

SCUOLA NORMALE SUPERIORE DI PISA

Tesi di Perfezionamento in Matematica

**Kolmogorov operators  
in spaces of continuous functions  
and equations for measures**

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- ▶ Introduction and setting of the problem

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- ▶ Contents of the Thesis

We consider the stochastic differential equation in separable Hilbert space  $H$  (norm  $|\cdot|$  and inner product  $\langle \cdot, \cdot \rangle$ )

$$\begin{cases} dX(t) = (AX(t) + F(X(t)))dt + BdW(t), & t \geq 0 \\ X(0) = x \in H, \end{cases} \quad (1)$$

where

- ▶  $A : D(A) \subset H \rightarrow H$  is the infinitesimal generator of a strongly continuous semigroup  $e^{tA}$ ,  $t \geq 0$ ;
- ▶  $F : D(F) \subset H \rightarrow H$  is nonlinear;
- ▶  $B : H \rightarrow H$  is linear and continuous;
- ▶  $(W(t))_{t \geq 0}$  is a cylindrical Wiener process, defined on a stochastic basis  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  and with values in  $H$ .

Assume that problem (1) has a solution  $X(t, x)$ , and consider the corresponding transition semigroup

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t, x))], \quad t \geq 0, x \in H$$

where  $\varphi : H \rightarrow \mathbb{R}$  is a suitable function.

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The function  $u(t, x) := P_t\varphi(x)$  is formally the solution of

$$D_t u(t, x) = K_0 u(t, x), \quad u(0, x) = \varphi(x),$$

where  $K_0$  is the *Kolmogorov* differential operator

$$K_0\varphi(x) = \frac{1}{2}\text{Tr}[BB^*D^2\varphi(x)] + \langle Ax + F(x), D\varphi(x) \rangle, \quad x \in H, \quad (2)$$

and  $B^*$  is the adjoint of  $B$ . The expression (2) is formal: it requires that  $\varphi$  is of class  $C^2$ , that  $BB^*D^2\varphi(x)$  is a trace class operator and that  $x \in D(A) \cap D(F)$ . So, it is convenient to define  $K_0$  in a suitable domain  $D(K_0)$ .

The main object of this Thesis consists in studying the relationships between the solution of the stochastic equation

$$\begin{cases} dX(t) = (AX(t) + F(X(t)))dt + BdW(t), & t \geq 0 \\ X(0) = x \in H, \end{cases}$$

and the Kolmogorov differential operator

$$K_0\varphi(x) = \frac{1}{2}\text{Tr}[BB^*D^2\varphi(x)] + \langle Ax + F(x), D\varphi(x) \rangle$$

in *spaces of continuous functions*.

For  $k \geq 0$  we consider the space  $C_{b,k}(H)$  of all continuous mappings  $\varphi : H \rightarrow \mathbb{R}$  such that

$$x \rightarrow \mathbb{R}, \quad x \mapsto \frac{\varphi(x)}{1 + |x|^k}$$

is uniformly continuous and

$$\|\varphi\|_{0,k} := \sup_{x \in H} \frac{|\varphi(x)|}{1 + |x|^k} < \infty.$$

We shall write  $C_b(H) := C_{b,0}(H)$ .

Under suitable conditions the semigroup  $P_t$ ,  $t \geq 0$  acts on  $C_{b,k}(H)$ .



# The abstract operator

$P_t$  is not a strongly continuous semigroup on  $C_{b,k}(H)$  with respect to the norm  $\|\cdot\|_{0,k}$ .

We follow the approach of  $\pi$ -convergence introduced in [Priola, 1999]. In this approach we define the infinitesimal generator  $K$  of  $P_t$  in the space  $C_{b,k}(H)$  as follows

$$K\varphi(x) = \lim_{t \rightarrow 0^+} \frac{P_t\varphi(x) - \varphi(x)}{t}, \quad \varphi \in D(K), x \in H,$$

where

$$D(K) = \left\{ \varphi \in C_{b,k}(H) : \sup_{t \in (0,1)} \left\| \frac{P_t\varphi - \varphi}{t} \right\|_{0,k} < \infty \text{ and} \right. \\ \left. \exists g \in C_{b,k}(H) : \lim_{t \rightarrow 0^+} \frac{P_t\varphi(x) - \varphi(x)}{t} = g(x), \quad \forall x \in H \right\}$$

The goal of this thesis is twofold.

- ▶ We show that the ‘abstract’ operator  $K$  is the closure (with respect to a suitable topology) of the ‘concrete’ Kolmogorov differential operator

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In other words, we characterize  $K$  on a *core*  $D(K_0) \subset D(K)$ , where  $K$  and  $K_0$  coincide.

- ▶ We study the following equation for measures

$$\frac{d}{dt} \int_H \varphi(x) \mu_t(dx) = \int_H K_0\varphi(x) \mu_t(dx), \quad \forall \varphi \in D(K_0), t \geq 0,$$

where the unknown is the family of measures  $\mu_t$ ,  $t \geq 0$  on  $H$  and  $\mu_0$  is given in advance.

Chapter 1 is devoted to introduce

- ▶ notations;
- ▶ gaussian measures on separable Hilbert spaces;
- ▶ regularity properties of the stochastic convolution.

In Chapter 2 we consider an abstract situation, where a general Markov semigroup  $P_t$ ,  $t \geq 0$  acts on the uniformly continuous and bounded functions  $C_b(E)$  on a Banach space  $E$ .

We study the semigroup and its infinitesimal generator in the framework of the  $\pi$ -convergence, as introduced in [Priola, 1999].

Finally, we solve the equation for measures

$$\frac{d}{dt} \int_E \varphi(x) \mu_t(dx) = \int_E K\varphi(x) \mu_t(dx), \quad \forall \varphi \in D(K), t \geq 0,$$

where  $(K, D(K))$  is the infinitesimal generator of  $P_t$  in the space  $C_b(E)$ .

We show that for any finite Borel measure  $\mu$  on  $E$  there exists a unique family of finite Borel measures  $\mu_t$ ,  $t \geq 0$  fulfilling

$$\int_0^T |\mu_t|_{TV}(E) dt < \infty, \quad \text{for any } T > 0;$$

$$\int_E \varphi(x) \mu_t(dx) - \int_E \varphi(x) \mu(dx) = \int_0^t \left( \int_E K\varphi(x) \mu_s(dx) \right) ds,$$

for any  $\varphi \in D(K)$ ,  $t \geq 0$ .

The solution is given by the measures  $P_t^* \mu$ ,  $t \geq 0$ , which are defined by

$$\int_E \varphi(x) P_t^* \mu(dx) = \int_E P_t \varphi(x) \mu(dx), \quad \text{for any } \varphi \in C_b(E).$$

In Chapter 3 we consider the case  $F = 0$ , that is the equation

$$\begin{cases} dX(t) = AX(t)dt + BdW(t), & t \geq 0 \\ X(0) = x \in H, \end{cases}$$

where

- ▶  $A : D(A) \subset H \rightarrow H$  is the infinitesimal generator of a strongly continuous semigroup  $e^{tA}$ ,  $t \geq 0$ ;
- ▶  $B : H \rightarrow H$  is linear and continuous;
- ▶ the linear operator  $Q_t$ , defined by

$$Q_t x = \int_0^t e^{sA} B B^* e^{sA^*} x \, ds, \quad x \in H, \quad t \geq 0$$

has finite trace.



In this case the associated transition semigroup reduces to the Ornstein-Uhlenbeck semigroup  $R_t$ ,  $t \geq 0$  given by

$$R_t \varphi(x) = \int_H \varphi(e^{tA}x + y) N_{Q_t}(dy), \quad \varphi \in C_b(H), \quad t \geq 0, \quad x \in H,$$

where  $N_{Q_t}$ ,  $t > 0$  is the Gaussian measure on  $H$  of zero mean and covariance operator  $Q_t$ .

We denote by  $(L, D(L))$  the infinitesimal generator of the OU semigroup in  $C_b(H)$ .

Let  $\mathcal{I}_A(H)$  be the linear span of the real and imaginary part of the functions

$$H \rightarrow \mathbb{C}, \quad x \mapsto \int_0^a e^{i\langle e^{sA}x, h \rangle - \frac{1}{2}\langle Q_s h, h \rangle} ds : a > 0, h \in D(A^*),$$

where  $D(A^*)$  is the domain of the adjoint operator of  $A$ .

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where  $D(A^*)$  is the domain of the adjoint operator of  $A$ .

We show that the set  $\mathcal{I}_A(H)$  is a core for the infinitesimal operator  $L$  and that

$$L\varphi(x) = \frac{1}{2}\text{Tr}[BB^*D^2\varphi(x)] + \langle x, A^*D\varphi(x) \rangle, \quad x \in H$$

for any  $\varphi \in \mathcal{I}_A(H)$ .

We also show that for any finite Borel measure  $\mu$  on  $H$  there exists a unique family of measures  $\mu_t$ ,  $t \geq 0$  such that

$$\int_0^T |\mu_t|(H) dt < \infty, \quad \text{for any } T > 0$$

and

$$\int_H \varphi(x) \mu_t(dx) - \int_H \varphi(x) \mu(dx) = \int_0^t \left( \int_H L_0 \varphi(x) \mu_s(dx) \right) ds,$$

$t \geq 0$ ,  $\varphi \in \mathcal{I}_A(H)$ , where

$$L_0 \varphi(x) = \frac{1}{2} \text{Tr}[BB^* D^2 \varphi(x)] + \langle x, A^* D \varphi(x) \rangle, \quad \varphi \in \mathcal{I}_A(H), x \in H.$$

Finally, the solution is given by  $\mu_t = R_t^* \mu$ .

In Chapter 4, we consider the case

$$\begin{cases} dX(t) = (AX(t) + F(X(t)))dt + BdW(t), & t \geq 0 \\ X(0) = x \in H, \end{cases}$$

where

- ▶  $A, B$  are as in Chapter 3;
- ▶  $F : H \rightarrow H$  is Lipschitz continuous and bounded.

There exists a unique solution  $X(t, x)$  of the equation. We denote by  $P_t, t \geq 0$  the associated transition semigroup and by  $(K, D(K))$  the infinitesimal generator on  $P_t, t \geq 0$  in  $C_b(H)$ .

We show that  $(K, D(K))$  is the closure in  $C_b(H)$  of

$$K_0\varphi(x) = \frac{1}{2}\text{Tr}[BB^*D^2\varphi(x)] + \langle x, A^*D\varphi(x) \rangle + \langle F(x), D\varphi(x) \rangle,$$

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$$\int_0^T |\mu_t|(H) dt < \infty, \quad \text{for any } T > 0$$

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$t \geq 0$ ,  $\varphi \in \mathcal{I}_A(H)$ . Finally, the solution is given by  $P_t^*\mu$ ,  $t \geq 0$ .

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$t \geq 0$ ,  $\varphi \in \mathcal{I}_A(H)$ . Finally, the solution is given by  $P_t^*\mu$ ,  $t \geq 0$ .

The results of chapters 2, 3, 4 are contained in the paper

[M., *Kolmogorov equations for measures*, to appear on J.Evol.Eq.]



In Chapter 5, we consider the case

$$\begin{cases} dX(t) = (AX(t) + F(X(t)))dt + BdW(t), & t \geq 0 \\ X(0) = x \in H, \end{cases}$$

where

- ▶  $A, B$  are as in Chapter 3;
- ▶  $F : H \rightarrow H$  is Lipschitz continuous with globally Lipschitz constant.

We consider the associated transition semigroup  $P_t, t \geq 0$  in  $C_{b,1}(H)$ .

Let us denote by  $(K, D(K))$  the infinitesimal generator of  $P_t, t \geq 0$  in  $C_{b,1}(H)$ .

We show that  $(K, D(K))$  is the closure in  $C_{b,1}(H)$  of the operator

$$K_0\varphi(x) = \frac{1}{2}\text{Tr}[BB^*D^2\varphi(x)] + \langle x, A^*D\varphi(x) \rangle + \langle F(x), D\varphi(x) \rangle,$$

$$\varphi \in \mathcal{E}_A(H), x \in H$$

The set of *exponential functions*  $\mathcal{E}_A(H)$  consists of the linear span of the real and imaginary part of the functions

$$H \rightarrow \mathbb{C}, \quad x \mapsto e^{i\langle x, h \rangle}, \quad h \in D(A^*),$$

where  $D(A^*)$  is the domain of the adjoint operator of  $A$ .

We also show that for any finite Borel measure  $\mu$  on  $H$  such that

$$\int_H |x| |\mu|_{TV}(dx) < \infty.$$

there exists a unique family of measures  $\mu_t$ ,  $t \geq 0$  satisfying

$$\int_0^T \left( \int_H (1 + |x|) |\mu_t|_{TV}(dx) \right) dt < \infty, \quad \forall T > 0$$

and

$$\int_H \varphi(x) \mu_t(dx) - \int_H \varphi(x) \mu(dx) = \int_0^t \left( \int_H K_0 \varphi(x) \mu_s(dx) \right) ds$$

for any  $t \geq 0$ ,  $\varphi \in \mathcal{E}_A(H)$ . The solution is given by  $P_t^* \mu$ ,  $t \geq 0$ .

In Chapter 6 we consider reaction diffusion equations of the form

$$\begin{cases} dX(t, \xi) = [\Delta_{\xi} X(t, \xi) + \lambda X(t, \xi) - p(X(t, \xi))]dt + BdW(t, \xi), \\ X(t, \xi) = 0, \quad t \geq 0, \xi \in \partial\mathcal{O}, \\ X(0, \xi) = x(\xi), \quad \xi \in \mathcal{O}, x \in H, \end{cases}$$

where  $\Delta_{\xi}$  is the Laplace operator,  $B \in \mathcal{L}(H)$  and  $p$  is an increasing polynomial with leading coefficient of odd degree  $d > 1$ . Here  $\mathcal{O} = [0, 1]^n$  and  $H = L^2(\mathcal{O})$ .

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where  $\Delta_{\xi}$  is the Laplace operator,  $B \in \mathcal{L}(H)$  and  $p$  is an increasing polynomial with leading coefficient of odd degree  $d > 1$ . Here  $\mathcal{O} = [0, 1]^n$  and  $H = L^2(\mathcal{O})$ .

The transition semigroup  $P_t$ ,  $t \geq 0$  acts on the space  $C_{b,d}(L^{2d}(\mathcal{O}))$  of all continuous functions  $\varphi : L^{2d}(\mathcal{O}) \rightarrow \mathbb{R}$  such that

$$\sup_{x \in L^{2d}(\mathcal{O})} \frac{|\varphi(x)|}{1 + |x|_{L^{2d}(\mathcal{O})}^d} < \infty.$$

Set

$$Ax = \Delta_{\xi}x, \quad x \in D(A) = H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}).$$

We show that the associated infinitesimal generator  $K$  is the closure in  $C_{b,d}(L^{2d}(\mathcal{O}))$  of

$$K_0\varphi(x) = \frac{1}{2}\text{Tr}[BB^*D^2\varphi(x)] + \langle x, AD\varphi(x) \rangle + \langle D\varphi(x), F(x) \rangle,$$
$$\varphi \in \mathcal{E}_A(H), \quad x \in L^{2d}(\mathcal{O}).$$

We show that for any finite Borel measure  $\mu$  on  $H$  such that

$$\int_H |x|_{L^{2d}(\mathcal{O})}^d |\mu|_{TV}(dx) < \infty$$

there exists a unique family of measures  $\mu_t$ ,  $t \geq 0$  satisfying

$$\int_0^T \left( \int_H |x|_{L^{2d}(\mathcal{O})}^d |\mu_t|_{TV}(dx) \right) dt < \infty, \quad \forall T > 0$$

and

$$\int_H \varphi(x) \mu_t(dx) - \int_H \varphi(x) \mu(dx) = \int_0^t \left( \int_H K_0 \varphi(x) \mu_s(dx) \right) ds,$$

for any  $\varphi \in \mathcal{E}_A(H)$ ,  $t \geq 0$ .

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for any  $\varphi \in \mathcal{E}_A(H)$ ,  $t \geq 0$ .

The results of chapters 5, 6 are contained in the submitted paper [M., *Measure-valued equations for Kolmogorov operators with unbounded coefficients*].



In Chapter 7 we consider the stochastic Burgers equation in the interval  $[0, 1]$  with Dirichlet boundary conditions perturbed by a space-time white noise

$$\begin{cases} dX = \left( D_{\xi}^2 X + \frac{1}{2} D_{\xi}(X^2) \right) dt + dW(t), & \xi \in [0, 1], t \geq 0, \\ X(t, 0) = X(t, 1) = 0 \\ X(0, \xi) = x(\xi), \quad \xi \in [0, 1], \end{cases}$$

where  $x \in L^2(0, 1)$ .

Existence and uniqueness of a solution has been proved in [Da Prato, Debussche, Temam, 1994].

In the paper [Da Prato, Debussche, 2006] has been shown several estimates on the solution.

The transition semigroup acts on the space  $C_{b,V}(L^6(0,1))$ , which consists of all the continuous functions  $\varphi : L^6(0,1) \rightarrow \mathbb{R}$  such that

$$\sup_{x \in L^6(0,1)} \frac{|\varphi(x)|}{1 + V(x)} < \infty$$

where

$$V(x) = |x|_{L^6(0,1)}^8 |x|_{L^4(0,1)}^2, \quad x \in L^6(0,1).$$

Let us denote by  $(K, D(K))$  the infinitesimal generator of  $P_t$ ,  $t \geq 0$  in  $\mathcal{C}_{b,V}(L^6(0, 1))$ .

We prove that  $K$  is the closure in  $\mathcal{C}_{b,V}(L^6(0, 1))$  of

$$K_0\varphi(x) = \frac{1}{2}\text{Tr}[D^2\varphi(x)] + \langle x, AD\varphi(x) \rangle - \frac{1}{2}\langle D_\xi D\varphi(x), x^2 \rangle,$$

$x \in L^6(0, 1)$ ,  $\varphi \in \mathcal{E}_A(H)$ , where

$$Ax = D_\xi^2 x, \quad x \in D(A) = H_0^1(0, 1) \cap H^2(0, 1)$$

We show that for any finite Borel measure  $\mu$  on  $L^6(0,1)$  such that

$$\int_{L^6(0,1)} V(x)|\mu|_{TV}(dx) < \infty$$

there exists a unique family of Borel measures  $\mu_t$ ,  $t \geq 0$  fulfilling

$$\int_0^T \left( \int_{L^6(0,1)} V(x)|\mu_t|_{TV}(dx) \right) dt < \infty, \quad \forall T > 0$$

and

$$\begin{aligned} \int_{L^6(0,1)} \varphi(x)\mu_t(dx) - \int_{L^6(0,1)} \varphi(x)\mu(dx) \\ = \int_0^t \left( \int_{L^6(0,1)} K_0\varphi(x)\mu_s(dx) \right) ds, \end{aligned}$$

$t \geq 0$ ,  $\varphi \in \mathcal{E}_A(H)$ . The solution is given by  $P_t^*\mu$ ,  $t \geq 0$ .

# The $\pi$ -convergence

A sequence  $(\varphi_n)_{n \in \mathbb{N}} \subset C_b(E)$  is said to be  $\pi$ -convergent to a function  $\varphi \in C_b(E)$  and we write

$$\lim_{n \rightarrow \infty} \varphi_n \stackrel{\pi}{=} \varphi$$

if

$$\lim_{n \rightarrow \infty} \varphi_n(x) = \varphi(x), \quad \text{for any } x \in E$$

and

$$\sup_{n \in \mathbb{N}} \|\varphi_n\|_0 < \infty.$$

The  $m$ -indexed sequence  $(\varphi_{n_1, \dots, n_m})_{n_1 \in \mathbb{N}, \dots, n_m \in \mathbb{N}} \subset C_b(E)$  is said to be  $\pi$ -convergent to  $\varphi \in C_b(E)$  if for any  $i \in \{2, \dots, m\}$  there exists an  $i - 1$ -indexed sequence  $(\varphi_{n_1, \dots, n_{i-1}})_{n_1 \in \mathbb{N}, \dots, n_{i-1} \in \mathbb{N}} \subset C_b(E)$  such that

$$\lim_{n_i \rightarrow \infty} \varphi_{n_1, \dots, n_i} \stackrel{\pi}{=} \varphi_{n_1, \dots, n_{i-1}}$$

and

$$\lim_{n_1 \rightarrow \infty} \varphi_{n_1} \stackrel{\pi}{=} \varphi.$$