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Cooperative Behaviour in Complex Systems

PhD defense of

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May 28, 2009

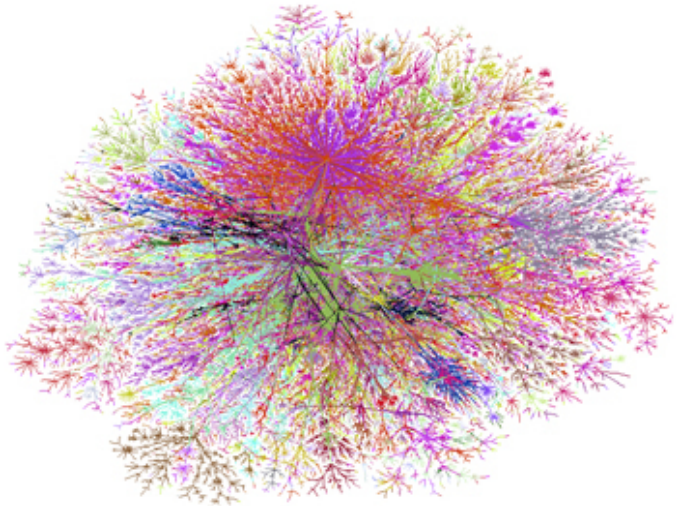
Introduction

- Cooperative behaviour could evaluate in interacting many-body systems
  - physics, sociology, economy, biology,...
- Large number of states $\Rightarrow$ relevant statistical description
- Phase transitions and critical phenomena
  - Critical exponents and universality classes
- Effects of disorder on the critical behaviour
- Influence of the system structure on the cooperation processes
- Develop and apply capable numerical methods
  - Monte Carlo methods, combinatorial optimization,...

Regular graphs and scale-free networks

- **Regular graphs** - each site has the same degree
  - Solid state physics, Crystallography
- **Scale-free (SF) networks** - degree distribution: $P(k) \sim k^{-\gamma}$
  - Self-organizing networks

- Only a few well connected node
- Short distance between any pair of sites
- WWW, Internet, collaboration networks, neural networks,...
- Real networks: $2 < \gamma < 3$
- **Barabási-Albert model**: $\gamma = 3$
  - Dynamical evolving networks
  - Preferential attachment



Aims

AIMS:

- Study interacting many-body systems in complex structures
- Analyze cooperative behavior in different macroscopic phases
- Describe their critical behaviour during phase transitions

SUBJECTS:

1. Non-equilibrium phase transitions and finite size scaling in weighted scale-free networks
2. Rounding of first-order phase transitions and optimal cooperation in scale-free networks
3. Density of critical clusters in a strongly disordered system
4. Non-equilibrium dynamics of triangular antiferromagnetic Ising model at $T = 0$

Non-equilibrium phase transitions and finite size scaling in weighted scale-free networks

- **Interactions between agents** are possible along the edges of the network - **cooperative behaviour** arises
- Possible some macroscopic phases
  - order parameter: strength of the interaction, external field,...
- **Scale-free networks** - the structure influences the process
- Homogeneous SF networks (edge weights $\lambda_{i,j} = const$)
  - conventional mean-field behavior is expected to hold if $\gamma > \gamma_u$
    - Ising model: $\gamma_u = 5$
    - Percolation, epidemic spreading: $\gamma_u = 4$
  - Phase transition occurs if $\gamma > 3$

- Inhomogeneous SF networks
  - weight degradation: $\lambda_{i,j} = \lambda \frac{(k_i k_j)^{-\mu}}{\langle k^{-\mu} \rangle^2}$
- Scales with the degradation exponent μ $(0 < \mu < 1)$
- Effective degree exponent: $\gamma' = \frac{\gamma - \mu}{1 - \mu}$
- Possible phase transition for $\gamma \leq 3$ if $\mu > (3 - \gamma)/2$
- It is true for equilibrium critical phenomena (Giuraniuc et. al.)

AIMS:

- Check this reparametrization working for non-equilibrium phase transitions
- Study the form and validity of finite-size scaling true in Euclidean lattices

- **Contact Process** - Model with non-equilibrium phase transition
- Infection spreading model - **Directed percolation universality class**
- Network sites $\equiv$ agents, edge $\equiv$ interaction, edge weights $\equiv$ strength of interaction
- Definition of dynamics: sites are **vacant** ($\oslash$) or **occupied** (A)
  - **Infection**: $\oslash_i A_j, A_i \oslash_j \longrightarrow A_i A_j$ with rate $\lambda_{i,j}$
  - **Immunization**: $A_i \longrightarrow \oslash_i$ with rate κ
- Control parameter: λ/κ (we choose $\kappa = 1$)
- Phase transition:
  - $\lambda < \lambda_c$ - **absorbing phase**
    - the number of occupied sites tends to zero and vanishes
  - $\lambda > \lambda_c$ - **active phase**
    - the number of occupied sites tends to a constant



- Expected to be exact due to long-range interactions
- Dynamical mean field (MF) equations for sites $i = 1, ..., N$:

$$\frac{\partial \rho_i}{\partial t} = \sum_j \lambda_{i,j}(1 - \rho_i)\rho_j - \rho_i \qquad (1)$$

- $\rho_i(t)$ - time dependent average active point density
- Three regimes for the critical behavior depends on γ' :
  - $\gamma' > 4$: **Conventional MF regime** (at $\gamma' = 4$ logarithmic corrections)
  - $3 < \gamma' < 4$: **Unconventional MF regime** - critical exponents depend on γ'
  - $\gamma' < 3$: The system for any λ value is in the active phase $\Rightarrow$ No phase transition

$$\rho = L^{-\beta/\nu^*}\tilde{\rho}(\delta L^{1/\nu^*}, hN^{\Delta/\nu^*}) \tag{2}$$

- where $\nu^* = 2/d$ and $\beta = 1$, $\Delta = 2$ (MF exponents)
- Scale-free networks: $N \leftrightarrow L^d$
- For **typical sites** ($k \sim \langle k \rangle$):

$$\rho_{typ} = N^{-\beta/2}\tilde{\rho}_{typ}(\delta N^{1/2}, hN^{\Delta/2}) \tag{3}$$

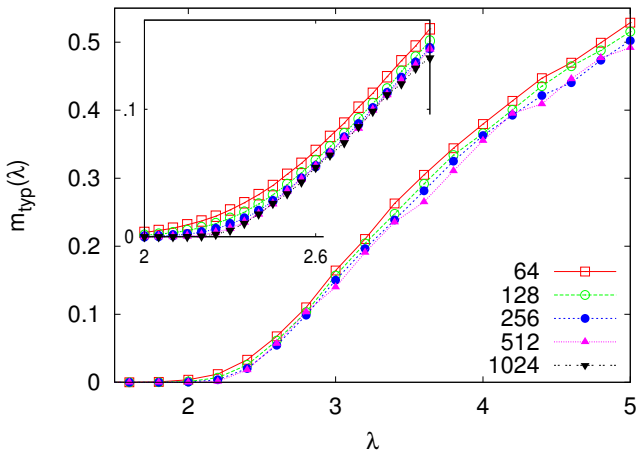
- For the **maximally connected site** ($k \sim N^{1/(\gamma-1)}$):

$$\rho_{max} = N^{-\beta/2+(1-\mu)/(\gamma-1)}\tilde{\rho}_{max}(\delta N^{1/2}, hN^{\Delta/2}) \tag{4}$$

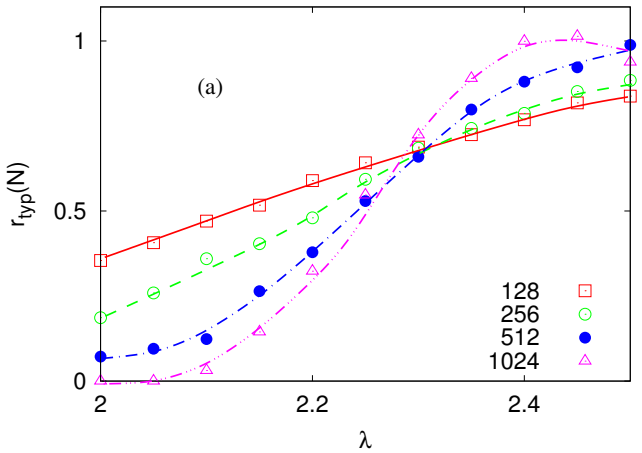
Numerical Simulations:

- Barabási-Albert network: $\gamma = 3$, $m_0 = m = 1$, $N = 2^6 ... 2^{12}$
- We choose $\mu = 1/2$ thus $\gamma' = 5 \Rightarrow$ conventional MF regime

The order parameter $m_{typ}(\lambda)$

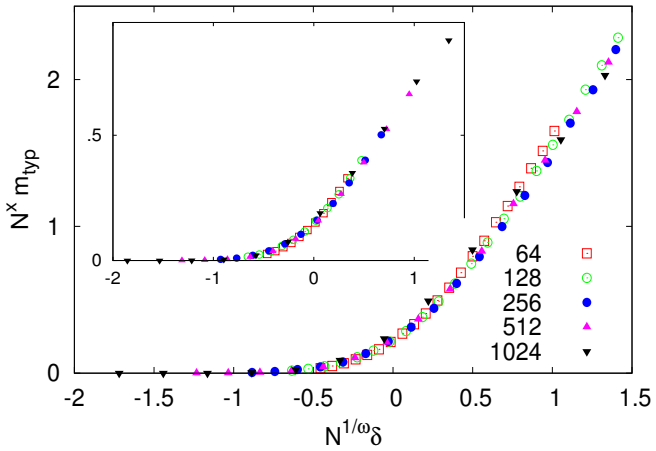


The order parameter ratio $r(N)$

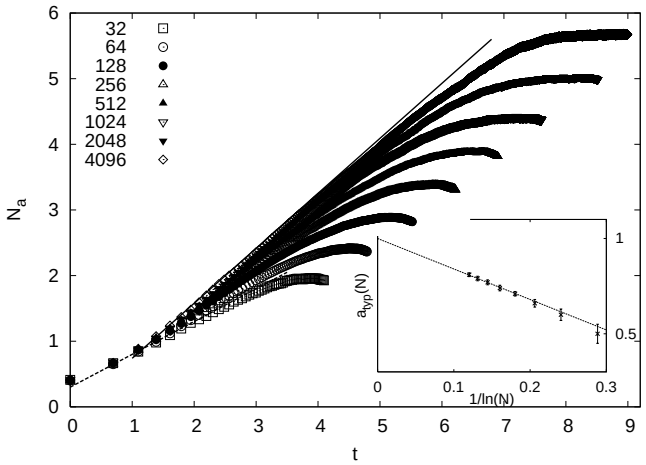


- Start the infection from a typical and maximally connected site (here only the typical case is presented)
- **Continuous phase transition** with finite-size effects at the criticality
- order parameter ratio:
$$r(N, \lambda = \lambda_c) = \frac{m(N)}{m(N/2)} = 2^{-x}$$
- **Critical point:**
  - $\lambda_c = 2.30(1)$ (in both cases)
- **Finite-size scaling exponent** x:
  - $x_{typ}^{Num} = 0.54(7) \leftrightarrow x_{typ}^{Theo} = 1/2$
  - $x_{max}^{Num} = 0.27(4) \leftrightarrow x_{max}^{Theo} = 1/4$

Scaling of the order parameter



Time dependence of $N_a(t)$



- **Correlation volume exponent:**
  - $\omega_{typ}^{Num} = 2.1(2) \leftrightarrow \omega_{typ}^{Theo} = 2$
  - $\omega_{max}^{Num} = 2.0(1) \leftrightarrow \omega_{max}^{Theo} = 2$

Dynamical scaling at $\lambda = \lambda_c$:

- **"Diffusion" exponent:** $N_a(t) \sim t^a$
  - $a_{typ}^{Num} = 0.95(1) \leftrightarrow a_{typ}^{Theo} = 1$
  - $a_{max}^{Num} = 0.52(4) \leftrightarrow a_{max}^{Theo} = 0.5$
  - Mean-field value: $a^{MF} = 0$
- **Dynamical exponent:**
  - $\zeta_{typ} = \zeta_{max} = 0.5 \leftrightarrow \zeta^{Theo} = 1/2$

M. Karsai, R. Juhász and F. Iglói, Phys.Rev.E **73** 036116 (2006)

Rounding of first-order phase transitions and optimal cooperation in scale-free networks

- Phase transition in complex networks with random interactions
- Study the combined effect of network topology and bond disorder

Optimal cooperation

- Each cooperating pair receive a benefit represented by the edge weights
- There is a unit support to each component (to each project)
- Optimal cooperation: the total sum of pair cooperation benefits and the supports of independent projects is maximal

- Potts model is a generalization of the Ising model

$$\mathcal{H} = -\sum_{\langle i,j \rangle} J_{ij}\delta(\sigma_i, \sigma_j) \quad , \quad \sigma_i = 0, \ldots, q-1 \qquad (5)$$

- $J_{ij} > 0$ - identical ferromagnetic random couplings
- Partition function: $q \to \infty$ random-cluster representation

$$Z = \sum_{G \subseteq E} q^{\phi(G)}, \quad \phi(G) = c(G) + \beta \sum_{ij \in G} J_{ij} \qquad (6)$$

  - inverse reduced temperature: $\beta \to \beta / \ln q$

- $\phi(G)$ is a **sub-modular function** related to the free-energy
  - calculated with a **combinatorial optimization** algorithm in strongly polynomial time
  - largest term $\phi^* = \max_G \phi(G)$ determines an **optimal graph**

- Phase transition occurs of different order in the thermodynamic limit depends the strength of disorder Δ

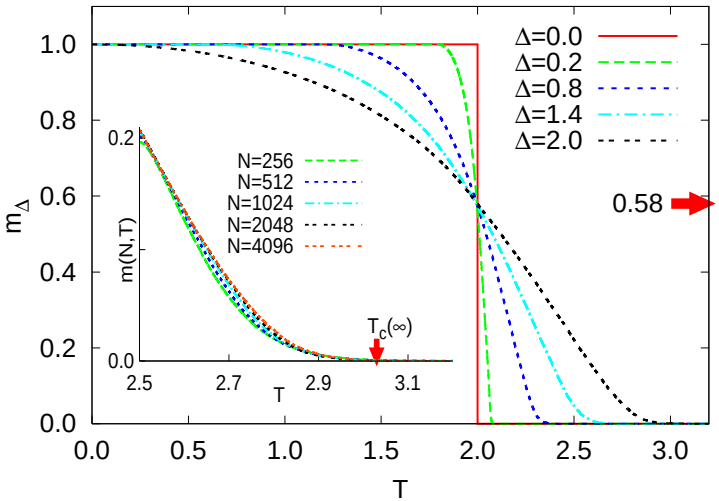
Exact solution for homogeneous $\Delta = 0$ case:

- Homogeneous regular graphs - first order phase transition
- Homogeneous evolving networks (random graphs, SF networks,...)
  - at each time step a new node is added with μ number of edges
- Two typical optimal set configuration:
  - $T < T_c(0)$: fully connected diagram
  - $T > T_c(0)$: empty diagram with isolated nodes
- Phase-transition point: $T_c(\Delta = 0) = J\mu$
- Maximally first-order phase transition

Numerical parameters:

- Barabási-Albert network with $\mu = \mu_0 = 2$ and $N = 2^6 \ldots 2^{12}$
- quasi-continuous distribution: $J(1 \pm \Delta/2)$ where $0 \leqslant \Delta \leqslant 2$
- For a given size, 100 generated networks and for each, 100 independent realizations of disordered couplings

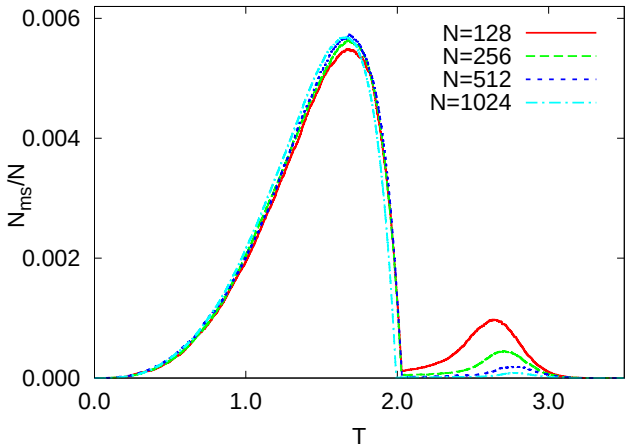
The magnetization:



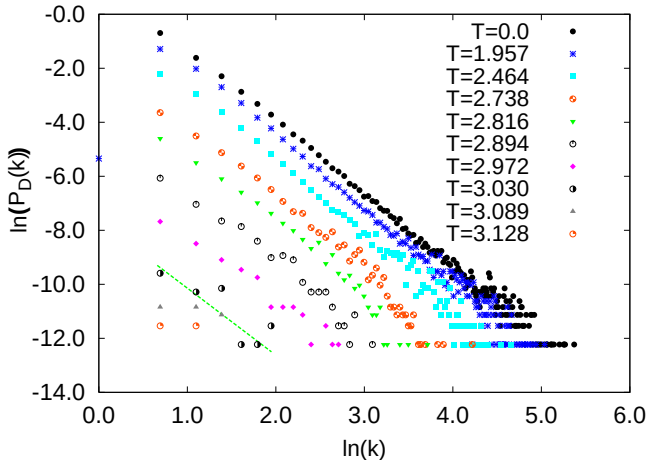
- $m(T)$: size of the largest cluster in the optimal set
- Homogeneous system ($\Delta = 0$): The phase transition is maximally first-order
- Random system ($\Delta \neq 0$): The phase transition rounded and becomes continuous even for weak disorder

Structure of the optimal set:

Fraction of intermediate size clusters



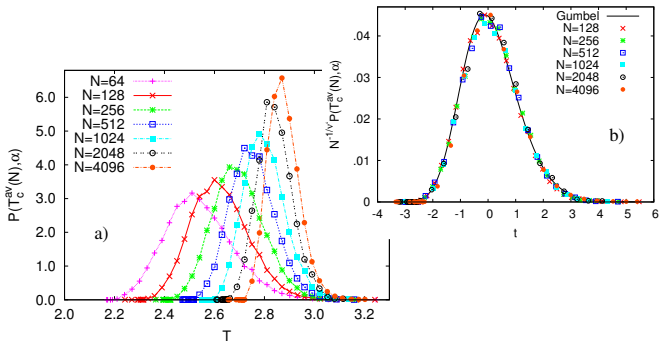
Degree distribution of the giant cluster



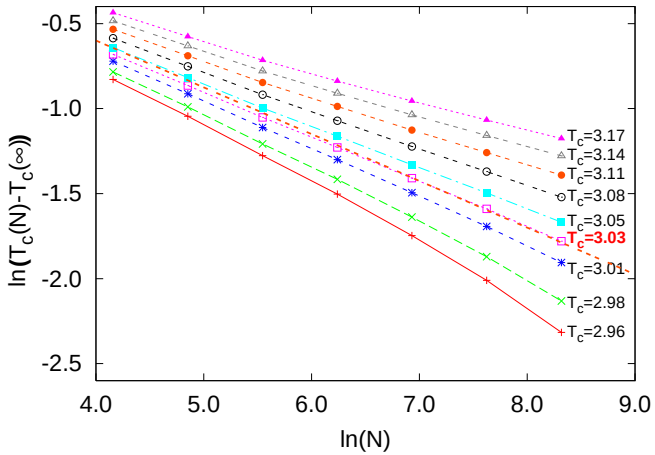
- The optimal set consist:
  - 1 giant cluster
  - large number of isolated sites
  - small number of clusters with intermediate size
- The fraction of intermediate size clusters is negligible
- The degree distribution of the giant cluster keeps the scale-free feature (with exponent $\gamma = 3$) for any $T < T_c$

Distribution of the finite-size transition temperatures:

Distribution of $T_c(\alpha, N)$



Shift of $T_c(\alpha, N)$



- Sample dependent finite-size transition temperature $T_c(\alpha, N)$
  - The largest cluster: $\mathcal{N}(T_c) \simeq AN^{1-x}$
  - Magnetization exponent: $x = 0.69$
- Distribution with two scaling exponents:
  - The shift of T_c^{av}: $\tilde{\nu}' = 3.8(2)$
  - The standard deviation: $\nu' = 5.6(2)$
- Scaling collapse: modified Gumbel distribution
- Critical temperature: $T_c(\infty) = 3.03(2)$
  - from $T_c^{av}(N) - T_c(\infty) \sim N^{1/\tilde{\nu}'}$

M. Karsai, J-Ch. Anglès d'Auriac and F. Iglói, Phys.Rev.E **76** 041107 (2007)

Density of critical clusters in strips of a strongly disordered system

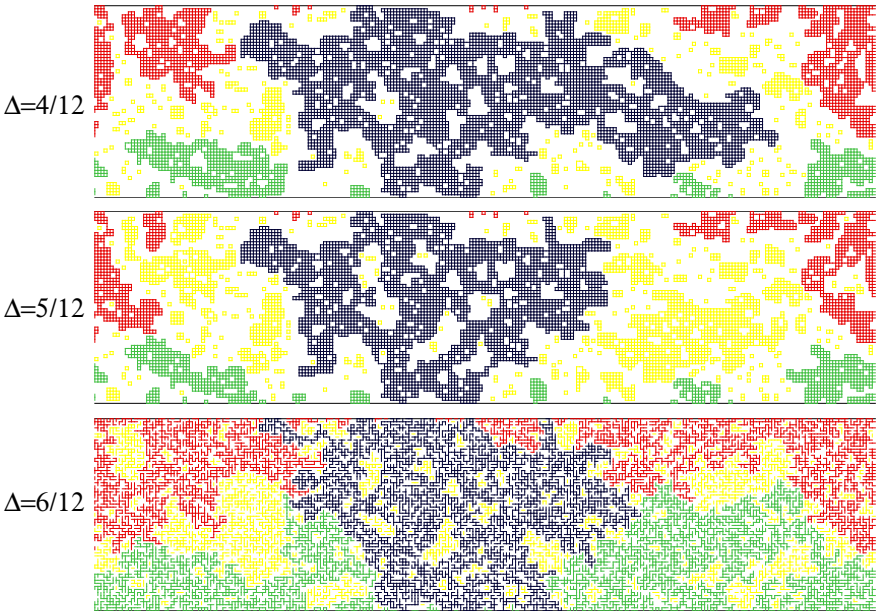
- In critical systems correlated clusters appear in all length scales
  - Cluster representations:
    - *Geometrical clusters*: Percolation, Ising, Potts clusters
    - *Fortuin-Kasteleyn clusters*: high-temperature expansion
  - Finite geometry: Restricted $L * 4L$ strip square lattice
  - *Density of clusters*: clusters touching one or more boundary
    - Exactly derivable using **conformal mapping**
    - Calculated for **critical percolation** (conformal invariance system with geometrical clusters)

AIMS: Check conformal results for **disordered systems**

- Fortuin-Kasteleyn clusters
- average quantities expected to hold conformal invariance

Model: Random bond Potts model with large-q state

- Magnetization scaling dimensions: $x_b = \frac{3-\sqrt{5}}{4}$, $x_s = \frac{1}{2}$
- Reduced random couplings:
  - Bimodal distribution: $K_{ij} = K \pm \Delta$ ($K_c = 1/2$, $0 \leq \Delta \leq 1/2$)
  - $\Delta = 0$ - *Homogeneous limit*
  - $\Delta = 1/2$ - *Percolation limit*



The breaking-up length:

- l_b: typical size of compact units

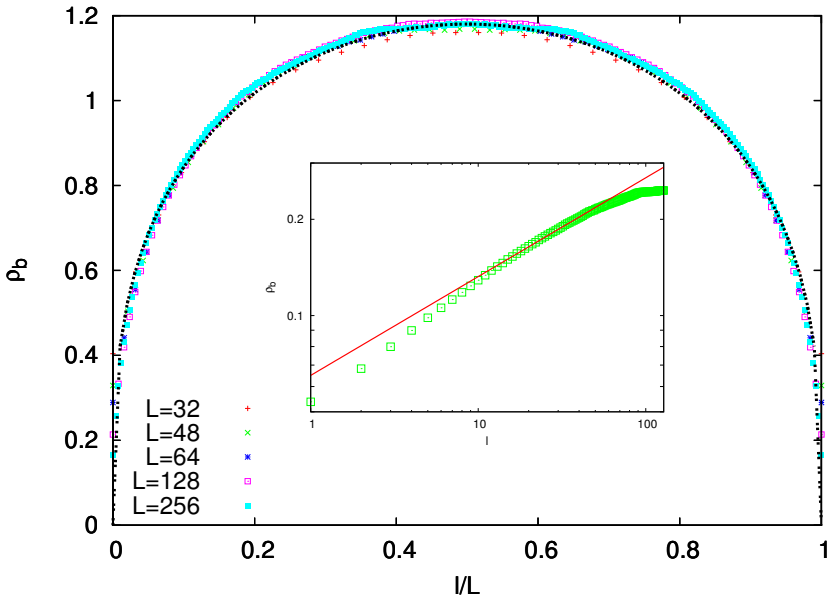
$$l_b \approx l_0 \exp\left[A\left(\frac{K}{\Delta}\right)^2\right]$$

- Choose an appropriate Δ:
  - Smaller than $1/2$
  - Large enough to $l_b \ll L$
- Optimal choice for $L = 256$:
  - $\Delta \approx 5/12 \Rightarrow l_b \sim 14$

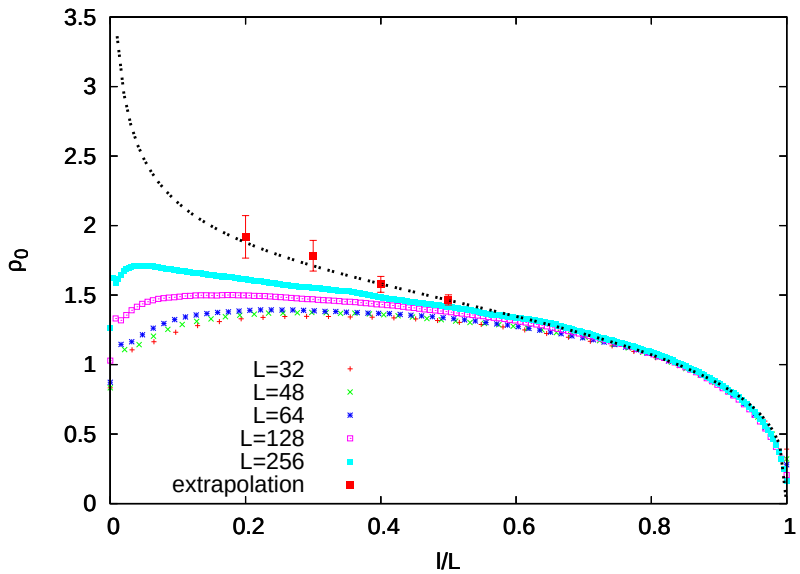
Definition: fraction of samples where a given point belongs to a cluster with prescribed property

- Study the densities along the y axis
- $y = l/L$ $(l = 0...L-1)$ - continuous limit if $l \gg 1$ and $L \gg 1$

The $\rho_b(y)$ density: $\rho_b(y) \propto (\sin \pi y)^{-x_b} \left[\left(\cos \frac{\pi y}{2}\right)^{x_s} + \left(\sin \frac{\pi y}{2}\right)^{x_s} - 1 \right]$



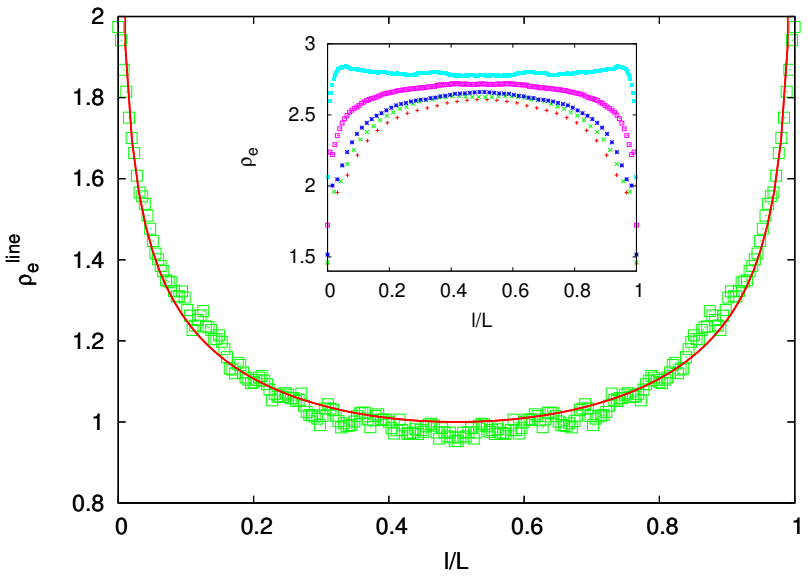
- Clusters which touch both boundaries
- Good fit between the numerical and analytical curves
- Inset: Fit close to the surface $x_s - x_b = 0.309$



The $\rho_0(y)$ density:

$$\rho_0(y) \propto (\sin \pi y)^{-x_b} \left(\cos \frac{\pi y}{2}\right)^{x_s}$$

- Clusters touch boundary at $y = 0$
- Good fit if $l/L \geq 0.5$
- Deviation: middle size clusters are underrepresented (long l_b)



The $\rho_e(y)$ density: $\rho_e(y) \propto (\sin \pi y)^{-x_b}$

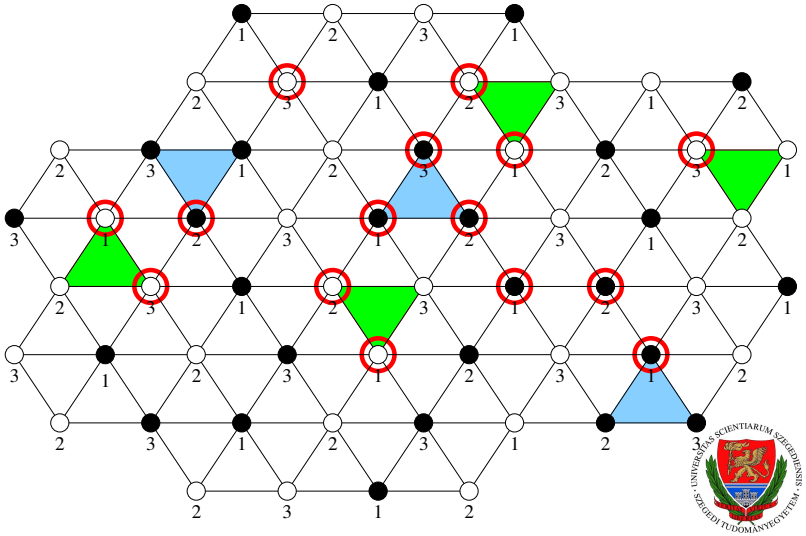
- Clusters touch either of boundaries
- Strongly influenced by l_b
- Measured along spanning lines
- Good fit but worse statistic

M. Karsai, I. A. Kovács, J-Ch. Anglès d'Auriac and F. Iglói, Phys.Rev.E **78** 061109 (2008)

Non-equilibrium dynamics of triangular antiferromagnetic Ising model (TAFIM) at $T = 0$

$\mathcal{H} = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i$ where $\sigma_i = \pm 1$ and $J < 0$

- geometrically frustrated system
  - One unsatisfied bond per triangle
- Reduced magnetic field: $H = h/k_B T$
  - if $T \to 0 \Rightarrow H = O(1)$



At $T = 0$:

- Non-zero residual entropy at the ground state
- Highly degenerated ground state - loose spins
- Expanded critical phase at $T_c = 0$ between $0 \leqslant H \leqslant H_{KT}$
- Kosterlitz-Thouless phase transition at $H_{KT} \simeq 0.266$

Non-equilibrium dynamics:

- Quench the system from $T = \infty$ to $T_c = 0$
- Defects of three neighbouring $\uparrow$ or $\downarrow$ spins
- Zero temperature random update Glauber dynamics
  - Accept spin a flip only if it does not increase the energy

Analog problems: XY model, fully frustrated Ising model

- diffusive non-equilibrium dynamics with logarithmic corrections

TAFIM: contradictory interpretations in the literature

- **Diffusive dynamics** with $z = 2$ dynamical exponent and **logarithmic corrections**
- **Subdiffusive dynamics** with an effective exponent $z = 2.33$
- All these results were derived from indirect considerations

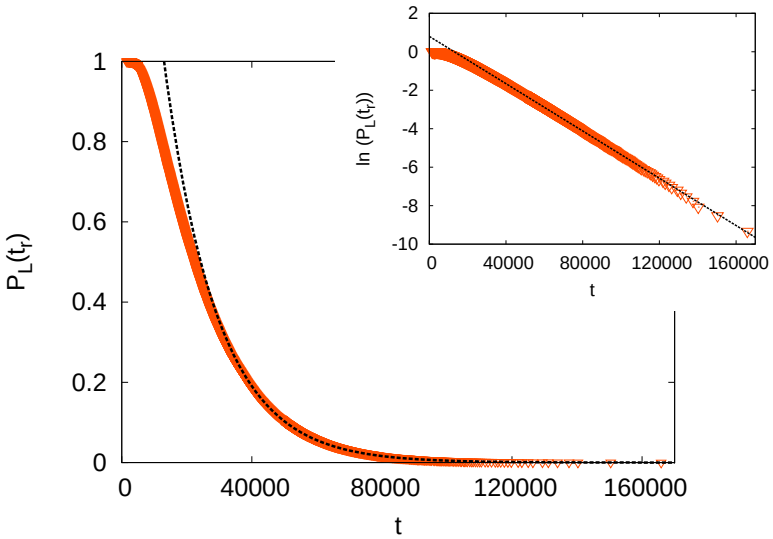


AIMS:

- Direct study of the non-equilibrium dynamics of TAFIM with a new parameter
- Find independent evidences to describe the relevant dynamical behavior

Non-equilibrium relaxation time:

- t_r: the time when the system reaches its ground state energy $E_{min} = -L^2$ after it was proceeded from a random initial state



Sample distribution of $t_r(L)$, $L = 192$

- t_r is size and sample dependent
- sample distribution of t_r

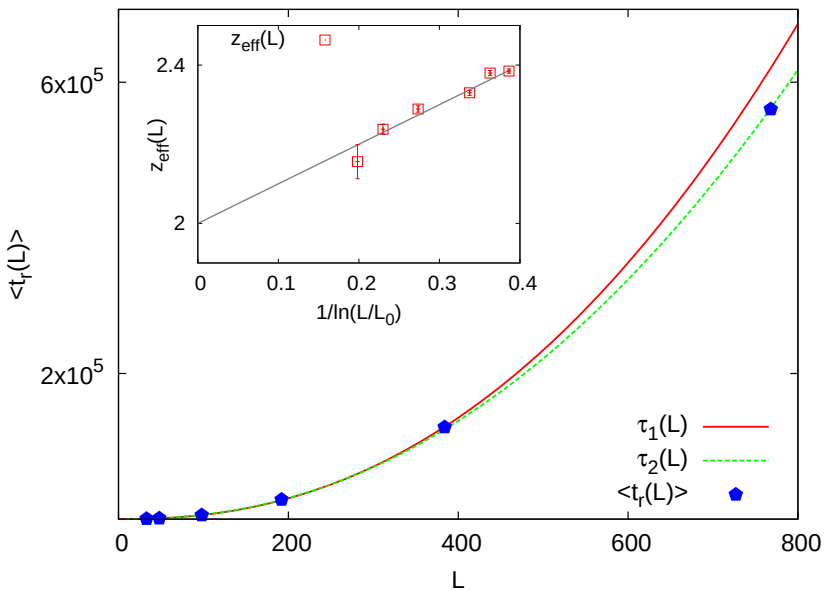
$$P_L(t_r) \propto e^{-t_r/\tau(L)}$$

- $\tau(L)$: size dependent characteristic time

- Two possible scenarios for the scaling of $\langle t_r \rangle$:

$$\tau_1(L) \sim L^z \qquad \text{and} \qquad \tau_2(L) \sim L^z \ln(L/L_0)$$

Fitting of $\langle t_r \rangle$ with $\tau_1(L)$ and $\tau_2(L)$

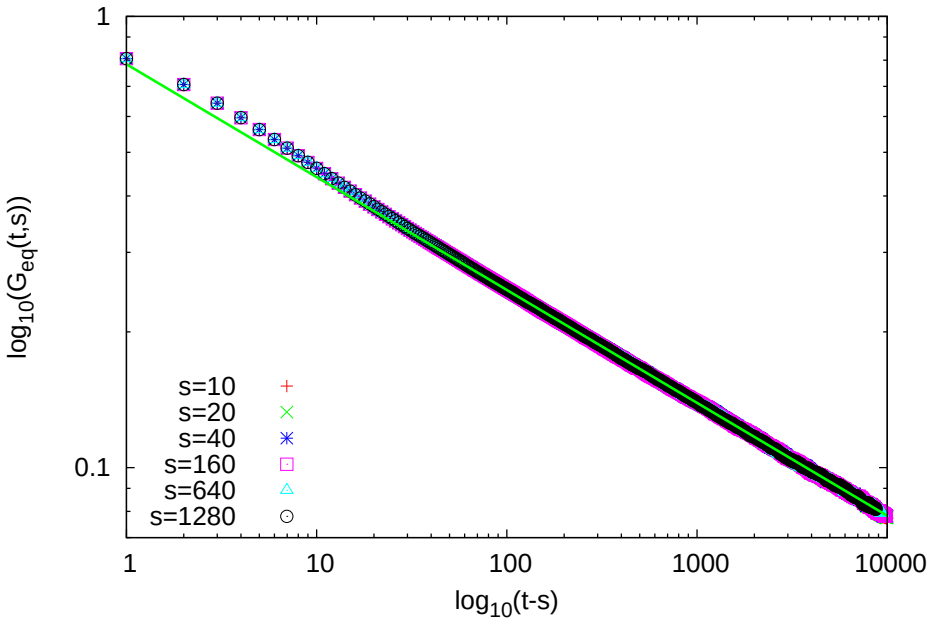


- z, L_0, A parameter fit:
  $\tau_1(L) = .155 L^{2.288}$
  $\tau_2(L) = .174 L^2 \ln(L/3.14)$
- $z = 2$ was obtained, but not fixed!!!
- inset: extrapolation of
  $z_{eff} = 2 + \frac{1}{\ln(L/L_0)} \Rightarrow z \simeq 2$

- For sufficiently large size the two scenarios are distinguishable
- The diffusive dynamical assumption with $z = 2$ and logarithmic correction has a much better fit

- Two time correlation function
  - t: execution time, s: waiting time
- Invariant under time-translation $\Rightarrow$ it depends only on the difference $(t-s)$

$$A_{eq}(t,s) \sim (t-s)^{-a_c} \quad \text{where} \quad a_c = 2\beta/\nu z \text{ és } 2\beta/\nu = 1/2$$



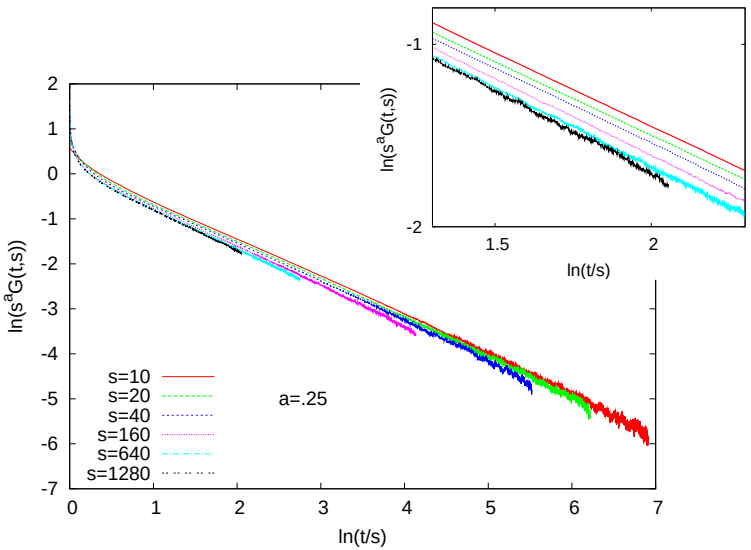
Equilibrium autocorrelation for different s

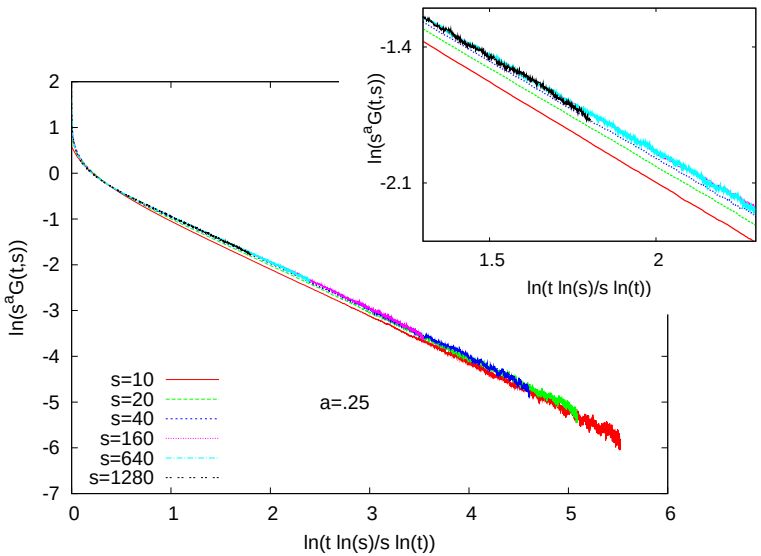
- Equilibrium state: thermalization from a random initial state
- Power-law behaviour for $t-s > 40$
- Fitted straight line:

$$a_c \simeq 0.25 \Rightarrow z \simeq 2$$

- Use the previously calculated $a_c \simeq 0.25$ exponent
- Scaling of the autocorrelation function following the two scenarios:

$$A_{ag}(s,t) \sim s^{-a_c} \widetilde{A}\left(\frac{t}{s}\right) \qquad A_{ag}(s,t) \sim s^{-a_c} \widetilde{A}\left(\frac{t\ln(s)}{s\ln(t)}\right)$$





- Logarithmically corrected scaling supply a better asymptotic collapse

M. Karsai, J-Ch. Anglès d'Auriac and F. Iglói, submitted to J. Stat. Mech.

Conclusions

- Cooperative behaviour and phase transitions in regular lattices and complex networks
- Completed studies in four different subjects
  - Non-equilibrium phase transitions in scale-free networks
  - Optimal cooperation in scale-free networks
  - Density of critical clusters in a strongly disordered system
  - Non-equilibrium dynamics of triangular antiferromagnetic Ising model at $T = 0$
- New results are published in referred scientific journals
  - M. Karsai, R. Juhász and F. Iglói, Phys.Rev.E **73** 036116 (2006)
  - M. Karsai, J-Ch. Anglès d'Auriac and F. Iglói, Phys.Rev.E **76** 041107 (2007)
  - M. Karsai, I. A. Kovács, J-Ch. Anglès d'Auriac and F. Iglói, Phys.Rev.E **78** 061109 (2008)
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THANK YOU FOR YOUR ATTENTION!