

# Robust Control and Observation of LPV Time-Delay Systems



C. Briat

PhD. defense, November 27th 2008

GIPSA-lab, Control Systems Department, Grenoble, France

Committee:

- Rapporteurs: [Sophie Tarbouriech](#) (Directeur de Recherche CNRS, LAAS, Toulouse)  
[Jean-Pierre Richard](#) (Professor, Ecole Centrale Lille)  
[Silviu-Iulian Niculescu](#) (Directeur de Recherche CNRS, LSS, Gif-sur-Yvette)
- Examineurs: [Erik I. Verriest](#) (Professor, Georgia Institute of Technology, USA)  
[Andrea Garulli](#) (Professor, Universita' degli Studi di Siena)
- Co-directeurs: [Olivier Sename](#) (Professor INPG, GIPSA-lab)  
[Jean-François Lafay](#) (Professor, Ecole Centrale Nantes)



## Administrative Context

### 3 years PhD thesis

- ▶ Advisors :
  - ▶ Olivier Sename (GIPSA-Lab)
  - ▶ Jean-François Lafay (IRCCyN)
- ▶ 6 months journey in GeorgiaTech (Rhone-Alpes Region scholarship)
  - ▶ Work with Erik Verriest
  - ▶ "Modeling and Control of Disease Epidemics by Vaccination"



## Scientific Context

Nonlinear System

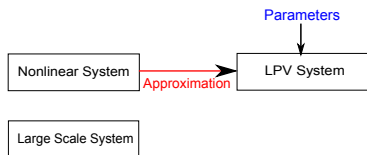


## Scientific Context



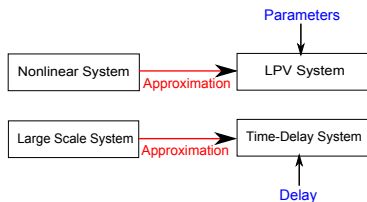


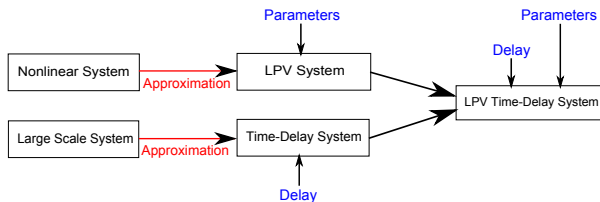
## Scientific Context





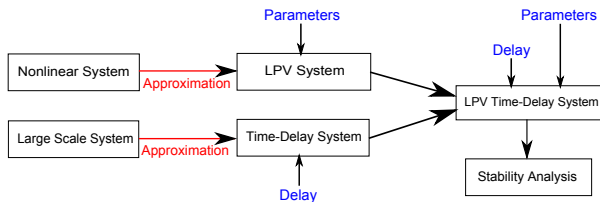
## Scientific Context



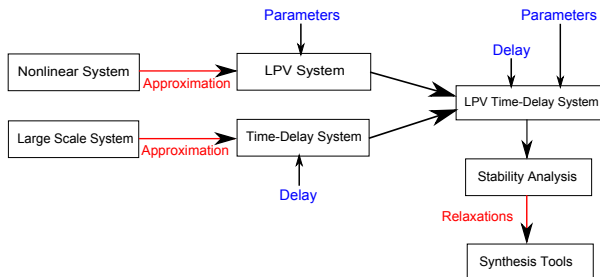


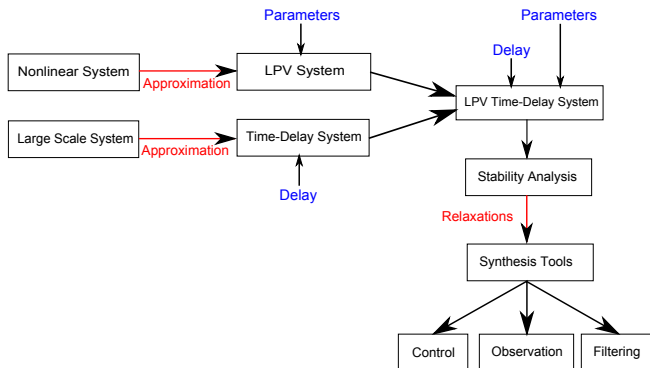


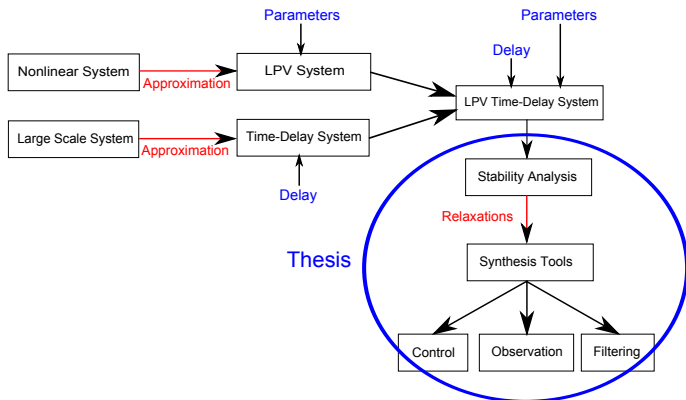
## Scientific Context













## Contributions of the Thesis

### Stability Results

- ▶ Conference publications [IFAC World Congress '08], [ECC07]
- ▶ Journal submissions IEEE TAC, Systems & Control Letters

### Design Methods

- ▶ Conference publications [IFAC World Congress '08], [ECC07], [IFAC SSSC'07]
- ▶ Conference submissions [ECC'09]
- ▶ Journal submissions IEEE TAC, Systems & Control Letters

### Modeling and Control of Disease Epidemics

- ▶ Conference publication [IFAC World Congress '08]
- ▶ Journal Submission [Biomedical Signal Processing and Control]



## Outline

1. Introduction
2. Stability of LPV Time-Delay Systems
3. Control of LPV Time-Delay Systems
4. Conclusion & Future Works



## Outline

### 1. Introduction

- ▶ Presentation of LPV systems
- ▶ Stability Analysis of LPV systems
- ▶ Control of LPV systems
- ▶ Presentation of time-delay systems
- ▶ Stability Analysis of time-delay systems
- ▶ Control of time-delay systems

### 2. Stability of LPV Time-Delay Systems

### 3. Control of LPV Time-Delay Systems

### 4. Conclusion & Future Works



## LPV Systems

- ▶ General expression [Packard 1993, Apkarian 1995, 1998]

$$\begin{cases} \dot{x}(t) &= A(\rho(t))x(t) + E(\rho(t))w(t) \\ \rho(t) &\in U_\rho \text{ compact} \\ \dot{\rho}(t) &\in co\{U_\nu\} \end{cases}$$

- + Approximation of nonlinear and LTV systems
- + Offer interesting solutions for control → gain scheduling
- + Semi-active suspensions [Poussot 2008], robotic systems [Kajiwara 1999], turbo-fan engines [Gilbert 2008], and so on. . .
- Eigenvalues computation of  $A(\rho)$



## LPV Systems

- ▶ General expression [Packard 1993, Apkarian 1995, 1998]

$$\begin{cases} \dot{x}(t) &= A(\rho(t))x(t) + E(\rho(t))w(t) \\ \rho(t) &\in U_\rho \text{ compact} \\ \dot{\rho}(t) &\in co\{U_\nu\} \end{cases}$$

- + Approximation of nonlinear and LTV systems
- + Offer interesting solutions for control → gain scheduling
- + Semi-active suspensions [Poussot 2008], robotic systems [Kajiwara 1999], turbo-fan engines [Gilbert 2008], and so on. . .
- Eigenvalues computation of  $A(\rho)$





## LPV Systems

- ▶ General expression [Packard 1993, Apkarian 1995, 1998]

$$\begin{cases} \dot{x}(t) &= A(\rho(t))x(t) + E(\rho(t))w(t) \\ \rho(t) &\in U_\rho \text{ compact} \\ \dot{\rho}(t) &\in co\{U_\nu\} \end{cases}$$

- + Approximation of nonlinear and LTV systems
- + Offer interesting solutions for control → gain scheduling
- + Semi-active suspensions [Poussot 2008], robotic systems [Kajiwara 1999], turbo-fan engines [Gilbert 2008], and so on. . .
- Eigenvalues computation of  $A(\rho)$



## Stability Analysis of LPV Systems

### Time vs. Frequency Domain Methods

- ▶ Frequency domain analysis 'inapplicable'
- ▶ Time Domain analysis → Lyapunov theory for LPV systems

$$V_q(x) = x^T P x(t) \quad V_r(x) = x^T P(\rho) x(t)$$

### Quadratic vs. Robust Stability

- ▶ Quadratic stability
  - ▶ Unbounded parameter variation rates  $\dot{\rho} \in (-\infty, +\infty)$
  - ▶ Necessary Condition :  $\text{Re}[\lambda(A(\rho))] < 0, \rho \in U_\rho$
- ▶ Robust stability
  - ▶ Bounded parameter variation rates
  - ▶ Necessary and sufficient condition :  $\text{Re}[\lambda(A(\rho))] < 0, \rho \in U_\rho$



## Stability Analysis of LPV Systems

### Time vs. Frequency Domain Methods

- ▶ Frequency domain analysis 'inapplicable'
- ▶ Time Domain analysis → Lyapunov theory for LPV systems

$$V_q(x) = x^T P x(t) \quad V_r(x) = x^T P(\rho) x(t)$$

### Quadratic vs. Robust Stability

- ▶ **Quadratic** stability
  - ▶ Unbounded parameter variation rates  $\dot{\rho} \in (-\infty, +\infty)$
  - ▶ Necessary Condition :  $\text{Re}[\lambda(A(\rho))] < 0, \rho \in U_\rho$
- ▶ **Robust** stability
  - ▶ Bounded parameter variation rates
  - ▶ Necessary and sufficient condition :  $\text{Re}[\lambda(A(\rho))] < 0, \rho \in U_\rho$



## Stability Analysis of LPV Systems

### Time vs. Frequency Domain Methods

- ▶ Frequency domain analysis 'inapplicable'
- ▶ Time Domain analysis → Lyapunov theory for LPV systems

$$V_q(x) = x^T \mathbf{P} x(t) \quad V_r(x) = x^T \mathbf{P}(\rho) x(t)$$

### Quadratic vs. Robust Stability

- ▶ **Quadratic** stability
  - ▶ Unbounded parameter variation rates  $\dot{\rho} \in (-\infty, +\infty)$
  - ▶ Necessary Condition :  $\text{Re}[\lambda(A(\rho))] < 0, \rho \in U_\rho$
- ▶ **Robust** stability
  - ▶ Bounded parameter variation rates
  - ▶ Necessary and sufficient condition :  $\text{Re}[\lambda(A(\rho))] < 0, \rho \in U_\rho$



## Stability Analysis of LPV Systems

### Time vs. Frequency Domain Methods

- ▶ Frequency domain analysis 'inapplicable'
- ▶ Time Domain analysis → Lyapunov theory for LPV systems

$$V_q(x) = x^T \mathbf{P} x(t) \quad V_r(x) = x^T \mathbf{P}(\rho) x(t)$$

### Quadratic vs. Robust Stability

- ▶ **Quadratic** stability
  - ▶ Unbounded parameter variation rates  $\dot{\rho} \in (-\infty, +\infty)$
  - ▶ Necessary Condition :  $\text{Re}[\lambda(A(\rho))] < 0, \rho \in U_\rho$
- ▶ Robust stability
  - ▶ Bounded parameter variation rates
  - ▶ Necessary and sufficient condition :  $\text{Re}[\lambda(A(\rho))] < 0, \rho \in U_\rho$



# Stability Analysis of LPV Systems



## Time vs. Frequency Domain Methods

- ▶ Frequency domain analysis 'inapplicable'
- ▶ Time Domain analysis → Lyapunov theory for LPV systems

$$V_q(x) = x^T \mathbf{P} x(t) \quad V_r(x) = x^T \mathbf{P}(\rho) x(t)$$

## Quadratic vs. Robust Stability

- ▶ **Quadratic** stability
  - ▶ Unbounded parameter variation rates  $\dot{\rho} \in (-\infty, +\infty)$
  - ▶ Necessary Condition :  $\mathbf{Re}[\lambda(A(\rho))] < 0, \rho \in U_\rho$
- ▶ **Robust** stability
  - ▶ Bounded parameter variation rates
  - ▶ Necessary and sufficient condition :  $\mathbf{Re}[\lambda(A(\rho))] < 0, \rho \in U_\rho$



## Control of LPV Systems

### Types of Controllers

	State-Feedback	Dynamic Output Feedback
Robust Controller	$u(t) = Kx(t)$	$\begin{bmatrix} \dot{x}_c(t) \\ u(t) \end{bmatrix} = K \begin{bmatrix} x_c(t) \\ y(t) \end{bmatrix}$
Gain-Scheduled Controller	$u(t) = K(\rho)x(t)$	$\begin{bmatrix} \dot{x}_c(t) \\ u(t) \end{bmatrix} = K(\rho) \begin{bmatrix} x_c(t) \\ y(t) \end{bmatrix}$

### Advantages and Drawbacks of LPV Controllers

- + Flexibility
- + Better performance
- Computation
- Implementation



## Control of LPV Systems

### Types of Controllers

	State-Feedback	Dynamic Output Feedback
Robust Controller	$u(t) = Kx(t)$	$\begin{bmatrix} \dot{x}_c(t) \\ u(t) \end{bmatrix} = K \begin{bmatrix} x_c(t) \\ y(t) \end{bmatrix}$
Gain-Scheduled Controller	$u(t) = K(\rho)x(t)$	$\begin{bmatrix} \dot{x}_c(t) \\ u(t) \end{bmatrix} = K(\rho) \begin{bmatrix} x_c(t) \\ y(t) \end{bmatrix}$

### Advantages and Drawbacks of LPV Controllers

- + Flexibility
- + Better performance
- Computation
- Implementation





## Control of LPV Systems

### Types of Controllers

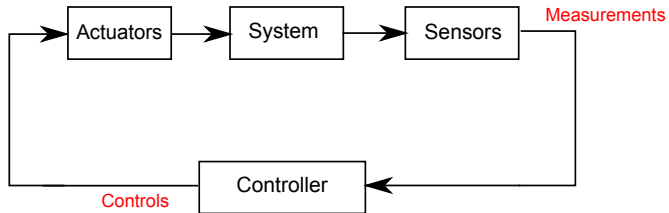
	State-Feedback	Dynamic Output Feedback
Robust Controller	$u(t) = Kx(t)$	$\begin{bmatrix} \dot{x}_c(t) \\ u(t) \end{bmatrix} = K \begin{bmatrix} x_c(t) \\ y(t) \end{bmatrix}$
Gain-Scheduled Controller	$u(t) = K(\rho)x(t)$	$\begin{bmatrix} \dot{x}_c(t) \\ u(t) \end{bmatrix} = K(\rho) \begin{bmatrix} x_c(t) \\ y(t) \end{bmatrix}$

### Advantages and Drawbacks of LPV Controllers

- + Flexibility
- + Better performance
- Computation
- Implementation

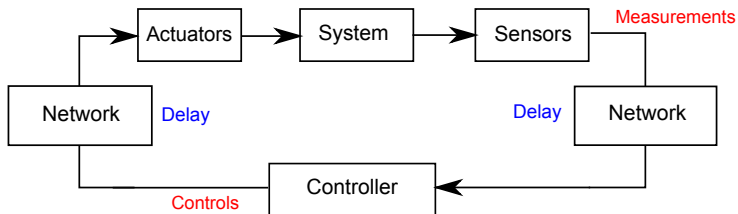


# Time-Delay Systems





# Time-Delay Systems





# Linear Time-Delay Systems

## General Expression

$$\left\{ \begin{array}{l} \dot{x}(t) \\ h(t) \\ h(t) \\ \frac{dh(t)}{dt} \end{array} \right. = \begin{array}{l} Ax(t) + \mathbf{A}_h \mathbf{x}(t - h(t)) + Ew(t) \\ \text{constant/time - varying} \\ \text{bounded/unbounded} \\ \text{bounded/unbounded} \end{array}$$

- + Approximation of systems with propagation, diffusion or memory phenomena
  - ▶ Networks, combustion processes, population growth, disease propagation, price fluctuations. . .
- Infinite number of eigenvalues
- Depend on the delay value



## Linear Time-Delay Systems

### General Expression

$$\left\{ \begin{array}{l} \dot{x}(t) \\ h(t) \\ h(t) \\ \frac{dh(t)}{dt} \end{array} \right. = \begin{array}{l} Ax(t) + \mathbf{A}_h \mathbf{x}(t - h(t)) + Ew(t) \\ \text{constant/time - varying} \\ \text{bounded/unbounded} \\ \text{bounded/unbounded} \end{array}$$

- + Approximation of systems with propagation, diffusion or memory phenomena
  - ▶ Networks, combustion processes, population growth, disease propagation, price fluctuations. . .
- Infinite number of eigenvalues
- Depend on the delay value



# Stability Analysis of Time-Delay Systems (1)

## Two notions of stability

- ▶ Delay-independent stability  $\rightarrow$  unbounded delay
- ▶ Delay-dependent stability  $\rightarrow$  bounded delay

## Frequency Domain Methods [Niculescu 2001, Gu 2003]

- ▶ LTI systems
- ▶ Constant delays

## Time-domain Methods [Fridman 2001, Gu 2003, Gouaisbaut 2006]

- ▶ LTV, LPV and Nonlinear systems
- ▶ Time-varying delays



# Stability Analysis of Time-Delay Systems (1)

## Two notions of stability

- ▶ Delay-independent stability  $\rightarrow$  unbounded delay
- ▶ Delay-dependent stability  $\rightarrow$  bounded delay

## Frequency Domain Methods [Niculescu 2001, Gu 2003]

- ▶ LTI systems
- ▶ Constant delays

## Time-domain Methods [Fridman 2001, Gu 2003, Gouaisbaut 2006]

- ▶ LTV, LPV and Nonlinear systems
- ▶ Time-varying delays



# Stability Analysis of Time-Delay Systems (1)

## Two notions of stability

- ▶ Delay-independent stability  $\rightarrow$  unbounded delay
- ▶ Delay-dependent stability  $\rightarrow$  bounded delay

## Frequency Domain Methods [Niculescu 2001, Gu 2003]

- ▶ LTI systems
- ▶ Constant delays

## Time-domain Methods [Fridman 2001, Gu 2003, Gouaisbaut 2006]

- ▶ LTV, **LPV** and Nonlinear systems
- ▶ Time-varying delays





## Stability Analysis of Time-Delay Systems (2)

### Extension of Lyapunov Theory

- ▶ Lyapunov-Krasovskii & Lyapunov-Razumikhin
- ▶ Large bestiary of Lyapunov-Krasovskii functionals [Fridman 2001, Han 2002, Gu 2003]

### Example

- ▶ Delay-independent stability [Verriest 1991, Bliman 2000, Gu 2003]

$$V_i = x(t)^T P x(t) + \int_{t-h}^t x(\theta)^T Q x(\theta) d\theta$$

- ▶ Delay-dependent stability [Han 2005, Gouaisbaut 2006]

$$V_d = V_i + \int_{-h}^0 \int_{t+\theta}^t \dot{x}(\eta)^T R \dot{x}(\eta) d\eta d\theta$$



## Stability Analysis of Time-Delay Systems (2)



### Extension of Lyapunov Theory

- ▶ Lyapunov-Krasovskii & Lyapunov-Razumikhin
- ▶ Large bestiary of Lyapunov-Krasovskii functionals [Fridman 2001, Han 2002, Gu 2003]

### Example

- ▶ Delay-independent stability [Verriest 1991, Bliman 2000, Gu 2003]

$$V_i = x(t)^T P x(t) + \int_{t-h}^t x(\theta)^T Q x(\theta) d\theta$$

- ▶ Delay-dependent stability [Han 2005, Gouaisbaut 2006]

$$V_d = V_i + \int_{-h}^0 \int_{t+\theta}^t \dot{x}(\eta)^T R \dot{x}(\eta) d\eta d\theta$$



## Control of Time-Delay Systems

### Controllers

	no memory		with memory
State Feedback	$u(t) = K_0 x(t)$	+	$K_h x(t-h)$
Dynamic Output Feedback	$\begin{bmatrix} \dot{x}_c(t) \\ u(t) \end{bmatrix} = K_0 \begin{bmatrix} x_c(t) \\ y(t) \end{bmatrix}$	+	$K_h x_c(t-h)$

### Advantages & Drawbacks of Memory Controllers

- + Flexibility
- + Better performances
- Needs more memory
- Delay supposed to be exactly known
- Problem of delay measurement/estimation [Belkoura]

Robust controllers with uncertain delay ?



## Control of Time-Delay Systems

### Controllers

	no memory		with memory
State Feedback	$u(t) = K_0 x(t)$	+	$K_h x(t - h)$
Dynamic Output Feedback	$\begin{bmatrix} \dot{x}_c(t) \\ u(t) \end{bmatrix} = K_0 \begin{bmatrix} x_c(t) \\ y(t) \end{bmatrix}$	+	$K_h x_c(t - h)$

### Advantages & Drawbacks of Memory Controllers

- + Flexibility
- + Better performances
- Needs more memory
- Delay supposed to be exactly known
- Problem of delay measurement/estimation [Belkoura]

Robust controllers with uncertain delay ?



## Control of Time-Delay Systems

### Controllers

	no memory		with memory
State Feedback	$u(t) = K_0 x(t)$	+	$K_h x(t - h)$
Dynamic Output Feedback	$\begin{bmatrix} \dot{x}_c(t) \\ u(t) \end{bmatrix} = K_0 \begin{bmatrix} x_c(t) \\ y(t) \end{bmatrix}$	+	$K_h x_c(t - h)$

### Advantages & Drawbacks of Memory Controllers

- + Flexibility
- + Better performances
- Needs more memory
- Delay supposed to be exactly known
- Problem of delay measurement/estimation [Belkoura]

Robust controllers with uncertain delay ?



## Control of Time-Delay Systems

### Controllers

	no memory		with memory
State Feedback	$u(t) = K_0 x(t)$	+	$K_h x(t - h)$
Dynamic Output Feedback	$\begin{bmatrix} \dot{x}_c(t) \\ u(t) \end{bmatrix} = K_0 \begin{bmatrix} x_c(t) \\ y(t) \end{bmatrix}$	+	$K_h x_c(t - h)$

### Advantages & Drawbacks of Memory Controllers

- + Flexibility
- + Better performances
- Needs more memory
- Delay supposed to be exactly known
- Problem of delay measurement/estimation [Belkoura]

Robust controllers with uncertain delay ?



## Delay-robust control of uncertain LPV time-delay systems

$$u(t) = K(\rho)x(t) + K_h(\rho)x(t - d(t))$$

- ▶ Choice of Lyapunov-Krasovskii functionals
- ▶ Derivation of design results
- ▶ Delay uncertainties

## Design of delay-scheduled state-feedback controllers

$$u(t) = K(d(t))x(t)$$

- ▶ Delay : parameter vs. operator
- ▶ Framework



## Delay-robust control of uncertain LPV time-delay systems

$$u(t) = K(\rho)x(t) + K_h(\rho)x(t - d(t))$$

- ▶ Choice of Lyapunov-Krasovskii functionals
- ▶ Derivation of design results
- ▶ Delay uncertainties

## Design of delay-scheduled state-feedback controllers

$$u(t) = K(d(t))x(t)$$

- ▶ Delay : parameter vs. operator
- ▶ Framework





## Delay-robust control of uncertain LPV time-delay systems

$$u(t) = K(\rho)x(t) + K_h(\rho)x(t - d(t))$$

- ▶ Choice of Lyapunov-Krasovskii functionals
- ▶ Derivation of design results
- ▶ Delay uncertainties

## Design of delay-scheduled state-feedback controllers

$$u(t) = K(d(t))x(t)$$

- ▶ Delay : parameter vs. operator
- ▶ Framework



## Delay-robust control of uncertain LPV time-delay systems

$$u(t) = K(\rho)x(t) + K_h(\rho)x(t - d(t))$$

- ▶ Choice of Lyapunov-Krasovskii functionals
- ▶ Derivation of design results
- ▶ Delay uncertainties

## Design of delay-scheduled state-feedback controllers

$$u(t) = K(d(t))x(t)$$

- ▶ Delay : parameter vs. operator
- ▶ Framework



## Delay-robust control of uncertain LPV time-delay systems

$$u(t) = K(\rho)x(t) + K_h(\rho)x(t - d(t))$$

- ▶ Choice of Lyapunov-Krasovskii functionals
- ▶ Derivation of design results
- ▶ Delay uncertainties

## Design of delay-scheduled state-feedback controllers

$$u(t) = K(d(t))x(t)$$

- ▶ Delay : parameter vs. operator
- ▶ Framework



## Delay-robust control of uncertain LPV time-delay systems

$$u(t) = K(\rho)x(t) + K_h(\rho)x(t - d(t))$$

- ▶ Choice of Lyapunov-Krasovskii functionals
- ▶ Derivation of design results
- ▶ Delay uncertainties

## Design of delay-scheduled state-feedback controllers

$$u(t) = K(d(t))x(t)$$

- ▶ Delay : parameter vs. operator
- ▶ Framework



## Delay-robust control of uncertain LPV time-delay systems

$$u(t) = K(\rho)x(t) + K_h(\rho)x(t - d(t))$$

- ▶ Choice of Lyapunov-Krasovskii functionals
- ▶ Derivation of design results
- ▶ Delay uncertainties

## Design of delay-scheduled state-feedback controllers

$$u(t) = K(d(t))x(t)$$

- ▶ Delay : parameter vs. operator
- ▶ Framework

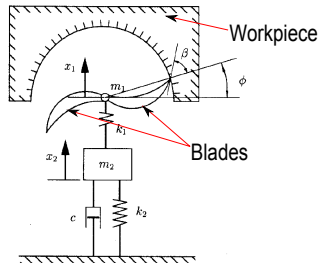


## Outline

1. Introduction
2. Stability of LPV Time-Delay Systems
  - ▶ Presentation of LPV time-delay systems
  - ▶ Choice of Lyapunov-Krasovskii functional
  - ▶ Reduction of conservatism
3. Control of LPV Time-Delay Systems
4. Conclusion & Future Works

## Example of LPV Time-Delay System

### Cutting Process [Zhang 2002]



- ▶ Nonlinearities
- ▶ Delay : time between two successive passes of the blades



# LPV Time-Delay Systems

## LPV Time-Delay System [Zhang 2002, Wu 2001, Zhang 2005]

$$\left\{ \begin{array}{l} \dot{x}(t) \\ \rho \\ \dot{\rho} \\ h(t) \\ \left| \frac{dh(t)}{dt} \right| \end{array} \right. \begin{array}{l} = A(\rho)x(t) + A_h(\rho)x(t - h(t)) + E(\rho)w(t) \\ \in U_\rho \\ \in \text{co}\{U_\nu\} \\ \in [0, h_{max}] \\ \leq \mu < 1 \end{array} \quad (1)$$

### Objectives

- ▶ Efficient stability tests
- ▶ Efficient design tools
- ▶ Tackle delay uncertainties





## LPV Time-Delay Systems

### LPV Time-Delay System [Zhang 2002, Wu 2001, Zhang 2005]

$$\left\{ \begin{array}{l} \dot{x}(t) \\ \rho \\ \dot{\rho} \\ h(t) \\ \left| \frac{dh(t)}{dt} \right| \end{array} \right. \begin{array}{l} = \\ \in \\ \in \\ \in \\ \leq \end{array} \begin{array}{l} A(\rho)x(t) + A_h(\rho)x(t - h(t)) + E(\rho)w(t) \\ U_\rho \\ \text{co}\{U_\nu\} \\ [0, h_{max}] \\ \mu < 1 \end{array} \quad (1)$$

### Objectives

- ▶ Efficient stability tests
- ▶ Efficient design tools
- ▶ Tackle delay uncertainties



## LPV Time-Delay Systems

### LPV Time-Delay System [Zhang 2002, Wu 2001, Zhang 2005]

$$\left\{ \begin{array}{l} \dot{x}(t) \\ \rho \\ \dot{\rho} \\ h(t) \\ \left| \frac{dh(t)}{dt} \right| \end{array} \right. \begin{array}{l} = \\ \in \\ \in \\ \in \\ \leq \end{array} \begin{array}{l} A(\rho)x(t) + A_h(\rho)x(t - h(t)) + E(\rho)w(t) \\ U_\rho \\ \text{co}\{U_\nu\} \\ [0, h_{max}] \\ \mu < 1 \end{array} \quad (1)$$

### Objectives

- ▶ Efficient stability tests
- ▶ Efficient design tools
- ▶ Tackle delay uncertainties



## LPV Time-Delay Systems

### LPV Time-Delay System [Zhang 2002, Wu 2001, Zhang 2005]

$$\left\{ \begin{array}{l} \dot{x}(t) \\ \rho \\ \dot{\rho} \\ h(t) \\ \left| \frac{dh(t)}{dt} \right| \end{array} \right. \begin{array}{l} = \\ \in \\ \in \\ \in \\ \leq \end{array} \begin{array}{l} A(\rho)x(t) + A_h(\rho)x(t - h(t)) + E(\rho)w(t) \\ U_\rho \\ \text{co}\{U_\nu\} \\ [0, h_{max}] \\ \mu < 1 \end{array} \quad (1)$$

### Objectives

- ▶ Efficient stability tests
- ▶ Efficient design tools
- ▶ Tackle delay uncertainties



## Choice of the Lyapunov-Krasovskii functional

### Criteria

- ▶ Simple form (few decision matrices, small size of LMIs)
- ▶ Avoid model-transformations
- ▶ 'Good' results (estimation of delay margin, system norms...)
- ▶ Stability over an interval of delay values
- ▶ Parameter dependent



## Stability Condition

Generalization of [Han 2005, Gouaisbaut 2006] to the LPV case

$$V = x(t)^T P(\rho)x(t) + \int_{t-h(t)}^t x(\theta)^T Qx(\theta)d\theta + h_{max} \int_{-h_{max}}^0 \int_{t+\theta}^t \dot{x}(s)^T R\dot{x}(s)dsd\theta$$

- Used along with Jensen's inequality [Han 2005, Gouaisbaut 2006]

### Theorem

The LPV Time-delay system (1) is asymptotically stable if there exists  $P(\rho), Q, R \succ 0$  such that the LMI

$$\begin{bmatrix} A(\rho)^T P(\rho) + P(\rho)A(\rho) + Q - R + \frac{\partial P(\rho)}{\partial \rho} \nu & P(\rho)A_h(\rho) + R & h_{max} A(\rho)^T R \\ * & -(1-\mu)Q - R & h_{max} A_h(\rho)^T R \\ * & * & -R \end{bmatrix} \prec 0$$

holds for all  $\rho \in U_\rho$  and  $\nu \in U_\nu$ .



## Stability Condition

Generalization of [Han 2005, Gouaisbaut 2006] to the LPV case

$$V = x(t)^T P(\rho)x(t) + \int_{t-h(t)}^t x(\theta)^T Qx(\theta)d\theta + h_{max} \int_{-h_{max}}^0 \int_{t+\theta}^t \dot{x}(s)^T R\dot{x}(s)dsd\theta$$

- Used along with Jensen's inequality [Han 2005, Gouaisbaut 2006]

### Theorem

The LPV Time-delay system (1) is asymptotically stable if there exists  $P(\rho), Q, R \succ 0$  such that the LMI

$$\begin{bmatrix} A(\rho)^T P(\rho) + P(\rho)A(\rho) + Q - R + \frac{\partial P(\rho)}{\partial \rho} \nu & P(\rho)A_h(\rho) + R & h_{max}A(\rho)^T R \\ * & -(1-\mu)Q - R & h_{max}A_h(\rho)^T R \\ * & * & -R \end{bmatrix} \prec 0$$

holds for all  $\rho \in U_\rho$  and  $\nu \in U_\nu$ .



## Stability Condition

Generalization of [Han 2005, Gouaisbaut 2006] to the LPV case

$$V = x(t)^T P(\rho)x(t) + \int_{t-h(t)}^t x(\theta)^T Qx(\theta)d\theta + h_{max} \int_{-h_{max}}^0 \int_{t+\theta}^t \dot{x}(s)^T R\dot{x}(s)dsd\theta$$

- Used along with Jensen's inequality [Han 2005, Gouaisbaut 2006]

### Theorem

The LPV Time-delay system (1) is asymptotically stable if there exists  $P(\rho), Q, R \succ 0$  such that the LMI

$$\begin{bmatrix} A(\rho)^T P(\rho) + P(\rho)A(\rho) + Q - R + \frac{\partial P(\rho)}{\partial \rho} \nu & P(\rho)A_h(\rho) + R & h_{max}A(\rho)^T R \\ * & -(1-\mu)Q - R & h_{max}A_h(\rho)^T R \\ * & * & -R \end{bmatrix} \prec 0$$

holds for all  $\rho \in U_\rho$  and  $\nu \in U_\nu$ .



## Example - LTI case

### LTI system with constant delay

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} x(t-h)$$

### Comparison with existing results

Method	$h_{max}$	nb. vars
Zhang et al. 2000	6.15	81
Han 2002	4.4721	9 or 18
Xu and Lam 2005	4.4721	17
This result	4.4721	9
Theoretical	6.17	–

- + Computational complexity
- + Competitive
- Gap → Conservatism







## Example - LTI case

### LTI system with constant delay

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} x(t-h)$$

### Comparison with existing results

Method	$h_{max}$	nb. vars
Zhang et al. 2000	6.15	81
Han 2002	4.4721	9 or 18
Xu and Lam 2005	4.4721	17
This result	4.4721	9
Theoretical	6.17	–

- + Computational complexity
- + Competitive
- Gap → Conservatism



## Example - LTI case

### LTI system with constant delay

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} x(t-h)$$

### Comparison with existing results

Method	$h_{max}$	nb. vars
Zhang et al. 2000	6.15	81
Han 2002	4.4721	9 or 18
Xu and Lam 2005	4.4721	17
This result	4.4721	9
Theoretical	6.17	—

- + Computational complexity
- + Competitive
- Gap → Conservatism



## Origin of Conservatism

- ▶ Constant matrices  $Q, R$

$$V(x_t) = x(t)^T P(\rho)x(t) + \int_{t-h(t)}^t x(\theta)^T Q x(\theta) d\theta + h_{max} \int_{-h_{max}}^0 \int_{t+\theta}^t \dot{x}(s)^T R \dot{x}(s) ds d\theta$$

- ▶ Jensen's inequality
  - ▶ Bound of an integral term over a finite interval
  - ▶ For illustration : Conservatism  $\equiv$  surface between curves



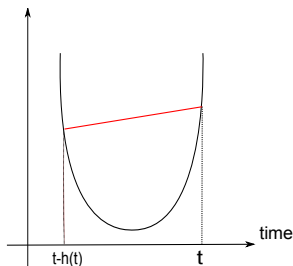
## Origin of Conservatism

- ▶ Constant matrices  $Q, R$

$$V(x_t) = x(t)^T P(\rho)x(t) + \int_{t-h(t)}^t x(\theta)^T Q x(\theta) d\theta + h_{max} \int_{-h_{max}}^0 \int_{t+\theta}^t \dot{x}(s)^T R \dot{x}(s) ds d\theta$$

- ▶ Jensen's inequality

- ▶ Bound of an integral term over a finite interval
- ▶ For illustration : Conservatism  $\equiv$  surface between curves





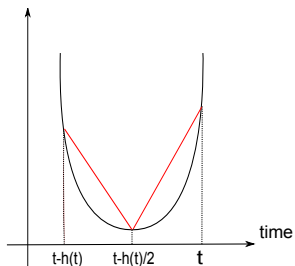
## Origin of Conservatism

- ▶ Constant matrices  $Q, R$

$$V(x_t) = x(t)^T P(\rho)x(t) + \int_{t-h(t)}^t x(\theta)^T Q x(\theta) d\theta + h_{max} \int_{-h_{max}}^0 \int_{t+\theta}^t \dot{x}(s)^T R \dot{x}(s) ds d\theta$$

- ▶ Jensen's inequality

- ▶ Bound of an integral term over a finite interval
- ▶ For illustration : Conservatism  $\equiv$  surface between curves





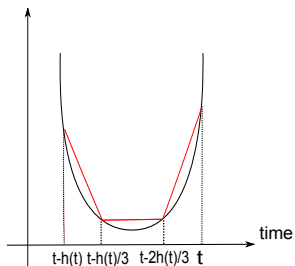
## Origin of Conservatism

- ▶ Constant matrices  $Q, R$

$$V(x_t) = x(t)^T P(\rho)x(t) + \int_{t-h(t)}^t x(\theta)^T Q x(\theta) d\theta + h_{max} \int_{-h_{max}}^0 \int_{t+\theta}^t \dot{x}(s)^T R \dot{x}(s) ds d\theta$$

- ▶ Jensen's inequality

- ▶ Bound of an integral term over a finite interval
- ▶ For illustration : Conservatism  $\equiv$  surface between curves





## Reduction of Conservatism



### Generalization of the functional [Han 2008]

$$V = x(t)^T P(\rho)x(t) + \int_{t-h(t)}^t x(\theta)^T Q(\theta)x(\theta)d\theta + \int_{-h_{max}}^0 \int_{t+\theta}^t \dot{x}(s)^T R(\theta)\dot{x}(s)dsd\theta$$

### Discretization

- ▶  $Q(\cdot)$ ,  $R(\cdot)$  : piecewise constant continuous [Gu 2001, Han 2008]

$$V = x(t)^T P(\rho)x(t) + \sum_{i=0}^{N-1} \int_{t-(i+1)h(t)/N}^{t-ih(t)/N} x(\theta)^T Q_i x(\theta)d\theta + \frac{h_{max}}{N} \sum_{i=0}^{N-1} \int_{-(i+1)h_{max}/N}^{-ih_{max}/N} \int_{t+\theta}^t \dot{x}(s)^T R_i \dot{x}(s)dsd\theta$$

Synergy of fragmentation and discretization



## Reduction of Conservatism

### Generalization of the functional [Han 2008]

$$V = x(t)^T P(\rho)x(t) + \int_{t-h(t)}^t x(\theta)^T Q(\theta)x(\theta)d\theta + \int_{-h_{max}}^0 \int_{t+\theta}^t \dot{x}(s)^T R(\theta)\dot{x}(s)dsd\theta$$

### Discretization

- ▶  $Q(\cdot)$ ,  $R(\cdot)$  : piecewise constant continuous [Gu 2001, Han 2008]

$$V = x(t)^T P(\rho)x(t) + \sum_{i=0}^{N-1} \int_{t-(i+1)h(t)/N}^{t-ih(t)/N} x(\theta)^T Q_i x(\theta)d\theta + \frac{h_{max}}{N} \sum_{i=0}^{N-1} \int_{-(i+1)h_{max}/N}^{-ih_{max}/N} \int_{t+\theta}^t \dot{x}(s)^T R_i \dot{x}(s)dsd\theta$$

Synergy of fragmentation and discretization





## Reduction of Conservatism

### Generalization of the functional [Han 2008]

$$V = x(t)^T P(\rho)x(t) + \int_{t-h(t)}^t x(\theta)^T Q(\theta)x(\theta)d\theta + \int_{-h_{max}}^0 \int_{t+\theta}^t \dot{x}(s)^T R(\theta)\dot{x}(s)dsd\theta$$

### Discretization

- ▶  $Q(\cdot)$ ,  $R(\cdot)$  : piecewise constant continuous [Gu 2001, Han 2008]

$$V = x(t)^T P(\rho)x(t) + \sum_{i=0}^{N-1} \int_{t-(i+1)h(t)/N}^{t-ih(t)/N} x(\theta)^T Q_i x(\theta)d\theta + \frac{h_{max}}{N} \sum_{i=0}^{N-1} \int_{-(i+1)h_{max}/N}^{-ih_{max}/N} \int_{t+\theta}^t \dot{x}(s)^T R_i \dot{x}(s)dsd\theta$$

Synergy of fragmentation and discretization



## Stability result for $N = 2$

### Theorem

The LPV Time-delay system (1) is asymptotically stable if there exists  $P(\rho), Q_1, Q_2, R_1, R_2 \succ 0$  such that the LMI

$$\begin{bmatrix} \mathcal{M}_{11}(\rho, \dot{\rho}) & R_1 & P(\rho)A_h(\rho) & \frac{h_{max}}{2}A(\rho)^T R_1 & \frac{h_{max}}{2}A(\rho)^T R_2 \\ * & U_1 & R_2 & 0 & 0 \\ * & * & U_2 & \frac{h_{max}}{2}A_h(\rho)^T R_1 & \frac{h_{max}}{2}A_h(\rho)^T R_2 \\ * & * & * & -R_1 & 0 \\ * & * & * & * & -R_2 \end{bmatrix} \prec 0$$

holds for all  $\rho \in U_\rho$  and  $\nu \in U_\nu$  with

$$\begin{aligned} \mathcal{M}_{11}(\rho, \dot{\rho}) &= A(\rho)^T P(\rho) + P(\rho)A(\rho) + Q_1 - R_1 + \frac{\partial P(\rho)}{\partial \rho} \nu \\ U_1 &= -Q_1 + Q_2 - R_1 - R_2 \\ U_2 &= -Q_2 - R_2 \end{aligned}$$



## Example - LTI case (cont'd)

### Comparison with existing method using fragmentation [Peaucelle et al. 2007]

Method	$h_{max}$	nb. vars	nb vars. Peaucelle et al.
Zhang et al. 2000	6.15	81	–
$N = 1$	4.4721	9	9
$N = 2$	5.1775	15	16
$N = 3$	5.9678	21	27
$N = 4$	6.0569	27	42
$N = 9$	6.149	57	177
$N = 30$	6.171	183	1836
Theoretical	6.172	–	–

- Zhang et al. 2000 : constant time-delays only
- Approach of Peaucelle et al. based on translation of the state



## Outline

1. Introduction
2. Stability of LPV Time-Delay Systems
3. Control of LPV Time-Delay Systems
  - ▶ Principle of delay-robust stabilization
  - ▶ Stabilization test - Relaxations
  - ▶ Example
4. Conclusion & Future Works



## Principle of delay-robust stabilization

### Nominal stabilization of LPV time-delay systems

- ▶ Gain-scheduled memoryless controller :

$$u(t) = K_0(\rho)x(t)$$

- ▶ Gain-scheduled exact memory controller :

$$u(t) = K_0(\rho)x(t) + K_h(\rho)x(t - h(t))$$

### Delay-robust stabilization of LPV time-delay systems

- ▶ Gain-scheduled approximate memory controller :

$$u(t) = K_0(\rho)x(t) + K_h(\rho)x(t - d(t)) \quad \text{with } |h(t) - d(t)| \leq \delta$$

Few studied in the literature



## Principle of delay-robust stabilization

### Nominal stabilization of LPV time-delay systems

- ▶ Gain-scheduled memoryless controller :

$$u(t) = K_0(\rho)x(t)$$

- ▶ Gain-scheduled exact memory controller :

$$u(t) = K_0(\rho)x(t) + K_h(\rho)x(t - h(t))$$

### Delay-robust stabilization of LPV time-delay systems

- ▶ Gain-scheduled approximate memory controller :

$$u(t) = K_0(\rho)x(t) + K_h(\rho)x(t - d(t)) \quad \text{with } |h(t) - d(t)| \leq \delta$$

Few studied in the literature



## Recall of stability test for $N = 1$

### Theorem

System (1) is asymptotically stable if there exists  $P(\rho), Q, R \succ 0$  such that the LMI

$$\begin{bmatrix} A(\rho)^T P(\rho) + P(\rho) A(\rho) + Q - R + \frac{\partial P(\rho)}{\partial \rho} \nu & P(\rho) A_h(\rho) + R & h_{max} A(\rho)^T R \\ * & -(1 - \mu) Q - R & h_{max} A_h(\rho)^T R \\ * & * & -R \end{bmatrix} \prec 0$$

holds for all  $\rho \in U_\rho$  and  $\nu \in U_\nu$ .

### Goal

- ▶ Derive efficient design results
- ▶ Tackle delay uncertainty



## Stabilization test for $N = 1$



### System and Controller

$$\begin{aligned}\dot{x}(t) &= A(\rho)x(t) + A_h(\rho)x(t - h(t)) + B(\rho)u(t) \\ u(t) &= K_0(\rho)x(t) + K_h(\rho)x(t - h(t))\end{aligned}$$

### Theorem

The system is asymptotically stabilizable by a control law with exact memory if there exists  $P(\rho), Q, R \succ 0$  and  $K_0(\rho), K_h(\rho)$  such that the LMI

$$\begin{bmatrix} A_{cl}^T P + P A_{cl} + Q - R + \dot{P} & P A_{hcl} + R & h_{max} A_{cl}^T R \\ * & -(1 - \mu)Q - R & h_{max} A_{hcl}^T R \\ * & * & -R \end{bmatrix} \prec 0$$

holds for all  $\rho \in U_\rho$  and  $\nu \in U_\nu$  with  $A_{cl} = A + BK_0$  and  $A_{hcl} = A_h + BK_h$ .

### Convexity

- Bilinear matrix inequality  $\rightarrow$  non-convex
- Single terms in  $R$  and multiple products  $\rightarrow$  Linearization not possible !!







## Stabilization test for $N = 1$



### System and Controller

$$\begin{aligned} \dot{x}(t) &= A(\rho)x(t) + A_h(\rho)x(t - h(t)) + B(\rho)u(t) \\ u(t) &= K_0(\rho)x(t) + K_h(\rho)x(t - h(t)) \end{aligned}$$

### Theorem

The system is asymptotically stabilizable by a control law with exact memory if there exists  $P(\rho), Q, R \succ 0$  and  $K_0(\rho), K_h(\rho)$  such that the LMI

$$\begin{bmatrix} A_{cl}^T P + P A_{cl} + Q - R + \dot{P} & P A_{hcl} + R & h_{max} A_{cl}^T R \\ * & -(1 - \mu)Q - R & h_{max} A_{hcl}^T R \\ * & * & -R \end{bmatrix} \prec 0$$

holds for all  $\rho \in U_\rho$  and  $\nu \in U_\nu$  with  $A_{cl} = A + BK_0$  and  $A_{hcl} = A_h + BK_h$ .

### Convexity

- Bilinear matrix inequality  $\rightarrow$  non-convex
- Single terms in  $R$  and multiple products  $\rightarrow$  Linearization not possible !!





## Stabilization test for $N = 1$



### System and Controller

$$\begin{aligned} \dot{x}(t) &= A(\rho)x(t) + A_h(\rho)x(t - h(t)) + B(\rho)u(t) \\ u(t) &= K_0(\rho)x(t) + K_h(\rho)x(t - h(t)) \end{aligned}$$

### Theorem

The system is asymptotically stabilizable by a control law with exact memory if there exists  $P(\rho), Q, R \succ 0$  and  $K_0(\rho), K_h(\rho)$  such that the LMI

$$\begin{bmatrix} A_{cl}^T P + P A_{cl} + Q - R + \dot{P} & P A_{hcl} + R & h_{max} A_{cl}^T R \\ * & -(1 - \mu)Q - R & h_{max} A_{hcl}^T R \\ * & * & -R \end{bmatrix} \prec 0$$

holds for all  $\rho \in U_\rho$  and  $\nu \in U_\nu$  with  $A_{cl} = A + BK_0$  and  $A_{hcl} = A_h + BK_h$ .

### Convexity

- Bilinear matrix inequality  $\rightarrow$  non-convex
- Single terms in  $R$  and multiple products  $\rightarrow$  Linearization not possible !!





## Preliminary Relaxations

### Common Relaxations

- ▶ Remove single terms in  $R$ 
  - ▶ Avoid Jensen's inequality
  - High increase of conservatism
- ▶ Set  $P(\rho) = \varepsilon(\rho)R$ 
  - Difficult choice of  $\varepsilon(\rho)$
  - High increase of conservatism
  - Increase of computational complexity

### Relaxation of [Briat. IFAC World Congress 2008]

- ▶ Use of adjoint system and projection lemma
  - + Non conservative
  - Nonlinear optimization problem (expensive, local convergence)

High increase of conservatism and/or computational complexity



## Preliminary Relaxations

### Common Relaxations

- ▶ Remove single terms in  $R$ 
  - ▶ Avoid Jensen's inequality
    - High increase of conservatism
- ▶ Set  $P(\rho) = \varepsilon(\rho)R$ 
  - Difficult choice of  $\varepsilon(\rho)$
  - High increase of conservatism
  - Increase of computational complexity

### Relaxation of [Briat. IFAC World Congress 2008]

- ▶ Use of adjoint system and projection lemma
  - + Non conservative
  - Nonlinear optimization problem (expensive, local convergence)

High increase of conservatism and/or computational complexity



## Preliminary Relaxations

### Common Relaxations

- ▶ Remove single terms in  $R$ 
  - ▶ Avoid Jensen's inequality
    - High increase of conservatism
- ▶ Set  $P(\rho) = \varepsilon(\rho)R$ 
  - Difficult choice of  $\varepsilon(\rho)$
  - High increase of conservatism
  - Increase of computational complexity

### Relaxation of [Briat. IFAC World Congress 2008]

- ▶ Use of adjoint system and projection lemma
  - + Non conservative
  - Nonlinear optimization problem (expensive, local convergence)

High increase of conservatism and/or computational complexity



## Preliminary Relaxations

### Common Relaxations

- ▶ Remove single terms in  $R$ 
  - ▶ Avoid Jensen's inequality
  - High increase of conservatism
- ▶ Set  $P(\rho) = \varepsilon(\rho)R$ 
  - Difficult choice of  $\varepsilon(\rho)$
  - High increase of conservatism
  - Increase of computational complexity

### Relaxation of [Briat. IFAC World Congress 2008]

- ▶ Use of adjoint system and projection lemma
  - + Non conservative
  - Nonlinear optimization problem (expensive, local convergence)

High increase of conservatism and/or computational complexity



## Proposed Relaxation

### Origin of the Problem

- ▶ Substitution of the closed-loop but convexity not preserved
- ▶ Relaxation done after substitution

### Proposed Methodology

- ▶ Test modification  $\rightarrow$  'convexity preserving' form
  - ▶ Relaxation done before substitution
  - ▶ Orientation of the relaxation

### Relaxation features

- ▶ Decoupling multiple matrix products
  - ▶ Introduction of a new variable



## Proposed Relaxation

### Origin of the Problem

- ▶ Substitution of the closed-loop but convexity not preserved
- ▶ Relaxation done after substitution

### Proposed Methodology

- ▶ Test modification → 'convexity preserving' form
  - ▶ Relaxation done before substitution
  - ▶ Orientation of the relaxation

### Relaxation features

- ▶ Decoupling multiple matrix products
  - ▶ Introduction of a new variable





## Proposed Relaxation

### Origin of the Problem

- ▶ Substitution of the closed-loop but convexity not preserved
- ▶ Relaxation done after substitution

### Proposed Methodology

- ▶ Test modification  $\rightarrow$  'convexity preserving' form
  - ▶ Relaxation done before substitution
  - ▶ Orientation of the relaxation

### Relaxation features

- ▶ Decoupling multiple matrix products
  - ▶ Introduction of a new variable



## Relaxed stability test for $N = 1$

### Theorem

System (1) is asymptotically stable if there exists  $P(\rho), Q, R \succ 0, X(\rho)$  and  $K_0(\rho), K_h(\rho)$  such that the LMI

$$\begin{bmatrix} -\mathbf{X}(\rho) - \mathbf{X}(\rho)^T & \mathbf{X}(\rho)^T A(\rho) + P(\rho) & \mathbf{X}(\rho)^T A_h(\rho) & \mathbf{X}(\rho)^T & h_{max} R \\ * & -P(\rho) + Q - R + \dot{P} & R & 0 & 0 \\ * & * & -(1 - \mu)Q - R & 0 & 0 \\ * & * & * & -P(\rho) & -h_{max} R \\ * & * & * & * & -R \end{bmatrix} \prec 0$$

holds for all  $\rho \in U_\rho$  and  $\nu \in U_\nu$ .

- ▶ Additional variable  $\mathbf{X}(\rho)$
- ▶ No multiple products anymore



## Relaxed stabilization test for $N = 1$

- Stabilization of system (1) by an exact memory control law :

$$u(t) = K_0(\rho)x(t) + K_h(\rho)x(t - h(t)) \quad (2)$$

- After some manipulations...

### Theorem

System (1) is stabilizable using (2) if there exists  $P(\rho), Q, R \succ 0, X$  and  $Y_0(\rho), Y_h(\rho)$  such that the LMI

$$\begin{bmatrix} -X - X^T & A(\rho)X + B(\rho)Y_0(\rho) + P(\rho) & A_h(\rho)X + B(\rho)Y_h(\rho) & X^T & \tilde{R} \\ * & -P(\rho) + Q - R + \dot{P}(\rho) & R & 0 & 0 \\ * & * & -(1 - \mu)Q - R & 0 & 0 \\ * & * & * & -P(\rho) & -\tilde{R} \\ * & * & * & * & -R \end{bmatrix} \prec 0$$

holds for all  $\rho \in U_\rho$  and  $\nu \in U_\nu$  with  $\tilde{R} = h_{max}R$ .

Suitable controller gains are given by  $K_0(\rho) = Y_0(\rho)X^{-1}$  and  $K_h(\rho) = Y_h(\rho)X^{-1}$ .





## Relaxed stabilization test for $N = 1$

- Stabilization of system (1) by an exact memory control law :

$$u(t) = K_0(\rho)x(t) + K_h(\rho)x(t - h(t)) \quad (2)$$

- After some manipulations...

### Theorem

System (1) is stabilizable using (2) if there exists  $P(\rho), Q, R \succ 0, X$  and  $Y_0(\rho), Y_h(\rho)$  such that the LMI

$$\begin{bmatrix} -X - X^T & A(\rho)X + B(\rho)Y_0(\rho) + P(\rho) & A_h(\rho)X + B(\rho)Y_h(\rho) & X^T & \tilde{R} \\ * & -P(\rho) + Q - R + \dot{P}(\rho) & R & 0 & 0 \\ * & * & -(1 - \mu)Q - R & 0 & 0 \\ * & * & * & -P(\rho) & -\tilde{R} \\ * & * & * & * & -R \end{bmatrix} \prec 0$$

holds for all  $\rho \in U_\rho$  and  $\nu \in U_\nu$  with  $\tilde{R} = h_{max}R$ .

Suitable controller gains are given by  $K_0(\rho) = Y_0(\rho)X^{-1}$  and  $K_h(\rho) = Y_h(\rho)X^{-1}$ .





## Example (1)

- ▶ LPV time-delay system [Zhang et al., 2005]

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 + 0.1\rho(t) \\ -2 & -3 + 0.2\rho(t) \end{bmatrix} x(t) + \begin{bmatrix} 0.2\rho(t) \\ 0.1 + 0.1\rho(t) \end{bmatrix} u(t) \\ &\quad + \begin{bmatrix} 0.2\rho(t) & 0.1 \\ -0.2 + 0.1\rho(t) & -0.3 \end{bmatrix} x(t - h(t)) + \begin{bmatrix} -0.2 \\ -0.2 \end{bmatrix} w(t) \\ z(t) &= \begin{bmatrix} 0 & 10 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u(t) \\ \rho(t) &= \sin(t) \end{aligned}$$

### Goal

- ▶ Find a controller such that such that the closed-loop system
  1. is asymptotically stable for all  $h(t) \in [0, h_{max}]$  with  $|\dot{h}(t)| \leq \mu < 1$  and
  2. satisfies

$$\|z\|_{\mathcal{L}_2} \leq \gamma \|w\|_{\mathcal{L}_2}$$



## Example (1)

- ▶ LPV time-delay system [Zhang et al., 2005]

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 + 0.1\rho(t) \\ -2 & -3 + 0.2\rho(t) \end{bmatrix} x(t) + \begin{bmatrix} 0.2\rho(t) \\ 0.1 + 0.1\rho(t) \end{bmatrix} u(t) \\ &\quad + \begin{bmatrix} 0.2\rho(t) & 0.1 \\ -0.2 + 0.1\rho(t) & -0.3 \end{bmatrix} x(t - h(t)) + \begin{bmatrix} -0.2 \\ -0.2 \end{bmatrix} w(t) \\ z(t) &= \begin{bmatrix} 0 & 10 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u(t) \\ \rho(t) &= \sin(t) \end{aligned}$$

### Goal

- ▶ Find a controller such that such that the closed-loop system
  1. is asymptotically stable for all  $h(t) \in [0, h_{max}]$  with  $|\dot{h}(t)| \leq \mu < 1$  and
  2. satisfies

$$\|z\|_{\mathcal{L}_2} \leq \gamma \|w\|_{\mathcal{L}_2}$$



## Example (2)



**Case 1 :**  $\dot{h}(t) \leq 0.5, h(t) \in [0, 0.5]$

- ▶ Design of a memoryless state-feedback control law

$$u(t) = K_0(\rho)x(t)$$

	minimal $\mathcal{L}_2$ gain
[Zhang et al. 2005]	$\gamma^* = 3.09$
[Briat et al. IFAC WC 2008]	$\gamma^* = 2.27$
$N = 1$	$\gamma^* = 1.90$

$$K_0(\rho) = \begin{bmatrix} -1.0535 - 2.9459\rho + 1.9889\rho^2 \\ -1.1378 - 2.6403\rho + 1.9260\rho^2 \end{bmatrix}^T$$

- ▶ Better performances
- ▶ Lower controller gains
- ▶ Lower numerical complexity



## Example (2)



**Case 1 :**  $\dot{h}(t) \leq 0.5, h(t) \in [0, 0.5]$

- ▶ Design of a memoryless state-feedback control law

$$u(t) = K_0(\rho)x(t)$$

	minimal $\mathcal{L}_2$ gain
[Zhang et al. 2005]	$\gamma^* = 3.09$
[Briat et al. IFAC WC 2008]	$\gamma^* = 2.27$
$N = 1$	$\gamma^* = 1.90$

$$K_0(\rho) = \begin{bmatrix} -1.0535 - 2.9459\rho + 1.9889\rho^2 \\ -1.1378 - 2.6403\rho + 1.9260\rho^2 \end{bmatrix}^T$$

- ▶ Better performances
- ▶ Lower controller gains
- ▶ Lower numerical complexity





## Example (3)

**Case 2 :**  $\dot{h}(t) \leq 0.9, h(t) \in [0, 10]$

- Synthesis of both memoryless and exact memory controllers

$$u(t) = K_0(\rho)x(t)$$

$$u(t) = K_0(\rho)x(t) + K_h(\rho)x(t - h(t))$$

	minimal $\mathcal{L}_2$ gain
Memoryless Controller	$\gamma^* = 12.8799$
Exact Memory Controller	$\gamma^* = 4.1641$

- + Delayed term important
- Needs the exact value of the delay at any time
- Problem of delay estimation [Belkoura et al. 2008]

Robust synthesis w.r.t. delay uncertainty on implemented delay



## Example (3)

**Case 2 :**  $\dot{h}(t) \leq 0.9, h(t) \in [0, 10]$

- Synthesis of both memoryless and exact memory controllers

$$u(t) = K_0(\rho)x(t)$$

$$u(t) = K_0(\rho)x(t) + K_h(\rho)x(t - h(t))$$

	minimal $\mathcal{L}_2$ gain
Memoryless Controller	$\gamma^* = 12.8799$
Exact Memory Controller	$\gamma^* = 4.1641$

+ Delayed term important

- Needs the exact value of the delay at any time
- Problem of delay estimation [Belkoura et al. 2008]

Robust synthesis w.r.t. delay uncertainty on implemented delay



## Example (3)

**Case 2 :**  $\dot{h}(t) \leq 0.9, h(t) \in [0, 10]$

- Synthesis of both memoryless and exact memory controllers

$$u(t) = K_0(\rho)x(t)$$

$$u(t) = K_0(\rho)x(t) + K_h(\rho)x(t - h(t))$$

	minimal $\mathcal{L}_2$ gain
Memoryless Controller	$\gamma^* = 12.8799$
Exact Memory Controller	$\gamma^* = 4.1641$

- + Delayed term important
- Needs the exact value of the delay at any time
- Problem of delay estimation [Belkoura et al. 2008]

Robust synthesis w.r.t. delay uncertainty on implemented delay



## Example (3)

**Case 2 :**  $\dot{h}(t) \leq 0.9, h(t) \in [0, 10]$

- Synthesis of both memoryless and exact memory controllers

$$u(t) = K_0(\rho)x(t)$$

$$u(t) = K_0(\rho)x(t) + K_h(\rho)x(t - h(t))$$

	minimal $\mathcal{L}_2$ gain
Memoryless Controller	$\gamma^* = 12.8799$
Exact Memory Controller	$\gamma^* = 4.1641$

- + Delayed term important
- Needs the exact value of the delay at any time
- Problem of delay estimation [Belkoura et al. 2008]

Robust synthesis w.r.t. delay uncertainty on implemented delay



## Delay-Robust Controllers (1)

### System and Controller

$$\begin{aligned}\dot{x}(t) &= A(\rho)x(t) + A_h(\rho)x(t - h(t)) + B(\rho)u(t) \\ u(t) &= K_0(\rho)x(t) + K_h(\rho)x(t - d(t))\end{aligned}$$

with  $|d(t) - h(t)| \leq \delta$ .

### Objectives

- ▶ Given maximal error  $\delta$  on the delay knowledge, find a controller such that the closed-loop system
  1. is asymptotically stable for all  $h(t) \in [0, h_{max}]$  with  $|\dot{h}(t)| \leq \mu < 1$ ,  $|d(t) - h(t)| \leq \delta$  and
  2. satisfies the input/output relationship

$$\|z\|_{\mathcal{L}_2} \leq \gamma \|w\|_{\mathcal{L}_2}$$



## Delay-Robust Controllers (1)

### System and Controller

$$\begin{aligned}\dot{x}(t) &= A(\rho)x(t) + A_h(\rho)x(t - h(t)) + B(\rho)u(t) \\ u(t) &= K_0(\rho)x(t) + K_h(\rho)x(t - d(t))\end{aligned}$$

with  $|d(t) - h(t)| \leq \delta$ .

### Objectives

- ▶ Given maximal error  $\delta$  on the delay knowledge, find a controller such that the closed-loop system
  1. is asymptotically stable for all  $h(t) \in [0, h_{max}]$  with  $|\dot{h}(t)| \leq \mu < 1$ ,  $|d(t) - h(t)| \leq \delta$  and
  2. satisfies the input/output relationship

$$\|z\|_{\mathcal{L}_2} \leq \gamma \|w\|_{\mathcal{L}_2}$$



## Delay-Robust Controllers (2)

### Closed-loop system

- ▶ System with **two constrained delays**

$$\begin{aligned}\dot{x}(t) &= A_{cl}(\rho)x(t) + A_h(\rho)x(t - h(t)) + B(\rho)K_h(\rho)x(t - d(t)) \\ A_{cl}(\rho) &= A(\rho) + B(\rho)K_0(\rho)\end{aligned}$$

with  $|d(t) - h(t)| \leq \delta$

How to consider the relation between the delays ?

### Model Transformation

$$\nabla(\eta) := \frac{1}{\delta} \int_{t-h(t)}^{t-d(t)} \eta(s) ds$$

- ▶ Linear dynamical time-varying operator  $\|\nabla\|_{\mathcal{L}_2-\mathcal{L}_2} \leq 1$

- ▶  $\nabla(\dot{x}) = \frac{1}{\delta}(x(t-d(t)) - x(t-h(t))) \Rightarrow x(t-h(t)) = x(t-d(t)) + \delta \nabla(\dot{x})$





## Delay-Robust Controllers (2)

### Closed-loop system

- ▶ System with **two constrained delays**

$$\begin{aligned}\dot{x}(t) &= A_{cl}(\rho)x(t) + A_h(\rho)x(t - h(t)) + B(\rho)K_h(\rho)x(t - d(t)) \\ A_{cl}(\rho) &= A(\rho) + B(\rho)K_0(\rho)\end{aligned}$$

with  $|d(t) - h(t)| \leq \delta$

**How to consider the relation between the delays ?**

### Model Transformation

$$\nabla(\eta) := \frac{1}{\delta} \int_{t-h(t)}^{t-d(t)} \eta(s) ds$$

- ▶ Linear dynamical time-varying operator  $\|\nabla\|_{\mathcal{L}_2-\mathcal{L}_2} \leq 1$

- ▶  $\nabla(\dot{x}) = \frac{1}{\delta}(x(t - d(t)) - x(t - h(t))) \Rightarrow x(t - h(t)) = x(t - d(t)) + \delta \nabla(\dot{x})$







## Delay-Robust Controllers (2)



### Closed-loop system

- ▶ System with **two constrained delays**

$$\begin{aligned}\dot{x}(t) &= A_{cl}(\rho)x(t) + A_h(\rho)x(t - h(t)) + B(\rho)K_h(\rho)x(t - d(t)) \\ A_{cl}(\rho) &= A(\rho) + B(\rho)K_0(\rho)\end{aligned}$$

with  $|d(t) - h(t)| \leq \delta$

How to consider the relation between the delays ?

### Model Transformation

$$\nabla(\eta) := \frac{1}{\delta} \int_{t-h(t)}^{t-d(t)} \eta(s) ds$$

- ▶ Linear dynamical time-varying operator  $\|\nabla\|_{\mathcal{L}_2-\mathcal{L}_2} \leq 1$

$$\nabla(\dot{x}) = \frac{1}{\delta} (x(t - d(t)) - x(t - h(t))) \Rightarrow x(t - h(t)) = x(t - d(t)) + \delta \nabla(\dot{x})$$





## Delay-Robust Controllers (2)



### Closed-loop system

- ▶ System with **two constrained delays**

$$\begin{aligned}\dot{x}(t) &= A_{cl}(\rho)x(t) + A_h(\rho)x(t - h(t)) + B(\rho)K_h(\rho)x(t - d(t)) \\ A_{cl}(\rho) &= A(\rho) + B(\rho)K_0(\rho)\end{aligned}$$

with  $|d(t) - h(t)| \leq \delta$

How to consider the relation between the delays ?

### Model Transformation

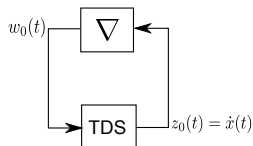
$$\nabla(\eta) := \frac{1}{\delta} \int_{t-h(t)}^{t-d(t)} \eta(s) ds$$

- ▶ Linear dynamical time-varying operator  $\|\nabla\|_{\mathcal{L}_2-\mathcal{L}_2} \leq 1$
- ▶  $\nabla(\dot{x}) = \frac{1}{\delta}(x(t - d(t)) - x(t - h(t))) \Rightarrow x(t - h(t)) = x(t - d(t)) + \delta\nabla(\dot{x})$





## Transformed Closed-Loop System



$$\begin{aligned}
 \dot{x}(t) &= A_{cl}(\rho)x(t) + A_{hcl}(\rho)x(t - d(t)) + \delta A_h w_0(t) \\
 z_0(t) &= \dot{x}(t) \\
 w_0(t) &= \nabla(z_0(t))
 \end{aligned}$$

- ▶ Uncertain system with one delay
- ▶ System stable for if
  - ▶ nominal system stable ( $\delta = 0$ )
  - ▶  $\|z_0\|_{\mathcal{L}_2} < \|w_0\|_{\mathcal{L}_2}$  for  $\delta \neq 0$  (small gain)



## Example (1)

- ▶ Previous results :  $\gamma = 12.8799$  (Memoryless),  $\gamma = 4.1641$  (Exact Memory)

### Delay-robust synthesis

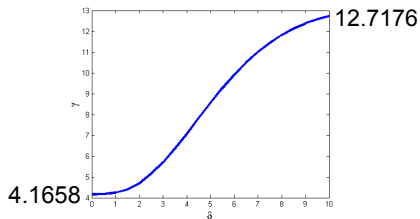


Fig.: Best  $\mathcal{L}_2$  performance  $\gamma$  vs. maximal error uncertainty  $\delta$

- ▶ Characterization of intermediate performances
- ▶ Direct generalization of the previous approach





## Example (1)

- ▶ Previous results :  $\gamma = 12.8799$  (Memoryless),  $\gamma = 4.1641$  (Exact Memory)

### Delay-robust synthesis

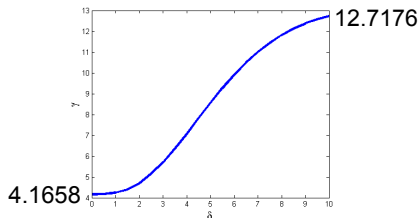


Fig.: Best  $\mathcal{L}_2$  performance  $\gamma$  vs. maximal error uncertainty  $\delta$

- ▶ Characterization of intermediate performances
- ▶ Direct generalization of the previous approach





## Towards Delay-Scheduled Controllers (1)

### Drawbacks of memory controllers

- ▶ Memory size (store past values)
- ▶ Implementing time-varying delays

### Delay-scheduled controllers

$$u(t) = K(\rho, h(t))x(t)$$

- + Still using delay information
- + Less memory
- Difficult synthesis



## Towards Delay-Scheduled Controllers (1)

### Drawbacks of memory controllers

- ▶ Memory size (store past values)
- ▶ Implementing time-varying delays

### Delay-scheduled controllers

$$u(t) = K(\rho, h(t))x(t)$$

- + Still using delay information
- + Less memory
- Difficult synthesis



## Towards Delay-Scheduled Controllers (1)

### Drawbacks of memory controllers

- ▶ Memory size (store past values)
- ▶ Implementing time-varying delays

### Delay-scheduled controllers

$$u(t) = K(\rho, h(t))x(t)$$

- + Still using delay information
- + Less memory
- Difficult synthesis

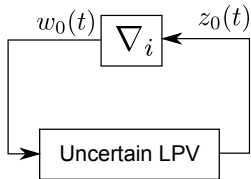




## Towards Delay-Scheduled Controllers (2)



### Model Transformations



$$[\text{ECC 07}] \quad \nabla_1(\eta) = \int_{t-h(t)}^t \frac{1}{h(s) + h_{max} + h_{min}} \eta(s) ds$$

### Comparison Models

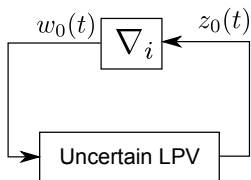
$$\begin{aligned} \dot{x}(t) &= (A + A_h)x(t) - A_h w_0(t) \\ z_0(t) &= (h(t) + h_{max} - h_{min})\dot{x}(t) \\ w_0(t) &= \nabla_1(\dot{x}(t)) \end{aligned}$$



## Towards Delay-Scheduled Controllers (2)



### Model Transformations



[IFAC WC 08] 
$$\nabla_2(\eta) = \sqrt{\frac{1}{h(t)h_{max}}} \int_{t-h(t)}^t \eta(s) ds$$

### Comparison Models

$$\begin{aligned} \dot{x}(t) &= (A + A_h)x(t) - A_h \sqrt{h(t)h_{max}} w_0(t) \\ z_0(t) &= \dot{x}(t) \\ w_0(t) &= \nabla_2(\dot{x}(t)) \end{aligned}$$





## Outline

1. Introduction
2. Stability of LPV Time-Delay Systems
3. Control of LPV Time-Delay Systems
4. Conclusion & Future Works



## Conclusion

- ▶ Methodology to derive stabilization results from stability results
- ▶ Based on LMI relaxation
- ▶ Generalizes to discretized versions of Lyapunov-Krasovskii functionals
- ▶ Synthesis of memoryless and memory controllers
- ▶ Synthesis of delay-robust controllers using either a adapted functional or (scaled) small gain results.



## Future works

- ▶ Improve the results for system with time-varying delays
- ▶ Generalize to system with non small-delays ( $h_{min} > 0$ )
- ▶ Develop new model transformations for delay-scheduled controller synthesis
- ▶ Enhance results on delay-scheduled controllers



Thank you for your attention

Vi ringrazio per l'attenzione

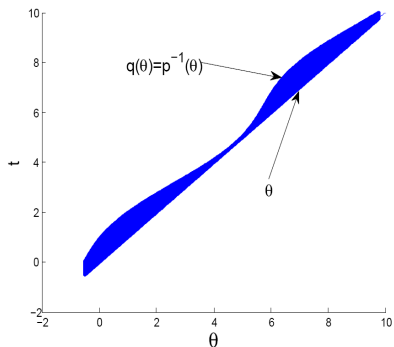
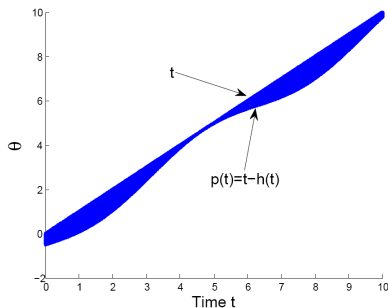
Merci de votre attention



## $\mathcal{L}_2$ induced norm of $\mathcal{D}_h$

$$\int_0^{+\infty} \int_{t-h(t)}^t \phi(t)\eta(s)dsdt = \int_0^{+\infty} \int_{q(s)}^s \phi(t)\eta(s)dt ds$$

with  $q := p^{-1}$ .





## Existence and Unicity of controller/observers (1)



### Synthesis problem

Find  $Z(\rho)$ ,  $\mathcal{X}(\rho)$  such that

$$\Psi(\rho, \dot{\rho}, \mathcal{X}(\rho)) + \mathcal{U}(\rho)Z(\rho)\mathcal{V}(\rho) + (\star)^T \prec 0$$

holds for all  $(\rho, \dot{\rho}) \in I_\rho \times \text{co}\{U_\nu\}$ .

### Controller existence - Projection Lemma

$$\text{Ker}[\mathcal{U}(\rho)]\Psi(\rho, \dot{\rho})\text{Ker}[\mathcal{U}(\rho)]^T \prec 0 \quad \text{Ker}[\mathcal{V}(\rho)]^T\Psi(\rho, \dot{\rho})\text{Ker}[\mathcal{V}(\rho)] \prec 0$$

### Controller construction

- ▶ Implicit
- ▶ Explicit [Iwasaki] :  $Z = f(\mathcal{U}, \mathcal{V}, \Psi, M)$  for every matrix  $M \in \mathcal{M}$  to be chosen