# Optimization in Graphs Under Degree Constraints.

## Application to Telecommunication Networks


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Mascotte – MA4

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# Outline of the talk

Traffic grooming - - - - - - - - - - - - - - - - - - - - - - Degree-constrained subgraph problems

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- Motivation
- Overview of the results

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# General idea

- WDM (Wavelength Division Multiplexing) networks
  - 1 wavelength (or frequency) = up to 40 Gb/s
  - 1 fiber = hundreds of wavelengths = Tb/s

- Traffic grooming consists in packing low-speed traffic flows into higher speed streams

  $\longrightarrow$ we allocate the same wavelength to several low-speed requests (TDM, Time Division Multiplexing)

- Objectives:
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# Definitions

- **Request** $(i, j)$: two vertices $(i, j)$ that want to exchange (low-speed) traffic

- **Grooming factor** $C$:

$$C = \frac{\text{Capacity of a wavelength}}{\text{Capacity used by a request}}$$

  ⋆ Typical values of the grooming factor:
  SDH: $4, 16, 64, 256, \ldots$
  SONET: $3, 12, 48, \ldots$

  **Example**:
  Capacity of one wavelength $= 2.5\ Gb/s$
  Capacity used by a request $= 640\ Mb/s$ $\Rightarrow C = 4$

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# ADM and OADM

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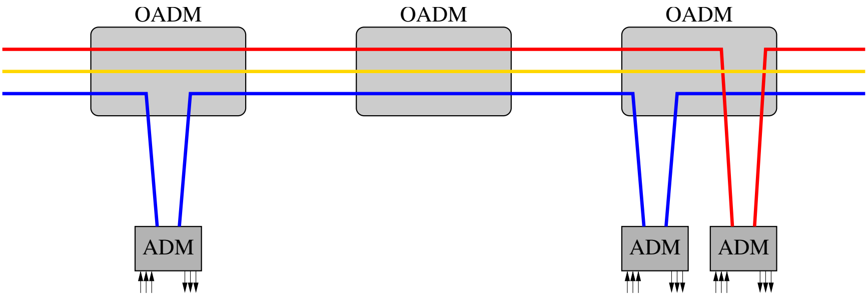
- **ADM** (Add/Drop Multiplexer)= insert/extract an OC/STM (electric low-speed signal) to/from a wavelength

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- We need to use an **ADM only at the endpoints of a request** (lightpaths) in order to save as many ADMs as possible

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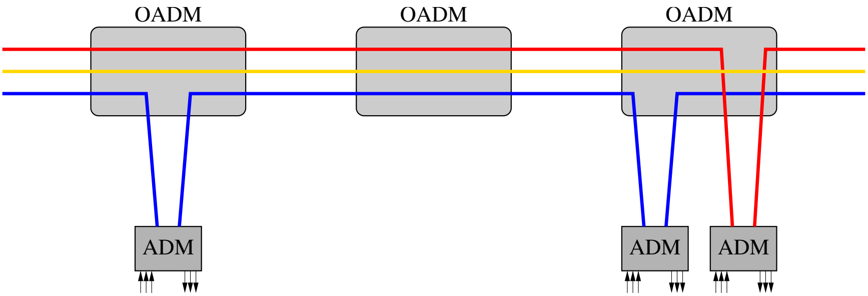


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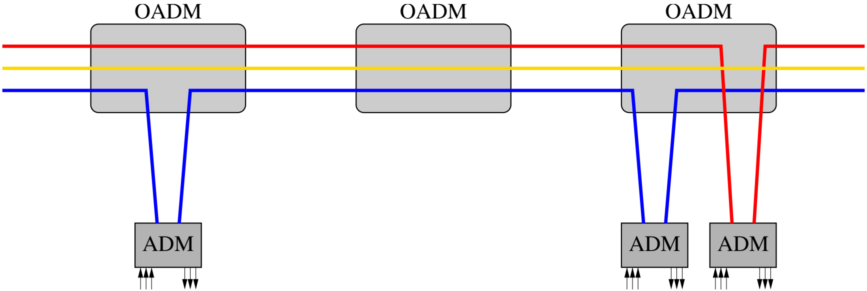


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# To fix ideas...

- Model:

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| Topology | $\rightarrow$ | graph $G$ |
| Request set | $\rightarrow$ | graph $R$ |
| Grooming factor | $\rightarrow$ | integer $C$ |
| **Wavelength** | $\rightarrow$ | **Subgraph** of $R$ |
| Requests in a **wavelength** | $\rightarrow$ | edges in a **subgraph** of $R$ |
| ADM in a wavelength | $\rightarrow$ | vertex in a subgraph of $R$ |

- A fundamental case is when $G = \overrightarrow{C}_n$ (**unidirectional ring**)
- It is also natural to consider **symmetric requests**

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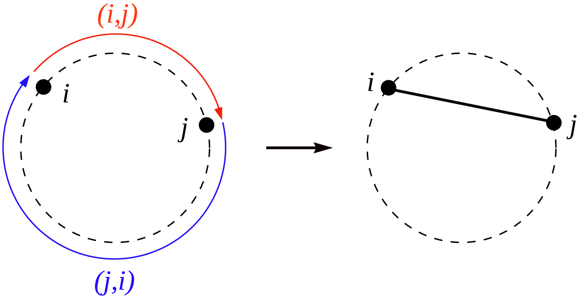
# Unidirectional ring with symmetric requests

- **Symmetric requests**: whenever there is the request $(i, j)$, there is also the request $(j, i)$.




- W.l.o.g. requests $(i, j)$ and $(j, i)$ are in the same subgraph
  $\rightarrow$ each pair of symmetric requests induces load 1
  $\rightarrow$ **grooming factor** $C$ $\Leftrightarrow$ **each subgraph has** $\leq C$ **edges.**

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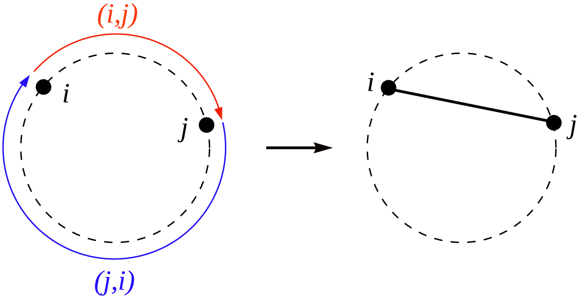


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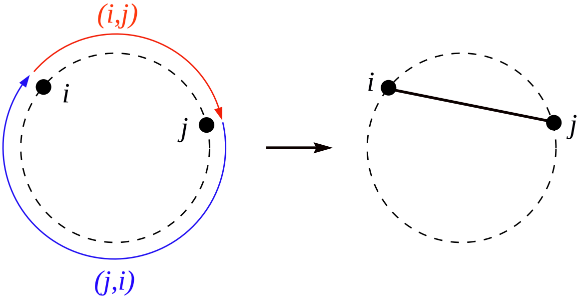


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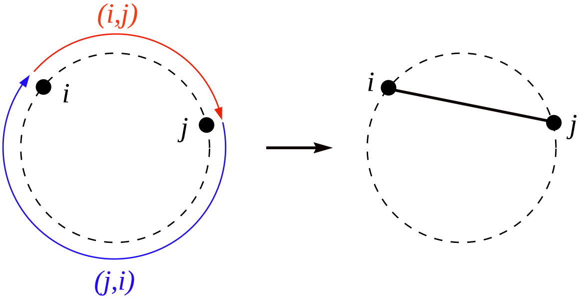


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# Statement of the problem

**Traffic Grooming in Unidirectional Rings** (with symmetric requests)

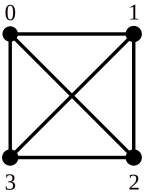
| | |
|---|---|
| **Input** | An *undirected* graph $R$ on $n$ nodes (request set);<br>A grooming factor $C$. |
| **Output** | A partition of $E(R)$ into subgraphs<br>$R_1, \ldots, R_W$ with $\vert E(R_i)\vert \leq C$, i=1,...,W. |
| **Objective** | Minimize $\sum_{i=1}^{W} \vert V(R_i)\vert$. |

# Example (unidirectional ring with symmetric requests)

$n = 4$

$R = K_4$

$C = 3$



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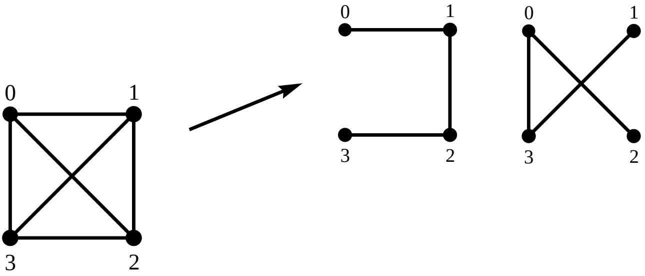

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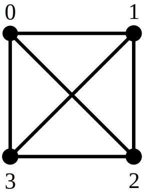


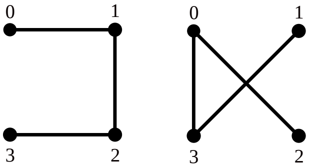


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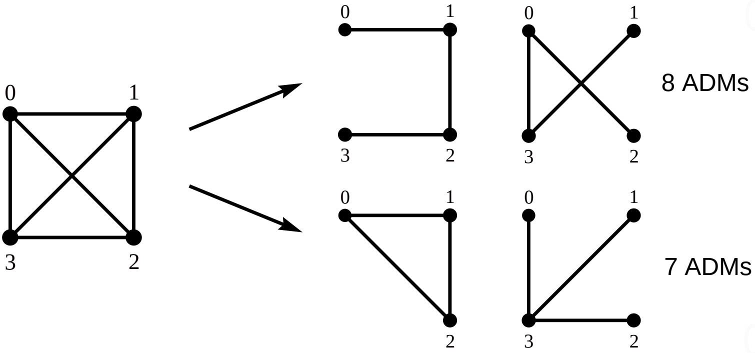

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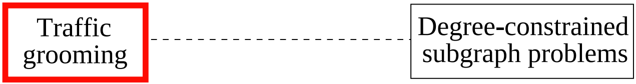

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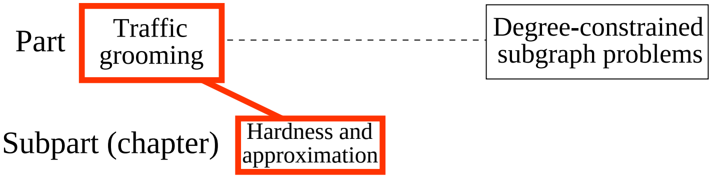

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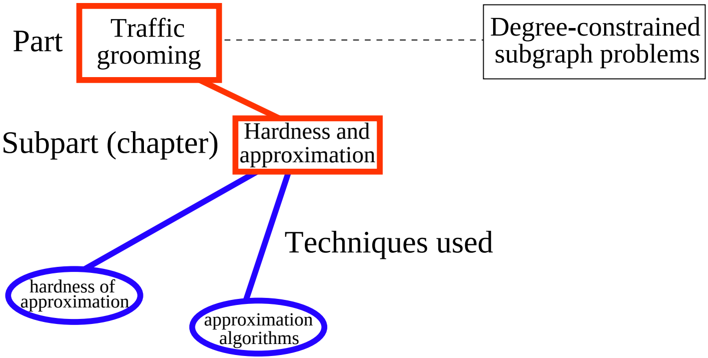

# Preliminaries: approximation algorithms

- Given a (typically NP-hard) **minimization** problem Π, ALG is an **$\alpha$-approximation algorithm** for Π (with $\alpha \geq 1$) if for any instance $I$ of Π,

$$ALG(I) \leq \alpha \cdot OPT(I).$$

- **Class APX (Approximable):**

  an NP-hard optimization problem is in APX if it can be approximated within a constant factor.

  **Example**: MINIMUM VERTEX COVER has a 2-approximation.

- **Class PTAS (Polynomial-Time Approximation Scheme):**

  an NP-hard optimization problem is in PTAS if it can be approximated within a constant factor $1 + \varepsilon$, for all $\varepsilon > 0$
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   [Chiu and Modiano. *IEEE JLT'00*]
2. **Not in APX** if $C$ is part of the input
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3. Remains **NP-complete** for fixed $C \geq 1$
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★ **Open problem:** inapproximability for fixed $C$?

**Conjecture**: Not in PTAS for fixed $C$.

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2. $\mathcal{O}(\log C)$-approximation algorithm, with running time $\mathcal{O}(n^C)$ [Flammini et al. *ISAAC'05, JDA'08*]
3. But in backbone networks, it is usually the case that $C \geq n$.

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*There is a polynomial-time approximation algorithm that approximates* RING TRAFFIC GROOMING *within a factor* $\mathcal{O}(n^{1/3}\log^2 n)$ *for any* $C \geq 1$.

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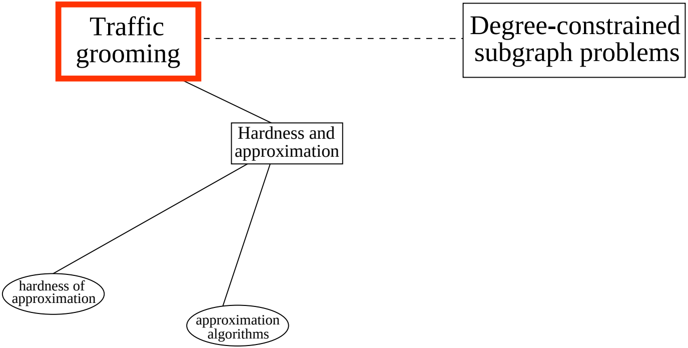
# Graph of the thesis



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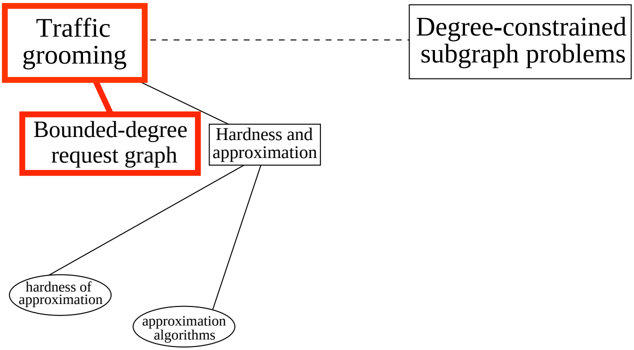

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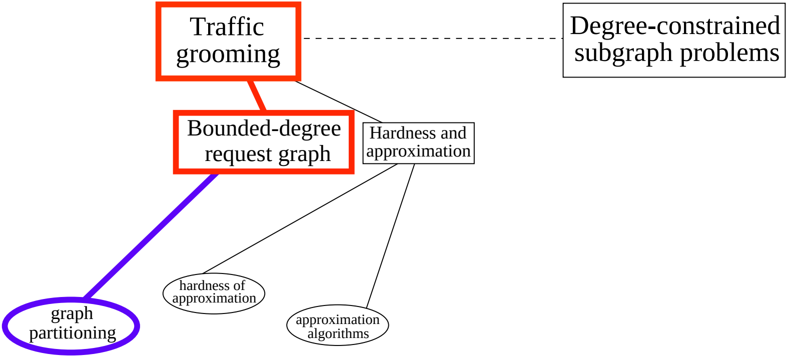

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- In the literature so far:
  place ADMs at nodes for a **fixed request graph.**
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# The parameter $M(C, \Delta)$

- $\Delta$-graph: graph with maximum degree at most $\Delta$.
- $C$-edge partition of $G$: partition of $E(G)$ into subgraphs with $\leq C$ edges.
- The problem is equivalent to determining the following parameter:

- Therefore, we focus on determining $M(C, \Delta)$.
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## Theorem (Li and S.)

*Let $\Delta \geq 2$ be even. Then for any $C \geq 1$, $M(C, \Delta) = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$.*

## Proof.

- We have just seen the lower bound. Construction:
  - Orient the edges of $G = (V, E)$ in an Eulerian tour.
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*Let $\Delta \geq 3$ be odd. Then for any $C \geq 1$, $M(C, \Delta) \leq \left\lceil \frac{C+1}{C}\frac{\Delta}{2} + \frac{C-1}{2C} \right\rceil$.*

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**Question**: is the lower bound $\left\lceil \frac{C+1}{C}\frac{\Delta}{2} \right\rceil$ always attained?

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# Open cases

Summarizing, we established the value of $M(C, \Delta)$ for "almost" all values of $C$ and $\Delta$, leaving **open** only the case where:

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- $C \geq 4$; and
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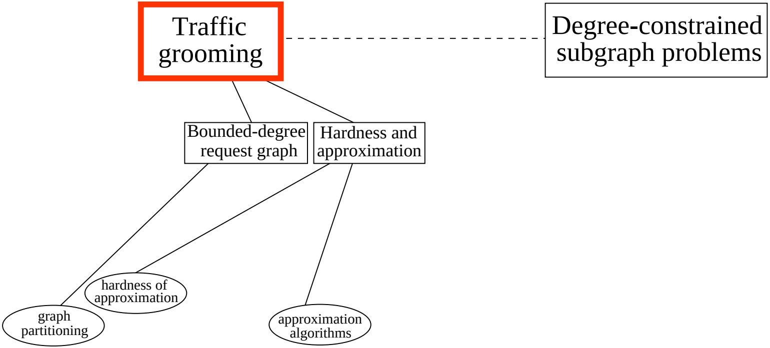

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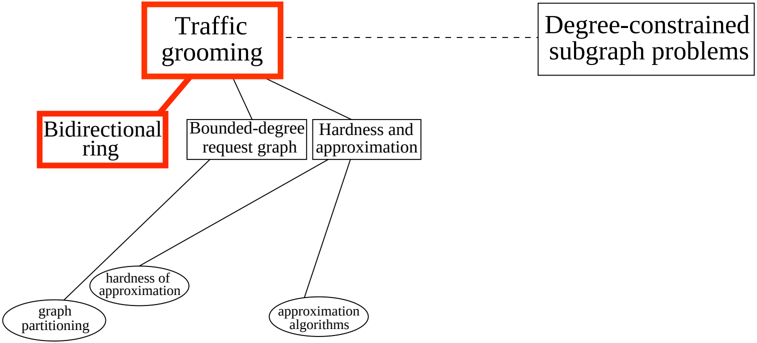

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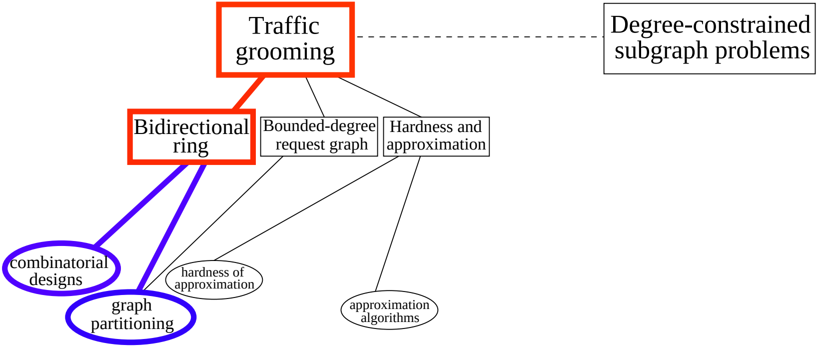

# Bidirectional rings

With *Jean-Claude Bermond* and *Xavier Muñoz*

- Most of the research had been done for **unidirectional rings.**
- We consider the bidirectional ring with
  - ⋆ all-to-all requests.
  - ⋆ shortest path routing.
- We provide:
  1. Statement of the problem and general lower bounds.
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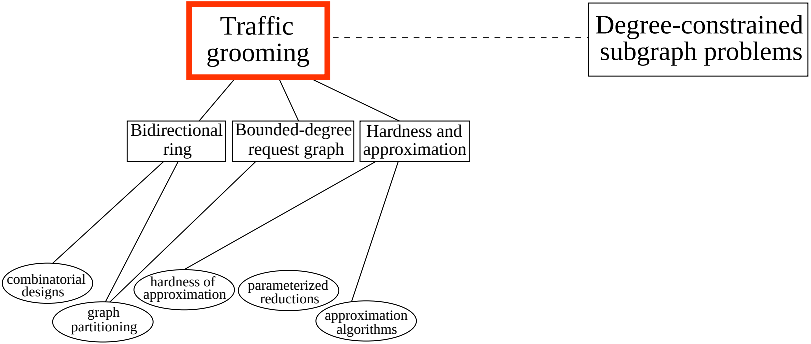

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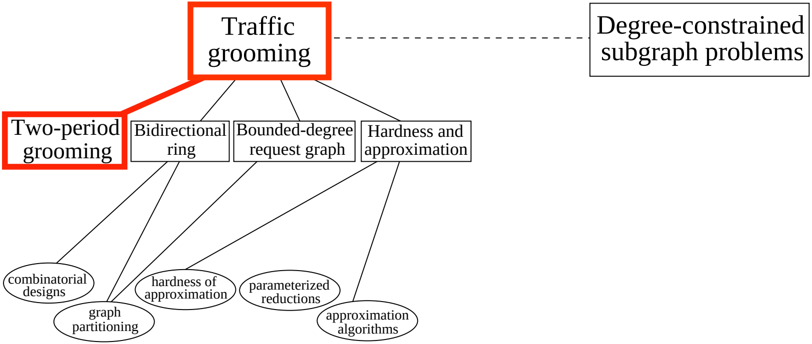

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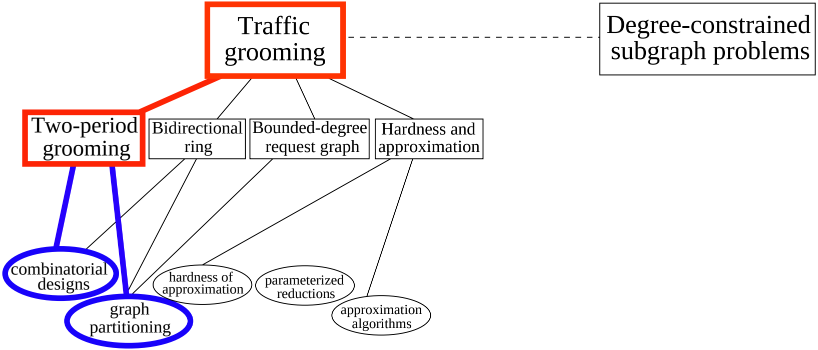

# 2-period traffic grooming in unidirectional rings

With *J-C. Bermond*, *C.J. Colbourn*, *L. Gionfriddo*, and *G. Quattrocchi*

- We consider a pseudo-dynamic scenario in unidirectional rings:
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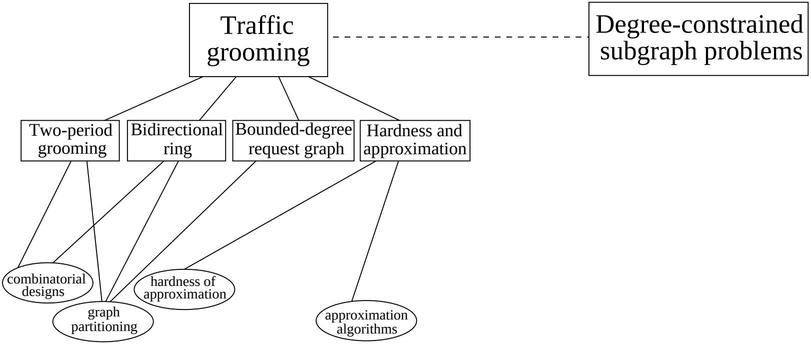

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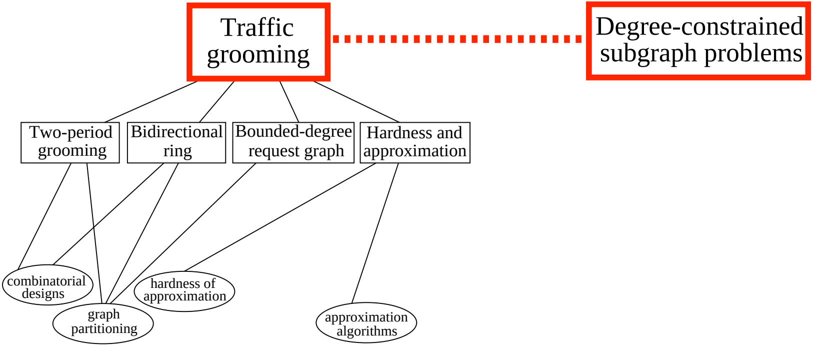

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**Theorem (Amini, Pérennes, and S.)**

*There is a polynomial-time approximation algorithm that approximates* RING TRAFFIC GROOMING *within a factor* $\mathcal{O}(n^{1/3} \log^2 n)$ *for any* $C \geq 1$.

1. partition the requests into groups of **similar length [factor** $\log n$**]**
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**Input:** An undirected graph $G = (V, E)$ and a positive integer $k$.
**Output:** A subset $S \subseteq V$, with $|S| = k$, such that $|E(G[S])|$ is maximized.

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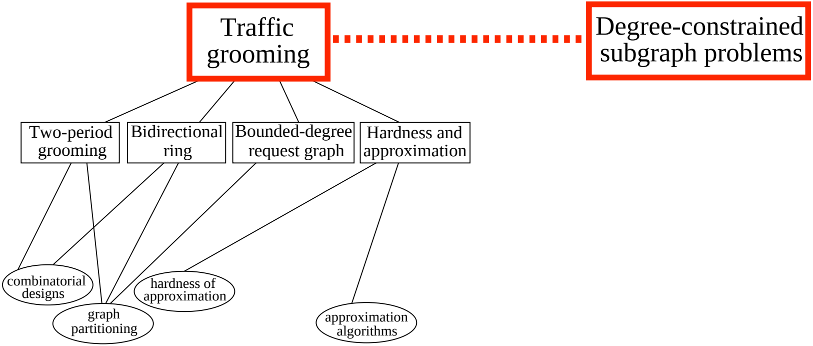

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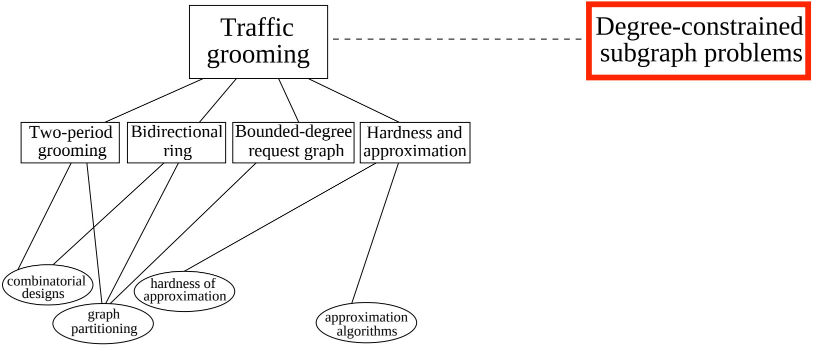

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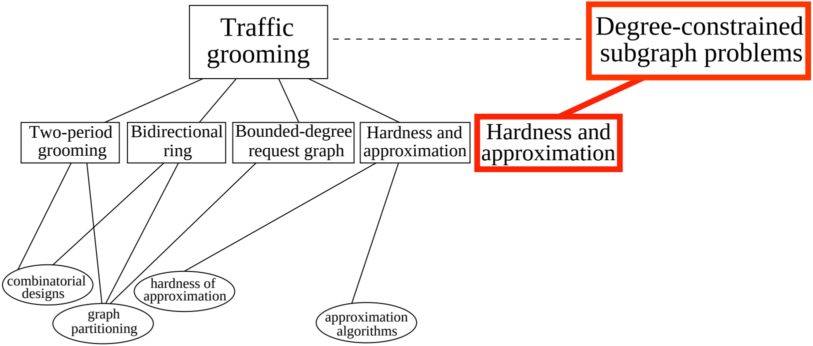

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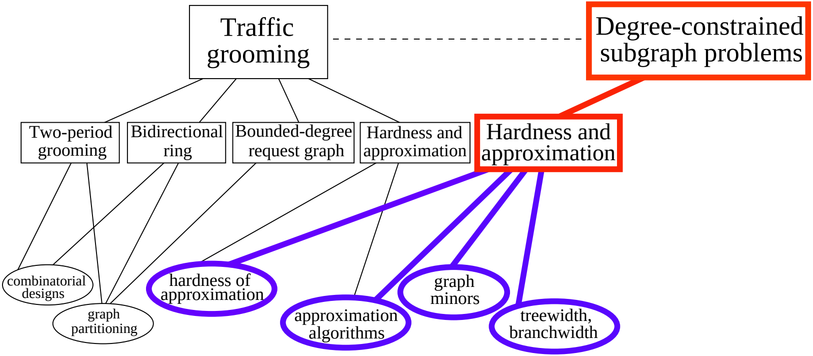

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**MINIMUM SUBGRAPH OF MINIMUM DEGREE $\geq d$ (MSMD$_d$)**:

**Input:** an undirected graph $G = (V, E)$ and an integer $d \geq 3$.

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With *Omid Amini*, *David Peleg*, *Stéphane Pérennes* and *Saket Saurabh*

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   - first we prove that $\text{MSMD}_d$ is not in PTAS (unless P=NP).
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**MAXIMUM $d$-DEGREE-BOUNDED CONNECTED SUBGRAPH (MDBCS$_d$):**

**Input:**

- an undirected graph $G = (V, E)$,
- an integer $d \geq 2$, and
- a weight function $\omega : E \rightarrow \mathbb{R}^+$.

**Output:**

a subset of edges $E' \subseteq E$ of **maximum weight**, s.t. $G' = (V, E')$

- is connected (except isolated vertices), and
- satisfies $\Delta(G') \leq d$.

- It is one of the classical **NP**-hard problems of *[Garey and Johnson, Computers and Intractability, 1979]*.
- If the output subgraph is not required to be connected, the problem is in **P** for any $d$ (using matching techniques). [Lovász, 70's]
- For *fixed* $d = 2$ it corresponds to the LONGEST PATH problem.

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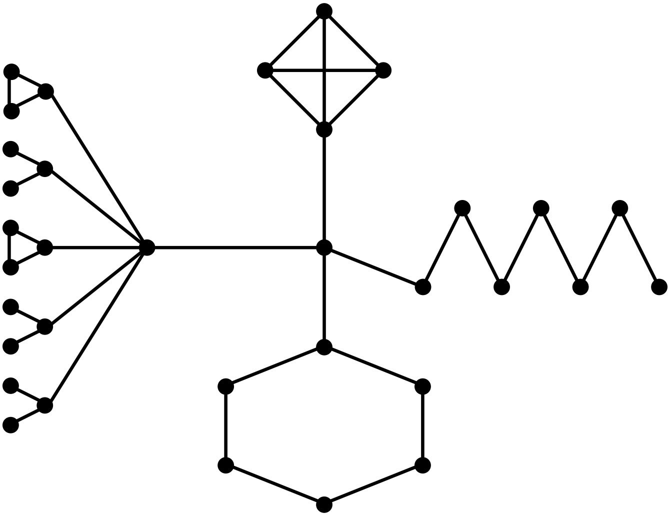

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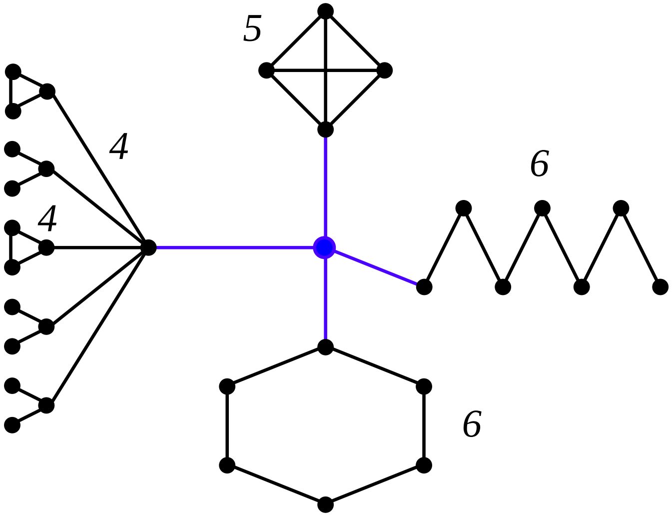

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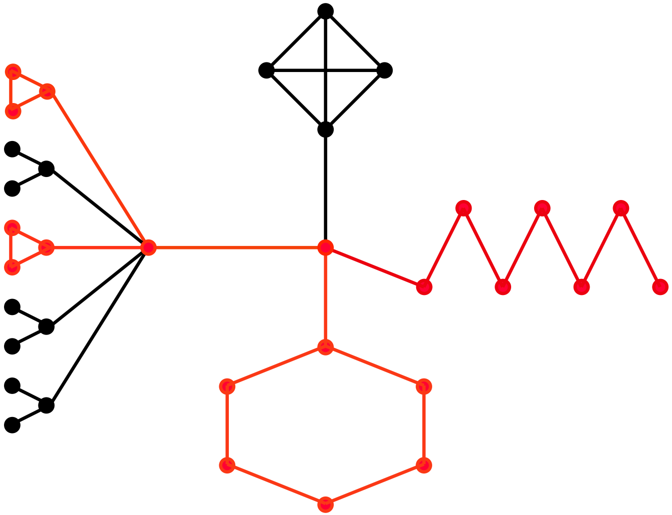

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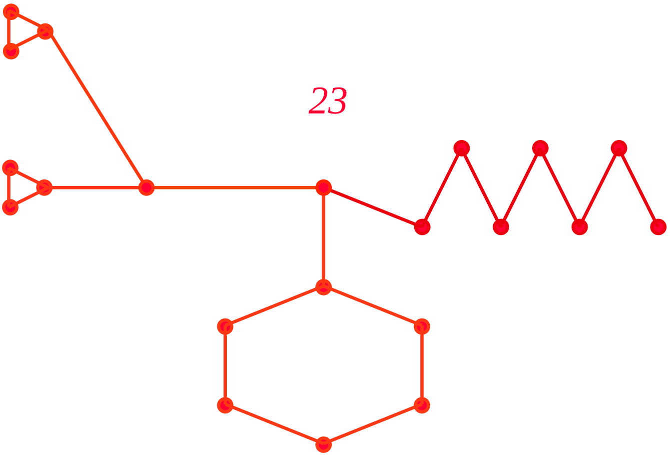

# Hardness and approximation

With Omid Amini, David Peleg, Stéphane Pérennes and Saket Saurabh

1. not in APX for any fixed $d \geq 2$.
2. if there is a polynomial time algorithm for $\text{MDBCS}_d$, $d \geq 2$, with performance ratio $2^{\mathcal{O}(\sqrt{\log n})}$, then $\text{NP} \subseteq \text{DTIME}(2^{\mathcal{O}(\log^5 n)})$.
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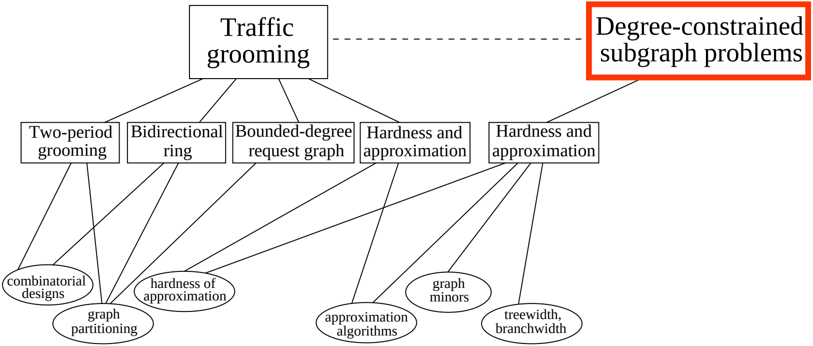

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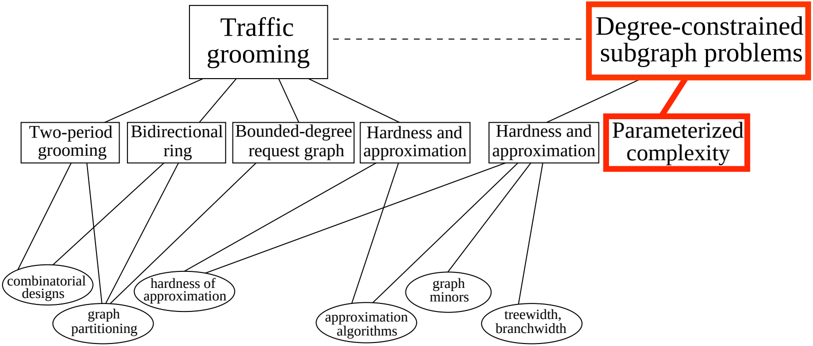

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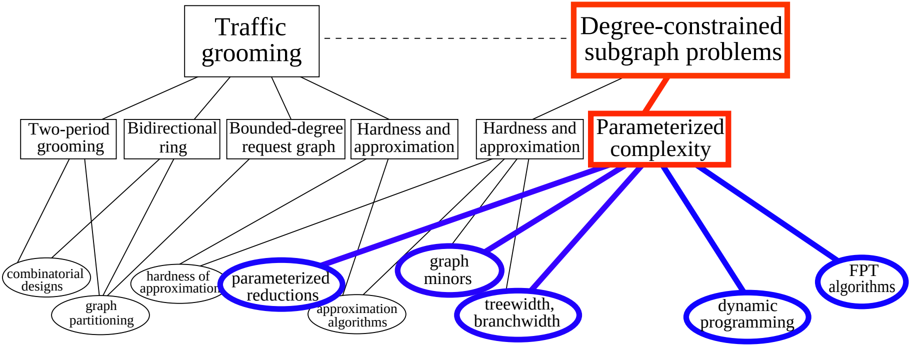

# Some words on parameterized complexity

- Idea: given an NP-hard problem, fix one parameter of the input to see if the problem gets more "tractable".

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- Given a (NP-hard) problem with input of size $n$ and a parameter $k$, a fixed-parameter tractable (FPT) algorithm runs in

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With *Omid Amini* and *Saket Saurabh*

- We have studied the parameterized complexity of finding degree-constrained subgraphs, with

  parameter $=$ number of vertices of the desired subgraph

- Namely, given two integers $d$ and $k$, the problems of finding
  1. a $d$-regular subgraph (induced or not) with at most $\leq k$ vertices.
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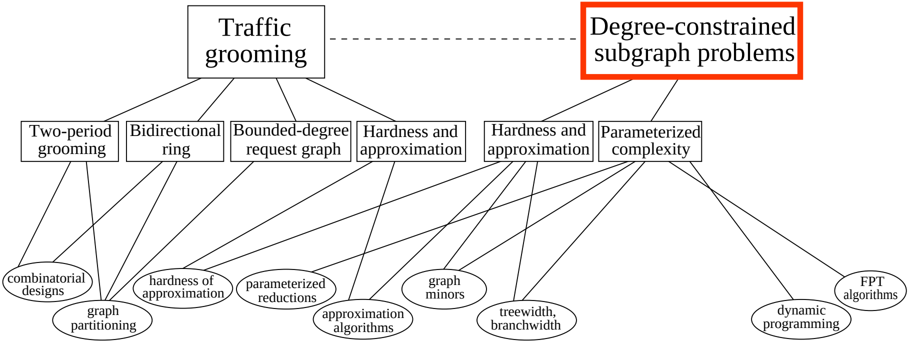

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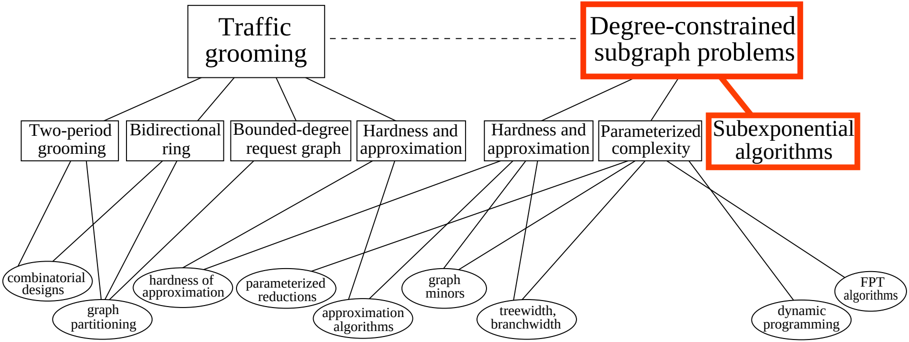

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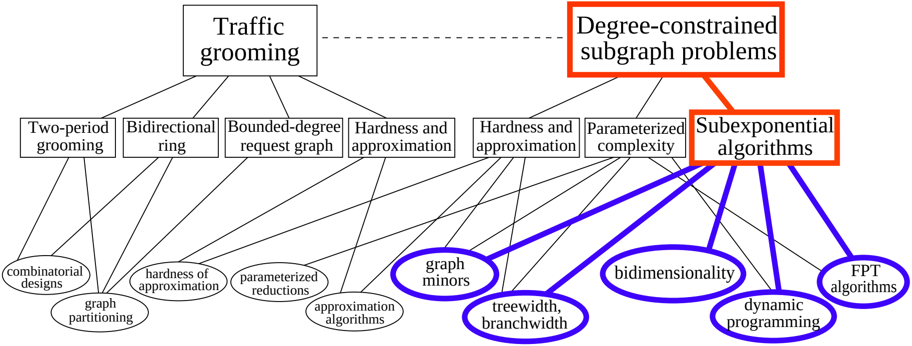

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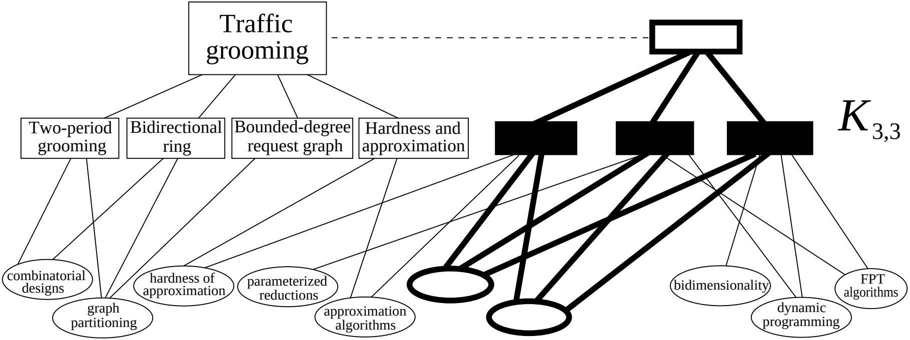

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Given a parameter **P** defined in a planar graph $G$, $\mathbf{P}(G) \leq k$ ?

First we compute $\mathbf{bw}(G)$. [Seymour and Thomas. *Combinatorica'94*]

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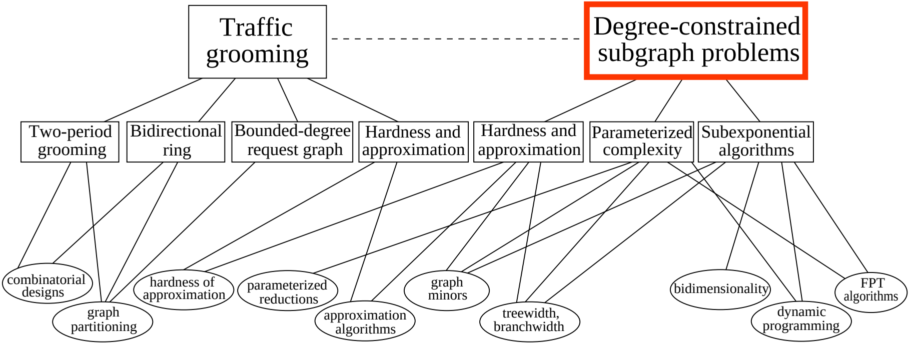
| |
|---|
| If **bw**$(G)$ is "small", we decide **P** by "fast" dynamic programming. |

- **Catalan structures**.
  [Dorn, Fomin, Thilikos. *ICALP'07, SODA'08*]

★ With D.M. Thilikos we have adapted this framework to $\text{MDBCS}_d$, as well as for a few variants, introducing some general techniques.

# Graph of the thesis



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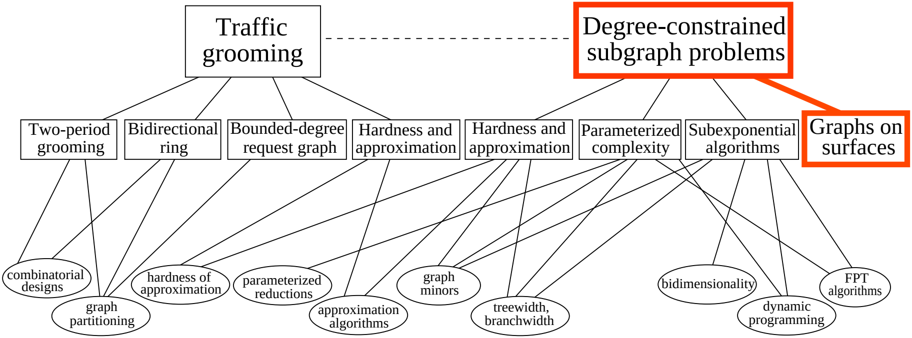

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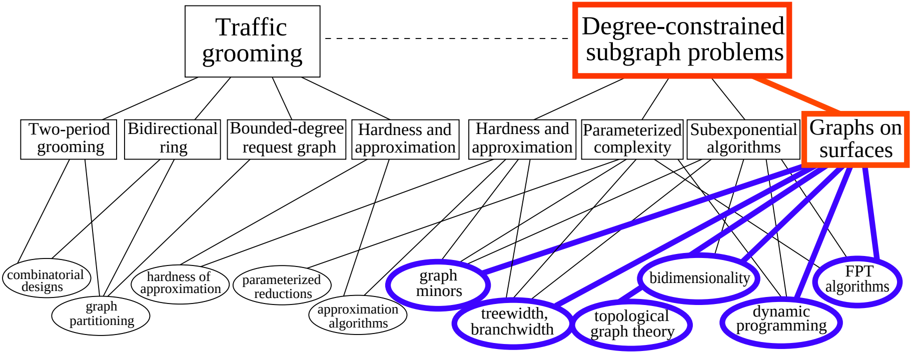
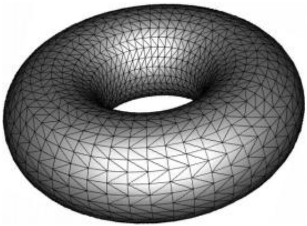
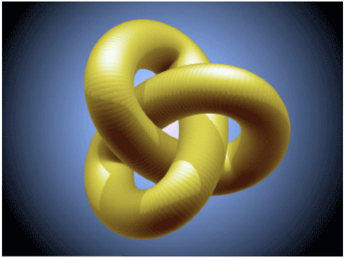


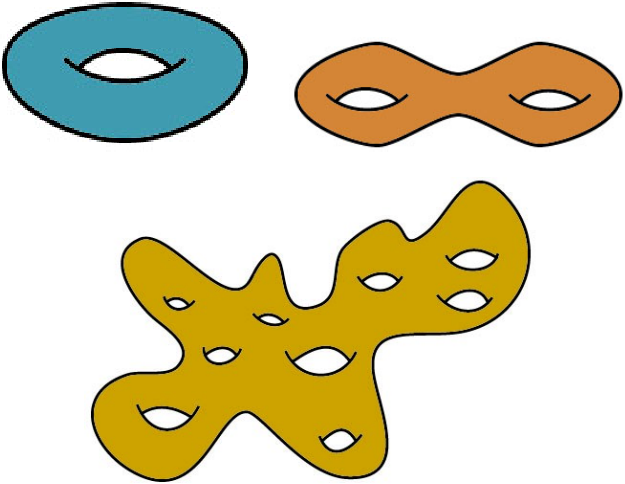
# Surfaces

- Surface: connected compact 2-manifold.

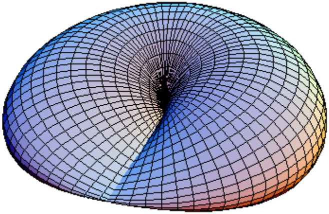


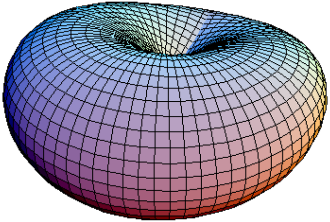


# Handles



# Cross-caps





# Genus of a surface

- The surface classification Theorem: any compact, connected and without boundary surface can be obtained from the sphere $\mathbb{S}^2$ by adding handles and cross-caps.
- Orientable surfaces: obtained by adding $g \geq 0$ *handles* to the sphere $\mathbb{S}^2$, obtaining the $g$-torus $\mathbb{T}_g$ with Euler genus $\mathbf{eg}(\mathbb{T}_g) = 2g$.
- Non-orientable surfaces: obtained by adding $h > 0$ *cross-caps* to the sphere $\mathbb{S}^2$, obtaining a non-orientable surface $\mathbb{P}_h$ with Euler genus $\mathbf{eg}(\mathbb{P}_h) = h$.

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# Graphs on surfaces

- An embedding of a graph $G$ on a surface $\Sigma$ is a drawing of $G$ on $\Sigma$ without edge crossings.
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# Dynamic programming for graphs on surfaces

With *Juanjo Rué* and *Dimitrios M. Thilikos*

- Let $G$ be a graph on $n$ vertices with branchwidth at most $k$.
- We consider graph problems for which dynamic programming uses tables encoding vertex partitions.

  For instance, our approach applies to MAXIMUM $d$-DEGREE-BOUNDED CONNECTED SUBGRAPH, MAXIMUM $d$-DEGREE-BOUNDED CONNECTED INDUCED SUBGRAPH and several variants, CONNECTED DOMINATING SET, CONNECTED $r$-DOMINATION, CONNECTED FVS, MAXIMUM LEAF SPANNING TREE, MAXIMUM FULL-DEGREE SPANNING TREE, MAXIMUM EULERIAN SUBGRAPH, STEINER TREE, MAXIMUM LEAF TREE, ...

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- We build a framework for the design of $2^{\mathcal{O}(k)} \cdot n$ step dynamic programming algorithms on **surface-embedded** graphs.
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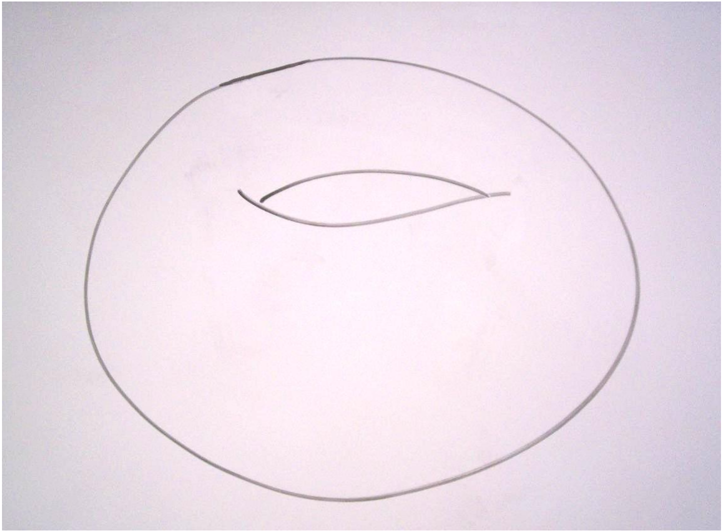

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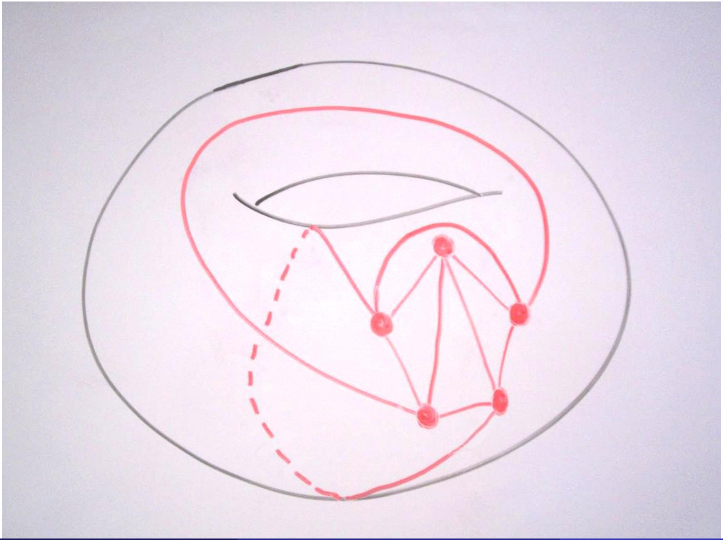

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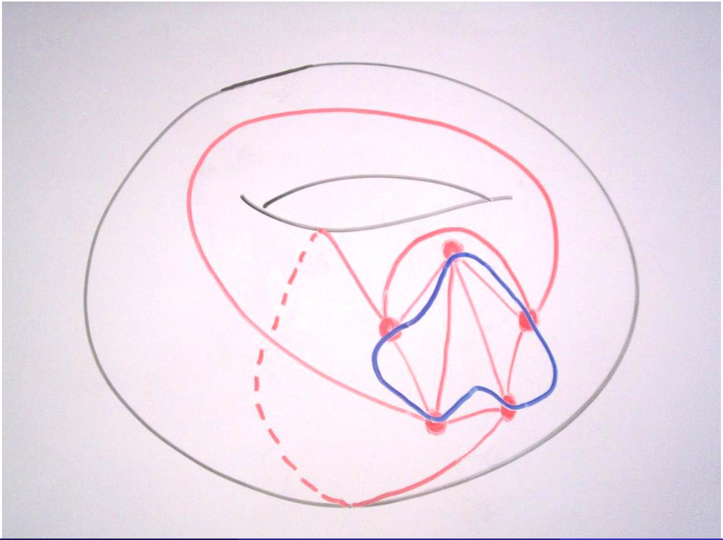

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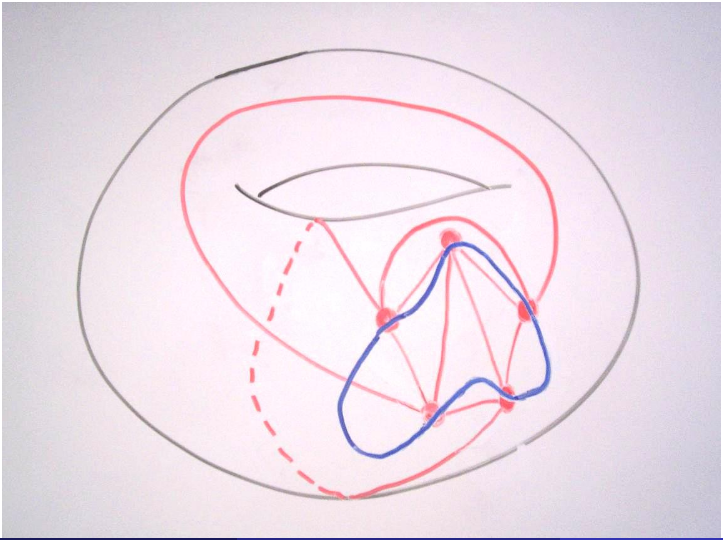

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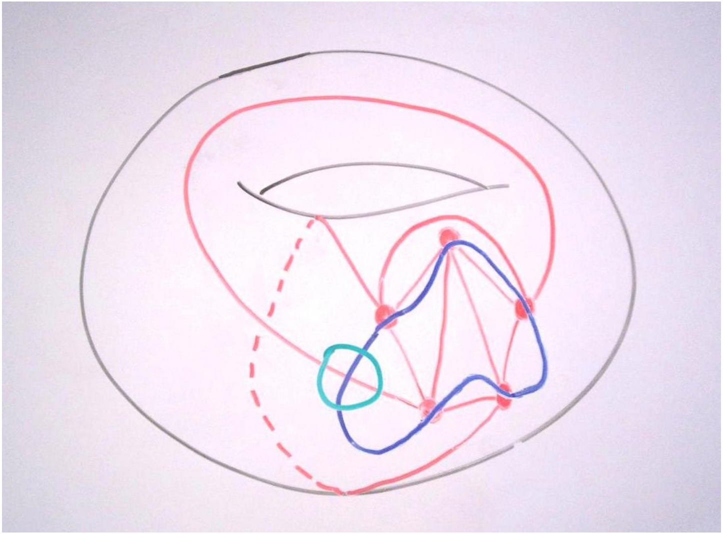

# Sphere cut decompositions

- *Sphere cut decomposition*: Branch decomposition where the vertices in each **mid**$(e)$ are situated around a noose.
- The size of the tables of a dynamic programming algorithm depend on how many ways a partial solution can intersect **mid**$(e)$.
- In how many ways we can draw polygons inside a circle such that they touch the circle only on its vertices and they do not intersect?
- Exactly the number of *non-crossing partitions* over $\ell$ elements, which is given by the $\ell$-th Catalan number:

$$\text{CN}(\ell) = \frac{1}{\ell+1}\binom{2\ell}{\ell} \sim \frac{4^\ell}{\sqrt{\pi}\ell^{3/2}} \approx 4^\ell.$$

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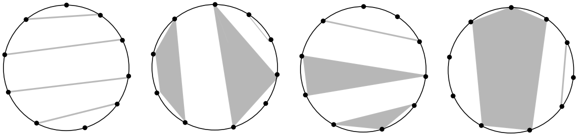

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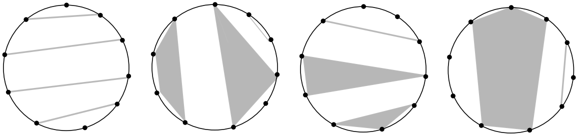

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# Surface cut decompositions (simplified version)

Let $G$ be a graph embedded in a surface $\Sigma$, with $\mathbf{eg}(\Sigma) = \mathbf{g}$.

A surface cut decomposition of $G$ is a branch decomposition $(T, \mu)$ of $G$ and a subset $A \subseteq V(G)$, with $|A| = \mathcal{O}(\mathbf{g})$, s.t. for all $e \in E(T)$

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# How to use surface cut decompositions?

Surface cut decompositions can be efficiently computed:

**Theorem (Rué, Thilikos, and S.)**

*Given a G on n vertices embedded in a surface of Euler genus* $\mathbf{g}$*, with* $\mathbf{bw}(G) \leq k$*, one can construct in* $2^{3k+\mathcal{O}(\log k)} \cdot n^3$ *time a surface cut decomposition* $(T, \mu)$ *of G of width at most* $27k + \mathcal{O}(\mathbf{g})$*.*

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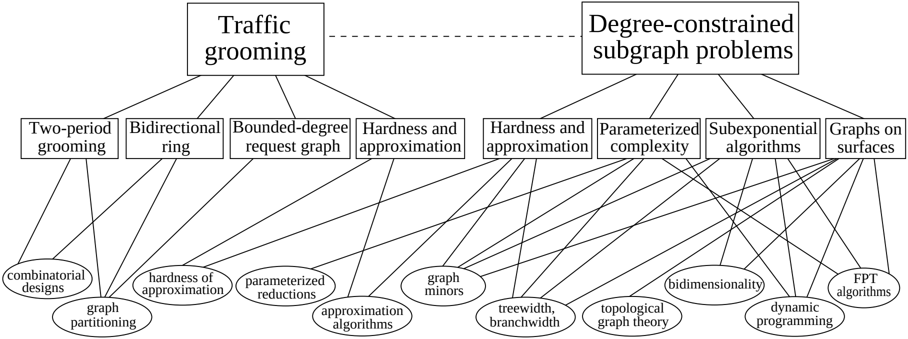

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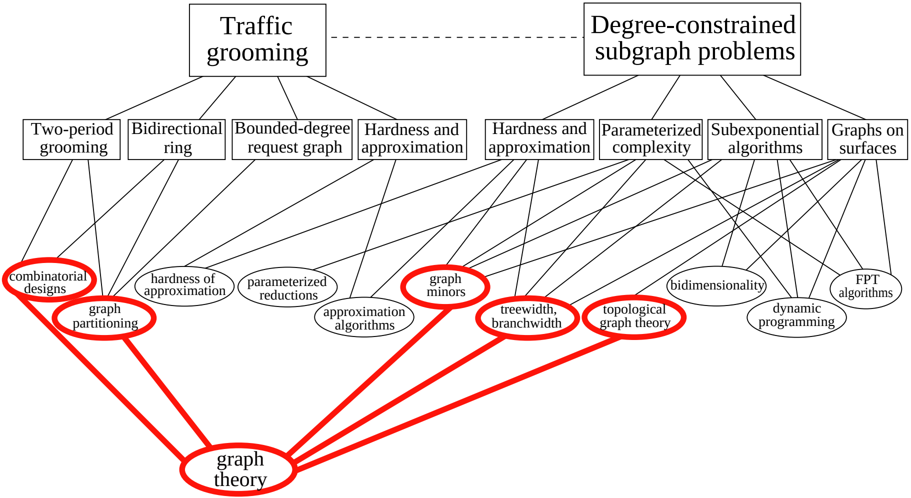

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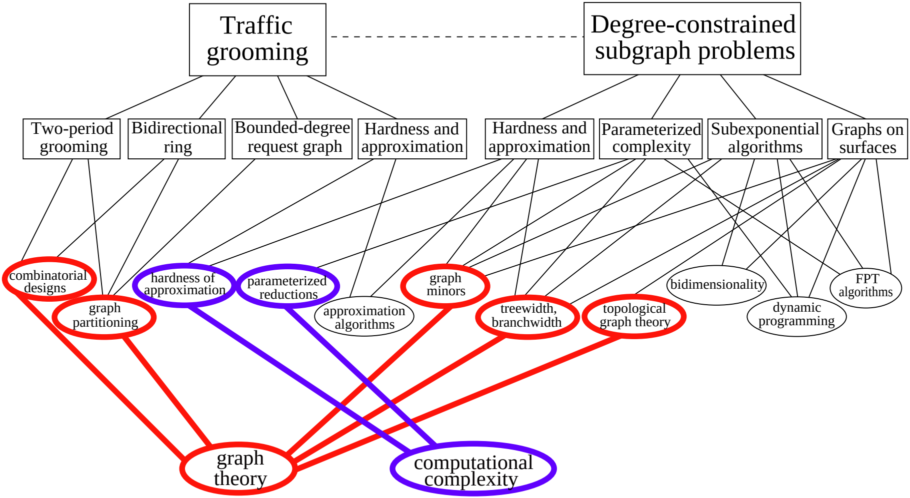

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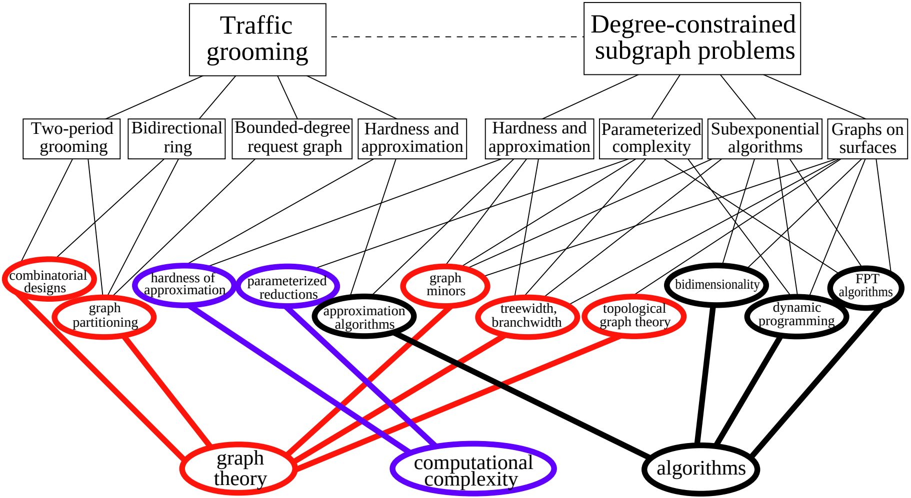

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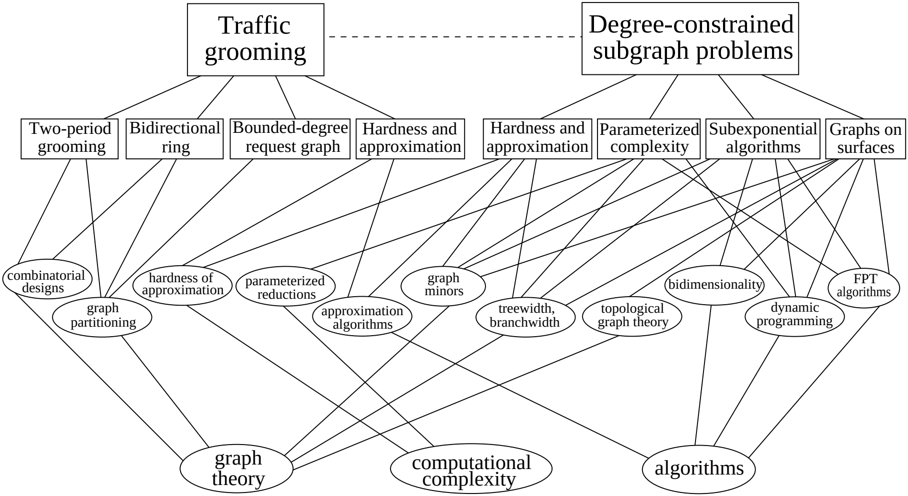

# Further research

- Open problems and conjectures in each chapter of the manuscript.
- Traffic grooming:
  - Close the complexity gap when $C$ is part of the input.
  - In rings, determine the best routing for each request graph.
  - Consider other physical topologies.
- Where is the limit of generalization? algorithmic meta-theorems
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- Graph coloring, probabilistic method, ...

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Gràcies!