mu$  is ergodic. However, in general, we do not have the equality  $h_{\text{top}}(G_\mu) = h_\mu$  (saturatedness, see Subsection 1.3.2) and it is even possible that  $G_\mu = \emptyset$ . Cajar ([30]) proved that full symbolic spaces are saturated. Concerning the  $\mu$ -measure of  $G_\mu$ , it is well known that  $\mu(G_\mu) = 1$  or  $0$  according as whether  $\mu$  is ergodic or not (see the book of Denker, Grillenberger and Sigmund ([39])).

## 2.4 Basic properties of continued fractions

In this section, we collect some known facts and establish some elementary properties of continued fractions that will be used later. For a wealth of classical results about continued fractions, see the books by J. Cassels ([32]), G. Hardy and E. Wright ([65]). The books by P. Billingsley ([16]), I. Cornfeld, S. Fomin and Ya. Sinai ([36]) contain an excellent introduction to the dynamics of the Gauss transformations and its connection with Diophantine approximation.

Let  $x = [a_1(x), a_2(x), \dots]$  be the continued fraction expansion of  $x \in [0, 1) \setminus \mathbb{Q}$ . Denote by  $p_n/q_n$  the usual  $n$ -th *convergent* of the continued fraction of  $x$ , defined by

$$\frac{p_n}{q_n} := [a_1(x), \dots, a_n(x)] := \frac{1}{a_1(x) + \frac{1}{a_2(x) + \dots + \frac{1}{a_n(x)}}}.$$

It is known (see [81] p.9) that  $p_n, q_n$  can be obtained by the recursive relation:

$$p_{n+1} = a_{n+1}p_n + p_{n-1}, \quad q_{n+1} = a_{n+1}q_n + q_{n-1}, \quad n \geq 1 \quad (2.1)$$

with the convention that  $p_{-1} = q_0 = 1, p_0 = q_{-1} = 0$ . Furthermore, we have

**Lemma 2.36** ([72] p.5). Let  $\varepsilon_1, \dots, \varepsilon_n \in \mathbb{R}^+$ . Define inductively

$$Q_{-1} = 0, \quad Q_0 = 1, \quad Q_n(\varepsilon_1, \dots, \varepsilon_n) = \varepsilon_n Q_{n-1}(\varepsilon_1, \dots, \varepsilon_{n-1}) + Q_{n-2}(\varepsilon_1, \dots, \varepsilon_{n-2}).$$

( $Q_n$  is commonly called a continuant.) Then we have

- i)  $Q_n(\varepsilon_1, \dots, \varepsilon_n) = Q_n(\varepsilon_n, \dots, \varepsilon_1)$ ,
- ii)  $q_n = Q_n(a_1, \dots, a_n)$ ,  $p_n = Q_{n-1}(a_2, \dots, a_n)$ .

As consequences, we have the following results.

**Lemma 2.37** ([81]). For any  $a_1, a_2, \dots, a_n, b_1, \dots, b_m \in \mathbb{N}$ , let  $q_n = q_n(a_1, \dots, a_n)$  and  $p_n = p_n(a_1, \dots, a_n)$ . We have

- (i)  $p_{n-1}q_n - p_nq_{n-1} = (-1)^n$ ;
- (ii)  $q_{n+m}(a_1, \dots, a_n, b_1, \dots, b_m) = q_n(a_1, \dots, a_n)q_m(b_1, \dots, b_m) + q_{n-1}(a_1, \dots, a_{n-1})p_{m-1}(b_1, \dots, b_{m-1})$ ;
- (iii)  $q_n \geq 2^{\frac{n-1}{2}}$ ,  $\prod_{k=1}^n a_k \leq q_n \leq \prod_{k=1}^n (a_k + 1)$ .

**Lemma 2.38** ([148]). For any  $a_1, a_2, \dots, a_n, b \in \mathbb{N}$ ,

$$\frac{b+1}{2} \leq \frac{q_{n+1}(a_1, \dots, a_j, b, a_{j+1}, \dots, a_n)}{q_n(a_1, \dots, a_j, a_{j+1}, \dots, a_n)} \leq b+1 \quad (\forall 1 \leq j < n).$$

For any  $a_1, a_2, \dots, a_n \in \mathbb{N}$ , let

$$I_n(a_1, a_2, \dots, a_n) = \{x \in [0, 1) : a_k(x) = a_k, 1 \leq k \leq n\}$$

which is called an  $n$ -th basic interval or a rank  $n$  fundamental interval. Sometimes, we also write it as  $I(a_1, a_2, \dots, a_n)$ .

**Lemma 2.39** ([72] p.18). For any  $a_1, a_2, \dots, a_n \in \mathbb{N}$ , the rank  $n$  fundamental interval  $I_n(a_1, a_2, \dots, a_n)$  is the interval with the endpoints  $p_n/q_n$  and  $(p_n + p_{n-1})/(q_n + q_{n-1})$ . As a consequence, the length of  $I_n(a_1, \dots, a_n)$  is equal to

$$\left| I_n(a_1, \dots, a_n) \right| = \frac{1}{q_n(q_n + q_{n-1})}. \quad (2.2)$$

Consequently, we have an estimate

$$\frac{1}{2q_n^2} \leq \left| I(a_1, \dots, a_n) \right| \leq \frac{1}{q_n^2}. \quad (2.3)$$

We will denote  $I_n(x)$  the rank  $n$  fundamental interval that contains  $x$ , i.e.  $I_n(x) = I_n(a_1(x), \dots, a_n(x))$ . Let  $B(x, r)$  denotes the ball centered at  $x$  with radius  $r$ . For any  $x \in I_n(a_1, \dots, a_n)$ , we have the following relationship between the ball  $B(x, |I_n(a_1, \dots, a_n)|)$  and  $I_n(a_1, \dots, a_n)$ , which is called *regular property* in [23].

**Lemma 2.40** ([23]). *Let  $x = [a_1, a_2, \dots]$ . We have:*

- (i) *When  $a_n \neq 1$ ,  $B(x, |I_n(x)|) \subset \bigcup_{j=-1}^3 I_n(a_1, \dots, a_n + j)$ .*
- (ii) *When  $a_n = 1$  and  $a_{n-1} \neq 1$ ,  $B(x, |I_n(x)|) \subset \bigcup_{j=-1}^3 I_{n-1}(a_1, \dots, a_{n-1} + j)$ .*
- (iii) *When  $a_n = 1$  and  $a_{n-1} = 1$ ,  $B(x, |I_n(x)|) \subset I_{n-2}(a_1, \dots, a_{n-2})$ .*

From Lemma 2.37 and Lemma 2.39, we deduce the following estimation:

**Lemma 2.41.** *For any irrational number  $x \in [0, 1)$  and  $n \geq 1$*

$$2^{-(2n+1)} \leq |I_n(x)| \left( \prod_{k=1}^n a_k(x) \right)^2 \leq 1.$$

Suppose  $a_n \neq 1$ . Let

$$I'_n(x) = I_n(a_1, \dots, a_{n-1}, j-1) \quad \text{and} \quad I''_n(x) = I_n(a_1, \dots, a_{n-1}, j+1)$$

be the fundamental intervals adjacent to  $I_n(x)$ . By (2.1) and Lemma 2.39, one has the following lemma.

**Lemma 2.42.** *If  $a_n(x) \geq 2$ , then the lengths of the fundamental interval  $I_n(x)$  and its two adjacent intervals  $I'_n(x)$  and  $I''_n(x)$  are related by*

$$|I_n(x)|/3 \leq |I''_n(x)| \leq |I_n(x)| \leq |I'_n(x)| \leq 3|I_n(x)|.$$

## Chapter 3

### Minimality of the Polynomials on $\mathbb{Z}_p$

In this chapter, a polynomial of degree  $\geq 2$  with coefficients in  $\mathbb{Z}_p$  is considered as dynamical system on  $\mathbb{Z}_p$ . It is proved that the dynamics of such a system is totally described by its minimal subsystems. For any quadratic polynomial on  $\mathbb{Z}_2$ , we exhibit all its minimal subsystems. <sup>1</sup>

#### 3.1 Introduction

Let  $\mathbb{Z}_p$  be the ring of  $p$ -adic integers ( $p$  being a prime number). Let  $f \in \mathbb{Z}_p[x]$  be a polynomial of coefficients in  $\mathbb{Z}_p$  and with degree  $\deg f \geq 2$ . It is well known that  $f : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$  is a 1-Lipschitz map. In this chapter we study the minimal decomposition of the topological dynamical system  $(\mathbb{Z}_p, f)$ .

One of interesting problems well studied in the literature is the minimality of the system  $f : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$  ([4, 5, 33, 40, 82, 84, 85, 86, 89, 95]). If the system is not minimal, we will prove the following result which says that the system admits at most countably many minimal subsystems which totally describe the dynamics of the system (see Theorem 3.18).

**Theorem A.** *Let  $f \in \mathbb{Z}_p[x]$  with  $\deg f \geq 2$ . We have the following decomposition*

$$\mathbb{Z}_p = A \sqcup B \sqcup C$$

where  $A$  is the finite set consisting of all periodic points of  $f$ ,  $B = \bigsqcup_i B_i$  is the union of all (at most countably many) clopen invariant sets such that each subsystem  $f : B_i \rightarrow B_i$  is minimal, and each point in  $C$  lies in the attracting basin of a periodic orbit or of a minimal subsystem.

Affine maps were fully studied in [50] (see also [34]). We will refer to the above decomposition as the minimal decomposition of the system  $f : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$ .

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<sup>1</sup>A. H. Fan and L. M. Liao, *On minimal decomposition of  $p$ -adic polynomial dynamical systems*, preprint.

A finite periodic orbit of  $f$  is trivially a minimal set. The theorem shows that there are only a finite number of periodic orbits. This is actually a consequence of the following theorem. The results are due to Pezda [125] and to Desjardins and Zieve [40] in the case  $p \geq 3$ . (see Theorem 3.17).

**Theorem B.** *Let  $f \in \mathbb{Z}_p[x]$ .*

- 1) *If  $p \geq 5$ , the periods of periodic orbits are of the form  $ab$  with  $a|(p-1)$  and  $1 \leq b \leq p$ .*
- 2) *If  $p = 3$ , the periods of periodic orbits must be 1, 2, 3, 4, 6 or 9.*
- 3) *If  $p = 2$ , the periods of periodic orbits must be 1, 2 or 4.*

In a recent work, Chabert, Fan and Fares [33] showed that each clopen set  $B_i$  must be a Legendre set and that in general, on any Legendre set there exist minimal 1-Lipschitz maps. We will show that for a polynomial system, the Legendre sets  $B_i$  are of special forms. Let  $(p_s)_{s \geq 1}$  be a sequence of positive integers such that  $p_s | p_{s+1}$  for every  $s \geq 1$ . We denote by  $\mathbb{Z}_{(p_s)}$  the inverse limit of  $\mathbb{Z}/p_s\mathbb{Z}$ , which is said to be an odometer. The map  $x \rightarrow x + 1$  is called the adding machine on  $\mathbb{Z}_{(p_s)}$ . We will prove the following Theorem (see Theorem 3.21).

**Theorem C.** *Let  $f \in \mathbb{Z}_p[x]$  with  $\deg f \geq 2$ . If  $E$  is a minimal clopen invariant set of  $f$ , then  $f : E \rightarrow E$  is conjugate to the adding machine on an odometer  $\mathbb{Z}_{(p_s)}$ , where*

$$(p_s) = (k, kd, kdp, kdp^2, \dots)$$

*with some  $k \in \mathbb{N}, k \leq p$  and  $d|(p-1)$ .*

There are few works on the minimal decomposition. Multiplications on  $\mathbb{Z}_p$  ( $p \geq 3$ ) were studied by Coelho and Parry [34] and general affine maps were studied by Fan, Li, Yao and Zhou [50]. In these cases and only in these cases, we have known the minimal decomposition.

It seems much more difficult to study higher order polynomials, even the quadratic polynomials. In this chapter, we try to attack the problem. The above mentioned results are obtained by using some idea coming from Desjardins and Zieve's work [40]. Let  $E$  be an  $f$ -invariant compact set. It is now well known that the subsystem  $(E, f)$  is minimal if and only if the induced map  $f_n : E/p^n\mathbb{Z} \rightarrow E/p^n\mathbb{Z}$  is transitive for any  $n \geq 1$  (see [5, 33]). The idea of Desjardins and Zieve

is to establish relations between  $f_n$ 's cycles and  $f_{n+1}$ 's cycles, by linearizing the map  $f_{n+1}^k$  on a  $k$ -cycle of  $f_n$ .

For an arbitrary 2-adic quadratic polynomial

$$f(x) = ax^2 + bx + c$$

on  $\mathbb{Z}_2$ , we find all its minimal components. In fact, such a system  $f : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$  is conjugate to one of the following quadratic polynomials

$$x^2 - \lambda, \quad x^2 + bx, \quad x^2 + x - d$$

where  $\lambda \in \mathbb{Z}_2$ ,  $b \equiv 1 \pmod{2}$  and  $\sqrt{d} \notin \mathbb{Z}_2$ . The obtained results are stated in Theorems 3.22-3.30. Let us state some of these results.

**Theorem D.** *Consider the polynomial  $x^2 - \lambda$  on  $\mathbb{Z}_2$ .*

1) *If  $\lambda \equiv 0 \pmod{4}$ , then there are two attracting fixed points, one in  $4\mathbb{Z}_2$  with basin  $2\mathbb{Z}_2$ , and the other one in  $1 + 4\mathbb{Z}_2$  with basin  $1 + 2\mathbb{Z}_2$ .*

2) *If  $\lambda \equiv 1 \pmod{4}$ , then the whole  $\mathbb{Z}_2$  is attracted into a periodic orbit of period 2 with one orbit point in  $4\mathbb{Z}_2$ , and the other one in  $3 + 4\mathbb{Z}_2$ .*

3) *If  $\lambda \equiv 2 \pmod{4}$ , then there are two attracting fixed points, one in  $2 + 4\mathbb{Z}_2$  with basin  $2\mathbb{Z}_2$ , and the other one in  $3 + 4\mathbb{Z}_2$  with basin  $1 + 2\mathbb{Z}_2$ .*

4) *If  $\lambda \equiv 3 \pmod{4}$ , then the whole  $\mathbb{Z}_2$  is attracted into a periodic orbit of period 2 with one orbit point in  $1 + 4\mathbb{Z}_2$ , and the other one in  $2 + 4\mathbb{Z}_2$ .*

**Theorem E.** *Consider the polynomial  $f(x) = x^2 + x$  on  $\mathbb{Z}_2[x]$ . There is one fixed point 0. We have  $f(1 + 2\mathbb{Z}_2) \subset 2\mathbb{Z}_2$  and we can decompose  $2\mathbb{Z}_2$  to be*

$$2\mathbb{Z}_2 = \{0\} \sqcup \left( \bigsqcup_{n \geq 2} 2^{n-1} + 2^n \mathbb{Z}_2 \right).$$

*Each  $2^{n-1} + 2^n \mathbb{Z}_2$  ( $n \geq 2$ ) consists of  $2^{n-2}$  pieces of minimal components:*

$$2^{n-1} + t2^n + 2^{2n-2} \mathbb{Z}_2, \quad t = 0, \dots, 2^{n-2} - 1.$$

**Theorem F.** *Consider the polynomial  $f(x) = x^2 + x - d$  with  $d \equiv 3 \pmod{4}$ . Then  $f(2\mathbb{Z}_2) \subset 1 + 2\mathbb{Z}_2$  and  $1 + 2\mathbb{Z}_2$  is the unique minimal component of  $f$ .*

This chapter is organized as follows. In Section 3.2, we give a detailed recall of the idea in [40] by studying the induced dynamical systems  $f_n$  on  $\mathbb{Z}/p^n\mathbb{Z}$  when  $p \geq 3$ . In Section 3.3, we consider the case of  $p = 2$  which was not developed in [40]. As we shall see, the case  $p = 2$  is not exactly the same as the case  $p \geq 3$ . In Section 3.4 and 3.5, we discuss how does a minimal component form by analyzing the reduced maps  $f_n$  ( $n \geq 1$ ) and prove the decomposition theorem. In Section 3.5, we discuss the possible forms of minimal components. In Section 3.6, we give a detailed description of the minimal decomposition for an arbitrary quadratic polynomial system on  $\mathbb{Z}_2$ .

### 3.2 Induced dynamics on $\mathbb{Z}/p^n\mathbb{Z}_p$ ( $p \geq 3$ )

The main core of this section follows Desjardins and Zieve [40]. We shall give more details and rewrite some proofs for reader's convenience. The case  $p = 2$ , which is a little bit special, will be fully discussed in the next section.

Let  $p \geq 3$  be a prime (we may replace 3 by 2 in many places). Let  $n \geq 1$  be a positive integer. Denote by  $f_n$  the induced mapping of  $f$  on  $\mathbb{Z}/p^n\mathbb{Z}$ , i.e.,

$$f_n(x \bmod p^n) = f(x) \bmod p^n.$$

Many properties of the dynamics  $f$  are linked to those of  $f_n$ . One is the following.

**Theorem 3.1** ([5], [33]). *Let  $f \in \mathbb{Z}_p[x]$  and  $E \subset \mathbb{Z}_p$  be a compact  $f$ -invariant set. Then  $f : E \rightarrow E$  is minimal if and only if  $f_n : E/p^n\mathbb{Z}_p \rightarrow E/p^n\mathbb{Z}_p$  is minimal for each  $n \geq 1$ .*

It is clear that if  $f_n : E/p^n\mathbb{Z}_p \rightarrow E/p^n\mathbb{Z}_p$  is minimal, then  $f_m : E/p^m\mathbb{Z}_p \rightarrow E/p^m\mathbb{Z}_p$  is also minimal for each  $1 \leq m < n$ . So, the above theorem shows that it is important to investigate under what condition, the minimality of  $f_n$  implies that of  $f_{n+1}$ .

Assume that  $\sigma = (x_1, \dots, x_k) \subset \mathbb{Z}/p^n\mathbb{Z}$  is a cycle of  $f_n$  of length  $k$  (also called  $k$ -cycle), i.e.,

$$f_n(x_1) = x_2, \dots, f_n(x_i) = x_{i+1}, \dots, f_n(x_k) = x_1.$$

In this case we also say  $\sigma$  is at level  $n$ . Let

$$X := \bigsqcup_{i=1}^k X_i \quad \text{where} \quad X_i := \{x_i + p^n t; t = 0, \dots, p-1\} \subset \mathbb{Z}/p^{n+1}\mathbb{Z}.$$

Then

$$f_{n+1}(X_i) \subset X_{i+1} \quad (1 \leq i \leq k-1) \quad \text{and} \quad f_{n+1}(X_k) \subset X_1.$$

In the following we shall study the behavior of the finite dynamics  $f_{n+1}$  on the  $f_{n+1}$ -invariant set  $X$  and determine all cycles in  $X$  of  $f_{n+1}$ , which will be called *lifts* of  $\sigma$ . Remark that the length of any lift  $\tilde{\sigma}$  of  $\sigma$  is a multiple of  $k$ .

Let  $g := f^k$  be the  $k$ -th iterate of  $f$ . Then, any point in  $\sigma$  is fixed by  $g_n$ , the  $n$ -th induced map of  $g$ . For  $x \in \sigma$ , denote

$$a_n(x) := g'(x) = \prod_{j=0}^{k-1} f'(f^j(x)) \quad (3.1)$$

$$b_n(x) := \frac{g(x) - x}{p^n} = \frac{f^k(x) - x}{p^n}. \quad (3.2)$$

The values on the cycle  $\sigma = (x_1, \dots, x_k)$  of the functions  $a_n$  and  $b_n$  are important for our purpose. With them, we define affine maps

$$\Phi(x, t) = b_n(x) + a_n(x)t \quad (x \in \sigma, t \in \mathbb{Z}/p\mathbb{Z}).$$

The 1-order Taylor expansion of  $g$  at  $x$  implies

$$g(x + p^n t) \equiv x + p^n b_n(x) + p^n a_n(x)t \equiv x + p^n \Phi(x, t) \pmod{p^{2n}}. \quad (3.3)$$

An important consequence of the last formula shows that  $g_{n+1} : X_i \rightarrow X_i$  is conjugate to the linear map

$$\Phi(x_i, \cdot) : \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z}.$$

We could call it the linearization of  $g_{n+1} : X_i \rightarrow X_i$ .

As we shall see in the following lemma, the coefficient  $a_n(x) \pmod{p}$  is always constant on  $X_i$  and the coefficient  $b_n(x) \pmod{p}$  is also constant on  $X_i$  but under the condition  $a_n(x) \equiv 1 \pmod{p}$ .

Denote by  $v_p(n)$  the  $p$ -valuation of  $n$ .

**Lemma 3.2.** *Let  $n \geq 1$  and  $\sigma = (x_1, \dots, x_k)$  be a  $k$ -cycle of  $f_n$ .*

(i) *For  $1 \leq i, j \leq k$ , we have*

$$a_n(x_i) \equiv a_n(x_j) \pmod{p^n}.$$

(ii) For  $1 \leq i \leq k$  and  $0 \leq t \leq p-1$ , we have

$$a_n(x_i + p^n t) \equiv a_n(x_i) \pmod{p^n}.$$

(iii) For  $1 \leq i \leq k$  and  $0 \leq t \leq p-1$ , we have

$$b_n(x_i + p^n t) \equiv b_n(x_i) \pmod{p^A},$$

where  $A := \min\{v_p(a_n(x_i) - 1), n\} = \min\{v_p(a_n(x_j) - 1), n\}$  for  $1 \leq i, j \leq k$ .

(iv) If  $a_n(x_i) \not\equiv 0 \pmod{p}$  for some  $1 \leq i \leq k$ , then for all  $1 \leq i, j \leq k$  we have

$$\min\{v_p(b_n(x_i)), n\} = \min\{v_p(b_n(x_j)), n\}.$$

Consequently,  $\min\{v_p(b_n(x_i)), A\} = \min\{v_p(b_n(x_j)), A\}$ .

*Proof.* Assertion (i) follows directly from the definition of  $a_n(x_i)$  and the fact that  $\sigma = (x_i, f_n(x_i), \dots, f_n^{k-1}(x_i))$ . The assertion (ii) is a direct consequence of

$$a_n(x_i + p^n t) \equiv \prod_{j=1}^k f'(f^j(x_i + p^n t)) \equiv \prod_{j=1}^k f'(f^j(x_i)) \pmod{p^n}.$$

The 1-order Taylor expansion of  $g$  at  $x_i$  gives

$$g(x_i + p^n t) - (x_i + p^n t) \equiv p^n \left( \frac{g(x_i) - x_i}{p^n} \right) + p^n t (g'(x_i) - 1) \pmod{p^{2n}}.$$

Hence

$$b_n(x_i + p^n t) \equiv b_n(x_i) + t(a_n(x_i) - 1) \pmod{p^n}.$$

Then (iii) follows.

Write

$$g(f(x_i)) - f(x_i) = f(f^k(x_i)) - f(x_i) = f(x_i + p^n b_n(x_i)) - f(x_i).$$

The 1-order Taylor expansion  $f$  at  $x_i$  leads to

$$g(f(x_i)) - f(x_i) \equiv p^n b_n(x_i) f'(x_i) \pmod{p^{2n}}.$$

Hence we have

$$b_n(f(x_i)) \equiv b_n(x_i) f'(x_i) \pmod{p^n}.$$

Thus we obtain (vi), because  $a_n(x_i) \not\equiv 0 \pmod{p}$  (for some  $1 \leq i \leq k$ ) implies  $f'(x_i) \not\equiv 0 \pmod{p}$  for all  $1 \leq i \leq k$ .  $\square$

According to Lemma 3.2 (i) and (ii), the value of  $a_n(x) \pmod{p^n}$  does not depend on  $x \in X$ . According to Lemma 3.2 (iii) and (iv), whether  $b_n(x) \equiv 0 \pmod{p}$  does not depend on  $x \in X$  if  $a_n(x) \equiv 1 \pmod{p}$ . For simplicity, sometimes we shall write  $a_n$  and  $b_n$  without mentioning  $x$ .

The above analysis allows us to distinguish the following four behaviors of  $f_{n+1}$  on  $X$ :

(a) If  $a_n \equiv 1 \pmod{p}$  and  $b_n \not\equiv 0 \pmod{p}$ , then  $\Phi$  preserves a single cycle of length  $p$ , so that  $f_{n+1}$  restricted to  $X$  preserves a single cycle of length  $pk$ . In this case we say  $\sigma$  *grows*.

(b) If  $a_n \equiv 1 \pmod{p}$  and  $b_n \equiv 0 \pmod{p}$ , then  $\Phi$  is the identity, so  $f_{n+1}$  restricted to  $X$  preserves  $p$  cycles of length  $k$ . In this case we say  $\sigma$  *splits*.

(c) If  $a_n \equiv 0 \pmod{p}$ , then  $\Phi$  is constant, so  $f_{n+1}$  restricted to  $X$  preserves one cycle of length  $k$  and the remaining points of  $X$  are mapped into this cycle. In this case we say  $\sigma$  *grows tails*.

(d) If  $a_n \not\equiv 0, 1 \pmod{p}$ , then  $\Phi$  is a permutation and the  $\ell$ -th iterate of  $\Phi$  reads

$$\Phi^\ell(x, t) = b_n(a_n^\ell - 1)/(a_n - 1) + a_n^\ell t$$

so that

$$\Phi^\ell(t) - t = (a_n^\ell - 1) \left( t + \frac{b_n}{a_n - 1} \right).$$

Thus,  $\Phi$  admits a single fixed point  $t = -b_n/(a_n - 1)$ , and the remaining points lie on cycles of length  $d$ , where  $d$  is the order of  $a_n$  in  $(\mathbb{Z}/p\mathbb{Z})^*$ . So,  $f_{n+1}$  restricted to  $X$  preserves one cycle of length  $k$  and  $\frac{p-1}{d}$  cycles of length  $kd$ . In this case we say  $\sigma$  *partially splits*.

Now let us study the relation between  $(a_n, b_n)$  and  $(a_{n+1}, b_{n+1})$ . Our aim is to see the change of nature from a cycle to its lifts.

**Lemma 3.3.** *Let  $\sigma = (x_1, \dots, x_k)$  be a  $k$ -cycle of  $f_n$  and let  $\tilde{\sigma}$  be a lift of  $\sigma$  of length  $kr$ , where  $r \geq 1$  is an integer. We have*

$$a_{n+1}(x_i + p^r t) \equiv a_n^r(x_i) \pmod{p^n}, \quad (1 \leq i \leq k, 0 \leq t \leq p-1) \quad (3.4)$$

$$\begin{aligned} & pb_{n+1}(x_i + p^r t) \\ & \equiv t(a_n(x_i)^r - 1) + b_n(x_i)(1 + a_n(x_i) + \dots + a_n(x_i)^{r-1}) \pmod{p^n}. \end{aligned} \quad (3.5)$$

*Proof.* The formula (3.4) follows from

$$a_{n+1} \equiv (g^r)'(x_i + p^n t) \equiv (g^r)'(x_i) \equiv \prod_{j=0}^{r-1} g'(g^j(x_i)) \equiv a_n^r \pmod{p^n}.$$

By repeating  $r$ -times of the linearization (3.3), we obtain

$$g^r(x_i + p^n t) \equiv x_i + \Phi^r(t, x_i)p^n \pmod{p^{2n}},$$

where  $\Phi^r$  means the  $r$ -th composition of  $\Phi$  as function of  $t$ . However,

$$\Phi^r(t, x_i) = ta_n(x_i)^r + b_n(x_i)(1 + a_n(x_i) + \cdots + a_n(x_i)^{r-1}).$$

Thus (3.5) follows from the definition of  $b_{n+1}$  and the above two expressions.  $\square$

By Lemma 3.3, we get immediately the following proposition.

**Proposition 3.4.** *Let  $n \geq 1$ . Let  $\sigma$  be a  $k$ -cycle of  $f_n$  and  $\tilde{\sigma}$  be a lift of  $\sigma$ . Then we have*

- 1) if  $a_n \equiv 1 \pmod{p}$ , then  $a_{n+1} \equiv 1 \pmod{p}$ ;
- 2) if  $a_n \equiv 0 \pmod{p}$ , then  $a_{n+1} \equiv 0 \pmod{p}$ ;
- 3) if  $a_n \not\equiv 0, 1 \pmod{p}$  and  $\tilde{\sigma}$  is of length  $k$ , then  $a_{n+1} \not\equiv 0, 1 \pmod{p}$ ;
- 4) if  $a_n \not\equiv 0, 1 \pmod{p}$  and  $\tilde{\sigma}$  is of length  $kd$  where  $d \geq 2$  is the order of  $a_n$  in  $(\mathbb{Z}/p\mathbb{Z})^*$ , then  $a_{n+1} \equiv 1 \pmod{p}$ .

This result is interpreted as follows in dynamical system language:

- 1) If  $\sigma$  grows or splits, then any lift  $\tilde{\sigma}$  grows or splits.
- 2) If  $\sigma$  grows tails, then the single lift  $\tilde{\sigma}$  also grows tails.
- 3) If  $\sigma$  partially splits, then the lift  $\tilde{\sigma}$  of the same length as  $\sigma$  partially splits, and the other lifts of length  $kd$  grow or split.

If  $\sigma = (x_1, \dots, x_k)$  is a cycle of  $f_n$  which grows tails, then  $f$  admits a  $k$ -periodic point  $x_0$  in the clopen set  $\mathbb{X} = \bigsqcup_{i=1}^k x_i + p^n \mathbb{Z}_p$  and  $\mathbb{X}$  is contained in the attracting basin of the periodic orbit  $x_0, f(x_0), \dots, f^{k-1}(x_0)$ .

With the preceding preparations, we are ready to prove the following Propositions 3.5-3.7 which predict the behavior of the lifts of a cycle  $\sigma$  by the properties of  $\sigma$ . We refer the reader to [40] for their proofs. Otherwise we can follow the similar proofs of Propositions 3.8-3.10 in the case  $p = 2$ .

**Proposition 3.5** ([40]). *Let  $\sigma$  be a growing cycle of  $f_n$  and  $\tilde{\sigma}$  be the unique lift of  $\sigma$ .*

- 1) *If  $p \geq 3$  and  $n \geq 2$  then  $\tilde{\sigma}$  grows.*
- 2) *If  $p > 3$  and  $n \geq 1$  then  $\tilde{\sigma}$  grows.*
- 3) *If  $p = 3$  and  $n = 1$ , then  $\tilde{\sigma}$  grows if and only if  $b_1(x) \not\equiv g''(x)/2 \pmod{p}$ .*

According to 1) and 2) of Proposition 3.5, in the cases  $p \geq 3, n \geq 2$  and  $p > 3, n \geq 1$ , if  $\sigma = (x_1, \dots, x_k)$  grows then its lift also grows, and the lift of the lift will grow and so on. So, the clopen set

$$\mathbb{X} = \bigsqcup_{i=1}^k x_i + p^n \mathbb{Z}_p$$

is a minimal set.

Let

$$A_n(x) := v_p(a_n(x) - 1), \quad B_n(x) := v_p(b_n(x)).$$

By Lemma 3.2, for a cycle  $\sigma = (x_1, \dots, x_k)$ ,  $\min\{A_n(x_i), n\}$  and  $\min\{B_n(x_i), n\}$  do not depend on the choice of  $x_i, 1 \leq i \leq k$ . Without misunderstanding, we will not mention  $x_i$  in  $A_n$  and  $B_n$ .

**Proposition 3.6** ([40]). *Let  $p \geq 3$  and  $n \geq 1$ . Let  $\sigma$  be a splitting cycle of  $f_n$ .*

- 1) *If  $\min\{A_n, n\} > B_n$ , every lift splits first  $B_n - 1$  times then grows.*
- 2) *If  $A_n \leq B_n$  and  $A_n < n$ , there is one lift which behaves the same as  $\sigma$  (i.e., this lift splits and  $A_{n+1} \leq B_{n+1}$  and  $A_{n+1} < n + 1$ ) and other lifts split  $A_n - 1$  times then grow.*
- 3) *If  $B_n \geq n$  and  $A_n \geq n$ , then all lifts split at least  $n - 1$  times.*

**Proposition 3.7** ([40]). *Let  $p \geq 3$  and  $n \geq 1$ . Let  $\sigma$  be a partially splitting  $k$ -cycle of  $f_n$  and  $\tilde{\sigma}$  be a lift of  $\sigma$  of length  $kd$ , where  $d$  is the order of  $a_n$  in  $\mathbb{Z}/p\mathbb{Z}$ .*

- 1) *If  $A_{n+1} < nd$ , then  $\tilde{\sigma}$  splits  $A_{n+1} - 1$  times then grows.*
- 2) *If  $A_{n+1} \geq nd$ , then  $\tilde{\sigma}$  splits at least  $nd - 1$  times.*

We remark that in the partially splitting case,  $\min\{A_{n+1}(x), nd\}$  depends only on the lifting cycle of  $f_{n+1}$  of length  $kd$  but not on  $x$  (see [40], Corollary 3).

### 3.3 Induced dynamics on $\mathbb{Z}/p^n\mathbb{Z}_p$ ( $p = 2$ )

In this section we focus on the special case  $p = 2$  which is not considered in [40]. The first part in the preceding section (where  $p \geq 3$  is not explicitly assumed) remain true for  $p = 2$ . Notice that when  $p = 2$ , there is no partially splitting cycles.

We only need to study how a cycle grow or split. We distinguish four cases. Let  $\sigma$  be a cycle of  $f_n$ . We say  $\sigma$  *strongly grows* if  $a_n \equiv 1 \pmod{4}$  and  $b_n \equiv 1 \pmod{2}$ ,  $\sigma$  *weakly grows* if  $a_n \equiv 3 \pmod{4}$  and  $b_n \equiv 1 \pmod{2}$ . We say  $\sigma$  *strongly splits* if  $a_n \equiv 1 \pmod{4}$  and  $b_n \equiv 0 \pmod{2}$ ,  $\sigma$  *weakly splits* if  $a_n \equiv 3 \pmod{4}$  and  $b_n \equiv 0 \pmod{2}$ .

The following results hold true when  $p = 2$ . Their proofs are postponed and get together at the end of this section.

**Proposition 3.8.** *Let  $\sigma$  be a cycle of  $f_n$  ( $n \geq 2$ ). If  $\sigma$  strongly grows then the lift of  $\sigma$  strongly grows. If  $\sigma$  weakly grows then the lift of  $\sigma$  strongly splits.*

The first assertion of Proposition 3.8 implies that if  $\sigma = (x_1, \dots, x_k)$  is a strongly growing cycle of  $f_n$  ( $n \geq 2$ ), then  $\bigsqcup x_i + p^n\mathbb{Z}_p$  is a minimal set.

Recall that

$$A_n(x) = v_2(a_n(x) - 1), \quad B_n(x) = v_2(b_n(x)).$$

**Proposition 3.9.** *Let  $\sigma$  be a strongly splitting cycle of  $f_n$  ( $n \geq 2$ ).*

- 1) *If  $\min\{A_n, n\} > B_n$ , then all lifts strongly split  $B_n - 1$  times, then strongly grow.*
- 2) *If  $A_n \leq B_n$  and  $A_n < n$ , then one lift behaves the same as  $\sigma$  (i.e., this lift strongly splits and  $A_{n+1} \leq B_{n+1}$  and  $A_{n+1} < n + 1$ ). The other one splits  $A_n - 1$  times, then strongly grows forever.*
- 3) *If  $B_n \geq n$  and  $A_n \geq n$ , then all lifts strongly split at least  $n - 1$  times.*

**Proposition 3.10.** *Let  $\sigma$  be a weakly splitting cycle of  $f_n$  ( $n \geq 2$ ). Then one lift behaves the same as  $\sigma$  and the other one weakly grows and then strongly splits.*

To prove these propositions, we need the following lemmas.

**Lemma 3.11.** *Let  $\sigma$  be a growing cycle of  $f_n$  ( $n \geq 2$ ). Then*

$$a_{n+1}(x_i) \equiv 1 \pmod{4}, \quad (3.6)$$

$$2b_{n+1}(x_i + p^n t) \equiv b_n(x_i)(1 + a_n(x_i)) \pmod{4}. \quad (3.7)$$

*Proof.* Taking  $p = 2$  and  $r = 2$  in (3.4), we get

$$a_{n+1}(x_i) \equiv a_n^2(x_i) \pmod{2^n}.$$

Since  $n \geq 2$  and  $a_n \equiv 1 \pmod{2}$ , we obtain (3.6).

Taking  $p = 2$  and  $r = 2$  in (3.5), we get

$$2b_{n+1}(x_i + 2^n t) \equiv t(a_n(x_i)^2 - 1) + b_n(x_i)(1 + a_n(x_i)) \pmod{2^n}.$$

Since  $n \geq 2$  and  $a_n \equiv 1 \pmod{2}$ , we obtain (3.7).  $\square$

**Lemma 3.12.** *Let  $\sigma$  be a splitting cycle of  $f_n$ .*

*If  $A_n < n$ , then  $A_{n+1} = A_n$  and if  $A_n \geq n$ , then  $A_{n+1} \geq n$ . Consequently,*

$$\min\{A_{n+1}, n\} = \min\{A_n, n\}. \quad (3.8)$$

*Proof.* We need only to notice that we have  $a_{n+1} \equiv a_n \pmod{2^n}$  since  $\sigma$  splits.  $\square$

**Lemma 3.13.** *Let  $\sigma = (x_1, \dots, x_k)$  be a splitting cycle of  $f_n$ . Then for  $1 \leq i \leq k$  and for  $t = 0$  or  $1$ , we have*

$$2b_{n+1}(x_i + 2^n t) \equiv b_n(x_i) + t(a_n(x_i) - 1) \pmod{2^n}. \quad (3.9)$$

*Consequently, we have*

$$B_{n+1}(x_i + 2^n t) = B_n(x_i) - 1 \quad \text{if } B_n(x_i) < \min\{A_n(x_i), n\}, \quad (3.10)$$

*Proof.* Since  $\sigma$  splits, taking  $p = 2$  and  $r = 1$  in (3.5), we obtain the result.  $\square$

The following lemma concerns an elementary property of polynomials on  $\mathbb{Z}_2$ .

**Lemma 3.14.** *Let  $h \in \mathbb{Z}_2[x]$ . If  $a \equiv b \pmod{2}$ , then  $h'(a) \equiv h'(b) \pmod{4}$ . Furthermore, if  $h'(a) \equiv 1 \pmod{2}$ , then  $h'(a)h'(b) \equiv 1 \pmod{4}$ .*

*Proof.* It suffices to notice that the coefficient of  $x^{2k+1}$  in  $h'(x)$  is equal to 0 (mod 2). □

**Lemma 3.15.** *Let  $\sigma$  be a growing  $k$ -cycle of  $f_n$  ( $n \geq 1$ ). Then its lift strongly grows or strongly splits.*

*Proof.* Let  $x_1$  be a point in  $\sigma$ . What we have to show is  $a_{n+1}(x_1) \equiv 1 \pmod{4}$ . Since  $\sigma$  is a growing  $k$ -cycle, we have

$$f^k(x_1) \equiv x_1 \pmod{2^n}, \quad a_n(x_1) = (f^k)'(x_1) \equiv 1 \pmod{2}.$$

So, by Lemma 3.14, we have

$$a_{n+1}(x_1) = (f^{2k})'(x_1) = (f^k)'(x_1)(f^k)'(f^k(x_1)) \equiv 1 \pmod{4}.$$

□

A direct consequence is the following result.

**Corollary 3.16.** *If a cycle grows twice (maybe between the two growths, it splits several times), then it will grow forever.*

*Proof.* Let  $\tilde{\sigma}$  be the lift of a growing cycle  $\sigma$ . Assume that after several times of splitting, one of lifts of  $\tilde{\sigma}$  grows. By Lemma 3.15, this growing lift at a level  $n \geq 2$  must strongly grow. Thus by Proposition 3.8, it will grow forever. □

We are now going to prove Propositions 3.8-3.10.

*Proof of Proposition 3.8.* If  $\sigma$  grows, then by (3.6), the lift of  $\sigma$  strongly grows or strongly splits. If  $\sigma$  strongly grows, then by (3.7), we have

$$2b_{n+1}(x_i + p^n t) \equiv 2b_n(x_i) \pmod{4}.$$

Thus

$$b_{n+1}(x_i + p^n t) \equiv b_n(x_i) \not\equiv 0 \pmod{2}.$$

Hence the lift of  $\sigma$  strongly grow.

If  $\sigma$  weakly grows, then by (3.7), we have

$$2b_{n+1}(x_i + p^n t) \equiv 0 \pmod{4}.$$

Thus

$$b_{n+1}(x_i + p^n t) \equiv 0 \pmod{2}.$$

Hence the lift of  $\sigma$  strongly splits.  $\square$

*Proof of Proposition 3.9.* First notice that if  $\sigma$  strongly splits then  $a_n \equiv 1 \pmod{4}$ . Since  $n \geq 2$ , by Lemma 3.2 we have  $a_\ell \equiv 1 \pmod{4}$  for all  $\ell > n$ . So, all the lifts strongly grow or strongly split. Recall that by Lemma 3.2, both  $\min\{A_n(x_i), n\}$  and  $\min\{B_n(x_i), n\}$  are independent of  $x_i$ . We will simply write  $A_n$  and  $B_n$  if there is no confusion.

Proposition 3.9 contains three cases which are defined by some conditions on  $A_n$  and  $B_n$ . If such a condition is satisfied, we say  $\sigma$  or  $(A_n, B_n)$  belongs to the corresponding case.

*Case 1:*  $\min\{A_n, n\} > B_n$ . By (3.10), we have  $B_{n+1} = B_n - 1$ . Thus by (3.8)

$$\min\{A_{n+1}, n+1\} \geq \min\{A_{n+1}, n\} = \min\{A_n, n\} > B_n > B_{n+1}.$$

Hence the lifts of  $\sigma$  still belongs to Case 1. By induction, we know that after  $\ell := B_n$  times,  $B_{n+\ell} = 0$  (i.e.  $b_{n+\ell} \not\equiv 0 \pmod{p}$ ). Since  $\sigma$  strongly splits, we have  $a_{n+\ell} \equiv 1 \pmod{4}$ . Thus the lifts at level  $n + \ell$  strongly grow. That is to say all lifts of  $\sigma$  split  $B_n - 1$  times, then strongly grow forever.

*Case 2:*  $A_n \leq B_n$  and  $A_n < n$ . By Lemma 3.12, we have  $A_{n+1} = A_n$ . Since  $B_n \geq A_n$ , there exists one  $t$  such that

$$b_n + t(a_n - 1) \equiv 0 \pmod{2^{A_n+1}},$$

and the other which we can write as  $1 - t$  such that

$$b_n + t(a_n - 1) \not\equiv 0 \pmod{2^{A_n+1}}.$$

Hence by (3.9), for one lift of  $\sigma$   $B_{n+1} \geq A_n$  and for the other one  $B_{n+1} = A_n - 1$ . Thus for one lift,  $A_{n+1} = A_n \leq B_{n+1}$ , and  $A_{n+1} = A_n < n + 1$ . Therefore, this lift belongs to Case 2. For the other one,  $B_{n+1} = A_n - 1 = A_{n+1} - 1 < A_{n+1}$ , and  $B_{n+1} = A_n - 1 < n + 1$ . Thus this lift belongs to Case 1. By induction, we know that one lift of  $\sigma$  behaves the same as  $\sigma$  (i.e., strongly splits and satisfies the condition of Case 2 at level  $n + 1$ ) and the other one splits  $A_n - 1$  times, then the lifts strongly grow.

*Case 3:  $B_n \geq n$  and  $A_n \geq n$ .* By the definition of  $b_n$ , if the cycle splits, the order of  $b_n$  decreases at most one when the level goes up one step. Since  $B_n \geq n$ , we have  $B_{n+1} \geq n - 1$ , and if  $n \geq 2$ , the lifts of  $\sigma$  still strongly split. Thus by induction, the lifts of  $\sigma$  split at least  $n - 2$  times. But after that we can not give any more information.  $\square$

*Proof of Proposition 3.10.* Since  $\sigma$  weakly splits,  $a_{n+1} \equiv a_n \equiv 3 \pmod{4}$ . Thus  $A_{n+1} = A_n = 1 < n$  and  $B_n \geq 1 = A_n$ . Thus  $(A_n, B_n)$  belongs to Case 2 in Proposition 3.9. By the proof of Proposition 3.9, we know that for one lift of  $\sigma$ ,  $B_{n+1} \geq A_n$  and then  $A_{n+1} = A_n \leq B_{n+1}$ . Thus this lift behaves the same as  $\sigma$ . For the other lift,  $B_{n+1} = A_n - 1 = 0$ . Hence this second lift weakly grows, and then strongly splits by Proposition 3.8. Therefore, we complete the proof.  $\square$

### 3.4 Minimal decomposition

If a cycle always grows then it will produce a minimal component of  $f$ . If a cycle always splits then it will produce a periodic orbit of  $f$ . If a cycle grows tails, it will produce an attracting periodic orbit with an attacking basin. We shall describe this more precisely.

Let  $\sigma = (x_1, \dots, x_k)$  be a cycle of  $f_n$ . We say  $\sigma$  is a cycle at level  $n$ . Let

$$\mathbb{X} := \bigsqcup_{i=1}^k (x_i + p^n \mathbb{Z}_p).$$

There are four special situations for the dynamical system  $f : \mathbb{X} \rightarrow \mathbb{X}$ .

(S1) Suppose  $\sigma$  grows tails. Then  $f$  admits a  $k$ -periodic orbit with one periodic point in each ball  $x_i + p^n \mathbb{Z}_p$  ( $1 \leq i \leq k$ ), and all other points in  $\mathbb{X}$  are attracted into this orbit. In this situation, if  $x$  is a point in the  $k$ -periodic orbit, then  $|(f^k)'(x)|_p < 1$  since  $(f^k)'(x) = a_m(x) \equiv 0 \pmod{p^m}$  for all  $m \geq n$ . The periodic orbit  $(x, f(x), \dots, f^{k-1}(x))$  is then attractive.

(S2) Suppose  $\sigma$  grows and its lifts always grow. Then  $f$  is transitive on each  $\mathbb{X}/p^m \mathbb{Z}_p$   $m \geq n$ . Thus, by Theorem 3.1,  $f$  is minimal on  $\mathbb{X}$ . In this case, we say that  $\sigma$  is a starting growing cycle at level  $n$ .

(S3) Suppose  $\sigma$  splits and there is a splitting lift on each level larger than  $n$ . Then there is a  $k$ -periodic orbit with one periodic point in each  $x_i + p^n \mathbb{Z}_p$  ( $1 \leq$

$i \leq k$ ). We say that  $\sigma$  is a starting splitting cycle at level  $n$ . In this situation, if  $x$  is a point in the  $k$ -periodic orbit, then  $(f^k)'(x) = 1$  since  $(f^k)'(x) = a_m(x) \equiv 1 \pmod{p^m}$  for all  $m \geq n$ . Thus the periodic orbit  $(x, f(x), \dots, f^{k-1}(x))$  is indifferent.

(S4) Suppose  $\sigma = (x_1, \dots, x_k)$  partially splits ( $p \geq 3$ ). Then by Proposition 3.7, there is one lift of length  $k$  which still partial splits like  $\sigma$ . Thus there is a  $k$ -periodic orbit with one periodic point in each  $x_i + p^n\mathbb{Z}_p$  ( $1 \leq i \leq k$ ). In this situation, if  $x$  is a point in the  $k$ -periodic orbit formed above, then  $|(f^k)'(x)|_p = 1$  since  $(f^k)'(x) = a_m(x) \not\equiv 0, 1 \pmod{p^m}$  for all  $m \geq n$ . Hence, the periodic orbit  $(x, f(x), \dots, f^{k-1}(x))$  is indifferent.

Now we can deduce all possible periods of the polynomial systems on  $\mathbb{Z}_p$ .

**Proposition 3.17** ([40], see also [125]). *Let  $f \in \mathbb{Z}_p$  with  $\deg f \geq 2$ .*

- 1) *If  $p \geq 5$ , the lengths of cycles are of the form  $ab$  with  $a|(p-1)$  and  $1 \leq b \leq p$ ;*
- 2) *If  $p = 3$ , the length of cycles must be 1, 2, 3, 4, 6 or 9;*
- 3) *If  $p = 2$ , the length of cycles must be 1, 2 or 4. If there is 4-periodic orbit, then  $f_1$  should be a permutation on  $\mathbb{Z}/2\mathbb{Z}$ .*

*Proof.* We only show 3), because the proofs of 1) and 2) are similar and can be found in [40] and [125]. The first assertion of 3) can also be found in [125].

Notice that any periodic orbit comes from an infinite sequence of splitting of some cycle, and that the length of the periodic orbit is the length of the cycle. So, what we want to study are all possible lengths of starting splitting cycles.

The possible lengths of cycles at the first level (i.e. the cycles of  $f_1$  on  $\mathbb{Z}/2\mathbb{Z}$ ) are 1 and 2. Notice that the growth of length must be multiplied 2, according to our discussion in the preceding sections. So, the possible lengths of cycles are  $2^k$  ( $k \geq 0$ ). However, by Corollary 3.16, if a cycle grows twice it will grow forever. There, any cycle of length  $2^k$  ( $k \geq 3$ ), which must have grown twice, can't be a starting splitting cycle. Hence the lengths of starting splitting cycles can only be 1, 2, 4.

If there is a periodic orbit of length 4, there must be a starting splitting cycle of length 4. This is possible only in the following case: at the first level,  $f_1$  admit

a 2-cycle. Otherwise it needs to grow twice and then its lifts will grow forever. This will produce a clopen minimal set not a periodic orbit.  $\square$

**Theorem 3.18.** *Let  $f \in \mathbb{Z}_p[x]$  with  $\deg f \geq 2$ . We have the following decomposition*

$$\mathbb{Z}_p = A \bigsqcup B \bigsqcup C$$

where  $A$  is the finite set consisting of all periodic points of  $f$ ,  $B = \bigsqcup_i B_i$  is the union of all (at most countably many) clopen invariant sets such that each subsystem  $f : B_i \rightarrow B_i$  is minimal, and each point in  $C$  lies in the attracting basin of a periodic orbit or of a minimal subsystem.

*Proof.* We first explain that there are only finitely many periodic points. In fact, by Proposition 3.17, there are only finitely many possible lengths of periods. Periodic points are solutions of the equations  $f^{q_i}(x) = x$  with  $\{q_i\}$  being one of possible length of period. Since  $\deg f \geq 2$ , each equation admits a finite number of solutions. So, there is only a finite number of periodic points.

We start from the second level. Decompose  $\mathbb{Z}_p$  into  $p^2$  balls with radius  $p^{-2}$ . Each ball is identified with a point in  $\mathbb{Z}/p^2\mathbb{Z}$ . The induced map  $f_2$  admits some cycles. It is possible that some point outside any cycle is mapped into a cycle. The ball corresponding to such a point will be put into the third part  $C$ . From now on, we really start our analysis with cycles at level  $n \geq 2$ . Let  $\sigma = (x_1, \dots, x_k)$  be a cycle at level  $n \geq 2$ . Let

$$\mathbb{X} = \bigsqcup_{i=1}^k (x_i + p^n \mathbb{Z}_p).$$

SUPPOSE  $p \geq 3$ . We distinguish four cases.

(P1)  $\sigma$  grows tails. Then by (S1), the clopen set  $\mathbb{X}$  consists of a  $k$ -periodic orbit and other points are attracted by this periodic orbit. So,  $\mathbb{X}$  contributes to the first part  $A$  and the third part  $C$ .

(P2)  $\sigma$  grows. Then by Proposition 3.5,  $\sigma$  is in the situation (S2). Therefore  $\mathbb{X}$  is a minimal component. So,  $\mathbb{X} \subset B$ .

(P3)  $\sigma$  splits. Then we shall apply Proposition 3.6. If  $\sigma$  belongs to Case 1 described by Proposition 3.6, then after finitely many times of splitting, the lifts will grow forever and so they are in the situation of (S2). Therefore we get a finite

number of minimal components, all belonging to  $B$ . If  $\sigma$  belongs to Case 2, then there is one lift of  $\sigma$  sharing the property (S3), and others lifts different from the cycle containing the periodic orbit (at any level  $m \geq n + 1$ ) find themselves in the situation (S2) after finitely many times of lifting. Therefore, we get a periodic orbit and an infinite number of minimal components. If  $\sigma$  belongs to Case 3, then  $\sigma$  splits into  $p^n$  cycles at level  $2n$ . These cycles at level  $2n$  may continue this procedure (P3). But this procedure cannot continue infinitely, because there is only a finite number of periodic points. So, all these cycle may continue to split but they must end with their lifts belonging either to Case 1 or Case 2. So,  $\mathbb{X}$  contributes to both  $A$  and  $B$ .

(P4)  $\sigma$  *partially splits*. Then  $\sigma$  is in the situation (S4). thus there comes out a periodic orbit. Suppose  $\sigma_m$  is the lift of  $\sigma$  containing the periodic orbit at level  $m \geq n + 1$ . If  $\sigma_m$  belongs Case 1 in Proposition 3.7, then the other lifts different from  $\sigma_{m+1}$ , will be in the situation (S1) after finite times. If  $\sigma_m$  belongs Case 2 in Proposition 3.7, then each of other lifts different from  $\sigma_{m+1}$ , splits as  $p^{nd-1}$  cycles at level  $nd$ . We do (P3) for these cycles at level  $nd$ .

SUPPOSE  $p \geq 3$ . We distinguish five cases.

(Q1)  $\sigma$  *grows tails*. Then  $\sigma$  is in the situation (S1). We have the same conclusion as (P1) above.

(Q2)  $\sigma$  *strongly grows*. Then by Proposition 3.8,  $\sigma$  is in the situation (S2). We have the same conclusion as (P2) above.

(Q3)  $\sigma$  *strongly splits*. By Proposition 3.9, the arguments are the same as (P3): The processes will be ended if the condition 1) or 2) in Proposition 3.9 is satisfied. If the condition 3) in Proposition 3.9 is satisfied, we repeat (Q3) for the lifts of  $\sigma$ . But the processes will be eventually ended with the condition 1) or 2), because there is only a finite number of periodic points.

(Q4)  $\sigma$  *weakly grows*. Then by Proposition 3.8, the lift of  $\sigma$  strongly splits. We are then in the situation (Q3).

(Q5)  $\sigma$  *weakly splits*. By Proposition 3.10, then one lift is in the situation (S3) which produces a periodic orbit, and the other lifts different from the cycle containing the periodic orbit, at any level  $m \geq n + 1$ , will weakly grow. Then we are in the situation (Q4).

All the above processes will stop. So, we get the decomposition in finite

steps. □

We have excluded the affine polynomials from the theorem. Exactly speaking the conclusion is false for affine polynomials. For example, every points in  $\mathbb{Z}_p$  are fixed by  $f(x) := x$ . Anyway, affine polynomials have been fully studied in [50].

**Corollary 3.19.** *Let  $f \in \mathbb{Z}_p[x]$  with  $\deg f \geq 2$ . If  $f$  admits an indifferent fixed point or a periodic orbit, then there exists a sequence of minimal components with their diameters and their distances from the fixed point or the periodic orbit tending to zero.*

*Proof.* Suppose  $(x_1, \dots, x_k)$  is an indifferent periodic orbit. Let  $x_j^{(n)} \in \mathbb{Z}/p^n\mathbb{Z}$  and  $x_j^{(n)} \equiv x_j \pmod{p^n}$  for  $1 \leq j \leq k$ . Then  $\sigma_n = (x_1^{(n)}, \dots, x_k^{(n)})$  is a splitting or partially splitting cycles at level  $n$ . By the processes of the decomposition, the cycles  $\sigma_n$  should be in the situation (S3) or (S4). That is to say  $\sigma_n$  splits for all  $n$  or  $\sigma_n$  partially splits for all  $n$ .

Since there are only finite number of periodic orbits, for any  $\epsilon > 0$  small enough, there is no other periodic orbits in the  $\epsilon$  neighborhood of the orbit  $(x_1, \dots, x_k)$ . Take  $n$  such that  $p^{-n} < \epsilon$ . Then the lifts of  $\sigma_n$  which are different to  $\sigma_{n+1}$  will never split infinitely. Hence they will grow after finite time. Then all the lifts of  $\sigma_n$  which are different to  $\sigma_{n+1}$ , considered as union of balls, are finite number of minimal components. Since these balls are contained in  $x_j + p^n\mathbb{Z}_p$  for each  $j$  respectively. Thus there is a minimal component such that the diameter and the distance to the orbit  $(x_1, \dots, x_k)$  are all less than  $p^{-n}$ . The result is obtained if we consider infinitely  $n$  and find one minimal component for each  $n$ . □

### 3.5 Conjugacy classes of Minimal subsystems

Recently, Chabert, Fan and Fares [33] proved that minimal sets of a 1-Lipschitz map are Legendre sets. We shall prove that minimal sets of a polynomial are special Legendre sets. A set  $E \subset \mathbb{Z}_p$  is a *Legendre set* if for any  $s \geq 1$  and any  $x \in E/p^s\mathbb{Z}_p$ , the number

$$q_s := \text{Card} \{y \in E/p^{s+1}\mathbb{Z}_p : y \equiv x \pmod{p^s}\}$$

is independent of  $x \in E/p^s\mathbb{Z}_p$ . Let

$$p_s := q_1 q_2 \cdots q_s \quad (\forall s \geq 1).$$

It is clear that  $p_s = \text{Card } E/p^s\mathbb{Z}_p$ . We call  $(p_s)_{s \geq 1}$  the *structure sequence* of  $E$ . Consider the inverse limit

$$\mathbb{Z}_{(p_s)} := \varprojlim \mathbb{Z}/p_s\mathbb{Z}.$$

This is a profinite group, usually called an *odometer*, and the map  $\tau : x \mapsto x + 1$  is called the *adding machine* on  $\mathbb{Z}_{(p_s)}$ .

**Theorem 3.20** ([33]). *Let  $E$  be a clopen set in  $\mathbb{Z}_p$  and  $f : E \rightarrow E$  be a 1-Lipschitz map. If the dynamical system  $(E, f)$  is minimal, then  $f$  is an isometry,  $E$  is a Legendre set and the system  $(E, f)$  is conjugate to the adding machine on  $(\mathbb{Z}_{(p_s)}, \tau)$  where  $(p_s)$  is the structure sequence of  $E$ . On the other hand, on any Legendre set there exists at least one minimal map.*

We improve the above result in the case of polynomials by giving more information on the structure sequence.

**Theorem 3.21.** *Let  $f \in \mathbb{Z}_p[x]$  with  $\deg f \geq 2$ . If  $E$  is a minimal clopen invariant set of  $f$ , then  $f : E \rightarrow E$  is conjugate to the adding machine on an odometer  $\mathbb{Z}_{(p_s)}$ , where*

$$(p_s) = (k, kd, kdp, kdp^2, \dots)$$

with some  $k \in \mathbb{N}, k \leq p$  and  $d|(p-1)$ .

*Proof.* By our previous discussion on the cycles of  $f_n$  on  $\mathbb{Z}/p^n\mathbb{Z}$ , a clopen minimal set  $E$  is formed when a cycle grows forever. If  $n$  is the starting level for the cycle to grow, then  $E$  is a union of some balls with radius  $p^{-n}$ . Therefore, for  $s \geq n$ , every nonempty intersection of  $E$  with a ball of radius  $p^{-s}$  contains  $p$  balls of radius  $p^{-(s+1)}$ . That is to say  $q_s = p$ . From the cycle at the first level to the starting cycle at level  $n$ , the growth of cycle length is multiplied by 1,  $p$  or some  $d$  satisfying  $d|(p-1)$ . That is to say for  $1 \leq s < n$ , every nonempty intersection of  $E$  with a ball of radius  $p^{-s}$  contains the same number (1,  $p$  or  $d$ ) of balls of radius  $p^{-(s+1)}$ . Thus  $E$  is a Legendre set. To determine  $p_s$  for  $1 \leq s < n$ , we distinguish three cases:  $p \geq 5$ ,  $p = 3$ ,  $p = 2$ .

*Case  $p \geq 5$ .* In this case, when a cycle grows, its lifts grow forever. So, a cycle at level 1 starts with growing, several times of splitting or several times of partial splitting. Then the lifted cycle grows forever. Therefore, there are three ways to form a minimal set. We show the three ways by the growth of cycle length as follows ( $k$  being the length of the cycle  $\sigma$  on the level 1).

Case 1.  $\sigma$  grows:

$$(k, kp, kp^2, \dots),$$

Case 2.  $\sigma$  splits:

$$(k, k, \dots, k, kp, kp^2, \dots),$$

Case 3.  $\sigma$  partially splits:

$$(k, kd, \dots, kd, kdp, kdp^2, \dots), \quad d|(p-1), d \geq 2.$$

The above three cases correspond to three kinds of adding machines. However, by the result of Buescu and Stewart [28], the adding machines in both Case 1 and Case 2 are conjugate to  $(\mathbb{Z}_{(p_s)}, \tau)$  where  $p_s = (k, kp, kp^2, \dots)$ . In Case 3, the adding machines are all conjugate to  $(\mathbb{Z}_{(p_s)}, \tau)$  where  $p_s = (k, kd, kdp, kdp^2, \dots)$  and  $d|(p-1), d \geq 2$ .

*Case  $p = 3$ .* We distinguish four cases.

Case 1.  $\sigma$  grows and its lift also grows:

$$(k, kp, kp^2, \dots),$$

Case 2.  $\sigma$  grows but its lift splits:

$$(k, kp, \dots, kp, kp^2, \dots),$$

Case 3.  $\sigma$  splits:

$$(k, k, \dots, k, kp, kp^2, \dots),$$

Case 4.  $\sigma$  partial splits:

$$(k, kd, \dots, kd, kdp, kdp^2, \dots), \quad d|(p-1), d \geq 2.$$

Then  $(E, f)$  is conjugate to  $(\mathbb{Z}_{(p_s)}, \tau)$  where  $p_s = (k, kd, kdp, kdp^2, \dots)$  with  $1 \leq k \leq p$ , and  $d|(p-1)$ .

*Case  $p = 2$ .* We distinguish twelve cases.

$$\begin{aligned}
& (1, \underbrace{1, 1, \dots, 1}_{\text{strongly split}}, 2, 2^2, 2^3, \dots), \\
& (1, \underbrace{1}_{\text{strongly grows}}, 2, 2^2, 2^3, \dots), \\
& (1, \underbrace{1}_{\text{weakly splits}}, \underbrace{1}_{\text{weakly grows}}, \underbrace{2, \dots, 2}_{\text{strongly split}}, 2^2, 2^3, \dots), \\
& (1, \underbrace{1}_{\text{weakly grows}}, \underbrace{2, \dots, 2}_{\text{strongly split}}, 2^2, 2^3, \dots), \\
& (1, \underbrace{2, \dots, 2}_{\text{strongly split}}, 2^2, 2^3, \dots), \\
& (1, \underbrace{2}_{\text{strongly grows}}, 2^2, 2^3, \dots), \\
& (2, \underbrace{2, \dots, 2}_{\text{strongly split}}, 2^2, 2^3, \dots), \\
& (2, \underbrace{2}_{\text{strongly grows}}, 2^2, 2^3, \dots), \\
& (2, \underbrace{2}_{\text{weakly splits}}, \underbrace{2}_{\text{weakly grows}}, \underbrace{2^2, \dots, 2^2}_{\text{strongly split}}, 2^3, \dots), \\
& (2, \underbrace{2}_{\text{weakly grows}}, \underbrace{2^2, \dots, 2^2}_{\text{strongly split}}, 2^3, \dots), \\
& (2, \underbrace{2^2, \dots, 2^2}_{\text{strongly split}}, 2^3, \dots), \\
& (2, \underbrace{2^2}_{\text{strongly grows}}, 2^3, \dots).
\end{aligned}$$

In any of these cases, the system  $(E, f)$  is conjugate to  $(\mathbb{Z}_2, x+1)$ . □

### 3.6 2-adic Quadratic Polynomials

In this section, we undertake a full investigation on the minimal decomposition of 2-adic quadratic polynomial systems on  $\mathbb{Z}_2$  of the form:

$$f(x) := ax^2 + bx + c \quad (a, b, c \in \mathbb{Z}_2, a \neq 0).$$

As we shall see, the system  $f(x) = ax^2 + bx + c$  is conjugate to one of the following quadratic polynomials

$$x^2 - \lambda, \quad x^2 + bx, \quad x^2 + x - d$$

where  $\lambda \in \mathbb{Z}_2$ ,  $b \equiv 1 \pmod{2}$  and  $\sqrt{d} \notin \mathbb{Z}_2$ .

Let us state our results on the minimal decomposition of  $(\mathbb{Z}_p, f)$ . The proofs are postponed at the end of this section. By the way, we shall discuss the behavior of  $f$  on the field  $\mathbb{Q}_p$ .

If  $a \equiv 1 \pmod{2}$ , then  $\lim_{n \rightarrow \infty} |f^n(x)| = \infty$  for any  $x \notin \mathbb{Z}_2$ . An elementary calculation shows that  $ax^2 + bx + c$  on  $\mathbb{Z}_2$  is conjugate to  $x^2 + bx + c$  on  $\mathbb{Z}_2$  through the conjugacy  $x \mapsto ax$ . If  $a \equiv 0 \pmod{2}$ , then  $\lim_{n \rightarrow \infty} |f^n(x)| = \infty$  for any  $x \notin \frac{1}{a}\mathbb{Z}_2$  and  $ax^2 + bx + c$  on  $\frac{1}{a}\mathbb{Z}_2$  conjugates to  $x^2 + ax + ac$  on  $\mathbb{Z}_2$  through the conjugacy  $x \mapsto ax$ . Thus without loss of generality, we need only to consider the quadratic polynomials of the form

$$x^2 + bx + c \quad (b, c \in \mathbb{Z}_2).$$

We distinguish two cases according to  $b \equiv 0 \pmod{2}$  or  $b \equiv 1 \pmod{2}$

If  $b \equiv 0 \pmod{2}$ ,  $x^2 + bx + c$  is conjugate to

$$x^2 - \lambda$$

with  $\lambda = \frac{b^2 - 4c - 2b}{4}$  through the conjugacy  $x \mapsto x + \frac{B}{2}$ .

**Theorem 3.22.** *Consider the polynomial  $f(x) = x^2 - \lambda$  on  $\mathbb{Z}_2$ .*

1) *If  $\lambda \equiv 0 \pmod{4}$ , then  $f$  admits two attracting fixed points, one in  $4\mathbb{Z}_2$  with  $2\mathbb{Z}_2$  as its attraction basin, and the other one in  $1 + 4\mathbb{Z}_2$  with  $1 + 2\mathbb{Z}_2$  as its attraction basin.*

2) *If  $\lambda \equiv 1 \pmod{4}$ , then the whole  $\mathbb{Z}_2$  is attracted into a periodic orbit of*

period 2 with one orbit point in  $4\mathbb{Z}_2$  and the other one in  $3 + 4\mathbb{Z}_2$ .

3) If  $\lambda \equiv 2 \pmod{4}$ , then  $f$  admits two attracting fixed points, one in  $2 + 4\mathbb{Z}_2$  as its attraction basin, and the other one in  $3 + 4\mathbb{Z}_2$  with  $1 + 2\mathbb{Z}_2$  as its attraction basin.

4) If  $\lambda \equiv 3 \pmod{4}$ , then the whole  $\mathbb{Z}_2$  is attracted into a periodic orbit of period 2 with one orbit point in  $1 + 4\mathbb{Z}_2$  and the other one in  $2 + 4\mathbb{Z}_2$ .

If  $b \equiv 1 \pmod{2}$ , then  $x^2 + bx + c$  is conjugate to

$$x^2 + x - d$$

where  $d = \frac{(b-1)^2 - 4c}{4} \in \mathbb{Z}_2$  through  $x \mapsto x + \frac{b-1}{2}$ . It is clear that  $x^2 + x - d$  admits fixed points if and if only  $\sqrt{d} \in \mathbb{Z}_2$ . Thus we need to study the case  $x^2 + x - d$  with  $\sqrt{d} \in \mathbb{Z}_2$  and the case  $x^2 + x - d$  with  $d \in \mathbb{Z}_2$  but  $\sqrt{d} \notin \mathbb{Z}_2$ .

If  $\sqrt{d} \in \mathbb{Z}_2$  (i.e.  $x^2 + x - d$  has a fixed point), then  $x^2 + x - d$  conjugates to

$$x^2 + bx$$

with  $b = 1 - 2\sqrt{d}$  through  $x \mapsto x + \sqrt{d}$ .

If  $b = 1$ , the minimal decomposition of  $x^2 + x$  is as follows.

**Theorem 3.23.** *Consider the polynomial  $f(x) = x^2 + x$  on  $\mathbb{Z}_2[x]$ . There is one fixed point 0. We have  $f(1 + 2\mathbb{Z}_2) \subset 2\mathbb{Z}_2$  and we can decompose  $2\mathbb{Z}_2$  into*

$$2\mathbb{Z}_2 = \{0\} \bigsqcup \left( \bigsqcup_{n \geq 2} 2^{n-1} + 2^n \mathbb{Z}_2 \right).$$

Each  $2^{n-1} + 2^n \mathbb{Z}_2$  ( $n \geq 2$ ) consists of  $2^{n-2}$  pieces of minimal components:

$$2^{n-1} + t2^n + 2^{2n-2} \mathbb{Z}_2, \quad t = 0, \dots, 2^{n-2} - 1.$$

If  $b \equiv 1 \pmod{2}$  but  $b \neq 1$ , we divide it into three subcases:  $b = 1 - 4m$ ,  $m \in \mathbb{Z}_2 \setminus \{0\}$ ;  $b = -1 - 4m$ ,  $m \in \mathbb{Z}_2$  with  $v_2(m) \in 1 + 2\mathbb{N}$ ;  $b = -1 - 4m$ ,  $m \in \mathbb{Z}_2$  with  $v_2(m) \in 2\mathbb{N}$ . If  $f(x) = x^2 + (-1 - 4m)x$  with  $v_2(m) = 0$ , then  $f$  is conjugate to  $g(x) = x^2 + (-1 - 4(-m - 1))x$  with  $v_2(-m - 1) = v_2(m + 1) \geq 1$  through  $x \mapsto x - 4m - 2$ . Thus the third case reduce to  $f(x) = x^2 + (-1 - 4m)x$  with  $v_2(m) \in 2\mathbb{N}^*$  with the notation  $\mathbb{N}^*$  being the numbers in  $\mathbb{N}$  different with 0.

Before the statement of the following results, we would like to give some terminology to simplify our statements.

We say a 1-cycle  $(x)$  at level  $n$  is of *type* I- $[k]$  if it splits  $k$  times then its lifts grow forever. In this case, the ball  $x + p^n\mathbb{Z}_p$  is decomposed into  $p^k$  pieces of minimal components. Such a component is a ball of radius  $p^{-n-k}$ . Sometimes the ball  $x + p^n\mathbb{Z}_p$  is said to be of type I- $[k]$ .

We say the a 2-cycle  $(x, y)$  at level  $n$  is of *type* II- $[k]$  if it splits  $k$  times then its lifts grow forever. In this case, the union of two balls  $(x + p^n\mathbb{Z}_p) \cup (y + p^n\mathbb{Z}_p)$  is decomposed into  $p^k$  pieces of minimal components. Such a component is a union of two balls of radius  $p^{-n-k}$ . The union  $(x + p^n\mathbb{Z}_p) \cup (y + p^n\mathbb{Z}_p)$  is sometimes said to be of type II- $[k]$ . Remark that it is possible for the union  $(x + p^n\mathbb{Z}_p) \cup (y + p^n\mathbb{Z}_p)$  to be a ball of radius  $p^{-n+1}$ .

If an invariant subset  $E \subset \mathbb{Z}_p$  is a union of invariant subsets  $F_n \subset \mathbb{Z}_p$   $n \in J \subset \mathbb{N}$  where each ball  $F_n$  is of type I- $[k]$ , we will denote it as

$$E = \bigsqcup_{n \in J} F_n - \{I-[k]\}.$$

If an invariant subset  $E \subset \mathbb{Z}_p$  is a union of invariant subsets  $F_n \subset \mathbb{Z}_p$   $n \in J \subset \mathbb{N}$  where each  $F_n$  is a union of two balls of type II- $[k]$ , we will denote it as

$$E = \bigsqcup_{n \in J} F_n - \{II-[k]\}.$$

Now we are ready to state the following theorems.

**Theorem 3.24.** *Consider  $f(x) = x^2 + (1 - 4m)x$ ,  $m \in \mathbb{Z}_2 \setminus \{0\}$ . Then  $f$  admits two fixed points  $0$  and  $4m$  and  $f(1 + 2\mathbb{Z}_2) \subset 2\mathbb{Z}_2$ , and we can decompose  $2\mathbb{Z}_2$  as*

$$2\mathbb{Z}_2 = \{0, 4m\} \bigsqcup E_1 \bigsqcup E_2 \bigsqcup E_3,$$

where

$$\begin{aligned} E_1 &= \bigsqcup_{2 \leq n < v_2(m)+3} (2^{n-1} + 2^n\mathbb{Z}_2) - \{I-[n-2]\}, \\ E_2 &= \bigsqcup_{n > v_2(m)+3} (2^{n-1} + 2^n\mathbb{Z}_2) - \{I-[v_2(m)+1]\}, \\ E_3 &= \bigsqcup_{n > v(m)+3} (4m + 2^{n-1} + 2^n\mathbb{Z}_2) - \{I-[v_2(m)+1]\}. \end{aligned}$$

**Theorem 3.25.** Consider  $f(x) = x^2 + (-1 - 4m)x$  with  $v_2(m) \in 1 + 2\mathbb{N}$ . Then  $f$  admits two fixed points  $0$  and  $4m + 2$ ,  $f(1 + 2\mathbb{Z}_2) \subset 2\mathbb{Z}_2$  and we can decompose  $2\mathbb{Z}_2$  as

$$2\mathbb{Z}_2 = \{0, 4m\} \bigsqcup E_1 \bigsqcup E_2 \bigsqcup E_3,$$

where

$$\begin{aligned} E_1 &= \bigsqcup_{n \geq 4} (4m + 2 + 2^{n-2} + 2^{n-1}\mathbb{Z}_2) - \{II-[1]\}, \\ E_2 &= \bigsqcup_{4 \leq n \leq \lfloor v_2(m)/2 \rfloor + 3} (2^{n-2} + 2^{n-1}\mathbb{Z}_2) - \{II-[2n - 5]\}, \\ E_3 &= \bigsqcup_{n > \lfloor v_2(m)/2 \rfloor + 3} (2^{n-2} + 2^{n-1}\mathbb{Z}_2) - \{II-[v_2(m) + 1]\}. \end{aligned}$$

**Theorem 3.26.** Consider  $f(x) = x^2 + (-1 - 4m)x$  with  $v_2(m) \in 2\mathbb{N}^*$ . Then  $f$  admits fixed points  $0$  and  $4m + 2$ , and  $f(1 + 2\mathbb{Z}_2) \subset 2\mathbb{Z}_2$ . The invariant set  $2\mathbb{Z}_2$  admits the following form

$$2\mathbb{Z}_2 = \{0, 4m\} \bigsqcup E_1 \bigsqcup E_2 \bigsqcup E_3 \bigsqcup (2^{v_2(m)/2+1} + 2^{v_2(m)/2+2}\mathbb{Z}_2),$$

where

$$\begin{aligned} E_1 &= \bigsqcup_{n \geq 4} (4m + 2 + 2^{n-2} + 2^{n-1}\mathbb{Z}_2) - \{II-[1]\}, \\ E_2 &= \bigsqcup_{4 \leq n < v_2(m)/2 + 3} (2^{n-2} + 2^{n-1}\mathbb{Z}_2) - \{II-[2n - 5]\}, \\ E_3 &= \bigsqcup_{n > v_2(m)/2 + 3} (2^{n-2} + 2^{n-1}\mathbb{Z}_2) - \{II-[v_2(m) + 1]\}. \end{aligned}$$

Denote  $E = 2^{v_2(m)/2+1} + 2^{v_2(m)/2+2}\mathbb{Z}_2$ .

- (1) If  $v_2(m) = 2$  and  $v_2(m - 4) = 3$ , then  $E$  is of type II-[4].
- (2) If  $v_2(m) = 2$  and  $v_2(m - 4) \geq 5$ , then  $E$  is of type II-[5].
- (3) If  $v_2(m) = 2$  and  $v_2(m - 4) = 4$ , then there exists a 2-periodic orbit with one point  $x_1 \in 4 + 16\mathbb{Z}_2$  and the other  $x_2 \in 12 + 16\mathbb{Z}_2$  and we can decompose  $E$  as  $E = \{x_1, x_2\} \bigsqcup E_4$ , where

$$E_4 = \bigsqcup_{k \geq 5} ((x_1 + 2^{k-1} + 2^k\mathbb{Z}_2) \cup (x_2 + 2^{k-1} + 2^k\mathbb{Z}_2)) - \{II-[5]\}.$$

(4) If  $v_2(m) \geq 4$  and  $v_2(m - 2^{v_2(m)}) < v_2(m) + 3$ , then  $E$  is of type II- $[v_2(m) - 2^{v_2(m)} + 1]$ .

(5) If  $v_2(m) \geq 4$  and  $v_2(m - 2^{v_2(m)}) \geq v_2(m) + 3$ , then there exists a 2-periodic orbit with one point  $x'_1 \in 2^{v_2(m)/2+1} + 2^{v_2(m)/2+3}\mathbb{Z}_2$  and the other  $x'_2 \in 2^{v_2(m)/2+1} + 2^{v_2(m)/2+2} + 2^{v_2(m)/2+3}\mathbb{Z}_2$ ; and we can decompose  $E$  as  $E = \{x'_1, x'_2\} \sqcup E'_4$ , where

$$E'_4 = \bigsqcup_{k \geq v_2(m)/2+3} ((x'_1 + 2^{k-1} + 2^k\mathbb{Z}_2) \cup (x'_2 + 2^{k-1} + 2^k\mathbb{Z}_2)) - \{II-[v_2(m) + 1]\}.$$

Now we are left to study the polynomials  $f(x) = x^2 + x - d$  with  $d \in \mathbb{Z}_2$  but  $\sqrt{d} \notin \mathbb{Z}_2$ .

We distinguish four cases.

**Theorem 3.27.** Consider  $f(x) = x^2 + x - d$  with  $d \equiv 0 \pmod{4}$  and  $\sqrt{d} \notin \mathbb{Z}_2$ . Then  $f(1 + 2\mathbb{Z}_2) \subset 2\mathbb{Z}_2$  and  $2\mathbb{Z}_2$  is decomposed as finite number of minimal components. Let  $n_0 = \lfloor v_2(d)/2 \rfloor + 1$ .

(1) If  $v_2(d) = 2$  and  $v_2(d - 4) = 3$ , then  $2\mathbb{Z}_2$  consists of three minimal components:  $4\mathbb{Z}_2$ ,  $2 + 8\mathbb{Z}_2$  and  $6 + 8\mathbb{Z}_2$ .

(2) If  $v_2(d) = 2$  and  $v_2(d - 4) = 4$ , then  $2\mathbb{Z}_2$  consists of five minimal components:  $4\mathbb{Z}_2$ ,  $2 + 8\mathbb{Z}_2$ ,  $6 + 8\mathbb{Z}_2$ ,  $10 + 8\mathbb{Z}_2$ , and  $14 + 8\mathbb{Z}_2$ .

(3) If  $v_2(d) \geq 3$  and  $v_2(d)$  is odd, then  $2\mathbb{Z}_2 = E_1 \sqcup E_2$ , where

$$E_1 = \bigsqcup_{2 \leq n \leq n_0} (2^{n-1} + 2^n\mathbb{Z}_2) - \{I-[n-2]\},$$

$$E_2 = 2^{n_0}\mathbb{Z}_2 - \{I-[n_0-1]\}.$$

(4) If  $v_2(d) \geq 3$  and  $v_2(d)$  is even, then  $2\mathbb{Z}_2 = E'_1 \sqcup E'_2 \sqcup E'_3$ , where

$$E'_1 = \bigsqcup_{2 \leq n \leq n_0-1} (2^{n-1} + 2^n\mathbb{Z}_2) - \{I-[n-2]\},$$

$$E'_2 = 2^{n_0}\mathbb{Z}_2 - \{I-[n_0-2]\},$$

$$E'_3 = 2^{n_0-1} + 2^{n_0}\mathbb{Z}_2 - \{I-[n_0+1]\}.$$

**Theorem 3.28.** Consider  $f(x) = x^2 + x - d$  with  $d = 1 \pmod{4}$  and  $\sqrt{d} \notin \mathbb{Z}_2$ . Then  $f(2\mathbb{Z}_2) \subset 1 + 2\mathbb{Z}_2$  and  $3 + 4\mathbb{Z}_2$  is of type II-1. Let  $d = 5 + 8t$  with  $t \in \mathbb{Z}_2$ . If  $v_2(t) \leq 1$ , then  $1 + 4\mathbb{Z}_2$  is of type II- $(v_2(t) + 1)$ . If  $v_2(t) \geq 2$ , then

$$1 + 4\mathbb{Z}_2 = \{x_1, x_2\} \bigsqcup E_1 \bigsqcup E_2 \bigsqcup E_3,$$

with the form

$$\begin{aligned} E_1 &= (a + 2^4\mathbb{Z}_2) \cup (f(a) + 2^4\mathbb{Z}_2) - \{II-[3]\}, \\ E_2 &= (b + 2^5\mathbb{Z}_2) \cup (f(b) + 2^5\mathbb{Z}_2) - \{II-[3]\}, \\ E_3 &= \bigsqcup_{n \geq 6} (x_1 + 2^n\mathbb{Z}_2) \cup (x_2 + 2^n\mathbb{Z}_2) - \{II-[3]\}, \end{aligned}$$

and  $x_1, x_2$  is a 2-periodic orbit such that  $x_1 \in c + 2^5\mathbb{Z}_2$  and  $x_2 \in f(c) + 2^5\mathbb{Z}_2$ . Precisely,

- (1) If  $v_2(t) = 2$  and  $v_2(t - 4) = 3$ , then  $a = 1, b = 25, c = 9$ .
- (2) If  $v_2(t) = 2$  and  $v_2(t - 4) \geq 4$ , then  $a = 1, b = 9, c = 25$ .
- (3) If  $v_2(t) = 3$ , then  $a = 9, b = 1, c = 17$ .
- (4) If  $v_2(t) \geq 4$ , then  $a = 9, b = 17, c = 1$ .

**Theorem 3.29.** Consider  $f(x) = x^2 + x - d$  with  $d = 2 \pmod{4}$  and  $\sqrt{d} \notin \mathbb{Z}_2$ . Then  $f(1 + 2\mathbb{Z}_2) \subset 2\mathbb{Z}_2$ .

- (1) If  $v_2(d - 2) = 2$ , then  $2\mathbb{Z}_2$  is of type II-1.
- (2) If  $v_2(d - 2) = 3$ , then  $8\mathbb{Z}_2 \cup (f(0) + 8\mathbb{Z}_2)$  is of type II-1,  $(4 + 8\mathbb{Z}_2) \cup (f(4) + 8\mathbb{Z}_2)$  consists of a 2-periodic orbit with one point  $x_1 \in 4 + 8\mathbb{Z}_2$  and the other  $x_2 \in f(4) + 8\mathbb{Z}_2$ , and for each  $n \geq 4$ ,  $(x_1 + 2^n\mathbb{Z}_2) \cup (x_2 + 2^n\mathbb{Z}_2)$  is of type II-2;
- (3) If  $v_2(d - 2) \geq 4$ , then  $4 + 8\mathbb{Z}_2 \cup (f(4) + 8\mathbb{Z}_2)$  is of type II-1,  $8\mathbb{Z}_2 \cup (f(0) + 8\mathbb{Z}_2)$  consists of a 2-periodic orbit with one point  $x_1 \in 8\mathbb{Z}_2$  and the other  $x_2 \in f(0) + 8\mathbb{Z}_2$ , and for each  $n \geq 4$ ,  $(x_1 + 2^n\mathbb{Z}_2) \cup (x_2 + 2^n\mathbb{Z}_2)$  is of type II-2.

**Theorem 3.30.** For  $f(x) = x^2 + x - d$  with  $d = 3 \pmod{4}$ , the ball  $2\mathbb{Z}_2$  is mapped into the ball  $1 + 2\mathbb{Z}_2$  which is the unique minimal component.

We prove Theorems 3.22-3.26. The proofs of Theorems 3.27-3.30 will be omitted since they are similar to those of Theorems 3.22-3.26.

*Proof of Theorem 3.22.* Let  $f(x) = x^2 - \lambda$ . Then  $f'(x) = 2x$  and  $(f^2)'(x) = 4x^3 - 4\lambda x$ .

1) If  $\lambda \equiv 0 \pmod{4}$ , then  $2 + 4\mathbb{Z}_2$  and  $3 + 4\mathbb{Z}_2$  are mapped into  $4\mathbb{Z}_2$  and  $1 + 4\mathbb{Z}_2$  respectively, and  $4\mathbb{Z}_2$  and  $1 + 4\mathbb{Z}_2$  are mapped into themselves respectively. Consider the cycles (0) and (1) of  $f_2$ . We have

$$a_2(0) = f'(0) \equiv 0 \pmod{2} \quad \text{and} \quad a_2(1) = f'(1) \equiv 0 \pmod{2}.$$

Thus cycles (0) and (1) grow tails, hence there will form two attracting fixed points, one in  $4\mathbb{Z}_2$  with basin  $2\mathbb{Z}_2$ , and the other one in  $1 + 4\mathbb{Z}_2$  with basin  $1 + 2\mathbb{Z}_2$ .

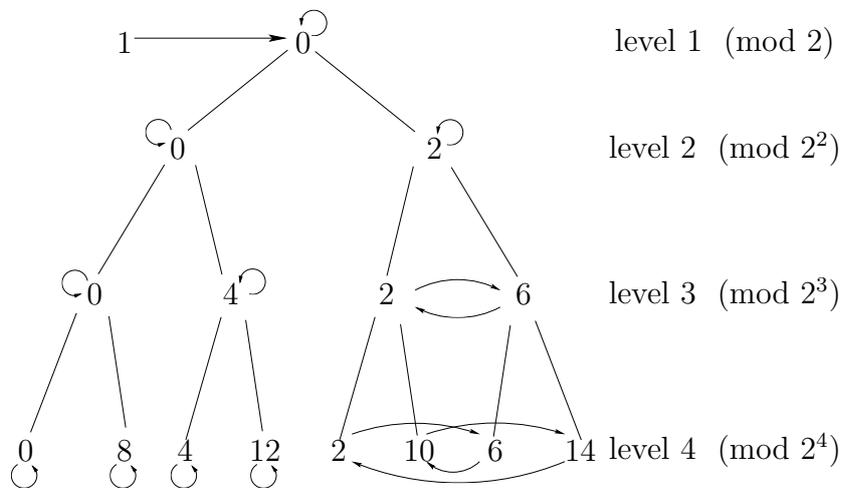
2) If  $\lambda \equiv 1 \pmod{4}$ , then  $1 + 4\mathbb{Z}_2$  and  $2 + 4\mathbb{Z}_2$  are mapped into  $4\mathbb{Z}_2$  and  $3 + 4\mathbb{Z}_2$  respectively, and  $4\mathbb{Z}_2$  and  $3 + 4\mathbb{Z}_2$  are mapped into  $3 + 4\mathbb{Z}_2$  and  $4\mathbb{Z}_2$  respectively. Consider the cycle (0, 3) of  $f_2$ . We have

$$a_2(0) = (f^2)'(0) \equiv 0 \pmod{2}.$$

Thus cycle the cycle (0, 3) grows tails, hence there will form an attracting 2-periodic orbit, with one point in  $4\mathbb{Z}_2$ , and the other one in  $3 + 4\mathbb{Z}_2$ . We also see that the attracting basin is the whole  $\mathbb{Z}_2$ .

The proofs of 3) and 4) are similar to the proofs of 1) and 2). □

*Proof of Theorem 3.23.* Let  $f(x) = x^2 + x$ . We will use a diagram to show the structure of the dynamics of  $f$ .



At level  $n$ , the " $\rightarrow$ " stands for the transformation of the elements of  $\mathbb{Z}/p^n\mathbb{Z}$  under  $f_n$ . Thus the diagram shows that

$$f_1(1) = 0, f_1(0) = 0, \quad i.e., f(1 + 2\mathbb{Z}_2) \subset 2\mathbb{Z}_2 \text{ and } f(2\mathbb{Z}_2) \subset 2\mathbb{Z}_2.$$

and

$$f_2(0) = 0, f_2(2) = 2, \quad i.e., f(4\mathbb{Z}_2) \subset 4\mathbb{Z}_2 \text{ and } f(2 + 4\mathbb{Z}_2) \subset 2 + 4\mathbb{Z}_2.$$

Since  $f(1 + 2\mathbb{Z}_2) \subset 2\mathbb{Z}_2$  and  $f^{-1}(1 + 2\mathbb{Z}_2) = \emptyset$ , we need only to consider  $2\mathbb{Z}_2$ . From the diagram, we also see that (0) is the only cycle of  $f_1$  with length 1, and (0), (2) are two lifts of (0).

We will start our examination from the level 2. Since

$$a_2(0) = f'(0) = 1 \quad \text{and} \quad b_2(0) = \frac{f(0) - 0}{2^2} = 0,$$

we have  $a_2(0) \equiv 1 \pmod{4}$  and

$$A_2(0) = \infty \quad \text{and} \quad B_2(0) = \infty.$$

Thus the cycle (0) strongly splits and by Proposition 3.9, the cycle (0) splits infinite times.

Since

$$a_2(2) = f'(2) = 5 \quad \text{and} \quad b_2(2) = \frac{f(2) - 2}{2^2} = 1,$$

we have  $a_2(2) \equiv 1 \pmod{4}$  and

$$A_2(2) = 2 \quad \text{and} \quad B_2(0) = 0.$$

Thus the cycle (2) strongly grows which implies that the lift of (2) still grows, and so on. Hence  $2 + 4\mathbb{Z}_2$  is a minimal component.

By induction we know that for all  $n \geq 2$

$$A_n(0) = \infty \quad \text{and} \quad B_n(0) = \infty.$$

Thus the cycle (0) of  $f_{n-1}$  always splits to be two cycles (0) and  $(2^{n-1})$  of  $f_n$ , and the number 0 should be a fixed point.

Now for  $n \geq 2$ , let us consider the cycle  $(2^{n-1})$  of  $f_n$ . With the same calculations,

$$a_n(2^{n-1}) = 2^n + 1 \quad \text{and} \quad b_n(2^{n-1}) = 2^{n-2}.$$

Thus  $a_n(2^{n-1}) \equiv 1 \pmod{4}$  and

$$A_n(2^{n-1}) = n \quad \text{and} \quad B_n(2^{n-1}) = n - 2.$$

Hence, the cycle  $(2^{n-1})$  strongly splits and  $B_n < \min\{A_n, n\}$ . By Proposition 3.9, the lift of  $(2^{n-1})$  splits  $B_n - 1 = n - 3$  times then all lifts strongly grow. Thus there are  $2^{n-2}$  pieces of minimal components which constitute  $2^{n-1} + 2^n\mathbb{Z}_2$ . They are

$$2^{n-1} + t2^n + 2^{2n-2}\mathbb{Z}_2, \quad t = 0, \dots, 2^{n-2} - 1.$$

This concludes the proof of Theorem 3.23.  $\square$

*Proof of Theorem 3.24.* Let  $f(x) = x^2 + (1 - 4m)x$ . We see that there are two fixed points 0 and  $4m$ , and  $1 + 2\mathbb{Z}_2$  is mapped into  $2\mathbb{Z}_2$ . We are concerned with the invariant subset  $2\mathbb{Z}_2$ .

Consider  $2^{n-1} + 2^n\mathbb{Z}_2$  ( $n \geq 1$ ). We study the cycle  $(2^{n-1})$  at level  $n$ . We have

$$a_n(2^{n-1}) = 2^n - 4m + 1 \quad b_n(2^{n-1}) = 2^{n-2} - 2m,$$

thus  $a_n(2^{n-1}) \equiv 1 \pmod{4}$  and if  $2 \leq n < v_2(m) + 3$ ,

$$A_n(2^{n-1}) \geq n \quad B_n(2^{n-1}) = n - 2.$$

Hence, the cycle  $(2^{n-1})$  strongly splits and  $B_n < \min\{A_n, n\}$ . By Proposition 3.9, the lift of  $(2^{n-1})$  strongly splits  $B_n - 1 = n - 3$  times then all lifts strongly grow. Thus we will obtain the part  $E_1$  in Theorem 3.24.

If  $n > v_2(m) + 3$ ,

$$A_n(2^{n-1}) = v_2(m) + 2 \quad B_n(2^{n-1}) = v_2(m) + 1.$$

Hence, the cycle  $(2^{n-1})$  strongly splits and  $B_n < \min\{A_n, n\}$ . By Proposition 3.9, the lift of  $(2^{n-1})$  strongly splits  $B_n - 1 = v_2(m)$  times then all lifts strongly grow. Hence we have the part  $E_2$ .

Consider  $4m + 2^{n-1} + 2^n\mathbb{Z}_2$  ( $n > v_2(m) + 3$ ). Let  $s_n \equiv 4m + 2^{n-1} \pmod{2^n}$  and  $0 \leq s_n < 2^n$ . We study the cycle  $(s_n)$  at level  $n$ . We have

$$a_n(s_n) = 2s_n - 4m + 1 \quad b_n(s_n) = \frac{s_n(s_n - 4m)}{2^n},$$

thus  $a_n(2^{n-1}) \equiv 1 \pmod{4}$  and

$$A_n(2^{n-1}) = v_2(m) + 2 \quad B_n(2^{n-1}) = v_2(m) + 1.$$

Hence,  $B_n < \min\{A_n, n\}$ . By Proposition 3.9, the cycle  $(s_n)$  strongly splits and the lift of  $(s_n)$  strongly splits  $B_n - 1 = v_2(m)$  times then all lifts strongly grow. Therefore, we have the part  $E_3$ . This completes the proof.  $\square$

*Proof of Theorem 3.25.* Let  $f(x) = x^2 + (-1 - 4m)x$  with  $v_2(m) \in 1 + 2\mathbb{N}$ . We see that there are two fixed points 0 and  $4m + 2$ , and  $1 + 2\mathbb{Z}_2$  is mapped into  $2\mathbb{Z}_2$ .

Since 0 and  $4m + 2$  are two fixed points, there are cycles (0) and  $(t_n)$  at each level, where  $t_n \equiv 4m + 2 \pmod{2^n}$  and  $0 \leq t_n < 2^n$ . Consider the cycles (0) and  $(t_{n-2})$  at level  $n - 2$ . By studying the  $a_{n-2}, b_{n-2}$  of these two cycles, we know that they weakly split. By Proposition 3.10, after splitting, half of lifts weakly grow. Thus we will obtain two 2-cycles:  $(2^{n-2}, 2^{n-2} + 2^{n-1})$  and  $(s_n, s_n + 2^{n-1})$  at level  $n$ , where  $s_n \equiv 4m + 2 + 2^{n-2} \pmod{2^{n-1}}$  and  $0 \leq s_n < 2^{n-1}$ .

For each  $n \geq 4$ , we study the cycle  $(s_n, s_n + 2^{n-1})$  at level  $n$ . We have

$$a_n(s_n) = 8 \left( 4 \left( \frac{s_n}{2} \right)^3 - 3(4m + 1) \left( \frac{s_n}{2} \right)^2 + m(4m + 1)s_n + 2m^2 + m \right) + 1$$

$$b_n(s_n) = \frac{1}{2^n} s_n (s_n - 4m - 2) (s_n^2 - 4ms_n - 4m),$$

thus  $a_n(s_n) \equiv 1 \pmod{4}$  and

$$A_n(s_n) = 3 \quad B_n(s_n) = 1.$$

Hence, the cycle  $(s_n, s_n + 2^{n-1})$  strongly splits and  $B_n < \min\{A_n, n\}$ . Therefore, by Proposition 3.9, the lift of  $(s_n, s_n + 2^{n-1})$  strongly splits  $B_n - 1 = 1 - 1 = 0$  times then all lifts strongly grow. Thus we obtain  $E_1$  in Theorem 3.25.

Now we study the cycle  $(2^{n-2}, 2^{n-2} + 2^{n-1})$  at level  $n \geq 4$ . We have

$$a_n(2^{n-2}) = 2^{3n-2} - 3(4m + 1)2^{2n-3} + m(4m + 1)2^{n+1} + 16m^2 + 8m + 1$$

$$b_n(2^{n-2}) = 2(2^{n-3} - 2m - 1)(2^{2n-6} - m2^{n-2} - m).$$

Thus  $a_n(2^{n-2}) \equiv 1 \pmod{4}$  and for each  $n > \lfloor \frac{v_2(m)}{2} \rfloor + 3$ ,

$$A_n(2^{n-2}) = v_2(m) + 3 \quad B_n(2^{n-2}) = v_2(m) + 1.$$

Hence, the cycle  $(2^{n-2}, 2^{n-1})$  strongly splits and  $B_n < \min\{A_n, n\}$ . Therefore, by Proposition 3.9, the lift of  $(2^{n-2}, 2^{n-1})$  strongly splits  $B_n - 1 = v_2(m) + 1 - 1 = v_2(m)$  times then all lifts strongly grow. Thus we have  $E_3$ .

For each  $4 \leq n \leq \lfloor \frac{v_2(m)}{2} \rfloor + 3$ ,

$$A_n(2^{n-2}) = 2n - 3 \quad B_n(2^{n-2}) = 2n - 5.$$

Hence, if  $n > 4$ , then the cycle  $(2^{n-2}, 2^{n-1})$  strongly splits and  $A_n > B_n \geq n$ . Therefore, the lift of  $(2^{n-2}, 2^{n-1})$  strongly splits at least  $n - 1$  times. But except this we do not obtain any further more information. Thus Proposition 3.9 is not sufficient for us. Now we do some calculations directly.

For any point  $2^{n-2} + t2^{n-1} \in 2^{n-2} + 2^{n-1}\mathbb{Z}_2$ , with  $t \in \mathbb{Z}_2$ , we have

$$f^2(2^{n-2} + t2^{n-1}) - (2^{n-2} + t2^{n-1}) = 2^{n+1}(1 + 2t) \cdot \Theta, \quad (3.11)$$

where

$$\Theta := (2^{n-3} + t2^{n-2} - 2m - 1) \left( (2^{n-3} + t2^{n-2})^2 - m(2^{n-2} + t2^{n-1}) - m \right).$$

Since  $4 \leq n \leq \lfloor \frac{v_2(m)}{2} \rfloor + 3$ , we have  $v_2(\Theta) = 2n - 6$ , and

$$\begin{aligned} f^2(2^{n-2} + t2^{n-1}) - (2^{n-2} + t2^{n-1}) &\equiv 0 \pmod{2^{3n-5}} \\ &\not\equiv 0 \pmod{2^{3n-4}} \end{aligned}$$

Thus the cycles grow at level  $3n - 5$ . By Corollary 3.16, the cycles grow always. Therefore we obtain the part  $E_2$  which completes the proof.  $\square$

*Proof of Theorem 3.26.* Let  $f(x) = x^2 + (-1 - 4m)x$  with  $v_2(m) \in 2\mathbb{N}^*$ . We see that there are two fixed points 0 and  $4m + 2$ , and  $1 + 2\mathbb{Z}_2$  is mapped into  $2\mathbb{Z}_2$ .

As the proof of Theorem 3.25, we study two 2-cycles:  $(2^{n-2}, 2^{n-2} + 2^{n-1})$  and  $(s_n, s_n + 2^{n-1})$  at level  $n$ , where  $s_n \equiv 4m + 2 + 2^{n-2} \pmod{2^{n-1}}$  and  $0 \leq s_n < 2^{n-1}$ . The existence of  $E_1, E_2, E_3$  are the same as that of Theorem 3.25.

We consider  $2^{n-2} + 2^{n-1}\mathbb{Z}_2$  with  $n = \frac{v_2(m)}{2} + 3$ . We are going to study the cycle  $(2^{n-2}, 2^{n-1})$  at level  $n$ . We study the points  $2^{n-2} + t2^{n-1} \in 2^{n-2} + 2^{n-1}\mathbb{Z}_2$ , with  $t \in \mathbb{Z}_2$ . With the same calculation in the proof of Theorem 3.25, we have the same equation (3.11). To continue the proof, we will distinguish two cases:  $v_2(m) = 2$  and  $v_2(m) \geq 4$ .

If  $v_2(m) = 2$ , then  $n = v_2(m)/2 + 3 = 4$  and

$$\Theta = (4t - 2m + 1)[(4 - m) + 16(t + t^2) - 4m(1 + 2t)].$$

Thus if  $v_2(m - 4) = 3$ , then  $v_2(\Theta) = 3$  and

$$\begin{aligned} f^2(2^{n-2} + t2^{n-1}) - (2^{n-2} + t2^{n-1}) &\equiv 0 \pmod{2^8} \\ &\not\equiv 0 \pmod{2^9}. \end{aligned}$$

If  $v_2(m - 4) \geq 5$ , then  $v_2(\Theta) = 4$  and

$$\begin{aligned} f^2(2^{n-2} + t2^{n-1}) - (2^{n-2} + t2^{n-1}) &\equiv 0 \pmod{2^9} \\ &\not\equiv 0 \pmod{2^{10}}. \end{aligned}$$

Hence we will obtain (1) and (2) of Theorem 3.26.

Since  $f^2(x) - x = x(x - 4m - 2)(x^2 - 4mx - 4m)$ ,  $f$  has 2-periodic orbit if and only if  $x^2 - 4mx - 4m = 0$  has solutions different to 0 and  $4m + 2$  in  $\mathbb{Z}_2$ . But  $x^2 - 4mx - 4m = 0$  has solution 0 or  $4m + 2$  only if  $m = 0$  or  $m = -1$ . Thus for the case  $v_2(m) \in \mathbb{N}^*$ ,  $f$  has 2-periodic orbit if and if only  $\Delta := 16m^2 + 16m$  has square roots in  $\mathbb{Z}_2$ . By the standard argument in number theory (see [133], p.18), this is equivalent to  $2^{-v_2(m)}m(m + 1) \equiv 1 \pmod{8}$ . By some basic calculations it is then equivalent to  $v_2(m - 4) = 4$ . This is nothing but the rest case we need to study. Thus for  $v_2(m - 4) = 4$ , there exists a 2-periodic orbit.

From the equation  $x^2 - 4mx - 4m = 0$ , the periodic point can be written as

$$x_1 = 4 \left( \frac{m}{2} + \sqrt{\frac{m(m+1)}{4}} \right), \quad x_2 = 4 \left( \frac{m}{2} - \sqrt{\frac{m(m+1)}{4}} \right).$$

Since  $v_2(m) = 2$ , we have  $x_1 \equiv 4 \pmod{8}$ . Recall we are concerned with  $2^{n-2} + 2^{n-1}\mathbb{Z}_2$  ( $n \geq 4$ ) which is the union of two balls  $2^{n-2} + 2^n\mathbb{Z}_2$  and  $2^{n-2} + 2^{n-1} + 2^n\mathbb{Z}_2$ , and we are studying the cycle  $(2^{n-2}, 2^{n-2} + 2^{n-1})$  at level  $n \geq 4$ . Thus we have  $x_1 \equiv 4 \pmod{16}$  and  $x_2 \equiv 12 \pmod{16}$ .

For each  $k \geq 5$ , we consider the union of the two balls  $(x_1 + 2^{k-1} + 2^k\mathbb{Z}_2) \cup (x_2 + 2^{k-1} + 2^k\mathbb{Z}_2)$ . We study the cycle  $(s_1, s_2)$  where  $s_1 \equiv x_1 + 2^{k-1} \pmod{2^k}$ ,  $s_2 \equiv x_2 + 2^{k-1} \pmod{2^k}$  and  $0 \leq s_1, s_2 < 2^k$ . For every point  $x_1 + 2^{k-1} + t2^k \in x_1 + 2^{k-1} + 2^k\mathbb{Z}_2$ , ( $t \in \mathbb{Z}_2$ ), we have

$$\begin{aligned} &f^2(x_1 + 2^{k-1} + t2^k) - (x_1 + 2^{k-1} + t2^k) \\ &= 2^3 \left( \frac{x_1}{4} + 2^{k-3} + t2^{k-2} \right) \left( \frac{x_1}{2} + 2^{k-2} + t2^{k-1} - 2m - 1 \right) \cdot \Phi, \end{aligned} \tag{3.12}$$

where

$$\Phi := 2x_1(2^{k-1} + t2^k) + (2^{k-1} + t2^k)^2 - 4m(2^{k-1} + t2^k).$$

Here we have used the property that  $x_1$  is a solution of the equation  $x^2 - 4mx - 4m = 0$ .

Since  $v_2(m) = 2$  and  $v_2(x_1) = 2$ , we get  $v_2(\Phi) = k + 2$ . Thus

$$\begin{aligned} f^2(x_1 + 2^{k-1} + t2^k) - (x_1 + 2^{k-1} + t2^k) &\equiv 0 \pmod{2^{k+5}} \\ &\not\equiv 0 \pmod{2^{k+6}}. \end{aligned}$$

Hence we have (3).

Now we are left to treat the case  $v_2(m) \geq 4$ . In this case the equation  $x^2 - 4mx - 4m = 0$  admits solutions if and only if  $v_2(m - 2^{v_2(m)}) \geq v_2(m) + 3$ .

We still consider  $2^{n-2} + 2^{n-1}\mathbb{Z}_2$  with  $n = \frac{v_2(m)}{2} + 3$ . If  $v_2(m - 2^{v_2(m)}) < v_2(m) + 3$ , then  $v_2(\Theta) = v_2(m - 2^{v_2(m)})$  and for any  $t \in \mathbb{Z}_2$

$$\begin{aligned} f^2(2^{n-2} + t2^{n-1}) - (2^{n-2} + t2^{n-1}) &\equiv 0 \pmod{2^{v_2(m-2^{v_2(m)})+n+1}} \\ &\not\equiv 0 \pmod{2^{v_2(m-2^{v_2(m)})+n+2}}. \end{aligned}$$

Then we will obtain (4).

If  $v_2(m - 2^{v_2(m)}) \geq v_2(m) + 3$ , then  $2^{n-2} + 2^{n-1}\mathbb{Z}_2$  consists of a 2-periodic orbit:

$$\begin{aligned} x'_1 &= 2^{\frac{v_2(m)}{2}+1} \left( 2^{-\frac{v_2(m)}{2}} m + \sqrt{2^{-v_2(m)} m(m+1)} \right), \\ x'_2 &= 2^{\frac{v_2(m)}{2}+1} \left( 2^{-\frac{v_2(m)}{2}} m - \sqrt{2^{-v_2(m)} m(m+1)} \right). \end{aligned}$$

For each  $k \geq \frac{v_2(m)}{2} + 3$ , we consider  $(x'_1 + 2^{k-1} + 2^k\mathbb{Z}_2) \cup (x'_2 + 2^{k-1} + 2^k\mathbb{Z}_2)$ . For every point  $x'_1 + 2^{k-1} + t2^k \in x'_1 + 2^{k-1} + 2^k\mathbb{Z}_2$ , ( $t \in \mathbb{Z}_2$ ), we have the same calculation as (3.12). Since  $k \geq \frac{v_2(m)}{2} + 3$  and  $v_2(x'_1) = \frac{v_2(m)}{2} + 1$ , we get  $v_2(\frac{x'_1}{4} + 2^{k-3} + t2^{k-2}) = \frac{v_2(m)}{2} - 2$  and  $v_2(\Phi) = \frac{v_2(m)}{2} + k$ . Thus

$$\begin{aligned} f^2(x'_1 + 2^{k-1} + t2^k) - (x'_1 + 2^{k-1} + t2^k) &\equiv 0 \pmod{2^{v_2(m)+k+1}} \\ &\not\equiv 0 \pmod{2^{v_2(m)+k+2}}. \end{aligned}$$

Hence we have (5). This completes the proof.  $\square$

## Chapter 4

### The $p$ -adic Repellers in $\mathbb{Q}_p$

In this chapter, we study the  $p$ -adic transitive weak repellers on the field  $\mathbb{Q}_p$  of  $p$ -adic numbers. It is proved that any such repeller is isometrically conjugate to a subshift of finite type where a suitable metric is defined. <sup>1</sup>

#### 4.1 Statement of results

Let  $p \geq 2$  be a prime number and  $\mathbb{Q}_p$  be the field of  $p$ -adic numbers. Let  $f : X \rightarrow \mathbb{Q}_p$  be a map from a compact open set  $X$  of  $\mathbb{Q}_p$  into  $\mathbb{Q}_p$ . We assume that (i)  $f^{-1}(X) \subset X$ ; (ii)  $X = \bigsqcup_{i \in I} B_{p^{-\tau}}(c_i)$  can be written as a finite disjoint union of balls of centers  $c_i$  and of the same radius  $p^{-\tau}$  (with some  $\tau \in \mathbb{Z}$ ) such that for each  $i \in I$  there is an integer  $\tau_i \in \mathbb{Z}$  such that

$$|f(x) - f(y)|_p = p^{\tau_i} |x - y|_p \quad (\forall x, y \in B_{p^{-\tau}}(c_i)). \quad (4.1)$$

For such a map  $f$ , define its Julia set by

$$J_f = \bigcap_{n=0}^{\infty} f^{-n}(X). \quad (4.2)$$

It is clear that  $f^{-1}(J_f) = J_f$  and then  $f(J_f) \subset J_f$ . We will study the dynamical system  $(J_f, f)$ .

The triple  $(X, J_f, f)$  is called a  $p$ -adic weak repeller if all  $\tau_i$  in (4.1) are non-negative, but at least one is positive. We call it a  $p$ -adic repeller if all  $\tau_i$  in (4.1) are positive. For later convenience, we will write  $\|f\| = p^{\tau_i}$  for any map having the property (4.1), which could be called the expanding ratio (resp. contractive ratio) of  $f$  on the ball  $B_{p^{-\tau}}(c_i)$  when  $\tau_i \geq 0$  (resp.  $\tau_i \leq 0$ ).

We study  $p$ -adic weak repellers with all  $\tau_i$  are nonnegative.

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<sup>1</sup>A. H. Fan, L. M. Liao, Y. F. Wang and D. Zhou,  $p$ -adic repellers in  $\mathbb{Q}_p$  are subshifts of finite type, C. R. Math. Acad. Sci. Paris **342** (2006), no. 2, 129-134.

For any  $i \in I$ , let

$$I_i := \{j \in I : B_j \cap f(B_i) \neq \emptyset\} = \{j \in I : B_j \subset f(B_i)\}.$$

(the second equality holds because of the expansiveness and of the ultrametric property). Then define a matrix  $A = (A_{i,j})_{I \times I}$ , called incidence matrix, by

$$A_{ij} = 1 \text{ if } j \in I_i; \quad A_{ij} = 0 \text{ otherwise.}$$

If  $A$  is irreducible, we say that  $(X, J_f, f)$  is *transitive*. That  $A$  is irreducible means, for any pair  $(i, j) \in I \times I$  there is positive integer  $m$  such that  $A_{ij}^m > 0$ .

Given  $I$  and the irreducible incidence matrix  $A$  as above. Let  $\Sigma_A$  be the corresponding subshift space and let  $\sigma$  be the shift transformation on  $\Sigma_A$ . We equip  $\Sigma_A$  with a metric  $d_f$  depending on the dynamics which is defined as follows. First for  $i, j \in I, i \neq j$ , let  $\kappa(i, j)$  be the integer such that  $|c_i - c_j|_p = p^{-\kappa(i,j)}$ . It clear that  $\kappa(i, j) < \tau$ . By the ultra-metric inequality, we have

$$|x - y|_p = |c_i - c_j|_p \quad (i \neq j, \forall x \in B_{p^{-\tau}}(c_i), \forall y \in B_{p^{-\tau}}(c_j)). \quad (4.3)$$

For  $x = (x_0, x_1, \dots, x_n, \dots)$  and  $y = (y_0, y_1, \dots, y_n, \dots)$  in  $\Sigma_A$ , define

$$\begin{aligned} d_f(x, y) &= p^{-\tau_{x_0} - \tau_{x_1} - \dots - \tau_{x_{n-1}} - \kappa(x_n, y_n)} \quad (\text{if } n \neq 0), \\ d_f(x, y) &= p^{-\kappa(x_0, y_0)} \quad (\text{if } n = 0) \end{aligned} \quad (4.4)$$

where  $n = n(x, y) = \min\{i \geq 0 : x_i \neq y_i\}$ . It is clear that  $d_f$  defines the same topology as the classical metric which is defined by  $d(x, y) = p^{-n(x,y)}$ .

**Theorem 4.1.** *Let  $(X, J_f, f)$  be a transitive  $p$ -adic weak repeller with incidence matrix  $A$ . Then the dynamics  $(J_f, f, |\cdot|_p)$  is isometrically conjugate to the shift dynamics  $(\Sigma_A, \sigma, d_f)$ .*

*Remark 1.* If the incidence matrix  $A$  is not irreducible, the index set  $I$  is partitioned into classes of indexes  $I^{(1)}, I^{(2)}, \dots, I^{(k)}$ . We can arrange  $I^{(1)}, I^{(2)}, \dots, I^{(k)}$  in an order so that  $A$  is written as lower triangular matrix with irreducible sub-matrixes  $A_1, A_2, \dots, A_k$  on its diagonal (see [98]). We can then apply Theorem 4.1 to  $f : X^{(t)} \rightarrow \mathbb{Q}$  for each  $1 \leq t \leq k$ , where  $X^{(t)} = \bigcup_{i \in I^{(t)}} B_{p^{-\tau}}(c_i)$ .

*Remark 2.* Theorem 4.1 holds not only for  $p$ -adic dynamics but also for any ultrametric dynamics which satisfy the two conditions (i) and (ii) listed at the beginning of the note and the property stated in Lemma 4.3 below.

The following result allows us to apply Theorem 4.1 to polynomial dynamics.

**Theorem 4.2.** *Let  $f \in \mathbb{Q}_p[x]$  be a polynomial with  $f'(x) \neq 0$  for all  $x \in \mathbb{Q}_p$ . There exists a compact-open set  $X \subset \mathbb{Q}_p$  and an integer  $\tau$  such that conditions (i) and (ii) are satisfied. Moreover, for  $x \notin J_f$ ,  $\lim_{n \rightarrow \infty} |f^n(x)|_p = \infty$ .*

Here is a class of polynomials to which the above results can apply. Let  $f(x) = p^{-m}P(x) \in \mathbb{Q}_p[x]$  with  $P \in \mathbb{Z}_p[x]$  and  $m > 0$  (otherwise  $f \in \mathbb{Z}_p[x]$ ). Assume first that

$$(a) \quad |P(x)|_p \geq |x|_p \quad \forall x \notin \mathbb{Z}_p.$$

Let

$$X = \bigcup_{i \in I_0} B_{p^{-m}}(i), \quad \text{where} \quad I_0 = \{0 \leq i < p^m : P(i) \equiv 0 \pmod{p^m}\}.$$

Assume further that

$$(b) \quad f'(x) \neq 0 \text{ for all } x \in X.$$

In order to apply Theorem 4.1 or Theorem 4.2 to this polynomial  $p^{-m}P(x)$ , in general, we have to find a finer partition of  $X$  such that (4.1) holds on each component of this new partition.

## 4.2 Proofs

**Proof of Theorem 4.1.** The proof consists of the following lemmas.

**Lemma 4.3.** *For each  $i \in I$ , the restricted map  $f : B_{p^{-\tau}}(c_i) \rightarrow B_{p^{-\tau+\tau_i}}(f(c_i))$  is a bijection.*

*Proof.* The injectivity and the inclusion  $f(B_{p^{-\tau}}(c_i)) \subseteq B_{p^{-\tau+\tau_i}}(f(c_i))$  are direct consequences of the hypothesis (4.1). Since  $f(B_{p^{-\tau}}(c_i))$  is the continuous image of a compact set and then is closed, for the surjectivity, it suffices to prove that  $f(B_{p^{-\tau}}(c_i))$  is dense in  $B_{p^{-\tau+\tau_i}}(f(c_i))$ . For an arbitrary integer  $n \geq 1$ , consider the  $p^n$  points  $c_i + kp^\tau$  ( $0 \leq k < p^n$ ) in the ball  $B_{p^{-\tau}}(c_i)$ . Any two such different points  $c_i + k'p^\tau$  and  $c_i + k''p^\tau$  has a distance strictly larger than  $p^{-n-\tau}$ . So, by the hypothesis (4.1),

$$|f(c_i + k'p^\tau) - f(c_i + k''p^\tau)|_p > p^{-n-\tau+\tau_i}.$$

Then the  $p^n$  image points  $f(c_i + kp^\tau)$  belong to  $p^n$  different balls of radius  $p^{-n-\tau+\tau_i}$ . Thus each ball of radius  $p^{-n-\tau+\tau_i}$  contained in  $B_{p^{-\tau+\tau_i}}(f(c_i))$  contains an image point. Since  $n$  is arbitrarily large, we get the density.  $\square$

Let  $I_{\text{exp}} \subset I$  (resp.  $I_{\text{iso}}$ ) be the subset of indexes  $i \in I$  such that  $\tau_i > 0$  (resp.  $\tau_i = 0$ ). A ball  $B_{p^{-\tau}}(c_i)$  is said to be expanding (resp. isometric) if  $i \in I_{\text{exp}}$  (resp.  $i \in I_{\text{iso}}$ ).

**Lemma 4.4.** *For each isometric ball  $B_{p^{-\tau}}(c_i)$ , there exists an integer  $n \geq 1$  such that  $f^n(B_{p^{-\tau}}(c_i))$  is an expanding ball.*

*Proof.* If  $B_{p^{-\tau}}(c_i)$  is an isometric ball, by Lemma 4.3, the image  $f(B_{p^{-\tau}}(c_i))$  is still a ball of the same size as  $B_{p^{-\tau}}(c_i)$ . Suppose that the conclusion of the lemma is not true. Then all balls  $f^k(B_{p^{-\tau}}(c_i))$  ( $k = 0, 1, \dots$ ) are isometric balls. Since there is a finite number of isometric balls, there are  $0 \leq k' < k''$  such that  $f^{k'}(B_{p^{-\tau}}(c_i)) = f^{k''}(B_{p^{-\tau}}(c_i))$ . This contradicts the irreducibility.  $\square$

**Lemma 4.5.** *For each  $x \in J_f$  there is a unique sequence  $(j_n)_{n \geq 0} \in \Sigma_A$  such that*

$$x \in B_{p^{-\tau}}(c_{j_0}), \quad f(x) \in B_{p^{-\tau}}(c_{j_1}), \quad \dots, \quad f^n(x) \in B_{p^{-\tau}}(c_{j_n}), \quad \dots$$

**Proof.** This is just because  $f(J_f) \subset J_f \subset X$  and  $\{B_{p^{-\tau}}(c_i)\}_{i \in I}$  is a partition of  $X$ .  $\square$

Denote by  $h : J_f \rightarrow \Sigma_A$  the map  $x \mapsto (j_n)_{n \geq 0}$  and call  $(j_n)_{n \geq 0}$  the code sequence of  $x$ .

**Lemma 4.6.** *For each sequence  $(j_n)_{n \geq 0} \in \Sigma_A$ , there are an infinite number of  $j_n$ 's belonging to  $I_{\text{exp}}$ .*

*Proof.* When  $A_{i,j} = 1$ , we say  $j$  is an issue of  $i$ . That  $i \in I_{\text{iso}}$  means  $i$  has only one issue. Using the same argument as in the proof of Lemma 4.4 but in a different presentation, we can prove that there is no sequence  $(j_n)_{n \geq 0}$  in  $\Sigma_A$  which ends with  $j_n$ 's having only one issue.  $\square$

If  $j$  is an issue of  $i$ , let  $T_{ij} : B_{p^{-\tau}}(c_j) \rightarrow B_{p^{-\tau}}(c_i)$  be the inverse map restricted on  $B_{p^{-\tau}}(c_j)$  of  $f : B_{p^{-\tau}}(c_i) \rightarrow f(B_{p^{-\tau}}(c_i))$ . Remark that if  $i \in I_{\text{iso}}$ , then  $T_{ij}$  is an isometry and if  $i \in I_{\text{exp}}$ , then  $T_{ij}$  is a contraction with  $p^{-\tau_i}$  as its contraction ratio.

**Lemma 4.7.** For each sequence  $(j_n)_{n \geq 0} \in \Sigma_A$  and for any choice  $(b_n)$  with  $b_n \in B_{p^{-\tau}}(c_{j_n})$  ( $n = 0, 1, 2, \dots$ ), the following limit

$$x = \lim_{n \rightarrow \infty} T_{j_0 j_1} T_{j_1 j_2} \cdots T_{j_{n-1} j_n}(b_n) \quad (4.5)$$

exists and is independent of the choice  $(b_n)_{n \geq 0}$ . The point  $x$  belongs to  $J_f$  and it has  $(j_n)$  as its code sequence.

*Proof.* Let  $x_n = T_{j_0 j_1} T_{j_1 j_2} \cdots T_{j_{n-1} j_n}(b_n)$ . We have

$$|x_{n+m} - x_n|_p \leq \|T_{j_0 j_1} T_{j_1 j_2} \cdots T_{j_{n-1} j_n}\| \text{diam}(X)$$

where  $\text{diam}(X)$  denotes the diameter of  $X$ . By Lemma 4.6,  $\|T_{j_0 j_1} T_{j_1 j_2} \cdots T_{j_{n-1} j_n}\|$  tends to zero. Thus we have proved the existence of the limit by the Cauchy criterion. Let  $(b'_n)$  is another choice. Let  $x'_n$  be the corresponding  $x_n$ . We have

$$|x'_n - x_n|_p \leq \|T_{j_0 j_1} T_{j_1 j_2} \cdots T_{j_{n-1} j_n}\| \text{diam}(X)$$

which proves the independence. Recall that  $f \circ T_{ij}(x) = x$ . Then the last assertions in the lemmas are obviously true.  $\square$

Denote by  $h^* : \Sigma_A \rightarrow J_f$  the map  $(j_n) \mapsto x$  where  $x$  is well determined by (4.5) in Lemma 4.7.

**Lemma 4.8.** The map  $h : (J_f, |\cdot|_p) \rightarrow (\Sigma_A, d_f)$  is an isometric homeomorphism.

*Proof.* Lemma 4.7 and the definition of code show  $h \circ h^* = \text{Id}_{\Sigma_A}$  and  $h^* \circ h = \text{Id}_{J_f}$ . So,  $h$  is a bijection. It remains to show that  $h$  is isometric. For  $\bar{x} = (j_0 j_1 \cdots j_n \cdots) \in \Sigma_A$  and  $\bar{y} = (j'_0 j'_1 \cdots j'_n \cdots) \in \Sigma_A$ , let  $x = h^*(\bar{x})$  and  $y = h^*(\bar{y})$ . Let  $n = \min\{i \geq 0 : j_i \neq j'_i\}$ . If  $n = 0$ , it is clear that  $d_f(\bar{x}, \bar{y}) = p^{-\kappa(j_0, j'_0)} = |x - y|_p$ . If  $n \neq 0$ ,

$$\begin{aligned} |x - y|_p &= |h^*(\bar{x}) - h^*(\bar{y})|_p \\ &= \left| \lim_{n \rightarrow \infty} T_{j_0 j_1} \cdots T_{j_{n-1} j_n}(b_n) - \lim_{n \rightarrow \infty} T_{j'_0 j'_1} \cdots T_{j'_{n-1} j'_n}(b'_n) \right|_p \\ &= \|T_{j_0 j_1} \cdots T_{j_{n-2} j_{n-1}}\| \cdot |T_{j_{n-1} j_n}(x') - T_{j_{n-1} j'_n}(y')|_p, \end{aligned}$$

where

$$x' = \lim_{k \rightarrow \infty} T_{j_n j_{n+1}} \cdots T_{j_{n+k-1} j_{n+k}}(b_{n+k}) \text{ and } y' = \lim_{k \rightarrow \infty} T_{j'_n j'_{n+1}} \cdots T_{j'_{n+k-1} j'_{n+k}}(b'_{n+k}).$$

Since

$$\|T_{j_0 j_1} \cdots T_{j_{n-2} j_{n-1}}\| = p^{-\tau_{j_0} - \cdots - \tau_{j_{n-2}}},$$

we have only to show that

$$|T_{j_{n-1} j_n}(x') - T_{j_{n-1} j'_n}(y')|_p = p^{-\tau_{j_{n-1}} - \kappa(j_n, j'_n)}.$$

In fact, since  $T_{j_{n-1} j_n}(x')$  and  $T_{j_{n-1} j'_n}(y')$  are both in the same ball  $B_{p^{-\tau}}(j_{n-1})$ , the expanding property (4.1) shows

$$|x' - y'|_p = |f(T_{j_{n-1} j_n}(x')) - f(T_{j_{n-1} j'_n}(y'))|_p = p^{\tau_{j_{n-1}}} |T_{j_{n-1} j_n}(x') - T_{j_{n-1} j'_n}(y')|_p.$$

Then we conclude by the definition of  $\kappa$  (see (4.3) and the fact that  $x' \in B_{p^{-\tau}}(j_n)$  and  $y' \in B_{p^{-\tau}}(j'_n)$  ( $j_n \neq j'_n$ ), which give  $|x' - y'|_p = p^{-\kappa(j_n, j'_n)}$ .  $\square$

**Lemma 4.9.** *We have  $h \circ f = \sigma \circ h$ .*

*Proof.* Recall that  $h(x)$  is the code sequence of  $x$  and that  $f \circ T_{ij}(x) = x$ . Using these facts and the expression (4.5) of  $x$ , we have

$$\begin{aligned} h \circ f(x) &= h \circ f(\lim_n T_{j_0 j_1} \cdots T_{j_{n-1} j_n}(b_n)) \\ &= h(\lim_n T_{j_1 j_2} \cdots T_{j_{n-1} j_n}(b_n)) = (j_1 \dots j_n \dots) = \sigma \circ h(x). \end{aligned}$$

$\square$

**Proof of Theorem 4.2.** Assume  $f(x) = \sum_{j=0}^n a_j x^j$  with  $a_n \neq 0$ . There exists an integer  $\ell$  such that if  $|x|_p \geq p^\ell$ , we have  $\frac{|a_{n-k}|_p}{|x|_p^k} < |a_n|_p$  ( $1 \leq k \leq n$ ) and  $|x|_p^{n-1} |a_n|_p \geq p$ , so that

$$|f(x)|_p = |x|_p^n |a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n}|_p = |x|_p^n |a_n|_p \geq p |x|_p. \quad (4.6)$$

Let  $X = \{x \in \mathbb{Q}_p : |x|_p < p^\ell\}$ . By (4.6), we have  $f(\mathbb{Q}_p \setminus X) \subset \mathbb{Q}_p \setminus X$ . Hence,  $f^{-1}(X) \subset X$ . Also by (4.6), we get  $\lim_{n \rightarrow \infty} |f^n(x)|_p = \infty$  for  $x \notin X$ . Furthermore, we have  $\lim_{n \rightarrow \infty} |f^n(x)|_p = \infty$  for  $x \notin J_f := \bigcap_{n=0}^{\infty} f^{-n}(X)$ . In fact, for  $x \notin J_f$ , there exists  $n_0 \geq 0$  such that  $f^{n_0}(x) \notin X$ , thus  $\lim_{n \rightarrow \infty} |f^{n+n_0}(x)|_p = \infty$ .

The strict differentiability of  $f \in \mathbb{Q}_p[x]$  (See [131], p.78) and the fact  $f'(x) \neq 0$  for all  $x \in \mathbb{Q}_p$  imply that for  $a \in X$ , there exists an integer  $\tau(a)$  such that  $|f(x) - f(y)|_p = |f'(a)|_p |x - y|_p$  for all  $x, y \in B_{p^{-\tau(a)}}(a)$ . Then the compactness of  $X$  implies that there is an integer  $\tau$  such that condition (ii) is satisfied.  $\square$

### 4.3 Examples.

**Example 1.** Let  $a \in \mathbb{Z}_p$  and  $m \geq 1$  be an integer. Consider the transformation  $f_{m,a} : \mathbb{Q}_p \rightarrow \mathbb{Q}_p$  given by

$$f_{m,a}(x) = \frac{x^p - ax}{p^m}.$$

In the case where  $m = 1$  and  $a \equiv 1 \pmod{p}$ , the map  $f_{m,a} : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$  was shown, by C. F. Woodcock and N. P. Smart [147], to be topologically conjugate to the full shift on the symbolic system with  $p$  symbols.

It is easy to see that  $|f'_{m,a}(x)|_p = p^m$ . We have even  $|f_{m,a}(x) - f_{m,a}(y)|_p = p^m|x - y|_p$  for all  $x, y \in \mathbb{Z}_p$ ,  $|x - y|_p < 1$  and  $|x^p - ax|_p \geq |x|_p$  for  $x \notin \mathbb{Z}_p$ . Let  $I_{m,a} = \{0 \leq k < p^m : k^p - ak \equiv 0 \pmod{p^m}\}$  and let  $X_{m,a} = \bigsqcup_{k \in I_{m,a}} (k + p^m\mathbb{Z}_p)$ . If  $a \equiv 1 \pmod{p}$ ,  $x^p - ax = 0$  has  $p$  solutions on  $\mathbb{Z}_p$ , by Hensel lemma. Then by Theorem 4.1, on the Julia set  $J_{m,a}$ ,  $f_{m,a}$  is conjugate to the full shift on the symbolic space of  $p$  symbols. If  $a \not\equiv 1 \pmod{p}$ , then  $I_{m,a} = \{0\}$  and  $J_{m,a} = \{0\}$  is the singleton consisting of the repeller fixed point 0. In both cases, for every  $x \notin J_{m,a}$  we have  $\lim_{n \rightarrow \infty} |f_{m,a}^n(x)|_p = \infty$ .

**Example 2.** Let  $c = \frac{c_0}{p^\tau} \in \mathbb{Q}_p$  with  $|c_0|_p = 1$  and  $\tau \geq 1$ . Consider the  $p$ -adic logistic map  $f_c : \mathbb{Q}_p \rightarrow \mathbb{Q}_p$  defined by

$$f_c(x) = cx(x - 1) = \frac{c_0x(x - 1)}{p^\tau}.$$

Let  $I_c = \{0 \leq k < p^\tau : k(k - 1) \equiv 0 \pmod{p^\tau}\}$ . It is clear that  $I_c = \{0, 1\}$ . Notice that

$$|f_c(x) - f_c(y)|_p = p^\tau|x - y|_p|1 - (x + y)|_p \quad (\forall x, y).$$

So, we get  $|f_c(x) - f_c(y)|_p = p^\tau|x - y|_p$  whenever  $x, y \in B_{p^{-\tau}}(0)$  or  $x, y \in B_{p^{-\tau}}(1)$ . Let  $J_c$  be the Julia set of  $f_c$ , by Theorem 4.1,  $(J_c, f_c)$  is always conjugate to the full shift on the symbolic space of two symbols.

**Example 3.** Consider the polynomial  $f : \mathbb{Q}_2 \rightarrow \mathbb{Q}_2$  defined by

$$f(x) = \frac{x(x - 1)(x + 1)}{2}.$$

It is easy to see  $|f'(x)|_p = 2$ ,  $|f''(x)/2|_p \leq 1$ ,  $|f'''(x)/6|_p = 2$  on  $2\mathbb{Z}_2$  and  $|f'(x)|_p = 1$ ,  $|f''(x)/2|_p = 2$ ,  $|f'''(x)/6|_p = 2$  on  $1 + 2\mathbb{Z}_2$ . Then by

$$f(x) - f(y) = (x - y)f'(y) + \frac{(x - y)^2}{2}f''(y) + \frac{(x - y)^3}{6}f'''(y),$$

we have  $|f(x) - f(y)|_p = |f'(y)|_p|x - y|_p$  for  $x, y \in \mathbb{Z}_2$ ,  $|x - y|_p \leq 1/4$ . Thus we can take  $X = \bigsqcup_{k=0}^3(k + 4\mathbb{Z}_2)$ . Then the conditions (i) and (ii) are satisfied and incidence matrix, which is irreducible, is equal to

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

In this case we have  $\tau_0 = \tau_2 = 2$  and  $\tau_1 = \tau_3 = 1$ . The topological entropy of  $(J_f, f)$  is equal to  $\log 1.6956\dots$  where  $1.6956\dots$  is the maximal eigenvalue of  $A$ .

## Chapter 5

### Level Sets of Birkhoff Averages in Saturated Systems

In this chapter, we will study compact topological dynamical systems with specification property. It is proved that any such system is saturated in the sense that the topological entropy of the set of generic points of any invariant measure is equal to the measure-theoretic entropy of the measure. Banach valued Birkhoff ergodic averages are studied and a variational principle for its topological entropy spectrum is obtained. As application, a particular example is fully examined. Such an example concerns with the set of real numbers for which the frequencies of occurrences in their dyadic expansions of infinitely many words are prescribed. <sup>1</sup>

#### 5.1 Introduction and statement of the results

Let  $(X, T)$  be a dynamical system which means a continuous transformation  $T : X \rightarrow X$  on a compact metric space  $X$  with metric  $d$ . We recall a list of notations:

- $T$ -invariant probability Borel measures on  $X$ :  $\mathcal{M}_{\text{inv}}$
- ergodic measures in  $\mathcal{M}_{\text{inv}}$ :  $\mathcal{M}_{\text{erg}}$
- Bowen's entropy of a subset  $E$ :  $h_{\text{top}}(E)$ ,
- measure-theoretic entropy:  $h_{\mu}$

We also recall some basic concepts. Recall that a dynamical system  $(X, T)$  is said to satisfy the *specification property* if for any  $\epsilon > 0$  there exists an integer  $m(\epsilon) \geq 1$  having the property that for any integer  $k \geq 2$ , for any  $k$  points  $x_1, \dots, x_k$  in  $X$ , and for any integers

$$a_1 \leq b_1 < a_2 \leq b_2 < \dots < a_k \leq b_k$$

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<sup>1</sup>A. H. Fan, L. M. Liao, and J. Peyrière, *Generic points in systems of specification and Banach valued Birkhoff ergodic average*, Discrete and Continuous Dynamical Systems, 21 (2008) 1103-1128.

with  $a_i - b_{i-1} \geq m(\epsilon)$  ( $\forall 2 \leq i \leq k$ ), there exists a point  $y \in X$  such that

$$d(T^{a_i+n}y, T^n x_i) < \epsilon \quad (\forall 0 \leq n \leq b_i - a_i, \quad \forall 1 \leq i \leq k).$$

The set  $G_\mu$  of  $\mu$ -generic points is defined by

$$G_\mu := \left\{ x \in X : \frac{1}{n} \sum_{j=0}^{n-1} \delta_{T^j x} \xrightarrow{w^*} \mu \right\},$$

where  $\xrightarrow{w^*}$  stands for the weak star convergence of the measures.

A dynamical system  $(X, T)$  is said to be *saturated* if for any  $\mu \in \mathcal{M}_{\text{inv}}$ , we have  $h_{\text{top}}(G_\mu) = h_\mu$ .

Our first result is stated as follows.

**Theorem 5.1.** *If the dynamical system  $(X, T)$  satisfies the specification property, then it is saturated.*

As application, we study Banach-valued Birkhoff averages for saturated systems. Let  $\mathbb{B}$  be a real Banach space and  $\mathbb{B}^*$  its dual space, their duality being denoted by  $\langle \cdot, \cdot \rangle$ . We consider  $\mathbb{B}^*$  as a locally convex topological space with the weak star topology  $\sigma(\mathbb{B}^*, \mathbb{B})$ . For any  $\mathbb{B}^*$ -valued continuous function  $\Phi : X \rightarrow \mathbb{B}^*$ , we consider its Birkhoff ergodic averages

$$A_n \Phi(x) = \frac{1}{n} \sum_{j=0}^{n-1} \Phi(T^j x) \quad (n \geq 1).$$

We would like to know the asymptotic behavior of  $A_n \Phi(x)$  in the  $\sigma(\mathbb{B}^*, \mathbb{B})$ -topology for different points  $x \in X$ .

Let us state the problem we are studying as follows. Fix a subset  $E \subset \mathbb{B}$ . For a sequence  $\{\xi_n\} \subset \mathbb{B}^*$  and a point  $\xi \in \mathbb{B}^*$ , we denote by  $\limsup_{n \rightarrow \infty} \xi_n \stackrel{E}{\leq} \xi$  the fact

$$\limsup_{n \rightarrow \infty} \langle \xi_n, w \rangle \leq \langle \xi, w \rangle \quad \text{for all } w \in E.$$

The meaning of " $\stackrel{E}{\leq}$ " is obvious. It is clear that  $\limsup_{n \rightarrow \infty} \xi_n \stackrel{\mathbb{B}}{\leq} \xi$ , or equivalently  $\limsup_{n \rightarrow \infty} \xi_n \stackrel{\mathbb{B}}{=} \xi$ , means  $\xi_n$  converges to  $\xi$  in the weak star topology  $\sigma(\mathbb{B}^*, \mathbb{B})$ .

Let  $\alpha \in \mathbb{B}^*$  and  $E \subset \mathbb{B}$ . The object of our study is the set

$$X_\Phi(\alpha; E) = \left\{ x \in X : \limsup_{n \rightarrow \infty} A_n \Phi(x) \stackrel{E}{\leq} \alpha \right\}.$$

The set  $X_\Phi(\alpha; \mathbb{B})$  will be simply denoted by  $X_\Phi(\alpha)$ . This is the set of points  $x \in X$  such that  $\lim_{n \rightarrow \infty} A_n \Phi(x) = \alpha$  in  $\sigma(\mathbb{B}^*, \mathbb{B})$ -topology. If  $\widehat{E}$  denotes the convex cone of  $E$  which consists of all  $aw' + bw''$  with  $a \geq 0, b \geq 0$  and  $w' \in E, w'' \in E$ , then  $X_\Phi(\alpha, E) = X_\Phi(\alpha, \widehat{E})$ . So we may always assume that  $E$  is a convex cone. If  $E$  is symmetric in the sense that  $E = -E$ , then we have

$$X_\Phi(\alpha, E) = \left\{ x \in X : \lim_{n \rightarrow \infty} A_n \Phi(x) \stackrel{E}{=} \alpha \right\}.$$

By entropy spectrum we mean the function

$$\mathcal{E}_\Phi^E(\alpha) := h_{\text{top}}(X_\Phi(\alpha; E)).$$

Invariant measures will be involved in the study of the entropy spectrum  $\mathcal{E}_\Phi^E(\alpha)$ . We set

$$\mathcal{M}_\Phi(\alpha; E) = \left\{ \mu \in \mathcal{M}_{\text{inv}} : \int \Phi d\mu \stackrel{E}{\leq} \alpha \right\}$$

where  $\int \Phi d\mu$  denotes the vector-valued integral in Pettis' sense (see [130]) and the inequality " $\stackrel{E}{\leq}$ " means

$$\int \langle \Phi, w \rangle d\mu \leq \langle \alpha, w \rangle \quad \text{for all } w \in E.$$

For saturated systems, we prove the following variational principle.

**Theorem 5.2.** *Suppose that the dynamical system  $(X, T)$  is saturated. Then*

- (a) *If  $\mathcal{M}_\Phi(\alpha; E) = \emptyset$ , we have  $X_\Phi(\alpha, E) = \emptyset$ .*
- (b) *If  $\mathcal{M}_\Phi(\alpha; E) \neq \emptyset$ , we have*

$$h_{\text{top}}(X_\Phi(\alpha; E)) = \sup_{\mu \in \mathcal{M}_\Phi(\alpha; E)} h_\mu. \quad (5.1)$$

When  $\mathbb{B}$  is a finite dimensional Euclidean space  $\mathbb{R}^d$  and  $E = \mathbb{R}^d$ , the variational principle (5.1) with  $E = \mathbb{R}^d$  was proved in [48, 49] for subshifts of finite type, then for conformal repellers ([58]) and later generalized to systems with specification property [135]. There are other works assuming that  $\Phi$  is regular (Hölder for example). See [15, 43] for classical discussions, [9, 10, 111, 112, 120, 136] for recent developments on Birkhoff averages, and [6, 11, 12, 27, 35] for the multifractal analysis of measures.

The study of infinite dimensional Birkhoff averages is a new subject. We point out that [121] provides another point of view, i.e. the thermodynamical point of view which was first introduced by physicists.

The above variational principle (5.1), when  $E = \mathbb{B}$ , is easy to generalize to the following setting. Let  $\Psi$  be a continuous function defined on the closed convex hull of the image  $\Phi(X)$  of  $\Phi$  into a topological space  $Y$ . For given  $\Phi$ ,  $\Psi$ , and  $\beta \in Y$ , we set

$$X_{\Phi}^{\Psi}(\beta) = \left\{ x \in X : \lim_{n \rightarrow \infty} \Psi(A_n \Phi(x)) = \beta \right\}$$

and

$$\mathcal{M}_{\Phi}^{\Psi}(\beta) = \left\{ \mu \in \mathcal{M}_{\text{inv}} : \Psi \left( \int \Phi d\mu \right) = \beta \right\}.$$

We also set

$$\widehat{X}_{\Phi}^{\Psi}(\beta) = \left\{ x \in X : \Psi \left( \lim_{n \rightarrow \infty} A_n \Phi(x) \right) = \beta \right\} = \bigcup_{\alpha: \Psi(\alpha)=\beta} X_{\Phi}(\alpha).$$

It is clear that  $\widehat{X}_{\Phi}^{\Psi}(\beta)$  is a subset of  $X_{\Phi}^{\Psi}(\beta)$ .

**Theorem 5.3.** *Suppose that the dynamical system  $(X, T)$  is saturated. Then*

- (1) if  $\mathcal{M}_{\Phi}^{\Psi}(\beta) = \emptyset$ , we have  $X_{\Phi}^{\Psi}(\beta) = \emptyset$ ,
- (2) if  $\mathcal{M}_{\Phi}^{\Psi}(\beta) \neq \emptyset$ , we have

$$h_{\text{top}}(X_{\Phi}^{\Psi}(\beta)) = h_{\text{top}}(\widehat{X}_{\Phi}^{\Psi}(\beta)) = \sup_{\mu \in \mathcal{M}_{\Phi}^{\Psi}(\beta)} h_{\mu}. \quad (5.2)$$

This generalized variational principle (5.2) allows us to study generalized ergodic limits like

$$\lim_{n \rightarrow \infty} \frac{\sum_{j=0}^{n-1} \Phi(T^j x)}{\sum_{j=0}^{n-1} g(T^j x)}, \quad (5.3)$$

where  $g : X \rightarrow \mathbb{R}^+$  is a continuous positive function. It suffices to apply (5.2) to  $\Phi$  replaced by  $(\Phi, g)$  and  $\Psi$  defined by  $\Psi(x, y) = x/y$ , with  $x \in \mathbb{B}^*$  and  $y \in \mathbb{R}^+$ .

It also allows us to study the set of points  $x \in X$  for which the limits  $A_{\infty} \Phi(x) = \lim_{n \rightarrow \infty} A_n \Phi(x)$  verify the equation

$$\Psi(A_{\infty} \Phi(x)) = \beta. \quad (5.4)$$

The present infinite dimensional version of variational principle would have many interesting applications. We will just illustrate the usefulness of the variational principle by the following study of frequencies of blocks in the dyadic development of real numbers. It can be reviewed as an infinitely multi-recurrence problem.

Let us state the question to which we can answer. All but a countable number of real numbers  $t \in [0, 1]$  can be uniquely developed as follows

$$t = \sum_{n=1}^{\infty} \frac{t_n}{2^n} \quad (t_n \in \{0, 1\}).$$

Let  $k \geq 1$ . We write  $0^k$  for the block of  $k$  consecutive zeroes and we define the  $0^k$ -frequency of  $t$  as the limit (if it exists)

$$f(t, k) = \lim_{n \rightarrow \infty} \frac{1}{n} \#\{1 \leq j \leq n : t_j t_{j+1} \cdots t_{j+k-1} = 0^k\}.$$

Let  $(a_1, a_2, \dots)$  be a sequence of non-negative numbers. We denote by  $S(a_1, a_2, \dots)$  the set of all numbers  $t \in [0, 1]$  such that  $f(t, k) = a_k$  for all  $k \geq 1$ . As a consequence of the variational principle (5.1), we prove

**Theorem 5.4.** *The set  $S(a_1, a_2, \dots)$  is non-empty if and only if the following condition is satisfied*

$$1 = a_0 \geq a_1 \geq a_2 \geq \dots; \quad a_i - 2a_{i+1} + a_{i+2} \geq 0 \quad (i \geq 0). \quad (5.5)$$

If the condition (5.5) is fulfilled, we have

$$h_{\text{top}}(S(a_1, a_2, \dots)) = -h(1 - a_1) + \sum_{j=0}^{\infty} h(a_j - 2a_{j+1} + a_{j+2}) \quad (5.6)$$

where  $h(x) = -x \log x$ .

Furthermore, it is proved that there is a unique maximal entropy measure, which is completely determined (see Lemma 5.19).

The rest of this chapter is organized as follows. In Section 5.2, we give some preliminaries. In Section 5.3, we prove Theorem 5.1. In Section 5.4, we prove the theorems 5.2 and 5.3 and examine the case where  $\mathbb{B} = \ell^1(\mathbb{Z})$ . In Section 5.5, we apply the variational principle (5.1) to the study of the recurrence into an infinite

number of cylinders of the symbolic dynamics. Especially, we study the set of orbits whose recurrences into infinitely many cylinders are prescribed. This relies on the explicit determination of a maximal entropy measure which, by definition, maximizes the supremum in (5.1).

## 5.2 Preliminary

The definitions and properties of entropy (measure-theoretic entropy and Bowen's generalized version of topological entropy) have been discussed in Section 2.3 of Chapter 2. We also remind the readers of Bowen's Lemma (Lemma 2.35) in Section 2.3 which is useful in this chapter. We now wish to recall two propositions about the measure-theoretic entropy.

We denote by  $C(X)$  the set of continuous functions on  $X$ , by  $\mathcal{M} = \mathcal{M}(X)$  the set of all Borel probability measures.

In the sequel, we fix a sequence  $(p_i)_{i \geq 1}$  such that  $p_i > 0$  for all  $i \geq 1$  and  $\sum_{i=1}^{\infty} p_i = 1$  (for example,  $p_i = 2^{-i}$  will do). Suppose that  $s_n = (s_{n,i})_{i \geq 1}$  ( $n = 1, 2, \dots$ ) is a sequence of elements in  $\ell^\infty$ . It is obvious that  $s_n$  converges to  $\alpha = (\alpha_i)_{i \geq 1} \in \ell^\infty$  in the weak star topology (i.e. each coordinate converges) is equivalent to

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} p_i |s_{n,i} - \alpha_i| = 0.$$

We also fix a sequence of continuous functions  $\{\Phi_1, \Phi_2, \dots\}$  which is dense in the unit ball of  $C(X)$ . Write  $\Phi = (\Phi_1, \Phi_2, \dots)$ . It is evident that  $\Phi : X \rightarrow \ell^\infty$  is continuous when  $\ell^\infty$  is equipped with its weak star topology. Fix an invariant measure  $\mu \in \mathcal{M}_{\text{inv}}$ . Let

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_i, \dots) \quad \text{where} \quad \alpha_i = \int \Phi_i d\mu.$$

The set of generic points  $G_\mu$  can be described as follows

$$G_\mu = \left\{ x \in X : \lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} p_i |A_n \Phi_i - \alpha_i| = 0 \right\} = X_\Phi(\alpha). \quad (5.7)$$

It is well known that the weak topology of  $\mathcal{M}$  is compatible with the topology induced by the metric

$$\tilde{d}(\mu, \nu) = \sum_{i=1}^{\infty} p_i \left| \int \Phi_i d\mu - \int \Phi_i d\nu \right| \quad (5.8)$$

where both  $(p_i)_{i \geq 1}$  and  $\{\Phi_i\}_{i \geq 1}$  are chosen as above.

The following two results will be useful for us.

**Proposition 5.5** (Young [150]). *For any  $\mu \in \mathcal{M}_{\text{inv}}$  and any numbers  $0 < \delta < 1$  and  $0 < \theta < 1$ , there exist an invariant measure  $\nu$  which is a finite convex combination of ergodic measures, i.e.*

$$\nu = \sum_{k=1}^r \lambda_k \nu_k, \quad \text{where } \lambda_k > 0, \quad \sum_{k=1}^r \lambda_k = 1, \quad \nu_k \in \mathcal{M}_{\text{erg}}, \quad r \in \mathbb{N}^+$$

such that

$$\tilde{d}(\mu, \nu) < \delta, \quad h_\nu \geq h_\mu - \theta.$$

This is a consequence of the following result due to Jacobs (see [140], p. 186). Let  $\mu \in \mathcal{M}_{\text{inv}}$  be an invariant measure which has the ergodic decomposition  $\mu = \int_{\mathcal{M}_{\text{erg}}} \tau d\pi(\tau)$  where  $\pi$  is a Borel probability measure on  $\mathcal{M}_{\text{erg}}$ . Then we have

$$h_\mu = \int_{\mathcal{M}_{\text{erg}}} h_\tau d\pi(\tau).$$

**Proposition 5.6** (Katok [77]). *Let  $\mu \in \mathcal{M}_{\text{erg}}$  be an ergodic invariant measure. For  $\epsilon > 0$  and  $\delta > 0$ , let  $r_n(\epsilon, \delta, \mu)$  denote the minimum number of  $\epsilon$ -balls in the Bowen metric  $d_n$  whose union has  $\mu$ -measure more than or equal to  $1 - \delta$ . Then for each  $\delta > 0$  we have*

$$h_\mu = \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log r_n(\epsilon, \delta, \mu) = \lim_{\epsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} \log r_n(\epsilon, \delta, \mu).$$

In [77], it was assumed that  $T : X \rightarrow X$  is a homeomorphism. But the proof in [77] works for the transformations we are studying.

### 5.3 Systems with specification property are saturated

In this section, we prove Theorem 5.1 which says that every system satisfying the specification property is saturated. Because of Bowen's lemma (Lemma 2.35), we have only to show  $h_{\text{top}}(G_\mu) \geq h_\mu$ . The idea of the proof appeared in [48, 49] and was developed in [135]. It consists of constructing the so-called dynamical Moran sets which approximate the set of generic points  $G_\mu$ .

### 5.3.1 Dynamical Moran sets and their entropies

Fix  $\epsilon > 0$ . Let  $\{m_k\}_{k \geq 1}$  be the sequence of integers defined by  $m_k = m(2^{-k}\epsilon)$  which is the constant appeared in the definition of the specification property ( $k = 1, 2, \dots$ ). Let  $\{W_k\}_{k \geq 1}$  be a sequence of finite sets in  $X$  and  $\{n_k\}_{k \geq 1}$  be a sequence of positive integers. Assume that

$$d_{n_k}(x, y) \geq 5\epsilon \quad (\forall x, y \in W_k \quad x \neq y). \quad (5.9)$$

Let  $\{N_k\}_{k \geq 1}$  be another sequence of positive integers with  $N_1 = 1$ . Using these data, we are going to construct a compact set of Cantor type, called a dynamical Moran set, which will be denoted by  $F = F(\epsilon, \{W_k\}, \{n_k\}, \{N_k\})$ . We will give an estimate for its topological entropy.

Denote

$$M_k = \#W_k.$$

Fix  $k \geq 1$ . For any  $N_k$  points  $x_1, \dots, x_{N_k}$  in  $W_k$  i.e.  $(x_1, \dots, x_{N_k}) \in W_k^{N_k}$ , we choose a point  $y(x_1, \dots, x_{N_k}) \in X$ , which does exist by the specification property, such that

$$d_{n_k}(x_s, T^{a_s}y) < \frac{\epsilon}{2^k} \quad (s = 1, \dots, N_k) \quad (5.10)$$

where

$$a_s = (s - 1)(n_k + m_k).$$

Both (5.9) and (5.10) imply that for two distinct points  $(x_1, \dots, x_{N_k})$  and  $(\bar{x}_1, \dots, \bar{x}_{N_k})$  in  $W_k^{N_k}$  we have

$$d_{t_k}(y(x_1, \dots, x_{N_k}), y(\bar{x}_1, \dots, \bar{x}_{N_k})) > 4\epsilon \quad (5.11)$$

where  $t_k = a_{N_k} + n_k$ , i.e.

$$t_k = (N_k - 1)m_k + N_k n_k.$$

In fact, let  $y = y(x_1, \dots, x_{N_k})$  and  $\bar{y} = y(\bar{x}_1, \dots, \bar{x}_{N_k})$ . Suppose  $x_s \neq \bar{x}_s$  for some  $s \in \{1, \dots, N_k\}$ . Then

$$\begin{aligned} d_{t_k}(y, \bar{y}) &\geq d_{n_k}(T^{a_s}y, T^{a_s}\bar{y}) \\ &\geq d_{n_k}(x_s, \bar{x}_s) - d_{n_k}(x_s, T^{a_s}y) - d_{n_k}(\bar{x}_s, T^{a_s}\bar{y}) \\ &> 5\epsilon - \epsilon/2 - \epsilon/2 = 4\epsilon. \end{aligned}$$

Let

$$D_1 = W_1, \quad D_k = \left\{ y(x_1, \dots, x_{N_k}) : (x_1, \dots, x_{N_k}) \in W_k^{N_k} \right\} \quad (\forall k \geq 2).$$

Now define recursively  $L_k$  and  $\ell_k$  as follows. Let

$$L_1 = D_1, \quad \ell_1 = n_1.$$

For any  $x \in L_k$  and any  $y \in D_{k+1}$  ( $k \geq 1$ ), by the specification property, we can find a point  $z(x, y) \in X$  such that

$$d_{\ell_k}(z(x, y), x) < \frac{\epsilon}{2^{k+1}}, \quad d_{t_{k+1}}(T^{\ell_k+m_{k+1}}z(x, y), y) < \frac{\epsilon}{2^{k+1}}.$$

We will choose one and only one such  $z(x, y)$  and call it the descend from  $x \in L_k$  through  $y \in D_{k+1}$ . Let

$$\begin{aligned} L_{k+1} &= \{z(x, y) : x \in L_k, y \in D_{k+1}\}, \\ \ell_{k+1} &= \ell_k + m_{k+1} + t_{k+1} = N_1 n_1 + \sum_{i=2}^{k+1} N_i (m_i + n_i). \end{aligned}$$

Observe that for any  $x \in L_k$  and for all  $y, \bar{y} \in D_{k+1}$  with  $y \neq \bar{y}$ , we have

$$d_{\ell_k}(z(x, y), z(x, \bar{y})) < \frac{\epsilon}{2^k} \quad (k \geq 1), \quad (5.12)$$

and for any  $x, \bar{x} \in L_k$  and  $y, \bar{y} \in D_{k+1}$  with  $(x, y) \neq (\bar{x}, \bar{y})$ , we have

$$d_{\ell_{k+1}}(z(x, y), z(\bar{x}, \bar{y})) > 3\epsilon \quad (k \geq 1). \quad (5.13)$$

The fact (5.12) is obvious. To prove (5.13), first remark that  $d_{\ell_1}(z, \bar{z}) \geq 5\epsilon > 4\epsilon$  for any  $z, \bar{z} \in L_1$  with  $z \neq \bar{z}$ , and that for any  $x, \bar{x} \in L_k$  and  $y, \bar{y} \in D_{k+1}$  with  $(x, y) \neq (\bar{x}, \bar{y})$  we have

$$d_{\ell_{k+1}}(z(x, y), z(\bar{x}, \bar{y})) \geq d_{\ell_k}(x, \bar{x}) - d_{\ell_k}(z(x, y), x) - d_{\ell_k}(z(\bar{x}, \bar{y}), \bar{x})$$

and

$$\begin{aligned} & d_{\ell_{k+1}}(z(x, y), z(\bar{x}, \bar{y})) \\ & \geq d_{t_{k+1}}(y, \bar{y}) - d_{t_{k+1}}(T^{\ell_k+m_{k+1}}z(x, y), y) - d_{t_{k+1}}(T^{\ell_k+m_{k+1}}z(\bar{x}, \bar{y}), \bar{y}). \end{aligned}$$

Now using the above two inequalities, we prove (5.13) by induction. For any  $x, \bar{x} \in L_1$  and  $y, \bar{y} \in D_2$  with either  $x \neq \bar{x}$  or  $y \neq \bar{y}$ , we have

$$d_{\ell_2}(z(x, y), z(\bar{x}, \bar{y})) > 4\epsilon - \frac{\epsilon}{2^2} - \frac{\epsilon}{2^2} = 4\epsilon - \frac{\epsilon}{2}.$$

Suppose we have obtained that

$$d_{\ell_k}(z(x, y), z(\bar{x}, \bar{y})) > 4\epsilon - \frac{\epsilon}{2} - \frac{\epsilon}{2^2} - \cdots - \frac{\epsilon}{2^{k-1}}.$$

Then for any  $x, \bar{x} \in L_k$  and  $y, \bar{y} \in D_{k+1}$  with  $(x, y) \neq (\bar{x}, \bar{y})$  we have

$$\begin{aligned} d_{\ell_{k+1}}(z(x, y), z(\bar{x}, \bar{y})) &> 4\epsilon - \frac{\epsilon}{2} - \frac{\epsilon}{2^2} - \cdots - \frac{\epsilon}{2^{k-1}} - \frac{\epsilon}{2^{k+1}} - \frac{\epsilon}{2^{k+1}} \\ &= 4\epsilon - \frac{\epsilon}{2} - \frac{\epsilon}{2^2} - \cdots - \frac{\epsilon}{2^{k-1}} - \frac{\epsilon}{2^k} > 3\epsilon \end{aligned}$$

Now define our dynamical Moran set

$$F = F(\epsilon, \{W_k\}, \{n_k\}, \{N_k\}) = \bigcap_{k=1}^{\infty} F_k,$$

where

$$F_k = \bigcup_{x \in L_k} \bar{B}_{\ell_k}(x, \epsilon 2^{-(k-1)})$$

( $\bar{B}(x, r)$  denoting the closed ball of center  $x$  and radius  $r$ ). The set  $F$  is Cantor-like because for any distinct points  $x', x'' \in L_k$ , by (5.13) we have

$$\bar{B}_{\ell_k}(x', \epsilon 2^{-(k-1)}) \cap \bar{B}_{\ell_k}(x'', \epsilon 2^{-(k-1)}) = \emptyset$$

and if  $z \in L_{k+1}$  descends from  $x \in L_k$ , by (5.12) we have

$$\bar{B}_{\ell_{k+1}}(z, \epsilon 2^{-k}) \subseteq \bar{B}_{\ell_k}(x, \epsilon 2^{-(k-1)}).$$

**Proposition 5.7** (Entropy of  $F$ ). *For any integer  $n \geq 1$ , let  $k = k(n) \geq 1$  and  $0 \leq p = p(n) < N_{k+1}$  be the unique integers such that*

$$\ell_k + p(m_{k+1} + n_{k+1}) < n \leq \ell_k + (p+1)(m_{k+1} + n_{k+1}).$$

We have

$$h_{\text{top}}(F) \geq \liminf_{n \rightarrow \infty} \frac{1}{n} (N_1 \log M_1 + \cdots + N_k \log M_k + p \log M_{k+1}).$$

*Proof.* For every  $k \geq 1$ , consider the discrete measure  $\sigma_k$  concentrated on  $F_k$

$$\sigma_k = \frac{1}{\#L_k} \sum_{x \in L_k} \delta_x.$$

It can be proved that  $\sigma_k$  converges in the weak star topology to a probability measure  $\sigma$  concentrated on  $F$ . Moreover, for sufficiently large  $n$  and every point  $x \in X$  such that  $B_n(x, \epsilon/2) \cap F \neq \emptyset$ , we have

$$\sigma(B_n(x, \epsilon/2)) \leq \frac{1}{\#(L_k)M_{k+1}^p} = \frac{1}{M_1^{N_1} \cdots M_k^{N_k} M_{k+1}^p}.$$

(see [135]). Then we apply the mass distribution principle to estimate the entropy.  $\square$

### 5.3.2 Box-counting of $G_\mu$

Recall that  $\alpha = (\alpha_i)_{i \geq 1} \in \ell^\infty$  and  $\Phi = (\Phi_i)$  is a dense sequence in the unit ball of  $C(X)$ . For  $\delta > 0$  and  $n \geq 1$ , define

$$X_\Phi(\alpha, \delta, n) = \left\{ x \in X : \sum_{i=1}^{\infty} p_i |A_n \Phi_i(x) - \alpha_i| < \delta \right\}.$$

For  $\epsilon > 0$ , let  $N(\alpha, \delta, n, \epsilon)$  denote the minimal number of balls  $B_n(x, \epsilon)$  to cover the set  $X_\Phi(\alpha, \delta, n)$ . Define

$$\Lambda_\Phi(\alpha) := \lim_{\epsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log N(\alpha, \delta, n, \epsilon) \quad (5.14)$$

By the same argument in [48] (p. 884-885), we can prove the existence of the limits, and the following equality:

$$\Lambda_\Phi(\alpha) = \lim_{\epsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} \log N(\alpha, \delta, n, \epsilon).$$

**Proposition 5.8.**  $\Lambda_\Phi(\alpha) \geq h_\mu$ .

*Proof.* It suffices to prove  $\Lambda_\Phi(\alpha) \geq h_\mu - 4\theta$  for any  $\theta > 0$ . For each  $i \geq 1$ , define the variation of  $\Phi_i$  by

$$\text{var}(\Phi_i, \epsilon) = \sup_{d(x,y) < \epsilon} |\Phi_i(x) - \Phi_i(y)|.$$

By the compactness of  $X$  and the continuity of  $\Phi_i$ ,  $\lim_{\epsilon \rightarrow 0} \text{var}(\Phi_i, \epsilon) \rightarrow 0$ . So

$$\lim_{\epsilon \rightarrow 0} \sum_{i=1}^{+\infty} p_i \text{var}(\Phi_i, \epsilon) \rightarrow 0.$$

This, together with (5.14), allows us to choose  $\epsilon > 0$  and  $\delta > 0$  such that

$$\sum_{i=1}^{+\infty} p_i \text{var}(\Phi_i, \epsilon) < \delta < \theta \quad (5.15)$$

and

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log N(\alpha, 5\delta, n, \epsilon) < \Lambda_{\Phi}(\alpha) + \theta. \quad (5.16)$$

For the measure  $\mu$ , take an invariant measure  $\nu = \sum_{k=1}^r \lambda_k \nu_k$  having the properties stated in Proposition 5.5. For  $1 \leq k \leq r$  and  $N \geq 1$ , set

$$Y_k(N) = \left\{ x \in X : \sum_{i=1}^{\infty} p_i \left| A_n \Phi_i(x) - \int \Phi_i d\nu_k \right| < \delta \quad (\forall n \geq N) \right\}.$$

Since  $\nu_k$  is ergodic, by the Birkhoff theorem, we have

$$\lim_{n \rightarrow \infty} \sum_{i=0}^{\infty} p_i \left| A_n \Phi_i(x) - \int \Phi_i d\nu_k \right| = 0 \quad \nu_k\text{-a.e.} \quad (5.17)$$

Then by the Egorov theorem, there exists a set with  $\nu_k$ -measure greater than  $1 - \theta$  on which the above limit (5.17) is uniform. Therefore, if  $N$  is sufficiently large, we have

$$\nu_k(Y_k(N)) > 1 - \theta \quad (\forall k = 1, \dots, r). \quad (5.18)$$

Apply the second equality in Proposition 5.6 to the triple  $(\nu_k, 4\epsilon, \theta)$  in place of  $(\mu, \epsilon, \delta)$ . When  $\epsilon > 0$  is small enough, we can find an integer  $N_k = N_k(\nu_k, 4\epsilon, \theta) \geq 1$  such that

$$r_n(4\epsilon, \theta, \nu_k) \geq \exp(n(h_{\nu_k} - \theta)) \quad (\forall n \geq N_k).$$

This implies that if  $n \geq N_k$ , then the minimal number of balls  $B_n(x, 4\epsilon)$  to cover  $Y_k(N)$  is greater than or equal to  $\exp(n(h_{\nu_k} - \theta))$ . Consequently, if we use  $C(n, 4\epsilon)$  to denote a maximal  $(n, 4\epsilon)$ -separated set in  $Y_k(N)$ , then

$$\#C(n, 4\epsilon) \geq \exp(n(h_{\nu_k} - \theta)) \quad (\forall n \geq N_k). \quad (5.19)$$

Choose a sufficiently large integer  $N_0$  such that

$$n_k := [\lambda_k n] \geq \max(N_1, \dots, N_k, N) \quad (\forall k = 1, \dots, r; \forall n \geq N_0)$$

( $[\cdot]$  denoting the integral part). By the specification property, for each  $r$  points  $x_1 \in C(n_1, 4\epsilon), \dots, x_r \in C(n_r, 4\epsilon)$ , there exist an integer  $m(\epsilon)$  depending on  $\epsilon$  and a point  $y = y(x_1, \dots, x_r) \in X$  such that

$$d_{n_k}(T^{a_k}y, x_k) < \epsilon \quad (1 \leq k \leq r) \quad (5.20)$$

where

$$a_1 = 0, \quad a_k = (k-1)m + \sum_{s=1}^{k-1} n_s \quad (k \geq 2).$$

Write  $\hat{n} = a_r + n_r$ , i.e.

$$\hat{n} = (r-1)m + \sum_{s=1}^r n_s.$$

We claim that for all such  $y = y(x_1, \dots, x_r)$ , we have

$$y = y(x_1, \dots, x_r) \in X_\Phi(\alpha, 5\delta, \hat{n}) \quad (5.21)$$

when  $n$  is sufficiently large, and that for two distinct points  $(x_1, \dots, x_r)$  and  $(x'_1, \dots, x'_r)$  in  $C(n_1, 4\epsilon) \times \dots \times C(n_r, 4\epsilon)$ , the points  $y = y(x_1, \dots, x_r)$  and  $y' = y(x'_1, \dots, x'_r)$  satisfy

$$d_{\hat{n}}(y, y') > 2\epsilon. \quad (5.22)$$

If we admit (5.21) and (5.22), we can conclude. In fact, the balls  $B_{\hat{n}}(y, \epsilon)$  are disjoint owing to (5.22) and hence there are  $\#C(n_1, 4\epsilon) \times \dots \times \#C(n_r, 4\epsilon)$  such balls. Therefore, because of (5.21), the minimal number of  $(\hat{n}, \epsilon)$ -balls needed to cover  $X_\Phi(\alpha, 5\delta, \hat{n})$  is greater than the number of such points  $y$ 's. That is to say

$$N(\alpha, 5\delta, \hat{n}, \epsilon) \geq \#C(n_1, 4\epsilon) \times \dots \times \#C(n_r, 4\epsilon)$$

Then by (5.19), we get

$$N(\alpha, 5\delta, \hat{n}, \epsilon) \geq \exp \sum_{k=1}^r [\lambda_k n] (h_{\nu_k} - \theta).$$

By noticing that  $\frac{[\lambda_k n]}{\hat{n}} \rightarrow \lambda_k$  as  $n \rightarrow \infty$  and  $\sum_{k=1}^r \lambda_k = 1$ , we get

$$\liminf_{\hat{n} \rightarrow \infty} \frac{1}{\hat{n}} \log N(\alpha, 5\delta, \hat{n}, \epsilon) \geq h_\mu - 3\theta$$

This, together with (5.16), implies  $\Lambda_\Phi(\alpha) \geq h_\mu - 4\theta$ .

Now return to prove (5.21) and (5.22). The proof of (5.22) is simple: suppose  $x_k \neq x'_k$  for some  $1 \leq k \leq r$ . By (5.20),

$$d_{\hat{n}}(y, y') \geq d_{n_k}(T^{a_k}y, T^{a_k}y') \geq d_{n_k}(x_k, x'_k) - 2\epsilon > 4\epsilon - 2\epsilon = 2\epsilon.$$

Now prove (5.21). Recall that  $\alpha_i = \int \Phi_i d\nu$  and  $\nu = \sum_{k=1}^r \lambda_k \nu_k$ . We have

$$|A_{\hat{n}}\Phi_i(y) - \alpha_i| \leq \left| A_{\hat{n}}\Phi_i(y) - \sum_{k=1}^r \lambda_k \int \Phi_i d\nu_k \right| + \left| \int \Phi_i d\nu - \int \Phi_i d\mu \right|.$$

Since  $\tilde{d}(\mu, \nu) < \delta$  i.e.  $\sum_{i=1}^{\infty} p_i |\int \Phi_i d\mu - \int \Phi_i d\nu| < \delta$ , we have only to show that

$$\sum_{i=1}^{\infty} p_i \left| A_{\hat{n}}\Phi_i(y) - \sum_{k=1}^r \lambda_k \int \Phi_i d\nu_k \right| < 4\delta. \quad (5.23)$$

Write

$$\begin{aligned} A_{\hat{n}}\Phi_i(y) &= \frac{1}{\hat{n}} \sum_{k=1}^r \sum_{j=0}^{[\lambda_k \hat{n}] - 1} \Phi_i(T^{a_k + j}y) + \frac{1}{\hat{n}} \sum_{k=2}^r \sum_{j=a_k - m}^{a_k - 1} \Phi_i(T^jy) \\ &= \sum_{k=1}^r \frac{[\lambda_k \hat{n}]}{\hat{n}} A_{[\lambda_k \hat{n}]} \Phi_i(T^{a_k}y) + \frac{1}{\hat{n}} \sum_{k=2}^r \sum_{j=a_k - m}^{a_k - 1} \Phi_i(T^jy). \end{aligned}$$

Then

$$\left| A_{\hat{n}}\Phi_i(y) - \sum_{k=1}^r \lambda_k \int \Phi_i d\nu_k \right| \leq I_1(i) + I_2(i) + I_3(i) + I_4(i)$$

with

$$\begin{aligned} I_1(i) &= \sum_{k=1}^r \frac{[\lambda_k \hat{n}]}{\hat{n}} \left| A_{[\lambda_k \hat{n}]} \Phi_i(T^{a_k}y) - A_{[\lambda_k \hat{n}]} \Phi_i(x_k) \right| \\ I_2(i) &= \sum_{k=1}^r \frac{[\lambda_k \hat{n}]}{\hat{n}} \left| A_{[\lambda_k \hat{n}]} \Phi_i(x_k) - \int \Phi_i d\nu_k \right| \\ I_3(i) &= \sum_{k=1}^r \left| \frac{[\lambda_k \hat{n}]}{\hat{n}} - \lambda_k \right| \int |\Phi_i| d\nu_k \\ I_4(i) &= \frac{1}{\hat{n}} \sum_{k=2}^r \sum_{j=a_k - m}^{a_k - 1} |\Phi_i(T^jy)|. \end{aligned}$$

Since  $[\lambda_k \hat{n}] \leq \lambda_k \hat{n}$  and  $x_k$  satisfies (5.20), by (5.15) we get

$$\sum_{i=1}^{\infty} p_i I_1(i) \leq \sum_{i=1}^{\infty} p_i \sum_{k=1}^r \lambda_k \text{var}(\Phi_i, \epsilon) = \sum_{i=1}^{\infty} p_i \text{var}(\Phi_i, \epsilon) < \delta.$$

Since  $x_k \in Y_k(N)$  and  $[\lambda_k n] \geq N$ , we have

$$\begin{aligned} \sum_{i=1}^{\infty} p_i I_2(i) &\leq \sum_{i=1}^{\infty} p_i \sum_{k=1}^r \lambda_k \left| A_{[\lambda_k n]} \Phi_i(x_k) - \int \Phi_i d\nu_k \right| \\ &= \sum_{k=1}^r \lambda_k \sum_{i=1}^{\infty} p_i \left| A_{[\lambda_k n]} \Phi_i(x_k) - \int \Phi_i d\nu_k \right| \leq \delta \sum_{k=1}^r \lambda_k = \delta. \end{aligned}$$

Since  $\|\Phi_i\| \leq 1$ , we have

$$\begin{aligned} \sum_{i=1}^{\infty} p_i I_3(i) &\leq \sum_{k=1}^r \left| \frac{[\lambda_k n]}{\hat{n}} - \lambda_k \right| < \delta \\ \sum_{i=1}^{\infty} p_i I_4(i) &\leq \frac{(r-1)m(\epsilon)}{\hat{n}} \sum_{i=1}^{\infty} p_i = \frac{(r-1)m(\epsilon)}{\hat{n}} < \delta \end{aligned}$$

when  $n$  is sufficiently large because  $\hat{n} \rightarrow \infty$  and  $\frac{[\lambda_k n]}{\hat{n}} \rightarrow \lambda_k$ .

By combining all these estimates, we obtain (5.23).  $\square$

### 5.3.3 Saturatedness of systems with specification

In this subsection we will finish our proof of Theorem 5.1, which says that systems satisfying the specification property are saturated. It remains to prove  $h_{\text{top}}(G_\mu) \geq \Lambda_\Phi(\alpha)$ . In fact, by Proposition 5.8, we will have  $h_{\text{top}}(G_\mu) \geq h_\mu$ . On the other hand, it was known to Bowen [24] that  $h_\mu \geq h_{\text{top}}(G_\mu)$ . So, we will get  $h_{\text{top}}(G_\mu) = h_\mu$ .

**Proposition 5.9.**  $h_{\text{top}}(G_\mu) \geq \Lambda_\Phi(\alpha)$

*Proof.* It suffices to prove  $h_{\text{top}}(G_\mu) \geq \Lambda_\Phi(\alpha) - \theta$  for any  $\theta > 0$ . To this end, we will construct dynamical Moran subsets of  $G_\mu = X_\Phi(\alpha)$ , which approach  $X_\Phi(\alpha)$ . The construction is based on separated sets of  $X_\Phi(\alpha, \delta, n)$ .

Let  $m_k = m(2^{-k}\epsilon)$  be the constants in the definition of specification. By the definition of  $\Lambda_\Phi(\alpha)$  (see (5.14)), when  $\epsilon > 0$  is small enough there exist a sequence of positive numbers  $\{\delta_k\}$  decreasing to zero and a sequence of integers  $\{n_k\}$  increasing to the infinity such that

$$n_k \geq 2^{m_k} \tag{5.24}$$

and that for any  $k \geq 1$  we can find a  $(n_k, 5\epsilon)$ -separated set  $W_k$  of  $X_\Phi(\alpha, \delta_k, n_k)$  with

$$M_k := \#W_k \geq \exp(n_k(\Lambda_\Phi(\alpha) - \theta)). \quad (5.25)$$

Choose a sequence of integers  $\{N_k\}$  such that

$$N_1 = 1 \quad (5.26)$$

$$N_k \geq 2^{n_{k+1} + m_{k+1}}, \quad k \geq 2 \quad (5.27)$$

$$N_{k+1} \geq 2^{N_1 n_1 + N_2(n_2 + m_2) \cdots + N_k(n_k + m_k)}, \quad k \geq 1 \quad (5.28)$$

Consider the dynamical Moran set  $F = F(\epsilon, \{W_k\}, \{n_k\}, \{N_k\})$  as we constructed in the last subsection. From (5.24)-(5.28), we get

$$\liminf_{n \rightarrow \infty} \frac{1}{n} (N_1 \log M_1 + \cdots + N_k \log M_k + p \log M_{k+1}) \geq \Lambda_\Phi(\alpha) - \theta.$$

By Proposition 5.7, we have

$$h_{\text{top}}(F) \geq \Lambda_\Phi(\alpha) - \theta.$$

Thus we have only to prove  $F \subseteq X_\Phi(\alpha)$ . Or equivalently

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{\infty} p_i |S_n \Phi_i(x) - n\alpha_i| = 0 \quad (x \in F). \quad (5.29)$$

Let us use the same notations as in the last subsection including  $\ell_k$ ,  $t_k$ ,  $D_k$  and  $L_k$  etc.

Fix  $n \geq 1$ . Let  $k \geq 1$  and  $0 \leq p < N_{k+1}$  be the integers, which depend on  $n$ , such that

$$\ell_k + p(m_{k+1} + n_{k+1}) < n \leq \ell_k + (p+1)(m_{k+1} + n_{k+1})$$

Write

$$q = n - (\ell_k + p(m_{k+1} + n_{k+1})), \quad b_s = (s-1)(m_{k+1} + n_{k+1}).$$

Decompose the interval  $[0, n)$  ( $\subset \mathbb{N}$ ) into small intervals

$$[0, n) = [0, \ell_k) \bigcup [\ell_k, \ell_k + p(m_{k+1} + n_{k+1})) \bigcup [\ell_k + p(m_{k+1} + n_{k+1}), n)$$

and decompose still  $[\ell_k, \ell_k + p(m_{k+1} + n_{k+1}))$  into intervals alternatively of lengths  $n_{k+1}$  and  $m_{k+1}$ . Then cut the sum  $\sum_{0 \leq j < n} \Phi_i(T^j x)$  into sums taken over small intervals. Thus we get

$$|S_n \Phi_i(x) - n\alpha_i| \leq J_1(i) + J_2(i) + J_3(i) + J_4(i)$$

where

$$\begin{aligned} J_1(i) &= |S_{\ell_k} \Phi_i(x) - \ell_k \alpha_i| \\ J_2(i) &= \sum_{s=1}^p |S_{m_{k+1}} \Phi_i(T^{\ell_k + b_s} x) - m_{k+1} \alpha_i| \\ J_3(i) &= \sum_{s=1}^p |S_{n_{k+1}} \Phi_i(T^{\ell_k + b_s + m_{k+1}} x) - n_{k+1} \alpha_i| \\ J_4(i) &= |S_q \Phi_i(T^{\ell_k + p(m_{k+1} + n_{k+1})} x) - q\alpha_i| \end{aligned}$$

Since  $\|\Phi_i\| \leq 1$  (hence  $|\alpha_i| \leq 1$ ), we have

$$J_2(i) \leq 2pm_{k+1}, \quad J_4(i) \leq 2q \leq 2(m_{k+1} + n_{k+1}).$$

By (5.27), we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{\infty} p_i J_2(i) = 0, \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{\infty} p_i J_4(i) = 0. \quad (5.30)$$

Now let us deal with  $J_1(i)$  and  $J_3(i)$ . We claim that for any  $x \in F$  there exists an  $\bar{x} \in L_k$  such that

$$d_{\ell_k}(\bar{x}, x) < \frac{\epsilon}{2^{k-1}}, \quad (5.31)$$

and that for all  $1 \leq s \leq p$ , there exists a point  $x_s \in W_{k+1}$  such that

$$d_{n_{k+1}}(x_s, T^{u_s} x) < \frac{\epsilon}{2^{k-1}} \quad (5.32)$$

where

$$u_s = \ell_k + b_s + m_{k+1}.$$

In fact, by the construction of  $F$ , there exists a point  $z \in L_{k+1}$  such that

$$d_{\ell_{k+1}}(z, x) \leq \frac{\epsilon}{2^k}. \quad (5.33)$$

Assume that  $z$  descends from some  $\bar{x} \in L_k$  through  $y \in D_{k+1}$ . Then

$$d_{\ell_k}(\bar{x}, z) < \frac{\epsilon}{2^{k+1}} \quad (5.34)$$

and

$$d_{t_{k+1}}(y, T^{\ell_k+m_{k+1}}z) < \frac{\epsilon}{2^{k+1}}. \quad (5.35)$$

On the other hand, according to the definition of  $D_{k+1}$ , there exists an  $x_s \in W_{k+1}$  such that

$$d_{n_{k+1}}(x_s, T^{b_s}y) < \frac{\epsilon}{2^{k+1}}. \quad (5.36)$$

Now by the trigonometric inequality, the fact  $d_{\ell_k}(z, x) \leq d_{\ell_{k+1}}(z, x)$  and (5.33) and (5.34) we get

$$d_{\ell_k}(\bar{x}, x) \leq \frac{\epsilon}{2^{k+1}} + \frac{\epsilon}{2^k} < \frac{\epsilon}{2^{k-1}}.$$

Thus (5.31) is proved. By (5.33), (5.35) and (5.36), we can similarly prove (5.34):

$$\begin{aligned} d_{n_{k+1}}(x_s, T^{u_s}x) &\leq d_{n_{k+1}}(x_s, T^{b_s}y) + d_{n_{k+1}}(T^{b_s}y, T^{u_s}z) + d_{n_{k+1}}(T^{u_s}z, T^{u_s}x) \\ &\leq d_{n_{k+1}}(x_s, T^{b_s}y) + d_{t_{k+1}}(y, T^{\ell_k+m_{k+1}}z) + d_{\ell_{k+1}}(z, x) \\ &< \frac{\epsilon}{2^{k+1}} + \frac{\epsilon}{2^{k+1}} + \frac{\epsilon}{2^k} \\ &= \frac{\epsilon}{2^{k-1}}. \end{aligned}$$

It is now easy to deal with  $J_3(i)$ , which is obviously bounded by

$$J_3(i) \leq \sum_{s=1}^p |S_{n_{k+1}}\Phi_i(T^{u_s}x) - S_{n_{k+1}}\Phi_i(x_s)| + \sum_{s=1}^p |S_{n_{k+1}}\Phi_i(x_s) - n_{k+1}\alpha_i|.$$

Using (5.32), we obtain

$$|S_{n_{k+1}}\Phi_i(x_s) - S_{n_{k+1}}\Phi_i(T^{u_s}x)| \leq n_{k+1}\text{var}(\Phi_i, \epsilon 2^{-(k-1)}).$$

On the other hand, since  $x_s \in W_{k+1} \subseteq X_\Phi(\alpha, \delta_{k+1}, n_{k+1})$ , we have, by definition,

$$\sum_{i=1}^{\infty} p_i |S_{n_{k+1}}\Phi_i(x_s) - n_{k+1}\alpha_i| \leq n_{k+1}\delta_{k+1}.$$

Then, combining the last three estimates, and using the facts  $\sum_{j=1}^{\infty} p_j = 1$  and  $pn_{k+1} \leq n$ , we get

$$\frac{1}{n} \sum_{i=1}^{\infty} p_i J_3(i) \leq \sum_{i=1}^{\infty} p_i \text{var}(\Phi_i, \epsilon 2^{-(k-1)}) + \delta_{k+1}.$$

Since  $k$  can be arbitrarily large, we finally get

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{\infty} p_i J_3(i) = 0. \quad (5.37)$$

Now it remains to prove

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{\infty} p_i J_1(i) = 0. \quad (5.38)$$

Observe that

$$J_1(i) \leq |S_{\ell_k} \Phi_i(x) - S_{\ell_k} \Phi_i(\bar{x})| + |S_{\ell_k} \Phi_i(\bar{x}) - \ell_k \alpha_i|.$$

By (5.31), we have  $|S_{\ell_k} \Phi_i(x) - S_{\ell_k} \Phi_i(\bar{x})| \leq \ell_k \text{var}(\Phi_i, \epsilon 2^{-(k-1)})$ . Then

$$J_1(i) \leq \ell_k \text{var}(\Phi_i, \epsilon 2^{-(k-1)}) + R_{k,i}$$

where

$$R_{k,i} = \max_{z \in L_k} |S_{\ell_k} \Phi_i(z) - \ell_k \alpha_i|.$$

Since  $\text{var}(\Phi_i, \epsilon 2^{-(k-1)})$  tends to zero as  $k \rightarrow \infty$ , the desired claim (5.38) is reduced to

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{\infty} p_i R_{k,i} = 0. \quad (5.39)$$

We need two lemmas to estimate  $R_{k,i}$ .

**Lemma 5.10.** *For any  $y \in D_{k+1}$ , we have*

$$\begin{aligned} & \sum_{i=1}^{\infty} p_i |S_{t_{k+1}} \Phi_i(y) - t_{k+1} \alpha_i| \\ & \leq \sum_{i=1}^{\infty} p_i N_{k+1} n_{k+1} \text{var}(\Phi_i, \epsilon 2^{-(k+1)}) + 2(N_{k+1} - 1)m_{k+1} + N_{k+1} n_{k+1} \delta_{k+1}. \end{aligned}$$

*Proof.* For any  $s = 1, \dots, N_{k+1}$ , there exists  $x_s \in W_{k+1}$  such that

$$d_{n_{k+1}}(x_s, T^{b_s} y) < \frac{\epsilon}{2^{k+1}} \quad (5.40)$$

where  $b_s = (s-1)(m_{k+1} + n_{k+1})$ . Write

$$S_{t_{k+1}} \Phi_i(y) = \sum_{s=1}^{N_{k+1}} S_{n_{k+1}} \Phi_i(T^{b_s} y) + \sum_{s=1}^{N_{k+1}-1} S_{m_{k+1}} \Phi_i(T^{b_s + n_{k+1}} y).$$

Then

$$\begin{aligned} & |S_{t_{k+1}}\Phi_i(y) - t_{k+1}\alpha_i| \\ \leq & \sum_{s=1}^{N_{k+1}} |S_{n_{k+1}}\Phi_i(T^{b_s}y) - n_{k+1}\alpha_i| + \sum_{s=1}^{N_{k+1}-1} |S_{m_{k+1}}\Phi_i(T^{b_s+n_{k+1}}y) - m_{k+1}\alpha_i|. \end{aligned}$$

Since  $x_s \in W_{k+1} \subseteq X_\Phi(\alpha, \delta_{k+1}, n_{k+1})$ , by (5.40), we have

$$\begin{aligned} & \sum_{i=1}^{\infty} p_i |S_{n_{k+1}}\Phi_i(T^{b_s}y) - n_{k+1}\alpha_i| \\ \leq & \sum_{i=1}^{\infty} p_i |S_{n_{k+1}}\Phi_i(T^{b_s}y) - S_{n_{k+1}}\Phi_i(x_s)| + \sum_{i=1}^{\infty} p_i |S_{n_{k+1}}\Phi_i(x_s) - n_{k+1}\alpha_i| \\ \leq & \sum_{i=1}^{\infty} p_i n_{k+1} \text{var}(\Phi_i, \epsilon 2^{-(k+1)}) + n_{k+1} \delta_{k+1}. \end{aligned}$$

On the other hand,

$$|S_{m_{k+1}}\Phi_i(T^{b_s+n_{k+1}}y) - m_{k+1}\alpha_i| \leq 2m_{k+1}.$$

Now it is easy to conclude.  $\square$

**Lemma 5.11.**

$$\sum_{i=1}^{\infty} p_i R_{k,i} \leq 2 \sum_{i=1}^{\infty} p_i \sum_{j=1}^k \ell_j \text{var}(\Phi_i, \epsilon 2^{-j}) + 2 \sum_{j=1}^k N_j m_j + \sum_{j=1}^k \ell_j \delta_j.$$

*Proof.* We prove it by induction on  $k$ . When  $k = 1$ , we have  $L_1 = D_1 = W_1 \subseteq X_\Phi(\alpha, \delta_1, n_1)$  and then

$$\sum_{i=1}^{\infty} p_i R_{1,i} \leq n_1 \delta_1 = \ell_1 \delta_1.$$

Suppose the lemma holds for  $k$ . For any  $z \in L_{k+1}$  there exist  $x \in L_k$  and  $y \in D_{k+1}$ , such that

$$d_{\ell_k}(x, z) < \frac{\epsilon}{2^{k+1}}, \quad d_{t_{k+1}}(y, T^{\ell_k+m_{k+1}}z) < \frac{\epsilon}{2^{k+1}}.$$

Write

$$S_{\ell_{k+1}}\Phi_i(z) = S_{\ell_k}\Phi_i(z) + S_{m_{k+1}}\Phi_i(T^{\ell_k}z) + S_{t_{k+1}}\Phi_i(T^{\ell_k+m_{k+1}}z).$$

Then  $|S_{\ell_{k+1}}\Phi_i(z) - \ell_{k+1}\alpha_i|$  is bounded by

$$|S_{\ell_k}\Phi_i(z) - \ell_k\alpha_i| + |S_{m_{k+1}}\Phi_i(T^{\ell_k}z) - m_{k+1}\alpha_i| + |S_{t_{k+1}}\Phi_i(T^{\ell_k+m_{k+1}}z) - t_{k+1}\alpha_i|.$$

Notice that

$$\begin{aligned} |S_{\ell_k} \Phi_i(z) - \ell_k \alpha_i| &\leq |S_{\ell_k} \Phi_i(z) - S_{\ell_k} \Phi_i(x)| + |S_{\ell_k} \Phi_i(x) - \ell_k \alpha_i| \\ &\leq \ell_k \text{var}(\Phi_i, \epsilon 2^{-(k+1)}) + R_{k,i}, \end{aligned}$$

$$|S_{m_{k+1}} \Phi_i(T^{\ell_k} z) - m_{k+1} \alpha_i| \leq 2m_{k+1}$$

and

$$\begin{aligned} &|S_{t_{k+1}} \Phi_i(T^{\ell_k + m_{k+1}} z) - t_{k+1} \alpha_i| \\ &\leq |S_{t_{k+1}} \Phi_i(T^{\ell_k + m_{k+1}} z) - S_{t_{k+1}} \Phi_i(y)| + |S_{t_{k+1}} \Phi_i(y) - t_{k+1} \alpha_i| \\ &\leq t_{k+1} \text{var}(\Phi_i, \epsilon 2^{-(k+1)}) + |S_{t_{k+1}} \Phi_i(y) - t_{k+1} \alpha_i|. \end{aligned}$$

By Lemma 5.10, we have

$$\begin{aligned} \sum_{i=1}^{\infty} p_i R_{k+1,i} &\leq \sum_{i=1}^{\infty} p_i R_{k,i} + \sum_{i=1}^{\infty} p_i (\ell_k + t_{k+1} + N_{k+1} n_{k+1}) \text{var}(\Phi_i, \epsilon / 2^{k+1}) \\ &\quad + 2N_{k+1} m_{k+1} + N_{k+1} n_{k+1} \delta_{k+1}. \end{aligned}$$

Then according to the induction hypothesis the Lemma holds for  $k + 1$ , because

$$\ell_k + t_{k+1} \leq \ell_{k+1}, \quad N_{k+1} n_{k+1} \leq \ell_{k+1}.$$

□

Let us finish the proof of Theorem 5.1 by showing (5.39). Since  $n_j \geq 2^{m_j}$ , we have

$$\frac{N_j m_j}{\ell_j} \leq \frac{N_j m_j}{N_j (n_j + m_j)} = \frac{m_j}{n_j + m_j} \rightarrow 0 \quad (j \rightarrow \infty).$$

Then the estimate in Lemma 5.11 can be written as

$$\sum_{i=1}^{\infty} p_i R_{k,i} \leq \sum_{j=1}^k \ell_j c_j$$

where  $c_j \rightarrow 0$  ( $j \rightarrow \infty$ ). By (5.28), we have  $\ell_k \geq 2^{\ell_k - 1}$ . It follows that

$$\frac{1}{\ell_k} \sum_{i=1}^{\infty} p_i R_{k,i} \leq c_k + \frac{1}{\ell_k} \sum_{i=1}^{k-1} c_j \ell_j.$$

This implies (5.39). □

## 5.4 Variational principle

In this section, we prove variational principles for saturated systems (Theorem 5.2 and Theorem 5.3).

### 5.4.1 Proofs of Theorems 5.2 and 5.3

The proof of Theorem 5.3 is similar to that of Theorem 5.2.

*Proof of Theorem 5.2 (a).* It suffices to prove that if there exists a point  $x \in X_\Phi(\alpha; E)$ , then  $\mathcal{M}_\Phi(\alpha; E) \neq \emptyset$ . That  $x \in X_\Phi(\alpha; E)$  means

$$\limsup_{n \rightarrow \infty} \langle A_n \Phi(x), w \rangle \leq \langle \alpha, w \rangle \quad (\forall w \in E). \quad (5.41)$$

Let  $\mu$  be a weak limit of  $n^{-1} \sum_{j=0}^{n-1} \delta_{T^j x}$ . That is to say, there exists a sequence  $n_m$  such that

$$\lim_{m \rightarrow \infty} \frac{1}{n_m} \sum_{j=0}^{n_m-1} f(T^j x) = \int f d\mu \quad (5.42)$$

for all scalar continuous functions  $f$ . We deduce from (5.41) and (5.42) that for all  $w \in E$  we have

$$\int \langle \Phi, w \rangle d\mu = \lim_{m \rightarrow \infty} \langle A_{n_m} \Phi(x), w \rangle \leq \limsup_{n \rightarrow \infty} \langle A_n \Phi(x), w \rangle \leq \langle \alpha, w \rangle.$$

So  $\mu \in \mathcal{M}_\Phi(\alpha; E)$ . □

*Proof of Theorem 5.2 (b).* Let  $t = \sup_{\mu \in \mathcal{M}_\Phi(\alpha; E)} h_\mu$ . What we have just proved above may be stated as follows: if  $x \in X_\Phi(\alpha; E)$ , then

$$V(x) \subset \mathcal{M}_\Phi(\alpha; E).$$

It follows that  $h_\mu \leq t$  for any  $\mu \in V(x)$ . Thus

$$\begin{aligned} X_\Phi(\alpha; E) &\subset \{x \in X : \forall \mu \in V(x) \text{ satisfying } h_\mu \leq t\} \\ &\subset \{x \in X : \exists \mu \in V(x) \text{ satisfying } h_\mu \leq t\}. \end{aligned}$$

Then, due to Lemma 2.35, we get  $h_{\text{top}}(X_\Phi(\alpha; E)) \leq t$ .

Now we prove the converse inequality. For any  $\mu \in \mathcal{M}_\Phi(\alpha; E)$ , consider  $G_\mu$  the set of generic points. We have

$$G_\mu \subset X_\Phi(\alpha; \mathbb{B}) \subset X_\Phi(\alpha; E).$$

The second inclusion is obvious and the first one is a consequence of the fact that  $x \in G_\mu$  implies  $\lim_{n \rightarrow \infty} A_n \Phi(x) = \int \Phi d\mu = \alpha$  in the  $\sigma(\mathbb{B}^*, \mathbb{B})$ -topology. Thus

$$h_{\text{top}}(X_\Phi(\alpha; E)) \geq h_{\text{top}}(G_\mu).$$

Since  $\mu$  is an arbitrary invariant measure in  $\mathcal{M}_\Phi(\alpha; E)$ , we can finish the proof because the system  $(X, T)$  is saturated (i.e.  $h_{\text{top}}(G_\mu) = h_\mu$ ).  $\square$

It is useful to point out the following facts appearing in the proof:

- (i) If  $x \in X_\Phi(\alpha, E)$ , then  $V(x) \subset \mathcal{M}_\Phi(\alpha, E)$ .
- (ii) We have

$$\bigcup_{\mu \in \mathcal{M}_\Phi(\alpha, E)} G_\mu \subset X_\Phi(\alpha, E) \subset \bigcup_{\mu \in \mathcal{M}_\Phi(\alpha, E)} \tilde{G}_\mu$$

with  $\tilde{G}_\mu = \{x \in X : V(x) \ni \mu\}$ . It is worth to notice the fact that all the  $G_\mu$  are disjoint.

It is clear that  $\mathcal{M}_\Phi(\alpha, E)$  is a compact convex subset of the space  $\mathcal{M}_{\text{inv}}$  of Borel probability invariant measures. If  $h_\mu$ , as a function of  $\mu$ , is upper semi-continuous on  $\mathcal{M}_{\text{inv}}$ , then the supremum in the variational principle is attained by some invariant measure, called the maximal entropy measure in  $\mathcal{M}_\Phi(\alpha, E)$ . Usually, the structure of  $\mathcal{M}_\Phi(\alpha, E)$  is complicated. But it is sometimes possible to calculate the maximal entropy.

*Proof of Theorem 5.3.* Let  $x \in X_\Psi^\Phi(\beta)$  and let  $\mu$  be a weak limit of  $n^{-1} \sum_{j=0}^{n-1} \delta_{T^j x}$ . Then there exists a subsequence of integers  $\{n_m\}$  such that  $A_{n_m} \Phi(x)$  tends to  $\int \Phi d\mu$  in the weak star topology as  $m \rightarrow \infty$  because we have an expression similar to (5.42) with  $f = \langle \Phi, w \rangle$  ( $w \in \mathbb{B}$  being arbitrary). Hence

$$\Psi \left( \int \Phi d\mu \right) = \lim_{m \rightarrow \infty} \Psi(A_{n_m} \Phi(x)) = \lim_{n \rightarrow \infty} \Psi(A_n \Phi(x)) = \beta.$$

Thus we have proved that  $\mu \in \mathcal{M}_\Phi^\Psi(\beta)$ . That is to say

$$V(x) \subset \mathcal{M}_\Phi^\Psi(\beta) \quad (\forall x \in X_\Phi^\Psi(\beta)).$$

It follows that (a) holds and that due to Lemma 2.35 we have

$$h_{\text{top}}(X_\Phi^\Psi(\beta)) \leq \sup_{\mu \in \mathcal{M}_\Phi^\Psi(\beta)} h_\mu.$$

The converse inequality is a consequence of the variational principle (5.1) and the relationship

$$X_{\Phi}^{\Psi}(\beta) \supset \widehat{X}_{\Phi}^{\Psi}(\beta) = \bigcup_{\alpha: \Psi(\alpha)=\beta} X_{\Phi}(\alpha).$$

In fact,

$$\begin{aligned} h_{\text{top}}(X_{\Phi}^{\Psi}(\beta)) &\geq h_{\text{top}}(\widehat{X}_{\Phi}^{\Psi}(\beta)) \geq \sup_{\alpha: \Psi(\alpha)=\beta} h_{\text{top}}(X_{\Phi}(\alpha)) \\ &= \sup_{\alpha: \Psi(\alpha)=\beta} \sup_{\mu \in \mathcal{M}_{\Phi}(\alpha)} h_{\mu} \\ &= \sup_{\mu \in \mathcal{M}_{\Phi}^{\Psi}(\beta)} h_{\mu}. \end{aligned}$$

□

#### 5.4.2 $\ell^{\infty}(\mathbb{Z})$ -valued ergodic average

Let us consider the special case where  $\mathbb{B} = \ell^1(\mathbb{Z})$ . Then  $\mathbb{B}^* = \ell^{\infty}(\mathbb{Z})$ . Any  $\ell^{\infty}(\mathbb{Z})$ -valued function  $\Phi$  can be written as

$$\Phi(x) = (\Phi_n(x))_{n \in \mathbb{Z}}, \quad \text{with} \quad \sup_n |\Phi_n(x)| < \infty.$$

Recall that  $\ell^{\infty}(\mathbb{Z})$  is equipped with the  $\sigma(\ell^{\infty}, \ell^1)$ -topology. A  $\ell^{\infty}(\mathbb{Z})$ -valued function  $\Phi$  is continuous if and only if all coordinate functions  $\Phi_n : X \rightarrow \mathbb{R}$  are continuous, because for any  $w = (w_n)_{n \in \mathbb{Z}} \in \ell^1$  we have

$$\langle \Phi(x), w \rangle = \sum_{n \in \mathbb{Z}} w_n \Phi_n(x).$$

Let us give an application of the variational principle in this setting. Let  $I$  be a finite or infinite subset of positive integers. Let  $\{\Phi_i\}_{i \in I}$  be a family of real continuous functions defined on  $X$ . We suppose that  $\sup_{i \in I} \|\Phi_i\|_{C(X)} < \infty$ . For two given sequences of real numbers  $\mathbf{a} = \{a_i\}_{i \in I}$  and  $\mathbf{b} = \{b_i\}_{i \in I}$ , we denote by  $S(\mathbf{a}, \mathbf{b})$  the set of points  $x \in X$  such that

$$a_i \leq \liminf_{n \rightarrow \infty} A_n \Phi_i(x) \leq \limsup_{n \rightarrow \infty} A_n \Phi_i(x) \leq b_i \quad (\forall i \in I).$$

**Corollary 5.12.** *Suppose that the system  $(X, T)$  is saturated. The topological entropy of  $S(\mathbf{a}, \mathbf{b})$  defined above is equal to the supremum of the measure-theoretical entropies  $h_{\mu}$  for those invariant measures  $\mu$  such that*

$$a_i \leq \int \Phi_i d\mu \leq b_i \quad (\forall i \in I).$$

*Proof.* For  $n \in \mathbb{Z}$ , let  $e_n$  be the  $n^{\text{th}}$  element of the canonical basis of  $\ell^1(\mathbb{Z})$ . Let  $\Phi$  be a function whose  $n^{\text{th}}$  coordinate and  $-n^{\text{th}}$  coordinate are respectively equal to  $\Phi_n$  and  $-\Phi_n$  for each  $n \in I$  (other coordinates may be taken to be zero). Take the set  $E \subset \ell^1$ , which consists of  $e_i$  and  $e_{-i}$  for  $i \in I$ . Take  $\alpha \in \ell^\infty$  such that  $\alpha_{-i} = -a_i$  and  $\alpha_i = b_i$  for  $i \in I$ . Now we can directly apply the variational principle by noticing that

$$\langle \Phi, e_i \rangle = \Phi_i, \quad \langle \Phi, e_{-i} \rangle = -\Phi_i \quad (i \in I).$$

□

The result contained in this corollary is new, even when  $I$  is finite. If  $I$  is finite and if  $a_i = b_i$  for  $i \in I$ , the preceding corollary allows one to recover the results in [49] and [135].

The validity of the variational principle is to some extent equivalent to the fact that the system  $(X, T)$  is saturated.

**Theorem 5.13.** *Let  $(X, T)$  be a compact dynamical system. The system is saturated if and only if the variational principle (Theorem 5.2 (b)) holds for all real Banach spaces  $\mathbb{B}$ .*

*Proof.* It remains to prove that the variational principle implies the saturation of the system.

Take a countable set  $\{f_i\}_{i \in \mathbb{N}}$  which is dense in the unit ball of  $C(X)$  ( $C(X)$  being the space of all real valued continuous functions on  $X$ ). Consider the function

$$\Phi = (f_1, f_2, \dots, f_n, \dots)$$

which takes values in  $\mathbb{B} = \ell^\infty(\mathbb{N})$ . For any invariant measure  $\mu \in \mathcal{M}_{\text{inv}}$ , define

$$\alpha = \left( \int f_1 d\mu, \int f_2 d\mu, \dots \right) \in \ell^\infty(\mathbb{N}).$$

It is clear that  $\mathcal{M}_\Phi(\alpha) = \{\mu\}$ . Then the variational principle implies  $h_{\text{top}}(X_\Phi(\alpha)) = h_\mu$ . This finishes the proof because  $X_\Phi(\alpha)$  is nothing but  $G_\mu$ . □

## 5.5 An example: recurrence in an infinite number of cylinders

We have got a general variational principle. In order to apply this principle, one of the main questions is to get information about the convex set  $\mathcal{M}_\Phi(\alpha, E)$  and the maximal entropy measures contained in it and to compute the maximal entropy. Let us consider the symbolic dynamical system  $(\{0, 1\}^\mathbb{N}, T)$ ,  $T$  being the shift. The structure of the space  $\mathcal{M}_{\text{inv}}$  is relatively simple. To illustrate the main result, we shall consider a special problem of recurrence into a countable number of cylinders.

### 5.5.1 Symbolic space

Let  $X = \{0, 1\}^\mathbb{N}$  and  $T$  be the shift transformation. As usual, an  $n$ -cylinder in  $X$  determined by a word  $w = x_1x_2 \cdots x_n$  is denoted by  $[w]$  or  $[x_1, \dots, x_n]$ . For any word  $w$ , define the recurrence to the cylinder  $[w]$  of  $x$  by

$$R(x, [w]) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} 1_{[w]}(T^j x)$$

if the limit exists.

Let  $\mathcal{W} = \{w_i\}_{i \in I}$  with  $I \subset \mathbb{N}$  be a finite or infinite set of words. Let  $\alpha = \{a_i\}_{i \in I}$  be a sequence of non-negative numbers. We are interested in the following recurrence set

$$R(\mathbf{a}; \mathcal{W}) = \{x \in X : R(x, [w_i]) = a_i \text{ for all } i \in I\}$$

whose topological entropy will be computed by the variational principle which takes a simpler form.

**Corollary 5.14.** *We have  $h_{\text{top}}(R(\mathbf{a}; \mathcal{W})) = \max_{\mu \in \mathcal{M}(\mathbf{a}, \mathcal{W})} h_\mu$  where*

$$\mathcal{M}(\mathbf{a}, \mathcal{W}) = \{\mu \in \mathcal{M}_{\text{inv}} : \mu([w_i]) = a_i \text{ for all } i \in I\}.$$

Remark that the shift transformation on the symbolic space is expansive. Thus the entropy function  $h_\mu$  is upper semi-continuous([140], p. 184). Hence we can obtain the supremum in Corollary 5.14.

Recall that any Borel probability measure on  $X$  is uniquely determined by its values on cylinders. In other words, a function  $\mu$  defined on all cylinders can be extended to be a Borel probability measure if and only if

$$\sum_{x_1, \dots, x_n} \mu([x_1, \dots, x_n]) = 1$$

and

$$\sum_{\epsilon \in \{0,1\}} \mu([x_1, \dots, x_{n-1}, \epsilon]) = \mu([x_1, \dots, x_{n-1}]).$$

Such a probability measure  $\mu$  is invariant if and only if

$$\sum_{\epsilon \in \{0,1\}} \mu([\epsilon, x_2, \dots, x_n]) = \mu([x_2, \dots, x_n]).$$

The entropy  $h_\mu$  of any invariant measure  $\mu \in \mathcal{M}_{\text{inv}}$  can be expressed as follows

$$h_\mu = \lim_{n \rightarrow \infty} \sum_{x_1, \dots, x_n} -\mu([x_1, \dots, x_n]) \log \frac{\mu([x_1, \dots, x_n])}{\mu([x_1, \dots, x_{n-1}])}.$$

The sum in the above expression which we will denote by  $h_\mu^{(n)}$  is nothing but a conditional entropy of  $\mu$  and it is also the entropy of an  $(n-1)$ -Markov measure  $\mu_n$ , which tends towards  $\mu$  as  $n$  goes to  $\infty$ .

A Markov measure of order  $k$  is an invariant measure  $\nu \in \mathcal{M}_{\text{inv}}$  having the following Markov property: for all  $n > k$  and all  $(x_1, \dots, x_n) \in \{0,1\}^n$

$$\frac{\nu([x_1, \dots, x_n])}{\nu([x_1, \dots, x_{n-1}])} = \frac{\nu([x_{n-k}, \dots, x_n])}{\nu([x_{n-k}, \dots, x_{n-1}])}.$$

A Markov measure of order  $k$  is uniquely determined by its values on the  $(k+1)$ -cylinders. The preceding approximating Markov measure  $\mu_n$  has the same values as  $\mu$  on  $n$ -cylinders.

To apply the above corollary, we have to maximize the entropy  $h_\mu$  among all invariant measures  $\mu$  with constraints  $\mu([w_i]) = a_i$  for  $i \in I$ . The entropy  $h_\mu$  is a function of an infinite number of variables  $\mu([w])$ . So we have to maximize a function of an infinite number of variables. However, in some cases it suffices to reduce the problem to maximize the conditional entropy which is a function of a finite number of variables.

Denote by  $|w|$  the length of the word  $w$ . Let  $\mathcal{W}_n := \{w \in \mathcal{W} : |w| \leq n\}$  and  $\mathcal{M}(\mathbf{a}, \mathcal{W}_n) := \{\mu \in \mathcal{M}_{\text{inv}} : \mu([w_i]) = a_i, w_i \in \mathcal{W}_n\}$ . Let  $\mu^*$  be a maximal entropy measure over  $\mathcal{M}(\mathbf{a}, \mathcal{W})$  and  $\mu_n^*$  be the  $(n-1)$ -Markov measure which converges to  $\mu^*$ . Let  $\mu^{(n)}$  be a maximal entropy measure over  $\mathcal{M}(\mathbf{a}, \mathcal{W}_n)$ . Then

$$h_{\mu^*} = \lim_{n \rightarrow \infty} h_{\mu_n^*} \leq \liminf_{n \rightarrow \infty} h_{\mu^{(n)}} \leq \limsup_{n \rightarrow \infty} h_{\mu^{(n)}} \leq h_{\mu^*}.$$

Hence

$$\lim_{n \rightarrow \infty} h_{\mu^{(n)}} = h_{\mu^*} = \max_{\mu \in \mathcal{M}(\mathbf{a}, \mathcal{W})} h_{\mu}.$$

However, for any measure  $\mu \in \mathcal{M}(\mathbf{a}, \mathcal{W}_n)$ , we have  $h_{\mu} = h_{\mu_n} = h_{\mu}^{(n)}$ , where  $\mu_n$  is the  $(n-1)$ -Markov measure which converges to  $\mu$  ([49]). Thus,  $\mu^{(n)}$  is the maximal point of the conditional entropy function  $h_{\mu}^{(n)}$ .

Thus we have proved the following proposition.

**Proposition 5.15.** *The maximal entropy over  $\mathcal{M}(\mathbf{a}, \mathcal{W})$  can be approximated by the maximal entropies over  $\mathcal{M}(\mathbf{a}, \mathcal{W}_n)$ 's.*

### 5.5.2 Example: Frequency of dyadic digital blocks

Let us consider a special example:

$$\mathcal{W} = \{[0], [0^2], \dots, [0^n], \dots\}$$

where  $0^k$  means the word with 0 repeated  $k$  times.

**Theorem 5.16.** *Let  $\mathcal{W} = \{[0^n]\}_{n \geq 1}$  and  $\mathbf{a} = \{a_n\}_{n \geq 1} \subset \mathbb{R}^+$ . We have*

(a)  $R(\mathbf{a}; \{[0^n]\}_{n \geq 1}) \neq \emptyset$  if and only if

$$1 = a_0 \geq a_1 \geq a_2 \geq \dots; \quad a_i - 2a_{i+1} + a_{i+2} \geq 0 \quad (i \geq 0). \quad (5.43)$$

(b) *If the above condition is satisfied, then*

$$h_{\text{top}}(R(\mathbf{a}; \mathcal{W})) = -h(1 - a_1) + \sum_{i=0}^{\infty} h(a_i - 2a_{i+1} + a_{i+2}) \quad (5.44)$$

where  $h(x) = -x \log x$ .

The proof of the above theorem is decomposed into several lemmas which actually allow us to find the unique invariant measure of maximal entropy and to compute its entropy.

Let  $\mu$  be an invariant measure. The consistence and the invariance of the measure imply that we may partition all  $(n+2)$ -cylinders into groups of the form

$$\{[0w0], [0w1], [1w0], [1w1]\}$$

such that the measures  $\mu([0w0]), \mu([0w1]), \mu([1w0]), \mu([1w1])$  are linked each other through the measures  $\mu([0w]), \mu([w0]), \mu([w1]), \mu([1w])$  of  $(n+1)$ -cylinders. More precisely, if we write  $p_w = \mu([w])$ , then for any word  $w$  of length  $n$ , we have

$$p_{0w0} + p_{0w1} = p_{0w}$$

$$p_{1w0} + p_{1w1} = p_{1w}$$

$$p_{0w0} + p_{1w0} = p_{w0}$$

$$p_{0w1} + p_{1w1} = p_{w1}$$

**Lemma 5.17.** *Suppose  $\mu \in \mathcal{M}(\mathbf{a}, \mathcal{W})$ . If  $w = 0^n$  with  $n \geq 0$ , we have*

$$p_{00^n0} = a_{n+2} \tag{5.45}$$

$$p_{00^n1} = a_{n+1} - a_{n+2} \tag{5.46}$$

$$p_{10^n0} = a_{n+1} - a_{n+2} \tag{5.47}$$

$$p_{10^n1} = a_n - 2a_{n+1} + a_{n+2}. \tag{5.48}$$

*Proof.* The relation (5.46) is a consequence of the consistence

$$p_{00^n1} + p_{00^n0} = p_{00^n}$$

and the facts  $p_{00^n} = a_{n+1}$  and  $p_{00^n0} = a_{n+2}$ ; the relation (5.47) is a consequence of the invariance

$$p_{10^n0} + p_{00^n0} = p_{0^n0}$$

and the same facts; to obtain the relation (5.48) we need both the invariance and the consistence:

$$p_{10^n1} + p_{00^n1} = p_{0^n1} = p_{0^n} - p_{0^n0}.$$

Then by (5.46) we get

$$p_{10^n1} = p_{0^n} - p_{0^n0} - p_{00^n1} = a_n - 2a_{n+1} + a_{n+2}.$$

□

Let  $a, b, c$  be three positive numbers such that  $a + b \geq c$ . Consider the function

$$F(t, u, v, w) = h(t) + h(u) + h(v) + h(w) \quad (5.49)$$

defined on  $\mathbb{R}^4$ , where  $h(x) = -x \log x$ .

**Lemma 5.18.** *Under the condition*

$$t + v = a, \quad u + w = b, \quad t + u = c$$

the function  $F$  defined by (5.49) attains its maximum at

$$t = \frac{ac}{a+b}, \quad u = \frac{bc}{a+b}, \quad v = \frac{a(a+b-c)}{a+b}, \quad w = \frac{b(a+b-c)}{a+b}. \quad (5.50)$$

*Proof.* From the condition we may solve  $t, u, v$  as functions of  $w$ :

$$t = c - b + w, \quad u = b - w, \quad v = a + b - c - w.$$

So, maximizing  $F(t, u, v, w)$  under the condition becomes maximizing the function

$$F(w) = h(c - b + w) + h(b - w) + h(a + b - c - w) + h(w)$$

which is strictly concave in its domain. Since  $h'(x) = -1 - \log x$ , we have

$$F'(w) = -\log(c - b + w) + \log(b - w) + \log(a + b - c - w) - \log w.$$

Solving  $F'(w) = 0$ , we get  $w = \frac{b(a+b-c)}{a+b}$ . The corresponding  $t, u, v$  are as announced in (5.50)  $\square$

**Lemma 5.19.** *Suppose that  $\{a_n\}_{n \geq 0}$  is a sequence of real numbers such that*

$$1 = a_0 \geq a_1 \geq a_2 \geq \dots; \quad a_i - 2a_{i+1} + a_{i+2} \geq 0 \quad (i \geq 0). \quad (5.51)$$

*There exists an invariant measure  $\mu$  such that if  $w$  is not a block of 0's, we have*

$$p_{ew\epsilon'} = \frac{p_{ew}p_{w\epsilon'}}{p_w} \quad (\forall \epsilon, \epsilon' \in \{0, 1\}). \quad (5.52)$$

*The above recursion relations (5.52) together with (5.45-5.48) completely determine the measure  $\mu$ , which is the unique maximal entropy measure among those invariant measures  $\nu$  such that  $\nu([0^n]) = a_n$  for  $n \geq 1$ .*

*Proof.* For any  $\mu \in \mathcal{M}(\mathbf{a}, \mathcal{W})$ , we must have  $\mu([0]) = a_1$  and  $\mu([1]) = 1 - a_1$ . By Proposition 5.15, we are led to find the measure  $\mu^{(n+2)}$  which maximizes  $h_\mu^{(n+2)}$  for each  $n \geq 0$ . Let  $\mu$  be an arbitrary invariant measure in  $\mathcal{M}(\mathbf{a}, \mathcal{W}_{n+2})$ ,  $n \geq 0$ . We identify  $\mu$  with the sequence  $p_w = \mu([w])$  indexed by finite words. By Lemma 5.17, we have

$$\begin{aligned} p_0 &= a_1, & p_1 &= 1 - a_1 \\ p_{00} &= a_2, & p_{01} &= p_{10} = a_1 - a_2, & p_{11} &= 1 - 2a_1 + a_2 \end{aligned}$$

So, we have

$$h_\mu^{(2)} = h(a_2) + 2h(a_1 - a_2) - h(a_1) - h(1 - a_1) + h(1 - 2a_1 + a_2). \quad (5.53)$$

Let us now consider the conditional entropy  $h_\mu^{(n+2)}$  for  $n \geq 1$ , which is a function of  $\mu$ -measures of  $(n+2)$ -cylinders, whose general form is  $[\epsilon w \epsilon']$  with  $w$  a word of length  $n$  and  $\epsilon, \epsilon' \in \{0, 1\}$ . If  $w = 0^n$ , by Lemma 5.17, the measures of the four cylinders  $[\epsilon 0^n \epsilon']$  with  $\epsilon, \epsilon' \in \{0, 1\}$  are determined by  $\{a_k\}_{k \geq 1}$ . If  $w \neq 0^n$ , the four quantities  $p_{\epsilon w \epsilon'}$  are linked to each other by

$$p_{0w0} + p_{0w1} = p_{0w}, \quad p_{1w0} + p_{1w1} = p_{1w}, \quad p_{0w0} + p_{1w0} = p_w$$

through measures of  $(n+1)$ -cylinders:  $a := p_{w0}, b := p_{1w}, c := p_{0w}$ . Consider the four measures  $p_{\epsilon w \epsilon'}$  as variables, there is only one free variable and three others are linked to it. Thus to any word  $w \neq 0^n$  of length  $n$  is associated a free variable.

In fact,  $h_\mu^{(n+2)}$  is the sum of all terms

$$- \sum_{\epsilon, \epsilon'} p_{\epsilon w \epsilon'} \log \frac{p_{\epsilon w \epsilon'}}{p_{\epsilon w}} \quad (w \in \{0, 1\}^n) \quad (5.54)$$

If  $w = 0^n$ , the corresponding term (5.54) is a constant depending on the sequence  $\{a_n\}$  (see Lemma 5.17). If  $w \neq 0^n$ , there is a free variable in the term (5.54). So, maximizing  $h_\mu^{(n+2)}$  is equivalent to maximizing all above terms, or equivalently to maximizing

$$- \sum_{\epsilon, \epsilon'} p_{\epsilon w \epsilon'} \log p_{\epsilon w \epsilon'}. \quad (5.55)$$

Applying Lemma 5.18 to the above function in (5.55) provides us with the maximal point  $\mu^{(n+2)}$  described by (5.52) with  $|w| = n$ . It is easy to check that the family

$\{p_w\}$  defined by (5.45-5.48) and (5.52) for all words  $w$ , verifies the consistency and the invariance conditions. The measure determined by the family is the unique measure of maximal entropy and  $\mu^{(n+2)}$  is its  $(n+1)$ -Markov measure.  $\square$

**Lemma 5.20.** *The entropy of the invariant measure  $\mu$  of maximal entropy determined in the above lemma is equal to*

$$h_\mu = -h(1 - a_1) + \sum_{j=0}^{\infty} h(a_j - 2a_{j+1} + a_{j+2})$$

where  $h(x) = -x \log x$ .

*Proof.* Recall that for any invariant measure  $\mu$  we have

$$h_\mu = \lim_{n \rightarrow \infty} h_\mu^{(n)} = \inf_n h_\mu^{(n)}$$

where

$$h_\mu^{(n)} = - \sum_{|w|=n-1} \sum_{\epsilon=0,1} p_{w\epsilon} \log \frac{p_{w\epsilon}}{p_w}.$$

Thus we may write

$$h_\mu = h_\mu^{(2)} + \sum_{n=1}^{\infty} (h_\mu^{(n+2)} - h_\mu^{(n+1)}) \quad (5.56)$$

For  $n \geq 0$ , write

$$\begin{aligned} h_\mu^{(n+2)} &= - \sum'_{|w|=n} \sum_{\epsilon, \epsilon'} p_{w\epsilon\epsilon'} \log \frac{p_{w\epsilon\epsilon'}}{p_{w\epsilon}} - \sum_{\epsilon, \epsilon'} p_{\epsilon 0^n \epsilon'} \log \frac{p_{\epsilon 0^n \epsilon'}}{p_{\epsilon 0^n}} \\ &=: I_1(n) + I_2(n) \end{aligned} \quad (5.57)$$

where  $\sum'$  means that the sum is taken over  $w \neq 0^n$  ( $0^0$  meaning the empty word so that  $I_1(0) = 0$ ). When  $n = 0$ , we get

$$h_\mu^{(2)} = h(a_2) + 2h(a_1 - a_2) - h(a_1) - h(1 - a_1) + h(1 - 2a_1 + a_2). \quad (5.58)$$

This coincides with (5.53).

Suppose  $n \geq 1$ . By the recursion relation (5.52), we have

$$\begin{aligned} I_1(n) &= - \sum'_{|w|=n} \sum_{\epsilon, \epsilon'} p_{w\epsilon\epsilon'} \log \frac{p_{w\epsilon\epsilon'}}{p_w} \\ &= - \sum_{|w|=n} \sum_{\epsilon, \epsilon'} p_{w\epsilon\epsilon'} \log \frac{p_{w\epsilon\epsilon'}}{p_w} + \sum_{\epsilon, \epsilon'} p_{\epsilon 0^n \epsilon'} \log \frac{p_{\epsilon 0^n \epsilon'}}{p_{0^n}} \\ &= h_\mu^{(n+1)} + I_3(n) \end{aligned} \quad (5.59)$$

where

$$I_3(n) = \sum_{\epsilon, \epsilon'} p_{\epsilon 0^n \epsilon'} \log \frac{p_{0^n \epsilon'}}{p_{0^n}}.$$

From (5.57) and (5.59) we get

$$h_\mu^{(n+2)} - h_\mu^{(n+1)} = I_2(n) + I_3(n). \quad (5.60)$$

On the one hand, by using the invariance and the consistence we can simplify  $I_3$  as follows

$$\begin{aligned} I_3(n) &= \sum_{\epsilon'} p_{0^n \epsilon'} \log \frac{p_{0^n \epsilon'}}{p_{0^n}} \\ &= \sum_{\epsilon'} p_{0^n \epsilon'} \log p_{0^n \epsilon'} - p_{0^n} \log p_{0^n} \\ &= a_{n+1} \log a_{n+1} + (a_n - a_{n+1}) \log(a_n - a_{n+1}) - a_n \log a_n. \end{aligned} \quad (5.61)$$

On the other hand, we have

$$\begin{aligned} I_2(n) &= - \sum_{\epsilon, \epsilon'} p_{\epsilon 0^n \epsilon'} \log p_{\epsilon 0^n \epsilon'} + \sum_{\epsilon} p_{\epsilon 0^n} \log p_{\epsilon 0^n} \\ &= -a_{n+2} \log a_{n+2} - 2(a_{n+1} - a_{n+2}) \log(a_{n+1} - a_{n+2}) \\ &\quad - (a_n - 2a_{n+1} + a_{n+2}) \log(a_n - 2a_{n+1} + a_{n+2}) \\ &\quad + a_{n+1} \log a_{n+1} + (a_n - a_{n+1}) \log(a_n - a_{n+1}). \end{aligned} \quad (5.62)$$

By combining (5.60), (5.61) and (5.62) we get

$$\begin{aligned} \varphi(n) &:= h_\mu^{(n+2)} - h_\mu^{(n+1)} \\ &= [h(a_{n+2}) - 2h(a_{n+1}) + h(a_n)] + 2[h(a_{n+1} - a_{n+2}) - h(a_n - a_{n+1})] \\ &\quad + h(a_n - 2a_{n+1} + a_{n+2}). \end{aligned}$$

Finally using (5.56) we get

$$h_\mu = h_\mu^{(2)} + \sum_{n=1}^{\infty} \varphi(n) = -h(1 - a_1) + \sum_{j=0}^{\infty} h(a_j - 2a_{j+1} + a_{j+2}).$$

□

Remark that the measure of maximal entropy in  $R(a, \mathcal{W})$  is not necessarily ergodic. Here is an example. If  $a_n = a$  ( $\forall n \geq 1$ ) is constant, then there is a unique

invariant measure in  $\mathcal{M}(a, \mathcal{W})$ , which is  $a\delta_{\bar{0}} + (1 - a)\delta_{\bar{1}}$ . In this case,  $R(a, \mathcal{W})$  is not empty but of zero entropy. Notice that  $R(a, \mathcal{W})$  contains no point in the support of the unique invariant measure. If  $a = \frac{1}{2}$ ,  $R(\frac{1}{2}, \mathcal{W})$  contains the following point

010011000111100001111...

(The terms in the two sequences  $\{0^k\}$  and  $\{1^k\}$  are alternatively appended.)

At the end of this chapter, we remark that Pfister and Sullivan ([122]) also obtained Theorem 5.1 with a different method.

## Chapter 6

### Khinchine and Lyapunov Spectra

In this chapter, we mainly study the Khinchine spectrum and Lyapunov spectrum which are in fact some kinds of Birkhoff spectra in continued fractions. We prove the remarkable fact that the Khinchine spectrum and Lyapunov spectrum are neither concave nor convex which signifies a new phenomenon from the usual point of view of multifractal analysis. We also study the fast Khinchine spectrum and fast Lyapunov spectrum which are proved to be constant functions.<sup>1</sup>

#### 6.1 Statement of the results

The continued fraction of a real number can be generated by the Gauss transformation  $T : [0, 1) \rightarrow [0, 1)$  which is defined by

$$T(0) := 0, \quad T(x) := \frac{1}{x} \pmod{1}, \quad \text{for } x \in (0, 1).$$

Every irrational number  $x$  in  $[0, 1)$  can be uniquely expanded as

$$x = \frac{1}{a_1(x) + \frac{1}{a_2 + \cdots + \frac{1}{a_n(x) + T^n(x)}}} = \frac{1}{a_1(x) + \frac{1}{a_2(x) + \frac{1}{a_3(x) + \cdots}}}$$

where  $a_1(x) = \lfloor 1/x \rfloor$  and  $a_n(x) = a_1(T^{n-1}(x))$  for  $n \geq 2$  are partial quotients of  $x$  ( $\lfloor x \rfloor$  denoting the integral part of  $x$ ). The continued fraction expansion is also denoted by  $x = [a_1, a_2, \cdots, a_n + T^n(x)] = [a_1, a_2, a_3, \cdots]$ .

We recall a list of definitions and notations we have discussed in Subsection **1.3.3**:

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<sup>1</sup>A. H. Fan, L. M. Liao, B. W. Wang and J. Wu, *On Khinchine exponents and Lyapunov exponents of continued fractions*, to appear in *Ergod. Th. Dynam. Sys.*.

- Khintchine exponents and the Lyapunov exponents of  $x \in [0, 1]$ :

$$\gamma(x) := \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \log a_j(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log a_1(T^j(x)),$$

$$\lambda(x) := \lim_{n \rightarrow \infty} \frac{1}{n} \log |(T^n)'(x)| = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |T'(T^j(x))|$$

- fast Khintchine exponent and fast Lyapunov exponent of  $x \in [0, 1]$ , relative to  $\varphi$  ( $\varphi : \mathbb{N} \rightarrow \mathbb{R}_+$ , and  $\lim_{n \rightarrow \infty} \frac{\varphi(n)}{n} = \infty$ ):

$$\gamma^\varphi(x) := \lim_{n \rightarrow \infty} \frac{1}{\varphi(n)} \sum_{j=1}^n \log a_j(x) = \lim_{n \rightarrow \infty} \frac{1}{\varphi(n)} \sum_{j=0}^{n-1} \log a_1(T^j(x)),$$

$$\lambda^\varphi(x) := \lim_{n \rightarrow \infty} \frac{1}{\varphi(n)} \log |(T^n)'(x)| = \lim_{n \rightarrow \infty} \frac{1}{\varphi(n)} \sum_{j=0}^{n-1} \log |T'(T^j(x))|.$$

- the level sets of Khintchine exponents and Lyapunov exponents ( $\xi, \beta \geq 0$ ):

$$E_\xi := \{x \in [0, 1) : \gamma(x) = \xi\}, \quad F_\beta := \{x \in [0, 1) : \lambda(x) = \beta\}.$$

- the level sets of fast Khintchine exponents and fast Lyapunov exponents ( $\xi, \beta \geq 0$ ):

$$E_\xi(\varphi) := \{x \in [0, 1) : \gamma^\varphi(x) = \xi\}, \quad F_\beta(\varphi) := \{x \in [0, 1) : \lambda^\varphi(x) = \beta\}.$$

- the Khintchine spectrum and the Lyapunov spectrum:

$$t(\xi) := \dim E_\xi \quad \tilde{t}(\beta) := \dim F_\beta.$$

- fast Khintchine spectrum and the fast Lyapunov spectrum relative to  $\varphi$ :

$$t^\varphi(\xi) := \dim E_\xi(\varphi) \quad \tilde{t}^\varphi(\beta) := \dim F_\beta(\varphi).$$

We start with the statement of the results on fast spectra.

**Theorem 6.1.** *Suppose  $(\varphi(n+1) - \varphi(n)) \uparrow \infty$  and  $\lim_{n \rightarrow \infty} \frac{\varphi(n+1)}{\varphi(n)} := b \geq 1$ . Then  $E_\xi(\varphi) = F_{2\xi}(\varphi)$  and  $\dim E_\xi(\varphi) = 1/(b+1)$  for all  $\xi \geq 0$ .*

In order to state the results on the Khintchine spectrum, let us first introduce some notation. Let

$$D := \{(t, q) \in \mathbb{R}^2 : 2t - q > 1\}, \quad D_0 := \{(t, q) \in \mathbb{R}^2 : 2t - q > 1, 0 \leq t \leq 1\}.$$

For  $(t, q) \in D$ , define

$$P(t, q) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\omega_1=1}^{\infty} \cdots \sum_{\omega_n=1}^{\infty} \exp \left( \sup_{x \in [0,1]} \log \prod_{j=1}^n \omega_j^q([\omega_j, \cdots, \omega_n + x])^{2t} \right).$$

It will be proved that  $P(t, q)$  is an analytic function in  $D$  (Proposition 6.19).

Moreover, for any  $\xi \geq 0$ , there exists a unique solution  $(t(\xi), q(\xi)) \in D_0$  to the equation

$$\begin{cases} P(t, q) = q\xi, \\ \frac{\partial P}{\partial q}(t, q) = \xi. \end{cases}$$

(Proposition 6.25).

**Theorem 6.2.** *Let  $\xi_0 = \int \log a_1(x) d\mu_G(x)$ . For  $\xi \geq 0$ , the set  $E_\xi$  is of Hausdorff dimension  $t(\xi)$ . Furthermore the dimension function  $t(\xi)$  has the following properties*

- 1)  $t(\xi_0) = 1$ ,  $t(+\infty) = 1/2$ .
- 2)  $t'(\xi) < 0$  for all  $\xi > \xi_0$ ;  $t'(\xi_0) = 0$  and  $t'(\xi) > 0$  for all  $\xi < \xi_0$ .
- 3)  $t'(0-) = +\infty$ ,  $t'(+\infty) = 0$ .
- 4)  $t''(\xi_0) < 0$ , but  $t''(\xi_1) > 0$  for some  $\xi_1 > \xi_0$ . So  $t(\xi)$  is neither convex nor concave.

See Figure 1 for the graph of  $t(\xi)$ .

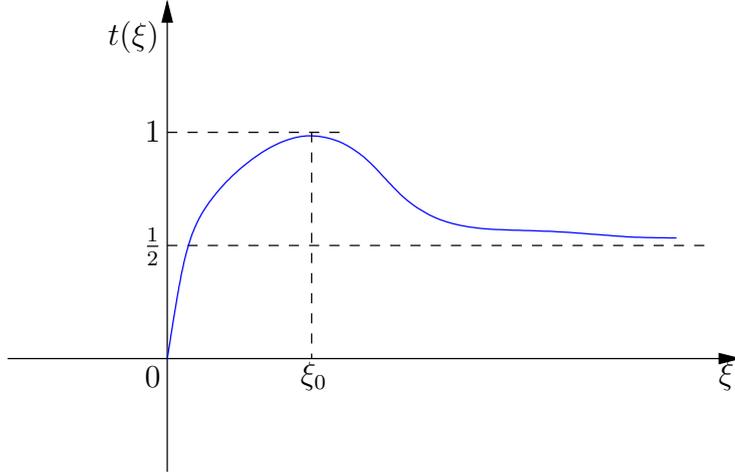


Figure 1. Khintchine spectrum

Let

$$\tilde{D} := \{(\tilde{t}, q) : \tilde{t} - q > 1/2\} \quad \tilde{D}_0 := \{(\tilde{t}, q) : \tilde{t} - q > 1/2, 0 \leq \tilde{t} \leq 1\}.$$

For  $(\tilde{t}, q) \in \tilde{D}$ , define

$$P_1(\tilde{t}, q) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\omega_1=1}^{\infty} \cdots \sum_{\omega_n=1}^{\infty} \exp \left( \sup_{x \in [0,1]} \log \prod_{j=1}^n ([\omega_j, \cdots, \omega_n + x])^{2(\tilde{t}-q)} \right).$$

In fact,  $P_1(\tilde{t}, q) = P(\tilde{t} - q, 0)$ , thus  $P_1(\tilde{t}, q)$  is analytic in  $\tilde{D}$ .

Denote  $\gamma_0 := 2 \log \frac{1+\sqrt{5}}{2}$ . For any  $\beta \in (\gamma_0, \infty)$ , the system

$$\begin{cases} P_1(\tilde{t}, q) = q\beta, \\ \frac{\partial P_1}{\partial q}(\tilde{t}, q) = \beta \end{cases}$$

admits a unique solution  $(\tilde{t}(\beta), q(\beta)) \in \tilde{D}_0$  (Proposition 6.33).

**Theorem 6.3.** *Let  $\lambda_0 = \int \log |T'(x)| d\mu_G$  and  $\gamma_0 = 2 \log \frac{1+\sqrt{5}}{2}$ . For any  $\beta \in [\gamma_0, \infty)$ , the set  $F_\beta$  is of Hausdorff dimension  $\tilde{t}(\beta)$ . Furthermore the dimension function  $\tilde{t}(\xi)$  has the following properties*

- 1)  $\tilde{t}(\lambda_0) = 1$ ,  $\tilde{t}(+\infty) = 1/2$ .
- 2)  $\tilde{t}'(\beta) < 0$  for all  $\beta > \lambda_0$ ;  $\tilde{t}'(\lambda_0) = 0$  and  $\tilde{t}'(\beta) > 0$  for all  $\beta < \lambda_0$ .

3)  $\tilde{t}'(\gamma_0-) = +\infty$ ,  $\tilde{t}'(+\infty) = 0$ .

4)  $\tilde{t}''(\lambda_0) < 0$ , but  $\tilde{t}''(\beta_1) > 0$  for some  $\beta_1 > \lambda_0$  i.e.  $\tilde{t}(\beta)$  is neither convex nor concave.

See Figure 2 for the graph of  $\tilde{t}(\beta)$ .

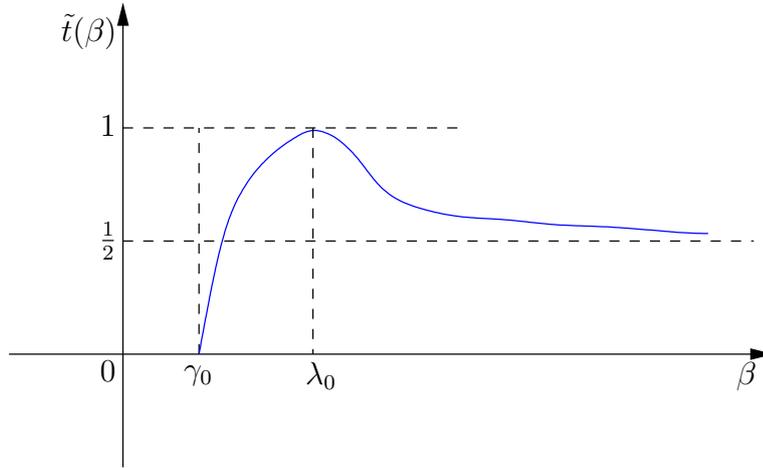


Figure 2. Lyapunov spectrum

The following of this chapter is organized as follows. In Section 6.2, we collect and establish some basic results that will be used later. Section 6.3 is devoted to proving the results about the fast Khintchine spectrum and fast Lyapunov spectrum (Theorem 6.1). In Section 6.4, we present a general Ruelle operator theory developed in [64] and then apply it to the Gauss transformation. Based on Section 6.4, we establish Theorem 6.2 in Section 6.5. The last section is devoted to the study of Lyapunov spectrum (Theorem 6.3).

## 6.2 Preliminary

The basic properties of continued fractions can be found in Section 2.4 of Chapter 2. We discuss the Khintchine and Lyapunov exponents and something about pointwise dimension in this section.

### 6.2.1 Jacobian estimate and the level sets of Lyapunov exponents

Recall that for any  $a_1, a_2, \dots, a_n \in \mathbb{N}$ , an  $n$ -th basic interval or a rank  $n$  fundamental interval is defined by

$$I_n(a_1, a_2, \dots, a_n) = \{x \in [0, 1) : a_k(x) = a_k, 1 \leq k \leq n\}.$$

An  $n$ -th basic interval containing  $x$  will be denoted by  $I_n(x)$ .

The Gauss transformation  $T$  admits the following *Jacobian estimate*.

**Lemma 6.4.** *There exists a positive number  $K > 1$  such that for all irrational number  $x$  in  $[0, 1)$ , one has*

$$0 < \frac{1}{K} \leq \sup_{n \geq 0} \sup_{y \in I_n(x)} \left| \frac{(T^n)'(x)}{(T^n)'(y)} \right| \leq K < \infty.$$

*Proof.* Assume  $x = [a_1, \dots, a_n, \dots] \in [0, 1) \setminus \mathbb{Q}$ . For any  $n \geq 0$  and  $y \in I_n(x) = I_n(a_1, \dots, a_n)$ , by the fact that  $T'(x) = -\frac{1}{x^2}$  we get

$$\sum_{j=0}^{n-1} \left| \log |T'(T^j(x))| - \log |T'(T^j(y))| \right| = 2 \sum_{j=0}^{n-1} \left| \log T^j(x) - \log T^j(y) \right|.$$

Applying the mean-value theorem, we have

$$\left| \log T^j(x) - \log T^j(y) \right| = \left| \frac{T^j(x) - T^j(y)}{T^j(z)} \right| \leq \frac{a_{j+1}}{q_{n-j}(a_{j+1}, \dots, a_n)},$$

where the assertion follows from the fact that all three points  $T^j(x)$ ,  $T^j(y)$  and  $T^j(z)$  belong to  $I_{n-j}(a_{j+1}, \dots, a_n)$ . By Lemma 2.37, we have

$$\sum_{j=0}^{n-1} \left| \log T^j(x) - \log T^j(y) \right| \leq \sum_{j=0}^{n-1} \frac{1}{q_{n-j-1}(a_{j+2}, \dots, a_n)} \leq \sum_{j=0}^{n-1} \left( \frac{1}{2} \right)^{n-j-2} \leq 4.$$

Thus the result is proved with  $K = e^4$ . □

The above Jacobian estimate property of  $T$  enables us to control the length of  $I_n(x)$  by  $|(T^n)'(x)|^{-1}$ , through the fact that  $\int_{I_n(x)} |(T^n)'(y)| dy = 1$ .

**Lemma 6.5.** *There exist a positive constant  $K > 0$  such that for all irrational numbers  $x$  in  $[0, 1)$ ,*

$$\frac{1}{K} \leq \frac{|I_n(x)|}{|(T^n)'(x)|^{-1}} \leq K.$$

We remark that from Lemma 2.39 and Lemma 6.5, we have

$$\frac{1}{2K}q_n^2(x) \leq |(T^n)'(x)| \leq Kq_n^2(x).$$

So the Lyapunov exponent  $\lambda(x)$  is nothing but the growth rate of  $q_n(x)$  up to a multiplicative constant 2:

$$\lambda(x) = \lim_{n \rightarrow \infty} \frac{2}{n} \log q_n(x).$$

For any irrational number  $x$  in  $[0, 1]$ , let  $N_n(x) := \{j \leq n : a_j(x) \neq 1\}$ . Set

$$\begin{aligned} A &:= \left\{ x \in [0, 1] : \lim_{n \rightarrow \infty} \frac{1}{n} \log q_n(x) = \frac{\gamma_0}{2} \right\}, \\ B &:= \left\{ x \in [0, 1] : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \log a_j(x) = 0 \right\}, \\ C &:= \left\{ x \in [0, 1] : \lim_{n \rightarrow \infty} \frac{1}{n} \#N_n(x) = 0 \right\}, \end{aligned}$$

where  $\#$  stands for the cardinal of a set. Then we have the following relationship.

**Lemma 6.6.** *With the notations given above, we have*

$$A = B \subset C.$$

*Proof.* It is clear that  $A \subset C$  and  $B \subset C$ . Let us prove  $A = B$ . First observe that, by Lemma 2.38, we have

$$\begin{aligned} \frac{1}{n} \log q_n(x) &\geq \frac{1}{n} \sum_{j \in N_n(x)} \log \frac{a_j(x) + 1}{2} + \frac{1}{n} \log q_{n-\#N_n}(1, \dots, 1) \\ &\geq \frac{1}{n} \sum_{j \in N_n(x)} \log a_j(x) - \frac{1}{n} \sum_{j \in N_n(x)} \log 2 + \frac{1}{n} \log q_{n-\#N_n}(1, \dots, 1). \end{aligned}$$

Assume  $x \in A$ . Since  $A \subset C$ , we have

$$-\frac{1}{n} \sum_{j \in N_n(x)} \log 2 + \frac{1}{n} \log q_{n-\#N_n}(1, \dots, 1) \longrightarrow 0 + \frac{\gamma_0}{2} \quad (n \rightarrow \infty).$$

Now by the assumption  $x \in A$ , it follows

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \log a_j(x) = \frac{1}{n} \sum_{j \in N_n(x)} \log a_j(x) = 0.$$

Therefore we have proved  $A \subset B$ . For the inverse inclusion, notice that

$$\frac{1}{n} \log q_n(x) \leq \frac{1}{n} \sum_{j \in N_n(x)} \log(a_j(x) + 1) + \frac{1}{n} \log q_{n-\#N_n}(1, \dots, 1).$$

Let  $x \in B$ . Since  $B \subset C$ , we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log q_{n-\#N_n}(1, \dots, 1) = \frac{\gamma_0}{2}.$$

Therefore by the assumption  $x \in B$ , we get

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log q_n(x) \leq \frac{\gamma_0}{2}.$$

Thus  $B \subset A$ . □

### 6.2.2 Exponents $\gamma(x)$ and $\lambda(x)$

In this subsection, we make a quick examination of the Khintchine exponent  $\gamma(x)$  and compare it with the Lyapunov exponent  $\lambda(x)$ . Our main concern is the possible values of both exponent functions.

A first observation is that for any  $x \in [0, 1)$ ,  $\gamma(x) \geq 0$  and  $\lambda(x) \geq \gamma_0 = 2 \log \frac{\sqrt{5}+1}{2}$ . By the Birkhoff ergodic theorem, we know that the Khintchine exponent  $\gamma(x)$  attains the value  $\xi_0$  for almost all points  $x$  with respect to the Lebesgue measure. We will show that every positive number is the Khintchine exponent  $\gamma(x)$  of some point  $x$ .

**Proposition 6.7.** *For any  $\xi \geq 0$ , there exists a point  $x_0 \in [0, 1)$  such that  $\gamma(x_0) = \xi$ .*

*Proof.* Assume  $\xi > 0$  ( for  $\xi = 0$ , we take  $x_0 = \frac{1+\sqrt{5}}{2}$  corresponding to  $a_n \equiv 1$ .) Take an increasing sequence of integers  $\{n_k\}_{k \geq 1}$  satisfying

$$n_0 = 1, \quad n_{k+1} - n_k \rightarrow \infty, \quad \text{and} \quad \frac{n_k}{n_{k+1}} \rightarrow 1, \quad \text{as } k \rightarrow \infty.$$

Let  $x_0 \in (0, 1)$  be a point whose partial quotients satisfy

$$e^{(n_k - n_{k-1})\xi} \leq a_{n_k} \leq e^{(n_k - n_{k-1})\xi} + 1; \quad a_n = 1 \text{ otherwise.}$$

Since for  $n_k \leq n < n_{k+1}$ ,

$$\frac{1}{n_{k+1}} \sum_{i=1}^k \log e^{(n_i - n_{i-1})\xi} \leq \frac{1}{n} \sum_{j=1}^n \log a_j \leq \frac{1}{n_k} \sum_{i=1}^k \log(e^{(n_i - n_{i-1})\xi} + 1),$$

we have

$$\gamma(x_0) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \log a_j(x) = \xi.$$

□

In the following, we will show that the set  $E_\xi$  and  $F_\lambda$  are never equal. So it is two different problems to study  $\gamma(x)$  and  $\lambda(x)$ . However, as we will see,  $E_\xi(\phi) = F_{2\xi}(\phi)$  when  $\phi$  is faster than  $n$ .

**Proposition 6.8.** *For any  $\xi \geq 0$  and  $\lambda \geq 2 \log \frac{\sqrt{5}+1}{2}$ , we have  $E_\xi \neq F_\lambda$ .*

*Proof.* Given  $\xi \geq 0$ . It suffices to construct two numbers with same Khintchine exponent  $\xi$  but different Lyapunov exponents.

For the first number, take just the number  $x_0$  constructed in the proof of Proposition 6.7. We claim that

$$\lambda(x_0) = 2\xi + 2 \log \frac{\sqrt{5} + 1}{2}. \quad (6.1)$$

In fact, by Lemma 2.38 we have

$$\prod_{j=1}^k \left( \frac{a_{n_j} + 1}{2} \right) q_{n_k - k}(1, \dots, 1) \leq q_{n_k}(a_1, \dots, a_{n_k}) \leq \prod_{j=1}^k (a_{n_j} + 1) q_{n_k - k}(1, \dots, 1).$$

Then by the assumption on  $n_k$ , we have

$$\lambda(x_0) = \lim_{n \rightarrow \infty} \frac{2}{n} \log q_n(x_0) = 2 \left( \xi + \log \frac{\sqrt{5} + 1}{2} \right).$$

Construct now the second number. Fix  $k \geq 1$ . Define  $x_1 = [\varsigma_1, \dots, \varsigma_n, \dots]$  where

$$\varsigma_n = \left( \underbrace{1, \dots, 1, \lfloor e^{k\xi} \rfloor, \dots, 1, \dots, 1, \lfloor e^{k\xi} \rfloor}_{kn}, \left\lfloor \left( \frac{e^{(k+1)\xi}}{\lfloor e^{k\xi} \rfloor} \right)^n \right\rfloor \right).$$

Notice that there are  $n$  small vectors  $(1, \dots, 1, \lfloor e^{k\xi} \rfloor)$  in  $\varsigma_n$  and the length of  $\varsigma_n$  is equal to  $N_k := kn + 1$ . We can prove

$$\gamma(x_1) = \xi, \quad \lambda(x_1) = \lambda \left( \left[ 1, \dots, \lfloor e^{k\xi} \rfloor \right] \right) + 2\xi - \frac{2}{k} \log \lfloor e^{k\xi} \rfloor$$

by the same arguments as in proving the similar result for  $x_0$ . It is clear that  $\lambda(x_0) \neq \lambda(x_1)$  for large  $k \geq 1$ . □

It is evident that Proposition 6.7 and the formula (6.1) yield the following result due to Pollicott and Weiss ([128]).

**Corollary 6.9** ([128]). *For any  $\lambda \geq 2 \log \frac{\sqrt{5}+1}{2}$ , there exists a point  $x_0 \in [0, 1)$  such that  $\lambda(x_0) = \lambda$ .*

### 6.2.3 Pointwise dimension

We are going to compare the pointwise dimension and the Markov pointwise dimension (corresponding to continued fraction system) of a Borel probability measure.

Let  $\mu$  be a Borel probability measure on  $[0, 1)$ . Define the pointwise dimension and the Markov pointwise dimension respectively by

$$d_\mu(x) := \lim_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r}, \quad \delta_\mu(x) := \lim_{n \rightarrow \infty} \frac{\log \mu(I_n(x))}{\log |I_n(x)|}$$

if the limits exist, where  $B(x, r)$  is the ball centered at  $x$  with radius  $r$ .

For two series  $\{u_n\}_{n \geq 0}$  and  $\{v_n\}_{n \geq 0}$ , we write  $u_n \asymp v_n$  which means that there exist absolute positive constants  $c_1, c_2$  such that  $c_1 v_n \leq u_n \leq c_2 v_n$  for  $n$  large enough. Sometimes, we need the following condition at a point  $x$ :

$$\mu(B(x, |I_n(x)|)) \asymp \mu(I_n(x)). \quad (6.2)$$

We have the following relationship between  $\delta_\mu(x)$  and  $d_\mu(x)$ .

**Lemma 6.10.** *Let  $\mu$  be a Borel measure.*

- (a) *Assume (6.2). If  $d_\mu(x)$  exists then  $\delta_\mu(x)$  exists and  $\delta_\mu(x) = d_\mu(x)$ .*
- (b) *If  $\delta_\mu(x)$  and  $\lambda(x)$  both exist, then  $d_\mu(x)$  exists and  $\delta_\mu(x) = d_\mu(x)$ .*

*Proof.* (a) If the limit defining  $d_\mu(x)$  exists, then the limit

$$\lim_{n \rightarrow +\infty} \frac{\log \mu(B(x, |I_n(x)|))}{\log |I_n(x)|}$$

exists and equals to  $d_\mu(x)$ . Thus by (6.2), the limit defining  $\delta_\mu(x)$  also exists and equals to  $d_\mu(x)$ .

(b) Since  $\lambda(x)$  exists, by Lemma 6.5 we have

$$\lim_{n \rightarrow \infty} \frac{\log |I_n(x)|}{\log |I_{n+1}(x)|} = \lim_{n \rightarrow \infty} \frac{1}{n} \log |I_n(x)| / \frac{1}{n+1} \log |I_{n+1}(x)| = 1. \quad (6.3)$$

For any  $r > 0$ , there exists an  $n$  such that  $|I_{n+1}(x)| \leq r < |I_n(x)|$ . Then by Lemma 2.40, we have  $I_{n+1}(x) \subset B(x, r) \subset I_{n-2}(x)$ . Thus

$$\frac{\log \mu(I_{n-2}(x))}{\log |I_{n+1}(x)|} \leq \frac{\log \mu(B(x, r))}{\log r} \leq \frac{\log \mu(I_{n+1}(x))}{\log |I_n(x)|}. \quad (6.4)$$

Combining (6.3) and (6.4) we get the desired result.  $\square$

Let us give some measures for which the condition (6.2) is satisfied. These measures will be used in the subsection 6.5.1. The existence of these measures  $\mu_{t,q}$  will be discussed in Proposition 6.19 and the subsection 6.5.1.

**Lemma 6.11.** *Suppose  $\mu_{t,q}$  is a measure satisfying*

$$\mu_{t,q}(I_n(x)) \asymp \exp(-nP(t, q)) |I_n(x)|^t \prod_{j=1}^n a_j^q,$$

where  $P(t, q)$  is a constant. Then (6.2) is satisfied by  $\mu_{t,q}$ .

*Proof.* Notice that when  $a_n(x) = 1$ ,  $\mu_{t,q}(I_n(x)) \asymp \mu_{t,q}(I_{n-1}(x))$ . Then in the light of Lemma 2.40, we can show that (6.2) is satisfied by  $\mu_{t,q}$ .  $\square$

## 6.3 Fast growth rate: proof of Theorem 6.1

### 6.3.1 Lower bound

We start with the mass distribution principle (see [45], Proposition 4.2), which will be used to estimate the lower bound of the Hausdorff dimension of a set.

**Lemma 6.12** ([45]). *Let  $E \subset [0, 1)$  be a Borel set and  $\mu$  be a measure with  $\mu(E) > 0$ . Suppose that*

$$\liminf_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} \geq s, \quad \forall x \in E$$

where  $B(x, r)$  denotes the open ball with center at  $x$  and radius  $r$ . Then  $\dim E \geq s$ .

Next we give a formula for computing the Hausdorff dimension for a class of Cantor sets related to continued fractions.

**Lemma 6.13.** Let  $\{s_n\}_{n \geq 1}$  be a sequence of positive integers tending to infinity with  $s_n \geq 3$  for all  $n \geq 1$ . Then for any positive number  $N \geq 2$ , we have

$$\dim\{x \in [0, 1) : s_n \leq a_n(x) < Ns_n \ \forall n \geq 1\} = \liminf_{n \rightarrow \infty} \frac{\log(s_1 s_2 \cdots s_n)}{2 \log(s_1 s_2 \cdots s_n) + \log s_{n+1}}.$$

*Proof.* Let  $F$  be the set in question and  $s_0$  be the liminf in the statement. We call

$$J(a_1, a_2, \dots, a_n) := Cl \bigcup_{a_{n+1} \geq s_{n+1}} I_{n+1}(a_1, \dots, a_n, a_{n+1})$$

a *basic CF-interval* of order  $n$  with respect to  $F$ , where  $s_k \leq a_k < Ns_k$  for all  $1 \leq k \leq n$ . Here  $Cl$  stands for the closure. Then it follows that

$$F = \bigcap_{n=1}^{\infty} \bigcup_{s_k \leq a_k < Ns_k, 1 \leq k \leq n} J(a_1, \dots, a_n). \quad (6.5)$$

By Lemma 2.39, we have

$$J(a_1, \dots, a_n) = \left[ \frac{p_n}{q_n}, \frac{s_{n+1}p_n + p_{n-1}}{s_{n+1}q_n + q_{n-1}} \right] \text{ or } \left[ \frac{s_{n+1}p_n + p_{n-1}}{s_{n+1}q_n + q_{n-1}}, \frac{p_n}{q_n} \right] \quad (6.6)$$

according to  $n$  is even or odd. Then by Lemma 2.39, Lemma 2.37 and the assumption on  $a_k$  that  $s_k \leq a_k < Ns_k$  for all  $1 \leq k \leq n$ , we have

$$\frac{1}{2N^n} \frac{1}{s_{n+1}(s_1 \cdots s_n)^2} \leq |J(a_1, \dots, a_n)| = \frac{1}{q_n(s_{n+1}q_n + q_{n-1})} \leq \frac{1}{s_{n+1}(s_1 \cdots s_n)^2}. \quad (6.7)$$

Since  $s_k \rightarrow \infty$  as  $k \rightarrow \infty$ , then

$$\lim_{n \rightarrow \infty} \frac{\log s_1 + \cdots + \log s_n}{n} = \infty.$$

This, together with the definition of  $s_0$ , implies that for any  $s > s_0$ , there exists a sequence  $\{n_\ell : \ell \geq 1\}$  such that for all  $\ell \geq 1$ ,

$$(N-1)^{n_\ell} < (s_{n_\ell+1}(s_1 \cdots s_{n_\ell})^2)^{\frac{s-s_0}{2}}, \quad \prod_{k=1}^{n_\ell} s_k \leq (s_{n_\ell+1}(s_1 \cdots s_{n_\ell})^2)^{\frac{s+s_0}{2}}.$$

Then, by (6.5), together with (6.7), we have

$$\begin{aligned} H^s(F) &\leq \liminf_{\ell \rightarrow \infty} \sum_{s_k \leq a_k < Ns_k, 1 \leq k \leq n_\ell} |J(a_1, \dots, a_{n_\ell})|^s \\ &\leq \liminf_{\ell \rightarrow \infty} \left( (N-1)^{n_\ell} \prod_{k=1}^{n_\ell} s_k \right) \left( \frac{1}{s_{n_\ell+1}(s_1 \cdots s_{n_\ell})^2} \right)^s \leq 1. \end{aligned}$$

Since  $s > s_0$  is arbitrary, we have  $\dim F \leq s_0$ .

For the lower bound, we define a measure  $\mu$  such that for any basic  $CF$ -interval  $J(a_1, a_2, \dots, a_n)$  of order  $n$ ,

$$\mu(J(a_1, a_2, \dots, a_n)) = \prod_{j=1}^n \frac{1}{(N-1)s_j}.$$

By the Kolmogorov extension theorem,  $\mu$  can be extended to a probability measure supported on  $F$ . In the following, we will check the mass distribution principle with this measure.

Fix  $s < s_0$ . By the definition of  $s_0$  and the fact that  $s_k \rightarrow \infty$  ( $k \rightarrow \infty$ ) and that  $N$  is a constant, there exists an integer  $n_0$  such that for all  $n \geq n_0$ ,

$$\prod_{k=1}^n (N-1)s_k \geq \left( s_{n+1} \left( \prod_{k=1}^n N s_k \right)^2 \right)^s. \quad (6.8)$$

We take  $r_0 = \frac{1}{2N^{n_0}} \frac{1}{s_{n_0+1}(s_1 \cdots s_{n_0})^2}$ .

For any  $x \in F$ , there exists an infinite sequence  $\{a_1, a_2, \dots\}$  with  $s_k \leq a_k < N s_k, \forall k \geq 1$  such that  $x \in J(a_1, \dots, a_n)$ , for all  $n \geq 1$ . For any  $r < r_0$ , there exists an integer  $n \geq n_0$  such that

$$|J(a_1, \dots, a_{n+1})| \leq r < |J(a_1, \dots, a_n)|.$$

We claim that the ball  $B(x, r)$  can intersect only one basic  $CF$ -interval of order  $n$ , which is just  $J(a_1, \dots, a_n)$ . We establish this only at the case that  $n$  is even, since for the case that  $n$  is odd, the argument is similar.

Case (1):  $s_n < a_n < N s_n - 1$ . The left and right adjacent basic  $CF$ -intervals (of order  $n$ ) to  $J(a_1, \dots, a_n)$  are  $J(a_1, \dots, a_n - 1)$  and  $J(a_1, \dots, a_n + 1)$  respectively. Then by (6.6) and the condition that  $s_n \geq 3$ , the gap between  $J(a_1, \dots, a_n)$  and  $J(a_1, \dots, a_n - 1)$  is

$$\frac{p_n}{q_n} - \frac{s_{n+1}(p_n - p_{n-1}) + p_{n-1}}{s_{n+1}(q_n - q_{n-1}) + q_{n-1}} = \frac{s_{n+1} - 1}{q_n(s_{n+1}(q_n - q_{n-1}) + q_{n-1})} \geq |J(a_1, \dots, a_n)|.$$

Hence  $B(x, r)$  can not intersect  $J(a_1, \dots, a_n - 1)$ . On the other hand, the gap  $J(a_1, \dots, a_n)$  and  $J(a_1, \dots, a_n + 1)$  is

$$\frac{p_n + p_{n-1}}{q_n + q_{n-1}} - \frac{s_{n+1}p_n + p_{n-1}}{s_{n+1}q_n + q_{n-1}} = \frac{s_{n+1} - 1}{(q_n + q_{n-1})(s_{n+1}q_n + q_{n-1})} \geq |J(a_1, \dots, a_n)|.$$

Hence  $B(x, r)$  can not intersect  $J(a_1, \dots, a_n + 1)$  either.

Case (2):  $a_n = s_n$ . The right adjacent  $n$ -th order basic CF-interval to  $J(a_1, \dots, a_n)$  is  $J(a_1, \dots, a_n + 1)$ . The same argument as in the case (1) shows that  $B(x, r)$  can not intersect  $J(a_1, \dots, a_n + 1)$ . On the other hand, the gap between the left endpoint of  $J(a_1, \dots, a_n)$  and that of  $I_{n-1}(a_1, \dots, a_{n-1})$  is

$$\frac{p_n}{q_n} - \frac{p_{n-1} + p_{n-2}}{q_{n-1} + q_{n-2}} = \frac{s_n - 1}{(q_{n-1} + q_{n-2})q_n} \geq \left| J(a_1, \dots, a_n) \right|.$$

It follows that  $B(x, r)$  can not intersect any  $n$ -th order  $CF$ -basic intervals on the left of  $J(a_1, \dots, a_n)$ . In general,  $B(x, r)$  can intersect no other  $n$ -th order  $CF$ -basic intervals than  $J(a_1, \dots, a_n)$ .

Case (3):  $a_n = Ns_n - 1$ . From the case (1), we know that  $B(x, r)$  can not intersect any  $n$ -th order  $CF$ -basic intervals on the left of  $J(a_1, \dots, a_n)$ . While for on the right, the gap between the right endpoint of  $J(a_1, \dots, a_n)$  and that of  $I_{n-1}(a_1, \dots, a_{n-1})$  is

$$\frac{p_{n-1}}{q_{n-1}} - \frac{s_{n+1}p_n + p_{n-1}}{s_{n+1}q_n + q_{n-1}} = \frac{s_{n+1}}{(s_{n+1}q_n + q_{n-1})q_{n-1}} \geq \left| J(a_1, \dots, a_n) \right|.$$

It follows that  $B(x, r)$  can not intersect any  $n$ -th order  $CF$ -basic intervals on the right of  $J(a_1, \dots, a_n)$ . In general,  $B(x, r)$  can intersect no other  $n$ -th order  $CF$ -basic intervals than  $J(a_1, \dots, a_n)$ .

Now we distinguish two cases to estimate the measure of  $B(x, r)$ .

*Case (i).*  $|J(a_1, \dots, a_{n+1})| \leq r < |I_{n+1}(a_1, \dots, a_{n+1})|$ . By Lemma 2.40 and the fact  $a_{n+1} \neq 1$ ,  $B(x, r)$  can intersect at most five  $(n + 1)$ -th order basic CF-intervals. As a consequence, by (6.8), we have

$$\mu(B(x, r)) \leq 5 \prod_{k=1}^{n+1} \frac{1}{(N-1)s_k} \leq 5 \left( \frac{1}{s_{n+2}(N^{n+1}s_1 \dots s_{n+1})^2} \right)^s. \quad (6.9)$$

Since

$$r > \left| J(a_1, \dots, a_{n+1}) \right| = \frac{1}{q_{n+1}(s_{n+2}q_{n+1} + q_n)} \geq \frac{1}{2s_{n+2}(N^{n+1}s_1 \dots s_{n+1})^2},$$

it follows that

$$\mu(B(x, r)) \leq 10r^s.$$

Case (ii).  $|I_{n+1}(a_1, \dots, a_{n+1})| \leq r < |J(a_1, \dots, a_n)|$ . In this case, we have

$$I_{n+1}(a_1, \dots, a_{n+1}) = \frac{1}{q_{n+1}(q_{n+1} + q_n)} \geq \frac{1}{2q_{n+1}^2} \geq \frac{1}{2N^{2(n+1)}} \left( \prod_{k=1}^{n+1} s_k \right)^2.$$

So  $B(x, r)$  can intersect at most a number  $8rN^{2(n+1)}(s_1 \cdots s_{n+1})^2$  of  $(n+1)$ -th basic CF-intervals. As a consequence,

$$\begin{aligned} \mu(B(x, r)) &\leq \min \left\{ \mu(J(a_1, \dots, a_n)), 8rN^{2(n+1)}(s_1 \cdots s_{n+1})^2 \prod_{k=1}^{n+1} \frac{1}{(N-1)s_k} \right\} \\ &\leq \prod_{k=1}^n \frac{1}{(N-1)s_k} \min \left\{ 1, 8rN^{2(n+1)}(s_1 \cdots s_{n+1})^2 \frac{1}{(N-1)s_{n+1}} \right\}. \end{aligned}$$

By (6.8) and the elementary inequality  $\min\{a, b\} \leq a^{1-s}b^s$  which holds for any  $a, b > 0$  and  $0 < s < 1$ , we have

$$\begin{aligned} \mu(B(x, r)) &\leq \left( \frac{1}{s_{n+1}(N^n s_1 \cdots s_n)^2} \right)^s \cdot \left( 8rN^{2(n+1)}(s_1 \cdots s_{n+1})^2 \frac{1}{(N-1)s_{n+1}} \right)^s \\ &\leq 16Nr^s. \end{aligned}$$

Combining these two cases, together with mass distribution principle, we have  $\dim F \geq s_0$ .  $\square$

Let

$$E' = \{x \in [0, 1) : e^{\varphi(n)-\varphi(n-1)} \leq a_n(x) \leq 2e^{\varphi(n)-\varphi(n-1)}, \forall n \geq 1\}.$$

It is evident that  $E' \subset E_\xi(\varphi)$ . Then applying Lemma 6.13, we have

$$E_\xi(\varphi) \geq \liminf_{n \rightarrow \infty} \frac{\varphi(n)}{\varphi(n+1) + \varphi(n)} = \frac{1}{b+1}.$$

### 6.3.2 Upper bound

We first give a lemma which is a little bit more than the upper bound for the case  $b = 1$ . Its proof uses a family of Bernoulli measures with an infinite number of states.

**Lemma 6.14.** *If  $\lim_{n \rightarrow \infty} \frac{\varphi(n)}{n} = \infty$ , then  $\dim E_\xi(\varphi) \leq \frac{1}{2}$ .*

*Proof.* For any  $t > 1$ , we introduce a family of Bernoulli measures  $\mu_t$ :

$$\mu_t(I_n(a_1, \dots, a_n)) = e^{-nC(t) - t \sum_{j=1}^n \log a_j(x)} \quad (6.10)$$

where  $C(t) = \log \sum_{n=1}^{\infty} \frac{1}{n^t}$ .

Fix  $x \in E_\xi(\varphi)$  and  $\epsilon > 0$ . If  $n$  is sufficiently large, we have

$$(\xi - \epsilon)\varphi(n) < \sum_{j=1}^n \log a_j(x) < (\xi + \epsilon)\varphi(n). \quad (6.11)$$

So,

$$E_\xi(\varphi) \subset \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} E_n(\epsilon),$$

where

$$E_n(\epsilon) = \{x \in [0, 1) : (\xi - \epsilon)\varphi(n) < \sum_{j=1}^n \log a_j(x) < (\xi + \epsilon)\varphi(n)\}.$$

Now let  $\mathcal{I}(n, \epsilon)$  be the family of all  $n$ -th basic intervals  $I_n(a_1, \dots, a_n)$  satisfying (6.11). For each  $N \geq 1$ , we select all those intervals in  $\bigcup_{n=N}^{\infty} \mathcal{I}(n, \epsilon)$  which are maximal ( $I \in \bigcup_{n=N}^{\infty} \mathcal{I}(n, \epsilon)$  is maximal if there is no other  $I'$  in  $\bigcup_{n=N}^{\infty} \mathcal{I}(n, \epsilon)$  such that  $I \subset I'$  and  $I \neq I'$ ). We denote by  $\mathcal{J}(N, \epsilon)$  the set of all maximal intervals in  $\bigcup_{n=N}^{\infty} \mathcal{I}(n, \epsilon)$ . It is evident that  $\mathcal{J}(N, \epsilon)$  is a cover of  $E_\xi(\varphi)$ . Let  $I_n(a_1, \dots, a_n) \in \mathcal{J}(N, \epsilon)$ , we have

$$\mu_t(I_n(a_1, \dots, a_n)) = e^{-nC(t) - t \sum_{j=1}^n \log a_j} \geq e^{-nC(t) - t(\xi + \epsilon)\varphi(n)}.$$

On the other hand,

$$\left| I_n(a_1, \dots, a_n) \right| \leq e^{-2 \log q_n} \leq e^{-2 \sum_{j=1}^n \log a_j} \leq e^{-2(\xi - \epsilon)\varphi(n)}.$$

Since  $\lim_{n \rightarrow \infty} \frac{\varphi(n)}{n} = \infty$ , for each  $s > t/2$  and  $N$  large enough, we have

$$\left| I_n(a_1, \dots, a_n) \right|^s \leq \mu_t(I_n(a_1, \dots, a_n)).$$

This implies  $\dim E_\xi(\varphi) \leq 1/2 = \frac{1}{b+1}$ . □

Now we return back to the proof of the upper bound.

*Case (i)*  $b = 1$ . Since  $(\varphi(n+1) - \varphi(n)) \uparrow \infty$ , Lemma 6.14 implies immediately  $\dim E_\xi(\varphi) \leq \frac{1}{2}$ .

*Case (ii)*  $b > 1$ . By (6.11), for each  $x \in E_\xi(\varphi)$  and  $n$  sufficiently large

$$(\xi - \epsilon)\varphi(n+1) - (\xi + \epsilon)\varphi(n) \leq \log a_{n+1}(x) \leq (\xi + \epsilon)\varphi(n+1) - (\xi - \epsilon)\varphi(n).$$

Take

$$L_{n+1} = e^{(\xi - \epsilon)\varphi(n+1) - (\xi + \epsilon)\varphi(n)}, \quad M_{n+1} = e^{(\xi + \epsilon)\varphi(n+1) - (\xi - \epsilon)\varphi(n)}.$$

Define

$$F_N = \{x \in [0, 1] : L_n \leq a_n(x) \leq M_n, \forall n \geq N\}.$$

Then we have

$$E_\xi(\varphi) \subset \bigcup_{N=1}^{\infty} F_N.$$

We can only estimate the upper bound of  $\dim F_1$ . Because  $F_N$  can be written as a countable union of sets with the same form as  $F_1$ , then by the  $\sigma$ -stability of Hausdorff dimension, we will have  $\dim F_N = \dim F_1$ . We can further assume that  $M_n \geq L_n + 2$ .

For any  $n \geq 1$ , define

$$D_n = \{(\sigma_1, \dots, \sigma_n) \in \mathbb{N}^n : L_k \leq \sigma_k \leq M_k, 1 \leq k \leq n\}.$$

It follows that

$$F_1 = \bigcap_{n \geq 1} \bigcup_{(\sigma_1, \dots, \sigma_n) \in D_n} J(\sigma_1, \dots, \sigma_n),$$

where

$$J(\sigma_1, \dots, \sigma_n) := Cl \bigcup_{\sigma \geq L_{n+1}} I(\sigma_1, \dots, \sigma_n, \sigma)$$

(called an admissible CF-interval of order  $n$ ). For any  $n \geq 1$  and  $s > 0$ , we have

$$\sum_{(\sigma_1, \dots, \sigma_n) \in D_n} \left| J(\sigma_1, \dots, \sigma_n) \right|^s \leq \sum_{(\sigma_1, \dots, \sigma_n) \in D_n} \left| \frac{1}{q_n^2 L_{n+1}} \right|^s \leq \frac{M_1 \cdots M_n}{\left( (L_1 \cdots L_n)^2 L_{n+1} \right)^s}.$$

It follows that

$$\dim F_1 \leq \liminf_{n \rightarrow \infty} \frac{\log M_1 + \cdots + \log M_n}{\sum_{k=1}^n \log L_k + \sum_{k=1}^{n+1} \log L_k} = \frac{\xi + \epsilon + \frac{2\epsilon}{b-1}}{(\xi - \epsilon)(b+1) - 2\epsilon - \frac{4\epsilon}{b-1}}.$$

Letting  $\epsilon \rightarrow 0$ , we get

$$\dim E_\xi(\varphi) \leq \frac{1}{b+1}.$$

## 6.4 Ruelle operator theory

There have been various works done on the Ruelle transfer operator for the Gauss dynamics. See Mayer ([107], [108], [109]), Jenkinson ([75]), Jenkinson and Pollicott ([74]), Pollicott and Weiss ([128]), Hanus, Mauldin and Urbański ([64]). In this section we will present a general Ruelle operator theory for conformal infinite iterated function system which was developed in [64] and then apply it to the Gauss dynamics. We will also prove some properties of the pressure function in the case of Gauss dynamics, which will be used later.

### 6.4.1 Conformal infinite iterated function systems

In this subsection, we present the conformal infinite iterated function systems which were studied by Hanus, Mauldin and Urbański in [64]. See also the book of Mauldin and Urbański ([106]).

Let  $X$  be a non-empty compact connected subset of  $\mathbb{R}^d$  equipped with a metric  $\rho$ . Let  $I$  be an index set with at least two elements and at most countable elements. An *iterated function system*  $S = \{\phi_i : X \rightarrow X : i \in I\}$  is a collection of injective contractions for which there exists  $0 < s < 1$  such that for each  $i \in I$  and all  $x, y \in X$ ,

$$\rho(\phi_i(x), \phi_i(y)) \leq s\rho(x, y). \quad (6.12)$$

Before further discussion, we are willing to give a list of notation.

- $I^n := \{\omega : \omega = (\omega_1, \dots, \omega_n), \omega_k \in I, 1 \leq k \leq n\}$ ,
- $I^* := \cup_{n \geq 1} I^n$ ,
- $I^\infty := \prod_{i=1}^\infty I$ ,
- $\phi_\omega := \phi_{\omega_1} \circ \phi_{\omega_2} \circ \dots \circ \phi_{\omega_n}$ , for  $\omega = \omega_1 \omega_2 \dots \omega_n \in I^n, n \geq 1$ ,
- $|\omega|$  denote the length of  $\omega \in I^* \cup I^\infty$ ,

- $\omega|_n = \omega_1\omega_2 \dots \omega_n$ , if  $|w| \geq n$ ,
- $[\omega|_n] = [\omega_1 \dots \omega_n] = \{x \in I^\infty : x_1 = \omega_1, \dots, x_n = \omega_n\}$ ,
- $\sigma : I^\infty \rightarrow I^\infty$  the shift transformation,
- $\|\phi'_\omega\| := \sup_{x \in X} |\phi'_\omega(x)|$  for  $\omega \in I^*$ ,
- $C(X)$  space of continuous functions on  $X$ ,
- $\|\cdot\|_\infty$  supremum norm on the Banach space  $C(X)$ .

For  $\omega \in I^\infty$ , the set

$$\pi(\omega) = \bigcap_{n=1}^{\infty} \phi_{\omega|_n}(X)$$

is a singleton. We also denote its only element by  $\pi(\omega)$ . This thus defines a coding map  $\pi : I^\infty \rightarrow X$ . The limit set  $J$  of the iterated function system is defined by

$$J := \pi(I^\infty).$$

Denote by  $\partial X$  the boundary of  $X$  and by  $\text{Int}(X)$  the interior of  $X$ .

We say that the iterated function system  $S = \{\phi_i\}_{i \in I}$  satisfies the *open set condition* if there exists a non-empty open set  $U \subset X$  such that  $\phi_i(U) \subset U$  for each  $i \in I$  and  $\phi_i(U) \cap \phi_j(U) = \emptyset$  for each pair  $i, j \in I, i \neq j$ .

An iterated function system  $S = \{\phi_i : X \rightarrow X : i \in I\}$  is said to be *conformal* if the following are satisfied:

- (1) The open set condition is satisfied for  $U = \text{Int}(X)$ .
- (2) There exists an open connected set  $V$  with  $X \subset V \subset \mathbb{R}^d$  such that all maps  $\phi_i, i \in I$ , extend to  $C^1$  conformal diffeomorphisms of  $V$  into  $V$ .
- (3) There exist  $h, \ell > 0$  such that for each  $x \in \partial X \subset \mathbb{R}^d$ , there exists an open cone  $\text{Con}(x, h, \ell) \subset \text{Int}(X)$  with vertex  $x$ , central angle of Lebesgue measure  $h$  and altitude  $\ell$ .
- (4) (Bounded Distortion Property) There exists  $K \geq 1$  such that  $|\phi'_\omega(y)| \leq K|\phi'_\omega(x)|$  for every  $\omega \in I^*$  and every pair of points  $x, y \in V$ .

The *topological pressure function* for a conformal iterated function systems  $S = \{\phi_i : X \rightarrow X : i \in I\}$  is defined as

$$\mathcal{P}(t) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{|\omega|=n} \|\phi'_\omega\|^t.$$

The system  $S$  is said to be *regular* if there exists  $t \geq 0$  such that  $\mathcal{P}(t) = 0$ .

Let  $\beta > 0$ . A *Hölder family of functions* of order  $\beta$  is a family of continuous functions  $F = \{f^{(i)} : X \rightarrow \mathbb{C} : i \in I\}$  such that

$$V_\beta(F) = \sup_{n \geq 1} V_n(F) < \infty,$$

where

$$V_n(F) = \sup_{\omega \in I^n} \sup_{x, y \in X} \{|f^{(\omega_1)}(\phi_{\sigma(\omega)}(x)) - f^{(\omega_1)}(\phi_{\sigma(\omega)}(y))|\} e^{\beta(n-1)}.$$

A family of functions  $F = \{f^{(i)} : X \rightarrow \mathbb{R}, i \in I\}$  is said to be *strong* if

$$\sum_{i \in I} \|e^{f^{(i)}}\|_\infty < \infty.$$

Define the *Ruelle operator* on  $C(X)$  associated to  $F$  as

$$\mathcal{L}_F(g)(x) := \sum_{i \in I} e^{f^{(i)}(x)} g(\phi_i(x)).$$

Denote by  $\mathcal{L}_F^*$  the dual operator of  $\mathcal{L}_F$ .

The *topological pressure* of  $F$  is defined by

$$P(F) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{|\omega|=n} \exp \left( \sup_{x \in X} \sum_{j=1}^n f^{\omega_j} \circ \phi_{\sigma^j \omega}(x) \right).$$

A measure  $\nu$  is called *F-conformal* if the following are satisfied:

- (1)  $\nu$  is supported on  $J$ .
- (2) For any Borel set  $A \subset X$  and any  $\omega \in I^*$ ,

$$\nu(\phi_\omega(A)) = \int_A \exp \left( \sum_{j=1}^n f^{(\omega_j)} \circ \phi_{\sigma^j \omega} - P(F)|\omega| \right) d\nu.$$

- (3)  $\nu(\phi_\omega(X) \cap \phi_\tau(X)) = 0 \quad \omega, \tau \in I^n, \omega \neq \tau, n \geq 1$ .

Two functions  $\phi, \varphi \in C(X)$  are said to be *cohomologous* with respect to the transformation  $T$ , if there exists  $u \in C(X)$  such that

$$\varphi(x) = \phi(x) + u(x) - u(T(x)).$$

The following two theorems are due to Hanus, Mauldin and Urbański [64].

**Theorem 6.15** ([64]). *For a conformal iterated function system  $S = \{\phi_i : X \rightarrow X : i \in I\}$  and a strong Hölder family of functions  $F = \{f^{(i)} : X \rightarrow \mathbb{C} : i \in I\}$ , there exists a unique  $F$ -conformal probability measure  $\nu_F$  on  $X$  such that  $\mathcal{L}_F^* \nu_F = e^{P(F)} \nu_F$ . There exists a unique shift invariant probability measure  $\tilde{\mu}_F$  on  $I^\infty$  such that  $\mu_F := \tilde{\mu}_F \circ \pi^{-1}$  is equivalent to  $\nu_F$  with bounded Radon-Nikodym derivative. Furthermore, the Gibbs property is satisfied:*

$$\frac{1}{C} \leq \frac{\tilde{\mu}_F([\omega|_n])}{\exp\left(\sum_{j=1}^n f^{(\omega_j)}(\pi(\sigma^j \omega)) - nP(F)\right)} \leq C.$$

Let  $\Psi = \{\psi^{(i)} : X \rightarrow \mathbb{R} : i \in I\}$  and  $F = \{f^{(i)} : X \rightarrow \mathbb{R} : i \in I\}$  be two families of real-valued Hölder functions. We define the *amalgamated functions* on  $I^\infty$  associated to  $\Psi$  and  $F$  as follows:

$$\tilde{\psi}(\omega) := \psi^{(\omega_1)}(\pi(\sigma\omega)), \quad \tilde{f}(\omega) := f^{(\omega_1)}(\pi(\sigma\omega)) \quad \forall \omega \in I^\infty.$$

**Theorem 6.16** ([64], see also [106], pp. 43-48). *Let  $\Psi$  and  $F$  be two families of real-valued Hölder functions. Suppose the sets  $\{i \in I : \sup_x(\psi^{(i)}(x)) > 0\}$  and  $\{i \in I : \sup_x(f^{(i)}(x)) > 0\}$  are finite. Then the function  $(t, q) \mapsto P(t, q) = P(t\Psi + qF)$ , is real-analytic with respect to  $(t, q) \in \text{Int}(D)$ , where*

$$D = \left\{ (t, q) : \sum_{i \in I} \exp(\sup(t\psi^{(i)} + qf^{(i)})) < \infty \right\}.$$

Furthermore, if  $t\Psi + qF$  is a strong Hölder family for  $(t, q) \in D$  and

$$\int (|\tilde{f}| + |\tilde{\psi}|) d\tilde{\mu}_{t,q} < \infty,$$

where  $\tilde{\mu}_{t,q} := \tilde{\mu}_{t\Psi + qF}$  is obtained by Theorem 6.15, then

$$\frac{\partial P}{\partial t} = \int \tilde{\psi} d\tilde{\mu}_{t,q} \quad \text{and} \quad \frac{\partial P}{\partial q} = \int \tilde{f} d\tilde{\mu}_{t,q}.$$

If  $t\tilde{\psi} + q\tilde{f}$  is not cohomologous to a constant function, then  $P(t, q)$  is strictly convex and

$$H(t, q) := \begin{pmatrix} \frac{\partial^2 P}{\partial t^2} & \frac{\partial^2 P}{\partial t \partial q} \\ \frac{\partial^2 P}{\partial t \partial q} & \frac{\partial^2 P}{\partial q^2} \end{pmatrix}$$

is positive definite.

### 6.4.2 Continued fraction dynamical system

We apply the theory in the precedent subsection to the continued fraction dynamical system. Let  $X = [0, 1]$  and  $I = \mathbb{N}$ . The continued fraction dynamical system can be viewed as an iterated function system:

$$S = \left\{ \psi_i(x) = \frac{1}{i+x} : i \in \mathbb{N} \right\}.$$

Recall that the projection mapping  $\pi : I^\infty \rightarrow X$  is defined by

$$\pi(\omega) := \bigcap_{n=1}^{\infty} \psi_{\omega|_n}(X), \quad \forall \omega \in I^\infty.$$

Notice that  $\psi_1'(0) = -1$ , thus (6.12) is not satisfied. However, this is not a real problem, since we can consider the system of second level maps and replace  $S$  by  $\tilde{S} := \{\psi_i \circ \psi_j : i, j \in \mathbb{N}\}$ . In fact, for any  $x \in [0, 1]$

$$(\psi_i \circ \psi_j)'(x) = \left( \frac{1}{i + \frac{1}{j+x}} \right)' = \left( \frac{1}{i(j+x) + 1} \right)^2 \leq \frac{1}{4}.$$

In the following, we will collect or prove some facts on the continued fraction dynamical system, which will be useful for applying Theorem 6.15 and 6.16.

**Lemma 6.17** ([104]). *The continued fraction dynamical system  $S$  is regular and conformal.*

For the investigation in the present work, our problems are tightly connected to the following two families of Hölder functions.

$$\Psi = \{\log |\psi_i'| : i \in \mathbb{N}\} \text{ and } F = \{-\log i : i \in \mathbb{N}\}.$$

*Remark 3.* We mention that our method used here is also applicable to other potentials than the two special families introduced here.

The families  $\Psi$  and  $F$  are Hölder families and their amalgamated functions are equal to

$$\tilde{\psi}(\omega) = -2 \log(\omega_1 + \pi(\sigma\omega)), \quad \tilde{f}(\omega) = -\log \omega_1 \quad \forall \omega \in \mathbb{N}^\infty.$$

For our convenience, we will consider the function  $t\Psi - qF$  instead of  $t\Psi + qF$ .

**Lemma 6.18.** *Let  $D := \{(t, q) : 2t - q > 1\}$ . For any  $(t, q) \in D$ , we have*

- (i) *The family  $t\Psi - qF := \{t \log |\psi'_i| + q \log i : i \in \mathbb{N}\}$  is Hölder and strong.*
- (ii) *The topological pressure  $P$  associated to the potential  $t\Psi - qF$  can be written as*

$$P(t, q) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\omega_1, \dots, \omega_n} \exp \left( \sup_x \log \prod_{j=1}^n \omega_j^q ([\omega_j, \dots, \omega_n + x])^{2t} \right).$$

*Proof.* The assertion on the domain  $D$  follows from

$$\frac{1}{4^t} \zeta(2t - q) = \mathcal{L}_{t\Psi - qF} 1 = \sum_{i=1}^{\infty} \frac{i^q}{(i+x)^{2t}} \leq \sum_{i=1}^{\infty} i^{q-2t} = \zeta(2t - q).$$

where  $\zeta(2t - q)$  is the Riemann zeta function, defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \forall s > 1.$$

- (i) For  $(t, q) \in D$ , write  $(t\Psi - qF)^{(i)} := t \log |\psi'_i| + q \log i$ . Then

$$\sum_{i \in I} \left\| \exp \{ (t\Psi - qF)^{(i)} \} \right\|_{\infty} = \sum_{i=1}^{\infty} \left\| \frac{i^q}{(i+x)^{2t}} \right\|_{\infty} = \sum_{i=1}^{\infty} i^{q-2t} = \zeta(2t - q) < \infty.$$

Thus  $t\Psi - qF$  is strong.

- (ii) It suffices to noticed that

$$\sup_x \left( \sum_{j=1}^n (t |\psi'_{\omega_j}| + q \log \omega_j) \circ \psi_{\sigma^j \omega}(x) \right) = \sup_x \log \prod_{j=1}^n \omega_j^q ([\omega_j, \dots, \omega_n + x])^{2t}.$$

□

Denote by  $\mathcal{L}_{t\Psi-qF}^*$  the conjugate operator of  $\mathcal{L}_{t\Psi-qF}$ . Applying Theorem 6.15 with the help of Lemma 6.17 and Lemma 6.18, we get

**Proposition 6.19.** *For each  $(t, q) \in D$ , there exists a unique  $t\Psi - qF$ -conformal probability measure  $\nu_{t,q}$  on  $[0, 1]$  such that  $\mathcal{L}_{t\Psi-qF}^* \nu_{t,q} = e^{P(t,q)} \nu_{t,q}$ , and a unique shift invariant probability measure  $\tilde{\mu}_{t,q}$  on  $\mathbb{N}^\infty$  such that  $\mu_{t,q} := \tilde{\mu}_{t,q} \circ \pi^{-1}$  on  $[0, 1]$  is equivalent to  $\nu_{t,q}$  and*

$$\frac{1}{C} \leq \frac{\tilde{\mu}_{t,q}([\omega|_n])}{\exp\left(\sum_{j=1}^n (t\Psi - qF)^{(\omega_j)}(\pi(\sigma^j \omega)) - nP(t, q)\right)} \leq C \quad \forall \omega \in \mathbb{N}^\infty.$$

**Lemma 6.20.** *For the amalgamated functions  $\tilde{\psi}(\omega) = -2\log(\omega_1 + \pi(\sigma\omega))$  and  $\tilde{f}(\omega) = -\log \omega_1$ , we have*

$$-\int \log |T'(x)|_{\mu_{t,q}} = \int \tilde{\psi} d\tilde{\mu}_{t,q} \quad \text{and} \quad \int \log a_1(x) d\mu_{t,q} = -\int \tilde{f} d\tilde{\mu}_{t,q} \quad (6.13)$$

and  $t\tilde{\psi} - q\tilde{f}$  is not cohomologous to a constant.

*Proof.* (i). Assertion (6.13) is just a consequence of the facts

$$-\log |T'(\pi(\omega))| = \tilde{\psi}(\omega), \quad \log a_1(\pi(\omega)) = -\tilde{f}(\omega) \quad \forall \omega \in I^\infty.$$

Suppose  $t\tilde{\psi} - q\tilde{f}$  was not cohomologous to a constant. Then there would be a bounded function  $g$  such that  $t\tilde{\psi} - q\tilde{f} = g - g \circ T + C$ , which implies

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} (t\tilde{\psi} - q\tilde{f})(\sigma^j \omega) = \lim_{n \rightarrow \infty} \frac{g - g \circ \sigma^n}{n} + C = C$$

for all  $\omega \in I^\infty$ . On the other hand, if we take  $\omega_1 = [1, 1, \dots]$ ,  $\omega_2 = [2, 2, \dots]$  and  $\omega_3 = [3, 3, \dots]$ , we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} (t\tilde{\psi} - q\tilde{f})(\sigma^j \omega_i) = C_i,$$

where

$$C_1 = 2t \log\left(\frac{\sqrt{5}-1}{2}\right), \quad C_2 = 2t \log\left(\frac{\sqrt{5}-2}{2}\right) + q \log 2, \quad C_3 = 2t \log\left(\frac{\sqrt{5}-3}{2}\right) + q \log 3.$$

Thus we get a contradiction.  $\square$

By Theorem 6.16 and the proof of Lemma 6.18, we know that  $D = \{(t, q) : 2t - q > 1\}$  is the analytic area of the pressure  $P(t, q)$ . Applying Lemma 6.20 and Theorem 6.16, we get more:

**Proposition 6.21.** *On  $D = \{(t, q) : 2t - q > 1\}$ ,*

(1)  *$P(t, q)$  is analytic, strictly convex.*

(2)  *$P(t, q)$  is strictly decreasing and strictly convex with respect to  $t$ . In other words,  $\frac{\partial P}{\partial t}(t, q) < 0$  and  $\frac{\partial^2 P}{\partial t^2}(t, q) > 0$ . Furthermore,*

$$\frac{\partial P}{\partial t}(t, q) = - \int \log |T'(x)| d\mu_{t,q}. \quad (6.14)$$

(3)  *$P(t, q)$  is strictly increasing and strictly convex with respect to  $q$ . In other words,  $\frac{\partial P}{\partial q}(t, q) > 0$  and  $\frac{\partial^2 P}{\partial q^2}(t, q) > 0$ . Furthermore,*

$$\frac{\partial P}{\partial q}(t, q) = \int \log a_1(x) d\mu_{t,q}. \quad (6.15)$$

(4)

$$H(t, q) := \begin{pmatrix} \frac{\partial^2 P}{\partial t^2} & \frac{\partial^2 P}{\partial t \partial q} \\ \frac{\partial^2 P}{\partial t \partial q} & \frac{\partial^2 P}{\partial q^2} \end{pmatrix}$$

*is positive definite.*

At the end of this subsection, we would like to quote some results by Mayer ([109]) (see also Pollicott and Weiss ([128])).

**Proposition 6.22** ([109]). *Let  $P(t) := P(t, 0)$  and  $\mu_t := \mu_{t,0}$ , then  $P(t)$  is defined in  $(1/2, \infty)$  and we have  $P(1) = 0$  and  $\mu_1 = \mu_G$ . Furthermore,*

$$P'(t) = - \int \log |T'(x)| d\mu_t(x). \quad (6.16)$$

*In particular*

$$P'(0) = - \int \log |T'(x)| d\mu_G(x) = -\lambda_0. \quad (6.17)$$

*Remark 4.* Since  $\mu_{1,0} = \mu_1 = \mu_G$ , by (6.15), we have

$$\frac{\partial P}{\partial q}(1, 0) = \int \log a_1(x) d\mu_G = \xi_0. \quad (6.18)$$

### 6.4.3 Further study on $P(t, q)$

We will use the following simple known fact of convex functions.

**Fact 6.23.** Suppose  $f$  is a convex continuously differentiable function on an interval  $I$ . Then  $f'(x)$  is increasing and

$$f'(x) \leq \frac{f(y) - f(x)}{y - x} \leq f'(y) \quad x, y \in I, x < y.$$

First we give an estimation for the pressure  $P(t, q)$  and show some behaviors of  $P(t, q)$  when  $q$  tends to  $-\infty$  and  $2t - 1$  ( $t$  being fixed).

**Proposition 6.24.** For  $(t, q) \in D$ , we have

$$-t \log 4 + \log \zeta(2t - q) \leq P(t, q) \leq \log \zeta(2t - q). \quad (6.19)$$

Consequently,

(1)  $P(0, q) = \log \zeta(-q)$ . For any point  $(t_0, q_0)$  on the line  $2t - q = 1$ ,

$$\lim_{(t, q) \rightarrow (t_0, q_0)} P(t, q) = \infty;$$

(2) for fixed  $t \in \mathbb{R}$ ,

$$\lim_{q \rightarrow 2t-1} \frac{\partial P}{\partial q}(t, q) = +\infty; \quad (6.20)$$

(3) for fixed  $t \in \mathbb{R}$ , we have

$$\lim_{q \rightarrow -\infty} \frac{P(t, q)}{q} = 0, \quad \lim_{q \rightarrow -\infty} \frac{\partial P}{\partial q}(t, q) = 0. \quad (6.21)$$

*Proof.* Notice that  $\frac{1}{\omega_{j+1}} \leq [\omega_j, \dots, \omega_n + x] \leq \frac{1}{\omega_j}$  for  $x \in [0, 1)$  and  $1 \leq j \leq n$ .

Thus we have

$$\frac{1}{4^{nt}} \sum_{\omega=1}^{\infty} (\omega^{q-2t})^n \leq \sum_{\omega_1, \dots, \omega_n} \prod_{j=1}^n \omega_j^q [\omega_j, \dots, \omega_n + x]^{2t} \leq \sum_{\omega=1}^{\infty} (\omega^{q-2t})^n.$$

Hence by Lemma 6.18 (ii), we get (6.19).

We get (1) immediately from (6.19).

Look at (2). For all  $q > q_0$ , by the convexity of  $P(t, q)$  and Fact 6.23, we have

$$\frac{\partial P}{\partial q}(t, q) \geq \frac{P(t, q) - P(t, q_0)}{q - q_0}.$$

Thus

$$\lim_{q \rightarrow 2t-1} \frac{\partial P}{\partial q}(t, q) \geq \lim_{q \rightarrow 2t-1} \frac{P(t, q_0) - P(t, q)}{q_0 - q} = \infty.$$

Here we use the fact that  $\lim_{q \rightarrow 2t-1} P(t, q) = +\infty$ . Hence we get (6.19).

In order to show (3), we consider  $P(t, q)/q$  as function of  $q$  on  $(-\infty, 2t - 1) \setminus \{0\}$ . Noticed that for fixed  $t \in \mathbb{R}$ ,  $\lim_{q \rightarrow -\infty} \zeta(2t - q) = 1$ . Thus

$$\lim_{q \rightarrow -\infty} \frac{\log \zeta(2t - q)}{q} = 0.$$

Then the first formula in (6.21) is followed from (6.19).

Fix  $q_0 < 2t - 1$ . Then for all  $q < q_0$ , by the convexity of  $P(t, q)$  and Fact 6.23, we have

$$\frac{\partial P}{\partial q}(t, q) \leq \frac{P(t, q_0) - P(t, q)}{q_0 - q}.$$

Thus

$$\lim_{q \rightarrow -\infty} \frac{\partial P}{\partial q}(t, q) \leq \lim_{q \rightarrow -\infty} \frac{P(t, q_0) - P(t, q)}{q_0 - q} = 0.$$

Hence by Proposition 6.21 (3), we get the second formula in (6.21).  $\square$

#### 6.4.4 Properties of $(t(\xi), q(\xi))$

Recall that  $\xi_0 = \int \log a_1(x) \mu_G$  and  $D_0 := \{(t, q) : 2t - q > 1, 0 \leq t \leq 1\}$ .

**Proposition 6.25.** *For any  $\xi \in (0, \infty)$ , the system*

$$\begin{cases} P(t, q) = q\xi, \\ \frac{\partial P}{\partial q}(t, q) = \xi \end{cases} \quad (6.22)$$

*admits a unique solution  $(t(\xi), q(\xi)) \in D_0$ . For  $\xi = \xi_0$ , the solution is  $(t(\xi_0), q(\xi_0)) = (1, 0)$ . The function  $t(\xi)$  and  $q(\xi)$  are analytic.*

*Proof. Existence and uniqueness of solution  $(t(\xi), q(\xi))$ .* Recall that  $P(1, 0) = 0$  and  $P(0, q) = \log \zeta(-q)$  (Proposition 6.24).

We start with a geometric argument which will followed by a rigorous proof. Consider  $P(t, q)$  as a family of function of  $q$  with parameter  $t$ . It can be seen from

the graph (see Figure 3) that for any  $\xi > 0$ , there exists a unique  $t \in (0, 1]$ , such that the line  $\xi q$  is tangent to  $P(t, \cdot)$ . This  $t = t(\xi)$  can be described as the unique point such that

$$\inf_{q < 2t(\xi) - 1} \left( P(t(\xi), q) - q\xi \right) = 0. \quad (6.23)$$

We denote by  $q(\xi)$  the point where the infimum in (6.23) is attained. Then the tangent point is  $(q(\xi), P(t(\xi), q(\xi)))$  and the derivative of  $P(t(\xi), q) - q\xi$  (with respect to  $q$ ) at  $q(\xi)$  equals 0, i.e.,

$$\left( P(t(\xi), q) - q\xi \right)' \Big|_{q(\xi)} = 0.$$

Thus we have  $\frac{\partial P}{\partial q}(t(\xi), q(\xi)) = \xi$ . By (6.23), we also have  $P(t(\xi), q(\xi)) - q(\xi)\xi = 0$ . Therefore  $(t(\xi), q(\xi))$  is a solution of (6.22). The uniqueness of  $q(\xi)$  follows by the fact that  $\frac{\partial P}{\partial q}$  is monotonic with respect to  $q$  (Proposition 6.21).

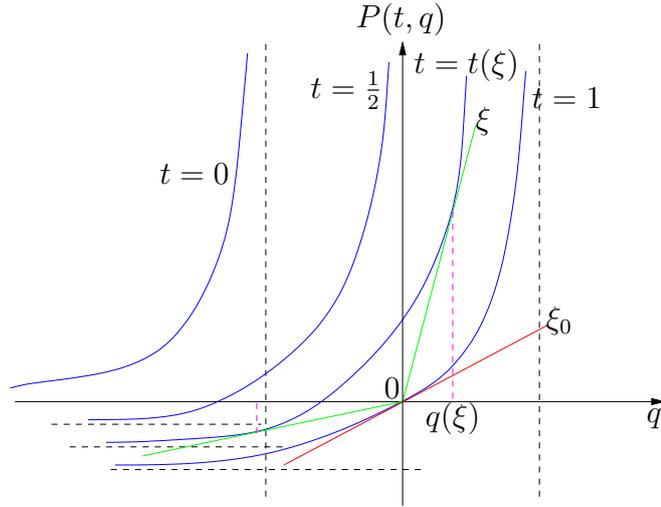


Figure 3. Solution of (6.22)

Let us give a rigorous proof. By (6.20), (6.21) and the mean-value theorem, for fixed  $t \in \mathbb{R}$  and any  $\xi > 0$ , there exists a  $q(t, \xi) \in (-\infty, 2t - 1)$  such that

$$\frac{\partial P}{\partial q}(t, q(t, \xi)) = \xi. \quad (6.24)$$

The monotonicity of  $\frac{\partial P}{\partial q}$  with respect to  $q$  implies the uniqueness of  $q(t, \xi)$  (Proposition 6.21).

Since  $P(t, q)$  is analytic, the implicit  $q(t, \xi)$  is analytic with respect to  $t$  and  $\xi$ . Fix  $\xi$  and set

$$W(t) := P(t, q(t, \xi)) - \xi q(t, \xi).$$

Since

$$\begin{aligned} W'(t) &= \frac{\partial P}{\partial t}(t, q(t, \xi)) + \frac{\partial P}{\partial q}(t, q(t, \xi)) \frac{\partial q}{\partial t}(t, \xi) - \xi \frac{\partial q}{\partial t}(t, \xi) \\ &= \frac{\partial P}{\partial t}(t, q(t, \xi)) \quad (\text{by (6.24)}) \\ &< 0 \quad (\text{by Proposition 6.21(2)}). \end{aligned}$$

Thus  $W(t)$  is strictly decreasing.

Since  $P(0, q) = \log \zeta(-q) > 0$  ( $q < -1$ ), for  $\xi > 0$  we have

$$W(0) = P(0, q(0, \xi)) - \xi q(0, \xi) > 0.$$

Since  $P(1, q)$  is convex and  $P(1, 0) = 0$ , by Fact 6.23 we have

$$\frac{P(1, q(1, \xi)) - 0}{q(1, \xi) - 0} \leq \frac{\partial P}{\partial q}(1, q(1, \xi)) = \xi, \quad \text{if } q(1, \xi) > 0$$

and

$$\frac{0 - P(1, q(1, \xi))}{0 - q(1, \xi)} \geq \frac{\partial P}{\partial q}(1, q(1, \xi)) = \xi, \quad \text{if } q(1, \xi) < 0.$$

If  $q(1, \xi) = 0$ , we have in fact  $\xi = \xi_0$  and  $P(1, q(1, \xi)) = 0$ . Hence, in any case we have

$$P(1, q(1, \xi)) - \xi q(1, \xi) \leq 0 \tag{6.25}$$

Therefore,  $W(1) = P(1, q(1, \xi)) - \xi q(1, \xi) \leq 0$ .

Thus by the mean-value theorem and the monotonicity of  $W(t)$ , there exists a unique  $t = t(\xi) \in (0, 1]$  such that  $W(t(\xi)) = 0$ , *i.e.*

$$P(t(\xi), q(t(\xi), \xi)) = \xi q(t(\xi), \xi). \tag{6.26}$$

If we write  $q(t(\xi), \xi)$  as  $q(\xi)$ , both (6.24) and (6.26) show that  $(t(\xi), q(\xi))$  is the unique solution of (6.22). For  $\xi = \xi_0$ , the assertion in Proposition 6.22 that  $P(0, 1) = 0 = 0 \cdot \xi_0$  and the assertion of Remark 4 that  $\frac{\partial P}{\partial q}(1, 0) = \xi_0$  imply that

$(0, 1)$  is a solution of (6.22). Then the uniqueness of the solution to (6.22) implies  $(t(\xi_0), q(\xi_0)) = (0, 1)$ .

*Analyticity of  $(t(\xi), q(\xi))$ .* Consider the map

$$F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} P(t, q) - q\xi \\ \frac{\partial P}{\partial q}(t, q) - \xi \end{pmatrix}.$$

Then the jacobian of  $F$  is equal to

$$J(F) =: \begin{pmatrix} \frac{\partial F_1}{\partial t} & \frac{\partial F_1}{\partial q} \\ \frac{\partial F_2}{\partial t} & \frac{\partial F_2}{\partial q} \end{pmatrix} = \begin{pmatrix} \frac{\partial P}{\partial t} & \frac{\partial P}{\partial q} - \xi \\ \frac{\partial^2 P}{\partial t \partial q} & \frac{\partial^2 P}{\partial q^2} \end{pmatrix}.$$

Consequently,

$$\det(J(F))|_{t=t(\xi), q=q(\xi)} = \frac{\partial P}{\partial t} \cdot \frac{\partial^2 P}{\partial q^2} \neq 0.$$

Thus by the implicit function theorem,  $t(\xi)$  and  $q(\xi)$  are analytic.  $\square$

Now let us present some properties on  $t(\xi)$ . Recall that  $\xi_0 = \frac{\partial P}{\partial q}(1, 0)$ .

**Proposition 6.26.**  $q(\xi) < 0$  for  $\xi < \xi_0$ ;  $q(\xi_0) = 0$ ;  $q(\xi) > 0$  for  $\xi > \xi_0$ .

*Proof.* Since  $P(1, q)$  is convex and  $P(1, 0) = 0$ , by Fact 6.23, we have

$$\begin{aligned} \frac{P(1, q) - 0}{q - 0} &\geq \frac{\partial P}{\partial q}(1, 0) = \xi_0, \quad (q > 0); \\ \frac{0 - P(1, q)}{0 - q} &\leq \frac{\partial P}{\partial q}(1, 0) = \xi_0, \quad (q < 0). \end{aligned}$$

Hence for all  $q < 1$ ,

$$P(1, q) \geq \xi_0 q \tag{6.27}$$

We recall that  $(t(\xi_0), q(\xi_0)) = (1, 0)$  is the unique solution of the system (6.22) for  $\xi = \xi_0$ . By the above discussion of the existence of  $t(\xi)$ ,  $t(\xi) = 1$  if and only if  $\xi = \xi_0$ . Now we suppose  $t \in (0, 1)$ . For  $\xi > \xi_0$ , using (6.27), we have

$$P(t, q) > P(1, q) \geq q\xi_0 \geq q\xi \quad (\forall q \leq 0).$$

Thus  $q(\xi) > 0$ . For  $\xi < \xi_0$ , using (6.27), we have

$$P(t, q) > P(1, q) \geq q\xi_0 \geq q\xi \quad (\forall q \geq 0).$$

Thus  $q(\xi) < 0$ .  $\square$

**Proposition 6.27.** For  $\xi \in (0, +\infty)$ , we have

$$t'(\xi) = \frac{q(\xi)}{\frac{\partial P}{\partial t}(t(\xi), q(\xi))}. \quad (6.28)$$

*Proof.* Recall that

$$\begin{cases} P(t(\xi), q(\xi)) = q(\xi)\xi, \\ \frac{\partial P}{\partial q}(t(\xi), q(\xi)) = \xi. \end{cases} \quad (6.29)$$

By taking the derivation with respect to  $\xi$  of the first equation in (6.29), we get

$$t'(\xi) \frac{\partial P}{\partial t}(t(\xi), q(\xi)) + q'(\xi) \frac{\partial P}{\partial q}(t(\xi), q(\xi)) = q'(\xi)\xi + q(\xi).$$

Taking into account the second equation in (6.29), we get

$$t'(\xi) \frac{\partial P}{\partial t}(t(\xi), q(\xi)) = q(\xi). \quad (6.30)$$

□

**Proposition 6.28.** We have  $t'(\xi) > 0$  for  $\xi < \xi_0$ ,  $t'(\xi_0) = 0$ , and  $t'(\xi) < 0$  for  $\xi > \xi_0$ . Furthermore,

$$t(\xi) \rightarrow 0 \quad (\xi \rightarrow 0), \quad (6.31)$$

$$t(\xi) \rightarrow 1/2 \quad (\xi \rightarrow +\infty). \quad (6.32)$$

*Proof.* By Propositions 6.26 and 6.27 and the fact  $\frac{\partial P}{\partial t} > 0$ ,  $t(\xi)$  is increasing on  $(0, \xi_0)$  and decreasing on  $(\xi_0, \infty)$ . Then by the analyticity of  $t(\xi)$ , we can obtain two analytic inverse functions on the two intervals respectively. For the first inverse function, write  $\xi_1 = \xi_1(t)$ . Then  $\xi_1'(t) > 0$  and

$$\xi_1(t) = \frac{P(t, q(t))}{q(t)} = \frac{\partial P}{\partial q}(t, q(t)).$$

(the equations (6.22) are considered as equations on  $t$ ). By Proposition 6.26, we have  $q(\xi_1(t)) < 0$  then  $P(t, q(\xi_1(t))) < 0$ . By Proposition 6.24 (1),  $\lim_{q \rightarrow 2t-1} P(t, q) = \infty$ . Thus there exists  $q_0(t)$  such that  $q_0(t) > q(t)$  and  $P(t, q_0(t)) = 0$ . Therefore

$$\xi_1(t) = \frac{\partial P}{\partial q}(t, q(t)) < \frac{\partial P}{\partial q}(t, q_0(t)).$$

Since  $P(0, q) = \log \zeta(-q)$ , we have  $\lim_{t \rightarrow 0} q_0(t) = \infty$ . Thus we get

$$\lim_{t \rightarrow 0} \frac{\partial P}{\partial q}(t, q_0(t)) = \lim_{q \rightarrow -\infty} \frac{\partial P}{\partial q}(0, q) = 0.$$

Hence by  $\xi_1(t) \geq 0$ , we obtain  $\lim_{t \rightarrow 0} \xi_1(t) = 0$  which implies (6.31).

Write  $\xi_2 = \xi_2(t)$  for the second inverse function. Then  $\xi_2'(t) < 0$  and

$$\xi_2(t) = \frac{P(t, q(t))}{q(t)} = \frac{\partial P}{\partial q}(t, q(t)) > \frac{\partial P}{\partial q}(t, 0) \rightarrow \infty \quad (t \rightarrow 1/2).$$

This implies (6.32). □

Let us summarize. We have proved that  $t(\xi)$  is analytic on  $(0, \infty)$ ,  $\lim_{\xi \rightarrow 0} t(\xi) = 0$  and  $\lim_{\xi \rightarrow \infty} t(\xi) = 1/2$ . We have also proved that  $t(\xi)$  is increasing on  $(0, \xi_0)$ , decreasing on  $(\xi_0, \infty)$  and  $t(\xi_0) = 1$ .

## 6.5 Khintchine spectrum

Now we are ready to study the Hausdorff dimensions of the level set

$$E_\xi = \{x \in [0, 1) : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \log a_j(x) = \xi\}.$$

Since  $\mathbb{Q}$  is countable, we need only to consider

$$\{x \in [0, 1) \setminus \mathbb{Q} : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \log a_j(x) = \xi\}.$$

which admits the same Hausdorff dimension with  $E_\xi$  and is still denoted by  $E_\xi$ .

### 6.5.1 Proof of Theorem 1.2 (1) and (2)

Let  $(t, q) \in D$  and  $\mu_{t,q}, \tilde{\mu}_{t,q}$  be the measures in Proposition 6.19. For  $x \in [0, 1) \setminus \mathbb{Q}$ , let  $x = [a_1, \dots, a_n, \dots]$  and  $\omega = \pi^{-1}(x)$ . Then  $\omega = a_1 \cdots a_n \cdots \in \mathbb{N}^{\mathbb{N}}$  and

$$\mu_{t,q}(I_n(x)) = \mu_{t,q}(I_n(a_1, \dots, a_n)) = \tilde{\mu}_{t,q}([\omega|_n]).$$

By the Gibbs property of  $\tilde{\mu}_{t,q}$ ,

$$\tilde{\mu}_{t,q}(\pi([\omega|_n])) \asymp \exp(-nP(t, q)) \prod_{j=1}^n \omega_j^q (\omega_j + \pi(\sigma^j \omega))^{-2t}.$$

In other words,

$$\mu_{t,q}(I_n(x)) \asymp \exp(-nP(t,q)) \prod_{j=1}^n a_j^q [a_j, \dots, a_n, \dots]^{2t}.$$

By Lemma 6.5,  $|I_n(x)| \asymp |(T^n)'(x)|^{-1} = \prod_{j=0}^{n-1} |T^j(x)|^2$ . Thus we have the following Gibbs property of  $\mu_{t,q}$ :

$$\mu_{t,q}(I_n(x)) \asymp \exp(-nP(t,q)) |I_n(x)|^t \prod_{j=1}^n a_j^q. \quad (6.33)$$

It follows that

$$\delta_{\mu_{t,q}}(x) = \lim_{n \rightarrow \infty} \frac{\log \mu_{t,q}(I_n(x))}{\log |I_n(x)|} = t + \lim_{n \rightarrow \infty} \frac{q \cdot \frac{1}{n} \sum_{j=1}^n \log a_j - P(t,q)}{\frac{1}{n} \log |I_n(x)|}.$$

The Gibbs property of  $\tilde{\mu}_{t,q}$  implies that  $\mu_{t,q}$  is ergodic. Therefore,

$$\delta_{\mu_{t,q}}(x) = t + \frac{q \int \log a_1(x) d\mu_{t,q} - P(t,q)}{- \int \log |T'(x)| d\mu_{t,q}} \quad \mu_{t,q} - a.e.$$

Using the formula (6.14) and (6.15) in Proposition 6.21, we have

$$\delta_{\mu_{t,q}}(x) = t + \frac{q \frac{\partial P}{\partial q}(t,q) - P(t,q)}{\frac{\partial P}{\partial t}(t,q)} \quad \mu_{t,q} - a.e. \quad (6.34)$$

Moreover, the ergodicity of  $\tilde{\mu}_{t,q}$  also implies that the Lyapunov exponents  $\lambda(x)$  exist for  $\mu_{t,q}$  almost every  $x$  in  $[0, 1)$ . Thus by (6.33), Lemma 6.10 and Lemma 6.11, we obtain

$$d_{\mu_{t,q}}(x) = \delta_{\mu_{t,q}}(x) = t + \frac{q \frac{\partial P}{\partial q}(t,q) - P(t,q)}{\frac{\partial P}{\partial t}(t,q)} \quad \mu_{t,q} - a.e. \quad (6.35)$$

For  $\xi \in (0, \infty)$ , choose  $(t, q) = (t(\xi), q(\xi)) \in D_0$  be the unique solution of (6.22). Then (6.35) gives

$$d_{\mu_{t(\xi), q(\xi)}}(x) = t(\xi) \quad \mu_{t,q} - a.e.$$

By the ergodicity of  $\tilde{\mu}_{t(\xi), q(\xi)}$  and (6.15), we have for  $\mu_{t(\xi), q(\xi)}$  almost every  $x$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \log a_j(x) = \int \log a_1(x) d\mu_{t(\xi), q(\xi)} = \frac{\partial P}{\partial q}(t(\xi), q(\xi)) = \xi.$$

So  $\mu_{t(\xi),q(\xi)}$  is supported on  $E_\xi$ . Hence

$$\dim(E_\xi) \geq \dim \mu_{t(\xi),q(\xi)} = t(\xi). \quad (6.36)$$

In the following we will show that

$$\dim(E_\xi) \leq t \quad (\forall t > t(\xi)). \quad (6.37)$$

Then it will imply that  $\dim(E_\xi) = t(\xi)$  for any  $\xi > 0$ . For any  $t > t(\xi)$ , take an  $\epsilon_0 > 0$  such that

$$0 < \epsilon_0 < \frac{P(t(\xi), q(\xi)) - P(t, q(\xi))}{q(\xi)} \quad \text{if } q(\xi) > 0.$$

and

$$0 < \epsilon_0 < \frac{P(t, q(\xi)) - P(t(\xi), q(\xi))}{q(\xi)} \quad \text{if } q(\xi) < 0.$$

(For the special case  $q(\xi) = 0$ , *i.e.*,  $\xi = \xi_0$ , we have  $\dim E_\xi = 1$  which is a well-known result). Such an  $\epsilon_0$  exists, for  $P(t, q)$  is strictly decreasing with respect to  $t$ . For all  $n \geq 1$ , set

$$E_\xi^n(\epsilon_0) := \left\{ x \in [0, 1) \setminus \mathbb{Q} : \xi - \epsilon_0 < \frac{1}{n} \sum_{j=1}^n \log a_j(x) < \xi + \epsilon_0 \right\}.$$

Then we have

$$E_\xi \subset \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} E_\xi^n(\epsilon_0).$$

Let  $\mathcal{I}(n, \xi, \epsilon_0)$  be the collection of all  $n$ -th order basic intervals  $I_n(a_1, \dots, a_n)$  such that

$$\xi - \epsilon_0 < \frac{1}{n} \sum_{j=1}^n \log a_j(x) < \xi + \epsilon_0.$$

Then

$$E_\xi^n(\epsilon_0) = \bigcup_{J \in \mathcal{I}(n, \xi, \epsilon_0)} J.$$

Hence  $\{J : J \in \mathcal{I}(n, \xi, \epsilon_0), n \geq 1\}$  is a cover of  $E_\xi$ . When  $q(\xi) > 0$ , by (6.33), we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \sum_{J \in \mathcal{I}(n, \xi, \epsilon_0)} |J|^t \\ & \leq \sum_{n=1}^{\infty} \sum_{(a_1 \dots a_n) > e^{n(\xi - \epsilon_0)}} \frac{e^{nP(t, q(\xi))}}{(a_1 \dots a_n)^{q(\xi)}} \cdot \frac{|J|^t (a_1 \dots a_n)^{q(\xi)}}{e^{nP(t, q(\xi))}} \\ & \leq C \cdot \sum_{n=1}^{\infty} e^{n(P(t, q(\xi)) - (\xi - \epsilon_0)q(\xi))} \cdot \sum_{J \in \mathcal{I}(n, \xi, \epsilon_0)} \mu_{t, q(\xi)}(J) < \infty \end{aligned}$$

where  $C$  is a constant. When  $q(\xi) < 0$ ,

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{J \in \mathcal{I}(n, \xi, \epsilon_0)} |J|^t \\
& \leq \sum_{n=1}^{\infty} \sum_{(a_1 \cdots a_n) < e^{n(\xi + \epsilon_0)}} \frac{e^{nP(t, q(\xi))}}{(a_1 \cdots a_n)^{q(\xi)}} \cdot \frac{|J|^t (a_1 \cdots a_n)^{q(\xi)}}{e^{nP(t, q(\xi))}} \\
& \leq C \cdot \sum_{n=1}^{\infty} e^{n(P(t, q(\xi)) - (\xi + \epsilon_0)q(\xi))} \cdot \sum_{J \in \mathcal{I}(n, \xi, \epsilon_0)} \mu_{t, q(\xi)}(J) < \infty.
\end{aligned}$$

Hence we get (6.37).

For the special case  $\xi = 0$ , we need only to show  $\dim(E_0) = 0$ . This can be induced by the same process. For any  $t > 0$ , since  $\lim_{\xi \rightarrow 0} t(\xi) = 0$ , there exists  $\xi > 0$  such that  $0 < t(\xi) < t$ . We can also choose  $\epsilon_0 > 0$  such that

$$\frac{P(t, q(\xi)) - P(t(\xi), q(\xi))}{q(\xi)} > \epsilon_0.$$

For  $n \geq 1$ , set

$$E_0^n(\epsilon_0) := \left\{ x \in [0, 1) \setminus \mathbb{Q} : \frac{1}{n} \sum_{j=1}^n \log a_j(x) < \xi + \epsilon_0 \right\}.$$

We have

$$E_0 \subset \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} E_0^n(\epsilon_0).$$

By the same calculation, we get  $\dim(E_0) \leq t$ . Since  $t$  can be arbitrary small, we obtain  $\dim(E_0) = 0$ .

By the discussion in the preceding subsection, we have proved Theorem 6.2 (1) and (2).

### 6.5.2 Proof of Theorem 1.2 (3) and (4)

We are going to investigate more properties of the functions  $q(\xi)$  and  $t(\xi)$ .

**Proposition 6.29.** *We have*

$$\lim_{\xi \rightarrow 0} q(\xi) = -\infty, \quad \lim_{\xi \rightarrow \infty} q(\xi) = 0.$$

*Proof.* We prove the first limit by contradiction. Suppose there exists a subsequence  $\xi_\delta \rightarrow 0$  such that  $q(\xi_\delta) \rightarrow M > -\infty$ . Then by (6.31) and Proposition 6.21 (3), we have

$$\lim_{\xi_\delta \rightarrow 0} \frac{\partial P}{\partial q}(t(\xi_\delta), q(\xi_\delta)) = \frac{\partial P}{\partial q}(0, M) > 0.$$

This contradicts with

$$\frac{\partial P}{\partial q}(t(\xi_\delta), q(\xi_\delta)) = \xi_\delta \rightarrow 0.$$

On the other hand, we know that  $q(\xi) \geq 0$  when  $\xi \geq \xi_0$ , and  $0 \leq q(\xi) < 2t(\xi) - 1$ . Then by (6.32), we have  $\lim_{\xi \rightarrow \infty} q(\xi) = 0$ .  $\square$

Apply this proposition and (6.28), combining (6.20) and (6.21). We get

$$\lim_{\xi \rightarrow 0} t'(\xi) = +\infty, \quad \lim_{\xi \rightarrow \infty} t'(\xi) = 0.$$

This is the assertion (3) of Theorem 6.2.

Now we will prove the last assertion of Theorem 6.2, *i.e.*,  $t''(\xi_0) < 0$  and there exists  $\xi_1 > \xi_0$  such that  $t''(\xi_1) > 0$ , basing on the following proposition.

**Proposition 6.30.** *For  $\xi \in (0, +\infty)$ , we have*

$$q'(\xi) = \frac{1 - t'(\xi) \frac{\partial^2 P}{\partial t \partial q}(t(\xi), q(\xi))}{\frac{\partial^2 P}{\partial q^2}(t(\xi), q(\xi))}; \quad (6.38)$$

$$t''(\xi) = \frac{t'(\xi)^2 \frac{\partial^2 P}{\partial t^2}(t(\xi), q(\xi)) - q'(\xi)^2 \frac{\partial^2 P}{\partial q^2}(t(\xi), q(\xi))}{-\frac{\partial P}{\partial t}(t(\xi), q(\xi))}. \quad (6.39)$$

*Proof.* Taking derivative of (6.30) with respect to  $\xi$ , we get

$$t'(\xi)^2 \frac{\partial^2 P}{\partial t^2}(t(\xi), q(\xi)) + q'(\xi) t'(\xi) \frac{\partial^2 P}{\partial q \partial t}(t(\xi), q(\xi)) + t''(\xi) \frac{\partial P}{\partial t}(t(\xi), q(\xi)) = q'(\xi). \quad (6.40)$$

Taking derivative of the second equation of (6.29) with respect to  $\xi$ , we get

$$t'(\xi) \frac{\partial^2 P}{\partial t \partial q}(t(\xi), q(\xi)) + q'(\xi) \frac{\partial^2 P}{\partial q^2}(t(\xi), q(\xi)) = 1, \quad (6.41)$$

which gives immediately (6.38).

Subtract (6.41) multiplied by  $q'(\xi)$  from (6.40), we get (6.39).  $\square$

We divide the proof of the assertion (4) of Theorem 6.2 into two parts.

*Proof of  $t''(\xi_0) < 0$ .* By Proposition 6.21,  $\frac{\partial P}{\partial t}(1, 0) < 0$ . Since  $q(\xi_0) = 0$ , by (6.28) we have  $t'(\xi_0) = 0$ . Also by Proposition 6.21, we get

$$0 < \frac{\partial^2 P}{\partial t^2}(t(\xi_0), q(\xi_0)) = \frac{\partial^2 P}{\partial t^2}(1, 0) < +\infty$$

and

$$0 \leq \frac{\partial^2 P}{\partial q^2}(t(\xi_0), q(\xi_0)) = \frac{\partial^2 P}{\partial q^2}(1, 0) < +\infty.$$

By (6.38) and (6.39), we have

$$t''(\xi) = \frac{t'(\xi)^2 \frac{\partial^2 P}{\partial t^2}(t(\xi), q(\xi)) \frac{\partial^2 P}{\partial q^2}(t(\xi), q(\xi)) - \left(1 - t'(\xi) \frac{\partial^2 P}{\partial t \partial q}(t(\xi), q(\xi))\right)^2}{-\frac{\partial P}{\partial t}(t(\xi), q(\xi)) \frac{\partial^2 P}{\partial q^2}(t(\xi), q(\xi))}. \quad (6.42)$$

Thus by  $t'(\xi_0) = 0$ , we have  $t''(\xi_0) < 0$ .  $\square$

*Proof of  $t''(\xi_1) > 0$ .* Proposition 6.29 shows  $\lim_{\xi \rightarrow \infty} q(\xi) = 0$  and we know that  $q(\xi_0) = 0$ . However,  $q(\xi)$  is not always equal to 0, so there exists a  $\xi_1 \in [\xi_0, +\infty)$ , such that  $q'(\xi_1) < 0$ . Write

$$H(t, q) := \begin{pmatrix} \frac{\partial^2 P}{\partial t^2} & \frac{\partial^2 P}{\partial t \partial q} \\ \frac{\partial^2 P}{\partial t \partial q} & \frac{\partial^2 P}{\partial q^2} \end{pmatrix}$$

and add (6.41) multiplied by  $q'(\xi)$  to (6.40), we get

$$\left(t'(\xi), q'(\xi)\right) H(t, q) \left(t'(\xi), q'(\xi)\right)^T + \frac{\partial P}{\partial t}(t(\xi), q(\xi)) t''(\xi) = 2q'(\xi). \quad (6.43)$$

Since  $H(t, q)$  is definite positive,  $\frac{\partial P}{\partial t}(t, q) < 0$  and  $q'(\xi_1) < 0$ , we have  $t''(\xi_1) > 0$ . This completes the proof.  $\square$

## 6.6 Lyapunov spectrum

In this last section, we follow the same procedure as in Section 4 and Section 5 to deduce the Lyapunov spectrum of the Gauss map. Kesseböhmer recently

pointed out that the Lyapunov spectrum was also studied by M. Kesseböhmer and B. Stratmann [80].

Take

$$F = \Psi = \{\log |\psi'_i| : i \in \mathbb{N}\}.$$

instead of  $F = \{-\log i : i \in \mathbb{N}\}$  and  $\Psi = \{\log |\psi'_i| : i \in \mathbb{N}\}$ . Then the strong Hölder family becomes  $(\tilde{t} - q)\Psi$  and  $D$  should be changed to

$$\tilde{D} := \{(\tilde{t}, q) : \tilde{t} - q > 1/2\}.$$

Here and in the rest of this section we will use  $\tilde{t}$  instead of  $t$  to distinguish the present situation from that of Khintchine exponents. What we have done in Section 4 is still useful. Denote by  $P_1(\tilde{t}, q)$  the pressure  $P((\tilde{t} - q)\Psi)$ . Then

$$P_1(\tilde{t}, q) = P(\tilde{t} - q), \quad \text{with } P(\cdot) = P(\cdot, 0)$$

where  $P(\cdot, \cdot)$  is the pressure function studied in the section 4. Hence  $P_1(\tilde{t}, q)$  is analytic and similar equations (6.14) and (6.15) are obtained just with  $\log |T'(x)|$  instead of  $\log a_1(x)$ .

To determine the Lyapunov spectrum, we begin with the following proposition which take the place of Proposition 6.24.

**Proposition 6.31.** *For  $(\tilde{t}, q) \in \tilde{D}$ , we have*

$$-(\tilde{t} - q) \log 4 + \log \zeta(2\tilde{t} - 2q) \leq P_1(\tilde{t}, q) \leq \log \zeta(2\tilde{t} - 2q). \quad (6.44)$$

*Consequently, the following are established.*

(1) *For any point  $(\tilde{t}_0, q_0)$  on the line  $\tilde{t} - q = 1/2$ ,*

$$\lim_{(\tilde{t}, q) \rightarrow (\tilde{t}_0, q_0)} P(\tilde{t}, q) = \infty.$$

(2) *For fixed  $\tilde{t} \in \mathbb{R}$ ,*

$$\lim_{q \rightarrow \tilde{t} - \frac{1}{2}} \frac{\partial P_1}{\partial q}(\tilde{t}, q) = +\infty.$$

(3) *Recall that  $\gamma_0 := 2 \log \frac{1+\sqrt{5}}{2}$ . For fixed  $\tilde{t} \in \mathbb{R}$ ,*

$$\lim_{q \rightarrow -\infty} \frac{P_1(\tilde{t}, q)}{q} = \gamma_0, \quad \lim_{q \rightarrow -\infty} \frac{\partial P_1}{\partial q}(\tilde{t}, q) = \gamma_0.$$

*Proof.*  $P_1(\tilde{t}, q)$  is defined as

$$P_1(\tilde{t}, q) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\omega_1=1}^{\infty} \cdots \sum_{\omega_n=1}^{\infty} \exp \left( \sup_{x \in [0,1]} \log \prod_{j=1}^n ([\omega_j, \cdots, \omega_n + x])^{2(\tilde{t}-q)} \right).$$

The proofs of (1) and (2) are the same as in the proof of Proposition 6.24.

To get (3), we follow another method. Since  $P_1(\tilde{t}, q) = P(\tilde{t} - q)$ , we need only to show

$$\lim_{\tilde{t} \rightarrow \infty} P'(\tilde{t}) = -\gamma_0, \quad P(\tilde{t}) + \tilde{t}\gamma_0 = o(\tilde{t}) \quad (\tilde{t} \rightarrow \infty).$$

By Proposition 6.22,  $P(\tilde{t})$  is analytic on  $(1/2, \infty)$ . Let  $E := \{P'(\tilde{t}) : \tilde{t} > 1/2\}$ , denote by  $\text{Int}(E)$  and  $\text{Cl}(E)$  the interior and closure of  $E$ . By a result in [76], we have

$$\text{Int}(E) \subset \left\{ - \int \log |T'(x)| d\mu : \mu \in \mathcal{M} \right\} \subset \text{Cl}(E),$$

where  $\mathcal{M}$  is the set of the invariant measures on  $[0, 1]$ . By Birkhoff's theorem, for any  $\mu \in \mathcal{M}$ , we have

$$\int \lambda(x) d\mu = \int \log |T'(x)| d\mu.$$

However, we know that  $\lambda(x) \geq \gamma_0 = 2 \log \frac{1+\sqrt{5}}{2}$ . Thus

$$- \int \log |T'(x)| d\mu \leq -\gamma_0 \quad \forall \mu \in \mathcal{M}. \quad (6.45)$$

Let  $\theta_0 = \frac{\sqrt{5}-1}{2}$ . Then  $T(\theta_0) = \theta_0$  and the Dirac measure  $\mu = \delta_{\theta_0}$  is invariant, and

$$- \int \log |T'(x)| d\delta_{\theta_0} = -\log |T'(\theta_0)| = -\gamma_0.$$

However, by the continuity of  $P'$ , we know that  $E$  is an interval. Therefore  $-\gamma_0$  is the right endpoint of  $E$ . Since  $P'(\tilde{t})$  is increasing, we get

$$\lim_{\tilde{t} \rightarrow \infty} P'(\tilde{t}) = -\gamma_0.$$

Let  $\{\beta_n\}_{n \geq 1}$  be such that  $\beta_n < -\gamma_0$  and  $\lim_{n \rightarrow \infty} \beta_n = -\gamma_0$ . There exist  $t_n \in \mathbb{R}$  such that  $t_n \rightarrow \infty$  and  $P'(t_n) = \beta_n$ . By the variational principle ([143], see also [109]), there exists an ergodic measure  $\mu_{t_n}$  such that

$$P(t_n) = h_{\mu_{t_n}} - t_n \int \log |T'(x)| d\mu_{t_n},$$

where  $h_{\mu_{t_n}}$  stands for the metric entropy of  $\mu_{t_n}$ . By the compactness of  $\mathcal{M}$  there exists an invariant measure  $\mu_\infty$  which is the weak limit of  $\mu_{t_n}$  (more precisely some subsequence of  $\mu_{t_n}$ . But, without loss of generality, we write it as  $\mu_{t_n}$ ). By the semi-continuity of metric entropy, for any  $\epsilon > 0$  we have  $h_{\mu_{t_n}} \leq h_{\mu_\infty} + \epsilon$  when  $t_n$  is large enough. Thus by (6.45),

$$P(t_n) \leq h_{\mu_\infty} + \epsilon - t_n \gamma_0.$$

We will show that  $h_{\mu_\infty} = 0$  (see the next lemma), which will imply

$$P(t_n) \leq \epsilon - t_n \gamma_0.$$

However, by the definition of  $P_1(\tilde{t}, q)$ ,  $P(\tilde{t})$  can be written as

$$P(\tilde{t}) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\omega_1=1}^{\infty} \cdots \sum_{\omega_n=1}^{\infty} \exp \left( \sup_{x \in [0,1]} \log \prod_{j=1}^n ([\omega_j, \cdots, \omega_n + x])^{2\tilde{t}} \right).$$

Thus if we just take one term in the summation, we have

$$P(\tilde{t}) \geq \lim_{n \rightarrow \infty} \frac{1}{n} \log \exp \left( \sup_{x \in [0,1]} \log \prod_{j=1}^n ([\underbrace{1, \cdots, 1}_{n-j}, 1 + x])^{2\tilde{t}} \right) = -\tilde{t} \gamma_0.$$

Hence we get

$$P(\tilde{t}) + \tilde{t} \gamma_0 = o(\tilde{t}) \quad (\tilde{t} \rightarrow \infty). \quad \square$$

Now we are led to show

**Lemma 6.32.**  $h_{\mu_\infty} = 0$ .

*Proof.* Let  $h_{\mu_\infty}(x)$  be the local entropy of  $\mu_\infty$  at  $x$  which is defined by

$$h_{\mu_\infty}(x) = \lim_{n \rightarrow \infty} \frac{\log \mu_\infty(I_n(x))}{n},$$

if the limit exists. Let  $\underline{D}_{\mu_\infty}(x)$  be the lower local dimension of  $\mu_\infty$  at  $x$  which is defined by

$$\underline{D}_{\mu_\infty}(x) := \liminf_{r \rightarrow 0} \frac{\log \mu_\infty(B(x, r))}{\log r}.$$

By the Shannon-McMillan-Breiman Theorem,  $h_{\mu_\infty}(x)$  exists  $\mu_\infty$ -almost everywhere. It is also known that  $\lambda(x)$  exists almost everywhere (by Birkhoff's theorem). So, by the definitions, we have

$$h_{\mu_\infty}(x) = \underline{D}_{\mu_\infty}(x) \lambda(x) \quad \mu_\infty - a.e.$$

By Birkhoff's theorem and (6.16),

$$\begin{aligned}\int \lambda(x) d\mu_\infty(x) &= \int \log |T'(x)| d\mu_\infty(x) \\ &= \lim_{n \rightarrow \infty} \int \log |T'(x)| d\mu_{t_n} \\ &= - \lim_{n \rightarrow \infty} P'(t_n) = \gamma_0 < \infty.\end{aligned}$$

Hence  $\lambda(x)$  is almost everywhere finite. Recall that [26]

$$h_{\mu_\infty} = \int h_{\mu_\infty}(x) d\mu_\infty(x).$$

Thus it suffices to prove

$$\underline{D}_{\mu_\infty}(x) = 0 \quad \mu_\infty - a.e.$$

That means ([47]) the upper dimension of  $\mu_\infty$  is zero, i.e.  $\mu_\infty$  is supported by a zero-dimensional set.

Since  $\int \lambda(x) d\mu_\infty(x) = \gamma_0$  and  $\lambda(x) \geq \gamma_0$  for any  $x$ , we have for  $\mu_\infty$  almost everywhere  $\lambda(x) = \gamma_0$ . Thus by Birkhoff's theorem,  $\mu_\infty$  is supported by the following set

$$\left\{ x \in [0, 1] : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |T'(T^j x)| = \gamma_0 \right\}. \quad (6.46)$$

Thus we need only to show that the Hausdorff dimension of this set is zero.

Recall

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |T'(T^j x)| = 2 \lim_{n \rightarrow \infty} \frac{1}{n} \log q_n(x).$$

By Lemma 6.6, (6.46) is in fact the following

$$\left\{ x \in [0, 1] : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \log a_j(x) = 0 \right\}. \quad (6.47)$$

However, the Hausdorff dimension of (6.47) is nothing but  $t(0)$ , the special case  $\xi = 0$  discussed in the subsection 5.1., which was proved to be zero. Thus the proof is completed.  $\square$

Recall that  $\lambda_0 = \int \log |T'(x)| d\mu_G$ . Let  $\tilde{D}_0 := \{(\tilde{t}, q) : \tilde{t} - q > 1/2, 0 \leq \tilde{t} \leq 1\}$ . We have a proposition similar to Proposition 6.25.

**Proposition 6.33.** For any  $\beta \in (\gamma_0, \infty)$ , the system

$$\begin{cases} P_1(\tilde{t}, q) = q\beta, \\ \frac{\partial P_1}{\partial q}(\tilde{t}, q) = \beta \end{cases} \quad (6.48)$$

admits a unique solution  $(\tilde{t}(\beta), q(\beta)) \in \tilde{D}_0$ . For  $\beta = \lambda_0$ , the solution is  $(\tilde{t}(\lambda_0), q(\lambda_0)) = (1, 0)$ . The functions  $\tilde{t}(\beta)$  and  $q(\beta)$  are analytic.

With the same argument, we can prove that  $\tilde{t}(\beta)$  is the spectrum of Lyapunov exponent. It is analytic, increasing on  $(\gamma_0, \lambda_0]$  and decreasing on  $(\lambda_0, \infty)$ . It is also neither concave nor convex. In other words, Theorem 6.3 can be similarly proved.

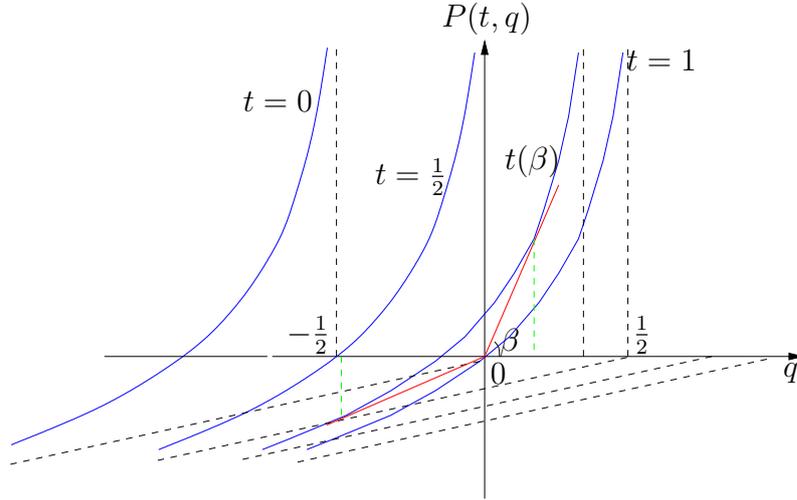


Figure 4. Solution of (6.48)

We finish this chapter by the observation that the Lyapunov spectrum can be stated as follows, which is similar to the classic formula, but with the difference that we have to divide the Legendre transform by  $\beta$ .

**Proposition 6.34.**

$$\tilde{t}(\beta) = \frac{P(-q(\beta))}{\beta} - q(\beta) = \frac{1}{\beta} \inf_q \{P(-q) - q\beta\}. \quad (6.49)$$

*Proof.* In fact, the family of functions  $P_1(\tilde{t}, q)$  with parameter  $\tilde{t}$  are just right translation of the function  $P(-q)$  with the length  $\tilde{t}$ . Write the system (6.48) as

follows

$$\begin{cases} P(\tilde{t} - q) = q\beta, \\ \frac{dP}{dq}(\tilde{t} - q) = \beta. \end{cases} \quad (6.50)$$

If we denote by  $\mu_q$ , the Gibbs measure with respect to potential  $q\Psi$ , then by a left translation the system (6.50) can be written as

$$\begin{cases} P(-q) = (\tilde{t} + q)\beta, \\ \frac{dP}{dq}(-q) = \beta. \end{cases}$$

Thus

$$\begin{cases} \tilde{t} = \frac{P(-q)}{\beta} - q, \\ \frac{dP}{dq}(-q) = \beta. \end{cases}$$

By using the second equation, we can write  $q$  as a function of  $\beta$ , hence we get (6.49).  $\square$

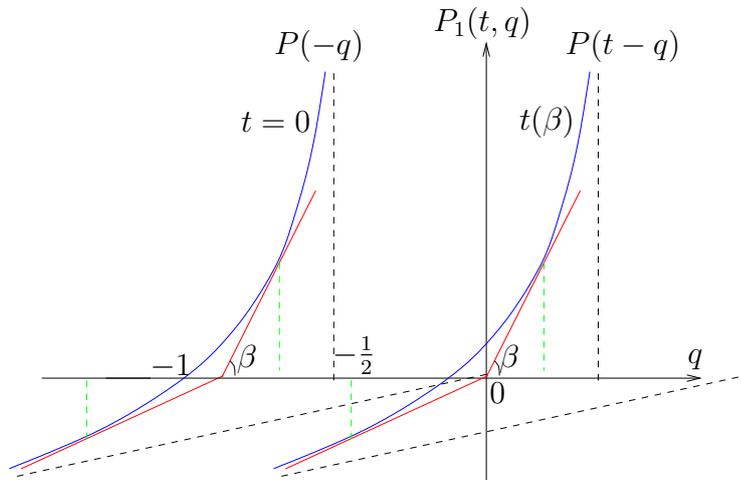


Figure 5. The other way to see  $t(\beta)$



# Chapter 7

## Non-normal Continued Fractions

In this chapter we consider certain sets of non-normal continued fractions for which the asymptotic frequencies of digit strings oscillate in one or other ways. The Hausdorff dimensions of these sets are shown to be the same value  $1/2$ . An example among them is the set of “extremely non-normal continued fractions” which was conjectured to be of Hausdorff dimension 0. <sup>1</sup>

### 7.1 Statement of the results

Every  $x \in [0, 1) \cap \mathbb{Q}^c$  can be expanded as a continued fraction expansion,  $x = [a_1(x), a_2(x), \dots]$  where  $a_n(x) \in \mathbb{N}$  for all  $n \geq 1$ . In what follows, the letter “ $x$ ” will denote both an infinite word  $(x_1, x_2, \dots) \in \mathbb{N}^{\mathbb{N}}$  and a real number  $x \in [0, 1) \cap \mathbb{Q}^c$  with continued fraction expansion  $[x_1, x_2, \dots]$ , where  $x_n = a_n(x)$  for all  $n \geq 1$ .

We recall the following list of definitions and notations which has appeared in Subsection 1.3.3.

- the set of finite words over  $\mathbb{N}$  (digit strings):  $\mathbb{N}^*$ .
- the number of occurrences of  $k$ -digit string  $W = (w_1, \dots, w_k) \in \mathbb{N}^*$  among the first  $n$  digits of  $x$ :

$$f(W, x, n) := \#\{1 \leq j \leq n - k + 1 : a_j(x) = w_1, \dots, a_{j+k-1}(x) = w_k\}.$$

- $\text{Freq}(W, x)$  denotes the collection of all accumulation points of the sequence  $\{\frac{1}{n}f(W, x, n) : n \geq 1\}$
- $\text{Freq}(W) = \bigcup_{x \in [0, 1) \cap \mathbb{Q}^c} \text{Freq}(W, x)$ .

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<sup>1</sup>L. M. Liao, J. H. Ma and B. W. Wang, *Dimension of some non-normal continued fraction sets*, to appear in Math. Proc. Camb. Phil. Soc..

The set of points of continued fractions with maximal frequency oscillation is defined by:

$$\mathbb{F} = \{x \in [0, 1) : \text{Freq}(W, x) = \text{Freq}(W) \text{ for all } W \in \mathbb{N}^*\}.$$

The first immediate result is

**Theorem 7.1.** *The set  $\mathbb{F}$  is residual in  $[0, 1)$ , namely, its complementary set is of the first category. As a consequence, the packing dimension of  $\mathbb{F}$  is 1.*

Let us remark that the packing dimension result follows from the fact (see [42, Exercise (1.8.4)]) that a residual subset of  $[0, 1)$  has packing dimension 1.

It is interesting to note that the Hausdorff dimension of  $\mathbb{F}$  is of “intermediate” size.

**Theorem 7.2.** *One has*

$$\dim(\mathbb{F}) = 1/2.$$

Here and in what follows, the symbol “dim” denotes the Hausdorff dimension.

To recall the definition of  $\mathbb{E}$  of “extremely non-normal continued fractions”, we recall the other list in Subsection 1.3.3:

- the simplex of probability vectors with index set  $\mathbb{N}^k$  ( $k \geq 1$ ):

$$\Delta_k = \left\{ (p(W))_{W \in \mathbb{N}^k} : p(W) \geq 0, \sum_W p(W) = 1 \right\}.$$

- 1-norm on  $\Delta_k$ :

$$\|\vec{p} - \vec{q}\|_1 = \sum_{W \in \mathbb{N}^k} |p(W) - q(W)|.$$

- the sub-simplex of shift invariant probability vectors in  $\Delta_k$ :

$$\mathbf{S}_k := \left\{ \vec{p} \in \Delta_k : \sum_i p(iV) = \sum_i p(Vi) \text{ for all } V \in \mathbb{N}^{k-1} \right\}.$$

- the vector of  $n$ -th asymptotic frequencies of the  $k$ -words occurring in  $x$ :

$$\Pi_k(x, n) := \left( \frac{1}{n} f(W, x, n) \right)_{W \in \mathbb{N}^k} \in \Delta_k.$$

- $\mathbf{A}_k(x)$  denotes the set of accumulation points of the sequence  $\{\Pi_k(x, n)\}_{n=1}^{\infty}$  with respect to the 1-norm  $\|\cdot\|_1$ .

It is known that  $\mathbf{A}_k(x) \subset \mathbf{S}_k$  (see [113]). We define

$$\mathbb{E}_k = \{x \in [0, 1) \cap \mathbb{Q}^c : \mathbf{A}_k(x) = \mathbf{S}_k\}$$

and  $\mathbb{E} = \bigcap_{k=1}^{\infty} \mathbb{E}_k$ , which is the set of “extremely non-normal continued fractions”.

The set  $\mathbb{E}$  is closely related to the set  $\mathbb{F}$  of continued fractions with maximal frequency oscillation. Indeed,

**Theorem 7.3.** *One has*

$$\mathbb{E} \subset \mathbb{F}.$$

It was proved that the set  $\mathbb{E}$  is residual in  $[0, 1)$  and has packing dimension 1 in [113]. Therefore, Theorem 7.1 follows directly from Theorem 7.3.

Olsen conjectured that  $\dim(\mathbb{E}) = 0$ . We show that

**Theorem 7.4.** *One has*

$$\dim(\mathbb{E}) = 1/2.$$

We remark that Theorem 7.2 and 7.4 will follow from more general results. Our results apply to a large kind of sets of continued fractions which are determined by various “frequency properties”. The techniques in the following might be useful in other parts of metric continued fraction theory. Let us describe these in more detail.

By virtue of  $\mathbb{E} \subset \mathbb{F}$ , to show that  $\dim(\mathbb{E}) = \dim(\mathbb{F}) = 1/2$ , it suffices to prove  $\dim(\mathbb{F}) \leq 1/2$  and  $\dim(\mathbb{E}) \geq 1/2$ . The upper and lower bound on the Hausdorff dimensions will be treated separately. First, by the definitions, for any  $\ell \geq 1$ , the set  $\mathbb{F}$  is included in

$$\mathbb{G}(\ell) = \{x \in [0, 1) : \limsup_{n \rightarrow \infty} \frac{f(\ell, x, n)}{n} = 1\}.$$

So the upper bound result will follow from

**Proposition 7.5.** *For any  $\ell \geq 1$ ,  $\dim(\mathbb{G}(\ell)) \leq 1/2$ .*

To deal with the lower bound, let us begin with the following

**Definition 1.** Let  $\mathbb{M} \subset \mathbb{N}$  and  $\mathbb{A} \subset [0, 1) \cap \mathbb{Q}^c$ . The set  $\mathbb{A}$  is said to be  $\mathbb{M}$ -free if

$$\{x \in [0, 1) \cap \mathbb{Q}^c : x_n = z_n \text{ for all } n \notin \mathbb{M}\} \subset \mathbb{A}$$

holds true for every  $z = [z_1, z_2, \dots] \in \mathbb{A}$ . The set  $\mathbb{A}$  is called “zero-density-free” if it is  $\mathbb{M}$ -free for any  $\mathbb{M}$  with zero density (i.e.  $\lim_{n \rightarrow \infty} \frac{1}{n} \#\{k \leq n : k \in \mathbb{M}\} = 0$ ).

It is then easy to see that both  $\mathbb{E}$  and  $\mathbb{F}$  are “zero-density-free” sets. Therefore, the lower bound results will follow from the following more general conclusion.

**Proposition 7.6.** *Let  $\mathbb{A} \subset [0, 1) \cap \mathbb{Q}^c$  be a “zero-density-free” set. If there exist an  $\alpha \in (0, 1)$  and at least one  $z = [z_1, z_2, \dots] \in \mathbb{A}$  such that*

$$z_n \leq \exp(\exp(n^\alpha)) \text{ for all } n \geq 1,$$

*then one has  $\dim(\mathbb{A}) \geq 1/2$ .*

In words, for a “zero-density-free” set, the existence of a “good seed” will ensure the set has Hausdorff dimension at least  $1/2$ .

*Remark 5.* It is shown in [101] that, for any  $b, c > 1$ , the set

$$\{x \in [0, 1) \cap \mathbb{Q}^c : a_n(x) \geq c^{b^n} \text{ infinitely often}\}$$

is of Hausdorff dimension  $\frac{1}{b+1} < 1/2$ . Therefore, essentially, the condition on the “seed” that  $z_n \leq \exp(\exp(n^\alpha))$  for some  $\alpha \in (0, 1)$  cant not be relaxed.

The following is organized as follows. In section 7.2, we recall some basic terminology on the combinatorics of words, and prove that  $\mathbb{E} \subset \mathbb{F}$ . In section 7.3, we verify the existence of a “good seed” for  $\mathbb{E}$ . The upper bound (Proposition 7.5) and lower bound (Proposition 7.6) results will be proved in the last two sections.

## 7.2 Some combinatorics of words and the relation between $\mathbb{E}$ and $\mathbb{F}$

We first recall some basic terminology on the combinatorics of words. Let  $\mathbb{N}^*$  denote the set of all finite words over the alphabet  $\mathbb{N}$ . For any  $k \geq 1$ , a word  $W = (w_1, \dots, w_k) \in \mathbb{N}^*$  of length  $|W| = k$  will be called a  $k$ -digit string alternatively. The concatenation of  $n$  consecutive  $W$  will be denoted by  $W^n$ .

**Definition 2.** The basic period of a word  $W = (w_1, \dots, w_k)$  is the integer

$$\text{per}(W) := \min\{p \leq k : w_{p+j} = w_j \text{ for } 1 \leq j \leq k-p\};$$

The basic factor of  $W$  is the word  $\widetilde{W} := (w_1, \dots, w_{\text{per}(W)})$ .

For example, let  $W = (1, 2, 3, 1, 2)$ , then  $\text{per}(W) = 3$ ,  $\widetilde{W} = (1, 2, 3)$ .

The following observation will be useful in the sequel.

**Lemma 7.7.** *Let  $W \in \mathbb{N}^*$  be a finite word with basic period  $p$  and basic factor  $\widetilde{W} := (w_1, \dots, w_p)$ . Then, for each  $n \geq 2$ , the number of occurrences of  $W$  in the  $pn$ -word  $(\widetilde{W})^n$  is  $n-1$  or  $n$  according as  $p < |W|$  or  $p = |W|$ . As a consequence,*

$$\lim_{n \rightarrow \infty} \frac{f(W, (\widetilde{W})^\infty, n)}{n} = 1/p.$$

Now let us recall that  $f(W, x, n)$  stands for the number of occurrences of  $W$  among the first  $n$  digits of the infinite words  $x \in \mathbb{N}^{\mathbb{N}}$ , that  $\text{Freq}(W, x)$  denotes the set of all accumulation points of  $\{f(W, x, n)/n : n \geq 1\}$ , and that

$$\text{Freq}(W) = \bigcup_{x \in \mathbb{N}^{\mathbb{N}}} \text{Freq}(W, x).$$

Since  $\lim_{n \rightarrow \infty} \left( \frac{f(W, x, n+1)}{n+1} - \frac{f(W, x, n)}{n} \right) = 0$ , one has

$$\text{Freq}(W, x) = \left[ \liminf_{n \rightarrow \infty} \frac{f(W, x, n)}{n}, \limsup_{n \rightarrow \infty} \frac{f(W, x, n)}{n} \right]. \quad (7.1)$$

As a corollary of this and Lemma 7.7, one has

**Corollary 7.8.** *For any  $W \in \mathbb{N}^*$  with basic period  $p$ ,  $\text{Freq}(W) = [0, 1/p]$ .*

Then we proceed to prove Theorem 7.3, namely,  $\mathbb{E} \subset \mathbb{F}$ . Let  $x \in \mathbb{E}$  which can be regarded as an infinite word  $(x_n)_{n=1}^\infty \in \mathbb{N}^{\mathbb{N}}$ . Recall that

$$\Pi_k(x, n) = \left( \frac{1}{n} f(W, x, n) \right)_{W \in \mathbb{N}^k},$$

that  $\mathbf{A}_k(x)$  denotes the set of accumulation points of the sequence  $\{\Pi_k(x, n)\}_{n=1}^\infty$  with respect to the 1-norm  $\|\cdot\|_1$ , and that  $\mathbf{A}_k(x) = \mathbf{S}_k$  for any  $x \in \mathbb{E}$ , where

$$\mathbf{S}_k := \left\{ \vec{p} \in \Delta_k : \sum_i p(iV) = \sum_i p(Vi) \text{ for all } V \in \mathbb{N}^{k-1} \right\}.$$

is the sub-simplex of shift invariant probability vectors.

Let  $V = (v_1, \dots, v_k) \in \mathbb{N}^k$  with basic period  $m$  (see Definition 2). By Corollary 7.8, to prove that  $x \in \mathbb{F}$ , it suffice to show that

$$\text{both of } 0 \text{ and } 1/m \text{ are limit points of the sequence } \left\{ \frac{f(V, x, n)}{n} \right\}_{n=1}^{\infty}.$$

For the first one, let  $d > \max\{v_i : 1 \leq i \leq k\}$  and  $\vec{p} \in \mathbf{S}_k$  with

$$p(\underbrace{d \cdots d}_{k \text{ times}}) = 1 \text{ and } p(W) = 0 \text{ for } W \neq \underbrace{d \cdots d}_{k \text{ times}}.$$

Given  $\varepsilon > 0$ , since  $\vec{p} \in \mathbf{A}_k(x)$ , we have  $\|\Pi_k(x, n) - \vec{p}\|_1 < \varepsilon$  for infinitely many  $n$ . It follows that  $\frac{f(V, x, n)}{n} < \varepsilon$  infinitely often. Therefore, 0 is a limit point. For the second one, we notice that  $y = (v_1, \dots, v_m)^\infty$  is a periodic point with minimal period  $m$  under the shift mapping  $T : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ . Let  $\mu$  be the periodic orbit measure (see [39] pp.195) which has mass  $1/m$  at each of the points  $\{y, Ty, \dots, T^{m-1}y\}$ . Then  $\mu$  is a  $T$ -invariant ergodic measure. It induces a shift-invariant probability vector

$$\vec{q} = (p(W))_{W \in \mathbb{N}^k} \in \mathbf{S}_k \text{ with } p(W) = \mu\{x : x_1 \cdots x_k = W\} \text{ for all } W \in \mathbb{N}^k.$$

In particular,  $p(V) = 1/m$ . Since  $\vec{q} \in \mathbf{A}_k(x)$ , we have  $\|\Pi_k(x, n) - \vec{q}\|_1 < \varepsilon$  for infinitely many  $n$ . It follows that  $|\frac{f(V, x, n)}{n} - \frac{1}{m}| < \varepsilon$  infinitely often. Therefore,  $1/m$  is a limit point. This completes the proof that  $x \in \mathbb{F}$ , and hence that  $\mathbb{E} \subset \mathbb{F}$ .

### 7.3 A “good seed” for $\mathbb{E}$

In this section, we shall verify the existence of a word  $z = (z_1, z_2, \dots) \in \mathbb{N}^{\mathbb{N}}$  such that

$$z = [z_1, z_2, \dots] \in \mathbb{E} \quad \text{and} \quad z_n \leq n \text{ for all } n \geq 1. \quad (7.2)$$

From this “good seed”, a Cantor-like subset of  $\mathbb{E}$  will be constructed in Section 5.

We shall use the terminology on combinatorics of words introduced in the last section. For a pair of finite words  $W = (w_1, \dots, w_k), V = (v_1, \dots, v_n) \in \mathbb{N}^*$  with  $k \leq n$ , we write

$$f(W, V) := \#\{1 \leq j \leq n - k + 1 : v_j = w_1, \dots, v_{j+k-1} = w_k\}$$

for the number of occurrences of  $W$  among  $V$ . Then for any  $x \in \mathbb{N}^{\mathbb{N}}$ , one has  $f(W, x, n) = f(W, (x_1, \dots, x_n))$ . A direct counting argument yields the following

**Lemma 7.9.** For any  $W, U, V \in \mathbb{N}^*$ ,

$$\left| \frac{f(W, UV)}{|U| + |V|} - \frac{f(W, V)}{|V|} \right| \leq \frac{|U|}{|U| + |V|}.$$

Recall that the vector of  $n$ -th asymptotic frequencies of the  $k$ -words occurring in  $x$  is denoted by

$$\Pi_k(x, n) := \left( \frac{1}{n} f(W, x, n) \right)_{W \in \mathbb{N}^k}.$$

For a finite word  $V$ , we use a similar notation  $\Pi_k(V)$  so that  $\Pi_k(x_1, \dots, x_n) = \Pi_k(x, n)$ . For ease of narration, we introduce the following

**Definition 3.** Given  $\varepsilon > 0$ , we say that a finite word  $V = (v_1, \dots, v_n)$  is  $\varepsilon$ -close to a vector  $\vec{p} \in \Delta_k$  if  $\|\Pi_k(V) - \vec{p}\|_1 < \varepsilon$ .

It is clear that the set of accumulation points of the sequence  $\{\Pi_k(x, n)\}_{n=1}^\infty$  can be expressed as

$$\mathbf{A}_k(x) = \{ \vec{p} \in \Delta_k : \forall \varepsilon > 0, (x_1, \dots, x_n) \text{ is } \varepsilon\text{-close to } \vec{p} \text{ for infinitely many } n \}.$$

Let us recall again that  $\mathbf{A}_k(x) = \mathbf{S}_k$  for any  $x \in \mathbb{E}$ , where

$$\mathbf{S}_k := \left\{ \vec{p} \in \Delta_k : \sum_i p(iV) = \sum_i p(Vi) \text{ for all } V \in \mathbb{N}^{k-1} \right\}$$

is the sub-simplex of shift invariant probability vectors.

For each  $d \geq 1$ , let  $\{1, \dots, d\}^*$  denote the collection of all finite words over the alphabet  $\{1, \dots, d\}$ . Consider the infinite lower triangular array of finite words

$$\begin{array}{l} W_1^{(1)}, \\ W_2^{(1)}, W_2^{(2)}, \\ W_3^{(1)}, W_3^{(2)}, W_3^{(3)}, \\ \dots \end{array}$$

where the  $d$ -th column consists of words from  $\{1, \dots, d\}^*$  for each  $d \geq 1$ . Juxtaposing the elements of the array row by row, we get an infinite word

$$z := W_1^{(1)} W_2^{(1)} W_2^{(2)} W_3^{(1)} W_3^{(2)} W_3^{(3)} \dots$$

In what follows, we'll specify the elements of the array, and show that the resulting  $z$  satisfies the condition (7.2) and hence is a “good seed” for  $\mathbb{E}$ .

We begin by defining, for each  $d \geq 1$ ,

$$\mathbf{S}_k^{(d)} := \{(p(W)) \in \mathbf{S}_k : p(W) > 0 \iff W \in \{1, \dots, d\}^k\}.$$

Suppose that  $\mathbf{S}^{(d)} = \left\{ \vec{p}_j^{(d)} \right\}_{j=1}^{\infty}$  is a dense subset of  $\bigcup_{k=1}^{\infty} \mathbf{S}_k^{(d)}$ . Since  $\bigcup_{d=1}^{\infty} \mathbf{S}_k^{(d)}$  is dense in  $\mathbf{S}_k$  and that  $\mathbf{A}_k(z)$  is a closed subset of  $\mathbf{S}_k$ . To verify that  $z = [z_1, z_2, \dots] \in \mathbb{E}$ , it suffices to show that for any  $d \geq 1$

$$\mathbf{S}^{(d)} \subset \mathbf{A}_k(z). \quad (7.3)$$

We need the following technical lemma which is similar to Lemma 2.1 in [113].

**Lemma 7.10.** *Let  $U \in \mathbb{N}^*$ ,  $\varepsilon > 0$  and  $\vec{p} \in \mathbf{S}_k^{(d)}$ . Then there exists a finite word  $V \in \{1, \dots, d\}^*$  such that the word  $UV$  is  $\varepsilon$ -close to  $\vec{p}$ .*

*Proof.* Let  $\vec{p} = (p(W))_{W \in \mathbb{N}^k}$ , then  $p(W) > 0$  if and only if  $W \in \{1, \dots, d\}^k$ . So we may identify  $\vec{p}$  with a shift-invariant probability vector over the symbolic space  $\{1, \dots, d\}^{\mathbb{N}}$  with index set  $\{1, \dots, d\}^k$ . By Theorem 2.3 in [113], there exists  $y \in \{1, \dots, d\}^{\mathbb{N}}$  such that

$$\lim_{n \rightarrow \infty} \|\Pi_k(y, n) - \vec{p}\|_1 = 0.$$

Let  $U = (u_1, \dots, u_m)$  and  $V = (y_1, \dots, y_n)$  with  $n$  being so large that

$$\frac{m^2 + md^k}{m+n} < \frac{\varepsilon}{2} \quad \text{and} \quad \|\Pi_k(y, n) - \vec{p}\|_1 < \frac{\varepsilon}{2}. \quad (7.4)$$

Denote by  $\mathcal{F}$  the collection of  $k$ -words that occur in  $UV$  but lie outside  $\{1, \dots, d\}^k$ , i.e.

$$\mathcal{F} = \{W \in \mathbb{N}^k \setminus \{1, \dots, d\}^k : W \text{ is a factor of } UV\}.$$

Since  $V \in \{1, \dots, d\}^n$ , we have

$$\#\mathcal{F} \leq m \text{ and } f(W, UV) \leq m \text{ for all } W \in \mathcal{F}.$$

In combination with Lemma 7.9 and (7.4), this yields

$$\begin{aligned} \|\Pi_k(UV) - \vec{p}\|_1 &= \sum_{W \in \{1, \dots, d\}^k} \left| \frac{1}{m+n} f(W, UV) - p(W) \right| + \sum_{W \in \mathcal{F}} \left| \frac{1}{m+n} f(W, UV) \right| \\ &\leq \sum_{W \in \{1, \dots, d\}^k} \left| \frac{1}{m+n} f(W, V) - p(W) \right| + \frac{md^k}{m+n} + \frac{m^2}{m+n} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This completes the proof of the lemma.  $\square$

Now we proceed to specify the elements of the array one by one. First, we observe that

$$\mathbf{S}_k^{(1)} = \{(p(W)) \in \mathbf{S}_k : p(1^k) = 1\} := \bar{p}_k^{(1)}$$

is a singleton. Let  $\mathbf{S}^{(1)} = \{\bar{p}_j^{(1)}\}_{j=1}^\infty$  and  $W_1^{(1)} = 1$ . It is clear that  $\|\Pi_1(1) - \bar{p}_1^{(1)}\|_1 = 0$ . Therefore,  $W_1^{(1)}$  is  $1/2$ -close to  $\bar{p}_1^{(1)}$ .

In general, for any  $d \geq 1$  and  $\ell \geq d$ , let  $U_\ell^{(d)}$  denote the prefix of  $z$  which is the concatenation of the words before  $W_\ell^{(d)}$  in the array. Suppose that  $U_\ell^{(d)}$  is determined. We proceed to specify the word  $W_\ell^{(d)} \in \{1, \dots, d\}^*$  which is at the  $\ell$ -th row and  $d$ -th column of the array. By Lemma 7.10, there exist  $V_i \in \{1, \dots, d\}^*$  such that  $U_\ell^{(d)} V_1 \cdots V_i$  is  $1/2^\ell$ -close to  $\bar{p}_i^{(d)}$  for  $1 \leq i \leq \ell - d + 1$ . Then we put  $W_\ell^{(d)} = V_1 \cdots V_{\ell-d+1}$ .

With this construction, it is clear that the relation (7.3) is satisfied and hence  $z \in \mathbb{E}$ . Now suppose that  $z_n$  lies in  $W_\ell^{(d)}$  with  $\ell \geq d$ . Then  $z_n \leq d \leq \ell \leq n$ . Therefore, the condition (7.2) is satisfied.

*Remark 6.* Let  $t_n$  be any real sequence satisfying  $\lim_{n \rightarrow \infty} t_n = +\infty$ , it is obvious from the construction that there exists a ‘‘seed’’  $z = (z_1, z_2, \dots) \in \mathbb{E}$  such that  $z_n \leq t_n$  for all but finitely many  $n$ 's.

## 7.4 Upper bound estimate

This section is devoted to the proof of Proposition 7.5. Let us fix some  $\ell \geq 1$  and denote by  $\mathbb{G}$  the set

$$\mathbb{G}(\ell) = \{x \in [0, 1) : \limsup_{n \rightarrow \infty} \frac{f(\ell, x, n)}{n} = 1\}.$$

To avoid complicated computation, we assume that  $\ell \geq 9$ . Now we are going to show that  $\dim(\mathbb{G}) \leq 1/2$ .

*Proof.* Fix  $0 < \varepsilon < 1$ . For each  $n \geq 1$ , put

$$A_n := \{(a_1, \dots, a_n) \in \mathbb{N}^n : \#\{j \leq n : a_j = \ell\} > (1 - \varepsilon)n\}.$$

By the definition of  $\mathbb{G}$ , we have

$$\mathbb{G} \subset \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{(a_1, \dots, a_n) \in A_n} I(a_1, \dots, a_n),$$

where we recall that  $I(a_1, \dots, a_n)$  is the rank  $n$  fundamental interval defined by  $I(a_1, \dots, a_n) := \{x \in [0, 1) : a_k(x) = a_k, 1 \leq k \leq n\}$ .

For any  $t > 1/2$ , let  $H^t$  signify the  $t$ -Hausdorff measure, then

$$\begin{aligned} H^t(\mathbb{G}) &\leq \liminf_{N \rightarrow \infty} \sum_{n=N}^{\infty} \sum_{(a_1, \dots, a_n) \in A_n} |I(a_1, \dots, a_n)|^t \\ &\leq \liminf_{N \rightarrow \infty} \sum_{n=N}^{\infty} \sum_{(a_1, \dots, a_n) \in A_n} \prod_{k=1}^n a_k^{-2t} \quad (\text{by Lemma 2.41}) \\ &= \liminf_{N \rightarrow \infty} \sum_{n=N}^{\infty} \sum_{(1-\varepsilon)n < m \leq n} \binom{n}{m} (\ell^{-2t})^m \left( \sum_{j \neq \ell} j^{-2t} \right)^{n-m}. \end{aligned}$$

Let  $t$  be close to  $1/2$  and then let  $\varepsilon$  be so small that

$$\ell^{-(1-\varepsilon)} \leq 1/8 \quad \text{and} \quad 1 < \left( \sum_{j \neq \ell} j^{-2t} \right)^\varepsilon < 2.$$

In combination with that  $\binom{n}{m} \leq 2^n$ , this implies

$$H^t(\mathbb{G}) \leq \liminf_{N \rightarrow \infty} \sum_{n=N}^{\infty} \varepsilon n 2^n 8^{-n} 2^n = 0.$$

Therefore,  $\dim(\mathbb{G}) \leq \frac{1}{2}$  as desired. □

## 7.5 Lower bound estimate

In order to get a lower bound on the Hausdorff dimension, we shall consider a class of Cantor sets which are interesting in themselves.

Let  $\alpha, \beta$  and  $\gamma$  be given with

$$0 < \alpha < \beta < \gamma < 1, \tag{7.5}$$

and  $\mathbb{M} = \{m_k : k \geq 1\}$  be a subsequence of  $\mathbb{N}$  such that

$$m_k = \lfloor k^{\gamma/\beta} \rfloor. \tag{7.6}$$

Here, as usual, the symbol “ $[t]$ ” means “the integer part of  $t$ ”.

Let  $z = [z_1, z_2, \dots, z_n, \dots] \in [0, 1) \cap \mathbb{Q}^c$  be a number such that

$$z_n \leq \exp(\exp(n^\alpha)) \text{ for all } n \geq 1. \tag{7.7}$$

and let  $\mathbb{B}(z)$  be the set of all  $x = [a_1(x), a_2(x), \dots] \in [0, 1)$  satisfying

$$a_n(x) = z_n \quad \text{if } n \notin \mathbb{M} \quad (7.8)$$

$$\lfloor \exp(\exp(k^\gamma)) \rfloor \leq a_n(x) < 2 \lfloor \exp(\exp(k^\gamma)) \rfloor \quad \text{if } n = m_k \in \mathbb{M}. \quad (7.9)$$

Since the set  $\mathbb{E}$  is “zero-density free” (see Definition 1) and contains a “seed” satisfying the condition (7.7), it contains the subset  $\mathbb{B}(z)$ . Therefore, the lower bound result is a consequence of Proposition 7.6, which, in turn, follows from the following

**Lemma 7.11.**  $\dim_H(\mathbb{B}(z)) \geq \frac{1}{2}$ .

*Proof.* We will construct a measure supported on  $\mathbb{B}(z)$  and then apply the mass distribution principle (see [46]) to get the dimension estimation.

Let

$$\mathbb{I}_n = \{I(a_1, \dots, a_n) : I(a_1, \dots, a_n) \cap \mathbb{B}(z) \neq \emptyset\}$$

and call the elements in  $\mathbb{I}_n$  admissible  $n$ -th order fundamental intervals.

We define a set function  $\mu$  on the collection of fundamental intervals as follows. Put  $\mu([0, 1)) = 1$ . If  $I(a_1, \dots, a_n) \notin \mathbb{I}_n$ , put

$$\mu(I(a_1, \dots, a_n)) = 0;$$

if  $I(a_1, \dots, a_n) \in \mathbb{I}_n$  and  $m_k \leq n < m_{k+1}$ , put

$$\mu(I(a_1, \dots, a_n)) = \prod_{j=1}^k \frac{1}{\lfloor \exp(\exp(j^\gamma)) \rfloor}.$$

By (7.8) and (7.9), the set function  $\mu$  is well-defined and can be extended to a unique probability measure supported on the set  $\mathbb{B}(z)$ .

Let  $x = [x_1, x_2, \dots] \in \mathbb{B}(z)$ . Denote by  $I_n(x)$  the rank  $n$  fundamental interval containing  $x$ . Consider the ball  $B(x, r)$  centered at  $x$  satisfying

$$|I_{n+1}(x)| \leq 3r < |I_n(x)| \quad \text{with } m_k \leq n < m_{k+1} \quad (7.10)$$

Since  $x \in I_{m_k}(x)$  and  $x_{m_k} \geq 2$ , by Lemma 2.42,  $B(x, r)$  is contained in the union of three adjacent  $m_k$ -rank fundamental intervals, namely

$$B(x, r) \subset I_{m_k}(x) \cup I'_{m_k}(x) \cup I''_{m_k}(x).$$

It follows that

$$\mu(B(x, r)) \leq 3 \prod_{j=1}^k \frac{1}{[\exp(\exp(j^\gamma))]}.$$

For convenience, we shall use the notation

$$f \sim g \iff \lim \frac{f}{g} = 1$$

in the sequel. On the one hand, one has

$$-\log(\mu(B(x, r))) \geq -\log 3 + \sum_{j=1}^k \log([\exp(\exp(j^\gamma))]) \sim \sum_{j=1}^k \exp(j^\gamma). \quad (7.11)$$

On the other hand, by (7.10), (2.41) and (7.8), one has

$$r \geq \frac{|I_{n+1}|}{3} \geq \frac{1}{2^{(2n+5)}} \left( \prod_{j=1}^{n+1} x_j \right)^{-2} \geq \frac{1}{2^{(3+2m_{k+1})}} \left( \prod_{j=1}^{m_{k+1}} z_j \right)^{-2} \left( \prod_{j=1}^{k+1} x_{m_j} \right)^{-2}.$$

In combination with (7.7), (7.9) and (7.6), this implies that

$$\begin{aligned} -\log r &\leq (3 + 2m_{k+1}) \log 2 + 2 \sum_{j=1}^{m_{k+1}} \exp(j^\alpha) + 2 \sum_{j=1}^{k+1} \log(2[\exp(\exp(j^\gamma))]) \\ &\sim 2 \sum_{j=1}^{\lfloor (k+1)^{\gamma/\beta} \rfloor} \exp(j^\alpha) + 2 \sum_{j=1}^{k+1} \exp(j^\gamma) \sim 2 \sum_{j=1}^{k+1} \exp(j^\gamma), \end{aligned}$$

where we have used the fact  $0 < \alpha < \beta < \gamma < 1$ . This and (7.11) together yield

$$\liminf_{r \rightarrow 0} \frac{\log(\mu(B(x, r)))}{\log r} \geq 1/2.$$

Since  $\mu$  is supported on  $\mathbb{B}(z)$ , one has  $\dim(\mathbb{B}(z)) \geq 1/2$  as desired.  $\square$

## Chapter 8

# The Frequency of Partial Quotients of Continued Fractions

In this chapter, we consider the frequency of partial quotients of continued fractions. The sets (Besicovitch-Eggleston sets) of points in  $[0, 1)$  with prescribed digit frequencies in their continued fraction expansions are studied. It is shown that the Hausdorff dimension of these sets, always bounded from below by  $1/2$ , are given by a modified variational principle.<sup>1</sup>

### 8.1 Introduction

Let  $\mathbb{Q}^c$  denote the set of irrational numbers. It is a well-known fact that each  $x \in [0, 1) \cap \mathbb{Q}^c$  possesses a unique continued fraction expansion of the form

$$x = \frac{1}{a_1(x) + \frac{1}{a_2(x) + \frac{1}{a_3(x) + \ddots}}}, \quad (8.1)$$

where  $a_k(x) \in \mathbb{N} =: \{1, 2, 3, \dots\}$  is called the  $k$ -th *partial quotients*. For each  $j \in \mathbb{N}$ , define the *frequency* of the digit  $j$  in the continued fraction expansion of  $x$  by

$$\tau_j(x) := \lim_{n \rightarrow \infty} \frac{\tau_j(x, n)}{n},$$

if the limit exists, where  $\tau_j(x, n) := \text{Card}\{k : a_k(x) = j, 1 \leq k \leq n\}$ .

This chapter is concerned with sets of real numbers with prescribed digit frequencies in their continued fraction expansions. To be precise, let  $\vec{p} = (p_1, p_2, \dots)$  be a probability vector with  $p_j \geq 0$  for all  $j \in \mathbb{N}$  and  $\sum_{j=1}^{\infty} p_j = 1$ , which will be called a *frequency vector* in the sequel. Our purpose is to determine the Hausdorff

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<sup>1</sup>A. H. Fan, L. M. Liao and J. H. Ma, *On the frequency of partial quotients of regular continued fractions*, preprint.

dimension of the set

$$\mathcal{E}_{\vec{p}} := \{x \in [0, 1) \cap \mathbb{Q}^c : \tau_j(x) = p_j \ \forall j \geq 1\}.$$

We begin with some notations. For any  $a_1, a_2, \dots, a_n \in \mathbb{N}$ , we call

$$I(a_1, a_2, \dots, a_n) := \{x \in [0, 1) : a_1(x) = a_1, a_2(x) = a_2, \dots, a_n(x) = a_n\}$$

a *rank  $n$  fundamental interval*. Let  $T : [0, 1) \rightarrow [0, 1)$  be the Gauss transformation defined by

$$T(0) = 0, \quad T(x) = 1/x \text{ for } x \in (0, 1).$$

For a given frequency vector  $\vec{p} = (p_1, p_2, \dots)$ , denote by  $\mathcal{N}(\vec{p})$  the set of  $T$ -invariant ergodic measures such that

$$\int |\log x| d\mu < \infty \text{ and } \mu(I(j)) = p_j \text{ for all } j \geq 1. \quad (8.2)$$

Let  $h_\mu$  stand for the measure-theoretical entropy of  $\mu$ , and  $\dim_H$  for the Hausdorff dimension. With the convention that  $\sup \emptyset = 0$ , the main result of this chapter can be stated as follows.

**Theorem 8.1.** *For any frequency vector  $\vec{p}$ , one has*

$$\dim_H(\mathcal{E}_{\vec{p}}) = \max \left\{ \frac{1}{2}, \sup_{\mu \in \mathcal{N}(\vec{p})} \frac{h_\mu}{2 \int |\log x| d\mu} \right\}.$$

Partial results of Theorem 8.1 were known long ago. In 1966, Kinney and Pitcher [88] proved that if  $\sum_{j=1}^{\infty} p_j \log j < \infty$  then

$$\dim_H \mathcal{E}_{\vec{p}} \geq \frac{-\sum_{j=1}^{\infty} p_j \log p_j}{2 \int |\log x| d\mu_{\vec{p}}},$$

where  $\mu_{\vec{p}}$  is the Bernoulli measure on  $[0, 1]$  defined by

$$\mu(I(a_1, a_2, \dots, a_n)) = \prod_{j=1}^n p_{a_j}.$$

This lower bound is in fact the Hausdorff dimension of the Bernoulli measure  $\mu_{\vec{p}}$ . However, by the result of Kifer, Peres and Weiss [87] in 2001, this lower bound is not optimal. Indeed, it is shown that

$$\dim_H \mu_{\vec{p}} \leq 1 - 10^{-7}$$

for any Bernoulli measure  $\mu_{\vec{p}}$ .

In 1975, Billingsley and Henningsen [17] proved that if  $\sum_{j=1}^{\infty} p_j \log j < \infty$  then

$$\dim_H(\mathcal{E}_{\vec{p}}) \geq \sup_{\mu \in \mathcal{N}(\vec{p})} \frac{h_{\mu}}{2 \int |\log x| d\mu}.$$

They also showed that this lower bound, which is a kind of variational principle, is just the exact Hausdorff dimension of the set  $\mathcal{E}_{\vec{p}}$  provided that all partial quotients are restricted to the set  $\{1, 2, \dots, N\}$  for some  $N \in \mathbb{N}$ . It is quite natural to guess that this should be true in the general case. However, as stated in Theorem 8.1, it is only a half part of the lower bound.

The other half of the lower bound viz.  $\dim_H(\mathcal{E}_{\vec{p}}) \geq 1/2$ , follows from Lemma 2.4 in [97]. However, we will give a direct proof.

The upper bound estimate is more difficult. In the proof, we will use the techniques of [97] and [17] in estimating the lengths of fundamental intervals. Not incidentally, an entropy-involved combinatorial lemma (Lemma 8.4) will play an important role.

This chapter is organized as follows. In Section 8.2, we give some preliminaries. In Section 8.3, we prove the upper bound of Theorem 8.1. In Section 8.4, we give a direct proof for  $\dim_H(\mathcal{E}_{\vec{p}}) \geq 1/2$  and show how we can drop the condition  $\sum_{j=1}^{\infty} p_j \log j < \infty$  in Billingsley and Henningsen's theorem and obtain the lower bound of Theorem 8.1.

## 8.2 Preliminary

In Section 2.4, we have given the basic properties of continued fractions. Here we want to discuss some estimates of the length of a rank  $n$  fundamental interval.

Recall  $p_n/q_n$  is the  $n$ -th convergent of continued fractions. Sometimes we denote by  $p_n(a_1, \dots, a_n)$  and  $q_n(a_1, \dots, a_n)$  the numerator and denominator of the  $n$ -th convergent of the points initiated with first  $n$  partial quotients  $a_1, \dots, a_n$ .

Recall that for any  $a_1, a_2, \dots, a_n \in \mathbb{N}$ ,

$$I(a_1, a_2, \dots, a_n) = \{x \in [0, 1) : a_1(x) = a_1, a_2(x) = a_2, \dots, a_n(x) = a_n\}$$

is a rank  $n$  fundamental interval.

We have

**Lemma 8.2.**

$$|I(x_1, \dots, j, \dots, x_n)| \leq \frac{8}{(j+1)^2} \left| I(x_1, \dots, \hat{j}, \dots, x_n) \right|,$$

where the notation  $\hat{j}$  means “deleting the digit  $j$ ”.

*Proof.* By (2.3) and Lemma 2.38, we have

$$\begin{aligned} |I(x_1, \dots, j, \dots, x_n)| &\leq \frac{1}{q_n^2(x_1, \dots, x_n)} \leq \left( \frac{2}{j+1} \right)^2 \frac{1}{q_{n-1}^2(x_1, \dots, \hat{j}, \dots, x_n)} \\ &\leq \frac{8}{(j+1)^2} \left| I(x_1, \dots, \hat{j}, \dots, x_n) \right|. \end{aligned}$$

□

For any  $x \in [0, 1] \setminus \mathbb{Q}$  and any word  $i_1 \cdots i_k \in \mathbb{N}^k$ , ( $k \geq 1$ ), denote by  $\tau_{i_1 \cdots i_k}(x, n)$  the number of  $j$ ,  $1 \leq j \leq n$ , for which

$$a_j(x) \cdots a_{j+k-1}(x) = i_1 \cdots i_k.$$

For  $N \in \mathbb{N}$ , define  $\Sigma_N := \{1, \dots, N\}$ . We shall use the following estimate in [17].

**Lemma 8.3** ([17]). *Let  $N \geq 1$  and  $n \geq 1$ . For any  $x = [x_1, x_2, \dots] \in [0, 1] \cap \mathbb{Q}^c$  with  $x_j \in \Sigma_N$  for  $1 \leq j \leq n$ . Then for any  $k \geq 1$ , we have*

$$\log |I_n(x)| \leq 2 \sum_{i_1 \cdots i_k \in \Sigma_N^k} \tau_{i_1 \cdots i_k}(x, n) \log \frac{p_k(i_1, \dots, i_k)}{q_k(i_1, \dots, i_k)} + 8 + \frac{8n}{2^k}. \quad (8.3)$$

Now we turn to the key combinatorial lemma which will be used in the upper bound estimate. Let  $\phi : [0, 1] \rightarrow \mathbb{R}$  denote the function

$$\phi(0) = 0, \quad \text{and} \quad \phi(t) = -t \log t \quad \text{for} \quad 0 < t \leq 1.$$

For every word  $\omega \in \Sigma_N^n$  of length  $n$  and every word  $u \in \Sigma_N^k$  of length  $k$ , denote by  $p(u|\omega)$  the frequency of appearances of  $u$  in  $\omega$ , i.e.

$$p(u|\omega) = \frac{\tau_u(\omega)}{n - k + 1},$$

where  $\tau_u(\omega)$  denote the number of  $j$ ,  $1 \leq j \leq n - k + 1$ , for which

$$\omega_j \cdots \omega_{j+k-1} = u.$$

Define

$$H_k(\omega) := \sum_{u \in \Sigma_N^k} \phi(p(u|\omega)).$$

We have the following counting lemma.

**Lemma 8.4** ([61]). *For any  $h > 0, \epsilon > 0$ , any  $k \in \mathbb{N}, k \geq 1$ , and any  $n \in \mathbb{N}$  large enough, we have*

$$\text{Card}\{\omega \in \Sigma_N^n : H_k(\omega) \leq kh\} \leq \exp(n(h + \epsilon)).$$

### 8.3 Upper bound

#### 8.3.1 Some Lemmas

Let  $(p(i_1, \dots, i_k))_{i_1 \dots i_k \in \mathbb{N}^k}$  be a probability vector with index set  $\mathbb{N}^k$ . Recall that  $q_k(i_1, \dots, i_k)$  is the denominator of the convergent of a real number with leading continued fraction digits  $i_1, \dots, i_k$ .

**Lemma 8.5.** *For any  $\epsilon > 0$ , there exists  $k \in \mathbb{N}$  such that for any probability vector  $(p(i_1, \dots, i_k))_{i_1 \dots i_k \in \mathbb{N}^k}$ , we have*

$$\begin{aligned} & -\frac{1}{k} \sum_{i_1 \dots i_k \in \mathbb{N}^k} p(i_1, \dots, i_k) \log p(i_1, \dots, i_k) \\ & \leq \frac{2}{k} \sum_{i_1 \dots i_k \in \mathbb{N}^k} p(i_1, \dots, i_k) \log q_k(i_1, \dots, i_k) + \epsilon. \end{aligned}$$

*Proof.* By Jensen's inequality, for any  $k \in \mathbb{N}$ , we have

$$\sum_{i_1 \dots i_k \in \mathbb{N}^k} p(i_1, \dots, i_k) \log \frac{q_k(i_1, \dots, i_k)^{-2}}{p(i_1, \dots, i_k)} \leq \log \sum_{i_1 \dots i_k \in \mathbb{N}^k} q_k(i_1, \dots, i_k)^{-2}.$$

By Lemma 2.39, we have

$$\sum_{i_1 \dots i_k \in \mathbb{N}^k} q_k(i_1, \dots, i_k)^{-2} \leq 2 \sum_{i_1 \dots i_k \in \mathbb{N}^k} |I(i_1 \dots i_k)| = 2.$$

Thus

$$\begin{aligned} & \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i_1 \dots i_k \in \mathbb{N}^k} p(i_1, \dots, i_k) \log \frac{q_k(i_1, \dots, i_k)^{-2}}{p(i_1, \dots, i_k)} \\ & \leq \lim_{k \rightarrow \infty} \frac{1}{k} \log \sum_{i_1 \dots i_k \in \mathbb{N}^k} q_k(i_1, \dots, i_k)^{-2} = 0, \end{aligned}$$

which completes the proof. □

**Lemma 8.6.** For any probability vector  $\vec{p} = (p_1, p_2, \dots)$  and any positive vector  $\vec{q} = (q_1, q_2, \dots)$ . Suppose  $-\sum_{j=1}^{\infty} p_j \log q_j = \infty$  and  $\sum_{j=1}^{\infty} q_j^s < \infty$  for some positive number  $s$ . Then

$$\limsup_{n \rightarrow \infty} \frac{-\sum_{j=1}^n p_j \log p_j}{-\sum_{j=1}^n p_j \log q_j} \leq s.$$

*Proof.* This is a consequence of the following inequality (see [140], p.217): for nonnegative numbers  $s_j$  ( $1 \leq j \leq m$ ) such that  $\sum_{j=1}^m s_j = 1$  and any real numbers  $t_j$  ( $1 \leq j \leq m$ ), we have

$$\sum_{j=1}^m s_j (t_j - \log s_j) \leq \log \left( \sum_{j=1}^m e^{t_j} \right). \quad (8.4)$$

For  $n \geq 1$ , put  $s_j = p_j$  for  $1 \leq j \leq n$  and  $s_{n+1} = \sum_{j=n+1}^{\infty} p_j$ . Take  $t_j = s \log q_j$  for  $1 \leq j \leq n$  and  $t_{n+1} = 0$ . Applying the above inequality (8.4) with  $m = n + 1$ , we get

$$s \sum_{j=1}^n p_j \log q_j - \sum_{j=1}^n p_j \log p_j - \left( \sum_{j=n+1}^{\infty} p_j \right) \log \left( \sum_{j=n+1}^{\infty} p_j \right) \leq \log \left( 1 + \sum_{j=1}^n q_j^s \right),$$

and consequently,

$$\frac{-\sum_{j=1}^n p_j \log p_j}{-\sum_{j=1}^n p_j \log q_j} \leq s + \frac{(\sum_{j=n+1}^{\infty} p_j) \log(\sum_{j=n+1}^{\infty} p_j)}{-\sum_{j=1}^n p_j \log q_j} + \frac{\log(1 + \sum_{j=1}^n q_j^s)}{-\sum_{j=1}^n p_j \log q_j}.$$

Using the facts  $-\sum_{j=1}^{\infty} p_j \log q_j = \infty$  and  $\sum_{j=1}^{\infty} q_j^s < \infty$ , we can finish the proof by letting  $n \rightarrow \infty$ .  $\square$

Lemma 8.6 implies the following Lemma. Recall that  $\Sigma_N^k = \{1, \dots, N\}^k$ .

**Lemma 8.7.** Let  $k \geq 1$  and  $(p(i_1, \dots, i_k))_{i_1, \dots, i_k \in \mathbb{N}^k}$  be a probability vector. If

$$\sum_{i_1, \dots, i_k \in \mathbb{N}^k} p(i_1, \dots, i_k) \log q_k(i_1, \dots, i_k) = \infty$$

then we have

$$\limsup_{N \rightarrow \infty} \frac{-\sum_{i_1, \dots, i_k \in \Sigma_N^k} p(i_1, \dots, i_k) \log p(i_1, \dots, i_k)}{2 \sum_{i_1, \dots, i_k \in \Sigma_N^k} p(i_1, \dots, i_k) \log q_k(i_1, \dots, i_k)} \leq \frac{1}{2}.$$

*Proof.* By Lemma 2.37, for any  $k \in \mathbb{N}$  and any  $s > 1/2$ , we have

$$\sum_{i_1, \dots, i_k} q_k(i_1, \dots, i_k)^{-2s} \leq \sum_{i_1, \dots, i_k} (i_1 \cdots i_k)^{-2s} = \left( \sum_{j=1}^{\infty} j^{-2s} \right)^k < \infty.$$

Thus we have the result by Lemma 8.6.  $\square$

### 8.3.2 Proof of the upper bound

To prove the upper bound, we shall make use of the multi-step Markov measures. Let  $k \geq 1$ , by a  $(k - 1)$ -step Markov measure, we mean a  $T$ -invariant probability measure  $P$  on  $[0, 1)$  satisfying the Markov property

$$\frac{P(I(a_1, \dots, a_n))}{P(I(a_1, \dots, a_{n-1}))} = \frac{P(I(a_{(n-k)}, \dots, a_n))}{P(I(a_{(n-k)}, \dots, a_{(n-1)}))} \quad (8.5)$$

for all  $n \geq 1$  and  $a_1, \dots, a_n \in \mathbb{N}$  (see [32] p.9). According to our conventions, we may regard Bernoulli measure as 0-step Markov measures.

Let us denote by  $\mathbb{P}_N^k = \mathbb{P}_N^k(\vec{p})$  the set of  $(k - 1)$ -step Markov measures satisfying

$$P(I(j)) = p_j \text{ for } 1 \leq j \leq N - 1 \text{ and } P(I(N)) = 1 - \sum_{j=1}^{N-1} p_j. \quad (8.6)$$

Put  $P([i_1, \dots, i_k]) = p(i_1, \dots, i_k)$  for all  $i_1 \cdots i_k \in \Sigma_N^k$  and

$$\beta_{N,k} := \sup_{P \in \mathbb{P}_N^k} \frac{-\frac{1}{k} \sum p(i_1, \dots, i_k) \log p(i_1, \dots, i_k)}{-2 \sum p(i_1, \dots, i_k) \log p_k(i_1, \dots, i_k)/q_k(i_1, \dots, i_k)}. \quad (8.7)$$

In [17], pp.171-172, the authors proved that the following limit exists

$$\beta_N := \lim_{k \rightarrow \infty} \beta_{N,k},$$

and it is equal to the following two limits which both exist:

$$\beta'_N := \lim_{k \rightarrow \infty} \sup_{P \in \mathbb{P}_N^k} \frac{-\sum p(i_1, \dots, i_k) \log p(i_1, \dots, i_k)}{2 \sum p(i_1, \dots, i_k) \log q_k(i_1, \dots, i_k)}$$

and

$$\beta''_N := \lim_{k \rightarrow \infty} \sup_{P \in \mathbb{P}_N^k} \frac{h_P}{2 \int |\log x| dP}.$$

Let

$$\beta := \limsup_{N \rightarrow \infty} \beta_N = \limsup_{N \rightarrow \infty} \beta'_N = \limsup_{N \rightarrow \infty} \beta''_N. \quad (8.8)$$

To show the upper bound, we need only to prove the following two propositions.

**Proposition 8.8.** *For any  $N \in \mathbb{N}$  large enough, we have*

$$\dim_H(\mathcal{E}_{\vec{p}}) \leq \max \left\{ \frac{1}{2}, \beta_N \right\},$$

**Proposition 8.9.** *We have*

$$\beta \leq \max \left\{ \frac{1}{2}, \sup_{\mu \in \mathcal{N}} \frac{h_\mu}{2 \int |\log x| d\mu} \right\}.$$

First, we prove Proposition 8.9.

*Proof of Proposition 8.9:* By (8.8),

$$\beta = \limsup_{N \rightarrow \infty} \lim_{k \rightarrow \infty} \sup_{P \in \mathbb{P}_N^k} \frac{-\sum p(i_1, \dots, i_k) \log p(i_1, \dots, i_k)}{2 \sum p(i_1, \dots, i_k) \log q_k(i_1, \dots, i_k)}.$$

If

$$\limsup_{N \rightarrow \infty} \lim_{k \rightarrow \infty} \sup_{P \in \mathbb{P}_N^k} 2 \sum p(i_1, \dots, i_k) \log q_k(i_1, \dots, i_k) = \infty,$$

then by Lemma 8.7, we have  $\beta \leq 1/2$ .

Now suppose that

$$\limsup_{N \rightarrow \infty} \lim_{k \rightarrow \infty} \sup_{P \in \mathbb{P}_N^k} 2 \sum p(i_1, \dots, i_k) \log q_k(i_1, \dots, i_k) < \infty,$$

which is equivalent to

$$\limsup_{N \rightarrow \infty} \lim_{k \rightarrow \infty} \sup_{P \in \mathbb{P}_N^k} \int |\log x| dP < \infty.$$

By (8.8),

$$\beta = \limsup_{N \rightarrow \infty} \lim_{k \rightarrow \infty} \sup_{P \in \mathbb{P}_N^k} \frac{h_P}{2 \int |\log x| dP}.$$

Without loss any generality, we suppose there is a sequences of measures  $P_{N,k} \in \mathbb{P}_N^k$  such that  $P_{N,k}$  converges weakly to a measure  $\mu \in \mathcal{N}(\vec{p})$  and

$$\beta = \limsup_{N \rightarrow \infty} \lim_{k \rightarrow \infty} \frac{h_{P_{N,k}}}{2 \int |\log x| dP_{N,k}}.$$

Then by the upper semi-continuous of the entropy function and the weak convergence, we have

$$\beta \leq \sup_{\mu \in \mathcal{N}(\vec{p})} \frac{h_\mu}{2 \int |\log x| d\mu}.$$

□

Now we are left to prove Proposition 8.8.

*Proof of Proposition 8.8:* For any fixed integer  $N$  which is large enough, and any  $\epsilon > 0$ , we have

$$E_{\vec{p}} \subset \bigcup_{\ell=1}^{\infty} \bigcap_{n=\ell}^{\infty} H_n(\epsilon, N),$$

where

$$H_n(\epsilon, N) := \left\{ x \in \mathbb{N}^{\mathbb{N}} := \left| \frac{\tau_j(x, n)}{n} - p_j \right| < \epsilon, 1 \leq j \leq N \right\}.$$

For any  $\gamma > \max\{1/2, \beta_N\}$ , and for any integer  $k \in \mathbb{N}$ , we have

$$\begin{aligned} & \mathcal{H}^{\gamma} \left( \bigcap_{n=\ell}^{\infty} H_n(\epsilon, N) \right) \\ & \leq \sum_{\substack{|\frac{\tau_j(x, n)}{n} - p_j| < \epsilon, 1 \leq j \leq N}} |I_n(x)|^{\gamma} \quad (\forall n \geq \ell) \\ & = \sum_{n(p_j - \epsilon) < m_j < n(p_j + \epsilon), 1 \leq j \leq N} \sum_{x_1 \cdots x_n \in A} |I(x_1, \dots, x_n)|^{\gamma}, \end{aligned}$$

where  $A := \{x_1 \cdots x_n \in \Sigma_N^n : \tau_j(x_1 \cdots x_n) = m_j, 1 \leq j \leq N\}$ . (We recall that the notion of  $\tau_j(x_1 \cdots x_n)$  denotes the times of the appearances of  $j$  in  $x_1 \cdots x_n$ .)

Let  $\tilde{n} := \sum_{j=1}^N m_j$ . By Lemma 8.2, we have the following estimate by deleting the digits  $j > N$  in the first  $n$  partial quotients  $x_1, \dots, x_n$  of  $x \in I_n(x)$ .

$$\sum_{x_1 \cdots x_n \in A} |I(x_1, \dots, x_n)|^{\gamma} \leq \left( \sum_{j=N+1}^{\infty} \frac{8}{(j+1)^{2\gamma}} \right)^{n-\tilde{n}} \sum_{x_1 \cdots x_{\tilde{n}} \in \tilde{A}} |I(x_1, \dots, x_{\tilde{n}})|^{\gamma},$$

where  $\tilde{A} := \{x_1 \cdots x_{\tilde{n}} \in \Sigma_N^{\tilde{n}} : \tau_j(x_1 \cdots x_{\tilde{n}}) = m_j, 1 \leq j \leq N\}$ . Since  $\gamma > 1/2$ , the term

$$\sum_{j=N+1}^{\infty} \frac{8}{(j+1)^{2\gamma}} \leq M,$$

where  $M$  is a constant.

By applying Lemma 8.3, and noticing that  $\tau_{i_1 \dots i_k}(x, \tilde{n}) \leq \tau_{i_1 \dots i_k}(x_1 \cdots x_{\tilde{n}}) + k$ , we have

$$\begin{aligned} & |I_{\tilde{n}}(x)| = \exp\{\log |I_{\tilde{n}}(x)|\} \\ & \leq \exp \left\{ 2 \sum_{i_1 \cdots i_k \in \Sigma_N^k} (\tau_{i_1 \dots i_k}(x_1 \cdots x_{\tilde{n}}) + k) \log \frac{p_k(i_1, \dots, i_k)}{q_k(i_1, \dots, i_k)} + 8 + \frac{8\tilde{n}}{2^k} \right\}. \end{aligned}$$

Thus

$$\begin{aligned} & \sum_{x_1 \cdots x_{\tilde{n}} \in \tilde{A}} |I(x_1, \cdots, x_{\tilde{n}})|^\gamma \\ & \leq \sum_{m_{i_1 \cdots i_k}} \sum_{x_1 \cdots x_{\tilde{n}} \in B} \exp \left\{ 2\gamma \sum_{i_1 \cdots i_k \in \Sigma_N^k} (m_{i_1 \cdots i_k} + k) \log \frac{p_k}{q_k} + 8\gamma + \frac{8\tilde{n}\gamma}{2^k} \right\}, \end{aligned}$$

where  $B := \{x_1 \cdots x_{\tilde{n}} \in \tilde{A} : \tau_{i_1 \cdots i_k}(x_1 \cdots x_{\tilde{n}}) = m_{i_1 \cdots i_k} \forall i_1 \cdots i_k \in \Sigma_N^k\}$ .

Take

$$h = \frac{1}{k} \sum_{i_1 \cdots i_k \in \mathbb{N}^k} \phi \left( \frac{m_{i_1 \cdots i_k}}{\tilde{n} - k + 1} \right) \quad (8.9)$$

in Lemma 8.4. We have for any  $\delta > 0$ , for  $\tilde{n}$  large enough

$$\begin{aligned} & \sum_{x_1 \cdots x_{\tilde{n}} \in B} \exp \left\{ 2\gamma \sum_{i_1 \cdots i_k \in \Sigma_N^k} (m_{i_1 \cdots i_k} + k) \log \frac{p_k}{q_k} + 8\gamma + \frac{8\tilde{n}\gamma}{2^k} \right\} \\ & \leq \exp \left\{ \tilde{n}(h + \delta) + 2\gamma \sum_{i_1 \cdots i_k \in \Sigma_N^k} (m_{i_1 \cdots i_k} + k) \log \frac{p_k}{q_k} + 8\gamma + \frac{8\tilde{n}\gamma}{2^k} \right\}. \end{aligned}$$

Rewrite the right side of the above inequality as

$$\exp \{ \tilde{n} (L(\gamma, k, m_{i_1 \cdots i_k})) \},$$

where

$$L(\gamma, k, m_{i_1 \cdots i_k}) := h + 2\gamma \sum_{i_1 \cdots i_k \in \Sigma_N^k} \frac{m_{i_1 \cdots i_k} + k}{\tilde{n}} \log \frac{p_k}{q_k} + \frac{8\gamma}{\tilde{n}} + \frac{8\gamma}{2^k} + \delta.$$

Since there are at most  $(\tilde{n} - k + 1)^{N^k}$  possible words of  $i_1 \cdots i_k$  in  $\Sigma_N^{\tilde{n}}$ , we have

$$\begin{aligned} & \sum_{x_1 \cdots x_{\tilde{n}} \in \tilde{A}} |I(x_1, \cdots, x_{\tilde{n}})|^\gamma \\ & \leq (\tilde{n} - k + 1)^{N^k} \exp \left\{ \tilde{n} \left( \sup_{m_{i_1 \cdots i_k}} L(\gamma, k, m_{i_1 \cdots i_k}) \right) \right\}. \end{aligned}$$

Notice that by the definition of  $\tilde{A}$  and  $B$ , the possible values of  $m_{i_1 \cdots i_k}$  are restricted to satisfy the condition that the frequency of digit  $j$  in  $x_1 \cdots x_{\tilde{n}}$  is about  $p_j$ . The vector

$$\left( \frac{m_{i_1 \cdots i_k}}{\tilde{n} - k + 1} \right)_{i_1 \cdots i_k}$$

then approximated to a probability measure  $P$  in  $\mathbb{P}_N^k$ .

Since  $\gamma > \beta_N$ , by (8.7) and (8.9), we can choose first  $\tilde{n}$  large enough then  $k$  large enough such that

$$\sup_{m_{i_1 \dots i_k}} L(\gamma, k, m_{i_1 \dots i_k}) < 0.$$

Hence finally we can obtain for any  $\gamma > \max\{1/2, \beta_N\}$ ,

$$\mathcal{H}^\gamma \left( \bigcap_{n=k}^{\infty} H_n(\epsilon, N) \right) < \infty.$$

This completes the proof of Proposition 8.8.  $\square$

At the end of this section, we would like to remark that by Proposition 8.8 and Lemma 8.5, we have  $\dim_H(\mathcal{E}_p) \leq 1$ .

## 8.4 Lower bound

In this section we first prove  $\dim_H(\mathcal{E}_{\bar{p}}) \geq 1/2$ , and then show how to delete the condition  $\sum_{j=1}^{\infty} p_j \log j < \infty$  in Billingsley and Henningsen's theorem to obtain the lower bound of Theorem 8.1.

The following lemma is important to prove  $\dim_H(\mathcal{E}_{\bar{p}}) \geq 1/2$ .

**Lemma 8.10.** *For any given sequence of positive integers  $\{c_n\}_{n \geq 1}$  tending to the infinity, there exists a point  $z = (z_1, z_2, \dots) \in \mathcal{E}_{\bar{p}}$  such that  $z_n \leq c_n$  for all  $n \geq 1$ .*

*Proof.* For any  $n \geq 1$ , we construct a probability vector  $(p_1^{(n)}, p_2^{(n)}, \dots, p_k^{(n)}, \dots)$  such that  $p_k^{(n)} > 0$  for all  $1 \leq k \leq c_n$  and  $\sum_{k=1}^{c_n} p_k^{(n)} = 1$ , and that for any  $k \geq 1$ ,

$$\lim_{n \rightarrow \infty} p_k^{(n)} = p_k. \quad (8.10)$$

Consider a product Bernoulli probability  $\mathbb{P}$  supported by  $\prod_{n=1}^{\infty} \{1, \dots, c_n\}$ . For each digit  $k \geq 1$ , consider the random variables of  $x \in \mathbb{N}^{\mathbb{N}}$ ,  $X_n(x) = \mathbb{1}_{\{k\}}(x_n)$  ( $n \geq 1$ ). By Kolmogorov's strong law of large numbers (see [134] p.388), we have for each digit  $k$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left( \sum_{i=1}^n \mathbb{1}_{\{k\}}(x_i) - \sum_{i=1}^n \mathbb{E}(\mathbb{1}_{\{k\}}(x_i)) \right) = 0 \quad \mathbb{P} - a.s.,$$

which implies

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{k\}}(x_i) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n p_k^{(i)} = p_k \quad \mathbb{P} - a.s.. \quad (8.11)$$

That is to say, for  $\mathbb{P}$  almost every point in the space  $\prod_{n=1}^{\infty} \{1, \dots, c_n\}$ , the digit  $k$  has the frequency  $p_k$ . Considering each point in  $\mathbb{N}^{\mathbb{N}}$  as a continued fraction expansion of a number in  $[0, 1]$ , we complete the proof.  $\square$

We begin to show  $\dim_H(\mathcal{E}_{\bar{p}}) \geq 1/2$ .

*Proof of  $\dim_H(\mathcal{E}_{\bar{p}}) \geq 1/2$ :* Take  $c_n = n$  in Lemma 8.10, we find a point  $z \in \mathcal{E}_{\bar{p}}$ , such that

$$z_n = a_n(z) \leq n \quad (\forall n \geq 1). \quad (8.12)$$

For a positive number  $b > 1$ , set

$$\mathcal{F}_z(b) := \{x \in [0, 1) : a_{k^2}(x) \in (b^{k^2}, 2b^{k^2}]; \quad a_k(x) = a_k(z) \text{ if } k \text{ is nonsquare}\}.$$

It is clear that  $\mathcal{F}_z(b) \subset \mathcal{E}_{\bar{p}}$  for all  $b > 1$ . We define a measure  $\mu$  on  $\mathcal{F}_z(b)$ . For  $n^2 \leq m < (n+1)^2$ , set

$$\mu(I_m(x)) = \prod_{k=1}^n \frac{1}{b^{k^2}}. \quad (8.13)$$

Denote by  $B(x, r)$  the ball centered at  $x$  with radius  $r$ . We will show that for any  $\theta > 0$ , there exists  $b > 1$ , such that for all  $x \in \mathcal{F}_z(b)$ ,

$$\liminf_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} \geq \frac{1}{2} - \theta. \quad (8.14)$$

In fact, for any positive number  $r$ , there exist integers  $m$  and  $n$  such that

$$|I_{m+1}(x)| < 3r \leq |I_m(x)| \quad \text{and} \quad n^2 \leq m < (n+1)^2. \quad (8.15)$$

By the construction of  $\mathcal{F}_z(b)$ ,  $a_{n^2}(x) > b^{n^2} > 1$ . Let  $x = [x_1, x_2, \dots]$ . By Lemma 2.42,  $B(x, r)$  is covered by the union of three adjacent rank  $n^2$  fundamental intervals, i.e.,

$$B(x, r) \subset I(x_1, x_2, \dots, x_{n^2} - 1) \cup I(x_1, x_2, \dots, x_{n^2}) \cup I(x_1, x_2, \dots, x_{n^2} + 1).$$

By the definition of  $\mu$ , the above three intervals admit the same measure. Hence by (8.15), we have

$$\frac{\log \mu(B(x, r))}{\log r} \geq \frac{\log 3\mu(I(x_1, x_2, \dots, x_{n^2}))}{\log \frac{1}{3}|I(x_1, x_2, \dots, x_{m+1})|}. \quad (8.16)$$

However, on the one hand, by (8.13)

$$-\log \mu(I(x_1, x_2, \dots, x_{n^2})) = -\log \prod_{k=1}^n \frac{1}{b^{k^2}} = \sum_{k=1}^n k^2 \log b.$$

On the other hand, by (2.3) and Lemma 2.37, we have

$$-\log |I(x_1, x_2, \dots, x_{m+1})| \leq \log 2 + \sum_{k=1}^{m+1} 2 \log(x_k + 1).$$

Let us estimate the second term of sum. First we have

$$\sum_{k=1}^{m+1} 2 \log(x_k + 1) \leq 2 \sum_{k=1}^{n+1} \log(x_{k^2} + 1) + 2 \sum_{k=1}^{m+1} \log(z_k + 1).$$

Since  $x_{k^2} \leq 2b^{k^2}$  for all  $k \geq 1$ , we deduce

$$\begin{aligned} \sum_{k=1}^{m+1} \log(x_{k^2} + 1) &\leq \sum_{k=1}^{n+1} \log(2b^{k^2} + 1) \leq \sum_{k=1}^{n+1} \log(3b^{k^2}) \\ &= (n+1) \log 3 + \sum_{k=1}^{n+1} k^2 \log b. \end{aligned}$$

By (8.12),  $z_n \leq n$ , for all  $n \geq 1$ , we know

$$\sum_{k=1}^{m+1} \log(z_k + 1) \leq \sum_{k=1}^{(n+1)^2} \log(k + 1).$$

Thus for any  $\theta > 0$ , take  $b > 1$  large enough, we have for all  $x \in \mathcal{F}_z(b)$

$$\liminf_{n \rightarrow \infty} \frac{\log \mu(I(x_1, x_2, \dots, x_{n^2}))}{\log |I(x_1, x_2, \dots, x_{m+1})|} \geq \frac{1}{2} - \theta.$$

Hence by (8.15) and (8.16), we obtain (8.14).

Therefore,

$$\dim_H(\mathcal{E}_{\bar{p}}) \geq \frac{1}{2}.$$

□

Now to complete the whole proof of Theorem 8.1, we are left to show how to cancel the condition  $\sum_{j=1}^{\infty} p_j \log j < \infty$  in Billingsley and Henningsen's theorem.

To this end, first notice that if  $\sum_{j=1}^{\infty} p_j \log j = \infty$ , then for any  $\mu \in \mathcal{M}_{\text{erg}}$  such that  $\mu(I(j)) = p_j$  for all  $j \geq 1$ , we have

$$\int \log x d\mu \geq \sum_{j=1}^{\infty} \mu(I(j)) \log j = \sum_{j=1}^{\infty} p_j \log j = \infty.$$

Thus  $\mathcal{N}(\vec{p}) = \emptyset$ . By the convention we have

$$\sup_{\mu \in \mathcal{N}} \frac{h_{\mu}}{2 \int |\log x| d\mu} = 0.$$

However, by Propositions 8.8 and 8.9, we have  $\dim_H(\mathcal{E}_{\vec{p}}) \leq 1/2$ . Since we always have  $\dim_H(\mathcal{E}_{\vec{p}}) \geq 1/2$ , we have finally proved that if  $\sum_{j=1}^{\infty} p_j \log j = \infty$ , then

$$\dim_H(\mathcal{E}_{\vec{p}}) = \frac{1}{2}.$$

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