SOME THRESHOLD SPECTRAL PROBLEMS OFSCHRODINGER OPERATORS

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1 - Summary

1.1 Semi-classical limits of the number and Riesz means of discrete eigenvalues of the N-body Schrödinger operators

In this section, we introduce the main work which is done in Chapter 2.

PROBLEMS. First, we introduce the problems we considered in Chapter 2. Let $H(h)$ denote the N-body Schrödinger operator obtained by removing the mass center of Hamilton of an N-particle system

$$-\sum_{j=1}^{N}\frac{h^2}{2m_i}\Delta_{r_i}+\sum_{1\le i<j\le N}V_{ij}(r_i-r_j). \tag{1.1}$$

Here $r_j\in\mathbb{R}^\nu(\nu\ge 3)$ and $m_j>0$ denote the position and the mass of the j-th particle respectively, and $-\Delta_{r_i}$ is the Laplacian in the r_i variables. Let $C_i(i=1,\cdots,k)$ be the subsets of $\{1,\cdots,N\}$. If $C_i\cap C_j=\emptyset$, $i\neq j$ and $\cup_{i=1}^{k}C_i=\{1,\cdots,N\}$, we say that $D=\{C_1,\cdots,C_k\}$ is a partition (or cluster decomposition) of $\{1,\cdots,N\}$. If $D=\{C_1,\cdots,C_k\}$, we denote $\#D=k$. If i,j are two numbers in $\{1,\cdots,N\}$, we write iDj if and only if i and j are in the same cluster C_l, and $\sim iDj$ if they are in the different clusters. Let $\sum_{iDj}$ (resp. $\sum_{\sim iDj}$) denote the sum over all pairs with i,j in the same (resp. different) cluster of D. We define

$$V_D=\sum_{iDj}V_{ij};\quad I_D=\sum_{\sim iDj}V_{ij};$$

$$H_D(h)=H(h)-I_D;\quad \Sigma_D(h)=\inf\sigma\left(H_D(h)\right);$$

$$a_D=\min_{\{x;x\in\mathbb{R}^{(N-1)\nu}\}}V_D(x);\quad \Sigma_{cl}=\min_{\{D;\#D\ge 2\}}a_D.$$

For each D, there is a natural decomposition of $\mathscr{H}=L^2\left(\mathbb{R}^{(N-1)\nu}\right)$ as $\mathscr{H}^D\otimes\mathscr{H}_D$ with $\mathscr{H}_D=$ functions of r_{ij} with iDj and $\mathscr{H}^D=$ functions of R_q-R_l where $R_q=\sum_{i\in C_q}m_ir_i/\sum_{i\in C_q}m_i$ is the center of the mass of C_q. Under this decomposition

$$H_D(h)=h_D(h)\otimes 1+1\otimes t_D(h).$$

By the well known HVZ-theorem [54, Section XIII.5],

$$\inf \sigma_{ess}\,(H(h)) = \Sigma_h, \tag{1.2}$$

where $\Sigma_h = \min_{\{D;\#D=2\}} \Sigma_D(h)$. We say $H(h)$ is unique two-cluster, if there is only one cluster decomposition D with $\#D = 2$, such that $\Sigma_h = \Sigma_D(h)$. The intuition is quite simple : if $H(h)$ is two-cluster, the threshold is basically due to a $\nu-$dimensional Laplacian, namely the relative kinetic energy of the two clusters. So one expects the coupling constant behavior to be a ν-dimensional problem. We concentrate on the unique two-cluster $H(h)$. We mainly work on two problems in Chapter 2. The first one is the semi-classical limit of the number of discrete eigenvalues of the N-body Schrödinger operator $H(h)$. The second is the semi-classical limit of Riesz means of discrete eigenvalues of $H(h)$.

KNOWN RESULTS. The semi-classical limit of the number of discrete eigenvalues of $H(h)$ has been studied by many authors for $N = 2$ (see [4], [31], [40], [44], [54], [68]). Roughly speaking, in the case of 3-dimensional space, $\mathbb{R}^3$, the result obtained by these authors can be formulated as following : assume that $V(x)$ is real and $V(x) \in L^{3/2}(\mathbb{R}^3)$, the number of the negative eigenvalues, $N(\lambda)$, of $-\Delta + \lambda V$ obeys the asymptotic formula :

$$N(\lambda) = (6\pi^2)^{-1} \int |V_-(x)|^{3/2}\, dx(1 + o(1)), \quad \lambda \to \infty.$$

Here $V_-(x) = \min\{0, V(x)\}$. The remainder term in the result of Tamura is $O(\lambda^{-1/2})$, under the condition $V(x) \in C^\infty(\mathbb{R}^3\backslash\{0\})$, $V(x) \sim |x|^{-d}, 0 \le d < 2$, $|\partial_x^\alpha V(x)| \le C_\alpha V(x)|x|^{-|\alpha|}$ for $|x| \le 1$ and $|V(x)| \le C\langle x\rangle^{-m}$, $m > 2$, $|\partial_x^\alpha V(x)| \le C_\alpha V(x)\langle x\rangle^{-l|\alpha|}$, $3 - m < l \le 1$, for $|x| \ge 1$. For the proof [4] and [44] use the min-max principle combined with a technique of Dirichlet-Neumann bracketing while [31],[40] and [54] use the Feyman-Kac formula. Tamura [68] uses pseudo-differential operators and Fourier integral operators to study this problem for $N = 2$. Klaus-Simon studied this problem (see [34]) for the 3-body Schrödinger operators. The main tool they used in that paper is Birman-Schwinger kernel. Their result is the following : assume $\nu \ge 3$, $V \le 0$, $V_{ij} \in C_0^\infty (1 \le i < j \le 3)$ and in the unique two-cluster case, the number of bound states, $N(\mu)$, of $-\mu\Delta + \sum_{1\le i<j\le 3} V_{ij}$ on $L^2(\mathbb{R}^{2\nu})$ obeys the asymptotic formula :

$$\lim_{\mu\to 0} \mu^\nu N(\mu) = \tau_{2\nu}\,(2\pi)^{-2\nu} \int_{V(x)\le\Sigma_{cl}} [\Sigma_{cl} - V(x)]^\nu dx.$$

Here τ_ν is the volume of the unit sphere in $\mathbb{R}^{2\nu}$.

Semi-classical limit of Riesz means of discrete eigenvalues of Schrödinger operator $P(h) = -h^2\Delta + V$ has been studies by Helffer and Robert [23], Bruneau [9]. Let $\Sigma = \underline{\lim}_{|x|\to\infty} V(x)$. They studied semi-classical limit of Riesz means of discrete eigenvalues less than λ_0. Here λ_0 is a constant that $\lambda_0 < \Sigma$. They used pseudo-differential operators to get the asymptotic expansion of

$$R_\gamma(h, \lambda) = \sum_{e_j(h)\le\lambda} (\lambda - e_j(h))^\gamma$$

for $\lambda < \lambda_0$ as $h \to 0$. They did not get the semi-classical limit of Riesz means of all bound states of $P(h)$.

OUR METHODS AND RESULTS. In Chapter 2, we concentrate on the unique two-cluster $N-$body Schrödinger operators. We also use the Birman-Schwinger kernel to get the semi-classical limit of the number of discrete eigenvalues of the N-body Schrödinger operator. Hence we recall some results of Birman-Schwinger kernel in Section 2.2. We use these results to get the leading term of the number of discrete eigenvalues of the $N-$body Schrödinger operator. We suppose that

$$V_{ij} \leq 0, \quad V_{ij} \in C^2(\mathbb{R}^\nu) \text{ and } |V_{ij}| \leq C\langle x\rangle^{-\rho} \text{ with some } \rho > 2. \tag{1.3}$$

First, we explain why we need the condition $V_{ij} \in C^2(\mathbb{R}^\nu)$. In fact, we do not use this condition in Section 2.2, since the condition $|V_{ij}| \leq C\langle x\rangle^{-\rho}$ with some $\rho > 2$ is enough. The condition $V_{ij} \in C^2(\mathbb{R}^\nu)$ is used to prove that $\Sigma_h - \Sigma_{cl} = O(h)$. The number of discrete eigenvalues in $(-\infty, \Sigma_{cl})$ is standard Dirichlet-Neumann bracket. The difficulty is to estimate the number of eigenvalues in the interval (Σ_{cl}, Σ_h).

Our result for the semi-classical limit of the number of discrete eigenvalues which is less than Σ_h, $N(h)$, of $H(h)$ is the following :

Theorem 1.1. *Let $\nu \geq 3$ and $H_0 = -h^2\Delta$ on $L^2\left(\mathbb{R}^{(N-1)\nu}\right)$. Suppose (1.3) holds. Then*

$$N(h) = h^{-(N-1)\nu}\tau_{(N-1)\nu}\,(2\pi)^{-(N-1)\nu}\int_{V(x)\leq\Sigma_{cl}} [\Sigma_{cl} - V(x)]^{(N-1)\nu/2}dx\,(1 + o(1)) \tag{1.4}$$

where $\tau_{(N-1)\nu}$ is the volume of the unit sphere in $\mathbb{R}^{(N-1)\nu}$.

In Section 2.4, we use Dirichlet-Neumann bracket to study the Riesz means of discrete eigenvalues of the N-body Schrödinger operator. We suppose that

$$V_{ij} \in C_0^\infty(\mathbb{R}^\nu) \text{ and } V_{ij} \leq 0. \tag{1.5}$$

Define the Riesz means of the N-body Schrödinger operator as

$$R_\gamma(h, \Sigma_h) = \sum_{e_j(h)\leq\Sigma_h} |e_j(h) - \Sigma_{cl}|^\gamma$$

for $\gamma \geq 0$. We first consider the semi-classical limit of Riesz means of discrete eigenvalues of the 2-body Schrödinger operator. Then we use this result to estimate $\sum_{e_j(h)\leq\Sigma_{cl}} |e_j(h) - \Sigma_{cl}|^\gamma$. Since

$$\sum_{\Sigma_{cl}\leq e_j(h)\leq\Sigma_h} |e_j(h) - \Sigma_{cl}|^\gamma \leq N_2(h)(\Sigma_h - \Sigma_{cl})^\gamma$$

and $\Sigma_h - \Sigma_{cl} = O(h)$, we can get the estimate of $\sum_{\Sigma_{cl}\leq e_j(h)\leq\Sigma_h} |e_j(h) - \Sigma_{cl}|^\gamma$. Here $N_2(h)$ is the number of eigenvalues in (Σ_{cl}, Σ_h).

We get the following result :

Theorem 1.2. *Let $\nu \geq 3$. Let $H_0 = -h^2\Delta$ on $L^2(\mathbb{R}^{(N-1)\nu})$. Suppose (1.5) holds. Then we have*

$$R_\gamma(h, \Sigma_h) = C(\gamma, \nu)h^{-(N-1)\nu} \int_{\{x;\ V(x)\leq\Sigma_{cl}\}} (\Sigma_{cl} - V(x))^{\gamma+\frac{(N-1)\nu}{2}}\, dx\, (1 + O(h^t)), \tag{1.6}$$

where $C(\gamma, \nu) = (2\pi)^{-(N-1)\nu}\tau_{(N-1)\nu}C_{\gamma,(N-1)\nu}$, $C_{\gamma,(N-1)\nu} = \gamma\int_0^1 \beta^{\gamma-1}(1-\beta)^{n/2}\, d\beta$ and $\tau_{(N-1)\nu}$ is the volume of the unit sphere in $\mathbb{R}^{(N-1)\nu}$, $t = \min\{\frac{1}{2},\ \gamma\}$.

NEW POINTS IN METHODS AND RESULTS. Our first result is the generalization of the result of Klaus-Simon's [34]. In their paper, the conditions on V_{ij} are too strong and they only consider the 3-body case. Our conditions on V are much weaker than theirs and we get the result for general N. In the second part of Chapter 2, we get the semi-classical limit of Riesz means of all discrete eigenvalues of $H(h)$.

1.2 Coupling constant limits and the asymptotic expansion of resolvent of Schrödinger operators with critical potentials

PROBLEMS. In Chapter 3, we consider a family of Schrödinger operators

$$P(\lambda) = P_0 + \lambda V$$

in $L^2(\mathbb{R}^n)$. Here $P_0 = -\Delta + \frac{q(\theta)}{r^2}$ on $L^2(\mathbb{R}^n)$ and (r, θ) is the polar coordinates on $\mathbb{R}^n$. $V \leq 0$ is a non-zero continuous function and satisfies

$$|V(x)| \leq C\langle x\rangle^{-\rho_0}, \qquad \text{for some } \rho_0 > 2. \tag{1.7}$$

In Section 2.5 (Chapter 2), we study the N-body Schrödinger operator with Coulomb potential, and we get that the effective potential has the following expansion for ρ large (see Section 2.5 (Chapter 2)) :

$$I_{eff}(\rho) = \frac{f_1(\hat{\rho})}{|\rho|^2} + \frac{f_2(\hat{\rho})}{|\rho|^3} + o(\frac{1}{|\rho|^4})$$

with $\hat{\rho} = \frac{\rho}{|\rho|}$, if $C_1 \equiv \sum\limits_{i\in a_1} e_i = 0$ or $C_2 \equiv \sum\limits_{i\in a_2} e_i = 0$. e_i is the charge of i-th particle. $f_j(\hat{\rho})(j = 1, 2)$ are the continuous functions of $\hat{\rho}$. Hence, we study the Schrödinger operators of the form $P(\lambda) = P_0 + \lambda V$. We suppose that

$$-\Delta_s + q(\theta) \geq -\frac{1}{4}(n-2)^2, \quad \text{on } L^2(\mathbb{S}^{n-1}). \tag{1.8}$$

Here $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$. Under this assumption, the operator P_0 is a positive operator on $L^2(\mathbb{R}^n)$ (see Chapter 3 for details). Set

$$\sigma_\infty = \{\nu; \nu = \sqrt{\lambda + \frac{1}{4}(n-2)^2},\ \lambda \in \sigma(-\Delta_s + q(\theta))\}, \quad \sigma_k = \sigma_\infty \cap [0, k],\ k \in \mathbb{N}.$$

If $q = 0$, then $P_0 = -\Delta$. In this case, σ_∞ consists of either only half-integers (n odd) or only integers (n even). In particular, for Laplace operator $-\Delta$, one has $\sigma_1 = \{0, 1\}$, $n = 2$; $\sigma_1 = \{\frac{1}{2}\}$, $n = 3$; $\sigma_1 = \{1\}$, $n = 4$; $\sigma_1 = \emptyset$, $n = 5$.

We show that the discrete eigenvalue, $e_i(\lambda)$, of $P(\lambda)$ converges to 0, as $\lambda \downarrow \lambda_0$ for some $\lambda_0 > 0$ in Chapter 3. We study the asymptotic behavior of $e_i(\lambda)$ as $\lambda \downarrow \lambda_0$.

Let $H^{r,s}(\mathbb{R}^n)$, $r \in \mathbb{Z}$, $s \in \mathbb{R}$, denote the weighted Sobolev space of order r with volume element $\langle x\rangle^{2s}\, dx$.

Definition 1.3. *Set $\mathcal{N}(\lambda) = \{u; P(\lambda)u = 0, u \in H^{1,-s}, \forall s > 1\}$, for $\lambda \geq \lambda_0$. If $\mathcal{N}(\lambda)\backslash L^2 \neq \{0\}$, we say that 0 is the resonance of $P(\lambda)$. A non-zero function $u \in \mathcal{N}(\lambda)\backslash L^2$ is called a resonant state of $P(\lambda)$ at zero.* $\dim \mathcal{N}(\lambda)\backslash L^2$ *is called the multiplicity of 0 as the resonance of $P(\lambda)$.*

Let λ_1 be the value at which $e_1(\lambda)$ converges to 0. $e_1(\lambda)$ is the smallest eigenvalue of $P(\lambda)$. In Chapter 3, we show that 0 is not the eigenvalue of $P(\lambda_1)$. The multiplicity of 0 as the resonance of $P(\lambda_1)$ is also studied in that chapter.

In Chapter 3, we also consider the Schrödinger operator $P = -\Delta + \tilde{V}$. Here $\tilde{V} = V_1 + V_2$. $V_1 \in C(\mathbb{R}^n)$ and $V_1(x) = \frac{q(\theta)}{|x|^2}$ for $|x| > R$. $R > 0$ is the constant. $q(\theta)$ satisfies (1.8). $V_2 \in C(\mathbb{R}^n)$ satisfies (1.7). Let

$$\tilde{P}_0 = \chi_1(-\Delta)\chi_1 + \chi_2 P_0 \chi_2.$$

Here $0 \leq \chi_1 \leq 1$ is a smooth function on $\mathbb{R}^n$ such that $\chi_1(x) = 1$ for $|x| \leq R$, $\mathrm{supp}\chi_1 \subset \{x;\ |x| \leq R_1\}$ for some $R_1 > R > 0$ and $\chi_1^2 + \chi_2^2 = 1$. Then P can be treated as the perturbation of $\tilde{P}_0$ and P also can be treated as the perturbation of P_0. One has

$$P = \tilde{P}_0 + \tilde{W} = P_0 + V.$$

Here

$$V = \tilde{V} - \frac{q(\theta)}{r^2},\ \tilde{W} = V + W,\ W = \frac{\chi_1^2}{r^2} q(\theta) + \sum_{i=1}^{2} |\nabla \chi_i|^2.$$

Then $\tilde{W}$ is a continuous function and satisfies $|\tilde{W}| \leq C\langle x\rangle^{-\rho_0}$. V has singularity at 0. We study the asymptotic expansion of $(P - z)^{-1}$ for z near 0, $\Im z \neq 0$ in Chapter 3.

KNOWN RESULTS. Klaus-Simon [33] studied the asymptotic behavior of eigenvalue, $e_i(\lambda)$, of $H(\lambda) = -\Delta + \lambda V$, λ near λ_0. λ_0 is the value at which $e_i(\lambda) \to 0$ as $\lambda \downarrow \lambda_0$. They got the leading term of $e_i(\lambda)$ using Birman-Schwinger kernel. The result depends on the dimension of the space. If the dimension of the space $n = 3$,

$$e_i(\lambda) = -c(\lambda - \lambda_0)^2 + O((\lambda - \lambda_0)^3)$$

or

$$e_i(\lambda) = -c(\lambda - \lambda_0) + O((\lambda - \lambda_0)^{3/2})$$

and the ground state is in the first case. If $n \geq 5$, and n is odd, then

$$e_i(\lambda) = a(\lambda - \lambda_0)^2 + O((\lambda - \lambda_0)^3).$$

If $n \geq 6$, and n is even, then

$$e_i(\lambda) = \sum_{n=2}^{\infty} \sum_{m=0}^{\infty} c_{nm}[(\lambda - \lambda_0)^{1/2}]^n[(\lambda - \lambda_0)^{1/2} \ln(\lambda - \lambda_0)]. \tag{1.9}$$

If $n = 4$, then $e_i(\lambda)$ obeys (1.9) or

$$e_i(\lambda) = \frac{q(\lambda - \lambda_0)}{[\ln(\lambda - \lambda_0)]^2} + \text{lower order term}$$

Fassri-Klaus [20] also studied this problem for Schrödinger operator $-\Delta + V + \lambda W$ with V periodic. They also used Birman-Schwinger kernel in their paper.

The operators of P_0 and P have been studied by Carron [14] and X.P. Wang [69, 70]. In [14], Carron gave some properties of $(P_0 - z)^{-1}$ for z near 0 and got the formula for the jump at zero of the spectral shift function associated with the pair (P_0, P). In [70], X.P. Wang gave the asymptotic expansion of $(P_0 - z)^{-1}$ for z near 0, $\Im z \neq 0$ which is used to study the asymptotic expansion of $(P - z)^{-1}$ for z near 0, $\Im z \neq 0$. In [69], asymptotic expansion of $(P - z)^{-1}$ has been studied for $\Im z \neq 0$, z near 0.

OUR METHODS AND RESULTS. In the first part of Chapter 3, we use Birman-Schwinger kernel to study the asymptotic behavior of $e_i(\lambda)$ as $\lambda \downarrow \lambda_0$. To get the leading term of $e_i(\lambda)$ as $\lambda \downarrow \lambda_0$, we need to know the asymptotic expansion of $(P_0 - \alpha)^{-1}$ for α near 0, $\alpha < 0$, which has been studied by X.P. Wang ([70]). We recall some results of P_0 and give some properties of Birman-Schwinger kernel, $|V|^{1/2}(P_0 - z)^{-1}|V|^{1/2}$, in Section 3.2(Chapter 3). For the technical reason, we let $V \leq 0$ when we study the asymptotic behavior of discrete eigenvalues. We show that there exists a one-to-one correspondence between the discrete eigenvalues of $P(\lambda)$ and the discrete eigenvalues of

$$K(\alpha) = |V|^{1/2}(P_0 - \alpha)^{-1}|V|^{1/2}, \quad \alpha < 0.$$

Therefore, we study discrete eigenvalues of $K(\alpha)$ in Section 3.3 (Chapter 3), and then get the asymptotic behavior of discrete eigenvalues of $P(\lambda)$. Our main result is the following :

Theorem 1.4. *Assume $0 \notin \sigma_\infty$ and $n \geq 3$. Suppose that $e_1(\lambda)$ is the ground state energy (the smallest eigenvalue) of $P(\lambda)$. ϕ is some function in $L^2(\mathbb{R}^n)$. If $\rho_0 > 6$, one of three exclusive situations holds :*

(a). If $\sigma_1 = \emptyset$, then $e_1(\lambda) = -c(\lambda - \lambda_0) + o(\lambda - \lambda_0)$, $c = (\lambda_0 \| F_0 |V|^{\frac{1}{2}} \phi \|)^{-2} \neq 0$;

(b). If $\nu_0 = 1$, then $e_1(\lambda) = -c\frac{\lambda - \lambda_0}{\ln(\lambda - \lambda_0)} + o(\frac{\lambda - \lambda_0}{\ln(\lambda - \lambda_0)})$, $c = \lambda_0^{-2}\langle \phi, |V|^{1/2} G_{1,0} \pi_1 |V|^{1/2} \phi \rangle^{-1} \neq 0$;

(c). If $\nu_0 < 1$, then $e_1(\lambda) = c((\lambda - \lambda_0)^{\frac{1}{\nu_0}}) + o((\lambda - \lambda_0)^{\frac{1}{\nu_0}})$, $c = \lambda_0^{-2}\langle \phi, |V|^{1/2} G_{\nu_0,0} \pi_{\nu_0} |V|^{1/2} \phi \rangle^{-1} \neq 0$.

Similarly, we can get the asymptotic behavior of the other discrete eigenvalues of $P(\lambda)$. In Section 3.4, we use the fact that discrete eigenvalues, $\mu_i(\alpha)$, of $K(\alpha)$ are continuous and monotonous with respect to α to get the result for the multiplicity of 0 as the resonance of $P(\lambda_0)$. Here λ_0 is the value at which the smallest eigenvalue of $P(\lambda)$ converges to 0.

Theorem 1.5. *Suppose* $\sum_{0<\nu\leq 1} n_\nu = m$. *If* $m > 0$, *then there exists* $\lambda_1 > \lambda_0$ *such that for all* $\lambda_0 < \lambda < \lambda_1$, *the number of eigenvalues, less than 0, of* $P(\lambda)$ *is equal to the dimension of* $\mathcal{N}(\lambda_0)$.

In Chapter 3, we also study the asymptotic expansion of $(P-z)^{-1}$ for z near 0, $\Im z \neq 0$, which is used to study the spectral shift function. X.P. Wang ([69]) has studied the asymptotic expansion of $(P_0+V-z)^{-1}$ with V satisfying $|V| \leq C_\alpha \langle x\rangle^{-\rho_0}$. Since our $V = \tilde{V} - \frac{q(\theta)}{r^2}$ with $\tilde{V}$ satisfying the condition (1.7), we can not use X.P. Wang's result directly. Note $\frac{\chi_1^2}{r^2}q(\theta) \in \mathcal{L}(1,-s;-1,s)$, $\forall s > 0$ for $n \geq 3$. We can use X.P. Wang's method to get the asymptotic expansion of $(P-z)^{-1}$. For $z \in \mathbb{C}\backslash\mathbb{R}$, z near 0, we have

$$(P_0-z)^{-1}(P-z) = 1 + F(z),$$

with

$$F(z) = (P_0-z)^{-1}V.$$

In Section 3.5 (Chapter 3), we prove that $1+F(0)$ is a Fredholm operator in $\mathcal{L}(1,-s;1,-s)$. Let $\mathcal{N} = \{u; Pu = 0,\ u \in H^{1,-s}, \forall s > 1\}$. If $\mathcal{N} = \{0\}$, then $(1+F(z))^{-1}$ exists for z near 0, $\Im z \neq 0$. We can get the asymptotic expansion of $(P-z)^{-1}$ by formula $(P-z)^{-1} = (1+F(z))^{-1}(P_0-z)^{-1}$. If $\mathcal{N} \neq \{0\}$, we solve a Grushin problem associated to $P-z$, and get that

$$(P-z)^{-1} = E(z) - E_+(z)E_{+-}(z)^{-1}E_-(z)$$

for z near 0. Here $E(z),\ E_+(z),\ E_-(z)$ are holomorphic near 0. Hence, the main task is to study the asymptotic expansion of $E_{+-}(z)^{-1}$ near 0. Note $\frac{\chi_1^2}{r^2}q(\theta) \notin \mathcal{L}(1,-s;1,-s)$ for $n = 2$. Therefore, we can not get the expansion of $(P-z)^{-1}$ by this method for $n = 2$.

1.3 Low-energy asymptotic of the spectral shift function for perturbation with critical decay

PROBLEMS. In Chapter 4, we study the spectral shift function. The spectral shift function was introduced in 1952 by the physicist I. M. Lifshitz in paper [41] as a trace perturbation formula in quantum mechanics. Its mathematical theory was created by M. G. Krein. Let P, P_0 be a pair of self-adjoint operators in some separable Hilbert space H. M. G. Krein proved in [36] that if

$V = P - P_0$ is a trace class operator, then $\forall f \in \mathcal{S}(\mathbb{R})$, $f(P) - f(P_0)$ is of trace class and there exists some function $\xi \in L^1(\mathbb{R})$, called spectral shift function, such that

$$\mathrm{Tr}\,(f(P) - f(P_0)) = -\int_{\mathbb{R}} f'(\lambda)\xi(\lambda)\,d\lambda, \quad \forall f \in \mathcal{S}(\mathbb{R}). \tag{1.10}$$

Then it was extended by him in [37] (see [38], for a more complete exposition) to operators $P_0,\ P$ with a trace class difference $R(z) - R_0(z)$. Here $R_0(z) = (P_0 - z)^{-1}$ and $R(z) = (P - z)^{-1}$. Yafaev ([73]) proved that if there exists some c such that $P + cI$ and $P_0 + cI$ are positive and there exists some $k \in \mathbb{N}^*$,

$$\|(P + cI)^{-k} - (P_0 + cI)^{-k}\|_{tr} < \infty. \tag{1.11}$$

then $f(P) - f(P_0)$ is of trace class and there exists some function $\xi \in L^1_{loc}(\mathbb{R})$, such that (1.10) holds. The right hand side of (1.10) can be interpreted as $\langle f, \xi' \rangle$, where ξ' is the derivative of ξ in the sense of distributions. For simplicity we assume that $P_0,\ P$ are bounded below. By the Birman-Krein theory ([6]), ξ is related with the scattering phase, $\rho(\lambda) = \arg\det S(\lambda)$, by the formula

$$\rho(\lambda) = 2\pi\xi(\lambda), \qquad \mathrm{mod}\ 2\pi\mathbb{Z},$$

and

$$\xi'(\lambda) = \frac{1}{2\pi}\mathrm{Tr}\,T(\lambda),$$

where $T(\lambda) = -iS(\lambda)^*\frac{d}{d\lambda}S(\lambda)$ is the Eisenbud-Wigner formula for the time-delay operator.

Let $0 \le \chi_1 \le 1$ be a smooth function on $\mathbb{R}^n$ such that $\chi_1(x) = 1$ for $|x| \le R$, $\mathrm{supp}\chi_1 \subset \{x;\ |x| \le R_1\}$ for some $R_1 > R > 0$ and $\chi_1^2 + \chi_2^2 = 1$. Let $P_0 = \chi_1(-\Delta)\chi_1 + \chi_2(-\Delta + \frac{q(\theta)}{r^2})\chi_2$ and $P = P_0 + V$ with V satisfying

$$|\partial_x^\alpha V(x)| \le C_\alpha \langle x\rangle^{-\rho-|\alpha|} \tag{1.12}$$

for some $\rho > 2$. We study the spectral shift function, $\xi(\lambda)$, of the pair (P_0, P).

We are mainly interested in the low-energy asymptotics of the derivative of the spectral shift function. We use the asymptotic expansion of $R_0(z) = (P_0 - z)^{-1}$ and $R(z) = (P - z)^{-1}$ for $\Im z \neq 0$, z near 0, to study the asymptotic behavior of $\xi'(\lambda)$ for λ near 0. After that, we use this result to prove the Levinson's Theorem. The Levinson's theorem is a fundamental theorem in quantum scattering theory, which shows the relation between the number of bound states and the phase shift at zero momentum. Levinson first established and proved this theorem in [39] for the Schrödinger equation with a spherically symmetric potential $V(r)$.

KNOWN RESULTS. The spectral shift function for Schrödinger operators has been studied by many authors (see for example [1],[11],[12],[47], [49],[50],[73]). High-energy asymptotics of the spectral shift function was studied in these paper. The result got by Robert in [49] is the following : assume $|\partial_x^\alpha V| \le C_\alpha\langle x\rangle^{-\rho-|\alpha|}$ with $\rho > n$, then the spectral shift function, $\xi(\lambda)$, for the pair $(-\Delta, \Delta + V)$ satisfying :

(i). $\xi(\lambda)$ is C^∞ in $(0,\infty)$.
(ii). $\frac{d^k}{d\lambda^k}\xi(\lambda)$ has a complete asymptotic expansion for $\lambda\to\infty$,

$$\frac{d^k}{d\lambda^k}\xi(\lambda)\sim\lambda^{n/2-k-1}\sum_{j\geq 0}\alpha_j^{(k)}\lambda^{-j}.$$

Levinson's theorem has been studied by many authors (see [8],[42], [32],[18] and references therein). In [8], Bolle got the Levinson's theorem for V satisfying $\langle x\rangle^n V\in L^1(\mathbb{R}^d)(d=1,2,3)$ for appropriate n, and $V\in L^{4/3}(\mathbb{R}^2)$ for $d=2$, such that the absence of positive embedded eigenvalues and of the singular continuous spectrum of $-\Delta+V$ is guaranteed. Levinson's theorem for the nonlocal interaction in one dimension was studied in [18].

OUR METHODS AND RESULTS. In Chapter 4, we use the asymptotic expansion of $(P_0-z)^{-1}$ and $(P-z)^{-1}$ for z near 0, $\Im z\neq 0$ to study the low-energy asymptotics of the derivative of the spectral shift function.

Our main result is the following :

Theorem 1.6. *Suppose $n\geq 3$ and $0\notin\sigma_\infty$. If (1.12) holds for $\rho>\max\{6,n+2\}$, one has*

$$\xi'(\lambda)=J_0\delta(\lambda)+g(\lambda),$$

with $|g(\lambda)|=O(\lambda^{-1+\epsilon_0})$ for some $\epsilon_0>0$, as $\lambda\downarrow 0$. Here $J_0=N_0+\sum_{j=1}^{k_0}\varsigma_j m_j$ where N_0 is the multiplicity of zero as eigenvalues of P and m_j the multiplicity of ς_j-resonance of zero.

Using this Theorem and the asymptotic expansion of $\xi'(\lambda)$ for λ large (see [49]), we can get the Levinson's Theorem.

Theorem 1.7. *Suppose $n\geq 3$ and $0\notin\sigma_\infty$. If (1.12) holds for $\rho>n+3$, one has*

$$\int_0^\infty(\xi'(\lambda)-\sum_{j=1}^{[\frac{n}{2}]}c_j\lambda^{[\frac{n}{2}]-1-j})\,d\lambda=-(N+J_0)+\beta_{n/2}. \tag{1.13}$$

$\beta_{n/2}$ depends on n and V and $\beta_{n/2}=0$ if n is odd.

1.4 Notation

We present some notations used is this thesis.

arg : the argument of a complex number
$C^\infty(\Omega)$: the set of infinitely differentiable functions on an open set Ω
$C_0^\infty(\Omega)$: the subset of $C^\infty(\Omega)$ consisting of functions with compact support
$\mathbb{C}$: the field of complex numbers

dim : the dimension of a linear set
$H^{r,s}(\mathbb{R}^n)$: weighted Sobolev space of order r with volume element $\langle x\rangle^{2s}\, dx$
$\mathfrak{I}$: the imaginary part of a complex number
inf : the infimum
$\mathcal{L}(H^{r,s}, H^{r',s'})(\mathcal{L}(r, s; r', s'))$: the set of bounded operator form $H^{r,s}$ to $H^{r',s'}$
$L^p(p \geq 1)$: the space of functions whose $p-$th power is integrable
$\ln z$: $\ln|z| + iargz$ with $0 \leq argz < 2\pi$
$\mathbb{R}^d$: Euclidean space of dimension d
$\mathbb{R}^+$: $[0, \infty)$
$\mathfrak{R}$; the real part of a complex number
$\mathcal{S}(\mathbb{R}^n)$: the set of Schrwarz function
$\mathscr{S}_p$: the set of operator whose $p-$th power is a trace class operator
$\mathbb{S}^{n-1}$: unit sphere in $\mathbb{R}^n$
sgn : sign
sup : supremum
$\sigma(\cdot)$: the spectra of an operator
$\sigma_{pp}(\cdot)$ the point spectra of an operator
Tr : the trace of an operator
$s-\lim$: the strong limit of vectors and operators
z^ν : $e^{\nu \ln z}$
$\mathbb{Z}$ the set of integers
$\langle\cdot,\cdot\rangle$: scalar product on $L^2(\mathbb{R}^n)$ or on $L^2(\mathbb{R}^+;\ r^{n-1}dr)$
$(\cdot,\cdot)$: scalar product on $L^2(\mathbb{S}^{n-1})$
$-\Delta_s$: the Laplace operator on the unit sphere in $\mathbb{R}^n$
$\|\cdot\|$: the norm of an operator
$\|\cdot\|_{tr}$: the trace norm of an operator
$\|\cdot\|_p$: norm in $\mathscr{S}_p$ or L^p
$\langle x\rangle$: $(1 + x^2)^{1/2}$

2 - Semi-classical limits of the number and Riesz means of discrete eigenvalues of the N-body Schrödinger operators

2.1 Introduction

In this chapter, we consider the N-body system of $\nu-$dimensional ($\nu \geq 3$) particles on $L^2(\mathbb{R}^{(N-1)\nu})$:

$$H = H_0 + V;$$
$$V = \sum_{i<j} V_{ij}(r_i - r_j),\ r_i \in \mathbb{R}^\nu.$$

Here H_0 is the operator resulting from removing the center of mass from $\sum_{i=1}^{N} -(2m_i)^{-1}\Delta_i$. m_i is the mass of the $i-$th particle, Δ_i is the Lpalacian in the r_i variables. We discuss the semi-classical limits of the number and the Riesz means of discrete eigenvalues of N-body Schrödinger operators $H(h) = h^2 H_0 + V$.

The semi-classical limit of the number of discrete eigenvalues has been studied by many authors for $N = 2$ (see [4], [31], [40], [44], [54], [68]). Roughly speaking, in the case of 3-dimensional space, $\mathbb{R}^3$, the result obtained by these authors can be formulated as following : assume that $V(x)$ is real and $V(x) \in L^{3/2}(\mathbb{R}^3)$, the number of discrete eigenvalues less than 0, $N(\lambda)$, of $-\Delta + \lambda V$ obeys the asymptotic formula :

$$N(\lambda) = (6\pi^2)^{-1} \int |V_-(x)|^{3/2}\, dx(1 + o(1)), \quad \lambda \to \infty.$$

Here $V_-(x) = \min\{0, V(x)\}$. The remainder term in the result of Tamura is $O(\lambda^{-1/2})$ under the condition $V(x) \in C^\infty(\mathbb{R}^3\backslash\{0\})$, $V(x) \sim |x|^{-d}, 0 \leq d < 2$, $|\partial_x^\alpha V(x)| \leq C_\alpha V(x)|x|^{-|\alpha|}$ for $|x| \leq 1$ and $|V(x)| \leq C\langle x\rangle^{-m}$, $m > 2$, $|\partial_x^\alpha V(x)| \leq C_\alpha V(x)\langle x\rangle^{-l|\alpha|}$, $3 - m < l \leq 1$, for $|x| \geq 1$. For the proof [4] and [44] use the min-max principle combined with a technique of Dirichlet-Neumann bracketing while [31],[40] and [54] use the Feyman-Kac formula. Tamura [68] uses pseudo-differential operators and Fourier integral operators to study this problem for $N = 2$. The semi-classical limit of discrete eigenvalues for the 3-body Schrödinger operator has been studied by

Klaus-Simon (see [34]). The main tool they used in that paper is Birman-Schwinger kernel. Their result is the following : Assume $\nu \geq 3$, $V \leq 0$, $V_{ij} \in C_0^\infty (1 \leq i < j \leq 3)$. Denote $a_{ij} = \min_{x \in \mathbb{R}^\nu} V_{ij}(x)$, $\Sigma_{cl} = \min_{i,j} a_{ij}$. If there exists only one a_{ij}, such that $a_{ij} = \Sigma_{cl}$, then the number of discrete eigenvalues, $N(\mu)$, of $-\mu\Delta + \sum_{1 \leq i < j \leq 3} V_{ij}$ on $L^2(\mathbb{R}^{2\nu})$ obeys the asymptotic formula :

$$\lim_{\mu \to 0} \mu^\nu N(\mu) = \tau_{2\nu} (2\pi)^{-2\nu} \int_{V(x) \leq \Sigma_{cl}} [\Sigma_{cl} - V(x)]^\nu dx.$$

Here τ_ν is the volume of the unit sphere in $\mathbb{R}^{2\nu}$.

Semi-classical limit of Riesz means of discrete eigenvalues of Schrödinger operator $P(h) = -h^2\Delta + V$ has been studied by Helffer and Robert [23], Bruneau [9]. Let $\Sigma = \varliminf_{|x| \to \infty} V(x)$. They studied semi-classical limit of Riesz means of discrete eigenvalues less than λ_0. Here λ_0 is a constant less than Σ. They used pseudo-differential operators to get the asymptotic expansion of

$$R_\gamma(h, \lambda) = \sum_{e_j(h) \leq \lambda} (\lambda - e_j(h))^\gamma$$

for $\lambda < \lambda_0$ as $h \to 0$. $e_j(h)$ is the discrete eigenvalue of $P(h)$. They did not get the semi-classical limit of Riesz means of all bound states of $P(h)$.

Here is the plan of this chapter. We use the Birman-Schwinger kernel to study the number of discrete eigenvalues of the N-body Schrödinger operator $H(h)$. For the unique two-cluster Schrödinger operator, we introduce a suitable Birman-Schwinger kernel in Section 2.2 and recall the results of the Birman-Schwinger kernel. In Section 2.3, we use these results to discuss the semi-classical limit of the number of discrete eigenvalues of the unique two-cluster Schrödinger operator $H(h)$. The result we get is the following : if V's lie in $C^2(\mathbb{R}^n)$, and are negative, $|V_{ij}| \leq C\langle x \rangle^{-\rho}$ with some $\rho > 2$, then

$$N(h) = h^{-(N-1)\nu} \tau_{(N-1)\nu} (2\pi)^{-(N-1)\nu/2} \int_{V(x) \leq \Sigma_{cl}} [\Sigma_{cl} - V(x)]^{(N-1)\nu/2} dx (1 + O(1)). \tag{2.1}$$

for $\nu \geq 3$. Here $N(h)$ is the number of eigenvalues of $H(h)$ in $(-\infty, \Sigma_h)$. The number of eigenvalues of $H(h)$ in $(-\infty, \Sigma_{cl})$ is standard Dirichlet-Neumann bracketing [53, Section 8.15]. The difficulty is to estimate the number of discrete eigenvalues in $(\Sigma_{cl},\ \Sigma_h)$. In Section 2.4, we use Dirichlet-Neumann bracketing to study the semi-classical of the Riesz means of discrete eigenvalues of $H(h)$. The result we get is following : Assume $\nu \geq 3$, $H_0 = -h^2\Delta$ on $L^2(\mathbb{R}^{(N-1)\nu})$ and (1.5) holds, then the Riesz means of discrete eigenvalues of $H(h)$ obeys the following estimate

$$R_\gamma(h, \Sigma_h) = C(\gamma, \nu) h^{-(N-1)\nu} \int_{\{x;\ V(x) \leq \Sigma_{cl}\}} (\Sigma_{cl} - V(x))^{\gamma + \frac{(N-1)\nu}{2}}\ dx\ (1 + O(h^t)), \tag{2.2}$$

where $C(\gamma, \nu) = (2\pi)^{-(N-1)\nu} \tau_{(N-1)\nu} C_{\gamma,(N-1)\nu}$, $C_{\gamma,(N-1)\nu} = \gamma \int_0^1 \beta^{\gamma-1} (1-\beta)^{n/2}\ d\beta$ and $\tau_{(N-1)\nu}$ is the volume of the unit sphere in $\mathbb{R}^{(N-1)\nu}$, $t = \min\{\frac{1}{2},\ \gamma\}$. In Section 2.5, we study the N-body Schrödinger operator with Coulomb potential. We get the asymptotic expansion of the effective potential, $I_{eff}(\rho)$, for $|\rho| \to \infty$.

2.2 A Birman-Schwinger kernel for the $N-$body Schrödinger operator

Consider a general N-body system of ν-dimensional particles. H is an operator on $L^2\left(\mathbb{R}^{\nu(N-1)}\right)$:

$$H = H_0 + V. \tag{2.3}$$

Here H_0 is the operator resulting from removing the center of mass from $\sum_{i=1}^{N} -(2m_i)^{-1}\Delta_{r_i}$ and

$$V = \sum_{i<j} V_{ij}\left(r_i - r_j\right). \tag{2.4}$$

Here r_j and $m_j > 0$ denote the position and the mass of the j-th particle respectively, $-\Delta_{r_i}$ is the Laplacian in the r_i variables. In this chapter, we suppose

$$V_{ij} \leqslant 0, \tag{2.5}$$

and normally we suppose

$$V_{ij} \in C^2\left(\mathbb{R}^\nu\right), \text{ and } |V_{ij}(x)| \leq C\langle x\rangle^{-\epsilon_0}, \text{ with } \epsilon_0 > 2. \tag{2.6}$$

Condition (2.5) considerably simplifies the arguments since the Birman-Schwinger kernel is self-adjoint when (2.5) holds. But most results should hold without (2.5). Condition (2.6) is similarly made for technical convenience. We begin this section with introducing some notations. **NOTATION :** Let $C_i (i = 1, \cdots, k)$ be the subsets of $\{1, \cdots, N\}$. If $C_i \cap C_j = \emptyset$, $i \neq j$ and $\cup_{i=1}^{k} C_i = \{1, \cdots, N\}$, we say that $D = \{C_1, \cdots, C_k\}$ is a partition (or cluster decomposition) of $\{1, \cdots, N\}$. If $D = \{C_1, \cdots, C_k\}$, we denote $\#D = k$. If i, j are two numbers in $\{1, \cdots, N\}$, we write iDj if and only if i and j are in the same cluster C_l, and $\sim iDj$ if they are in the different clusters. Let $\sum_{iDj}$ (resp. $\sum_{\sim iDj}$) denote the sum over all pairs with i, j in the same (resp. different) cluster of D. We define

$$V_D = \sum_{iDj} V_{ij}; \quad I_D = \sum_{\sim iDj} V_{ij};$$

$$H_D = H - I_D; \quad \Sigma_D = \inf \sigma\left(H_D\right);$$

$$a_D = \min_{\{x; x\in\mathbb{R}^{(N-1)\nu}\}} V_D(x); \quad \Sigma_{cl} = \min_{\{D;\#D\geq 2\}} a_D.$$

For each D, there is a natural decomposition of $\mathscr{H} = L^2\left(\mathbb{R}^{(N-1)\nu}\right)$ as $\mathscr{H}^D \otimes \mathscr{H}_D$ with $\mathscr{H}_D =$ functions of r_{ij} with iDj and $\mathscr{H}^D =$ functions of $R_q - R_l$ where $R_q = \sum_{i\in C_q} m_i r_i / \sum_{i\in C_q} m_i$ is the center of the mass of C_q. Under this decomposition

$$H_D = h_D \otimes 1 + 1 \otimes t_D.$$

By the HVZ-theorem [54, Section XIII.5],

$$\inf \sigma_{ess}(H) = \Sigma, \tag{2.7}$$

where $\Sigma = \min_D \Sigma_D$. We say that H is unique two-cluster, if there is only one D with $\#D \geq 2$, such that $\Sigma = \Sigma_D$. The intuition is quite simple : if the continuum is two-cluster, the threshold is basically due to a $\nu-$dimensional Laplacian, namely the relative kinetic energy of the two clusters. So one expects the coupling constant behavior to be a ν-dimensional problem. We will concentrate on the unique two-cluster N-body Schrödinger operators.

By the hypothesis of unique two-cluster, we know that there exists a decomposition D with $\#(D) \equiv k \neq 1$ such that

$$\Sigma_D < \Sigma_{D'} \tag{2.8}$$

for all $D' \neq D$ with $\#(D') \neq 1$. It is easy to see that (2.8) can only hold if $\#(D) = 2$ (since $\Sigma_{D_1} \leq \Sigma_{D_2}$ if D_2 is a refinement of D_1 written $D_1 < D_2$). If (2.8) holds, then h_D must have an eigenvalue at the bottom of its spectrum ([54, Section XIII.12] since $\inf \sigma_{ess}(h_D) = \min(\Sigma_{D'} | D < D', D \neq D')$ by the HVZ-Theorem) and this eigenvalue will be simple. Thus, we pick once and for all a vector $\eta \in \mathscr{H}_D$ with $\| \eta \| = 1$ and

$$h_D \eta = \Sigma_D \eta. \tag{2.9}$$

Let p be the projection in $\mathscr{H}_D$ onto η and $P = p \otimes 1$, the projection in $\mathscr{H}$. We define $q = 1 - p, Q = 1 - P$. It follows that

$$\sigma(H_D \upharpoonright Q\mathscr{H}) = [\Sigma', \infty). \tag{2.10}$$

with

$$\Sigma' > \Sigma \equiv \min_{D'} \Sigma_{D'} = \Sigma_D \tag{2.11}$$

since Σ_D is the simple eigenvalue of h_D.

We define the Birman-Schwinger kernel by :

$$K(E) = | I_D |^{1/2} (H_D - E)^{-1} | I_D |^{1/2} \tag{2.12}$$

for $E < \Sigma$. In the following we recall some results for the Birman-Schwinger kernel. These results have been studied by Klaus-Simon for $V_{ij} \in C_0^\infty$. In fact, these results are also true for V_{ij} satisfying (2.6).

Proposition 2.1. *(Proposition 2.1 and Proposition 2.2 [34]) Let $E < \Sigma$. Then $E \in \sigma(H_D + \mu I_D)$ if and only if $\mu^{-1} \in \sigma(K(E))$. This result remains true if σ is replaced by σ_{ess} in both places. Moreover, the multiplicity of E as an eigenvalue of $H_D + \mu_0 I_D$ is exactly the multiplicity of μ_0^{-1} as an eigenvalue of $K(E)$.*

Remark 2.2. *This result is from Proposition 2.1 and Proposition 2.2 [34]. In fact, they did not use the condition $V_{ij} \in C_0^\infty$ in the proof.*

Proof. Write

$$H_D + \mu I_D - E = (H_D - E)\left(1 + \mu\,(H_D - E)^{-1}\, I_D\right).$$

We conclude that (since $E \notin \sigma\,(H_D)$), $E \in \sigma\,(H_D + \mu I_D)$ if and only if

$$\mu^{-1} \in \sigma\left(-\,(H_D - E)^{-1}\, I_D\right) = \sigma((H_D - E)^{-1}|I_D|) = \sigma\,(K\,(E))\,.$$

The last equality follows from the well-known fact (see eg.[16]) that $\sigma\,(AB) \setminus \{0\} = \sigma\,(BA) \setminus \{0\}$ for any bounded operators A, B.

If $E \in \sigma_{ess}\,(H_D + \mu_0 I_D)$, then since $\sigma_{ess}\,(H_D + \mu I_D) = [\Sigma_\mu, \infty)$ with Σ_μ decreasing in μ, $E \in \sigma_{ess}\,(H_D + \mu I_D)$ for all $\mu \geqslant \mu_0$, so $[0, \mu_0^{-1}) \subset \sigma\,(K\,(E))$ which implies that $\mu_0^{-1} \in \sigma_{ess}\,(K\,(E))$. If $E \notin \sigma_{ess}\,(H_D + \mu_0 I_D)$, then, we claim that for some $\delta > 0$, $E \notin \sigma\,(H_D + \mu I_D)$ for $\mu \in (\mu_0 - \delta, \mu_0 + \delta) \setminus \{\mu_0\}$. This is obvious if $E \notin \sigma\,(H_D + \mu_0 I_D)$ and if $E \in \sigma_{disc}\,(H_D + \mu_0 I_D)$, $E \notin \sigma_{disc}\,(H_D + \mu I_D)$ for μ near μ_0, since discrete eigenvalues of $H_D + \mu I_D$ are strictly monotonous in μ. It follows that μ^{-1} is either not in $\sigma\,(K\,(E))$ or is an isolated point of $\sigma(K\,(E))$. In the later case, we must show that the multiplicity of μ_0^{-1} as an eigenvalue of $K\,(E)$ is finite. But if $K\,(E)\,\psi = \mu_0^{-1}\psi$, then $(H_D + \mu I_D - E)\,\phi = 0$ with

$$\phi = (H_D - E)^{-1}\,|I_D|^{1/2}\psi,$$

since

$$(H_D + \mu I_D - E)\,\phi = |I_D|^{1/2}\,(1 - \mu K\,(E))\,\psi.$$

Since $(H_D - E)^{-1}\,|I_D|^{1/2} \upharpoonright \{\psi \mid K\,(E)\,\psi = \mu_0^{-1}\psi\}$ has no kernel, this shows that the multiplicity of μ_0^{-1} is at most the multiplicity of E as an eigenvalue of $H_D + \mu I_D$. Notice that if $(H_D + \mu I_D - E)\,\phi = 0$, then

$$\psi = |I_D|^{1/2}\phi$$

satisfies $K(E)\psi = \mu_0^{-1}\psi$. Since $|I_D|^{1/2}$ is non-vanishing on such ϕ (since $\ker\,(H_D - E) = \{0\}$). This ends the proof. □

In the proof of the proposition, we noted that $\sigma_{ess}\,(H_D + \mu I_D) = [\Sigma_\mu, \infty)$ with Σ_μ monotone in μ. It follows that :

Proposition 2.3. *(Proposition 2.3 [34]) Let $E < \Sigma$. Then $\sigma_{ess}\,(K\,(E)) = [0, \Lambda\,(E)]$ where $\Lambda\,(E) = \sup_\lambda \left\{\, \lambda;\ E \in \sigma_{ess}\left(H_D + \lambda^{-1} I_D\right)\right\}$.*

For later reference, it is useful to show that $K\,(E)$ has a norm limit as $E \uparrow \Sigma$. To see this, we decompose $K(E)$ as $K(E) = K_P(E) + K_Q\,(E)$ with

$$K_P\,(E) = |I_D|^{1/2}\,(H_D - E)^{-1}\,P\,|I_D|^{1/2};$$

$$K_Q\,(E) = |I_D|^{1/2}\,(H_D - E)^{-1}\,Q\,|I_D|^{1/2}.$$

where P, Q are the projections introduced before (2.10). Before analyzing $K\,(E)$, we need to introduce two definitions first.

Definition 2.4. *Let A be a compact operator, and all of the singular values of A are denoted by $\{\mu_i(A)\}$ with $\mu_1(A) \geqslant \mu_2(A) \geqslant \mu_3(A) \geqslant \cdots$. We say that $A \in \mathscr{S}_r,\ r \geq 1$, if*

$$\|A\|_r = (\sum_i \mu_i(A)^r)^{1/r} \ < \ \infty.$$

We say that $A \in \mathscr{S}_p^w,\ p > 1$, if

$$\|A\|_{p,w} = \sup_n n^{(-1+1/p)} \sum_{j=1}^{n} \mu_j(A) < \infty.$$

From the definition, it is easy to see that $\|A\| \leq \|A\|_r$ and $\|A\| \leq \|A\|_{p,w}$.

We want to consider the operators on $L^2(\mathbb{R}^\nu)$ formally given by $f(x)g(-i\nabla)$. First, we need to the give the definition of these operators (see [60] for details). If f and g are a.e. finite measurable function on $\mathbb{R}^\nu$, the sets D_f and D_g in L^2 given by $D_f = \{h \in L^2; fh \in L^2\}$, $D_g = \{h \in L^2; gh \in L^2\}$ are dense in L^2. For $\phi \in D_f$ and $\mathscr{F}(\psi) \in D_g$, ($\mathscr{F}$ denote Fourier transform), we can unambiguously define $\langle \phi, A\psi\rangle$ with A given formally by $f(x)g(-i\nabla)$ as the inner product of $f\phi$ and $\mathscr{F}^{-1}(g\mathscr{F}\psi)$ ($\mathscr{F}^{-1}$ denote inverse Fourier transform). Some of the most celebrated estimates in analysis assert that A is a bounded operator (to be pedantic, there is a bounded operator A with $\langle \phi, \tilde{A}\psi\rangle = \langle \phi, A\psi\rangle$ for $\phi \in D_f$, $\psi \in D_g$; henceforth, we will be sloppy about this point). For example, if $g(-i\nabla) = (-\Delta)^{-1}$, then $\mathscr{F}^{-1}(g\mathscr{F}\psi) = h * \psi$ with $h = c_\nu |x|^{-\nu+2}$ for $\nu \geq 3$. Thus using Holder's inequality ($2 < p < \infty$),

$$\|f(h * \psi)\|_2 \leq C_p \|f\|_p \|h\|_{p',w} \|\psi\|_2. \tag{2.13}$$

Here $p' = \frac{p}{p-1}$ and $\|\cdot\|_{p',w}$ is the norm of L^p_w. Under this definition, we have the following result.

Theorem 2.5. *([60, Theorem 4.2],) If $f \in L^p$ and $g \in L^p_w$ with $2 < p < \infty$, then $f(x)g(-i\nabla)$ is in $\mathscr{S}_p^w$ and*

$$\|f(x)g(-i\nabla)\|_{p,w} \leq C_p \|f\|_p \|g\|_{p,w}.$$

We use this theorem to study $K_P(E)$.

Proposition 2.6. *(1).$K_Q(E)$ has an analytic continuation to $\mathbb{C} \setminus [\Sigma', +\infty)$. In particular, on account of (2.11), $K_Q(E)$ has a norm limit as $E \uparrow \Sigma$.*

(2). Let $\nu \geq 3$ and let $r > \nu/2$. Then for all $E < \Sigma$, $K_P(E)$ is in the trace ideal $\mathscr{S}_r^w$ and as $E \uparrow \Sigma$, $K_P(E)$ converges to an operator $K_P(\Sigma)$ in $\mathscr{S}_r^w$ norm.

Proof : (1). This result is directly from (2.10) and (2.11).

(2). We use Theorem 2.5 to prove this result. We first prove the operator $K_P(\Sigma)$ is in $\mathscr{S}_r^w$. $K_P(E) \in \mathscr{S}_r^w$ with $E < \Sigma$ can be proved in the similar way. We need only to prove $(H_D - \Sigma)^{-1/2} P|I_D|^{1/2}$ is in $\mathscr{S}_{2r}^w$. Since $|I_D|^{1/2}\left(\sum_{\sim iDj}\left(V_{ij}^{1/2}\right)^{-1}\right)$ is multiplication by a function bounded by 1, it suffices to show the required fact for $(H_D - E)^{-1/2} P|V_{ij}|^{1/2}$ with $\sim iDj$. Since

$$(H_D - \Sigma)^{-1/2} P|V_{ij}|^{1/2} = t_D^{-1/2} P|V_{ij}\left(r_i - r_j\right)|^{1/2}$$

and $r_i - r_j = R + \rho$ with $R = R_{c_1} - R_{c_2}$ and ρ an "internal coordinate". By Theorem 2.5, we know the operator $(t_D)^{-1/2}\langle R\rangle^{-s} \in \mathscr{S}^w_\nu(L^2(\mathbb{R}^\nu; R))$ if $s > 1$. So $(t_D)^{-1/2}\langle R\rangle^{-s}P \in \mathscr{S}^w_t$ for all $t > \nu$ by the definition of $\mathscr{S}^w_p$. We need only to prove that $\langle R\rangle^s P|V_{ij}|^{1/2}$ is a bounded operator. Then it suffices to show that $(\langle R\rangle^s P|V_{ij}|^{1/2})^* = |V_{ij}|^{1/2}\langle R\rangle^s P$ is a bounded operator.

For each $u \in L^2(\mathbb{R}^{(N-1)\nu})$,

$$|V_{ij}|^{1/2}\langle R\rangle^s Pu(\zeta,\rho) = \int |V_{ij}|^{1/2}(\zeta,\rho)\langle R\rangle^s \eta(\zeta)\eta(\zeta')u(\zeta', R)d\zeta'.$$

(i) If $R \sim \rho$, we have

$$|\langle R\rangle^s \eta(\zeta')\eta(\zeta)|V_{ij}(R+\rho)|^{1/2}| \leq |C\langle\rho\rangle^s \eta(\zeta')\eta(\zeta)|$$

with C be a constant independent of R.

(ii) If $R \gtrsim \rho$, since $|V_{ij}(x)| \leq C\langle x\rangle^{-\epsilon_0}$, we know $|V_{ij}(R+\rho)|^{1/2} \leq C\langle R\rangle^{-\epsilon_0/2}$. Then, we can get

$$|\langle R\rangle^s \eta(\zeta')\eta(\zeta)|V_{ij}(R+\rho)|^{1/2}| \leq C\langle R\rangle^{(s-\epsilon_0/2)}|\eta(\zeta')\eta(\zeta)|.$$

Choose $1 < s < \epsilon_0/2$, one has

$$|\langle R\rangle^s \eta(\zeta')\eta(\zeta)|V_{ij}(R+\rho)|^{1/2}| \leq C|\eta(\zeta')\eta(\zeta)|.$$

(iii) If $R \lesssim \rho$,

$$|\langle R\rangle^s \eta(\zeta')\eta(\zeta)|V_{ij}(R+\rho)|^{1/2}| \leq C|\eta(\zeta')\eta(\zeta)|.$$

From the argument above, one has $\langle R\rangle^s P|V_{ij}(R+\rho)|^{1/2}$ is a bounded operator, since η has an exponential decay at infinity. It follows $(H_D - \Sigma)^{-1/2}P|I_D|^{1/2} \in \mathscr{S}^w_{2r}$. Then we get $K_P(\Sigma) \in \mathscr{S}^w_r$. In the similar way, we can get $K_P(E) \in \mathscr{S}^w_r$, and

$$\begin{aligned} K_P(\Sigma) - K_P(E) &= |I_D|^{1/2}((H_D - E)^{-1} - (H_D - \Sigma)^{-1})P|I_D|^{1/2} \\ &= (\Sigma - E)|I_D|^{1/2}((H_D - E)^{-1}(H_D - \Sigma)^{-1})P|I_D|^{1/2} \\ &\leq (\Sigma - E)^\delta |I_D|^{1/2}(t_D)^{-1-\delta}P|I_D|^{1/2} \end{aligned}$$

with $0 < \delta \leq 1$. Then, one has $K_P(\Sigma) - K_P(E)$ is in $\mathscr{S}^w_{2r}$ and tends to zero when $E \to \Sigma$. □

Theorem 2.7. *(Theorem 2.8 [34])*
$K(E)$ has a norm limit $K(\Sigma)$ as $E \uparrow \Sigma$ and moreover :
(i). $\sigma_{ess}(K(E)) = [0, \Lambda(\Sigma)]$, where $\Lambda(\Sigma) = \sup_\lambda \left\{ \lambda;\ \sigma_{ess}(H_D + \lambda^{-1}I_D) \cap (-\infty, \Sigma] \neq \emptyset \right\} < 1$;
(ii). $K(E) \leq K(\Sigma)$ for all $E \leq \Sigma$.

Remark 2.8. *This theorem is from Theorem 2.8 [34]. In their proof, they use that $K_P(E) \to K_P(\Sigma)$ in norm as $E \uparrow \Sigma$(Proposition 2.6 [34]). But they only proved that $K_P(E)$ were uniformly bounded for $E < \Sigma$, E near Σ, they did not prove $K_P(E) \to K_P(\Sigma)$. So we rewrite the proof of Proposition 2.6.*

Proof. Propositions 2.6 imply the existence of the norm limit. The identification of $\sigma_{ess}(K(\Sigma))$ follows form Proposition 2.3 and Lemma 2.9 below. That $\Lambda(\Sigma) < 1$ follows from the fact that Σ is unique two cluster. Finally (*ii*) is obvious since

$$(H_D - E)^{-1} \leq (H_D - \Sigma)^{-1},$$

for $E \leq \Sigma$. □

The following lemma is the result of [34].

Lemma 2.9. *(Lemma 2.9 [34]) Let $A_n \geq 0$, $\sigma_{ess}(A_n) = [0, a_n]$ and suppose $A_n \to A$ in norm. Then $a = \lim_{n\to\infty} a_n$ exists and $\sigma_{ess}(A) = [0, a]$.*

Proof. Let $\bar{a}$ (resp. a) be $\overline{\lim} a_n$ (resp. $\underline{\lim} a_n$). Let $\lambda < \overline{\lim} a_n$. If $\lambda \notin \sigma(A)$, then $\lambda \notin \sigma(A_n)$ for all large n, so $[0, \bar{a}] \subset \sigma_{ess}(A)$. Now let $\lambda > a$. Pick $\delta > 0$ so that $\lambda - \delta > a$. Pick n so large that $\| A - A_n \| \leq \delta/3$ and that $\lambda - 2\delta/3 > a_n$. Since $[\lambda - 2\delta/3, \lambda + 2\delta/3] \cap \sigma(A_n) = \emptyset$, we can find F finite rank so that $[\lambda - 2\delta/3, \lambda + 2\delta/3] \cap \sigma(A_n + F) = \emptyset$. Thus $[\lambda - \delta/3, \lambda + \delta/3] \cap \sigma(A + F) = \emptyset$. So $\lambda \notin \sigma_{ess}(A)$, i.e. $\sigma(A) = [0, a]$. □

By Theorem 2.7, we can get the following result.

Theorem 2.10. *(Theorem 2.9 [34])) Let H be the Hamiltonian of an N-body system with potentials satisfying (2.5) and (2.6). Let Σ, the infimum of the essential spectrum of H be unique two cluster. Then* dim *ran* $E_{(-\infty,\Sigma)}(H) < \infty$ *(i.e. there are finitely many "bound states").*

Proof.

$$\begin{aligned}
\text{dim } ran\, E_{(\infty,\Sigma)}(H) &= \lim_{n\to\infty} \text{dim } ran\, E_{(-\infty,\Sigma-\frac{1}{n})}(H) \\
&= \lim_{n\to\infty} \#\left\{\lambda \mid \lambda > 1, \lambda \in \sigma\left(K\left(\Sigma - \frac{1}{n}\right)\right)\right\} \qquad (2.14) \\
&\leq \#\{\lambda \mid \lambda > 1, \lambda \in \sigma(K(\Sigma))\} \qquad (2.15) \\
&< \infty.
\end{aligned}$$

In (2.14) we use the standard Birman [5]-Schwinger [56] argument ; in the next step, we use $K(E) \leq K(\Sigma)$, in the last step, we use that $[1, \infty) \cap \sigma_{ess}(K(\Sigma)) = \emptyset$ which can be get from Lemma 2.9 and Theorem 2.7. □

2.3 Semi-classical limit of the number of discrete eigenvalues

Let $N(h)$ denote the number of discrete eigenvalues, less than Σ_h, of $H(h) = h^2 H_0 + V$. From the last section, we can see that $N(h)$ is finite for any $h > 0$. In this section, we want to discuss the small h behavior of $N(h)$. For the two-body case the result is well-known (see [4, 43, 66] for original work or [54, 65] for further discussion). Here we consider the unique

two-cluster N-body Schrödinger operators. First we will show that if (2.5) and (2.6) hold, then $|(V-\Sigma_{cl})_-| \in L^{(N-1)\nu/2}$. Here $(V-\Sigma_{cl})_-$ is the negative part of $V-\Sigma_{cl}$. It suffices to show that

$$|(V(x)-\Sigma_{cl})_-| \le C\langle x\rangle^{-\epsilon_0}, \quad \epsilon_0 > 2. \tag{2.16}$$

Let $\mathcal{A}$ denote the set of all cluster decompositions of the $N-$particle system. For each $D = \{C_1, \cdots, C_k\} \in \mathcal{A}$, there is a natural decomposition of $\mathscr{H} = L^2\left(\mathbb{R}^{(N-1)\nu}\right)$ as $\mathscr{H}^D \otimes \mathscr{H}_D$ with

$$\begin{aligned}
&\mathscr{H}^D = \{x \in \mathscr{H}; \sum_{l\in C_j} m_l x_l = 0;\ j = 1, \cdots, k\};\\
&\mathscr{H}_D = \{x \in \mathscr{H}; x_i = x_j, \text{ if } (i,j) \in C_m, \text{ for some } m = \{1, \cdots, k\}\}.
\end{aligned}$$

Here m_j is the mass of the $j-$th particle. Under these notations, one has for each $x \in \mathscr{H}$, $x = x^D + x_D$ with $x^D \in \mathscr{H}^D$, $x_D \in \mathscr{H}_D$.

$$V_D = \sum_{\#D'=N-1,\ D'\subset D} V_{D'}(x^{D'}), \quad I_D = \sum_{\#D'=N-1,\ D'\not\subset D} V_{D'}(x^{D'})$$

Lemma 2.11. *For some $\delta > 0$ small, define*

$$J_D = \{x \in \mathscr{H}; |x^D| > \delta|x|, \forall D' \not\subset D\},$$

then $\bigcup\limits_{\#D=2} J_D = \mathscr{H}\backslash\{0\}$.

Proof. It is easy check that $J_{D'} \subset J_D$, if $D' \subset D$. Therefore, it suffices to prove that $\bigcup\limits_{\#D\in\mathcal{A};\#D\ge 2} J_D = \mathscr{H}\backslash\{0\}$. Let $\mathbb{S}$ denote the unite sphere in $\mathscr{H}$ and

$$\mathbb{S}_D = \{x \in \mathbb{S} \cap \mathscr{H}_D; x \notin \mathscr{H}_{D'},\ \forall D' \not\subset D\}.$$

For any $x \in \mathbb{S}$, the set $A_x = \{D \in \mathcal{A};\ x \in \mathscr{H}_D\}$ is non-empty. Then $x \in \mathscr{H}_{D_0}$ with $D_0 = \bigcap\limits_{D\in A_x} D$. It can be easily checked that $x \notin \mathscr{H}_{D'}$, $\forall D' \not\subset D_0$. For any $x \in \mathbb{S}_D$, we can take some $\delta_0 = \delta_0(x) > 0$, such that

$$|x^{D'}| \ge \delta_0 > 0, \qquad \forall D' \not\subset D.$$

If $\Omega(x, \epsilon_x)$ is a small conic neighborhood of x in $\mathscr{H}$,

$$\Omega(x,\epsilon_x) = \{y \in \mathscr{H}\backslash\{0\};\ |\hat{y} - x| \le \epsilon_x\},\ (\epsilon_x \le \delta_0/4)$$

with $\hat{y} = \frac{y}{|y|}$, then for $y \in \Omega(x, \epsilon_x)$, we have

$$|y^{D'}| \ge |y|(|x^{D'}| - |(\hat{y}-x)^{D'}|) > |y^{D'}|\epsilon_x.$$

for $D' \not\subset D$. The family of open sets $\{y \in \mathbb{S};\ |y-x| \le \epsilon_x\} = O_x$, $x \in \mathbb{S}$ forms a covering for $\mathbb{S}$ and we can extract a finite covering of $\mathbb{S}$ from it, denoted by $\{O_{x_j};\ j = 1, \cdots, N\}$. Put

$\delta = \min_j \epsilon_{x_j} > 0$. Define J_D as in lemma. We derive that $\bigcup_{\#D\in\mathcal{A};\#D\geq 2} J_D \supseteq \cup_j \Omega(x_j,\delta) = \mathscr{H}\backslash\{0\}$. □

By the above lemma, we know that there exists a family of non-negative smooth function $\{\chi_D;\ D\in\mathcal{A}, \#D=2\}$ such that

$$\sum_{D\in\mathcal{A},\#D=2} \chi_D = 1, \quad \text{and} \quad \text{supp}\chi_D \subset B(0,1)\cup J_D.$$

Here $B(0,1)$ is the unite ball in $\mathscr{H}$. Then

$$V-\Sigma_{cl} = \sum_{D\in\mathcal{A},\#D=2} \chi_D(V_D-\Sigma_{cl}) + \sum_{D\in\mathcal{A},\#D=2} \chi_D I_D.$$

Since $\Sigma_{cl} = \min_{\#D=2} a_D$ with $a_D = \min_{x\in\mathscr{H}^D} V_D(x)$, one has $V_D(x)\geq\Sigma_{cl}$. It follows that $\chi_D(V_D-\Sigma_{cl})\geq 0$, since $\chi_D\geq 0$. Note $\chi_D I_D = \chi_D \sum_{\#D'=N-1,\ D'\not\subset D} V_{D'}(x^{D'})$, and $|V_{D'}(x^{D'})|\leq C\langle x\rangle^{-\epsilon_0}$. One has $|\chi_D V_{D'}(x^{D'})|\leq C\langle x^{D'}\rangle^{-\epsilon_0}\leq C_\delta\langle x\rangle^{-\epsilon_0}$. Therefore, we get (2.16).

The main result of this section is the following.

Theorem 2.12. *Let $\nu\geq 3$. Let $H_0 = -h^2\Delta$ on $L^2\left(\mathbb{R}^{(N-1)\nu}\right)$ and let V be given by (2.4) with V_{ij} satisfying (2.5),(2.6). Suppose that (2.8) holds. Then*

$$N(h) = h^{-(N-1)\nu}\tau_{(N-1)\nu}(2\pi)^{-(N-1)\nu}\int_{V(x)\leq\Sigma_{cl}} [\Sigma_{cl}-V(x)]^{(N-1)\nu/2}dx\,(1+o(1)) \tag{2.17}$$

where $\tau_{(N-1)\nu}$ is the volume of the unit sphere in $\mathbb{R}^{(N-1)\nu}$.

Before proving this theorem, we first give a useful lemma.

Lemma 2.13. *Let $\Sigma_h = \min_D\{\min\sigma(h_D(h))\}$. Σ_{cl} is defined as before. Under the condition of Theorem 2.12, one has $\Sigma_h-\Sigma_{cl} = O(h)$.*

Proof. By the definition of Σ_h, we know that there exists a cluster decomposition D, such that

$$\Sigma_h = \min\sigma(h_D) = \min\sigma(-h^2\Delta+V_D).$$

Note that $\Sigma_h-\Sigma_{cl}>0$, it suffices to prove $\Sigma_h-\Sigma_{cl}\leq Ch$ for some constant C. By the definition of V, we know that there exists a point $x_0\in\mathbb{R}^{(N-2)\nu}$, such that $V_D(x_0)=\Sigma_{cl}$. Let $\psi\in C_0^\infty(\mathbb{R}^{(N-2)\nu})$ be a normalized function, which is supported in $B(x_0,1)(=\{\,x\in\mathbb{R}^{(N-2)\nu};\ |x-x_0|\leq 1\})$. Define

$$\psi^h(x) = h^{-s(N-2)\nu/2}\psi(x/h^s),$$

where s is a constant and will be fixed later. Then $\|\psi^h\|=1$. Note that $\nabla V(x_0)=0$, by Taylor expansion, we have

$$V(x)-\Sigma_{cl} = \sum_{i=1}^n\sum_{j=1}^n (x_i-x_{0i})(x_j-x_{0j})\frac{\partial^2 V}{\partial x_i\partial x_j}(x_0) + O(|x-x_0|^3).$$

It follows

$$((-h^2\Delta + V_D - \Sigma_{cl})\psi^h, \psi^h) \leq h^{2-2s}\|\nabla\psi\|^2 + Ch^{2s} + O(h^{3s}).$$

Choose $s = 1/2$, we have $\min\sigma(-h^2\Delta + V_D - \Sigma_{cl}) \leq Ch$. This ends the proof. □

Proof of Theorem 2.12 : Write

$$N(h) = N_1(h) + N_2(h) \tag{2.18}$$

with $N_1(h)$ the number of eigenvalues of $H(h)$ in $(-\infty, \Sigma_{cl}]$ and $N_2(h)$ the number of eigenvalues of $H(h)$ in $[\Sigma_{cl}, \Sigma_h)$. We know that (see [65]) :

$$N_1(h) = h^{-(N-1)\nu}\tau_{(N-1)\nu}\,(2\pi)^{-(N-1)\nu}\int_{V(x)\leq\Sigma_{cl}}[\Sigma_{cl} - V(x)]^{(N-1)\nu/2}dx\,(1 + o(1))\,. \tag{2.19}$$

Thus, it suffices to prove that

$$N_2\,(h) = o(h^{-(N-1)\nu}). \tag{2.20}$$

Let $K_h\,(E)$ be the Birman-Schwinger kernel of Section 2.2 for $H\,(h)$. Below, we will prove that for any $\epsilon > 0$:

$$h^{(N-1)\nu}[\#\{e.v.\ of\ [K_h(\Sigma_h) - K_h(\Sigma_{cl})] \geq \epsilon\}] \to 0. \tag{2.21}$$

We first note that :

Lemma 2.14. *Under the condition of Theorem 2.12, one has (2.21) implies (2.20), and hence the theorem.*

Proof. Let $V^\epsilon = V_D + \epsilon I_D$. Then for ϵ small enough, $\Sigma_{cl}^\epsilon = \underline{\lim}_{|x|\to\infty} V^\epsilon(x) = \Sigma_{cl}$. It follows that (2.19) continuous to hold for V replaced by V^ϵ in the integral on the right and with $N_1(h)$ replaced by $N_1^\epsilon(h)$, the number of eigenvalues for $H(h) + \epsilon I_D$ in $(-\infty, \Sigma_{cl}]$. Thus

$$\lim_{\epsilon\downarrow 0}\lim_{h\downarrow 0} h^{(N-1)\nu}[N_1^\epsilon(h) - N_1(h)] = 0. \tag{2.22}$$

Since

$$\begin{aligned} N_1^\epsilon(h) &= \#\{E;\ E \in \sigma(H(h) + \epsilon I_D),\ E < \Sigma_{cl}\} \\ &= \#(\,of\ e.v.\ K_h(\Sigma_{cl}) > (1+\epsilon)^{-1}))\,, \end{aligned}$$

and

$$\begin{aligned} N_2(h) &= N(h) - N_1(h) \\ &= \lim_{\epsilon\downarrow 0}\lim_{h\to 0} \#(\,of\ e.v.\ K_h(\Sigma_h) > 1) - \#(\,of\ e.v.\ K_h(\Sigma_{cl}) \geq (1+\epsilon)^{-1}), \end{aligned}$$

then we have,

$$\begin{aligned} & \lim_{h\to 0} h^{(N-1)\nu} N_2(h) \\ = & \lim_{h\to 0} h^{(N-1)\nu} [\#(e.v.\ of\ K_h(\Sigma_h) > 1) - \#\,(e.v.\ of\ K_h(\Sigma_{cl}) > (1+\epsilon)^{-1})] \\ \leq & \lim_{h\to 0} h^{(N-1)\nu} \#(e.v.\ of\ (K_h(\Sigma_h) - K_h(\Sigma_{cl})) > (1+\epsilon)^{-1} - 1). \end{aligned}$$

In the above, we use that for $A, B \geq 0$,

$$\#(\ e.v.\ of\ A + B > 1) - \#(\ e.v.\ of\ A > 1 - \delta) \leq \#(\ e.v.\ of\ B > \delta).$$

(2.21) implies (2.20) . □

Hence, we need only to show that (2.21) holds. Before that, we recall one result.

Proposition 2.15. *([22]) Let $P(h) = -h^2\Delta + \Sigma V_a(x^a)$, V_a continuous, and $|V_a(x^a)| \leq C\langle x^a\rangle^{-\rho}$ with $\rho > 0$. Let $\delta > 0$, $E(h) < \Sigma_{cl} - \delta$ $(\forall h > 0)$, $(P(h) - E(h))\psi(h) = 0$, $\|\psi(h)\| = 1$, then $\forall \epsilon > 0$, $\exists C_\epsilon > 0$ and the continuous function $d(\cdot)$, $d(x) \sim \sqrt{\Sigma_{cl} - E(h) - \epsilon}|x|$, $x \to \infty$, such that*

$$h\|\nabla(e^{d/h}\psi(h))\| + \|e^{d/h}\psi(h)\| \leq C_\epsilon,$$

uniformly hold for all h small enough.

Lemma 2.16. *Under the condition of Theorem 2.12, (2.21) holds.*

Proof. Let

$$\tilde{\Sigma}_{cl} = \min\{a_{D'};\ a_{D'} = \min V_{D'},\ D' \text{ is the refinement of } D\}.$$

Let p_h be the projection onto those eigenvalues of $h_D(h)$ less than $\alpha = (1/2)(\Sigma_{cl} + \tilde{\Sigma}_{cl})$. Let $P_h = p_h \otimes 1$, $Q_h = 1 - P_h$. Then

$$[K_h(\Sigma_h) - K_h(\Sigma_{cl})] = \alpha_{P_h} + \alpha_{Q_h},$$

where

$$\alpha_A = (\Sigma_h - \Sigma_{cl})|I_D|^{1/2}[A\,(H_D(h) - \Sigma_{cl})^{-1}\,(H_D(h) - \Sigma_h)^{-1}]|I_D|^{1/2}.$$

Now, clearly

$$\|\alpha_{Q_h}\| \leq \|I_D\|_\infty(\Sigma_h - \Sigma_{cl})(\frac{1}{4}(\Sigma_{cl} - \tilde{\Sigma}_{cl}))^{-2} \leq C\|I_D\|_\infty h(\frac{1}{4}(\Sigma_{cl} - \tilde{\Sigma}_{cl}))^{-2}.$$

It follows $\|\alpha_{Q_h}\| < \epsilon/2$, as $h > 0$ small enough. Hence, (2.21) follows from

$$h^{(N-1)\nu}[\#\{e.v.\ of\ \alpha_{P_h} \geq \epsilon/2\}] \to 0.$$

Write $P_h = P_{1h} + P_{2h}$, with

$$\begin{aligned} P_{1h} &= \chi(|\zeta| \geq R+1)P_h + \chi(|\zeta| \leq R+1)P_h\chi(|\zeta| \geq R+1); \\ P_{2h} &= \chi(|\zeta| \leq R+1)P_h\chi(|\zeta| \leq R+1). \end{aligned}$$

where $|\zeta|$ is a measure of the total size of the internal coordinates of D and R is defined by

$$\{\ \zeta;\ V_D(\zeta) < \tilde{\Sigma}_{cl}\ \} \subset \{\ \zeta;\ |\zeta| < R\ \}.$$

Then

$$\begin{aligned} \alpha_{P_h} &= (\Sigma_h - \Sigma_{cl})|I_D|^{1/2}P_h(H_D(h) - \Sigma_h)^{-1}(H_D(h) - \Sigma_{cl})^{-1}|I_D|^{1/2} \\ &= (\Sigma_h - \Sigma_{cl})|I_D|^{1/2}(P_{1h} + P_{2h})(H_D(h) - \Sigma_h)^{-1}(H_D(h) - \Sigma_{cl})^{-1}(P_{1h} + P_{2h})|I_D|^{1/2} \\ &\leq 2(\Sigma_h - \Sigma_{cl})|I_D|^{1/2}P_{1h}(H_D(h) - \Sigma_h)^{-1}(H_D(h) - \Sigma_{cl})^{-1}P_{1h}|I_D|^{1/2} \\ &\quad +2(\Sigma_h - \Sigma_{cl})|I_D|^{1/2}P_{2h}(H_D(h) - \Sigma_h)^{-1}(H_D(h) - \Sigma_{cl})^{-1}P_{2h}|I_D|^{1/2} \\ &= \alpha^{(1)}_{P_h} + \alpha^{(2)}_{P_h}. \end{aligned}$$

Since

$$\begin{aligned} & (\Sigma_h - \Sigma_{cl})(H_D(h) - \Sigma_h)^{-1}(H_D(h) - \Sigma_{cl})^{-1} \\ = \ & (\Sigma_h - \Sigma_{cl})(H_D(h) - \Sigma_h)^{-1}(H_D(h) - \Sigma_h + \Sigma_h - \Sigma_{cl})^{-1} \\ = \ & (H_D(h) - \Sigma_h)^{-1}(1 + (\Sigma_h - \Sigma_{cl})^{-1}(H_D(h) - \Sigma_h)^{-1}\ )^{-1} \\ \leq \ & t_D^{-1}, \end{aligned}$$

then we get

$$\alpha^{(1)}_{P_h} \leq 2|I_D|^{1/2}P_{1h}t_D^{-1}P_{1h}|I_D|^{1/2}.$$

From Proposition 2.15, we can conclude that

$$\| \chi(|\zeta| > R+1)\varphi(h) \| \leq C_2 \exp\left(-C_3 h^{-1}\right), \tag{2.23}$$

where $C_3 > 0$ and C_2, C_3 is independent of h and $\varphi(h)$ is the eigenvector of $h_D(h)$ correspondent to eigenvalue $E(h) < \alpha$. From the one body result, one has

$$\dim ran P_h \leq Ch^{-(N-2)\nu},$$

so $\alpha^{(1)}_{P_h}$ is a sum of at most $Ch^{-(N-2)\nu}$ terms, and each of them of the form controlled by Proposition 2.6. Using the similar methods used in Proposition 2.6 and by (2.23), we can get that each of these terms has a norm bounded by

$$Ch^{-2}\exp(-C_2h^{-1}). \tag{2.24}$$

It follows that $\|\alpha^{(1)}_{P_h}\| \leq \epsilon/4$, for $h > 0$ small enough. Thus, it suffices to prove that

$$h^{(N-1)\nu}[\#\{e.v.\ of\ \alpha_{P_h^{(2)}} \geq \epsilon/4\}] \to 0$$

as $h \to 0$.

If x is a positive number, A is a positive operator, and $0 < \delta < 1$, one has $xA^{-1}(A+x)^{-1} \leq x^{\delta}A^{1+\delta}$. It follows that

$$\begin{aligned}\alpha^{(2)}_{P_h} &= 2(\Sigma_h - \Sigma_{cl})|I_D|^{1/2}P_{2h}(H_D(h)-\Sigma_h)^{-1}(H_D(h)-\Sigma_{cl})^{-1}P_{2h}|I_D|^{1/2}\\ &\leq 2|I_D|^{1/2}P_{2h}(\Sigma_h-\Sigma_{cl})^{\delta}(H_D(h)-\Sigma_h)^{-1-\delta}P_{2h}|I_D|^{1/2}\\ &\leq 2(\Sigma_h-\Sigma_{cl})^{\delta}|I_D|^{1/2}P_{2h}(t_D)^{-1-\delta}P_{2h}|I_D|^{1/2}.\end{aligned}$$

Because $\chi(|\zeta| < R+1)|I_D|^{1/2} \leq C\langle\rho\rangle^{-n/2}$, one has

$$\alpha^{(2)}_{P_h} \leq (\Sigma_h-\Sigma_{cl})^{\delta}h^{-2-2\delta}P_h\langle\rho\rangle^{-n/2}t_D^{-1-\delta}\langle\rho\rangle^{-n/2}.$$

Thus,

$$\begin{aligned}&\#\{e.v.\ of\ \alpha^{(2)}_{P_h} \geq \frac{\epsilon}{4}\}\\ \leq\ &\dim ran(P_h)\ \#\{e.v.\ of\ (\Sigma_h-\Sigma_{cl})^{\delta}h^{-2-2\delta}P_h\langle\rho\rangle^{-n/2}t_D^{-1-\delta}\langle\rho\rangle^{-n/2} \geq \frac{\epsilon}{4}\}\\ =\ &\dim ran(P_h)\ \#\{e.v.\ of\ \langle\rho\rangle^{-n/2}t_D^{-1-\delta}\langle\rho\rangle^{-n/2} \geq \frac{\epsilon}{4}(\Sigma_h-\Sigma_{cl})^{-\delta}h^{2+2\delta}\}\\ =\ &\dim ran(P_h)\ \#\{e.v.\ of\ t_D^{1+\delta}+4\epsilon^{-1}h^{-2-2\delta}(\Sigma_h-\Sigma_{cl})^{\delta}\langle\rho\rangle^{-n} \leq 0\}\\ \leq\ &C\dim ran(P_h)h^{-\nu}h^{\frac{\delta\nu}{2(\delta+1)}}\epsilon^{\frac{-\nu}{2(\delta+1)}} \qquad (2.25)\end{aligned}$$

in (2.25), we use Birman-Schwinger principle and in the last step, we use $n > 2$. Choose $\delta > 0$ small enough. We complete the proof. □

2.4 Semi-classical limit of Riesz means of discrete eigenvalues

In this section, we want to discuss the small h behavior of Riesz means of discrete spectrum of Schrödinger operators. First, we consider the Riesz means of the two-body Schrödinger operators $P(h) = -h^2\Delta + V$ on $L^2(\mathbb{R}^n),\ n \geq 3$ with

$$V \in C^1(\mathbb{R}^n), \qquad (2.26)$$

and

$$\{x \in \mathbb{R}^n;\ V(x) < 0\}\ \text{be a bounded set.} \qquad (2.27)$$

We define the Riesz means of order $\gamma \geq 0$ of $P(h)$ by

$$R_{\gamma}(h;\lambda) = \sum_{e_j(h)\leq\lambda}(\lambda - e_j(h))^{\gamma},$$

where $\lambda < \sigma_{ess}(P(h))$. Let $N_h(\lambda)$ be the number of discrete eigenvalues of $P(h)$ less than λ, then $R_0(h;\lambda) = N_h(\lambda)$.

Then the main result for Riesz means of the two-body Schrödinger operator is the following :

Theorem 2.17. *Let $n \geq 3$. If (2.26) and (2.27) hold, then*

$$R_\gamma(h;0) = (2\pi)^{-n}\tau_n h^{-n} C_{\gamma,n} \int_{\{x;\ V(x)\leq 0\}} (-V(x))^{\gamma+\frac{n}{2}}\, dx(1+O(h^{1/2})), \tag{2.28}$$

where $C_{\gamma,n} = \gamma \int_0^1 \beta^{\gamma-1}(1-\beta)^{\frac{n}{2}}\, d\beta$ and τ_n is the volume of the unit sphere in $\mathbb{R}^n$.

Before proving the theorem, we give a lemma first which can be proved by Dirichlet-Neumann bracket. We begin with recalling some results of Dirichlet-Neumann bracket (See [54] for details).

Let Ω be an open region of $\mathbb{R}^n$ with connected components $\Omega_1, \cdots$ (finite or infinite). The Dirichlet Laplacian for Ω, $-\Delta_D^\Omega$ is the unique self-adjoint operator on $L^2(\Omega)$ whose quadratic form is the closure of the form $q(f,g) = \int \overline{\nabla f} \cdot \nabla g\, dx$ with the domain $C_0^\infty(\Omega)$. The Neumann Laplacian for Ω, $-\Delta_N^\Omega$ is the unique self-adjoint operator on $L^2(\Omega)$ whose quadratic form is the closure of the form $q(f,g) = \int \overline{\nabla f} \cdot \nabla g\, dx$ with the domain $H^1(\Omega) = \{f \in L^2(\Omega);\ \nabla f \in L^2(\mathbb{R}^n)\}$.

Proposition 2.18. *([54])*

(1). Let $N_D(a,\lambda)$ (respectively, $N_N(a,\lambda)$) denote the dimension of the spectral projection $P_{[0,\lambda)}$ for $-\Delta_D$ (respectively, $-\Delta_N$) on $(-a,a)^m$. Then for all $a,\ \lambda$, we have

$$|N_D(a,\lambda) - \tau_m(2a/2\pi)^m \lambda^{m/2}| \leq C(1+(a^2\lambda)^{(m-1)/2});$$
$$|N_N(a,\lambda) - \tau_m(2a/2\pi)^m \lambda^{m/2}| \leq C(1+(a^2\lambda)^{(m-1)/2}).$$

Here τ_m is the volume of the unit ball in $\mathbb{R}^n$ and C is a suitable constant independent of a and λ.

(2). Let $N_D(\Omega,\lambda)$ (respectively, $N_N(\Omega,\lambda)$) be the dimension of the spectral projection $P_{[0,\lambda)}$ for $-\Delta_D$ (respectively, $-\Delta_N$). Then if $\Omega_1, \cdots, \Omega_k$ are disjoint,

$$N_D(\cup_{i=1}^k \Omega_i, \lambda) = \sum_{i=1}^k N_D(\Omega_i,\lambda);$$
$$N_N(\cup_{i=1}^k \Omega_i, \lambda) = \sum_{i=1}^k N_N(\Omega_i,\lambda).$$

(3). For any Ω, $0 \leq -\Delta_N^\Omega \leq -\Delta_D^\Omega$. If $\Omega_1,\ \Omega_2$ be disjoint open subsets of an open set Ω so that $\overline{(\Omega_1 \cup \Omega_2)}^{int} = \Omega$ and $\Omega\backslash(\Omega_1 \cup \Omega_2)$ has measure 0, then

$$0 \leq -\Delta_D^\Omega \leq -\Delta_D^{\Omega_1\cup\Omega_2};$$
$$0 \leq -\Delta_N^{\Omega_1\cup\Omega_2} \leq -\Delta_N^\Omega.$$

Lemma 2.19. *If (2.26) and (2.27) hold, then*

$$N_h(0) = (2\pi)^{-n}\tau_n h^{-n} \int_{\{x;\ V(x)\leq 0\}} (-V(x))^{\frac{n}{2}}\, dx(1+O(h^{1/2})), \tag{2.29}$$

where $O(h^{1/2})$ depends only on $|\{x \in \mathbb{R}^n;\ V(x) < 0\}|$ ($|A|$ denotes the Lebesgue measure of set A).

Proof : Let $h^2 = \lambda^{-1}$. Then we consider the number of the negative eigenvalues of $-\Delta + \lambda V$. Let $A = \{x \in \mathbb{R}^n;\ V(x) < 0\}$. Let A_i be the cube of the form

$$[a_1, a_1 + d) \times \cdots \times [a_n, a_n + d),$$

and $\mathbb{R}^n = \bigcup_{i=1}^{\infty} A_i$, $A_i \cap A_j = \emptyset$, if $i \neq j$. Let $A_i (i = 1, \cdots, N)$ be the cubes such that $A_i \cap A \neq \emptyset$ for $1 \leq i \leq N$, and $A \subset \bigcup_{i=1}^{N} A_i$. Let

$$V_i^+ = \sup_{x \in A_i} V(x);\ V_i^- = \inf_{x \in A_i} V(x).$$

Let

$$V^+(x) = V_i^+;\ V^-(x) = V_i^-,\ \text{if } x \in A_i.$$

Let $-\Delta^+$ be the Laplace operator on $\mathbb{R}^n$ but with Dirichlet boundary conditions on the boundaries of all the A_i, and let $-\Delta^-$ be the Laplace operator on $\mathbb{R}^n$ but with Neumann boundary conditions on the boundaries of all the A_i. Now, since $V^- \leq V \leq V^+$, and $-\Delta^- \leq -\Delta \leq -\Delta^+$ by Proposition 2.18, we have that

$$-\Delta^- + \lambda V^- \leq -\Delta + \lambda V \leq -\Delta^+ + \lambda V^+.$$

Let $N(\lambda)$ be the number of the negative eigenvalues of $-\Delta + \lambda V$, and $N^+(\lambda)$ (respectively, $N^-(\lambda)$) be the number of the negative eigenvalues of $-\Delta^+ + \lambda V^+$ (respectively, $-\Delta^- + \lambda V^-$). Then by Proposition 2.18, one has

$$\begin{aligned} N(\lambda) &\geq N^+(\lambda) \\ &\geq \sum_{\{i;\ V_i^+ \leq 0\}} \tau_n (2\pi)^{-n} d^n (\lambda |V_i^+|)^{n/2} - \sum_{\{i;\ V_i^+ \leq 0\}} C_n (1 + d^{n-1} (\lambda |V_i^+|)^{(n-1)/2}) \\ &\geq \tau_n (2\pi)^{-n} \lambda^{n/2} \int_{\{x \in A_i;\ V_i^+ \leq 0\}} |V_i^+|^{n/2}\, dx \\ &\quad - \frac{1}{d} C_n \lambda^{(n-1)/2} \int_{\{x \in A_i;\ V_i^+ \leq 0\}} |V_i^+|^{(n-1)/2}\, dx - C_n C (\frac{1}{d})^n, \end{aligned}$$

where C_n depends only on n. For $x \in A_i$ with i satisfying $V_i^+ \leq 0$, one has

$$|V(x)|^{n/2} - |V_i^+|^{n/2} \leq C(|V(x)| - |V_i^+|)(|V|^{n/2-1} + |V_i^+|^{n/2-1}) \leq Cd|V(x)|^{n/2-1}. \tag{2.30}$$

Here C depends on $\|\nabla V\|_{L^\infty}$. In the last step, we use that $|V_i^+| \leq |V|$, since $V_i^+ \leq 0$. It follows

$$\begin{aligned} N(\lambda) &\geq \tau_n (2\pi)^{-n} \lambda^{n/2} \int_{\{x; V \leq 0\}} (-V)^{n/2} dx - Cd\lambda^{n/2} \int_{\{x; V \leq 0\}} (-V)^{n/2-1} dx \\ &\quad - C_n (\frac{c}{d})^n - \frac{C_n}{d} \lambda^{(n-1)/2} \int_{\{x; V \leq 0\}} (-V)^{(n-1)/2} dx - C_n \lambda^{(n-1)/2} \int_{\{x; V \leq 0\}} (-V)^{(n-3)/2} dx. \end{aligned}$$

Let $d = c\lambda^{-1/4}$. One has

$$N(\lambda V) \geq \tau_n (2\pi)^{-n} \lambda^{n/2} \int_{\{x; V\leq 0\}} (-V)^{n/2} dx - C\lambda^{\frac{n}{2}-\frac{1}{4}}$$

with C depending on $|A|$ and $\|\nabla V\|_{L^\infty}$ only.

Similarly, we can get

$$\begin{aligned} N(\lambda) &\leq N^-(\lambda) \\ &\leq \tau_n (2\pi)^{-n} \lambda^{n/2} \int_{\{x\in A_i;\ V_i^-\leq 0\}} |V_i^-|^{n/2}\, dx \\ &\quad -\frac{1}{d} C_n \lambda^{(n-1)/2} \int_{\{x\in A_i;\ V_i^-\leq 0\}} |V_i^-|^{(n-1)/2}\, dx - C_n C(\frac{1}{d})^n. \end{aligned}$$

Note that for $x \in A_i$, $\nu \geq 1$, one has

$$|\,|V_i^-|^\nu - |V(x)|^\nu| \leq C_\nu [|V(x) - V_i^-|\,|V(x)|^{\nu-1} + |V(x) - V_i^-|^\nu].$$

We can obtain that

$$\begin{aligned} N(\lambda) &\leq \tau_n (2\pi)^{-n} \lambda^{n/2} \int_{\{x;V\leq 0\}} (-V)^{n/2} dx + C_n \frac{\lambda^{(n-1)/2}}{d} \int_{\{x;V\leq 0\}} (-V)^{n/2-1/2} dx \\ &\quad +\lambda^{n/2} d \int_{\{x;V\leq 0\}} |V|^{n/2-1} dx + \lambda^{(n-1)/2} d \int_{\{x;V\leq 0\}} |V|^{n/2-3/2} \\ &\quad +G_n d^{n/2} \lambda^{n/2-1} |A| + H_n \lambda^{n/2-1} d^{1/n-1/2} |A| + C_n (\frac{C}{d})^n. \end{aligned}$$

where $C_n,\ H_n,\ G_n$ are the constants depending on n only. If we choose $d = c\lambda^{-1/4}$, then we get

$$N(\lambda) \leq \frac{\tau_n}{(2\pi)^n} \lambda^{n/2} \int_{\{x;\ V(x)\leq 0\}} |V_-|^{n/2} + C\lambda^{n/2-1/4},$$

where C depends only on $|A|$, $\|\nabla V\|_{L^\infty}$and n. This ends the proof. □

Proof of Theorem 2.17 : By Lemma 2.19, one has, for $\mu \leq 0$,

$$N_h(\mu) = (2\pi)^{-n} \tau_n h^{-n} \int_{\{x;\ V(x)\leq \mu\}} (\mu - V(x))^{\frac{n}{2}}\, dx(1 + O(h^{1/2})), \tag{2.31}$$

and $O(h^{1/2})$ uniformly hold for $\mu \leq 0$. From the definition of $R(h,\lambda)$, we can compute

$$\begin{aligned} & R_\gamma(h,0) \\ = \; & -\gamma \int_{-\infty}^{0} (-\mu)^{\gamma-1} N_h(\mu) d\mu \\ = \; & -\gamma \int_{-\infty}^{0} (-\mu)^{\gamma-1} h^{-n} (2\pi)^{-n} \tau_n \int_{\{x;\ V(x)<\mu\}} (\mu - V(x))^{n/2} dx \, (1 + O(h^{1/2})) \, d\mu \\ = \; & \gamma (2\pi)^{-n} h^{-n} \tau_n \int_0^1 \int_{\{x;\ V(x)<0\}} (-V(x))^{\gamma+n/2} \, dx \, \beta^{\gamma-1} (1-\beta)^{n/2} \, d\beta \, (1 + O(h^{1/2})) \\ = \; & \gamma (2\pi)^{-n} h^{-n} \tau_n \int_0^1 \beta^{\gamma-1} (1-\beta)^{n/2} \, d\beta \int_{\{x;\ V(x)\leq 0\}} (-V(x))^{\gamma+n/2} \, dx \, (1 + O(h^{1/2})) \\ = \; & (2\pi)^{-n} h^{-n} \tau_n C_{\gamma,n} \int_{\{x;\ V(x)\leq 0\}} (-V(x))^{\gamma+n/2} \, dx \, (1 + O(h^{1/2})), \end{aligned} \tag{2.32}$$

with

$$C_{\gamma,n} = \gamma \int_0^1 \beta^{\gamma-1} (1-\beta)^{n/2} \, d\beta. \tag{2.33}$$

This ends the proof. □

We use Theorem 2.17 to study the Riesz means of the $N-$body Schrödinger operators $H(h) = h^2 H_0 + \sum_{i<j} V_{ij}$ with

$$V_{ij} \in C_0^\infty(\mathbb{R}^\nu), \tag{2.34}$$

and

$$V_{ij} \leq 0. \tag{2.35}$$

For a technical reason, we have changed a little the usual definition of Riesz means. We define

$$R_\gamma(h, \Sigma_h) = \sum_{e_j(h) \leq \Sigma_h} |\Sigma_{cl} - e_j(h)|^\gamma.$$

Then the result for semi-classical limit of Riesz means of the N-body Schrödinger operator is the following :

Theorem 2.20. *Let $\nu \geq 3$. Let $H_0 = -h^2\Delta$ on $L^2(\mathbb{R}^{(N-1)\nu})$. Suppose (2.34) and (2.35) hold. Then, we have*

$$R_\gamma(h, \Sigma_h) = C(\gamma,\nu) h^{-(N-1)\nu} \int_{\{x;\ V(x)\leq \Sigma_{cl}\}} (\Sigma_{cl} - V(x))^{\gamma + \frac{(N-1)\nu}{2}} \, dx \, (1 + O(h^t)), \tag{2.36}$$

where $C(\gamma,\nu) = (2\pi)^{-(N-1)\nu} \tau_{(N-1)\nu} C_{\gamma,(N-1)\nu}$, $C_{\gamma,(N-1)\nu}$ is given by (2.33), and $\tau_{(N-1)\nu}$ is the volume of the unit sphere in $\mathbb{R}^{(N-1)\nu}$, $t = \min\{\frac{1}{2},\ \gamma\}$.

Proof. From the definition of $R_\gamma(h,\Sigma_h)$, one has

$$R_\gamma(h,\Sigma_h) \quad = \quad \sum_{e_j(h)\le\Sigma_{cl}} (\Sigma_{cl}-e_j(h))^\gamma + \sum_{\Sigma_{cl}\le e_j(h)\le\Sigma_h} |\Sigma_{cl}-e_j(h)|^\gamma.$$

By Theorem 2.12 and Lemma 2.13, one has

$$\sum_{\Sigma_{cl}\le e_j(h)\le\Sigma_h} |\Sigma_{cl}-e_j(h)|^\gamma = o(h^{\gamma-(N-1)\nu}).$$

Let $\tilde{V} = V-\Sigma_{cl}$. First, we will show that $\{x\in\mathbb{R}^{(N-1)\nu};\ \tilde{V}(x)<0\}$ is a bounded set. Let $\{\chi_D;\ D\in\mathcal{A},\#D=2\}$ be a family of non-negative smooth functions such that

$$\sum_{D\in\mathcal{A},\#D=2}\chi_D = 1, \quad \text{and} \quad \mathrm{supp}\chi_D\subset B(0,1)\cup J_D.$$

Here $B(0,1)$ is the unite ball in $\mathscr{H}$. Then

$$V-\Sigma_{cl} = \sum_{D\in\mathcal{A},\#D=2}\chi_D(V_D-\Sigma_{cl}) + \sum_{D\in\mathcal{A},\#D=2}\chi_D I_D.$$

Note $\chi_D I_D = \chi_D \sum\limits_{\#D'=N-1,\ D'\not\subset D} V_{D'}(x^{D'})$, and $V_{D'}(x^{D'})\in C_0^\infty$. It is easy to check that $\chi_D V_{D'}(x^{D'})=0$ for $D'\not\subset D, \#D'=N-1$, $|x|$ large enough. It follows that $\chi_D I_D\in C_0^\infty(\mathscr{H})$. Since $\Sigma_{cl} = \min\limits_{\#D=2} a_D$ with $a_D = \min\limits_{x\in\mathscr{H}^D} V_D(x)$, one has $V_D(x)\ge\Sigma_{cl}$. It follows that $\chi_D(V_D-\Sigma_{cl})\ge 0$, since $\chi_D\ge 0$. Therefore, we drive that $(V(x)-\Sigma_{cl})_-$ has compact support.

Then $\{x\in\mathbb{R}^{(N-1)\nu};\ \tilde{V}(x)<0\}$ is a bounded set. Using Theorem 2.17 for Schrödinger operator $-h^2\Delta+\tilde{V}$, one has

$$\sum_{e_j(h)\le\Sigma_{cl}} (\Sigma_{cl}-e_j(h))^\gamma = C(\gamma,\nu)\int_{\{x;\ V(x)\le\Sigma_{cl}\}} (\Sigma_{cl}-V(x))^{\gamma+\frac{(N-1)\nu}{2}}\,dx\,(1+O(h^{1/2})).$$

This ends the proof. □

Remark 2.21. *From the proof of Theorem 2.17, one can see that the uniform holding of the remainder term of $N(h,\lambda)$ with respected to $\lambda\le 0$ play an important role. For the two-body case, we can only get that for the V's satisfying (2.27). If we can extend this results to general V, then we can get the estimate of the Riesz means for general V. In the $N-$body case, we need (2.34) and (2.35), because these two conditions imply that $\{x;\ V<\Sigma_{cl}\}$ is a bound set.*

2.5 The N-body Schrödinger operators with Coulomb potentials

In Chapter 3 and Chapter 4, we consider the Schrödinger operators of the form $-\Delta+V$ with $V = \frac{q(\theta)}{|x|^2}+o(\langle x\rangle^{-\rho})$ for $|x|$ large, with some $\rho>2$. Here $\theta=\frac{x}{|x|}$, and $q(\theta)$ is a real continuous

function. We explain why we consider these Schrödinger operators in this section. We consider the N-body system of ν-dimensional particles with Coulomb potential. Let

$$\tilde{H} = \tilde{H}_0 + V$$

with

$$\tilde{H}_0 = \sum_{i=1}^{N} \frac{1}{2m_i}(-\Delta_{r_i});$$

and

$$V = \sum_{1\le i<j\le N} \frac{e_i e_j}{|r_i - r_j|}\,.$$

Here m_j is the mass of j-th particle and e_j is the charge of j-th particle. $-\Delta_{r_j}$ is the Laplacian in the r_j variables. Let H_0 be the operator resulting from removing the center of mass from $\tilde{H}_0$ and $H = H_0 + V$. For each cluster decomposition D, there is a natural decomposition of $\mathscr{H} = L^2\left(\mathbb{R}^{(N-1)\nu}\right)$ as $\mathscr{H}^D \otimes \mathscr{H}_D$ with $\mathscr{H}_D$ = functions of r_{ij} with iDj and $\mathscr{H}^D$ = functions of $R_q - R_l$ where $R_q = \sum\limits_{i\in a_q} m_i r_i / \sum\limits_{i\in a_q} m_i$ is the center of the mass of a_q. By HVZ-Theorem, we know that there exists a decomposition D, such that

$$E_0 \equiv \inf \sigma_{ess}(H) = \min_{D'} \{\Sigma_{D'};\ D' \text{ is the decomposition of } \{\,1, 2, \cdots, N\,\}\} = \inf \sigma_d(H_D)$$

and E_0 is the simple eigenvalue of H_D. Obviously, $\#(D) = 2$, since $\sigma_{D'} \le \sigma_D$, if D' is the refinement of D. Without loss of generality, we can suppose that $D = \{a_1, a_2\}$ with $a_1 = \{\,1, 2, \cdots, k\,\}$ and $\{\,k+1, k+2, \cdots, N\,\}$. Let η be the normlized function such that

$$H_D\eta = E_0\eta.$$

Let $\rho = R_1 - R_2$. Define the effective potential as

$$I_{eff}(\rho) = \langle \eta, I_D\eta\rangle(\rho). \tag{2.37}$$

Let

$$\begin{aligned}
&\xi_i = r_i - r_k \text{ for } \ i = 1, \cdots, k-1;\\
&\xi_i = r_{i+1} - r_N \text{ for } \ i = k, \cdots, N-2;\\
&\rho = R_1 - R_2.
\end{aligned}$$

Then one can check that

$$H_D = -(2M_c)\Delta_\rho + \sum_{i=1,2} H(a_i)$$

with $M_c^{-1} = m_{a_1}^{-1} + m_{a_2}^{-1}$ and $H(c_i)$ be the sub-Hamilton to $\sum\limits_{j\in a_i} \frac{1}{2m_j}(-\Delta_j) + \sum\limits_{l,j\in a_i} \frac{e_l e_j}{r_l - r_j}$ with its center of mass removed. Then one has,

$$\begin{aligned}
&r_k = R_1 + l(\xi_1, \xi_2, \cdots, \xi_{k-1});\\
&r_i = r_k + \xi_i = R_1 + l_i(\xi_1, \xi_2, \cdots, \xi_{k-1}), \ \text{ for } i = 1, 2, \cdots, k-1.
\end{aligned}$$

Here l, and l_i are linear combination of $\xi_j (j = 1, 2, \cdots, k-1)$. Similarly, we can get

$$r_i = r_N + \xi_{i-1} = R_2 + l_i(\xi_{k+1}, \xi_{k+2}, \cdots, \xi_{N-1}), \quad \text{for } i = k+1, k+2, \cdots, N-1$$

with l_i the linear combination of $\xi_j (j = k+1, k+2, \cdots, N-1)$. It follows that, for $i \in a_1,\ j \in a_2$,

$$r_i - r_j = \rho + l_{i,j}(\xi_1, \xi_2, \cdots, \xi_N - 2)$$

with $l_{i,j} = l_i - l_j$ be the linear combination of $\xi_i (i = 1, 2, \cdots, N-2)$.

Lemma 2.22. *Let $\hat{\rho} = \frac{\rho}{|\rho|}$. Then I_{eff} has the following expansion :*

$$I_{eff}(\rho) = \frac{f_1(\hat{\rho})}{|\rho|^2} + \frac{f_2(\hat{\rho})}{|\rho|^3} + O(\frac{1}{|\rho|^4}),$$

if $C_1 \equiv \sum\limits_{i \in a_1} e_i = 0$ or $C_2 \equiv \sum\limits_{i \in a_2} e_i = 0$. Here

$$f_1(\hat{\rho}) = \int \sum_{i \in a_1, j \in a_2} e_i e_j \hat{\rho} \cdot l_{i,j} |\eta|^2 d\xi_1 \cdots d\xi_{N-2};$$

$$f_2(\hat{\rho}) = \int \sum_{i \in a_1, j \in a_2} e_i e_j (|l_{i,j}|^2 - 3|\hat{\rho} \cdot l_{ij}|^2) d\xi_1 \cdots d\xi_{N-2}.$$

Moreover, if $C_1 = C_2 = 0$, $f_1(\hat{\rho}) = 0$.

Proof. One has

$$\begin{aligned} I_{eff}(\rho) &= \langle \eta, I_a \eta \rangle(\rho) \\ &= \int \frac{\sum\limits_{i \in a_i,\ j \in a_2} e_i e_j |\eta|^2}{|\rho + l_{i,j}(\xi_1, \xi_2, \cdots, \xi_{N-2})|} d\xi_1 \cdots \xi_{N-2}. \end{aligned}$$

By Taylor expansion at zero of the function

$$f : \mathbb{R} \ni r \to |u_1 + r u_2|^{-1}$$

for non-zero vectors $u_1, u_2 \in \mathbb{R}^\nu$, for each $r \in \mathbb{R}$, there exists some $\theta \in (0, 1)$ such that

$$f(r) = f(0) + r f'(0) + \frac{r^2}{2} f''(r) + \frac{r^3}{6} f'''(r\theta).$$

It follows that,

$$\begin{aligned} & |\rho + l_{i,j}(\xi_1, \xi_2, \cdots, \xi_{N-2})|^{-1} \\ = \quad & |\rho|^{-1} - |\rho|^{-2} \hat{\rho} \cdot l_{i,j} - |\rho|^{-3}(|l_{i,j}|^2 - 3|\hat{\rho} \cdot l_{ij}|^2) \\ & + |\rho|^{-4} \left(\frac{3|l_{ij}|^2 (\hat{\rho} + |\rho|^{-1} \theta l_{ij}) \cdot l_{ij}}{|\hat{\rho} + |\rho|^{-1} \theta l_{i,j}|^5} - \frac{15 |(\hat{\rho} + |\rho|^{-1} \theta l_{ij}) \cdot l_{ij}|^3}{|\hat{\rho} + |\rho|^{-1} \theta l_{i,j}|^7} \right). \end{aligned}$$

Then, one has

$$I_{eff} = \frac{1}{|\rho|}\int \sum_{i\in a_1, j\in a_2} e_i e_j |\eta|^2 d\xi_1 \cdots d\xi_{N-2} + \frac{1}{|\rho|^2}\int \sum_{i\in a_1, j\in a_2} e_i e_j \hat{\rho}\cdot l_{i,j}|\eta|^2 d\xi_1 \cdots d\xi_{N-2}$$
$$+\frac{1}{|\rho|^3}\int \sum_{i\in a_1, j\in a_2} e_i e_j (|l_{i,j}|^2 - 3|\hat{\rho}\cdot l_{ij}|^2) d\xi_1 \cdots d\xi_{N-2} + O(\frac{1}{|\rho|^4})\ .$$

The first term of the right hand side is equal to 0 if $C_1 = 0$ or $C_2 = 0$. Note that $l_{ij} = l_i - l_j$ for $i \in a_1,\ j \in a_2$. It follows that $\sum_{i\in a_1, j\in a_2} e_i e_j l_{i,j} = 0$, if $C_1 = C_2 = 0$. This ends the proof. □

3 - Coupling constant limits of Schrödinger operators with critical potentials and the asymptotic expansion of resolvent of Schrödinger operators

3.1 Introduction

In Section 2.5 (Chapter 2), we consider the N-body System of ν-dimensional particles with Coulomb potentials. The effective potential has the form $\frac{f_1(\hat{\rho})}{|\rho|^2} + \frac{f_2(\hat{\rho})}{|\rho|^3} + O(\frac{1}{|\rho|^4})$ as $|\rho|$ large. Here $\hat{\rho} = \frac{\rho}{|\rho|}$. Hence, we study Schrödinger operators $P(\lambda) = P_0 + \lambda V$ and $P(\lambda) = \tilde{P}_0 + \lambda V$ in $L^2(\mathbb{R}^n)$ in this chapter. Here P_0 and $\tilde{P}_0$ are perturbation of $-\Delta$ in the form $-\Delta + f(x)$ on $L^2(\mathbb{R}^n)$ with $f(x) = \frac{q(\theta)}{|x|^2}$ when $|x|$ is large enough. Here $\theta = \frac{x}{|x|}$ and $q(\theta)$ is a real continuous function on unit sphere $\mathbb{S}^{n-1}$. Assume that $V(x) = O(|x|^{-2-\epsilon})$ with some $\epsilon > 0$, when $|x|$ large enough.

This chapter is composed of two parts. In the first part, we consider a family of Schrödinger operators, $P(\lambda)$, which are the perturbation of P_0 in the form

$$P(\lambda) = P_0 + \lambda V, \text{ for } \lambda \geq 0$$

on $L^2(\mathbb{R}^n)$, $n \geq 2$. Here $P_0 = -\Delta + \frac{q(\theta)}{r^2}$. (r, θ) is the polar coordinates on $\mathbb{R}^n$, $q(\theta)$ is a real continuous function. $V \leq 0$ is a non-zero continuous function and satisfies

$$|V(x)| \leq C\langle x\rangle^{-\rho_0}, \qquad \text{for some } \rho_0 > 2. \tag{3.1}$$

Let Δ_s denote Laplace operator on the sphere $\mathbb{S}^{n-1}$. Assume

$$-\Delta_s + q(\theta) \geq -\frac{1}{4}(n-2)^2, \quad \text{on } L^2(\mathbb{S}^{n-1}). \tag{3.2}$$

We will show that P_0 and $P(\lambda)$ are self-adjoint operators in $L^2(\mathbb{R}^n)$ with the form domain $Q(P_0)$, when $-\Delta_s + q(\theta) > -\frac{1}{4}(n-2)^2$, on $L^2(\mathbb{S}^{n-1})$ (see Section 3.2), especially $Q(P_0) = H^1$ if $n \geq 3$. If (3.2) holds, we also have $P_0 \geq 0$ in $L^2(\mathbb{R}^n)$(see Section 3.2). Set

$$\sigma_\infty = \{\nu; \nu = \sqrt{\lambda + \frac{1}{4}(n-2)^2},\ \lambda \in \sigma(-\Delta_s + q(\theta))\}, \quad \sigma_k = \sigma_\infty \cap [0, k],\ k \in \mathbb{N}.$$

If $q = 0$, then $P_0 = -\Delta$. In this case, σ_∞ consists of either only half-integers (n odd) or only integers (n even). In particular, for Laplace operator $-\Delta$, one has $\sigma_1 = \{0, 1\}$, $n = 2$; $\sigma_1 = \{\frac{1}{2}\}$, $n = 3$; $\sigma_1 = \{1\}$, $n = 4$; $\sigma_1 = \emptyset$, $n = 5$.

$P(\lambda)$ has continuous spectrum $[0, \infty)$ for $\lambda \geq 0$, because $\lim_{|x|\to\infty} V(x)$ exists and equals to 0 (See [3]). We claim that when λ large enough, $P(\lambda)$ has discrete spectrum less than 0. In fact, we need only to show that there exists a function $\psi \in L^2(\mathbb{R}^n)$ such that $\langle \psi, P(\lambda)\psi \rangle < 0$.

From the assumption on V, we know that there exists a point $x_0 \in \mathbb{R}^n$ such that $V(x_0) = \inf_{x\in\mathbb{R}^n} V(x)$. Choose $\delta > 0$ small enough such that for all $x \in B(x_0, \delta)$, $V(x) < \frac{1}{2}V(x_0)$. For $\psi \in C_0^\infty(\mathbb{R}^n), \|\psi(x)\| = 1$, $\text{supp}\psi \subset B(x_0, \delta)$, one has

$$\langle \psi, P(\lambda)\psi \rangle = \langle \psi, P_0\psi \rangle + \lambda\langle \psi, V\psi \rangle < \langle \psi, P_0\psi \rangle + \frac{\lambda}{2}V(x_0),$$

when λ large enough , one has $\langle \psi, P(\lambda)\psi \rangle < 0$.

We also know that $\sigma(P(0)) = \sigma(P_0) = [0, \infty)$. Hence, from the continuity of discrete spectrum of $P(\lambda)$, we know there exists some λ_0 such that when $\lambda > \lambda_0$, $P(\lambda)$ has eigenvalues less then 0, and when $\lambda \leq \lambda_0$, $\sigma(P(\lambda)) = [0, \infty)$. So $P(\lambda)$ has an eigenvalue $e_1(\lambda) < 0$ at the bottom of its spectrum for $\lambda > \lambda_0$. By Proposition 3.5, one has $e_1(\lambda)$ is simple and the corresponding eigenfunction can be chosen to be positive everywhere. (There are many results about the simplicity of the smallest eigenvalue of Schrödinger operator with potential without singularity, but we did not find the result which can be used directly, because the potential of our Schrödinger operator has singularity at 0. Theorem XIII.48 [54] can treat the Schrödinger operator with the potential with singularity at 0, but the positivity of potential is demanded. Hence we give this result.) In this paper, we suppose $\lambda_0 > 0$. We study the 0 resonances of $P(\lambda_0)$ and the asymptotic behavior of discrete eigenvalues of $P(\lambda)$ at value λ_0. λ_0 is the value at which some eigenvalue converges to 0. In [33], Klaus and Simon studied the convergent rate of discrete eigenvalues of $P(\lambda) = -\Delta + \lambda V$, when $\lambda \to \lambda_0$. Here λ_0 is the value at which some discrete eigenvalue $e_i(\lambda) \uparrow 0$ of $P(\lambda)$. Fassri-Klaus [20] also studied this problem for Schrödinger operator $-\Delta + V + \lambda W$ with V periodic. They used Birman-Schwinger kernel in their papers. We also use Birman-Schwinger kernel to study this problem. In order to use the Birman-Schwinger technic to $P(\lambda)$, we need to know the asymptotic expansion of $(P_0 - \alpha)^{-1}$ for α near 0, $\alpha < 0$, which has been studied by X.P. Wang ([70]).

In the second part, we consider Schrödinger operator P, which is the perturbation of $-\Delta$ in the form

$$P = -\Delta + \tilde{V}, \text{ for } \lambda \geq 0 \tag{3.3}$$

on $L^2(\mathbb{R}^n)$, $n \geq 3$. Here $\tilde{V} = V_1 + V_2$. V_1, V_2 are continuous function such that $|V_1| = \frac{q(\theta)}{r^2}$ when r large enough, $V_2 \leq 0$ is a non-zero function and satisfies $|V_2| \leq C\langle x \rangle^{-\rho_0}$ with some $\rho_0 > 2$. $q(\theta)$ is a real function on sphere $\mathbb{S}^{n-1}$ such that (3.4) holds. We study the asymptotic expansion of $(P - z)^{-1}$ for z near 0, $\Im z > 0$. Let

$$\tilde{P}(\lambda) = \tilde{P}_0 + \lambda V$$

with $\tilde{P}_0 = \chi_1(-\Delta)\chi_1 + \chi_2(-\Delta + \frac{q(\theta)}{r^2})\chi_2$. (r, θ) is the polar coordinates on $\mathbb{R}^n$, V is given by (3.1). $\chi_1 \in C_0^\infty(\mathbb{R})$ such that $\chi_1(r) = 1$ for $|r| \leq 1$ and $\chi_1^2 + \chi_2^2 = 1$. We study the asymptotic behavior of the smallest eigenvalue of $\tilde{P}(\lambda)$ as $\lambda \to \lambda_0$. λ_0 is the value, at which the smallest discrete eigenvalue of $\tilde{P}(\lambda)$ converges to 0.

Here is the plan of this chapter. In section 3.2, we recall some results of P_0([70]). In section 3.3, we study the asymptotic behavior of $e_1(\lambda)$ of $P(\lambda)$ when $0 \notin \sigma_\infty$. We get the leading term of $e_1(\lambda)$. The multiplicity of 0 as the resonance of $P(\lambda_0)$ is studied in Section 3.4. In Section 3.5, we study the asymptotic expansion of $(P - z)^{-1}$, z near 0, $\Im z > 0$ (P is given by (3.3)). In section 3.6, we study the asymptotic behavior of the smallest eigenvalue of $\tilde{P}(\lambda)$.

3.2 Some results for P_0

In this section, we consider the operator

$$P_0 = -\Delta + \frac{q(\theta)}{r^2}$$

on $L^2(\mathbb{R}^n)$ $n \geq 2$. Here (r, θ) is the polar coordinates on $\mathbb{R}^n$, $q(\theta)$ is a real continuous function. Let Δ_s denote Laplace operator on the sphere $\mathbb{S}^{n-1}$. Assume

$$-\Delta_s + q(\theta) \geq -\frac{1}{4}(n-2)^2, \quad \text{on } L^2(\mathbb{S}^{n-1}). \tag{3.4}$$

We begin this section with studying the form domain of P_0. We will show that $-\Delta_s + q(\theta) \geq -\frac{1}{4}(n-2)^2$, on $L^2(\mathbb{S}^{n-1})$ implies

$$\langle \phi, (-\Delta + \frac{q(\theta)}{r^2})\phi \rangle \geq 0, \quad \text{for} \quad \phi \in D. \tag{3.5}$$

Here $D = C_0^\infty(\mathbb{R}^n \backslash \{0\})$. Let $\tilde{D}$ be the set of functions in D which is the set of finite linear combination of products $f(r)g(\theta)$. Note that $\tilde{D}$ is dense in $L^2(\mathbb{R}^n)$. One has

$$P_0 = -\frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}(-\Delta_s + q(\theta)).$$

Let $U : L^2(\mathbb{R}_+, r^{n-1}dr) \to L^2(\mathbb{R}_+, dr)$ be the unitary operator, $U : \phi \mapsto r^{(n-1)/2}\phi$. Then

$$UP_0U^{-1} = -\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}[\frac{(n-1)(n-3)}{4} + (-\Delta_s + q(\theta))].$$

It follows that $\langle \phi, UP_0U^{-1}\phi \rangle \geq \langle \phi, (-\frac{\partial^2}{\partial r^2} - \frac{1}{4r^2})\phi \rangle$ for $U^{-1}\phi \in \tilde{D}$. To prove $P_0 \geq 0$ on $\tilde{D}$, it suffices to show that $\langle \psi, (-\frac{\partial^2}{\partial r^2} - \frac{1}{4r^2})\psi \rangle \geq 0$ for $\psi \in C_0^\infty(\mathbb{R}\backslash\{0\}, dr)$. Note that

$$(r^{1/2}\psi)' = r^{1/2}\psi' + \frac{1}{2}r^{-1/2}\psi,$$

then one has

$$|\psi'|^2 = r^{-1}|\,(r^{1/2}\psi)\,' - \frac{1}{2}r^{-1/2}\psi|^2 \geq \frac{1}{4r^2}|\psi|^2 - r^{-3/2}\psi(r^{1/2}\psi)'.$$

It follows

$$\int_0^\infty |\psi'|^2 dr \geq \int_0^\infty \frac{1}{4r^2}|\psi|^2 - r^{-3/2}\psi(r^{1/2}\psi)' dr = \int_0^\infty \frac{1}{4r^2}|\psi|^2 dr\,.$$

Then we get that, for all $\phi \in \tilde{D}$, $\langle \phi, P_0\phi\rangle \geq 0$. By a density argument, we can get (3.5). Write

$$q(\theta) = q^+(\theta) - q^-(\theta) \;\text{ with }\; q^+(\theta) \geq 0,\; q^-(\theta) \geq 0\,. \tag{3.6}$$

Note that $-\Delta$ is a self-adjoint operator in $L^2(\mathbb{R}^n)$ with the form domain $H^1(\mathbb{R}^n)$. If $n \geq 3$, by Hardy inequality, one has $\langle \phi, \frac{q^+(\theta)}{r^2}\phi\rangle \leq C\langle \phi, -\Delta\phi\rangle$ with some $C > 0$, since $q(\theta)$ is a continuous function on $\mathbb{S}^{n-1}$. Hence, we can define

$$\gamma(\phi,\phi) = \langle \phi, -\Delta\phi\rangle + \langle \phi, \frac{q^+(\theta)}{r^2}\phi\rangle$$

for $\phi \in H^1(\mathbb{R}^n)$. To prove that $-\Delta + \frac{q^+(\theta)}{r^2}$ is a self-adjoint operator in $L^2(\mathbb{R}^n)$ with the form domain $H^1(\mathbb{R}^n)$, we need only to show that $\gamma(\cdot,\cdot)$ is a closed semi-bounded quadratic form (see Theorem VIII 15 [52]). Since $\langle \phi, -\Delta\phi\rangle + \langle \phi, \frac{q^+(\theta)}{r^2}\phi\rangle \geq 0$ for $\phi \in H^1(\mathbb{R}^n)$, then γ is a semi-bounded quadratic form. Note that

$$\langle \phi, -\Delta\phi\rangle + \|\phi\|^2 \leq \langle \phi, -\Delta\phi\rangle + \langle \phi, \frac{q^+(\theta)}{r^2}\phi\rangle + \|\phi\|^2 \leq (1+C)\,(\langle \phi, -\Delta\phi\rangle + \|\phi\|^2),$$

then $H^1(\mathbb{R}^n)$ is closed with respect to the norm $\|\cdot\|_1 \equiv \sqrt{\langle \cdot, -\Delta\cdot\rangle + \langle \cdot, \frac{q^+(\theta)}{r^2}\cdot\rangle + \|\cdot\|^2}$. This means γ is closed. It follows that γ is the quadratic form of a unique self-adjoint operator with form domain $H^1(\mathbb{R}^n)$. Then $-\Delta + \frac{q^+(\theta)}{r^2}$ is a self-adjoint operator in $L^2(\mathbb{R}^n)$ with the form domain $H^1(\mathbb{R}^n)$. If $n = 2$, let $Q(P_0)$ be the completion of D under the norm $\|\cdot\|_1 \equiv \sqrt{\langle \cdot, -\Delta\cdot\rangle + \langle \cdot, \frac{q^+(\theta)}{r^2}\cdot\rangle + \|\cdot\|^2}$. Then $Q(P_0)$ is the form domain of $-\Delta + \frac{q^+(\theta)}{r^2}$. If

$$-\Delta_s + q(\theta) > -\frac{1}{4}(n-2)^2, \qquad \text{on } L^2(\mathbb{S}^{n-1}),$$

there exists a constant $c > 0$ such that

$$-\Delta_s + q(\theta) \geq -\frac{(n-2)^2}{4} + c, \qquad \text{on } L^2(\mathbb{S}^{n-1}).$$

Set $q'(\theta) = q(\theta) - c$, then

$$-\Delta_s + q'(\theta) \geq -\frac{(n-2)^2}{4}, \qquad \text{on } L^2(\mathbb{S}^{n-1}).$$

Let $Q(P_0) = H^1(\mathbb{R}^n)$ for $n \geq 3$. By (3.5), one has, for $\phi \in Q(P_0)$,

$$\langle \phi, (-\Delta + \frac{q'(\theta)}{r^2})\phi\rangle \geq 0. \tag{3.7}$$

Write

$$q'(\theta) = q'(\theta)^+ - q'(\theta)^- \text{ with } q'(\theta)^+ \geq 0, q'(\theta)^- \geq 0.$$

By (3.7), one has

$$\langle \phi, \frac{q'(\theta)^-}{r^2}\phi\rangle \leq \langle \phi, (-\Delta + \frac{q'(\theta)^+}{r^2})\phi\rangle.$$

Then $q^-(\theta) \leq \alpha_0 q'(\theta)^-$. Here $\alpha_0 = 0$ when $q(\theta) \geq 0$ and $\alpha_0 = \max\limits_{\{\ \theta;\ q(\theta)<0\ \}} \frac{q^-(\theta)}{q'(\theta)^-}$ when $q(\theta) < 0$. Since $q'(\theta) < q(\theta)$, one can get $\alpha_0 < 1$. It follows

$$\langle \phi, \frac{q^-(\theta)}{r^2}\phi\rangle \leq \alpha_0\langle \phi, \frac{q'(\theta)^-}{r^2}\phi\rangle \leq \alpha_0\langle \phi, (-\Delta + \frac{q'(\theta)^+}{r^2})\phi\rangle \leq \alpha_0\langle \phi, (-\Delta + \frac{q^+(\theta)}{r^2})\phi\rangle. \tag{3.8}$$

Then we have $-\Delta + \frac{q^+(\theta)}{r^2} + \frac{q^-(\theta)}{r^2} = -\Delta + \frac{q(\theta)}{r^2}$ is a self-adjoint in $L^2(\mathbb{R}^n)$ with the form domain $Q(P_0)$ for $n \geq 2$.

Now, we recall some results on the resolvent and the Schrödinger group for the unperturbed operator P_0. Let

$$\sigma_\infty = \Big\{\nu; \nu = \sqrt{\lambda + \frac{(n-2)^2}{4}}, \lambda \in \sigma(-\Delta_s + q(\theta))\Big\}.$$

Denote

$$\sigma_k = \sigma_\infty \cap [0, k],\ k \in \mathbb{N}.$$

For $\nu \in \sigma_\infty$, let n_ν denote the multiplicity of $\lambda_\nu = \nu^2 - \frac{(n-2)^2}{4}$ as the eigenvalue of $-\Delta_s + q(\theta)$. Let $\varphi_\nu^{(j)}, \nu \in \sigma_\infty, 1 \leq j \leq n_\nu$ denote an orthonormal basis of $L^2(\mathbb{S}^{n-1})$ consisting of eigenfunctions of $-\Delta_s + q(\theta)$:

$$(-\Delta_s + q(\theta))\varphi_\nu^{(j)} = \lambda_\nu \varphi_\nu^{(j)},\ (\varphi_\nu^{(i)}, \varphi_\nu^{(j)}) = \delta_{ij}.$$

Let π_ν denote the orthogonal projection in $L^2(\mathbb{S}^{n-1})$ onto the subspace spanned by the eigenfunctions of $-\Delta_s + q(\theta)$ associated with the eigenvalue λ_ν, and $\pi_\nu^{(i)}$ denote the orthogonal projection in $L^2(\mathbb{S}^{n-1})$ onto eigenfunction $\varphi_\nu^{(i)}$:

$$\pi_\nu f = \sum_{j=1}^{n_\nu} (f, \varphi_\nu^{(j)}) \otimes \varphi_\nu^{(j)},\ f \in L^2(\mathbb{S}^{n-1}).$$

$$\pi_\nu^{(i)} f = (f, \varphi_\nu^{(i)}) \otimes \varphi_\nu^{(i)},\ f \in L^2(\mathbb{S}^{n-1}), 1 \leq i \leq n_\nu.$$

Let

$$Q_\nu = -\frac{d^2}{dr^2} - \frac{n-1}{r}\frac{d}{dr} + \frac{\nu^2 - \frac{(n-2)^2}{4}}{r^2},\ \text{in } L^2(\mathbb{R}_+; r^{n-1}dr).$$

Then we have the orthogonal decomposition for the resolvent $R_0(z) = (P_0 - z)^{-1}$,

$$R_0(z) = \sum_{\nu \in \sigma_\infty} (Q_\nu - z)^{-1}\pi_\nu,\ z \notin \mathbb{R}$$

Set

$$f(s;r,\tau,\nu) = D_\nu(r,\tau)\int_{-1}^{1} e^{i(\rho+\theta/2)s}(1-\theta^2)^{\nu-1/2}d\theta,\ \nu \geq 0,$$

with

$$\begin{aligned}\rho &= \rho(r,\tau) \equiv \frac{r^2+\tau^2}{4r\tau},\\ D_\nu &= a_\nu(r\tau)^{-\frac{(n-2)}{2}},\ a_\nu = -\frac{e^{-i\pi\nu/2}}{2^{2\nu+1}\pi^{1/2}\Gamma(\nu+1/2)}.\end{aligned}$$

Then

$$f(s;r,\tau,\nu) = \sum_{j=0}^{\infty} s^j f_j(r,\tau,\nu),\ s \in \mathbb{R},$$

with

$$f_j(r,\tau,\nu) = (r\tau)^{-\frac{1}{2}(n-2)}P_{j,\nu}(\rho),$$

with $P_{j,\nu}(\rho)$ a polynomial in ρ of degree j :

$$P_{j,\nu}(\rho) = \frac{i^j a_\nu}{j!}\int_{-1}^{1}(\rho+\frac{1}{2}\theta)^j(1-\theta^2)^{\nu-\frac{1}{2}}d\theta.$$

In particular,

$$\begin{aligned}f_0(r,\tau,\nu) &= d_\nu(r\tau)^{-\frac{n-2}{2}},\ d_\nu = -\frac{e^{-\frac{1}{2}i\pi\nu}}{2^{2\nu+1}\Gamma(\nu+1)};\\ f_1(r,\tau,\nu) &= id_\nu(r\tau)^{-\frac{n-2}{2}}\rho.\end{aligned}$$

Denote J_ν the Bessel function of the first kind of order ν, J_ν can be represented as

$$J_\nu(\lambda) = \frac{1}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}(\frac{\lambda}{2})^\nu\int_{-1}^{1}e^{i\lambda t}(1-t^2)^{\nu-\frac{1}{2}}dt,\ \Re\nu > -\frac{1}{2}.$$

Let $O_N(g(s))$ denote the remainder in Taylor expansion of g up to the $N-$th order :

$$O_N(g(s)) = g(s) - \sum_{j=0}^{N}\frac{g^{(k)}(0)}{k!}s^k = \frac{1}{N!}\int_0^1(1-\theta)^N s^{(N+1)}g^{(N+1)}(s\theta)\,d\theta.$$

For $\Im z > 0$, $t > 0$, one has $|O_N(e^{izt})| \leq C_N(|z|t)^{N+1}$. Using $O_N(e^{izt}) = O_{N-1}(e^{izt}) - \frac{1}{N!}(izt)^N$, we can drive that $|O_N(e^{izt})| \leq C_N(|z|t)^N$. It follows that for any $0 \leq \theta \leq 1$,

$$|O_N(e^{izt})| \leq C_{N,\theta}(|z|t)^{N+\theta} \quad \text{and} \tag{3.9}$$

$$|\frac{d}{dz}O_N(e^{izt})| = t|O_{N-1}(e^{izt})| \leq C_{N,\theta}t^{N+\theta}|z|^{N+\theta-1} \tag{3.10}$$

for $t > 0$, $\Im z > 0$. For $0 < \nu < 1$, denote

$$C_{\nu,j} = \frac{i^j}{j!}\int_1^\infty t^{j-\nu-1}O_j(f(\frac{1}{t}))dt + \frac{i^j}{j!}\sum_{k=0}^{j}\frac{f_k}{k-j+\nu} \tag{3.11}$$

$$b_{\nu,j} = -\frac{i^j e^{-i\nu\pi/2}\Gamma(1-\nu)}{\nu(\nu+1)\cdots(\nu+j)}, \quad j \geq 0 \tag{3.12}$$

$$\begin{aligned}\tilde{R}_{\nu,N,2}(\zeta) = &\int_1^\infty O_N(e^{i\zeta t})t^{-1-\nu}O_N(f(\frac{1}{t})dt \\ &-\sum_{j=0}^{N-1}f_j\frac{(i\zeta)^{j+1}}{j!}\int_0^1\int_0^1 O_{N-j-1}(e^{i\zeta\theta t})(1-\theta)^j t^{-\nu}dtd\theta.\end{aligned} \tag{3.13}$$

For $\nu = 0$, denote

$$C_{0,j} = \frac{i^j}{j!}\int_1^\infty t^{j-1}O_j(f(\frac{1}{t}))dt + i^j f_j b_j + \frac{i^j}{j!}\sum_{k=0}^{j-1}\frac{f_k}{k-j}, \quad j \geq 0, \tag{3.14}$$

$$b_0 = \int_1^\infty e^{it}\frac{dt}{t} + \int_0^1 (e^{it}-1)\frac{dt}{t} \tag{3.15}$$

$$b_j = -\frac{1}{(j-1)!}\int_0^1 (1-\theta)^{j-1}\ln\theta d\theta + \frac{b_0}{j!}, \quad j \geq 1 \tag{3.16}$$

$$\begin{aligned}\tilde{R}_{0,N,2}(\zeta) = &\int_1^\infty t^{-1}O_N(e^{i\zeta t})O_{N+1}(f(\frac{1}{t}))dt - f_0\int_0^1 t^{-1}O_N(e^{i\zeta t})dt \\ &-\sum_{j=1}^{N}f_j\frac{(i\zeta)^j}{(j-1)!}\int_0^1\int_0^1 (1-\theta)^{j-1}O_{N-j}(e^{it\zeta\theta})\frac{dt}{t})d\theta.\end{aligned} \tag{3.17}$$

Then we have following

Proposition 3.1. *(Proposition 2.1 [70])Let $\nu \in \sigma_\infty$ and $l \in \mathbb{N}$ with $l \leq \nu < l+1$. Set $\nu' = \nu - l \in [0,1)$,*

(a) If $l < \nu < l+1$, one has

$$(Q_\nu - z)^{-1} = \sum_{j=0}^{N} z^j F_{\nu,j} + z^{\nu'}\sum_{j=l}^{N-1} G_{\nu,j} + R_{\nu,N}(z)$$

with

$$F_{\nu,j} = -(r\tau)^{-\frac{n-2}{2}+j}\frac{i^j}{j!}\int_0^\infty e^{i\frac{\rho}{t}-i\frac{\pi\nu}{2}}t^j J_\nu(\frac{1}{2t})\frac{dt}{2t} \tag{3.18}$$

for $0 \leq j \leq l$ and for $l+1 \leq j \leq N$,

$$\begin{aligned}F_{\nu,j} &= \frac{i^l(j-l)!}{j!}(r\tau)^j C_{\nu',j-l} + \frac{(ir\tau)^j}{j!}\int_0^1 t^{j-\nu-1}f(\frac{1}{t};r,\tau,\nu)dt, \\ G_{\nu,j} &= (r\tau)^{j+\nu'}b_{\nu',j}f_{j-l}\ l \leq j \leq N,\end{aligned}$$

$$R_{\nu,N}(z) = z^{\nu'+N}G_{\nu+N} + \frac{(ir\tau z)^l}{(l-1)!}\int_0^1 (1-\theta)^{l-1}\Big\{\widetilde{R}_{\nu',N-l,2}(\theta z r\tau) + \int_0^1 O_{N-l}(e^{it\theta z r\tau})t^{l-1-\nu}f(\frac{1}{t};r,\tau,\nu)dt\Big\}d\theta.$$

When $l = 0$, the integral in θ is absent.

(b) If $\nu = l \in \mathbb{N}$, then

$$(Q_\nu - z)^{-1} = \sum_{j=0}^{N} z^j F_{\nu,j} + \ln z \sum_{j=l}^{N} z^j G_{\nu,j} + R_{\nu,N}(z)$$

with $F_{\nu,j}$ was given by (3.18) for $0 \le j \le l-1$ and

$$\begin{aligned} F_{\nu,j} &= (r\tau)^j\Big\{\frac{i^l(j-l)!}{j!}C_{0,j-l} - \ln(r\tau)\frac{i^j f_{j-l}}{j!} - c_{l,j}f_{j-l}\Big\} \\ &\quad + \frac{(ir\tau)^j}{j!}\int_0^1 t^{j-l-1}f(\frac{1}{t};r,\tau,\nu)dt,\ l \le j \le N, \\ G_{\nu,j} &= -(ir\tau)^j\frac{f_{j-l}}{j!},\ l \le j \le N, \end{aligned}$$

$$\begin{aligned} R_{0,N}(z) &= (ir\tau z)^{N+1}f_{N+1}b_{N+1} + \widetilde{R}_{0,N,2}(zr\tau) + \int_0^1 O_N(e^{itzr\tau})t^{-1}f(\frac{1}{t};r,\tau,\nu)dt \text{ and} \\ R_{l,N}(z) &= \frac{(ir\tau z)^{N+1}(N-l+1)!}{(N+1)!}f_{N+1-l}b_{N+1-l} + \int_0^1 (1-\theta)^{l-1}\Big\{\widetilde{R}_{0,N-l,2}(\theta z r\tau) \\ &\quad + \int_0^1 O_{N-l}(e^{it\theta z r\tau})t^{-1}f(\frac{1}{t};r,\tau,\nu)dt\Big\}d\theta \end{aligned}$$

for $\nu = l \ge 1$. Here $c_{0,j} = 0$ for all j and

$$c_{l,j} = -\frac{i^j}{(l-1)!(j-l)!}\int_0^1 (1-\theta)^{l-1}\theta^{j-l}\ln\theta d\theta,\ l \ge 1,\ j \ge l.$$

Here

$$C_{\nu',j} = \frac{i^j}{j!}\int_1^\infty t^{j-\nu'-1}O(f(\frac{1}{t};r,\tau,\nu))dt + \frac{i^j}{j!}\sum_{k=0}^{j}\frac{f_k}{k-j+\nu'};$$

$$b_{\nu',j} = -\frac{i^j e^{-i\nu'\pi/2}\Gamma(1-\nu')}{\nu'(\nu'+1)\cdots(\nu'+j)}, \tag{3.19}$$

for $0 \le \nu' < 1$.

Denote for $\nu \in \sigma_\infty$,

$$z_\nu = \begin{cases} z^{\nu'}, & \text{if } \nu \notin \mathbb{N}, \\ z \ln z, & \text{if } \nu \in \mathbb{N}. \end{cases}$$

For $\nu > 0$, let $[\nu]_-$ be the largest integer strictly less than ν. When $\nu = 0$, set $[\nu]_- = 0$. Define δ_ν by $\delta_\nu = 1$, if $\nu \in \sigma_\infty \cap \mathbb{N}$, $\delta_\nu = 0$, otherwise. One has $[\nu] = [\nu]_- + \delta_\nu$.

Theorem 3.2. *(Theorem 2.2 [70]) The following asymptotic expansion holds for z near 0 with $\Im z > 0$,*

$$R_0(z) = \delta_0 \ln z G_{0,0}\pi_0 + \sum_{j=0}^{N} z^j F_j + \sum_{\nu\in\sigma_N} z_\nu \sum_{j=[\nu]_-}^{N-1} z^j G_{\nu,j+\delta_\nu}\pi_\nu + R_0^{(N)}(z)$$

in $\mathcal{L}(-1,s;1,-s)$, $s > 2N+1$. Here

$$G_{\nu,j}(r,\tau) = \begin{cases} b_{\nu',j}(r\tau)^{j+\nu'} f_{j-[\nu]}(r,\tau;\nu'), & \nu \notin \mathbb{N} \\ -\dfrac{(ir\tau)^j}{j!} f_{j-[\nu]}(r,\tau;0), & \nu \in \mathbb{N} \end{cases}$$

$$F_j = \sum_{\nu\in\sigma_\infty} F_{\nu,j}\pi_\nu \in \mathcal{L}(-1,s;1,-s),\ s > 2j+1$$

$$R_0^{(N)}(z) = O(|z|^{N+\epsilon}) \in \mathcal{L}(-1,s;1,-s),\ s > 2N+1, \epsilon > 0.$$

Here $b_{\nu',j}$ is given by (3.19).

Let $V \le 0$ be a non-zero continuous function and satisfying

$$|V(x)| \le C\langle x\rangle^{-\rho_0}, \qquad \text{for some } \rho_0 > 2. \tag{3.20}$$

Let $P(\lambda) = P_0 + \lambda V$.

Definition 3.3. *Set $\mathcal{N}(\lambda) = \{u; P(\lambda)u = 0, u \in H^{1,-s}, \forall s > 1\}$, for $\lambda \ge \lambda_0$. If $\mathcal{N}(\lambda)\backslash L^2 \neq \{0\}$, we say that 0 is the resonance of $P(\lambda)$. A non-zero function $u \in \mathcal{N}(\lambda)\backslash L^2$ is called a resonant state of $P(\lambda)$ at zero.* $\dim \mathcal{N}(\lambda)\backslash L^2$ *is called the multiplicity of 0 as the resonance of $P(\lambda)$.*

Let $K(z) = |V|^{1/2}(P_0 - z)^{-1}|V|^{1/2}$ for $z \notin \sigma(P_0)$, and $K(0) = |V|^{1/2}F_0|V|^{1/2}$. Then we have the following

Proposition 3.4. *Let $\alpha < 0$. Then $\alpha \in \sigma_d(P(\lambda))$ if and only if $\lambda^{-1} \in \sigma_d(K(\alpha))$. Moreover, the multiplicity of α as the eigenvalue of $P(\lambda)$ is exactly the multiplicity of λ^{-1} as the eigenvalue of $K(\alpha)$.*

Proof. Since $P_0 \ge 0$ in $L^2(\mathbb{R}^n)$ and $\alpha < 0$, then $(P_0 - \alpha)^{-1}$ exists in $L^2(\mathbb{R}^n)$.

$$P(\lambda) - \alpha = (P_0 - \alpha)(I + \lambda(P_0 - \alpha)^{-1}V(x)).$$

Therefore $\alpha \in \sigma_d(P(\lambda))$ if and only if $\lambda^{-1} \in \sigma_d(-(P_0 - \alpha)^{-1}V(x)) = \sigma_d(K(\alpha))$. In the last equality, we use that for bounded operators A, B, $\sigma(AB)\backslash\{0\} = \sigma(BA)\backslash\{0\}$(See [16]).

For fixed $\alpha < 0$, let

$$A = \{ \psi \in L^2(\mathbb{R}^n);\ (P(\lambda) - \alpha)\psi = 0 \},$$
$$B = \{\phi \in L^2(\mathbb{R}^n) : K(\alpha)\phi = \lambda^{-1}\phi\}.$$

It suffices to prove $\dim A = \dim B$. First, we will prove that $|V|^{1/2}$ is injective from A to B. Note that if $\psi \in A$, then

$$K(\alpha)\phi = \lambda^{-1}\phi$$

with $\phi = |V|^{1/2}\psi$. And if $\phi = 0$, then

$$\psi = -\lambda(P_0 - \alpha)^{-1}V\psi = \lambda(P_0 - \alpha)^{-1}|V|^{1/2}\phi = 0.$$

It follows that $|V|^{1/2}$ is injective from A to B. On the other hand, we can show that $(P_0 - \alpha)^{-1}|V|^{1/2}$ is injective from B to A. If $\phi \in B$, then

$$(P(\lambda) - \alpha)\psi = 0, \text{ with } \psi = (P_0 - \alpha)^{-1}|V|^{1/2}\phi.$$

And if $\psi = 0$, then

$$0 = |V|^{1/2}\psi = K(\alpha)\phi = \lambda^{-1}\phi.$$

It follows that $(P_0 - \alpha)^{-1}|V|^{1/2}$ is injective from B to A. This ends the proof. □

From this proposition, we can see that there is a one-to-one corresponding between discrete eigenvalues of $P(\lambda)$ and discrete eigenvalues of $K(\alpha)$. Thus we study discrete eigenvalue of $K(\alpha)$ in the next section.

3.3 The case $0 \notin \sigma_\infty$

If P_0 and V are defined as before, we will show that if $P_0 + V$ has the eigenvalue less than 0, then the smallest eigenvalue of $P_0 + V$ is simple. We use Theorem XIII.45 [54] to prove that.

Proposition 3.5. *Suppose P_0+V has an eigenvalue at the bottom of its spectrum. If $0 \notin \sigma_\infty$, then this eigenvalue is simple and the corresponding eigenfunction can be chosen to be a positive function.*

Proof. Let $0 \le \chi(t) \le 1$ be a smooth function such that $\chi(t) = 1$ if $|t| < 1$ and $\chi(t) = 0$ if $t > 2$. Let $\chi_n(t) = \chi(nt)$. Let

$$P = P_0 + V, \quad V_n = (1 - \chi_n(r))\frac{q(\theta)}{r^2} + V,$$
$$P_n = -\Delta + (1 - \chi_n(r))\frac{q(\theta)}{r^2} + V.$$

By Theorem XIII45 [54], we need only to prove that P_n converges to P, and $P - V_n$ converges to $-\Delta$ in the strong resolvent norm sense. By min-max principle, we know that if $\mu < P$, then

$\mu < P_n$, because $0 \le \chi(t) \le 1$. It suffices to show that there exists some $\mu_0 < \sigma(P)$, such that $\forall \phi \in D(P)$, the domain of P,

$$\|(P_n - \mu_0)^{-1}\phi - (P - \mu_0)^{-1}\phi\| \to 0 \tag{3.21}$$

$$\|(P - V_n - \mu_0)^{-1}\phi - (-\Delta - \mu_0)^{-1}\phi\| \to 0 \tag{3.22}$$

when $n \to \infty$. The proof of (3.21) and the proof of (3.22) are similar. So we need only to prove (3.21). $q^+(\theta)$ and $q^-(\theta)$ are given by (3.6), then

$$\begin{aligned}
& (P_n - \mu_0)^{-1} \\
= \; & (-\Delta + 1 + V_n - \mu_0 - 1)^{-1} \\
= \; & (-\Delta + \frac{q^+(\theta)}{r^2} + 1)^{-1/2}[I + (-\Delta + \frac{q^+(\theta)}{r^2} + 1)^{-1/2}(\chi_n(r)\frac{q^+(\theta)}{r^2} - (1 - \chi_n(r))\frac{q^-(\theta)}{r^2})(-\Delta + \frac{q^+(\theta)}{r^2} + 1)^{-1/2} \\
& +(-\Delta + \frac{q^+(\theta)}{r^2} + 1)^{-1/2}(V - \mu_0 - 1)(-\Delta + \frac{q^+(\theta)}{r^2} + 1)^{-1/2}]^{-1}(-\Delta + \frac{q^+(\theta)}{r^2} + 1)^{-1/2} \\
\equiv \; & A_n(-\Delta + \frac{q^+(\theta)}{r^2} + 1)^{-1/2}.
\end{aligned}$$

From (3.8), we know that $(-\Delta + \frac{q^+(\theta)}{r^2} + 1)^{-1/2}(1 - \chi_n(r))\frac{q^-(\theta)}{r^2}(-\Delta + \frac{q^+(\theta)}{r^2} + 1)^{-1/2} \le \alpha_0 I$, and choose μ_0 so negative such that $V - \mu_0 - 1 > 0$, then A_n are uniform bounded operators. Similarly, one has

$$(P - \mu_0)^{-1} \equiv (-\Delta + \frac{q^+(\theta)}{r^2} + 1)^{-1/2} B$$

with B is a bounded operator. Then

$$\begin{aligned}
& (P_n - \mu_0)^{-1}\phi - (P - \mu_0)^{-1}\phi \\
= \; & (P_n - \mu_0)^{-1}\chi_n(r)\frac{q(\theta)}{r^2}(P - \mu_0)^{-1}\phi \\
= \; & A_n(-\Delta + \frac{q^+(\theta)}{r^2} + 1)^{-1/2}\chi_n(r)\frac{q(\theta)}{r^2}(-\Delta + \frac{q^+(\theta)}{r^2} + 1)^{-1/2}B\phi
\end{aligned}$$

by (3.8),

$$(-\Delta + \frac{q^+(\theta)}{r^2} + 1)^{-1/2}\frac{q^-(\theta)}{r^2}(-\Delta + \frac{q^+(\theta)}{r^2} + 1)^{-1/2} \le \alpha_0$$

and

$$(-\Delta + \frac{q^+(\theta)}{r^2} + 1)^{-1/2}\frac{q^+(\theta)}{r^2}(-\Delta + \frac{q^+(\theta)}{r^2} + 1)^{-1/2} \le 1$$

Therefore,

$$\|(P_n - \mu_0)^{-1}\phi - (P - \mu_0)^{-1}\phi\| \to 0$$

as $n \to \infty$.

Similarly, we can prove (3.22). This ends the proof. □

Lemma 3.6. *Assume* $0 \notin \sigma_\infty$, *and* $\rho_0 > 3$. *If* $\sigma_1 \neq \emptyset$, *then 0 is not the eigenvalue of* $P(\lambda_0)$.

Proof. If 0 is the eigenvalue of $P(\lambda_0)$. Then there exists a function $u \in L^2(\mathbb{R}^n)$ such that $P(\lambda)u = 0$. By Proposition 3.5, one has that u can be chosen to be a positive function. By Theorem 3.1 [69],

$$u = \sum_{0<\nu\le 1}\sum_{j=1}^{n_\nu} -\frac{1}{2\nu}\langle Vu, |y|^{-\frac{1}{2}(n-1)+\nu}\varphi_\nu^{(j)}\rangle \frac{\varphi_\nu^{(j)}(\theta)}{r^{\frac{1}{2}(n-2)+\nu}} + \tilde{u}$$

with $\tilde{u} \in L^2(\mathbb{R}^n)$. Since u and $\varphi_\nu^{(1)}$ are positive, $V \le 0$ is a non-zero function, we have $u \notin L^2(\mathbb{R}^n)$. This is contradictory to that u is the eigenfunction of $P(\lambda_0)$. □

In the following of this section, we suppose that $n \ge 3$.

Proposition 3.7. *Assume* $0 \notin \sigma_\infty$ *and* $n \ge 3$. $P_0F_0u = u$, *for any* $u \in H^{-1,s}$, $s > 1$; $F_0P_0u = u$ *for* $u \in H^{1,-s}$ *and* $P_0u \in H^{-1,s}$, $s > 1$.

Proof. If $u \in H^{-1,s}$, then $F_0u \in H^{1,-s}$. For any test function $\phi \in C_0^\infty(\mathbb{R}^n)$, we have $\langle P_0F_0u, \phi\rangle = \langle u, F_0P_0\phi\rangle$. If $0 \notin \sigma_\infty$, we have $\lim_{z\to 0}(P_0 - z)^{-1} = F_0$ in $H^{-1,s}$ for $\Im z > 0$. It follows $\langle u, F_0P_0\phi\rangle = \lim_{z\to 0}\langle u, (P_0 - z)^{-1}P_0\phi\rangle = \lim_{z\to 0}\langle u, \phi - z(P_0 - z)^{-1}\phi\rangle = \langle u, \phi\rangle$, because ϕ and $P_0\phi$ belong to $H^{-1,s}$. Hence, $P_0F_0u = u$ in $H^{-1,s}$.

On the other hand, if $P_0u \in H^{-1,s}$, we have $P_0F_0P_0u = P_0u$ in $H^{-1,s}$. It follows $P_0(F_0P_0u - u) = 0$. Then $F_0P_0u = u$, because $F_0P_0u - u \in H^{1,-s}$ and P_0 has no kernel in $H^{1,-s}$, $s > 1$. □

Proposition 3.8. *Assume* $0 \notin \sigma_\infty$, $n \ge 3$ *and* $\rho_0 > 2$. *For* $\lambda \ge \lambda_0$, $\dim \mathcal{N}(\lambda)$ *is equal to the multiplicity of* λ^{-1} *as the eigenvalue of* $K(0)$.

Proof. We use the method used in the proof of Proposition 3.4. First, for $u \in \mathcal{N}(\lambda)$, one has $(1+\lambda F_0V)u = 0$, by Proposition 3.7. Then $K(0)\tilde{u} = \lambda^{-1}\tilde{u}$ with $\tilde{u} = |V|^{1/2}u$. Due to $|V| \le C\langle x\rangle^{-\rho_0}$ with some $\rho_0 > 2$, one has $\tilde{u} \in L^2(\mathbb{R}^n)$. By the same argument as in Proposition 3.4, one can show that $|V|^{1/2}$ is injective from $\mathcal{N}(\lambda)$ to $\{\ \psi \in L^2(\mathbb{R}^n);\ K(0)\psi = \lambda^{-1}\psi\ \}$. This means that $\dim \mathcal{N}(\lambda)$ is at most the multiplicity of λ^{-1} as the eigenvalue of $K(0)$.

On the other hand, if λ^{-1} is the eigenvalue of $K(0)$. Then there exists a function $\tilde{u} \in L^2(\mathbb{R}^n)$ such that $K(0)\tilde{u} = \lambda^{-1}\tilde{u}$. Let $u = F_0|V|^{1/2}\tilde{u}$. Note that $|V|^{1/2}\tilde{u} \in H^{-1,\rho_0/2}$, since $|V| \le C\langle x\rangle^{-\rho_0}$ with some $\rho_0 > 2$. It follows $u \in H^{1,-\rho_0/2}$ because F_0 is a bounded operator in $\mathcal{L}(-1, s; 1, -s)$ for $s > 1$. Then

$$P(\lambda)u = P_0F_0|V|^{1/2}\tilde{u} + \lambda VF_0|V|^{1/2}\tilde{u} = |V|^{1/2}\tilde{u} - \lambda|V|^{1/2}|V|^{1/2}F_0|V|^{1/2}\tilde{u} = 0.$$

This implies $u \in \mathcal{N}(\lambda)$. As in Proposition 3.4, we can get that $F_0|V|^{1/2}$ is the injective from $\{\ \psi \in L^2(\mathbb{R}^n);\ K(0)\psi = \lambda^{-1}\psi\ \}$ to $\mathcal{N}(\lambda)$. It follows that the multiplicity of λ^{-1} as the eigenvalue of $K(0)$ is at most $\dim \mathcal{N}(\lambda)$. □

Proposition 3.9. *Assume $0 \notin \sigma_\infty$. $K(\alpha)$ is a compact operator for $\alpha \le 0$. And $K(\alpha)$ convergent to $K(0)$ in operator norm sense.*

Proof. For $\alpha < 0$, $K(\alpha) = |V|^{1/2}(P_0 - \alpha)^{-1}|V|^{1/2}$. Since $(P_0 - \alpha)^{-1}$ is a bounded operator from $L^2(\mathbb{R}^n)$ to $H^1(\mathbb{R}^n)$, and V is a compact operator from $H^1(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. Then $V(P_0 - \alpha)^{-1}$ is a compact operator on $L^2(\mathbb{R}^n)$. So is $K(\alpha)$. Because

$$K(\alpha) - K(0) = |V|^{1/2}[(P_0 - \alpha)^{-1} - F_0]|V|^{1/2} = |V|^{1/2}R_0^{(0)}|V|^{1/2}$$

and if $\rho_0 > 2$, then $|V|^{1/2}R_0^{(0)}|V|^{1/2} = o(|\alpha|^\epsilon)$ in $L^2(\mathbb{R}^n)$. Hence, $K(\alpha) \to K(0)$ in operator norm sense, as $\alpha \to 0$. This means $K(0)$ is a compact operator. □

Lemma 3.10. *Suppose A_1, A_2 are two bounded self-adjoint operators on Hilbert space H. Set*

$$\mu_n(A_i) = \sup_{\phi_1,\cdots,\phi_n} \inf_{\|\psi\|=1,\psi\in[\phi_1,\cdots,\phi_n]^\perp} (\psi, A_i\psi),$$

then $|\mu_n(A_1) - \mu_n(A_2)| \le \|A_1 - A_2\|$.

Proof. By the definition of $\mu_n(A_i)$, one has

$$\begin{aligned}
& \mu_n(A_1) - \mu_n(A_2) \\
= & \sup_{\phi_1,\cdots,\phi_n} \inf_{\|\psi\|=1,\psi\in[\phi_1,\cdots,\phi_n]^\perp} (\psi, A_1\psi) - \sup_{\phi_1,\cdots,\phi_n} \inf_{\|\psi\|=1,\psi\in[\phi_1,\cdots,\phi_n]^\perp} (\psi, A_2\psi) \\
\le & \sup_{\phi_1,\cdots,\phi_n} [\inf_{\|\psi\|=1,\psi\in[\phi_1,\cdots,\phi_n]^\perp} (\psi, A_1\psi) - \inf_{\|\psi\|=1,\psi\in[\phi_1,\cdots,\phi_n]^\perp} (\psi, A_2\psi)] \\
= & \sup_{\phi_1,\cdots,\phi_n} [\sup_{\|\psi\|=1,\psi\in[\phi_1,\cdots,\phi_n]^\perp} (\psi, -A_1\psi) - \sup_{\|\psi\|=1,\psi\in[\phi_1,\cdots,\phi_n]^\perp} (\psi, -A_2\psi)] \\
\le & \sup_{\phi_1,\cdots,\phi_n} \sup_{\|\psi\|=1,\psi\in[\phi_1,\cdots,\phi_n]^\perp} [(\psi, -A_1\psi) - (\psi, -A_2\psi)] \\
\le & \|A_1 - A_2\|.
\end{aligned}$$

Similarly, we have $\mu_n(A_2) - \mu_n(A_1) \le \|A_1 - A_2\|$. This ends the proof. □

Lemma 3.11. *Suppose $T(\alpha)$ is a family of compact self-adjoint operators on some separable Hilbert space H , and $T(\alpha) = T_0 + o(|\alpha|^\epsilon)$, for α near 0. Set*

$$\mu_i(\alpha) = \inf_{\phi_1,\cdots,\phi_i} \sup_{\|\psi\|=1,\psi\in[\phi_1,\cdots,\phi_i]^\perp} (\psi, T(\alpha)\psi).$$

Then

(a). $\mu_i(\alpha)$ is the eigenvalue of $T(\alpha)$ and $\mu_i(\alpha)$ converges when $\alpha \to 0$. Moreover, if $\mu_i(\alpha) \to \mu_i$, then μ_i is the eigenvalue of T_0.

(b). Suppose $E_0 \neq 0$ is the eigenvalue of T_0 of multiplicity m. Then there are m eigenvalues (counting multiplicity), $E_i(\alpha)$ $(1 \le i \le m)$, of $T(\alpha)$ near E_0. Moreover, we can choose $\{ \phi_i(\alpha); 1 \le i \le m \}$ such that $(\phi_i(\alpha), \phi_j(\alpha)) = \delta_{ij}$ $(1 \le i, j \le m)$, $\phi_i(\alpha)$ is the eigenvector of $T(\alpha)$ corresponding to $E_i(\alpha)$ ($E_i(\alpha) \to E_0$), and $\phi_i(\alpha)$ converges as $\alpha \to 0$. If $\phi_i(\alpha)$ converges to ϕ_i, then ϕ_i is the eigenvector of T_0 corresponding to E_0.

Proof. (a). By min-max principle, we know that $\mu_i(\alpha)$ is an eigenvalue of $T(\alpha)$. By Lemma 3.10, one has

$$|\mu_i(\alpha) - \mu_i(0)| \leq \|T(\alpha) - T_0\| = O(|\alpha|^\epsilon).$$

It follows $\mu_i(\alpha)$ converges to the eigenvalue of T_0.

(b). Because T_0 is a compact operator, and $E_0 \neq 0$ is the eigenvalue of T_0, then E_0 is a discrete spectrum of T_0. Then there exists a constant $\delta > 0$ small enough, such that T_0 has only one eigenvalue, E_0, in $B(E_0, \delta)(= \{ z \in \mathbb{C};\ |z - E_0| < \delta \})$. For α small enough, $T(\alpha)$ has exactly m eigenvalues (counting multiplicity) in $B(E_0, \delta)$ because the eigenvalues of $T(\alpha)$ converge to the eigenvalues of T_0, by part (a) of the lemma. Suppose the m eigenvalues, near E_0, of $T(\alpha)$ are $E_1(\alpha), E_2(\alpha), \cdots, E_m(\alpha)$, and the corresponding eigenvectors are $\psi_1(\alpha), \psi_2(\alpha), \cdots, \psi_m(\alpha)$ such that $(\psi_i(\alpha), \psi_j(\alpha)) = \delta_{ij}$. Let

$$P_\alpha = -\frac{1}{2\pi i} \oint_{|E-E_0|=\delta} (T(\alpha) - E)^{-1} dE.$$

Then, $P_\alpha = \sum_{i=1}^{m} (\cdot, \psi_i(\alpha))\, \psi_i(\alpha)$. Let $P_\alpha^{(i)} = (\cdot, \psi_i(\alpha))\, \psi_i(\alpha)$, then $P_\alpha = \sum_{i=1}^{m} P_\alpha^{(i)}$. For α near 0, one has

$$\begin{aligned} \|P_\alpha - P_0\| &= \| - \frac{1}{2\pi i} \oint_{|E-E_0|=\delta} (T(\alpha) - E)^{-1} - (T_0 - E)^{-1} dE \| \\ &= \| - \frac{1}{2\pi i} \oint_{|E-E_0|=\delta} (T(\alpha) - E)^{-1} (T_0 - T_\alpha)(T_0 - E)^{-1} dE \| \\ &= O(|\alpha|^\epsilon). \end{aligned}$$

It follows that, there are $\phi_i,\ 1 \leq i \leq m$, such that $(\phi_i, \phi_j) = \delta_{ij}$, $\phi_i \in \mathrm{Ran} P_0$ and $\|P_\alpha^{(i)} \phi_i - \phi_i\| = O(|\alpha|^\epsilon)$. Let $\phi_i(\alpha) = \frac{P_\alpha^{(i)} \phi_i}{\|P_\alpha^{(i)} \phi_i\|}$. Then, $(\phi_i(\alpha), \phi_j(\alpha)) = 0$ for $i \neq j$, because $P_\alpha^{(i)} P_\alpha^{(j)} = 0$ if $i \neq j$, and $\|\phi_i(\alpha) - \phi_i\| \leq \|P_\alpha^{(i)} \phi_i - \phi_i\| + \|(1 - \frac{1}{\|P_\alpha^{(i)} \phi_i\|}) P_\alpha^{(i)} \phi_i\| = O(|\alpha|^\epsilon)$.

This ends the proof. □

Let $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_i < \cdots < \alpha_n$ and

$$T(\beta) = T_0 + \sum_{i=1}^{n} \beta^{\alpha_i} (\ln \beta)^{\delta_i} T_i + T_r(\beta).$$

Here, $\delta_i = 0$ or 1, $T_0 \geq 0, T_i(\ 1 \leq i \leq m\ )$ are compact self-adjoint operators, $T_r(\beta)$ are compact operators, and $T_r(\beta) = O(|\beta|^{\alpha_m + \epsilon})$ for β near 0. Set

$$e_i = \inf_{\phi_1, \cdots, \phi_i} \sup_{\|\psi\|=1, \psi \in [\phi_1, \cdots, \phi_i]^\perp} (\psi, T_0 \psi).$$

Then by min-max principle, e_i is the eigenvalue of T_0. Moreover, if $e_i \neq 0$, then e_i is a discrete eigenvalue of T_0, because T_0 is a compact operator. If $e_i \neq 0$ is an eigenvalue of T_0 of multiplicity m, without loss, we can suppose that $e_i = e_{i+1} = \cdots = e_{i+m-1}$. Then there exist exactly m eigenvalues (counting multiplicity), $e_i(\beta), e_{i+1}(\beta), \cdots, e_{i+m-1}(\beta)$, of $T(\beta)$ near e_i. By Lemma 3.11,

we know that there exists a family of normalized eigenvector $\{\phi_j(\beta);\ j = i, i+1, \cdots, i+m-1\}$ of $T(\beta)$ such that $T(\beta)\phi_j(\beta) = e_j(\beta)\phi_j(\beta)$, $(\phi_j(\beta), \phi_k(\beta)) = \delta_{jk}$ ($j,k = i, i+1, \cdots, i+m-1$), and $\phi_j(\beta)$ ($j = i, i+1, \cdots, i+m-1$) converge as $\beta \to 0$. Suppose $\phi_j(\beta)$ converge to ϕ_j for all j such that $e_j \neq 0$. Then $(\phi_i, \phi_j) = \delta_{ij}$. $\{\phi_i\}$ can be extended to a standard orthogonal basis. Set

$$T_1(\beta) = \sum_{i=1}^{n} \beta^{\alpha_i} T_i + T_r(\beta),\ T_{ij}(\beta) = (\phi_i, T_1(\beta)\phi_j).$$

Then we having the following result.

Lemma 3.12. *$T(\beta)$, e_i are given as before. Then the eigenvalue of $T(\beta)$, $e_j(\beta)$ ($j = i, i+1, \cdots, i+m-1$), have the following form*

$$e_j(\beta) = e_i + \frac{\sum\limits_{n=0}^{\infty} a_n^{(j)}(\beta)}{\sum\limits_{n=0}^{\infty} b_n^{(j)}(\beta)}.$$

Here

$$\begin{aligned}
a_0^{(j)}(\beta) &= T_{jj}(\beta),\\
a_1^{(j)}(\beta) &= -\sum_{\{k;\ e_k \neq e_i\}} (e_k - e_i)^{-1} T_{jk}(\beta) T_{kj}(\beta),\\
a_2^{(j)}(\beta) &= \sum_{k \neq j \neq l} (e_k - e_i)^{-1}(e_l - e_i)^{-1} T_{jk}(\beta) T_{kl}(\beta) T_{lj}(\beta)\\
&\quad - 2 \sum_{\{k;\ e_k \neq e_i\}} (e_k - e_i)^{-1} T_{jk}(\beta) T_{kj}(\beta) T_{jj}(\beta),\\
a_n^{(j)}(\beta) &= -\frac{(-1)^n}{2\pi i} \oint_{|E-e_i|=\delta} (e_i - E)^{-1} \sum_{i_1, i_2, \cdots, i_n} (e_{i_1} - 1)^{-1} \cdot \cdots \cdot (e_{i_n} - 1)^{-1}\\
&\qquad T_{ji_1} T_{i_1 i_2} \cdot \cdots \cdot T_{i_n j} dE, \qquad \text{for } n > 2,
\end{aligned}$$

$$\begin{aligned}
b_0^{(j)}(\beta) &= 1,\\
b_1^{(j)}(\beta) &= 0,\\
b_2^{(j)}(\beta) &= \sum_{\{k;\ e_k \neq e_i\}} (e_i - e_0)^{-2} T_{1i}(\beta) T_{i1}(\beta),\\
b_n^{(j)}(\beta) &= -\frac{(-1)^n}{2\pi i} \oint_{|E-e_i|=\delta} (e_i - E)^{-2} \sum_{i_1, i_2, \cdots, i_{n-1}} (e_{i_1} - E)^{-1} \cdot \cdots \cdot (e_{i_n} - E)^{-1}\\
&\qquad T_{ji_1} T_{i_1 i_2} \cdot \cdots \cdot T_{i_{n-1} j} dE, \qquad \text{for } n > 2.
\end{aligned}$$

Proof. If $e_i \neq 0$, then e_i is the discrete spectrum of T_0. Suppose the multiplicity of e_i is m, and suppose $e_i = e_{i+1} = \cdots = e_{i+m-1}$ as before. Hence, we can choose $\delta > 0$ small enough,

such that there is only one eigenvalue, e_i, in $B(e_i, \delta) = \{ z \in \mathbb{C};\ |z - e_i| < \delta \}$. We know that $e_i(\beta)$ converge to e_i. It follows that if δ small enough, there are exactly m eigenvalue (counting multiplicity) of $T(\beta)$ in $B(e_i, \delta)$ for β small. Set

$$P(\beta) \triangleq -\frac{1}{2\pi i} \oint_{|E-e_i|=\delta} (T(\beta) - E)^{-1} dE.$$

Then

$$\begin{aligned} e_j(\beta) &= \frac{\langle \phi_j, T(\beta)P(\beta)\phi_j \rangle}{\langle \phi_j, P(\beta)\phi_j \rangle} \\ &= \frac{\langle \phi_j, T_0 P(\beta)\phi_j \rangle}{\langle \phi_j, P(\beta)\phi_j \rangle} + \frac{\langle \phi_j, T_1(\beta)P(\beta)\phi_j \rangle}{\langle \phi_j, P(\beta)\phi_j \rangle} \\ &= e_j + \frac{\langle \phi_j, T_1(\beta)P(\beta)\phi_j \rangle}{\langle \phi_j, P(\beta)\phi_j \rangle}. \end{aligned}$$

One has

$$\begin{aligned} (T(\beta) - E)^{-1} &= (T_0 - E)^{-1}(I + T_1(\beta)(T_0 - E)^{-1})^{-1} \\ &= (T_0 - E)^{-1} \sum_{n=0}^{\infty} (-1)^n [T_1(\beta)(T_0 - E)^{-1}]^n. \end{aligned}$$

It follows

$$\langle \phi_j, P(\beta)\phi_j \rangle = -\frac{1}{2\pi i} \oint_{|E-e_i|=\delta} \langle \phi_j, (T_0 - E)^{-1} \sum_{n=0}^{\infty} (-1)^n [(T_1(\beta))(T_0 - E)^{-1}]^n \phi_j \rangle dE.$$

Then

$$b_n^{(j)}(\beta) = -\frac{(-1)^n}{2\pi i} \oint_{|E-e_i|=\delta} \langle \phi_j, (T_0 - E)^{-1} [(T_1(\beta))(T_0 - E)^{-1}]^n \phi_j \rangle dE$$

In particular,

$$\begin{aligned}
b_0^{(j)}(\beta) &= -\frac{1}{2\pi i}\oint_{|E-e_i|=\delta}\langle\phi_j,(T_0-E)^{-1}\phi_j\rangle dE \\
&= 1; \\
b_1^{(j)}(\beta) &= \frac{1}{2\pi i}\oint_{|E-e_i|=\delta}\langle\phi_j,(T_0-E)^{-1}[(T_1(\beta))(T_0-E)^{-1}]\phi_j\rangle dE \\
&= \frac{1}{2\pi i}\oint_{|E-e_i|=\delta}(e_i-E)^{-2}\langle\phi_j,T_1(\beta)\phi_j\rangle dE \\
&= 0; \\
b_2^{(j)}(\beta) &= -\frac{(-1)^2}{2\pi i}\oint_{|E-e_i|=\delta}\langle\phi_j,(T_0-E)^{-1}[(T_1(\beta))(T_0-E)^{-1}]^2\phi_j\rangle dE \\
&= -\frac{1}{2\pi i}\oint_{|E-e_i|=\delta}(e_i-E)^{-2}\langle\phi_j,[(T_1(\beta))(T_0-E)^{-1}(T_1(\beta))]\phi_j\rangle dE \\
&= -\frac{1}{2\pi i}\oint_{|E-e_i|=\delta}(e_i-E)^{-2}\sum_k\langle\phi_j,(T_1(\beta))\phi_k\rangle(e_k-E)^{-1}\langle\phi_j,(T_1(\beta))\phi_k\rangle dE \\
&= -\frac{1}{2\pi i}\oint_{|E-e_i|=\delta}(e_i-E)^{-2}\sum_{\{k;e_k\neq e_i\}}\langle\phi_j,(T_1(\beta))\phi_k\rangle(e_k-E)^{-1}\langle\phi_j,(T_1(\beta))\phi_k\rangle dE \\
&\quad -\frac{1}{2\pi i}\oint_{|E-e_i|=\delta}(e_i-E)^{-2}\sum_{\{k;e_k=e_i\}}\langle\phi_j,(T_1(\beta))\phi_k\rangle(e_k-E)^{-1}\langle\phi_j,(T_1(\beta))\phi_k\rangle dE \\
&\equiv I_1+I_2.
\end{aligned}$$

For I_2, note that there are only finite term in the summation, because T_0 is a compact operator and $e_i \neq 0$. It is easy to check $I_2 = 0$. From the choice of δ, we know that $|(e_k - E)^{-1}| \leq C$ for $e_k \neq e_i$ and $|E - e_i| = \delta$. Here C is independent of E. Therefore,

$$\begin{aligned}
&\sum_{\{k;e_k\neq e_i\}}|(e_k-e_j)^{-2}T_{jk}(\beta)T_{kj}(\beta)(e_k-E)^{-1}| \\
&\leq C\sum(|T_{jk}(\beta)|^2+|T_{kj}(\beta)|^2) \leq C(\|T_1^*(\beta)\phi_j\|^2+\|T_1(\beta)\phi_j\|^2)
\end{aligned} \tag{3.23}$$

with C independent of E. It follows

$$\begin{aligned}
I_1(z) &= -\frac{1}{2\pi i}\sum_{\{k;e_k\neq e_i\}}\oint_{|E-e_i|=\delta}(e_i-E)^{-2}\langle\phi_j,(T_1(\beta))\phi_k\rangle(e_k-E)^{-1}\langle\phi_j,(T_1(\beta))\phi_k\rangle dE \\
&= \sum_{\{k;e_k\neq e_i\}}(e_i-e_k)^{-2}T_{jk}(\beta)T_{kj}(\beta).
\end{aligned}$$

By (3.23), $\sum_{\{k;e_k\neq e_i\}} (e_i - e_k)^{-2} T_{jk}(\beta) T_{kj}(\beta)$ is finite. Similarly, we can get

$$b_n^{(j)}(\beta) = -\frac{(-1)^n}{2\pi i} \oint_{|E-e_i|=\delta} (e_i - E)^{-2} \sum_{i_1, i_2, \cdots, i_{n-1}} (e_{i_1} - E)^{-1} \cdot \cdots \cdot (e_{i_n} - E)^{-1} \\ T_{ji_1} T_{i_1 i_2} \cdot \cdots \cdot T_{i_{n-1} j} dE;$$

and

$$\begin{aligned} a_0^{(j)}(\beta) &= T_{jj}(\beta) \\ a_1^{(j)}(\beta) &= -\sum_{\{k;\ e_k\neq e_i\}} (e_k - e_i)^{-1} T_{jk}(\beta) T_{kj}(\beta); \\ a_2^{(j)}(\beta) &= \sum_{\{l,k;\ e_k\neq e_j\neq e_l\}} (e_j - e_k)^{-1} (e_j - e_l)^{-1} T_{jk}(\beta) T_{kl}(\beta) T_{lj}(\beta) \\ &\qquad -2 \sum_{\{k;\ e_k\neq e_i\}} (e_j - e_k)^{-1} T_{jk}(\beta) T_{kj}(\beta) T_{jj}(\beta); \end{aligned}$$

$$\begin{aligned} a_n^{(j)}(\beta) &= -\frac{(-1)^n}{2\pi i} \oint_{|E-e_i|=\delta} \langle \phi_j, [(T_1(\beta))(T_0 - E)^{-1}]^{n+1} \phi_j \rangle dE \\ &= -\frac{(-1)^n}{2\pi i} \oint_{|E-e_i|=\delta} (e_i - E)^{-1} \langle \phi_j, [(T_1(\beta))(T_0 - E)^{-1}]^n T_1(\beta) \phi_j \rangle dE \\ &= -\frac{(-1)^n}{2\pi i} \oint_{|E-e_i|=\delta} (e_i - E)^{-1} \sum_{i_1} \langle \phi_j, [(T_1(\beta))(T_0 - E)^{-1}]^{n-1} \\ &\qquad T_1(\beta) \phi_{i_1} \rangle (e_i - E)^{-1} T_{i_1 j}(\beta) dE \\ &= \cdots \\ &= -\frac{(-1)^n}{2\pi i} \oint_{|E-e_1|=\delta} (e_1 - E)^{-1} \sum_{i_1, i_2, \cdots, i_n} (e_{i_1} - E)^{-1} \cdot \cdots \cdot (e_{i_n} - E)^{-1} \\ &\qquad T_{ji_1} T_{i_1 i_2} \cdot \cdots \cdot T_{i_n j}\, dE. \end{aligned}$$

□

Set $\nu_0 = \min\{\ \nu;\ \nu \in \sigma_\infty\ \}$. Let $u(\lambda)$ be the ground state of $P(\lambda)$. $u(\lambda)$ can be chosen to be a positive function. Then

$$\tilde{u}(\lambda) \equiv |V|^{1/2} u(\lambda) \in L^2(\mathbb{R}^n).$$

Choose appropriate $u(\lambda)$ such that $\|\tilde{u}(\lambda)\|_{L^2(\mathbb{R}^n)} = 1$.

Lemma 3.13. *Assume $0 \notin \sigma_\infty$, $n \geq 3$. $u(\lambda)$ and $\tilde{u}(\lambda)$ are define as above. Then $\tilde{u}(\lambda)$ converges in $L^2(\mathbb{R}^n)$, as $\lambda \to \lambda_0$. If $\tilde{u}(\lambda)$ converges to ϕ, then ϕ is the eigenvalue of $K(0)$, and $\langle \phi, |V|^{1/2} G_{\nu_0,0} \pi_{\nu_0} |V|^{1/2} \phi \rangle \neq 0$.*

Proof. By the assumption of $\tilde{u}(\lambda)$, one has $K(e_1(\lambda))\tilde{u}(\lambda) = \lambda^{-1}\tilde{u}(\lambda)$ ($e_1(\lambda)$ is the smallest eigenvalue of $P(\lambda)$). One has that $\tilde{u}(\lambda)$ converges to some function ϕ in $L^2(\mathbb{R}^n)$ as $\lambda \to \lambda_0$ by Lemma 3.11. By Lemma3.11, we know that ϕ is the normalized eigenfunction of $K(0)$ corresponding to E_0. ϕ is a positive function, since $\tilde{u}(\lambda)$ is positive. Let $u = F_0|V|^{1/2}\phi$, then $P(\lambda_0)u = 0$ and u is a positive function, because $|V|^{1/2}u = |V|^{1/2}F_0|V|^{1/2}\phi = K(0)\phi = \lambda_0\phi$.

Then

$$\begin{aligned} & \langle \phi, |V|^{1/2} G_{\nu_0,0}\pi_{\nu_0}|V|^{1/2}\phi\rangle \\ = \; & \lambda_0^2\langle |V|^{1/2}u, |V|^{1/2}G_{\nu_0,0}\pi_{\nu_0}|V|^{1/2}|V|^{1/2}u\rangle \\ = \; & \lambda_0^2\langle Vu, G_{\nu_0,0}\pi_{\nu_0}Vu\rangle \\ = \; & \lambda_0^2 C_{\nu_0}|\, \langle Vu, |y|^{-\frac{n-2}{2}+\nu_0}\varphi_{\nu_0}\rangle|^2 \\ \neq \; & 0. \end{aligned}$$

This ends the proof. □

Let

$$\mu_i(\alpha) = \inf_{\phi_1,\cdots,\phi_i} \sup_{\|\psi\|=1,\psi\in[\phi_1,\cdots,\phi_i]^\perp} (\psi, K(\alpha)\psi).$$

Then $\mu_i(\alpha)$ is the eigenvalue of $K(\alpha)$. Because $K(\alpha) \to K(0)$ as $\alpha \to 0$, one has $\mu_i(\alpha)$ converges to the eigenvalue of $K(0)$ by Lemma 3.11. Suppose $\mu_i(\alpha) \to \mu_i$, and suppose $\mu_1 = \cdots = \mu_m$, and $\mu_1 \neq \mu_{m+1}$, then μ_1 is an eigenvalue of $K(0)$ of multiplicity m. By Lemma 3.11, one can choose $\phi_i(\alpha)$ $(1 \le i \le m)$, which is the eigenfunction of $K(\alpha)$ corresponding to $\mu_i(\alpha)$ such that $\langle \phi_i(\alpha), \phi_j(\alpha)\rangle = \delta_{ij}$ and $\phi_i(\alpha)$ converges. Suppose $\phi_i(\alpha) \to \phi_i$ as $\alpha \to 0$, then $\phi_i(1 \le i \le m)$ is the eigenfunction of $K(0)$ corresponding to μ_1, and $\langle \phi_i, \phi_j\rangle = \delta_{ij}$ for $1 \le i, j \le m$. Note that $\mu_1 = \lambda_0^{-1}$, one has $P(\lambda_0)\psi_i = 0$ $(1 \le i \le m)$ with $\psi_i = F_0|V|^{1/2}\phi_i \in H^{1,-s}$, $s > 1$. By Lemma 3.6, we know that 0 is not the eigenvalue of $P(\lambda_0)$, then ψ_i $(1 \le i \le m)$ is the 0 resonance of $P(\lambda_0)$. Suppose that ψ_i $(1 \le i \le m)$ is ν_i-resonant state of $P(\lambda_0)$. Then $0 < \nu_i \le 1$ by Theorem 3.1 [70]. Then, we have

Lemma 3.14. *Assume $n \ge 3$, $\sigma_1 \neq \emptyset$ and $0 \notin \sigma_\infty$. $\psi_i, \phi_i, \mu_i(\alpha), \mu_i$ are defined as above. Then $\langle \phi_i, |V|^{1/2}G_{\nu_i,0}\pi_{\nu_i}|V|^{1/2}\phi_i\rangle \neq 0$, and $\langle \phi_i, |V|^{1/2}G_{\nu,0}\pi_\nu|V|^{1/2}\phi_i\rangle = 0$ for $\nu < \nu_i$. Moreover, if $\nu_j < 1$, $\mu_j(\alpha) = c\alpha^{\nu_j} + o(\alpha^{\nu_j})$ with some $c \neq 0$; if $\nu_j = 1$, $\mu_j(\alpha) = c\alpha\ln\alpha + o(\alpha)$ with some $c \neq 0$ for $1 \le j \le m$.*

Proof. From the definition of ϕ_j, one has

$$\begin{aligned} & \langle \phi_j, |V|^{1/2}G_{\nu,0}\pi_\nu|V|^{1/2}\phi_k\rangle \\ = \; & \langle \lambda_0|V|^{1/2}\psi_j, \lambda_0|V|^{1/2}G_{\nu,0}\pi_\nu|V|^{1/2}|V|^{1/2}\psi_k\rangle \\ = \; & C_\nu\lambda_0^2\langle V\psi_j, \frac{1}{|x|^{\frac{1}{2}(n-2)-\nu}}\varphi_\nu\rangle\overline{\langle V\psi_k, \frac{1}{|x|^{\frac{1}{2}(n-2)-\nu}}\varphi_\nu\rangle}, \end{aligned}$$

where $C_\nu \neq 0$ is a constant dependent on ν. Hence, if $\langle \phi_j, |V|^{1/2} G_{\nu,0} \pi_\nu |V|^{1/2} \phi_j \rangle = 0$, one has $\langle V\psi_j, \frac{1}{|x|^{\frac{1}{2}(n-2)-\nu}} \varphi_\nu \rangle = 0$. It follows $\langle \phi_j, |V|^{1/2} G_{\nu,0} \pi_\nu |V|^{1/2} \phi_k \rangle = 0$. If ϕ_j is ν_j- resonant state of $P(\lambda_0)$, then by Theorem 3.1 [70], one has $\langle \phi_j, |V|^{1/2} G_{\nu,0} \pi_\nu \ |V|^{1/2} \phi_j \rangle \neq 0$, and $\langle \phi_j, |V|^{1/2} G_{\nu,0} \pi_\nu |V|^{1/2} \phi_j \rangle = 0$ for all $\nu \in \sigma_\infty, \ \nu < \nu_j$. Use Lemma 3.12 to computer $\mu_j(\alpha)$. If $\nu_j < 1$,

$$\begin{aligned} \mu_j(\alpha) &= \mu_1 + \frac{\alpha^{\nu_j} \langle \phi_j, |V|^{1/2} G_{\nu_j,0} \pi_{\nu_j} |V|^{1/2} \phi_j \rangle + o(\alpha^{\nu_j})}{1 + o(\alpha^{\nu_j})} \\ &= \mu_1 + c\alpha^{\nu_j} + o(\alpha^{\nu_j}) \end{aligned}$$

with $c = \langle \phi_j, |V|^{1/2} G_{\nu,0} \pi_\nu |V|^{1/2} \phi_j \rangle$, for $1 \leq j \leq m$. If $\nu_j = 1$,

$$\begin{aligned} \mu_j(\alpha) &= \mu_1 + \frac{\alpha \ln \alpha \langle \phi_j, |V|^{1/2} G_{1,0} \pi_1 |V|^{1/2} \phi_j \rangle + o(\alpha^{\nu_j})}{1 + o(\alpha^{\nu_j})} \\ &= \mu_1 + c\alpha \ln \alpha + o(\alpha^{\nu_j}) \end{aligned}$$

with $c = \langle \phi_j, |V|^{1/2} G_{1,0} \pi_1 |V|^{1/2} \phi_j \rangle$.

Theorem 3.15. *Assume $0 \notin \sigma_\infty$ and $n \geq 3$. Suppose that $e_1(\lambda)$ is the ground state energy (the smallest eigenvalue) of $P(\lambda)$. ϕ is defined in Lemma 3.13. If $\rho_0 > 6$, one of three exclusive situations holds :*

(a). If $\sigma_1 = \emptyset$, then $e_1(\lambda) = -c(\lambda - \lambda_0) + o(\lambda - \lambda_0)$, $c = (\lambda_0 \| F_0 |V|^{\frac{1}{2}} \phi \|)^{-2} \neq 0$;

(b). If $\nu_0 = 1$, then $e_1(\lambda) = -c\frac{\lambda - \lambda_0}{\ln(\lambda - \lambda_0)} + o(\frac{\lambda - \lambda_0}{\ln(\lambda - \lambda_0)})$, $c = \lambda_0^{-2} \langle \phi, |V|^{1/2} G_{1,0} \pi_1 |V|^{1/2} \phi \rangle^{-1} \neq 0$;

(c). If $\nu_0 < 1$, then $e_1(\lambda) = c((\lambda - \lambda_0)^{\frac{1}{\nu_0}}) + o((\lambda - \lambda_0)^{\frac{1}{\nu_0}})$, $c = \lambda_0^{-2} \langle \phi, |V|^{1/2} G_{\nu_0,0} \pi_{\nu_0} |V|^{1/2} \phi \rangle^{-1} \neq 0$.

Proof. (a)By Theorem 3.2, one has

$$R_0(\alpha) = F_0 + \alpha F_1 + R_0^{(1)}(\alpha), \text{ in } \mathcal{L}(-1, s; 1, -s) \ s > 3.$$

Then if $\rho_0 > 6$, we can get $K(\alpha) = K(0) + |V|^{1/2}(\alpha F_1 + R_0^{(1)}(\alpha))|V|^{1/2}$ in $\mathcal{L}(0, 0; 0, 0)$. Because $e_1(\lambda)$ is the simple eigenvalue of $P(\lambda)$, then λ^{-1} is the simple eigenvalue of $K(e_1(\lambda))$. And we also have that λ^{-1} is the biggest eigenvalue of $K(e_1(\lambda))$. If not, suppose that $a > \lambda^{-1}$ is the eigenvalue of $K(e_1(\lambda))$, then by the continuous and monotonous of the eigenvalue of $K(e_1(\lambda))$ with respect to λ, we know that there exists a constant $\lambda' < \lambda$ such that $\lambda \in \sigma(K(e_1(\lambda')))$. It follows that $e_1(\lambda') < e_1(\lambda)$ is the eigenvalue of $P_0 + \lambda V$. This is contradictory to that $e_1(\lambda)$ is the smallest eigenvalue. By Lemma 3.13, we know the normalized eigenfunction ,$\tilde{u}(\lambda)$, of $K(e_1(\lambda))$ converges to ϕ. It follows $\tilde{u}(\lambda) = \frac{P_\lambda \phi}{\|P_\lambda \phi\|}$. Then

$$\mu(e_1(\lambda)) = \langle \tilde{u}(\lambda), K(e_1(\lambda)) \tilde{u}(\lambda) \rangle = \frac{\langle \phi, K(e_1(\lambda) P_\lambda) \phi \rangle}{\langle \phi, P_\lambda \phi \rangle}.$$

By Lemma 3.12, we should compute $\langle \phi, |V|^{1/2} F_1 |V|^{1/2} \phi \rangle$.

From the definition of ϕ, one has

$$(P_0 + \lambda_0 V)\psi = 0 \text{ with } \psi = F_0|V|^{1/2}\phi,$$

Since $\nu_0 > 1$, we have $\psi \in L^2(\mathbb{R}^n)$ by Theorem 3.1 [70]. So ψ is the ground state of $P(\lambda_0)$.

We also have

$$|V|^{1/2}\psi = K(0)\phi = \lambda_0^{-1}\phi,$$
$$(P_0 - \alpha)^{-1}|V|\psi = \lambda_0^{-1}(\psi + \alpha R_0(\alpha)\psi).$$

Hence,

$$\begin{aligned} |V|^{1/2}F_1|V|^{1/2}\phi &= \lambda_0|V|^{1/2}F_1|V|^{1/2}|V|^{1/2}\psi \\ &= \lambda_0\alpha^{-1}|V|^{1/2}(R_0(\alpha) - F_0)|V|\psi + O(|\alpha|^\epsilon) \\ &= \lambda_0\alpha^{-1}|V|^{1/2}\lambda_0^{-1}(\psi + \alpha R_0(\alpha)\psi - \psi) + O(|\alpha|^\epsilon) \\ &= |V|^{1/2}R_0(\alpha)\psi + O(|\alpha|^\epsilon)\ . \end{aligned}$$

It follows

$$\begin{aligned} \langle\phi, |V|^{1/2}F_1|V|^{1/2}\phi\rangle &= \lim_{\alpha\to 0}\langle\psi, R_0(\alpha)\psi\rangle \\ &= \langle F_0|V|^{1/2}\phi, \psi\rangle \\ &= \|\psi\|^2 \neq 0. \end{aligned}$$

So, $\mu(\alpha) = \lambda_0^{-1} + c_1\alpha + o(|\alpha|^{1+\epsilon})$, with $c_1 = \||F_0|V|^{\frac{1}{2}}\phi\|^2$. By the Proposition 3.4, one has $\mu(e(\lambda)) = \lambda^{-1}$. It follows

$$\lambda^{-1} = \lambda_0^{-1} + c_1 e_1(\lambda) + O(|e(\lambda)|^{1+\epsilon}).$$

Since $\lambda^{-1} = \lambda_0^{-1} - \lambda_0^{-2}(\lambda - \lambda_0) + O(|\lambda - \lambda_0|^2)$, we can get the leading term of $e_1(\lambda)$ is $-c(\lambda - \lambda_0)$, with $c = (\lambda_0\||F_0|V|^{\frac{1}{2}}\phi\|)^{-2}$.

(b). If $\nu_0 = 1$, then

$$K(\alpha) = K(0) + \alpha\ln\alpha|V|^{1/2}G_{1,0}\pi_1|V|^{1/2} + O(\alpha).$$

By Lemma 3.13, one has

$$\langle\phi, |V|^{1/2}G_{1,0}\pi_1|V|^{1/2}\phi\rangle \neq 0.$$

Then we have

$$\mu(\alpha) = \lambda_0^{-1} + c_1\alpha\ln\alpha + o(\alpha)$$

with $c_1 = \langle\phi, |V|^{1/2}G_{1,0}\pi_1|V|^{1/2}\phi\rangle$. As in (a), using $\mu(e_1(\lambda)) = \lambda^{-1}$, and

$$\lambda^{-1} = \lambda_0^{-1} - \lambda_0^{-2}(\lambda - \lambda_0) + O(|\lambda - \lambda_0|),$$

one has

$$-\lambda^{-2}(\lambda - \lambda_0) + O(|\lambda - \lambda_0|) = ce_1(\lambda)\ln e_1(\lambda) + O(e_1(\lambda)).$$

To get the leading term of $e_1(\lambda)$, we can suppose that $e_1(\lambda) = (\lambda-\lambda_0)f(\lambda-\lambda_0)$ with $f(\lambda-\lambda_0) = O(1)$. Then by comparing the leading term, we can get $f(\lambda-\lambda_0) = 1/\ln(\lambda-\lambda_0)$. It follows

$$e_1(\lambda) = -c\frac{\lambda-\lambda_0}{\ln(\lambda-\lambda_0)} + o(\frac{\lambda-\lambda_0}{\ln(\lambda-\lambda_0)})$$

with $c = \lambda_0^{-2}\langle\phi, |V|^{1/2}G_{1,0}\pi_1|V|^{1/2}\phi\rangle^{-1}$.

(c). If $\nu_0 < 1$, one has

$$K(\alpha) = K(0) + \sum_{0<\nu\leq 1}\alpha^\nu|V|^{1/2}G_{\nu,0}\pi_\nu|V|^{1/2} + O(|\alpha|).$$

By Lemma 3.13, we know that $\langle\phi, |V|^{1/2}G_{\nu_0,0}\pi_{\nu_0}|V|^{1/2}\phi\rangle^{-1} \neq 0$. Using the same argument as before, we can conclude

$$\mu(\alpha) = \lambda_0^{-1} + c_1\alpha^{\nu_0} + o(|\alpha|^{\nu_0})$$

with $c_1 = \langle\phi, |V|^{1/2}G_{\nu_0,0}\pi_{\nu_0}|V|^{1/2}\phi\rangle$. As above, we can get that

$$e_1(\lambda) = c(\lambda-\lambda_0)^{\frac{1}{\nu_0}} + o((\lambda-\lambda_0)^{\frac{1}{\nu_0}})$$

with $c = \lambda_0^{-2}\langle\phi, |V|^{1/2}G_{\nu_0,0}\pi_{\nu_0}|V|^{1/2}\phi\rangle^{-1}$. □

Let

$$e_i(\lambda) = \sup_{\phi_1,\cdots,\phi_i}\inf_{\|\psi\|=1,\psi\in[\phi_1,\cdots,\phi_i]^\perp,\psi\in H^1}(\psi, P(\lambda)\psi).$$

Then $e_i(\lambda)$ is the eigenvalue of $P(\lambda)$ by min-max principle. We say that there are m eigenvalues of $P(\lambda)$ converge to 0 at λ_0, if there exist some k such that $\lim_{\lambda\downarrow\lambda_0} e_i(\lambda) = 0$ for $k+1 \leq i \leq k+m$; $e_i(\lambda) \neq 0$ $(i = k+1, \cdots, i = k+m)$ for any $\lambda > \lambda_0$; $e_i(\lambda_0) \neq 0$ for some $i \leq k$. Let $u_i(\lambda)$ be the eigenfunction of $P(\lambda)$, then $\tilde{u}_i(\lambda) \equiv |V|^{1/2}u_i(\lambda)$ is the eigenfunction of $K(e_i(\lambda))$ corresponding to λ^{-1}. Because $K(e_i(\lambda))$ converges to $K(0)$ as $\lambda \to 0$, by Lemma 3.11, one can choose $u_i(\lambda)$ such $\tilde{u}_i(\lambda)$ converges as $\lambda \to \lambda_0$ and $\langle\tilde{u}_i(\lambda), \tilde{u}_j(\lambda)\rangle = \delta_{ij}$. Let $\tilde{u}_i(\lambda) \to \phi_i$ in $L^2(\mathbb{R}^n)$. Then, $K(0)\phi_i = \lambda_0^{-1}\phi_i$ and $\langle\phi_i, \phi_j\rangle = \delta_{ij}$.

Theorem 3.16. *Assume $0 \notin \sigma_\infty$ and $n \geq 3$. $e_i(\lambda), \phi_i (1 \leq i \leq m)$ are defined as above, ν_i is defined as in Lemma 3.14. If $\rho_0 > 6$, one of three exclusive situations holds :*

(a). If $\sigma_1 = \emptyset$, then $m = 1$ and $e_1(\lambda) = -c(\lambda-\lambda_0) + o(\lambda-\lambda_0)$, $c = (\lambda_0\||F_0|V|^{\frac{1}{2}}\phi_1\|)^{-2} \neq 0$;

(b). If $\nu_i = 1$, then $e_i(\lambda) = -c\frac{(\lambda-\lambda_0)}{\ln(\lambda-\lambda_0)} + o(\lambda-\lambda_0)$, $c = \lambda_0^{-2}\langle\phi_i, |V|^{1/2}G_{1,0}\pi_1|V|^{1/2}\phi_i\rangle^{-1} \neq 0$;

(c). If $\nu_i < 1$, then $e_i(\lambda) = c((\lambda-\lambda_0)^{\frac{1}{\nu_i}}) + o((\lambda-\lambda_0)^{\frac{1}{\nu_i}})$, $c = \lambda_0^{-2}\langle\phi_i, |V|^{1/2}G_{\nu_i,0}\pi_{\nu_i}|V|^{1/2}\phi_i\rangle^{-1} \neq 0$.

Proof. (a). If $\sigma_1 = \emptyset$, by Theorem 3.2, one has

$$R_0(\alpha) = F_0 + \alpha F_1 + R_0^{(1)}(\alpha), \text{ in } \mathcal{L}(-1, s; 1, -s)\ s > 3.$$

Then if $\rho_0 > 6$, we can get $K(\alpha) = K(0) + |V|^{1/2}(\alpha F_1 + R_0^{(1)}(\alpha))|V|^{1/2}$ in $\mathcal{L}(0,0;0,0)$. By definition of ϕ_i, one has $P(\lambda)\psi_i = 0$ with $\phi_i = F_0|V|^{1/2}\phi_i$. Because $\sigma_1 = \emptyset$, by Theorem 3.1

[70], one has $\psi_i \in L^2$. This means that ϕ_i is the eigenfunction of $P(\lambda_0)$ corresponding to 0. Because 0 is the simple eigenvalue of $P(\lambda_0)$, then $m = 1$. It is clearly, this case is the same as part(a) of Theorem 3.15.

(b). If $\nu_i = 1$, by Lemma 3.14, one has

$$\mu_i(\alpha) = c\alpha \ln \alpha + o(\alpha)$$

with $c_1 = \langle \phi_i, |V|^{1/2} G_{1,0} \pi_1 |V|^{1/2} \phi_i \rangle^{-1}$. By the same argument as in Theorem 3.15, we can get

$$e_i(\lambda) = -c \frac{\lambda - \lambda_0}{\ln(\lambda - \lambda_0)} + o(\frac{\lambda - \lambda_0}{\ln(\lambda - \lambda_0)}),$$

$c = \lambda_0^{-2} \langle \phi_i, |V|^{1/2} G_{1,0} \pi_1 |V|^{1/2} \phi_i \rangle^{-1} \neq 0$.

(c). If $\nu_i < 1$, by Lemma 3.14, one has

$$\mu_i(\alpha) = c\alpha^{\nu_i} + o(|\alpha|^{\nu_i})$$

with $c_1 = \langle \phi_i, |V|^{1/2} G_{\nu_i,0} \pi_{\nu_i} |V|^{1/2} \phi_i \rangle^{-1}$. By the same argument as in Theorem 3.15, we can get

$$e_i(\lambda) = c(\lambda - \lambda_0)^{\frac{1}{\nu_i}} + o((\lambda - \lambda_0)^{\frac{1}{\nu_i}}),$$

$c = \lambda_0^{-2} \langle \phi_i, |V|^{1/2} G_{\nu_i,0} \pi_{\nu_i} |V|^{1/2} \phi_i \rangle^{-1} \neq 0$. □

3.4 Zero resonance in coupling constant limit

In this section, we study the multiplicity of zero as the resonance of P. We have the following result. Suppose $\sum_{0<\nu\leq 1} n_\nu = m$. If $m > 0$, then $P(\lambda_0)$ has at most m linear independent zero resonance solutions, by Corollary 4.2[70].

Theorem 3.17. *Assume $0 \notin \sigma_\infty$ and $n \geq 3$. Suppose $\sum_{0<\nu\leq 1} n_\nu = m$. If $m > 0$, then there exists $\lambda_1 > \lambda_0$ such that for all $\lambda_0 < \lambda < \lambda_1$, the multiplicity of zero as the resonance of $P(\lambda_0)$ is equal to the number of eigenvalues, less than 0, of $P(\lambda)$.*

Proof.Let $n(\lambda, \alpha)$ denote the multiplicity of λ^{-1} as the eigenvalue of $K(\alpha)$. Let $\lambda_1 > \lambda_0$ such that for any $\lambda_0 < \lambda < \lambda_1$, λ^{-1} is not the eigenvalue of $K(0)$. Then,

$$\begin{aligned}
& \dim\Big\{ u \in H^{1,-s}(\mathbb{R}^n) \backslash L^2(\mathbb{R}^n); P(\lambda_0)u = 0 \Big\} \\
= \; & \mathcal{N}(\lambda_0) \\
= \; & \dim\Big\{ \phi \in L^2(\mathbb{R}^n);\ K(0)\phi = \lambda_0 \phi \Big\} \\
= \; & \#\Big\{ \mu_i(\lambda) \in \sigma(K(\alpha));\ \mu_i(\alpha) \to \lambda_0^{-1} \text{ as } \alpha \to 0 \Big\} \text{ (counting multiplicity)} \\
= \; & \sum_\alpha n(\lambda, \alpha) \qquad (\ \lambda_0 < \lambda < \lambda_1\ ) \\
= \; & \#\Big\{ \alpha;\ \alpha \in \sigma(P_0 + \lambda V);\ \Big\} \text{ (counting multiplicity)}
\end{aligned}$$

In the first step, we use that 0 is not the eigenvalue of $P(\lambda_0)$, by Lemma 3.6. In the second step, we use Proposition 3.8. In the third step, we use Proposition 3.9 and Lemma 3.11. In the fourth step, we use that the eigenvalue of $K(\alpha)$ is continuous and monotonous increasing with respect to α, and the eigenvalue of $K(\alpha) \to \lambda_0^{-1}$. Hence, for any $\lambda_0 < \lambda < \lambda_1$, there exist α_i such that $\mu_i(\alpha_i) = \lambda^{-1}$ for all i such that $\mu_i(\alpha) \to \lambda_0^{-1}$. In the last step, we use Proposition 3.4. □

3.5 Asymptotic expansion of resolvent of Schrödinger operator with critical potential

This section is concerned with the Schrödinger operator $P = -\Delta + \tilde{V}$ in $L^2(\mathbb{R}^n)$ for some $n \geq 2$. First we make some assumptions on $\tilde{V}$.

Assumption on $\tilde{V}$:

(1). $\tilde{V} = V_1 + V_2$;

(2). $V_1 \in C(\mathbb{R}^n)$, and there exists a constant R such that $|V_1(x)| = \frac{q(\theta)}{r^2}$ when $|x| \geq R$, where (r, θ) is the polar coordinate of $\mathbb{R}^n$, $q(\theta)$ is a real continuous function on sphere $\mathbb{S}^{n-1}$.

(3). $V_2 \in C(\mathbb{R}^n)$, and there exists some $\rho_0 > 2$ such that $|V_2(x)| \leq c\langle x\rangle^{-\rho_0}$.

Assume $-\Delta_s + q(\theta) \geq -\frac{1}{4}(n-2)^2$ on $L^2(\mathbb{S}^{n-1})$ in this section. Here $-\Delta_s$ is the Laplace operator on sphere $\mathbb{S}^{n-1}$. In this section, we want to study the asymptotic expansion of $(P-z)^{-1}$ for $z \in \mathbb{C}\backslash\mathbb{R}$, z near 0, in $\mathcal{L}(-1, s; 1, -s)$ for appropriate s.

Set $R(z) = (P-z)^{-1}$. Let $R_1 > R$. Let $0 \leq \chi_j \leq 1,\ j = 1, 2$ be smooth functions on $\mathbb{R}^n$ such that $\text{supp}\chi_1 \subset B(0, R_1), \chi_1(x) = 1$ when $|x| < R$ and

$$\chi_1(x)^2 + \chi_2(x)^2 = 1.$$

Set

$$P_0 = -\Delta + \frac{q(\theta)}{r^2}\ ; \qquad \tilde{P_0} = \chi_1(-\Delta)\chi_1 + \chi_2 P_0 \chi_2\ .$$

Then

$$\tilde{P_0} = P_0 - \frac{\chi_1^2}{r^2}q(\theta) - \sum_{i=1}^{2}|\nabla\chi_i|^2 = P_0 - W\ ,$$
$$P = P_0 + (\tilde{V} - \frac{q(\theta)}{r^2}) = P_0 + V,$$

where

$$W = \frac{\chi_1^2}{r^2}q(\theta) + \sum_{i=1}^{2}|\nabla\chi_i|^2,\ V = -\frac{\chi_1^2}{r^2}q(\theta) + \tilde{W}.$$

$\tilde{W}$ is a continuous function and satisfies $|\tilde{W}| \leq C\langle x\rangle^{-\rho_0}$. Under the above notation, we can see that the Schrödinger operator P can be considered as the perturbation of the model operator P_0. In the following we use the asymptotic expansion of $(P_0 - z)^{-1}$ to get the expansion of $R(z)$. This

problem has been studied by X.P. Wang (see [69]). Note that our potential V has singularity at 0. We can not use his result directly. We use his method to get the asymptotic expansion of $R(z)$ for $\Im z \neq 0$ and z near 0.

Definition 3.18. *Set* $\mathcal{N} = \{ u;\ Pu = 0,\ u \in H^{1,-s},\ \forall s > 1\}$. *We say that* 0 *is the regular point of* P *if* $\mathcal{N} = \{0\}$.

As in [69], we need to get the asymptotic expansion of 0 resonant states of P. Set $\nu_0 = \min \left\{ \nu;\ \nu \in \sigma_\infty \right\}$. Let $0 \leq \tilde{\chi}_1, \tilde{\chi}_2 \leq 1$ belong to $C^\infty(\mathbb{R}_+)$ such that $\tilde{\chi}_1(r) = 1$ on $r \leq 1$, and $\tilde{\chi}_1 + \tilde{\chi}_2 = 1$.

Theorem 3.19. *Assume* $\rho_0 > 3$ *and that* $0 \notin \sigma_1$. *Let* $u \in \mathcal{N}$. *Then,*

$$u(r\theta) = \sum_{0<\nu\leq 1} \sum_{j=1}^{n_\nu} -\frac{1}{2\nu} < Vu, |y|^{-\frac{n-2}{2}+\nu} \varphi_\nu^{(j)} > \frac{\tilde{\chi}_2(r)\varphi_\nu^{(j)}(\theta)}{r^{\frac{n-2}{2}+\nu}} + \tilde{u}, \tag{3.24}$$

where $\tilde{u} \in L^2$, *and* $< \cdot, \cdot >$ *is the scalar product in* $L^2(\mathbb{S}^{n-1})$. *In particular,*

$$u \in L^2 \Longleftrightarrow < Vu, |y|^{-\frac{n-2}{2}+\nu} \varphi_\nu^{(j)} >= 0, \quad \forall \nu \in \sigma_1, 1 \leq j \leq n_\nu. \tag{3.25}$$

Let C *denote the linear span of all vectors of the form*

$$c(u) = (\frac{1}{2\nu} < Vu, -|y|^{-\frac{n-2}{2}+\nu} \varphi_\nu^{(j)} >; \nu \in \sigma_1, 1 \leq j \leq n_\nu) \in \mathbb{C}^\kappa,$$

with $u \in \mathcal{N}$, $\kappa = \sum_{\nu\in\sigma_1} n_\nu$. *Then,*

$$\dim(\mathcal{N}/(\ker_{L^2} P)) = \dim C. \tag{3.26}$$

Remark 3.20. *This theorem has been proved for* $P = -\Delta + \frac{q(\theta)}{r^2} + V$ *with* $|V| \leq C\langle x\rangle^{-\rho}$ *by X.P. Wang (see [70]). We use the same method with a little change.*

Proof. For $u \in \mathcal{N}$, set

$$u = \sum_{0<\nu\leq 1} \sum_{j=1}^{n_\nu} u_{\nu,j} \otimes \varphi_\nu^{(j)}(\theta) + u',$$

where $u' = \pi' u$ with $\pi' = \sum_{\nu>1} \pi_\nu$ and $u_{\nu,j} = (u, \varphi_\nu^{(j)})_{\mathbb{S}^{n-1}}$. Then

$$u = (\tilde{\chi}_1(r) + \tilde{\chi}_2(r))u = \tilde{\chi}_1(r)u + \tilde{\chi}_2(r)(\sum_{0<\nu\leq 1} \sum_{j=1}^{n_\nu} u_{\nu,j} \otimes \varphi_\nu^{(j)}(\theta) + u').$$

One has $\tilde{\chi}_1(r)u \in L^2(\mathbb{R}^n)$, since $u \in H^{1,-s}$. We have

$$P_0 u' = -\pi'(Vu).$$

Let $(Vu)_{\nu,j} = (Vu, \varphi_\nu^{(j)})_{\mathbb{S}^{n-1}}$. Let $\tilde{u}_{\nu,j} = |x|^{\frac{n-2}{2}} u_{\nu,j}$. In the cylindrical coordinates (t,θ) ($x = r\theta$, $r = e^t$), one has

$$(-\partial_t^2 + \nu^2)\tilde{u}_{\nu,j} = -e^{\frac{n+2}{2}t}(Vu)_{\nu,j}\ . \tag{3.27}$$

By (3.27), $\tilde{u}_{\nu,j}$ can be represented as

$$\tilde{u}_{\nu,j}(e^t) \quad = \quad C_+e^{\nu t} + C_-e^{-\nu t} - \frac{1}{2\nu}\int_{\mathbb{R}} e^{-\nu|t-s|}e^{\frac{n+2}{2}s}(Vu)_{\nu,j}(e^s)ds.$$

Since $u_{\nu,j} = -F_{\nu,0}(Vu)_{\nu,j}$, we see that $|u_{\nu,j}(r)| \le Cr^{-\frac{n-2}{2}}$ for all $r > 0$. This shows $C_+ = C_- = 0$. For $0 < \nu \le 1$, set $\tilde{u}_{\nu,j}(e^t) = \tilde{u}^{(0)}_{\nu,j} + \tilde{u}^{(1)}_{\nu,j} + \tilde{u}^{(2)}_{\nu,j}$ with

$$\begin{aligned}
\tilde{u}^{(0)}_{\nu,j}(e^t) &= -\frac{1}{2\nu}\int_{-\infty}^{\infty} e^{-\nu(t-s)}e^{\frac{n+2}{2}s}(Vu)_{\nu,j}(e^s)ds;\\
\tilde{u}^{(1)}_{\nu,j}(e^t) &= \frac{1}{2\nu}\int_{t}^{\infty} e^{-\nu(t-s)}e^{\frac{n+2}{2}s}(Vu)_{\nu,j}(e^s)ds;\\
\tilde{u}^{(2)}_{\nu,j}(e^t) &= -\frac{1}{2\nu}\int_{t}^{\infty} e^{\nu(t-s)}e^{\frac{n+2}{2}s}(Vu)_{\nu,j}(e^s)ds.
\end{aligned}$$

For $0 < \nu \le 1$, one has that $G_{\nu,0}\pi_\nu \in \mathcal{L}(-1, s; 1, -s)$ for $s > 1 + \nu$. One has

$$-\frac{1}{2\nu}\int_0^\infty \tau^{-\frac{n-2}{2}+\nu}(Vu)_{\nu,j}(\tau)\tau^{n-1}d\tau = \frac{1}{2\nu}\langle Vu, -|\tau|^{-\frac{1}{2}(n-2)+\nu}\varphi_\nu^{(j)}\rangle$$

is finite, because $V|\tau|^{-\frac{1}{2}(n-2)+\nu}\varphi_\nu^{(j)} \in H^{-1,\rho_0-1-\epsilon}$. Hence, $\tilde{\chi}_2(e^t)\tilde{u}^{(0)}_{\nu,j}$ gives the desired leading term.

Note that, if $1 < \nu' < \rho_0 - 2$,

$$\begin{aligned}
r^{\nu'+1}(\tilde{W}u)_{\nu,j} &\in L^{2,\rho_0-2-\epsilon-\nu'}(\mathbb{R}_+, r^{n-1}dr);\\
r^{\nu'+1}(-\frac{\chi_1^2}{r^2}q(\theta)u)_{\nu,j} &\in L^2(\mathbb{R}_+, r^{n-1}dr).
\end{aligned}$$

It follows

$$r^{\nu'+1}(Vu)_{\nu,j} = r^{\nu'+1}(-\frac{\chi_1^2}{r^2}q(\theta)u)_{\nu,j} + r^{\nu'+1}(\tilde{W}u)_{\nu,j} \in L^2(\mathbb{R}_+, r^{n-1}dr).$$

Then $\tilde{u}^{(1)}_{\nu,j}(e^t)$ and $\tilde{u}^{(2)}_{\nu,j}(e^t)$ can be bounded by

$$\begin{aligned}
&\frac{1}{2\nu}\int_t^\infty e^{\nu(s-t)}e^{\frac{n+2}{2}s}|(Vu)_{\nu,j}(e^s)|ds\\
&\le \frac{1}{2\nu}\left\{\int_t^\infty e^{2\nu(s-t)-2\nu's}ds\right\}^{1/2}\|e^{(\nu'+\frac{n+2}{2})s}(Vu)_{\nu,j}\|_{L^2(\mathbb{R};ds)}\\
&\le Ce^{-\nu't}\|r^{\nu'+1}(Vu)_{\nu,j}\|_{L^2(\mathbb{R}+;r^{n-1}dr)}
\end{aligned}$$

with $1 < \nu' < \rho_0 - 2$. $\tilde{\chi}_2 v^{(1)}_{\nu,j}$ and $\tilde{\chi}_2 v^{(2)}_{\nu,j}$ give rise to the L^2-remainder term of $u_{\nu,j}$.

For $\nu > 1$, set $\tilde{u}_{\nu,j}(e^t) = \tilde{u}^{(1)}_{\nu,j} + \tilde{u}^{(2)}_{\nu,j}$ with

$$\begin{aligned}\tilde{u}^{(1)}_{\nu,j}(e^t) &= -\frac{1}{2\nu}\int_{-\infty}^{t} e^{-\nu(t-s)}e^{\frac{n+2}{2}s}(Vu)_{\nu,j}(e^s)ds;\\ \tilde{u}^{(2)}_{\nu,j}(e^t) &= -\frac{1}{2\nu}\int_{t}^{\infty} e^{\nu(t-s)}e^{\frac{n+2}{2}s}(Vu)_{\nu,j}(e^s)ds.\end{aligned}$$

Then,

$$\begin{aligned}&\frac{1}{2\nu}\int_{-\infty}^{t} e^{\nu(s-t)}e^{\frac{n+2}{2}s}|(Vu)_{\nu,j}(e^s)|ds\\ &\le \frac{1}{2\nu}\left\{\int_{-\infty}^{t} e^{2\nu(s-t)-2\nu' s}ds\right\}^{1/2}\|e^{(\nu'+\frac{n+2}{2})s}(Vu)_{\nu,j}\|_{L^2(\mathbb{R};ds)}\\ &\le Ce^{-\nu' t}\|r^{\nu'+1}(Vu)_{\nu,j}\|_{L^2(\mathbb{R}+;r^{n-1}dr)}\end{aligned}$$

for $1 < \nu' < \min\{\rho_0 - 2, \min\{\nu \in \sigma_\infty\}\backslash\sigma_1\}$, and

$$\begin{aligned}&\frac{1}{2\nu}\int_{t}^{\infty} e^{\nu(t-s)}e^{\frac{n+2}{2}s}|(Vu)_{\nu,j}(e^s)|ds\\ &\le \frac{1}{2\nu}\left\{\int_{t}^{\infty} e^{2\nu(t-s)+2\nu'(t-s)-2\nu' t}ds\right\}^{1/2}\|e^{(\nu'+\frac{n+2}{2})s}(Vu)_{\nu,j}\|_{L^2(\mathbb{R};ds)}\\ &\le Ce^{-\nu' t}\|r^{\nu'+1}(Vu)_{\nu,j}\|_{L^2(\mathbb{R}+;r^{n-1}dr)}\end{aligned}$$

for $1 < \nu' < \min\{\rho_0 - 2, \min\{\nu \in \sigma_\infty\}\backslash\sigma_1\}$. It follows that $\tilde{\chi}_2 u_{\nu,j} \in L^2(\mathbb{R}^n)$. □

Proposition 3.21. *Assume $\rho_0 > 3$ and that $0 \notin \sigma_1$. 0 is a regular point of $\tilde{P}_0$.*

Proof.By Theorem 3.1 [69], we deduce that $u \in L^{2,-s'}(\mathbb{R}^n)$ with $s' = 1 - \nu_0/2$, because $\nu_0 > 0$, and $u \in H^{1,-s}(\mathbb{R}^n), \forall s > 1$. Let $0 \le \tilde{\chi}(t) \le 1$ be a smooth function on $\mathbb{R}$ such that $\mathrm{supp}\tilde{\chi} \subset B(0,2)$, $\tilde{\chi}(t) = 1$ on $|t| < 1$. Let $\tilde{\chi}_m(t) = \tilde{\chi}(\frac{t}{m})$. Then $\langle\tilde{\chi}_m u, \tilde{P}_0\tilde{\chi}_m u\rangle \ge c\langle\tilde{\chi}_m u, \frac{1}{r^2}\tilde{\chi}_m u\rangle$ with some $c > 0$. It suffices to prove

$$\tilde{\chi}_m(r)u \to u \ \text{ in } \ H^{1,-s} \tag{3.28}$$

$$\tilde{P}_0\tilde{\chi}_m(r)u \to \tilde{P}_0 u \ \text{ in } \ H^{-1,s} \tag{3.29}$$

for some $s > 1$. (3.28) is trivial.

Set $s_0 = 1 + \frac{\nu_0}{4}$. We will show that (3.29) is correct for $s = s_0$. Since

$$\tilde{P}_0\tilde{\chi}_m(r)u - \tilde{P}_0 u = \tilde{\chi}_m\tilde{P}_0 u - \tilde{P}_0 u + [\tilde{P}_0, \tilde{\chi}_m]u,$$

and $\|\tilde{\chi}_m\tilde{P}_0 u - \tilde{P}_0 u\|_{H^{-1,s_0}} \to 0$ is trivial, thus we need only to show that $[\tilde{P}_0, \tilde{\chi}_m]u \to 0$ in H^{-1,s_0}. One has

$$[\tilde{P}_0, \tilde{\chi}_m]u = -2\chi_1(\nabla\tilde{\chi}_m)\cdot\nabla(\chi_1 u) - \chi_1(\Delta\tilde{\chi}_m)(\chi_1 u) - 2\chi_2(\nabla\tilde{\chi}_m)\cdot\nabla(\chi_2 u) - \chi_2(\Delta\tilde{\chi}_m)(\chi_2 u).$$

Note that

$$-2\chi_1(\nabla\tilde{\chi}_m)\cdot\nabla(\chi_1 u)-\chi_1(\Delta\tilde{\chi}_m)(\chi_1 u)=0$$

for m large enough, since $\mathrm{supp}\chi_1(x)\cap\mathrm{supp}(\nabla\tilde{\chi}_m(r))=\emptyset$. Since $u\in L^{2,-s'}$, then

$$-\Delta u=(\frac{\chi_2^2}{r^2}q(\theta)+\sum_{i=1}^{2}|\nabla\chi_i|^2)u\in L^{2,2-s'}.$$

It follows that $\langle u,-\Delta u\rangle$ is finite, moreover, by Parseval's formula, one has that

$$\langle u,-\Delta u\rangle=\langle\langle x\rangle^{-(2-s')}u,\langle x\rangle^{2-s'}(-\Delta)u\rangle=\int|\xi|^2|\hat{u}|^2d\xi.$$

This means that $|\nabla u|\in L^2(\mathbb{R}^n)$. Let $f\in C_0^\infty(\mathbb{R}^n)$ and $f(x)=1$ for $|x|\le 1$. Since $-\Delta u\in L^{2,2-s'}$ and $u\in L^{2,-s'}$, then

$$\langle\langle x\rangle^{2s_0-2}u,(-\Delta)u\rangle=\lim_{R\to\infty}\langle f(\frac{x}{R})\langle x\rangle^{2s_0-2}u,(-\Delta)u\rangle.$$

Using integration by part, we conclude that

$$\begin{aligned}\langle f(\frac{x}{R})\langle x\rangle^{2s_0-2}u,-\Delta u\rangle&=\int|\nabla u|^2f(\frac{x}{R})\langle x\rangle^{2s_0-2}\,dx\\&+(2s_0-2)\int f(\frac{x}{R})\langle x\rangle^{2s_0-4}(x\cdot\nabla)|u|^2\,dx+\int\langle x\rangle^{2s_0-2}(\nabla f(\frac{x}{R})\cdot\nabla)|u|^2\,dx.\end{aligned}\tag{3.30}$$

Note that $\nabla f(\frac{x}{R})\le C\langle x\rangle^{-1}$ with C independent of R. Let $R\to\infty$ in both side of (3.30), we get that $|\nabla u|\in L^{2,s_0-1}(\mathbb{R}^n)$. Hence

$$\|(\nabla\tilde{\chi}_m)\cdot\nabla(\chi_2 u)\|_{H^{-1,s_0}(\mathbb{R}^n)}\le C\|\langle x\rangle^{s_0}\frac{1}{|x|}\nabla(\chi_2 u)\|_{L^2(m\le|x|\le 2m)}\to 0$$

and

$$\|\Delta\tilde{\chi}_m\chi_2 u\|_{H^{-1,s_0}(\mathbb{R}^n)}\le C\|\langle x\rangle^{s_0}\frac{1}{|x|^2}\chi_2 u\|_{L^2(m\le|x|\le 2m)}\to 0$$

when $m\to\infty$. This implies that $\langle u,\tilde{P}_0u\rangle=\lim_{m\to 0}\langle\tilde{\chi}_m u,\tilde{P}_0\tilde{\chi}_m u\rangle\ge\langle u,\frac{1}{r^2}u\rangle$ with some $c>0$. It follows that $\ker\tilde{P}_0=\{0\}$ in $H^{1,-s}(\mathbb{R}^n)$. □

For $z\in\mathbb{C}\backslash\mathbb{R}^+$, z near 0, one has

$$[\chi_1(-\Delta+1-z)^{-1}\chi_1+\chi_2(P_0-z)^{-1}\chi_2](\tilde{P}_0-z)=1-K(z)\,,$$

where

$$K(z)\quad=\quad\chi_1(-\Delta+1-z)^{-1}A_1+\chi_2(P_0-z)^{-1}A_2,$$

with

$$\begin{aligned} A_1 &= -[-\Delta, \chi_1^2]\chi_1 + \chi_1 + \frac{q(\theta)}{r^2}\chi_1\chi_2^2 - [\chi_1\chi_2, -\Delta]\chi_2; \\ A_2 &= [\chi_1\chi_2, -\Delta]\chi_1 + \frac{q(\theta)}{r^2}\chi_1^2\chi_2 - [\chi_2^2, -\Delta]\chi_2. \end{aligned}$$

Assume $0 \notin \sigma_\infty$. Then one has $(P_0 - z)^{-1} = F_0 + o(1)$ in $\mathcal{L}(-1, s; 1, -s)$ with $s > 1$, for z near 0, $\Im z \neq 0$. $K(0) \in \mathcal{L}(-1, s; 1, -s)$, $s > 1$ and close to 1.

Lemma 3.22. *Assume* $0 \notin \sigma_\infty$, $n \geq 3$. $\ker(1 - K(0)) = \{0\}$ *and* $1 - K(0)$ *is a Fredhlom operator in* $\mathcal{L}(-1, s; 1, -s)$, $s > 1$.

Proof. If $u \in H^{1,-s}$ belong to $\ker(1 + K(0))$, then $\tilde{P}_0 u = \tilde{P}_0 K(0)u$. It is easy to check that $\tilde{P}_0 K(0)u \in H^{-1,s}$, for some $s > 1$. This means that $\tilde{u} = \tilde{P}_0 K(0)u$ satisfies

$$[\chi_1(-\Delta + 1)^{-1}\chi_1 + \chi_2 F_0 \chi_2]\tilde{u} = 0.$$

Taking the dual product of the above equation with $\tilde{u}$,we deduce that $\chi_i\tilde{u} = 0, (i = 1, 2)$, because $(-\Delta+1)^{-1}$ and F_0 are positive. Therefore $\tilde{u} = \tilde{P}_0 u = 0$. By Proposition 3.21, we get that $u = 0$. It is easy to check that $K(0)$ is a compact operator. It follows that $1 + K(0)$ is a Fredholm operator. This ends the proof. □

By Lemma 3.22 and Proposition 3.21, one has $1 - K(0)$ has bounded inverse on $H^{1,-s}$. It follows that $(1 - K(z))^{-1}$ exists for z small. We can use Theorem 2.2 [69] to get the asymptotic expansion of $(\tilde{P}_0 - z)^{-1}$. For $z \in \mathbb{C}\backslash\mathbb{R}^+$, z near 0, one has

$$(\tilde{P}_0 - z)^{-1} = (1 - K(z))^{-1}[\chi_1(-\Delta + 1 - z)^{-1}\chi_1 + \chi_2(P_0 - z)^{-1}\chi_2].$$

It follows that $(\tilde{P}_0 - z)^{-1} = \tilde{F}_0 + O(|z|^\epsilon)$ in $\mathcal{L}(-1, s; 1, -s)$, $s > 1$ with some $\epsilon > 0$. Here $\tilde{F}_0 = (1 - K(0))^{-1}[\chi_1(-\Delta + 1)^{-1}\chi_1 + \chi_2 F_0 \chi_2]$.

We write

$$P = P_0 + V = \tilde{P}_0 - \tilde{W}$$

where $V = \tilde{V} - \frac{q(\theta)}{r^2}$ and $\tilde{W} = V - W$. It's easy to see that $\tilde{W} \in C(\mathbb{R}^n)$, and $|\tilde{W}| \leq C\langle x\rangle^{-\rho_0}$.

For $z \in \mathbb{C}\backslash\mathbb{R}$, z near 0, we have

$$(\tilde{P}_0 - z)^{-1}(P - z) = 1 + \tilde{F}(z), \qquad (P_0 - z)^{-1}(P - z) = 1 + F(z),$$

where

$$\tilde{F}(z) = (\tilde{P}_0 - z)^{-1}\tilde{W}, \quad F(z) = (P_0 - z)^{-1}V.$$

In the following, we use these two formulas to study the asymptotic expansion of $R(z)$.

Lemma 3.23. *Assume* $0 \notin \sigma_\infty$, $n \geq 3$. $\ker(1 + F(0))$ *and* $\ker(1 + \tilde{F}(0))$ *coincide with the kernel,* $\mathcal{N}$*, of* P *in* $H^{1,-s}$. $1 + \tilde{F}(0)$ *and* $1 + F(0)$ *are Fredholm operators in* $\mathcal{L}(1, -s; 1, -s)$, $s > 1$.

Proof. It is clear that $\mathcal{N} \subseteq \ker(1+\tilde{F}(0))$. If $u \in H^{1,-s}$ is in $\ker(1+\tilde{F}(0))$, then $Pu = -P\tilde{F}(0)u \in H^{-1,s}$ for some $s > 1$, because $P = \tilde{P}_0 + O(r^{-\rho_0})$ for some $\rho_0 > 2$. This means that $\tilde{u} = Pu$ satisfies $\tilde{F}_0\tilde{u} = 0$ in $H^{1,-s}$. It follows that $\tilde{u} = 0$ and $u \in \mathcal{N}$. Similarly, we can get $\ker(1+F(0)) = \mathcal{N}$.

It is easy to check that $\tilde{F}(0)$ is a compact operator. It follows that $1+\tilde{F}(0)$ is a Fredholm operator. Note that due to the local singularity and the second order perturbation, $F(0)$ is not a compact operator. Since $\ker(1+F(0)) = \mathcal{N} = \ker(1+\tilde{F}(0))$, and $1+\tilde{F}(0)$ is a Fredholm operator, one has that $\ker(1+F(0))$ is of finite dimension. In the following we will show that Ran $(1+F(0))$ is closed.

Suppose that $f \in \overline{\text{Ran}\,(1+F(0))}$, then there exist $u_n \in H^{1,-s}$ such that $f_n = (1+F(0))u_n \to f$ in $H^{1,-s}$. Then

$$P_0 f_n = P_0 u_n + V u_n = (\tilde{P}_0 + \tilde{W})u_n$$

it follows

$$(1+\tilde{F}_0 W)f_n = (1+\tilde{F}(0))u_n.$$

Because $1+\tilde{F}_0 W$ is a bounded operator, and $f_n \to f$ in $H^{1,-s}$, one has

$$(1+\tilde{F}_0 W)f_n \to (1+\tilde{F}_0 W)f.$$

It follows that

$$(1+\tilde{F}(0))u_n \to (1+\tilde{F}(0))u$$

in $H^{1,-s}$, because Ran $1+\tilde{F}(0)$ is closed. Then $(1+\tilde{F}_0 W)f = (1+\tilde{F}(0))u$. It follows $f = (1+F(0))u$. This proves that the Ran $(1+F(0))$ is closed. By Lemma 3.25 (a), we can derive that dim coker$(1+F(0))$ is of finite, since $\mathcal{N}$ is a finite dimension subspace of $H^{1,-s}$. Then, we obtain that $1+F(0)$ is a Fredholm operator. □

Denote

$$\vec{\nu} = (\nu_1, \cdots, \nu_k) \in (\sigma_N)^k, \quad z_{\vec{\nu}} = z_{\nu_1} \cdots z_{\nu_k},$$

$$\{\vec{\nu}\} = \sum_{j=1}^{k} \nu'_j, \quad [\vec{\nu}]_- = \sum_{j=1}^{k} [\nu_j]_-, \quad [\vec{\nu}] = \sum_{j=1}^{k} [\nu_j].$$

Here $\nu'_j = \nu_j - [\nu_j]_-$ for $\nu_j > 0$.

First, we need to study the operator $(1-F(z))^{-1}$. If $\mathcal{N} = \{0\}$, then $1+F(0)$ has a bounded inverse on $H^{1,-s}$, by Lemma 3.23. It follows that $(1+F(z))^{-1}$ exists for z small. We can use the formula

$$R(z) = (1-F(z))^{-1}(P_0 - z)^{-1}$$

to calculate its asymptotic expansion. By Theorem 3.2, we can get the asymptotic expansion of $(P_0 - z)^{-1}$ and $(1-F(z))^{-1}$. Therefore we can get the following result for $R(z)$.

Corollary 3.24. *Assume $\mathcal{N} = \{0\}$. Let $N \in \mathbb{N}$ and $\rho_0 > 4N+2$, $R(z)$ has the following expansion in $\mathcal{L}(-1, s; 1, -s)$ with $s > 2N + 1$*

$$R(z) = \sum_{j=0}^{N} z^j R_j + \sum_{k=1}^{N_0} \sum_{,} z_{\vec{v}} \sum_{j=[\vec{v}]_-}^{N-1} z^j R_{\vec{v},j} + O(|z|^{N+\epsilon}) \tag{3.31}$$

Here N_0 is some integer large enough depending on σ_∞ and N,

$$R_0 = (1 + F_0 V)^{-1} F_0.$$

and R_j (resp., $R_{\vec{v};j}$) are in $\mathcal{L}(1, -s; 1, -s)$ for $s > 2j + 1$ (resp., for $s > 2j + \{\vec{v}\} + 1$).

When $\mathcal{N} \neq \{0\}$, we use the Grushin's method to study $(1 + F(z))^{-1}$ as in [69]. Set

$$\begin{aligned} &\mathcal{N} = \ker(1 + F_0 V) \subset H^{1,-s}, \\ &\mathcal{N}^* = \ker(1 + F_0 V)^* \subset H^{-1,s}, \quad 1 < s < \rho_0 - 1. \end{aligned}$$

Since $V(1 + F_0 V) = (1 + V F_0)V = (1 + F_0 V)^* V$, one can check that V is injective from $\mathcal{N}$ into $\mathcal{N}^*$ and $V^* = V$ is injective from $\mathcal{N}^*$ into $\mathcal{N}$. Consequently, V is bijective from $\mathcal{N}$ onto $\mathcal{N}^*$. This shows that $\mathcal{N}$ is independent of s with $1 < s < \rho_0 - 1$, $\dim \mathcal{N} = \dim \mathcal{N}^*$, and the quadratic form

$$\phi \to < \phi, -V\phi >$$

is non-degenerate on $\mathcal{N}$. Since $P_0 \geq 0$, this quadratic form is positive definite. Let

$$\mu = \dim \mathcal{N}, \quad \mu_r = \dim \mathcal{N}/(\ker_{L^2} P).$$

We can choose a basis $\{\phi_1, \cdots, \phi_\mu\}$ of $\mathcal{N}$ such that

$$< \phi_i, -V\phi_j >= \delta_{ij}. \tag{3.32}$$

Here ϕ_j, $1 \leq j \leq \mu_r$, are resonant states. To get the asymptotic expansion of $W^{-1}(z) = (1 + R_0(z)V)^{-1}$ for z near zero and $\Im z > 0$, we study as in [71] a Grushin problem associated to the operator

$$A(z) = \begin{pmatrix} W(z) & T \\ S & 0 \end{pmatrix} : H^{1,-s} \times \mathbb{C}^\mu \to H^{1,-s} \times \mathbb{C}^\mu,$$

where $s > 1$, T and S are given by

$$\begin{aligned} Tc &= \sum_{j=1}^{\mu} c_j \phi_j, \quad c = (c_1, \cdots, c_\mu) \in \mathbb{C}^\mu, \\ Sf &= (< f, -V\phi_1 >, \cdots, < f, -V\phi_\mu >) \in \mathbb{C}^\mu, \quad f \in H^{1,-s}. \end{aligned}$$

Define $Q : H^{1,-s} \to H^{1,-s}$ by

$$Qf = \sum_{j=1}^{\mu} < f, -V\phi_j > \phi_j. \tag{3.33}$$

Then,

$$TS = Q \text{ on } H^{1,-s} \text{ and } ST = I_\mu \text{ on } \mathbb{C}^\mu.$$

Decompose Q as $Q = Q_r + Q_e$ where $Q_r = \sum_{j=1}^{\mu_r} < \cdot, -V\phi_j > \phi_j$. Then,

$$Q_r^2 = Q_r, \quad Q_e^2 = Q_e, \quad Q_r Q_e = Q_e Q_r = 0. \tag{3.34}$$

As in [69], we first show that $(1 + F_0V)^{-1}$ exists in some space. We need to decompose the space $H^{1,-s}$ first. We have the following result.

Lemma 3.25. *(a). One has the decomposition*

$$H^{1,-s} = \mathcal{N} \oplus \ Ran\ (1 + F_0V). \tag{3.35}$$

Q is the projection from $H^{1,-s}$, $s > 1$, onto $\mathcal{N}$ with ker $Q =$ Ran $(1 + F_0V)$.

(b). Let $Q' = 1 - Q$. Then, $Q'(1 + F_0V)Q'$ is invertible on the range of Q' and $(Q'(1 + F_0V)Q')^{-1}Q' \in \mathcal{L}(1, -s; 1, -s)$, $s > 1$.

Remark 3.26. *This lemma has been proved by X.P. Wang [69] for V without singularity. In fact, these two results are also true for $V = -\frac{\chi_1^2}{r^2}q(\theta) + \tilde{W}$, because $-\frac{\chi_1^2}{r^2}q(\theta)$ is a bounded operator from $H^{1,-s}$ to $H^{-1,s}$ for any $s > 1$, if $n \geq 3$. Here $|\tilde{W}| \leq C\langle x\rangle^{-\rho_0}$.*

Proof. (a). It is easy to check that $\mathcal{N} \cap$ Ran $(1 + F_0V) = \{0\}$. Since $1 + F_0V$ is continuous on $H^{1,-s}$, Ran $(1 + F_0V)$ is closed and is therefore equal to (ker $(1 + F_0V)^*)^\perp$. For any $u \in H^{1,-s}$, one has $u = Qu + (u - Qu)$ with

$$u - Qu \in (\text{ Ker } (1 + F_0V)^*)^\perp = \text{ Range } (1 + F_0V).$$

This proves $H^{1,-s} = \mathcal{N} \oplus$ Ran $(1 + F_0V)$. It is easy to verify that Q is the projection onto $\mathcal{N}$ w.r.t. this decomposition of $H^{1,-s}$.

(b). $Q' = 1 - Q$ is a projection from $H^{1,-s}$ onto Ran $(1 + F_0V) = F$. For $u \in F$ such that $Q'(1 + F_0V)Q'u = 0$, we have $Q'u = u$ and $0 = Q'(1 + F_0V)Q'u = (1 + F_0V)u - Q(1 + F_0V)u = (1 + F_0V)u$. This means $u \in \mathcal{N}$. By (a), $u = 0$. This proves that $Q'(1 + F_0V)Q'$ is injective on Range $(1 + F_0V)$. Since Range $Q' =$ Range $(1 + F_0V)$, we can show also that $Q'(1 + F_0V)Q'$ is surjective on Ran $(1 + F_0V)$. Therefore, $Q'(1 + F_0V)Q'$ is bijective on Ran $(1 + F_0V)$. Since Ran $(1 + F_0V)$ is closed, $Q'(1 + F_0V)Q'$ is invertible on Ran $(1 + F_0V)$ and

$$D_0 = (Q'(1 + F_0V)Q')^{-1}Q' \in \mathcal{L}(1, -s; 1, -s), \quad s > 1.$$

□

By this Lemma, one has that $Q'(1+F_0V)Q'$ is invertible on Ran $(1+F_0V)$. It follows that $Q'W(z)Q'$ is invertible on Ran $(1+F_0V)$ with bounded by the asymptotic expansion of $R_0(z)$. Let

$$D(z) = (Q'(1+R_0(z)V)Q')^{-1}Q', \quad D_0 = (Q'(1+F_0V)Q')^{-1}Q'.$$

We construct an approximate inverse of $A(z)$ as in [69] to prove that for $z \in U_\delta$, the operator $A(z)$ is invertible from $H^{1,-s}\times\mathbb{C}^\mu$ to $H^{1,-s}\times\mathbb{C}^\mu$. Write the inverse of $A(z)$ in the form

$$A(z)^{-1} = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{+-}(z) \end{pmatrix},$$

Then by a simply computation, we can get

$$\begin{aligned} E(z) &= D(z), \qquad E_+(z) = T - D(z)Y, \\ E_-(z) &= S - XD(z), \qquad E_{+-}(z) = -S\,W(z)T + XD(z)Y, \end{aligned}$$

with

$$X = S\,W(z)Q', \quad Y = X^*.$$

Note that $A(z)A(z)^{-1} = 1$ and $A(z)^{-1}A(z) = 1$. We can obtain the inverse of $W(z)$

$$W(z)^{-1} = E(z) - E_+(z)E_{+-}(z)^{-1}E_-(z). \tag{3.36}$$

We use this formula to get the asymptotic expansion of $W(z)^{-1}$. First, we need to get the asymptotic expansion of $D(z)$. Note that V is a bounded operator from $H^{1,-s}$ to $H^{-1,s}$ for $s > \rho/2$ if $\rho > 0$, because $\frac{\chi_1^2}{r^2}q(\theta)$ is a bounded operator from $H^{1,-s}$ to $H^{-1,s}$ for any $s > 1$, $n \geq 3$. Using the asymptotic expansion of $R_0(z)$, we can get that, if $k \geq 1$ and $\rho_0 > 4N+2$, the following asymptotic expansion holds

$$D(z) \quad = \quad \sum_{j=0}^{N} z^j D_j + \sum_{k=1}^{N_0}\sum_{\vec{\nu}\in(\sigma_N)^k} z_{\vec{\nu}} \sum_{j=[\vec{\nu}]_-}^{N-1} z^j D_{\vec{\nu},j} + O(|z|^{N+\epsilon}) \tag{3.37}$$

in $\mathcal{L}(-1,s;1,-s)$, $s > 2N+1$. Here D_j (resp., $D_{\vec{\nu};j}$) are in $\mathcal{L}(1,-s;1,-s)$ for $s > 2j+1$ (resp., for $s > 2j+\{\vec{\nu}\}+1$) and $D_{\vec{\nu},j}$ are operators of finite rank. Here and in the following, N_0 is some integer large enough depending on σ_∞ and N. Here N_0 can be taken as the largest integer such that $N_0\nu_0 \leq N$, where $\nu_0 = \min\{\nu \in \sigma_\infty\} > 0$. Since the terms with $\{\vec{\nu}\} + j > N$ can be put into the remainder, (3.37) can be rewritten as

$$D(z) \quad = \quad \sum_{j=0}^{N} z^j D_j + \sum_{\{\vec{\nu}\}+j\leq N}^{(1)} z_{\vec{\nu}} z^j D_{\vec{\nu},j} + O(|z|^{N+\epsilon}) \tag{3.38}$$

where $D_{\vec{\nu},j} = 0$ if $j < [\vec{\nu}]_-$, and for $\ell \geq 1$, $\sum^{(\ell)}_{\{\vec{\nu}\}+j\leq N}$ stands for the sum over all $\vec{\nu} \in (\sigma_N)^k, \ell \leq k \leq N_0$ and $[\vec{\nu}]_- \leq j \leq N$ with $\{\vec{\nu}\} + j \leq N$. In particular,

$$D_1 = -D_0 F_1 V D_0, \quad D_{\nu_0,0} = -D_0 G_{\nu_0,\delta_{\nu_0}} \pi_{\nu_0} V D_0. \tag{3.39}$$

Similarly, we can obtain the asymptotic expansion of $E_+(z)$ (resp., $E_-(z)$) in $\mathcal{L}(\mathbb{C}^\kappa; H^{1,-s})$ (resp., in $\mathcal{L}(H^{-1,s}; \mathbb{C}^\kappa)$) for $s > 2k + 1$ if $\rho_0 > 2k + 2$. Therefore the asymptotic expansion of $W(z)^{-1}$ depends on the asymptotic expansion of $E_{+-}(z)^{-1}$.

Proposition 3.27. *(Proposition 4.4 [69]) Let $\rho_0 > 3$ if $\mu_r = 0$, and $\rho_0 > 4$ if $\mu_r \neq 0$. $E_{+-}(z)$ is invertible for z small enough and its inverse is given by*

$$\begin{aligned} E_{+-}(z)^{-1} \quad = \quad & \begin{pmatrix} (\mathcal{T}^* \mathcal{D}_1(z) \mathcal{T})^{-1} & -(\mathcal{T}^* \mathcal{D}_1(z) \mathcal{T})^{-1} C \Phi_e^{-1} \\ -\Phi_e^{-1} C^* (\mathcal{T}^* \mathcal{D}_1(z) \mathcal{T})^{-1} & z^{-1} \Phi_e^{-1} \end{pmatrix} \\ & \times \left\{ I_\nu + \begin{pmatrix} O(|z|/|z_{\varsigma_{\kappa_0}}|) + O(|z|^\epsilon) & O(|z|^\epsilon) \\ O(|z|^\epsilon) & O(|z|^\epsilon) + O(|z|/|z_{\varsigma_{\kappa_0}}|) \end{pmatrix} \right\} \end{aligned}$$

Here

$$\Phi_e = \Big(< \phi_i, \phi_j > \Big)_{\mu_r < i,j \leq \mu}, \quad C = \Big(< F_1 V \phi_j, V \phi_i > \Big)_{1 \leq i \leq \mu_r, \mu_r < j \leq \mu}$$

$\mathcal{T}$ is an invertible matrix,

$$\mathcal{D}_1(z) = \begin{pmatrix} (c'_{\varsigma_1} z_{\varsigma_1}) I_{m_1} & & 0 \\ & \ddots & \\ 0 & & (c'_{\varsigma_1} z_{\varsigma_{\kappa_0}}) I_{m_{\kappa_0}} \end{pmatrix}$$

with $c'_\nu = 4\nu^2 c_\nu \neq 0$, and $0 < \varsigma_1 < \cdots < \varsigma_{\kappa_0} \leq 1$ are those of $\nu \in \sigma_1$ for which there exist m_j linearly independent ς_j-resonant states with $\sum_{j=1}^{\kappa_0} m_j = \mu_r$.

Remark 3.28. *This proposition has been studied by X.P. Wang for Schrödinger operator with potentials without singularity.*

Proof. By the formula of $E_{+-}(z)$, one has

$$E_{+-}(z) = \Big(< (W(z) - W(z) Q' D(z) Q' W(z)) \phi_j, V \phi_i > \Big)_{1 \leq i,j \leq \mu}$$

with $W(z) = 1 + R_0(z) V$. Set

$$L_1(z) = \sum_{\nu \in \sigma_1} z_\nu G_{\nu,\delta_\nu} \pi_\nu.$$

By expansion of $R_0(z)$, one has that for $\rho_0 > 4$, $3 < s < \rho_0 - 1$, the following expansion holds in $H^{1,-s}$,

$$W(z)\phi_i = (L_1(z) + z F_1) V \phi_i + O(|z|^{1+\epsilon}), \quad \epsilon > 0.$$

And for $\mu_r < j \le \mu$, $\phi_i \in L^2$ and the above expansion remains true for $\rho_0 > 3$. By Theorem 3.19, one has $L_1(z)V\phi_i = 0$ if ϕ_i is an eigenfunction. Therefore $W(z)\phi_i$ is of the order $O(|z|)$ if ϕ_i is an eigenfunction and of the order $O((|z|^\epsilon)$ if ϕ_i is a resonant state. Since $D(z)$ is uniformly bounded in $\mathcal{L}(1,-s;1,-s)$ for $s > 1$, the (i,j)-th entry of $E_{+-}(z)$ has the asymptotic expansion

$$(E_{+-}(z))_{ij} = z < F_1 V\phi_j, V\phi_i > +O(|z|^{1+\epsilon}) \tag{3.40}$$

if at least one of ϕ_i and ϕ_j is an eigenfunction and $\rho_0 > 4$. In the case both ϕ_i and ϕ_j are eigenfunctions, since $V\phi_i$, $V\phi_j$ are in H^{-1,ρ_0}, we can prove as in [71] that $< F_1 V\phi_i, V\phi_j >=< \phi_i, \phi_j >$ for $\rho_0 > 3$. Since $V\phi_i \in H^{-1,\rho_0}$ and $(1 + F_0 V)\phi_i = 0$, then $(P_0 + V)\phi = 0$. It follows $\phi_i + (P_0 - z)^{-1}\phi_i = z(P_0 - z)^{-1}\phi_i$ for z near 0, $z \notin \mathbb{R}^+$. Then

$$F_1 V\phi_i = z^{-1}[(P_0 - z)^{-1} - F_0]V\phi_i + O(|z|^\epsilon) = (P_0 - z)^{-1}\phi_i + O(|z|^\epsilon).$$

It follows

$$\langle F_1 V\phi_i, V\phi_j\rangle = \langle (P_0 - z)^{-1}\phi_i, V\phi_j\rangle + O(|z|^\epsilon) = \langle \phi_i, \phi_j\rangle + O(|z|^\epsilon).$$

Thus we obtain that if $\rho_0 > 3$,

$$(E_{+-}(z))_{ij} = z < \phi_j, \phi_i > +O(|z|^{1+\epsilon}), \quad \mu_r < i, j \le \mu. \tag{3.41}$$

For $1 \le i, j \le \mu_r$, $W(z)\phi_i = L_1(z)V\phi_i + O(|z|)$ in $H^{1,-s}$, for $3 < s < \rho_0 - 1$ and

$$\begin{aligned} &(E_{+-}(z))_{ij} \\ &= < L_1(z)V\phi_j, V\phi_i > - < L_1(z)VD(z)W(z)\phi_j, V\phi_i > +O(|z|) \\ &= \sum_\nu c'_\nu z_\nu \sum_{l=1}^{n_\nu} u^{(l)}_{\nu,j}\overline{u^{(l)}_{\nu,i}} + \sum_\nu c'_\nu z_\nu \sum_{l=1}^{n_\nu} \tilde{u}^{(l)}_{\nu,j}(z)\overline{u^{(l)}_{\nu,i}} + O(|z|) \end{aligned} \tag{3.42}$$

with

$$\begin{aligned} u^{(l)}_{\nu,i} &= < V\phi_i, -|x|^{-\frac{n-2}{2}+\nu}\varphi^{(l)}_\nu > /(2\nu); \\ \tilde{u}^{(l)}_{\nu,j}(z) &= < VD(z)W(z)\phi_j, -|x|^{-\frac{n-2}{2}+\nu}\varphi^{(l)}_\nu > /(2\nu); \\ c'_\nu &= 4\nu^2 c_\nu. \end{aligned}$$

Here

$$c_\nu = -\frac{e^{-i\pi\nu}\Gamma(1-\nu)}{\nu 2^{2\nu+1}\Gamma(\nu+1)}, \quad \text{for } 0 < \nu < 1, \text{ and } \quad c_1 = \frac{1}{8}. \tag{3.43}$$

Let $\kappa = \sum_\nu n_\nu$ and let $\mathcal{U}, \mathcal{V}(z)$ denote the $\kappa \times \mu_r$ matrices with entries $u^{(l)}_{\nu,j}$ and $\tilde{u}^{(l)}_{\nu,j}(z)$, $1 \le j \le \mu_r$, respectively. Let $\mathcal{D}(z)$ denote the diagonal $\kappa \times \kappa$ matrix : $\mathcal{D}(z) =$ Diag $(c'_\nu z_\nu I_{n_\nu})$. Then,

$$(E_{+-}(z))_{1\le i,j\le\mu_r} = \mathcal{U}^*\mathcal{D}(z)(\mathcal{U} + \mathcal{V}(z)) + O(|z|). \tag{3.44}$$

Remark that the j-th column of $\mathcal{U}$ is just $c(\phi_j)$ defined in Theorem 3.19. Since $\phi_1, \cdots, \phi_{\mu_r}$ are linearly independents as resonant states, by Theorem 3.1 [69], $\mathcal{U}$ is of maximum rank μ_r. $\mathcal{U}^*\mathcal{D}(z)(\mathcal{U} + \mathcal{V}(z))$ is the matrix of the Hermitian form

$$\Phi(\cdot, \cdot) =< L_1(z)V(1 - D(z)W(z))\cdot, V\cdot >$$

in the basis $\{\phi_1, \cdots, \phi_{\mu_r}\}$ of $\mathcal{N}/(\ker_{L^2} P)$.

It is not clear from (3.44) whether the inverse of $\mathcal{U}^*\mathcal{D}(z)\mathcal{U}$ gives the leading term of the inverse of $E_{+-}(z)$, because due to different values of ν, not all of the entries in $\mathcal{U}^*\mathcal{D}(z)\mathcal{V}(z)$ are of higher order in z than those in $\mathcal{U}^*\mathcal{D}(z)\mathcal{U}$. To prove that $\mathcal{U}^*\mathcal{D}(z)(\mathcal{U} + \mathcal{V}(z))$ is invertible for $z \in U_\delta$ with an explicit leading term, we compute the matrix of the Hermitian form $\Phi(\cdot, \cdot)$ in another basis $\{\psi_j; 1 \le j \le \mu_r\}$ constructed in the following way. Let $0 < \varsigma_1 < \varsigma_2 < \cdots < \varsigma_{\kappa_0} \le 1$ be those of $\nu \in \sigma_\infty$ for which there are m_{ς_j} linearly independent ς_j-resonant states with $m_{\varsigma_j} \ge 1$ and $\sum_{j=1}^{\kappa_0} m_{\varsigma_j} = \mu_r$. Let $\{\varphi^{(l)}_{\varsigma_j}(\theta); l = 1, \cdots, n_{\varsigma_j}\}$ $(n_{\varsigma_j} \ge m_{\varsigma_j})$ be an orthonormal basis of the eigenspace of $-\Delta_h + q(\theta)$ associated with the eigenvalue $\varsigma_j^2 - (n-2)^2/4$. Modifying the orthonormal basis $\varphi^{(l)}_\nu$ used before if necessary, we can assume that there are m_{ς_j} linearly independent ς_j-resonant states in the form

$$\frac{\varphi^{(l)}_{\varsigma_j}(\theta)}{r^{\frac{n-2}{2}+\varsigma_j}} + O(r^{-\frac{n-2}{2}-\varsigma_j-\epsilon}), \quad 1 \le l \le m_{\varsigma_j}.$$

By an induction on j, we can construct from these resonant states m_{ς_j} linearly independent ς_j-resonant states such that

$$\psi^{(l)}_{\varsigma_j}(r\theta) = \frac{\varphi^{(l)}_{\varsigma_j}(\theta)}{r^{\frac{n-2}{2}+\varsigma_j}} + \sum_{\nu>\varsigma_j, 1\le l'\le n_\nu} c_{\nu,l';j,l}\frac{\varphi^{(l')}_\nu(\theta)}{r^{\frac{n-2}{2}+\nu}} + O_{L^2}(1). \tag{3.45}$$

Here

$$c_{\nu,l';j,l} =< V\psi^{(l)}_{\varsigma_j}, -\frac{1}{2\varsigma_j}\frac{\varphi^{(l')}_\nu}{r^{\frac{n-2}{2}-\nu}} > .$$

Subtracting if necessary a suitable multiple of $\psi^{(l')}_{\varsigma_i}$ from $\psi^{(l)}_{\varsigma_j}$ which leaves unchanged the leading term of $\psi^{(l)}_{\varsigma_j}$, one can assume without loss that

$$c_{\nu,l';j,l} = 0, \text{ for } \nu = \varsigma_i, i > j, 1 \le l' \le m_{\varsigma_i}. \tag{3.46}$$

Let $\{\varphi_m; 1 \le m \le \kappa\}$ be a rearrangement of the basis $\{\varphi^{(l)}_\nu, \nu \in \sigma_1, 1 \le l \le n_\nu\}$ such for $1 \le m \le \mu_r$

$$\varphi_m = \varphi^{(l)}_{\varsigma_j}, \quad \text{if } m = \sum_{s=1}^{j-1} m_{\varsigma_s} + l.$$

Correspondingly, set $\psi_m = \psi_{\varsigma_j}^{(l)}$, $1 \le m \le \mu_r$. The matrix of $\Phi(\cdot,\cdot)$ in this new basis $\{\psi_m\}$ is given by $\mathcal{M}(z) = \mathcal{U}'^*\mathcal{D}'(z)(\mathcal{U}' + \mathcal{V}'(z))$, where $\mathcal{V}'(z) = O(|z|^\epsilon)$, $\mathcal{D}'(z) = \text{ Diag } (c'_{\nu_{\sigma(j)}} z_{\nu_{\sigma(j)}})$, with σ an appropriate permutation of $\{1,\cdots,\kappa\}$, c'_ν being defined in Proposition 3.27 and

$$\mathcal{U}' = (u'_{i,j})_{1\le i\le\kappa, 1\le j\le\mu_r},$$

with $u'_{ij} = \delta_{ij}$ for $1 \le i, j \le \mu_r$ and for $i > \mu_r$, $u'_{ij} = 0$ if $\nu_{\sigma(i)} \le \nu_{\sigma(j)}$. In fact, u'_{ij} is given by

$$u'_{ij} = -\frac{1}{2\nu_{\sigma(i)}} < V\psi_j, |x|^{-\frac{n-2}{2}+\nu_{\sigma(i)}}\varphi_i >$$

and these properties follow from (3.45) and (3.46). Write the matrices in blocks

$$\mathcal{U}' = \begin{pmatrix} I_{\mu_r} \\ \mathcal{U}_2 \end{pmatrix}, \quad \mathcal{V}'(z) = \begin{pmatrix} \mathcal{V}_1(z) \\ \mathcal{V}_2(z) \end{pmatrix}, \quad \mathcal{D}'(z) = \begin{pmatrix} \mathcal{D}_1(z) & 0 \\ 0 & \mathcal{D}_2(z) \end{pmatrix}.$$

One has :

$$\mathcal{D}_1(z) = \text{ Diag } (c'_{\varsigma_1} z_{\varsigma_1} I_{m_{\varsigma_1}}, \cdots, c'_{\varsigma_{\kappa_0}} z_{\varsigma_{\kappa_0}} I_{m_{\varsigma_{\kappa_0}}}) \tag{3.47}$$

and

$$\begin{aligned} \mathcal{M}(z) &= \mathcal{D}_1(z) + \mathcal{U}_2^*\mathcal{D}_2(z)\mathcal{U}_2 + \mathcal{D}_1(z)\mathcal{V}_1(z) + \mathcal{U}_2^*\mathcal{D}_2(z)\mathcal{V}_2(z) \\ &= \mathcal{D}_1(z)\Big(1 + \mathcal{V}_1(z) + \mathcal{D}_1(z)^{-1}\mathcal{U}_2^*\mathcal{D}_2(z)(\mathcal{U}_2 + \mathcal{V}_2(z))\Big). \end{aligned}$$

$\mathcal{D}_1(z)^{-1}\mathcal{U}_2^*\mathcal{D}_2(z)$ is a $\mu_r \times (\kappa - \mu_r)$ matrix whose entries are

$$\Big[\mathcal{D}_1(z)^{-1}\mathcal{U}_2^*\mathcal{D}_2(z)\Big]_{ij} = \overline{u'_{\mu_r+j,i}}\frac{c'_{\nu_{\sigma(\mu_r+j)}} z_{\nu_{\sigma(\mu_r+j)}}}{c'_{\nu_{\sigma(i)}} z_{\nu_{\sigma(i)}}},$$

for $1 \le i \le \mu_r$, $1 \le j \le \kappa - \mu_r$. Since $u'_{ij} = 0$ if $\nu_{\sigma(i)} \le \nu_{\sigma(j)}$, $\Big[\mathcal{D}_1(z)^{-1}\mathcal{U}_2^*\mathcal{D}_2(z)\Big]_{ij} \ne 0$ only when $\nu_{\sigma(\mu_r+j)} > \nu_{\sigma(i)}$. This proves that

$$\mathcal{D}_1(z)^{-1}\mathcal{U}_2^*\mathcal{D}_2(z) = O(|z|^\epsilon).$$

Consequently, $\mathcal{M}(z)$ is invertible and its inverse is given by

$$\begin{aligned} \mathcal{M}(z)^{-1} &= \Big(1 + \mathcal{V}_1(z) + \mathcal{D}_1(z)^{-1}\mathcal{U}_2^*\mathcal{D}_2(z)(\mathcal{U}_2 + \mathcal{V}_2(z))\Big)^{-1}\mathcal{D}_1(z)^{-1} \\ &= (1 + O(|z|^\epsilon))\mathcal{D}_1(z)^{-1}. \end{aligned} \tag{3.48}$$

Since $\mathcal{U}^*\mathcal{D}(z)(\mathcal{U} + \mathcal{V}(z))$ is related to $\mathcal{M}(z)$ by

$$\mathcal{U}^*\mathcal{D}(z)(\mathcal{U} + \mathcal{V}(z)) = \mathcal{T}^*\mathcal{M}(z)\mathcal{T}$$

where $\mathcal{T}$ is the transfer matrix from $\{\psi_1,\cdots,\psi_{\mu_r}\}$ to $\{\phi_1,\cdots,\phi_{\mu_r}\}$, it is also invertible. The leading term of its inverse is $(\mathcal{T}^*\mathcal{D}_1(z)\mathcal{T})^{-1}$ which is of the order $O(|z_{\zeta_{\kappa_0}}|^{-1})$. This proves Proposition 3.27 when zero is not an eigenvalue of P under the assumption $\rho_0 > 4$.

When zero is an eigenvalue of P, we obtain with $\rho_0 > 3$

$$E_{+-}(z) = \begin{pmatrix} T^*\mathcal{M}(z)T + O(|z|) & zC + O(|z|^{1+\epsilon}) \\ zC^* + O(|z|^{1+\epsilon}) & z\Phi_e + O(|z|^{1+\epsilon}) \end{pmatrix}, \tag{3.49}$$

where Φ_e and C are given in Proposition 3.27. Let $S(z) = (T^*\mathcal{M}(z)T)^{-1}$. One has

$$\begin{aligned} & E_{+-}(z)\begin{pmatrix} S(z) & -S(z)C\Phi_e^{-1} \\ -\Phi_e^{-1}C^*S(z) & z^{-1}\Phi_e^{-1} \end{pmatrix} \\ = \; & I_\nu + \begin{pmatrix} O(|z/z_{\varsigma_{\kappa_0}}| + |z|^\epsilon) & O(|z|^\epsilon) \\ O(|z|^\epsilon) & O(|z/z_{\varsigma_{\kappa_0}}| + |z|^\epsilon) \end{pmatrix}. \end{aligned}$$

This proves that $E_{+-}(z)$ is invertible for $z \in U_\delta$ with δ small enough. This ends the proof. □

In the following, we use Proposition 3.27 and the formula $R(z) = (1 + F(z))^{-1}R_0(z)$ to study the asymptotic expansion of $R(z)$. Let $0 < \varsigma_1 < \cdots < \varsigma_{\kappa_0} \leq 1$ be the points in σ_1 such that P has m_j linearly independent ς_j-resonant states with $\sum_{j=1}^{\kappa_0} m_j = \mu_r$. Then there exists a basis of ς_j-resonant states, $u_j^{(i)}$, $i = 1, \cdots, m_j$ verifying

$$|c_{\varsigma_j}|^{1/2} < Vu_j^{(l)}, -|x|^{-\frac{n-2}{2}+\varsigma_j}\varphi_j^{(l')} >= \delta_{ll'}, \quad 1 \leq l, l' \leq m_j, \quad 1 \leq j \leq \kappa_0, \tag{3.50}$$

where c_{ς_j} is given by (3.43) and $\delta_{ll'} = 1$ if $l = l'$; 0 otherwise. Then we have the following result.

Theorem 3.29. *(Theorem 4.6 [69]) Assume $0 \notin \sigma_\infty$. Let $\mu = \dim \mathcal{N} \neq 0$.Assume*

$\rho_0 > \max\{4N - 6, 2N + 1\}$ *if* $\mu_r = 0$ *and*

$\rho_0 > \max\{4N - 6, 2N + 2\}$ *if* $\mu_r \neq 0$.

One has the following asymptotic expansion for $R(z)$ in $\mathcal{L}(-1, s; 1, -s)$, $s > \max\{2N - 3, 2\}$:

$$R(z) = \sum_{j=0}^{N-2} z^j T_j + \sum_{\{\vec{\nu}\}+j\leq N-2}^{(1)} z_{\vec{\nu}} z^j T_{\vec{\nu},j} + T_e(z) + T_r(z) + T_{er}(z) + O(|z|^{N-2+\epsilon}) \tag{3.51}$$

Here T_j (resp., $T_{\vec{\nu},j}$) is in $\mathcal{L}(1, -s; -1, s)$ for $s > 2j + 1$ (resp., for $s > 2j + 1 + \{\vec{\nu}\}$),

$$T_0 = AF_0, T_1 = -AF_1VA^*.$$

with $A = (1 + F_0V)^{-1}$ The sum $\sum_{\{\vec{\nu}\}+j\leq N}^{(1)}$ has the same meaning as in (3.38) and the first singular term in this sum is z_{ν_0} with coefficient $T_{\nu_0,0}$ given by

$$T_{\nu_0,0} = AG_{\nu_0,\delta_{\nu_0}}\pi_{\nu_0}A^*,$$

where ν_0 is the smallest value of $\nu \in \sigma_\infty$. $T_e(z)$, $T_r(z)$ describe the contributions up to the order $O(|z|^{N-2+\epsilon})$ from eigenfunctions and resonant states, respectively, and $T_{er}(z)$ the interaction between eigenfunctions and resonant states. One has

$$\begin{aligned}
T_e(z) &= -z^{-1}\Pi_0 + \sum_{j\geq -1,\,\{\vec{\nu}\}+j\leq N-2}^{(1)} z_{\vec{\nu}} z^j T_{e;\vec{\nu};j} \\
T_r(z) &= \sum_{j=1}^{\kappa_0} z_{\varsigma_j}^{-1}(\Pi_{r,j} + \sum_{\alpha,\beta,\vec{\nu},l}^{+,N-1} z_{\vec{\nu}} z^{|\beta|}(z_{\vec{\varsigma}})^{-\alpha-\beta} z^l T_{r;\vec{\nu},\alpha,\beta,l,j}), \qquad \text{with} \\
\Pi_{r,j} &= e^{i\pi\varsigma_j}\sum_{l=1}^{m_j} <\cdot, u_j^{(l)}> u_j^{(l)}, \quad j=1,\cdots,\kappa_0 \\
T_{er}(z) &= \sum_{j=1}^{\kappa_0} z_{\varsigma_j}^{-1}(\Pi_0 V Q_e F_1 V \Pi_{r,j} + \Pi_{r,j} V Q_r F_1 V \Pi_0 + \sum_{\alpha,\beta,\vec{\nu},l}^{+,N-1} z_{\vec{\nu}} z^{|\beta|}(z_{\vec{\varsigma}})^{-\alpha-\beta} z^l T_{er;\vec{\nu},\alpha,\beta,l,j}).
\end{aligned}$$

Π_0 is the spectral projection of P at 0, and $T_e(z)$ is of rank not exceeding Rank Π_0 with leading singular parts given by $\nu_j \in \sigma_2$:

$$T_{e;\vec{\nu};-1} = (-1)^{k'+1}(\Pi_0 V G_{\nu_1,1+\delta_{\nu_1}}\pi_{\nu_1} V)\cdots(\Pi_0 V G_{\nu_{k'},1+\delta_{\nu_{k'}}}\pi_{\nu_{k'}} V)\Pi_0, \tag{3.52}$$

for $\vec{\nu} = (\nu_1,\cdots,\nu_{k'}) \in \sigma_2^{k'}$ with $\{\vec{\nu}\} \leq 1$, $(z_{\vec{\varsigma}})^{-\alpha} = (z_{\varsigma_1})^{-\alpha_1}\cdots(z_{\varsigma_{\kappa_0}})^{-\alpha_{\kappa_0}}$. The summation $\sum_{\alpha,\beta,\vec{\nu},l}^{+,N-1}$ is taken over all possible $\alpha,\beta \in \mathbb{N}^{\kappa_0}$ with $1 \leq |\alpha| \leq N_0$, $|\beta| \geq 1$, $\vec{\nu} = (\nu_1,\cdots,\nu_{k'}) \in \sigma_N^{k'}$, $k' \geq 2|\alpha|$, for which there are at least α_k values of ν_j's belonging to σ_1 with $\nu_j \geq \varsigma_k$, for $1 \leq k \leq \kappa_0$, $l \in \mathbb{N}$, satisfying

$$|\beta| + \{\vec{\nu}\} + l - \sum_{k=1}^{\kappa_0}(\alpha_k+\beta_k)\varsigma_k \leq N-1.$$

Remark 3.30. *This theorem has been studied by X.P. Wang [69] for the Schrödinger operator $P = P_0 + V$ with V satisfying $|V| \leq C\langle x\rangle^{-\rho_0}$. Note our $V = -\frac{\chi_1^2}{r^2}q(\theta) + \tilde{W}$ with $\tilde{W}$ be a continuous function and satisfying $|\tilde{W}| \leq C\langle x\rangle^{-\rho_0}$. We can not use X.P. Wang's result directly.*

Proof. We only give the proof of (3.51) based on the representation formula $R(z) = (E(z) - E_+(z)E_{+-}(z)^{-1}E_-(z))R_0(z)$ in the case $N = 2$ and $\rho_0 > 6$. The proof for general case is the same. It is clear that the asymptotic expansion of $E(z)R_0(z)$ gives arise to the first two sums in (3.51). Let us study the leading singularities and the form of asymptotic expansion related to the term $E_+(z)E_{+-}(z)^{-1}E_-(z)R_0(z)$ which is of rank $\leq \mu$. One has

$$\begin{aligned}
&-E_+(z)E_{+-}(z)^{-1}E_-(z) \\
&= -(T - D(z)Y)(-S\,W(z)T + S\,XY)^{-1}(S - S\,X) \\
&= -(1 - D(z)W(z))Q\,(Q(-W(z) + W(z)D(z)W(z))Q)^{-1}\,Q(1 - W(z)D(z))
\end{aligned}$$

and

$$\begin{aligned} QW(z) &= Q((L_1(z) + zF_1 + L_2(z) + z^2F_2)V + O(|z|^{2+\epsilon})), \\ D(z) &= D_0 + D_1(z) + D_2(z) + O(|z|^{2+\epsilon}), \end{aligned}$$

in $\mathcal{L}(1, -s; 1, -s), 5 < s < \rho_0 - 1$, where

$$L_1(z) = \sum_{\nu\in\sigma_1} z_\nu G_{\nu,\delta_\nu}\pi_\nu, \qquad L_2(z) = \sum_{\nu\in\sigma_2\setminus\sigma_1} z_\nu z G_{\nu,\delta_\nu}\pi_\nu + \sum_{\nu\in\sigma_1} z_\nu z G_{\nu,1+\delta_\nu}\pi_\nu$$
$$D_1(z) = zD_1 + \sum_{\nu\in\sigma_1} z_\nu D_{\nu,0}, \qquad D_2(z) = z^2D_2 + \sum_{\nu\in\sigma_1} z_\nu z D_{\nu,1} + \sum_{\nu\in\sigma_2\setminus\sigma_1} z_\nu z D_{\nu,0}$$

with $\sigma_j = \sigma_\infty \cap [0, j]$. It follows that

$$\begin{aligned} &Q(-W(z) + W(z)D(z)W(z))Q \\ &= Q\{-[L_1(z) + zF_1 + L_2(z)]V + [L_1(z) + zF_1 + L_2(z)]V(D_0 + D_1(z))L_1(z)V \\ &\quad + L_1(z)V[D_2(z)L_1(z) + (D_0 + D_1(z))(zF_1 + L_2(z))]V + O(|z|^2)\}Q \end{aligned}$$

Let $S(z) = Q(-W(z) + W(z)D(z)W(z))Q$. Set

$$\nu_0 = \min\{\min\{\nu \in \sigma_1\}, \min\{\nu - 1; \nu \in \sigma_2 \setminus \sigma_1\}\}.$$

Assume first that 0 is not a resonance of P. Then $QL_1(z)V = L_1(z)VQ = 0$ and $QW(z) = Q(zF_1 + L_2(z))V + O(|z|^2))$. We have for $\rho_0 > 5$

$$\begin{aligned} S(z) &= Q(-(zF_1 + L_2(z))V + O(|z|^2))Q \\ &= -Q(QF_1VQ)[z + Q(QF_1VQ)^{-1}L_2(z)V + O(|z|^2)]Q \end{aligned}$$

As in [71], it can be shown that $(QF_1VQ)^{-1}Q = \Pi_0V$, where Π_0 is the orthogonal projection onto the zero eigenspace of P. Note that $Q = -TT^*$, it suffices to show $(QF_1VQ)^{-1}(-TT^*) = \Pi_0$. Since $\{\phi_i; i = 1, 2, \cdots, \mu\}$ is a basis of the zero eigenspace of P, and TT^*, QF_1VQ, are linear operator no zero eigenspace of P, we need only to computer the corresponding matrixes of TT^*, QF_1VQ under the basis $\{\phi_i; i = 1, 2, \cdots, \mu_r\}$. we calculate the matrix corresponding to QF_1VQ first. The (i, j) entry of the matrix is

$$\begin{aligned} &\langle QF_1VQ\phi_j, V\phi_i\rangle \\ = &\langle QF_1V\phi_j, V\phi_i\rangle \\ = &\sum_{k=1,\cdots,\mu} \langle F_1V\phi_j, -V\phi_k\rangle\langle\phi_k, V\phi_i\rangle \\ = &\langle\phi_j, \phi_i\rangle. \end{aligned}$$

Similarly, we can get the (i, j) entry of the matrix of TT^* under the basis $\{\phi_i; i = 1, 2, \cdots, \mu\}$. $\langle QF_1VQ\phi_j, V\phi_i\rangle = \langle\phi_j, \phi_i\rangle$. Hence, one has $(QF_1VQ)^{-1}Q = \Pi_0V$. Since $z^{-1}L_2(z) = O(z_{\nu_0})$ is

small, we obtain

$$S(z)^{-1} = -z^{-1}\left(1+\sum_{j\geq 1}^{N_0}(-1)^j(z^{-1}\Pi_0 V L_2(z) V Q)^j\right)\Pi_0 V + O(1),$$

with N_0 large enough so that $N_0\nu_0 \geq 1$.

$$\begin{aligned}&-E_+(z)E_{+-}(z)^{-1}E_-(z)\\ &= z^{-1}\left(1+\sum_{j\geq 1}^{N_0}(-1)^j(z^{-1}\Pi_0 V L_2(z) V)^j\right)\Pi_0 V + O(1).\end{aligned}$$

Since $R_0(z) = F_0 + L_1(z) + O(z)$ and $\Pi_0 V L_1(z) = 0$, we obtain that

$$\begin{aligned}&-E_+(z)E_{+-}(z)^{-1}E_-(z)R_0(z)\\ &= -z^{-1}\left(1+\sum_{j\geq 1}^{N_0}(-1)^j(z^{-1}\Pi_0 V L_2(z) V)^j\right)\Pi_0 + O(1).\end{aligned}$$

This gives the formula for $T_{e,\vec{\nu},-1}$ in the case when there is no resonant state. Since $Q' W(z) Q = zQ' F_1 V Q + O(|z|^{1+\nu_0})$, it follows that

$$-(1-\Pi_0)E_+(z)E_{+-}(z)^{-1}E_-(z)R_0(z) = -(1-\Pi_0)D_0 F_1 V \Pi_0(z^{-1}L_2(z))V\Pi_0 + O(|z|) \quad (3.53)$$

This shows that $(1-\Pi_0)T_e(z) = O(|z|^{\nu_0})$.

Assume now that zero is a resonance of P. One has in $H^{1,-s}$ with $s > 1$ sufficiently close to 1 :

$$\begin{aligned}
S(z) &= S_r(z) + S_e(z) + S_{re}(z) + S_{er}(z) + O(|z|^2) && \text{with}\\
S_e(z) &= -Q_e(zF_1 + L_2(z))VQ_e\\
S_r(z) &= Q_r\{-[L_1(z) + zF_1 + L_2(z)] + [L_1(z) + zF_1 + L_2(z)]V(D_0 + D_1(z))L_1(z)\\
&\quad + L_1(z)V[D_2(z)L_1(z) + (D_0 + D_1(z))(zF_1 + L_2(z))]\}VQ_r\\
S_{re}(z) &= Q_r\{-(zF_1 + L_2(z)) + L_1(z)V(D_2(z)L_1(z) + (D_0 + D_1(z))[zF_1 + L_2(z)])\}VQ_e\\
S_{er}(z) &= Q_e\{-(zF_1 + L_2(z)) + [zF_1 + L_2(z)]V[D_0 + D_1(z)]L_1(z)\}VQ_r.
\end{aligned}$$

From the proof of Proposition 3.27,

$$I_r(z) \equiv S_r(z)^{-1}Q_r, \quad I_e(z) \equiv S_e(z)^{-1}Q_e$$

exist. We have

$$S(z)(I_r(z) + I_e(z)) = Q + S_{er}(z)I_r(z) + S_{re}(z)I_e(z) + O(|z|)$$

Note that $I_e(z) = O(|z|^{-1})$, $I_r(z) = O(|z_{\varsigma_{\kappa_0}}|^{-1})$. It follows that $S_{er}(z)I_r(z) \to 0$, $S_{re}(z)I_e(z)) = O(1)$ as $z \to 0$, which implies

$$(S_{er}(z)I_r(z) + S_{re}(z)I_e(z))^2 = S_{er}(z)I_r(z)S_{re}(z)I_e(z) + S_{re}(z)I_e(z)S_{er}(z)I_r(z) \to 0$$

as $z \to 0$. Therefore, $Q + S_{er}(z)I_r(z) + S_{re}(z)I_e(z)$ is invertible on the range of Q and we have the convergent expansion :

$$(Q + S_{er}(z)I_r(z) + S_{re}(z)I_e(z))^{-1}Q = Q + \sum_{j=1}^{\infty}(-1)^j(S_{er}(z)I_r(z) + S_{re}(z)I_e(z))^j$$

in $\mathcal{L}(1, -s; 1, -s)$ for $s > 1$. $S(z)^{-1}$ is then given by

$$(I_e(z) + I_r(z))(Q + \sum_{j=1}^{\infty}(-1)^j(S_{er}(z)I_r(z) + S_{re}(z)I_e(z))^j) + O(1).$$

It follows that

$$\begin{aligned} &-E_+(z)E_{+-}(z)^{-1}E_-(z) \\ &= (1 - (D_0 + D_1(z))L_1(z)V)I_r(z)Q_r(1 - L_1(z)V(D_0 + D_1(z))) + I_e(z) + I_{er}(z) + O(1). \end{aligned}$$

Here, $I_{er}(z)$ is defined by

$$\begin{aligned} I_{er}(z) &= (1 - (D_0 + D_1(z))L_1(z)V)(I_r(z) + I_e(z)) \\ &\quad \times\left(\sum_{j=1}^{\infty}(-1)^j(S_{er}(z)I_r(z) + S_{re}(z)I_e(z))^j\right)(Q_r(1 - L_1(z)V(D_0 + D_1(z))) + Q_e), \end{aligned}$$

$I_{er}(z)$ is the contribution from the interaction between resonant states and eigenfunctions. $I_e(z)$ has the same asymptotic expansion as in the case $\mu_r = 0$. The contribution from resonant states is given by

$$(1 - (D_0 + D_1(z))L_1(z)V)I_r(z)Q_r(1 - L_1(z)V(D_0 + D_1(z))).$$

By the analysis made in Proposition 3.27, $Q_r(-L_1(z)V + L_1(z)V(D_0 + D_1(z))L_1(z)V)Q_r$ is invertible on the range of Q_r. Let $I_{r,0}(z)$ denote its inverse. By (3.48),

$$I_{r,0}(z) = T(\mathcal{T}^*\mathcal{D}_1(z)\mathcal{T})^{-1}S(1 + O(|z|^{\epsilon})),$$

where $\mathcal{T}$ is the transfer matrix from $\{\psi_1, \cdots, \psi_{\mu_r}\}$ to $\{\phi_1, \cdots, \phi_{\mu_r}\}$ and $\mathcal{D}_1(z)$ is given in Proposition 3.27. Note that $S = -T^*V$, where $T^* : H^{1,-s} \to \mathbb{C}^{\mu}$ is the formal adjoint of T. Let

$$\Pi_r(z) = T(\mathcal{T}^{-1}\mathcal{D}_1(z)^{-1}(\mathcal{T}^{-1})^*)T^*$$

One can verify that

$$\Pi_r(z) = \sum_{j=1}^{\kappa_0}(z_{\varsigma_j})^{-1}\sum_{l=1}^{m_j}\frac{1}{4\varsigma_j^2 c_{\varsigma_j}} < \cdot, \psi_j^{(l)} > \psi_j^{(l)}. \tag{3.54}$$

$\mathcal{T}^{-1}$ is the transfer matrix from $\{\phi_1, \cdots, \phi_{\mu_r}\}$ to $\{\psi_1, \cdots, \psi_{\mu_r}\}$, and suppose the (i,j) entry of the this matrix is $a_{i,j}$. Then one has $\sum_{i=1,\cdots,\mu_r} a_{i,j}\phi_i = \psi_j$. It follows

$$\begin{aligned}
& T(\mathcal{T}^{-1}\mathcal{D}_1(z)^{-1}(\mathcal{T}^{-1})^*)T^* f \\
= \; & T(\mathcal{T}^{-1}\mathcal{D}_1(z)^{-1}(\mathcal{T}^{-1})^*)\begin{pmatrix} \langle f, \phi_1\rangle \\ \cdots \\ \langle f, \phi_{\mu_r}\rangle \end{pmatrix} \\
= \; & T(\mathcal{T}^{-1}\mathcal{D}_1(z)^{-1})\begin{pmatrix} \sum_j a_{j,1}\langle f, \phi_j\rangle \\ \cdots \\ \sum_j a_{j,\mu_r}\langle f, \phi_j\rangle \end{pmatrix} \\
= \; & T(\mathcal{T}^{-1}\mathcal{D}_1(z)^{-1})\begin{pmatrix} \langle f, \psi_1\rangle \\ \cdots \\ \langle f, \psi_{\mu_r}\rangle \end{pmatrix} \\
= \; & T\mathcal{T}^{-1}\begin{pmatrix} (4c_{\varsigma_1}\varsigma_1^2 z_{\varsigma_1})^{-1}\langle f, \phi_j\rangle \\ \cdots \\ (4c_{\varsigma_{\kappa_0}}\varsigma_{\kappa_0}^2 z_{\varsigma_{\kappa_0}})^{-1}\langle f, \phi_j\rangle \end{pmatrix} \\
= \; & \sum_{j=1}^{\kappa_0}(z_{\varsigma_j})^{-1}\sum_{l=1}^{m_j}\frac{1}{4\varsigma_j^2 c_{\varsigma_j}} < f, \psi_j^{(l)} > \psi_j^{(l)}.
\end{aligned}$$

Since $I_{r,0}(z) = -\Pi_r(z)V(1 + O(|z|^\epsilon))$, we obtain

$$I_{r,0}(z)R_0(z) = \Pi_r(z)(1 + O(|z|^\epsilon)).$$

By Theorem 3.1 [69] and (3.45), $\psi_j^{(l)}$ satisfies

$$< V\psi_j^{(l)}, -\frac{1}{2\varsigma_j}|y|^{-\frac{n-2}{2}+\varsigma_j}\varphi_{\varsigma_j}^{(l')} >= \delta_{ll'}.$$

It suffices to take

$$u_j^{(l)} = \frac{1}{2\varsigma_j|c_{\varsigma_j}|^{\frac{1}{2}}}\psi_j^{(l)} \tag{3.55}$$

in order to obtain the leading part of the singularity from resonant states as stated in Theorem 3.29. For z small enough, $I_{r,0}(z)$ has a convergent expansion

$$I_{r,0}(z) = -\left(1 + \sum_{j=1}^{\infty}(\Pi_r(z)V(L_1(z)V(D_0 + D_1(z))L_1(z)V)Q_r)^j\right)\Pi_r(z)V. \tag{3.56}$$

We need only to sum up to $j = N_0$ for some N_0 large enough such that the remainder is $O(|z|^{N-2+\epsilon})$. By Theorem 3.1 [69]and (3.45), $\Pi_{r,j}G_{\nu,\delta_\nu}\pi_\nu = 0$ if $\nu < \varsigma_j$. Therefore, $\Pi_r(z)V(L_1(z)V(D_0 + D_1(z))L_1(z)V)Q_r$ can be written as

$$\sum_{j=1}^{\kappa_0} z_{\varsigma_j}^{-1}\Pi_{r,j}\left(\sum_{\vec{\nu}\in(\sigma_1)^s, s=2,3, k=0,1}^{+} z_{\vec{\nu}}z^k J_{r;\vec{\nu},k,j}\right)$$

where the notation $\sum^{+}_{\vec{\nu}\in(\sigma_1)^l}$, $l = 2, 3$, means that the summation is taken over those $\vec{\nu} = (\nu_1, \cdots, \nu_l)$, which has at least one component, say ν_1, verifying $\nu_1 \geq \varsigma_j$ and $J_{r;\vec{\nu},k,j} = 0$ for $\vec{\nu} \in (\sigma_1)^3$ and $k = 1$. It follows that $I_{r,0}(z)$ can be expanded as

$$\begin{aligned} I_{r,0}(z) &= (1 + \sum_{l=1}^{N_0}(\sum_{j=1}^{\kappa_0} z_{\varsigma_j}^{-1}\Pi_{r,j} \sum_{\vec{\nu}\in(\sigma_1)^s, s=2,3, k=0,1, \nu_1\geq\varsigma_j} z_{\vec{\nu}} z^k J_{r;\vec{\nu},k,j})^l)\Pi_r(z)V + O(|z|^{N-2+\epsilon}) \\ &= -\Pi_r(z)V + \sum_{\alpha\in\mathcal{N}^{\kappa_0}, 1\leq|\alpha|\leq N_0} \sum_{\vec{\nu}\in\sigma_1^s, 2|\alpha|\leq s\leq 3|\alpha|; k\leq|\alpha|}^{+} z_{\vec{\nu}} z^k (z_{\vec{\varsigma}})^{-\alpha} I_{r;\vec{\nu},\alpha,k,j}\Pi_r(z)V + O(|z|^{N-2+\epsilon}). \end{aligned}$$

Here $(z_{\vec{\varsigma}})^{-\alpha} = (z_{\varsigma_1})^{-\alpha_1}\cdots(z_{\varsigma_{\kappa_0}})^{-\alpha_{\kappa_0}}$ and the summation $\sum^{+}$ is taken over all possible $\vec{\nu} = (\nu_1, \cdots, \nu_s) \in \sigma_1^s$ for which there are at least α_l of the ν_j's belonging to σ_1 with $\nu_j \geq \varsigma_l$ for all $1 \leq l \leq \kappa_0$.

Since $I_{r,0}(z)S_{r,1}(z) = O(|\frac{z}{z_{\varsigma_{\kappa_0}}}|) = O(|\frac{1}{\ln z}|)$, one has the following convergent series in $\mathcal{L}(1, -s; 1, -s)$, $s > 1$, for $z \in U_\delta$ with $\delta > 0$ small enough,

$$I_r(z) = S_r(z)^{-1}Q_r = I_{r,0}(z) + \sum_{j=1}^{\infty}(-1)^j(I_{r,0}(z)S_{r,1}(z))^j I_{r,0}(z)$$

where

$$\begin{aligned} S_{r,1}(z) &= Q_r(-[zF_1 + L_2(z)] + [zF_1 + L_2(z)]V[D_0 + D_1(z)]L_1(z) \\ &\quad + L_1(z)V[D_2(z)L_1(z) + (D_0 + D_1(z))(zF_1 + L_2(z))])VQ_r \\ &= zQ_r(-F_1 + \sum_{\vec{\nu}\in(\sigma_2)^s, 1\leq s\leq 3, j=0,1} z_{\vec{\nu}} z^j S_{r;\vec{\nu},j})VQ_r. \end{aligned}$$

Inserting the expansions of $I_{r,0}(z)$ and $S_{r,1}(z)$ into $I_r(z)$ and rearranging the terms, we obtain

$$I_r(z) = -\Pi_r(z)V + \sum_{j,\alpha,\beta,\vec{\nu}\in\sigma_2^k, k\geq 2|\alpha|}^{+,N-1} z_{\vec{\nu}} z^{|\beta|+j}(z_{\vec{\varsigma}})^{-\alpha-\beta} I_{r;\vec{\nu},\alpha,\beta,j}\Pi_r(z)V + O(|z|^{N-2+\epsilon}).$$

Note that here $N = 2$ and only $\nu\sigma_2$ is needed. In the case $\varsigma_{\kappa_0} < 1$, a finite sum on β is sufficient in order to obtain an asymptotic expansion of $R(z)$ up to $O(|z|^{N-2+\epsilon})$. In the case $\varsigma_{\kappa_0} = 1$, $z_{\varsigma_{\kappa_0}} = z\ln z$. It is then necessary first to sum over all β in order to expand $R(z)$ up to $O(|z|^{N-2+\epsilon})$. It is now clear that

$$(1 - (D_0 + D_1(z))L_1(z)V)I_r(z)Q_r(1 - L_1(z)V(D_0 + D_1(z)))R_0(z)$$

has the asymptotic expansion of $T_r(z)$.

For the interaction between resonant states and eigenfunctions, note that $S_{er}(z) = -zQ_e(F_1V + O(|z|^\epsilon))Q_r$ and $S_{re}(z) = -zQ_r(F_1V + O(|z|^\epsilon))Q_e$. It follows that

$$\begin{aligned} I_e(z)S_{er}(z)I_r(z) &= -\Pi_0 VQ_eF_1V\Pi_r(z)V + O(|z|^\epsilon/|z_{\varsigma_{\kappa_0}}|) + O(|z|/(|z_{\varsigma_{\kappa_0}}|)^2) \\ I_r(z)S_{re}(z)I_e(z) &= -\Pi_r(z)VQ_rF_1V\Pi_0 V + O(|z|^\epsilon/|z_{\varsigma_{\kappa_0}}|) + O(|z|/(|z_{\varsigma_{\kappa_0}}|)^2) \end{aligned}$$

which gives

$$I_{er}(z) \quad = \quad -(\Pi_0 V Q_e F_1 V \Pi_r(z) V + \Pi_r(z) V Q_e F_1 V \Pi_0 V) + O(|z|^{\epsilon}/|z_{\varsigma_{\kappa_0}}|) + O(|z|/(|z_{\varsigma_{\kappa_0}}|)^2)$$

The remainder terms have asymptotic expansions of the form of $T_{er}(z)$. Theorem 3.29 is proved. □

Theorem 3.29 shows that the asymptotic expansion of $R(z)$ may contain any terms of the form $z_{\vec{\nu}} z^l$, $(\frac{z_\varsigma}{z^{\varsigma_j}})^k$, $\varsigma \in \sigma_1$ with $\varsigma_j < \nu \leq 1$, and $(\frac{1}{\ln z})^m$. If P has only 1-resonant states (*i.e.*, $\kappa_0 = 1$ and $\varsigma_1 = 1$) which may, however, still have an arbitrarily large multiplicity, α is absent in the summation $\sum_{\alpha,\beta,\vec{\nu},l}^{+,N-1}$ and the sum on β is infinite and gives rise to convergent series in $\frac{1}{\ln z}$. In this case, $T_r(z)$ in Theorem 3.29 can be written in the form

$$T_r(z) = \frac{1}{z \ln z}\{\Pi_{r,1} + \sum_{\vec{\nu}\in(\sigma_N)^s, s\leq N_0, l\geq 0, \{\vec{\nu}\}+l\leq N-1} z_{\vec{\nu}} z^l \Psi_{\vec{\nu},l}(z)\}$$

where $\Psi_{\vec{\nu},l}(z)$ is a convergent series of the form

$$\Psi_{\vec{\nu},l}(z) = \sum_{k=1}^{\infty} \frac{1}{\ln^k z} T_{r;\vec{\nu},k,l}.$$

3.6 Asymptotic behavior of the smallest eigenvalue of $\tilde{P}(\lambda)$

In this section, we consider a family of Schrödinger operators, $\tilde{P}(\lambda)$, which are the perturbation of $\tilde{P}_0$ in the form

$$\tilde{P}(\lambda) = \tilde{P}_0 + \lambda V(x), \text{ for } \lambda \geq 0$$

on $L^2(\mathbb{R}^n)$, $n \geq 3$. Here

$$\tilde{P}_0 = -\Delta + V_1(x).$$

$V_1(x) = \chi_2^2(x)\frac{q(\theta)}{r^2}$, $0 \leq \chi_2 \leq 1$ be a smooth function such that $\chi_2(x) = 1$ when $|x| > R_1$, $\chi_2 = 0$ when $|x| < R$. Let χ_1 be a non-negative function such that $\chi_1^2 + \chi_2^2 = 1$. V and $q(\theta)$ are defined in Section 2, (r, θ) is the polar coordinates on $\mathbb{R}^n$.

By min-max principle, one has $\tilde{P}_0 \geq 0$, because $P_0 \geq 0$. As in Section 1, we can also show that $\tilde{P}(\lambda)$ has negative eigenvalue when λ large enough and there exists some λ_0 such that when $\lambda > \lambda_0$, $\tilde{P}(\lambda)$ has eigenvalues less then 0, and when $\lambda \leq \lambda_0$, $\sigma(\tilde{P}(\lambda)) = [0, \infty)$. As in Section 3, to study the asymptotic behavior of the smallest eigenvalue of $\tilde{P}(\lambda)$, we need to know the asymptotic expansion of $(\tilde{P}_0 - z)^{-1}$.

Assume $0 \notin \sigma_\infty$. Let $N \in \mathbb{N}$ and $\rho_0 > 4N + 2$. For $z \in \mathbb{C}\backslash\mathbb{R}$, z near 0, $(\tilde{P}_0 - z)^{-1}$ has the following expansion in $\mathcal{L}(-1, s; 1, -s)$ with $s > 2N + 1$

$$(\tilde{P}_0 - z)^{-1} = \sum_{j=0}^{N} z^j R_j + \sum_{k=1}^{N_0} \sum_{\vec{\nu}\in(\sigma_N)^k} z_{\vec{\nu}} \sum_{j=[\vec{\nu}]_-}^{N-1} z^j R_{\vec{\nu},j} + O(|z|^{N+\epsilon}).$$

Here N_0 is some integer large enough depending on σ_∞ and N, and

$$R_0 = AF_0; \quad R_1 = AF_1A^*; \quad R_{\nu_i,0} = AG_{\nu_i,0}\pi_{\nu_i}A^*;$$
$$R_{(\nu_i,\nu_j),0} = AG_{\nu_i,0}\pi_{\nu_i}\frac{\chi_1^2}{r^2}q(\theta)AG_{\nu_j,0}\pi_{\nu_j}A^*;$$
$$R_{(\nu_i,\nu_j,\nu_k),0} = AG_{\nu_i,0}\pi_{\nu_i}\frac{\chi_1^2}{r^2}q(\theta)AG_{\nu_j,0}\pi_{\nu_j}\frac{\chi_1^2}{r^2}q(\theta)AG_{\nu_k,0}\pi_{\nu_k}A^*,$$

where $A = (1 - F_0\frac{\chi_1^2}{r^2}q(\theta))^{-1}$.

Definition 3.31. *Set $\tilde{\mathcal{N}}(\lambda) = \{u; \tilde{P}(\lambda)u = 0, u \in H^{1,-s}, \forall s > 1\}$, for $\lambda \geq \lambda_0$. A function $u \in \tilde{\mathcal{N}}(\lambda)\backslash L^2$ is called a resonant state of $\tilde{P}(\lambda)$ at zero.*

Set $\tilde{K}(z) = |V|^{1/2}(\tilde{P}_0 - z)^{-1}|V|^{1/2}$ for $z \notin \sigma(\tilde{P}_0)$, and $\tilde{K}(0) = |V|^{1/2}AF_0|V|^{1/2}$.

There are some results for $\tilde{P}(\lambda)$ similar to those for $P(\lambda)$. We state these results without proof (see Section 3 for details).

Proposition 3.32. *Let $\alpha < 0$. Then $\alpha \in \sigma(\tilde{P}(\lambda))$ if and only if $\lambda^{-1} \in \sigma(\tilde{K}(\alpha))$. Moreover, the multiplicity of α as the eigenvalue of $\tilde{P}(\lambda)$ is exactly the multiplicity of λ^{-1} as the eigenvalue of $\tilde{K}(\alpha)$.*

Proposition 3.33. *$\tilde{K}(\alpha)$ is a compact operator for $\alpha \leq 0$. And $\tilde{K}(\alpha)$ converges to $\tilde{K}(0)$ as $\alpha \to 0$ in operator norm sense.*

Let

$$\mu_i(\alpha) = \inf_{\phi_1,\cdots,\phi_i} \sup_{\|\psi\|=1,\psi\in[\phi_1,\cdots,\phi_i]^\perp} (\psi, \tilde{K}(\alpha)\psi).$$

Then, $\mu_i(\alpha)$ is the eigenvalue of $\tilde{K}(\alpha)$. Because $\tilde{K}(\alpha) \to \tilde{K}(0)$ as $\alpha \to 0$, one has $\mu_i(\alpha)$ converges to the eigenvalue of $\tilde{K}(0)$ by Lemma 3.11. Suppose $\mu_i(\alpha) \to \mu_i$, and suppose $\mu_1 = \cdots = \mu_m$, $\mu_1 \neq \mu_{m+1}$, then μ_1 is an eigenvalue of $\tilde{K}(0)$ of multiplicity m. By Lemma 3.11, one can choose $\phi_i(\alpha)$ $(1 \leq i \leq m)$, which is the eigenfunction of $\tilde{K}(\alpha)$ corresponding to $\mu_i(\alpha)$ such that $\langle\phi_i(\alpha), \phi_j(\alpha)\rangle = \delta_{ij}$ and $\phi_i(\alpha)$ converges. Suppose $\phi_i(\alpha) \to \phi_i$ as $\alpha \to 0$, then $\phi_i(1 \leq i \leq m)$ is the eigenfunction of $\tilde{K}(0)$ corresponding to μ_1, and $\langle\phi_i, \phi_j\rangle = \delta_{ij}$ for $1 \leq i, j \leq m$. Note that $\mu_1 = \lambda_0^{-1}$, one has $\tilde{P}(\lambda_0)\psi_i = 0$ $(1 \leq i \leq m)$ with $\psi_i = AF_0|V|^{1/2}\phi_i \in H^{1,-s}, s > 1$. Because 0 is the simple eigenvalue of $\tilde{P}(\lambda_0)$, then there is at most one $\psi_i \in L^2(\mathbb{R}^n)$ and the other ψ_i is the 0 resonant solution of $\tilde{P}(\lambda_0)$. Suppose that ψ_i $(1 \leq i \leq m)$ which is not belongs to $L^2(\mathbb{R}^n)$ is ν_i-resonant state of $\tilde{P}(\lambda_0)$, $0 < \nu_i \leq 1$. Then we have

Lemma 3.34. *Assume $0 \notin \sigma_\infty$. Let $\phi_i, \psi_i, \mu_i, \mu_i(\alpha), \nu_i$ are defined as above. If $\psi_i \notin L^2$, and ψ_i is ν_i-resonant state. If $\nu_i < 1$, then $\mu_i(\alpha) = c\alpha^{\nu_i} + o(\alpha^{\nu_i})$ with some $c \neq 0$; if $\nu_i = 1$ then $\mu_i(\alpha) = c\alpha\ln\alpha + o(\alpha)$ with some $c \neq 0$.*

Proof. The proof of this lemma is similar to that of lemma3.14. Because $\psi_i \in H^{1,-s},\ s>1$ satisfies $(\tilde{P}_0+\lambda_0 V)\psi_i=0$, then $(P_0-\frac{\chi_1^2}{r^2}q(\theta))\psi_i=0$. It follows that

$$(\,(1-\frac{\chi_1^2}{r^2}q(\theta)F_0)P_0+\lambda_0 V)\psi_i=0,$$

since

$$P_0\psi_i=\frac{\chi_1^2}{r^2}q(\theta)\psi_i-\lambda_0 V\psi_i\in H^{-1,s}.$$

Then $(P_0+\lambda_0(1-\frac{\chi_1^2}{r^2}q(\theta)F_0)^{-1}V)\psi_i=0$. By Theorem 3.19, we have

$$\begin{aligned}
\psi_i &= \sum_{0<\nu\le 1}\sum_{j=1}^{n_\nu}-\frac{1}{2\nu}\langle(\lambda_0 V-\frac{\chi_1^2}{r^2}q(\theta))\psi_i,|y|^{-\frac{1}{2}(n-1)+\nu}\varphi_\nu^{(j)}\rangle\frac{\varphi_\nu^{(j)}(\theta)}{r^{\frac{1}{2}(n-2)+\nu}}+\tilde{u}\\
&= \sum_{0<\nu\le 1}\sum_{j=1}^{n_\nu}-\frac{1}{2\nu}-\langle P_0\psi_i,|y|^{-\frac{1}{2}(n-1)+\nu}\varphi_\nu^{(j)}\rangle\frac{\varphi_\nu^{(j)}(\theta)}{r^{\frac{1}{2}(n-2)+\nu}}+\tilde{u}\\
&= \sum_{0<\nu\le 1}\sum_{j=1}^{n_\nu}-\frac{1}{2\nu}\langle\lambda_0 A^*V\psi_i,|y|^{-\frac{1}{2}(n-1)+\nu}\varphi_\nu^{(j)}\rangle\frac{\varphi_\nu^{(j)}(\theta)}{r^{\frac{1}{2}(n-2)+\nu}}+\tilde{u}.
\end{aligned}$$

One has

$$\begin{aligned}
&\langle\phi_j,|V|^{1/2}AG_{\nu,0}\pi_\nu^{(i)}A^*|V|^{1/2}\phi_k\rangle\\
=\ &\lambda_0^2\langle A^*V\psi_j,\frac{1}{|x|^{\frac{1}{2}(n-2)-\nu}}\varphi_\nu^{(i)}\rangle\overline{\langle A^*V\psi_k,\frac{1}{|x|^{\frac{1}{2}(n-2)-\nu}}\varphi_\nu^{(i)}\rangle}.
\end{aligned}$$

As the proof of Lemma 3.14, we can get the result. □

Using this lemma, we can get the following result.

Theorem 3.35. *Assume $0\notin\sigma_\infty$. Suppose that $e_1(\lambda)$ is the ground state of $\tilde{P}(\lambda)$, and ϕ_1,ψ_1 are defined as in Lemma 3.34. Then,*

(a). If $\psi_1\in L^2$, then $e_1(\lambda)=-c(\lambda-\lambda_0)+o(\lambda-\lambda_0)$, with some $c\neq 0$;

(b). If $\psi_1\notin L^2$, and ψ_1 is ν-resonant state of $\tilde{P}(\lambda_0)$, then if $\nu'=1$, $e_1(\lambda)=c\frac{\lambda-\lambda_0}{\ln(\lambda-\lambda_0)}+o(\lambda-\lambda_0)$, and if $\nu'<1$, $e_1(\lambda)=c((\lambda-\lambda_0)^{\frac{1}{\nu'}})+o((\lambda-\lambda_0)^{\frac{1}{\nu'}})$.

Moreover, if the eigenvalue of $\tilde{P}(\lambda)$, $e(\lambda)$, which is not ground state, approaches to 0 as $\lambda\downarrow\lambda_0$, then $e(\lambda)$ has the similar asymptotic expansion as $e_1(\lambda)$.

The proof of this Theorem is similar to that of Theorem 3.15. We omit the proof.

4 - Low-energy asymptotics of the spectral shift function for perturbation with critical decay

4.1 Introduction

In this chapter, we study Schrödinger operator $P = -\Delta + V(x)$ on $L^2(\mathbb{R}^n)$ with $V(x)$ a real smooth function on $\mathbb{R}^n$, $n \geq 3$, satisfying

$$V(x) = \frac{q(\theta)}{r^2} + O(\langle x\rangle^{-\rho}), \quad |x| \to \infty. \tag{4.1}$$

for some $q \in C^\infty(\mathbb{S}^{n-1})$ and $\rho > 2$. Here $x = r\theta$ with $r = |x|$ and $\theta = \frac{x}{|x|}$. $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$.

Let $0 \leq \chi_j \leq 1$ $(j = 1, 2)$ be smooth functions on $\mathbb{R}^n$ such that $\text{supp}\chi_1 \subset B(0, R_1), \chi_1(x) = 1$ when $|x| < R_0$ and

$$\chi_1(x)^2 + \chi_2(x)^2 = 1.$$

Consider the operator

$$P_0 = \chi_1(-\Delta)\chi_1 + \chi_2\tilde{P}_0\chi_2,$$

on $L^2(\mathbb{R}^n)$, where $\tilde{P}_0 = -\Delta + \frac{q(\theta)}{r^2}$. Let Δ_s denote the Laplacian on the sphere $\mathbb{S}^{n-1}$. Assume that λ_1 is the smallest eigenvalue of $-\Delta_s + q(\theta)$ on the sphere $\mathbb{S}^{n-1}$ and verifies

$$\lambda_1 > -\frac{1}{4}(n-2)^2. \tag{4.2}$$

Under this assumption, one has $P_0 \geq 0$ on $L^2(\mathbb{R}^n)$. The operator P can be considered as a perturbation of model operator P_0. We are mainly interested in the low-energy asymptotics of the derivative of the spectral shift function.

The spectral shift function was introduced in 1952 by the physicist I. M. Lifshitz in paper [41] as a trace perturbation formula in quantum mechanics. Its mathematical theory was created by M. G. Krein. Let $H,\ H_0$ be a pair of self-adjoint operators in some separable Hilbert space

$\mathcal{H}$. Krein proved in [36] that if $V = H - H_0$ is a trace class operator, then $f(H) - f(H_0)$ is of trace class and there exists some function $\xi \in L^1(\mathbb{R})$, called spectral shift function, such that

$$\mathrm{Tr}\,(f(H) - f(H_0)) = -\int_{\mathbb{R}} f'(\lambda)\xi(\lambda)\,d\lambda, \quad \forall f \in \mathcal{S}(\mathbb{R}). \tag{4.3}$$

Then it was extended by him in [37] (see [38], for a more complete exposition) to operators $H_0,\ H$ with a trace class difference $R(z) - R_0(z)$. Here $R_0(z) = (H_0 - z)^{-1}$ and $R(z) = (H - z)^{-1}$. Yafaev ([73]) proved that if there exists some c such that $P + cI$ and $P_0 + cI$ are positive and there exists some $k \in \mathbb{N}^*$,

$$\|(P + cI)^{-k} - (P_0 + cI)^{-k}\|_{tr} < \infty. \tag{4.4}$$

then $f(P) - f(P_0)$ is of trace class and there exists some function $\xi \in L^1_{loc}(\mathbb{R})$, such that (4.3) holds. The right hand side of (4.3) can be interpreted as $\langle f, \xi' \rangle$, where ξ' is the derivative of ξ in the sense of distributions.

The spectral shift function of Schrödinger operator has been studied by many authors (see for example [1],[47],[49],[50],[73]). High-energy asymptotics of the spectral shift function was studied in these paper. The result got by Robert in [49] is the following : assume $|\partial_x^\alpha V| \leq C_\alpha \langle x \rangle^{-\rho-|\alpha|}$ with $\rho > n$, then the spectral shift function, $\xi(\lambda)$, for the pair $(-\Delta, \Delta + V)$ satisfying :

(i). $\xi(\lambda)$ is C^∞ in $(0, \infty)$.

(ii). $\frac{d^k}{d\lambda^k}\xi(\lambda)$ has a complete asymptotic expansion for $\lambda \to \infty$,

$$\frac{d^k}{d\lambda^k}\xi(\lambda) \sim \lambda^{n/2-k-1} \sum_{j\geq 0} \alpha_j^{(k)} \lambda^{-j}.$$

In this chapter, we use the asymptotic expansion of $(P_0 - z)^{-1}$ and $(P - z)^{-1}$ for z near 0, $\Im z \neq 0$ to study the low-energy asymptotics of the derivative of the spectral shift function. The main result we get is the following : assume $V = \frac{q(\theta)}{r^2} + W$, and $|\partial_x^\alpha W| \leq C_\alpha \langle x \rangle^{-\rho-|\alpha|}$ with $\rho > \max\{6, n+2\}$ for $|x|$ large, then

$$\xi'(\lambda) = J_0 \delta(\lambda) + g(\lambda),$$

with $|g(\lambda)| = O(\lambda^{-1+\epsilon_0})$ for some $\epsilon_0 > 0$, as $\lambda \to 0$, J_0 depends on the multiplicity of 0 as the eigenvalue of P and the multiplicity of 0 as the resonance of P. Then we use this result and Robert's result to study Levinson's theorem. If $\rho > n + 3$, we can get that

$$\int_0^\infty (\xi'(\lambda) - \sum_{j=1}^{[\frac{n}{2}]} c_j \lambda^{[\frac{n}{2}]-1-j})\,d\lambda = -(N + J_0) + \beta_{n/2}. \tag{4.5}$$

Here $\beta_{n/2}$ depends on the dimension n and V. $\beta_{n/2} = 0$ if n is odd. N is the number of discrete eigenvalues of P.

Here is the plan of this chapter. In Section 4.2, we study a representation formula of the spectral shift function which is used to study Levinson's theorem. In Section 4.3, we use the

asymptotic expansion of $(\tilde{P}_0 - z)^{-1}$ to get the asymptotic expansion of $(P_0 - z)^{-1}$. The residues of $Tr(R(z) - R_0(z))f(P)$ is studied in Section 4.4. Here f is some smooth function with compact support. This result is used to study the low-energy asymptotics of the derivative of the spectral shift function in Section 4.5. The Levinson's theorem is also studied in this section.

4.2 A representation formula

Let P, P_0 be a pair of self-adjoint operators, semi-bounded from below, in some separable Hilbert space H. Assume that for some $k \in \mathbb{N}^*$,

$$\|(P - i)^{-k} - (P_0 - i)^{-k}\|_{tr} < \infty. \tag{4.6}$$

Then for any $f \in \mathcal{S}(\mathbb{R})$, $f(P) - f(P_0)$ is of trace class and there exists some function $\xi \in L^1_{loc}(\mathbb{R})$, called spectral shift function, such that

$$\mathrm{Tr}\,(f(P) - f(P_0)) = -\int_{\mathbb{R}} f'(\lambda)\xi(\lambda)\, d\lambda, \quad \forall f \in \mathcal{S}(\mathbb{R}). \tag{4.7}$$

The right hand side can be interpreted as $\langle f, \xi' \rangle$, where ξ' is the derivative of ξ in the sense of distributions. By the Birman-Krein theory, ξ is related with the scattering phase, $\rho(\lambda) = \arg\det S(\lambda)$, by the formula

$$\rho(\lambda) = 2\pi\xi(\lambda), \qquad \mathrm{mod}\ 2\pi\mathbb{Z},$$

and

$$\xi'(\lambda) = \frac{1}{2\pi}\mathrm{Tr}\, T(\lambda),$$

where $T(\lambda) = -iS(\lambda)^* \frac{d}{d\lambda} S(\lambda)$ is the Eisenbud-Wigner formula for the time-delay operator. We make the following assumptions.

- The spectra of P and P_0 are purely absolutely continuous in $]0, +\infty[$

$$\sigma(P_0) = \sigma_{ac}(P_0) = [0, +\infty[, \tag{4.8}$$
$$\sigma_{ac}(P) = [0, +\infty[. \tag{4.9}$$

  In particular, there are no embedded eigenvalues of P and P_0 in $]0, +\infty[$.

- Let $f \in C_0^\infty(\mathbb{R})$. There exists some $\epsilon_0 > 0$ and $C_f > 0$ such that

$$|\ \mathrm{Tr}\ [(R(z) - R_0(z))f(P)]\ | \leq C_f \frac{1}{|z|^{1+\epsilon_0}}, \tag{4.10}$$

  uniformly in $z \in \mathbb{C}$ with $|z|$ large and $z \notin \sigma(P)$. For any $\delta > 0$, and $\lambda > \delta$, $\lim_{\epsilon \downarrow 0} Tr[(R(\lambda \pm i\epsilon) - R_0(\lambda \pm i\epsilon))f(P)]$ exists. Moreover, there exists $C_{\delta,f} > 0$ such that

$$|\lim_{\epsilon \downarrow 0} \mathrm{Tr}\ [(R(\lambda \pm i\epsilon) - R_0(\lambda \pm i\epsilon))f(P)]| \leq C_{\delta,f},$$

  uniformly in λ with $\lambda > \delta$.

- The residue of the function $z \to \mathrm{Tr}\,[(R(z) - R_0(z))f(P)]$ at $z = 0$ is finite in the following sense :

$$J_0 = -\frac{1}{2\pi i} \lim_{\epsilon\to 0} \lim_{\delta\to 0} \int_{|z|=\epsilon, |\Im z|\geq\delta} \mathrm{Tr}\,[(R(z) - R_0(z))f(P)]\, dz < \infty. \tag{4.11}$$

- $\xi'(\lambda) \in L^1_{loc}(]0, \infty[)$.
- The total number, N, of negative eigenvalues of P is finite.

We have the following representation formula for the spectral shift function $\xi(\lambda)$.

Theorem 4.1. *Under the above assumptions, suppose in addition that $\xi(\cdot)$ is absolutely continuous in $]0, \infty[$. Let $f \in C_0^\infty(\mathbb{R})$ such that $f(\lambda) = 1$ for λ in neighborhood of $\sigma_{pp}(P) \cup \{0\}$. Under the above assumptions, the limit*

$$\int_0^\infty f(\lambda)\xi'(\lambda)d\lambda = \lim_{\delta\to 0_+} \lim_{R\to\infty} \int_\delta^R f(\lambda)\xi'(\lambda)d\lambda$$

exists and one has

$$Tr(f(P) - f(P_0)) - \int_0^\infty \xi'(\lambda)f(\lambda)d\lambda = N + J_0. \tag{4.12}$$

Proof. $(R(z) - R_0(z))f(P)$ can be written as

$$(R(z) - R_0(z))f(P) = (R(z)f(P) - R_0(z)f(P_0)) - (R_0(z)(f(P) - f(P_0))).$$

Under the condition (4.6), $(R(z) - R_0(z))f(P)$ is of trace class for any $f \in \mathcal{S}(\mathbb{R})$ and $z \notin \sigma(P)$ and the function

$$F(z) = Tr(R(z) - R_0(z)f(P))$$

is holomorphic outside $\sigma(P)$. We want to deduce (4.12) from Cauchy's formula applied to $F(z)$.

Let N be the total number of discrete eigenvalues of P (counted with the multiplicity). Let $E_i, 1 \leq i \leq k$, be the distinct eigenvalues of P with multiplicity m_i, such that $E_i < E_j$ if $i < j$. Then, $\sum_{i=1}^k m_i = N$. Denote for $z_0 \in \mathbb{C}$ and $\delta > 0$

$$\gamma(z_0; \delta) = \{\, z \in \mathbb{C};\ |z - z_0| = \delta \,\}; \quad D(z_0; \delta) = \{\, z \in \mathbb{C};\ |z - z_0| \leq \delta \,\}.$$

Then if $\delta > 0$ small enough, $\sigma(P) \cap D(E_i; \delta) = \{E_i\}$ and $\sigma(P_0) \cap D(E_i; \delta) = \emptyset$ for $1 \leq i \leq k$.

Set $E_0 = 0$. For $R >> 1$ and $0 < \epsilon << \delta$, denote

$$\begin{aligned}
&\gamma_{R,\epsilon} = \{z \in \mathbb{C}; |z| = R, \mathrm{dist}(z, \mathbb{R}_+) \geq \epsilon\}\\
&\gamma(1, \delta, \epsilon) = \{z \in \mathbb{C}; |z - E_1| = \delta, \mathrm{dist}(z,]E_1, \infty[) \geq \epsilon\}\\
&\gamma(i, \delta, \epsilon) = \{\, z \in \gamma(E_i, \delta), |\Im z| \geq \epsilon\}, 0 \leq i \leq k \text{ with } i \neq 1\\
&d_j^\pm = [E_j + \sqrt{\delta^2 - \epsilon^2} \pm i\epsilon, E_{j+1} - \sqrt{\delta^2 - \epsilon^2} \pm i\epsilon], \quad 1 \leq j \leq k, E_{k+1} = E_0 = 0\\
&d_R^\pm(\delta, \epsilon) = [\sqrt{\delta^2 - \epsilon^2} \pm i\epsilon, R \pm i\epsilon].
\end{aligned}$$

We denote by $\Gamma_{\delta,\epsilon,R}$ the closed curve defined by

$$\Gamma_{\delta,\epsilon,R} = (\cup_{j=0}^{k}\gamma(j;\delta,\epsilon)) \cup (\cup_{j=1}^{k} d_j^+ \cup d_j^-) \cup d_R^+(\delta,\epsilon) \cup d_R^-(\delta,\epsilon) \cup \gamma_{R,\epsilon}.$$

$\Gamma_{\delta,\epsilon,R}$ is positively oriented according to the anti-clockwise orientation of the big circle $\gamma_{R,\epsilon}$. Since $F(z)$ is holomorphic in the domain limited by $\Gamma_{\delta,\epsilon,R}$, the Cauchy integral formula gives

$$\frac{1}{2\pi i}\oint_{\Gamma_{\delta,\epsilon,R}} F(z)dz = 0.$$

We split the integral into four terms

$$\frac{1}{2\pi i}\oint_{\Gamma_{\delta,\epsilon,R}} F(z)dz \quad = \quad \sum_{j=1}^{4} I_j \quad \text{with} \tag{4.13}$$

$$I_1 = \frac{1}{2\pi i}\oint_{\gamma(R,\epsilon)} F(z)\,dz, \qquad I_2 = \sum_{j=0}^{k}\frac{1}{2\pi i}\oint_{\gamma(j;\delta,\epsilon)} F(z)\,dz$$

$$I_3 = -\frac{1}{2\pi i}\sum_{j=1}^{k}\oint_{d_j^+\cup d_j^-} F(z)\,dz, \qquad I_4 = \frac{1}{2\pi i}\oint_{d_R^+(\delta,\epsilon)\cup d_R^-(\delta,\epsilon)} F(z)\,dz$$

By condition (4.10), one has $I_1 = O(R^{-\epsilon_0})$ uniformly in $\epsilon > 0$. For z in $D(E_j,\delta)$, $j = 1,\cdots,k$, one has

$$R(z) = \Pi_j(E_j - z)^{-1} + R_j(z)$$

where Π_j is the spectral projection onto the eigenspace of P associated with the eigenvalue E_j and $R_j(z)$ is holomorphic in $D(E_j,\delta)$. $(R_j(z) - R_0(z))f(P)$ is of trace class and $z \to \mathrm{Tr}(R_j(z) - R_0(z))f(P)$ is holomorphic near $z = E_j$. Therefore, for $j = 1,\cdots,k$

$$\frac{1}{2\pi i}\oint_{\gamma(j;\delta,\epsilon)} F(z)\,dz \to \frac{1}{2\pi i}\oint_{\gamma(j;\delta)} \mathrm{Tr}(\Pi_j f(P))(E_j - z)^{-1}\,dz = -m_j f(E_j) = -m_j.$$

as $\epsilon \to 0$, for each $\delta > 0$ sufficiently small. For the integral over $\gamma(0;\delta,\epsilon)$, the assumption (4.11) and the choice of orientation on $\Gamma_{\delta,\epsilon,R}$ gives

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0}\frac{1}{2\pi i}\oint_{\gamma(0;\delta,\epsilon)} F(z)\,dz = -J_0.$$

Therefore

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0} I_2 = N + J_0. \tag{4.14}$$

For $\delta > 0$ small enough and for $j = 1,\cdots,k$, $[E_j + \delta, E_{j+1} - \delta] \cap \sigma(P) = \emptyset$ and $F(z)$ is holomorphic in a connected domain containing $[E_j + \delta, E_{j+1} - \delta]$. Thus,

$$\lim_{\epsilon\to 0}\oint_{d_j^+\cup d_j^-} F(z)\,dz = \lim_{\epsilon\to 0}\int_{E_j+\delta}^{E_{j+1}-\delta}(F(\lambda + i\epsilon) - F(\lambda - i\epsilon))d\lambda = 0$$

which shows that $\lim_{\epsilon\to 0} I_3 = 0$. We obtain that

$$\lim_{R\to\infty,\delta\to 0}\lim_{\epsilon\to 0} I_4 = -N - J_0. \tag{4.15}$$

Decompose $F(z)$ as

$$F(z) = F_1(z) + F_2(z)$$

where

$$F_1(z) = \mathrm{Tr}(R(z)f(P) - R_0(z)f(P_0)), \quad F_2(z) = \mathrm{Tr}(R_0(z)(f(P_0) - f(P))$$

By the definition of the spectral shift function,

$$F_1(z) = -\int_{\mathbb{R}} \xi(\lambda) f_z'(\lambda) d\lambda = \int_{\mathbb{R}} \xi'(\lambda) f_z(\lambda) d\lambda$$

with $f_z(\lambda) = (\lambda - z)^{-1} f(\lambda)$, f_z' is the derivative of f_z with respect to λ and $\xi'(\cdot)$ denotes the derivative of $\xi(\cdot)$ in the sense of distributions. It is elementary to check that

$$\lim_{\epsilon\to 0} \frac{1}{2\pi i} \oint_{d_R^+(\delta,\epsilon)\cup d_R^-(\delta,\epsilon)} f_z(\lambda)\, dz = \begin{cases} f(\lambda), & \text{if } \lambda \in]\delta, R[; \\ \frac{f(\lambda)}{2}, & \text{if } \lambda = \delta \text{ or } R; \\ 0, & \text{if } \lambda \in \mathbb{R} \setminus [\delta, R] \end{cases} \tag{4.16}$$

Note that $\xi'(\lambda)$ is in $L^1_{loc}(]0,\infty[)$. Making use of Fubini and the dominated convergence theorems, one derives that

$$\lim_{\epsilon\to 0} \frac{1}{2\pi i} \oint_{d_R^+(\delta,\epsilon)\cup d_R^-(\delta,\epsilon)} F_1(z)\, dz = \int_\delta^R \xi'(\lambda) f(\lambda) d\lambda. \tag{4.17}$$

For $F_2(z)$, making use of the Stone's formula for P_0 and the assumption (4.8), one has

$$\lim_{\epsilon\to 0} \frac{1}{2\pi i} \oint_{d_R^+(\delta,\epsilon)\cup d_R^-(\delta,\epsilon)} F_2(z)\, dz = \mathrm{Tr}(E_0(\delta, R)(f(P) - f(P_0))$$

where $E_0(\delta, R)$ is the spectral projection of P_0 onto the interval $[\delta, R]$. Since

$$s - \lim_{\delta\to 0, R\to\infty} E_0(\delta, R) = I$$

and $f(P) - f(P_0)$ is of trace class, one can deduce that

$$\lim_{\delta\to 0, R\to\infty} \mathrm{Tr} E_0(\delta, R)(f(P_0) - f(P)) = \mathrm{Tr}\,(f(P_0) - f(P))$$

See Lemma 4.2 below. (4.15) shows that the limit

$$\int_0^\infty \xi'(\lambda) f(\lambda) d\lambda = \lim_{\delta\to 0, R\to\infty} \int_\delta^R \xi'(\lambda) f(\lambda) d\lambda$$

exists and

$$\mathrm{Tr}(f(P) - f(P_0)) - \int_0^\infty \xi'(\lambda) f(\lambda) d\lambda = N + J_0. \tag{4.18}$$

□

Lemma 4.2. *Suppose A is of trace class and $f(\lambda)$ is an operator valued function such that $\|f(\lambda)\| \leq C$ with C independent of λ. If $B = s - \lim_{\lambda\to\lambda_0} f(\lambda)$ exists, then $f(\lambda)A$ converges to BA in $\mathscr{S}^1$ as $\lambda \to \lambda_0$. In particular,*

$$\lim_{\lambda\to\lambda_0} Trf(\lambda)A = TrBA.$$

Proof. For any $\epsilon > 0$, let F be a finite rank operator such that $\|A - F\|_1 < \epsilon$. Then

$$\|f(\lambda)A - BA\|_1 \leq \|(f(\lambda) - B)F\|_1 + \|(f(\lambda) - B)(F - A)\|_1.$$

The first term on the right hand side can be controlled by $C\epsilon$ when $|\lambda - \lambda_0| \leq \delta$ with some $\delta > 0$ small enough, since $s - \lim_{\lambda\to\lambda_0} f(\lambda) = B$ and F is a finite rank operator. The second term also can be controlled by $C\epsilon$, since $f(\lambda) - B$ is a bounded operator and $\|A - F\|_1 \leq \epsilon$. This ends the proof. □

Remark. In many cases, the high energy asymptotics of the spectral shift function is known. For example, if $P_0 = -\Delta$ and $P = -\Delta_g + V(x)$ on $\mathbb{R}^n$ with g a smooth metric g and a smooth potential $V(x)$ satisfying

$$|\partial_x^\alpha(g(x) - I)| + |\partial_x^\alpha V(x)| \leq C_\alpha\langle x\rangle^{-\rho-|\alpha|}$$

for some $\rho > n$ and the metric g has no trapped geodesics. Then, the asymptotic expansion of $\xi'(\lambda)$ as $\lambda \to +\infty$ is given in [49] :

$$\xi'(\lambda) \sim \lambda^{\frac{n}{2}-1} \sum_{j\geq 0} c_j\lambda^{-j} \tag{4.19}$$

where

$$c_0 = \frac{4\pi^{(n+1)/2}}{\Gamma(n/2+1)} \int_{\mathbb{R}^n} \sqrt{\det g(x)} - 1 \, dx.$$

If one can show that $\xi'(\lambda)$ is integrable in $]0, 1]$, then one can take a family of functions $f_R(\lambda) = \chi(\frac{\lambda}{R})$, where χ is smooth and $0 \leq \chi(s) \leq 1$, $\chi(s) = 1$ for s near 0, $\chi(s) = 0$ for $s > 1$ and expand both the terms

$$\int_0^\infty \xi'(\lambda)f_R(\lambda)\, d\lambda, \text{ and } \mathrm{Tr}(f_R(P) - f_R(P_0))$$

in $R \to \infty$. Theorem 4.1 will give a generalized Levinson's theorem.

The remaining part of this work is to apply Theorem 4.1 to Schrödinger operator, using the known results on the asymptotic expansion of $\xi'(\lambda)$ as $\lambda \to \infty$. The main task is to study $\xi'(\lambda)$ as $\lambda \to 0$.

4.3 Some results for P_0 and P

In this section, we will study the operator $P = -\Delta + V_1 + V_2$. Here $V_i(x) \in C^\infty(\mathbb{R}^n)$, $(i = 1, 2)$. $V_1(x) = \frac{q(\theta)}{|x|^2}$, if $|x| > R_0$ for some R_0 large enough. $V_2(x)$ satisfies

$$|\partial_x^\alpha V_2(x)| \leq C_\alpha \langle x\rangle^{-\rho-|\alpha|} \tag{4.20}$$

for some $\rho > 2$. Let $0 \leq \chi_j \leq 1$ $(j = 1, 2)$ be smooth functions on $\mathbb{R}^n$ such that $\text{supp}\chi_1 \subset B(0, R_1), \chi_1(x) = 1$ when $|x| < R_0$ and

$$\chi_1(x)^2 + \chi_2(x)^2 = 1.$$

Consider the operator

$$P_0 = \chi_1(-\Delta)\chi_1 + \chi_2\tilde{P}_0\chi_2,$$

on $L^2(\mathbb{R}^n)$, where $\tilde{P}_0 = -\Delta + \frac{q(\theta)}{r^2}$. (r, θ) is the polar coordinates on $\mathbb{R}^n$, $q(\theta)$ is a real continuous function. Let Δ_s denote Laplace operator on the sphere $\mathbb{S}^{n-1}$. Assume

$$-\Delta_s + q(\theta) \geq -\frac{1}{4}(n-2)^2, \quad \text{on } L^2(\mathbb{S}^{n-1}). \tag{4.21}$$

We have that $\tilde{P}_0 \geq 0$ in $L^2(\mathbb{R}^n)$, if (4.21) holds. By the above notation, we can see that operator P can be treated as the perturbation of P_0, and if we denote $V = \sum_{i=1}^{2} |\nabla\chi_i|^2 + V_2 + \chi_1^2 V_1$, then $P = P_0 + V$. V satisfies

$$|\partial_x^\alpha V(x)| \leq C_\alpha \langle x\rangle^{-\rho-|\alpha|}. \tag{4.22}$$

Let $\tilde{R}_0(z) = (\tilde{P}_0 - z)^{-1}$ for $z \notin \sigma(P_0)$. For the later purpose, we should establish the asymptotic expansion of $\frac{d}{dz}\tilde{R}_0(z)$ for z near 0. Let

$$Q_\nu = -\frac{d^2}{dr^2} - \frac{n-1}{r}\frac{d}{dr} + \frac{\nu^2 - \frac{(n-2)^2}{4}}{r^2}, \qquad \text{in } L^2(\mathbb{R}_+; r^{n-1}dr).$$

Then we have the orthogonal decomposition for the resolvent $\tilde{R}_0(z)$,

$$\tilde{R}_0(z) = \sum_{\nu \in \sigma_\infty} (Q_\nu - z)^{-1}\pi_\nu, \ z \notin \mathbb{R}$$

As in [70], we first expand each $\frac{d}{dz}(Q_\nu - z)^{-1}$ and estimate the remainder term. First, we will give the kernel of $\frac{d}{dz}(Q_\nu - z)^{-1}$. The Schwartz kernel of $(Q_\nu - z)^{-1}$, $\Im z > 0$ is (see [70])

$$K_\nu(r, \tau; z) = -(r\tau)^{-\frac{1}{2}(n-2)} \int_0^\infty e^{i\frac{\rho}{t} + iz r\tau t - i\frac{\pi\nu}{2}} J_\nu(\frac{1}{2t})\,\frac{dt}{2t}.$$

Here J_ν is the Bessel function of the first kind of order ν and

$$\rho = \rho(r, \tau) \equiv \frac{r^2 + \tau^2}{4r\tau}.$$

Lemma 4.3. *The Schwartz kernel of* $\frac{d}{dz}(Q_\nu - z)^{-1}$, $\Im z > 0$ *on* $L^2(\mathbb{R}_+; r^{n-1}dr)$ *is* $\frac{d}{dz}K_\nu(r, \tau; z)$.

Proof. By the definition of $\frac{d}{dz}(Q_\nu - z)^{-1}$, one has that for $\phi \in L^2(\mathbb{R}_+, r^{n-1}dr)$,

$$\frac{d}{dz}(Q_\nu - z)^{-1}\phi(r) = \frac{d}{dz}\int_0^\infty K_\nu(r, \tau; z)\phi(\tau)\tau^{n-1}\, d\tau. \tag{4.23}$$

Let U be a bounded set such that $\Im z \geq \epsilon_0 > 0$, for all $z \in U$. To show that the kernel of $\frac{d}{dz}(Q_\nu - z)^{-1}$ is $\frac{d}{dz}K_\nu(r, \tau; z)$, it suffices to show that for $z \in U$, fixed $r > 0$, there exists a function $g(\tau) \in L^1(\mathbb{R}_+)$, such that $|\frac{d}{dz}K_\nu(r, \tau; z)\phi(\tau)\tau^{n-1}| \leq g(\tau)$. Since $\Im z > 0$, and $J_\nu(\frac{1}{2t}) = O(t^{1/2})$ as $t \to 0$, $J_\nu(\frac{1}{2t}) = O(t^{-\nu})$ as $t \to \infty$, one has

$$\frac{d}{dz}K_\nu(r, \tau; z) = A(r, \tau; \nu)\int_0^\infty e^{i\frac{\rho}{t} + iz r\tau t} J_\nu(\frac{1}{2t})\, dt,$$

where

$$A(r, \tau; \nu) = -(r\tau)^{-\frac{n}{2}+2} e^{-i\frac{\pi\nu}{2}}\frac{i}{2}.$$

For $\tau \leq (\epsilon_0 r)^{-1}$, one has

$$\begin{aligned} |\frac{d}{dz}K_\nu(r, \tau; z)| &\leq C(r\tau)^{-\frac{n}{2}+2}|\int_0^{(\epsilon_0 r\tau)^{-1}} e^{-\Im z t r\tau} J_\nu(\frac{1}{2t})\, dt + \int_{(\epsilon_0 r\tau)^{-1}}^\infty e^{-\Im z t r\tau} J_\nu(\frac{1}{2t})\, dt| \\ &\leq C\epsilon_0^{-3/2}(r\tau)^{-\frac{n-1}{2}} + C(\epsilon_0^{\nu-1} + 1)(r\tau)^{-\frac{n}{2}+1+\nu}. \end{aligned} \tag{4.24}$$

For $\tau \geq (\epsilon_0 r)^{-1}$, note that

$$\int_0^\infty e^{i\frac{\rho}{t} + iz r\tau t} J_\nu(\frac{1}{2t})\, dt = \int_0^\infty e^{i\rho\lambda + i\frac{z r\tau}{\lambda}} J_\nu(\frac{\lambda}{2})\lambda^{-2}\, d\lambda,$$

and $\rho \neq 0$. One has, for any $N > 0$ large enough,

$$\begin{aligned} &\int_0^\infty e^{i\rho\lambda + i\frac{z r\tau}{\lambda}} J_\nu(\frac{\lambda}{2})\lambda^{-2}\, d\lambda \\ =\ & (\frac{1}{i\rho})^N(-1)^N\int_0^\infty e^{i\rho\lambda}\sum_{i_1+i_2+i_3=N} C(i_1, i_2, i_3)\frac{d^{i_1}}{d\lambda^{i_1}}(e^{i\frac{z r\tau}{\lambda}})\frac{d^{i_2}}{d\lambda^{i_2}}(J_\nu(\frac{\lambda}{2}))\frac{d^{i_3}}{d\lambda^{i_3}}(\lambda^{-2})\, d\lambda \\ =\ & (\frac{i}{\rho})^N\int_0^\infty e^{i\rho\lambda + i\frac{z r\tau}{\lambda}}\sum_{i_1+i_2+i_3=N}\sum_{0\leq k\leq i_1} C'(i_1, i_2, i_3, k)\frac{d^{i_2}}{d\lambda^{i_2}}(J_\nu(\frac{\lambda}{2}))(iz r\tau\, \lambda^{-2})^k\lambda^{k-i_1}\lambda^{-2-i_3}\, d\lambda \\ =\ & (\frac{i}{\rho})^N\int_0^\infty e^{i\frac{\rho}{t} + iz r\tau t}\sum_{i_1+i_2+i_3=N}\sum_{0\leq k\leq i_1} C'(i_1, i_2, i_3, k)\frac{d^{i_2}}{dt^{i_2}}(J_\nu)(\frac{1}{2t})(iz r\tau)^k t^{i_1+i_3+k}\, dt. \end{aligned}$$

One has $\frac{d^{i_2}}{dz^{i_2}}J_\nu(z) = O(|z|^{\nu-i_2})$ when $z \to 0$, and $\frac{d^{i_2}}{dz^{i_2}}J_\nu(z) = O(|z|^{-1/2})$ when $z \to \infty$, since $J'_\nu(z) = \nu J_\nu(z)/z - J_{\nu+1}(z)$ (see [72, P.45]). It follows that

$$\begin{aligned} &|\frac{d}{dz}K_\nu(r, \tau; z)| \\ \leq\ & C(r\tau)^{-\frac{n}{2}+2}\sum_{i_1+i_2+i_3=N}\sum_{0\leq k\leq i_1} C'(i_1, i_2, i_3, k)|z|^k\frac{(4r\tau)^N}{(r^2+\tau^2)^N}|\int_0^\infty e^{-\Im z t r\tau} t^{i_1+i_3+k} J^{(i_2)}(\frac{1}{2t})\, dt| \\ \leq\ & C(\epsilon_0^{-(N+2k-\nu+1)} + 1)\frac{(4r\tau)^N(r\tau)^{-\frac{n}{2}+2}}{(r^2+\tau^2)^N} \end{aligned}$$

where C is independent of r and z. Let

$$g(\tau) = \begin{cases} C\epsilon_0^{-3/2}(r\tau)^{-\frac{n-1}{2}} + C(\epsilon_0^{\nu-1}+1)(r\tau)^{-\frac{n}{2}+1+\nu}, & \text{if} \tau \le (\epsilon_0 r)^{-1}\,; \\ C(\epsilon_0^{-(N+2k-\nu+1)}+1)\dfrac{(4r\tau)^N(r\tau)^{-\frac{n}{2}+2}}{(r^2+\tau^2)^N}, & \text{if } \tau \ge (\epsilon_0 r)^{-1}. \end{cases}$$

Then for all $z \in U$, one has $|\frac{d}{dz}K_\nu(r,\tau;z)\phi(\tau)\tau^{(n-1)}| \le g(\tau)$, and $g(\tau) \in L^1(\mathbb{R}_+)$ if N large enough. This ends the proof. □

In the following, we suppose that $0 \notin \sigma_\infty$. We use the method used in [70] to get the asymptotic expansion of $\frac{d}{dz}(\tilde{P}_0 - z)^{-1}$. In [70], X.P. Wang got the expansion of $(\tilde{P}_0 - z)^{-1}$, we wish to take the derivative in that formula.

Proposition 4.4. *Suppose that* $0 \notin \sigma_\infty$. *The following asymptotic expansion holds for* z *near* 0 *with* $\Im z > 0$.

$$\frac{d}{dz}\tilde{R}_0(z) = \sum_{j=1}^{N} jz^{j-1}F_j + \sum_{\nu\in\sigma_N}\frac{d}{dz}(z_\nu \sum_{j=[\nu]_-}^{N-1} z^j)G_{\nu,j+\delta_\nu}\pi_\nu + \frac{d}{dz}\tilde{R}_0^{(N)}(z), \tag{4.26}$$

in $\mathcal{L}(0,s;0,-s)$, $s > 2N+1$. *Here*

$$F_j = \sum_{\nu\in\sigma_\infty} F_{\nu,j}\pi_\nu \in \mathcal{L}(0,s;0,-s), \quad s > 2j+1 \tag{4.27}$$

$$\frac{d}{dz}\tilde{R}_0^{(N)}(z) = O(|z|^{N-1+\epsilon}) \in \mathcal{L}(0,s;0,-s), \quad s > 2N+1, \quad \epsilon > 0. \tag{4.28}$$

Proof. The proof of the first part is similarly as that in [70]. First, we will show that

$$\frac{d}{dz}R_{\nu,N}(z) = O(|z|^{N-1+\epsilon}) \in \mathcal{L}(0,s;0,-s), \quad s > 2N-\nu+1.$$

By Lemma 4.3 and Proposition 3.1, one has, for $l < \nu < l+1$, and $l \ge 1$,

$$\begin{aligned} \frac{d}{dz}R_{\nu,N}(z) &= (\nu'+N)z^{\nu'+N-1}G_{\nu,N} + \frac{1}{(l-1)!}\frac{d}{dz}\int_0^1 (1-\theta)^{l-1}(ir\tau z)^l \tilde{R}_{\nu',N-l,2}(\theta z r\tau)d\theta \\ &\quad + \frac{1}{(l-1)!}\frac{d}{dz}\int_0^1 (1-\theta)^{l-1}\int_0^1 (ir\tau z)^l O_{N-l}(e^{it\theta z r\tau})t^{l-1-\nu}f(\frac{1}{t};r,\tau,\nu)dtd\theta \\ &\equiv I + II + III. \end{aligned}$$

By the definition of $G_{\nu,N}$, one has

$$|I| \le C|z|^{\nu'+N-1}(r\tau)^{N+\nu'-\frac{n-2}{2}}F_{N-l}(\rho).$$

Here $F_{N-l}(\rho)$ is a polynomial of degree $N-l$ of ρ. It is easy to see that if $s > 2N-l+\nu'+1$, $(1+r)^{-s}G_{\nu,N}(1+\tau)^{-s}$ defines a Hilbert-Schmidt operator on $L^2(\mathbb{R}_+;r^{n-1}dr)$. Hence $I = O(|z|^{\nu'+N-1})$

in $\mathcal{L}(0,s;0,-s),\ s>2N-l+\nu'+1$.

$$\begin{aligned} III &= \frac{l}{(l-1)!}\int_0^1(1-\theta)^{l-1}\int_0^1(ir\tau)^l z^{l-1}t^{l-1-\nu}O_{N-l}(e^{ir\tau t\theta z})f(\frac{1}{t};r,\tau,\nu)\,dtd\theta \\ &\quad +\frac{1}{(l-1)!}\int_0^1(1-\theta)^{l-1}\int_0^1(ir\tau z)^l\frac{d}{dz}O_{N-l}(e^{ir\tau t\theta z})t^{l-1-\nu}f(\frac{1}{t};r,\tau,\nu)\,dtd\theta \\ &\equiv III_1+III_2. \end{aligned}$$

Using (3.9) and (3.10), we can get $|III_1|\le C_N|z|^N(r\tau)^{N+1-\frac{n-2}{2}}$, and $|III_2|\le C_N|z|^N(r\tau)^{N+1-\frac{n-2}{2}}$. It follows that $III=O(|z|^N)$ in $\mathcal{L}(0,s;0,-s),\ s>N+2$.

$$\begin{aligned} II &= \frac{1}{(l-1)!}\int_0^1(1-\theta)^{l-1}\frac{d}{dz}(\tilde{R}_{\nu',N-l,2}(ir\tau z)^l(\theta zr\tau)d\theta \\ &= \frac{1}{(l-1)!}\int_0^1(1-\theta)^{l-1}\frac{d}{dz}\int_1^\infty(ir\tau z)^l O_{N-l}(e^{it\theta zr\tau})t^{-1-\nu}O_{N-l}(f(\frac{1}{t}))\,dtd\theta \\ &- \frac{1}{(l-1)!}\int_0^1(1-\theta)^{l-1}\frac{d}{dz}(\sum_{j=0}^{N-l-1}f_j\frac{(i\theta zr\tau t)^{j+1}}{j!}\int_0^1\int_0^1(ir\tau z)^l O_{N-l-1-j}(e^{it\theta zr\tau})\,dtd\theta')d\theta \\ &\equiv II_1+II_2 \end{aligned}$$

As above, we can get that $II_1=O(|z|^N)$ in $\mathcal{L}(0,s;0,-s),\ s>N+2$ and $II_2=O(|z|^N)$ in $\mathcal{L}(0,s;0,-s),\ s>2N-l+1$.

Summing up the estimate of $I,\ II$ and III, we get that

$$\frac{d}{dz}R_{\nu,N}(z)=O(|z|^{N+\nu'-1})\in\mathcal{L}(0,s;0,-s),\quad s>2N-l+1.$$

Note that $\tilde{R}_{\nu,N}(z)$ can be expressed in terms of $R_{\nu,N-1}(z)$ and $F_{\nu,N},G_{\nu,N-1}$. It follows

$$\frac{d}{dz}R_{\nu,N}(z)=O(|z|^{N+\nu'-2})\in\mathcal{L}(0,s;0,-s),\quad s>2N-l-1.$$

Lemma 4.8 gives that

$$\frac{d}{dz}R_{\nu,N}(z)=O(|z|^{N-1+\epsilon})\in\mathcal{L}(0,s;0,-s),\quad s>2N-\nu+1,$$

for some $\epsilon>0$.

By the similar argument, we can also prove that $\frac{d}{dz}R_{\nu,N}(z)$ has the same estimate for $0<\nu<1$ and $\nu=l\in\mathbb{N}$.

For fixed N, let $0<\epsilon<\frac{1}{2}\min\{\nu';\nu\in\sigma_N\}$. By the above computation, we can get that for $\nu\in\sigma_N$, $\frac{d}{dz}R_{\nu,N}(z)=O(|z|^{N-1+\epsilon})\in\mathcal{L}(0,s;0,-s),\ s>2N-\nu+1$. For $\nu>N$, one has that

$$R_{\nu,N}(z)=-(r\tau)^{-\frac{n-2}{2}}\int_0^\infty e^{i\frac{\rho}{t}-i\frac{\pi\nu}{2}}O_N(e^{ir\tau zt})J_\nu(\frac{1}{2t})\,\frac{dt}{2t}.$$

By (3.10), we know that for any $\nu > N$, $0 \le \theta \le 1$, $|\frac{d}{dz}O_N(e^{ir\tau zt})| \le C_{N,\theta}(r\tau t)^{N+\theta}|z|^{N-1+\theta}$, where $C_{N,\theta}$ is independent of ν. Thus, we can get that

$$\frac{d}{dz}R_{\nu,N}(z) = O(|z|^{N-1+\epsilon}) \in \mathcal{L}(0,s;0,-s), \quad s > N+\epsilon+1,$$

uniformly hold for $\nu > N$. Summing up $\nu \in \sigma_\infty$, one can get the expansion of $\frac{d}{dz}\tilde{R}_0(z)$ in $\mathcal{L}(0,s;0,-s)$ for appropriate s. □

Definition 4.5. *Let X be a complex vector space. Two norms $\|\cdot\|^{(0)}$ and $\|\cdot\|^{(1)}$ on X are called consistent if any sequence $\{x_n\}$ that converges to zero in one norm and which is Cauchy in the other norm converges to zero in both norms. If $\|\cdot\|^{(0)}$ and $\|\cdot\|^{(1)}$ are consistent, we define*

$$\|x\|_+ = \inf\{\|y\|^{(0)} + \|z\|^{(1)} | x = y+z\}.$$

Let S denote the closed strip $\{z \in \mathbb{C} \mid 0 \le \Re z \le 1\}$, S the interior of S, and let $\|\cdot\|^{(0)}$ and $\|\cdot\|^{(1)}$ be two consistent norms on a complex vector space X. We define $\mathscr{F}(X)$ to be the set of continuous functions f from S to X_+ which are analytic in S and which satisfy :

(1). if $\Re z = 0$, then $f(z) \in X_0$, and if $\Re z = 1$, then $f(z) \in X_1$;

(2). $\sup_{z\in S} \|f(z)\|_+ < \infty$;

(3). $|||f||| \equiv \sup_{t\in\mathbb{R}}\{\|f(it)\|^{(0)}, \|f(it)\|^{(0)}\} < \infty$.

Proposition 4.6. *([55, IX.4])*

(a). $\mathscr{F}(X)$ with the norm $|||\cdot|||$ is a Banach space.

(b). For each $t \in [0,1]$, the subspace

$$K_t = \{f \in \mathscr{F}(X) | f(t) = 0\}$$

is $|||\cdot|||$-closed.

$\forall x \in X$, let $\|x\|^{(t)} = \inf\{|||f||| \mid f \in \mathscr{F}(X), f(t) = x\}$. Let X_t be the completion of X in the norm $\|\cdot\|^{(t)}$. From appendix of IX.4 [55], we know that $X_t = \mathscr{F}(X)/K_t$.

Theorem 4.7. *(Calderon-Lions interpolation theorem) Let X and Y be complex vector spaces with given consistent norms $\|\cdot\|_X^{(0)}$ and $\|\cdot\|_X^{(1)}$ on X and $\|\cdot\|_Y^{(0)}$ and $\|\cdot\|_Y^{(1)}$ on Y. Suppose that $T(\cdot)$ is an analytic, uniformly bounded, continuous, $\mathscr{L}(X_+, Y_+)$-valued function on the strip S with the following properties :*

(1). $T(t) : X \to Y$ for each $t \in (0,1)$.

(2). For all $y \in \mathbb{R}$, $T(iy) \in \mathscr{L}(X_0, Y_0)$ and

$$M_0 = \sup_{y\in\mathbb{R}} \|T(iy)\|_{\mathscr{L}(X_0,Y_0)} < \infty.$$

(3). For all $y \in \mathbb{R}$, $T(1+iy) \in \mathscr{L}(X_1, Y_1)$ and

$$M_1 = \sup_{y\in\mathbb{R}} \|T(1+iy)\|_{\mathscr{L}(X_1,Y_1)} < \infty.$$

Then for any $t \in (0,1)$,

$$T(t)[X_t] \subset Y_t$$

and

$$\|T(t)\|_{\mathscr{L}(X_t,Y_t)} \leq M_0^{1-t} M_1^t.$$

We use Calderon-Lions interpolation theorem to get the interpolation spaces between $L^{2,s_1}(\mathbb{R}_n)$ and $L^{2,s_2}(\mathbb{R}_n)$.

Lemma 4.8. *(a). s_1, s_2 are two real number. Then the interpolation spaces between $X_0 = L^{2,s_1}(\mathbb{R}_n)$ and $X_1 = L^{2,s_2}(\mathbb{R}_n)$, $X_t (0 < t < 1)$ are L^{2,p_t} with $p_t = (1-t)s_1 + ts_2$.*

(b). Suppose $F(z) = O(|z|^{m_1}) \in \mathcal{L}(0, s_1; 0, -s_1)$ and $F(z) = O(|z|^{m_2}) \in \mathcal{L}(0, s_2; 0, -s_2)$. Then $F(z) = O(|z|^{(1-t)m_1 + tm_2}) \in \mathcal{L}(0, p_t; 0, -p_t)$.

Proof. (a). Let $X = L^{2,s_1}(\mathbb{R}_n) \cap L^{2,s_2}(\mathbb{R}_n)$. Since $C_0^\infty(\mathbb{R}_n)$ is dense in X, it suffices to show that $\|\cdot\|^{(t)}-$norm on $C_0^\infty(\mathbb{R}_n)$ coincides with $\|\cdot\|_{p_t}-$norm on $C_0^\infty(\mathbb{R}_n)$. Let $t \in (0,1)$ and $\phi \in C_0^\infty(\mathbb{R}_n)$ and define

$$f(z) = \langle x \rangle^{\frac{p_t - s_1 + z(s_1 - s_2)}{2}} \phi.$$

Then for each $z \in S$, $f(z) \in X$, and

$$\|f(iy)\|_{L^{2,s_1}} = \|\langle x \rangle^{\frac{p_t - s_1}{2}} \phi\|_{L^{2,s_1}} = \|\phi\|_{L^{2,p_t}},$$

$$\|f(1+iy)\|_{L^{2,s_2}} = \|\langle x \rangle^{\frac{p_t - s_2}{2}} \phi\|_{L^{2,s_2}} = \|\phi\|_{L^{2,p_t}}.$$

Thus $|||f||| = \|\phi\|_{p_t}$, so $\|f\|^{(t)} = \|f\|_{\mathscr{F}(X)/K_t} \leq \|\phi\|_{L^{2,p_t}}$. To prove $\|\phi\|_{L^{2,p_t}} \leq \|f\|^{(t)}$, let $f \in \mathscr{F}(X)$ and let $\psi \in C_0^\infty(\mathbb{R}^n)$. Let $g(z) = \langle x \rangle^{-\frac{p_t - s_1 + z(s_1 - s_2)}{2}} \psi$, $H(z) = \int_{\mathbb{R}^n} f(z) g(z)\, dx$. Then $H(z)$ is analytic and bounded in S, and $H(t) = \int_{\mathbb{R}^n} f(t)\psi\, dx$. By the three line theorem, one has

$$\begin{aligned} |H(t)| &\leq \sup_{y \in \mathbb{R}} \{|H(iy)|,\ |H(1+iy)|\} \\ &\leq \sup_{y \in \mathbb{R}} \{\|f(iy)\|_{L^{2,s_1}},\ \|f(1+iy)\|_{L^{2,s_2}}\} \sup_{y \in \mathbb{R}} \{\|g(iy)\|_{L^{2,-s_1}},\ \|g(1+iy)\|_{L^{2,-s_2}}\} \\ &= |||f||| \cdot \|\psi\|_{L^{2,-p_t}}\ . \end{aligned}$$

It follows that $f \in L^{2,p_t}$ and $\|f(t)\|_{L^{2,p_t}} \leq |||f|||$. Thus, for any $\phi \in C_0^\infty(\mathbb{R}^n)$, and $f \in \phi + K_t$, $\|\phi\|^{(t)} = \inf_{f \in \phi + K_t} |||f||| \geq \|\phi\|_{L^{2,p_t}}$. Thus the norms $\|\cdot\|_{L^{2,p_t}}$ and $\|\phi\|^{(t)}$ agree on C_0^∞. Since C_0^∞ is dense in X, we conclude that $X_t = L^{2,p_t}$.

(b). Let $T(\lambda; z) = |z|^{m_1(\lambda-1) - m_2\lambda} F(z)$, for $\lambda \in S$. Apply Calderon-lions interpolation theorem to $T(\lambda; z)$, we conclude (b). □

Definition 4.9. *Set $\mathcal{N} = \{\, u;\ Pu = 0,\ u \in H^{1,-s},\ \forall s > 1\}$. A function $u \in \mathcal{N} \backslash L^2$ is called a resonant state of P at zero. If $\mathcal{N} = \{0\}$, we say that 0 is the regular point of P.*

Denote

$$\vec{\nu} = (\nu_1, \cdots, \nu_k) \in (\sigma_N)^k, \quad z_{\vec{\nu}} = z_{\nu_1} \cdots z_{\nu_k},$$
$$\{\vec{\nu}\} = \sum_{j=1}^{k} \nu'_j, \quad [\vec{\nu}]_- = \sum_{j=1}^{k} [\nu_j]_-, \quad [\vec{\nu}] = \sum_{j=1}^{k} [\nu_j].$$

Here $\nu'_j = \nu_j - [\nu_j]_-$ for $\nu_j > 0$. If $\vec{\nu}_i \in (\sigma_N)^{m_i}$, $i = 1, \cdots, k$, and $\vec{\nu}_i = (\nu_{i,1}, \cdots, \nu_{i,m_i})$, denote

$$(\vec{\nu}_1, \cdots, \vec{\nu}_k) = (\nu_{1,1}, \cdots, \nu_{1,m_1}, \nu_{2,1}, \cdots, \nu_{k,m_k}).$$

Let $R_0(z) = (P_0 - z)^{-1}$ and $\tilde{R}_0(z) = (\tilde{P}_0 - z)^{-1}$. Let $W = \frac{\chi_1^2}{r^2} q(\theta) + \sum_{i=1}^{2} |\nabla \chi_i|^2$, then $P_0 = \tilde{P}_0 - W$. By Proposition 3.21 and Lemma 3.23, we know that 0 is the regular point of P_0 and $1 - F_0 W$ is a Fredholm operator in $H^{1,-s}$, $s > 1$. It follows that $(1 - \tilde{R}_0(z)W)^{-1}$ exists for z near 0 and $z \notin \sigma(P_0)$. Then $(P_0 - z)^{-1}$ exists, and

$$R_0(z) = (1 - F(z))^{-1} \tilde{R}_0(z) \quad \text{with } F(z) = \tilde{R}_0(z) W. \tag{4.29}$$

We use this formula to get the asymptotic expansions of $R_0(z)$ and $\frac{d}{dz} R_0(z)$ for z near 0.

Proposition 4.10. *Suppose that $0 \notin \sigma_\infty$. The following asymptotic expansions hold for z near 0 with $\Im z > 0$.*

(a).

$$R_0(z) = \sum_{j=0}^{N} z^j R_j + \sum_{\{\vec{\nu}\} + [\vec{\nu}] \le N} z_\nu \sum_{j=[\vec{\nu}]_-}^{N-1} z^j R_{\vec{\nu},j} + R_0^{(N)}(z), \tag{4.30}$$

in $\mathcal{L}(-1, s; 1, -s)$, $s > 2N + 1$. Here

$$R_0 = AF_0; \qquad R_1 = AF_1A^*;$$
$$R_{\vec{\nu},0} = AG_{\nu_1,\delta_{\nu_1}} \pi_{\nu_1} W A G_{\nu_2,\delta_{\nu_2}} \pi_{\nu_2} W \cdots A G_{\nu_k,\delta_{\nu_k}} \pi_{\nu_k} A^*$$

for $\vec{\nu} = (\nu_1, \nu_2, \cdots, \nu_k)$ with $A = (1 - F_0 W)^{-1}$. R_j (resp. $R_{\vec{\nu},j}$) are in $\mathcal{L}(-1, s; 1, -s)$ for $s > 2j + 1$ (resp. for $s > 2j + \{\vec{\nu}\} + 1$), and $R_0^{(N)}(z) = O(|z|^{N+\epsilon})$ in $\mathcal{L}(-1, s; 1, -s)$, $s > 2N + 1$.

(b).

$$\frac{d}{dz} R_0(z) = \sum_{j=0}^{N} j z^{j-1} R_j + \sum_{\{\vec{\nu}\} + [\vec{\nu}] \le N} \frac{d}{dz} (z_{\vec{\nu}} \sum_{j=[\vec{\nu}]_-}^{N-1} z^j R_{\vec{\nu},j}) + O(|z|^{N-1+\epsilon}), \tag{4.31}$$

in $\mathcal{L}(0, s; 0, -s)$, $s > 2N + 1$, with some $\epsilon > 0$.

Proof. Since W is a bounded operator in $\mathcal{L}(1, -s; -1, s)$, $\forall s > 0$, thus by Theorem 3.2, one has for z near 0, $\Im z > 0$,

$$F(z) = \sum_{j=0}^{N} z^j F_j W + \sum_{\nu \in \sigma_N} z_\nu \sum_{j=[\nu]_-}^{N-1} z^j G_{\nu, j+\delta_\nu} \pi_\nu W + \tilde{R}_0^{(N)}(z) W \tag{4.32}$$

in $\mathcal{L}(1,-s;1,-s), s > 2N+1$. Since $(1-F_0W)^{-1}$ exists on $H^{1,-s}$, $s>1$, we can get the asymptotic expansion of $(1-F(z))^{-1}$. For z near 0, $\Im z > 0$,

$$(1-F(z))^{-1} = \sum_{j=0}^{N} z^j S_j + \sum_{\{\vec{\nu}\}+[\vec{\nu}]\le N} z_{\vec{\nu}} \sum_{j=[\vec{\nu}]_-}^{N-1} z^j S_{\vec{\nu},j} + O(|z|^{N+\epsilon})$$

in $\mathcal{L}(1,-s;1,-s), s>2N+1$. Here

$$S_0 = A \in \mathcal{L}(1,-s;1,-s), s>1;$$
$$S_{\vec{\nu},0} = AG_{\nu_1,\delta_{\nu_1}}\pi_{\nu_1}WAG_{\nu_2,\delta_{\nu_2}}\pi_{\nu_2}W\cdots AG_{\nu_k,\delta_{\nu_k}}\pi_{\nu_k}WA \in \mathcal{L}(1,-s;1,-s), s>2N+1$$

with $\vec{\nu}=(\nu_1,\nu_2,\cdots,\nu_k)$. Using (4.29) and the asymptotic expansion of $(1-F(z))^{-1}$ and $\tilde{R}_0(z)$, we can get (4.30).

(b). We only give the proof of (4.31) for $N=1$. The general case can be proved similarly. By Proposition 4.4, one has $\frac{d}{dz}\tilde{R}_0^{(1)}(z) = O(|z|^{\epsilon})$ in $\mathcal{L}(0,-s;0,s), s>3$. Take the derivative in both sides of (4.32), we drive that

$$\frac{d}{dz}F(z) = F_1W + \sum_{\nu\in\sigma_1}\frac{d}{dz}(z_\nu G_{\nu,\delta_\nu}\pi_\nu W) + \frac{d}{dz}\tilde{R}_0^{(1)}(z)W \tag{4.33}$$

in $\mathcal{L}(0,-s;0,-s), s>3$. It follows that $\frac{d}{dz}(1-F(z))^{-1} = (1-F(z))^{-1}\frac{d}{dz}F(z)(1-F(z))^{-1} \in \mathcal{L}(0,-s;0,-s), s>3$. Moreover,

$$\frac{d}{dz}(1-F(z))^{-1} = AF_1WA + \sum_{i=1}^{4} L_i(z) + O(|z|^{\epsilon}) \tag{4.34}$$

in $\mathcal{L}(0,-s;0,-s), s>3$. Here

$$L_1(z) = \sum_{\nu\in\sigma_1}\frac{d}{dz}z_\nu S_{\nu,0}; \quad L_2(z) = \sum_{\nu+\{\vec{\nu}_1\}+\{\vec{\nu}_2\}\le 1}(\frac{d}{dz}z_\nu)z_{\vec{\nu}_1}z_{\vec{\nu}_2}S_{(\vec{\nu}_1,\nu,\vec{\nu}_2),0};$$
$$L_3(z) = \sum_{\nu+\{\vec{\nu}\}\le 1}(\frac{d}{dz}z_\nu)z_{\vec{\nu}}S_{(\nu,\vec{\nu}),0}; \quad L_4(z) = \sum_{\nu+\{\vec{\nu}\}\le 1}(\frac{d}{dz}z_\nu)z_{\vec{\nu}}S_{(\vec{\nu},\nu),0}.$$

Thus

$$\frac{d}{dz}(1-F(z))^{-1}\tilde{R}_0(z) = AF_1WAF_0 + \sum_{i=1}^{5} I_i(z) + O(|z|^{\epsilon})$$

in $\mathcal{L}(0,s;0,-s), s>3$, where

$$I_1(z) = \sum_{\nu_1+\nu_2\le 1}(\frac{d}{dz}z_{\nu_1})z_{\nu_2}S_{\nu_1,0}G_{\nu_2,\delta_{\nu_2}}\pi_{\nu_2};$$

$$I_2(z) = \sum_{\nu_1+\{\vec{\nu}_1\}+\{\vec{\nu}_2\}+\nu_2\le 1}(\frac{d}{dz}z_{\nu_1})z_{\nu_2}z_{\vec{\nu}_1}z_{\vec{\nu}_2}S_{(\vec{\nu}_1,\nu,\vec{\nu}_2),0}G_{\nu_2,\delta_{\nu_2}}\pi_{\nu_2};$$

$$I_3(z) = \sum_{\nu_1+\{\vec{\nu}\}+\nu_2\le 1} (\frac{d}{dz} z_{\nu_1}) z_{\nu_2} z_{\vec{\nu}} S_{(\nu,\vec{\nu}),0} G_{\nu_2,\delta_{\nu_2}} \pi_{\nu_2};$$

$$I_4(z) = \sum_{\nu_1+\{\vec{\nu}\}+\nu_2\le 1} (\frac{d}{dz} z_{\nu_1}) z_{\nu_2} z_{\vec{\nu}} S_{(\vec{\nu},\nu),0} G_{\nu_2,\delta_{\nu_2}} \pi_{\nu_2}; \quad I_5(z) = \sum_{i=1}^{4} L_i(z) F_0.$$

Similarly, we can get that

$$(1-F(z))^{-1}\frac{d}{dz}\tilde{R}_0(z) = AF_1 + \sum_{\nu\in\sigma_1} \frac{d}{dz} z_\nu A G_{\nu,\delta_\nu}\pi_\nu + \sum_{\nu+\{\vec{\nu}\}\le 1} z_{\vec{\nu}} \frac{d}{dz} z_\nu S_{\vec{\nu},0} G_{\nu,\delta_\nu}\pi_\nu + O(|z|^\epsilon)$$

in $\mathcal{L}(0,s;0,-s)$, $s>3$. Using

$$\frac{d}{dz}R_0(z) = \frac{d}{dz}(1-F(z))^{-1}\tilde{R}_0(z) + (1-F(z))^{-1}\frac{d}{dz}\tilde{R}_0(z),$$

we can get the asymptotic expansion of $\frac{d}{dz}R_0(z)$ in $\mathcal{L}(0,s;0,-s)$, $s>3$. For $\vec{\nu}=(\nu_1,\cdots,\nu_k)$ and $\sum_{i=1}^k \nu_i \le 1$, let $\vec{\nu}_1, \vec{\nu}_{i,1}\vec{\nu}_{i,2}, (i=2,\cdots,k-1), \vec{\nu}_k$ satisfy

$$\vec{\nu} = (\nu_1,\vec{\nu}_1) = (\vec{\nu}_{i,1},\nu_i,\vec{\nu}_{i,2}) = (\vec{\nu}_k,\nu_k).$$

It is easy to check that

$$\sum_{\nu\in\sigma_1}\frac{d}{dz}z_\nu A G_{\nu,\delta_\nu}\pi_\nu + \sum_{\nu+\{\vec{\nu}\}\le 1} z_{\vec{\nu}}\frac{d}{dz}z_\nu S_{\vec{\nu},0} G_{\nu,\delta_\nu}\pi_\nu + \sum_{i=1}^{5} I_i(z) = \sum_{\{\vec{\nu}\}+[\vec{\nu}]\le 1} (I_{\vec{\nu},1}(z) + I_{\vec{\nu},2}(z)),$$

where

$$I_{\vec{\nu},1}(z) = (\frac{d}{dz}z_{\nu_1}) z_{\vec{\nu}_1} S_{\vec{\nu},0} F_0 + \sum_{i=2}^{k-1} (\frac{d}{dz} z_{\nu_i}) z_{\vec{\nu}_{i_1}} z_{\vec{\nu}_{i_2}} S_{\vec{\nu},0} F_0 + (\frac{d}{dz} z_{\nu_k})\, z_{\vec{\nu}_k} S_{\vec{\nu},0} F_0$$

$$I_{\vec{\nu},2}(z) = (\frac{d}{dz}z_{\nu_1}) z_{\vec{\nu}_1} S_{\vec{\nu}_k,0} G_{\nu_k,\delta_{\nu_k}}\pi_{\nu_k} + \sum_{i=2}^{k-1} (\frac{d}{dz} z_{\nu_i}) z_{\vec{\nu}_{i_1}} z_{\vec{\nu}_{i_2}} S_{\vec{\nu}_k,0} G_{\nu_k,\delta_{\nu_k}}\pi_{\nu_k}$$
$$+ (\frac{d}{dz} z_{\nu_k}) z_{\vec{\nu}_k} S_{\vec{\nu}_k,0} G_{\nu_k,\delta_{\nu_k}}\pi_{\nu_k}.$$

By a simple computation, one has

$$I_{\vec{\nu},1}(z) = \frac{d}{dz} z_{\vec{\nu}} S_{\vec{\nu},0} F_0,$$
$$I_{\vec{\nu},2}(z) = \frac{d}{dz} z_{\vec{\nu}} S_{\vec{\nu}_k,0} G_{\nu_k,\delta_{\nu_k}}\pi_{\nu_k}.$$

Note that $WAF_0 + 1 = A^*$ and $AF_1 + AF_1WAF_0 = AF_1A^*$. It follows that $I_{\vec{\nu},1}(z) + I_{\vec{\nu},2}(z) = \frac{d}{dz} z_{\vec{\nu}} R_{\vec{\nu},0}$. Then we prove (4.31) for $N=1$. The general case is the same. This ends the proof. □

4.4 Residues of the trace at zero

Let $f \in C_0^\infty(\mathbb{R})$ and $f(t) = 1$ for t near 0. Then $(R(z)-R_0(z))f(P)$ is in trace class for $z \notin \sigma(P)$ and $z \to T(z) = \mathrm{Tr}\,[(R(z) - R_0(z))f(P)]$ is meromorphic on $\mathbb{C}\backslash\mathbb{R}_+$. We want to calculate the residues of $T(z)$ at $z = 0$. First, we recall some results (see [55, IX.4] for details).

Definition 4.11. *Let A be a compact operator on a separable Hilbert space H. We say that $A \in \mathscr{S}_p(H)$, $1 \le p < \infty$, if $|A|^p$ is a trace class operator and set $\|A\|_p = (Tr\,|A|^p)^{1/p}$.*

Proposition 4.12. *([55, IX.4]) Let $1 \le p < \infty$ and $p^{-1} + q^{-1} = 1$.*
(a).If $A \in \mathscr{S}_p$ and $B \in \mathscr{S}_q$, then $AB \in \mathscr{S}_1$ and $\|AB\|_1 \le \|A\|_p \cdot \|B\|_q$.
(b). $\mathscr{S}_p$ is a Banach space with norm $\|\cdot\|_p$.
(c). $\mathscr{S}_1 \subset \mathscr{S}_p$.
(d).If $A \in \mathscr{S}_p$, then $A^* \in \mathscr{S}_p$ and $\|A^*\|_p = \|A\|_p$.

By Proposition 4.10, one has for z near 0, $\Im z \neq 0$,

$$R_0(z) = R_0 + R_0^{(0)}(z)$$

in $\mathcal{L}(-1, s; 1 - s)$, $s > 1$.

Lemma 4.13. *Let $m > n/2$, $s > 3$. For z near 0, $\Im z \neq 0$, $\langle x\rangle^{-s} R_0(z)\langle x\rangle^{-s} \in \mathscr{S}_m$ and there exists a constant C independent of z, such that*

$$\|\langle x\rangle^{-s} R_0(z)\langle x\rangle^{-s}\|_m \le C.$$

Moreover, $\langle x\rangle^{-s} R_0\langle x\rangle^{-s} \in \mathscr{S}_m$.

Proof. Let $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi(r) = 1$ for $|r| < 1$. Then $\langle x\rangle^{-s} R_0(z)\langle x\rangle^{-s}$ can be written as

$$\langle x\rangle^{-s} R_0(z)\langle x\rangle^{-s} = F_1(z) + F_2(z)$$

with

$$F_1(z) = \langle x\rangle^{-s} R_0(z)\chi(P_0)\langle x\rangle^{-s}; \quad F_2(z) = \langle x\rangle^{-s} R_0(z)(1 - \chi(P_0))\langle x\rangle^{-s}.$$

$F_2(z)$ can be decomposed as $F_2(z) = F_{21} + F_{22}(z)$, where

$$\begin{aligned} F_{21} &= \langle x\rangle^{-s} R_0(-1)(1 - \chi(P_0))\langle x\rangle^{-s}; \\ F_{22}(z) &= (1 + z)\langle x\rangle^{-s} R_0(z)\langle x\rangle^{-s'}\langle x\rangle^{s'} R_0(-1)(1 - \chi(P_0))\langle x\rangle^{-s}. \end{aligned}$$

Here $s' > 1$ is a constant very close to 1. It is easy to check F_{21} is in $\mathscr{S}_m$ and $\langle x\rangle^{s'} R_0(-1)(1 - \chi(P_0))\langle x\rangle^{-s}$ is in $\mathscr{S}_m$ for some $s' > 1$. By Proposition 4.10, one has $\langle x\rangle^{-s} R_0(z)\langle x\rangle^{-s'}$ is uniformly bounded for z near 0, $\Im z \neq 0$. Then we deduce that $F_2(z) \in \mathscr{S}_m$, and $\|F_2(z)\|_m \le C$ with some constant C independent of z. Moreover from the above argument, we can see that $\lim_{z\to 0} F_2(z)$

exists in $\mathscr{S}_m$. One has $F_1(z) = \langle x\rangle^{-s}R_0(z)\langle x\rangle^{-s'}\langle x\rangle^{s'}\chi(P_0)\langle x\rangle^{-s}$ for $s' > 1$. Using the similar argument as above, we can get that $\|F_1(z)\|_m \leq C$ with some constant C independent of z, and $\lim_{z\to 0} F_1(z)$ exists in $\mathscr{S}_m$. Therefore $\langle x\rangle^{-s}R_0(z)\langle x\rangle^{-s} \in \mathscr{S}_m$, $\langle x\rangle^{-s}R_0\langle x\rangle^{-s} \in \mathscr{S}_m$ and

$$\|\langle x\rangle^{-s}R_0(z)\langle x\rangle^{-s}\|_m \leq \|F_1(z)\|_m + \|F_2(z)\|_m \leq C$$

for some C independent of z. □

Note that if $u \in \mathcal{N}$, then $\tilde{u} \equiv |V|^{1/2}u$ satisfies

$$\tilde{u} + \mathrm{sgn}V\ |V|^{1/2}AF_0|V|^{1/2}\tilde{u} = 0. \tag{4.35}$$

If $u \in \mathcal{N}$, then $(\tilde{P}_0 + \tilde{V})u = 0$ with $A = (1 - F_0W)^{-1}$. It follows that $(\tilde{P}_0 + A^*V)u = 0$. Then we get

$$A^*Vu = \tilde{V}u. \tag{4.36}$$

Theorem 4.14. *Assume $0 \notin \sigma_\infty$ and $\rho > \max\{6, n+2\}$. Suppose f satisfies the condition of Theorem 4.1, then the residue of $T(z) = Tr\,[(R(z) - R_0(z))f(P)]$ at 0 is given by*

$$J_0 = N_0 + \sum_{j=1}^{k_0} \varsigma_j m_j \tag{4.37}$$

where N_0 is the multiplicity of zero as the eigenvalue of P and m_j the multiplicity of ς_j-resonance of zero.

Proof. Let $k \in \mathbb{N}$ with $k > \frac{n}{2} - 1$. We decompose $T(z) = -\mathrm{Tr}\,[R_0(z)VR(z)f(P)]$ as

$$T(z) = T_1(z) + T_2(z), \tag{4.38}$$

where

$$T_1(z) = -\mathrm{Tr}\,[R_0(z)Vf(P)(R_0(z) + \sum_{j=1}^{k-1}(-1)^j(R_0(z)V)^jR_0(z))] \tag{4.39}$$

$$T_2(z) = (-1)^{k+1}\mathrm{Tr}\,[R_0(z)Vf(P)R(z)(VR_0(z))^k)]. \tag{4.40}$$

One has

$$\mathrm{Tr}\,R_0(z)Vf(P)R_0(z) = -\mathrm{Tr}\langle x\rangle^{-s}\frac{d}{dz}R_0(z)\langle x\rangle^{-s}\langle x\rangle^{s}Vf(P)\langle x\rangle^{s}$$

$s > 1$. By Proposition 4.10, we can deduce that if $\rho > n + 2$,

$$\mathrm{Tr}\,R_0(z)Vf(P)R_0(z) = O(|z|^{-1+\epsilon})$$

with some $\epsilon > 0$. Similarly, we can get that if $\rho > n + 2$, the other terms of $T_1(z)$ are $O(|z|^{-1+\epsilon})$. It follows that $T_1(z) = O(|z|^{-1+\epsilon})$. Thus the residue of $T_1(z)$ at 0 is zero. Let

$$T_3(z) = (-1)^{k+1}\mathrm{Tr}\,[R_0(z)VR(z)(VR_0(z))^k)].$$

Then

$$\begin{aligned} T_2(z) - T_3(z) &= (-1)^{k+1}\mathrm{Tr}\,[R_0(z)VR(z)(1-f(P))(VR_0(z))^k)] \\ &= (-1)^k\mathrm{Tr}\,[\langle x\rangle^{-s}\frac{d}{dz}R_0(z)\langle x\rangle^{-s}\langle x\rangle^s VR(z)(1-f(P))(VR_0(z))^{k-1})V\langle x\rangle^s] \end{aligned}$$

for $s > 1$. Since $1 - f(t)$ is equal to 0 for t near 0, $R(z)(1 - f(P))$ is continuous in weighted Sobolev spaces. By Proposition 4.10, and Lemma 4.13, we conclude that, if $k > \frac{n}{2} + 1$,

$$T_2(z) - T_3(z) = O(|z|^{-1+\epsilon}),$$

for some $\epsilon > 0$, for z near 0 and $\Im z \neq 0$. Therefore the residue of $T_2(z) - T_3(z)$ at 0 is zero. Now, we apply Proposition 4.10, Theorem 3.19 and 3.29 to compute the residues of $T_3(z)$.

First, we assume that 0 is not an eigenvalue of P. To begin with, remark that

$$T_3(z) = (-1)^k\mathrm{Tr}\,[(\frac{d}{dz}R_0(z))VR(z)V(R_0(z)V)^{k-1}].$$

By Proposition 4.10, one has

$$R_0(z) = R_0 + R_0^{(0)}(z) \tag{4.41}$$

in $\mathcal{L}(-1, s; 1, -s)$, $s > 1$, and

$$\frac{d}{dz}R_0(z) = R_1 + \sum_{\{\vec{\nu}\}+[\vec{\nu}]\leq 1}\frac{d}{dz}z_{\vec{\nu},0}R_{\vec{\nu},0} + O(|z|^{-1+\epsilon}) \tag{4.42}$$

in $\mathcal{L}(0, s; 0, -s)$, $s > 3$. Denote

$$sgn(V)|V|^{-1/2} = U_1; \quad |V|^{-1/2} = U_2; \quad sgn(V)|V|^{-1/2}R_0|V|^{-1/2} = \tilde{R}_0.$$

Let

$$S_1(z) = U_1\frac{d}{dz}R_0(z)U_2; \quad S_2(z) = U_1R(z)U_2; \quad S_3(z) = U_1R_0(z)U_2.$$

Then

$$T_3(z) = (-1)^k\mathrm{Tr}\,S_1(z)S_2(z)S_3^{k-1}(z).$$

By Lemma 4.13, one has that, if $k > \frac{n}{2} + 1$, $S_3^{k-1}(z)$ is of trace class, and $\|S_3^{k-1}(z)\|_1 = O(1)$ for z near 0 with $\Im z \neq 0$. By Proposition 4.10, one has

$$S_1(z) = S_{11}(z) + S_{12}(z) \quad \text{in } L^2(\mathbb{R}^n)$$

with

$$S_{11}(z) = U_1R_1U_2 + \sum_{[\vec{\nu}]+\{\vec{\nu}\}\leq 1}\frac{d}{dz}z_{\vec{\nu}}U_1R_{\vec{\nu},0}U_2; \qquad S_{12}(z) = O(|z|^\epsilon)$$

in $L^2(\mathbb{R}^n)$ for some $\epsilon > 0$. By Theorem 3.29, we can decompose $S_2(z)$ as

$$S_2(z) = S_{21}(z) + S_{22}(z) + S_{23}(z) \quad \text{in } L^2(\mathbb{R}^n)$$

with

$$S_{21}(z) = U_1 \sum_{j=1}^{k_0} z_{\varsigma_j}^{-1} \Pi_{r,j} U_2 \qquad S_{23}(z) = O(|z|^\epsilon),$$
$$S_{22}(z) = \sum_{j=1}^{k_0} \sum_{\alpha,\beta,\vec{\nu},l}^{+,1} z_{\varsigma_j}^{-1} z_{\vec{\nu}} z^{|\beta|} (z_{\vec{\varsigma}})^{-\alpha-\beta} z^l U_1 T_{r;\vec{\nu},\alpha,\beta,l,j} U_2$$

in $L^2(\mathbb{R}^n)$, for some $\epsilon > 0$. It follows

$$T_3(z) = T_{31}(z) + T_{32}(z) + T_{33}(z)$$

where

$$T_{31}(z) = (-1)^k \mathrm{Tr}\, S_{11}(z) S_{21}(z) S_3^{k-1}(z); \quad T_{32}(z) = (-1)^k \mathrm{Tr}\, S_{11}(z) S_{22}(z) S_3^{k-1}(z);$$
$$T_{33}(z) = (-1)^k \mathrm{Tr}\, [(S_{11}(z) S_{23}(z) + S_{12}(z) S_{21}(z) + S_{12}(z) S_{22}(z) + S_{12}(z) S_{23}(z)) S_3^{k-1}(z)].$$

Using the fact $\|S_3^{k-1}(z)\|_1 = O(1)$ for z near 0 and $\Im z \neq 0$, $k > \frac{n}{2} + 1$, it is easy to see that $T_{33}(z) = O(|z|^{-1+\epsilon})$ for some $\epsilon > 0$. For $k > \frac{n}{2} + 1$, we have

$$\begin{aligned}
& (-1)^k T_{31}(z) \\
= \;& \mathrm{Tr}\; [U_1 \sum_{[\vec{\nu}]+\{\vec{\nu}\}\leq 1} \frac{d}{dz} z_{\vec{\nu}} R_{\vec{\nu},0} U_2 \sum_{j=1}^{k_0} z_{\varsigma_j}^{-1} U_1 \Pi_{r,j} U_2 (S_3(z))^{k-1}] + O(|z|^{-1+\epsilon}) \\
= \;& \sum_{[\vec{\nu}]+\{\vec{\nu}\}\leq 1, j, l} z_{\varsigma_j}^{-1} e^{i\pi\varsigma_j} \frac{d}{dz} z_{\vec{\nu}} \mathrm{Tr}\; [U_1 R_{\vec{\nu},0} U_2 \langle U_2 (S_3(z))^{k-1} \cdot, u_j^{(l)} \rangle U_1 u_j^{(l)}] + O(|z|^{-1+\epsilon}) \quad (4.43) \\
= \;& \sum_{[\vec{\nu}]+\{\vec{\nu}\}\leq 1, j, l} z_{\varsigma_j}^{-1} e^{i\pi\varsigma_j} \frac{d}{dz} z_{\vec{\nu}} \langle U_1 R_{\vec{\nu},0} U_2 U_1 u_j^{(l)}, (S_3^*(z))^{k-1} U_2 u_j^{(l)} \rangle + O(|z|^{-1+\epsilon}).
\end{aligned}$$

Since $(S_3^*(z))^{k-1} = \tilde{R}_0^{k-1} + O(|z|^{-1+\epsilon})$ in $L^2(\mathbb{R}^n)$ and $\tilde{R}_0^{k-1} U_2 u_j^{(l)} = (-1)^{k-1} U_2 u_j^{(l)}$ by (4.35), one has

$$T_{31}(z) = - \sum_{[\vec{\nu}]+\{\vec{\nu}\}\leq 1, j, l} z_{\varsigma_j}^{-1} e^{i\pi\varsigma_j} \frac{d}{dz} z_{\vec{\nu}} \langle U_1 R_{\vec{\nu},0} U_2 U_1 u_j^{(l)}, U_2 u_j^{(l)} \rangle + O(|z|^{-1+\epsilon}).$$

For $\vec{\nu} = (\nu_1, \ldots, \nu_k)$ and $\sum_{i=1}^{k} \nu_i > \varsigma_j$, one has $z_{\varsigma_j}^{-1} \frac{d}{dz} z_{\vec{\nu}} = O(|z|^{-1+\epsilon})$ for some $\epsilon > 0$. For $\sum_{i=1}^{k} \nu_i \leq \varsigma_j$, if $\nu_k = \varsigma_j$, then $\vec{\nu} = \varsigma_j$. We use the normalization condition of resonant states given in Theorem

3.19 to calculate

$$\sum_{l=1}^{m_j}\langle R_{\vec{\nu},0}Vu_j^{(l)}, Vu_j^{(l)}\rangle = \sum_{l=1}^{m_j} c_{\varsigma_j}|\langle A^*Vu_j^{(l)}, -|y|^{-\frac{n-2}{2}+\varsigma_j}\varphi_{\varsigma_j}^{(l')}\rangle|^2$$
$$= \sum_{l=1}^{m_j} c_{\varsigma_j}|\langle \tilde{V}u_j^{(l)}, -|y|^{-\frac{n-2}{2}+\varsigma_j}\varphi_{\varsigma_j}^{(l')}\rangle|^2 = -e^{-i\pi\varsigma_j}m_j. \tag{4.44}$$

In the second step, we use (4.36), and in the last step, we use (3.50). If $\nu_k < \varsigma_j$, one has

$$\langle R_{\vec{\nu},0}Vu_j^{(l)}, Vu_j^{(l)}\rangle = \langle B_\nu G_{\nu_k,\delta_{\nu_k}}\pi_{\nu_k}A^*Vu_j^{(l)}, A^*Vu_j^{(l)}\rangle$$
$$= \sum_{l'} c_{\nu_k}\langle \tilde{V}u_j^{(l)}, -|y|^{-\frac{n-2}{2}+\nu_k}\varphi_{\nu_k}^{(l')}\rangle\overline{\langle \tilde{B}_\nu^* Vu_j^{(l)}, -|y|^{-\frac{n-2}{2}+\nu_k}\varphi_{\nu_k}^{(l')}\rangle} = 0. \tag{4.45}$$

Here $B_\nu \in \mathcal{L}(-1,s;1,-s)$, $s>3$. In the last step, we use that $u_j^{(l)}$ is ς_j-resonant state and (3.50). Trivially,

$$z_{\varsigma_j}^{-1}\frac{d}{dz}z_{\varsigma_j} = \begin{cases} \dfrac{\varsigma_j}{z}, & \text{if } \varsigma_j \in (0,1), \\ \dfrac{1}{z}+\dfrac{1}{z\ln z}, & \text{if } \varsigma_j = 1. \end{cases}$$

Summing up, we proved that

$$T_{31}(z) = \frac{1}{z}\sum_j \varsigma_j m_j + \frac{m}{z\ln z} + O(|z|^{-1+\epsilon}), \tag{4.47}$$

with m the multiplicity of 1−resonance of zero, as $z\to 0$. For $T_{32}(z)$, from the proof of Theorem 3.29, we know that $T_{r;\vec{\nu}_1,\alpha,\beta,l,j}$ has the form

$$\Pi_{r,j}B_{r;\vec{\nu}_1,\alpha,\beta,l,j} \quad \text{or} \quad A_{r;\vec{\nu}_1,\alpha,\beta,l,j}\Pi_{r,j}\tilde{V}G_{\nu,\delta_\nu}\pi_\nu B_{r;\vec{\nu}_1,\alpha,\beta,l,j}.$$

Here $A_{r;\vec{\nu}_1,\alpha,\beta,l,j}$ is a bounded operator in $\mathcal{L}(1,-s;1,-s)$, $s>3$ and $B_{r;\vec{\nu}_1,\alpha,\beta,l,j}$ is a bounded operator in $\mathcal{L}(-1,s;-1,s)$ for $s>3$ and $G_{\nu,\delta_\nu}\pi_\nu B_{r;\vec{\nu}_1,\alpha,\beta,l,j}$ comes from the expansion of $L_1(z)V(D_0+D_1(z))\tilde{R}_0(z)$. Note that the summation $\sum_{\alpha,\beta,\vec{\nu}_1,l}^{+,1}$ is taken over all possible $\alpha,\beta\in\mathbb{N}^{\kappa_0}$ with $1\le|\alpha|\le N_0$, $|\beta|\ge 1$, $\vec{\nu}_1=(\nu_1,\cdots,\nu_{k'})\in\sigma_1^{k'}$, $k'\ge 2|\alpha|$, for which there are at least α_k values of ν_j's belonging to σ_1 with $\nu_j\ge\varsigma_k$, for $1\le k\le\kappa_0$ $l\in\mathbb{N}$, satisfying

$$|\beta|+\{\vec{\nu}_1\}+l-\sum_{k=1}^{\kappa_0}(\alpha_k+\beta_k)\varsigma_k \le 1.$$

It follows that $z^{|\beta|}(z_{\vec{\varsigma}})^{-\beta} = O(\frac{1}{|\ln z|^\beta})$ and $z_{\vec{\nu}_1}(z_{\vec{\varsigma}})^{-\alpha} = O(|z|^\epsilon)$ for some $\epsilon>0$. For $\vec{\nu}=(\nu_1,\ldots,\nu_k)$ and $\sum_{i=1}^k \nu_i \ge \varsigma_j$, one has

$$\sum_{[\vec{\nu}]+\{\vec{\nu}\}\le 1,j,l}\ \sum_{\alpha,\beta,\vec{\nu}_1,l}^{+,1} z_{\varsigma_j}^{-1}z_{\vec{\nu}_1}z^{|\beta|}(z_{\vec{\varsigma}})^{-\alpha-\beta}z^l\frac{d}{dz}z_{\vec{\nu}} = O(|z|^{-1+\epsilon})$$

for some $\epsilon > 0$. When $\sum_{i=1}^{k} \nu_i < \varsigma_j$, if $T_{r;\vec{\nu},\alpha,\beta,l,j} = \Pi_{r,j} B_{r;\vec{\nu},\alpha,\beta,l,j}$, using the same computation made for $T_{31}(z)$, we can get

$$\mathrm{Tr}\,[U_1 R_{\vec{\nu},0} U_2 U_1 T_{r;\vec{\nu}_1,\alpha,\beta,l,j} U_2] = \langle U_1 R_{\vec{\nu},0} U_2 U_1 u_j^{(l)}, (S_3^*(z))^{k-1} U_2 B^*_{r;\vec{\nu}_1,\alpha,\beta,l,j} u_j^{(l)} \rangle.$$

Since $u_j^{(l)}$ is ς_j-resonant state and $\nu_k < \varsigma_j$, as (4.45), one has

$$\begin{aligned} & \langle U_1 R_{\vec{\nu},0} U_2 U_1 u_j^{(l)}, (S_3^*(z))^{k-1} U_2 B^*_{r;\vec{\nu}_1,\alpha,\beta,l,j} u_j^{(l)} \rangle \\ = & \sum_{l'} \langle c_{\nu_k} \tilde{V} u_j^{(l)}, -|y|^{-\frac{n-2}{2}+\nu_k} \varphi_{\nu_k}^{(l')} \rangle \overline{\langle B u_j^{(l)}, -|y|^{-\frac{n-2}{2}+\nu_k} \varphi_{\nu_k}^{(l')} \rangle} = 0. \end{aligned}$$

Here B is a operator belongs to $\mathcal{L}(1,-s,-1,s)$, $s > 3$. If

$$T_{r;\vec{\nu}_1,\alpha,\beta,l,j} = A_{r;\vec{\nu}_1,\alpha,\beta,l,j} \Pi_{r,j} \tilde{V} G_{\nu,\delta_\nu} \pi_\nu B_{r;\vec{\nu}_1,\alpha,\beta,l,j},$$

since $G_{\nu,\delta_\nu}\pi_\nu\ B_{r;\vec{\nu}_1,\alpha,\beta,l,j}$ is from the expansion of $L_1(z)\tilde{V}(D_0 + D_1(z))\tilde{R}_0(z)$, the coefficient of $G_{\nu,\delta_\nu}\pi_\nu\ B_{r;\vec{\nu}_1,\alpha,\beta,l,j}$ is $O(z^\nu)$. Note $A_{r;\vec{\nu}_1,\alpha,\beta,l,j}\Pi_{r,j}\tilde{V}$ comes from the the expansion of $(1 - (D_0 + D_1(z))L_1(z)\tilde{V})I_r(z)Q_r$. From the proof of Theorem 3.29, we can see that the coefficient of $A_{r;\vec{\nu}_1,\alpha,\beta,l,j}\Pi_{r,j}\tilde{V}$ is $o(|z|^{-\varsigma_j+\epsilon})$. It follows that the coefficient of $T_{r;\vec{\nu}_1,\alpha,\beta,l,j}$ is $O(|z|^{\epsilon+\nu-\varsigma_j})$ with some $\epsilon > 0$. By Theorem 3.19 and (3.50), $\Pi_{r,j}\tilde{V}G_{\nu,\delta_\nu}\pi_\nu = 0$ if $\nu < \varsigma_j$. If $\nu \geq \varsigma_j$, one has

$$\sum_{[\vec{\nu}]+\{\vec{\nu}\}\leq 1,j,l} \sum_{\alpha,\beta,\vec{\nu}_1,l}^{+,1} z_{\varsigma_j}^{-1} z_{\vec{\nu}_1} z^{|\beta|} (z_{\vec{\varsigma}})^{-\alpha-\beta} z^l \frac{d}{dz} z_{\vec{\nu}} = O(|z|^{-1+\epsilon}).$$

Therefore $T_{32}(z) = O(|z|^{-1+\epsilon})$. Note that

$$\frac{1}{2\pi i} \lim_{\epsilon\to 0} \lim_{\delta\to 0} \int_{|z|=\epsilon, |\Im z|\geq\delta} \frac{1}{z \ln z}\, dz = 0.$$

Therefore we drive (4.37) when 0 is not the eigenvalue of P.

From now on, we assume that 0 is the eigenvalue of P. $S_1(z), S_2(z), S_3(z), S_{11}(z), S_{12}(z)$ are the same as before. $S_2(z)$ can be decomposed as

$$S_2(z) = S_{21}(z) + S_{22}(z) + S_{23}(z) + S_{24}(z) + S_{25}(z) \quad \text{in } L^2(\mathbb{R}^n)$$

with

$$\begin{aligned} S_{21}(z) &= \sum_{j=1}^{k_0} z_{\varsigma_j}^{-1} U_1 \Pi_{r,j} U_2 + \sum_{j=1}^{k_0} \sum_{\alpha,\beta,\vec{\nu},l}^{+,1} z_{\varsigma_j}^{-1} z_{\vec{\nu}} z^{|\beta|} (z_{\vec{\varsigma}})^{-\alpha-\beta} z^l U_1 T_{r;\vec{\nu},\alpha,\beta,l,j} U_2 \\ S_{22}(z) &= -z^{-1} U_1 \Pi_0 U_2, \quad S_{23}(z) = \sum_{\{\vec{\nu}\}\leq 1}^{(1)} z_{\vec{\nu}} z^{-1} U_1 T_{e;\vec{\nu},j} U_2 \\ S_{24}(z) &= U_1 \sum_{j=1}^{\kappa_0} z_{\varsigma_j}^{-1} (\Pi_0 \tilde{V} Q_e F_1 \tilde{V} \Pi_{r,j} + \Pi_{r,j} \tilde{V} Q_r F_1 \tilde{V} \Pi_0 + \sum_{\alpha,\beta,\vec{\nu},l}^{+,1} z_{\vec{\nu}} z^{|\beta|} (z_{\vec{\varsigma}})^{-\alpha-\beta} z^l T_{er;\vec{\nu},\alpha,\beta,l,j}) U_2, \\ S_{25}(z) &= O(|z|^\epsilon) \end{aligned}$$

in $L^2(\mathbb{R}^n)$, for some $\epsilon > 0$. It follows

$$T_3(z) = T_{31}(z) + T_{32}(z) + T_{33}(z) + T_{34}(z) + T_{35}(z)$$

where

$$\begin{aligned}
&T_{31}(z) = (-1)^k \mathrm{Tr}\, S_{11}(z) S_{21}(z) S_3^{k-1}(z); \quad T_{32}(z) = (-1)^k \mathrm{Tr}\, S_{11}(z) S_{22}(z) S_3^{k-1}(z);\\
&T_{33}(z) = (-1)^k \mathrm{Tr}\, S_{11}(z) S_{23}(z) S_3^{k-1}(z); \quad T_{34}(z) = (-1)^k \mathrm{Tr}\, S_{11}(z) S_{24}(z) S_3^{k-1}(z);\\
&T_{35}(z) = (-1)^k \mathrm{Tr}\, [(S_{11}(z) S_{25}(z) + S_{12}(z) S_{21}(z) + S_{12}(z) S_{22}(z)\\
&\qquad\qquad + S_{12}(z) S_{23}(z) + S_{12}(z) S_{24}(z) + S_{12}(z) S_{25}(z)) S_3^{k-1}(z)].
\end{aligned}$$

Since $\|S_3^{k-1}(z)\|_1 = O(1)$ for z near 0 with $\Im z \neq 0$, $k > \frac{n}{2} + 1$, we can conclude that $T_{35}(z) = O(|z|^{-1+\epsilon})$ for some $\epsilon > 0$. Let $\{\phi_j; j = 1, \cdots, N_0\}$ be the basis of eigenspace of P associated with the eigenvalue 0. One has

$$(-1)^k T_{32}(z) \quad = \quad I_1(z) + I_2(z)$$

with

$$\begin{aligned}
I_1(z) &= -\sum_{[\vec{\nu}]+\{\vec{\nu}\}\leq 1} z^{-1} \frac{d}{dz} z_{\vec{\nu}} \mathrm{Tr}\ [U_1 R_{\vec{\nu},0} U_2 \sum_{j=1}^{k_0} U_1 \Pi_0 U_2 (S_3(z))^{k-1}]\\
I_2(z) &= -z^{-1} \mathrm{Tr}\ [U_1 R_1 U_2 U_1 \Pi_0 U_2 (S_3(z))^{k-1}]
\end{aligned}$$

By the similar computation as (4.43), we can get that

$$I_1(z) = -\sum_{j=1}^{N_0} \sum_{[\vec{\nu}]+\{\vec{\nu}\}\leq 1, j, l} \frac{d}{dz} z_{\vec{\nu}} z^{-1} \langle U_1 R_{\vec{\nu},0} U_2 U_1 \phi_j, (S_3^*(z))^{k-1} U_2 \phi_j \rangle.$$

We drive that $I_1(z) = 0$ using the similar argument as (4.45), since $\phi_j \in L^2(\mathbb{R}^n)$. One has

$$\mathrm{Tr}\ [U_1 R_1 U_2 U_1 \Pi_0 U_2 (S_3(z))^{k-1}] = -\langle U_1 R_1 U_2 U_1 \phi_j, U_2 \phi_j \rangle + O(|z|^{\epsilon})$$

and

$$\langle U_1 R_1 U_2 U_1 \phi_j, U_2 \phi_j \rangle = \langle U_1 R_1 U_2 U_1 \phi_j, U_2 \phi_j \rangle = \langle F_1 \tilde{V} \phi_j, \tilde{V} \phi_j \rangle.$$

In the last step, we use (4.36). It is easy to check that $\langle F_1 \tilde{V} \phi_i, \tilde{V} \phi_j \rangle = \delta_{ij}$. It follows that

$$I_2(z) = (-1)^k \frac{N_0}{z} + O(|z|^{-1+\epsilon}).$$

Therefore the residues of $T_{32}(z)$ is N_0. Note $T_{e;\vec{\nu};j} = \Pi_0 A_{e;\vec{\nu};j}$ with $A_{e;\vec{\nu};j}$ be a bounded operator in $\mathcal{L}(-1, s; -1, s)$, $s > 3$, and $z_{\vec{\nu}} z^{-1} = O(|z|^{-1+\epsilon})$ for some $\epsilon > 0$. Making use a similar computation as $T_{32}(z)$, we can get $T_{33} = O(|z|^{-1+\epsilon})$. Decompose $T_{34}(z)$

$$T_{34}(z) = J_1(z) + J_2(z) + J_3(z)$$

where

$$J_1(z) = \mathrm{Tr}\,[U_1 \sum_{[\vec{\nu}]+\{\vec{\nu}\}\le 1} \frac{d}{dz} z_{\vec{\nu}} R_{\vec{\nu},0} U_2 U_1 \sum_{j=1}^{\kappa_0} z_{\varsigma_j}^{-1} \Pi_0 \tilde{V} Q_e F_1 \tilde{V} \Pi_{r,j} U_2 (S_3(z))^{k-1}]$$

$$J_2(z) = \mathrm{Tr}\,[U_1 \sum_{[\vec{\nu}]+\{\vec{\nu}\}\le 1} \frac{d}{dz} z_{\vec{\nu}} R_{\vec{\nu},0} U_2 U_1 \sum_{j=1}^{\kappa_0} z_{\varsigma_j}^{-1} \Pi_{r,j} \tilde{V} Q_r F_1 \tilde{V} \Pi_0 U_2 (S_3(z))^{k-1}]$$

$$J_3(z) = \mathrm{Tr}\,[U_1 \sum_{[\vec{\nu}]+\{\vec{\nu}\}\le 1} \frac{d}{dz} z_{\vec{\nu}} R_{\vec{\nu},0} U_2 U_1 \sum_{j=1}^{\kappa_0} z_{\varsigma_j}^{-1} \sum_{\alpha,\beta,\vec{\nu}_1,l}^{+,1} z_{\vec{\nu}} z^{|\beta|} (z_{\vec{\varsigma}})^{-\alpha-\beta} z^l T_{er;\vec{\nu}_1,\alpha,\beta,l,j} U_2 (S_3(z))^{k-1}]$$

We only computer $J_1(z)$. $J_2(z)$, $J_3(z)$ can be computed in a similar way. By a similar computation as (4.43), we can get that

$$J_1(z) = \sum_{j=1}^{N_0} \sum_{[\vec{\nu}]+\{\vec{\nu}\}\le 1, j, l} \frac{d}{dz} z_{\vec{\nu}} z_{\varsigma_j}^{-1} \langle U_1 R_{\vec{\nu},0} U_2 U_1 \phi_j, (S_3^*(z))^{k-1} U_2 P_{r,j} \tilde{V} F_1 Q_e \tilde{V} \phi_j \rangle.$$

Note that $\phi_j \in L^2(\mathbb{R}^n)$. We can get that $J_1(z) = 0$ as (4.45). Similarly, we can conclude that $J_2(z) = O(|z|^{-1+\epsilon})$ and $J_3(z) = O(|z|^{-1+\epsilon})$. $T_{31}(z)$ has been computed in the case 0 is not the eigenvalue of P. This ends the proof. □

4.5 Levinson's theorm

In this section, we use Theorem 4.1 to prove Levinson's theorem. First, we should verify that the conditions (4.6) and (4.10) hold. We start this section with recalling a result.

Theorem 4.15. *(Theorem 2.2 [25]) Assume that $n \ge 3$, $V \in L^{n/2}_{loc}$ and there exists some $q \in [\frac{n}{2}, \infty]$ such that*

$$\lim_{R\to\infty} R^{\beta(q)} \|V\|_{L^q(\{x;|x|\in[R,2R]\})} = 0 \tag{4.48}$$

Here $\beta(q) = (2q-n)/(2q)$. Assume that u belongs to the Sobolev space $W^{2,1}_{loc}$ satisfies the decay

$$(1+|x|)^{-1/2+\delta_0} u(x) \in L^2$$

for some $\delta_0 > 0$. If $-\Delta u + Vu = Eu$ for some $E > 0$, then $u \equiv 0$.

Lemma 4.16. *(a). $f \in C_0^\infty(\mathbb{R})$. Let $F(z) = Tr\,[(R(z) - R_0(z))f(P)]$. Then, if $\rho > n+1$,*

$$|F(z)| \le C_f \frac{1}{|z|^{1+\epsilon}}, \quad \text{if } |z| > R_0, \quad \Im z \ne 0.$$

C_f is the constant independent of z. For any $\delta > 0$, if $\rho > n+3$, $\lim_{\epsilon\downarrow 0} Tr[(R(\lambda\pm i\epsilon) - R_0(\lambda\pm i\epsilon))f(P)]$ exists if $\lambda > \delta$. Moreover, there exists $C_{\delta,f} > 0$ such that

$$|\lim_{\epsilon\downarrow 0} Tr\,[(R(\lambda \pm i\epsilon) - R_0(\lambda \pm i\epsilon))f(P)]| \le C_{\delta,f}$$

uniformly hold for λ with $\lambda > \delta$.

(b).Assume $0 \notin \sigma_\infty$ and $n \geq 3$. The total number, N, of negative eigenvalues of P is finite.

(c). $(P-i)^{-k} - (P_0 - i)^{-k}$ is of trace class, if $k > n/2 - 1$ and $\rho > n$.

(d). $\sigma_{ac}(P_0) = \sigma_{ac}(P) = [0, +\infty[$.

(e). $\xi'(\lambda) \in L^1_{loc}(]0, \infty[)$.

Proof. (a). Let f_1 be the smooth function such that $f_1 f = f$. For $z \notin \sigma(P) \cup \sigma(P_0)$,

$$R_0(z) - R(z) = R_0(z)VR(z),$$

one has

$$\begin{aligned} F(z) &= -\mathrm{Tr}\,[R_0(z)VR(z)f(P)] \\ &= -\mathrm{Tr}\,[\langle x\rangle^{-s}R_0(z)\langle x\rangle^{-s} \cdot \langle x\rangle^{s}Vf(P)\langle x\rangle^{s} \cdot \langle x\rangle^{-s}f_1(P)R(z)\langle x\rangle^{s}] \end{aligned}$$

For any $s > 1/2$, one has $\langle x\rangle^{-s}R_0(z)\langle x\rangle^{-s} = O(|z|^{-1/2})$ ([51]) for $|z|$ large, $\Im z \neq 0$. $\langle x\rangle^{s}Vf(P)\langle x\rangle^{s}$ is a trace class operator, if $s < \frac{\rho-n}{2}$. Note that the principal symbol of $\langle x\rangle^{-s}f_1(P)R(z)\langle x\rangle^{s}$ is $\frac{f(|\xi|^2+V_1+V_2)}{|\xi|^2+V_1+V_2-z}$, and f_1 is a smooth function with compact support. By Calderon-Vaillancourt theorem, one has that $\|\langle x\rangle^{-s}f_1(P)R(z)\langle x\rangle^{s}\| = O(|z|^{-1})$ for $|z|$ large enough. Choose $1/2 < s < \frac{\rho-n}{2}$, then

$$|F(z)| \leq \|\langle x\rangle^{-s}R_0(z)\langle x\rangle^{-s}\| \cdot \|\langle x\rangle^{s}Vf(P)\langle x\rangle^{s}\|_1 \cdot \|\langle x\rangle^{-s}f_1(P)R(z)\langle x\rangle^{s}\| \leq C_f \frac{1}{|z|^{3/2}}.$$

For any $\lambda > \delta$, $F(\lambda + i\epsilon) = F_1(\lambda + i\epsilon) + F_2(\lambda + i\epsilon)$ with

$$\begin{aligned} F_1(\lambda + i\epsilon) &= -Tr(R_0(\lambda + i\epsilon)Vf(P)R_0(\lambda + i\epsilon)); \\ F_2(\lambda + i\epsilon) &= Tr(R_0(\lambda + i\epsilon)Vf(P)R(\lambda + i\epsilon)VR_0(\lambda + i\epsilon)). \end{aligned}$$

One has

$$F_1(\lambda + i\epsilon) = -Tr(\langle x\rangle^{-s}R_0^2(\lambda + i\epsilon)\langle x\rangle^{-s} \cdot \langle x\rangle^{s}Vf(P)\langle x\rangle^{s}).$$

If $s > 3/2$, there exists some $C_\delta > 0$ such that $\|\lim_{\epsilon\downarrow 0}\langle x\rangle^{-s}R_0^2(\lambda + i\epsilon)\langle x\rangle^{-s}\| \leq C_\delta$ for $\lambda > \delta$. Since $\langle x\rangle^{s}Vf(P)\langle x\rangle^{s}$ is a trace class operator, if $\rho - 2s > n$. It follows that if $\rho > n + 3$, $|\lim_{\epsilon\downarrow 0} F_1(\lambda + i\epsilon)| \leq C_{\delta,f}$ for some $C_{\delta,f} > 0$. Similarly, we can get $|\lim_{\epsilon\downarrow 0} F_2(\lambda + i\epsilon)| \leq C_{\delta,f}$. It follows that $|\lim_{\epsilon\downarrow 0} F(\lambda + i\epsilon)| \leq C_{\delta,f}$. $|\lim_{\epsilon\downarrow 0} F(\lambda - i\epsilon)| \leq C_{\delta,f}$ can be proved in the same way.

(b). One has $P = P_0 + V$ with V satisfying $|V| \leq C\langle x\rangle^{-\rho}$, $\rho > 2$. Let $q(\theta) = q^+(\theta) - q^-(\theta)$. Here $q^+(\theta)$ is the positive part of $q(\theta)$. By (3.8), one has that if $0 \notin \sigma_\infty$, there exists a constant $0 < \alpha_0 < 1$ such that

$$\frac{q^-(\theta)}{r^2} \leq \alpha_0(-\Delta + \frac{q^+(\theta)}{r^2}) \quad \text{in } H^1(\mathbb{R}^n).$$

It follows that

$$P_0 \geq (1-\alpha_0)(\chi_1(-\Delta)\chi_1 + \chi_2(-\Delta + \frac{q^+(\theta)}{r^2})\chi_2) \geq (1-\alpha_0)(-\Delta) - (1-\alpha_0)(\sum_{i=1}^{2}|\nabla\chi_i|^2).$$

Then

$$P \geq (1-\alpha_0)(-\Delta) + \tilde{W},$$

with $\tilde{W} = V - (1-\alpha_0)(\sum_{i=1}^{2}|\nabla\chi_i|^2)$ in $H^1(\mathbb{R}^n)$. Then the number of negative eigenvalues of P is less then the number of negative eigenvalues of $(1-\alpha_0)(-\Delta)+\tilde{W}$. Since $\rho > 2$, $\tilde{W} \in L^{n/2}(\mathbb{R}^n)$ for $n \geq 3$. By Cwikel-Lieb-Rosenbljum formula ([54, Theorem XIII.12]), one has that the number of bound states, $N(\tilde{W})$, of $(1-\alpha_0)(-\Delta) + \tilde{W}$ has the following estimate

$$N(\tilde{W}) \leq c_n \int |(1-\alpha_0)^{-1}\tilde{W}|^{n/2}\, dx.$$

Here c_n is a constant only depending on n. It follows that the number of negative eigenvalues, N, of P is finite.

(c). By a simple computation, one has

$$(P-i)^{-k} - (P_0-i)^{-k} = -\sum_{j=0}^{k-1}(P-i)^{-j+1}V(P_0-i)^{-k+j}.$$

It is easy to check that $(P-i)^{-j+1}V(P_0-i)^{-k+j}$ is a trace class operator if $\rho > n$, and $k > n/2-1$.

(d). We need only to show that P_0 and P have no positive eigenvalues. Suppose that $u \in L^2$ such that $P_0 u = Eu$ for some $E > 0$. It is easy to check that the conditions of Theorem 4.15 are satisfied for $q = n$ and $0 < \delta_0 < 1/2$. Then by Theorem 4.15, one has that $u = 0$. This means that P_0 has no positive eigenvalues. Similarly, we can get that P has no positive eigenvalues.

(e). By Theorem 1.2 [49], one has that $\xi'(\lambda) \in C^\infty([\lambda_0, \infty)$ for any $\lambda_0 > 0$. This ends the proof. □

Let $b_0(x,\xi) \in C_0^\infty(\mathbb{R}^n)$ and $b_\pm(x,\xi) \in C^\infty(\mathbb{R}^n)$ be non-negative functions such that

$$\begin{aligned} &|\partial_x^\alpha \partial_\xi^\beta b_\pm(x,\xi)| \leq C_{\alpha,\beta}\langle x\rangle^{-|\alpha|}\langle\xi\rangle^{-|\beta|}; \\ &b_0(x,\xi) + b_+(x,\xi) + b_-(x,\xi) = 1 \end{aligned} \tag{4.49}$$

and for some $0 < \delta < 1$, $\text{supp}b_\pm(x,\xi) \subset \{(x,\xi);\ \pm\hat{x}\cdot\hat{\xi} > -(1-\delta)\}$. Let $\tilde{b}_\pm(x,\xi) \in C^\infty$ satisfy (4.49) and $b_0(x,\xi) + \tilde{b}_+(x,\xi) + \tilde{b}_-(x,\xi) = 1$, and there exists some $0 < \delta' < 1$ such that $\text{supp}\tilde{b}_\pm(x,\xi) \subset \{(x,\xi);\ \pm\hat{x}\cdot\hat{\xi} > -(1-\delta')\}$, and $b_-\tilde{b}^+ = 0$. Denote by $a(x,D)$ by the pseudo-differential operator with the symbol $a(x,\xi)$ defined by

$$a(x,D)u(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}a(x,\xi)\hat{u}(\xi)\, d\xi$$

where $u \in \mathcal{S}(\mathbb{R}^n)$ and $\hat{u}$ is the Fourier transform of u.

Theorem 4.17. *Under the condition of Theorem 4.14. One has*

$$\xi'(\lambda) = J_0\delta(\lambda) + g(\lambda),$$

with $|g(\lambda)| = O(\lambda^{-1+\epsilon_0})$ *for some* $\epsilon_0 > 0$, *as* $\lambda \downarrow 0$.

Proof. Let $f \in C_0^\infty(\mathbb{R}_+)$. Then one has $f(P) - f(P_0) \in \mathscr{S}^1$, and

$$\mathrm{Tr}\,(f(P) - f(P_0)) = \int_{\mathbb{R}} f(\lambda)\xi'(\lambda)\,d\lambda.$$

Let $\delta > 0$, $R_0 > 0$, such that $\mathrm{supp} f \in [\delta, R_0]$. Let $f_1 \in C_0^\infty(\mathbb{R})$ such that $f_1(t) = 1$ for $t \in [-\delta', R_0]$. Here $\delta' > 0$. Then $f(P) = f(P)f_1(P)E(\delta, R_0)$, and $f(P_0) = f(P_0)f_1(P_0)E_0(\delta, R_0)$. Here $E(\delta, R_0)$ (respectively, $E_0(\delta, R_0)$) is the spectral projection of P (respectively, P_0) onto the interval $[\delta, R_0]$. Let $0 \le \chi \le 1$ be a smooth function with compact support such that $\chi(x) = 1$ for $|x| \le 1$. Let $\chi_R(x) = \chi(\frac{x}{R})$. By (3.7) [49], one has

$$\mathrm{Tr}[\chi_R(f(P) - f(P_0))\chi_R] = \frac{1}{2\pi}\int_{\mathbb{R}} f(\lambda)\tau_R(\lambda)\,d\lambda$$

and by Theorem 3.2 [49], one has $\xi'(\lambda) = \frac{1}{2\pi}\lim\limits_{R\to\infty}\tau_R(\lambda)$. By functional calculus, one has that

$$s - \lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_\delta^{R_0} f(\lambda)(R(\lambda + i\epsilon) - R(\lambda - i\epsilon))\,d\lambda = f(P)E(\delta, R_0);$$
$$s - \lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_\delta^{R_0} f(\lambda)(R_0(\lambda + i\epsilon) - R_0(\lambda - i\epsilon))\,d\lambda = f(P_0)E_0(\delta, R_0).$$

It follows that

$$\begin{aligned}\chi_R(f(P) - f(P_0))\chi_R &= s - \lim_{\epsilon\to 0}\frac{1}{2\pi i}[\int_\delta^{R_0} f(\lambda)\chi_R(R(\lambda + i\epsilon)f_1(P) - R_0(\lambda + i\epsilon)f_1(P_0))\chi_R\,d\lambda \\ &\qquad - \int_\delta^{R_0} f(\lambda)\chi_R(R(\lambda - i\epsilon)f_1(P) - R_0(\lambda - i\epsilon)f_1(P_0))\chi_R\,d\lambda] \\ &= F_1(R) + F_2(R).\end{aligned}$$

Here

$$F_1(R) = s - \lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_\delta^{R_0} f(\lambda)\chi_R(R(\lambda + i\epsilon) - R_0(\lambda + i\epsilon) - R(\lambda - i\epsilon) + R_0(\lambda - i\epsilon))f_1(P)\chi_R\,d\lambda,$$
$$F_2(R) = s - \lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_\delta^{R_0} f(\lambda)\chi_R(R_0(\lambda + i\epsilon) - R_0(\lambda - i\epsilon))(f_1(P) - f_1(P_0))\chi_R\,d\lambda.$$

Since $f_1(P) - f_1(P_0) \in \mathscr{S}^1$ and $\langle x\rangle^{-s}R_0(\lambda \pm i\epsilon)\langle x\rangle^{-s}$ converge to $\langle x\rangle^{-s}R_0(\lambda \pm i0)\langle x\rangle^{-s}$ as $\epsilon \downarrow 0$, $s > 1$, by Lemma 4.2, one has $\chi_R(R_0(\lambda + i\epsilon) - R_0(\lambda - i\epsilon))(f_1(P) - f_1(P_0))\chi_R$ converges to $\chi_R f(P_0)(R_0(\lambda + i0) - R_0(\lambda - i0))(f_1(P) - f_1(P_0))\chi_R$ in $\mathscr{S}^1$. Hence,

$$\mathrm{Tr}\,F_2(R) = \int_\delta^{R_0} f(\lambda)\lim_{\epsilon\to 0}\Im\mathrm{Tr}[\chi_R R_0(\lambda + i\epsilon)(f_1(P) - f_1(P_0))\chi_R]\,d\lambda.$$

Since

$$\chi_R(R(\lambda \pm i\epsilon) - R_0(\lambda \pm i\epsilon))f_1(P)\chi_R = \chi_R R_0(\lambda \pm i\epsilon)VR(\lambda \pm i\epsilon)f_1(P)\chi_R,$$

and $\chi_R R_0(\lambda \pm i\epsilon)\langle x\rangle^{-s}$ converge to $\chi_R R_0(\lambda \pm i0)\langle x\rangle^{-s}$ in norm, $\langle x\rangle V^s f_1(P)\langle x\rangle^s$ is of trace class, if $s > 1$ and $\rho > n + 2s$, then

$$\chi_R(R(\lambda \pm i\epsilon) - R_0(\lambda \pm i\epsilon))f_1(P)\chi_R \longrightarrow \chi_R R_0(\lambda \pm i0)Vf_1(P)R(\lambda \pm i0)\chi_R$$

in $\mathscr{S}^1$ as $\epsilon \downarrow 0$, if $s > 1$ and $\rho > n + 2s$. It follows that

$$\text{Tr } F_1(R) = \int_\delta^{R_0} f(\lambda) \lim_{\epsilon\downarrow 0} \Im\text{Tr}[\chi_R(R_0(\lambda + i\epsilon)Vf_1(P)R(\lambda + i\epsilon))\chi_R]\, d\lambda.$$

Then we obtain

$$\begin{aligned}\tau_R(\lambda) \quad = \quad & \lim_{\epsilon\downarrow 0} \Im\text{Tr}[\chi_R R_0(\lambda + i\epsilon)(f_1(P) - f_1(P_0))\chi_R] \\ & + \lim_{\epsilon\downarrow 0} \Im\text{Tr}[\chi_R(R_0(\lambda + i\epsilon)Vf_1(P)R(\lambda + i\epsilon))\chi_R].\end{aligned}$$

Thus

$$\xi'(\lambda) = \lim_{R\to\infty}\lim_{\epsilon\to 0} \Im\text{Tr } F_1(R,\epsilon) + \lim_{R\to\infty}\lim_{\epsilon\to 0} \Im\text{Tr } F_2(R,\epsilon)$$

with

$$\begin{aligned}F_1(R,\epsilon) &= \chi_R R_0(\lambda + i\epsilon)(f_1(P) - f_1(P_0))\chi_R; \\ F_2(R,\epsilon) &= \chi_R R_0(\lambda + i\epsilon)Vf_1(P)R(\lambda + i\epsilon)\chi_R.\end{aligned}$$

By (2.10)[24], one has

$$f_1(P) - f_1(P_0) = \sum_{m=1}^{M} \frac{(-1)^m X_m f^{(m)}(P_0)}{m!} + \frac{1}{\pi}\int \bar\partial\tilde f_1(z)R(z)X_{M+1}R_0(z)^{M+1}L(dz).$$

Here $L(dz)$ is the Lebesgue measure in $\mathbb{C}$, and $\tilde f_1 \in C_0^\infty(\mathbb{C})$ is an almost analytic extension of f_1 with support close to that of f_1 (see Chapter 8 in [17] and the references given there). $X_1 = -V$, $X_m = -VX_{m-1} + [X_{m-1}, H_0]$. A simply proof by induction shows that X_m can be written in the following form

$$X_m = \sum_{|\alpha|\le m-1} b_{m\alpha}(x)D_x^\alpha,$$

with $b_{m\alpha}(x)$ satisfying $|\partial_x^\alpha b_{m\alpha}(x)| \le C_{m\alpha}\langle x\rangle^{-\rho-m+1-|\alpha|}$. By Lemma 2.3 [24], one has that $(-\Delta + V - z)^{-1}X_{M+1}(-\Delta - z)^{-(M+1)}$ is of trace class operators on $L^2(\mathbb{R}^n)$ for $\Im z \neq 0$ for M large enough, of the trace norm

$$O(1)\frac{\langle z\rangle^{M/2+M+1}}{|\Im z|^{M+2+(M+1)/2}}.$$

For $\Im z \neq 0$,

$$\begin{aligned} & R(z)X_{M+1}R_0(z)^{M+1} \\ = \; & R(z)(-\Delta+V-z)(-\Delta+V-z)^{-1}X_{M+1}(-\Delta-z)^{-(M+1)}(-\Delta-z)^{M+1}R_0(z)^{M+1}. \end{aligned}$$

Since the principal symbol of $(-\Delta-z)^{M+1}R_0^M(z)$ is $(\frac{\xi^2-z}{\xi^2+g(x)-z})^{M+1}$, with $g(x)=\frac{q(\theta)}{r^2}(1-\chi_1^2(r))$, then by Calderon-Vaillancourt theorem

$$\|(-\Delta-z)^{M+1}R_0(z)^{M+1}\| \leq C(1+\frac{1}{|\Im z|})^{A_M}.$$

Here A_m is a constant depending on M. Similarly, one has $\|(-\Delta+V-z)(-\Delta+V-z)^{-1}\| \leq C(1+\frac{1}{|\Im z|})^C$. It follows that $R(z)X_{M+1}R_0(z)^{M+1}$ is of trace class operators on $L^2(\mathbb{R}^n)$ for $\Im z \neq 0$, of the trace norm

$$O(1)\frac{\langle z\rangle^{M/2+M+1}}{|\Im z|^{M+2+(M+1)/2}}(1+\frac{1}{|\Im z|})^{A_M}.$$

According to the fact $\bar\partial \tilde f_1(z) = O(|\Im z|^\infty)$, we have $\langle x\rangle^s(f_1(P)-f_1(P_0))\langle x\rangle^s$ is a trace class operator if $\rho > n+2s$. Hence,$\forall s>1$,

$$\text{Tr } F_1(R,\epsilon) = \text{Tr } \langle x\rangle^{-s}\chi_R^2 R_0(\lambda+i\epsilon)\langle x\rangle^{-s}\langle x\rangle^s(f_1(P)-f_1(P_0))\langle x\rangle^s.$$

Since $s-\lim \chi_R^2 = 1$ on $L^2(\mathbb{R}^n)$, and $\langle x\rangle^s(f_1(P)-f_1(P_0))\langle x\rangle^s$ is of trace class, by Lemma 4.2, one has

$$\lim_{R\to\infty}\lim_{\epsilon\to 0}\Im\text{Tr } F_1(R,\epsilon) = \Im\text{Tr}[\langle x\rangle^{-s}R_0(\lambda+i0)\langle x\rangle^{-s}\cdot\langle x\rangle^s(f_1(P)-f_1(P_0))\langle x\rangle^s] = O(1)$$

for $\lambda \geq 0, \rho > n+2$.

For $s > 3/2$, one has

$$\text{Tr } F_2(R,\epsilon) = \text{Tr } I_1(R,\epsilon) + \text{Tr } I_2(R,\epsilon)$$

with

$$\begin{aligned} &\text{Tr } I_1(R,\epsilon) = \text{Tr }[\langle x\rangle^{-s}R_0(\lambda+i\epsilon)\chi_R^2R_0(\lambda+i\epsilon)\langle x\rangle^{-s}\cdot\langle x\rangle^s V f(P)\langle x\rangle^s]; \\ &\text{Tr } I_2(R,\epsilon) = \text{Tr }[\langle x\rangle^{-s}R_0(\lambda+i\epsilon)\chi_R^2R_0(\lambda+i\epsilon)\langle x\rangle^{-s}\cdot\langle x\rangle^s V f(P)\langle x\rangle^s\cdot\langle x\rangle^{-s}R(\lambda+i\epsilon)V\langle x\rangle^s]. \end{aligned}$$

Let $b_0,\ b_\pm,\ \tilde b_\pm$be the functions introduced before this theorem. Then

$$\text{Tr } I_1(R,\epsilon) = \text{Tr } I_{11}(R,\epsilon) + \text{Tr } I_{12}(R,\epsilon) + \text{Tr } I_{13}(R,\epsilon).$$

Here

$$\begin{aligned} &\text{Tr } I_{11}(R,\epsilon) = \text{Tr }[\langle x\rangle^{-s}R_0(\lambda+i\epsilon)b_+(x,D)\chi_R^2R_0(\lambda+i\epsilon)\langle x\rangle^{-s}\cdot\langle x\rangle^s V f(P)\langle x\rangle^s]; \\ &\text{Tr } I_{12}(R,\epsilon) = \text{Tr }[\langle x\rangle^{-s}R_0(\lambda+i\epsilon)b_-(x,D)\chi_R^2R_0(\lambda+i\epsilon)\langle x\rangle^{-s}\cdot\langle x\rangle^s V f(P)\langle x\rangle^s]; \\ &\text{Tr } I_{13}(R,\epsilon) = \text{Tr }[\langle x\rangle^{-s}R_0(\lambda+i\epsilon)b_0(x,D)\chi_R^2R_0(\lambda+i\epsilon)\langle x\rangle^{-s}\cdot\langle x\rangle^s V f(P)\langle x\rangle^s]. \end{aligned}$$

By Theorem 1 [26], one has $\langle x\rangle^{-s}R_0(\lambda+i\epsilon)b_+(x,D)\langle x\rangle^{s-1}$ converges to $\langle x\rangle^{-s}R_0(\lambda+i0)b_+(x,D)\langle x\rangle^{s-1}$ as $\lambda\to 0$ if $s>1/2$. If $s>3/2$, one has

$$s-\lim_{R\to\infty}\lim_{\epsilon\downarrow 0}\langle x\rangle^{1-s}\chi_R^2R_0(\lambda+i\epsilon)\langle x\rangle^{-s}=\langle x\rangle^{1-s}R_0(\lambda+i0)\langle x\rangle^{-s}.$$

It follows that $I_{11}(R,\epsilon)$ converges to $\langle x\rangle^{-s}R_0(\lambda+i0)b_+(x,D)R_0(\lambda+i0)\langle x\rangle^{-s}\cdot\langle x\rangle^sVf(P)\langle x\rangle^s$ in $\mathscr{S}^1$ by Lemma 4.2, since $\langle x\rangle^sVf(P)\langle x\rangle^s$ is a trace class operator if $\rho>n+2s$. Similarly, we can show that $I_{13}(R,\epsilon)$ converges to $\langle x\rangle^{-s}R_0(\lambda+i0)b_0(x,D)R_0(\lambda+i0)\langle x\rangle^{-s}\cdot\langle x\rangle^sVf(P)\langle x\rangle^s$ in $\mathscr{S}^1$, since $b_0(x,\xi)\in C_0^\infty$. Decompose $I_{12}(R,\epsilon)$ as

$$I_{12}(R,\epsilon)=T_1(R,\epsilon)+T_2(R,\epsilon)+T_3(R,\epsilon)$$

with

$$\begin{aligned}
T_1(R,\epsilon)&=\langle x\rangle^{-s}R_0(\lambda+i\epsilon)b_-(x,D)\chi_R^2\tilde{b}_-(x,D)R_0(\lambda+i\epsilon)\langle x\rangle^{-s}\cdot\langle x\rangle^sVf(P)\langle x\rangle^s;\\
T_2(R,\epsilon)&=\langle x\rangle^{-s}R_0(\lambda+i\epsilon)b_-(x,D)\chi_R^2\tilde{b}_+(x,D)R_0(\lambda+i\epsilon)\langle x\rangle^{-s}\cdot\langle x\rangle^sVf(P)\langle x\rangle^s\\
T_3(R,\epsilon)&=\langle x\rangle^{-s}R_0(\lambda+i\epsilon)b_-(x,D)\chi_R^2b_0(x,D)R_0(\lambda+i\epsilon)\langle x\rangle^{-s}\cdot\langle x\rangle^sVf(P)\langle x\rangle^s.
\end{aligned}$$

Since $b_-(x,\xi)\tilde{b}_+(x,\xi)=0$, then we can get that $T_2(R,\epsilon)=0$. Note that $b_0(x,\xi)\in C_0^\infty$. We can get that

$$T_3(R,\epsilon)\longrightarrow\langle x\rangle^{-s}R_0(\lambda+i0)b_-(x,D)\tilde{b}_0(x,D)R_0(\lambda+i0)\langle x\rangle^{-s}\cdot\langle x\rangle^sVf(P)\langle x\rangle^s$$

in $\mathscr{S}^1$, as $\epsilon\to 0$ and $R\to\infty$. Using Theorem 1 [26] again, we can deduce that

$$T_1(R,\epsilon)\longrightarrow\langle x\rangle^{-s}R_0(\lambda+i0)b_-(x,D)\tilde{b}_-(x,D)R_0(\lambda+i0)\langle x\rangle^{-s}\cdot\langle x\rangle^sVf(P)\langle x\rangle^s.$$

in $\mathscr{S}^1$, as $\epsilon\to 0$ and $R\to\infty$, if $\rho>n+2s$. It follows that $I_{12}(R,\epsilon)$ converges to $\langle x\rangle^{-s}R_0(\lambda+i0)b_-(x,D)R_0(\lambda+i0)\langle x\rangle^{-s}\cdot\langle x\rangle^sVf(P)\langle x\rangle^s$ in $\mathscr{S}^1$, as $\epsilon\to 0$ and $R\to\infty$. Therefore, we obtain

$$I_1(R,\epsilon)\longrightarrow\langle x\rangle^{-s}R_0(\lambda+i0)^2\langle x\rangle^{-s}\cdot\langle x\rangle^sVf(P)\langle x\rangle^s$$

in $\mathscr{S}^1$, as $\epsilon\to 0$ and $R\to\infty$ if $\rho>n+2s$ and $s>3/2$. It follows that

$$\lim_{R\to\infty}\lim_{\epsilon\to 0}\operatorname{Tr}I_1(R,\epsilon)=\lim_{\epsilon\to 0}\operatorname{Tr}\langle x\rangle^{-s}R_0^2(\lambda+i\epsilon)\langle x\rangle^{-s}\cdot\langle x\rangle^sVf(P)\langle x\rangle^s.$$

Similarly, we can prove that

$$\lim_{R\to\infty}\lim_{\epsilon\to 0}\operatorname{Tr}I_2(R,\epsilon)=\lim_{\epsilon\to 0}\operatorname{Tr}\left[\langle x\rangle^{-s}R_0^2(\lambda+i\epsilon)\langle x\rangle^{-s}\cdot\langle x\rangle^sVf(P)\langle x\rangle^s\cdot\langle x\rangle^{-s}R(\lambda+i\epsilon)V\langle x\rangle^s\right]$$

if $s>3/2$ and $\rho>n+2s$. Thus, one has

$$\lim_{R\to\infty}\lim_{\epsilon\to 0}\Im\operatorname{Tr}F_1(R,\epsilon)=\lim_{\epsilon\to 0}\Im\operatorname{Tr}\left[R_0(\lambda+i\epsilon)Vf_1(P)R(\lambda+i\epsilon)\right].$$

Hence, one has that in the distributions sense, for $\lambda > 0$,

$$\xi'(\lambda) = \frac{1}{\pi} \lim_{\epsilon \downarrow 0} \Big[\Im \mathrm{Tr}\, [(R(\lambda + i\epsilon) - R_0(\lambda + i\epsilon)) f_1(P)] + \Im \mathrm{Tr}\, [R_0(\lambda + i\epsilon)(f_1(P) - f_1(P_0))] \Big].$$

Note that $\|\langle x\rangle^{-s} R_0(\lambda + i\epsilon)\langle x\rangle^{-s} \cdot \langle x\rangle^s (f_1(P) - f_1(P_0))\langle x\rangle^s\|_1 \leq C$ for $0 < \lambda < 1$. From the proof of Theorem 4.14, we have for $0 < \lambda < 1$,

$$\mathrm{Tr}\, [(R(\lambda + i\epsilon) - R(\lambda + i\epsilon)) f(P)] = \frac{J_0}{\lambda + i\epsilon} + \frac{m}{(\lambda + i\epsilon)\ln(\lambda + i\epsilon)} + O(|\lambda + i\epsilon|^{-1+\epsilon_0}).$$

For fixed $\lambda \neq 0$,

$$\lim_{\epsilon \downarrow 0} \Im \frac{m}{(\lambda + i\epsilon)\ln(\lambda + i\epsilon)} = \lim_{\epsilon \downarrow 0} \frac{-\lambda\theta - \epsilon \ln \sqrt{\lambda^2 + \epsilon^2}}{(\lambda^2 + \epsilon^2) \cdot (\theta^2 + \ln \sqrt{\lambda^2 + \epsilon^2})} = 0.$$

Here $\theta = \arctan \frac{\epsilon}{\lambda}$. Note that $\lim_{\epsilon \downarrow 0} \frac{J_0}{\lambda + i\epsilon} = \pi J_0 \delta(\lambda)$. It follows that

$$\xi'(\lambda) = J_0 \delta(\lambda) + O(|\lambda|^{-1+\epsilon_0}).$$

This ends the proof. □

Theorem 4.18. *Assume that $0 \notin \sigma_\infty$, $\rho > n + 3$ and $n \geq 3$. One has*

$$\int_0^\infty (\xi'(\lambda) - \sum_{j=1}^{[\frac{n}{2}]} c_j \lambda^{[\frac{n}{2}]-1-j})\, d\lambda = -(N + J_0) + \beta_{n/2}. \tag{4.50}$$

$\beta_{n/2}$ depends on n and V. $\beta_{n/2} = 0$ if n is odd.

Proof. Let $\chi \in C_0^\infty$ and $\chi(r) \equiv 1$ for $r \leq 1$. Then for $R >> 1$, $\chi(\frac{\cdot}{R})$ satisfies the conditions of Theorem 4.1.

Case 1. The dimension n is odd.

We first compute the second term on the left hand side of (4.12) for $f = \chi(\frac{\cdot}{R})$.

$$\begin{aligned}
& \int_0^\infty \chi(\frac{\lambda}{R}) g(\lambda)\, d\lambda \\
= & \int_0^\infty \chi(\frac{\lambda}{R})[g(\lambda) - \sum_{j=1}^{[\frac{n}{2}]} c_j \lambda^{\frac{n}{2}-1-j}]\, d\lambda + \sum_{j=1}^{[\frac{n}{2}]} c_j \int_0^\infty \chi(\frac{\lambda}{R}) \lambda^{\frac{n}{2}-1-j}\, d\lambda \\
= & \int_0^\infty \chi(\frac{\lambda}{R})[g(\lambda) - \sum_{j=1}^{[\frac{n}{2}]} c_j \lambda^{\frac{n}{2}-1-j}]\, d\lambda + \sum_{j=1}^{[\frac{n}{2}]} \tilde{c}_j R^{\frac{n}{2}-j},
\end{aligned}$$

with $\tilde{c}_j = c_j \int_0^\infty \chi(t) t^{\frac{n}{2}-1-j}\, dt$. By Theorem 1.1 [49], one has

$$\mathrm{Tr}\, (\chi(\frac{P}{R}) - \chi(\frac{P_0}{R})) \sim \sum_{j \geq 1} \beta_j R^{\frac{n}{2}-j}, \tag{4.51}$$

as $R \to \infty$. It follows

$$\int_0^\infty \chi(\frac{\lambda}{R})[g(\lambda) - \sum_{j=1}^{[\frac{n}{2}]} c_j \lambda^{\frac{n}{2}-1-j}]\, d\lambda = -(N + J_0) + \sum_{j=1}^{[\frac{n}{2}]} (\beta_j - \tilde{c}_j) R^{\frac{n}{2}-j} + O(R^{-\epsilon}) \tag{4.52}$$

where $\epsilon = -1 + \frac{n}{2} - [\frac{n}{2}] > 0$. $\int_0^\infty \chi(\frac{\lambda}{R})[g(\lambda) - \sum_{j=1}^{[\frac{n}{2}]} c_j \lambda^{\frac{n}{2}-1-j}]\, d\lambda$ can be written as

$$\begin{aligned} &\int_0^\infty \chi(\frac{\lambda}{R})[g(\lambda) - \sum_{j=1}^{[\frac{n}{2}]} c_j \lambda^{\frac{n}{2}-1-j}]\, d\lambda \\ = &\int_0^1 \chi(\frac{\lambda}{R})[g(\lambda) - \sum_{j=1}^{[\frac{n}{2}]} c_j \lambda^{\frac{n}{2}-1-j}]\, d\lambda + \int_1^\infty \chi(\frac{\lambda}{R})[g(\lambda) - \sum_{j=1}^{[\frac{n}{2}]} c_j \lambda^{\frac{n}{2}-1-j}]\, d\lambda. \end{aligned}$$

Since $g \in C^\infty(0,1)$ and g is integrable at 0, thus

$$\lim_{R\to+\infty} \int_0^1 \chi(\frac{\lambda}{R})[g(\lambda) - \sum_{j=1}^{[\frac{n}{2}]} c_j \lambda^{\frac{n}{2}-1-j}]\, d\lambda = \int_0^1 [g(\lambda) - \sum_{j=1}^{[\frac{n}{2}]} c_j \lambda^{\frac{n}{2}-1-j}]\, d\lambda.$$

By Theorem 1.2 [49], one has $\xi'(\lambda) \in C^\infty(0,\infty)$ and

$$\xi'(\lambda) \sim \sum_{j\geq 1} c_j \lambda^{\frac{n}{2}-j-1}, \quad \text{as } \lambda \to +\infty. \tag{4.53}$$

By Lemma 4.17 and the above formula, we get $|g(\lambda) - \sum_{j=1}^{[\frac{n}{2}]} c_j \lambda^{\frac{n}{2}-1-j}| \leq C\lambda^{-\nu}$ with $\nu = 2 - (\frac{n}{2} - [\frac{n}{2}]) > 1$. Therefore,

$$\lim_{R\to+\infty} \int_1^\infty \chi(\frac{\lambda}{R})[g(\lambda) - \sum_{j=1}^{[\frac{n}{2}]} c_j \lambda^{\frac{n}{2}-1-j}]\, d\lambda = \int_1^\infty [g(\lambda) - \sum_{j=1}^{[\frac{n}{2}]} c_j \lambda^{\frac{n}{2}-1-j}]\, d\lambda.$$

It follows that

$$\begin{aligned} &\lim_{R\to+\infty} \int_0^\infty \chi(\frac{\lambda}{R})[g(\lambda) - \sum_{j=1}^{[\frac{n}{2}]} c_j \lambda^{\frac{n}{2}-1-j}]\, d\lambda \\ = &\int_0^\infty [g(\lambda) - \sum_{j=1}^{[\frac{n}{2}]} c_j \lambda^{\frac{n}{2}-1-j}]\, d\lambda = \int_0^\infty [\xi'(\lambda) - \sum_{j=1}^{[\frac{n}{2}]} c_j \lambda^{\frac{n}{2}-1-j}]\, d\lambda. \end{aligned}$$

Let $R \to +\infty$ in both sides of (4.52) we get that $\beta_j = \tilde{c}_j$. Thus, we get the Levinson's Theorem for odd dimension.

Case 2 : The dimension n is even. Suppose $n = 2p$.

As in case 1, we can drive that

$$\begin{aligned}&\int_0^\infty \chi(\frac{\lambda}{R})[g(\lambda) - \sum_{j=1}^{p-1} c_j \lambda^{p-1-j}]\, d\lambda \\ = \quad & -(N + J_0) + \sum_{j=1}^{p-1} (\beta_j - \tilde{c}_j) R^{\frac{n}{2}-j} + \beta_p + O(R^{-1}),\end{aligned} \tag{4.54}$$

with $\tilde{c}_j = \int_0^\infty \chi(t) t^{p-1-j}\, dt$. The first term in (4.54) can be written as :

$$\int_0^1 \chi(\frac{\lambda}{R})[g(\lambda) - \sum_{j=1}^{p-1} c_j \lambda^{\frac{n}{2}-1-j}]\, d\lambda + \int_1^\infty \chi(\frac{\lambda}{R})[g(\lambda) - \sum_{j=1}^{p-1} c_j \lambda^{\frac{n}{2}-1-j}]\, d\lambda.$$

Note that

$$\int_1^\infty \chi(\frac{\lambda}{R})[g(\lambda) - \sum_{j=1}^{p-1} c_j \lambda^{\frac{n}{2}-1-j}]\, d\lambda = \int_1^\infty \chi(\frac{\lambda}{R})[\frac{c_p}{\lambda} + h(\lambda)]\, d\lambda.$$

with $h(\lambda) = O(\frac{1}{\lambda^2})$. Formula (4.54) can be written as :

$$\begin{aligned}&\int_0^1 \chi(\frac{\lambda}{R})[g(\lambda) - \sum_{j=1}^{p-1} c_j \lambda^{\frac{n}{2}-1-j}]\, d\lambda + \int_1^\infty \chi(\frac{\lambda}{R}) h(\lambda)\, d\lambda \\ = \quad & -(N + J_0) + \sum_{j=1}^{p-1} (\beta_j - \tilde{c}_j) R^{p-j} - c_p \int_1^\infty \chi(\frac{\lambda}{R})\, \frac{d\lambda}{\lambda} + \beta_p + O(\frac{1}{R}).\end{aligned}$$

As before, we can conclude that

$$\int_0^1 \chi(\frac{\lambda}{R})[g(\lambda) - \sum_{j=1}^{p-1} c_j \lambda^{\frac{n}{2}-1-j}]\, d\lambda + \int_1^\infty \chi(\frac{\lambda}{R}) h(\lambda)\, d\lambda$$

converges to

$$\int_0^1 [\xi'(\lambda) - \sum_{j=1}^{p-1} c_j \lambda^{p-1-j}]\, d\lambda + \int_1^\infty h(\lambda)\, d\lambda$$

as $R \to +\infty$. It follows $\beta_j = \tilde{c}_j$. Since $\int_1^\infty \chi(\frac{\lambda}{R})\, \frac{d\lambda}{\lambda} = \int_{\frac{1}{R}}^\infty \chi(t)\, \frac{dt}{t} \sim \log R$, we can drive that $c_p = 0$. Hence

$$h(\lambda) = \xi'(\lambda) - \sum_{j=1}^{p-1} c_j \lambda^{p-1-j}.$$

Therefore, we get the Levinson's theorem for even dimension :

$$\int_0^\infty (\xi'(\lambda) - \sum_{j=1}^{p-1} c_j \lambda^{p-1-j})\, d\lambda = -(N + J_0) + \beta_p.$$

□

Remark 4.19. *The values of β_p is independent of χ. We use the formula in [15] to compute β_p. Let $V_0 = \frac{\chi_2^2}{r^2} q(\theta) - \sum_{i=1}^{2} |\nabla \chi_i|^2$ and $\tilde{V}_0 = V_0 + V$. Then $P_0 = -\Delta + V_0$ and $P = -\Delta + \tilde{V}_0$.*

- *n=2 (i.e. p=1) :*

$$\beta_1 = \frac{1}{(2\pi)^2} \gamma_2 \int_{\mathbb{R}^2} \tilde{V}_0 \, dx \int_{\mathbb{R}_+} \chi'(\rho) \, d\rho.$$

  Using $\int_{\mathbb{R}_+} \chi'(\rho) \, d\rho = 1$, we get that

$$\beta_1 = C \int_{\mathbb{R}^2} \tilde{V}_0 \, dx$$

  with $C = \frac{1}{(2\pi)^2} \gamma_2$.

- *n=4 (i.e. p=2) :*

$$\beta_2 = \frac{1}{(2\pi)^4} \frac{\gamma_4}{2} \int_{\mathbb{R}^4} \tilde{V}_0^2(x) + 2\tilde{V}_0(x) V_0(x) \, dx \int_{\mathbb{R}_+} \rho \chi''(\rho) \, d\rho.$$

  Using $\int_{\mathbb{R}_+} \rho \chi''(\rho) \, d\rho = 1$, we get that

$$\beta_2 = C \int_{\mathbb{R}^4} \tilde{V}_0^2(x) + 2\tilde{V}_0(x) V_0(x) \, dx$$

  with $C = \frac{1}{(2\pi)^4} \frac{\gamma_4}{2}$.

- *n=6 (i.e. p=3) :*

$$\beta_3 = C \int_{\mathbb{R}^6} \tilde{V}_0^3(x) + 3\tilde{V}_0^2(x) V_0(x) + 3\tilde{V}_0(x) V_0^2(x) + \frac{1}{4} |\nabla \tilde{V}_0(x)|^2 + \frac{1}{2} \nabla \tilde{V}_0(x) \cdot \nabla V_0(x) \, dx \int_{\mathbb{R}_+} \rho^2 \chi'''(\rho) \, d\rho.$$

  Using $\int_{\mathbb{R}_+} \rho^2 \chi'''(\rho) \, d\rho = 2$,

$$\beta_3 = C \int_{\mathbb{R}^6} \tilde{V}_0^3(x) + 3\tilde{V}_0^2(x) V_0(x) + 3\tilde{V}_0(x) V_0^2(x) + \frac{1}{4} |\nabla V(x)|^2 + \frac{1}{2} \nabla \tilde{V}_0(x) \cdot \nabla V_0(x) \, dx.$$

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Résumé : On étudie dans cette thèse certains problèmes spectraux pour des opérateurs de Schrödinger. On s'intéresse d'abord à la limite semi-classique pour le nombre d'états propres de l'opérateur de Schrödinger à N corps. On utilise ensuite le crochet de Dirichlet-Neumann pour obtenir la limite semi-classique des moyennes de Riesz des valeurs propres discrètes pour l'opérateur de Schrödinger à N corps. On considère également le potentiel effectif de l'opérateur de Schrödinger à N corps avec potentiel de Coulomb et on obtient qu'il a une décroissance critique à l'infini. On étudie donc l'opérateur de Schrödinger à potentiel critique. On s'intéresse au seuil pour la constante de couplage et au développement asymptotique de la résolvante de l'opérateur de Schrödinger, puis on utilise ce développement pour étudier la limite à basse énergie de la dérivée de la fonction de décalage spectral pour une perturbation à décroissance critique. Finalement, on utilise ce résultat avec le résultat connu pour le développement asymptotique à haute énergie de cette fonction de décalage spectral pour obtenir le théorème de Levinson.

Mots clé : limite semi-classique ; moyenne Riesz ; opérateur de Schrödinger à N-corps ; limite de la constante du couplage ; résolvante ; état résonnants ; décroissance critique ; fonction de décalage spectral ; théoème de Levinson.

Summary : This PhD thesis deals with some spectral problems of Schrödinger operators. We first consider the semi-classical limit of the number of bound states of unique two-cluster N-body Schrödinger operator. Then we use Dirichlet-Neumann bracket to get semi-classical limit of Riesz means of the discrete eigenvalues of N-body Schrödinger operator. The effective potential of N-body Schrödinger operator with Coulomb potential is also considered and we find that the effective potential has critical decay at infinity. Thus, the Schrödinger operator with critical potential is studied in this thesis. We study the coupling constant threshold of Schrödinger operator with critical potential and the asymptotic expansion of resolvent of Schrödinger operator with critical potential. We use that expansion to study low-energy asymptotics of derivative of spectral shift function for perturbation with critical decay. After that, we use this result and the known result for high-energy asymptotic expansion of spectral shift function to obtain the Levinson theorem.

Key words : semi-classical limit, Riesz means, N-body Schrödinger operator, coupling constant limit, resolvent, resonant states, critical decay, spectrum shift function, Levinson theorem.