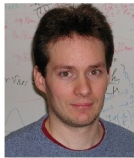


THE ANDERSON TRANSITION WITH ATOMIC MATTER WAVES



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Outline

Quantum transport/localization in disordered or chaotic systems

Anderson localization and the Anderson metal-insulator transition in disordered media

The Kicked Rotor with cold atoms: a very practical tool for studying Anderson localization

Experimental observation of the Anderson transition with the quasiperiodic Kicked Rotor

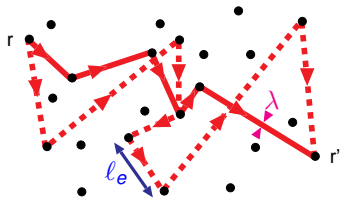
Critical regime

Anderson localization
and the Anderson metal-insulator transition
in disordered media

Interplay between disorder and interference effects

Classical

- Multiple scattering \Rightarrow random walk \Rightarrow at long times, large distances: diffusive behavior



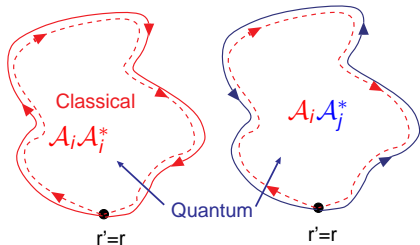
l_e mean free path
 λ wave length

Quantum with time-reversal symmetry

- Coherent backscattering \Rightarrow slows down the diffusive behavior

$l_e/\lambda \gg 1$: weak disorder

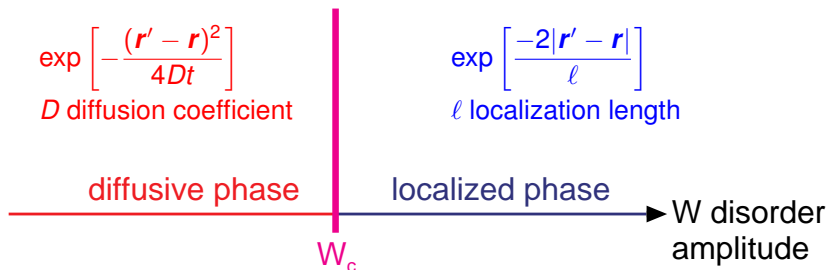
- $P(\mathbf{r}, \mathbf{r}') = \left| \sum_{\text{path } i} \mathcal{A}_i \right|^2$
- phase \sim length of i^{th} path/ λ
- if $i \neq j$ uncorrelated phases?
- $P(\mathbf{r}, \mathbf{r}') = \sum_i \mathcal{A}_i \mathcal{A}_i^* + \sum_{i \neq j} \mathcal{A}_i \mathcal{A}_j^*$
 $\approx \sum_i \mathcal{A}_i \mathcal{A}_i^*$



Essential features of Anderson localization

- If return probability sufficiently strong \Rightarrow can stop the diffusion = Anderson localization
- Return probability = highly dependent on the dimensionality
- In 1D, always localized (\forall disorder amplitude !)
- In 2D, localization length exponentially large for weak disorder \Rightarrow marginally localized
- In 3D, **ANDERSON metal-insulator TRANSITION**

At long times, large distances: $P(\mathbf{r}, \mathbf{r}'; t) \sim ?$

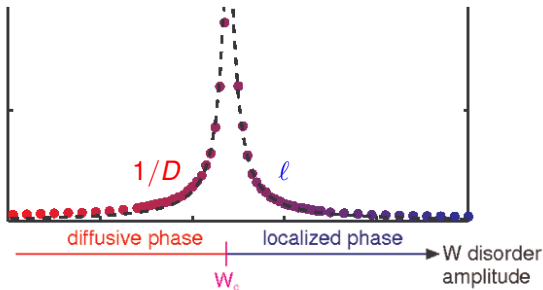


The Anderson metal-insulator transition

A second-order continuous phase transition

- At the threshold of the transition, **no characteristic quantity** (length, energy, ...) \neq **first-order discontinuous** transition (latent heat for liquid-gas transition)
- $\ell \sim (W - W_c)^{-\nu}$, $W > W_c$, localized
- $D \sim (W_c - W)^s$, $W < W_c$, diffusive
- $s = \nu$ **critical exponents = universal** (do not depend on microscopic details)

Numerics: $\nu = 1.57 \pm 0.02$ [Slevin et al., PRL (1999)]



Experimental studies of the Anderson transition

Disordered solids

- Relies on global transport measurements like conductance = transmission \Rightarrow localized phase: $G(L) \sim e^{-2L/\ell}$
- **!!! Interactions between electrons** \Rightarrow non-trivial effects
 \Rightarrow **affect the critical behavior: $\nu \approx 1$**

Classical waves in disordered media

- Anderson localization = interference phenomenon
- light, microwaves, acoustics, ... \Rightarrow no interactions
- **!!! Localization/Absorption = same signature on the transmission**
= exponential decay
- \Rightarrow Statistics, temporal response, transverse localization
[Genack et al., PRL (1991); Wiersma et al., Nat. (1997); Störzer et al., PRL (2006); Hu et al., Nat. Phys. (2008)]
- **No reliable determination of the critical exponents**

Anderson localization with atomic matter waves

Cold atoms physics

- Interference phenomena with atomic matter waves
- Cooling and trapping methods \Rightarrow very large coherence length and λ of atomic matter waves
- Manipulation with optical potentials
- Watching the wavefunction

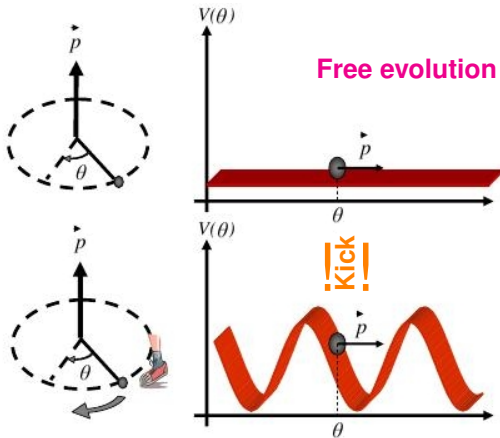
Cold atoms in a 1D disordered light potential

- Dilute Bose-Einstein condensate \Rightarrow negligible interactions
- Speckle \Rightarrow 1D disordered light potential
- Observation of Anderson localization in 1D [Billy et al., Nat. (2008)]
- No 2D/3D disordered light potential yet!

The Kicked Rotor with cold atoms: a very practical tool for studying Anderson localization

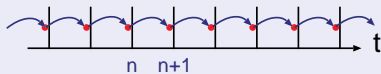
The 1D periodically Kicked Rotor

$$H = \frac{p^2}{2} + K \cos \theta \sum_n \delta(t - n)$$



Classical dynamics

Standard map



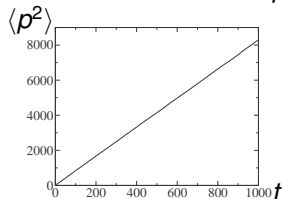
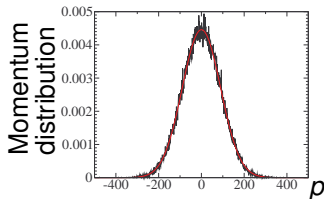
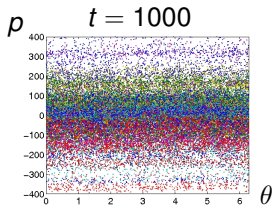
$$p_{n+1} = p_n + K \sin \theta_n$$

$$\theta_{n+1} = \theta_n + p_{n+1}$$

- K stochasticity parameter
- $K \gg 1$: pseudo-random (deterministic) walk in **momentum space** with short memory

Chaotic diffusion

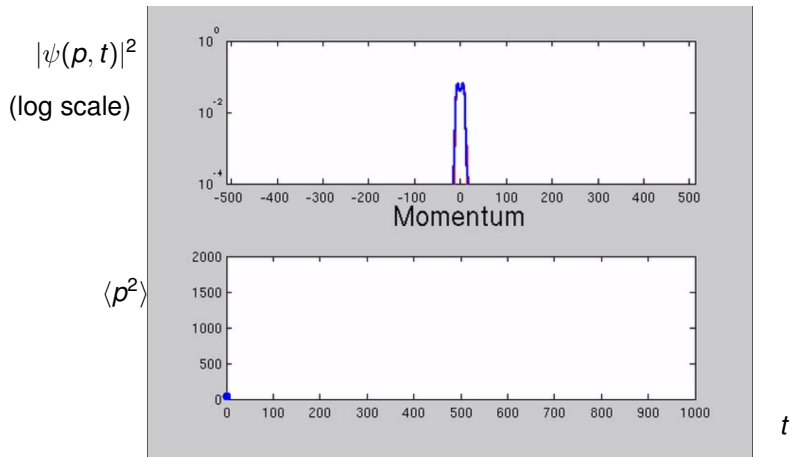
- On average: **diffusion in momentum space p**
- Assuming no correlations between successive positions ($K \gg 1$):
$$\langle p_{n+1}^2 \rangle \approx \langle p_n^2 \rangle + K^2 \langle \sin^2 \theta_n \rangle \approx \langle p_n^2 \rangle + K^2/2$$
- **Diffusion constant: $D = K^2/2$**



Quantum dynamics vs. Classical dynamics

Interplay between chaos and interference effects

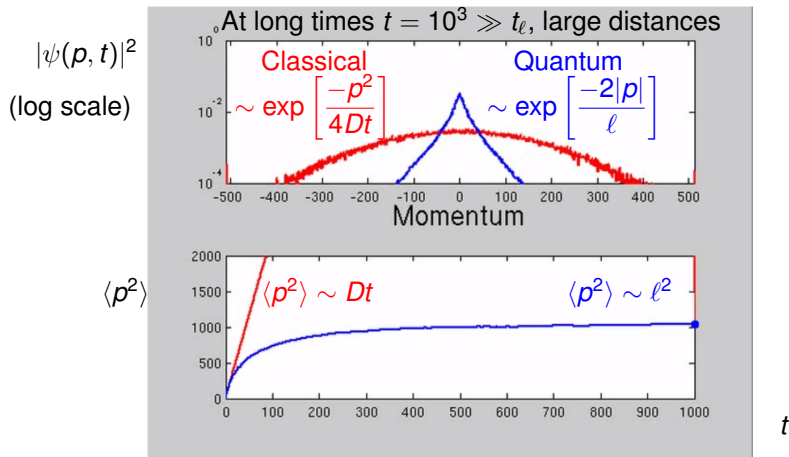
- Start from an initially peaked state \Rightarrow Chaotic diffusive expansion?
- $t > t_\ell$, **dynamical localization** [G. Casati et al., Springer (1979)]



Quantum dynamics vs. Classical dynamics

Interplay between chaos and interference effects

- Quantum localization (due to interference effects) in the presence of chaos (disorder)
- Dynamical localization = Anderson localization?

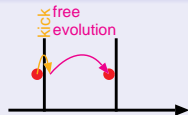


The Kicked Rotor: a 1D pseudo-disordered system

Quantum dynamics

- Evolution operator:

$$\hat{U} = \hat{U}_{\text{free}} \hat{U}_{\text{kick}} = e^{-i\hat{p}^2/2\hbar} e^{-iK \cos \hat{\theta}/\hbar}$$



- 2π periodicity in $\theta \Rightarrow$ **discrete momentum basis** $|p = \hbar l\rangle$

Evolution operator = pseudo-random banded matrix

- Evolution operator in the discrete momentum basis:

$$\left(U_{\beta l, l'} \right) = \left(e^{-i\hbar l^2/2} \delta_{l, l'} \right) \times \left(i^{l' - l} J_{l' - l}(K/\hbar) \right)$$

- \hbar/π irrational $\Rightarrow \phi_l = \hbar \frac{l^2}{2}$ modulo $[2\pi] =$ **pseudo-random phases**
- $J_{l' - l}(K/\hbar) =$ **Bessel function** $\neq 0$ for $|l' - l| \lesssim K/\hbar \Rightarrow$ **hopping amplitudes controlled by K/\hbar**

The Kicked Rotor: a 1D pseudo-disordered system (2)

Kicked Rotor \equiv 1D Anderson-like model [Fishman et al., PRL (1982)]

$$\nu_l a_l + \sum_{l' \neq 0} t_{l'} a_{l-l'} = -t_0 a_l$$

- Pseudo-random on-site energies $\nu_l = \tan \left[\frac{1}{2}(\omega - l^2 \hbar/2) \right]$
- Hopping amplitudes $t_{l'}$ increase with K/\hbar

Other rigorous arguments supporting the equivalence:

**dynamical localization in the 1D Kicked Rotor
= Anderson localization in 1D disordered systems**

- Diagrammatic approach
- Self-consistent theory
- Supersymmetry

Quasi-periodicity and effective dimensionality

The quasiperiodic Kicked Rotor [Shepelyansky, Phys. D (1987)]

$$H_{\text{qp}} = \frac{\hat{p}^2}{2} + \mathcal{K}(t) \cos \hat{\theta} \sum_n \delta(t - n)$$

- quasi-periodic modulation with **two new frequencies**, ε modulation amplitude:

$$\mathcal{K}(t) = K [1 + \varepsilon \cos(\omega_2 t + \varphi_2) \cos(\omega_3 t + \varphi_3)]$$

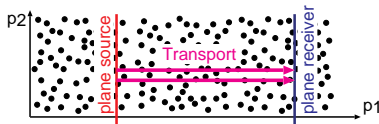


- dynamics **strictly identical** to that of a 3D Kicked **pseudo-Rotor**

$$H_3 = \frac{p_1^2}{2} + \omega_2 p_2 + \omega_3 p_3 + K \cos \theta_1 [1 + \varepsilon \cos \theta_2 \cos \theta_3] \sum_n \delta(t - n)$$

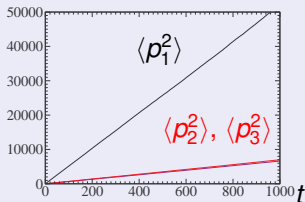
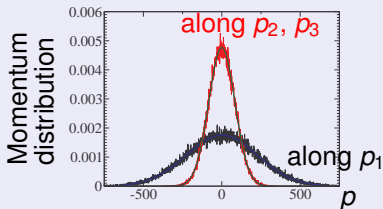
with **transverse kinetic energy linear in momentum** and initial condition taken as a **plane source**

$$\psi_3(\theta_1, \theta_2, \theta_3; t = 0) = \psi_{\text{qp}}(\theta_1, t = 0) \delta(\theta_2 - \varphi_2) \delta(\theta_3 - \varphi_3)$$



Quasi-periodicity and effective dimensionality (2)

Diffusive classical dynamics in all 3 directions, anisotropy driven by ε



Floquet states of the 3D Kicked "Rotor" in a 3D Anderson-like model

- On-site energies $\mathcal{V}_l = \tan \left[\frac{1}{2} (\omega - \hbar l_1^2 / 2 + \omega_2 l_2 + \omega_3 l_3) \right]$
 $(\hbar, \omega_2, \omega_3, \pi)$ incommensurate quadruplet \Rightarrow pseudo-random

Numerical observation of the Anderson transition [Casati et al., PRL **62**, 345 (1989)]

- at very long time ($t \approx 10^6$) transition from localized to diffusive distribution for $K > K_c$

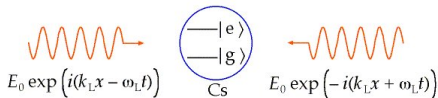
Experimental observation
of the Anderson transition with the quasiperiodic
Kicked Rotor

Experimental realization with cold atoms [Moore et al., PRL (1995)]

Quantum chaos group of PHLAM laboratory, Lille: JC Garreau, P Szriftgiser, J Chabé, H Lignier

Thermal distribution (MOT)

- few recoils
- no interactions

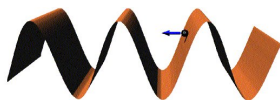


1D quantum dynamics using the external motion of cold atoms

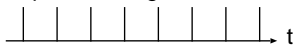
- Quasi-resonant laser field $\omega_L = \omega_0 + \Delta_L$
 \Rightarrow light-shift \Rightarrow effective potential
- With a standing wave (Ω : Rabi frequency):

$$\hat{\mathcal{H}} = \frac{\hat{P}^2}{2M} + \frac{\hbar\Omega^2}{8\Delta_L} \cos(2k_L \hat{X}) f(t)$$

- Temporal modulation of the laser intensity (with δ -kicks) \Rightarrow Kicked Rotor with $\hbar = 8\omega_R T_1$ (ω_R : atomic recoil frequency, T_1 : kicking period)



temporal forcing



Watching the wave-function

- time of flight
- velocity selective Raman technique

Limitations of the experimental setup

Limitations

- Atoms fall down out of the standing wave because of gravitational field \Rightarrow **150 kicks maximum**

Decoherence effects: condition $t \ll \tau_\phi$

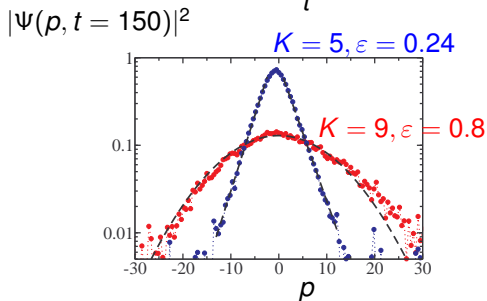
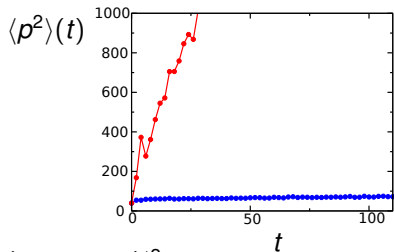
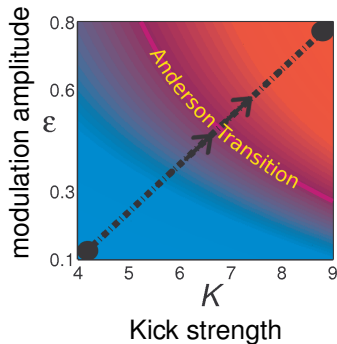
Already known:

- **Atomic collisions** between cold atoms $\Rightarrow \tau_\phi \approx 600$ kicks
- **Spontaneous emission** \Rightarrow gives a random recoil \Rightarrow breaks phase-coherence of the atomic wave function $\Rightarrow \tau_\phi \approx 500$ kicks

New:

- **Deviation of the standing wave from strict horizontality** (angle α) \Rightarrow residual gravitational field along the laser axis \Rightarrow non-trivial deterministic “decoherence” effect $\Rightarrow \alpha < 0.1^\circ \Rightarrow \tau_\phi > 350$ kicks.

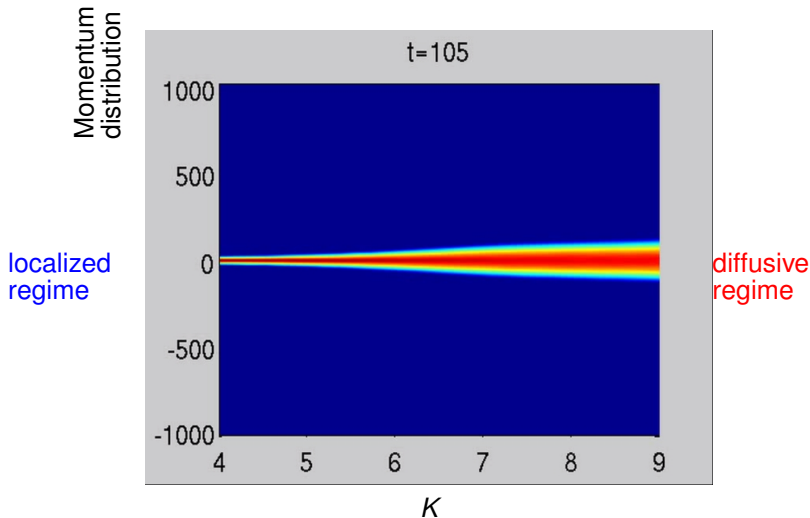
Experimental observation of localized/diffusive dynamics



Finite-time limitations on a continuous transition

How to determine K_c at 150 kicks? seems easier at long times!

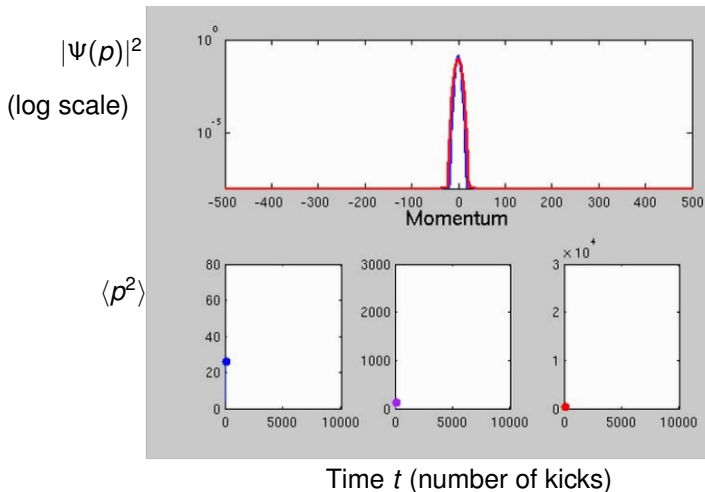
- for $t \ll t_\ell$ not yet localized (\approx “not yet diffusive” distribution) but t_ℓ diverges at the transition



How to unambiguously identify the transition?

What characterizes the criticality?

- **No characteristic time** \Rightarrow algebraic dependence of $\langle p^2 \rangle \sim t^\gamma$



$K=9$
diffusive
regime

$$\langle p^2 \rangle \sim Dt$$

$K = K_c$
critical
regime

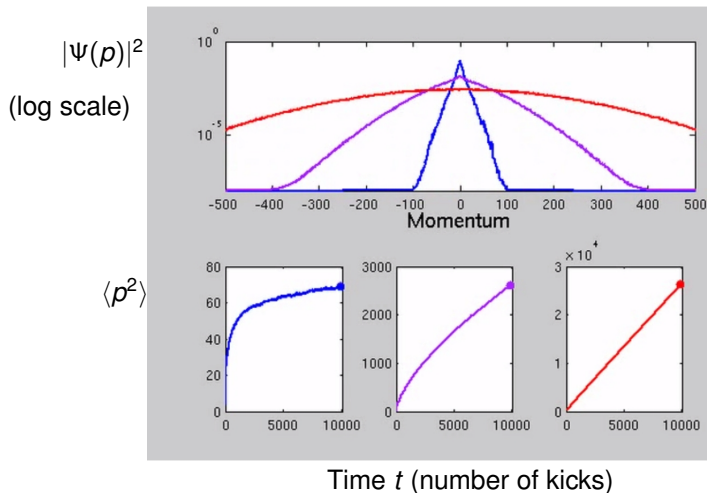
$K=4$
localized
regime

$$\langle p^2 \rangle \sim t^2$$

How to unambiguously identify the transition?

What characterizes the criticality?

- Algebraic dependence of $\langle p^2 \rangle \sim t^\gamma$, i.e. anomalous diffusion



$K=9$
diffusive
regime
 $\langle p^2 \rangle \sim Dt$

$K = K_c$
critical
regime

$K=4$
localized
regime
 $\langle p^2 \rangle \sim t^2$

Scaling law in the vicinity of the Anderson transition

Characteristic lengths

- Localization length diverges at K_c^- : $\ell \sim (K_c - K)^{-\nu}$
- Diffusion constant vanishes at K_c^+ : $D \sim (K - K_c)^\nu$

Continuous phase transition

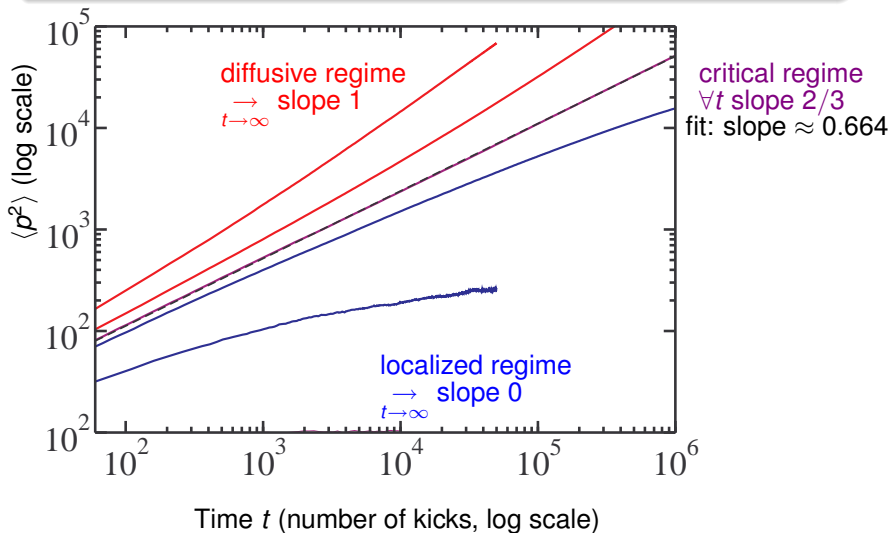
- Unified description of the localized and diffusive regimes
- **One parameter scaling hypothesis** when $K \simeq K_c$ and at long times: $\langle p^2 \rangle \sim t^\gamma F[(K - K_c) t^{k_2}]$
- When $t \rightarrow \infty$ we must recover:
 - a localized behavior for $K < K_c$: $\gamma - 2\nu k_2 = 0$
 - a diffusive behavior for $K > K_c$: $\gamma + \nu k_2 = 1$
- $\Rightarrow \gamma = 2/3$ and $k_2 = 1/3\nu$

Scaling hypothesis

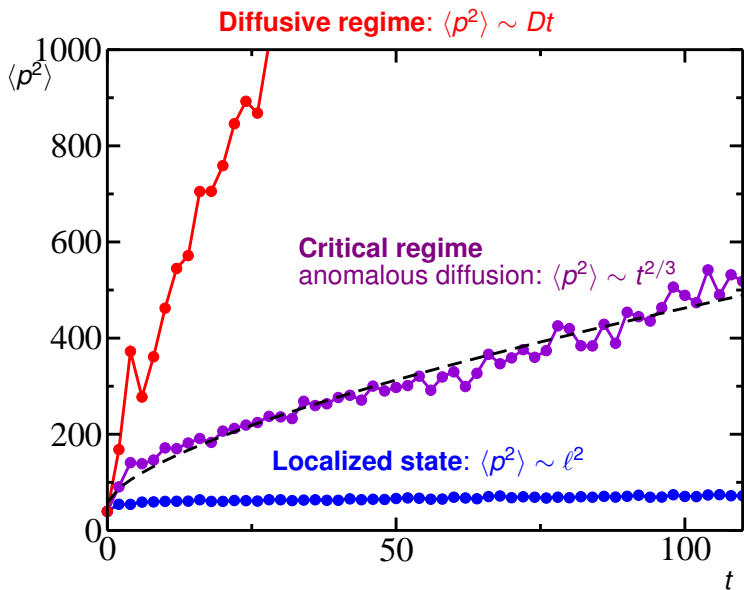
$$\langle p^2 \rangle \sim t^{2/3} F[(K - K_c) t^{1/3\nu}]$$

Critical anomalous diffusion

• $\langle p^2 \rangle \sim t^{2/3} F[(K - K_c) t^{1/3\nu}] \Rightarrow \langle p^2 \rangle \sim t^{2/3}$ at $K = K_c$



Critical anomalous diffusion: experimental observation

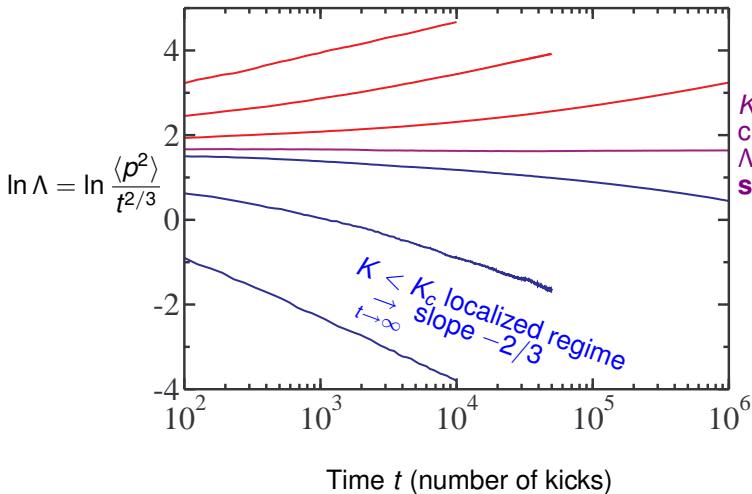


Rescaled dynamics: $\Lambda = \langle p^2 \rangle / t^{2/3}$

$K > K_c$, diffusive regime
 \rightarrow slope $+1/3$
 $t \rightarrow \infty$

$K = K_c$
critical regime
 $\Lambda = \text{cste}$
scale invariance

$K < K_c$ localized regime
 \rightarrow slope $-2/3$
 $t \rightarrow \infty$



Finite-time scaling

Directly transpose the ideas of finite-size scaling (in usual configuration space) to the temporal dynamics of the Kicked Rotor
⇒ overcome the finite-time limitation ⇒ determine ℓ for $t \rightarrow \infty$
⇒ **determination of the critical exponents** $\nu = s$: $\ell \sim |K_c - K|^{-\nu}$

How to verify the scaling hypothesis $\langle p^2 \rangle \sim t^{2/3} F[(K - K_c) t^{1/3\nu}]$?

- More general scaling hypothesis:
- Existence of $\xi(K)$ **scaling parameter depending only on K** such that $\Lambda = \frac{\langle p^2 \rangle}{t^{2/3}} = \mathcal{F} \left[\frac{\xi(K)}{t^{1/3}} \right]$?
- $\xi(K)$ unknown
- Scaling function \mathcal{F} unknown
- We can determine both! Does $\xi(K) \sim |K_c - K|^{-\nu}$?

Finite-time scaling

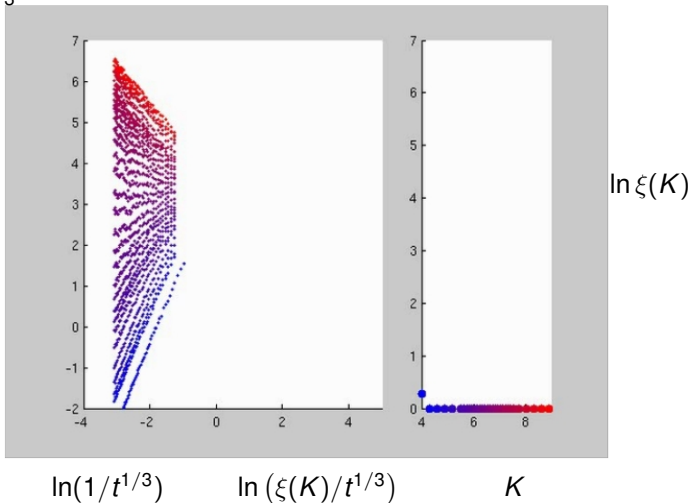
- Existence of $\xi(K)$ such that: $\Lambda = \frac{\langle p^2 \rangle}{t^{2/3}} = \mathcal{F} \left[\frac{\xi(K)}{t^{1/3}} \right]$?

$\ln \Lambda = \ln \frac{\langle p^2 \rangle}{t^{2/3}}$ The "displacement" is $\ln \xi(K)$

diffusive

critical

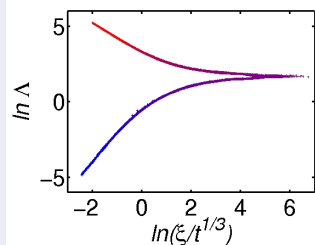
localized



Finite-time scaling analysis of numerical results

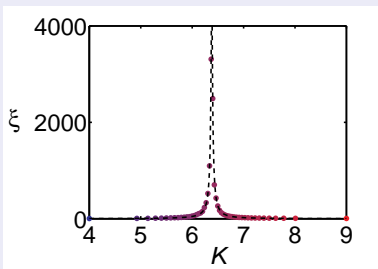
Scaling function \mathcal{F}

$$\Lambda(K, t) = \frac{\langle p^2 \rangle}{t^{2/3}} \sim \mathcal{F} \left[\frac{\xi(K)}{t^{1/3}} \right]$$



Scaling parameter ξ

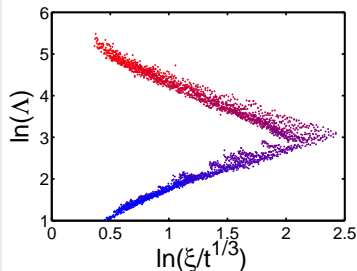
- $\xi \sim \ell$ for $K < K_c$
- $\xi \sim 1/D$ for $K > K_c$



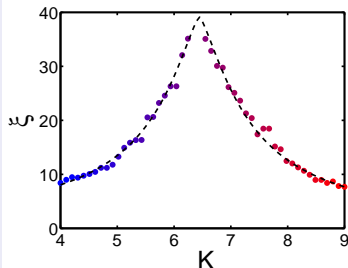
- well fitted by: $\xi \sim |K - K_c|^{-\nu}$
- Critical point: $K_c \simeq 6.4$
- Critical exponent: $\nu \simeq 1.6 \pm 0.2$

Finite-time scaling analysis of experimental results

Scaling function \mathcal{F}



Scaling parameter ξ



Experimental determination of the critical exponent ν

- $1/\xi = \alpha|K - K_c|^\nu + \beta$
- β accounts for experimental imperfections
- **Critical point:** $K_c \simeq 6.4$
- **Critical exponent:** $\nu \simeq 1.4 \pm 0.3$
- Excellent agreement with numerics (**no adjustable parameter**)

Does the quasiperiodic Kicked Rotor have the same critical behavior as the 3D Anderson model?

Universality of the Anderson transition

[Slevin et al., PRL (1997); Slevin et al., PRL (1999)]

- Critical exponents should be **universal**, i.e. not depend on microscopic details but only on the symmetries of the system
- 3D Anderson model belongs to the orthogonal universality class (time-reversal symmetry): $\nu = 1.57 \pm 0.02$
- Unitary class when the time-reversal symmetry is broken (e.g. magnetic field): $\nu = 1.43 \pm 0.04$

Universality class of the quasiperiodic Kicked Rotor?

- Kicked Rotor = time-reversal symmetric \Rightarrow orthogonal
- Microscopic details of the quasiperiodic Kicked Rotor controlled by $(\hbar, \omega_2, \omega_3, \pi)$ (incommensurate):
$$\nu_I = \tan \left[\frac{1}{2} (\omega - \hbar l_1^2 / 2 + \omega_2 l_2 + \omega_3 l_3) \right]$$

YES! The quasiperiodic Kicked Rotor has the same critical behavior as the 3D Anderson model.

	\hbar	ω_2	ω_3	K	ϵ
\mathcal{A}	2.85	$2\pi\sqrt{5}$	$2\pi\sqrt{13}$	6.24 \rightarrow 6.58	0.413 \rightarrow 0.462
\mathcal{B}	2.85	$2\pi\sqrt{7}$	$2\pi\sqrt{17}$	5.49 \rightarrow 5.57	0.499 \rightarrow 0.514
\mathcal{C}	2.2516	$1/\eta$	$1/\eta^2$	4.98 \rightarrow 5.05	0.423 \rightarrow 0.436
\mathcal{D}	3.5399	\hbar/η	\hbar/η^2	7.9 \rightarrow 8.3	0.425 \rightarrow 0.485

Table: The four sets of parameters considered: \hbar , ω_2 and ω_3 control the microscopic details of the disorder, while ϵ drives the anisotropy of the hopping amplitudes. $\eta = 1.324717\dots$

	K_c	$\ln \Lambda_c$	ν	y
\mathcal{A}	6.36 ± 0.02	1.60 ± 0.04	1.58 ± 0.01	0.71 ± 0.28
\mathcal{B}	5.53 ± 0.03	1.08 ± 0.09	1.60 ± 0.03	0.33 ± 0.30
\mathcal{C}	5.00 ± 0.03	1.19 ± 0.15	1.60 ± 0.02	0.23 ± 0.29
\mathcal{D}	8.09 ± 0.01	1.64 ± 0.03	1.59 ± 0.01	0.43 ± 0.23

Critical regime

Predictions of the self-consistent theory

Self-consistent theory for the quasi-periodic Kicked Rotor

$$D(\omega) = D - 2D(\omega) \int \frac{d^3\Phi}{(2\pi)^3} \frac{1}{-i\omega + D(\omega)\Phi^2}$$

- Anisotropy \Rightarrow diffusion tensor
- $f(\varphi) = \text{plane source} = \delta(\varphi_2 = 0)\delta(\varphi_3 = 0)$

$$|\psi(\mathbf{p}, t)|^2 = \int \frac{d\omega}{2\pi} e^{-i\omega t} \int \frac{d^3\varphi}{(2\pi)^3} e^{i\varphi \cdot \mathbf{p}} \frac{f(\varphi)}{-i\omega + D(\omega)\varphi^2}$$

- Anderson transition with $\nu = 1 \Rightarrow$ fluctuations are not taken into account
- **At criticality:** $D(\omega) \underset{\omega \rightarrow 0}{\sim} \omega^{1/3}$, thus $\langle p^2 \rangle \sim t^{2/3}$

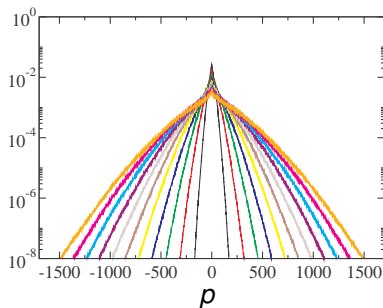
Analytic prediction for the critical wave function

$$t^{1/3} |\psi(\mathbf{p}, t)|^2 = \mathcal{G}(pt^{-1/3})$$

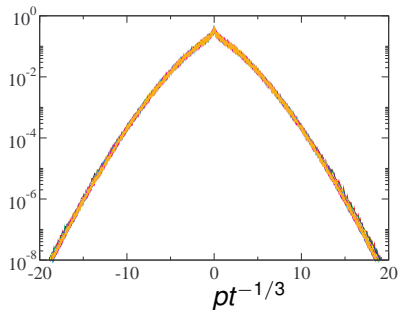
$$\text{with } \mathcal{G}(x) = \frac{3}{2} (3\rho^{3/2})^{-1/3} \text{Ai} \left[(3\rho^{3/2})^{-1/3} |x| \right]$$

Scale invariance of the critical wave function

$$|\Psi(p, t)|^2$$



$$t^{1/3}|\Psi(p, t)|^2$$

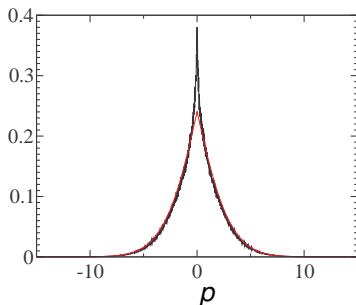
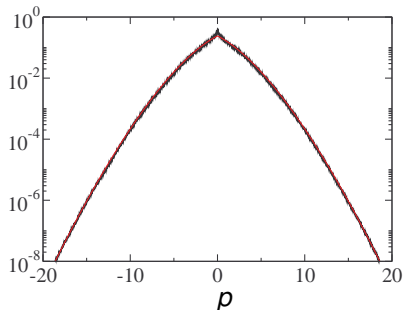


Scaling $t^{1/3}|\psi(p, t)|^2 = \mathcal{G}(pt^{-1/3})$?

Rescaling of all critical wave functions for t from $t = 10^3$ to $t = 10^6$!

Confrontation with numerics

$$|\Psi(\rho, t = 10^6)|^2$$

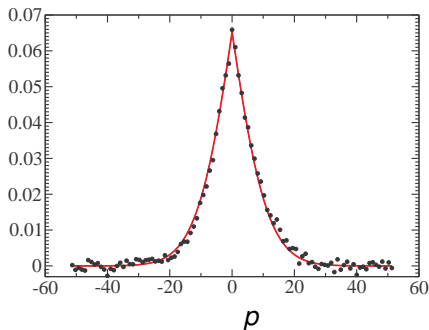


$$\mathcal{G}(x) = \frac{3}{2} (3\rho^{3/2})^{-1/3} \text{Ai} \left[(3\rho^{3/2})^{-1/3} |x| \right] ?$$

- YES! **No adjustable parameter**: $\rho = \frac{\Gamma(2/3)}{3} \Lambda_c$
- deviations at $\rho \approx 0 \Rightarrow$ multifractality \equiv fluctuations (not taken into account in the self-consistent theory)

Confrontation with the experimental results

$$|\Psi(\rho, t = 150)|^2$$



$$\mathcal{G}(x) = \frac{3}{2} (3\rho^{3/2})^{-1/3} \text{Ai} \left[(3\rho^{3/2})^{-1/3} |x| \right] ?$$

YES! No adjustable parameter

Conclusion

First experimental observation of the Anderson transition with atomic matter waves (Chabé et al., 2008; Lemarié et al., 2009)

Full characterization of the transition

- Finite-Time scaling to overcome the experimental limitations
- First experimental unambiguous determination of the critical exponent $\nu \simeq 1.4 \pm 0.3 \simeq \nu_{\text{Anderson}} = 1.57 \pm 0.02$

New data: $\nu \simeq 1.5 \pm 0.2$

Universality of the Anderson transition with the quasiperiodic Kicked Rotor (Lemarié et al., 2009)

- The quasiperiodic Kicked Rotor belongs to the same universality class as for the Anderson model: $\nu = 1.59 \pm 0.01 = \nu_{\text{Anderson}}$

Critical wave function

- Analytical prediction for the critical wave-function from the self-consistent theory
- Very good agreement with numerical data and experimental data

The Anderson transition in 4D and 2D

- Quasi-periodic modulation with 3 frequencies or 1 frequency
- Critical exponent in 4D? Critical dimension?

Symmetries

- Possibility to break the Time-Reversal Symmetry! \equiv effective magnetic field
- \Rightarrow Anderson localization and transition in the Unitary class

Interactions

- Kicked Rotor with BEC
- Interactions controlled by Feshbach resonances