

Minimisation d'énergie sous contraintes, applications en algèbre linéaire et en contrôle linéaire

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Outline

Zolotarev problem

Applications

Constrained logarithmic potential theory

Asymptotics of discrete Zolotarev quantity

Explicit expression

Numerical examples

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Our problem

Consider

1. E_N, F_N disjoint sets of at most N elements in \mathbb{C}
2. $\mathcal{R}_n = \{p/q : p, q \text{ polynomials of degree } \leq n\}$
3. $\|f\|_{L^\infty(E)} = \sup_{z \in E} |f(z)|$

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Find N -th root asymptotics for

$$Z_n(E_N, F_N) := \inf_{r \in \mathcal{R}_n} \|r\|_{L^\infty(E_N)} \|1/r\|_{L^\infty(F_N)}$$

provided that $n, N \rightarrow \infty$ such that $n/N \rightarrow t > 0$, and

$$\nu_N(E_N) := \frac{1}{N} \sum_{\lambda \in E_N} \delta_\lambda \xrightarrow{*} \sigma_1 \quad \nu_N(F_N) := \frac{1}{N} \sum_{\lambda \in F_N} \delta_\lambda \xrightarrow{*} \sigma_2$$

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What is known ?

- E_N, F_N replaced by larger infinite sets E, F of positive capacity, [Gonchar 78] :

$$Z_n(E_N, F_N)^{1/n} \leq Z_n(E, F)^{1/n} \rightarrow \exp\left(-\frac{1}{\text{cap}(E, F)}\right)$$

- $\text{cap}(E, F)$ capacity of condenser with plates E (positive unit charge) and F (negative unit charge) in \mathbb{R}^2
- Extensions known for different degrees of numerator/denominator [Levin & Saff 01]

Zolotarev's findings

- For $E = [\alpha, \beta]$, $F = [-\beta, -\alpha]$: explicit formulas

$$\text{cap}([\alpha, \beta], [-\beta, -\alpha]) = \frac{K'(\alpha/\beta)}{2\pi K(\alpha/\beta)}$$

- Elliptic functions

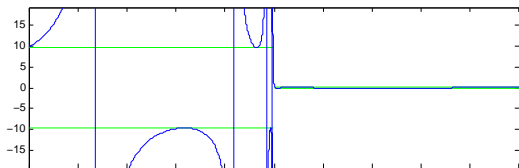
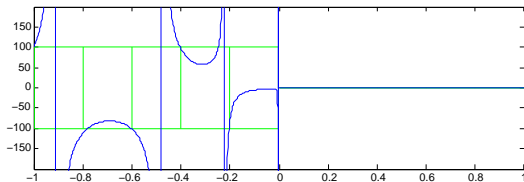
$$K(k) := \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \quad K'(k) = K(\sqrt{1-k^2})$$

- Then

$$Z_n([\alpha, \beta], [-\beta, -\alpha])^{1/n} \rightarrow \exp\left(-2\pi \frac{K(\alpha/\beta)}{K'(\alpha/\beta)}\right)$$

- Inequalities [Braess 80]

An example



- $E = 1/200 + \{0, 1/5, 2/5, 3/5, 4/5, 1\}$
- Top : $Z_4(E, -E) \sim 10^{-4}$
- Bottom : $Z_4(\text{conv}(E), \text{conv}(-E)) \sim 10^{-2}$

Main finding for real E_N, F_N

Under suitable smoothness, separation assumptions and assumptions on the form of limit measures σ_1, σ_2

$$Z_n(E_N, F_N)^{1/N} \rightarrow \exp\left(-\int_0^t 2\pi \frac{K(a_\tau/b_\tau)}{K'(a_\tau/b_\tau)} d\tau\right)$$

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Notice :

1. $\tau \mapsto a_\tau$ increasing, $\tau \mapsto b_\tau$ decreasing (defined implicitly).
2. If both functions constant : we recover result of Zolotarev.
3. Intervals $[a_\tau, b_\tau]$ shrinking implies superlinear rate.

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Sylvester/Lyapunov equation

- ADI -alternating direction implicit- method [Peaceman & Rachford 1955] : solve the Sylvester equation : $AX - XB = C$, $X \in \mathbb{C}^{N \times N}$, $A, B, C \in \mathbb{C}^{N \times N}$, A, B normal.

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- Symmetric case : $B = -A^*$ and C hermitian, the equation becomes $AX + XA^* = C$, known as Lyapunov equation.
- Unique solution if $\Lambda(A) \cap \Lambda(B) = \emptyset$.
- True in Lyapunov case if A symmetric positive definite (which we consider for the moment).
- Can be formulated as N^2 by N^2 linear system, but generally not efficient.

The ADI iteration for $AX - XB = C$

- With an initial guess $X_0 \in \mathbb{C}^{N \times N}$ and free parameters $p_k, q_k \in \mathbb{C}$
- Written at two steps :

$$(A - p_k)X_{k+1/2} = X_k(B - p_k) + C$$

$$X_{k+1}(B - q_k) = (A - q_k)X_{k+1/2} - C$$

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- Thus

$$\frac{\|X - X_n\|}{\|X - X_0\|} \leq \|r_n(A)\| \|r_n(B)^{-1}\| \quad r_n(z) := \prod_{k=0}^{n-1} \frac{z - q_k}{z - p_k}.$$

- Bound $Z_n(\Lambda(A), \Lambda(B))$ in case of optimal parameters
- Around ADI : [Smith 68], [Penzl 00], [Sabino 06], [Benner 08]

ADI bound $Z_n(\Lambda(A), \Lambda(B))$ in case of optimal parameters

CLASSICAL APPROACH : replace $\Lambda(A), \Lambda(B)$ by larger "nice" sets E, F and use parameters p_k, q_k for optimal rational function for $Z_n(E, F)$.

PROBLEM : we do forget about that there are perhaps only relatively few eigenvalues close to 0.

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But for many sequences of matrices $(A_N)_{N \geq 1}$ coming from discretization of PDE with varying "stepsize", the limit distribution σ_1 for eigenvalues $E_N = \Lambda(A_N)$ is known, even if we do not know so much about the eigenvalues.

Other applications

- SV of matrices with small displacement rank

$$\sigma_{1+n\rho}(X) \leq Z_n(\Lambda(A), \Lambda(B)) \sigma_1(X), \quad \rho = \text{rg}(AX - XB).$$

- Approximation of signum function : $E \subset \mathbb{R}_+$, $F \subset \mathbb{R}_-$ discrete disjoint sets,

$$S_n(E, F) = \frac{\sqrt{Z_n(E, F)}}{1 + Z_n(E, F)}$$

with

$$S_n(E, F) := \inf \{S_R, R \in \mathcal{R}_n, \text{ real coefficients}\}$$

and

$$S_R := \max_{x \in E \cup F} |R(x) - s(x)|.$$

Link with linear control theory

- Model reduction for continuous stationary linear dynamical system.
- Σ square system realizing

$$\mathcal{H}(z) = C(zI - A)^{-1}B$$

Error bound : $\|\Sigma - \Sigma_r\| \leq (\sigma_{r+1} + \dots + \sigma_n)$ where Σ_r obtained by model reduction and balanced truncation, (σ_ℓ) Hankel SV of Σ and eigenvalues of X solution of

$$AX + XA = -BC.$$

- To predict the quality of the reduction : solve (large) Sylvester equations.

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Continuous potential theory

- α, β finite positive Borel measures on \mathbb{C} with compact support
- Logarithmic potential :

$$U^\alpha(z) := \int \log \frac{1}{|z-x|} d\alpha(x)$$

- Logarithmic energy :

$$I(\alpha, \beta) := \int \int \log \frac{1}{|x-y|} d\alpha(x) d\beta(y) = \int U^\alpha(y) d\beta(y)$$

- Signed measures : $\mu = \mu_1 - \mu_2$ (Jordan decomposition)

$$I_Q(\mu) := I(\mu_1, \mu_1) - 2I(\mu_1, \mu_2) + I(\mu_2, \mu_2) + 2 \int Q(z) d\mu(z)$$

Discrete potential theory

- Normalized counting measure

$$\nu_N(E_N) = \frac{1}{N} \sum_{\lambda \in E_N} \delta_\lambda$$

- Discrete energy

$$I^*(\nu_N(E_N), \nu_N(F_N)) := \frac{1}{N^2} \sum_{\substack{x \in E_N, y \in F_N, \\ x \neq y}} \log \frac{1}{|x - y|}$$

and

$$I^*(\nu_N(E_N) - \nu_N(F_N)) := I^*(\nu_N(E_N)) + I^*(\nu_N(F_N)) \\ - 2I^*(\nu_N(E_N), \nu_N(F_N))$$

Why potential theory?

$$p(z) = \prod_{\lambda \in E_N} (z - \lambda) \implies -\frac{1}{N} \log |p(z)| = U^{\nu_N(E_N)}(z)$$

Each absolutely continuous measure can be approached by zero counting measures

Difficulties while passing to the limit : $\nu_N(E_N) \xrightarrow{*} \mu$ implies

$$\liminf U^{\nu_N(E_N)}(z) \geq U^\mu(z), \quad \liminf I^*(\nu_N(E_N)) \geq I(\mu).$$

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Rational functions :

$$r(z) = \prod_{\lambda \in E_N} (z - \lambda) / \prod_{\lambda \in F_N} (z - \lambda) \implies -\frac{1}{N} \log |r(z)| = U^{\nu_N(E_N) - \nu_N(F_N)}(z)$$

signed measures instead of positive measures.

Logarithmic potential theory

	Positive measures	Signed measures
Q	[Gauss 1830], [Frostman 36], [Rakhmanov 84], [Mhaskar & Saff 84]	[Gonchar & Rakhmanov 84]
σ	[Dragnev & Saff 97], [Beckermann & Kuijlaars 01]	

Potential theory with constraint

For $t \in (0, \min(\sigma_1(\mathbb{C}), \sigma_2(\mathbb{C})))$, we define

$$\mathcal{M}_\sigma^t := \{\mu := \mu_1 - \mu_2, 0 \leq \mu_j \leq \sigma_j, \mu_j \text{ measure of mass } t\}$$

and the constrained energy problem :

among all the measures μ which belongs to \mathcal{M}_σ^t , determine μ^t such that

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Theorem 1

Unique minimizer μ^t characterized by the following variational conditions : there are two constants F_1^t and F_2^t such that

$$U^{\mu^t}(z) = F_1^t \quad z \in \text{supp}(\sigma_1 - \mu_1^t) =: S_1^t$$

$$U^{\mu^t}(z) \leq F_1^t \quad z \in \mathbb{C}$$

$$-U^{\mu^t}(z) = F_2^t \quad z \in \text{supp}(\sigma_2 - \mu_2^t) =: S_2^t$$

$$-U^{\mu^t}(z) \leq F_2^t \quad z \in \mathbb{C}$$

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$$\lim_{N \rightarrow +\infty} I^*(\nu_N(E_N) + \nu_N(F_N)) = I(\sigma_1 + \sigma_2)$$

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5.

$$\lim_{N \rightarrow +\infty} \text{dist}(E_N, F_N)^{1/N} = 1$$

Precise statement of asymptotics

With $F^t := F_1^t + F_2^t$

Theorem 2

Under the assumptions 1, 2, 3,

$$\limsup_{\substack{n, N \rightarrow \infty \\ n/N \rightarrow t}} Z_n(E_N, F_N)^{1/N} \leq \exp(-F^t) \quad (1)$$

and with 4, 5,

$$\lim_{\substack{n, N \rightarrow \infty \\ n/N \rightarrow t}} Z_n(E_N, F_N)^{1/N} = \exp(-F^t) \quad (2)$$

Upper bound

Given for $\ell = 1, 2$

$$S_\ell(t) := \left\{ U^{\mu^t}(z) = (-1)^{\ell+1} F_\ell^t \right\}$$

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Construct good rational function with

- Zeros : every point in $E_N \setminus S_1(t)$
- Poles : every point in $F_N \setminus S_2(t)$
- Method : discretizing μ^t

Lower bound

- Consider **rational Fekete points** : subsets $E_n^* \subset E_N$, $F_n^* \subset F_N$ both with $n + 1$ elements and minimal $I_N(E_n^*, F_n^*)$.
- Show that $\nu_N(E_n^*) - \nu_N(F_n^*) \stackrel{*}{\sim} \mu^t$, and

$$\frac{1}{(n+1)^2} Z_n^*(E_n^*, F_n^*) \leq Z_n(E_N, F_N)$$

- Method : singular values of a Cauchy-type matrix built on Fekete points.

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A more explicit expression for F^t

From now on, $\sigma_1(\mathbb{C}) = \sigma_2(\mathbb{C})$.

The formula for $\triangleright F^t$ is not very explicit, link with superlinear convergence? In the polynomial case, explicit formula given by [Kuijlaars & Beckermann 01] via [Buyarov & Rakhmanov 99].

Theorem 3

With $S_\ell^t = \text{supp}(\sigma_\ell - \mu_\ell^t)$, the families $(S_1^t)_t$, $(S_2^t)_t$ and $(\text{cap}(S_1^t, S_2^t))_t$ decrease in t .

Moreover,

$$F^t = \int_0^t \frac{d\tau}{\text{cap}(S_1^\tau, S_2^\tau)}$$

Sketched proof

- Duality with vector equilibrium problem with external field
- Monotonicity of sets follows from [Lapik 06]
- Characterization by generalization of Rakhmanov-Maskhar-Saff functional
- $t \mapsto F^t$ **convex**, with derivative $1/\text{cap}(S_1^t, S_2^t)$

Sufficient conditions for interval case

Consider constraint with symmetry $\sigma_1'(x) = -\sigma_2'(-x)$.

E.g. $A = A^* = -B \geq 0$ in ADI.

Same symmetry for μ^t .

Theorem 4

Let $\text{supp}(\sigma_1) = [a, b] \subset [0, +\infty]$. If

$$x \mapsto \sqrt{(x^2 - a^2)(b^2 - x^2)}\sigma_1'(x)$$

is increasing then

$$S_1^t = -S_2^t = [\alpha, b].$$

Computation of S_ℓ^t

- Interval symmetric case : $S_1^t = [\alpha, b] = -S_2^t$.
- Minimization of RMS functional.
- Integral equation :

$$t = \frac{2}{b\pi} \int_{[0,\alpha] \cup [b,+\infty]} \Pi \left(1, 1 - \frac{\alpha^2}{z^2}, \sqrt{1 - \frac{\alpha^2}{b^2}} \right) z \sqrt{\left| \frac{z^2 - b^2}{z^2 - \alpha^2} \right|} d\sigma(z)$$

with

$$\Pi(z, \nu, k) := \int_0^z \frac{dt}{(1 - \nu t^2) \sqrt{(1 - t^2)(1 - k^2 t^2)}}$$

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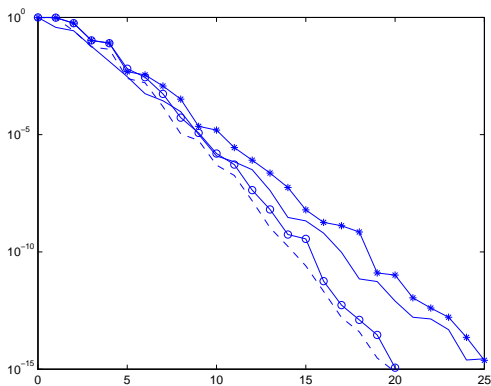
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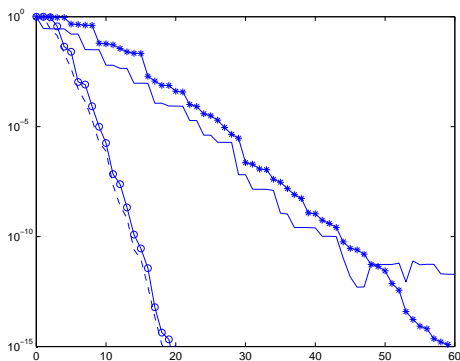
- ADI for Lyapounov equation $AX + XA = B$
- Choose X randomly, then compute B and solve the equation with ADI, initial guess $X_0 = 0$
- x-axis : number of iterations
- y-axis : error and upper bound for both continuous and discrete parameters
- Parameters : built with Leja-Bagby rational points

Equidistant eigenvalues



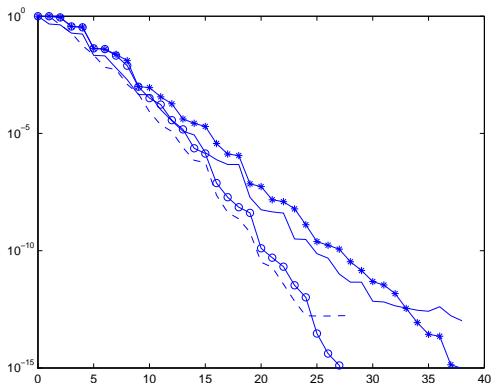
$$E_N^{equi} := \left\{ \frac{2k-1}{2N} : 1 \leq k \leq N \right\}, \quad N = 100$$

Toy example



$$E_N^{\cos} := \left\{ \cos\left(\frac{\pi k}{2N}\right) + \frac{1}{N^4} : 1 \leq k \leq N \right\}, \quad N = 100$$

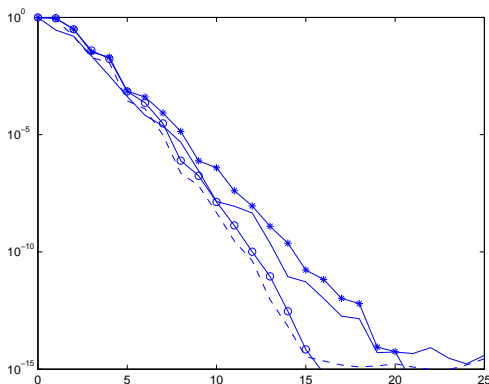
2D Laplacian



$$E_N^{2D} = \left\{ 2 - 2 \cos \left(\frac{k\pi}{N+1} \right) \right\}_{1 \leq k \leq N}, \quad N = 100$$

In this case : [Sabino 06] : 40 iterations, [Penzl 00] : 70 iterations.

4D Laplacian



$$E_N^{4D} = \left\{ 4 - 2 \left(\cos \left(\frac{k\pi}{N+1} \right) + \cos \left(\frac{j\pi}{N+1} \right) \right) \right\}_{1 \leq j, k \leq m}, \quad m = 10$$

Conclusion, future work

- Potential theory learns where the discrete nature is important and in which case the discrete nature is important.
- More general necessary condition for the interval case ?
- Asymptotics of Leja-Bagby points for discrete sets ?
- Computation of the Ritz values
- Strong asymptotics ?
- Other choice of ratio ?

Leja-Bagby points



$$q_{m+1} = \arg \min_{z \in E} |w(z)r_m(z)|, \quad r_m(z) = \prod_{j=1}^m \frac{z - q_j}{z + q_j}, \quad r_0(z) = 1.$$

- Suggested in the context of ADI in [Levenberg & Reichel 93]
- Asymptotic distribution : described by equilibrium measure ?