Normalization properties of symmetric logical calculi

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Introduction

The theory of the λ-calculus was introduced by Church in two subsequent papers (cf. Church [5], Church [6]). His work was part of the vast efforts aiming to give a theoretical foundation for logic and mathematics at the end of the nineteenth and in the beginning of the twentieth century. The λ-calculus was invented for two main purposes: to develop a general theory of functions and to extend this theory with logical notions providing a foundation for logic and mathematics. After the appearance of Gödel's papers (see Gödel [23]) it turned out that the foundational issues have their limits (Church [7]), but the calculus has still proved to be a good tool from the computational point of view: it is strong enough to describe all mechanically computable functions (cf. Turing [63]) in spite of the fact that it has a very simple syntax. Therefore it has become a model for later programming languages.

Shortly after Church's papers, also in the spirit of these foundational investigations, in 1935 Gentzen has published a proof of the consistency of arithmetic ([21], [22]), which naturally, by Gödel's results, cannot be formalized in first order Peano arithmetic. In these papers Gentzen presented two formalizations of logic, on one hand the sequent calculus and on the other hand the natural deduction, which have been the key tools for investigating logics ever since. Gentzen's method was based on normalizing proofs in sequent calculus with cut eliminations. It was not known until Prawitz (cf. [50]) how to define normal proofs in natural deduction and how to construct them. And it turned out even more later, though some preliminary results were known (see e.g. Curry and Feys [10], de Bruijn [4]), that there is a close connection between the calculus introduced by Church and the formulation of logic invented by Gentzen. In his paper ([31]) Howard described a direct relation between proofs for intuitionistic predicate logic in natural deduction style and terms of typed λ-calculus, with normalization steps of proofs given by Prawitz corresponding to reductions in typed λ-calculus. The correspondence presented in Howard [31] has came to be known as the Curry-Howard isomorphism.

Parallel to this a new concept called continuations emerged from the 1960s (cf. Reynolds [54], Fischer [19], Wadsworth and Strachey [60]). It proved to be an efficient tool for defining the denotational semantics of numerous control facilities. The first syntactic description of control-like devices appeared owing to the work of Felleisen et al. [18], where the CPS-translation was also extended to a λ-calculus with control operators such as $\mathcal{C}$ and $\mathcal{A}$. In their semantical analysis of the CPS-translation [40], Meyer and Wand has associated a typing with the translation, this established a connection with implicational propositional logic. This connection was further expanded by Griffin, who gave typing rules for a λ-calculus with $\mathcal{C}$ and $\mathcal{A}$ and proved the termination of evaluations of such terms w.r.t. call-by-value β-conversion and conversion rules for $\mathcal{C}$ and $\mathcal{A}$. The remarkable observation in Griffin's work was the discovery that the control operator $\mathcal{C}$ can be given a type $\neg\neg A \to A$ for some A, though in the cost of sacrificing the expectation for a program of producing evidence for a formula in the sense of the Brouwer-Heyting-Kolmogorov- (BHK-) interpretation established for intuitionistic predicate logic (cf. [3], [8], [31]). This

drawback was eliminated by Murthy in [41], he amalgamated a method of transforming classical proofs into constructive ones, known as Friedman's A-translation (see [20]), with Griffin's typing of the λ-terms formulated with control operators. He augments a programming language construction equivalent to Heyting-arithmetic with the control operator $\mathcal{C}$, thus obtaining a typed-calculus containing a rule for double-negation elimination. He shows for the resulting calculus that it is a suitable interpretation for Peano-arithmetic: type-soundness is preserved w.r.t a specific reduction strategy and a well-known property of PA (cf. [34]) is retained: it provides evidence for proofs of Π^0_2-sentences in the sense of the BHK-interpretation.

These investigations highlighted the possibility of finding logical calculi which, by the Curry-Howard isomorphism, can provide direct computational interpretations for classical logic. Several calculi have been invented for this purpose.

The $\lambda\mu$-calculus presented by Parigot in [45] finds its origin in the so called Free Deduction (FD) [48]. The calculus of Free Deduction is a calculus of multiple conclusions, moreover, in its formulation, Parigot resolves the deterministic nature of intuitionistic natural deduction: unlike in the case of intuitionistic natural deduction, when eliminating an appearance of a cut in FD, there can be several choices of picking out the subdeductions to be transformed. This is expressed by Parigot by saying that FD is symmetric in the sense of having left and right inputs at the same time, while natural deduction only has left inputs. He obtains the $\lambda\mu$-calculus in [45] by restricting the system of FD to left inputs only. Furthermore, the role of indicating which one is the active formula in a deduction rule is solved by the introduction of a new kind of variables, the so called μ-variables. They mark formulas which are not active at the moment but the current continuation can be passed over to them. Besides the usual β-reduction he introduces a new reduction rule called the μ-rule corresponding to structural cut eliminations made necessary by the occurrence of new forms of cuts due to the rule in connection with the μ-variables, which can be corresponded to the classical absurdity rule. In addition, some more simplification rules are introduced. The result is a calculus, the $\lambda\mu$-calculus, which is closely related to classical natural deduction by the Curry-Howard isomorphism and which can be considered as the classical extension of Krivine's second order λ-calculus called AF_2 (cf. Krivine [35]). In particular, Felleisen's control operator $\mathcal{C}$ can be expressed in this calculus.

The μ'-rule is the symmetric counterpart of the μ-rule. The reason for its introduction is the following. The $\lambda\mu$-calculus possesses many nice properties habitual in the case of the λ-calculus: subject reduction property, confluence and strong normalization. However, a property important from computer science point of view is not present: the unicity of the representation of data is lost. In typed λ-calculus, if N is the type of the integers, any normal term of type N is of the form $\overline{n} = \lambda x \lambda f (f)^n x$ for some natural number n, that is, equal to the n-th Church-integer. This is no more true here: we can find normal terms of type N other than Church-integers (see Parigot [46], Nour [42]). Parigot noticed in [46] that the addition of the μ'-rule can solve this problem. This is achieved, however, at the cost of losing confluence and, when additional simplification rules are present, even the strong normalization property disappears. This is examined in detail in Chapter 1 of the present work.

Another calculus expressing classical meaning and showing much resemblance to Parigot's $\lambda\mu$-calculus is the $\lambda_\triangle$-calculus of Rehof and Sørensen (cf. [53]). They define reduction rules very similar to those of the $\lambda\mu$-calculus, but in a calculus only having one set of variables. This yields that the simplification rule corresponding to the ρ-rule of the $\lambda\mu$-calculus is defined in a rather circumstantial way, which results that the problem of unicity of the representation of data, being present concerning $\lambda_\triangle$ also, cannot be solved as elegantly as in the case of the $\lambda\mu$-calculus by defining simply the symmetric counterpart of the reduction rule in $\lambda_\triangle$ corresponding to the μ-reduction.

In [36] Krivine gives a method for expressing terms of type $\neg\neg A \to A$ in the λ-calculus. He adds a constant c of type $\forall X(\neg\neg X \to X)$ to the language of AF_2 together with its head-reduction rule. Krivine's method resembles the ones of Griffin and Murthy in two ways: first, terms for classical proofs will be usual λ-terms possibly with c inside, since classical proofs can be obtained from intuitionistic proofs supplemented with the axiom $\forall X(\neg\neg X \to X)$. Secondly, the reduction-rules of the λ_c-calculus are defined in a call-by-name manner, only head-reduction sequences are allowed, which prevents of utilizing the calculus for modelling the reductions in the proofs of the classical formulas corresponding to the terms by reductions of the terms themselves. Instead, Krivine applies the λ_c-calculus for proving a result of the simulation of call-by-value with call-by-name via the so-called storage operators (see Krivine [37] and [38]). This gives another way of calculating the value of a classical integer (cf. Nour [42]).

Curien and Herbelin introduced the $\overline{\lambda}\mu\tilde{\mu}$-calculus (cf. [11]), which was designed to establish a correspondence, via the Curry-Howard isomorphism, between reductions in proofs written in classical Gentzen-style sequent calculus and simplifications of terms of a logical calculus. The $\overline{\lambda}\mu\tilde{\mu}$-calculus possesses a rather strong symmetry: it has right-hand side and left-hand side terms (also referred to as environments). The strong normalization of $\overline{\lambda}\mu\tilde{\mu}$ was shown by Polonovski ([49]). David and Nour gave an arithmetical proof of this result (cf. [14]).

The calculus defined by Wadler (cf. [64]) is very similar to the $\overline{\lambda}\mu\tilde{\mu}$-calculus of Curien and Herbelin. The Wadler-calculus was also designed to provide an extension of the Curry-Howard-isomorphism between the calculus and classical Gentzen-style sequent calculus. Moreover, it has a clear notion of duality, making it possible to establish a convenient correspondence between call-by-name and call-by-value evaluations (see also [65], [17]).

The λ^{Sym}-calculus presented by Berardi and Barbanera ([1]) differs in some respect from the ones discussed above. Namely, it has an involutive negation and $\neg A$ is not derived from A as $A \to \bot$. As atomic types they have negated and unnegated types, and they apply $\wedge$ and $\vee$ for connectors instead of $\to$. Hence, in defining negation, they rather make use of the natural symmetry of classical logic expressed by the de Morgan laws.

The present work is concerned with proof theoretical properties, more exactly mostly with normalization and standardization properties, of the propositional parts of some of the calculi mentioned above. As a common feature of the investigations continued in the sequel we note that all the proofs presented in this work are arithmetical, that is, they can be formalized in first order Peano-arithmetic.

The $\lambda\mu$-calculus

The $\lambda\mu$-calculus was introduced by Parigot in [45]. There are two kinds of variables: $\{x, y, z, \ldots\}$ denote λ-variables and $\{\alpha, \beta, \gamma, \ldots\}$ denote μ-variables, respectively. In the formation of terms named terms and unnamed terms are distinguished. Basically, a named term arises by the application of a μ-variable to an unnamed term, moreover, a μ-operator must always be followed by a named term.

Definition 1 *The term formation rules are:*

$$\mathcal{T} := \mathcal{T}_u \cup \mathcal{T}_n,$$

where

$$\mathcal{T}_u := x \mid \mu\alpha\mathcal{T}_n \mid \lambda x\mathcal{T}_u \mid (\mathcal{T}_u\ \mathcal{T}_u), \qquad \mathcal{T}_n := (\alpha\ \mathcal{T}_u).$$

The calculus examined by us is the simply typed one. The types are built from atomic formulas (or, in other words, atomic types) with the connector $\rightarrow$. In the definition below Γ denotes a (possibly empty) context, that is, a set of declarations of the form $x : A$ (resp. $\alpha : \neg A$) for a λ-variable x (resp. a μ-variable α) and type A such that a λ-variable x (resp. a μ-variable α) occurs at most once in an expression $x : A$ (resp. $\alpha : A$) of Γ.

Definition 2 *The typing rules are:*

$$\frac{}{\Gamma, x : A \ \vdash\ x : A}\ ax$$

$$\frac{\Gamma, x : A \ \vdash\ M : B}{\Gamma \ \vdash\ \lambda x M : A \rightarrow B}\ \rightarrow_i \qquad \frac{\Gamma \ \vdash\ M : A \rightarrow B \quad \Gamma \ \vdash\ N : A}{\Gamma \ \vdash\ (M\ N) : B}\ \rightarrow_e$$

$$\frac{\Gamma, \alpha : \neg A, \beta : \neg B \ \vdash\ M : B}{\Gamma, \beta : \neg B \ \vdash\ \mu\alpha(\beta\ M) : A}\ \mu$$

We will say that M is typable with A, if there is a set of declarations Γ such that $\Gamma \ \vdash\ M : A$ holds.

Definition 3 *The reduction rules are as follows:*

- *The β-reduction is:*
$$(\lambda x M\ N) \rightarrow_\beta M[x := N].$$
- *The μ-reduction rule is*
$$(\mu\alpha M\ N) \rightarrow_\mu \mu\alpha M[\alpha :=_r N],$$
where $M[\alpha :=_r N]$ is obtained from M by replacing every subterm in M of the form $(\alpha\ U)$ by $(\alpha\ (U\ N))$.
- *The ρ-reduction is*
$$(\alpha\ \mu\beta M) \rightarrow_\rho M[\beta := \alpha],$$
where $M[\beta := \alpha]$ is obtained by exchanging in M every free occurrence of β for α.

- *And finally the θ-reduction is*

$$\mu\alpha(\alpha\ M) \rightarrow_\theta M,$$

provided α is not free in M.

As it is customary, by a reduction step we mean the closure of the reduction relation compatible with respect to the term formation rules. In general, as it will be defined in Chapter 1, $\rightarrow$ denotes the compatible closure of a reduction relation, or that of the union of some set of relations, while by $\twoheadrightarrow$ we mean the reflexive, transitive closure of $\rightarrow$. Observe that in the calculus there are two different kinds of substitutions: $M[x := N]$ is a substitution for a λ-variable, or λ-substitution, while $M[\alpha =_r N]$ is a substitution for a μ-variable, or μ-substitution. Unless otherwise stated, by substitutions we mean the usual capture avoiding substitutions.

Remark 4 *We regard the $\lambda\mu$-calculus as the calculus defined above, together with the β- and μ-rules. However, as an abuse of notation, we may also call some extended versions of the calculus the $\lambda\mu$-calculus. If ambiguity should occur, we indicate explicitly the reduction rules considered besides the β- and the μ-rules.*

The proof-theoretical meaning of the reduction rules

Explanation 5 *The reductions of terms correspond to reductions in the proofs in the following sense.*

- *The β-rule serves for eliminating a $\rightarrow_i$ followed by a $\rightarrow_e$.*
- *The μ-reduction: Assume $\mu\alpha(\beta\ M)$ corresponds to a proof of $A \rightarrow B$ and N corresponds to a proof of A yielding together, as $(\mu\alpha(\beta\ M)\ N)$, a proof of B. Then $\mu\alpha(\beta\ M)[\alpha :=_r N]$ corresponds to a proof of B in the following way. Every step in the deduction where a μ-rule is used to obtain $\mu\gamma(\alpha\ U) : C$ for some $U : A \rightarrow B$ with $\alpha : \neg(A \rightarrow B)$ is exchanged for a μ-rule with $(U[\alpha =_r N]\ N) : B$ and $\alpha : \neg B$ to obtain a proof of $\mu\gamma(\alpha\ (U[\alpha =_r N]\ N)) : C$.*
- *The ρ-reduction: Assume $\mu\alpha(\beta\ M)$ corresponds to a proof of A, where the last rule applied was a μ-rule, and suppose $\beta : \neg B$. Let $M = \mu\gamma M_1$. This implies $\gamma : \neg B$. Now, $\mu\alpha M_1[\gamma := \beta]$ corresponds to a proof of A as follows. Every step in the deduction where a μ-rule is used to obtain $\mu\delta(\gamma\ U) : C$ for some $U : B$ with $\gamma : \neg B$ is substituted with μ-rule with $U[\gamma := \beta] : B$ and $\beta : \neg B$ to obtain a deduction of $\mu\delta(\beta\ U[\gamma := \beta]) : C$.*
- *The θ-reduction: Assume $\mu\alpha(\alpha\ M)$ corresponds to a proof of A assuming $\alpha : \neg A$, and $\alpha : \neg A$ is not free in M. Then the last rule applied must have been a μ-rule with $M : A$ and $\alpha : \neg A$. But α is not free in M, thus we also have a deduction of $\Gamma \vdash M : A$ for some Γ such that $\alpha : \neg A \notin \Gamma$.*

Figures 1 and 2 give an account of the correspondences described above.

Main theoretical properties of the $\lambda\mu$-calculus

- **Church-Rosser property**

  Let $\rightarrow$ denote one of the relations $\rightarrow_\beta, \rightarrow_\mu, \rightarrow_\rho, \rightarrow_\theta$. Let $\twoheadrightarrow$ be the reflexive, transitive closure of $\rightarrow$. Then we have:

  Theorem 6 *Let M_1, M_2 and M_3 be $\lambda\mu$-terms such that $M_1 \twoheadrightarrow M_2$ and $M_1 \twoheadrightarrow M_3$. Then there exists an M_4 for which $M_2 \twoheadrightarrow M_4$ and $M_3 \twoheadrightarrow M_4$.*

A proof of the above assertion can be found in Parigot [45], in Py [51] or in Rozière [56]. In Py [51] the question is expounded to a greater extent together with the results belonging to the theme.

- **Type preservation property**

Proposition 7 *Let M, N and A, Γ be such that $\Gamma \vdash M : A$ and $M \twoheadrightarrow N$. Then $\Gamma \vdash N : A$.*

The property can be verified by double induction on the length of the reduction sequence $M \twoheadrightarrow N$ and the complexity of M.

- **Subformula property**

Let Γ be a context, A be a type. Let $A^* = B$ if $A = \neg B$ for some B, otherwise let $A^* = A$. We write $\Gamma^* = \{A^* \mid x : A \in \Gamma\}$. Intuitively, we say that a formula A occurs in a derivation Π if there is a sequent $\Gamma \vdash M : B$ of Π such that either $A = B$ or there exists a declaration in Γ the type belonging to which is A.

Proposition 8 *Let M, A, Γ be such that*

$$\Gamma \vdash M : A, \qquad (*)$$

and let Π be a derivation of $(*)$. If M is in $\lambda\mu$-normal form, then every type occurring in Π is either a type occurring in Γ, or a subformula of a type occurring in Γ^*, or a subformula of A.

The proof of this property can be found in De Groote [29], though stated in a slightly different form.

- **Strong normalization**

Theorem 9 *Let M be a term typable with some type A. Then every reduction sequence starting from M is finite.*

There are several proofs of this result in the literature. Consider, for example, Parigot [47], David and Nour [12]. In [29] de Groote proves the strong normalization of the simply typed $\lambda\mu$-calculus extended with terms of conjunctive and disjunctive types, respectively. He does not consider the ρ- and θ-rules in his calculus.

The $\lambda\mu\mu'$-calculus

The $\lambda\mu\mu'$-calculus arises from the $\lambda\mu$-calculus by adding to the set of reduction rules the symmetric analogue of the μ-rule, called the μ'-rule.

Definition 10 *The μ'-reduction is the rule*

$$(M\ \mu\alpha N) \rightarrow_{\mu'} \mu\alpha N[\alpha :=_l M],$$

where $N[\alpha :=_l M]$ is obtained from N by replacing every subterm in N of the form $(\alpha\ U)$ by $(\alpha\ (M\ U))$.

Parigot defined the $\lambda\mu\mu'$-calculus in [46] because of the following reasons. Though the $\lambda\mu$-calculus is confluent, enjoys the subject reduction property and strongly normalizes, it has the drawback that the uniqueness of the representation of data is lost. In λ-calculus it is true, that any term of type $N = \forall X(X \to ((X \to X) \to X))$ (which is the propositional trace of the usual type of integers) is β-equivalent to a Church integer. In $\lambda\mu$-calculus this is no more true. The admittance of the μ'-rule in the system solves the problem, and even the subject reduction property is retained, at least for the simply typed calculus. However, confluence is destroyed. In Nour [42] a detailed exposition is given about the possibilities of finding the values of classical integers in the $\lambda\mu$-calculus. Figure 3 shows two simple terms demonstrating the non-confluence of $\lambda\mu\mu'$. As for the terminology, the same stipulation should be valid as in Remark 4.

The proof-theoretical meaning of the μ'-rule

We can also establish a correspondence between the reductions of terms by means of the μ'-rule and the reductions in the proofs.

Explanation 11 *Assume $\mu\alpha(\beta\ N)$ corresponds to a proof of A and M corresponds to a proof of $A \to B$ yielding together as $(M\ \mu\alpha(\beta\ N))$ a proof of B. Then $\mu\alpha(\beta\ N)[\alpha :=_l M]$ corresponds to a proof of B in the following way. Every step in the deduction where a μ-rule is used to obtain $\mu\gamma(\alpha\ U) : C$ for some $U : A$ with $\alpha : \neg A$ is exchanged for a μ-rule with $(M\ U[\alpha =_l M]) : B$ and $\alpha : \neg B$ to obtain a proof of $\mu\gamma(\alpha\ (M\ U[\alpha =_l M])) : C$.*

Figure 4 illustrates this correspondence.

The $\lambda\mu$-calculus in de Groote-style

In [28] de Groote has proposed a new version of the $\lambda\mu$-calculus by modifying its syntax. Namely, in the construction of terms the distinction between named and unnamed terms has disappeared and the term forming rules became more flexible: a μ-operator can be followed now by any kind of term (in the untyped version), not necessarily by a term beginning with a μ-variable. Thus the term formation rules are:

Definition 12

$$\mathcal{T} := x \mid \lambda x\mathcal{T} \mid \mu\alpha\mathcal{T} \mid (\alpha\ \mathcal{T}) \mid (\mathcal{T}\ \mathcal{T})$$

A new type called $\bot$ is introduced. The types are built from atomic formulas (or atomic types) with the constant symbol $\bot$ and the connector $\to$. The typing rules and the reduction rules remain the same except for the following modification. The μ-rule is replaced by the new rules $\bot_e$ and $\bot_i$. Let Γ denote a context, a set of declarations of the form $x : A$ (resp. $\alpha : \neg A$) for a λ-variable x (resp. a μ-variable α) and type A such that a λ-variable x (resp. a μ-variable α) occurs at most once in a context $x : A$ (resp. $\alpha : \neg A$) of Γ. The new typing rules of the de Groote-style $\lambda\mu$-calculus are as follows.

Definition 13

$$\frac{\Gamma, \alpha : \neg A \ \vdash\ M : \bot}{\Gamma \ \vdash\ \mu\alpha M : A}\ \bot_e \qquad \frac{\Gamma, \alpha : \neg A \ \vdash\ M : A}{\Gamma, \alpha : \neg A \ \vdash\ (\alpha\ M) : \bot}\ \bot_i$$

The reduction rules are the same as in the Parigot-style $\lambda\mu$- and $\lambda\mu\mu'$-calculi, adjusted to the new syntax of the terms.

The justification of the new presentation

There can arise the question what was the reason behind the modification of the syntax of the $\lambda\mu$-calculus. In Parigot [45] the following example for a deduction of $\neg\neg A \to A$ is given.

$$\dfrac{\dfrac{\alpha : \neg A, \gamma : \neg\bot, y : \neg\neg A \ \vdash \ y : \neg\neg A \qquad \dfrac{\dfrac{\alpha : \neg A, \beta : \neg\bot, \gamma : \neg\bot, x : A, y : \neg\neg A \ \vdash \ x : A}{\alpha : \neg A, \gamma : \neg\bot, x : A, y : \neg\neg A \ \vdash \ \mu\beta(\alpha\ x) : \bot}}{\alpha : \neg A, \gamma : \neg\bot, y : \neg\neg A \ \vdash \ \lambda x\mu\beta(\alpha\ x) : \neg A}}{\alpha : \neg A, \gamma : \neg\bot, y : \neg\neg A \ \vdash \ (y\ \lambda x\mu\beta(\alpha\ x)) : \bot}}{\gamma : \neg\bot, y : \neg\neg A \ \vdash \ \mu\alpha(\gamma\ (y\ \lambda x\mu\beta(\alpha\ x))) : A}}{\gamma : \neg\bot \ \vdash \ \lambda y\mu\alpha(\gamma\ (y\ \lambda x\mu\beta(\alpha\ x))) : \neg\neg A \to A}$$

The disadvantage of this proof is that the declaration $\gamma : \neg\bot$ does not disappear from the context, owing to which the variable γ remains free in the term representing the proof. This situation seems to be rather unsatisfactory, since the term obtained is intended to be the code of a proof of a closed formula. The reason behind this phenomenon is the fact that the formation of unnamed terms beginning with a μ is closely connected with that of named terms.

In the de Groote-style formulation this proof can be written as follows.

$$\dfrac{\dfrac{\dfrac{\alpha : \neg A, y : \neg\neg A \ \vdash \ y : \neg\neg A \qquad \dfrac{\dfrac{\alpha : \neg A, x : A, y : \neg\neg A \ \vdash \ x : A}{\alpha : \neg A, x : A, y : \neg\neg A \ \vdash \ (\alpha\ x) : \bot}}{\alpha : \neg A, y : \neg\neg A \ \vdash \ \lambda x(\alpha\ x) : \neg A}}{\alpha : \neg A, y : \neg\neg A \ \vdash \ (y\ \lambda x(\alpha\ x)) : \bot}}{y : \neg\neg A \ \vdash \ \mu\alpha(y\ \lambda x(\alpha\ x)) : A}}{\vdash \ \lambda y\mu\alpha(y\ \lambda x(\alpha\ x)) : \neg\neg A \to A}$$

The $\overline{\lambda}\mu\tilde{\mu}$-calculus

The $\overline{\lambda}\mu\tilde{\mu}$-calculus was introduced by Curien and Herbelin (cf. [30] and [11]). We examine here the calculus defined in [11], which is a simply typed one. The $\overline{\lambda}\mu\tilde{\mu}$-calculus was invented for representing proofs in classical Gentzen-style sequent calculus: under the Curry-Howard correspondence a version of Gentzen-style sequent calculus is obtained as a system of simple types for the $\overline{\lambda}\mu\tilde{\mu}$-calculus. Moreover, the system presents a clear duality between call-by-value and call-by-name evaluations.

Besides the notions of contexts and terms there is an other syntactic category called commands. The set of commands are denoted by p below, the other two sets constitute the sets of terms.

Definition 14 *The set of commands, terms and contexts:*

$$\begin{array}{lcccccc} p & ::= & \lfloor t, e \rfloor & & & & \\ t & ::= & x & | & \lambda x t & | & \mu a p \\ e & ::= & a & | & (t.e) & | & \tilde{\mu} x p \end{array}$$

In the sequel, we use the terminology right-term for a term and left-term for a context. The types are built from atomic formulas (or, in other words, atomic types) with the connectors $\to$. The typing system is a sequent calculus based on judgements of the following form:

$$p : (\Gamma \ \vdash \ \triangle) \qquad \Gamma \ \vdash \ t : A \mid \triangle \qquad \Gamma \mid e : A \ \vdash \ \triangle,$$

where Γ (resp. $\triangle$) is a set of declarations of the form $x : A$ (resp. $a : A$), x (resp. a) denoting a t-variable (resp. an e-variable) and A representing a type, such that x (resp. a) occurs at most once in an expression of Γ ($\triangle$) of the form $x : A$ (resp. $a : A$).

Definition 15 *The typing rules are as follows:*

$$\frac{}{\Gamma, x : A \vdash x : A \mid \triangle} \qquad \frac{}{\Gamma \mid a : A \vdash a : A, \triangle}$$

$$\frac{\Gamma, x : A \vdash t : B \mid \triangle}{\Gamma \vdash \lambda x t : A \to B \mid \triangle} \qquad \frac{\Gamma \vdash t : A \mid \triangle \quad \Gamma \mid e : B \vdash \triangle}{\Gamma \mid (t.e) : A \to B \vdash \triangle}$$

$$\frac{\Gamma \vdash t : A \mid \triangle \quad \Gamma \mid e : A \vdash \triangle}{\lfloor t, e \rfloor : (\Gamma \vdash \triangle)}$$

$$\frac{p : (\Gamma \vdash a : A, \triangle)}{\Gamma \vdash \mu a p : A \mid \triangle} \qquad \frac{p : (\Gamma, x : A \vdash \triangle)}{\Gamma \mid \tilde{\mu} x p : A \vdash \triangle}$$

Definition 16 *The reduction rules of the calculus:*

- $\lfloor \lambda x t, (t'.e) \rfloor \to_{\lambda} \lfloor t', \tilde{\mu} x \lfloor t, e \rfloor \rfloor$
- $\lfloor \mu a p, e \rfloor \to_{\mu} p[a := e]$
- $\lfloor t, \tilde{\mu} x p \rfloor \to_{\tilde{\mu}} p[x := t]$
- $\mu a \lfloor t, a \rfloor \to_{s_l} t$ *if* $a \notin Fv(t)$
- $\tilde{\mu} x \lfloor x, e \rfloor \to_{s_r} e$ *if* $x \notin Fv(e)$

The reduction rules of the calculus are corresponded to reductions on the proofs on the logical side. Figure 5 illustrates the correspondences between the reductions of the terms and reductions in the proofs concerning the β- and s_l-rules. Figure 6 deals with the μ-rule together with the more interesting steps of the transformation of a proof represented by the term $\lfloor \mu a p, e \rfloor$ into a proof represented by the term $p[a := e]$.

Main theoretical properties of the $\overline{\lambda}\mu\tilde{\mu}$-calculus

- **Type-preservation property**

  Proposition 17 *Let t, t' (or e, e' or p, p', resp.) and Γ, $\triangle$ be such that $\Gamma \vdash p : A \mid \triangle$ (or $\Gamma \mid e : A \vdash \triangle$ or $p : (\Gamma \vdash \triangle)$, resp.) and $t \twoheadrightarrow t'$ (or $e \twoheadrightarrow e'$ or $p \twoheadrightarrow p'$). Then $\Gamma \vdash t' : A \mid \triangle$ (or $\Gamma \mid e' : A \vdash \triangle$ or $p' : (\Gamma \vdash \triangle)$, resp.).*

  The property can be verified by double induction on the length of the reduction sequence $t \twoheadrightarrow t'$ (or $e \twoheadrightarrow e'$ or $p \twoheadrightarrow p'$) and the complexity of t (or e or p, resp.).

- **Subformula property**

Proposition 18 *Let t (or e or p, resp.), A, Γ, $\triangle$ be such that*

$$\Gamma \vdash t : A \mid \triangle$$

(or $\Gamma \mid e : A \vdash \triangle$ or $p : (\Gamma \vdash \triangle)$, resp.), and let Π be a derivation of the sequent under discussion. If t (resp. e) is in $\overline{\lambda}\mu\tilde{\mu}$-normal form, then every type occurring in Π is a subformula of a type occurring in $\Gamma \cup \triangle$, or a subformula of A. In the case of p we have if p is in $\overline{\lambda}\mu\tilde{\mu}$-normal form, then every type occurring in Π is a subformula of a type in $\Gamma \cup \triangle$.

The proof can be accomplished in a way similar to the one applied in the case of the $\lambda\mu$-calculus (cf. de Groote [29]).

- **Strong normalization property**

Theorem 19 *1. Let t (resp. e) be a right-term (resp. a left-term) typable with some type A. Then every reduction sequence starting from t (resp. e) is finite.*

2. Let $p = \langle t, e \rangle$ be a command such that t and e are typable with some type A. Then every reduction sequence starting from p is finite.

The proof of this property can be found in Polonovski [49], as well as in David and Nour [15], where an arithmetical proof is presented.

The λ^{Sym}_{Prop}-calculus

The λ^{Sym}-calculus defined by Berardi and Barbanera in [1] differs a bit in its approach from the calculi presented so far. It makes full use of the duality in classical logic, by this reason it has a negation "built-in", that is, it contains an involutive negation such that $\neg A$ is not defined as $A \to \bot$ with some type $\bot$. Instead, each type has its natural negated type as each formula of classical logic has its dual as its negated formula. We are going to consider here only the propositional part of the calculus, denoted by λ^{Sym}_{Prop}, since all the other calculi treated by us in this work are concerned with propositional logic.

The set of types of our system are built from two sets of base types: $\mathcal{A} = \{a, b, \ldots\}$ (atomic types) and $\mathcal{A}^{\perp} = \{a^{\perp}, b^{\perp}, \ldots\}$ (negated atomic types).

Definition 20 *1. The set of m-types is defined by the following grammar:*

$$A := \alpha \mid \alpha^{\perp} \mid A \wedge A \mid A \vee A,$$

where α ranges over $\mathcal{A}$ and $\alpha^{\perp}$ over $\mathcal{A}^{\perp}$.

2. The set of types is defined by the following grammar:

$$C := A \mid \bot.$$

In effect, Definition 20 yields that the set of m-types does not contain the absurdity proposition $\bot$ as a proper subtype. We also prevent $\bot$ to be used as assumption in a derivation, that is we do not have variables of type $\bot$ in the calculus. It can be checked that none of these conditions is a restriction at all, the proof of the strong normalization of the λ^{Sym}_{Prop}-calculus in Chapter 3 also works without these stipulations.

Definition 21 *We define the negation of an m-type as follows:*
$1.(\alpha)^{\perp} = \alpha^{\perp}$ $\quad 2.(\alpha^{\perp})^{\perp} = \alpha$
$3.(A \wedge B)^{\perp} = A^{\perp} \vee B^{\perp}$ $\quad 4.(A \vee B)^{\perp} = A^{\perp} \wedge B^{\perp}$.

In this way we get a calculus with involutive negation.

Lemma 22 $(A^{\perp})^{\perp} = A$.

Proof By induction on A, using Definition 21. □

The terms of λ^{Sym}_{Prop} together with their typing rules are defined as follows. In the definition below the type of a variable must be an m-type. Γ denotes a context.

Definition 23 *The typing rules are:*

$$var) \quad \Gamma, x : A \vdash x : A$$

$$\langle\,,\rangle) \quad \frac{\Gamma \vdash P_1 : A \quad \Gamma \vdash P_2 : A}{\Gamma \vdash \langle P_1, P_2\rangle : A_1 \wedge A_2} \qquad \sigma_i) \quad \frac{\Gamma \vdash P_i : A_i}{\Gamma \vdash \sigma_i{}^{A_1,A_2}\,(P_i) : A_1 \vee A_2}\ (i = 1, 2)$$

$$\lambda) \quad \frac{\Gamma, x : A \vdash P : \perp}{\Gamma \vdash \lambda x P : A^{\perp}} \qquad \star) \quad \frac{\Gamma \vdash P_1 : A^{\perp} \quad \Gamma \vdash P_2 : A}{\Gamma \vdash (P_1 \star P_2) : \perp}$$

The λ^{Sym}_{Prop} reduction rules are enumerated below.

Definition 24 *The reduction rules of the calculus:*

$$\begin{cases} \beta) & (\lambda x P \star Q) & \to_{\beta} & P[x := Q] \\ \beta^{\perp}) & (Q \star \lambda x P) & \to_{\beta^{\perp}} & P[x := Q] \end{cases}$$

$$\begin{cases} \eta) & \lambda x (P \star x) & \to_{\eta} & P \quad (1) \\ \eta^{\perp}) & \lambda x (x \star P) & \to_{\eta^{\perp}} & P \quad (1) \end{cases}$$

$$\begin{cases} \pi) & (\langle P_1, P_2\rangle \star \sigma_i(Q_i)) & \to_{\pi} & (P_i \star Q_i) \quad (i = 1, 2) \\ \pi^{\perp}) & (\sigma_i(Q_i) \star \langle P_1, P_2\rangle) & \to_{\pi^{\perp}} & (Q_i \star P_i) \quad (i = 1, 2) \end{cases}$$

$$Triv) \quad E[P] \quad \to_{Triv} \quad P \quad (2)$$

(1) if $x \notin FV(P)$.
(2) if $E[-]$ is a context with type $\perp$ and $E[-] \neq [-]$, then P has type $\perp$ and $E[-]$ does not bind any free variables in P.

The correspondences between the reductions in the proofs and the reductions of the terms are similar to the ones already detailed in respect of the calculi treated before. Figure 7 illustrates some of them.

Main theoretical properties of the λ^{Sym}_{Prop}-calculus

- **Type-preservation property**

  Proposition 25 *Let P, Q and A, Γ be such that $\Gamma \vdash P : A$ and $P \twoheadrightarrow Q$. Then $\Gamma \vdash Q : A$.*

  The property can be verified by double induction on the length of the reduction sequence $P \twoheadrightarrow Q$ and the complexity of P.

- **Subformula property**

Proposition 26 *Let P, A, Γ be such that*

$$\Gamma \vdash P : A, \qquad (*)$$

and let Π be a derivation of $(*)$. If P is in λ^{Sym}_{Prop}-normal form, then every type occurring in Π is a subformula of a type occurring in Γ, or a subformula of A.

The proof is analogous to that of the $\lambda\mu$-calculus (see de Groote [29]).

- **Strong normalization**

Theorem 27 *Let P be a term typable with some type A. Then every reduction sequence starting from P is finite.*

Berardi and Barbanera proved this result for the λ^{Sym}-calculus, which is an extension of the λ^{Sym}_{Prop}-calculus equivalent to first-order Peano-arithmetic. The proof of this result in Berardi-Barbanera [1] is based on reducibility candidates, but the definition of the interpretation of a type relies on non-arithmetical fixed-point constructions. In Chapter 3 we give an arithmetical proof for the strong normalization of the λ^{Sym}_{Prop}-calculus.

An outline of the present work

The general structure of the document is as follows.

- In Chapter 1 we examine the normalization properties of the $\lambda\mu\mu'$-calculus supplied with the ρ- and θ-rules. We implement these investigations both for the Parigot-style and for the de Groote-style (or, in other words, the extended) versions of the $\lambda\mu\mu'$-calculus. More exactly, we establish first that the Parigot-style $\lambda\mu\mu'\rho\theta$-calculus is strongly normalizing. We achieve this by showing that the ρ- and θ-rules can be strongly postponed with respect to the other rules. From this the assertion follows by a result of David and Nour ([14]) stating the strong normalization of the de Groote-style $\lambda\mu\mu'$-calculus. After this we continue with the investigations of the de Groote-style $\lambda\mu\mu'$-calculus. We present a short proof of the strong normalization of the untyped de Groote-style $\mu\mu'$-calculus. The proof is based on a norm strictly decreasing on certain reduction sequences. It turns out, however, that the de Groote-style $\mu\mu'$-calculus extended with the ρ-rule is not strongly normalizing. Namely, we give a simple counterexample for the strong normalization of the $\mu\mu'\rho$-calculus. In the second half of the chapter we are concerned with establishing the weak normalization of the $\mu\mu'\rho$-calculus and then we apply this result to the typed case, that is, we prove weak normalization for the $\lambda\mu\mu'\rho$-calculus. As an easy consequence we obtain then the weak normalization of the $\lambda\mu\mu'\rho\theta$-calculus as well. The proofs of the weak normalizations are entirely constructive, they consist of explicitly defining normalizing algorithms and proving that these algorithms really have the required properties. Finally, we formulate a standardization theorem for the $\lambda\mu\mu'\rho\theta$-calculus. In the concluding remarks we give a brief account of the normalization properties of the calculi obtained by adding some other simplification rules, like the ε- or the ν-rules, to the $\mu\mu'$- or $\lambda\mu\mu'$-calculi.

- Chapter 2 is devoted to finding a bound for the lengths of the reduction sequences in the $\lambda\mu\rho\theta$-calculus. The proof is based on a result of Xi (cf. [66]) concerning the lengths of the reduction sequences in the simply typed λ-calculus. The argument relies on the facts that in the $\lambda\mu I$-calculus the length of a reduction sequence is less than that of its standardization and we can establish a bound for the length of a standard reduction sequence in $\lambda\mu I$. Then the result is extended for the general case as well. As a side-issue, during the proof we obtain a demonstration for the standardization in the $\lambda\mu\rho\theta$-calculus, together with a bound for the length of the standardized reduction sequence in terms of some properties concerning the original one.

- In Chapter 3 we give an arithmetical proof for the strong normalization of the λ^{Sym}_{Prop}-calculus defined by Berardi and Barbanera ([1]). The proof is an adaptation of the method applied for the $\lambda\mu\mu'$- and for the $\overline{\lambda}\mu\tilde{\mu}$-calculi by David and Nour in [14] and in [15], respectively.

- In Chapter 4 the connection between the λ^{Sym}_{Prop}-calculus and the $\overline{\lambda}\mu\tilde{\mu}^*$-calculus, which is an extension of the $\overline{\lambda}\mu\tilde{\mu}$-calculus defined in Chapter 4, is studied. In the literature there already exist results about the connections between the $\lambda\mu$- and the $\overline{\lambda}\mu\tilde{\mu}$-calculus (cf. [11], [55]). We give a method how to simulate the λ^{Sym}_{Prop}-calculus in the $\overline{\lambda}\mu\tilde{\mu}^*$-calculus and vice versa. In the concluding remarks we give a proof of the strong normalization of the $\overline{\lambda}\mu\tilde{\mu}^*$-calculus by making use of the method applied in David and Nour [15] for the $\overline{\lambda}\mu\tilde{\mu}$-calculus.

β-reduction:

$$\dfrac{\dfrac{\vdots}{\dfrac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B}} \qquad \dfrac{\vdots}{\Gamma \vdash N : A}}{\Gamma \vdash (\lambda x.M\ N) : B} \quad \rightsquigarrow \quad \dfrac{\vdots}{\Gamma \vdash M[x := N] : B}$$

μ-reduction:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma, \alpha : \neg(A \to B), \beta : \neg C \vdash M : C}}{\Gamma, \beta : \neg C \vdash \mu\alpha(\beta\ M) : A \to B} \qquad \dfrac{\vdots}{\Gamma, \beta : \neg C \vdash N : A}}{\Gamma, \beta : \neg C \vdash (\mu\alpha(\beta\ M)\ N) : B} \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\vdots}{\Gamma, \alpha : \neg B, \beta : \neg C \vdash M[\alpha :=_r N] : C}}{\Gamma, \beta : \neg C \vdash \mu\alpha(\beta\ M[\alpha :=_r N]) : B}, \quad \text{provided } \alpha \neq \beta.$$

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma, \alpha : \neg(A \to B) \vdash M : A \to B}}{\Gamma \vdash \mu\alpha(\alpha\ M) : A \to B} \qquad \dfrac{\vdots}{\Gamma, \alpha : \neg(A \to B) \vdash N : A}}{\Gamma \vdash (\mu\alpha(\alpha\ M)\ N) : B} \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma, \alpha : \neg B \vdash M[\alpha :=_r N] : A \to B} \qquad \dfrac{\vdots}{\Gamma, \alpha : \neg B \vdash N : B}}{\Gamma, \alpha : \neg B \vdash (M[\alpha :=_r N]\ N) : B}}{\Gamma \vdash \mu\alpha(\alpha\ (M[\alpha :=_r N]\ N)) : B}, \quad \text{when } \alpha = \beta.$$

locally:

$$\dfrac{\dfrac{\vdots}{\Gamma, \gamma : \neg C, \alpha : \neg(A \to B) \vdash U : A \to B}}{\Gamma, \alpha : \neg(A \to B) \vdash \mu\gamma(\alpha\ U) : C} \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma, \gamma : \neg C, \alpha : \neg B \vdash U[\alpha :=_r N] : A \to B} \qquad \dfrac{\vdots}{\Gamma, \gamma : \neg C, \alpha : \neg B \vdash N : A}}{\Gamma, \gamma : \neg C, \alpha : \neg B \vdash (U[\alpha :=_r N]\ N) : B}}{\Gamma, \alpha : \neg B \vdash \mu\gamma(\alpha\ (U[\alpha :=_r N]\ N)) : C}$$

Figure 1: The reductions in the proofs corresponding to the β- and μ-rules of the $\lambda\mu$-calculus

ρ-reduction:
$$\dfrac{\dfrac{\vdots}{\Gamma, \alpha : \neg A, \beta : \neg B \ \vdash \ \mu\gamma M : B}}{\Gamma \ \vdash \ \mu\alpha(\beta\ \mu\gamma M) : A} \ \rightsquigarrow \ \dfrac{\vdots}{\Gamma \ \vdash \ \mu\alpha M[\gamma := \beta]}$$

locally:
$$\dfrac{\dfrac{\vdots}{\Gamma, \delta : \neg C, \gamma : \neg B \ \vdash \ U : B}}{\Gamma \ \vdash \ \mu\delta(\gamma\ U) : C} \ \rightsquigarrow \ \dfrac{\dfrac{\vdots}{\Gamma, \delta : \neg C, \beta : \neg B \ \vdash \ U[\gamma := \beta] : B}}{\Gamma, \beta : \neg B \ \vdash \ \mu\delta U[\gamma := \beta] : C}$$

θ-reduction:
$$\dfrac{\dfrac{\vdots}{\Gamma, \alpha : \neg A \ \vdash \ M : A}}{\Gamma \ \vdash \ \mu\alpha(\alpha\ M) : A} \rightsquigarrow \dfrac{\vdots}{\Gamma \ \vdash \ M : A}\,,$$

$(\mu\alpha_1(\alpha_2\ x)\ \mu\beta_1(\beta_2\ y)) \xrightarrow{\#\ Fv(M)} \mu\alpha_1(\alpha_2\ x)$ where $\alpha \notin Fv(M)$

μ' ↓

$\mu\beta_1(\beta_2\ y)$

and

$(\lambda xz\ \mu\alpha(\beta\ y)) \xrightarrow{\beta} z$

μ' ↓

$\mu\alpha(\beta\ y)$

Figure 2: The reductions in the proofs corresponding to the ρ- and θ-rules of the $\lambda\mu$-calculus

Figure 3: The $\lambda\mu\mu'$-calculus is not confluent

μ'-reduction:
$$\dfrac{\dfrac{\vdots}{\Gamma, \beta : \neg C \ \vdash \ M : A \to B} \qquad \dfrac{\dfrac{\vdots}{\Gamma, \alpha : \neg A, \beta : \neg C \ \vdash \ N : C}}{\Gamma, \beta : \neg C \ \vdash \ \mu\alpha(\beta\ N) : A}}{\Gamma, \beta : \neg C \ \vdash \ (M\ \mu\alpha(\beta\ N)) : B} \ \rightsquigarrow$$

$$\dfrac{\dfrac{\vdots}{\Gamma, \alpha : \neg B, \beta : \neg C \ \vdash \ N[\alpha :=_l M] : C}}{\Gamma, \beta : \neg C \ \vdash \ \mu\alpha(\beta\ N[\alpha :=_l M]) : B}\,, \qquad \text{provided } \alpha \neq \beta.$$

$$\dfrac{\dfrac{\vdots}{\Gamma, \alpha : \neg A \ \vdash \ M : A \to B} \qquad \dfrac{\dfrac{\vdots}{\Gamma, \alpha : \neg A \ \vdash \ N : A}}{\Gamma \ \vdash \ \mu\alpha(\alpha\ N) : A}}{\Gamma \ \vdash \ (M\ \mu\alpha(\alpha\ N)) : B} \ \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma, \alpha : \neg B \ \vdash \ M : A \to B} \qquad \dfrac{\vdots}{\Gamma, \alpha : \neg B \ \vdash \ N[\alpha :=_l M] : A}}{\Gamma, \alpha : \neg B \ \vdash \ (M\ N[\alpha :=_l M]) : B}}{\Gamma \ \vdash \ \mu\alpha(\alpha\ (M\ N[\alpha :=_l M])) : B}\,, \qquad \text{when } \alpha = \beta.$$

locally:
$$\dfrac{\dfrac{\vdots}{\Gamma, \gamma : \neg C, \alpha : \neg A \ \vdash \ U : A}}{\Gamma, \alpha : \neg A \ \vdash \ \mu\gamma(\alpha\ U) : C} \ \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma, \gamma : \neg C, \alpha : \neg B \ \vdash \ M : A \to B} \qquad \dfrac{\vdots}{\Gamma, \gamma : \neg C, \alpha : \neg B \ \vdash \ U[\alpha :=_l M] : A}}{\Gamma, \gamma : \neg C, \alpha : \neg B \ \vdash \ (M\ U[\alpha :=_l M]) : B}}{\Gamma, \alpha : \neg B \ \vdash \ \mu\gamma(\alpha\ (M\ U[\alpha :=_l M])) : C}$$

Figure 4: The reductions in the proofs corresponding to the μ'-rule

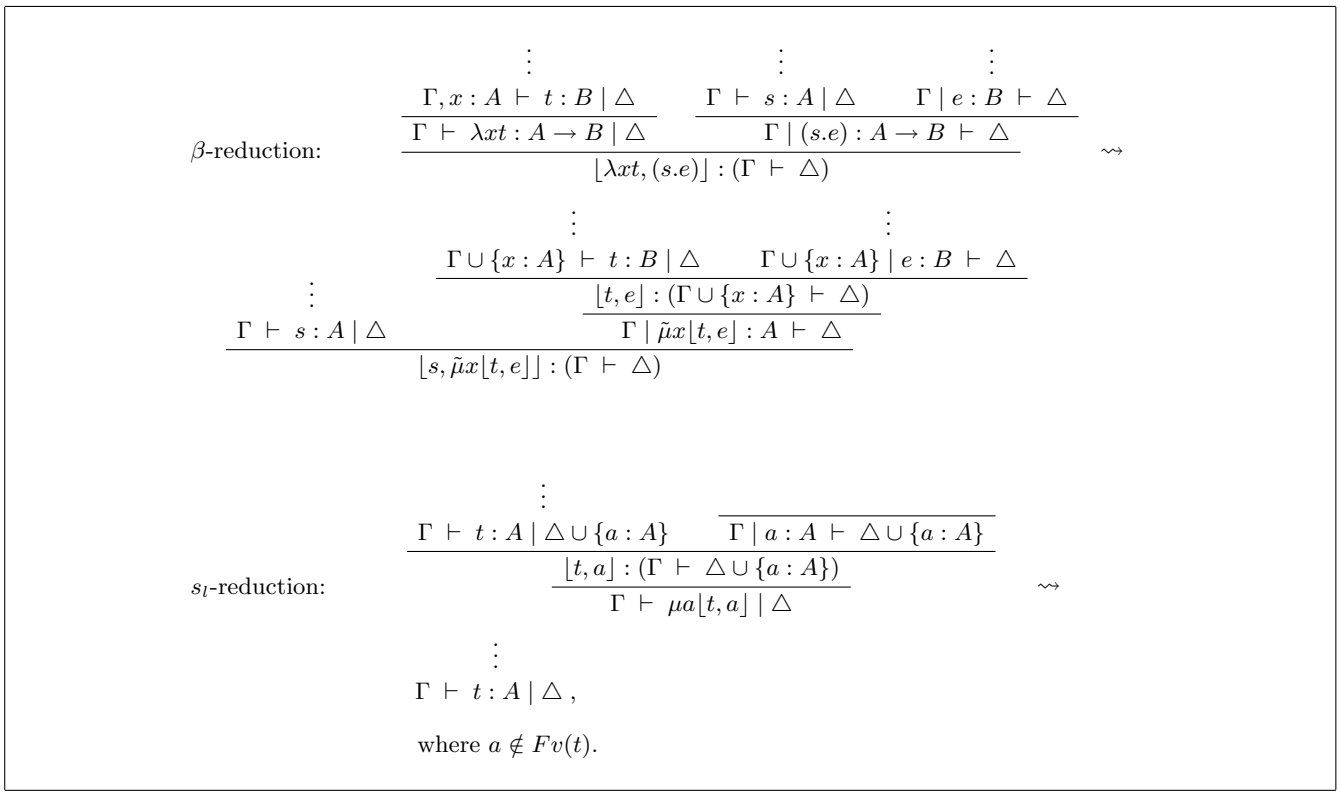

Figure 5: The reductions of proofs corresponding to the β- and s_l-rules in the $\overline{\lambda}\mu\tilde{\mu}$-calculus

μ-reduction:

$$\dfrac{\dfrac{\dfrac{\vdots}{p : (\Gamma \vdash \triangle \cup \{\alpha : A\})}}{\Gamma \vdash \mu ap : A \mid \triangle} \qquad \dfrac{\vdots}{\Gamma \mid e : A \vdash \triangle}}{\lfloor \mu ap, e \rfloor : (\Gamma \vdash \triangle)} \rightsquigarrow \dfrac{\vdots}{p[a := e] : (\Gamma \vdash \triangle)}$$

The more interesting steps required in the transformation:

$$\dfrac{\dfrac{\vdots}{\Gamma' \vdash t' : A \mid \triangle'} \qquad \dfrac{}{\Gamma' \mid a : A \vdash \triangle'}}{\lfloor t', a \rfloor : (\Gamma' \vdash \triangle')} \rightsquigarrow \dfrac{\dfrac{\vdots}{\Gamma' \vdash t'[a := e] : A \mid \triangle'} \qquad \dfrac{\vdots}{\Gamma' \mid e : A \vdash \triangle'}}{\lfloor t'[a := e], e \rfloor : (\Gamma' \vdash \triangle')}$$

$$\dfrac{\dfrac{\vdots}{\Gamma' \vdash t' : B \mid \triangle'} \qquad \dfrac{}{\Gamma' \mid a : A \vdash \triangle'}}{\Gamma' \mid (t'.a) : B \to A \vdash \triangle'} \rightsquigarrow \dfrac{\dfrac{\vdots}{\Gamma' \vdash t'[a := e] : B \mid \triangle'} \qquad \dfrac{\vdots}{\Gamma' \mid e : A \vdash \triangle'}}{\Gamma' \mid (t'[a := e].e) : B \to A \vdash \triangle'},$$

where, in both cases, $\triangle' \supseteq \triangle \cup \{a : A\}$.

Figure 6: The reductions in the proofs corresponding to the μ-rule of the $\overline{\lambda}\mu\tilde{\mu}$-calculus

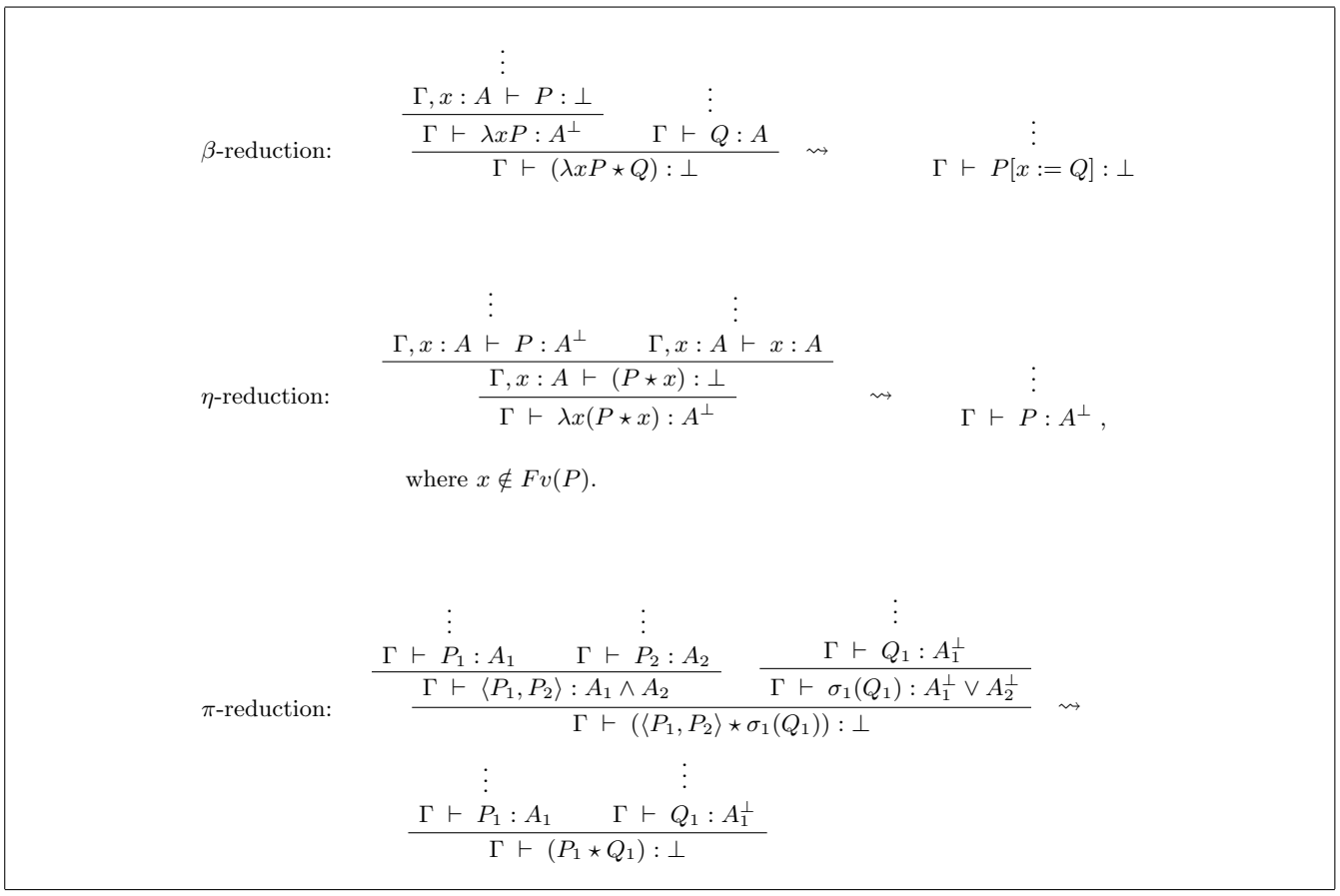

Figure 7: The reductions in the proofs in respect of the β-, η- and π-rules of the λ^{Sym}_{Prop}-calculus

Chapter 1

Normalization properties of the $\lambda\mu\mu'$-calculus

Before turning our attention to the main subject of the present chapter we list some of the basic definitions and notation which will be used in the sequel. Most of the definitions are taken from Klop [32].

Definition 28 *An Abstract Reduction System (ARS) is a pair $\mathcal{A} = \{\Sigma, (\rightarrow_i)_{i\in I}\}$, where Σ is a set and $(\rightarrow_i)_{i\in I}$ is a set of binary relations defined on Σ. The relations $(\rightarrow_i)_{i\in I}$ are called reduction relations.*

Remark 29 *The sets we are concerned are sets of terms of some calculi. In every case it is supposed that a reduction relation $\rightarrow$ is compatible with the term formation rules.*

Definition 30 *Let $\rightarrow$ be a reduction relation. Then $\twoheadrightarrow$ is its reflexive, transitive, $\rightarrow^+$ is its transitive, $\rightarrow^=$ is its reflexive closure. $\rightarrow^n$ $(n \geq 0)$ denotes the n-step reduction (the n-th product relation obtained from $\rightarrow$). Moreover, if $\rightarrow_1$, $\rightarrow_2$ are reductions, then $\rightarrow_{1,2}$ means the union of the two reductions as relations.*

Remark 31 *Let $\mathcal{A} = \{\Sigma, (\rightarrow_i)_{i\in I}\}$ be an ARS. In the definitions below we denote by $\rightarrow$ the union of the set of reductions $(\rightarrow_i)_{i\in I}$.*

Definition 32 *Let $\mathcal{A} = \{\Sigma, (\rightarrow_i)_{i\in I}\}$ be an ARS. Then $a \in \Sigma$ is in normal form, if there is no $b \in \Sigma$ such that $a \rightarrow b$. In notation: $a \in NF$. Further, a has a normal form, if there is a b in normal form for which $a \twoheadrightarrow b$. In this case, the reduction sequence starting from a and yielding a $b \in NF$ is called a normalizing reduction sequence for a.*

Definition 33 *The reduction relation $\rightarrow$ on an ARS $\mathcal{A}$ is weakly normalizing, if every a has a normal form. In this case we also say that $\mathcal{A}$ is weakly normalizing (in notation: $\mathcal{A}$ is WN).*

Definition 34 *Let $\mathcal{A}$ be an ARS. We say that an element $a \in \mathcal{A}$ strongly normalizes ($a \in SN$), if every reduction sequence starting from a is finite. $\mathcal{A}$ is strongly normalizing ($\mathcal{A}$ is SN), if, for every $a \in \mathcal{A}$, $a \in SN$.*

Definition 35 *An ARS $\mathcal{A}$ is finitely branching, if, for all a, the set of one-step reducts of a, $\{b \mid a \rightarrow b\}$, is finite.*

Remark 36 *If we have a finitely branching $\mathcal{A}$, then, by König-lemma, Definition 34 is equivalent to saying that the length of each reduction sequence starting from a is bounded uniformly by some number n.*

Thus, the following definition makes sense:

Definition 37 *Suppose $\mathcal{A}$ is SN, let $a \in \mathcal{A}$. Then $\eta(a)$ denotes the length of the longest reduction sequence starting from a.*

Definition 38 *A reduction relation is confluent (or Church-Rosser), if it is self-commuting. That is, for every a, b, $c \in \Sigma$ there is a d such that $a \twoheadrightarrow b$ and $a \twoheadrightarrow c$ imply $b \twoheadrightarrow d$ and $c \twoheadrightarrow d$.*

Definition 39 *Let $\mathcal{A}$ be an ARS. The relation $\rightarrow_2$ in $\mathcal{A}$ can be postponed w.r.t. $\rightarrow_1$ if, for every a, $b \in \Sigma$, $a \twoheadrightarrow_{1,2} b$ implies $a \twoheadrightarrow_1 c \twoheadrightarrow_2 b$ for some c.*

Definition 40 *Let $\mathcal{A}$ be an ARS. The relation $\rightarrow_2$ in $\mathcal{A}$ can be strongly postponed w.r.t. $\rightarrow_1$ if for every a, $b \in \Sigma$ such that $a \twoheadrightarrow_{1,2} b$ and the reduction sequence contains at least one $\rightarrow_1$-reduction we have $a \rightarrow_1^+ c \twoheadrightarrow_2 b$ for some c.*

1.1 The Parigot-style $\lambda\mu\mu'$-calculus

1.1.1 The θ-rule can be postponed

In the sequel, we are concerned with the untyped $\mu\mu'\rho$- and the simply typed $\lambda\mu\mu'\rho$- and $\lambda\mu\mu'\rho\theta$-calculi. In the first subsection of the chapter we examine the Parigot-style formulation and in the second one the de Groote-style formulation of the calculi. As for the basic definitions and notations we refer to the introduction. We prove in this subsection that the θ-rule can be postponed w.r.t. the β-, μ- and μ'-rules. Though, the proof is implemented in the Parigot-style $\lambda\mu$-calculus, it remains valid in the de Groote-style calculus as well. Prior to this, we present some more definitions frequently used later on.

Definition 41 *The complexity of a term is defined inductively as follows.*

1. $cxty(x) = 1$,
2. $cxty((\alpha\ M)) = cxty(M) + 1$,
3. $cxty(\lambda x M) = cxty(M) + 1$,
4. $cxty(\mu\alpha M) = cxty(M) + 1$,
5. $cxty((M\ N)) = cxty(M) + cxty(N) + 1$.

The complexity of a term in the de Groote-style $\lambda\mu\mu'$-calculus is understood in an analogous way, with the necessary notational changes.

Notation 42
1. *We use the notation $N \leq M$ (or $M \geq N$) if N is a subterm of M, and the notation $N < M$ (or $M > N$) if N is a subterm of M other than M.*
2. *By $\eta c(M)$ we denote the lexicographically ordered pair $\langle \eta(M), cxty(M) \rangle$ for a term M.*

Definition 43 *Let M, N be terms. $N \prec M$ (or $M \succ N$) will denote the fact that there is an M' such that $M \twoheadrightarrow M' \geq N$ holds and either $M \rightarrow^+ M'$ or $M > N$ is valid. $\preceq$ will be the reflexive closure of $\prec$.*

Remark 44
1. *$\prec$ and $\preceq$ are transitive. Moreover, $N \preceq M$ iff there is an M' such that $M \twoheadrightarrow M' \geq N$.*

2. $N \prec M$ *implies* $\eta c(N) < \eta c(M)$.

Having settled the necessary terminology we prove that the ρ- and the θ-rules can be strongly postponed w.r.t. the β-, μ- and μ'-rules. More precisely we show something more:

Theorem 45 *Let* M, N *be such that* $M \twoheadrightarrow_{\beta\mu\mu'\rho\theta} N$ *and the reduction sequence contains at least one* β- *or* μ- *or* μ'*-reduction. Then there is a* P *for which*

$$M \to^{+}_{\beta\mu\mu'} P \twoheadrightarrow_{\rho\theta} N.$$

We prove the theorem in two steps. First we establish that the θ-reduction can be strongly postponed w.r.t. all of the rules β, μ, μ', ρ, then we show that the ρ-rule can be strongly postponed w.r.t. the remaining three rules.

Definition 46 *Let* $\to_{\mu_0}$ *and* $\to_{\mu'_0}$ *be defined as follows:*

$$\begin{array}{ll} (\mu\alpha M\ N) \to_{\mu_0} \mu\alpha M[\alpha :=_r N], & \text{if } \alpha \text{ occurs at most once in } M \\ (M\ \mu\alpha N) \to_{\mu'_0} \mu\alpha N[\alpha :=_l M], & \text{if } \alpha \text{ occurs at most once in } N. \end{array}$$

Lemma 47 *Assume* $M \to_\theta P \to_\beta N$. *Then either we have a* Q *such that*

$$M \to_\beta Q \twoheadrightarrow_\theta N,$$

or there are R, Q *for which*

$$M \to_{\mu_0} R \to_\beta Q \to_\theta N.$$

Proof By induction on $cxty(M)$. Let us only treat the case $M = (M_1\ M_2)$.

1. $M_1 \to_\theta M'_1$. The only nontrivial case is $M'_1 = \lambda x M_3$ and $M_1 = \mu\alpha(\alpha\ M'_1)$. Now we obtain

$$\begin{array}{rcl} M & = & (\mu\alpha(\alpha\ \lambda x M_3)\ M_2) \ \to_{\mu_0}\ \mu\alpha(\alpha\ (\lambda x M_3\ M_2)) \\ & \to_\beta & \mu\alpha(\alpha\ M_3[x := M_2]) \ \to_\theta\ M_3[x := M_2]. \end{array}$$

2. $M_2 \to_\theta M'_2$. Analogous to the above one. The only difference is the case $M_1 = \lambda x M_3$, $N = M_3[x := M'_2]$. Then, with $Q = M_3[x := M_2]$, we have $M \to_\beta Q \twoheadrightarrow_\theta N$.

The remaining cases follow easily from the induction hypothesis. □

Lemma 48 *Let* $M \to_\theta P \to_\mu N$. *Then either*

$$M \to_\mu Q \twoheadrightarrow_\theta N$$

for some Q *or there are* R, Q *such that*

$$M \to_{\mu_0} R \to_\mu Q \to_\theta N.$$

Proof By induction on $cxty(M)$. Let $M = (M_1\ M_2)$.

1. $M_1 = \mu\alpha(\alpha\ M'_1) \to_\theta M'_1$. The only interesting case is $M'_1 = \mu\beta M_3$. Let $Q = (\mu\beta M_3\ M_2)$, then, applying $\alpha \notin M'_1$, we have

$$\begin{array}{rcl} M & = & (\mu\alpha(\alpha\ \mu\beta M_3)\ M_2) \ \to_{\mu_0}\ \mu\alpha(\alpha\ (\mu\beta M_3\ M_2)) \\ & \to_\mu & \mu\alpha(\alpha\ \mu\beta M_3[\beta :=_r M_2]) \ \to_\theta\ \mu\beta M_3[\beta :=_r M_2]. \end{array}$$

2. $M_2 = \mu\alpha(\alpha\ M'_2) \to_\theta M'_2$. Analogously. The only difference is $M_1 = \mu\beta M_3$, $N = \mu\beta M_3[\beta :=_r M'_2]$, where with $Q = \mu\beta M_3[\beta :=_r M_2]$ we obtain $M \to_\mu Q \to^*_\theta N$.

The remaining cases can be verified easily. □

Lemma 49 *Let $M \to_\theta P \to_{\mu_0} N$. Then either*

$$M \to_{\mu_0} Q \to_\theta N$$

for some Q or there are R, Q such that

$$M \to_{\mu_0} R \to_{\mu_0} Q \to_\theta N.$$

Proof The proof proceeds as that of Lemma 48 with the necessary changes implemented. □

Lemma 50 *Let $M \to_\theta P \to_{\mu'} N$. Then either*

$$M \to_{\mu'} Q \twoheadrightarrow_\theta N$$

for some Q or there are R, Q such that

$$M \to_{\mu'_0} R \to_{\mu'} Q \to_\theta N.$$

Proof Similar to the proof of Lemma 48. □

Lemma 51 *Let $M \to_\theta P \to_{\mu'_0} N$. Then either we have*

$$M \to_{\mu'_0} Q \to_\theta N$$

for some Q or there are R, Q such that

$$M \to_{\mu'_0} R \to_{\mu'_0} Q \to_\theta N.$$

Proof Analogous to that of Lemma 48. □

Lemma 52 *Let $M \to_\theta P \to_\rho N$. Then we have a Q such that*

$$M \to_\rho Q \to_\theta N.$$

Proof An obvious induction on $cxty(M)$. The only interesting case is $M = \mu\alpha(\alpha\ M_1) \to_\theta M_1 \to_\rho M_2$, but then $M = \mu\alpha(\alpha\ M_1) \to_\rho \mu\alpha(\alpha\ M_2) \to_\theta M_2$ is true. □

Lemma 53 *Let $M \to_\theta^n P \to_{\mu_0} N$ (resp. $M \to_\theta^n P \to_{\mu'_0} N$). Then there is a Q such that*

$$M \twoheadrightarrow_{\mu_0} Q \to_\theta^n N \quad (\textit{resp. } M \twoheadrightarrow_{\mu'_0} Q \to_\theta^n N).$$

Proof Follows from Lemma 49 by induction on n. □

Lemma 54 *Let $M \twoheadrightarrow_\theta P \to_{\beta\mu\mu'\rho} N$. Then there is a Q such that*

$$M \to^+_{\beta\mu\mu'\rho} Q \twoheadrightarrow_\theta N.$$

Proof Let us suppose first $M \to_\theta^n P \to_\mu N$. The proof proceeds by induction on n. We may assume $n \geq 1$, that is, $M \to_\theta^{n-1} U \to_\theta P$ holds for some U. By Lemma 48 we have either $U \to_\mu R \twoheadrightarrow_\theta N$ or $U \to_{\mu_0} R \to_\mu V \to_\theta N$. In the former case the induction hypothesis applies to $M \to_\theta^{n-1} U \to_\mu R$. In the latter case by Lemma 53 there exists an S such that $M \twoheadrightarrow_{\mu_0} S \to_\theta^{n-1} R$ is true, and we can apply the induction hypothesis again to $S \to_\theta^{n-1} R \to_\mu V$.
The case for $P \to_{\mu'} N$ is analogous.
The proof for $P \to_\beta N$ is similar to that for $P \to_\mu N$, and, finally, the case $P \to_\rho N$ is straightforward by Lemma 52. □

Theorem 55 *Let $M \twoheadrightarrow_\theta P \twoheadrightarrow_{\beta\mu\mu'\rho} N$ such that the reduction sequence $P \twoheadrightarrow_{\beta\mu\mu'\rho} N$ is not empty. Then there is a Q for which*

$$M \to^+_{\beta\mu\mu'\rho} Q \twoheadrightarrow_\theta N.$$

Proof By Lemma 54. □

Remark 56 *The above proof, thus Theorem 55, is valid in the de Groote-style $\lambda\mu\mu'$-calculus as well.*

1.1.2 The ρ-rule can be postponed

In this subsection we show that the ρ-reduction can be strongly postponed w.r.t. the β-, μ- and μ'-reductions in the Parigot-style $\lambda\mu\mu'$-calculus. The proofs of this subsection can be found in Py [51] as well. The presentation there, however, may differ in some places from the one applied here. We also outline the proofs here so that the material is self contained.

Theorem 57 *Let $M \twoheadrightarrow_\rho P \twoheadrightarrow_{\beta\mu\mu'} N$ such that the reduction sequence $P \twoheadrightarrow_{\beta\mu\mu'} N$ is not empty. Then there is a Q for which*

$$M \to^+_{\beta\mu\mu'} Q \twoheadrightarrow_\rho N.$$

Lemma 58 *Let $M \to_\rho P \to_\beta N$. Then we have a Q such that*

$$M \to_\beta Q \twoheadrightarrow_\rho N.$$

Proof By induction on $cxty(M)$. Let us only treat the case $M = (M_1\ M_2)$.

1. $M_1 \to_\rho M_1'$. The only nontrivial case is $M_1' = \lambda x M_3$. Then either $M_1 = (\alpha\ \mu\beta M_4) \to_\rho M_1' = M_4[\beta := \alpha]$ or $M_1 = \lambda x M_4$ and $M_4 \to_\rho M_3$. The first case is impossible, since then $M_4 = (\gamma\ M_4')$ for some M_4', and in the second case $(\lambda x M_4\ M_2) \to_\beta M_4[x := M_2] \to_\rho M_3[x := M_2]$, which gives the result.

2. $M_2 \to_\rho M_2'$. Analogous to the above one. The only difference is the case $M_1 = \lambda x M_3$, $N = M_3[x := M_2']$. Then, with $Q = M_3[x := M_2]$, $M \to_\beta Q \twoheadrightarrow_\rho N$.

The remaining cases follow easily from the induction hypothesis. □

Lemma 59 *Let M, M', N be given such that $M \to_\rho M'$ and $\alpha \notin Fv(N)$. Then either $M[\alpha :=_r N] \to_\rho M'[\alpha :=_r N]$ (resp. $M[\alpha :=_l N] \to_\rho M'[\alpha :=_l N]$) or $M[\alpha :=_r N] \to_\mu P \to_\rho M'[\alpha :=_r N]$ (resp. $M[\alpha :=_l N] \to_{\mu'} P \to_\rho M'[\alpha :=_l N]$) for some P.*

Proof By induction on $cxty(M)$. The only interesting case is $M = (\gamma\ \mu\beta M_1)$.

1. $\gamma = \alpha$. Then

$$\begin{array}{rcl} (\alpha\ \mu\beta M_1)[\alpha :=_r N] & = & (\alpha\ (\mu\beta M_1[\alpha :=_r N]\ N)) \\ & \to_\mu & (\alpha\ \mu\beta M_1[\alpha :=_r N][\beta :=_r N]) \\ & \to_\rho & M_1[\alpha :=_r N][\beta :=_r N][\beta := \alpha] \\ & = & M_1[\beta := \alpha][\alpha :=_r N]. \end{array}$$

2. $\gamma \neq \alpha$. In this case $(\gamma\ \mu\beta M_1)[\alpha :=_r N] = (\gamma\ \mu\beta M_1[\alpha :=_r N]) \to_\rho M_1[\beta := \gamma][\alpha :=_r N]$.

□

Lemma 60 *Let $M \to_\rho P \to_\mu N$. Then there exists a Q for which*

$$M \to_\mu Q \twoheadrightarrow_\rho N.$$

Proof By induction on $cxty(M)$. Assume $M = (M_1\ M_2)$.

1. $M_1 \to_\rho M_1'$. The only interesting case is $M_1' = \mu\gamma M_4$. In this case the only possibility is $M_1 = \mu\gamma M_3$, $M_3 \to_\rho M_4$. This is the point where we have made use of the fact we are in the realm of the Parigot-style $\lambda\mu\mu'$-calculus. By Lemma 59 we have

$$\begin{array}{rcl} \mu\gamma M_3\ M_2 & \to_\mu & \mu\gamma M_3[\gamma :=_r M_2] \\ & \twoheadrightarrow_\mu & Q \to_\rho \mu\gamma M_4[\gamma :=_r M_2] \end{array}$$

   for some Q.

2. $M = (M_1\ M_2)$, $M_2 \to_\rho M_2'$. Similar to the above one. The only additional case is $M_1 = \mu\alpha M_3$, $N = \mu\alpha M_3[\alpha :=_r M_2']$. Then, with $Q = \mu\alpha M_3[\alpha :=_r M_2]$, $M \to_\mu Q \twoheadrightarrow_\rho N$ holds.

The other cases can be verified easily. □

Lemma 61 *Let $M \to_\rho P \to_{\mu'} N$. Then there exists a Q for which*

$$M \to_{\mu'} Q \twoheadrightarrow_\rho N.$$

Proof Analogous to the above one. □

Lemma 62 *Let $M \twoheadrightarrow_\rho P \to_\beta N$. Then there exists a Q for which*

$$M \to_\beta Q \twoheadrightarrow_\rho N.$$

Proof Let $M \to_\rho^n P \to_\beta N$. The proof goes by induction on n using Lemma 58. □

Lemma 63 *Let $M \twoheadrightarrow_\rho P \to_\mu N$ (resp. $M \twoheadrightarrow_\rho P \to_{\mu'} N$). Then there is a Q such that*

$$M \to_\mu Q \twoheadrightarrow_\rho N \quad (\textit{resp. } M \to_{\mu'} Q \twoheadrightarrow_\rho N).$$

Proof Let $M \to_\rho^n P \to_\mu N$. The statement follows from Lemmas 60 and 61 by induction on n. □

We obtain Theorem 57 as a consequence of Lemmas 62 and 63. Theorems 55 and 57 together yield Theorem 45.

1.1.3 The Parigot-style $\lambda\mu\mu'$-calculus strongly normalizes

Theorem 64 *The Parigot-style $\lambda\mu\mu'$-calculus with the ρ- and θ-rules is strongly normalizing.*

Proof It is proved in David and Nour [14] that the de Groote-style $\lambda\mu\mu'$-calculus is strongly normalizing. Since the Parigot-style $\lambda\mu\mu'$-calculus is contained in the de Groote-style one, we have the strong normalization of the Parigot-style $\lambda\mu\mu'$-calculus as well. Let M be a $\lambda\mu$-term, let $\eta_{\beta\mu\mu'}(M)$ denote the length of the longest $\beta\mu\mu'$-reduction sequence starting from M. We prove by induction on $\eta_{\beta\mu\mu'}(M)$ that M has no infinite $\beta\mu\mu'\rho\theta$-reduction sequence. Let M be a term, assume we have an infinite $\beta\mu\mu'\rho\theta$-reduction sequence σ starting from M. Since this cannot consist entirely of ρ- and θ-reductions, we may suppose σ begins with $M \twoheadrightarrow_{\rho\theta} M' \to_{\beta\mu\mu'} M''$ for some M' and M''. By Theorem 45 we have an N such that $M \twoheadrightarrow^+_{\beta\mu\mu'} N \twoheadrightarrow_{\rho\theta} M''$. This means N would have an infinite $\beta\mu\mu'\rho\theta$-reduction sequence, contradicting the induction hypothesis. □

1.2 The de Groote-style $\lambda\mu\mu'$-calculus

In respect of normalization properties the de Groote-style $\lambda\mu\mu'$-calculus manifests a somewhat different behaviour than the version treated in the previous section.

We state first the following theorem:

Theorem 65 *The de Groote-style $\lambda\mu\mu'$-calculus without the ρ-rule is strongly normalizing.*

Proof For the proof see David and Nour [14]. □

The proof in David and Nour [14] is arithmetical in the sense that it can be formalized in Peano arithmetic. In fact, proofs for the strong normalization of the $\overline{\lambda}\mu\tilde{\mu}$- and λ^{Sym}_{Prop}-calculi, which turned out in the meantime to be equivalent with the extended $\lambda\mu\mu'$-calculus (cf. Rocheteau [55] for the equivalence of $\overline{\lambda}\mu\tilde{\mu}$ and $\lambda\mu\mu'$, and Chapter 5 in this work for the equivalence of $\overline{\lambda}\mu\tilde{\mu}$ and λ^{Sym}_{Prop}), were already known. Polonovski has proved that $\overline{\lambda}\mu\tilde{\mu}$ is strongly normalizing (Polonovski [49]), Berardi and Barbanera proved the same for λ^{Sym}_{Prop}, but none of these proofs were arithmetical, that is, they could not be formalized in first order Peano arithmetic. Thus, the proof in David and Nour [14] can be considered as the first arithmetical proof of the strong normalization of a symmetric logical calculus which can be related to classical logic. Proceeding in this direction all proofs in the present work are arithmetical.

The proof of Theorem 65 consists of two parts: proving first that the untyped $\mu\mu'$-calculus is strongly normalizing, the demonstration of the strong normalization of the typed case can be regarded as an extension of the result for the untyped one. Keeping this method in mind, in the following investigations we also treat the untyped and typed cases separately. In the sequel, unless otherwise stated, by referring to an instance of the $\mu\mu'$- or the $\lambda\mu\mu'$-calculi we understand the de Groote-style versions of the calculi.

First, we present a proof of the strong normalization of the $\mu\mu'$-calculus, probably somewhat simpler than the one given in David and Nour [14]. It seems to be an appealing task to find a proof for the strong normalization of the typed case along the same lines.

1.2.1 A strong normalization proof for the $\mu\mu'$-calculus

Let us consider now the untyped de Groote-style $\mu\mu'$-calculus. Its set of terms is defined by the following grammar.

$$\mathcal{T} := x \mid \mu\alpha.\mathcal{T} \mid (\alpha\ \mathcal{T}) \mid (\mathcal{T}\ \mathcal{T}).$$

Let us call the elements of the $\mathcal{T}$ above μ-terms. The reduction rules of the calculus are the μ- and the μ'-rules.

Lemma 66 *Let $(M\ N) \twoheadrightarrow \mu\alpha P$. Then either $M \twoheadrightarrow \mu\alpha M_1$, $M_1[\alpha :=_r N] \twoheadrightarrow P$ or $N \twoheadrightarrow \mu\alpha N_1$, $N_1[\alpha :=_l M] \twoheadrightarrow P$.*

Proof By induction on the length of the reduction $(M\ N) \twoheadrightarrow \mu\alpha P$. □

Notation 67 *A simultaneous substitution of the form $\sigma = [\alpha_1 :=_{s_1} N_1, \ldots, \alpha_k :=_{s_k} N_k]$, where $s_i \in \{l, r\}$ $(1 \leq i \leq k)$, is called a μ-substitution.*

Lemma 68 *Let M be a term, σ be a μ-substitution. If $M\sigma \twoheadrightarrow \mu\alpha P$ for some P, then there exists a Q such that $M \twoheadrightarrow \mu\alpha Q$ and $Q\sigma \twoheadrightarrow P$.*

Proof By induction on $cxty(M)$. The only possibilities are $M = \mu\alpha M_1$ and $M = (M_1\ M_2)$. The former case is trivial, in the latter we have $(M_1\sigma\ M_2\sigma) \twoheadrightarrow \mu\alpha P$. By Lemma 66 either $M_1\sigma \twoheadrightarrow \mu\alpha N_1$, $N_1[\alpha :=_r M_2\sigma] \twoheadrightarrow P$ or $M_2\sigma \twoheadrightarrow \mu\alpha N_2$, $N_2[\alpha :=_l M_1\sigma] \twoheadrightarrow P$. Suppose the former case holds. By the induction hypothesis there is an R such that $M_1 \twoheadrightarrow \mu\alpha R$ and $R\sigma \twoheadrightarrow N_1$. Then our assertion holds with $Q = R[\alpha :=_r M_2]$. □

Lemma 69 *Let M, N be terms of the $\mu\mu'$-calculus such that M, $N \in SN$ and $(M\ N) \notin SN$. Then either $M \twoheadrightarrow \mu\alpha M'$ such that $\mu\alpha M'[\alpha =_r N] \notin SN$ or $N \twoheadrightarrow \mu\beta N'$ such that $\mu\beta N'[\beta =_l M] \notin SN$.*

Proof By induction on $\eta(M) + \eta(N)$. If $M \to M'$ and $(M'\ N) \notin SN$, then $\eta(M) < \eta(M')$ and the induction hypothesis applies. The situation is similar when $N \to N'$ and $(M\ N') \notin SN$. If $M = \mu\alpha M_1$, $\mu\alpha M_1[\alpha :=_r N] \notin SN$, or $N = \mu\beta N_1$, $\mu\beta N_1[\beta :=_l M] \notin SN$, then the result is obvious. □

Lemma 70 *Let M, $N \in SN$ such that $M[\alpha :=_r N] \notin SN$ (resp. $M[\alpha :=_l N] \notin SN$) for some α. Then there is an $(\alpha\ M_1) \preceq M$ for which $M_1[\alpha :=_r N] \in SN$ (resp. $M_1[\alpha :=_l N] \in SN$) and $(M_1[\alpha :=_r N]\ N) \notin SN$ (resp. $(M_1[\alpha :=_l N]\ N) \notin SN$).*

Proof The proof goes by induction on $\eta c(M)$. Let us only treat the case of $M[\alpha :=_r N] \notin SN$.

1. $M = (M_1\ M_2)$. By Lemma 69 either $M_1[\alpha :=_r N] \twoheadrightarrow \mu\beta M_1'$ and $\mu\beta M_1'[\beta :=_r M_2[\alpha :=_r N]] \notin SN$ or $M_2[\alpha :=_r N] \twoheadrightarrow \mu\gamma M_2'$ and $\mu\gamma M_2'[\gamma :=_l M_1[\alpha :=_r N]] \notin SN$. Suppose the former case holds, the latter being similar. Then, by Lemma 68, there is an M_3 such that $M_1 \twoheadrightarrow \mu\beta M_3$ and $M_3[\alpha :=_r N] \twoheadrightarrow M_1'$. By this we have
$$\begin{aligned}\mu\beta M_3[\beta :=_r M_2][\alpha :=_r N] &= \mu\beta M_3[\alpha :=_r N][\beta :=_r M_2[\alpha :=_r N]] \\ &\twoheadrightarrow \mu\beta M_1'[\beta :=_r M_2[\alpha :=_r N]] \notin SN.\end{aligned}$$
But then, since $\eta(\mu\beta M_3[\beta :=_r M_2]) < \eta(M)$, we can apply the induction hypothesis.

2. $M = \mu\beta M_1$. Trivial.

3. $M = (\beta\ M_1)$.
   - $\beta = \alpha$. If $M_1[\alpha :=_r N] \notin SN$, then the induction hypothesis applies. Otherwise our assertion follows with the M_1 under discussion.
   - $\beta \neq \alpha$. The induction hypothesis gives the result.

□

Definition 71 *Let M be a term. Let us define a norm for M:*

$$|M| = \begin{cases} 0 & \text{if } M = x, \\ |M_1| + |M_2| & \text{if } M = (M_1\ M_2), \\ max\{|M_2| \mid (\alpha\ M_2) \leq M_1\} + 1 & \text{if } M = \mu\alpha M_1 \text{ and } \alpha \in Fv(M_1), \\ 0 & \text{if } M = \mu\alpha M_1 \text{ and } \alpha \notin Fv(M_1), \\ 0 & \text{if } M = (\alpha\ M_1). \end{cases}$$

Remark 72 *For every M the norm of M is a natural number.*

Remark 73 *In the sequel, if σ is a μ-substitution consisting of one element, then we are going to omit the subscript l or r, if it is clear from the context. Moreover, in this case, if σ can equally be a μ- or a μ'-substitution, then we are going to suppose that σ is a μ-substitution, unless otherwise stated.*

Lemma 74 *Let M be a term and σ be a substitution. Then*

$$|M\sigma| = |M|.$$

Proof Obviously, it is enough to prove the statement for $\sigma = [\alpha := N]$, supposing, by the above remark, that σ is a μ-substitution. The proof goes by induction on $cxty(M)$. The only interesting case is $M = \mu\beta M_1$. If $\beta \in Fv(M_1)$, then, applying the induction hypothesis,

$$\begin{aligned} |\mu\beta M_1[\alpha := N]| &= max\{|M_1'| \mid (\beta\ M_1') \leq M_1[\alpha := N]\} + 1 \\ &= max\{|M_2[\alpha := N]| \mid (\beta\ M_2) \leq M_1\} + 1\ =\ |\mu\beta M_1|. \end{aligned}$$

In case of $\beta \notin Fv(M_1)$ the equation $|\mu\beta M_1[\alpha := N]| = |\mu\beta M_1| = 0$ is valid. □

Lemma 75 *Let M and N be terms. Then $|(\mu\alpha M\ N)| \geq |\mu\alpha M[\alpha :=_r N]|$ (resp. $|(M\ \mu\alpha N)| \geq |\mu\alpha N[\alpha :=_l M]|$).*

Proof We deal with the case of the μ-reduction only. Let us suppose first $\alpha \in Fv(M)$. By Definition 71 and Lemma 74

$$\begin{aligned} |\mu\alpha M[\alpha := N]| &= max\{|M_1| \mid (\alpha\ M_1) \leq M[\alpha := N]\} + 1 \\ &= max\{|(M_2[\alpha := N]\ N)| \mid (\alpha\ M_2) \leq M\} + 1 \\ &= max\{|M_2| \mid (\alpha\ M_2) \leq M\} + |N| + 1 \\ &= |\mu\alpha M| + |N|\ =\ |(\mu\alpha M\ N)|. \end{aligned}$$

If $\alpha \notin Fv(M)$, then, by Definition 71, we have $|(\mu\alpha M\ N)| = |\mu\alpha M| + |N| = |N| \geq |\mu\alpha M[\alpha :=_r N]| = 0$. □

Lemma 76 *Let $M \twoheadrightarrow N$. Then*

$$|M| \geq |N|.$$

Proof It is enough to show that $M \to N$ implies $|M| \geq |N|$. The proof goes by induction on $cxty(M)$. The only interesting case is $M = (M_1\ M_2)$.

1. $M_1 \to M_1'$ or $M_2 \to M_2'$. Obvious.
2. $M_1 = \mu\alpha M_3$, $N = \mu\alpha M_3[\alpha :=_r M_2]$, or $M_2 = \mu\beta M_3$, $N = \mu\beta M_3[\alpha :=_l M_1]$. Applying Lemma 75 we obtain the result.

□

Lemma 77 *Let M, N be terms, assume $(\alpha\ M_1) \leq M$. Then we have*

$$|\mu\alpha M| > |M_1[\alpha := N]|.$$

Proof By Definition 71 and Lemma 74

$$\begin{aligned} |\mu\alpha M| &= max\{|P| \mid (\alpha\ P) \leq M\} + 1 \\ &> |M_1| = |M_1[\alpha := N]|. \end{aligned}$$

□

Theorem 78 *The $\mu\mu'$-calculus strongly normalizes.*

Proof It is enough to prove that, for arbitrary M, $N \in SN$, $(M\ N) \in SN$ as well. Let M, $N \in SN$ such that $(M\ N) \notin SN$, with the property that $\langle |(M\ N)|, \eta(M) + \eta(N)\rangle$ is minimal. Then by Lemma 69 either $M \twoheadrightarrow \mu\alpha M'$, $\mu\alpha M'[\alpha =_r N] \notin SN$ or $N \twoheadrightarrow \mu\beta N'$, $\mu\beta N'[\beta =_l M] \notin SN$. Suppose the former is valid. On account of Lemma 76, $M \to^+ \mu\alpha M'$ contradicts the minimality of $(M\ N)$, so we may assume $M = \mu\alpha M'$. In accordance with Lemma 70 there exists an $(\alpha\ M_1) \preceq M'$ such that $M_1[\alpha :=_r N] \in SN$ and $(M_1[\alpha :=_r N]\ N) \notin SN$. By Lemma 77 this contradicts the minimality of $(M\ N)$ again. □

1.2.2 The $\mu\mu'\rho$-calculus is not strongly normalizing

On the grounds of the example of the Parigot-style $\lambda\mu\mu'$-calculus, the question naturally arises in this case also what can be said about the extended $\lambda\mu\mu'$-calculus with the ρ-rule added. We would remind the reader that the de Groote-style $\lambda\mu\mu'$-calculus coincides with the original one except for the facts that in the construction of the terms the distinction between named and unnamed terms has disappeared and the term forming rules have become more flexible: a μ-operator can now be followed by any kind of term (in the untyped version), not necessarily by a term beginning with a μ-variable.

First, we are going to have a closer look at the $\mu\mu'$-calculus expanded with the ρ-rule. It turns out that, unlike the untyped Parigot-style $\mu\mu'\rho$-calculus, this calculus is not strongly normalizing, as the following assertion states.

Proposition 79 *The untyped $\mu\mu'\rho$-calculus is not strongly normalizing.*

Proof Let $U = \mu\alpha(\alpha\ (\alpha\ x))$, and $V = \mu\beta U$. Then the following reductions show that there exists an infinite reduction sequence starting from $(V\ U)$:

$$\begin{array}{rcl} (V\ U) & \to_{\mu'} & \mu\alpha(\alpha\ (V\ (\alpha\ \underline{(V\ x)}))) \\ & \to_{\mu} & \mu\alpha(\alpha\ (V\ \underline{(\alpha\ \mu\beta U)})) \\ & \to_{\rho} & \mu\alpha(\alpha\ (V\ U[\beta := \alpha])) \\ & = & \mu\alpha(\alpha\ (V\ U)). \end{array}$$

□

We remark that in case the θ-rule is allowed, the example above serves as a complete loop giving back the initial term in the end. Hereupon, what remains as a natural question to be examined is the weak normalization of the calculus.

1.2.3 The $\mu\mu'\rho$-calculus is weakly normalizing

In this subsection we present a proof for the weak normalization of the $\mu\mu'\rho$-calculus. The proof consists of determining an algorithm by which every term can be reduced to a normal form and, at the same time, verifying that this algorithm really provides a weak normalization process, that is, it terminates. The description of the algorithm and the proof for its termination take place in Theorem 101, which is the main statement of this section. Beforehand, we are concerned with introducing the notions and assertions used in the proof of this theorem.

Definition 80 *An address is a finite list of the symbols $\{l, r\}$. The empty address will be denoted as $[]$. Let a be an address, $s \in \{l, r\}$, then $[s :: a]$ denote the concatenation of $[s]$ and a.*
If M is a term and a an address, we can define the subterm of M at address a in the following way.

1. *$M_{[]} = M$,*
2. *$M_{[l::b]} = P_b$ (resp. $M_{[r::b]} = Q_b$), if $M = (P\ Q)$ and P_b (resp. Q_b) is defined. Otherwise, $M_{[s::b]}$ $(s \in \{l, r\})$ is undefined.*

We say that M' can be addressed in M with an address a if $M_a = M'$ holds.

Definition 81 *Let M be a term and a be an address, such that M_a is defined. Then $M\langle a = N\rangle$ is the term where M_a is exchanged by N in M.*
With the above notation let $N[\alpha :=_a M]$ denote the term which is obtained from N by replacing each subterm of the form $(\alpha\ U)$ in N by $(\alpha\ M\langle a = U\rangle)$.

Example 82 *1. Let* $N = \mu\beta(\alpha\ (\mu\gamma(z\ (\alpha\ y))))$, $M = (x_1\ (x_2\ x_3))$, $a = [r :: l]$. *Then* $N[\alpha :=_a M] = \mu\beta(\alpha\ (x_1\ ((\mu\gamma(z\ (\alpha\ (x_1\ (y\ x_3)))))\ x_3)))$.

2. $M[\alpha :=_l N] = M[\alpha :=_{[r]} (N\ U)]$ *and* $M[\alpha :=_r N] = M[\alpha :=_{[l]} (U\ N)]$, *where* U *can be arbitrary.*

3. Suppose $M\langle a := N\rangle$ *is defined. Let* M' *be the term obtained from* M *by exchanging* M_a *for the new variable* x_a. *Then* $M\langle a := N\rangle = M'[x_a := N]$. *This relation will be used in the sequel.*

Definition 83 *1. An occurrence of a* λ*-variable* x *in* M *is said to be correct with respect to an occurrence of a* μ*-variable* α, *if there is an* N *and an address* a, *such that* $(\alpha\ N) \leq M$ *and* $(N)_a = x$ *for the occurrences under discussion.*

2. We call an occurrence of a variable x *correct if there is a (necessarily unique)* α *such that* x *is correct w.r.t.* α.

3. M *is* x*-correct, if all the occurrences of* x *are correct such that different occurrences of* x *are corresponded to different occurrences of* μ*-variables of* M. *In notation:* $M \in \mathcal{C}(x)$.

With an abuse of terminology, in the sequel we may omit the references to occurrences and simply speak of the variables itself.

Definition 84 *We call a term* M *admissible, if, for all* $x \in Fv(M)$, *we have* M *is* x*-correct and, for every occurrence of* x, *if* x *is correct with respect to an occurrence of an* α, *then* $\alpha \in Fv(M)$.

Example 85 *1. Let* $M = (\alpha\ (x\ (\beta\ (x\ (\mu\gamma y\ x)))))$. *Then the first occurrence of* x *is correct w.r.t.* α, *the second and the third are correct w.r.t.* β, *but* M *is not* x*-correct.* y *is not correct in* M.

2. Let $N = (\alpha\ (x\ (\beta\ (x\ (\gamma\ (y\ x))))))$. *Then* $N \in \mathcal{C}(x)$, $N \in \mathcal{C}(y)$, *moreover* N *is even admissible.*

3. Let $N = \mu\beta(\gamma\ ((x\ y)\ (\beta\ x)))$. *Then* $N \in \mathcal{C}(x)$, $N \in \mathcal{C}(y)$ *but, because of the second occurrence of* x, N *is not admissible.*

Definition 86 *Let* U *be a term. Let us define a function* h_x *inductively on the subterms of* U.

1. $h_x((U_1\ U_2)) = max\{h_x(U_1), h_x(U_2)\}$,

2. $h_x((\alpha\ U_1)) = \begin{cases} h_x(U_1) + 1 & \text{if } x \text{ has a correct occurrence in } (U_1) \text{ with respect to } \alpha, \\ h_x(U_1) & \text{otherwise,} \end{cases}$

3. $h_x(\mu\alpha U_1) = h_x(U_1)$,

4. $h_x(y) = 0$, *if* $y \in Var$.

Example 87 *Let* M *and* N *be as in Example 85. Then* $h_x(M) = 2$, $h_y(M) = 0$. *Furthermore,* $h_x(N) = 3$, $h_y(N) = 1$.

Lemma 88 *Let* U *be a term. Then* $h_x(U)$ *equals the length of the longest sequence of the form* $U \geq (\alpha_1\ U_1) \geq \ldots \geq (\alpha_m\ U_m)$ *such that each* $(\alpha_i\ U_i)$ $(1 \leq i \leq m)$ *contains an occurrence of* x *correct with respect to* α_i.

Proof Let $U' \leq U$ and $l \in \mathbb{N}$. We are going to prove by induction on $cxty(U')$ that

$$h_x(U') \geq l \Leftrightarrow \text{ there is a sequence } U' \geq (\alpha_1\ U_1) \geq \ldots \geq (\alpha_l\ U_l)$$

such that each $(\alpha_i\ U_i)$ $(1 \leq i \leq m)$ contains an occurrence of x correct with respect to α_i.
($\Rightarrow$) Let $h_x(U') \geq l$.

1. $U' = (U'_1\ U'_2)$. Then $h_x(U'_i) \geq l$ for some $i \in \{1, 2\}$. The result follows from the induction hypothesis.

2. $U' = (\alpha\ U'')$.

   - x has a correct occurrence in U'' w.r.t. α. Then $h_x(U'') \geq l - 1$, the induction hypothesis applies.

   - x has no correct occurrences in U'' w.r.t. α. In this case $h_x(U'') \leq l$, again the induction hypothesis gives the result.

3. $U' = \mu\alpha U''$. $h_x(U') = h_x(U'')$, the induction hypothesis applies.

($\Leftarrow$) Let $U' \geq (\alpha_1\ U_1) \geq \ldots \geq (\alpha_l\ U_l)$ be a sequence of the above form. By induction on i we have $h_x((\alpha_{l-i}\ U_{l-i})) \geq i + 1$, from which $h_x(U') \geq l$ follows. □

Corollary 89 *If U is x-correct and $h_x(U) = m$, then there is a sequence $U \geq (\alpha_1\ U_1) \geq \ldots \geq (\alpha_m\ U_m)$ such that each $(\alpha_i\ U_i)$ $(1 \leq i \leq m)$ contains exactly one occurrence of x which is correct w.r.t. α_i.*

Definition 90 *Let us define sets of terms for all $n \in \mathbb{N}$ in the following way:*
$\mathcal{S}_n := \{M \mid M \in NF$ and M contains at most n μ-s$\}$.
We are going to write $V \in \mathcal{S}_{n^+}$, if $V \in \mathcal{S}_n$ and V contains at least one μ.

Remark 91 *1. Observe that $M \in NF$ implies that $M = \vec{\mu}M_1$ for some sequence of μ-s and an $M_1 \in \mathcal{S}_0$.*

2. In the sequel in $\vec{\mu}W$ the notation $\vec{\mu}$ will stand for an arbitrary (possibly empty) sequence of μ-s and, in this context, W denotes a term in $\mathcal{S}_0$.

Definition 92 *We say that $U \in \mathcal{S}_0$ is almost admissible (in short a.a.), if for all $x \in Fv(U)$ either $U \in \mathcal{C}(x)$ or there exists exactly one occurrence of x which is not correct in U. Moreover, U is such that, for every x satisfying $U \notin \mathcal{C}(x)$, if we replace the non-correct occurrence of x in question by a new variable x', then, for the resulting term U', we have $U' \in \mathcal{C}(x)$.*

Example 93 *1. Let $N = (x\ (y\ (\beta\ (x\ z))))$, then N is not admissible, since neither x nor y are correct in N. However, replacing x by x', y by y' we obtain for the resulting term N' that $N' \in \mathcal{C}(x)$ and $N' \in \mathcal{C}(y)$. Thus N is almost admissible.*

2. Let $N = \mu\beta((\beta\ x)\ x)$. Then N is neither almost admissible, nor admissible, since $N \notin \mathcal{C}(x)$ but, for the term $N' = \mu\beta((\beta\ x)\ x')$, we still have $N' \notin \mathcal{C}(x)$.

Lemma 94 *Let $U \in \mathcal{S}_0$. Let us take a non-correct occurrence of a variable x in U. Then there is a (possibly empty) address a such that*

$$U_a = x$$

for that occurrence of x.

Proof By induction on $cxty(U)$.

1. The cases $U = y$ and $U = (U_1\ U_2)$ are obvious.

2. $U = (\alpha\ U_1)$. This case is impossible. Namely, since the underlying occurrence of x is non-correct in U, it is non-correct in U_1 as well. So, by induction hypothesis, we would get $(U_1)_b = x$ for some b, which would imply the correctness of this occurrence of x.

□

Definition 95 *Let $s \in \{l, r\}$. We denote by $\overline{s}$ the complement of s, which is defined as r if $s = l$, and as l otherwise.*

Lemma 96 *Let $(\alpha\ U)$ be admissible, $U_a = x$ for some $x \in Fv(U)$ and address $a = [a_1, \ldots, \overline{a}_k]$. Assume $h_x((\alpha\ U)) = m$. Furthermore let $U' = U_{[a_1,\ldots,\overline{a}_i]}$ with an $1 \leq i \leq k$, suppose $(\beta\ V) \leq U'$. Then $(\beta\ V)$ is admissible and $h_x((\beta\ V)) \leq m - 1$.*

Proof Let us take an occurrence of a variable $x' \in Fv(V)$. Since $(\alpha\ U) \in \mathcal{C}(x')$ there are $(\alpha'\ W) \leq (\alpha\ U)$ and an address b such that $W_b = x'$ for this occurrence. Then $(\alpha'\ W) > (\beta\ V)$, together with $W_b = x' \in V$, is impossible. Thus $(\alpha'\ W) \leq (\beta\ V)$, that is, the underlying occurrence of x' is correct in $(\beta\ V)$ w.r.t. the same variable as in $(\alpha\ U)$. It follows that $(\beta\ V)$ is admissible.
Let $(\alpha_1\ V_1) \geq \ldots \geq (\alpha_l\ V_l)$ be a sequence of subterms of $(\beta\ V)$ forming an element of the set determining $h_x((\beta\ V))$ as in Lemma 88. But $U_a = x \notin (\beta\ V)$, so $(\alpha\ U) \geq (\alpha_1\ V_1) \geq \ldots \geq (\alpha_l\ V_l)$ is also a suitable sequence for $h_x((\alpha\ U))$, which means $l + 1 \leq m$, that is, $h_x((\beta\ V)) \leq m - 1$. □

Lemma 97 *Let U, V be admissible. Then $U[x := V]$ also is admissible.*

Proof Let $y \in U[x := V]$. Then either y comes from from U or from V. In the former case $U \in \mathcal{C}(y)$ implies there is a $(\alpha\ U') \leq U$ and an a such that $U'_a = y$. Then $(\alpha\ U'[x := V]) \leq U[x := V]$ and $(U'[x := V])_a = y$ is valid. In the latter case $V'_b = y$ for some b and $(\beta\ V') \leq V$, but this relation remains true for the occurrence of y in the substituted instance of V also. By this reasoning the admissibility of $U[x := V]$ follows. □

Lemma 98 *Let $U \in \mathcal{S}_0$, $V = \mu\gamma V_1 \in \mathcal{S}_{n^+}$, and suppose U_a is defined. Then*

$$U\langle a := V\rangle \twoheadrightarrow \mu\gamma V_1[\gamma :=_a U] \in \mathcal{S}_{n^+}.$$

Proof Let $a = [a_1, \ldots, a_k]$, the proof goes by induction on k. (For $k = 0$ the statement is obvious.)

1. Let $k = 1$, say $a = [r]$. Then $U = (U_1\ U_2)$ and

$$U\langle a = V\rangle = (U_1\ V) \to \mu\gamma V_1[\gamma :=_l U_1] = \mu\gamma V_1[\gamma :=_{[r]} (U_1\ U_2)].$$

2. Suppose $a := [a_1 :: b]$, say $a_1 = l$. Then $U = (U_1\ U_2)$ and

$$U\langle a = V\rangle = (U_1\langle b = V\rangle\ U_2) \twoheadrightarrow (\mu\gamma V_1[\gamma :=_b U_1]\ U_2) \to \mu\gamma V_1[\gamma :=_a (U_1\ U_2)].$$

□

Lemma 99 *Let $U \in \mathcal{S}_0$ be almost admissible, and let $V = \mu\gamma\vec{\mu}V_1 \in \mathcal{S}_{n^+}$ be admissible. Assume x is such that the only occurrence of x in U is a non-correct one. Then we have an admissible $W \in \mathcal{S}_0$ such that*

$$U[x := V] \twoheadrightarrow \mu\gamma\vec{\mu}W.$$

Moreover, let $x' \in Fv(\mu\gamma\vec{\mu}W)$ be correct w.r.t. α. Then either α is a free μ-variable of $\mu\gamma\vec{\mu}W$ or $\alpha = \gamma$.

Proof Applying Lemma 94 we have $U_a = x$ for some address a. By Lemma 98,

$$U[x := V] \twoheadrightarrow \mu\gamma\vec{\mu}V_1[\gamma :=_a U].$$

Let $W := V_1[\gamma :=_a U] \in \mathcal{S}_0$. For the admissibility of W it must be checked first that for all $x' \in Fv(W)$ we have $W \in \mathcal{C}(x')$.

1. Let $x' \in Fv(W)$, suppose first $x' \in V_1$ holds. Since $V \in \mathcal{C}(x')$, we have a $(\beta\ V_2) \leq V_1$ and a b such that $(V_2)_b = x'$. By the admissibility of V, $\beta \neq \gamma$. Then we have
$$(\beta\ V_2)[\gamma :=_a U] = (\beta\ V_2[\gamma :=_a U])$$
and $(V_2[\gamma :=_a U])_b = x'$, so x' is correct in W w.r.t. the same variable as in V_1.

2. Suppose $x' \in Fv(W)$ comes from U, let $a = [a_1, \ldots, a_k]$. Then there is some instance of U and $(\gamma\ V') \leq V_1$ such that $x' \in U\langle a := V'[\gamma :=_a U]\rangle$ and $x' \notin V'[\gamma :=_a U]$. This implies that there is an $1 \leq i \leq k$ such that $a' = [a_1, \ldots, \overline{a_i}]$ and $x' \in U_{a'}$ for that instance of U.
   - Assume $x' \in U$ is correct. Then we have $(\beta\ U') \leq U$ and $U'_b = x'$ for some $(\beta\ U')$ and b. By induction on i we obtain $(\beta\ U') \leq U_{a'}$ for the $U_{a'}$ as above. Thus x' is correct in W w.r.t. the same variable as in U.
   - Let the occurrence of x' in U be non-correct. Then, since $U \in \mathcal{S}_0$, by Lemma 94 we have a $b \neq a$ such that $U_b = x'$ for that instance of U. In this case $(U\langle a = V'[\gamma :=_a U]\rangle)_b = x'$, which means that x' is correct w.r.t. γ.

By the above argument it can easily be checked also that different occurrences of $x' \in Fv(W)$ are correct w.r.t. different occurrences of μ-variables of W, so admissibility holds. The last statement is again straightforward, since the argument shows that for any $x' \in Fv(\mu\gamma\vec{\mu}W)$ the occurrences of x' are correct either w.r.t. free μ-variables of $\mu\gamma\vec{\mu}W$, or μ-variables of U, or γ. □

Lemma 100 *Let U, U_1, x, x_1 be such that $x \in Fv(U)$, $x_1 \notin Fv(U)$ and $U_1 \in \mathcal{C}(x_1)$. Then $U[x := U_1] \in \mathcal{C}(x_1)$ and*

$$h_{x_1}(U_1) = h_{x_1}(U[x := U_1]).$$

Proof The claim $U[x := U_1] \in \mathcal{C}(x_1)$ follows at once from the facts that $U_1 \in \mathcal{C}(x_1)$ and $x_1 \notin Fv(U)$.
Let us prove $h_{x_1}(U_1) = h_{x_1}(U[x := U_1])$.
The direction $h_{x_1}(U_1) \leq h_{x_1}(U[x := U_1])$ is trivial. For the reverse direction let $(\alpha_1\ W_1) \geq \ldots \geq (\alpha_m\ W_m)$ be a sequence of subterms of $U[x := U_1]$ as in Lemma 88. Let us take an occurrence of x_1 in W_1 correct w.r.t. α_1. Since $x_1 \notin U$, x_1 is in some substituted instance of U_1. But $U_1 \in \mathcal{C}(x_1)$ and x_1 is correct in $U[x := U_1]$ with respect to α_1, these together involve $(\alpha_1\ W_1) \leq U_1$, from which $h_{x_1}(U[x := U_1]) \leq h_{x_1}(U_1)$ follows. □

Theorem 101 *Let U, V be admissible. Assume $U \in \mathcal{S}_0$ and $V \in \mathcal{S}_n$ for some $n \in \mathbb{N}$. Then there is an admissible $W \in \mathcal{S}_0$ such that*

$$U[x := V] \twoheadrightarrow \vec{\mu}W.$$

Proof We may assume $n \geq 1$ and $x \in Fv(U)$, since otherwise the statement is trivial. If $x \in Fv(U)$, we have $h_x(U) = m$ for some $m > 0$.
The proof goes by induction with respect to the lexicographically ordered pair $\langle n, m\rangle$. Let us examine the general case, that is, $n, m > 1$ first, since the cases $n = 1$ or $m = 1$ follow as special instances of the general one.
Let us enumerate in U all the maximal subterms of the form $(\alpha'\ U')$ such that U' contains an occurrence of x correct with respect to α'. Let them be $(\alpha_1\ U_1), \ldots, (\alpha_p\ U_p)$. Let us take $(\alpha_1\ U_1)$, suppose $(U_1)_a = x$ for some $a = [a_1, \ldots, a_k]$. Assume temporarily $k \geq 1$, that is $a \neq []$. Let us assign a term $\phi(U_1, V)$ to U_1 in the following manner. Let us take all the maximal subterms of the terms $U_{[a_1,\ldots,\overline{a}_i]}$ $(1 \leq i \leq k)$ such that they are of the form $(\alpha'\ U')$ and U' contains a correct occurrence of x w.r.t. α'. Let $W_1, \ldots, W_s$ be an enumeration of them. By Lemma 96 we can assert that W_j is admissible and $h_x(W_j) \leq m - 1$ $(1 \leq j \leq s)$. Exchange the terms $W_1, \ldots, W_s$ with the new variables $x_1, \ldots, x_s$, respectively. Denote the new term by $\pi(U_1)$. In case of $a = []$ let $\pi(U_1) = V$. Write

$$\eta_1 := [x_1 := W_1[x := V], \ldots, x_s := W_s[x := V]].$$

Now we have

$$U_1[x := V] = \pi(U_1)[x := V]\eta_1.$$

Let $\phi(U_1, V) = \pi(U_1)[x := V]$. Observe that $x \notin \phi(U_1, V)$ and $\phi(U_1, V)$ contains a μ only in the subterm $(\phi(U_1, V))_a = V$. By hypothesis the conditions of Lemma 99 fulfil, so

$$\phi(U_1, V) \twoheadrightarrow \mu\gamma\vec{\mu}W,$$

where $V = \mu\gamma\vec{\mu}V_1$ and $W \in \mathcal{S}_0$. Implementing a ρ-reduction with α_1 we arrive at

$$(\alpha_1\ \phi(U_1, V)) \twoheadrightarrow \vec{\mu}W[\gamma := \alpha_1].$$

Denote the term on the right hand side by $\psi(U_1, V)$. We should remark that, since $n > 1$, $\psi(U_1, V) \in \mathcal{S}_{(n-1)^+}$ and by Lemma 99 we also have the admissibility of $\psi(U_1, V)$. As a summary we have obtained that

$$U_1[x := V] \twoheadrightarrow \psi(U_1, V)\eta_1.$$

Let us perform the same process for every $(\alpha_i\ U_i)$, taking care that the sets $dom(\eta_i)$ $(1 \leq i \leq p)$ are pairwise disjoint sets of new variables. Let $\pi^*(U)$ denote the term obtained form U by exchanging every U_i by $\pi(U_i)$, moreover let $\chi(U, V)$ be the result of exchanging every $(\alpha_i\ U_i)$ $(1 \leq i \leq p)$ for $\psi(U_i, V)$ in the original U. Denote $\eta := \bigcup_{i=1}^{p} \eta_i$. Then we can conclude

$$U[x := V] = \pi^*(U)[x := V]\eta \twoheadrightarrow \chi(U, V)\eta. \tag{1.1}$$

Let $\{y_1, \ldots, y_p\}$ be a set of new variables. Let us replace in $\chi(U, V)$ every subterm $\psi(U_i, V)$ $(1 \leq i \leq p)$ by the variables $y_1, \ldots, y_p$, respectively. Denote the resulting term of $\mathcal{S}_0$ by U'. Observe that U' is almost admissible and the only variables which may not be correct in U' are $y_1, \ldots, y_p$. Let

$$\sigma := [y_1 := \psi(U_1, V), \ldots, y_p := \psi(U_p, V)].$$

With this notation we get

$$\chi(U, V) = U'\sigma.$$

We have two possible cases.

1. For all y_i $(1 \leq i \leq p)$ $U' \in C(y_i)$ holds. In this case, since all variable occurrences are correct in U' and $U' \in \mathcal{S}_0$, U' is admissible. Moreover, $\sigma(y_i) = \psi(U_i, V) \in \mathcal{S}_{(n-1)^+}$ are also admissible. Thus, successively applying the induction hypothesis to U' and the one-element substitutions constituting σ, we obtain the relation
$$U'\sigma \twoheadrightarrow \vec{\mu}U'',$$
where $U'' \in \mathcal{S}_0$ is admissible.

2. There is a $y_j \in dom(\sigma)$ such that $U' \notin \mathcal{C}(y_j)$. Then, by Lemma 99, there is an admissible $\tilde{U} \in \mathcal{S}_0$ such that $U'[y_j := \psi(U_j, V)] \twoheadrightarrow \vec{\mu}_1\tilde{U}$. Let σ' be the substitution obtained from σ by omitting y_j from $dom(\sigma)$. Successively applying the induction hypothesis to $\tilde{U}$ and σ' we acquire an admissible $U'' \in \mathcal{S}_0$ such that $\tilde{U}\sigma' \twoheadrightarrow \vec{\mu}_2U''$, which implies
$$U'\sigma \twoheadrightarrow \vec{\mu}U''.$$

Let $U'' \in \mathcal{S}_0$ be the term obtained by any of the above arguments such that
$$\chi(U, V) = U'\sigma \twoheadrightarrow \vec{\mu}U''. \tag{1.2}$$
We should recall that x does not occur in U''. Take $x_1 \in dom(\eta)$. We may suppose $x_1 \in Fv(U'')$. Then
$$U''[x_1 := W_1[x := V]] = U''[x_1 := W_1][x := V]$$
and by Lemma 100 we have
$$h_x(U''[x_1 := W_1]) = h_x(W_1) \leq m - 1.$$
Lemma 97 gives the admissibility of $U''[x_1 := W_1]$. Thus $U''[x_1 := W_1] \in \mathcal{S}_0$ is admissible, $h_x(U''[x_1 := W_1]) \leq m - 1$, hence the induction hypothesis applied to $U''[x_1 := W_1]$ and V yields
$$U''[x_1 := W_1[x := V]] = U''[x_1 := W_1][x := V] \twoheadrightarrow \vec{\mu}U_1^*,$$
where $U_1^* \in \mathcal{S}_0$ is admissible and does not contain x. We can repeat the above process with U_1^* and $[x_2 := W_2[x := V]]$ instead of U'' and $[x_1 := W_1[x := V]]$ to obtain an admissible $U_2^* \in \mathcal{S}_0$ such that $x \notin Fv(U_2^*)$ and
$$U_1^*[x_2 := W_2[x := V]] = U_1^*[x_2 := W_2][x := V] \twoheadrightarrow \vec{\mu}U_2^*.$$
Continuing in this way for the remaining elements of $dom(\eta)$ we can conclude that
$$U''\eta \twoheadrightarrow \vec{\mu}U^* \tag{1.3}$$
with an admissible $U^* \in \mathcal{S}_0$. Then (1.3), together with (1.1) and (1.2), yields the result.

Let us treat the case $m = 1$ or $n = 1$. Let us suppose first $m = 1$ and $n > 1$. In this case the η of the previous process is empty. Thus, with the notations as above,
$$U[x := V] \twoheadrightarrow \chi(U, V).$$
Applying for $\chi(U, V)$ the method described in the second part of the algorithm we obtain the result.
If $n = 1$, then we can demonstrate by induction on m, that there is an admissible $U' \in \mathcal{S}_0$ such that $U[x := V] \twoheadrightarrow U'$. For $n, m = 1$ the statement is obvious. If $n = 1$ and $m > 1$, then
$$U[x := V] \twoheadrightarrow \chi(U, V)\eta,$$
where, in this case, $\chi(U, V) \in \mathcal{S}_0$ is admissible and for all $x' \in dom(\eta)$ we have $\eta(x') = W'[x := V]$ for some admissible $W' \in \mathcal{S}_0$ such that $h_x(W') \leq m - 1$. Then we can apply the induction hypothesis to every $\eta(x')$ with $x' \in dom(\eta)$, by which the assertion follows. □

Corollary 102 *The $\mu\mu'\rho$-calculus is weakly normalizing.*

Proof It is enough to prove that for $U,\ V \in NF$ we have $(U\ V) \in NF$. We may suppose $U,\ V \in \mathcal{S}_{n+}$ and, in the beginning, U and V contain no λ-variables. Let $U = \mu\alpha\overrightarrow{\mu}_1 U'$ for some $U' \in \mathcal{S}_0$. Let U^* be the term obtained from U' by exchanging every subterm $(\alpha\ U'') \leq U'$ for $(\alpha\ (U''\ x))$. Then, applying Theorem 101, we have

$$\begin{aligned}(U\ V) &= (\mu\alpha\overrightarrow{\mu}_1 U'\ V) \to \mu\alpha\overrightarrow{\mu}_1 U'[\alpha := V] \\ &= \mu\alpha\overrightarrow{\mu}_1 U^*[x := V] \twoheadrightarrow \mu\alpha\overrightarrow{\mu}_1\overrightarrow{\mu}_2 W = \overrightarrow{\mu}W\end{aligned}$$

for some $W \in \mathcal{S}_0$. □

Example 103 *The following easy example gives some insight into the operation of the algorithm. Let*

$$U = (\alpha\ (x_0\ u))(\alpha\ (x_0\ (\alpha\ (x_0\ v)))), \quad V = \mu\beta_1\mu\beta_2(\beta_1\ (w\ (\beta_1\ w)))(\beta_2\ (w\ (\beta_1\ (w\ (\beta_2\ w))))).$$

We may suppose u, v and w are constants. Then $h_{x_0}(U) = 2$, $V \in \mathcal{S}_2$ and

$$U[x_0 := V] = (\alpha\ (V\ u))(\alpha\ (V\ (\alpha\ (V\ v)))).$$

Applying the notation of the algorithm and writing $V = \mu\beta_1\mu\beta_2 V'$ we have

$$\begin{array}{ll} U_1 = (x_0\ u), & U_2 = (x_0\ (\alpha\ (x_0\ v))), \\ \pi(U_1) = (x_0\ u), & \pi(U_2) = (x_0\ x_1), \\ \phi(U_1, V) = (V\ u), & \phi(U_2, V) = (V\ x_1), \\ \psi(U_1, V) = \mu\beta_2(\alpha\ ((w\ (\alpha\ (w\ u)))\ u))(\beta_2\ (w(\alpha\ ((w\ (\beta_2\ w))\ u))) & \\ \quad = \mu\beta_2 V'[\beta_1 := u][\beta_1 := \alpha], & \\ \psi(U_2, V) = \mu\beta_2(\alpha\ ((w\ (\alpha\ (w\ x_1)))\ x_1))(\beta_2\ (w(\alpha\ ((w\ (\beta_2\ w))\ x_1))) & \\ \quad = \mu\beta_2 V'[\beta_1 := x_1][\beta_1 := \alpha], & \\ \eta = \eta_1 = [x_1 := (\alpha\ (x_0\ v))[x_0 := V]]. & \end{array}$$

Figure 1.1 illustrates the operation of the algorithm until this point, the notation applied there is the one used here and in Theorem 101. Now, writing y_i for $\psi(U_i, V)$ $(i \in \{1, 2\})$, we arrive at $U' = (y_1\ y_2)$, thus

$$U'[y_1 := \psi(U_1, V)] \to \mu\beta_2(\alpha\ ((w\ (\alpha\ (w\ u)))\ u))(\beta_2\ ((w\ (\alpha\ ((w\ (\beta_2\ (w\ y_2)))\ u)))\ y_2)).$$

We have $\psi(U_2, V) \in \mathcal{S}_1$, so making use of the case $n = 1$ in Theorem 101, we obtain

$$U'[y_1 := \psi(U_1, V), y_2 := \psi(U_2, V)] \to^* \mu\beta_2 U''$$

with some admissible $U'' \in \mathcal{S}_0$. Returning back to η, since $x_0 \notin U''$, $W_1 = (\alpha\ (x_0\ v)) \in \mathcal{C}(x_0)$, Lemma 100 yields

$$h_{x_0}(U''[x_1 := (\alpha\ (x_0\ v))]) = h_{x_0}((\alpha\ (x_0\ v))) = 1.$$

We are now dealing with the case of $m = 1$. By the process described in Theorem 101

$$(U''[x_1 := (\alpha\ (x_0\ v))])[x_0 := V] \to^* \chi(U''[x_1 := (\alpha\ (x_0\ v))], V) = M$$

such that $M = M'[y_3 := N_1, y_4 := N_2]$ for some $M' \in \mathcal{S}_0$ and $N_1,\ N_2 \in \mathcal{S}_1$. But then, successively applying the algorithm for $n = 1$, we obtain a normal form of M, by which the normalization process is completed.

1.2.4 The $\lambda\mu\mu'\rho$-calculus is weakly normalizing

Let us consider the example in Proposition 79. We can observe that the terms $U = \mu\alpha(\alpha\ (\alpha\ x))$ and $V = \mu\beta U$ can even be typed. Let $x : \bot$, then $\alpha : \neg\bot$ and $\beta : \neg(\bot \to \bot)$ yield the typing relations $U : \bot$, $V : \bot \to \bot$. Thus, we have obtained that the $\lambda\mu\mu'\rho$-calculus is not strongly normalizing either. In possession of the weak normalization of the untyped calculus we can prove, however, that the typed one is weakly normalizing also.
The general idea of the proof is as follows: starting from a term M_1 in $\mu\mu'\rho$-normal form, i.e. from a term in which there are no μ-, μ'- or ρ-redexes, we begin eliminating all its β-redexes by any weak normalization algorithm for the typed λ-calculus until we arrive at a term M_2 in β-normal form. Having done this, we apply the $\mu\mu'\rho$-weak normalization algorithm to M_2 to obtain an M_3 in $\mu\mu'\rho$-normal form. We state and prove in Theorem 111 that the maximum of the rank of the β-redexes of M_1 is strictly greater than that of M_3. By this, the process must terminate.

Definition 104 *1.* $NF_\mu := \{M \mid M$ *is in* $\mu\mu'\rho$*-normal form*$\}$

2. $NF_\beta := \{M \mid M$ *is in* β*-normal form*$\}$

Definition 105 *Let* M, N *be terms such that* $M \twoheadrightarrow^\nu N$. *The notation* $M \twoheadrightarrow^\nu_\beta N$ *(or simply* $M \twoheadrightarrow_\beta N$*) means that* ν *entirely consists of* β*-reductions, analogously,* $M \twoheadrightarrow^\nu_\mu N$ *(or simply* $M \twoheadrightarrow_\mu N$*) indicates that* ν *consists of* μ*-,* μ'*- and* ρ*-reductions.*

Definition 106 *The length of a type* A *is defined as the number of the arrows in* A.

Definition 107 *Let* M *be a* $\lambda\mu$*-term, and* r *a redex in* M.

1. *The rank of* r *is:*

$$rank(r, M) = \begin{cases} lh(type(\lambda x M_1)) & \text{if } r = (\lambda x M_1\ M_2), \\ lh(type(\mu\alpha M_1)) & \text{if } r = (\mu\alpha M_1\ M_2), \\ lh(type(\mu\beta M_2)) & \text{if } r = (M_1\ \mu\beta M_2) \\ & \text{and } r \text{ is neither a } \beta\text{- nor a } \mu\text{-redex}, \\ lh(type(\mu\beta M_1)) & \text{if } r = (\alpha\ \mu\beta M_1). \end{cases}$$

2. *The rank of* M *is:*

$$rank(M) = max\{rank(r, M) \mid r \text{ is a redex in } M\}.$$

Remark 108 *When* r *is both a* μ*- and a* μ'*-redex of* M*, then we calculate* $rank(r, M)$ *as if* r *were a* μ*-redex only. Similarly, for the case, when* r *is both a* β*- and a* μ'*-redex in* M*. This causes no changes when determining the value* $rank(M)$*, since in both of the cases above* $rank(r, M)$ *is greater when* r *is considered as a* β*- or as a* μ*-redex, respectively, than as a* μ'*-redex.*

Lemma 109 *Let* M, N *be* $\lambda\mu$*-terms. Then*

$$rank(M[x := N]) \leq max\{rank(M), rank(N), lh(type(x))\}.$$

Proof By induction on $cxty(M)$. The only interesting case is $M = (M_1\ M_2)$. We examine some of the cases.

1. $M[x := N]$ is not a redex. Applying the induction hypothesis we have

$$\begin{aligned} rank(M[x := N]) &= max\{rank(M_1[x := N]), rank(M_2[x := N])\} \\ &\leq max\{rank(M_1), rank(M_2), rank(N), lh(type(x))\} \\ &\leq max\{rank(M), rank(N), lh(type(x))\}, \end{aligned}$$

where we have made use of the fact that in this case M is not a redex either.

2. $M[x := N]$ is a β-redex. The proof for the other redexes is similar.

   - If $M_1 = \lambda y M_3$, then

$$\begin{aligned} rank(M[x := N]) &= max\{rank(M_3[x := N]), rank(M_2[x := N]), \\ &\quad lh(type(M_1[x := N]))\} \\ &\leq max\{rank(M_3), rank(M_2), rank(N), \\ &\quad lh(type(M_1)), lh(type(x))\} \\ &= max\{rank(M), rank(N), lh(type(x))\}. \end{aligned}$$

   - Assume $M_1 = x$. Then

$$\begin{aligned} rank(M[x := N]) &= max\{rank(N), rank(M_2), lh(type(N))\} \\ &\leq max\{rank(M), rank(N), lh(type(x))\}. \end{aligned}$$

□

Lemma 110 *Let $M \twoheadrightarrow_\beta M'$. Then $rank(M) \geq rank(M')$.*

Proof It is enough to prove if $M \rightarrow^r M'$ such that r is a β-redex, then $rank(M) \geq rank(M')$. The proof goes by induction on $cxty(M)$. The only interesting case is $M = (M_1\ M_2)$.

1. $M = (M_1\ M_2) \rightarrow^r (M_1'\ M_2) = M'$.

   - M' is not a redex. Then $rank(M') = max\{rank(M_1'), rank(M_2)\} \leq max\{rank(M_1), rank(M_2)\} \leq rank(M)$ by the induction hypothesis.
   - M' is a redex. The only non-trivial case is $M_1 = r \rightarrow_\beta M_1'$. Then, by the induction hypothesis and since $lh(type(M_1')) < rank(r, M) \leq rank(M)$, we obtain

$$\begin{aligned} rank(M') &= max\{rank(M_1'), rank(M_2), lh(type(M_1'))\} \\ &\leq max\{rank(M_1), rank(M_2), rank(M)\} \\ &= rank(M). \end{aligned}$$

2. $M = (M_1\ M_2) \rightarrow^r (M_1\ M_2') = M'$. The induction hypothesis applies.

3. $r = M \rightarrow^r M'$. Assume $M = (\lambda x M_3\ M_2)$. By Lemma 109, $rank(M') \leq max\{rank(M_3), rank(M_2), lh(type(x))\} \leq rank(M)$.

□

Theorem 111 *Let $M_1, M_3 \in NF_\mu$ and $M_2 \in NF_\beta$ such that*

$$M_1 \twoheadrightarrow_\beta M_2 \twoheadrightarrow_\mu M_3.$$

Then

$$rank(M_1) > rank(M_3).$$

We are going to proceed with the proof of the theorem by disassembling it into several lemmas which will altogether yield the result.

Definition 112 *The function occ is such that if M is a term, then $occ(M, \pi)$ is the subterm of M at address π provided it is defined.*

1. $occ(M, []) = M$,

2. $occ((M_1\ M_2), [l :: s]) = occ(M_1, s)$,

3. $occ((M_1\ M_2), [r :: s]) = occ(M_2, s)$,

4. $occ(\lambda x M_1, [\lambda x :: s]) = occ(M_1, s)$,

5. $occ(\mu\alpha M_1, [\mu\alpha :: s]) = occ(M_1, s)$,

6. $occ((\alpha\ M_1), [\alpha :: s]) = occ(M_1, s)$.

Let the expression $occ(M, \pi)$ be undefined if it is not in one of the forms indicated on the left-hand side of the above equations. We say that π is an occurrence of N in M, if $occ(M, \pi) = N$.

With an abuse of terminology we may refer simply to a subterm N of M, rather than mentioning N together with its occurrence π.

Notation 113 *Let M be a term.*

1. *In this subsection let Tm denote the set of $\lambda\mu$-terms.*

2. *Denote by $Sbt(M)$ the set of subterms of M.*

3. *Denote by Occ the set of all possible addresses, that is,*

$$Occ = \big(\{\lambda x \mid x \in \mathcal{V}\} \cup \{\alpha \mid \alpha \in \mathcal{W}\} \cup \{\mu\alpha \mid \alpha \in \mathcal{W}\} \cup \{l, r\}\big)^{<\infty},$$

   where $\mathcal{V}$ (resp. $\mathcal{W}$) denotes the set of λ-variables (resp. μ-variables).

4. *If S is an arbitrary set, let $\mathcal{P}(S)$ stand for the set of its subsets.*

Remark 114 *In order to facilitate the presentation, in this section we ignore the stipulation that all terms are in pure variable forms. Thus, when performing a substitution we disregard the renaming of bound variables.*

Definition 115 *Let $lst(\xi)$ denote the last element of ξ provided $\xi \neq []$, undefined otherwise.*

Definition 116 *Let M be a term. The function $bn : Tm \times Occ \to Tm$ defined below has the following property: if $occ(M, \xi) = (\alpha\ N)$ (resp. $occ(M, \xi) = x$), then $bn(M, \xi)$ is the smallest subterm of M containing $occ(M, \xi)$ in which the variable α (resp. the variable x) becomes bound.*

1. *If $occ(M, \xi) = (\alpha\ P)$, then let $bn(M, \xi) = occ(M, \zeta)$, where ζ is the longest initial segment of ξ such that $occ(M, \zeta) = \mu\alpha Q$ if such a Q exists.*

2. *If $occ(M, \xi) = x$, then $bn(M, \xi) = occ(M, \zeta)$, where ζ is the longest initial segment of ξ such that $occ(M, \zeta) = \lambda x Q$ if such a Q exists.*

3. *Let $bn(M, \xi)$ be undefined otherwise.*

Notation 117 *An address ξ is said to be an occurrence of a μ-variable α (resp. λ-variable x) if $occ(M, \xi) = (\alpha\ P)$ for some P (resp. $occ(M, \xi) = x$).*

Definition 118 *An occurrence ξ of α (resp. x) is free in M if $bn(M, \xi)$ is undefined.*

Definition 119 *Let M be a term and $\xi = [a_1, \ldots, a_n]$. Assume $occ(M, \xi)$ is defined. We denote by $r(M, \alpha, \xi)$ (resp. $l(M, \alpha, \xi)$) the address obtained from ξ by inserting simultaneously an element r (resp. l) after each a_i for which $a_i = \alpha$ and $\xi_{i-1} = [a_1, \ldots, a_{i-1}]$ is a free occurrence of α in M (where $\xi_0 = []$). We write briefly $l(\xi)$ (resp. $r(\xi)$) if M and α are clear from the context.*

Example 120 *Let* $M = (\alpha\ (x\ \mu\alpha(\alpha\ y)))\ (\alpha\ \lambda xy)$. *Let* $\xi_1 = [l, \alpha, l]$, $\xi_2 = [l, \alpha, r, \mu\alpha, \alpha]$ *and* $\xi_3 = [r, \alpha]$. *Then* $l(M, \xi_1, \alpha) = [l, \alpha, l, l]$, $r(M, \xi_2, \alpha) = [l, \alpha, r, r, \mu\alpha, \alpha]$ *and* $l(M, \xi_1, \alpha) = [r, \alpha, l]$, *respectively.*

Notation 121 *We write* $\pi_2 \leq \pi_1$ *if* π_2 *is an initial subsequence of* π_1. *The notation* $\pi_1 \nsim \pi_2$ *indicates the fact that neither* $\pi_1 \leq \pi_2$ *nor* $\pi_2 \leq \pi_1$ *holds.*

Remark 122 *Let* M *be a term. We can observe that* $\pi_2 \leq \pi_1$ *iff* $occ(M, \pi_1) \leq occ(M, \pi_2)$.

Definition 123 *Let* r *be a redex of* M, *assume* $occ(M, \pi) = r$. *The function* $adr : Tm \times Occ \times Occ \to \mathcal{P}(Occ)$ *is defined in a way such that its value* $adr(M, \pi, \xi)$ *gives the set of addresses of the residuals of the subterm* $occ(M, \xi)$ *after reducing in* M *with the redex* r.

1. $r = (\lambda x P\ Q)$.
   (a) $adr(M, \pi, \xi) = \{\xi\}$ *if* $\pi \nsim \xi$,
   (b) $adr(M, \pi, \xi) = \{\pi \# \zeta\}$ *if* $\xi = [\pi :: l :: \lambda x] \# \zeta$,
   (c) $adr(M, \pi, \xi) = \{\pi \# \varepsilon \# \zeta \mid occ(P, \varepsilon) = x\}$ *if* $\xi = [\pi :: r] \# \zeta$,
   (d) $adr(M, \pi, \xi) = \{\xi\}$ *if* $\xi < \pi$.
2. $r = (\mu\alpha P\ Q)$.
   (a) $adr(M, \pi, \xi) = \{\xi\}$ *if* $\pi \nsim \xi$,
   (b) $adr(M, \pi, \xi) = \{[\pi :: \mu\alpha] \# l(occ(M, \pi'), \alpha, \zeta)\}$, *if* $\xi = \pi' \# \zeta$ *with* $\pi' = [\pi :: l :: \mu\alpha]$ *and* $l(occ(M, \pi'), \alpha, \zeta)$ *is defined according to Definition 119,*
   (c) $adr(M, \pi, \xi) = \{[\pi :: \mu\alpha] \# l(occ(M, \pi'), \alpha, \varepsilon) \# [\alpha :: r :: \zeta] \mid occ(P, \varepsilon) = (\alpha\ V),$ *for some* $(\alpha\ V) \leq P\}$ *if* $\xi = [\pi :: r] \# \zeta$ *and* $\pi' = [\pi :: l :: \mu\alpha]$,
   (d) $adr(M, \pi, \xi) = \{\pi\}$ *if* $\xi = [\pi :: l]$,
   (e) $adr(M, \pi, \xi) = \{\xi\}$ *if* $\xi < \pi$.
3. $r = (P\ \mu\alpha Q)$.
   (a) $adr(M, \pi, \xi) = \{\xi\}$ *if* $\pi \nsim \xi$,
   (b) $adr(M, \pi, \xi) = \{[\pi :: \mu\alpha] \# r(occ(M, \pi'), \alpha, \zeta)\}$ *if* $\xi = \pi' \# \zeta$ *with* $\pi' = [\pi :: r :: \mu\alpha]$ *and* $r(occ(M, \pi'), \alpha, \zeta)$ *is defined according to Definition 119,*
   (c) $adr(M, \pi, \xi) = \{[\pi :: \mu\alpha] \# r(occ(M, \pi'), \alpha, \varepsilon) \varepsilon \# [\alpha :: l :: \zeta] \mid occ(P, \varepsilon) = (\alpha\ V),$ *for some* $(\alpha\ V) \leq P\}$ *if* $\xi = [\pi :: l] \# \zeta$ *and* $\pi' = [\pi :: r :: \mu\alpha]$,
   (d) $adr(M, \pi, \xi) = \{\pi\}$ *if* $\xi = [\pi :: r]$,
   (e) $adr(M, \pi, \xi) = \{\xi\}$ *if* $\xi < \pi$.
4. $r = (\alpha\ \mu\beta P)$.
   (a) $adr(M, \pi, \xi) = \{\xi\}$ *if* $\pi \nsim \xi$,
   (b) $adr(M, \pi, \xi) = \{[\pi] \# \zeta'\}$ *if* $\xi = [\pi :: \alpha :: \mu\beta] \# \zeta$, *where* ζ' *is obtained from* ζ *by exchanging every occurrence of* β *in* ζ *for* α,
   (c) $adr(M, \pi, \xi) = \{\xi\}$ *if* $\xi < \pi$.

Definition 124 *We define the function* $des_0 : Tm \times Occ \times Occ \to \mathcal{P}(Tm)$ *as follows. Let* $M \to^r N$, *assume* $occ(M, \pi) = r$. *The value* $des_0(M, \pi, \xi)$ *determines the set of descendants of the subterm of* M *at address* ξ *after reducing in* M *with the redex at address* π. *That is,*

$$des_0(M, \pi, \xi) = \{occ(N, \zeta) \mid \zeta \in adr(M, \pi, \xi)\}.$$

Remark 125 *The above function is not everywhere defined. More exactly, for every address π there are terms which has no descendants w.r.t. the reduction with the redex r at address π. In detail:*

1. *If $r = (\lambda xP\ Q)$, then the terms which have no residuals are those with addresses π and $[\pi :: l]$, namely r itself and λxP.*
2. *If $r = (\mu\alpha P\ Q)$, then r only has no residual.*
3. *The situation is analogous to the above one, if $r = (P\ \mu\beta Q)$.*
4. *If $r = (\alpha\ \mu\beta P)$, then r and $\mu\beta P$, that is, the terms with addresses π and $[\pi :: \alpha]$, have no residuals.*

Definition 126 *Let σ be a sequence of addresses. We define the values $des(M, \sigma, \xi)$ of the function $des : Tm \times Occ \times Occ \to \mathcal{P}(Tm)$ by induction on $|\sigma|$.*

1. *$des(M, [\pi], \xi) = des_0(M, \pi, \xi)$ if $\sigma = [\pi]$.*
2. *$des(M, [\pi :: \sigma_1], \xi) = \bigcup\{des(N, \sigma_1, \zeta) \mid \zeta \in adr(M, \pi, \xi)\}$, where $M \to^r N$, in other words: $N = des(M, \pi, [])$, and $occ(M, \pi) = r$.*

Notation 127 *Let $M \geq N$ and $M \twoheadrightarrow^\sigma M'$, suppose $N = occ(M, \xi)$. We say that N' is a descendant of N w.r.t. σ, if $N' \in des(M, \sigma, \xi)$. We denote this by $N \gg^\sigma N'$. In the case when σ is a sequence of β-reductions we say that N' is a β-descendant of N, in notation $N \gg_\beta N'$. If σ contains no β-reductions we speak of μ-descendants. In notation: $N \gg_\mu N'$.*

Remark 128 *The reduction relation, considered as a binary relation in the sense of Definition 28, conveys information about what redex occurrence is being reduced and which is the actual reduction rule being applied. In simply typed λ-calculus either of these two pieces of information determine the other. Here, a redex, considered as a subterm occurrence, does not necessarily indicate which reduction rule takes place actually. Throughout the present work, by a reduction $M \to^r N$ we not only mean an address π, where $occ(M, \pi) = r$ is a redex occurrence in M, but rather a pair carrying the piece of information of what kind of reduction is being applied together with the address of the redex occurrence in the term. Thus, the terminology redex is mostly reserved for denoting a subterm occurrence rather than an application of a reduction rule. The name reduction or reduction rule is intended to refer to the reduction step actually implemented together with knowing which redex occurrence is under discussion. The same stipulation should hold for the notion of reduction sequences as well. If ambiguity should emerge, we indicate explicitly what kind of reduction rule we are concerned with.*

The definition below goes round this notation a little further.

Definition 129 *Let $C[]$ be a (possibly empty) one-holed context. Assume $M = C[r] \to^r N = C[r']$. Then the functional part of r is defined as follows.*

1. *If $r = (\lambda xP\ Q)$, $r' = P[x := Q]$, then the functional part of r is λxP.*
2. *If $r = (\mu\alpha P\ Q)$, $r' = \mu\alpha P[\alpha :=_r Q]$, then the functional part of r is $\mu\alpha P$.*
3. *If $r = (P\ \mu\alpha Q)$, $r' = \mu\alpha Q[\alpha :=_l P]$, then the functional part of r is $\mu\alpha Q$.*
4. *If $r = (\alpha\ \mu\beta P)$, $r' = P[\beta := \alpha]$, then the functional part of r is $\mu\beta P$.*

Lemma 130 *Let $M \twoheadrightarrow^\sigma M'$ and $M \geq N$. Assume $N \gg^\sigma N'$ for some N'. Then the following assertions are valid.*

1. *If $N = \lambda x P$, then $type(N) = type(N')$.*

2. *If $N = (\alpha\ P)$, then $type(N) = type(N')$.*

3. *If $N = (P_1\ P_2)$, then $type(N) = type(N')$.*

4. *If $N = \mu\alpha P$ and σ is a β-reduction sequence, then $type(N) = type(N')$.*

Proof A straightforward induction on $|\sigma|$. Note that in Case 4 the stipulation that σ is a β-reduction sequnce is actually needed. For, let $M = (Q\ \mu\alpha P)$, where $Q : A \to B$ and $N = \mu\alpha P : A$ are such that $lh(A) < lh(B)$. Assume σ consists of the single μ'-reduction M. Then $N \gg^{\sigma} N' = \mu\alpha P[\alpha :=_l Q]$ and $lh(type(N')) > lh(type(N))$. □

Lemma 131 *Let $M \in NF_{\mu}$, suppose $occ(M, \xi) = \mu\alpha N < M$. Then one of the following cases holds:*

1. *$lst(\xi) = \mu\beta$ for some β,*

2. *$lst(\xi) = \lambda x$ for some x.*

Proof Trivial. □

Lemma 132 *Let $M \in NF_{\beta}$, suppose $occ(M, \xi) = \lambda x N < M$. Then one of the following cases holds:*

1. *$lst(\xi) = \lambda y$ for some y,*

2. *$lst(\xi) = r$,*

3. *$lst(\xi) = \alpha$ for some α.*

Proof Trivial. One should observe that, since $\mu\alpha S \leq M$ implies $S : \bot$ for an arbitrary $\mu\alpha S$, the case $lst(\xi) = \mu\alpha$ is impossible. □

Remark 133 *Intuitively, Lemma 132 has the following meaning. Let $M \in NF_{\beta}$ and $\lambda x N < M$. Then either $\lambda y \lambda x N \leq M$, or there exits an S such that $(S\ \lambda x N) \leq M$, or $(\alpha\ \lambda x N) \leq M$ for the occurrence of $\lambda x N$ in question. An analogous statement is true in relation with Lemma 131.*

Definition 134 *Let σ be $M = M_0 \to^{r_1} M_1 \to^{r_2} \ldots \to^{r_n} M_n$, assume $occ(M, \xi) = (\alpha\ P)$. We say that $(\alpha\ P)$ is used in σ, if there is an $1 \leq i < n$ such that $(\alpha\ P) \gg^{[r_1,\ldots,r_i]} occ(M_i, \zeta)$ for some ζ and $bn(M_i, \zeta)$ is the functional part of r_{i+1}.*

Definition 135 *Let σ be $M = M_0 \to^{r_1} M_1 \to^{r_2} \ldots \to^{r_n} M_n$, assume $occ(M, \xi) = \lambda x P$. We say that $\lambda x P$ is used in σ, if there is an $1 \leq i < n$ such that $\lambda x P \gg^{[r_1,\ldots,r_i]} occ(M_i, \zeta)$ for some ζ and $\lambda x P'$ is the functional part of r_{i+1}.*

Lemma 136 *Let $M \twoheadrightarrow^{\sigma} N$, $N' \leq N$. Then there is a unique $M' \leq M$ such that $M' \gg^{\sigma} N'$.*

Proof Straightforward. □

Lemma 137 *Let $M \twoheadrightarrow^{\sigma} N$, $P \leq M$. Assume $P \gg^{\sigma} Q$. Then the following assertions are valid.*

1. *If $P = \lambda x P_1$, then $Q = \lambda x Q_1$.*

2. *If $P = \mu\alpha P_1$, then $Q = \mu\alpha Q_1$.*

3. *If $P = (\alpha\ P_1)$, then $Q = (\beta\ Q_1)$ for some β.*

4. *If $P = (P_1\ P_2)$, then $Q = (Q_1\ Q_2)$.*

Proof Straightforward. □

Remark 138 *Observe that by Lemma 137 in Definition 134 (resp. in Definition 135)* $(\alpha\ P) \gg^{[r_1,\ldots,r_i]} occ(M_i,\zeta)$ *(resp.* $\lambda xP \gg^{[r_1,\ldots,r_i]} occ(M_i,\zeta)$*) imply* $occ(M_i,\zeta) = (\beta\ P')$ *for some* P' *(resp.* $occ(M_i,\zeta) = \lambda xP'$ *for some* P'*).*

Lemma 139 *Let* $M \twoheadrightarrow^{\sigma} N$, $P \leq M$. *Assume* $P \gg^{\sigma} Q$. *Then the following assertions are valid.*

1. *If* $Q = \lambda xQ_1$, *then* $P = \lambda xP_1$.
2. *If* $Q = \mu\alpha Q_1$, *then* $P = \mu\alpha P_1$.
3. *If* $Q = (\beta\ Q_1)$, *then* $P = (\alpha\ P_1)$ *for some* α.
4. *If* $Q = (Q_1\ Q_2)$, *then* $P = (P_1\ P_2)$.

Proof Combine Lemmas 136 and 137. □

Lemma 140 *Let* $M \twoheadrightarrow^{\sigma} N$, $occ(M,\xi) = (\alpha\ P)$. *Assume* $(\alpha\ P)$ *is not used in* σ. *Then, for any* $occ(N,\zeta) = (\beta\ Q)$, $(\alpha\ P) \gg^{\sigma} (\beta\ Q)$ *iff* $P \gg^{\sigma} Q$ *with* $occ(N,[\zeta :: \beta]) = Q$.

Proof The proof goes by induction on $|\sigma|$. The case for $|\sigma| = 0$ is trivial. Assume $M \to^{r} M' \twoheadrightarrow^{\sigma'} N$ such that $\sigma = [r]\#\sigma'$ for some σ'. Let $r = occ(M,\pi) = (\mu\gamma S\ T)$ such that r is a μ-reduction in σ. We prove that $(\alpha\ P) \gg^{r} (\alpha\ R)$ for some $occ(M',\delta) = (\alpha\ R)$ iff $P \gg^{r} R$ with $occ(M',\delta') = R$ such that $\delta' = [\delta :: \alpha]$. From this, by the induction hypothesis, the result follows. We have to distinguish the subcases forming Definition 123.

1. $\pi \nsim \xi$. Then $\pi \nsim [\xi :: \alpha]$: the result obviously holds.
2. $\xi = [\pi :: l :: \mu\gamma]\#\xi'$. Then $\delta = [\pi :: \mu\gamma]\#l(S,\gamma,\xi')$, where $l(S,\gamma,\xi')$ is defined in accordance with Definition 119. Moreover, $\delta' = [\pi :: \mu\gamma]\#l(S,\gamma,[\xi' :: \alpha])$. Since $(\alpha\ P)$ is not used in $[r]$ we have either α is not free in S or $\alpha \neq \gamma$. This means $l(\xi') = [l(\xi') :: \alpha]$, that is, $\delta' = [\delta :: \alpha]$, which proves our assertion.
3. $\xi = [\pi :: r]\#\xi'$. In this case $\delta = \pi\#[l(S,\gamma,\varepsilon) :: \gamma :: r]\#\xi'$, where $occ(\mu\gamma S,\varepsilon) = (\gamma\ U)$ for some U and $l(S,\gamma,\varepsilon)$ is defined in accordance with Definition 119. Moreover, we have $\delta' = \pi\#[l(S,\gamma,\varepsilon) :: \gamma :: r]\#[\xi' :: \alpha]$ for some ε, where $occ(\mu\gamma S,\varepsilon) = (\gamma\ U)$ for some U. This implies again that for every δ there is a $\delta' = [\delta :: \alpha]$ and vice versa.
4. $\xi = [\pi :: l]$. Since $occ(M,\xi) = (\alpha\ P)$, this case is impossible.
5. $\xi < \pi$. Since, by hypothesis, both $(\alpha\ R)$ and R exist, we have $[\xi :: \alpha] < \pi$. Then, $\delta' = [\xi :: \alpha] = [\delta :: \alpha]$, which gives the result.

□

Lemma 141 *Let* $M \twoheadrightarrow^{\sigma} N$, $occ(M,\xi) = \lambda xP$. *Assume* λxP *is not used in* σ. *Then, for any* $occ(N,\zeta) = \lambda xQ$, $\lambda xP \gg^{\sigma} \lambda xQ$ *iff* $P \gg^{\sigma} Q$ *with* $occ(N,[\zeta :: \lambda x]) = Q$.

Proof Similar to the above one. □

Lemma 142 *Let $M \twoheadrightarrow_{\mu}^{\sigma} N$, $occ(M,\xi) = \lambda xS$ such that $lst(\xi) = r$. Assume $\lambda xS \gg_{\mu}^{\sigma} \lambda xU$ for some $occ(N,\zeta) = \lambda xU$. Then $lst(\zeta) = r$.*

Proof The proof goes by induction on $|\sigma|$. The case for $|\sigma| = 0$ is trivial. Assume $M \to^{r} M' \twoheadrightarrow^{\sigma'} N$ such that $\sigma = [r]\#\sigma'$ for some σ'. Let $r = (\mu\alpha P\ Q)$ be a μ-redex. The other cases are similar. Assume $occ(M,\pi) = r$ and $occ(M,\xi) = \lambda xS$ such that $lst(\xi) = r$. We prove that $\lambda xS \gg^{r} \lambda xV$ with $occ(M',\zeta') = \lambda xV$ implies $lst(\zeta') = r$, which, by the induction hypothesis, yields the result. We have to distinguish the cases constituting Definition 123.

1. $\xi \nsim \pi$. Then $\zeta' = \xi$, which gives the result.
2. $\xi = [\pi :: l :: \mu\alpha]\#\tau$. In this case $\zeta = \pi\#[l(P,\alpha,\tau)]$, where $l(P,\alpha,\tau)$ is defined in accordance with Definition 119. Then, since $lst(\tau) = r$, we necessarily have $lst(l(P,\alpha,\tau)) = r$.
3. $\xi = [\pi :: r]\#\tau$. Then $\zeta' = [\pi :: \mu\alpha]\#[l(P,\alpha,\varepsilon) :: \alpha :: r]\#\tau$, where $occ(\mu\alpha P,\varepsilon) = (\alpha\ U)$ for some U and $l(P,\alpha,\varepsilon)$ is defined in accordance with Definition 119. If $\tau \neq \emptyset$, then $lst(\xi) = lst(\tau) = lst(\zeta')$ and we are done. Otherwise, the result is again obvious.
4. $\xi = [\pi :: l]$. Since $occ(M,\xi) = \lambda xP$, this case is impossible.
5. $\xi < \pi$. Then the relation $\zeta' = \xi$ yields the result again.

□

Remark 143 *Informally, Lemma 142 can be stated as follows. Let $M \twoheadrightarrow_{\mu}^{\sigma} N$ and $(T\ \lambda xS) \leq M$. Assume $\lambda xS \gg_{\mu}^{\sigma} \lambda xU$ for some $\lambda xU \leq N$. Then there is a V such that*

$$(V\ \lambda xU) \leq N$$

for the same occurrence of λxU.

Remark 144 *Before giving a formal proof of Theorem 111 we present an informal explanation of the main argument of the proof. Let M_1, M_2 and M_3 be given such that M_1, $M_3 \in NF_{\mu}$ and $M_2 \in NF_{\beta}$, moreover*

$$M_1 \twoheadrightarrow_{\beta}^{\sigma} M_2 \twoheadrightarrow_{\mu}^{\nu} M_3$$

for some σ and ν. Let $(\lambda xP\ Q) \leq M_3$ be such that $rank((\lambda xP\ Q), M_3) = rank(M_3)$. We prove that there is a $\lambda zW \leq M_1$ such that it is used in σ and $lh(type(\lambda zW)) > lh(type(\lambda xP))$. In this case, by Lemmas 110 and 130, we have

$$rank(M_1) \geq lh(type(\lambda zW)) > lh(type(\lambda xP)) = rank(M_3),$$

which is the result. Starting from $(\lambda xP\ Q)$, we find λzW in the following way. By Lemma 139 there is a $\lambda xS \leq M_2$ for which

$$\lambda xS \gg^{\nu} \lambda xP.$$

Lemma 130 gives $type(\lambda xS) = type(\lambda xP)$. Furthermore, since also $\lambda xS < M_2$, Lemma 132 yields three possibilities. Either $\lambda y\lambda xS \leq M_2$ for some y, or $(T\ \lambda xS) \leq M_2$ for some T, or $(\alpha\ \lambda xS) \leq M_2$ for some α. We prove that only the latter case is possible. Then, we show that $(\alpha\ \lambda xS)$ is used in ν. Applying this, we find a $\mu\beta V < M_1$ such that

$$type(\mu\beta V) = type(\lambda xS).$$

By Lemma 131, either $\mu\gamma\mu\beta V \leq M_1$ for some γ or $\lambda y\mu\beta V \leq M_1$ for some y. We prove finally that only the latter case is possible and $\lambda y\mu\beta V$ is exactly the λ-term needed.

Proof of Theorem 111 Assume we have M_1, $M_3 \in NF_\mu$ and $M_2 \in NF_\beta$ such that

$$M_1 \twoheadrightarrow_\beta^\sigma M_2 \twoheadrightarrow_\mu^\nu M_3$$

for some σ and ν.
Let $occ(M_3, \xi') = (\lambda x P\ Q)$ with $rank((\lambda x P\ Q), M_3) = rank(M_3)$, furthermore we write $\xi = [\xi' :: l]$. Then, by Lemmas 136 and 139, there is an $occ(M_2, \zeta) = \lambda x S$ such that

$$\lambda x S \gg^\nu \lambda x P. \tag{1.4}$$

The relation $\lambda x S = M_2$, together with (1.4), would yield $\lambda x P = M_3$, which is impossible. Hence, $\lambda x S < M_2$. By Lemma 132, there are three possibilities.

1. $lst(\zeta) = \lambda y$. Since ν is a $\mu\mu'\rho$-reduction sequence, $\lambda y \lambda x S$ cannot be used in ν, hence, by Lemma 141, $lst(\zeta) = lst(\xi) = \lambda y$ would also hold, which is impossible.

2. $lst(\zeta) = r$. Then, by Lemma 142, $lst(\xi) = r$, which is impossible again.

3. $lst(\zeta) = \alpha$. In this case $(\alpha\ \lambda x S)$ is used in ν. Otherwise, by Lemma 140, we have $lst(\xi) = \beta$ for some β, which is not possible. Now, let $M_2 = N_0 \to^{r_1} N_1 \to^{r_2} \ldots \to^{r_n} N_n = M_3$ with $\nu := [r_1, \ldots, r_n]$. There exists an $1 \leq i < n$ such that

$$(\alpha\ \lambda x S) \gg^{[r_1,\ldots,r_i]} (\beta\ S')$$

for some $occ(N_i, \zeta') = (\beta\ S')$, and $\mu\beta U = bn(N_i, \zeta')$ is the functional part of the μ- or μ'-reduction r_{i+1}. We use the notation $occ(N_i, \varepsilon) = \mu\beta U$. We may suppose that $(\alpha\ \lambda x S)$ is not used in $[r_1, \ldots, r_i]$, so, applying Lemmas 140 and 137, we can conclude

$$\lambda x S \gg^{[r_1,\ldots,r_i]} \lambda x S'' = S' \tag{1.5}$$

for some S''. Let $\sigma' := \sigma \# [r_1, \ldots, r_i]$. By Lemmas 139 and 136 there exists a unique ε' such that $occ(M_1, \varepsilon') = \mu\beta V$ and

$$\mu\beta V \gg_\beta^{\sigma'} \mu\beta U.$$

Additionally, by a reasoning as before, $\mu\beta V < M_1$. Then, making use of Lemma 131, we have two possibilities.

- $lst(\varepsilon') = \mu\gamma$. Since $\mu\beta U = bn(N_i, \varepsilon)$ and $S' = \lambda x S''$, we have

$$type(\mu\beta U) = type(\lambda x S''),$$

thus, Lemma 130 yields

$$type(\mu\beta V) = type(\mu\beta U) = type(\lambda x S'') \neq \perp . \tag{1.6}$$

In the same time, because of typing rules, we have $type(\mu\beta V) = \perp$, a contradiction.

- $lst(\varepsilon') = \lambda y$. Then $\lambda y \mu\beta V$ is used in σ. Assume the contrary. Then $\lambda y \mu\beta V$ is not used in σ' either, so, by Lemma 141, $lst(\varepsilon) = \lambda y$ would hold as well. But then $\mu\beta U$ cannot be the functional part of the μ- or μ'-reduction r_{i+1}. If $\lambda y \mu\beta V$ is used in $\sigma = [s_1, \ldots, s_m]$, we have a $1 \leq j < m$ such that $\lambda y \mu\beta V \gg^{[s_1,\ldots,s_j]} \lambda y V'$ and $\lambda y V'$ is the functional part of s_{j+1}. By Lemmas 110 and 130 this implies

$$rank(M_1) \geq lh(type(\lambda y \mu\beta V)) = lh(type(\lambda y V')). \tag{1.7}$$

By (1.4), (1.5) and Lemma 130 we have

$$type(\lambda xS) = type(\lambda xS'') = type(\lambda xP).$$

Combining this with (1.6) and (1.7) we obtain

$$\begin{array}{lcl} rank(M_1) & \geq & lh(type(\lambda y \mu \beta V)) > lh(type(\mu \beta V)) \\ & = & lh(type(\lambda xP)) = rank(M_3). \end{array}$$

□

1.2.5 The $\lambda\mu\mu'\rho\theta$-calculus

We can extend the algorithm presented for the weak normalization of the $\lambda\mu\mu'\rho$-calculus to the case of the $\lambda\mu\mu'\rho\theta$-calculus in a straightforward way. In the sequel we apply the notation of the previous section.

Lemma 145 *Let $M \in NF_{\beta\mu\mu'\rho}$, suppose $occ(M, \xi) = \mu\alpha N < M$. Then one of the following cases is valid.*

1. *$lst(\xi) = \lambda x$ for some x,*
2. *$lst(\xi) = \mu\beta$ for some β.*

Proof Trivial. □

Lemma 146 *Let $M \in NF_{\beta\mu\mu'\rho}$, assume $r = \mu\alpha(\alpha\ N) \leq M$ is a θ-redex and $M \to^r M'$. Then we have $M' \in NF_{\beta\mu\mu'\rho}$.*

Proof Obvious by Lemma 145. □

Theorem 147 *Let $M \in NF_{\beta\mu\mu'\rho}$. Then there is an M^* such that $M \twoheadrightarrow_\theta M^*$ and $M^* \in NF_{\beta\mu\mu'\rho\theta}$.*

Proof We prove the assertion by induction on $cxty(M)$. Let $M \to^r M'$, where r is a θ-redex. By the previous lemma $M' \in NF_{\beta\mu\mu'\rho}$. But $cxty(M') < cxty(M)$, which gives the result. □

1.2.6 Standardization for the $\lambda\mu\mu'\rho\theta$-calculus

In this section we present a standardization theorem for the $\lambda\mu\mu'\rho\theta$-calculus.

Definition 148 *The sequence $(M_i)_{1\leq i\leq n}$ is $\beta\mu\mu'\rho\theta$-standard if one of the following holds.*

1. *$M_i = \lambda x N_i$ $(1 \leq i \leq n)$ and $(N_i)_{1\leq i\leq n}$ is standard.*
2. *$M_i = \mu\alpha N_i$ $(1 \leq i \leq n)$ and $(N_i)_{1\leq i\leq n}$ is standard.*
3. *$M_i = (\alpha\ N_i)$ $(1 \leq i \leq n)$ and $(N_i)_{1\leq i\leq n}$ is standard.*
4. *There are standard sequences $(N_i)_{1\leq i\leq k}$ and $(P_i)_{k\leq i\leq n}$ such that $M_i = (N_i\ P_k)$ for $1 \leq i \leq k$ and $M_i = (N_k\ P_i)$ for $k \leq i \leq n$.*
5. *There is a standard sequence $(N_i)_{1\leq i\leq k}$ and a Q such that*
   (a) *either $M_i = (N_i\ Q)$ for $1 \leq i \leq k$, $N_k = \lambda xP$ and $(N_i)_{1\leq i\leq k-1}$ does not begin with a λ, $M_{k+1} = P[x := Q]$ and the sequence $(M_i)_{k+1\leq i\leq n}$ is standard,*
   (b) *or $M_i = (N_i\ Q)$ for $1 \leq i \leq k$, $N_k = \mu\alpha P$ and $(N_i)_{1\leq i\leq k-1}$ does not begin with a μ, $M_{k+1} = \mu\alpha P[\alpha := Q]$ and the sequence $(M_i)_{k+1\leq i\leq n}$ is standard,*

(c) *or* $M_i = (Q\ N_i)$ *for* $1 \leq i \leq k$, $N_k = \mu\alpha P$ *and* $(N_i)_{1\leq i\leq k-1}$ *does not begin with a* μ, $M_{k+1} = \mu\alpha P[\alpha := Q]$ *and the sequence* $(M_i)_{k+1\leq i\leq n}$ *is standard.*

6. *There is a standard sequence* $(N_i)_{1\leq i\leq k}$ *such that* $M_i = (\alpha\ N_i)$ $(1 \leq i \leq k)$, $N_k = \mu\beta P$ *and* N_i *does not begin with a* μ *for* $1 \leq i \leq k-1$, $M_{k+1} = P[\beta := \alpha]$ *and the sequence* $(M_i)_{k+1\leq i\leq n}$ *is standard.*

7. *There is a standard sequence* $(N_i)_{1\leq i\leq k}$ *such that* $M_i = \mu\alpha N_i$ $(1 \leq i \leq k)$, $M_k = \mu\alpha(\alpha\ P)$ *with* $\alpha \notin Fv(P)$ *and* M_i *is not a* θ*-redex for* $1 \leq i \leq k-1$, $M_{k+1} = P$ *and the sequence* $(M_i)_{k+1\leq i\leq n}$ *is standard.*

Definition 149 *We say that* $M \twoheadrightarrow_{st} M'$ *if there is a standard reduction sequence* $(M_i)_{1\leq k\leq n}$ *such that* $M = M_1$ *and* $M' = M_n$.

To handle the case of the $\lambda\mu\mu'\rho\theta$-calculus in relation with standardization assertions, we need some additional notions, these are the notions of overlapping redexes and critical pairs. However, the definition of these notions are rather technical in the literature (cf. [33]), so we use these concepts intuitively only. Consider the following example. Let an *ARS*, $\mathcal{A}$, be given as $\mathcal{A} = \{\Sigma, \rightarrow_1, \rightarrow_2\}$. Suppose Σ, as the set of terms for $\mathcal{A}$, contains two term forming operators F and G. Let $a, b \in \Sigma$ be terms in normal forms other than variables, suppose $b \neq G(c)$ for any c. Assume the relations

$$F(G(x), y) \rightarrow_1 y,$$

$$G(a) \rightarrow_2 b$$

define the reduction rules $\rightarrow_1$ and $\rightarrow_2$. Then $F(G(a), x) \rightarrow_1 x$ and, on the other hand, $F(G(a), x) \rightarrow_2 F(b, x)$. Observe that x and $F(b, x)$ are both in normal forms and cannot be reduced to each other. The cause of the phenomenon was the fact that $F(G(a), x)$ contains a redex w.r.t both the $\rightarrow_1$ and the $\rightarrow_2$ reductions, however, the two redexes have a subterm in common, namely $G(a)$. Now, the contraction of one of the two redexes destroys the other by removing the subterm in common. This situation is called overlapping. Moreover, contracting the overlapping redexes in $F(G(a), x)$ supplies the critical pair $\langle x, F(b, x)\rangle$.

Remark 150 *Definition 148 is formulated on the model of the definition of a standard* $\lambda\mu\mu'$*-reduction sequence in David and Nour [13]. Restricted to the* λ*-,* μ*- and* ρ*-reductions, this definition is equivalent to Definition* **??** *of Chapter 2. In this case, we cannot define standard reduction sequences in a way like there, since the representation of Lemma 176, together with the notion of the head-redex defined in Definition 184, is no more applicable here. Moreover, in contrast to the* $\lambda\mu\mu'$*-calculus in David and Nour [13], we have to cope with some additional problems. Namely, there are several critical pairs induced by the reduction rules.*

1. *A* θ*-redex can overlap with a* μ*- and a* μ'*-redex, respectively. For example, let* $\mu\alpha(\alpha\ M)$ *be a* θ*-redex,* N *be a term. Then* $(\mu\alpha(\alpha\ M)\ N) \rightarrow_\theta (M\ N)$ *and* $(\mu\alpha(\alpha\ M)\ N) \rightarrow_\mu \mu\alpha(\alpha\ (M\ N))$. *Similarly for the* θ*- and the* μ'*-reductions.*

2. *A* θ*-redex can overlap with a* ρ*-redex. For example, let* $\mu\alpha(\alpha\ M)$ *be a* θ*-redex. If* $M = \mu\beta M_1$, *then* $\mu\alpha(\alpha\ \mu\beta M_1) \rightarrow_\theta \mu\beta M_1$ *and* $\mu\alpha(\alpha\ \mu\beta M_1) \rightarrow_\rho \mu\alpha M_1[\beta := \alpha]$. *Similarly for the case when, concerning the overlapping redexes, the* ρ*-redex is inside and the* θ*-redex is outside.*

The following examples illustrate how we can manage the cases of overlapping redexes.

Example 151 *1. The overlapping redexes are a θ- and a ρ-redex. The θ-redex is inside the ρ-redex. The original sequence:*

$$(\alpha\ \mu\beta(\beta\ \mu\gamma x)) \to_\theta (\alpha\ \mu\gamma x) \to_\rho x.$$

The standardization:

$$(\alpha\ \mu\beta(\beta\ \mu\gamma x)) \to_\rho (\alpha\ \mu\gamma x) \to_\rho x.$$

2. The overlapping redexes are a θ- and a ρ-redex. The ρ-redex is inside the θ-redex. The original sequence:

$$\mu\alpha(\alpha\ \mu\beta(\lambda x(\beta\ y)\ y)) \to_\rho \mu\alpha(\lambda x(\alpha\ y)\ y) \to_\beta \mu\alpha(\alpha\ y) \to_\theta y.$$

The standardization:

$$\mu\alpha(\alpha\ \mu\beta(\lambda x(\beta\ y)\ y)) \to_\theta \mu\beta(\lambda x(\beta\ y)\ y) \to_\beta \mu\beta(\beta\ y) \to_\theta y.$$

3. The overlapping redexes are a μ- and a θ-redex. The original sequence:

$$(\mu\alpha(\alpha\ x)\ y) \to_\theta (x\ y).$$

The standardization:

$$(\mu\alpha(\alpha\ x)\ y) \to_\mu \mu\alpha(\alpha\ (x\ y)) \to_\theta (x\ y).$$

4. Though, this is not overlapping, the case when a μ-reduction is obtained by implementing a θ-reduction before is interesting also. The original sequence:

$$(\mu\alpha(\alpha\ \mu\beta(x\ (\beta\ x)))\ y) \to_\theta (\mu\beta(x\ (\beta\ x))\ y) \to_\mu \mu\beta(x\ (\beta\ (x\ y))).$$

The standardization:

$$\begin{array}{lll}(\mu\alpha(\alpha\ \mu\beta(x\ (\beta\ x)))\ y) & \to_\mu & \mu\alpha(\alpha\ (\mu\beta(x\ (\beta\ x))\ y)) \to_\theta (\mu\beta(x\ (\beta\ x))\ y) \\ & \to_\mu & \mu\beta(x\ (\beta\ (x\ y))).\end{array}$$

Definition 152 *Let σ be $M_1 \to^{r_1} \to^{r_2} \ldots \to^{r_n} M_{n+1}$ for some $n \geq 0$. We call the terms $M_1, \ldots, M_{n+1}$ the elements of σ.*

Lemma 153 *Assume $M \twoheadrightarrow_{st} P$ and $N \twoheadrightarrow_{st} Q$. Then we have:*

1. *$M[x := N] \twoheadrightarrow_{st} P[x := Q]$,*
2. *$M[\alpha := N] \twoheadrightarrow_{st} P[\alpha := Q]$,*
3. *$M[\beta := \alpha] \twoheadrightarrow_{st} P[\beta := \alpha]$.*

Proof In each case the proof goes by induction on $\langle lh(M \twoheadrightarrow_{st} P), cxty(M)\rangle$. Let us consider the case of $M[\alpha := N]$. Let $M \twoheadrightarrow^\sigma P$ and $N \twoheadrightarrow^\nu Q$ such that σ, $\nu \in St$.

1. $|\sigma| = 0$: straightforward.
2. $|\sigma| \geq 1$. We treat some of the cases, distinguishing the various possibilities according to Definition 148.
   (a) $M = (M_1\ M_2)$.
   - Let $P = (M_1'\ M_2)$ with $M_1 \to M_1'$ or $P = (M_1\ M_2')$ with $M_2 \to M_2'$. Then the induction hypothesis applies.

- Let $M = (M_1\ M_2) \twoheadrightarrow^{\sigma_1} (\lambda y U\ M_2) \to (U[y := M_2]) \twoheadrightarrow^{\sigma_2} P$, where $\lambda y U$ is the first element of σ_1 beginning with a λ. Then, by the induction hypothesis, the sequence

$$\begin{array}{rcl}(M_1\ M_2)[\alpha := N] & \twoheadrightarrow & (\lambda y U[\alpha := N]\ M_2[\alpha := N]) \\ & \to & U[\alpha := N][y := M_2[\alpha := N]] \\ & = & U[y := M_2][\alpha := N] \ \twoheadrightarrow_{st}\ P[\alpha := Q]\end{array}$$

  is standard also.
- Let $M = (M_1\ M_2) \twoheadrightarrow^{\sigma_1} (\mu\beta U\ M_2) \to (\mu\beta U[\beta := M_2]) \twoheadrightarrow^{\sigma_2} P$, where $\mu\beta U$ is the first element of σ_1 beginning with a μ. Then, by the induction hypothesis, the sequence

$$\begin{array}{rcl}(M_1\ M_2)[\alpha := N] & \twoheadrightarrow & (\mu\beta U[\alpha := N]\ M_2[\alpha := N]) \\ & \to & \mu\beta U[\alpha := N][\beta := M_2[\alpha := N]] \\ & = & \mu\beta U[\beta := M_2][\alpha := N] \\ & \twoheadrightarrow & P[\alpha := Q]\end{array}$$

  is standard.

(b) $M = (\beta\ M_1)$.

- $M = (\alpha\ M_1)$. If $(\alpha\ M_1) \twoheadrightarrow_{st} (\alpha\ P_1)$, then, by the induction hypothesis, the sequence $(\alpha\ (M_1[\alpha := N]\ N)) \twoheadrightarrow_{st} (\alpha\ (P_1[\alpha := Q]\ N)) \twoheadrightarrow_{st} (\alpha\ (P_1[\alpha := Q]\ Q)) = (\alpha\ P_1)[\alpha := Q]$ is standard.
  Assume $M \twoheadrightarrow^{\sigma_1} (\alpha\ \mu\gamma U) \to_\rho U[\gamma := \alpha] \twoheadrightarrow^{\sigma_2} P$, where $\mu\gamma U$ is the first element in σ_1 beginning with a μ. Then, by the induction hypothesis, the reduction sequence

$$\begin{array}{rcl}M[\alpha := N] & = & (\alpha\ (M_1[\alpha := N]\ N)) \\ & \twoheadrightarrow & (\alpha\ (\mu\gamma U[\alpha := N]\ N)) \\ & \to_\mu & (\alpha\ \mu\gamma U[\alpha := N][\gamma := N]) \\ & \to_\rho & U[\alpha := N][\gamma := N][\gamma := \alpha] \\ & = & U[\gamma := \alpha][\alpha := N] \\ & \twoheadrightarrow & P[\alpha := Q]\end{array}$$

  is standard.
- $M = (\beta\ M_1)$ such that $\beta \neq \alpha$. Similar to the above one.

(c) $M = \mu\beta M_1$. Analogous to the above case.

□

Lemma 154 *Let $M \twoheadrightarrow_{st} P \to Q$. Then we have $M \twoheadrightarrow_{st} Q$.*

Proof The proof goes by induction on $\langle lh(M \twoheadrightarrow_{st} P), cxty(M)\rangle$, distinguishing the cases of Definition 148. We only treat some of the cases.

1. Assume $M = (M_1\ M_2) \twoheadrightarrow_{st}^{\sigma_1} (N_1\ M_2)^{\sigma_2} \twoheadrightarrow_{st} (N_1\ N_2) = P$. If $Q = (N_1'\ N_2)$ with $N_1 \to N_1'$ or $Q = (N_1\ N_2')$ with $N_2 \to N_2'$, by the induction hypothesis we have the result. Otherwise we have obtained Q by reducing P. We examine some of the cases.

   (a) $P = (\lambda x N_3\ N_2)$. Let R be the first element of σ_1 being of the form $\lambda x R_1$. Then by Lemma 153 the reduction sequence

$$M \twoheadrightarrow (\lambda x R_1\ M_2) \to R_1[x := M_2] \twoheadrightarrow N_3[x := N_2] = Q$$

   is standard.

(b) $P = (\mu\alpha N_3\ N_2)$.

- Let $M_1 \twoheadrightarrow^{\sigma_1'} \mu\alpha M_1' \twoheadrightarrow^{\sigma_1''} \mu\alpha N_3$ such that $\sigma_1 = \sigma_1'\#\sigma_1''$ and $\mu\alpha M_1'$ is the first element of σ_1' beginning with a μ. An argument similar to the above one gives the result.
- Assume $M_1 \twoheadrightarrow^{\sigma_1'} \mu\beta M_1' \twoheadrightarrow^{\sigma_1''} \mu\alpha N_3$, where $\mu\beta M_1'$ is the first element of the sequence σ_1' beginning with a μ. Then we necessarily have $\mu\beta M_1' \twoheadrightarrow_{st} \mu\beta(\beta\ M_1'') \to_\theta M_1'' \twoheadrightarrow_{st} \mu\alpha N_3$ for some M_1'' such that $\beta \notin Fv(M_1'')$. In this case, by Lemma 153 and the induction hypothesis, we obtain the standard reduction sequence

$$\begin{array}{rcl} M & = & (M_1\ M_2) \twoheadrightarrow (\mu\beta M_1'\ M_2) \to_\mu \mu\beta M_1'[\beta := M_2] \\ & \twoheadrightarrow & \mu\beta(\beta\ M_1'')[\beta := M_2] = \mu\beta(\beta\ (M_1''\ M_2)) \\ & \to_\theta & (M_1''\ M_2) \twoheadrightarrow Q. \end{array}$$

2. Let $M = (M_1\ M_2) \twoheadrightarrow^\sigma (\lambda y P\ M_2) \to_\beta P[y := M_2] \twoheadrightarrow^\nu P$, where $\sigma, \nu \in St$ and $\lambda y P$ is the first element of σ beginning with a λ. Then the induction hypothesis applied to $P[y := M_2] \twoheadrightarrow^\nu P$ gives a standard ν' such that $P[y := M_2] \twoheadrightarrow^{\nu'} Q$. The sequence $M = (M_1\ M_2) \twoheadrightarrow^\sigma (\lambda y P\ M_2) \to_\beta P[y := M_2] \twoheadrightarrow^{\nu'} Q$ is standard again.

□

Theorem 155 *Let M be a term, assume $M \twoheadrightarrow N$. Then we have $M \twoheadrightarrow_{st} N$.*

Proof Follows from Lemma 154 by induction on $lh(M \twoheadrightarrow N)$. □

1.2.7 Concluding remarks

The $\lambda\mu\mu'\rho'\theta$-calculus

In the paper de Groote [28] the $\lambda\mu$-calculus is expanded with a simplification rule differing from the ones treated so far. The syntax of the $\lambda\mu$-calculus is based on the de Groote-style formalism, that is, the set of terms is defined as in Definition 12. The main difference with our presentation is the fact that the ρ-rule is defined in a context dependent manner. The new rule, which we denote by ρ', is as follows.

$$\mu\alpha(\beta\ \mu\gamma M) \to_{\rho'} \mu\alpha M[\gamma := \beta]$$

The justification of this presentation is closely related to the introduction of a new rule called the ε-rule, which will be treated in the next subsection in detail. De Groote proves the strong normalization of the $\lambda\mu$-calculus equipped with the rules ρ', ε and θ in de Groote [28] in the following way. By a reducibility argument similar to the ones presented in [61] and in [24], he demonstrates that the $\lambda\mu$-calculus is strongly normalizing. Then, with a similar reasoning as in Section 1.1, he shows that the ρ'-, ε- and θ-rules can be strongly postponed w.r.t. the λ- and μ-rules.

Let us consider now the de Groote-style $\lambda\mu\mu'$-calculus expanded with the ρ'-, θ-rules. We call it the $\lambda\mu\mu'\rho'\theta$-calculus. It is straightforward to check, following the reasoning in de Groote [28], that the ρ'- and the θ-rules can be strongly postponed w.r.t. the μ'-rule as well. As an immediate consequence we obtain the following assertion.

Theorem 156 *The $\lambda\mu\mu'$-calculus expanded with the ρ'- and θ-rules is strongly normalizing.*

Proof By the above remark and by the strong normalizability of the $\lambda\mu\mu'$-calculus (cf. David and Nour [14]). □

Adjoining the ε-rule to the calculus raises some problems, namely, the ε-rule cannot be postponed w.r.t. the μ'-rule and this anticipates that the strong normalization may no longer be valid for a calculus where both the μ'- and the ε-rules are present.

The ε-rule

Let us restrict our presentation to the typed $\lambda\mu\mu'$-calculus first.

Definition 157 *Let M be a term, let $\alpha : \neg\bot$. Define a transformation by $cxty(M)$ rendering M_α to M.*

1. $x_\alpha = x$,
2. $(M'\ M'')_\alpha = (M'_\alpha\ M''_\alpha)$,
3. $\mu\beta M'_\alpha = \mu\beta M'_\alpha$,
4. $(\beta\ M') = (\beta\ M'_\alpha)$, *if* $\beta \neq \alpha$,
5. $(\alpha\ M')_\alpha = M'_\alpha$,
6. $\lambda x M'_\alpha = \lambda x M'_\alpha$.

In effect, given a term M and an $\alpha : \neg\bot$, M_α is the result of replacing, starting from the innermost ones, every subterm $(\alpha\ N)$ in M by N. Now let the ε-rule be defined as follows:

$$\mu\gamma\mu\alpha M \to_\varepsilon \mu\gamma M_\alpha.$$

Observe that, since $\mu\alpha M : \bot$, the type-condition $\alpha : \neg\bot$ is satisfied. Figure 1.2 justifies the introduction of the ε-rule from the side of reductions in the proofs. We note that the ε-rule simply eliminates in a proof of $\Gamma \vdash \mu\alpha M : \bot$ the superfluous derivation steps which lead from a conclusion $N : \bot$ to $(\alpha\ N) : \bot$, where N is a term occurring in the right hand side of the derivation of $\Gamma \vdash \mu\alpha M : \bot$.

As for the untyped case, Definition 157 makes sense without changes if we delete the type restriction imposed on α. With this modification the definition for the ε-rule in the typed setting can be accepted for this case also. Observe that the context dependent presentation of the rule ensures for the untyped setting that the α eliminated by the rule behaves like a variable $\alpha : \neg\bot$ in the typed calculus.

The reason for the introduction of a new rule ρ' in de Groote [28] was the following. For a moment, consider the $\lambda\mu$-calculus with the original ρ-rule and the ε-rule just defined. In contrary to the $\lambda\mu\rho$-calculus, the new calculus does not satisfy the Church-Rosser property as the following counterexample shows.

Example 158

$$\begin{array}{lll} (\alpha\ \mu\beta\mu\gamma M) & \to_\rho & \mu\gamma M[\beta := \alpha] \\ (\alpha\ \mu\beta\mu\gamma M) & \to_\varepsilon & (\alpha\ \mu\beta M_\gamma) \end{array}$$

If we define ρ' as in the previous subsection, then confluence is restored. But, as we have mentioned before, adding the ε-rule in the presence of μ' raises some other problems. Namely, even the weak normalization property is not retained in the $\mu\mu'\varepsilon$-calculus. In what follows let $\to$ denote the union of the μ-, μ'- and ε-reductions.

Definition 159 *We define a function ϕ assigning to a typed μ-term an untyped μ-term as follows. Let u be a fixed λ-variable.*

1. $\phi(y) = u$ *if* y *is a* λ*-variable,*
2. $\phi((\alpha\ M)) = (\alpha\ \phi(M))$,
3. $\phi((M_1\ M_2)) = \phi(M_1)$, *if* M_2 *is a* λ*-variable or* M_1 *and* M_2 *both are* λ*-variables,*
4. $\phi((M_1\ M_2)) = \phi(M_2)$, *if* M_1 *is a* λ*-variable,*
5. $\phi((M_1\ M_2)) = (\phi(M_1)\ \phi(M_2))$, *if neither of* M_1 *and* M_2 *is a* λ*-variable,*
6. $\phi(\mu\alpha M) = \mu\alpha\phi(M)$.

Intuitively, $\phi(M)$ is obtained from M by erasing all the left- or right-hand side parts of the applications of M which are merely λ-variables, not counting subterms of the form $(\alpha\ M')$, for some M', as applications and then replacing every λ-variable remaining in the resulting term by u.

Example 160
1. $\phi((y_1\ (y_2\ (y_3\ y_4)))) = u$, *where* $y_i : \bot \to \bot$ $(1 \leq i \leq 3)$ *and* $y_4 : \bot$,
2. $\phi((((y_1\ y_2)\ y_3)\ y_4)) = u$, *where* $y_1 : \bot \to (\bot \to (\bot \to \bot))$ *and* $y_i : \bot$ $(2 \leq i \leq 4)$,
3. $\phi(\mu\gamma(\mu\beta(\beta\ (z\ (\beta\ y)))\ \mu\alpha(\alpha\ (\alpha\ x)))) = \mu\gamma(\mu\beta(\beta\ (\beta\ u))\ \mu\alpha(\alpha\ (\alpha\ u)))$, *where* $x : \bot$, $y : \bot \to \bot$, $z : \bot \to (\bot \to \bot)$ *and* $\alpha : \neg\bot$, $\beta : \neg(\bot \to \bot)$.

Lemma 161 *Let* M, N *be typed* μ*-terms.*
1. $\phi(M[\alpha :=_r N]) = \phi(\phi(M)[\alpha :=_r \phi(N)])$ *(resp.* $\phi(N[\alpha :=_l M]) = \phi(\phi(N)[\alpha :=_l \phi(M)])$*).*
2. $\phi(M)_\alpha = \phi(M_\alpha)$.
3. $\phi(\phi(M)) = \phi(M)$.

Proof Obvious. □

Lemma 162 *Let* M *be a typed* μ*-term. Then* $\phi(M[\alpha := z]) = \phi(M)$.

Proof Straightforward. □

Lemma 163 *Let* M *be a typed* μ*-term. Assume* $M \to M'$. *Then either* $\phi(M) = \phi(M')$ *or there exists an* N *such that* $\phi(M) \to N$ *and* $\phi(N) = \phi(M')$.

Proof By induction on $cxty(M)$, applying Lemmas 161 and 162. □

Lemma 164 *Let* M *be a typed* μ*-term. If* M *is in normal form, then* $\phi(M)$ *is in normal form either.*

Proof Obvious. □

Lemma 165 *In the $\mu\mu'\varepsilon$-calculus there exists a typed μ-term which does not reduce to a normal form.*

Proof Let $U = \mu\alpha(\alpha\ (\alpha\ x))$, $V = \mu\beta(\beta\ (z\ (\beta\ y)))$, where $\alpha : \neg\bot$, $\beta : \neg(\bot \to \bot)$, $x : \bot$, $y : \bot \to \bot$ and $z : \bot \to (\bot \to \bot)$. Assume $M = \mu\gamma(V\ U)$, where $\gamma : \neg\bot$. Let $M = M_0 \to M_1 \to \ldots \to M_n \to \ldots$ be a reduction sequence starting from M. Let us define a set of terms by induction on n. Let $\mathcal{S}_0 = \{\phi(M)\}$. Assume $\mathcal{S}_i$ is defined, then let $\mathcal{S}_{i+1} = \phi(R(\mathcal{S}_i))$, where $R(\mathcal{S}) = \{N' \mid N \to N' \text{ and } N \in \mathcal{S}\}$ and $\phi(\mathcal{S}) = \{\phi(N) \mid N \in \mathcal{S}\}$ for an arbitrary set of terms $\mathcal{S}$. By Lemma 163, $\phi(M_i) \subseteq \bigcup_{j=0}^{i} \mathcal{S}_j$, where $0 \leq i$ is a natural number. By Lemma 164 it is enough to prove that no element of $\mathcal{S}_j$ $(j \geq 0)$ is in normal form. Let $W = \mu\alpha(\alpha\ (\alpha\ u))$, then we have:

$$
\begin{aligned}
\mathcal{S}_0 &= \{\mu\gamma(W\ W)\},\\
\mathcal{S}_1 &= \{\mu\gamma\mu\alpha(\alpha\ ((\alpha\ W)\ W)),\ \mu\gamma\mu\alpha(\alpha\ (W\ (\alpha\ W)))\},\\
\mathcal{S}_2 &= \{\mu\gamma(W\ W),\ \mu\gamma\mu\alpha(\alpha\ \mu\beta(\beta\ ((\beta\ (\alpha\ W))\ (\alpha\ W)))),\ \mu\gamma\mu\alpha(\alpha\ \mu\beta(\beta\ ((\alpha\ W)\ (\beta\ (\alpha\ W)))))\},\\
\mathcal{S}_3 &= \{\mu\gamma\mu\beta(\beta\ ((\beta\ W)\ W)),\ \mu\gamma\mu\beta(\beta\ (W\ (\beta\ W)))\} = \mathcal{S}_1,\\
&\ldots
\end{aligned}
$$

The above equations prove our assertion. □

The following lemma refers to the ν-reduction rule, which is defined in the subsection below.

Lemma 166 *There are terms of the typed $\lambda\mu\mu'\varepsilon$- and $\lambda\mu\mu'\nu\varepsilon$-calculi which have no normalizing reduction sequences.*

Proof It is enough to consider the term in the proof of Lemma 165. □

Theorem 167 *The typed $\lambda\mu\mu'\rho\varepsilon$-calculus is weakly normalizing.*

Proof Let M be a typed $\lambda\mu$-term. By Theorem 111 we have an M' in $\beta\mu\mu'\rho$-normal form such that $M \twoheadrightarrow M'$. It is straightforward to check that implementing an ε-reduction on a term in $\beta\mu\mu'\rho$-normal form creates no additional redexes except for ε- and θ-redexes. Likewise, implementing a θ-reduction on a term in $\beta\mu\mu'\rho$-normal form creates no additional redexes except for θ-redexes. But a reduction sequence consisting entirely of θ- and ε-reductions must necessarily terminate. This yields the result. □

The ν-rule

There is one more simplification rule defined in Parigot [46], this is the so-called ν-rule. The ν-rule is as follows:

$$\mu\alpha M \to_\nu \lambda x \mu\alpha M[\alpha :=_r x],$$

if M contains a subterm of the form $(\alpha\ \lambda y N)$. The ν-rule obtains its importance in relation with determining the general form of normal, classical integers, that is, normal $\lambda\mu$-terms of type $\forall X(X \to ((X \to X) \to X))$ (cf. Parigot [46]).

We define the untyped $\mu\mu'\nu$-calculus as the set of $\lambda\mu$-terms equipped with the μ-, μ'- and ν-reduction rules. As an application of the method of Section 1.2.1 we can show that the untyped $\mu\mu'\nu$-calculus is strongly normalizing. To this end it is enough to modify the norm of Section 1.2.1 in the following way.

Definition 168 *Let M be a $\lambda\mu$-term. Let us define a norm for M as follows:*

$$|M| = \begin{cases} 0 & \text{if } M = x, \\ |M_1| + |M_2| & \text{if } M = (M_1\ M_2), \\ max\{|M_2| \mid (\alpha\ M_2) \leq M_1\} + 1 & \text{if } M = \mu\alpha M_1 \text{ and } \alpha \in Fv(M_1), \\ 0 & \text{if } M = \mu\alpha M_1 \text{ and } \alpha \notin Fv(M_1), \\ |M_1| & \text{if } M = \lambda x M_1, \\ 0 & \text{if } M = (\alpha\ M_1). \end{cases}$$

It is straightforward to check that all the assertions of Section 1.2.1 remain true, making the slight changes in the proofs where necessary. We state without proof the result:

Theorem 169 *The untyped $\mu\mu'\nu$-calculus is strongly normalizing.*

As for the typed $\lambda\mu\mu'\nu$-calculus, it can be verified that the reasoning for the case of the $\lambda\mu\mu'$-calculus in David and Nour [14] can be modified to give the strong normalization of the $\lambda\mu\mu'\nu$-calculus as well. We state the assertion without proof:

Theorem 170 *The typed $\lambda\mu\mu'\nu$-calculus is strongly normalizing.*

We have not examined the normalization properties of the $\lambda\mu\mu'\rho\nu$-calculus yet. We conjecture that the $\lambda\mu\mu'\rho\nu$-calculus is weakly normalizing.

The $\lambda\mu\mu'$-calculus with type-restrictions

The following construction provides a strongly normalizing version of the typed $\lambda\mu\mu'$-calculus without requiring the ρ-rule to be defined in a context dependent way, and it also eludes the problem of non-termination emerged before when trying to adjoin the ε-rule. Let us consider the following version of the $\lambda\mu\mu'$-calculus denoted by $\lambda\mu_\varepsilon\mu'_\varepsilon$. Let the terms of the calculus be as follows. Let $\mathcal{V} = \{x, y, z, \ldots\}$ be the set of λ-variables and let $\mathcal{W}_\varepsilon = \{\alpha, \beta, \gamma, \ldots\}$ be the set of μ-variables such that, for every $\alpha \in \mathcal{W}_\varepsilon$, if $\alpha : \neg A$, then we have $A \neq \bot$. If $\alpha \in \mathcal{W}_\varepsilon$, we say that α is a μ_ε-variable.

Definition 171 *The set of terms, $\mathcal{T}_\varepsilon$, of the $\lambda\mu_\varepsilon\mu'_\varepsilon$-calculus is defined as follows.*

$$\mathcal{T}_\varepsilon := \mathcal{V} \mid \lambda\mathcal{V}\mathcal{T}_\varepsilon \mid \mu\mathcal{W}_\varepsilon\mathcal{T}_\varepsilon \mid (\mathcal{W}_\varepsilon\ \mathcal{T}_\varepsilon) \mid (\mathcal{T}_\varepsilon\ \mathcal{T}_\varepsilon).$$

The reduction rules are the β-rule, the μ_ε- and the μ'_ε-rules. The μ_ε- and μ'_ε-rules are defined as follows. In the definition below let the term M_α be obtained from M as given in Definition 157.

Definition 172 *1.*

$$(\mu\alpha M\ N) \to_{\mu_\varepsilon} \begin{cases} \mu\alpha M[\alpha :=_r N], & \text{if } (\mu\alpha M\ N) : A \text{ such that } A \neq \bot, \\ M[\alpha :=_r N]_\alpha, & \text{if } (\mu\alpha M\ N) : \bot. \end{cases}$$

2.

$$(M\ \mu\alpha N) \to_{\mu'_\varepsilon} \begin{cases} \mu\alpha N[\alpha :=_l M], & \text{if } (M\ \mu\alpha N) : A \text{ such that } A \neq \bot, \\ N[\alpha :=_l M]_\alpha, & \text{if } (M\ \mu\alpha N) : \bot. \end{cases}$$

In other words, the μ_ε- and the μ'_ε-rules differ form the μ- and the μ'-rules in the following respect. Let $(\mu\alpha M\ N)$ (resp. $(N\ \mu\alpha M)$) be a μ_ε-redex (resp. a μ'_ε-redex) such that $(\mu\alpha M\ N) : \bot$ (resp. $(N\ \mu\alpha M) : \bot$). Then $(\mu\alpha M\ N) \to_{\mu_\varepsilon} M[\alpha :=_r N]_\alpha$ (resp. $(M\ \mu\alpha N) \to_{\mu'_\varepsilon} N[\alpha :=_l M]_\alpha$). We should note that the μ_ε-rule (resp. the μ'_ε-rule) emerges as if we implemented successively a μ-rule (resp. a μ'-rule)

together with a context independent version of the ε-rule. Observe that the proof-theoretical meaning of the change in the presentation in comparison with the $\lambda\mu\mu'$-calculus is that we omit the otherwise superfluous inference steps from a proof of $\Gamma, \alpha : \neg\bot \vdash M : \bot$ to $\Gamma, \alpha : \neg\bot \vdash (\alpha\ M) : \bot$. Since have no variables $\alpha : \neg\bot$ in this case, we had to alter the definitions of the μ- and μ'-reductions accordingly.

Now, we add the ρ-rule unchanged to the $\lambda\mu_\varepsilon\mu'_\varepsilon$-calculus. Note that the ε-rule is meaningless in this calculus. If we examine the proof of the strong normalization of the $\lambda\mu\mu'$-calculus in David and Nour [14], we can find that the argument applied there can be adapted to our case as well. The core of the reasoning is the fact that the following property remains true in the $\mu_\varepsilon\mu'_\varepsilon\rho$-calculus: let M be a term, σ be a μ-substitution such that $M\sigma \twoheadrightarrow \mu\alpha P$ for some $\mu\alpha P$. Then there exists an M_1 such that $M \twoheadrightarrow \mu\alpha M_1$ and $M_1\sigma \twoheadrightarrow P$. The validity of this property is ensured by our type restrictions. We state without proof the following assertion:

Theorem 173 *The $\lambda\mu_\varepsilon\mu'_\varepsilon\rho$-calculus is strongly normalizing.*

Proof By modifying the proof related to the $\lambda\mu\mu'$-calculus in David and Nour [14]. □

Remark 174 *It is straightforward to check that the θ-rule can be postponed w.r.t. the β-, μ_ε-, μ'_ε- and ρ-rules. Thus, we can state as a simple corollary that the $\lambda\mu_\varepsilon\mu'_\varepsilon$-calculus expanded with the ρ- and θ-rules is strongly normalizing.*

Remark 175 *If we intend to adapt the above method to the untyped case also, the solutions being at the author's disposal at present do not appear to be as handy as for the typed case. The restrictions on the formation of terms do not seem to be so strongly justified as in the untyped setting. If we aim to find a method without defining any of the rules in a context dependent manner, the following definition for the ρ-rule might be a solution. Let $\widetilde{\rho}$ be defined as*

$$(\alpha\ \mu\beta M) \rightarrow_{\widetilde{\rho}} (\varepsilon\ M[\beta := \alpha]),$$

where ε is a new variable. This means roughly, if we borrow Parigot's terminology here, that a named term cannot reduce to anything else but a named term. As before, this involves that the $\lambda\mu\mu'\widetilde{\rho}$-calculus is strongly normalizing. The drawback of the $\widetilde{\rho}$-rule is that in a typed environment the derivation of the typing relation $(\varepsilon\ M[\beta := \alpha]) : \bot$ contains an unnecessary step from the sequent $\Gamma, \varepsilon : \neg\bot \vdash M[\beta := \alpha] : \bot$ to the sequent $\Gamma, \varepsilon : \neg\bot \vdash (\varepsilon\ M[\beta := \alpha]) : \bot$. On the other hand it decreases the number of detours in the proof in the sense that it eliminates a part of the original proof tree where a $\bot$-elimination rule is followed by a $\bot$-introduction rule.

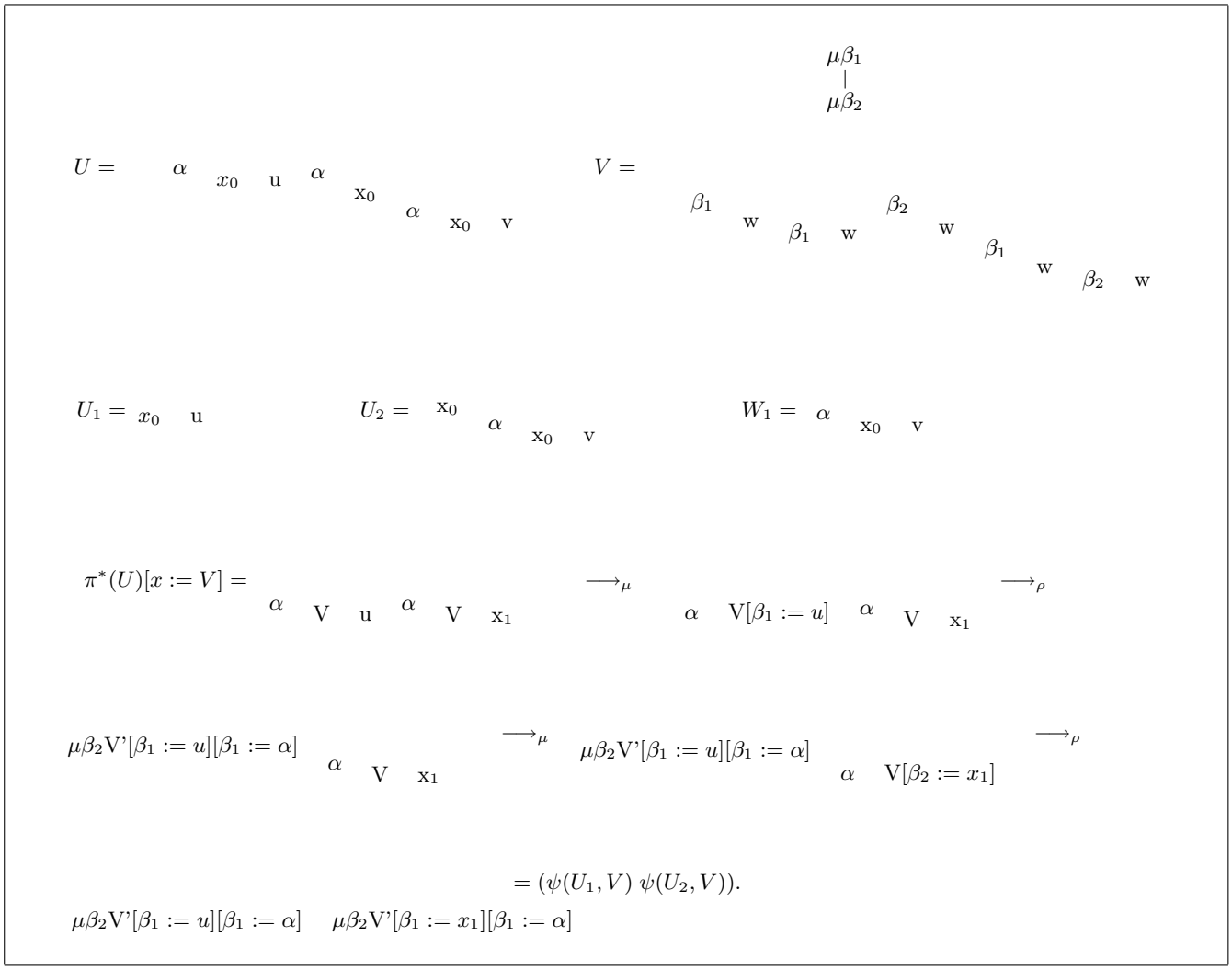

Figure 1.1: The first steps of the algorithm of Theorem 101 for the U and V of Example 103.

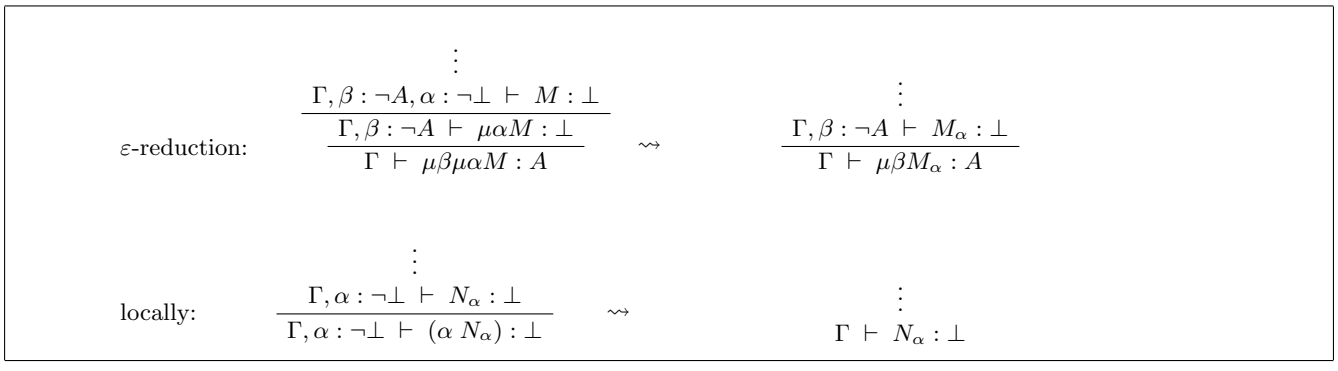

Figure 1.2: The reductions in the proofs in relation with the ε-rule

Chapter 2

An estimation for the lengths of reduction sequences of the $\lambda\mu\rho\theta$-calculus

The present chapter is concerned with the simply typed $\lambda\mu\rho\theta$-calculus. It is known that the $\lambda\mu$-calculus is strongly normalizing (see e.g. Parigot[47], or David and Nour [12]). This chapter contributes to the existing results by finding an upper bound for the lengths of the $\beta\mu\rho\theta$-reduction sequences, from which, as a consequence, another proof of the strong normalization of the $\lambda\mu$-calculus extended with the ρ- and θ-rules is acquired. For the simply-typed λ-calculus several results have appeared concerning the lengths of reduction sequences (cf. Schwichtenberg [57], Schwichtenberg [58], Xi [66], Beckmann [2]). The following proof is based on the estimations for the simply-typed λ-calculus published by Xi (cf. [66]).

First of all, let us get acquainted with some definitions and notation.

Lemma 176 *Every term of the simply typed $\lambda\mu\rho\theta$-calculus can be written uniquely in one of the following forms. In the definition below let $\overrightarrow{P}$ denote a (possibly empty) sequence of arguments.*

1. *M is a variable,*
2. *$M = \lambda x M_1$,*
3. *$M = \mu\alpha M_1$ and M is not a θ-redex,*
4. *$M = (\alpha\ M_1)$ and M is not a ρ-redex,*
5. *$M = (x\ M_1\ \overrightarrow{P})$,*
6. *$M = (\lambda x M_1\ M_2\ \overrightarrow{P})$,*
7. *$M = (\mu\alpha M_1\ M_2\ \overrightarrow{P})$,*
8. *$M = (\alpha\ \mu\beta M_1)$,*
9. *$M = \mu\alpha(\alpha\ M_1)$ and $\alpha \notin Fv(M_1)$.*

Proof Obvious. □

In the sequel, for the sake of simplicity, redexes can also be denoted by lowercase letters. In this chapter we modify the definition of the complexity of a term as follows:

Definition 177 *1.* $comp(x) = 1$,

2. $comp((\alpha\ M)) = comp(M) + 1$,

3. $comp(\lambda x M) = comp(M) + 1$,

4. $comp(\mu\alpha M) = comp(M) + 1$,

5. $comp((M\ N)) = comp(M) + comp(N)$.

Definition 178 *Let* $\sigma = [r_1, \ldots, r_n]$ $(n \geq 1)$. *We say that* σ' *is an initial subsequent of* σ, *if* $\sigma' = [r_1, \ldots, r_i]$ *for some* $1 \leq i \leq n$. *We denote it by* $\sigma' \leq \sigma$.

The definitions and notations in relation with the notion of the descendant are taken from Section 1.2.4 of the previous chapter.

We have to supplement it with the notion of descendant w.r.t. the θ-reduction. First we extend the function *adr* of Definition 123 for the case when its first argument is a θ-redex.

Definition 179 *Let* $occ(M, \pi) = \mu\alpha(\alpha\ P)$. *Assume* $\alpha \notin Fv(P)$.

1. *If* $\xi \nsim \pi$, *then* $adr(M, \pi, \xi) = \{\xi\}$.

2. *If* $\xi = [\pi :: \mu\alpha :: \alpha]\#\xi'$, *then* $adr(M, \pi, \xi) = \{\pi\#\xi'\}$.

3. *If* $\xi < \pi$, *then* $adr(M, \pi, \xi) = \{\xi\}$.

Definition 180 *Let* $M \to^r N$, *assume* $occ(M, \pi) = r$. *Then*

$$des_0(M, \pi, \xi) = \{occ(N, \zeta) \mid \zeta \in adr(M, \pi, \xi)\}.$$

We should observe that if $occ(M, \pi) = \mu\alpha(\alpha\ P)$, the terms $\mu\alpha(\alpha\ P)$ and $(\alpha\ P)$ have no descendants w.r.t. the reduction of $\mu\alpha(\alpha\ P)$ in M.

The function *des* is defined in a way analogous to that of Definition 126, applying the function des_0 of Definition 180.

Definition 181 *Let* $M_1 \to^{r_1} M_2 \to^{r_2} \ldots \to^{r_n} M_{n+1}$, *suppose* $occ(M_1, \pi) = r$ *is a redex of* M_1. *Then* r *is involved in* σ, *if there is a* $\sigma_i = [r_1, \ldots, r_i]$ $(1 \leq i < n)$ *such that* $r_{i+1} \in des(M_1, \sigma_i, \pi)$.

Definition 182 *Let* $M_1 \to^{r_1} M_1 \to^{r_2} \ldots \to^{r_n} M_{n+1}$, $\sigma_i = [r_1, \ldots, r_i]$ $(1 \leq i \leq n)$ *and* $N = occ(M_1, \xi)$. *We say that* N *disappears in* $\sigma = \sigma_n$ *if one of the following conditions holds.*

1. $N = \mu\alpha P$ *and* $N' = \mu\alpha P'$, *where* $N' \in des(M_1, \sigma_i, \xi)$ *and* N' *is the only descendant of* N *w.r.t.* σ_i, *and* $r_{i+1} = N'$.

2. $N = (\alpha\ P)$, $N' = (\beta\ P')$, *where* $N' \in des(M_1, \sigma_i, \xi)$ *and* N' *is the only descendant of* N *w.r.t.* σ_i, *and* $r_{i+1} = N'$.

Remark 183 *1. Intuitively, if* $\mu\alpha P$ *disappears in a reduction sequence* σ, *then, for every element of* σ, *it has at most one descendant, and there is a descendant of it, being of the form* $\mu\alpha P'$, *which is reduced as a* θ*-redex in* σ.

2. *Likewise, if* $(\alpha\ P)$ *disappears in a reduction sequence* σ*, then, for every element of* σ*, it has at most one descendant, and there is a descendant of it, being of the form* $(\beta\ P')$*, which is reduced as a* ρ*-redex in* σ*.*

Definition 184 *The head-redex of a term* M *is defined as follows. The numbering of the cases refers to the numbering in Lemma 176.*

- *Case 1:* M *has no head-redex.*
- *Cases 2-4: The head-redex of* M *is that of* M_1*, if it exists.*
- *Case 5:* M *has no head-redex.*
- *Case 6: The head-redex of* M *is* $(\lambda x M_1\ M_2)$*.*
- *Case 7: The head-redex of* M *is* $(\mu\alpha M_1\ M_2)$*.*
- *Case 8-9: The head-redex of* M *is* M *itself.*

Remark 185 *If* $M = (\mu\alpha M_1\ M_2\ \overrightarrow{P})$*, which is Case 7, then there can appear a critical pair of redexes provided* $\mu\alpha M_1$ *is a* θ*-redex as well. In this situation we always choose the* μ*-redex* $(\mu\alpha M_1\ M_2)$ *as the head-redex of* M*.*

Definition 186 *Let* $M_1 \to^{r_1} M_2 \to^{r_2} \ldots \to^{r_n} M_{n+1}$*. Then* $\sigma = [r_1, \ldots, r_n]$ *is a head-reduction sequence, if, for each* $1 \leq i \leq n$*,* r_i *is the head-redex of* M_i*.*

Lemma 187 *Let* $M \twoheadrightarrow^{\sigma} M'$ *such that the head-redex* r *of* M *exists and is not involved in* σ*. Moreover, assume that if* $r = (\mu\alpha M_1\ M_2)$ *for some* M_1 *and* M_2*, then* $\mu\alpha M_1$ *does not disappear in* σ*. Then the head-redex* r' *of* M' *exists and it is the unique residual of* r *w.r.t.* σ*.*

Proof By induction on $|\sigma|$, taking into account the various cases of Definition 184. □

In the definition below, as an abuse of notation, we speak of a subterm of a given term rather than an occurrence in a term, although we understand the notation $N \leq M$ as specifying an occurrence in M. In the sequel, we resort to this shorthand in terminology several times.

Definition 188 *Let* $M_1 \leq M$*. The number of arguments of* M_1 *in* M *is the maximal value of* n *for which* $(M_1 \ldots M_{n+1}) \leq M$ *for the same occurrence of* M_1 *in* M*. In case of* $n = 0$ *we say that* M_1 *has no arguments.*

Definition 189

1. *The* θ*-redex* $\mu\alpha(\alpha\ M_1) \leq M$ *is called regular if it has no arguments in* M*. Otherwise,* $\mu\alpha(\alpha\ M_1)$ *is called irregular.*
2. *The reduction sequence* $M_1 \to^{r_1} M_2 \to^{r_2} \ldots \to^{r_n} M_{n+1}$ *is regular if it contains no reductions with irregular* θ*-redexes.*

Notation 190

1. *Let* σ*,* ν *be (possibly empty) sequences of reductions. Then* $\sigma\#\nu$ *denotes their concatenation.*
2. *Let* $\sigma = [r_1, \ldots, r_n]$*. We denote by* $\sigma[x := M]$ *(resp.* $\sigma[\alpha :=_s M]$ *with* $s \in \{l, r\}$*) the reduction sequence* $[r_1[x := M], \ldots, r_n[x := M]]$ *(resp.* $[r_1[\alpha :=_s M], \ldots, r_n[\alpha :=_s M]]$*). Moreover, let* $\sigma[\alpha := \beta]$ *denote the reduction sequence* $[r_1[\alpha := \beta], \ldots, r_n[\alpha := \beta]]$*.*

Notation 191 *Let* $M = (M_1\ M_2 \ldots M_n) = (M_1\ M_2\ \overrightarrow{P})$*, with a possibly empty sequence of arguments* $\overrightarrow{P}$*. Then, for* $2 < i \leq n$*, we write* $M_i \in \overrightarrow{P}$*. In this case we call the* M_i $(2 < i \leq n)$ *the components of* $\overrightarrow{P}$*.*

Definition 192 *Let us define the components of a term M which is different from a variable.*

1. *If $M = \lambda x M_1$ or $M = \mu\alpha M_1$ or $M = (\alpha\ M_1)$, then the component of M is M_1.*
2. *If $M = (\mu\alpha M_1\ M_2\ \overrightarrow{P})$, then the components of M are $\mu\alpha M_1$, M_2 and the components of $\overrightarrow{P}$.*
3. *If $M = (\lambda x M_1\ M_2\ \overrightarrow{P})$, then the components of M are M_1, M_2 and the components of $\overrightarrow{P}$.*
4. *If $M = (x\ M_1\ \overrightarrow{P})$, then the components of M are x, M_1 and the components of $\overrightarrow{P}$.*

2.0.8 Standardization in the $\lambda\mu\rho\theta$-calculus

In the present subsection we inspect some assertions concerning estimations for the lengths of standard reduction sequences in the $\lambda\mu\rho\theta$-calculus. Many of the proofs are the adaptations of the ones related to the simply typed λ-calculus in Xi [66] to the case of the $\lambda\mu\rho\theta$-calculus. We give here a detailed account of them, however, since in our opinion the proofs presented below are perhaps formulated in a bit more distinct way. The definition below coincides with Definition 148 except for the fact that the case for the μ'-reduction is omitted.

Definition 193 *The sequence $(M_i)_{1\leq i\leq n}$ is $\beta\mu\rho\theta$-standard if one of the following holds.*

1. *$M_i = \lambda x N_i$ $(1 \leq i \leq n)$ and $(N_i)_{1\leq i\leq n}$ is standard.*
2. *$M_i = \mu\alpha N_i$ $(1 \leq i \leq n)$ and $(N_i)_{1\leq i\leq n}$ is standard.*
3. *$M_i = (\alpha\ N_i)$ $(1 \leq i \leq n)$ and $(N_i)_{1\leq i\leq n}$ is standard.*
4. *There are standard sequences $(N_i)_{1\leq i\leq k}$ and $(P_i)_{k\leq i\leq n}$ such that $M_i = (N_i\ P_k)$ for $1 \leq i \leq k$ and $M_i = (N_k\ P_i)$ for $k \leq i \leq n$.*
5. *There is a standard sequence $(N_i)_{1\leq i\leq k}$ and a Q such that*
   (a) *either $M_i = (N_i\ Q)$ for $1 \leq i \leq k$, $N_k = \lambda x P$ and $(N_i)_{1\leq i\leq k-1}$ does not begin with a λ, $M_{k+1} = P[x := Q]$ and the sequence $(M_i)_{k+1\leq i\leq n}$ is standard,*
   (b) *or $M_i = (N_i\ Q)$ for $1 \leq i \leq k$, $N_k = \mu\alpha P$ and $(N_i)_{1\leq i\leq k-1}$ does not begin with a μ, $M_{k+1} = \mu\alpha P[\alpha := Q]$ and the sequence $(M_i)_{k+1\leq i\leq n}$ is standard,*
6. *There is a standard sequence $(N_i)_{1\leq i\leq k}$ such that $M_i = (\alpha\ N_i)$ $(1 \leq i \leq k)$, $N_k = \mu\beta P$ and N_i does not begin with a μ for $1 \leq i \leq k-1$, $M_{k+1} = P[\beta := \alpha]$ and the sequence $(M_i)_{k+1\leq i\leq n}$ is standard.*
7. *There is a standard sequence $(N_i)_{1\leq i\leq k}$ such that $M_i = \mu\alpha N_i$ $(1 \leq i \leq k)$, $M_k = \mu\alpha(\alpha\ P)$ with $\alpha \notin Fv(P)$ and M_i is not a θ-redex for $1 \leq i \leq k-1$, $M_{k+1} = P$ and the sequence $(M_i)_{k+1\leq i\leq n}$ is standard.*

Definition 194 *We say that $M \twoheadrightarrow_{st} M'$ if there is a standard reduction sequence $(M_i)_{1\leq k\leq n}$ such that $M = M_1$ and $M' = M_n$.*

Lemma 195 *Let $M \twoheadrightarrow^{\sigma} M'$ such that σ is standard. Assume the head-redex r_h of M, if it exists, is not involved in σ. Then the following statements are true.*

1. *If* $M = \lambda x M_1$, *then* $M' = \lambda x M_1'$, *where* $M_1 \twoheadrightarrow^{\sigma} M_1'$.

2. *If* $M = (\alpha\ M_1)$, *then* $M' = (\alpha\ M_1')$, *where* $M_1 \twoheadrightarrow^{\sigma} M_1'$.

3. *If* $M_1, \ldots, M_n$ *are the components of* M, *then there are standard* $\sigma_1, \ldots, \sigma_n$ *and terms* $M_1', \ldots, M_n'$ *such that* $M_i \twoheadrightarrow^{\sigma_i} M_i'$ $(1 \leq i \leq n)$, $M' = (M_1' \ldots M_n')$ *and* $\sigma = \sigma_1 \# \ldots \# \sigma_n$.

Proof By induction on $\langle \sigma, comp(M) \rangle$. Let us only consider only Case 3. Assume $M = (\mu\alpha M_1\ M_2\ \overrightarrow{P})$. The other cases are similar.

1. If σ is standard by virtue of Point 4 of Definition 193 and $\overrightarrow{P}$ is not empty, then the induction hypothesis applies. If $\overrightarrow{P}$ is empty, the result is again obvious.

2. Assume σ is standard by reason of Point 5 (b) of Definition 193. We prove that this situation is impossible. The case of Point 5 (a) of Definition 193 is similar. If $\overrightarrow{P}$ is not empty, we have the result induction hypothesis. Let $\overrightarrow{P}$ be empty. Then σ must be of the form $(\mu\alpha M_1\ M_2) \rightarrow \mu\alpha M_1[\alpha := M_2] \twoheadrightarrow^{\sigma'} M'$ for some $\sigma' \in St$, which, by assumption, is again impossible.

□

Remark 196 *The analogue of the above assertion is not valid for* $M = \mu\alpha M_1$. *Let* $M = \mu\alpha(\alpha\ (x\ (\lambda xy\ (\alpha\ x))))$. *Assume* σ *is*

$$\mu\alpha(\alpha\ (x\ (\lambda xy\ (\alpha\ x)))) \rightarrow_{\beta} \mu\alpha(\alpha\ (x\ y)) \rightarrow_{\theta} (x\ y).$$

Then $\sigma \in St$ *and* M *has no head-redex, but* M *is not of the form* $\mu\alpha M_1$.

Lemma 197 *Let* $M \twoheadrightarrow^{\sigma'} M' \twoheadrightarrow^{\sigma''} M''$ *such that* σ' *is a head-reduction sequence and* σ'' *is standard. Then* $\sigma = \sigma' \# \sigma''$ *is standard.*

Proof Let $\sigma' = [r] \# \sigma$. We prove the result by induction on $\langle |\sigma|, comp(M) \rangle$, taking into account the various points of Definition 193.

1. $|\sigma| = 0$. Assume, for example, $M = (\lambda x M_1\ M_2\ \overrightarrow{P}) = (\lambda x M_1\ M_2 \ldots M_n)$, where $\overrightarrow{P}$ might be empty. Then $M \rightarrow^{r} (M_1[x := M_2] \ldots M_n) \twoheadrightarrow^{\sigma''} M''$ such that $\sigma'' \in St$. Let us examine some of the cases.

   - σ'' is standard by virtue of Points 1-3 of Definition 193. Then, necessarily, $\overrightarrow{P}$ is empty and $[r] \# \sigma'' \in St$ by Point 5 (a) of Definition 193.
   - σ'' is standard by virtue of Point 4 of Definition 193. If $\overrightarrow{P}$ is empty, we have the result by Point 5 (a) of the same definition. Otherwise we have $(M_1[x := M_2] \ldots M_{n-1}) \twoheadrightarrow^{\sigma_1''} N_1$ and $M_n \twoheadrightarrow^{\sigma_2''} N_2$ with some standard σ_1'' and σ_2'' such that $\sigma'' = \sigma_1'' \# \sigma_2''$. We can apply the induction hypothesis to $[r] \# \sigma_1''$, which yields the result.
   - σ'' is standard by virtue of Point 5 (a) of Definition 193. We may suppose again that $\overrightarrow{P}$ is not empty. Then we have standard σ_1'' and σ_2'' for which $\sigma'' = \sigma_1'' \# \sigma_2''$ and $\sigma_1'' = [r_1, \ldots, r_{k+1}]$ for some $k \geq 0$ such that $(M_1[x := M_2] \ldots M_{n-1}) = N_1 \rightarrow^{r_1} N_2 \rightarrow^{r_2} \ldots \rightarrow^{r_k} N_{k+1} = \lambda y Q$ for some $\lambda y Q$ and $(\lambda y Q\ M_n) \rightarrow^{r_{k+1}} Q[y := M_n] \twoheadrightarrow^{\sigma_2''} M''$. Moreover, none of the terms $N_1, \ldots, N_k$ begins with a λ. Then, by the induction hypothesis, $[r] \# \sigma_1'' \in St$ and, obviously, no element of the sequence $M, N_1, \ldots, N_k$ begins with a λ. This means that $[r] \# \sigma''$ is standard.

2. The cases when $|\sigma| \geq 1$ follow from the induction hypothesis.

□

Lemma 198 *Let $M \twoheadrightarrow^{\sigma} M'$ such that σ is standard, regular and the head-redex r_h of M is involved in σ. Then $\sigma = [r_h]\#\sigma'$ for some σ'.*

Proof The proof goes by induction on $|\sigma|$, considering the various cases of Definition 193.

1. $|\sigma| = 0$: there is nothing to prove.
2. $|\sigma| \geq 1$. Assume $r \neq r_h$, where $\sigma = [r]\#\sigma'$. Let $M \to^{r} M'' \twoheadrightarrow^{\sigma'} M'$. By Lemma 187, the head-redex r'' of M'' is involved in σ', thus, by the induction hypothesis, we have $\sigma' = [r'']\#\sigma''$. Now, by examining the various forms of M according to Lemma 176, we can check easily that the above situation is impossible. In case of $M = (\mu\alpha M_1\ M_2 \dots M_n)$ we have to use the fact that, by the regularity of σ, $\mu\alpha M_1$ does not disappear.

□

Remark 199 *1. The above lemma is not valid without the regularity assumption, as the following reduction sequence shows:*

$$(\mu\alpha(\alpha\ \lambda x x)\ y) \to_{\theta} (\lambda x x\ y) \to_{\beta} y.$$

2. Taking into account the form of Definition 193, it seems to be promising to formulate the lemma above in the following way. Let the standard, regular reduction sequence σ be defined as $M_1 \to^{r_1} M_2 \to \dots \to M_n \to^{r_n} M_{n+1}$. Assume $M_n = r_n$. Then σ is a head-reduction sequence. But in the presence of θ, this statement is false, as it is demonstrated by the reduction sequence in Remark 196.

Definition 200 *Let M be a term, x be a λ-variable (resp. α be a μ-variable). Denote by $|M|_x$ (resp. $|M|_\alpha$) the number of occurrences of x (resp. α) in M.*

Notation 201 *Let σ be the reduction sequence $M_1 \to^{r_1} M_1 \to^{r_2} \dots \to^{r_n} M_{n+1}$ for some terms $M_1, \dots, M_{n+1}$. Let $\langle\sigma\rangle_{(\rho,\alpha)}$ denote the number of ρ-reductions of the form $(\alpha\ \mu\beta P)$ in σ if $\alpha \in Fv(M)$, otherwise let $\langle\sigma\rangle_{(\rho,\alpha)}$ be 0.*

Lemma 202 *Let $M \twoheadrightarrow^{\sigma} M'$. Then there exists a ν such that $M[x := N] \twoheadrightarrow^{\nu} M'[x := N]$ and $|\nu| = |\sigma|$. Moreover, if σ is standard, then ν is standard, and if σ is regular, then ν is regular.*

Proof The proof goes by a straightforward induction on $\langle|\sigma|, comp(M)\rangle$. □

Lemma 203 *Let M, M' be $\lambda\mu$-terms such that $M \twoheadrightarrow^{\sigma} M'$. Assume $N_1, \dots, N_k$ are $\lambda\mu$-terms for which $\alpha \notin Fv(N_i)$ $(1 \leq i \leq k)$. Then there exists a ν such that $M[\alpha := N_1] \dots [\alpha := N_k] \twoheadrightarrow^{\nu} M'[\alpha := N_1] \dots [\alpha := N_k]$ and $|\nu| = |\sigma| + k \cdot \langle\sigma\rangle_{(\rho,\alpha)}$. Moreover, if σ is standard, then ν is standard, and if σ is regular, then ν is regular.*

Proof By induction on $\langle|\sigma|, comp(M)\rangle$.

1. $|\sigma| = 0$: trivial.
2. $\sigma = r\#\sigma'$, where $M \to^{[r]} M'' \twoheadrightarrow^{\sigma'} M'$. The only interesting case is $M = (\gamma\ \mu\beta M_1) \to_{\rho} M_1[\beta := \gamma] = M'' \twoheadrightarrow^{\sigma'} M'$. If $\alpha \neq \gamma$, then the result follows from the induction hypothesis. Otherwise, we have

$$\begin{aligned} M[\alpha := N_1] \dots [\alpha := N_k] \ &= \ (\alpha\ (\mu\beta M_1[\alpha := N_1] \dots [\alpha := N_k]\ N_1 \dots N_k)) \\ &\twoheadrightarrow^{k}_{\mu} \ (\alpha\ \mu\beta M_1[\alpha := N_1] \dots [\alpha := N_k][\beta := N_1] \dots [\beta := N_k]) \\ &\to_{\rho} \ M_1[\beta := \alpha][\alpha := N_1] \dots [\alpha := N_k], \end{aligned}$$

which, by the induction hypothesis, yields the result. In the case when σ is standard, the standardness of ν is ensured by Lemma 197.

□

Lemma 204 *1. Let $N \twoheadrightarrow^{\sigma} N'$. Then there exists a ν such that $M[x := N] \twoheadrightarrow^{\nu} M[x := N']$ and $|\nu| = |\sigma| \cdot |M|_x$. Moreover, if σ is standard, then ν is standard, and if σ is regular, then ν is regular.*

2. Let $N \twoheadrightarrow^{\sigma} N'$. Then there exists a ν such that $M[\alpha := N] \twoheadrightarrow^{\nu} M[\alpha := N']$ and $|\nu| = |\sigma| \cdot |M|_{\alpha}$. Moreover, if σ is standard, then ν is standard, and if σ is regular, then ν is regular.

Proof Let us only deal with Case 1. The proof goes by a straightforward induction on $comp(M)$. For example, let us consider two of the cases.

- $M = x$: $\tau = \sigma$ is appropriate.
- $M = (M_1\ M_2)$: The induction hypothesis gives $\tau_i \in St$ such that $M_i[x := N] \twoheadrightarrow^{\tau_i} M_i[x := N']$ and $|\tau_i| = |\sigma| \cdot |M_i|_x$ $(i \in \{1, 2\})$. Then we can choose $\tau = \tau_1 \# \tau_2$.

□

Notation 205 *1. Let $M \twoheadrightarrow^{\sigma} M'$ for some terms M and M'. Let us denote by $\langle \sigma \rangle_{\theta}$ the number of θ-reductions occurring in σ.*

2. Let M be a term, assume $occ(M, \pi) = N$. Let us denote by $arg(M, \pi)$ the number of arguments of the occurrence of N in M. We write

$$sumarg(M, N) = \sum \{arg(M, \pi) \mid occ(M, \pi) = N\}$$

if N has at least one occurrence in M, otherwise let $sumarg(M, N) = 0$.

Lemma 206 *Let $M \twoheadrightarrow^{\sigma} M'$, $N \twoheadrightarrow^{\nu} N'$. Assume σ and ν are standard and regular. Then there is a standard, regular reduction sequence τ such that*

$$M[x := N] \twoheadrightarrow^{\tau} M'[x := N']$$

and $|\tau| = |\sigma| + |M'|_x \cdot |\nu| + sumarg(M', x) \cdot \langle \nu \rangle_{\theta}$.

Proof The proof goes by induction on $\langle |\sigma|, comp(M) \rangle$, taking into account the various points of Definition 193.

A, $|\sigma| = 0$: This case is treated by Lemma 204.

B, $\sigma = [r] \# \sigma'$.

1. $M = \lambda y M_1$ or $M = \mu \alpha M_1$ or $M = (\alpha\ M_1)$ such that $M_1 \twoheadrightarrow^{\sigma} M_1'$: the induction hypothesis gives the result.
2. $M = (M_1\ M_2) \twoheadrightarrow^{\sigma_1} (M_1'\ M_2) \twoheadrightarrow^{\sigma_2} (M_1'\ M_2') = M'$ such that $\sigma = \sigma_1 \# \sigma_2$ and σ_1, σ_2 are standard, regular.
   (a) If the head-redex r_h of M_1 is involved in σ_1, then, by Lemma 198, $r = r_h$, furthermore $r_h[x := N]$ is again the head-redex of $M_1[x := N]$. By the regularity of σ, r_h is not a θ-redex. This means that $[r_h[x := N]] \# \sigma'[x := N]$ is standard and regular, and, by applying the induction hypothesis to σ', we obtain the result.
   (b) Assume the head-redex r_h of M_1 is not involved in σ_1 or does not exist.
   - If M_1 is of the form other than $(x\ \overrightarrow{P})$, for some (possibly empty) $\overrightarrow{P}$, then, taking into consideration Lemma 195, we can apply the induction hypothesis. Observe that in the case of $M = (\mu \alpha N_1\ N_2\ \overrightarrow{P})$, by regularity, $\mu \alpha N_1$ cannot disappear, thus the sequence $(\mu \alpha N_1[x := N]\ N_2[x := N]\ \overrightarrow{P}[x := N]) \twoheadrightarrow (\mu \alpha N_1'[x := N']\ N_2'[x := N']\ \overrightarrow{P}'[x := N'])$ obtained by the induction hypothesis is again standard and regular.

- Finally, assume $M_1 = (x\ \overrightarrow{P})$. Suppose $\overrightarrow{P}$ is not empty, that is, $M_1 = (x\ P_1 \dots P_k)$ for some $P_1, \dots, P_k$. We write $M^* = (y\ P'_1 \dots P'_k\ M'_2)$, where y is a new variable and $M' = (x\ P'_1 \dots P'_k\ M'_2)$. Let $N \twoheadrightarrow^{\nu''} N'' \twoheadrightarrow^{\nu'} N'$, where ν' and ν'' are such that $\nu = \nu''\#\nu'$. Now, we prove by induction on $|\nu'|$ that a standard, regular τ' can be given for which $(N''\ P_1[x := N] \dots P_k[x := N]\ M_2[x := N]) \twoheadrightarrow^{\tau'} M'[x := N']$ and $|\tau'| = |\sigma| + |\nu'| + (k+1) \cdot \langle \nu' \rangle_\theta + |M^*|_x \cdot |\nu| + sumarg(M^*, x) \cdot \langle \nu \rangle_\theta$. Assume $\nu' = [s]\#\zeta$, and we already have the result for ζ. If s is other than a θ-redex, then the induction hypothesis applies. Assume $N'' = \mu\alpha(\alpha\ Q) \to^{[s]} Q \twoheadrightarrow^{\zeta} N'$. Then we can choose as τ' the reduction sequence below:

$$\begin{array}{rl} & (\mu\alpha(\alpha\ Q)\ P_1[x := N] \dots P_k[x := N]\ M_2[x := N]) \\ \to_\mu^{k+1} & \mu\alpha(\alpha\ (Q\ P_1[x := N] \dots M_2[x := N])) \\ \to_\theta & (Q\ P_1[x := N] \dots M_2[x := N]) \\ \twoheadrightarrow^{\tau''} & M'[x := N'], \end{array}$$

where τ'' is obtained by the induction hypothesis. For the length of τ' we have

$$\begin{array}{rcl} |\tau'| & = & 1 + (k+1) + |\tau''| \\ & = & 1 + (k+1) + |\sigma| + |\zeta| + (k+1) \cdot \langle \zeta \rangle_\theta \\ & & + |M^*|_x \cdot |\nu| + sumarg(M^*, x) \cdot \langle \nu \rangle_\theta \\ & = & |\sigma| + |\nu'| + (k+1) \cdot \langle \nu' \rangle_\theta + |M^*|_x \cdot |\nu| + sumarg(M^*, x) \cdot \langle \nu \rangle_\theta. \end{array}$$

Taking into account the facts that $|M'|_x = |M^*|_x + 1$ and $sumarg(M', x) = sumarg(M^*, x) + (k+1)$, we obtain the desired result with $\nu = \nu'$.

3. Let σ be $M = (M_1\ M_2) \twoheadrightarrow^{\sigma_1} (\lambda y P\ M_2) \to P[y := M_2] \twoheadrightarrow^{\sigma_2} M'$. Let us write $s = (\lambda y P\ M_2)$. We show that $r = r_h$, where $\sigma = [r]\#\sigma'$ and r_h is the head-redex of M. Assume that the head-redex r_h of M is not involved in σ_1. By the regularity of σ, $M_1 = \mu\alpha M_3$ is not possible. Thus, applying Lemma 195, we obtain that $M_1 = \lambda y M_3$ for some M_3. Therefore, σ_1 is empty and σ begins with r_h. If r_h is involved in σ, then, by Lemma 198, we again have the result. Since $r_h[x := N]$ is the head-redex of $M[x := N]$, by Lemma 197 and the induction hypothesis, we obtain our assertion.
4. $M = \mu\alpha M_1 \to^{r_1} \mu\alpha M_2 \to \dots \to \mu\alpha M_k \to^{r_k} \mu\alpha(\alpha\ M_{k+1}) \to_\theta M_{k+1} \twoheadrightarrow^{\sigma_2} M'$, where none of $\mu\alpha M_1, \dots, \mu\alpha M_k$ is a θ-redex and $\sigma = \sigma_1\#[r_{k+1}]\#\sigma_2$ with $\sigma_1 = [r_1, \dots, r_k]$ and $r_{k+1} = \mu\alpha(\alpha\ M_{k+1})$. Then, applying Lemma 202 for σ_1 and the induction hypothesis for σ_2, we obtain the result for the standard, regular reduction sequence $M[x := N] \twoheadrightarrow^{\sigma_1[x:=N]} \mu\alpha(\alpha\ M_{k+1}[x := N]) \to_\theta M_{k+1}[x := N] \twoheadrightarrow^{\tau_2} M'[x := N']$.

The remaining cases are proved analogously.

□

Lemma 207 *Let $M \twoheadrightarrow^{\sigma} M'$, $N \twoheadrightarrow^{\nu} N'$. Assume σ and ν are standard and regular. Then there is a standard, regular reduction sequence τ such that*

$$M[\alpha := N] \twoheadrightarrow^{\tau} M'[\alpha := N']$$

and $|\tau| = |\sigma| + \langle \sigma \rangle_{(\rho,\alpha)} + |M'|_\alpha \cdot |\nu|$.

Proof The proof goes by induction on $\langle |\sigma|, comp(M)\rangle$, similarly to that of the previous lemma. We consider some of the cases according to Definition 193.

1. σ=0: This case is treated in Lemma 204.
2. $\sigma = [r]\#\sigma'$ for some σ'.
   (a) $M = (M_1\ M_2) \twoheadrightarrow^{\sigma_1} (M_1'\ M_2) \twoheadrightarrow^{\sigma_2} (M_1'\ M_2') = M'$, where σ_1 and σ_2 are standard and regular such that $\sigma = \sigma_1\#\sigma_2$. By the induction hypothesis we have the standard and regular reduction sequences $M_1[\alpha := N] \twoheadrightarrow^{\tau_1} M_1'[\alpha := N']$ and $M_2[\alpha := N] \twoheadrightarrow^{\tau_2} M_2'[\alpha := N']$ with appropriate lengths. Moreover, since, for any M and N, $M[\alpha := N] = \mu\beta N_1$ implies $M = \mu\beta M_1$ and $N_1 = M_1[\alpha := N]$ we also have the regularity of $\tau = \tau_1\#\tau_2$.
   (b) $M = (\alpha\ M_1) \twoheadrightarrow^{\sigma_1} (\alpha\ (\mu\beta M_2)) \to_\rho M_2[\beta := \alpha] \twoheadrightarrow^{\sigma_2} M'$, where $\sigma = \sigma_1\#[s]\#\sigma_2$ with $s = (\alpha\ (\mu\beta M_2))$ and σ_1, σ_2 standard, regular. Then, by the induction hypothesis and Lemmas 203 and 197, we have the standard and regular τ such that

$$\begin{array}{lll} (\alpha\ (M_1[\alpha := N]\ N)) & \twoheadrightarrow^{\tau_1} & (\alpha\ (\mu\beta M_2[\alpha := N]\ N)) \\ & \to_\mu & (\alpha\ \mu\beta M_2[\alpha := N][\beta := N]) \\ & \to_\rho & M_2[\beta := \alpha][\alpha := N] \\ & \twoheadrightarrow^{\tau_2} & M'[\alpha := N']. \end{array}$$

   For the length of τ we have $|\tau| = 2 + |\tau_1| + |\tau_2| = 2 + |\sigma_1| + \langle\sigma_1\rangle_{(\rho,\alpha)} + |\sigma_2| + \langle\sigma_2\rangle_{(\rho,\alpha)} + |M'|_\alpha \cdot |\nu| = |\sigma| + \langle\sigma\rangle_{(\rho,\alpha)} + |M'|_\alpha \cdot |\nu|$.

All the remaining cases are proved in a similar way, by applying the induction hypothesis. □

Before we continue with our estimations we prove the following lemma.

Lemma 208 *Let M be a term. We have $sumarg(M, x) \leq comp(M) - 1$.*

Proof By induction on $comp(M)$. The only interesting case is $M = (x\ \overrightarrow{P})$ for some $\overrightarrow{P}$. We may assume $\overrightarrow{P}$ is non-empty. Let $M_1, \ldots, M_n$ be the components of $\overrightarrow{P}$. Then $sumarg(M, x) = n + \sum_{i=1}^n sumarg(M_i, x) \leq n + \sum_{i=1}^n (comp(M_i) - 1) = comp(M) - 1$. □

For our estimations we also need the following notion.

Definition 209 *The set of $\lambda\mu I$-terms is defined inductively as follows:*

1. *x is a $\lambda\mu I$-term,*
2. *$\lambda x M$ is a $\lambda\mu I$-term provided M is a $\lambda\mu I$-term and $x \in Fv(M)$,*
3. *$\mu\alpha M$ is a $\lambda\mu I$-term provided M is a $\lambda\mu I$-term and $\alpha \in Fv(M)$,*
4. *$(M\ N)$ is a $\lambda\mu I$-term if M and N are $\lambda\mu I$-terms.*

Lemma 210 *Let $M \twoheadrightarrow^\sigma M'$ such that M is a $\lambda\mu I$-term. Then M' is a $\lambda\mu I$-term also.*

Proof By a straightforward induction on $\langle |\sigma|, comp(M)\rangle$. □

Definition 211 *Let r be a redex, the number $m(r, M)$ is defined by:*

$$m(r,M) = \begin{cases} |P|_\alpha & \text{if } r = (\mu\alpha P\ Q), \\ |P|_x + comp(P) & \text{if } r = (\lambda x P\ Q), \\ 1 & \text{if } r = (\alpha\ \mu\beta P), \\ 2n+1 & \text{if } r = \mu\alpha(\alpha\ P) \leq M \text{ and } \mu\alpha(\alpha\ P) \text{ has } n \text{ arguments in } M. \end{cases}$$

Notation 212 *Let us write $|\sigma|^* = max\{|\sigma|, 1\}$.*

Lemma 213 *Let $M \twoheadrightarrow^\sigma M' \to^r M''$, where σ is a regular, standard reduction sequence. Then we can construct a regular, standard $M \twoheadrightarrow^\tau M''$ such that*

$$|\tau| \leq 1 + max\{m(r, M'), 2\} \cdot |\sigma|^*.$$

Moreover, if M is a $\lambda\mu I$-term, then $1 + |\sigma| \leq |\tau|$ holds as well.

Proof The proof goes by induction $\langle |\sigma|, comp(M)\rangle$. The case of $|\sigma| = 0$ is obvious, thus we may assume $\sigma = [s]\#\sigma'$ for some σ'.

1. The head-redex r_h of M does not exist or is not involved in σ. We consider some of the cases.

   (a) $M = (\mu\alpha M_1\ M_2\ \overrightarrow{P}) = (\mu\alpha M_1\ M_2 \dots M_n)$. Since σ is regular, $\mu\alpha M_1$ cannot disappear. Thus, by Lemma 195, we have $M' = (\mu\alpha M_1'\ M_2' \dots M_n')$ such that $M_i \twoheadrightarrow^{\sigma_i} M_i'$ and $\sigma = \sigma_1\# \dots \#\sigma_n$, where σ_i $(1 \leq i \leq n)$ are standard and regular. If $r \leq M_i'$ for some $1 \leq i \leq n$, then the induction hypothesis applies.

   - $r = \mu\alpha M_1'$. Then $M_1' = (\alpha\ M_1'')$ for some $\alpha \notin Fv(M_1'')$. Now, we can define τ_1 as

   $$\begin{array}{rcl} (\mu\alpha M_1\ M_2 \dots M_n) & \twoheadrightarrow_\mu^{n-1} & \mu\alpha M_1[\alpha := M_2] \dots [\alpha := M_n] \\ & \to^{\sigma_1'} & \mu\alpha(\alpha\ M_1'')[\alpha := M_2] \dots [\alpha := M_n] \\ & = & \mu\alpha(\alpha\ (M_1''\ M_2 \dots M_n)) \to_\theta (M_1''\ M_2 \dots M_n), \end{array}$$

   where σ_1' is obtained from σ_1 by Lemma 203. Then

   $$\begin{array}{rcl} |\tau_1| & = & (n-1) + |\sigma_1'| + 1 \ = \ 1 + (n-1) + |\sigma_1| + (n-1) \cdot \langle\sigma_1\rangle_{(\rho,\alpha)} \\ & \leq & 1 + (2n-1) \cdot |\sigma_1|^*. \end{array}$$

   By this $|\tau| \leq 1 + max\{m(r, M'), 2\} \cdot |\sigma|^*$ follows. The above equation also yields for τ_1 the relation $|\tau_1| \geq |\sigma_1| + 1$, which gives $|\tau| \geq |\sigma| + 1$ concerning the second part of our assertion.

   - $r = (\mu\alpha M_1'\ M_2')$. By Lemmas 207 and 197, we have a standard, regular τ_1 such that $(\mu\alpha M_1\ M_2) \to_\mu M_1[\alpha := M_2] \twoheadrightarrow M_1'[\alpha := M_2']$ with $|\tau_1| = 1 + |\sigma_1| + \langle\sigma_1\rangle_{(\rho,\alpha)} + |M_1'|_\alpha \cdot |\sigma_2|$. This means $|\tau_1| \leq 1 + 2 \cdot |\sigma_1|^* + |M_1'|_\alpha \cdot |\sigma_2|^* \leq 1 + max\{m(r, M'), 2\} \cdot (|\sigma_1|^* + |\sigma_2|^*)$, by which the result follows for $\tau = \tau_1\# \dots \#\sigma_n$. If M is a $\lambda\mu I$-term, we also have $|\tau_1| \geq 1 + |\sigma_1| + |\sigma_2|$, which, in this case, implies $|\tau| \geq 1 + |\sigma|$.

   (b) $M = (\lambda x M_1\ M_2\ \overrightarrow{P}) = (\lambda x M_1\ M_2 \dots M_n)$. Similar to the second subcase of the case above. In the course of the argument we have to apply Lemma 206 and, by the estimation for $m(r, M')$, we also make use of Lemma 208.

   (c) $M = (\alpha\ M_1)$. By Lemma 195, we have $M' = (\alpha\ M_1')$ and $M_1 \twoheadrightarrow^\sigma M_1'$. If $r \leq M_1'$, the induction hypothesis applies. Assume $r = (\alpha\ M_1') = (\alpha\ \mu\beta M_1'')$. Since r_h, if it exists, is not involved in σ, Lemma 195 implies that the only possibility is $M_1 = \mu\gamma M_2$.

- $\gamma = \beta$: We have $M_2 \twoheadrightarrow^{\sigma} M_1''$, and the reduction sequence $M \rightarrow_{\rho} M_2[\beta := \alpha] \twoheadrightarrow^{\sigma[\beta:=\alpha]} M_1''[\beta := \alpha]$ is appropriate.
- $\gamma \neq \beta$: In this case $\mu\gamma M_2$ must disappear in σ. There exists standard and regular σ_1 and σ_2 such that $\mu\gamma M_2 \twoheadrightarrow^{\sigma_1} \mu\gamma(\gamma\ M_3) \rightarrow_{\theta} M_3 \twoheadrightarrow^{\sigma_2} \mu\beta M_1''$. The reduction sequence $(\alpha\ \mu\gamma M_2) \rightarrow_{\rho} M_2[\gamma := \alpha] \twoheadrightarrow^{\sigma_1[\gamma:=\alpha]} (\gamma\ M_3)[\gamma := \alpha] = (\alpha\ M_3) \twoheadrightarrow^{\tau_1} M_1''[\beta := \alpha]$ is a standard and regular reduction sequence of appropriate length, where τ_1 is obtained from $\sigma_2\#[r]$ by the induction hypothesis.

2. The head-redex r_h of M is involved in σ. We have, by Lemma 198, $s = r_h$. Applying the induction hypothesis and Lemma 197, we obtain the result.

□

Theorem 214 *Let σ be the reduction sequence $M_1 \rightarrow^{r_1} M_2 \rightarrow^{r_2} \ldots \rightarrow^{r_n} M_{n+1}$. Then there is a standard, regular reduction sequence $st(\sigma)$ such that $M_1 \twoheadrightarrow^{st(\sigma)} M_{n+1}$ and the following relations are valid.*

1. *$|st(\sigma)| \leq (1 + max\{m(r_1, M_1), 2\}) \cdot \ldots \cdot (1 + max\{m(r_n, M_n), 2\})$.*
2. *Moreover, if M is a $\lambda\mu I$-term, then $|st(\sigma)| \geq |\sigma|$.*

Proof Both cases are proved by induction on $|\sigma|$, applying Lemma 213. We examine only Case 1.

1. $|\sigma| = 1$: The only interesting case is $M_1 = (\mu a(\alpha\ N_1)\ N_2 \ldots N_k) \rightarrow_{\theta} (N_1\ N_2 \ldots N_k) = M_2$. Then $M_1 = (\mu a(\alpha\ N_1)\ N_2 \ldots N_k) \twoheadrightarrow_{\mu}^{n-1} \mu\alpha(\alpha\ (N_1\ N_2 \ldots N_k)) \rightarrow_{\theta} (N_1\ N_2 \ldots N_k) = M_2$ is appropriate for $st(\sigma)$.
2. $\sigma = \sigma'\#[r_n]$, where $|\sigma'| \geq 1$: By the induction hypothesis we can find a standard and regular $st(\sigma')$ with appropriate length such that $M_1 \twoheadrightarrow^{st(\sigma')} M_n$. Moreover, $|st(\sigma')|^* = |st(\sigma')|$. Then, by Lemma 213, there is a standard and regular $M_1 \twoheadrightarrow^{\tau} M_{n+1}$ such that $|\tau| \leq 1 + max\{m(r_n, M_n), 2\} \cdot |st(\sigma')|^* \leq (1 + max\{m(r_n, M_n), 2\}) \cdot |st(\sigma')|^*$, which yields the result.

□

Remark 215 *Theorem 214 asserts for the case of the $\lambda\mu\rho\theta$-calculus the standardization result well-known in the simply-typed λ-calculus. Moreover, as in the case for the λ-calculus examined in Xi [66], it furnishes us with an upper bound concerning the length of the standard, regular reduction sequence constructed from the original one.*

Definition 216 *Let σ be the reduction sequence $M_1 \rightarrow^{r_1} M_2 \rightarrow^{r_2} \ldots \rightarrow^{r_n} M_{n+1}$. Denote by $\mathcal{M}(\sigma)$ (the measure of σ) the number*

$$\mathcal{M}(\sigma) = \prod_{i=1}^{n} (1 + max\{m(r_i, M_i), 2\}).$$

2.0.9 On the lengths of standard reduction sequences

Definition 217 *Let M be a term. The leftmost redex of M, in notation $lr(M)$, is defined as follows.*

1. *$lr(\lambda x M) = lr(M)$,*
2. *$lr(\mu\alpha M) = \mu\alpha M$ if $\mu\alpha M$ is a θ-redex, and $lr(\mu\alpha M) = lr(M)$ otherwise.*

3. $lr((\alpha\ M)) = (\alpha\ M)$ *if* $(\alpha\ M)$ *is a* ρ*-redex, and* $lr((\alpha\ M)) = lr(M)$ *otherwise.*

4. $lr((\mu\alpha M_1\ M_2\ \overrightarrow{P})) = (\mu\alpha M_1\ M_2)$.

5. $lr((\lambda x M_1\ M_2\ \overrightarrow{P})) = (\lambda x M_1\ M_2)$.

6. $lr((x\ M_1\ \overrightarrow{P})) = lr(M_i)$ *provided* $M_i \notin NF$ *and* $M_j \in NF$ $(1 \leq j \leq i-1)$.

Definition 218 *A reduction sequence* $M_1 \to^{r_1} M_2 \to^{r_2} \ldots \to^{r_n} M_{n+1}$ *is the leftmost reduction sequence from* M_1 *to* M_{n+1} *if* r_i *is the leftmost redex of* M_i $(1 \leq i \leq n)$. *We denote by* $M \twoheadrightarrow_{lrs} N$ *the fact that* M *reduces to* N *via a leftmost reduction sequence. Then the reduction sequence itself is denoted by* $lrs(M \twoheadrightarrow N)$.

Lemma 219 *Let* M *be a* $\lambda\mu I$*-term. Assume* M *has a normal form* M'. *Then every standard and regular reduction sequence* $M \twoheadrightarrow M'$ *is of the same length.*

Proof We prove something more. Let σ be standard, regular such that either $M \twoheadrightarrow^{\nu} M'' \to^{r} M'$, where $\sigma = \nu\#[r]$ and r is the leftmost redex of M'', or $M \twoheadrightarrow^{\sigma} M'$, where $M' \in NF$. Then we have $M \twoheadrightarrow_{lrs} M'$ such that $|\sigma| = |lrs(M \twoheadrightarrow M')|$. The two cases are proven simultaneously by induction on $\langle |\sigma|, comp(M)\rangle$. The case of $|\sigma| = 0$ is trivial. Hence, we may assume that $\sigma = [s]\#\sigma'$.

1. Let $M = (\mu\alpha M_1\ M_2\ \overrightarrow{P}) = (\mu\alpha M_1\ M_2 \ldots M_n)$.

   - Let us assume first $M \twoheadrightarrow^{\sigma} M'$ with a standard and regular σ such that $M' \in NF$. It is straightforward to check by induction on $|\sigma|$, and distinguishing the cases of Definition 193, that in this case $r_h = (\mu\alpha M_1\ M_2)$ must be involved in σ. It follows, by Lemma 198, that $s = r_h$. Thus, the induction hypothesis applies.
   - On the other hand, let $M \twoheadrightarrow^{\nu} M'' \to^{r} M'$ as above. Assume that r_h is not involved in ν. Then, making use of the regularity of ν and Lemma 195, we obtain immediately that $M'' = (\mu\alpha M_1''\ M_2'' \ldots M_n'')$ for some M_i'' $(1 \leq i \leq n)$. Then $r = (\mu\alpha M_1''\ M_2'')$ would follow, contradicting the standardness of σ. Therefore, r_h is involved in ν and Lemma 197 yields again $s = r_h$, by which the result follows.

2. The case of $M = (\lambda x M_1\ M_2\ \overrightarrow{P})$ is analogous to the above one.

3. $M = \mu\alpha M_1$. Assume any of the premisses are valid, that is either $M \twoheadrightarrow^{\nu} M'' \to^{r} M'$ with $\sigma = \nu\#[r]$ standard, regular and $r = lr(M'')$ or $M \twoheadrightarrow^{\sigma} M'$ with σ standard, regular and $M' \in NF$.

   (a) $M = \mu\alpha M_1 \twoheadrightarrow^{\sigma} \mu\alpha M_1' = M'$. By the induction hypothesis, we have $|\sigma| = |lrs(M_1 \twoheadrightarrow M_1')|$. Let $lrs(M_1 \twoheadrightarrow M_1') = M_1 \to^{r_1} M_2 \to^{r_2} \to \ldots \to M_{n+1}$. We show $lrs(M_1 \twoheadrightarrow M_1') = lrs(\mu\alpha M_1 \twoheadrightarrow \mu\alpha M_1')$ by induction on n.

   - The sequence $(\mu\alpha M_i)_{1\leq i\leq n+1}$ does not contain a θ-redex. Then $lrs(M_1 \twoheadrightarrow M_1') = lrs(\mu\alpha M_1 \twoheadrightarrow \mu\alpha M_1')$, and we are done.
   - The sequence $(\mu\alpha M_i)_{1\leq i\leq n+1}$ contains a θ-redex. Assume $\mu\alpha M_j = \mu\alpha(\alpha\ N)$ is the first θ-redex in the sequence under discussion. Let $\mu\alpha M_1 \twoheadrightarrow^{\sigma_1} \mu\alpha(\alpha\ N) \twoheadrightarrow^{\sigma_2} \mu\alpha M_1'$. Then $(\alpha\ N)$ must disappear in σ_2, otherwise, in both cases, a contradiction would emerge either by the standardness of σ or by the fact that $M' \in NF$. Thus, we have $\mu\alpha(\alpha\ N) \twoheadrightarrow^{\sigma_3} \mu\alpha(\alpha\ \mu\beta P) \to_{\rho} \mu\alpha P[\beta := \alpha] \twoheadrightarrow^{\sigma_4} \mu\alpha M_1'$ for some σ_3 and σ_4. Then $\mu\alpha M_1 \twoheadrightarrow^{\sigma_1} \mu\alpha(\alpha\ N) \to_{\theta} N \twoheadrightarrow^{\sigma_3} \mu\beta P$ is the leftmost reduction sequence from M to $\mu\beta P$, and, applying the induction hypothesis to $\mu\beta P \twoheadrightarrow^{\sigma_4[\alpha:=\beta]} \mu\beta M_1'[\alpha := \beta]$, we obtain the result.

(b) $M = \mu\alpha M_1 \to^{r_1} \mu\alpha M_2 \to \ldots \to \mu\alpha M_k \to^{r_k} \mu\alpha M_{k+1} = \mu\alpha(\alpha\ M'_{k+1}) \to_\theta M'_{k+1} = M_{k+2} \twoheadrightarrow^{\sigma_2} M'$, where $\mu\alpha(\alpha\ M'_{k+1})$ is the first θ-redex in the sequence $(\mu\alpha M_i)_{1\leq i\leq k+1}$. We may assume $\sigma_1 = [r_1, \ldots, r_k]$ is not empty, otherwise the result follows by applying the induction hypothesis to σ_2. In what follows, we show that $r_k = lr(\mu\alpha M_k)$. Obviously, this implies that the induction hypothesis can be applied to $M_1 \twoheadrightarrow^{\sigma_1} M_{k+1}$, that is, we have $M_1 \twoheadrightarrow_{lrs} M_{k+1}$ such that $|lrs(M_1 \twoheadrightarrow M_{k+1})| = |\sigma_1|$. Then, as in the above subcase of Case 3, we can prove that $|lrs(\mu\alpha M_1 \twoheadrightarrow \mu\alpha M_{k+1})| = |lrs(M_1 \twoheadrightarrow M_{k+1})|$, which yields the result.

- $M_k = (\beta\ M'_k)$. Suppose $\beta = \alpha$. Then, by assumption, $\alpha \in Fv(M'_k)$. Since M_k is a $\lambda\mu I$ term, this would imply $\alpha \in Fv(M_{k+1})$, a contradiction. Hence, $\beta \neq \alpha$. In this case $M_k = (\beta\ M'_k) \to M_{k+1} = (\alpha\ M'_{k+1})$ means $M'_k = \mu\gamma M''_k$ such that $\mu\alpha M_k \to_\rho \mu\alpha M''_k[\gamma := \beta] = \mu\alpha M_{k+1}$. This proves our assertion.
- $M_k = \mu\beta M'_k$. Then, from $M_k \to M_{k+1} = (\alpha\ M'_{k+1})$, it follows that $M'_k = (\beta\ M''_k)$ and $r_k = M_k \to_\theta M''_k = M_{k+1}$.
- $M_k = \lambda x M'_k$ is impossible.
- $M_k = (M'_k\ M''_k)$. Then, necessarily, $r_k = (M'_k\ M''_k)$, and the proof is finished.

□

Remark 220 *As it is expected, Lemma 219 is not valid in the general case. Let* $M = \mu\alpha(\alpha\ (\lambda x(x\ x)\ (\lambda xy\ (\alpha\ y))))$. *The reduction sequences* σ_1 *defined as*

$$\begin{array}{rll} \mu\alpha(\alpha\ (\lambda x(x\ x)\ (\lambda xy\ (\alpha\ y)))) & \to_\beta & \mu\alpha(\alpha\ (\lambda x(x\ x)\ y)) \\ & \to_\theta & (\lambda x(x\ x)\ y) \\ & \to_\beta & (y\ y) \end{array}$$

and σ_2 *defined as*

$$\begin{array}{rll} \mu\alpha(\alpha\ (\lambda x(x\ x)\ (\lambda xy\ (\alpha\ y)))) & \to_\beta & \mu\alpha(\alpha\ ((\lambda xy\ (\alpha\ y))\ (\lambda xy\ (\alpha\ y)))) \\ & \to_\beta & \mu\alpha(\alpha\ (y\ (\lambda xy\ (\alpha\ y)))) \\ & \to_\beta & \mu\alpha(\alpha\ (y\ y)) \\ & \to_\theta & (y\ y) \end{array}$$

are both standard and regular, leading from M *to its normal form. In spite of all these they have different lengths.*

Corollary 221 *Let* M *be a* $\lambda\mu I$*-term. Assume* M *has a normal form* M'*. Then the leftmost reduction sequence is the longest one leading from* M *to* M'*.*

Proof Follows from Theorem 214 and Lemma 219. □

2.0.10 The estimation for the $\lambda\mu\rho\theta I$-calculus

In this section we give an estimation for the lengths of the reduction sequences in the $\lambda\mu\rho\theta I$-calculus.

Let A be a type. We apply the notion of the length of A in correspondence with Definition 106 in the previous chapter. The rank of a redex, given in Definition 107, needs to be supplemented with the case of a θ-redex, the other cases remaining the same as in the definition mentioned.

Definition 222 *Let M be a $\lambda\mu I$-term, and $r = \mu\alpha(\alpha\ M_1) \leq M$ a θ-redex. Then the rank of r is:*

$$rank(r, M) = lh(type(\mu\alpha(\alpha\ M_1))).$$

The rank of a term M is defined again as the maximum of the rank of its redexes.

Definition 223 *1. We say that a reduction sequence ν is a k-reduction sequence, if every redex in ν is of rank k.*

2. A reduction sequence is a k-normalization for a given term M, if it is a k-reduction sequence and eliminates all redexes of rank k in M. Define

$$NF_k = \{M \mid rank(M) < k\}.$$

3. Let M be a $\lambda\mu$-term, suppose $rank(M) = k$. A reduction sequence starting from M is good, if it has the following properties. It begins with eliminating redexes of rank k from M in a certain order: it eliminates first the leftmost, innermost redex of rank k, that is, the redex containing no other redexes of rank k and stands in the leftmost position among these redexes. After eliminating all the redexes of rank k it continues in the same way with the set of redexes having maximal rank in the term obtained.

Lemma 224 *The algorithm described in Definition 223 is a normalization algorithm.*

Proof We only give a sketch of the proof. Let M be a term of rank k. Let r be the leftmost, innermost k-redex of M.

1. $k \geq 1$.

   - $r = (\lambda x P\ Q)$. Then, as in the case of the simply typed λ-calculus, it can be checked easily that reducing with r creates no k-redexes.
   - Let $r = (\mu\alpha P\ Q)$ with $\mu\alpha P : A \to B$ and $Q : A$. Let $(\alpha\ U) \leq P$. Assume $(U[\alpha := Q]\ Q) \leq r' = \mu\alpha P[\alpha := Q]$ is a k-redex. Then, since r is innermost, the only possibility is $U = \lambda x U_1$ for some U_1. But reducing with $(\lambda x U_1[\alpha := Q]\ Q)$ creates no k-redexes, thus we can eliminate subsequently the k-redexes from r'.
   - Let $r = (\alpha\ \mu\beta P)$ be the leftmost, innermost k-redex of M with $k \geq 1$. Since $(\alpha\ \mu\beta P) : \bot$, the case when $((\alpha\ \mu\beta P)\ U) \leq M$ for some U is impossible. Furthermore, since r is the leftmost redex, the case $\mu\gamma(\alpha\ \mu\beta P) \leq M$ and $P = (\gamma\ P')$ for some $\gamma \notin P'$, that is, the situation when reducing with r would create a θ-redex, is again impossible. Hence the reduction with r could create a ρ-redex only. This happens when $(\gamma\ (\alpha\ \mu\beta P)) \leq M$ and $P = \mu\delta P'$. But this involves $\mu\delta P' : \bot$, that is, $(\gamma\ \mu\delta P'[\beta := \alpha])$ is not a k-redex. Therefore, when reducing with $r = (\alpha\ \mu\beta P)$, the number of k-redexes decrease.
   - $r = \mu\alpha(\alpha\ P)$. By Definition 217, if r is leftmost, then there exists no U such that $(\mu\alpha(\alpha\ P)\ U) \leq M$. This means that reducing with r can create a k-redex only if $P = (\beta\ P')$ such that $\beta \notin Fv(P')$ and $\mu\beta\mu\alpha(\alpha\ (\beta\ P')) \leq M$. But then $\mu\alpha(\alpha\ P) : \bot$, which is impossible.

2. $k = 0$. In this case the only redexes in M are ρ- or θ-redexes of type $\bot$. It is straightforward to check that the reducing of these redexes can only create ρ- or θ-redexes. But any reduction sequence consisting entirely of ρ- or θ-redexes must terminate. This proves our assertion.

□

Lemma 225 *Let $rank(\mu\alpha P\ Q) = k$, assume $x \notin Fv(P)$. Let $(\mu\alpha P\ Q) \twoheadrightarrow^{\nu} U$, where ν is a good k-normalization sequence. Then there are terms P', Q', U' and good k-normalization sequences ν_1, ν_2, ν_3 such that*

$$P \twoheadrightarrow^{\nu_1} P', \qquad Q \twoheadrightarrow^{\nu_2} Q', \qquad \mu\alpha P'\ x \twoheadrightarrow^{\nu_3} U'$$

together with $U = U'[x = Q']$, $\nu = \nu_1 \# \nu_2 \# \nu_3[x := Q']$.

Proof The algorithm proceeds by eliminating the innermost k-redexes from left to right, that is we have (possibly empty) ν_1 and ν_2- both being k-normalization sequences- such that $\nu_1 \# \nu_2 \leq \nu$ and $P \twoheadrightarrow^{\nu_1} P' \in NF_k$, $Q \twoheadrightarrow^{\nu_2} Q' \in NF_k$. Then ν continues with reducing $(\mu\alpha P'\ Q')$ and the redexes created by this reduction. It is immediate to check that when reducing $(\mu\alpha P'\ Q')$, the created k-redexes can only be redexes of the form $(\lambda y V[\alpha := Q']\ Q')$ for some $\lambda y V$ of rank k such that $(\alpha\ \lambda y V) \leq P'$, so for every k-redex r in $\mu\alpha P'[\alpha := Q']$ there is an r' in $\mu\alpha P'[\alpha := x]$ such that $r = r'[x := Q']$. Reducing with these β-redexes in $\mu\alpha P'[\alpha := Q']$, no more k-redexes are created. This proves our assertion. □

Lemma 226 *Let $rank(\lambda y P\ Q) = k$, assume $x \notin Fv(P)$. Let $(\lambda y P\ Q) \twoheadrightarrow^{\nu} U$, where ν is a good k-normalization sequence. Then there are terms P', Q', U' and good k-normalization sequences ν_1, ν_2 such that*

$$P \twoheadrightarrow^{\nu_1} P', \qquad Q \twoheadrightarrow^{\nu_2} Q', \qquad (\lambda y P'\ x) \rightarrow^{\nu_3} P'[y := x] = P''$$

and $\nu = \nu_1 \# \nu_2 \# \nu_3[x := Q']$. Moreover, $U = P''[x := Q']$.

Proof Analogous to the proof of the above lemma. □

Remark 227 *A result very similar in nature to those of Lemmas 225 and 226 can be found in Xi [67].*

Lemma 228 *Let $r = (\mu\alpha P\ x)$, $\mu\alpha P \in NF_k$, $rank(r) = k$ and $x \notin Fv(P)$. Assume $(\mu\alpha P\ x) \twoheadrightarrow^{\nu} U \in NF_k$, where ν is a good k-normalization sequence. Then*

1. *$|\nu| \leq comp(P)$,*
2. *$comp(U) \leq 2 \cdot comp(P)$.*

Proof

1. Since $\mu\alpha P \in NF_k$, in $\mu\alpha P[\alpha := x]$ k-redexes of the form $(\lambda y Q[\alpha := x]\ x)$ can only occur, where $(\alpha\ \lambda y Q) \leq P$ and $rank(\lambda y Q) = k$. Subsequently reducing these redexes gives U, which means that U can be obtained in at most $|P|_\alpha + 1 \leq comp(P)$ steps.

2. Considering the above argument, since x is a variable, the β-reduction steps in ν does not increase the size of the term, so

$$comp(U) \leq comp(\mu\alpha P[\alpha := x]) = 1 + comp(P) + |P|_\alpha \leq 2 \cdot comp(P).$$

□

Lemma 229 *Let M be a term, suppose $rank(M) = k$. Let $M \twoheadrightarrow^{\nu} M'$ by a good k-normalization sequence ν. Then we have*

1. $comp(M') \leq 2^{comp(M)-1}$,
2. $|\nu| \leq 2^{comp(M)-1}$.

Proof

1. The proof goes by induction on $comp(M)$.
   (a) $M = x$ or $M = \lambda x M_1$. Obvious.
   (b) $M = \mu\alpha M_1$. If $M = \mu\alpha(\alpha\ M_1)$ is a θ-redex of rank k, then, since the algorithm eliminates k-redexes from bottom to up and from left to right, we have a $\nu' \leq \nu$ such that $\mu\alpha(\alpha\ M_1) \twoheadrightarrow^{\nu'} \mu\alpha(\alpha\ M_1') \rightarrow^{r} M_1' = M'$. But in this case $M \rightarrow_{\theta} M_1 \twoheadrightarrow^{\nu'} M'$ is valid as well, thus by the induction hypothesis
   $$comp(M') \leq 2^{comp((M_1)-1)} < 2^{comp(M)-1}.$$
   If $\mu\alpha M_1$ is not a θ-redex, but reduces to a θ-redex of rank k in the course of the process, then a reasoning analogous to the above one works. Finally, if $\mu\alpha M_1$ is not a θ-redex and it neither reduces to a θ-redex, then the induction hypothesis applies.
   (c) $M = (M_1\ M_2)$.
   - M is not a k-redex. Then M cannot reduce to a k-redex. Suppose on the contrary that there is some initial subsequent of ν such that it reduces M to a k-redex, take ν' as the shortest such reduction sequence. Suppose M reduces to a μ-redex. (The case of a β-redex is similar.) In this case we have $M \twoheadrightarrow^{\nu'} (\mu\beta N_1\ N_2)$, where $M_1 \twoheadrightarrow \mu\beta N_1$ and $M_2 \twoheadrightarrow N_2$. Then $M \twoheadrightarrow^{\nu''} (N_3\ N_2) \rightarrow^{r'} (\mu\beta N_1\ N_2)$ must hold for some r', ν'' such that $\nu' = \nu''\#[r']$ and for some N_3, N_3 not beginning with a μ. This means $N_3 = r'$ would be again a k-redex, but a straightforward examination of the possible cases shows it is impossible. Hence we have $M' = (M_1'\ M_2')$, $\nu = \nu_1\#\nu_2$ for some k-reduction sequences sequences ν_1, ν_2 and $M_i \twoheadrightarrow^{\nu_i} M_i'$ $(i \in \{1, 2\})$. Thus by the induction hypothesis
   $$\begin{array}{rcl} comp(M') & = & comp(M_1') + comp(M_2') \leq 2^{comp(M_1)-1} + 2^{comp(M_2)-1} \\ & \leq & 2^{comp(M)-1}. \end{array}$$
   - M is a k-redex. Suppose $M = (\mu\alpha M_1\ M_2)$. Then M is involved in ν as a μ-redex. By Lemma 225 we have M_1', M_2', M'' and ν_1, ν_2, ν_3 such that
   $$M_1 \twoheadrightarrow^{\nu_1} M_1', \qquad M_2 \twoheadrightarrow^{\nu_2} M_2', \qquad (\mu\alpha M_1'\ x) \twoheadrightarrow^{\nu_3} M''$$
   together with $M' = M''[x := M_2']$, $\nu = \nu_1\#\nu_2\#\nu_3[x := M_2']$, provided $x \notin Fv(M_1)$. From this, by Lemma 228 and by the induction hypothesis,
   $$\begin{array}{rcl} comp(M') & = & comp(M''[x := M_2']) \\ & = & comp(M'') + |M''|_x \cdot (comp(M_2') - 1) \\ & < & comp(M'') \cdot comp(M_2') \leq 2 \cdot comp(M_1') \cdot comp(M_2') \\ & \leq & 2 \cdot 2^{comp(M_1)-1} \cdot 2^{comp(M_2)-1} < 2^{comp(M)-1} \end{array}$$
   follows.
   If $M = (\lambda x M_1\ M_2)$: similar to the case of the μ-redex.

(d) $M = (\alpha\ M_1)$.

- M does not reduce to a k-redex.
- M is either a k-redex, or reduces to a k-redex. That is, there is a ν' and a $\mu\beta M_2 \in NF_k$ such that $(\alpha\ M_1) \twoheadrightarrow^{\nu'} (\alpha\ \mu\beta M_2) \rightarrow^{r} M_2[\beta := \alpha]$ and $\nu'\#[r] = \nu$. The induction hypothesis for M_1 gives the result.

2. By induction on $comp(M)$ we prove $|\nu| \leq 2^{comp(M)-1}$. The only interesting case is when M is a redex of rank k. Let, for example, $M = (\mu\alpha M_1\ M_2)$. Since ν is a k-normalization sequence we can assume again that M is involved in ν. By Lemma 225 we have M_1', M_2' and k-normalization sequences ν_1, ν_2, ν_3 such that

$$M_1 \twoheadrightarrow^{\nu_1} M_1', \qquad M_2 \twoheadrightarrow^{\nu_2} M_2', \qquad (\mu\alpha M_1'\ x) \twoheadrightarrow^{\nu_3} M''$$

together with $M' = M''[x = M_2']$, $\nu = \nu_1\#\nu_2\#\nu_3[x := M_2']$, provided $x \notin Fv(M_1)$. Then, using Lemma 228 and the induction hypothesis, we obtain

$$\begin{array}{rcl} |\nu| & = & |\nu_1| + |\nu_2| + |\nu_3[x := M_2']| = |\nu_1| + |\nu_2| + |\nu_3| \\ & \leq & 2^{comp(M_1)-1} + 2^{comp(M_2)-1} + 2^{comp(M_1)-1} \\ & = & 2^{comp(M_1)} + 2^{comp(M_2)-1} \leq 2^{comp(M)-1}. \end{array}$$

□

Definition 230 *Let* tower *be defined as follows.*

$$\text{tower}(n, m) = \begin{cases} m & \text{if } n = 0, \\ 2^{\text{tower}(n-1,m)} & \text{if } n > 0. \end{cases}$$

Theorem 231 *Let M be a term such that $rank(M) = k$. Suppose $M \twoheadrightarrow^{\sigma} N \in NF$, where σ is a good reduction sequence. Then*

$$\mathcal{M}(\sigma) < \text{tower}(k + 1, comp(M)).$$

Proof We first we prove by induction on k that

$$\mathcal{M}(\sigma) < \text{tower}(1, \text{tower}(1, comp(M)) + \sum_{i=2}^{k} \text{tower}(i, comp(M) - 1)).$$

Let $k = 1$. Then σ is a 1-normalization sequence. Suppose σ is $M = M_1 \rightarrow^{r_1} M_2 \rightarrow^{r_2} \ldots \rightarrow^{r_{n-1}} M_n \rightarrow^{r_n} M_{n+1}$ for some $n \geq 1$. We have, by Lemma 229, $1 + max\{m(r_i, M_i), 2\} \leq 2 \cdot comp(M_i) - 1 \leq 2 \cdot 2^{comp(M)-1} - 1 < 2^{comp(M)}$. Again, by Lemma 229,

$$\begin{array}{rcl} \mathcal{M}(\sigma) & = & \prod_{i=1}^{n}(1 + max\{m(r_i, M_i), 2\}) \ < \ \prod_{i=1}^{n} 2^{comp(M)} \\ & = & 2^{n \cdot comp(M)}. \end{array}$$

Applying Lemma 229, we obtain $n = |\sigma| \leq 2^{comp(M)-1}$, so $\mathcal{M}(\sigma) < 2^{comp(M) \cdot 2^{comp(M)-1}} \leq 2^{2^{comp(M)}} = \text{tower}(1, \text{tower}(1, comp(M)))$.
Let $rank(M) = k + 1$, $k \geq 1$. Assume $M \twoheadrightarrow^{\sigma'} M' \twoheadrightarrow^{\sigma''} N \in NF$, where σ' is a $k + 1$-normalization sequence starting from M. By the induction hypothesis we have

$$\mathcal{M}(\sigma'') < \text{tower}(1, \text{tower}(1, comp(M')) + \sum_{i=2}^{k} \text{tower}(i, comp(M') - 1)).$$

As above, we obtain again $\mathcal{M}(\sigma') < 2^{2^{comp(M)}}$. Then, using the multiplicity of $\mathcal{M}$ and Lemma 229, we can assert

$$\begin{aligned}
\mathcal{M}(\sigma) &= \mathcal{M}(\sigma') \cdot \mathcal{M}(\sigma'') \\
&< 2^{2^{comp(M)}} \cdot \mathrm{tower}\Big(1, \mathrm{tower}(1, comp(M')) \ + \ \sum_{i=2}^{k} \mathrm{tower}(i, comp(M') - 1)\Big) \\
&< 2^{2^{comp(M)}} \cdot \mathrm{tower}\Big(1, \mathrm{tower}(1, \mathrm{tower}(1, comp(M) - 1)) \\
&\quad + \sum_{i=2}^{k} \mathrm{tower}(i, \mathrm{tower}(1, comp(M) - 1))\Big) \\
&= 2^{2^{comp(M)}} \cdot \underbrace{2^{2^{comp(M)-1}} + \ldots + 2^{\cdot^{\cdot^{\cdot^{2^{comp(M)-1}}}}}}_{k} \\
&= \mathrm{tower}(1, \mathrm{tower}(1, comp(M)) + \sum_{i=2}^{k+1} \mathrm{tower}(i, comp(M) - 1)).
\end{aligned}$$

Finally, we prove by induction on k that

$$\mathrm{tower}(1, comp(M)) + \sum_{i=2}^{k} \mathrm{tower}(i, comp(M) - 1) \leq \mathrm{tower}(k, comp(M)),$$

by which the assertion of the lemma follows. Let $k = 1$. Then we have

$$2^{comp(M)} = 2^{comp(M)}.$$

Let $k = n + 1$, $n \geq 1$. Applying the induction hypothesis, we obtain

$$\begin{aligned}
&\mathrm{tower}(1, comp(M)) + \sum_{i=2}^{n+1} \mathrm{tower}(i, comp(M) - 1) \\
&= \underbrace{2^{comp(M)} + 2^{2^{comp(M)-1}} + \ldots + 2^{\cdot^{\cdot^{\cdot^{2^{comp(M)-1}}}}}}_{n+1} \\
&\leq \mathrm{tower}(n, comp(M)) + \mathrm{tower}(n+1, comp(M) - 1) \\
&< \mathrm{tower}(n+1, comp(M)).
\end{aligned}$$

By this the proof is finished. □

Corollary 232 *Let M be a $\lambda\mu I$-term of rank k. Then every reduction sequence starting from M has length less than* $\mathrm{tower}(k+1, comp(M))$.

Proof Let N be the normal-form of M. By Lemma 224 and by Theorem 231 there exists a σ such that $M \twoheadrightarrow^{\sigma} N$ and $\mathcal{M}(\sigma) < \mathrm{tower}(k+2, comp(M))$. By Theorem 214 there is a standard and regular σ' such that $M \twoheadrightarrow^{\sigma'} N$ and $|\sigma'| < \mathcal{M}(\sigma)$. The result follows now from Corollary 221. □

2.0.11 The general case

In what follows we transform every $\lambda\mu$-term M into a $\lambda\mu I$-term $[|M|]_k$ with some $k \geq 0$ such that $\eta(M) \leq \eta([|M|]_k)$, by which, using Corollary 232, we can obtain a bound for $\eta(M)$ also.

Notation 233 *Let $type(\alpha) = \neg A$. Then we write $type^*(\alpha) = A$.*

Lemma 234 *Let M, N be terms. Then*

$$rank(M[\alpha := N]) \leq max\{rank(M), rank(N), lh(type^*(\alpha))\}.$$

Proof By induction on $comp(M)$. The only interesting case is $M = (\alpha\ M_1)$ for some M_1.

1. $M_1 = \lambda x M_2$. By induction hypothesis:

$$\begin{array}{rcl} rank(M[\alpha := N]) & = & rank((\alpha\ (\lambda x M_2[\alpha := N]\ N))) \\ & = & max\{rank(M_2[\alpha := N]), rank(N), \\ & & lh(type(\lambda x M_2[\alpha := N]))\} \\ & \leq & max\{rank(M), rank(N), lh(type^*(\alpha))\}. \end{array}$$

2. $M_1 = \mu\beta M_2$. Similar to the previous one.

3. M_1 is neither of the above forms. Then we have

$$\begin{array}{rcl} rank(M[\alpha := N]) & = & rank((\alpha\ (M_1[\alpha := N]\ N))) \\ & = & max\{rank(M_1[\alpha := N]), rank(N)\} \\ & \leq & max\{rank(M), rank(N), lh(type^*(\alpha))\}. \end{array}$$

□

Lemma 235 *If $M \twoheadrightarrow M'$, then $rank(M') \leq rank(M)$.*

Proof It is enough to show $rank(M') \leq rank(M)$ provided $M \to^r M'$.

1. r is a β-redex. This case was already dealt with in Lemma 110 in Chapter 1. The addition of the θ-reduction to the set of the reductions in the proof of Lemma 110 only means slight changes.

2. $r = (\mu\alpha M_1\ M_2)$.

   - r is not the head-redex of M. Clearly, the induction hypothesis applies.
   - $M = (\mu\alpha M_1\ M_2 \ldots M_n)$. Then $M' = (\mu\alpha M_1[\alpha := M_2] \ldots M_n)$. By Lemma 234 we have

$$\begin{array}{rcl} rank(M') & \leq & max\{rank(\mu\alpha M_1[\alpha := M_2]), \ldots, rank(M_n), \\ & & lh(type(\mu\alpha M_1[\alpha := M_2]))\} \\ & \leq & max\{rank(M_1), rank(M_2), \ldots, rank(M_n), lh(type^*(\alpha))\} \\ & = & rank(M). \end{array}$$

3. $r = (\alpha\ \mu\beta M_1)$.

   - r is not the head-redex of M. The induction hypothesis applies again.
   - $M = (\alpha\ \mu\beta M_1)$. In this case $M' = M_1[\beta := \alpha]$, and our assertion obviously follows.

□

Let $\mathcal{V} = \{v_{(A,B)} \mid A, B \text{ are types}\}$ be a set of distinguished variables such that for all A, B we have $v_{(A,B)} : A \to (B \to A)$. Let $M : A$ and $N : B$ be typed $\lambda\mu$-terms. We denote the term $((v_{(A,B)}\ M)\ N)$ by $\langle M, N\rangle$.

Definition 236 *Let M be a $\lambda\mu$-term, $k \geq 0$. The $\lambda\mu I$-term, $[|M|]_k$, assigned to M is defined as follows.*

- $[|M|]_k = M$, *if M is a variable,*
- $[|M|]_k = \lambda x \lambda y_1 \ldots \lambda y_m \langle([|M_1|]_k\ y_1 \ldots y_m), x\rangle$, *if $M = \lambda x M_1$ such that $lh(type(M)) \leq k$ and $type(M_1) = A_1 \to \ldots \to A_m \to B$, $type(y_i) = A_i$ $(1 \leq i \leq m)$ and B is atomic,*
- $[|M|]_k = \lambda x \langle [|M_1|]_k, x)\rangle$, *if $M = \lambda x M_1$ and $lh(type(M)) > k$,*
- $[|M|]_k = \mu\alpha\langle [|M_1|]_k, (\alpha\ z)\rangle$, *if $M = \mu\alpha M_1$, where $\alpha \notin Fv(M_1)$ and z is a new variable such that $type(M) = type(z)$,*
- $[|M|]_k = \mu\alpha [|M_1|]_k$, *if $M = \mu\alpha M_1$ and $\alpha \in Fv(M_1)$,*
- $[|M|]_k = (\alpha\ [|M_1|]_k)$, *if $M = (\alpha\ M_1)$,*
- $[|M|]_k = ([|M_1|]_k\ [|M_2|]_k)$, *if $M = (M_1\ M_2)$.*

Remark 237 *The above definition differs slightly from the one applied in Xi [66]. Apart from the fact that it is stated for the case of the $\lambda\mu\rho\theta$-calculus, even its restriction to the case of the simply-typed λ-calculus is modified a little. The reason of the modification was the observation that Definition 5.11 in [66] is apparently not enough for proving Corollary 5.13. Xi defines a transformation $[|.|]$ from the simply-typed λ- to the λI-calculus in Definition 5.11 of [66] as follows. Let the set of variables V be defined as above. Let $((v_{(A,B)}\ M)\ N) = \langle M, N\rangle$ provided $M : A$, $N : B$ and $v_{(A,B)} : A \to (B \to A)$.*

- $[|M|] = M$, *if M is a variable,*
- $[|M|] = \lambda x \lambda y_1 \ldots \lambda y_m \langle([|M_1|]\ y_1 \ldots y_m), x\rangle$, *if $M = \lambda x M_1$ such that $type(M_1) = A_1 \to \ldots \to A_m \to B$, $type(y_i) = A_i$ $(1 \leq i \leq m)$ and B is atomic,*
- $[|M|] = ([|M_1|]\ [|M_2|])$, *if $M = (M_1\ M_2)$.*

In his estimation concerning the length of a reduction sequence from M (cf. Xi [66], Corollary 5.13) he uses the inequality $comp([|M|]) \leq (2k+3) \cdot comp(M)$, where $rank(M) = k$. When M contains a subterm of the form $\lambda x M_1$ with $lh(type(\lambda x M_1)) > k$- in this case $\lambda x M_1$ is not the functional part of β-redex-, this inequality can become false. He tries to remedy the problem by saying if $\lambda x M_1 \leq M$ such that $lh(type(\lambda x M_1)) > k$, then let $[|\lambda x M_1|] = \lambda x [|M_1|]$. This gives rise to another problem, since, with this stipulation, $[|M|]$ can cease to be a λI-term. The following definition, which is a restriction of Definition 236, can be a solution. Let M be a λ-term, let $k \geq 0$.

- $[|M|]_k = M$, *if M is a variable,*
- $[|M|]_k = \lambda x \lambda y_1 \ldots \lambda y_m \langle([|M_1|]_k\ y_1 \ldots y_m), x\rangle$, *if $M = \lambda x M_1$ such that $lh(type(M)) \leq k$ and $type(M_1) = A_1 \to \ldots \to A_m \to B$, $type(y_i) = A_i$ $(1 \leq i \leq m)$ and B is atomic,*
- $[|M|]_k = \lambda x \langle [|M_1|]_k, x)\rangle$, *if $M = \lambda x M_1$ and $lh(type(M)) > k$,*
- $[|M|]_k = ([|M_1|]_k\ [|M_2|]_k)$, *if $M = (M_1\ M_2)$.*

We have to make sure that the definition is "stable" concerning reductions, that is, if $N \leq M$ and $M \to^r M'$, then, if N' is a descendant of N w.r.t. r, $[|N'|]_k$ is calculated by applying the same point of the above definition as the one when determining $[|N|]_k$ before. This is ensured by Lemma 235. The same thing must also be proved for the $\lambda\mu$-calculus in relation with Definition 236, this is done in the sequel.

Lemma 238 *Let M be a $\lambda\mu$-term of type A, assume $k \geq 0$.*

1. *$[|M|]_k$ is a $\lambda\mu I$-term of type A,*
2. *$[|M|]_k[x := [|N|]_k] = [|M[x := N]|]_k$, provided $x \notin Fv([|M|]_k)$,*
3. *$[|M|]_k[\alpha := [|N|]_k] = [|M[\alpha := N]|]_k$,*
4. *$[|M|]_k[\beta := \alpha] = [|M[\beta := \alpha]|]_k$.*

Proof By induction on $comp(M)$. □

Our next aim is to prove $\eta(M) \leq \eta([|M|]_k)$. For this purpose we prove some additional lemmas.

Lemma 239 *Let $M = (\lambda x M_1\ M_2\ \overrightarrow{P})$ and $N = (M_1[x := M_2]\ \overrightarrow{P})$. Assume N and M_2 are strongly normalising. Then M is also strongly normalizable and $\eta(M) \leq \eta(N) + \eta(M_2) + 1$.*

Proof Let $M \twoheadrightarrow^\sigma U$ be an arbitrary reduction sequence, we are going to show that $|\sigma| \leq \eta(N) + \eta(M_2) + 1$, from which the result follows. We may suppose that $(\lambda x M_1\ M_2)$ is involved in σ. Then σ is of the following form for some σ_1 and σ_2:

$$\begin{array}{rcl} M & = & (\lambda x M_1\ M_2\ \overrightarrow{P}) \twoheadrightarrow^{\sigma_1} M' = (\lambda x M_1'\ M_2'\ \overrightarrow{P'}) \\ & \to & (M_1'[x := M_2']\ \overrightarrow{P'}) \twoheadrightarrow^{\sigma_2} U, \end{array}$$

where $M_i \twoheadrightarrow^{\nu_i} M_i'$ $(i \in \{1, 2\})$, $\overrightarrow{P} \twoheadrightarrow^{\nu_3} \overrightarrow{P'}$ and $\sigma_1 = \nu_1 \# \nu_2 \# \nu_3$. Let σ' denote the reduction sequence as below:

$$\begin{array}{rcl} M & = & (\lambda x M_1\ M_2\ \overrightarrow{P}) \twoheadrightarrow^{\nu_2} (\lambda x M_1\ M_2'\ \overrightarrow{P}) \\ & \to & (M_1[x := M_2']\ \overrightarrow{P}) \twoheadrightarrow^{\sigma^*} U, \end{array}$$

where $\sigma^* = \nu_1' \# \nu_3 \# \sigma_2$ and ν_1' is constructed from ν_1 by Lemma 202. Then

$$|\sigma| = |\sigma'| \leq \eta(M_2) + 1 + \eta(N),$$

which is the desired result. □

Lemma 240 *Let $M = (\alpha\ \mu\beta M_1)$, $N = M_1[\beta := \alpha]$, suppose N is strongly normalising. Then M is also strongly normalizable and $\eta(M) = \eta(N) + 1$.*

Proof Assume σ is a reduction sequence starting from $(\alpha\ \mu\beta M_1)$. We prove $\sigma \leq \eta(N) + 1$, from which the result follows. Let $\sigma = [r] \# \sigma'$ for some σ'. We distinguish the various cases according to the form of σ.

1. $(\alpha\ \mu\beta M_1) \to_\rho^r M_1[\beta := \alpha] \twoheadrightarrow^{\sigma'} M_2$, where $\sigma = [r] \# \sigma'$. Then the result obviously follows.
2. $(\alpha\ \mu\beta M_1) \to^r M_2 \twoheadrightarrow^{\sigma'} M_3$, where $M_2 \neq N$ and $\mu\beta M_1$ does not disappear in σ. Then $M_3 = (\alpha\ \mu\beta M_3')$ and $M_1 \twoheadrightarrow^\sigma M_3'$, which yields the result.
3. $(\alpha\ \mu\beta M_1) \to^r M_2 \twoheadrightarrow^{\sigma'} M_3$, where $M_2 \neq N$ and $\mu\beta M_1$ disappears in σ. Then $(\alpha\ \mu\beta M_1) \twoheadrightarrow^{\sigma''} (\alpha\ \mu\beta(\beta\ M_k)) \to_\theta (\alpha\ M_k) \twoheadrightarrow^{\sigma'''} M_3$, where $\mu\beta M_1$ does not disappear in σ''. We have $(\alpha\ \mu\beta M_1) \to_\rho M_1[\beta := \alpha] \twoheadrightarrow^{\sigma''[\beta:=\alpha]} (\beta\ M_k)[\beta := \alpha] = (\alpha\ M_k) \twoheadrightarrow^{\sigma'''} M_3$, and the latter reduction sequence is equal in length to σ. By this the result follows.

The reverse direction is obvious. □

Lemma 241 *Let $M = \mu\alpha(\alpha\ M_1)$ be a θ-redex, suppose M_1 is strongly normalising. Then M is also strongly normalizable and $\eta(M) = \eta(M_1) + 1$.*

Proof Similar to the above one. □

Lemma 242 *Let $M = (\mu\alpha M_1\ M_2\ \overrightarrow{P})$ and $N = (\mu\alpha M_1[\alpha := M_2]\ \overrightarrow{P'})$. Assume $\alpha \in Fv(M_1)$. Then*

$$\eta(M) = \eta(N) + 1.$$

Proof Let $M \to^{\sigma} M^*$. We prove $\sigma \leq \eta(N) + 1$, from this $\eta(M) \leq \eta(N) + 1$ follows.

1. $r = (\mu\alpha M_1\ M_2)$ is involved in σ.

   - $\mu\alpha M_1$ does not disappear in σ: $(\mu\alpha M_1\ M_2\ \overrightarrow{P}) \to^{\sigma'} (\mu\alpha M_1'\ M_2'\ \overrightarrow{P'}) \to_{\mu} (\mu\alpha M_1'[\alpha := M_2']\ \overrightarrow{P'}) \twoheadrightarrow^{\sigma''} M^*$. Then, since $\alpha \in Fv(M_1)$, by Lemmas 204 and 203, the reduction sequence $(\mu\alpha M_1\ M_2\ \overrightarrow{P}) \to^{r} (\mu\alpha M_1[\alpha := M_2]\ \overrightarrow{P}) \twoheadrightarrow (\mu\alpha M_1[\alpha := M_2']\ \overrightarrow{P}) \twoheadrightarrow (\mu\alpha M_1'[\alpha := M_2']\ \overrightarrow{P'}) \twoheadrightarrow^{\sigma''} M^*$ has length at least $|\sigma|$, by which the assertion follows.
   - $\mu\alpha M_1$ disappears in σ: $(\mu\alpha M_1\ M_2\ \overrightarrow{P}) \to^{\sigma'} (\mu\alpha(\alpha\ M_1')\ M_2'\ \overrightarrow{P'}) \to_{\theta} (M_1'\ M_2'\ \overrightarrow{P'}) \twoheadrightarrow^{\sigma''} M^*$. Then, since $\alpha \in Fv(M_1)$, by Lemmas 204 and 203 the sequence $(\mu\alpha M_1\ M_2\ \overrightarrow{P}) \to_{\mu} (\mu\alpha M_1[\alpha := M_2]\ \overrightarrow{P}) \twoheadrightarrow (\mu\alpha(\alpha\ M_1')[\alpha := M_2']\ \overrightarrow{P'}) = (\mu\alpha(\alpha\ (M_1'\ M_2'))\ \overrightarrow{P'}) \to_{\theta} (M_1'\ M_2'\ \overrightarrow{P'}) \twoheadrightarrow^{\sigma''} M^*$ has length at least $|\sigma|$, which yields the result.

2. $r = (\mu\alpha M_1\ M_2)$ is not involved in σ.

   - $\mu\alpha M_1$ does not disappear in σ: That is, $(\mu\alpha M_1\ M_2\ \overrightarrow{P}) \twoheadrightarrow M^* = (\mu\alpha M_1'\ M_2'\ \overrightarrow{P'})$. Then, since $\alpha \in Fv(M_1)$, we can apply Lemmas 204 and 203 to assert that $(\mu\alpha M_1\ M_2\ \overrightarrow{P}) \to^{r} (\mu\alpha M_1[\alpha := M_2]\ \overrightarrow{P}) \twoheadrightarrow (\mu\alpha M_1'[\alpha := M_2']\ \overrightarrow{P}')$ has length at least $|\sigma| + 1$.
   - $\mu\alpha M_1$ disappears in σ: $(\mu\alpha M_1\ M_2\ \overrightarrow{P}) \twoheadrightarrow (\mu\alpha(\alpha\ M_1')\ M_2'\ \overrightarrow{P'}) \to_{\theta} (M_1'\ M_2'\ \overrightarrow{P'}) \twoheadrightarrow M^*$. By Lemmas 203 and 204 the sequence $(\mu\alpha M_1\ M_2\ \overrightarrow{P}) \to^{r} (\mu\alpha M_1[\alpha := M_2]\ \overrightarrow{P}) \twoheadrightarrow (\mu\alpha(\alpha\ M_1')[\alpha := M_2']\ \overrightarrow{P'}) = (\mu\alpha(\alpha\ (M_1'\ M_2'))\ \overrightarrow{P'}) \to_{\theta} (M_1'\ M_2'\ \overrightarrow{P'}) \twoheadrightarrow M^*$ has length at least $|\sigma|+1$, which proves the assertion.

The reverse direction is obvious. □

Lemma 243 *Let $M = (\mu\alpha M_1\ M_2\ \overrightarrow{P})$, $\alpha \notin Fv(M_1)$, and $N = (\mu\alpha M_1\ \overrightarrow{P})$. Assume N and M_2 are strongly normalising. Then M is also strongly normalizable and $\eta(M) = \eta(N) + \eta(M_2) + 1$.*

Proof The proof of $\eta(M) \leq \eta(N) + \eta(M_2) + 1$ is similar to the first part of the proof of Lemma 242. In this case the verification is made easier by the fact that, since $\alpha \notin Fv(M_1)$, $\mu\alpha M_1$ does not disappear in a reduction sequence starting from M.
For the converse: Let $N \twoheadrightarrow^{\sigma} N'$ and $M_2 \twoheadrightarrow^{\nu} M_2'$. Then $(\mu\alpha M_1\ M_2\ \overrightarrow{P}) \to^{\nu} (\mu\alpha M_1\ M_2'\ \overrightarrow{P}) \to_{\mu} (\mu\alpha M_1\ \overrightarrow{P}) \to^{\sigma} N'$ is a reduction sequence starting from M, which means that $\eta(N) + \eta(M_2) + 1 \leq \eta(M)$. □

Lemma 244 *Let M, N be $\lambda\mu I$-terms and $\overrightarrow{P}$ be a sequence of arguments consisting of $\lambda\mu I$-terms. Assume $\alpha \notin Fv(N)$. Then $\eta((\mu\alpha\langle M, (\alpha\ z)\rangle\ \overrightarrow{P})) + \eta(N) \leq \eta((\mu\alpha\langle M, (\alpha\ (z\ N))\rangle\ \overrightarrow{P}))$.*

Proof Let $U = (\mu\alpha\langle M, (\alpha\ z)\rangle\ \overrightarrow{P})$, $V = (\mu\alpha\langle M, (\alpha\ (z\ N))\rangle\ \overrightarrow{P})$. If $\overrightarrow{P}$ is empty, the result is trivial, so may assume $\overrightarrow{P}$ is not empty and its components are $M_1, \ldots, M_n$. We are going to prove if $U \twoheadrightarrow^{\sigma_1} U'$, $N \twoheadrightarrow^{\sigma_2} N'$ for some σ_1, σ_2, U', N', then we have a reduction sequence ν of V such that $|\sigma_1|+|\sigma_2| \leq |\nu|$. By the second part of Theorem 214, it is enough to restrict our attention to the case when σ_1 and σ_2 are standard. We may assume that the head-redex of U is involved in σ_1, otherwise the result is trivial. Furthermore, we may suppose that $\mu\alpha\langle M, (\alpha\ z)\rangle$ is reduced in $|\sigma_1|$ with all of its arguments $M_1, \ldots, M_n$. Then σ_1 is of the form

$$\begin{array}{lll} U & \twoheadrightarrow^{\xi} & \mu\alpha\langle M[\alpha := M_1]\ldots[\alpha := M_n], (\alpha\ (z\ M_1 \ldots M_n))\rangle \\ & \twoheadrightarrow^{\zeta} & \mu\alpha\langle M', (\alpha\ (z\ M_1 \ldots M_n))\rangle\ \twoheadrightarrow^{\zeta^*}\ \mu\alpha\langle M', (\alpha\ (z\ M_1' \ldots M_n'))\rangle, \end{array}$$

where $M[\alpha := M_1]\ldots[\alpha := M_n] \twoheadrightarrow^{\zeta} M'$ and $\zeta^* = \zeta_1 \# \ldots \# \zeta_n$ with $M_i \twoheadrightarrow^{\zeta_i} M_i'$ for $1 \leq i \leq n$. Let ξ' be $V \twoheadrightarrow^{\xi'} \mu\alpha\langle M[\alpha := M_1]\ldots[\alpha := M_n], (\alpha\ (z\ N\ M_1 \ldots M_n))\rangle$, then choosing ν as

$$\nu = \xi' \# \zeta \# \sigma_2 \# \zeta^*$$

is appropriate. □

Lemma 245 *Let M be a $\lambda\mu$-term, α be a μ-variable. Then $\alpha \in Fv(M) \Leftrightarrow \alpha \in Fv([|M|]_k)$.*

Proof Trivial. □

Lemma 246 *Let $M \to M'$, assume $rank(M) \leq k$. Then $\eta([|M'|]_k) + 1 \leq \eta([|M|]_k)$.*

Proof By induction on $comp(M)$.

1. $M = \lambda x M_1$. The induction hypothesis applies.

2. $M = (\lambda x M_1\ M_2 \ldots M_n)$. We have $lh(type(\lambda x M_1)) \leq k$. Let $type(M_1) = A_1 \to \ldots A_m \to B$, where B is atomic. Let $M' = (M_1[x := M_2]\ldots M_n)$, otherwise the induction hypothesis applies. Since B is atomic, we have $m \geq n-2$. From this

$$\begin{array}{lll} [|M|]_k & \to & \lambda y_1 \ldots \lambda y_m \langle ([|M_1|]_k[x := [|M_2|]_k]\ y_1 \ldots y_m), [|M_2|]_k \rangle \ldots [|M_n|]_k \\ & \twoheadrightarrow & \lambda y_{n-1} \ldots \lambda y_m \langle ([|M_1|]_k[x := [|M_2|]_k] \ldots [|M_n|]_k\ y_{n-1} \ldots y_m), [|M_2|]_k \rangle. \end{array}$$

   Lemma 238 gives

$$([|M_1|]_k[x := [|M_2|]_k]\ [|M_3|]_k \ldots [|M_n|]_k) = ([|M_1[x := M_2]|]_k\ [|M_3|]_k \ldots [|M_n|]_k)$$

$$= ([|M_1[x := M_2]\ M_3 \ldots M_n|]_k) = [|M'|]_k,$$

   by which the result follows.

3. $M = (\mu\alpha M_1\ M_2 \ldots M_n)$. We may assume again that $M \to^r M'$, where $r = (\mu\alpha M_1\ M_2)$.

   - $\alpha \in Fv(M_1)$. Let $M' = (\mu\alpha M_1[\alpha := M_2]\ldots M_n)$. We have by Lemma 238:

$$\begin{array}{lll} [|M|]_k & = & (\mu\alpha[|M_1|]_k\ [|M_2|]_k \ldots [|M_n|]_k) \ \to\ (\mu\alpha[|M_1|]_k[\alpha := [|M_2|]_k]\ldots[|M_n|]_k) \\ & = & (\mu\alpha[|M_1[\alpha := M_2]|]_k \ldots [|M_n|]_k) \ =\ [|M'|]_k. \end{array}$$

- $\alpha \notin Fv(M_1)$. In this case $M' = \mu\alpha M_1\ M_3 \ldots M_n$. Then

$$[|M|]_k \quad \rightarrow \quad (\mu\alpha\langle[|M_1|]_k, (\alpha\ (z\ [|M_2|]_k))\rangle\ [|M_3|]_k \ldots [|M_n|]_k).$$

Lemma 244 gives $\eta((\mu\alpha\langle[|M_1|]_k, (\alpha\ (z\ [|M_2|]_k))\rangle\ [|M_3|]_k \ldots [|M_n|]_k)) \geq \eta((\mu\alpha\langle[|M_1|]_k, (\alpha\ z)\rangle\ [|M_3|]_k \ldots [|M_n|]_k)) + \eta([|M_2|]_k) + 1$. Moreover, by induction on n, we obtain that $\eta([|M'|]_k) \leq \eta((\mu\alpha\langle[|M_1|]_k, (\alpha\ z)\rangle\ [|M_3|]_k \ldots [|M_n|]_k))$, by which the result follows.

4. $M = (\alpha\ M_1)$. The only interesting case is $M = (\alpha\ \mu\beta M_1') \rightarrow M_1'[\beta := \alpha]$. If $\beta \in Fv(M_1')$, then $[|M|]_k = (\alpha\ \mu\beta[|M_1'|]_k)$. Otherwise, $[|M|]_k = (\alpha\ \mu\beta\langle[|M_1'|]_k, (\beta\ z)\rangle)$. Applying Lemma 238, in both cases we obtain the result.

5. $M = \mu\alpha M_1$. Analogous to the previous one.

6. If $M = (x\ M_1\ \overrightarrow{P})$, the induction hypothesis applies.

□

Lemma 247 *Let M be a $\lambda\mu$-term, assume $rank(M) \leq k$. Then $\eta(M) \leq \eta([|M|]_k)$.*

Proof By induction on $\langle\eta([|M|]_k), comp(M)\rangle$.

1. If $M = \lambda x M_1$, by the induction hypothesis we have the result.

2. Let $M = (x\ M_1 \ldots M_n)$. Then, by the induction hypothesis,

$$\eta(M) = \eta(M_1) + \ldots + \eta(M_n) \leq \eta([|M_1|]_k) + \ldots + \eta([|M_n|]_k) = \eta([|M|]_k).$$

3. $M = (\lambda x M_1\ M_2 \ldots M_n)$. Let $M' = (M_1[x := M_2] \ldots M_n)$. If follows from Lemma 235 that $rank(M') \leq k$. Thus, by Lemmas 239, 246 and the induction hypothesis,

$$\eta(M) \leq \eta(M') + \eta(M_2) + 1 \leq \eta([|M'|]_k) + \eta([|M_2|]_k) + 1 \leq \eta([|M|]_k).$$

4. $M = (\mu\alpha M_1\ M_2 \ldots M_n)$.

   - $\alpha \in Fv(M_1)$. Let $M' = (\mu\alpha M_1[\alpha := M_2] \ldots M_n)$. Then $rank(M') \leq k$ by Lemma 235 again. We have by Lemmas 246, 242 and the induction hypothesis

$$\eta(M) = \eta(M') + 1 \leq \eta([|M'|]_k) + 1 = \eta([|M|]_k).$$

   - $\alpha \notin Fv(M_1)$. Let $M' = (\mu\alpha M_1\ M_3 \ldots M_n)$. We have

$$[|M|]_k \quad \rightarrow \quad (\mu\alpha\langle[|M_1|]_k, (\alpha\ (z\ [|M_2|]_k))\rangle\ [|M_3|]_k \ldots [|M_n|]_k),$$

   which, together with Lemmas 243, 244, 235 and the induction hypothesis, yields that

$$\eta(M) \leq \eta(M') + \eta(M_2) + 1 \leq \eta([|M'|]_k) + \eta([|M_2|]_k) + 1 \leq \eta([|M|]_k).$$

5. $M = \mu\alpha M_1$.

- $\alpha \in Fv(M_1)$. If $\mu\alpha M_1 = \mu\alpha(\alpha\ M_2)$ is a θ-redex, then, by Lemmas 241, 235 and the induction hypothesis,

$$\eta(M) = \eta(M_2) + 1 \leq \eta([|M_2|]_k) + 1 = \eta([|M|]_k).$$

Otherwise, let $\mu\alpha M_1 \rightarrow M'$. Since $\mu\alpha M_1$ is not a θ-redex, we have $M' = \mu\alpha M_1'$ together with $rank(M') \leq k$. By Lemma 246, we can apply the induction hypothesis to M', that is,

$$\eta(\mu\alpha M_1') + 1 \leq \eta([|\mu\alpha M_1'|]_k) + 1 \leq \eta([|\mu\alpha M_1|]_k).$$

But M' was arbitrary and $\eta(M) = max\{\eta(M') + 1 \mid M \rightarrow M'\}$, which proves our assertion.

- $\alpha \notin Fv(M_1)$. Then we can apply the induction hypothesis to M_1.

6. Let $M = (\alpha\ \mu\beta M')$. Similar to the previous case.

□

Lemma 248 *Let M be a $\lambda\mu$-term. Assume $rank(M) \leq k$ for some $k \geq 0$. Then $comp([|M|]_k) \leq (2k+3) \cdot comp(M)$.*

Proof The only nontrivial case is $M = \lambda x M_1$. Let $lh(type(\lambda x M_1)) = l$.

1. Assume $k < l$. Then

$$comp([|M|]_k) = comp(\lambda x\langle [|M_1|]_k, x\rangle) = comp([|M_1|]_k) + 3 \leq (2k+3) \cdot comp(M).$$

2. Let $k \geq l$. Then, for some $m \leq l$, we obtain by the induction hypothesis

$$\begin{aligned} comp([|M|]_k) &= comp(\lambda x \lambda y_1 \ldots \lambda y_m \langle ([|M_1|]_k\ y_1 \ldots y_m), x\rangle) \\ &= comp([|M_1|]_k) + 2m + 3 \leq (2k+3) \cdot comp(M). \end{aligned}$$

□

Lemma 249 *Let M be a term. Then $rank([|M|]_k) = rank(M)$.*

Proof Obvious. □

Theorem 250 *Let M be a $\lambda\mu$-term, suppose $rank(M) = k$. Then every $\beta\mu\rho\theta$-reduction sequence starting from M is of length less than*

$$tower(k+1, (2k+3) \cdot comp(M)).$$

Proof We obtain, by Lemma 248, $comp([|M|]_k) \leq (2k+3) \cdot comp(M)$ and, by Lemma 249, $rank([|M|]_k) = rank(M)$. These, together with Corollary 232 and Lemma 247, imply

$$\begin{aligned} \eta(M) &\leq \eta([|M|]_k) < tower(k+1, comp([|M|]_k)) \\ &\leq tower(k+1, (2k+3) \cdot comp(M)). \end{aligned}$$

□

2.1 Concluding remarks

2.1.1 A possible attempt to calculate an upper bound for the $\lambda\mu\rho$-calculus

In the following observations we confine our attention to the case of the $\lambda\mu\rho$-calculus. For establishing a bound for the lengths of reduction sequences of the $\lambda\mu\rho$-calculus it seems to be a natural idea to try to transform a reduction sequence of the $\lambda\mu\rho$-calculus into a reduction sequence of the λ-calculus. We go round this approach a little bit more detailed: we present the CPS-translation from the simply-typed $\lambda\mu\rho$-calculus to the simply-typed λ-calculus introduced by de Groote [29], and then we give an account of the possibilities of finding an appropriate bound with this method. The notation for the CPS-translation is taken from de Groote [29]. As to a bound for the simply-typed λ-calculus we regard the one presented in Xi [66].

Definition 251 *Let o be some distinguished atomic type. Then $\overline{A} = {\sim\sim} A^o$, where ${\sim} A = A \to o$ and where*

1. $\bot^o = o$,
2. $A^o = A$, *if A is atomic,*
3. $(A \to B)^o = \overline{A} \to \overline{B}$.

To facilitate reading, in the following definition a is used as a μ-variable and as a λ-variable at the same time.

Definition 252 *The CPS-translation $\overline{M}$ of a $\lambda\mu\rho$-term M is defined as follows.*

1. $\overline{x} = \lambda k.(x\ k)$,
2. $\overline{\lambda x M} = \lambda k.(k\ (\lambda x \overline{M}))$,
3. $\overline{M\ N} = \lambda k.(\overline{M}\ (\lambda m.(m\ \overline{N}\ k)))$,
4. $\overline{\mu a M} = \lambda a(\overline{M}\ (\lambda k.k))$,
5. $\overline{a\ M} = \lambda k.(\overline{M}\ a)$.

Proposition 253 *Let Γ (resp. $\triangle$) denote a λ-context (resp. μ-context), that is, a finite (possibly empty) set of declarations of the form $x : A$ (resp. $a : \neg A$). Let $M : A$ be a typable term with λ-context Γ and μ-context Δ. Then its CPS-translation, $\overline{M}$, is typable with contexts $\overline{\Gamma}$ and ${\sim}\triangle^o$.*

Let $=_\lambda$ (resp. $=_\mu$) denote the relation defined as the reflexive, symmetric, transitive closure of the β-reduction (resp. that of the union of the β-, μ- and ρ-reductions). As usual, we consider terms differing in renaming of bound variables as equals. Then, in [26], de Groote proves the following:

Proposition 254 *$M =_\mu N$ iff $\overline{M} =_\lambda \overline{N}$.*

Unfortunately, in Proposition 254 $\overline{M} \twoheadrightarrow_\lambda \overline{N}$ does not hold generally, even if $M \twoheadrightarrow_\mu N$. So on one hand we cannot use the CPS-translation to imitate the reduction sequences in the $\lambda\mu\rho$-calculus by reduction sequences in the λ-calculus. On the other hand there can be another drawback of this approach.
In general, we could make use of the CPS-translation for estimating bounds of reduction sequences if for any $M \twoheadrightarrow^\sigma_\mu nf(M)$ we could find a ν with $\overline{M} \twoheadrightarrow^\nu_\lambda nf(\overline{M})$ such that $|\sigma| \leq c \cdot |\nu|$ with some constant c, where $nf(M)$ and $nf(\overline{M})$ denote the (unique) normal form of M in the $\lambda\mu\rho$- and of $\overline{M}$ in the λ-calculus, respectively. (In fact, we even know that $nf(\overline{M}) = \overline{\overline{nf(M)}}$, where $\overline{\overline{M}}$ stands for the so called modified CPS-translation of the term M (cf. de Groote [26]).)

Definition 255 *Let comp be defined for λ-terms as follows.*

1. $comp(x) = 1$,
2. $comp(\lambda x M) = comp(M) + 1$,
3. $comp((M\ N)) = comp(M) + comp(N)$.

For the moment suppose for every reduction sequence $M \twoheadrightarrow^{\sigma} nf(M)$ we can find a reduction sequence ν such that $\overline{M} \twoheadrightarrow^{\nu}_{\lambda} nf(\overline{M})$ with $|\sigma| \leq c \cdot |\nu|$. By the result for the β-normalization in Xi [66], we would have for any ν as above

$$|\nu| < c \cdot \mathrm{tower}(rank(\overline{M}) + 1, (2 \cdot rank(\overline{M}) + 3) \cdot comp(\overline{M})),$$

where the *rank* of a λ-term is understood as the maximum of the ranks of its β-redexes. On the other hand we have the following estimations.

Proposition 256 *Let M be a $\lambda\mu$-term. Then*

1. $rank(\overline{M}) = 3 \cdot rank(M)$,
2. $2 \cdot comp(M) < comp(\overline{M})$.

This means that the best estimation for the lengths of the reductions with this method would be greater than $c \cdot \mathrm{tower}(3 \cdot rank(M)+1, (12 \cdot rank(M)+6) \cdot comp(M))$, and by the direct method this upper bound is $\mathrm{tower}(rank(M)+1, (2 \cdot rank(M)+3) \cdot comp(M))$. At present, no CPS-translation which could yield a significantly better estimation is known to the author.

2.1.2 A translation of the $\lambda\mu$-calculus into the λ^*_c-calculus

Recently, a new translation of the $\lambda\mu$-calculus into a version of the λ-calculus formulated with recursive equations for types was discovered by David and Nour (cf. [16]). This is somewhat simpler than the CPS-translation and provides an easy method of finding an estimation for the lengths of reduction sequences in the $\lambda\mu$-calculus. We present a version of it, establishing a connection between the $\lambda\mu$-calculus and a variant of the λ-calculus enlarged with some constants. Enhance the set of types of the simply typed λ-calculus with an element $\bot$ and define $\neg A$ as $A \to \bot$. Let X be an atomic type, add for each X a new constant c_X of type $\neg\neg X \to X$. Let us call the new calculus as λ^*_c. The method traces back to Krivine [36], where he supplemented the typed-calculus with a constant of type $\forall X(\neg\neg X \to X)$.

Definition 257 *We define for each type A a closed λ^*_c-term T_A such that $T_A : \neg\neg A \to A$.*

1. $T_{\bot} = \lambda y(yI)$, *where* $I = \lambda x x$,
2. $T_X = c_X$, *where X is a base type,*
3. $T_{A \to B} = \lambda x \lambda y(T_B\ \lambda z(x\ \lambda t(z\ (t\ y))))$.

Definition 258 *Let $k \geq 0$. We define a translation of the set of $\lambda\mu$-terms into the set of terms of the λ^*_c-calculus as follows.*

1. $|x|_k = x$,
2. $|\lambda x M|_k = \lambda x |M|_k$,
3. $|(M\ N)|_k = (|M|_k\ |N|_k)$,

4. $|\mu aM|_k = (T_A\ \lambda a|M|_k)$, *if* a *has type* $\neg A$ *and* $lh(A) \leq k$,

5. $|\mu aM|_k = (z\ |M|_k)$, *if* a *has type* $\neg A$ *and* $lh(A) > k$ *and where* $z : \bot \to A$ *is a new variable,*

6. $|(a\ M)|_k = (a\ |M|_k)$.

In the above definition the μ-variables and its translated counterparts were denoted with the same letters. Let $\vdash_{\lambda\mu}$ and $\vdash_{\lambda^*_c}$ denote the typing relations in the $\lambda\mu$- and in the λ^*-calculus, respectively. We have the following assertions.

Lemma 259 *Let* $k \geq 0$. *If* $\Gamma \vdash_{\lambda\mu} M : A$, *then* $\Gamma \vdash_{\lambda^*_c} |M|_k : A$.

Proof Straightforward. □

Lemma 260 *Let* M, N *be typed* $\lambda\mu$*-terms. Assume* $k \geq rank(M)$. *If* $M \to_{\lambda\mu} N$, *then* $|M|_k \twoheadrightarrow^+_\lambda |N|_k$.

Proof Obviously, it is enough to check the relation $|(\mu aM_1\ M_2)|_k \twoheadrightarrow^+_\lambda |\mu aM_1[a := M_2]|_k$, where, necessarily, $k \geq lh(A)$ provided $type(a) = \neg A$. □

Lemma 261 *Let* M, N *be typed* $\lambda\mu$*-terms. Assume* $k \geq rank(M)$. *Let* $M \twoheadrightarrow^n N$. *Then* $|M|_k \twoheadrightarrow^m |N|_k$ *for some* $m \geq n$.

Proof Follows from Lemmas 235 and 260. □

Since no reduction rules are added to λ when defining λ^*_c, the method of Xi [66] for estimating the lengths of reduction sequences is also applicable to λ^*_c without any changes. We state without proof the following theorem:

Theorem 262 *Let* M *be a* λ^*_c*-term, assume* $rank(M) = k$. *Then every reduction sequence starting from* M *has length less than*

$$tower(k + 1, (2k + 3) \cdot comp(M)).$$

In order to establish a bound for the lengths of $\lambda\mu$-reduction sequences we have to estimate the size of the translated terms as well.

Lemma 263 *Let* A *be a type. Assume* $lh(A)$ *is defined as in Definition 106. Then*

$$comp(T_A) \geq 8 \cdot lh(A) + 3.$$

Proof Obvious. □

Lemma 264 *Let* M *be a* $\lambda\mu$*-term. Assume* $rank(M) = k$. *Then*

$$comp(|M|_k) \leq (8k + 4) \cdot comp(M).$$

Proof By induction on $comp(M)$. We only check one of the cases. Let $M = (\mu aM_1\ M_2)$. Assume $type(a) = \neg A$. Then, since $k \geq lh(A)$, we have by Lemma 263 and the induction hypothesis

$$\begin{aligned} comp(|M|_k) &= comp(|\mu aM_1|_k\ |M_2|_k) \\ &= comp((T_A\ \lambda a|M_1|_k)) + comp(|M_2|_k) \\ &\leq (8k + 4) + comp(|M_1|_k) + comp(|M_2|_k) \\ &\leq (8k + 4) \cdot comp(M). \end{aligned}$$

□

Theorem 265 *Let* M *be a* $\lambda\mu$*-term, assume* $rank(M) = k$. *Then every reduction sequence starting from* M *has length less than*

$$tower(k + 1, (2k + 3) \cdot (8k + 4) \cdot comp(M)).$$

Proof Follows from Theorem 262 and Lemma 264. □

This method, however, is not applicable to the $\lambda\mu\rho\theta$-calculus, since, in the case of the ρ-reduction, Lemma 260 is not valid.

Chapter 3

Strong normalization of the λ^{Sym}_{Prop}-calculus

The λ^{Sym}-calculus was introduced by Berardi and Barbanera in [1]. It is organized entirely around the duality in classical logic, by this reason, it has a negation "built-in", that is, it contains an involutive negation for which $\neg A$ is not defined as $A \to \bot$ with some type $\bot$. Instead, each type is related to its natural negated type on the model of the notion of duality introduced by negation in classical logic. In fact, Berardi and Barbanera defined a calculus equivalent to first order Peano arithmetic, however, we only consider here its propositional part, denoted by λ^{Sym}_{Prop}, since all the other calculi treated by us in this work are concerned with propositional logic.

The main objective of this chapter is to give an arithmetical proof for the strong normalization of the λ^{Sym}_{Prop}-calculus. Our exposition is based on the methods of the strong normalization proofs presented in the papers of David and Nour (cf. [14] and [15], respectively). As regards the basic definitions and notation for the calculus, we refer to the introduction.

3.1 The $\lambda_{\beta\pi}$-calculus is strongly normalizing

The present chapter is devoted to giving an arithmetical proof for the following theorem.

Theorem 266 *The typed λ^{Sym}_{Prop}-calculus is strongly normalizing.*

We accomplish this task in several substeps. In this section we prove the strong normalization of a restricted version of the typed λ^{Sym}_{Prop}-calculus, then extend the result for the remaining reduction rules also. In what follows we assume that we are in the realm of the typed calculus.
Let $\to$ stand for the union of $\to_\beta$, $\to_{\beta_\bot}$, $\to_\pi$, $\to_{\pi_\bot}$ and, as usual, $\twoheadrightarrow$ denote the reflexive, symmetric and transitive closure of $\to$. The calculus with this reduction will be termed as $\lambda_{\beta\beta_\bot\pi\pi_\bot}$. The notions of reduction sequence, normal form and normalization are defined with respect to $\to$. Let SN denote the set of strongly normalizable terms of $\lambda_{\beta\beta_\bot\pi\pi_\bot}$ concerning $\to$ settled as above. In the sequel we detail the proofs for the β- and π-reductions only, all the proofs below can be extended with the cases of the $\beta_\bot$- and $\pi_\bot$-reduction rules in a straightforward way.

Definition 267 *The complexity of a term is defined as follows.*

1. $cxty(x) = 1$,
2. $cxty(\langle P_1, P_2\rangle) = cxty(P_1) + cxty(P_2)$,
3. $cxty(\sigma_i(P)) = cxty(P) + 1$, *for* $i \in \{1, 2\}$,
4. $cxty(\lambda xP) = cxty(P) + 1$,
5. $cxty((P_1 \star P_2)) = cxty(P_1) + cxty(P_2)$.

Definition 268 *Let $M \in SN$. Then $\eta c(M)$ stands for $\langle \eta(M), cxty(M)\rangle$, where $\eta(M)$ denotes the length of the longest reduction sequence starting from M.*

Lemma 269 *Let us suppose $M \in SN$, $N \in SN$ and $(M \star N) \notin SN$. Then either $M \twoheadrightarrow \lambda xP$, and $P[x := N] \notin SN$ (resp. $N \twoheadrightarrow \lambda yQ$ and $Q[y := M] \notin SN$), or there are P, Q, R such that $M \twoheadrightarrow P$, $N \twoheadrightarrow Q$ and $(M \star N) \twoheadrightarrow (P \star Q) \rightarrow_\pi R \notin SN$ (resp. $(M \star N) \twoheadrightarrow (P \star Q) \rightarrow_{\pi^\perp} R \notin SN$).*

Proof By induction on $\eta c(M) + \eta c(N)$.
Let us consider an infinite reduction sequence of $(M \star N)$. If the first reduction takes place in M or in N, then the induction hypothesis applies. Otherwise, if the first reduction is a β (or $\beta_\perp$) the first case holds, if the first step is a π (or $\pi_\perp$), then the second case is valid. □

Notation 270 1. *A proper term is a term differing from a variable.*

2. *Let Σ_A denote the set of simultaneous substitutions of the form $\sigma = [x_1 := N_1, \ldots, x_k := N_k]$ for some $k \geq 1$, where every N_i has type A.*
   σ is said to be in SN, if, for every $x \in dom(\sigma)$, $\sigma(x) \in SN$ holds.

In the sequel we suppose that if ρ is a simultaneous substitution and $x \in dom(\rho)$, then $\rho(x)$ is proper.

Lemma 271 *Let $M \twoheadrightarrow N$.*

1. *If $N = \lambda xP$, then $M = \lambda xP_1$ with $P_1 \twoheadrightarrow P$.*
2. *If $N = \langle P, Q\rangle$, then $M = \langle P_1, Q_1\rangle$ with $P_1 \twoheadrightarrow P$, $Q_1 \twoheadrightarrow Q$.*
3. *If $N = \sigma(P)$, then $M = \sigma(P_1)$ with $P_1 \twoheadrightarrow P$.*

Proof Straightforward. □

Lemma 272 *Let $M\rho \twoheadrightarrow N$, assume ρ is a simultaneous substitution.*

1. *If $N = \lambda xP$, then either $M = \lambda yP_1$ and $P_1\rho \twoheadrightarrow P$ or $M = x \in dom(\rho)$, $\rho(x) = \lambda yP_1$ and $P_1 \twoheadrightarrow P$.*
2. *If $N = \langle P_1, P_2\rangle$, then either $M = \langle M_1, M_2\rangle$, $M_i\rho \twoheadrightarrow P_i$ or $M = x \in dom(\rho)$, $\rho(x) = \langle M_1, M_2\rangle$ and $M_i \twoheadrightarrow P_i$ $(i \in \{1, 2\})$.*
3. *If $N = \sigma_i(P)$ for $i \in \{1, 2\}$, then either $M = \sigma_i(P_1)$ and $P_1\rho \twoheadrightarrow P$ or $M = x \in dom(\rho)$, $\rho(x) = \sigma_i(P_1)$ and $P_1 \twoheadrightarrow P$.*

Proof We consider only Case 1, the other cases being similar. By Lemma 271 $M\rho \twoheadrightarrow \lambda yP$ involves $M\rho = \lambda yP_1$, for some P_1, such that $P_1 \twoheadrightarrow P$. Then either $M = \lambda yP_2$ and $P_2\rho \twoheadrightarrow P$ or $M = x \in dom(\rho)$, $\rho(x) = \lambda yP_1$ and $P_1 \twoheadrightarrow P$. □

Lemma 273 *If $M \in SN$, then $(M \star x) \in SN$ (resp. $(x \star M) \in SN$).*

Proof Let us suppose $M \in SN$ and $(M \star x) \notin SN$. Applying Lemma 269, there exist an M_1 such that $M \twoheadrightarrow \lambda y M_1$ and $M_1[y := x] \notin SN$. Assume $\eta c(M_1)$ is minimal. Because of being non-SN, M_1 must be an application, but in this case, by Lemma 269, either $M_1 = (\lambda z M_2 \star M_3)$ and $(M_2[y := x])[z := M_3[y := x]] = (M_2[z := M_3])[y := x] \notin SN$, or $M_1 = (\langle M_3, M_4\rangle \star \sigma_1(M_5))$ (resp. $M_1 = (\langle M_3, M_4\rangle \star \sigma_2(M_5))$) and $(M_3 \star M_5)[y := x] \notin SN$ (resp. $(M_4 \star M_5)[y := x] \notin SN$). Each of the cases contradict the minimality of $\eta(M_1)$. □

Definition 274 *The complexity of a type is defined inductively as follows.*

1. $c(A) = 0$, *if A is atomic or negated atomic type,*
2. $c(A) = 0$, *if $A = \bot$,*
3. $c(A_1 \wedge A_2) = c(A_1 \vee A_2) = c(A_1) + c(A_2) + 1$.

Lemma 275 *For every type $A \neq \bot$:*

$$c(A) = c(A^\bot).$$

Proof By induction on $cxty(A)$. □

Lemma 276 *Let M, P be proper terms, assume $type(P) = A$. Let $\sigma = [x_1 := N_1, \ldots, x_k := N_k] \in \Sigma_A$ and y is a variable not occurring in N_i $(1 \leq i \leq k)$. Suppose $M\sigma \in SN$ and $M\sigma[y := P] \notin SN$.*
Assume, for every $U, V \in SN$ such that $c(type(V)) < c(type(P)) = n$, we have $U[x := V] \in SN$.
Then there exists a proper $M' \prec M$ and a substitution $\sigma' \in SN$ such that $M'\sigma' \in SN$ and $(P \star M'\sigma') \notin SN$ (or $(M'\sigma' \star P) \notin SN$).

Proof Let us denote by C the conclusion of the lemma, and by H the assumption that for every $U, V \in SN$ such that $c(type(V)) < c(A) = n$ we have $U[x := V] \in SN$. Let $\mathcal{U} := \{U \mid U \preceq M$ and U is proper$\}$ and $\mathcal{V} := \{V \mid V \preceq N_i$ for some $1 \leq i \leq k$ and V is proper$\}$. Define inductively the following sets of substitutions:

$\rho \in \Sigma$ iff $\rho = \emptyset$ or $\rho = \rho' + [x := V\tau]$, where $\rho' \in \Sigma$, $\tau \in \Theta$, $V \in \mathcal{V}$ and x is of type A,

$\tau \in \Theta$ iff $\tau = \emptyset$ or $\tau = \tau' + [z := U\rho]$, where $\tau' \in \Theta$, $\rho \in \Sigma$, $U \in \mathcal{U}$ and z is of type $A^\bot$.

We are going to prove something more general.
(1) If $U \in \mathcal{U}$ and $\rho \in \Sigma$, $U\rho \in SN$ and $U\rho[y := P] \notin SN$, then C holds.
(2) If $V \in \mathcal{V}$ and $\tau \in \Theta$, $V\tau \in SN$ and $V\tau[y := P] \notin SN$, then C holds.
The statements are proved simultaneously by induction on $\eta c(U\rho)$ and $\eta c(V\tau)$. We are going to deal with case (1) only, case (2) can be treated in a similar way.
The only nontrivial case is $U = (U_1 \star U_2)$. Assume $U\rho \in SN$ and $U\rho[y := P] \notin SN$. We may suppose $U_i\rho[y := P] \in SN$ $(i \in \{1, 2\})$, otherwise the statement follows.

A, $U_1\rho[y := P] \twoheadrightarrow \lambda u U'$, $U'[u = U_2\rho[y := P]] \notin SN$.

1. If U_1 is proper, then by Lemma 272 there exists U_1' such that $U_1 = \lambda u U_1'$ and $U_1'\rho[y := P] \twoheadrightarrow U'$. Then $U_1'\rho[y := P][u = U_2\rho[y := P]] = (U_1'[u = U_2])\rho[y := P] \notin SN$, $U_1'[u := U_2] \in \mathcal{U}$ and $\eta((U_1'[u := U_2])\rho) < \eta(U\rho)$. The result follows by the induction hypothesis.
2. $U_1 \in Var$:
   (a) If $U_1 = x \in dom(\rho)$:

- $\rho(x) = V'\tau \twoheadrightarrow \lambda z V$, for some $V' \in \mathcal{V}$, $\tau \in \Theta$. Since V' is proper, by Lemma 272 there is a V_1 such that $\lambda z V_1 = V'$ and $V_1\tau \twoheadrightarrow V$. Then V_1 is proper, since $type(V_1) = \bot$. Thus $V_1 \in \mathcal{V}$, $(V_1\tau[z := U_2\rho])[y := P] \notin SN$, so the induction hypothesis applies, since $type(z) = A^\bot$ and $\eta c(V_1\tau[z := U_2\rho]) < \eta c(U\rho)$.
- $\rho(x) = V\tau$, $U_2\rho[y := P] \twoheadrightarrow \lambda z U'$, U_2 is proper and $U'[z := V\tau[y := P]] \notin SN$. Then by Lemma 272 there is an U_3 such that $U_2 = \lambda z U_3$ and $U_3\rho[y := P] \twoheadrightarrow U'$. By the above argument U_3 is proper, thus, since $type(z) = A$ and $\eta c(U_3\rho[z := V\tau]) < \eta c(U\rho)$, the induction hypothesis yields C.
- $U_1 = x \in dom(\rho)$ and $U_2 \in Var$: In this case, since $U_1\rho[y := P] \in SN$, by Lemma 273 $U_2 \in Var/(dom(\rho) \cup \{y\})$ is impossible. If $U_2 \in dom(\rho)$ or $U_2 = y$, then this would involve $type(U_1) = type(U_2)$, which is impossible again.

(b) $U_1 = y$: Then $(P \star U_2\rho[y := P]) \notin SN$. Since, for every $x \in dom(\rho)$, $type(x) = type(P) = A$, $U_2 \notin dom(\rho)$, and for the same reason $U_2 \neq y$. Moreover, by Lemma 273, $U_2 \in Var/(dom(\rho) \cup \{y\})$ together with $(P \star U_2\rho[y := P]) \notin SN$ would involve $P \notin SN$, thus U_2 is proper. This gives the conclusion of the lemma.

(c) $U_1 = z \notin dom(\rho) \cup \{y\}$: By Lemma 273, $U_2\rho[y := P] \in SN$ implies $(z \star U_2\rho[y := P]) \in SN$, which would mean $U\rho[y := P] \in SN$, contradicting our hypothesis.

B, $U_1\rho[y := P] \twoheadrightarrow \langle U', U''\rangle$ and either $U_2\rho[y := P] \twoheadrightarrow \sigma_1(W)$ and $(U' \star W) \notin SN$ or $U_2\rho[y := P] \twoheadrightarrow \sigma_2(W)$ and $(U'' \star W) \notin SN$. Assume the former is valid.

1. If U_1, U_2 are proper, then, by Lemma 272, $U_1 = \langle U_3, U_4\rangle$, $U_2 = \sigma_1(U_5)$, $U_3\rho[y := P] \twoheadrightarrow U'$ and $U_5\rho[y := P] \twoheadrightarrow W$. The induction hypothesis applies with $(U_3 \star U_5)$.
2. $U_1 \in Var$:
   (a) $U_1 = x \in dom(\rho)$: Then there exist $V \in \mathcal{V}$ and $\tau \in \Theta$ such that $\rho(x) = V\tau$. $V\tau \twoheadrightarrow \langle U', U''\rangle$, so, by Lemma 272, $V = \langle V_1, V_2\rangle$, $V_1\tau \twoheadrightarrow U'$, $V_2\tau \twoheadrightarrow U''$. By hypothesis $(V_1\tau \star W) = (z \star W[z := V_1\tau]) \notin SN$, where $U_2\rho[y := P] \twoheadrightarrow \sigma_1(W) \in SN$. But then $(z \star W) \in SN$, and $c(type(V_1\tau)) < c(type(x))$, contradicting H.
   (b) $U_1 = y$: $(P \star U_2\rho[y := P]) \notin SN$. Since, for every $x \in dom(\rho)$, $type(x) = type(P)$, we have $U_2 \notin dom(\rho)$. For the same reason $U_2 \neq y$. Because of Lemma 273 and $P \in SN$, $U_2 \notin Var/(dom(\rho) \cup \{y\})$, so $U_2 \in \mathcal{U}$, which yields the conclusion of the lemma again.
3. The proof for $U_2 \in Var$ is similar to part B,2., by which the proof of the lemma is completed.

□

Lemma 277 *If $M, N \in SN$, then $M[x := N] \in SN$.*

Proof We are going to prove a bit more general statement. Suppose $M, N_i \in SN$ are proper, $type(N_i) = A$ $(1 \leq i \leq k)$. Let $\tau_i \in \Sigma_{A^\bot}$ are such that $\tau_i \in SN$ $(1 \leq i \leq k)$ and let $\rho = [x_1 := N_1\tau_1, \ldots, x_k := N_k\tau_k]$. Then we have $M\rho \in SN$.

The proof is by induction on $(c(A), \eta(M), cxty(M), \Sigma\, \eta(N_i), \Sigma\, cxty(N_i))$ where, in $\Sigma\, \eta(N_i)$ and $\Sigma\, cxty(N_i)$, we count each occurrence of the substituted variable. For example if $k = 1$ and x_1 has n occurrences, then $\Sigma\, \eta(N_i) = n \cdot \eta(N_1)$.

The only nontrivial case is when $M = (M_1 \star M_2)$ and $M\rho \notin SN$. By the induction hypothesis $M_i\rho \in SN$ $(i \in \{1, 2\})$.

A, $M_1\rho \twoheadrightarrow \lambda z M'$, $M_2\rho \twoheadrightarrow M''$ and $M'[z := M''] \notin SN$:

1. M_1 is proper, then there is an M_3 such that $M_1 = \lambda z M_3$ and $M_3\rho \twoheadrightarrow M'$. In this case $(M_3[z := M_2])\rho \notin SN$ and since $\eta(M_3[z := M_2]) < \eta(M)$, the induction hypothesis gives the result.
2. $M_1 \in Var$: Then $M_1 = x \in dom(\rho)$, $\rho(x) = N_j\tau_j \twoheadrightarrow \lambda z M'$ for some $(1 \leq j \leq k)$. Since N_j is proper, there is an N' such that $N_j = \lambda z N'$, $N'\tau_j \twoheadrightarrow M'$. Then $N'\tau_j[z := M_2\rho] \notin SN$ and $type(z) = type(N_j)^\perp = type(\tau_j)$, so, by the previous lemma, we have an $N'' \prec N'$ and a τ' such that $(N''\tau' \star M_2\rho) \notin SN$.
   Now we have $(N''\tau'\star M_2\rho) = (y\star M_2\rho)[y := N''\tau']$, $type(N'') = type(\tau')^\perp = A$ and $\eta c(N'') < \eta c(N_j)$, which contradicts the induction hypothesis.

B, $M_1\rho \twoheadrightarrow \langle M', M''\rangle$ and either $M_2 \twoheadrightarrow \sigma_1(M''')$, $(M' \star M''') \notin SN$ or $M_2 \twoheadrightarrow \sigma_2(M''')$, $(M'' \star M''') \notin SN$. Suppose the former.

1. M_1, M_2 are proper, then there are M_3, M_4, M_5 such that $M_1 = \langle M_3, M_4\rangle$, $M_2 = \sigma_1(M_5)$ and $M_3\rho \twoheadrightarrow M'$, $M_4\rho \twoheadrightarrow M''$, $M_5\rho \twoheadrightarrow M'''$. We have $(M_3 \star M_5)\rho \notin SN$ and $\eta((M_3 \star M_5)) < \eta(M)$, a contradiction.
2. $M_1 = x \in dom(\rho)$: $\rho(x) = N_j\tau_j \twoheadrightarrow \langle M', M''\rangle$, N_j is proper. Then by Lemma 272 $N_j = \langle U, V\rangle$, $U\tau_j \twoheadrightarrow M'$, $V\tau_j \twoheadrightarrow M''$. Now $(U\tau_j \star M''') = (y\star M''')[y := U\tau_j] \notin SN$, but $c(type(U)) < c(type(N_j))$, a contradiction again.
3. $M_2 \in Var$: This is similar to the previous case. By the same argument as in part A,2.,(a) of the proof of the previous lemma M_1 and M_2 cannot be both variables. This completes the proof of the lemma.

□

Theorem 278 *The $\lambda_{\beta\beta_\perp\pi\pi_\perp}$-calculus is strongly normalizing.*

Proof It is enough to show that, for every term, M, $N \in SN$ implies $(M \star N) \in SN$. Supposing $M, N \in SN$, Lemma 273 gives $(M \star x) \in SN$, which yields, by the previous lemma, $(M \star N) = (M \star x)[x := N] \in SN$. □

3.2 The cases of the η- and $\eta^\perp$-reductions

In this section we prove that the η- and $\eta^\perp$-reductions can be postponed w.r.t. $\beta, \beta^\perp, \pi, \pi^\perp$. Let $\rightarrow_{\beta\pi}$ stand for the union of $\rightarrow_\beta, \rightarrow_{\beta_\perp}, \rightarrow_\pi, \rightarrow_{\pi_\perp}$ and let $M \rightarrow_e N$ denote the fact that $M \rightarrow_\eta N$ or $M \rightarrow_{\eta^\perp} N$. Furthermore, we denote by β_0 (resp. by $\beta_0^\perp$) the β-reduction $(\lambda x M \star N) \rightarrow_\beta M[x := N]$ (resp. the $\beta^\perp$-reduction $(N \star \lambda x M) \rightarrow_{\beta_\perp} M[x := N]$), where x occurs at most once in M. The reasoning applied in this section is analogous to the one already used in Section 1.1 of Chapter 1.

Lemma 279 *If $U \rightarrow_e V \rightarrow_\beta W$, then there is a V' such that $U \rightarrow_\beta V' \twoheadrightarrow_e W$ or $U \rightarrow_{\beta_0} V' \rightarrow_\beta W$.*

Proof It is sufficient to outline the proof for the case when $\rightarrow_e$ stands for an η-reduction. The proof is by induction on $cxty(U)$. The only interesting case is $U = (U_1 \star U_2)$. We only deal with the nontrivial subcases.

1. $U_1 = \lambda x(U_3 \star x)$, with $x \notin Fv(U_3)$, and $V = (U_3 \star U_2) \rightarrow_\beta U_4[y := U_2] = W$, where $U_3 = \lambda y U_4$. In this case $U = (\lambda x(U_3 \star x) \star U_2) \rightarrow_{\beta_0} (U_3 \star U_2) \rightarrow_\beta U_4[y := U_2] = W$, so $\rightarrow_\eta\rightarrow_\beta$ turns into $\rightarrow_{\beta_0}\rightarrow_\beta$.

2. $U_1 = \lambda x U_3$, $U_3 \to_\eta U_4$ and $V = (\lambda x U_4 \star U_2) \to_\beta U_4[x := U_2] = W$. Then $U \to_\beta V' = U_3[x := U_2] \to_\eta U_4[x := U_2] = W$.

3. $U_1 = \lambda x U_3$, $U_2 \to_\eta U_4$ and $V = (\lambda x U_3 \star U_4) \to_\beta U_3[x := U_4] = W$. Then $U \to_\beta V' = U_3[x := U_2] \twoheadrightarrow_\eta U_3[x := U_4]$.

□

Lemma 280 *If $U \to_e V \to_{\beta_0} W$, then $U \to_{\beta_0} W$ or there is a V' such that $U \to_{\beta_0} V' \to_e W$ or $U \to_{\beta_0} V' \to_{\beta_0} W$.*

Proof The proof is completely similar to that of Lemma 279. □

Lemma 281 *If $U \twoheadrightarrow_e V \to_{\beta_0} W$, then $U {\to_{\beta_0}}^+ V' \twoheadrightarrow_e W$ for some V', and $lg(U {\to_{\beta_0}}^+ V' \twoheadrightarrow_e W) \leq lg(U \twoheadrightarrow_e V \to_{\beta_0} W)$.*

Proof By induction on $lg(U {\to_e}^* V \to_{\beta_0} W)$, using Lemma 280. □

Lemma 282 *If $U \twoheadrightarrow_e V \to_\beta W$, then $U {\to_\beta}^+ V' \twoheadrightarrow_e W$ for some V'.*

Proof By induction on $lg(U \twoheadrightarrow_e V \to_\beta W)$. Use Lemmas 279, 281. □

Analogous statements can also be formulated for $\to_{\beta_\perp}$. So we have on the pattern of Lemma 282:

Lemma 283 *If $U \twoheadrightarrow_e V \to_{\beta_\perp} W$, then $U {\to_{\beta_\perp}}^+ V' \twoheadrightarrow_e W$ for some V'.*

Proof Similar to that of the previous lemma. □

Lemma 284 *If $U \to_e V \to_\pi W$ (resp. $U \to_e V \to_{\pi_\perp} W$), then there is a V' such that $U \to_\pi V' \to_e W$ (resp. $U \to_{\pi_\perp} V' \to_e W$).*

Proof Obvious. □

Lemma 285 *If $U \twoheadrightarrow_e V \to_{\beta\pi} W$, then there exists a V' such that $U \to^+_{\beta\pi} V' \twoheadrightarrow_e W$.*

Proof By Lemmas 282, 283 and 284. □

Lemma 286 *The η- and the $\eta_\perp$-reductions are strongly normalizing.*

Proof Implementing an η- or an $\eta_\perp$-reduction on M reduces the complexity of M. □

Corollary 287 *Let $\lambda_{\beta\pi\eta}$ denote the calculus obtained from $\lambda_{\beta\beta_\perp\pi\pi_\perp}$ by adding to it the η- and $\eta^\perp$-reductions. Then, $\lambda_{\beta\pi\eta}$ is strongly normalizing.*

Proof Let $\to_{\beta\pi\eta}$ denote the union of $\{\to_\beta, \to_{\beta^\perp}, \to_\pi, \to_{\pi^\perp}, \to_\eta, \to_{\eta^\perp}\}$. Let M be a term, we prove by induction on $\eta_{\beta\pi}(M)$ that $M \in SN$. Assume σ is an infinite $\beta\pi\eta$-reduction sequence starting from M. If σ begins with a $\to_{\beta\pi}$ or contains only $\to_e$-reductions, then we are done. Otherwise there is an initial subsequent $M \twoheadrightarrow^{\sigma'}_e M' \to_{\beta\pi} N$ with $|\sigma'| \geq 1$. By Lemma 285, we have $M \to^+_{\beta\pi} M'' \twoheadrightarrow_e N$. Thus, we can apply the induction hypothesis to M''. □

3.3 The λ^{Sym}_{Prop}-calculus is strongly normalizing

What has remained is to augment the calculus treated so far with the rule $Triv$. Let $\to_{\beta\pi\eta}$ denote one of the reductions in the set $\{\beta, \beta^\perp, \pi, \pi^\perp, \eta, \eta^\perp\}$. For the strong normalization it is enough to show that $\to_T riv$ can be postponed w.r.t. $\to_{\beta\pi\eta}$.

Lemma 288 *If $U \to_{Triv} V \to_{\beta\pi\eta} W$, then there exists a V' such that $U \to_{\beta\pi\eta} V' \to_{Triv} W$.*

Proof From $U = E[V] \to_{Triv} V \to_{\beta\pi\eta} W$ it follows that $U = E[V] \to_{\beta\pi\eta} E[W] \to_{Triv} W$, since $W : \perp$ necessarily holds. □

Lemma 289 *If $U \twoheadrightarrow_{Triv} V \rightarrow_{\beta\pi\eta} W$, then there exists a V' such that $U \rightarrow^{+}_{\beta\pi\eta} V' \twoheadrightarrow_{Triv} W$.*

Proof Follows from the previous lemma. □

Lemma 290 *The reduction $\rightarrow_{Triv}$ is strongly normalizing.*

Proof Implementing $\rightarrow_{Triv}$ on M reduces the complexity of M. □

Corollary 291 *The $\lambda_{\beta\pi\eta}$-calculus extended with the rule $Triv$ is strongly normalizing.*

Proof By Corollary 287 and Lemmas 289 and 290. □

By this the proof of Theorem 266 is completed.

Chapter 4

Translations between the λ^{Sym}_{Prop}-calculus and the $\overline{\lambda}\mu\tilde{\mu}^*$-calculus

After realizing that the Curry-Howard isomorphism can also be extended to classical logic, several logical calculi have been invented to answer this purpose. It appears to be a natural question how these logical calculi relate to each other.
In respect of the relation between the $\lambda\mu$- and the $\overline{\lambda}\mu\tilde{\mu}$-calculus the question was answered by Curien and Herbelin (cf. [11]) and by Rocheteau ([55]). Curien and Herbelin defined a translation both for the call-by-value and the call-by-name part of the $\lambda\mu$-calculus into the $\overline{\lambda}\mu\tilde{\mu}$-calculus. Rocheteau finished this work by defining simulations between the two calculi in both directions. In the first part of this chapter we define the $\overline{\lambda}\mu\tilde{\mu}^*$-calculus, which is the $\overline{\lambda}\mu\tilde{\mu}$-calculus extended with negation, and we describe translations between the $\overline{\lambda}\mu\tilde{\mu}^*$-calculus and the λ^{Sym}_{Prop}-calculus. In the remaining part we present a proof of the strong normalization of the $\overline{\lambda}\mu\tilde{\mu}^*$-calculus based on the method applied in David and Nour [15] for proving the strong normalization of the $\overline{\lambda}\mu\tilde{\mu}$-calculus.

4.1 Relating the λ^{Sym}_{Prop}-calculus to the $\overline{\lambda}\mu\tilde{\mu}^*$-calculus

4.1.1 The $\overline{\lambda}\mu\tilde{\mu}^*$-calculus

The typing rules in the $\overline{\lambda}\mu\tilde{\mu}$-calculus do not handle negation. This means, that for a full treatment of propositional logic we have to introduce rules concerning it. Since commands, which could have been candidates for objects of type $\bot$, are distinctly separated from terms, it seems to be the most convenient way to define negation by new term- and type-forming operators. In accordance with this, the terms and commands of the $\overline{\lambda}\mu\tilde{\mu}^*$-calculus are defined as follows:

Definition 292 *The commands and terms of the $\overline{\lambda}\mu\tilde{\mu}^*$-calculus.*

$$\begin{array}{lcccccccc} p & ::= & \lfloor t, e \rfloor & & & & & & \\ t & ::= & x & | & \lambda x\, t & | & \mu a\, p & | & \overline{e} \\ e & ::= & a & | & (t.e) & | & \tilde{\mu} x\, p & | & \widetilde{t} \end{array}$$

As an abuse of terminology, in the sequel when speaking about the syntactic elements of the $\overline{\lambda}\mu\tilde{\mu}^*$-calculus, we may not distinguish terms from commands and we regard commands as terms also. The type inference rules are the same as in the $\overline{\lambda}\mu\tilde{\mu}$-calculus with two extra rules added for the types of the complemented terms. Below, Γ (resp. $\triangle$) is a set of declarations of the form $x : A$ (resp. $a : A$) where x (resp. a) is a t-variable (resp. an e-variable) and A is a type such that an expression of the form $x : A$ (resp. $a : A$) occurs at most once in Γ (resp. in $\triangle$). Moreover, we introduce an equation between types to ensure that our negation is an involutive one.

Definition 293 *The type inference rules related to the negation.*

$$\frac{\Gamma \mid e : A \vdash \triangle}{\Gamma \vdash \overline{e} : A^{\perp} \mid \triangle} \qquad \frac{\Gamma \vdash t : A \mid \triangle}{\Gamma \mid \widetilde{t} : A^{\perp} \vdash \triangle}$$

Besides the reduction rules already present in $\overline{\lambda}\mu\tilde{\mu}$ we endow the calculus with some more new rules to handle the larger set of terms. In what follows cl stands for the name: complementer rule.

Definition 294 *The reduction rules concerning negation.*

- $\overline{\widetilde{t}} \rightarrow_{cl_{1,l}} t$,
- $\widetilde{\overline{e}} \rightarrow_{cl_{1,r}} e$,
- $\lfloor \overline{e}, \widetilde{t} \rfloor \rightarrow_{cl_2} \lfloor t, e \rfloor$.

We shall denote the $cl_{1,l}$- and $cl_{1,r}$-rules with a common notation as the cl_1-rules.

The new rules correspond to straightforward transformations of proofs. Figure 4.1 gives a description of them. In order to have the subject reduction in the new calculus, we define an equational theory on types by adding to the usual equations which ensure that the equality is an equivalence relation the set of equations

$$(A^{\perp})^{\perp} = A.$$

With this, the negation in the $\overline{\lambda}\mu\tilde{\mu}^*$-calculus becomes an involutive one.

4.1.2 A translation of the $\overline{\lambda}\mu\tilde{\mu}^*$-calculus into the λ^{Sym}_{Prop}-calculus

Preparatory to the presentation of the translation let us introduce some definitions and notation below.

Notation 295
1. *In what follows, let $\vdash_1, \vdash_2$ stand for deducibility in $\overline{\lambda}\mu\tilde{\mu}^*$ and in λ^{Sym}_{Prop}, respectively.*
2. *Likewise $\rightarrow_1$ and $\rightarrow_2$ will be applied to denote an arbitrary reduction step in $\overline{\lambda}\mu\tilde{\mu}^*$ and in λ^{Sym}_{Prop}, respectively. As usual $\rightarrow_i^*$ will mean the reflexive, transitive and and $\rightarrow_i^+$ the transitive closure of $\rightarrow_i$ $(i \in \{1, 2\})$.*
3. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ denote the set of terms of the $\overline{\lambda}\mu\tilde{\mu}^*$- and the λ^{Sym}_{Prop}-calculi, respectively.*

4. *Let us consider the* λ^{Sym}_{Prop}*-calculus. We write* $\pi_i(y) = \lambda z(y \star \sigma_i(z))$ $(i \in \{1,2\})$*. Then, we can observe that*

$$\dfrac{\dfrac{y : A \wedge B, z : A^{\perp} \vdash y : A \wedge B \qquad \dfrac{y : A \wedge B, z : A^{\perp} \vdash z : A^{\perp}}{y : A \wedge B, z : A^{\perp} \vdash \sigma_1(z) : A^{\perp} \vee B^{\perp}}}{y : A \wedge B, z : A^{\perp} \vdash (y \star \sigma_1(z)) : \perp}}{y : A \wedge B \vdash \lambda z(y \star \sigma_1(z)) : A}$$

is a deduction of $\pi_1(y) : A$ *from* $y : A \wedge B$ *in* λ^{Sym}_{Prop}*. Similarly, we can establish a deduction of* $\pi_2(y) : B$ *from* $y : A \wedge B$*.*

Definition 296 *We define a translation* $.^{\mathfrak{e}} : \mathcal{T}_1 \to \mathcal{T}_2$ *as follows.*

$$p^{\mathfrak{e}} = (u^{\mathfrak{e}} \star v^{\mathfrak{e}}) \quad \textit{if} \quad p = \lfloor v, u \rfloor.$$

$$t^{\mathfrak{e}} = \begin{cases} x & \textit{if} \quad t = x, \\ \lambda y(\lambda x(\pi_2(y) \star u^{\mathfrak{e}}) \star \pi_1(y)) & \textit{if} \quad t = \lambda x u, \\ \lambda x(e^{\mathfrak{e}} \star t^{\mathfrak{e}}) & \textit{if} \quad t = \tilde{\mu} x \lfloor t, e \rfloor, \\ u^{\mathfrak{e}} & \textit{if} \quad t = \widetilde{u}. \end{cases}$$

$$e^{\mathfrak{e}} = \begin{cases} a & \textit{if} \quad e = a, \\ \langle t^{\mathfrak{e}}, h^{\mathfrak{e}} \rangle & \textit{if} \quad e = t.h, \\ \lambda a(e^{\mathfrak{e}} \star t^{\mathfrak{e}}) & \textit{if} \quad e = \mu a \lfloor t, e \rfloor, \\ h^{\mathfrak{e}} & \textit{if} \quad e = \overline{h}. \end{cases}$$

The translation $.^{\mathfrak{e}} : \mathcal{T}_1 \to \mathcal{T}_2$ extends to the types also.

Definition 297 1. $A^{\mathfrak{e}} = A$*, where* A *is an atomic type,*

2. $(A^{\perp})^{\mathfrak{e}} = (A^{\mathfrak{e}})^{\perp}$*,*

3. $(A \to B)^{\mathfrak{e}} = (A^{\mathfrak{e}})^{\perp} \vee B^{\mathfrak{e}}$*.*

Lemma 298 *The translation* $.^{\mathfrak{e}}$ *maps equal types to equal types.*

Proof It is enough to check $(A^{\perp\perp})^{\mathfrak{e}} = A^{\mathfrak{e}}$, which obviously holds. □

Notation 299 *Let* Γ*,* $\triangle$ *be contexts of the* $\overline{\lambda}\mu\tilde{\mu}^*$*-calculus. Then* $\Gamma^{\mathfrak{e}} = \{x : A^{\mathfrak{e}} \mid x : A \in \Gamma\}$ *and* $(\triangle^{\mathfrak{e}})^{\perp} = \{a : (A^{\mathfrak{e}})^{\perp} \mid a : A \in \triangle\}$*.*

Lemma 300 1. $\Gamma \vdash_1 t : A \mid \triangle \Longrightarrow \Gamma^{\mathfrak{e}}, (\triangle^{\mathfrak{e}})^{\perp} \vdash_2 t^{\mathfrak{e}} : A^{\mathfrak{e}}$*,*

2. $\Gamma \mid t : A \vdash_1 \triangle \Longrightarrow \Gamma^{\mathfrak{e}}, (\triangle^{\mathfrak{e}})^{\perp} \vdash_2 t^{\mathfrak{e}} : (A^{\mathfrak{e}})^{\perp}$*,*

3. $c : (\Gamma \vdash_1 \triangle) \Longrightarrow \Gamma^{\mathfrak{e}}, (\triangle^{\mathfrak{e}})^{\perp} \vdash_2 c^{\mathfrak{e}} : \perp$*.*

Proof The above statements are proved simultaneously according to the length of the $\overline{\lambda}\mu\tilde{\mu}^*$-deduction. Let us examine some of the more interesting cases.

1. Suppose

$$\dfrac{\Gamma, x : A \vdash_1 u : B \mid \triangle}{\Gamma \vdash_1 \lambda x u : A \to B \mid \triangle}$$

Then we have by the induction hypothesis and Notation 295

$$\Gamma^{\mathfrak{e}}, (\triangle^{\mathfrak{e}})^{\perp}, x : A^{\mathfrak{e}}, y : A^{\mathfrak{e}} \wedge (B^{\mathfrak{e}})^{\perp} \vdash_2 u^{\mathfrak{e}} : B^{\mathfrak{e}},$$

$$\Gamma^{\mathfrak{e}}, (\triangle^{\mathfrak{e}})^{\perp}, y : A^{\mathfrak{e}} \wedge (B^{\mathfrak{e}})^{\perp} \vdash_2 \pi_1(y) : A^{\mathfrak{e}},$$

and likewise

$$\Gamma^{\mathfrak{e}}, (\triangle^{\mathfrak{e}})^{\perp}, y : A^{\mathfrak{e}} \wedge (B^{\mathfrak{e}})^{\perp} \vdash_2 \pi_2(y) : (B^{\mathfrak{e}})^{\perp}.$$

Thus we can conclude

$$\frac{\dfrac{\Gamma^{\mathfrak{e}}, (\triangle^{\mathfrak{e}})^{\perp}, x : A^{\mathfrak{e}}, y : A^{\mathfrak{e}} \wedge (B^{\mathfrak{e}})^{\perp} \vdash_2 u^{\mathfrak{e}} : B^{\mathfrak{e}} \qquad \Gamma^{\mathfrak{e}}, (\triangle^{\mathfrak{e}})^{\perp}, x : A^{\mathfrak{e}}, y : A^{\mathfrak{e}} \wedge (B^{\mathfrak{e}})^{\perp} \vdash_2 \pi_2(y) : (B^{\mathfrak{e}})^{\perp}}{\Gamma^{\mathfrak{e}}, (\triangle^{\mathfrak{e}})^{\perp}, x : A^{\mathfrak{e}}, y : A^{\mathfrak{e}} \wedge (B^{\mathfrak{e}})^{\perp} \vdash_2 (\pi_2(y) \star u^{\mathfrak{e}}) : \perp}}{\Gamma^{\mathfrak{e}}, (\triangle^{\mathfrak{e}})^{\perp}, y : A^{\mathfrak{e}} \wedge (B^{\mathfrak{e}})^{\perp} \vdash_2 \lambda x(\pi_2(y) \star u^{\mathfrak{e}}) : (A^{\mathfrak{e}})^{\perp}}$$

From which

$$\frac{\Gamma^{\mathfrak{e}}, (\triangle^{\mathfrak{e}})^{\perp}, y : A^{\mathfrak{e}} \wedge (B^{\mathfrak{e}})^{\perp} \vdash_2 \lambda x(\pi_2(y) \star u^{\mathfrak{e}}) : (A^{\mathfrak{e}})^{\perp} \qquad \Gamma^{\mathfrak{e}}, (\triangle^{\mathfrak{e}})^{\perp}, y : A^{\mathfrak{e}} \wedge (B^{\mathfrak{e}})^{\perp} \vdash_2 \pi_1(y) : A^{\mathfrak{e}}}{\Gamma^{\mathfrak{e}}, (\triangle^{\mathfrak{e}})^{\perp} \vdash_2 \lambda y(\lambda x(\pi_2(y) \star u^{\mathfrak{e}}) \star \pi_1(y)) : (A^{\mathfrak{e}})^{\perp} \vee B^{\mathfrak{e}}}$$

follows.

2. Assume now

$$\frac{\Gamma \vdash_1 t : A \mid \triangle \qquad \Gamma \mid e : B \vdash_1 \triangle}{\Gamma \mid t.e : a \to B \vdash_1 \triangle}$$

Then we have

$$\frac{\Gamma^{\mathfrak{e}}, (\triangle^{\mathfrak{e}})^{\perp} \vdash_2 t^{\mathfrak{e}} : A^{\mathfrak{e}} \qquad \Gamma^{\mathfrak{e}}, (\triangle^{\mathfrak{e}})^{\perp} \vdash_2 e^{\mathfrak{e}} : (B^{\mathfrak{e}})^{\perp}}{\Gamma^{\mathfrak{e}}, (\triangle^{\mathfrak{e}})^{\perp} \vdash_2 \langle t^{\mathfrak{e}}, e^{\mathfrak{e}} \rangle : A^{\mathfrak{e}} \wedge (B^{\mathfrak{e}})^{\perp}}$$

3. From

$$\frac{\Gamma \vdash_1 t : A \mid \triangle \qquad \Gamma \mid e : A \vdash_1 \triangle}{\lfloor t, e \rfloor : (\Gamma \vdash_1 \triangle)}$$

we obtain

$$\frac{\Gamma^{\mathfrak{e}}, (\triangle^{\mathfrak{e}})^{\perp} \vdash_2 t^{\mathfrak{e}} : A^{\mathfrak{e}} \qquad \Gamma^{\mathfrak{e}}, (\triangle^{\mathfrak{e}})^{\perp} \vdash_2 e^{\mathfrak{e}} : (A^{\mathfrak{e}})^{\perp}}{\Gamma^{\mathfrak{e}}, (\triangle^{\mathfrak{e}})^{\perp} \vdash_2 (e^{\mathfrak{e}} \star t^{\mathfrak{e}}) : \perp}$$

4. Moreover

$$\frac{\lfloor t, e \rfloor : (\Gamma \vdash_1 \triangle)}{\Gamma \vdash_1 \lambda a \lfloor t, e \rfloor : A \mid \triangle \backslash \{a : A\}}$$

yields

$$\frac{\Gamma^{\mathfrak{e}}, (\triangle^{\mathfrak{e}})^{\perp} \vdash_2 (e^{\mathfrak{e}} \star t^{\mathfrak{e}}) : \perp}{\Gamma^{\mathfrak{e}}, (\triangle^{\mathfrak{e}})^{\perp} \backslash \{a : (A^{\mathfrak{e}})^{\perp}\} \vdash_2 \lambda a(e^{\mathfrak{e}} \star t^{\mathfrak{e}}) : A^{\mathfrak{e}}}$$

5. Finally

$$\frac{\Gamma \vdash_1 t : A \mid \triangle}{\Gamma \mid \widetilde{t} : A^{\perp} \vdash_1 \triangle}$$

gives

$$\frac{\Gamma^{\mathfrak{e}}, (\triangle^{\mathfrak{e}})^{\perp} \vdash_2 t^{\mathfrak{e}} : A^{\mathfrak{e}}}{\Gamma^{\mathfrak{e}}, (\triangle^{\mathfrak{e}})^{\perp} \vdash_2 t^{\mathfrak{e}} : A^{\mathfrak{e}}}$$

□

Our next aim is to prove that $\overline{\lambda}\mu\tilde{\mu}^*$ can be simulated by the λ^{Sym}_{Prop}-calculus. To this end we introduce a new notion of equality in the λ^{Sym}_{Prop}-calculus.

Definition 301 *We define a relation $\sim$ on the terms of the λ^{Sym}_{Prop}-calculus.*

1. $x \sim x$,
2. $\lambda x u \sim \lambda x u'$ *if* $u \sim u'$,
3. $\langle p, q\rangle \sim \langle p', q'\rangle$ *if* $p \sim p'$ *and* $q \sim q'$,
4. $\sigma_i(p) \sim \sigma_i(p')$ $(i \in \{1, 2\})$ *if* $p \sim p'$,
5. $(p \star q) \sim (p' \star q')$ *and* $(p \star q) \sim (q' \star p')$ *if* $p \sim p'$ *and* $q \sim q'$.

We say that p and q are equal up to symmetry provided $p \sim q$.

Lemma 302 *The relation $\sim$ is an equivalence relation on the terms of the λ^{Sym}_{Prop}-calculus.*

Proof Obvious. □

Lemma 303 *Let u, u', v and v' be terms of the λ^{Sym}_{Prop}-calculus. Assume $u \sim u'$ and $v \sim v'$. Then we have:*

1. $u[x := v] \sim u'[x := v]$,
2. $u[x := v] \sim u[x := v']$.

Proof Both assertions are proved by induction on $cxty(u)$. □

The relation $\sim$ is interchangeable with the reductions of the λ^{Sym}_{Prop}-calculus, as the following lemma shows.

Lemma 304 *Let p, p', q be λ^{Sym}_{Prop}-terms such that $p \sim p' \to q$. Then there is a q' for which $p \to q' \sim q$.*

Proof We can apply Lemmas 302 and 303. □

We also need the lemma below.

Lemma 305 *Let u be a either a command or a term, and t, e be terms of the $\overline{\lambda}\mu\tilde{\mu}^*$-calculus. Then*

1. $(u[x := t])^{\mathfrak{e}} = u^{\mathfrak{e}}[x := t^{\mathfrak{e}}]$,
2. $(u[a := e])^{\mathfrak{e}} = u^{\mathfrak{e}}[a := e^{\mathfrak{e}}]$.

Proof By induction on u. □

Now we can formulate our assertion about the simulation of $\overline{\lambda}\mu\tilde{\mu}^*$ by λ^{Sym}_{Prop}.

Theorem 306 *1.* $v \to_1 w \implies v^\mathfrak{e} \to_2^+ w^\mathfrak{e}$, *if* $\to_1 \in \{\to_\beta, \to_\mu, \to_{\tilde{\mu}}, \to_{s_l}, \to_{s_r}\}$.

2. $v \to_1 w \implies v^\mathfrak{e} \sim w^\mathfrak{e}$, *if* $\to_1 \in \{\to_{cl_{1,l}}, \to_{cl_{1,r}}, \to_{cl_2}\}$.

Proof

1. Let us only treat the typical cases.

   (a) $v = \lfloor \lambda x u, (t.e) \rfloor \to_\beta \lfloor t, \tilde{\mu} x \lfloor u, e \rfloor \rfloor = w$. Then

   $$\begin{aligned} v^\mathfrak{e} &= (\langle t^\mathfrak{e}, e^\mathfrak{e} \rangle \star \lambda y(\lambda x(\pi_2(y) \star u^\mathfrak{e}) \star \pi_1(y))) \to_{\beta^\perp} \\ &\quad (\lambda x(\pi_2(\langle t^\mathfrak{e}, e^\mathfrak{e} \rangle) \star u^\mathfrak{e}) \star \pi_1(\langle t^\mathfrak{e}, e^\mathfrak{e} \rangle)) \twoheadrightarrow \\ &\quad (\lambda x(e^\mathfrak{e} \star u^\mathfrak{e}) \star t^\mathfrak{e}) = w^\mathfrak{e}. \end{aligned}$$

   (b) $v = \lfloor \mu a p, e \rfloor \to_\mu p[a := e] = w$. Then, by Lemma 305,

   $$v^\mathfrak{e} = (e^\mathfrak{e} \star \lambda a p^\mathfrak{e}) \to_{\beta^\perp} p^\mathfrak{e}[a = e^\mathfrak{e}] = w^\mathfrak{e}.$$

   (c) $v = \mu a \lfloor w, a \rfloor \to_{s_l} w$, $a \notin w$. We obtain

   $$v^\mathfrak{e} = \lambda a(a \star w^\mathfrak{e}) \to_{\eta^\perp} w^\mathfrak{e}.$$

2. (a) $v = \overline{\widetilde{u}} \to_{cl_{1,l}} u = w$. Then

   $$v^\mathfrak{e} = (\overline{\widetilde{u}})^\mathfrak{e} = u^\mathfrak{e} = w^\mathfrak{e}.$$

   (b) $v = \lfloor \overline{v}, \widetilde{u} \rfloor \to_{cl_2} \lfloor u, v \rfloor = w$. We have

   $$v^\mathfrak{e} = \lfloor \overline{v}, \widetilde{u} \rfloor^\mathfrak{e} = (u^\mathfrak{e} \star v^\mathfrak{e}) \sim w^\mathfrak{e}.$$

□

Corollary 307 *If* λ^{Sym}_{Prop} *strongly normalizes, then the same is true for* $\overline{\lambda}\mu\tilde{\mu}^*$.

Proof Let σ be an infinite reduction sequence in $\overline{\lambda}\mu\tilde{\mu}^*$. By Theorem 306 and Lemma 304, σ cannot contain an infinite number of β-, μ-, $\tilde{\mu}$-, s_l- and s_r-reductions. Thus, there would exist an infinite reduction sequence consisting entirely of $cl_{1,l}$-, $cl_{1,r}$- and cl_2-reductions, which is impossible. □

4.1.3 A translation of the λ^{Sym}_{Prop}-calculus into the $\overline{\lambda}\mu\tilde{\mu}^*$-calculus

Now we are going to deal with the converse relation. That is we will present a translation of λ^{Sym}_{Prop} into $\overline{\lambda}\mu\tilde{\mu}^*$ which faithfully reflects the typability relations of one calculus in the other one. Then we prove that our translation is in fact a simulation of λ^{Sym}_{Prop} in $\overline{\lambda}\mu\tilde{\mu}^*$. As in the previous subsection, let $\mathcal{T}_1$ and $\mathcal{T}_2$ denote the set of terms of the $\overline{\lambda}\mu\tilde{\mu}^*$- and the λ^{Sym}_{Prop}-calculi, respectively.

Definition 308 *The translation* $.^\mathfrak{f} : \mathcal{T}_2 \to \mathcal{T}_1$ *is defined as follows.*

$$t^\mathfrak{f} = \begin{cases} x & \text{if } t = x, \\ \lfloor v^\mathfrak{f}, \widetilde{u^\mathfrak{f}} \rfloor & \text{if } t = (u \star v), \\ \overline{\tilde{\mu} x u^\mathfrak{f}} & \text{if } t = \lambda x u, \\ \overline{(u^\mathfrak{f}.\widetilde{v^\mathfrak{f}})} & \text{if } t = \langle u, v \rangle, \\ \lambda a \mu b \lfloor u^\mathfrak{f}, \widetilde{a} \rfloor & \text{if } t = \sigma_1(u) \text{ provided } b \notin Fv(\lfloor u^\mathfrak{f}, \widetilde{a} \rfloor), \\ \lambda a u^\mathfrak{f} & \text{if } t = \sigma_2(u). \end{cases}$$

Definition 309 *The translation extends to the types:*

1. $\alpha^{\mathsf{f}} = \alpha$,
2. $(\alpha^{\perp})^{\mathsf{f}} = \alpha^{\perp}$,
3. $(A \wedge B)^{\mathsf{f}} = (A^{\mathsf{f}} \to (B^{\mathsf{f}})^{\perp})^{\perp}$,
4. $(A \vee B)^{\mathsf{f}} = (A^{\mathsf{f}})^{\perp} \to B^{\mathsf{f}}$.

We remark that $.^{\mathsf{f}}$ maps the terms of the λ^{Sym}_{Prop}-calculus with type $\perp$ to commands of the $\overline{\lambda}\mu\tilde{\mu}^*$-calculus, which have no types.

Lemma 310 *For all types A we have $(A^{\perp})^{\mathsf{f}} = (A^{\mathsf{f}})^{\perp}$.*
Proof By induction on $c(A)$. □

Lemma 311 *The translation $.^{\mathsf{f}}$ maps equal types to equal types.*
Proof By the lemma above. □

As before, let $\vdash_1$, $\vdash_2$ denote deducibility in $\overline{\lambda}\mu\tilde{\mu}^*$ and in λ^{Sym}_{Prop} , respectively. Then we can assert:

Lemma 312 1. $\Gamma \vdash_2 t : A \Longrightarrow \Gamma^{\mathsf{f}} \vdash_1 t^{\mathsf{f}} : A^{\mathsf{f}}$, *if* $A \neq \perp$,

2. $\Gamma \vdash_2 t : \perp \Longrightarrow t^{\mathsf{f}} : (\Gamma^{\mathsf{f}} \vdash_1)$ *otherwise.*

Proof The proof proceeds by a simultaneous induction on the length of the derivation in λ^{Sym}_{Prop}. Let us only examine some of the typical cases of the first assertion.

1. Suppose

$$\frac{\Gamma, x : A \vdash_2 u : \perp}{\Gamma \ \vdash_2 \ \lambda x u : A^{\perp}}$$

Then, applying the induction hypothesis,

$$\frac{\dfrac{u^{\mathsf{f}} : (\Gamma^{\mathsf{f}}, x : A^{\mathsf{f}} \vdash_1)}{\Gamma^{\mathsf{f}} \mid \tilde{\mu} x u^{\mathsf{f}} : A^{\mathsf{f}} \ \vdash_1}}{\Gamma^{\mathsf{f}} \ \vdash_1 \ \overline{\tilde{\mu} x u^{\mathsf{f}}} : (A^{\mathsf{f}})^{\perp}}$$

2. By the induction hypothesis and Lemma 310

$$\frac{\Gamma \vdash_2 u : A^{\perp} \qquad \Gamma \vdash_2 v : A}{\Gamma \vdash_2 (u \star v) : \perp}$$

yields

$$\frac{\dfrac{\Gamma^{\mathsf{f}} \vdash_1 u^{\mathsf{f}} : (A^{\mathsf{f}})^{\perp}}{\Gamma^{\mathsf{f}} \mid \widetilde{u^{\mathsf{f}}} : A^{\mathsf{f}} \vdash_1} \qquad \Gamma^{\mathsf{f}} \vdash_1 v^{\mathsf{f}} : A^{\mathsf{f}}}{\lfloor v^{\mathsf{f}}, \widetilde{u^{\mathsf{f}}} \rfloor : (\Gamma^{\mathsf{f}} \vdash_1)}$$

3. The deduction

$$\frac{\Gamma \vdash_2 u : A}{\Gamma \vdash_2 \sigma_1(u) : A \vee B}$$

gives

$$\dfrac{\dfrac{\dfrac{\Gamma^{\mathsf{f}}, a : (A^{\mathsf{f}})^{\perp} \vdash_1 u^{\mathsf{f}} : A^{\mathsf{f}} \mid b : B^{\mathsf{f}} \qquad \dfrac{\Gamma^{\mathsf{f}}, a : (A^{\mathsf{f}})^{\perp} \vdash_1 a : (A^{\mathsf{f}})^{\perp} \mid b : B^{\mathsf{f}}}{\Gamma^{\mathsf{f}}, a : (A^{\mathsf{f}})^{\perp} \mid \widetilde{a} : A^{\mathsf{f}} \vdash_1 b : B^{\mathsf{f}}}}{\lfloor u^{\mathsf{f}}, \widetilde{a} \rfloor : (\Gamma^{\mathsf{f}}, a : (A^{\mathsf{f}})^{\perp} \vdash_1 b : B^{\mathsf{f}})}}{\Gamma^{\mathsf{f}}, a : (A^{\mathsf{f}})^{\perp} \vdash_1 \mu b \lfloor u^{\mathsf{f}}, \widetilde{a} \rfloor : B^{\mathsf{f}}}}{\Gamma^{\mathsf{f}} \vdash_1 \lambda a \mu b \lfloor u^{\mathsf{f}}, \widetilde{a} \rfloor : (A^{\mathsf{f}})^{\perp} \to B^{\mathsf{f}}}$$

The remaining cases can be proved in a similar way. □

Now we turn to the proof of the simulation of λ^{Sym}_{Prop} in $\overline{\lambda}\mu\tilde{\mu}^*$.

Lemma 313 *Let u, v be λ^{Sym}_{Prop}-terms. Then*

$$(u[x := v])^{\mathsf{f}} = u^{\mathsf{f}}[x := v^{\mathsf{f}}].$$

Proof By induction on $cxty(u)$. □

Theorem 314

$$t \to_2 s \Longrightarrow t^{\mathsf{f}} \to_1^+ s^{\mathsf{f}}$$

provided $\to_1$ and $\to_2$ denote the reducibility relations in $\overline{\lambda}\mu\tilde{\mu}^*$ and in λ^{Sym}_{Prop}, respectively.

Proof Let us prove some of the more interesting cases.

1. $t = (\lambda x u \star v) \to_{\beta} u[x := v] = s$. Then, applying Lemma 313,

$$\begin{array}{rcl} t^{\mathsf{f}} & = & \lfloor v^{\mathsf{f}}, \widetilde{\tilde{\mu} x u^{\mathsf{f}}} \rfloor \to_{cl_1} \\ & & \lfloor v^{\mathsf{f}}, \tilde{\mu} x u^{\mathsf{f}} \rfloor \to_{\tilde{\mu}} \\ & & u^{\mathsf{f}}[x := v^{\mathsf{f}}] = s^{\mathsf{f}}. \end{array}$$

2. $t = (v \star \lambda x u) \to_{\beta_\perp} u[x := v] = s$. In this case

$$\begin{array}{rcl} t^{\mathsf{f}} & = & \lfloor \overline{\tilde{\mu} x u^{\mathsf{f}}}, \widetilde{v^{\mathsf{f}}} \rfloor \to_{cl_2} \\ & & \lfloor v^{\mathsf{f}}, \tilde{\mu} x u^{\mathsf{f}} \rfloor \to_{\tilde{\mu}} \\ & & u^{\mathsf{f}}[x := v^{\mathsf{f}}] = s^{\mathsf{f}}. \end{array}$$

3. $t = (\langle u, v \rangle \star \sigma_1(w)) \to_{\pi} (u \star w) = s$. We have

$$\begin{array}{rcl} t^{\mathsf{f}} & = & \lfloor \lambda z \mu b \lfloor w^{\mathsf{f}}, \widetilde{z} \rfloor, \widetilde{(u^{\mathsf{f}}.v^{\mathsf{f}})} \rfloor \to_{cl_1} \\ & & \lfloor \lambda z \mu b \lfloor w^{\mathsf{f}}, \widetilde{z} \rfloor, (u^{\mathsf{f}}.\widetilde{v^{\mathsf{f}}},) \rfloor \to_{\beta} \\ & & \lfloor u^{\mathsf{f}}, \tilde{\mu} z \lfloor \mu b \lfloor w^{\mathsf{f}}, \widetilde{z} \rfloor, \widetilde{v^{\mathsf{f}}} \rfloor \rfloor \to_{\tilde{\mu}} \\ & & \lfloor \mu b \lfloor w^{\mathsf{f}}, \widetilde{u^{\mathsf{f}}} \rfloor, \widetilde{v^{\mathsf{f}}} \rfloor \to_{\mu} \\ & & \lfloor w^{\mathsf{f}}, \widetilde{u^{\mathsf{f}}} \rfloor = s^{\mathsf{f}}. \end{array}$$

□

Corollary 315 *If the $\overline{\lambda}\mu\tilde{\mu}^*$-calculus is strongly normalizable, then the same is true for the λ^{Sym}_{Prop}-calculus as well.*

Proof By Theorem 314. □

4.1.4 The connection between the two translations

In this section we examine the compositions of the translations $.^{\mathfrak{e}}$ and $.^{\mathfrak{f}}$. In order to establish a smooth connection between a $\overline{\lambda}\mu\tilde{\mu}^*$-term u and its corresponding term $u^{\mathfrak{e}\mathfrak{f}}$, we have to enhance both the $\overline{\lambda}\mu\tilde{\mu}^*$- and the λ^{Sym}_{Prop}-calculi with terms and equations for surjective pairs. Let us deal first with the $\overline{\lambda}\mu\tilde{\mu}^*$-calculus. We denote the calculus arising by the modifications below by $\overline{\lambda}\mu\tilde{\mu}^*_{pair}$.

Definition 316 *The commands and terms of the extended calculus are defined as follows.*

$$\begin{array}{lcl} p & ::= & \lfloor t, e \rfloor \\ t & ::= & x \mid \lambda x\, t \mid \mu a\, p \mid p_1(t) \mid p_2(t) \mid \overline{e} \\ e & ::= & a \mid (t.e) \mid \tilde{\mu} x\, p \mid \widetilde{t} \end{array}$$

Let $\mathcal{T}_1$ denote the set of terms and commands of the $\overline{\lambda}\mu\tilde{\mu}^*_{pair}$-calculus.

Notation 317 *We abbreviate the term $\overline{(u.\widetilde{v})}$ by $\langle u, v\rangle$. Observe that $\langle u, v\rangle$ has type $(A \to B^{\perp})^{\perp}$ provided $u : A$ and $v : B$.*

The typing rules below are added to the calculus, where Γ (resp. $\triangle$) is a set of declarations of the form $x : A$ (resp. $a : A$) where x (resp. a) is a t-variable (resp. an e-variable) and A is a type such that an expression of the form $x : A$ (resp. $a : A$) occurs at most once in Γ (resp. in $\triangle$).

$$\frac{\Gamma \vdash t : (A \to B^{\perp})^{\perp} \mid \triangle}{\Gamma \vdash p_1(t) : A \mid \triangle} \qquad \frac{\Gamma \vdash t : (A \to B^{\perp})^{\perp} \mid \triangle}{\Gamma \vdash p_2(t) : B \mid \triangle}$$

Definition 318 *We introduce equations for the new terms.*

- $p_1(\langle u, v\rangle) =_{p_1} u$,
- $p_2(\langle u, v\rangle) =_{p_2} v$,
- $\langle p_1(t), p_2(t)\rangle) =_p t$.

The union of the equations, compatible with term formation rules, will be denoted by $=_{pair}$.

Now, consider the λ^{Sym}_{Prop}-calculus. The result of the modifications described below is called the $\lambda^{Sym}_{Prop+Pair}$-calculus. calculus The set of terms of the $\lambda^{Sym}_{Prop+Pair}$-calculus are those of λ^{Sym}_{Prop} with two new term forming rules added.

Definition 319 *The terms of the calculus are:*

$$\mathcal{T} := x \mid \lambda x\mathcal{T} \mid (\mathcal{T} \star \mathcal{T}) \mid \langle \mathcal{T}, \mathcal{T}\rangle \mid \sigma_1(\mathcal{T}) \mid \sigma_2(\mathcal{T}) \mid p_1(\mathcal{T}) \mid p_2(\mathcal{T}).$$

We denote the set of terms for the $\lambda^{Sym}_{Prop+Pair}$-calculus by $\mathcal{T}_2$.

The typing rules of the $\lambda^{Sym}_{Prop+Pair}$-calculus are those of λ^{Sym}_{Prop} supplemented by the following two rules. As usual, let Γ denote a context, that is, a possibly empty set of declarations of the form $x : A$.

$$p_1)\ \frac{\Gamma\ \vdash\ u : A \wedge B}{\Gamma\ \vdash\ p_1(u) : A} \qquad p_2)\ \frac{\Gamma\ \vdash\ u : A \wedge B}{\Gamma\ \vdash\ p_2(u) : B}$$

Definition 320 *The new equations of $\lambda^{Sym}_{Prop+Pair}$ are as follows.*

- $p_1(\langle u, v\rangle) =_{p_1} u$,
- $p_2(\langle u, v\rangle) =_{p_2} v$,
- $\langle p_1(u), p_2(u)\rangle =_p u$.

The union of the equations, compatible with term formation rules, will be denoted by $=_{pair}$.

By relating the terms of $\overline{\lambda}\mu\tilde{\mu}^*_{pair}$ of the forms $p_1(u)$, $p_2(u)$ and $\langle u, v\rangle$ to the terms of $\lambda^{Sym}_{Prop+Pair}$ of the forms $p_1(u)$, $p_2(u)$ and $\langle u, v\rangle$, respectively, the transformations $.^{\mathfrak{e}}$ and $.^{\mathfrak{f}}$ straightforwardly extend to the new calculi preserving *pair*-equations. Thus, we have the counterparts of Lemmas 300, 306 and 312 and 314. In the following statements, we understand by $.^{\mathfrak{e}}$ and $.^{\mathfrak{f}}$ the transformations between the extended calculi.

Lemma 321 *Let π_1 and π_2 be defined as in Notation 295. Then we have:*

1. $\pi_1(y)^{\mathfrak{f}} =_{pair} q_1 \twoheadrightarrow p_1(y)$ *for some* q_1,
2. $\pi_2(y)^{\mathfrak{f}} =_{pair} q_2 \twoheadrightarrow p_2(y)$ *for some* q_2.

Proof A straightforward computation yields the result. □

To describe the effect of the composition $.^{\mathfrak{e}\mathfrak{f}} : \mathcal{T}_1 \to \mathcal{T}_1$ we need the following function T on $\mathcal{T}_1$.

Definition 322 *We define a function T assigning a $\overline{\lambda}\mu\tilde{\mu}^*_{pair}$-term to a $\overline{\lambda}\mu\tilde{\mu}^*_{pair}$-term.*

1. $T(x) = x$,
2. $T(\lambda x u) = \overline{\tilde{\mu}y\lfloor u[x := p_1(y)], \widetilde{p_2(y)}\rfloor}$,
3. $T(\tilde{\mu}xp) = \overline{\tilde{\mu}xT(p)}$,
4. $T(\overline{u}) = T(u)$,
5. $T(p_1(u)) = p_1(T(u))$,
6. $T(p_2(u)) = p_2(T(u))$.
7. $T(a) = a$,
8. $T((u.v)) = \langle T(u), T(v)\rangle$,
9. $T(\mu a p) = \overline{\tilde{\mu}aT(p)}$,
10. $T(\widetilde{h}) = T(h)$,
11. $T(\lfloor t, e\rfloor) = \lfloor T(t), \widetilde{T(e)}\rfloor$.

Lemma 323 *Let u be a term of the $\overline{\lambda}\mu\tilde{\mu}^*_{pair}$-calculus, then we have*

$$u^{\mathfrak{e}\mathfrak{f}} =_{pair} u' \twoheadrightarrow T(u)$$

for some term u'. Moreover, if u is a $\overline{\lambda}\mu\tilde{\mu}^*$-term, then $u^{\mathfrak{e}\mathfrak{f}}$ is a $\overline{\lambda}\mu\tilde{\mu}^*$-term also.

Proof Straightforward. In the case of $u = \lambda x v$, we can apply Lemma 321. □

The other direction gives an easier solution. In this case we return to our original calculi $\overline{\lambda}\mu\tilde{\mu}^*$ and λ^{Sym}_{Prop}.

Lemma 324 *Let u be a λ^{Sym}_{Prop}-term. Then $u^{\mathfrak{f}\mathfrak{e}} \twoheadrightarrow u$.*

Proof Obvious. □

4.2 Concluding remarks

We present a proof of the strong normalization of the $\overline{\lambda}\mu\tilde{\mu}^*$-calculus, which is a reformulation of the proof of the strong normalization of the $\overline{\lambda}\mu\tilde{\mu}$-calculus given in David and Nour [15]. The main assertions are the same as in David and Nour [15], only the reasoning leading to the result differs somewhat. In this proof Lemmas 331 and 345 play more emphatic roles, the main lemmas, Lemmas 338 and 354, are deduced from them. To illustrate the essential argument of the proof we begin with the case of the untyped $\mu\tilde{\mu}$-calculus.

4.2.1 The strong normalization of the $\mu\tilde{\mu}$-calculus

Let us consider the set of terms of the $\overline{\lambda}\mu\tilde{\mu}^*$-calculus. In this section let $\rightarrow$ denote a μ- or a $\tilde{\mu}$-reduction. We call the resulting calculus the untyped $\mu\tilde{\mu}$-calculus.

Definition 325 *A term is called proper, if it differs from a variable.*

Notation 326 *Let Σ_l (resp. Σ_r) denote the set of substitutions of the form $[x_1 := t_1, \ldots, x_k = t_k]$ ($[a_1 := e_1, \ldots, a_k = e_k]$, resp.), where, for all $1 \leq i \leq k$, t_i is proper (resp. e_i is proper). We call an element of Σ_l (resp. Σ_r) a left- or l-substitution (resp. right- or r-substitution). Moreover, let A be a type. If we require $type(t_i) = A$ (resp. $type(e_i) = A$) for $1 \leq i \leq k$, then we denote the set of such substitutions by $\Sigma_{A,l}$ (resp. $\Sigma_{A,r}$).*

Lemma 327 *Let t be a left-, e be a right term, assume t and e are proper. Let ρ be a left-substitution (resp. a right-substitution). Then the following assertions are valid.*

1. *If $t\rho \twoheadrightarrow \mu\alpha t'_1$, then $t = \mu\alpha t_1$ for some t_1 such that $t_1\rho \twoheadrightarrow t'_1$.*
2. *If $e\rho \twoheadrightarrow \tilde{\mu}x e'_1$, then $e = \tilde{\mu}x e_1$ for some e_1 such that $e_1\rho \twoheadrightarrow e'_1$.*

Proof Obvious. □

Lemma 328 *Assume t, $e \in SN$, $\lfloor t, e \rfloor \notin SN$. Then we have the following possibilities:*

1. *$t = \mu\alpha t_1$ and $t_1[\alpha := e] \notin SN$,*
2. *$e = \tilde{\mu}x e_1$, $e_1[x := t] \notin SN$.*

Proof By induction on $\eta(t) + \eta(e)$. □

Notation 329 *In what follows, u and v denote arbitrary terms or commands. Moreover, in the sequel we may loosen the convention of the denotations for l- and r-terms, that is, a t may as well denote an r-term and likewise for e.*

Definition 330 *Let ξ be $M_1 \rightarrow^{r_1} M_2 \rightarrow \ldots \rightarrow M_n \rightarrow^{r_n} M_{n+1}$. Then ξ is called zoom-in if, for every $1 \leq i < n$, there is an r_i' such that $r_i \rightarrow^{r_i} r_i' \geq r_{i+1}$.*

Lemma 331 *Let u, $s \in SN$ (resp. u, $h \in SN$), suppose $u[x := s] \notin SN$ (resp. $u[a := h] \notin SN$). Then $u \twoheadrightarrow^{\xi} v \geq \lfloor x, e \rfloor$ (resp. $u \twoheadrightarrow^{\xi} v \geq \lfloor t, a \rfloor$) for some $\lfloor x, e \rfloor$ (resp. $\lfloor t, a \rfloor$) such that ξ is zoom-in and $e[x := s] \in SN$ and $\lfloor x, e \rfloor [x := s] \notin SN$ (resp. $t[a := h] \in SN$ and $\lfloor t, a \rfloor [a := h] \notin SN$).*

Proof By induction on $\eta c(u)$, applying Lemmas 327 and 328. □

Definition 332 *A set $\mathcal{A}$ of proper terms is called $\preceq$-closed from below if, for every u and u', $u' \preceq u \in \mathcal{A}$ and u' is proper implies $u' \in \mathcal{A}$.*

Definition 333 *Let $\mathcal{A}$ and $\mathcal{B}$ be sets $\preceq$-closed from below. We define simultaneously two sets of substitutions $\Pi(\mathcal{B}) \subseteq \Sigma_r$ and $\Theta(\mathcal{A}) \subseteq \Sigma_l$ as follows.*

1. *(a) $\emptyset \in \Pi(\mathcal{B})$,*
   (b) $[a_1 := e_1\tau_1, \ldots, a_n = e_n\tau_n] \in \Pi(\mathcal{B})$ if $n \geq 0$ and $e_i \in \mathcal{B}$ and $\tau_i \in \Theta(\mathcal{A})$ for $1 \leq i \leq n$.
2. *(a) $\emptyset \in \Theta(\mathcal{A})$.*
   (b) $[x_1 := t_1\rho_1, \ldots, x_n = t_n\rho_n] \in \Theta(\mathcal{A})$ if $n \geq 0$ and $t_i \in \mathcal{A}$ and $\rho_i \in \Pi(\mathcal{B})$ for $1 \leq i \leq n$.

Definition 334 *Let $\mathcal{S}(\mathcal{A}, \mathcal{B})$ be a set of terms as follows.*

$$\mathcal{S}(\mathcal{A}, \mathcal{B}) = \{u\rho \mid u \in \mathcal{A} \text{ and } \rho \in \Pi(\mathcal{B})\} \cup \{v\tau \mid v \in \mathcal{B} \text{ and } \tau \in \Theta(\mathcal{A})\}.$$

Definition 335 *Let $u \in \mathcal{S}(\mathcal{A}, \mathcal{B})$. We define*

$$ht(u) = \begin{cases} 0 & \text{if } u \in \mathcal{A} \text{ or } u \in \mathcal{B}, \\ max\{ht(\rho(a)) \mid a \in dom(\rho)\} + 1 & \text{if } u = u_1\rho \text{ with } u_1 \in \mathcal{A} \text{ and } \rho \in \Pi(\mathcal{B}), \\ max\{ht(\tau(x)) \mid x \in dom(\tau)\} + 1 & \text{if } u = u_1\tau \text{ with } u_1 \in \mathcal{B} \text{ and } \tau \in \Theta(\mathcal{A}). \end{cases}$$

Lemma 336 *Let $u \leq v \in \mathcal{S}(\mathcal{A}, \mathcal{B})$, then $u \in \mathcal{S}(\mathcal{A}, \mathcal{B})$.*

Proof A straightforward induction on $ht(v)$. □

Lemma 337 *Let $\sigma = [r_1, \ldots, r_n]$ be a zoom-in reduction sequence. Assume $r_1 \in \mathcal{S}(\mathcal{A}, \mathcal{B})$. Then $r_n \in \mathcal{S}(\mathcal{A}, \mathcal{B})$.*

Proof It is enough to prove the assertion for $n = 2$. Let $r_1 \in \mathcal{S}(\mathcal{A}, \mathcal{B})$, $r_1 \rightarrow r_1' \geq r_2$. We may assume $r_1 = u\rho$ for some $u \in \mathcal{A}$ and $\rho \in \Pi(\mathcal{B})$. Since u is proper we have $u = \lfloor u_1, u_2 \rfloor$. Let $u_2\rho = \tilde{\mu}xu_2'$, the proof for $u_1\rho = \mu a u_1'$ being similar. If u_2 is proper, then $u_2 = \tilde{\mu}xu_2''$ and $r_1' = u_2''[x := u_1]\rho \in \mathcal{S}(\mathcal{A}, \mathcal{B})$. Otherwise $u_2 = a \in dom(\rho)$ and $\rho(a) = \tilde{\mu}xu_2' = \tilde{\mu}xv\tau$ for some $v \in \mathcal{B}$ and $\tau \in \Theta(\mathcal{A})$. We have $r_1' = u_2'[x := u_1\rho] = v\tau[x := u_1\rho] \in \mathcal{S}(\mathcal{A}, \mathcal{B})$. The statement follows from Lemma 336. □

Lemma 338 *1. Let $t\sigma$, $e \in SN$ for some term t, r-term e and $\sigma \in \Sigma_r$. Assume $t\sigma[a := e] \notin SN$, where $a \notin Im(\sigma)$. Then there exists an $\lfloor t', a \rfloor \preceq t$ and a $\sigma' \in \Sigma_r$ such that $t'\sigma' \in SN$ and $\lfloor t'\sigma', e \rfloor \notin SN$.*

2. Let $t\delta$, $s \in SN$ for some term t, l-term s and $\delta \in \Sigma_l$. Assume $t\delta[x := s] \notin SN$, where $x \notin Im(\delta)$. Then there exists an $\lfloor x, t' \rfloor \preceq t$ and a $\delta' \in \Sigma_l$ such that $t'\delta' \in SN$ and $\lfloor s, t'\delta' \rfloor \notin SN$.

Proof We regard only the case $t\sigma$, $e \in SN$ for some r-term e and a $\sigma \in \Sigma_r$ such that $t\sigma[a := e] \notin SN$, where $a \notin Im(\sigma)$. Let us define

$$\mathcal{U} := \{u \mid u \preceq t \text{ and } u \text{ is proper}\},$$

$$\mathcal{V} := \{v \mid v \preceq \sigma(b) \text{ for some } b \in dom(\sigma) \text{ and } v \text{ is proper}\}.$$

Define $\Pi(\mathcal{V})$, $\Theta(\mathcal{U})$ and $\mathcal{S}(\mathcal{U}, \mathcal{V})$ as in Definitions 333 and 334. We have $t\sigma \in \mathcal{S}(\mathcal{U}, \mathcal{V})$. By Lemma 331 there is an $\lfloor t^*, a \rfloor \preceq t\sigma$ and a zoom-in reduction sequence $t\sigma \to^{r_1} t_1 \to^{r_2} t_2 \to^{r_3} \ldots \to^{r_n} t_n$ such that $\lfloor t^*, a \rfloor \leq r_n$ and $t^*[a := e] \in SN$ and $\lfloor t^*[a := e], e \rfloor \notin SN$. Lemmas 337 and 336 give $\lfloor t^*, a \rfloor \in \mathcal{S}(\mathcal{U}, \mathcal{V})$. Since $a \notin Im(\sigma)$ and the elements of $\mathcal{U}$ are proper we have $\lfloor t^*, a \rfloor = \lfloor t', a \rfloor\rho$ for some $\lfloor t', a \rfloor \preceq t$ and $\rho \in \Sigma_r$, which yields the result. □

Theorem 339 *The $\mu\tilde{\mu}$-calculus is strongly normalizing.*

Proof See Theorem 3.2 in David and Nour [15]. □

4.2.2 The strong normalization of the $\overline{\lambda}\mu\tilde{\mu}^*$-calculus

We continue with the proof of the strong normalization of the $\overline{\lambda}\mu\tilde{\mu}^*$-calculus. The proof is analogous to that of the previous section, so we indicate only the main differences below. As in David and Nour [15], it can be verified that the s_l- and s_r-rules can be postponed w.r.t. the other rules of the $\overline{\lambda}\mu\tilde{\mu}^*$-calculus, hence we ignore them in our examinations. In this section let $\to$ denote the union of the β-, μ-, $\tilde{\mu}$-, $cl_{1.l}$-, $cl_{1,r}$-, and cl_2-reductions. The following lemma will be useful.

Lemma 340 *Let $t \in SN$ (resp. $e \in SN$). Then $\lfloor t, a \rfloor \in SN$ (resp. $\lfloor x, e \rfloor \in SN$).*

Proof Straightforward. □

Lemma 341 *Let t be a left-, e be a right term, assume t and e are proper. Let ρ be a left-substitution (resp. a right-substitution). Then the following assertions are valid.*

1. *If $t\rho \twoheadrightarrow \mu\alpha t_1'$, then $t = \mu\alpha t_1$ for some t_1 such that $t_1\rho \twoheadrightarrow t_1'$.*
2. *If $t\rho \twoheadrightarrow \lambda x t_1'$, then $t = \lambda x t_1$ for some t_1 such that $t_1\rho \twoheadrightarrow t_1'$.*
3. *If $e\rho \twoheadrightarrow \tilde{\mu}x e_1'$, then $e = \tilde{\mu}x e_1$ for some e_1 such that $e_1\rho \twoheadrightarrow e_1'$.*
4. *If $e\rho \twoheadrightarrow (t_1'.e_1')$, then $e = (t_1.e_1)$ for some t_1, e_1 such that $t_1\rho \twoheadrightarrow t_1'$ and $e_1\rho \twoheadrightarrow e_1'$.*

Proof Obvious. □

Lemma 342 *Assume t, $e \in SN$, $\lfloor t, e\rfloor \notin SN$. Then we have the following possibilities:*

1. *$t = \mu\alpha s$ and $s[\alpha := e] \notin SN$,*
2. *$e = \tilde{\mu}xh$, $h[x := t] \notin SN$,*
3. *$t = \lambda x u$, $e = (s.h)$ and $\lfloor s, \tilde{\mu}x\lfloor u, h\rfloor\rfloor \notin SN$,*
4. *$t = \overline{\widetilde{s}}$ and $\lfloor s, e\rfloor \notin SN$,*
5. *$e = \widetilde{\overline{h}}$ and $\lfloor t, h\rfloor \notin SN$,*
6. *$t = \overline{h}$, $e = \widetilde{s}$ and $\lfloor s, h\rfloor \notin SN$.*

Proof By induction on $\eta(t) + \eta(e)$. □

The presence of the complementer rules makes the definition of a zoom-in strategy a little more elaborate. In the following definition let $\rightarrow_1$ stand for the union of the β-, μ-, $\tilde{\mu}$-, and cl_2-reductions. .

Definition 343 *Let $\pi = [p_1, \ldots, p_n]$ $(n \geq 1)$ be a sequence of commands, assume $p_i = \lfloor q_i, h_i\rfloor$ $(1 \leq i \leq n)$. Then π is a zoom-in sequence if there is a reduction sequence such that, for every $1 \leq i \leq n$, $p_i \rightarrow_1^{p_i} p_i' \geq p_{i+1}$ or $q_i = \overline{\widetilde{q_i'}}$ and $p_i \rightarrow_{cl_{1,l}} \lfloor q_i', h_i\rfloor \geq p_{i+1}$ or $h_i = \widetilde{\overline{h_i'}}$ and $p_i \rightarrow_{cl_{1,r}} \lfloor q_i, h_i'\rfloor \geq p_{i+1}$. Furthermore, we say that π is a minimal non-sn sequence if q_i, $h_i \in SN$ and $p_i \notin SN$ $(1 \leq i \leq n)$.*

Definition 344 *Let u be a λ^{Sym}_{Prop}-term. The function ϕ gives the associated command of a redex $r \leq u$, if it exists. If r is a λ-, μ-, $\tilde{\mu}$-, or cl_2-redex, then $\phi(r) = r$. If $r = \overline{\widetilde{t}}$ (resp. $r = \widetilde{\overline{e}}$) such that there exists an v for which $\lfloor r, v\rfloor \leq u$ (resp. $\lfloor\rfloor v, r \leq u$), then $\phi(r) = \lfloor r, v\rfloor$ (resp. $\phi(r) = \lfloor v, r\rfloor$). Let $\phi(r)$ be undefined otherwise. Let $\xi = [r_1, \ldots, r_n]$, then $\phi(\xi) = [\phi(r_1), \ldots, \phi(r_n)]$ if $\phi(r_i)$ is defined for every $1 \leq i \leq n$ and set $\phi(\xi)$ as undefined otherwise.*

Lemma 345 *Let u, $s \in SN$ (resp. u, $h \in SN$), suppose $u[x := s] \notin SN$ (resp. $u[a := h] \notin SN$). Then $u \twoheadrightarrow^{\xi} v \geq \lfloor x, e\rfloor$ (resp. $u \twoheadrightarrow^{\xi} v \geq \lfloor t, a\rfloor$) for some $\lfloor x, e\rfloor$ (resp. $\lfloor t, a\rfloor$) such that $\phi(\xi)$ is a zoom-in, minimal non-sn sequence of commands and $e[x := s] \in SN$ and $\lfloor x, e\rfloor[x := s] \notin SN$ (resp. $t[a := h] \in SN$ and $\lfloor t, a\rfloor[a := h] \notin SN$).*

Proof The proof goes by induction on $\eta c(u)$ applying Lemmas 341 and 342. □

The complementer rules also necessitate a more involved definition of closure from below.

Definition 346 *We introduce a relation on the terms of the $\overline{\lambda}\mu\tilde{\mu}^*$-calculus. Let Var_l (resp. Var_r) denote the set of l-variables (resp. r-variables) of the $\overline{\lambda}\mu\tilde{\mu}^*$-calculus. Let $Var = Var_l \cup Var_r$. Assume u is a term and $Fv(u) = \{x_1, \ldots, x_n, a_1, \ldots, a_m\}$. Then $u \approx u'$ iff there exists a $\sigma : Var \rightarrow Var$ such that $\sigma(x_i) \in Var_l$ if $1 \leq i \leq n$, $\sigma(a_j) \in Var_r$ provided $1 \leq j \leq m$ and $u' = u\sigma$. In this case we say that u' is almost equal to u.*

It is immediate to see that $\approx$ is an equivalence relation on the set of terms of the $\overline{\lambda}\mu\tilde{\mu}^*$-calculus.

Definition 347 *Let u, u' be terms. We write $u' \precapprox u$ if there is an $u'' \preceq u$ such that $u' \approx u''$.*

Lemma 348 *Let u and u' be such that $u' \precapprox u$. Then $\eta c(u') \leq \eta c(u)$.*

Proof Obvious. □

Definition 349 *A set $\mathcal{A}$ of proper terms is called $\precsim$-closed from below if, for every u and u', $u' \precsim u \in \mathcal{A}$ and u' is proper implies $u' \in \mathcal{A}$.*

Definition 350 *Let $\mathcal{A}$ and $\mathcal{B}$ be sets $\precsim$-closed from below. Let A be a type. We define simultaneously two sets of substitutions $\Pi_A(\mathcal{B}) \subseteq \Sigma_{A,r}$ and $\Theta_A(\mathcal{A}) \subseteq \Sigma_{A,l}$ as follows.*

1. *(a) $\emptyset \in \Pi_A(\mathcal{B})$,*
   (b) $[a_1 := e_1\tau_1, \ldots, a_n = e_n\tau_n] \in \Pi_A(\mathcal{B})$ if $n \geq 0$ and $e_i \in \mathcal{B}$ such that $type(e_i) = A$ and $\tau_i \in \Theta_A(\mathcal{A})$ for $1 \leq i \leq n$.
2. *(a) $\emptyset \in \Theta_A(\mathcal{A})$.*
   (b) $[x_1 := t_1\rho_1, \ldots, x_n = t_n\rho_n] \in \Theta_A(\mathcal{A})$ if $n \geq 0$ and $t_i \in \mathcal{A}$ such that $type(t_i) = A$ and $\rho_i \in \Pi_A(\mathcal{B})$ for $1 \leq i \leq n$.

The set $\mathcal{S}_A(\mathcal{A}, \mathcal{B})$ is defined as in Definition 334 using $\precsim$ instead of $\preceq$, where the subscript A indicates the common type of the substitutions involved.

Lemma 351 *Let $u \leq v \in \mathcal{S}_A(\mathcal{A}, \mathcal{B})$, then $u \in \mathcal{S}_A(\mathcal{A}, \mathcal{B})$.*

Proof Analogous to the proof of Lemma 336. □

Definition 352 *The length of a type A of the $\overline{\lambda}\mu\tilde{\mu}^*$-calulus is defined as follows.*

1. *$lh(A) = 0$ if A is an atomic type,*
2. *$lh(A \to B) = lh(A) + lh(B) + 1$,*
3. *$lh(A^\perp) = lh(A)$.*

We remark that this definition results that the length of a type A is the number of arrows in A in this calculus also. Thus, equal types have the same lengths.

Lemma 353 *Let n be an integer. Let H be the property as follows: if $u, v \in SN$ and $lh(type(v)) < n$, then $u[x := v] \in SN$ (resp. $u[a := v] \in SN$). Assume H holds. Let $\pi = [p_1, \ldots, p_n]$ be a zoom-in, minimal non-sn sequence of commands. Let A be a type of length n. Assume $p_1 \in \mathcal{S}_A(\mathcal{A}, \mathcal{B})$ for some sets $\mathcal{A}$ and $\mathcal{B}$ $\precsim$-closed from below. Then $p_n \in \mathcal{S}_A(\mathcal{A}, \mathcal{B})$.*

Proof The proof is similar to that of Lemma 337. We only indicate the more interesting changes. Is is enough to consider the case of $n = 2$.

1. $p_1 = \lfloor \lambda xt, (s.e) \rfloor \to_\lambda p_1' = \lfloor s, \tilde{\mu}x\lfloor t, e \rfloor \rfloor \geq p_2$. Assume $p_1 \in \mathcal{S}_A(\mathcal{A}, \mathcal{B})$, we may suppose $p_1 = u\rho$ for some $u \in \mathcal{A}$ and $\rho \in \Pi_A(\mathcal{B})$. Then $u = \lfloor u_1, u_2 \rfloor$ with $u_1\rho = \lambda xt$ and $u_2\rho = (s.e)$. Assume u_1 and u_2 are proper. Then, by Lemma 341, we have $t = t_1\rho$ and $(s.e) = (s_1.e_1)\rho$ for some t_1, s_1 and e_1. Now, $p_1' = \lfloor s_1, \tilde{\mu}x\lfloor t_1, e_1 \rfloor \rfloor \rho \in \mathcal{S}_A(\mathcal{A}, \mathcal{B})$, which yields, by Lemma 351, $p_2 \in \mathcal{S}_A(\mathcal{A}, \mathcal{B})$. Assume either u_1 or u_2 is a variable. Since $\rho \in \Pi_A(\mathcal{B}) \subseteq \Sigma_{A,r}$, this must be u_2. Then $u_2 = \rho(a) = (u_2'.u_2'')$ for some $a \in dom(\rho)$. We have $p_1' = \lfloor u_2', \tilde{\mu}x\lfloor t, u_2'' \rfloor \rfloor$. Since $lh(type(u_2')) < lh(A)$ and $lh(type(u_2'')) < lh(A)$, by assumption H and Lemma 340, we obtain $p_1' \in SN$, which contradicts the hypothesis $p_2 \notin SN$. Similarly, u_1 cannot be a variable either.

2. $p_1 = \lfloor \overline{\widetilde{q}}, e \rfloor \to_{cl_{1,l}} \lfloor q, e \rfloor = p_1' \geq p_2$ or $p_1 = \lfloor q, \widetilde{\overline{e}} \rfloor \to_{cl_{1,r}} \lfloor q, e \rfloor = p_1' \geq p_2$. Assume the latter holds. Let $p_1 = u\rho$ for some $u \in \mathcal{A}$ and $\rho \in \Pi_A(\mathcal{B})$. Then $u = \lfloor u_1, u_2 \rfloor$. If u_2 is proper, we have the result since $\mathcal{A}$ is $\precsim$-closed from below. Otherwise, $\widetilde{\overline{e}} = \rho(a) = v\tau$ for some $v \in \mathcal{B}$ and $\tau \in \Theta_A(\mathcal{A})$. This means $v = \widetilde{\overline{v_1}}$ for some $v_1 \in \mathcal{B}$. Let $\rho_1 := \rho + [a_1 := v_1\tau]$. Then $p_1' = \lfloor u_1\rho, v_1\tau \rfloor = \lfloor u_1, a_1 \rfloor \rho_1 \in \mathcal{S}_A(\mathcal{A}, \mathcal{B})$, since $\lfloor u_1, a_1 \rfloor \precsim u$ and $\rho_1 \in \Pi_A(\mathcal{B})$. □

Lemma 354 *Let n be an integer. Let H be the property as follows: if u, $v \in SN$ and $lh(type(v)) < n$, then $u[x := v] \in SN$ (resp. $u[a := v] \in SN$). Assume H holds. Let A be a type such that $lh(A) = n$.*

1. *Let $t\sigma$, $e \in SN$ for some term t, r-term e and $\sigma \in \Sigma_{A,r}$. Assume $t\sigma[a := e] \notin SN$, where $a \notin Im(\sigma)$. Then there exists an $\lfloor t', a \rfloor \precsim t$ and a $\sigma' \in \Sigma_{A,r}$ such that $t'\sigma' \in SN$ and $\lfloor t'\sigma', e \rfloor \notin SN$.*

2. *Let $t\delta$, $s \in SN$ for some term t, l-term s and $\delta \in \Sigma_{A,l}$. Assume $t\delta[x := s] \notin SN$, where $x \notin Im(\delta)$. Then there exists an $\lfloor x, t' \rfloor \precsim t$ and a $\delta' \in \Sigma_{A,l}$ such that $t'\delta' \in SN$ and $\lfloor s, t'\delta' \rfloor \notin SN$.*

Proof Similar to that of Lemma 338, applying Lemmas 345 and 353. □

Lemma 355 *If t, s, $e \in SN$, then $t[x := s]$, $t[a := e] \in SN$.*

Proof As in Lemma 3.11 of David and Nour [15]. □

Theorem 356 *The $\overline{\lambda}\mu\tilde{\mu}^*$-calculus is strongly normalizing.*

Proof Cf. Theorem 3.3 in David and Nour [15]. The cases induced by the complementer rules in the enumeration in Lemma 342, and thus not appearing in the $\overline{\lambda}\mu\tilde{\mu}$-calculus, can be handled in a straightforward way. □

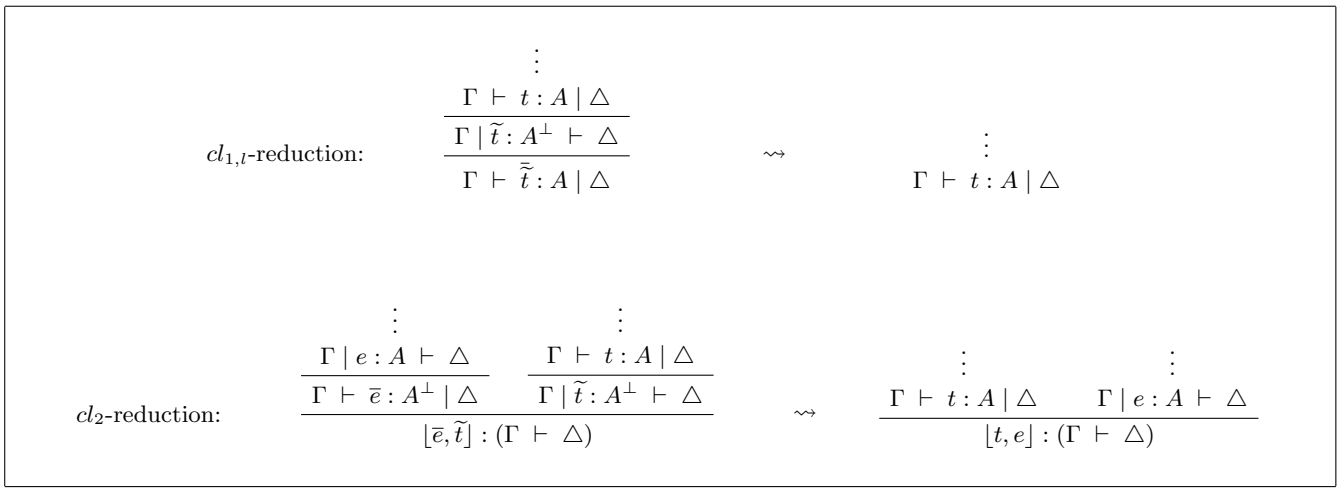

Figure 4.1: The reductions in the proofs corresponding to the $cl_{1,l}$- and cl_2-rules of the $\overline{\lambda}\mu\tilde{\mu}^*$-calculus.

Conclusions

Overview

In this work we continued investigations in the following directions.

1. We examined in Chapter 1 the $\lambda\mu\mu'$-calculus. In the first part of the chapter we proved that the Parigot-style $\lambda\mu\mu'$-calculus extended with the ρ- and the θ-rules is strongly normalizing. For this purpose, knowing, by the result [14] of David and Nour, the strong normalization of the Parigot-style $\lambda\mu\mu'$-calculus also, it was enough to establish the following two assertions.

   - In the Parigot-style $\lambda\mu\mu'$-calculus expanded with the ρ- and the θ-rules, the θ-reduction can be postponed w.r.t. all the other rules.
   - The ρ-reduction can be postponed in the Parigot-style $\lambda\mu\mu'$-calculus.

   The second half of the chapter was devoted to the de Groote-style $\lambda\mu\mu'$-calculus.

   - A new proof of the strong normalization of the $\mu\mu'$-calculus was presented.
   - A counterexample was given to refute the strong normalization of the $\mu\mu'\rho$-calculus.
   - The weak normalization of $\mu\mu'\rho$ was established by finding a normalizing algorithm for the calculus.
   - The above result was applied to obtain the weak normalization of the $\lambda\mu\mu'\rho$-calculus. The algorithm trivially extends to the case of the $\lambda\mu\mu'\rho\theta$-calculus also.
   - A standardization theorem was formulated and proved for the $\lambda\mu\mu'\rho\theta$-calculus.
   - In the end of the chapter the effect of the addition of some other simplification rules to the $\mu\mu'$- or the $\lambda\mu\mu'$-calculi were examined.

2. In Chapter 2 we were concerned with estimations about the lengths of reductions in the $\lambda\mu\rho\theta$-calculus.

   - An upper bound was found for the lengths of reductions in $\lambda\mu\rho\theta$-calculus.
   - In regard to the $\lambda\mu\rho\theta$-calculus a standardization theorem was asserted and proved, together with finding a bound for the length of the standardization of a given $\lambda\mu\rho\theta$-reduction sequence.

3. In Chapter 3 an arithmetical proof for the strong normalization of the λ^{Sym}_{Prop}-calculus was presented.

4. In Chapter 4 we defined translations between the λ^{Sym}_{Prop}- and the $\overline{\lambda}\mu\tilde{\mu}^*$-calculi preserving strong normalization.

Open questions

1. In Section 1.2.2 a counterexample for the strong normalization of the $\mu\mu'\rho$-calculus was given. In the realm of the untyped λ-calculus there is a well-known result supplying a necessary condition for a term not to be strongly normalizing, this is the so-called Ω-theorem (cf. Raamsdonk et al. [52]). Let us use the notation $\Omega = (\omega\omega)$, where $\omega = \lambda x(x\ x)$. The Ω-theorem roughly states that if a term is not strongly normalizing, it must contain in some sense the non-strongly normalizing term Ω. We conjecture that a very similar theorem also holds for the $\mu\mu'\rho$-calculus with $(U\ V)$ and $(V\ U)$, respectively, where $U = \mu\alpha(\alpha\ (\alpha\ x))$ and $V = \mu\beta U$ are the terms introduced in Section 1.2.2. For this purpose we clarify a little more in what sense the term $(U\ V)$ (resp. $(V\ U)$) is contained in a non-strongly normalizing term of the $\mu\mu'\rho$-calculus. The treatment below follows that of the case of Ω in Raamsdonk et al. [52].

   Definition 357 *We define a relation called substring on the set of all terms of the $\mu\mu'\rho$-calculus.*

   1. $x \trianglelefteq y$ *for arbitrary variables* x *and* y,
   2. $M \trianglelefteq \mu\alpha N$, *if* $M \trianglelefteq N$ *and* $\alpha \notin Fv(N)$,
   3. $M \trianglelefteq (N\ P)$, *if* $M \trianglelefteq N$,
   4. $M \trianglelefteq (P\ N)$, *if* $M \trianglelefteq N$,
   5. $\mu\alpha M \trianglelefteq \mu\beta N[\alpha := \beta]$, *if* $M \trianglelefteq N$ *and* $\beta \notin Fv(N)$,
   6. $M \trianglelefteq (\alpha\ N)$, *if* $M \trianglelefteq N$,
   7. $(\alpha\ M) \trianglelefteq (\alpha\ N)$, *if* $M \trianglelefteq N$,
   8. $(M_1\ M_2) \trianglelefteq (N_1\ N_2)$, *if* $M_1 \trianglelefteq N_1$ *and* $M_2 \trianglelefteq N_2$.

   Example 358 1. $\mu\alpha(\alpha\ (\alpha\ x)) \trianglelefteq \mu\beta(\beta\ (\beta\ (x\ y)))$,
   2. $\mu\beta\mu\alpha(\alpha\ x) \trianglelefteq \mu\beta\mu\alpha\mu\gamma(\gamma\ x)$,
   3. $\mu\alpha(\alpha\ x) \trianglelefteq \mu\beta\mu\gamma((\beta\ x)\ (\beta\ x))$,
   4. $\mu\alpha(\alpha\ (\alpha\ x)) \not\trianglelefteq \mu\alpha(\alpha\ (\beta\ x))$,
   5. $\mu\alpha(\alpha\ (\alpha\ x)) \not\trianglelefteq \mu\beta(\beta\ \mu\alpha(\alpha\ x))$,
   6. $\mu\alpha(\alpha\ (\alpha\ x)) \not\trianglelefteq (\mu\alpha(\alpha\ x)\ \mu\alpha(\alpha\ x))$.

   The set Λ_U is the set of all terms not containing multiple occurrences of a bound variable. In the definition below we use the notation $|M|_\alpha$ defined in Definition 200.

   Definition 359 *The set Λ_U is defined as follows.*

   1. $x \in \Lambda_U$,
   2. $\mu\alpha M \in \Lambda_U$, *if* $M \in \Lambda_U$ *and* $|M|_\alpha \leq 1$,
   3. $(\alpha\ M) \in \Lambda_U$, *if* $M \in \Lambda_U$,
   4. $(M_1\ M_2) \in \Lambda_U$, *if* $M_1 \in \Lambda_U$ *and* $M_2 \in \Lambda_U$.

   We can observe that $M \in \Lambda_U$ iff $U = \mu\alpha(\alpha\ (\alpha\ x)) \not\trianglelefteq M$.

   Definition 360 *We define the set Λ_{UV}.*

1. $x \in \Lambda_{UV}$,
2. $\mu\alpha M \in \Lambda_{UV}$, *if* $M \in \Lambda_{UV}$,
3. $(\alpha\ M) \in \Lambda_{UV}$, *if* $M \in \Lambda_{UV}$,
4. $(M_1\ M_2) \in \Lambda_{UV}$, *if* $M_1 \in \Lambda_{UV}$ *and* $M_2 \in \Lambda_U$,
5. $(M_1\ M_2) \in \Lambda_{UV}$, *if* $M_1 \in \Lambda_U$ *and* $M_2 \in \Lambda_{UV}$.

We can see easily that $M \in \Lambda_{UV}$ iff $(U\ V) \not\trianglelefteq M$ (or $(V\ U) \not\trianglelefteq M$, respectively) with U and V defined as above. We can formulate the following conjecture now.

Conjecture 361 *If* $M \in \Lambda_{UV}$, *then* $M \in SN$.

Examining the proof of the Ω-theorem, we find that for proving Conjecture 361 it is enough to establish the assertion below.

Definition 362 *We define two sets of terms,* Λ_X *and* Λ_Y, *simultaneously.*

(a)
1. $x \in \Lambda_X$,
2. $(\alpha\ M) \in \Lambda_X$, *if* $M \in \Lambda_X$,
3. $\mu\alpha M \in \Lambda_X$, *if* $M \in \Lambda_Y$,
4. $(M_1\ M_2) \in \Lambda_X$, *if* M_1 *and* $M_2 \in \Lambda_X$.

(b)
1. $x \in \Lambda_Y$,
2. $(\alpha\ M) \in \Lambda_Y$, *if* $M \in \Lambda_X$,
3. $\mu\alpha M \in \Lambda_Y$, *if* $M \in \Lambda_Y$ *and* $|M|_\alpha \leq 1$,
4. $(M_1\ M_2) \in \Lambda_Y$, *if* M_1 *and* $M_2 \in \Lambda_Y$.

Intuitively, if $M \in \Lambda_X$, then M has the following property. Let $\mu\alpha M_1 \leq M$ such that $|M|_\alpha > 1$. Then either there is an address a for which $M_a = \mu\alpha M_1$ or there is a $(\beta\ N) \leq M$ and an address b such that $N_b = \mu\alpha M_1$. We may observe that the latter condition means that $\mu\alpha M_1$ is correct in M in the sense of Definition 83. Now we can state the following conjecture.

Conjecture 363 *If* $M \in \Lambda_X$, *then* $M \in SN$.

2. In Section 1.2.1 we presented a new proof for the strong normalization of the $\mu\mu'$-calculus by finding a norm strictly decreasing on the possible non-terminating reduction sequences. It seems to be promising to extend the proof for $\lambda\mu\mu'$ also. This necessitates an appropriate extension of the norm to $\lambda\mu$-terms also, which can be a future work for us. We remark that it is a long-standing open problem to find a norm for the simply typed λ-calculus which is strictly decreasing at each reduction step.

3. In Chapter 2 we established a bound for the lengths of the reduction sequences in the $\lambda\mu\rho\theta$-calculus. It is an interesting question also to find a bound for the lengths of the reduction sequences in the $\mu\mu'$- and in the $\lambda\mu\mu'$-calculus, respectively. It appears that the norm defined in Section 1.2.1 can be modified in a way such that it also supplies a bound for the lengths of reductions in the $\mu\mu'$-calculus. Probably, a similar approach could also be suitable for the $\lambda\mu\mu'$-calculus: having found a norm establishing the strong normalization of the $\lambda\mu\mu'$-calculus, we might try to extract with the help of this norm a bound for the lengths of the reduction sequences in the calculus.

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