Quelques problèmes mathématiques en thermodynamique des fluides visqueux et compressibles

Jan Brezina

Charles University in Prague
Faculty of Mathematics and Physics
and
Université du Sud Toulon-Var

DOCTORAL THESIS

Jan Březina

Selected mathematical problems in the thermodynamics of viscous compressible fluids

Advisor: Eduard Feireisl,
Institute of Mathematics of the AS CR

Advisor: Antonín Novotný,
Laboratoire d'Analyse Non Linéaire Appliquée, l'USTV

Title: Selected problems in the thermodynamics of viscous compressible fluids

Author: Jan Březina
Institute of Mathematics of the Academy of Sciences, of the Czech Republic
Žitná 25, 115 67, Praha 1, Czech Republic
e-mail: jan.brezina@matfyz.cz
phone: +420 485 353 032

Advisor: Eduard Feireisl,
Institute of Mathematics of the Academy of Sciences of the Czech Republic
Žitna 25, 115 67, Praha 1, Czech Republic
e-mail: feireisl@math.cas.cz
phone: +420 222 090 737

Advisor: Antonín Novotný,
ANLA, Université du Sud Toulon-Var
Avenue de l'Université, BP 20132, 83957 La Garde, France
e-mail: novotny@univ-tln.fr
phone: +33 4 94 14 36 25

Abstract

We present a complete existence theory for the physical system consisting of a viscous compressible fluid and a number of rigid bodies in it. We assume a bounded domain and homogeneous Dirichlet boundary conditions for the velocity. Both the fluid and the bodies are allowed to be heat-conducting and share the heat. The existence of global-in-time variational solutions is proved via the viscosity penalization method due to San Martin, Starovoitov, Tucsnak [30], whereas the existence theory for a viscous compressible fluid developed by Feireisl [14] is used in the approximations as well as in the last high-viscosity limit.

The second subject is an improvement of the existence theory for steady barotropic flows. We use L^∞ estimates for the inverse Laplacian of the pressure introduced by Plotnikov, Sokolowski [39] and Frehse, Goj, Steinhauer [19] together with the non-linear potential theory due to Adams and Hedberg [1], to get a priori estimates and to prove existence of weak solutions. Our approach admits physically relevant adiabatic constants $\gamma > \frac{1}{3}(1 + \sqrt{13}) \approx 1.53$ for the flows powered by volume non-potential forces and $\gamma > \frac{1}{8}(3+\sqrt{41}) \approx 1.175$ in the case of potential volume forces and arbitrary non-volume forces. The solutions are constructed in a rectangular domain with periodic boundary conditions.

Keywords

Navier-Stokes-Fourier system, fluid-solid interaction, rigid bodies, variational solutions, steady isentropic flow

Acknowledgments

I am very grateful to all those who supported me during my doctoral studies. Especially, I would like to thank my advisors Eduard Feireisl and Antonín Novotný for valuable counsels and hours of discussion as well as for their encouragement and patience. My thanks go also to Josef Málek, Milan Pokorný, Jiří Neustupa and Šárka Nečasová for the motivation and for the various help they provided me.

I would also like to thank French Government for the scholarship, which allowed me to spend eightteen fruitful months at the university in Toulon. I am indebted to my french advisor Antonín Novotný and his wife Stáňa, whose hospitality and support made that stay possible at all. I would also like to thank Françoise Lade and Maison d'Amour Trinitaire for the accommodation during the last year.

Last but not least, my thanks are due to my parents, my brother and my girlfriend Pavlína for the moral and material support.

Contents

Preface

Throughout our life, we are in continuous contact with fluids. We can not live without breathing the air and drinking the water. Despite of this lifelong experience, we are still far from the complete understanding of their dynamics. At the end of the nineteenth century, Navier and Stokes have independently derived a system of equations describing the evolution of a viscous incompressible fluid based on the classical conservation laws for the mass, the linear and the angular momentum. In 1934, Leray introduced a concept of weak solutions and he proved the existence of global-in-time solutions in this class. Throngs of mathematicians have investigated the Navier-Stokes system since and there is a vast number of practical applications of this model. Yet, the existence of strong global-in-time solution is still one of the most challenging open problems of contemporary mathematics.

In the case of compressible fluids, the full system consists of equations for the density, the velocity, and the temperature, which are based on the conservation of the mass, the linear momentum, and the energy, respectively. If heat conduction and heat production caused by dissipation can be neglected, the complete system splits into the smaller system for the density and the velocity and the heat equation. Then we speak of a barotropic flow. The first existence result for a compressible fluid is due to Lions (1998). Although he assumed a barotropic flow with a physically unrealistic adiabatic constant, his work contains already two fundamental tools for the mathematical theory of compressible fluids, namely compactness properties of the effective viscous pressure and the renormalized solutions to the continuity equation. Developing further these ideas, Feireisl (2003) presented the existence theory for the complete system with physically relevant constitutive equations.

This thesis deals with two distinct topics. The main subject is the existence result for the problem of rigid bodies in a viscous compressible fluid, where both are heat-conducting materials. Using the penalization method developed by Conca, San Martin, Tucsnak, and Starovoitov (1999, 2002) for an incompressible fluid, we have shown that the problem of rigid bodies is a limit case of a heterogeneous compressible fluid as the viscosity tends to infinity on the solid region. This limit could be of independent interest, though it turns out that possibly better result can be proved, if the high viscosity limit is made previously in the chain of approximations. The high viscosity limit is performed in Chapter 4, while an existence theory for the heterogeneous fluid is presented in Chapter 3. Some further ideas and open problems are collected in Section 4.9.

The second topic are steady state solutions for the barotropic flows. The first existence result has been achieved again by Lions (1998). Despite several improvements made by Novotný and Novo (2002) the existence theory in three dimensions was applicable only for non-physical adiabatic constants at least for a general external force. The problem was a

lack of sufficient a priori estimates for the density. In 2005, Frehse, Goj, Steinhauer and Plotnikov, Sokolowski have presented new estimates based on the potential theory. However the both works have assumed a priori L^1-estimate for the quantity $\varrho \boldsymbol{u}^2$. In Chapter 4, we use a bootstrapping technique together with the potential estimates to prove the existence of weak solutions, without any a priory knowledge about their regularity. Our result is applicable for the monatomic gas and general external force or for the considerably wider interval of adiabatic constants for a potential force.

Appropriate physical background for the both topics is explained in Chapter 1. The mathematical tools and the notation we will use, are summarized in Chapter 2.

Chapter 1

Physical background

In this chapter, we present the physical origin of the equations that describe a motion of the rigid bodies in the viscous compressible heat-conducting fluid, fluid-solid interaction, and thermodynamics of this system.

A fluid as well as solid bodies consist of particles. Matter is quantized and discontinuous. Nevertheless, if there is a sufficient number of particles small enough with respect to the volume of our interest, the distribution of the mass could be considered continuous, at least if no macroscopic discontinuities are present. We can imagine that the density at the point $\boldsymbol{x}$ is an average density of an elementary volume located at $\boldsymbol{x}$. The elementary volume should be nearly point from macroscopic view, but still contain enough particles. Similarly, the macroscopic velocity is average velocity of the particles in the elementary volume, while the kinetic energy of the particles could be represented by the temperature. The existence of the temperature is derived only for a system in thermodynamical equilibrium, therefore we have to assume that microscopical events are fast enough with respect to the macroscopic velocity.

This deliberation justify the concept of *continuum*. A fluid or a solid body in three dimensional space is represented by a domain in the Euclidean space $\mathbf{R}^3$. Their state in the time t and at the point $\boldsymbol{x}$ is given by the density $\varrho(t, \boldsymbol{x})$ and the temperature $\vartheta(t, \boldsymbol{x})$ functions. The evolution of the continuum can be described by a *displacement mapping*

$$\boldsymbol{\eta}[t] : \boldsymbol{x}_0 \in \mathbf{R}^3 \to \boldsymbol{x} \in \mathbf{R}^3, \tag{1.1}$$

where an elementary volume that was at the point $\boldsymbol{x}_0$ in the time 0 moves to the new position $\boldsymbol{x} = \boldsymbol{\eta}[t](\boldsymbol{x}_0)$ in the time t. The displacement mapping is at least absolutely continuous in time and $\boldsymbol{\eta}[t]$ in the time t is a diffeomorphism. It has to satisfy $\det(\nabla \boldsymbol{\eta}[t]) \neq 0$, which means that an elementary volume can not degenerate to surface. This condition implies $\det(\nabla \boldsymbol{\eta}[t]) > 0$. Equivalently, the motion of the continuum can be described by the velocity field $\boldsymbol{u}(t, \boldsymbol{x})$, which is interrelated with the displacement mapping through a differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\eta}[t](\boldsymbol{x}_0) = \boldsymbol{u}(t, \boldsymbol{\eta}[t](\boldsymbol{x}_0)), \quad \boldsymbol{\eta}[0](\boldsymbol{x}_0) = \boldsymbol{x}_0. \tag{1.2}$$

For description of a fluid, we rather use the velocity, since the fluid is invariant with respect to deformations. In this case, unknowns are the density $\varrho(t, \boldsymbol{x})$, the velocity $\boldsymbol{u}(t, \boldsymbol{x})$, and the temperature $\vartheta(t, \boldsymbol{x})$ on a space-time domain $Q = I \times \Omega$, where $I = (0, T)$ is a finite time interval and $\Omega \subset \mathbf{R}^3$ a spatial domain.

The rigid body is a physical model of an ideal solid-state object that does not deform at all. Formally, it shall be represented by a connected compact subset of $\mathbf{R}^3$. Using the displacement mapping (1.1), the body in the time t is a set $S(t) = \boldsymbol{\eta}[t](S_0)$, where S_0 is the body in the initial position. Compactness of the body is preserved because $\boldsymbol{\eta}[t]$ is a diffeomorphism. Since the body does not deform, we shall choose a local Cartesian system connected with it. In particular, we can choose the local system in the time $t = 0$ parallel to the system of ambient space with the origin $X^s(0)$. We will write $\boldsymbol{r}$ for the local coordinates. Body itself as well as its physical features, namely the density, the weight

$$m^s := \int_S \varrho(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r}$$

and its thermal properties, remain constant in the local system. Therefore the motion of the body is given by motion of the local system, which consists only of rotation and translation. In other words, $\boldsymbol{\eta}[t]$ restricted to the body is an *affine isometry* $\boldsymbol{\eta}_S[t]$, namely

$$\boldsymbol{\eta}[t](\boldsymbol{x}_0) = \boldsymbol{\eta}_S[t](\boldsymbol{x}_0) := \boldsymbol{X}^s(t) + \mathbb{O}[t](\boldsymbol{x}_0 - \boldsymbol{X}^s(0)) \quad \text{for any } \boldsymbol{x}_0 \in S_0, \tag{1.3}$$

where the rotation is represented by an orthonormal tensor $\mathbb{O}[t]$ and the translation is given by the motion of the local origin $\boldsymbol{X}_S(t)$. It is convenient to identify the origin $\boldsymbol{X}_S(0)$ with the *center of mass*. Then we have

$$\boldsymbol{X}^s(t) = \frac{1}{m^s}\int_{S(t)} \varrho(t,\boldsymbol{x})\boldsymbol{x}\ \mathrm{d}\boldsymbol{x} = \boldsymbol{\eta}[t]\big(\boldsymbol{X}^s(0)\big) \qquad \forall\, t \in I.$$

The mapping $\boldsymbol{\eta}^s[t]$ extended to the whole $\mathbf{R}^3$ can also be viewed as a transformation from the local system of the body to the system of the ambient space

$$\boldsymbol{x} = \boldsymbol{\eta}^s[t](\boldsymbol{r} + \boldsymbol{X}^s(0)) = \boldsymbol{X}^s(t) + \mathbb{O}[t]\boldsymbol{r}.$$

In order to simplify notation, we shall define $\boldsymbol{\eta}$ and $\boldsymbol{\eta}_S$ for the negative times as a corresponding inverse mappings

$$\boldsymbol{\eta}[-t] := (\boldsymbol{\eta}[t])^{-1}, \quad \boldsymbol{\eta}^s[-t] := (\boldsymbol{\eta}^s[t])^{-1}.$$

Taking a time derivative of (1.3), we obtain condition for a *rigid velocity*

$$\begin{aligned}\dot{\overline{\boldsymbol{\eta}}}^s[t](\boldsymbol{x}_0) &= \dot{\overline{\boldsymbol{X}}}^s(t) + \dot{\overline{\mathbb{O}}}[t]\boldsymbol{r}\\ &= \boldsymbol{V}(t) + \mathbb{Q}[t]\mathbb{O}[t]\boldsymbol{r}\\ &= \boldsymbol{V}(t) + \boldsymbol{\omega}(t) \times \big(\boldsymbol{\eta}^s[t](\boldsymbol{x}_0) - \boldsymbol{X}^s(t)\big),\end{aligned} \tag{1.4}$$

where $\boldsymbol{V} = \dot{\overline{\boldsymbol{X}^s}}$ is the velocity of the center of mass. The tensor of angular velocity $\mathbb{Q}[t]$ acts in the actual system, therefore $\dot{\overline{\mathbb{O}}} = \mathbb{Q}\mathbb{O}$. Furthermore, as $\mathbb{O}[t]$ is orthogonal, we have

$$0 = \dot{\mathbb{I}} = \dot{\overline{\mathbb{O}^T\mathbb{O}}} = \mathbb{Q}^T + \mathbb{Q}.$$

Accordingly, tensor $\mathbb{Q}[t]$ is antisymmetric and can be written in terms of the vector product with the *angular velocity* $\boldsymbol{\omega}$. From the right-hand side of (1.4), we can read the velocity field on the body

$$\begin{aligned}\boldsymbol{u}^s(t,\boldsymbol{x}) &:= \boldsymbol{V}(t) + \boldsymbol{\omega} \times (\boldsymbol{x} - \boldsymbol{X}^s(t)) \quad \forall \boldsymbol{x} \in S(t),\\ \boldsymbol{u}^s(t,\boldsymbol{r}) &:= \boldsymbol{u}^s(t,\boldsymbol{x}), \quad \boldsymbol{x} = \boldsymbol{\eta}^s[t](\boldsymbol{r} + \boldsymbol{X}^s).\end{aligned}$$

Thus in problem of evolution of a heat conducting rigid body, the unknowns are the velocity of the center of mass $\boldsymbol{V}(t)$, the angular velocity $\boldsymbol{\omega}(t)$ in the time interval $I = (0,T)$, and the temperature $\vartheta(t,\boldsymbol{x})$ on the domain Ω.

1.1 Balance laws

After we have clarified what are the unknowns, we turn our attention to the equations. These are based on the balance of the mass, the linear momentum, the angular momentum, and the total energy, which have a common form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{B(t)} q \,\mathrm{d}\boldsymbol{x} = \text{boundary flux} + \text{volume sources}, \tag{1.5}$$

where q is a balanced quantity and $B(t) = \boldsymbol{\eta}[t](B)$ is evolution of some volume B. In order to compute the left-hand side of (1.5), we apply Reynold's transport theorem.

Theorem 1.1.1. *Let I be a finite time interval and $\Omega \subset \mathbf{R}^3$ a bounded domain. Let the velocity field $\boldsymbol{u} \in W^{1,\infty}(\Omega;\mathbf{R}^3)$ and the displacement mapping $\boldsymbol{\eta}$ according to* (1.2) *be given. Finally, let $q \in C^1(I \times \Omega)$. Then for any $B[t] \subset \Omega$, $B[t] = \boldsymbol{\eta}[t](B[0])$ it holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{B[t]} q \,\mathrm{d}\boldsymbol{x} = \int_{B[t]} \partial_t q + \operatorname{div}(q\boldsymbol{u}) \,\mathrm{d}\boldsymbol{x}. \tag{1.6}$$

Proof. Take $f \in \mathcal{D}(B[t])$ and define $\chi(t,\boldsymbol{x})$ by $\chi(t,\boldsymbol{\eta}[t](\boldsymbol{y})) = f(\boldsymbol{y})$. By definition, we have

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\chi = \partial_t\chi + \boldsymbol{u}\cdot\nabla\chi.$$

Then, we compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{B[t]} q\chi \,\mathrm{d}\boldsymbol{x} = \int_{\mathbf{R}^3} \partial_t q\chi - q\boldsymbol{u}\cdot\nabla\chi \,\mathrm{d}\boldsymbol{x} = \int_{B[t]} \big(\partial_t q + \operatorname{div}(q\boldsymbol{u})\big)\chi \,\mathrm{d}\boldsymbol{x}.$$

Let the f tend to the characteristic function of $B[t]$ and the statement follows. □

1.1.1 Continuity equation

If we consider fluid enclosed in an impermeable vessel, there are no boundary fluxes nor internal sources of the mass. Thus applying Reynold's transport theorem, we get so called *continuity equation*

$$\partial_t\varrho + \operatorname{div}(\varrho\boldsymbol{u}) = 0. \tag{1.7}$$

For a rigid body the same equation holds. Indeed, a direct calculation yields

$$\partial_t\varrho^s + \operatorname{div}(\varrho^s\boldsymbol{u}^s) = \frac{\mathrm{d}}{\mathrm{d}t}\varrho^s(t,\boldsymbol{x}(t)) + \varrho^s\operatorname{div}\boldsymbol{u}^s = 0, \tag{1.8}$$

where $\operatorname{div}\boldsymbol{u}^s = 0$, since $\boldsymbol{u}^s$ is a rigid velocity.

1.1.2 Equations of motion

By the virtue of *Newton's second law of motion*, we deduce following balance of the linear momentum

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{B[t]} [\varrho\boldsymbol{u}](t,\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} = \int_{B[t]} \boldsymbol{F}(t,\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} + \int_{\partial B[t]} \boldsymbol{T}(t,\boldsymbol{x},\boldsymbol{n}) \,\mathrm{d}\sigma. \tag{1.9}$$

The first term on the right-hand side represents external volume forces $\boldsymbol{F} = \varrho \boldsymbol{f}$, while the second is integral of surface forces $\boldsymbol{T}$. According to the *Cauchy law* [45, Chapter 2] the surface force has a form

$$\boldsymbol{T}(t, \boldsymbol{x}, \boldsymbol{n}) = \mathbb{T}(t, \boldsymbol{x})\boldsymbol{n},$$

where $\mathbb{T}$ is called the *Cauchy stress tensor*. Then using Theorem 1.1.1 and the Green theorem we we obtain a pointwise form of the balance law (1.9)

$$\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) = \operatorname{div}(\mathbb{T}) + \varrho \boldsymbol{f} \tag{1.10}$$

called the *momentum equation*.

A similar balance law holds for the *angular momentum*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{B(t)} \boldsymbol{r} \times \varrho \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} = \int_{B(t)} \boldsymbol{r} \times \boldsymbol{F}(t, \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} + \int_{\partial B(t)} \boldsymbol{r} \times (\mathbb{T}\boldsymbol{n}) \, \mathrm{d}\sigma. \tag{1.11}$$

If we subtrace (1.10) multipied by $\boldsymbol{r}$ and integrated over B, we obtain symmetry condition for $\mathbb{T}$:

$$0 = \int_B \varepsilon_{ijk}(\partial_l r_j) T_{kl} \, \mathrm{d}\boldsymbol{x} + \int_B \varepsilon_{ijk} r_j (\partial_l T_{kl}) \, \mathrm{d}\boldsymbol{x} - \int_B \varepsilon_{ijk} r_j (\partial_l T_{kl}) \, \mathrm{d}\boldsymbol{x} = \int_B \varepsilon_{ijk} T_{kj} \, \mathrm{d}\boldsymbol{x}.$$

The motion of the body is given by the translation velocity V and the angular velocity $\boldsymbol{\omega}$. The former is governed by the conservation law for the linear momentum of the whole body:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{S[t]} \varrho^s \boldsymbol{u}^s \, \mathrm{d}\boldsymbol{x} = m^s \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{V}^s(t) = \boldsymbol{F}^s(t) := \int_{\partial S[t]} \mathbb{T}\boldsymbol{n} \, \mathrm{d}\sigma + \int_{S[t]} \varrho^s \boldsymbol{f} \, \mathrm{d}\boldsymbol{x}. \tag{1.12}$$

For the later one, we have conservation of the angular momentum

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{S[t]} \boldsymbol{r} \times (\varrho^s \boldsymbol{u}^s) \, \mathrm{d}\boldsymbol{r} = \frac{\mathrm{d}}{\mathrm{d}t} \int_S \boldsymbol{r} \times \big(\varrho^s \boldsymbol{r} \times \boldsymbol{\omega}(t)\big) \, \mathrm{d}\boldsymbol{r} =$$
$$\boldsymbol{M}^s(t) := \int_{\partial S[t]} \boldsymbol{r} \times \mathbb{T}\boldsymbol{n} \, \mathrm{d}\sigma + \int_{S[t]} \boldsymbol{r} \times \big(\varrho^s \boldsymbol{f}(t, \boldsymbol{x})\big) \, \mathrm{d}\boldsymbol{x}. \tag{1.13}$$

On the right-hand side of (1.12) and (1.13), there appears a surface force $\mathbb{T}\boldsymbol{n}$, which represents mechanical interaction between the fluid and the body.

1.1.3 Energy balance

The total energy E of an elementary volume consist of the kinetic energy $\frac{1}{2}\varrho|\boldsymbol{u}|^2$ and the internal energy $\varrho e(\varrho, \vartheta)$. In accordance with the *first law of thermodynamics*, which we will discuss in the next section, the change of the total energy in the volume $B[t]$ is caused partly by the work of volume and surface forces and partly by the heat flux $\boldsymbol{q}$. The energy balance reads

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{B[t]} \frac{1}{2}\varrho|\boldsymbol{u}|^2 + \varrho e(\varrho, \vartheta) \, \mathrm{d}\boldsymbol{x} = \int_{B[t]} \varrho \boldsymbol{f} \cdot \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} + \int_{\partial B[t]} (\mathbb{T}\boldsymbol{n}) \cdot \boldsymbol{u} \, \mathrm{d}\sigma - \int_{\partial B[t]} \boldsymbol{q} \cdot \boldsymbol{n} \, \mathrm{d}\sigma. \tag{1.14}$$

Using again the transport theorem 1.1.1, we infer the pointwise form of the *total energy equation*

$$\partial_t(\frac{1}{2}\varrho|\boldsymbol{u}|^2 + \varrho e(\varrho,\vartheta)) + \operatorname{div}(\frac{1}{2}\varrho|\boldsymbol{u}|^2\boldsymbol{u} + \varrho e(\varrho,\vartheta)\boldsymbol{u}) + \operatorname{div}\boldsymbol{q} = \operatorname{div}(\mathbb{T}\boldsymbol{u}) + \varrho\boldsymbol{f}\cdot\boldsymbol{u}. \tag{1.15}$$

Multiplying (1.10) by $\boldsymbol{u}$ and using (1.7), we arrive at the *kinetic energy equation*

$$\partial_t\big(\frac{1}{2}\varrho|\boldsymbol{u}|^2\big) + \operatorname{div}\big(\frac{1}{2}\varrho|\boldsymbol{u}|^2\boldsymbol{u}\big) - \operatorname{div}(\mathbb{T}\boldsymbol{u}) = -\mathbb{T} : \nabla\boldsymbol{u} + \varrho\boldsymbol{f}\cdot\boldsymbol{u}. \tag{1.16}$$

It can be subtracted from (1.15), which yields the *internal energy equation*

$$\partial_t(\varrho e) + \operatorname{div}(\varrho e\boldsymbol{u}) + \operatorname{div}\boldsymbol{q} = \mathbb{T} : \nabla\boldsymbol{u}. \tag{1.17}$$

As we will see later, for the fluid, the stress tensor has a form $\mathbb{T} = \mathbb{S} - p\mathbb{I}$, where p is the *pressure* and $\mathbb{S}$ the *viscous stress tensor*. Then there appears a new term $p\operatorname{div}\boldsymbol{u}$ in (1.17), which can be used to expres the energy balance also in terms of the *entropy*. Using continuity equation (1.7) and Gibbs equation (1.24), one gets

$$\begin{aligned}\partial_t(\varrho e) + \operatorname{div}(\varrho e\boldsymbol{u}) + p\operatorname{div}\boldsymbol{u} - \underbrace{\big[\partial_t\varrho + \operatorname{div}\varrho\boldsymbol{u}\big]}_{0}\Big(\frac{p}{\varrho} + e\Big) = \\ \overset{\text{C.E.}}{=}\quad \varrho\big[\partial_t e - p\varrho^{-2}\partial_t\varrho\big] + \varrho\boldsymbol{u}\cdot\big[\nabla e - p\varrho^{-2}\nabla\varrho\big] = \\ \overset{\text{Gibbs}}{=}\quad \vartheta\big[\varrho\partial_t s + \varrho\boldsymbol{u}\cdot\nabla s\big] = \vartheta\big[\partial_t(\varrho s) + \operatorname{div}(\varrho s\boldsymbol{u})\big].\end{aligned} \tag{1.18}$$

Thus dividing (1.17) by ϑ, we obtain *entropy equation*

$$\partial_t(\varrho s) + \operatorname{div}(\varrho s\boldsymbol{u}) + \operatorname{div}\Big(\frac{\boldsymbol{q}}{\vartheta}\Big) = \frac{1}{\vartheta}\Big(\mathbb{S} : \nabla\boldsymbol{u} - \frac{\boldsymbol{q}\cdot\nabla\vartheta}{\vartheta}\Big) = \sigma, \tag{1.19}$$

where σ is called the *entropy production rate*.

On the solid region, the equations (1.15), (1.17), and (1.19) remain valid in particular since the symmetric part of the velocity gradient $\mathbb{D}\boldsymbol{u}$ is zero for the rigid velocity (c.f. Lemma 4.1.3). For example, (1.17) reduces to the usual heat equation

$$\varrho c^v\,\partial_t\vartheta + \varrho c^v\,\nabla\vartheta\cdot\boldsymbol{u} + \operatorname{div}\boldsymbol{q} = 0,$$

where $c^v = \frac{\partial e}{\partial\vartheta}$ is the *specific heat at constant volume*.

1.2 Thermodynamics and constitutive equations

The equations we have derived up to now are valid for the general continuum, in particular they do not reflect any material properties of matter in the question. On the other hand the system of equations (1.7), (1.10), (1.15) is not complete. We have to determinate the stress $\mathbb{T}$, the heat flux $\boldsymbol{q}$, as well as the pressure p, the internal energy e, and the entropy s in terms of of the state variables ϱ, $\boldsymbol{u}$, ϑ.

Widely accepted definition of the fluid says that a fluid deforms as long as the shear stress is applied. It means that the shear stress is independent of the deformation. Usually the

fluid is also *isotropic* that is invariant with respect to the rotation of the coordinate system. Symbolically:

$$\mathbb{O}^T\mathbb{T}(\mathbb{E})\mathbb{O} = \mathbb{T}(\mathbb{O}^T\mathbb{E}\mathbb{O}), \quad \text{for any orthonormal matrix } \mathbb{O}, \tag{1.20}$$

whenever $\mathbb{T}$ is a function of a tensor $\mathbb{E}$. Consequently, the stress tensor has a form

$$\mathbb{T} = \mathbb{S}(\vartheta, \varrho, \nabla \boldsymbol{u}) - p(\vartheta, \varrho)\mathbb{I}, \tag{1.21}$$

where the pressure p is a general function of the temperature and the density and it represents a bulk deformation. The *viscous stress tensor* $\mathbb{S}$ depends on the rate of deformation, i.e. on the gradient of the velocity, and possibly on the temperature and the density. As a consequence of the isotropy condition (1.20), the viscous stress $\mathbb{S}$ depends only on the symmetric part of the velocity gradient

$$\mathbb{D}\boldsymbol{u} = \frac{1}{2}\big(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T\big).$$

In this work, we consider a *Newtonian fluid*, which is characterized by the linearity of the viscous stress with respect to $\mathbb{D}\boldsymbol{u}$. The constitutive relation then reads

$$\mathbb{S} = 2\mu\big(\mathbb{D}\boldsymbol{u} - \frac{1}{3}\mathrm{div}\boldsymbol{u}\mathbb{I}\big) + \zeta \mathrm{div}\boldsymbol{u}\mathbb{I}. \tag{1.22}$$

The *shear viscosity* μ and the *bulk viscosity* ζ are in general functions of ϑ and ϱ. However, because our mathematical theory does not cope with the viscosities dependent on the density, we assume they are only functions of the temperature.

For the heat flux $\boldsymbol{q}$, we consider *Fourier's law*

$$\boldsymbol{q} = -\kappa(\vartheta)\nabla\vartheta. \tag{1.23}$$

In general, the *heat conductivity coefficient* κ is a function of both the density and the temperature, but for the simplicity, we assume only the dependence only on temperature.

The heat transfer is caused by two different mechanisms. On the one hand the heat spreads by the chaotic motion of the particles. This process is called *heat advection* and it is significant mainly at low temperatures. On the other hand, at high temperatures, collisions between particles have enough energy to change their internal quantum states, which is accompanied by the release of photons. This radiation is absorbed elsewhere and effectively produce the heat transfer. According to these considerations, the heat conductivity compose of two parts

$$\kappa = \kappa_G + \kappa_R,$$

where the classical conductivity κ_G of the prefect gas is of the same order as the viscosity $\kappa(\vartheta) \approx \mu(\vartheta)$, while the radiation part behaves like $\kappa_R \approx \vartheta^3$, see for example [3].

In accordance with the second law of thermodynamics, the entropy production is non-negative

$$\sigma = \frac{1}{\vartheta}\Big(\mathbb{S} : \nabla \boldsymbol{u} - \frac{\boldsymbol{q}\cdot\nabla\vartheta}{\vartheta}\Big) \geq 0.$$

This is also called the *Clausius-Duhem* inequality. Consequently, in view of relations (1.22) and (1.23), it has to be

$$\mu, \zeta, \kappa \geq 0.$$

1.2.1 Equation of state

In order to obtain constitutive relations for the remaining functions $p(\vartheta, \varrho)$, $e(\vartheta, \varrho)$ and $s(\vartheta, \varrho)$, we assume an elementary volume in thermodynamical equilibrium, so that we can apply the classical thermodynamics. According to the *first law of thermodynamic*, the change of the internal energy is equal to the absorbed heat minus the work performed by the gas. In the language of differentials, this can be written as the so-called the *Gibbs equation*

$$\mathrm{d}e = \vartheta \,\mathrm{d}s - p \,\mathrm{d}V. \tag{1.24}$$

Equivalently, for the *free energy* $A = e - s\vartheta$, we have

$$\mathrm{d}A = \vartheta \,\mathrm{d}s - p \,\mathrm{d}V - \,\mathrm{d}s\vartheta - s \,\mathrm{d}\vartheta = -s \,\mathrm{d}\vartheta - p \,\mathrm{d}V.$$

Differentiating with respect to ϑ and V, we obtain one of the *Maxwell relations*

$$\Big(\frac{\partial s}{\partial V}\Big)_{\vartheta} = \Big(\frac{\partial p}{\partial T}\Big)_{V} = -\frac{\partial^2 A}{\partial V \,\partial \vartheta}. \tag{1.25}$$

The equations (1.24) and (1.25) imply general constrains for the choice of the functions $p(\vartheta, \varrho)$, $e(\varrho, \vartheta)$, $s(\varrho, \vartheta)$, namely

$$\frac{\partial e}{\partial \varrho} = -\frac{1}{\varrho^2}\Big(\frac{\partial e}{\partial V}\Big)_{\vartheta} = \frac{1}{\varrho^2}\Big(p - \vartheta \frac{\partial p}{\partial \vartheta}\Big) \tag{1.26}$$

$$\frac{\partial s}{\partial \varrho} = -\frac{1}{\varrho^2}\Big(\frac{\partial s}{\partial V}\Big)_{\vartheta} = -\frac{1}{\varrho^2}\frac{\partial p}{\partial \vartheta} \tag{1.27}$$

$$\frac{\partial s}{\partial \vartheta} = \Big(\frac{\partial s}{\partial e}\Big)_{V}\Big(\frac{\partial e}{\partial \vartheta}\Big)_{V} = \frac{1}{\vartheta}\frac{\partial e}{\partial \vartheta} = \frac{1}{\vartheta}c_v. \tag{1.28}$$

Through the last equation, we have also defined c_v — the *specific heat at the constant volume*.

Similarly as in the case of the heat conductivity, we assume that the pressure consist from the pressure of the *perfect gas* p_G and the pressure caused by radiation p_R,

$$p = p_G + p_R.$$

For the perfect gas the pressure is related to the specific internal energy by a relation

$$p_G = \frac{2}{3}\varrho e_G; \tag{1.29}$$

one can consult for example [31, Chapter 4]. On the other hand, referring to [11, Chapter 15], the radiation part has a form

$$p_R = \frac{a}{3}\vartheta^4. \tag{1.30}$$

Having a constitutive relation for the pressure, the equation (1.26) can be to determine e up to the function of the temperature. We consider the specific internal energy

$$e(\varrho, \vartheta) = e_G(\varrho, \vartheta) + e_R(\vartheta),$$

where in accordance with (1.26) the radiation part is given by

$$e_R = \frac{a}{\varrho}\vartheta^4,$$

while e_G as well as p_G has to be resolved from (1.29) and (1.26). For $p = p_G$, we obtain an equation

$$p = \frac{3}{5}\varrho\frac{\partial p}{\partial \varrho} + \frac{2}{5}\vartheta\frac{\partial p}{\partial \vartheta}.$$

It can be integrated as follows. On the line $\varrho = 0$, one has $p(0,\vartheta) = C\vartheta^{\frac{5}{2}}$, which suggest solution in the form $p(\varrho,\vartheta) = P(\varrho,\vartheta)\vartheta^{\frac{5}{2}}$. Such a substitution yields

$$0 = \frac{3}{2}\varrho\frac{\partial P}{\partial \varrho} + \vartheta\frac{\partial P}{\partial \vartheta}. \tag{1.31}$$

One can see that P is constant on the lines given by equation

$$\frac{\partial \varrho}{\partial \vartheta} = \frac{3}{2}\frac{\varrho}{\vartheta}.$$

Its solution is satisfies $\varrho\vartheta^{-\frac{3}{2}} = const.$, thus the pressure $p = p_G$ is

$$p_G(\varrho,\vartheta) = \vartheta^{\frac{5}{2}}P(Y), \quad Y = \varrho\vartheta^{-\frac{3}{2}} \tag{1.32}$$

and the internal energy

$$e_G(\varrho,\vartheta) = \frac{3}{2}\vartheta P(Y)Y^{-1},$$

where $P(Y)$ is a suitable $C^1[0,\infty)$ function.

In accordance with principles of statistical physics, the pressure is positive non-decreasing function of the density, and it should be zero for the vanishing density. Consequently the function P has to satisfy

$$P(0) = 0, \text{ and } P'(Y) \geq 0 \text{ on } [0,\infty). \tag{1.33}$$

The later condition is called *thermodynamics stability condition*. Furthermore, the specific heat at constant volume c_v is always positive, which leads to

$$0 < c_v(Y) := \frac{\partial e}{\partial \vartheta} = \frac{3}{2}\frac{1}{\varrho}\frac{\partial p}{\partial \vartheta} = \frac{9}{4Y}\Big(\frac{5}{3}P(Y) - P'(Y)Y\Big). \tag{1.34}$$

Hence we deduce

$$\big(P(Y)Y^{-\frac{5}{3}}\big)' = Y^{-\frac{8}{3}}\big(P'(Y)Y - \frac{5}{3}P(Y)\big) < 0$$

and therefore

$$\inf_{Y>0} P(Y)Y^{-\frac{5}{3}} = \lim_{Y\to\infty} P(Y)Y^{-\frac{5}{3}} \geq 0. \tag{1.35}$$

However, when $Y = \varrho\vartheta^{-\frac{3}{2}}$ approaches infinity, the gas exhibits degeneration phenomena (see [11, Chapter 15] and [22, Chapters 2, 3]). For example, so-called *Fermi gas* (see [11, Chapters 6, 15] and [31, Chapter 4]) keeps the pressure positive as the temperature tends to the absolute zero, thus

$$\lim_{\vartheta\to 0+} p(\varrho,\vartheta) = p_c(\varrho) > 0 \quad \text{for any } \varrho > 0.$$

This limit always exists, since the pressure is non-negative and non-decreasing function of the temperature in accordance with (1.33). On the other hand

$$p_c(\varrho) = \varrho^{\frac{5}{3}}\lim_{\vartheta\to 0+}(\varrho^{-\frac{2}{3}}\vartheta)^{\frac{5}{2}}P\Big((\varrho^{-\frac{2}{3}}\vartheta)^{-\frac{3}{2}}\Big) = \varrho^{\frac{5}{3}}p_c(1), \quad \text{for any } \varrho > 0.$$

Hence, comparing to (1.35), we obtain even a sharp inequality

$$\lim_{Y\to\infty} P(Y)Y^{-\frac{5}{3}} = p_c(1) = P_\infty > 0 \tag{1.36}$$

as is required later in the hypotheses (3.15), (4.19). The growth condition present in these hypotheses seems to be also physically relevant as according to [41], one has

$$P(Y) \approx aY^{\frac{5}{3}} + bY^{\frac{1}{3}} + \text{ lower order terms.}$$

The specific entropy again consist from the classical and the radiation part

$$s = s_R + s_G,$$

where due to (1.28)

$$s_R(\vartheta) = \frac{4}{3}\frac{a}{\varrho}\vartheta^3,$$

while for $s = s_G$, one can use also (1.27) and (1.29) to derive the equation

$$\frac{\partial s}{\partial \varrho} = -\frac{1}{\varrho^2}\,\frac{2}{3}\varrho\frac{\partial e}{\partial \vartheta} = -\frac{2}{3}\,\frac{\vartheta}{\varrho}\,\frac{\partial s}{\partial \vartheta},$$

In fact, this is very same equation as (1.31), thus the specific entropy s_G has to be a function of Y, namely

$$s(\varrho,\vartheta) = S(Y), \quad Y = \frac{\varrho}{\vartheta^{\frac{3}{2}}}. \tag{1.37}$$

Moreover due to (1.28), one has

$$\frac{\partial s}{\partial \vartheta} = S'(Y)\varrho\Big(-\frac{3}{2}\Big)\vartheta^{-\frac{5}{2}}, = \frac{1}{\vartheta}c_v(Y),$$

which implies

$$S'(Y) = -\frac{2}{3}c_v(Y)Y^{-1} = -\frac{3}{2}Y^{-2}\Big(\frac{5}{3}P(Y) - P'(Y)Y\Big). \tag{1.38}$$

For a constant c_v, the function $S(Y)$ behave like $-\log(Y)$ and it is natural to fix an additive constant by $S(1) = 0$. Doing the same in the general case, we can write

$$S(Y) = -\frac{2}{3}\int_1^Y c_v(s)s^{-1}\,\mathrm{d}s. \tag{1.39}$$

Another possible normalization of the entropy follows form the *third law of thermodynamics*, which states that the entropy tends to zero as $\vartheta \to 0$. Then a natural normalization is given by condition

$$\lim_{Y\to\infty} S(Y) = 0.$$

1.2.2 Barotropic flows

Because of an enormous complexity of the model based on the general constitutive laws, the first mathematical theory for a compressible fluid [29] was done for the case of the *barotropic* flow, where one assumes the pressure to be a function of the sole density. Possible physical explanation and mathematical consequences of this simplification is the topic of this section.

Let us consider the flow of a compressible fluid. If the velocity gradient and/or viscosities are small, the dissipation (i.e. transformation of the kinetic energy into heat) may be neglected. Similarly, in the case of small heat conductivity of the gas and/or small temperature gradients, the heat flux $\boldsymbol{q}$ may be neglected, as well. Such a flow called *adiabatic*. Taking into account the entropy equation (1.19), it appears that in the adiabatic case, the entropy production is zero and the specific entropy is constant along trajectories of fluid particles. This implies that the pressure has a particular form

$$p(\varrho) = a\varrho^\gamma, \quad a > 0, \quad \gamma > 1, \tag{1.40}$$

where $\gamma \geq 1$ is the so called *adiabatic constant* and $a > 0$ is a constant along any trajectory. In the sequel, we will assume that the flow is *isentropic*, which means that a is constant across all trajectories. This is a particular case of the barotropic flow.

In the barotropic case, the pressure is independent of the temperature. Consequently the equations (1.7), (1.10) form an enclosed system, while, once $(\varrho, \boldsymbol{u})$ is known, (1.17) can be used to determine the temperature field.

It is shown in statistical physics that the adiabatic constant γ in (1.40) depends on the number M of the degrees of freedom of the molecules of the gas. One has $\gamma = \frac{5}{3} \approx 1.66$ for the mono-atomic gas, $\gamma = \frac{7}{5} = 1.4$ for the air and in general $\gamma = \frac{M+2}{M}$. Parameters similar to γ appear in the complete theory of the viscous compressible fluids described by the full Navier-Stokes-Fourier system $(1.7 - 1.17)$, and from the mathematical point of view, they determine the quality of density estimates. That is why the simplified isentropic model for compressible fluids is important, in spite of its slightly contradictory physical background. Some new advances in the existence theory for the steady isentropic flow are presented in Chapter 5.

1.2.3 Constitutive equations for solid state

On the solid region that correspond to the rigid bodies it is enough to prescribe constitutive equations for the thermal quantities, namely the internal energy e^s, the entropy s^s and the heat conductivity κ^s. From the physical point of view, the simplest model is a homogeneous linear material with constitutive relations

$$e^s = e^s(\vartheta) = C^v\vartheta, \ s^s = s^s(\vartheta) = C^v \log \vartheta, \ \kappa^s = \kappa.$$

Nevertheless, the mathematical theory we are going to use needs presence of the radiation part. Therefore, we assume

$$\begin{aligned} e^s = e^s(\varrho, \vartheta) &= e_G(\varrho, \vartheta) + e_R(\vartheta), \ \varrho e_R = a\vartheta^4, \\ s^s = s^s(\varrho, \vartheta) &= s_G(\varrho, \vartheta) + s_R(\vartheta), \ \varrho s_R = a\vartheta^4, \\ \kappa^s = \kappa^s(\vartheta) &= \kappa^s_G(\vartheta) + \kappa^s_R, \ \kappa^s_G \approx \vartheta, \ \kappa^s_R = a^s\vartheta^4. \end{aligned}$$

Still, this can be a physically relevant model, if we could allow different e, s on the fluid and on the rigid region. Unfortunately this is not the case because of the technical difficulties in the existence theory explained briefly in Section 4.9. Consequently, we have to assume the same e, s on the both regions. However the heat conductivity is allowed to be different.

1.3 Fluid-solid interaction

In order to obtain mathematically well-posed problem, we have to specify the *boundary conditions* on $\partial\Omega$ as well as on the body surface ∂S. Let us denote $\boldsymbol{u}^f$, $\boldsymbol{u}^s$ the velocity field on the fluid region and on the rigid region, respectively. Similarly, $\boldsymbol{q}^f$, $\boldsymbol{q}^s$ shall be the heat flux on the fluid and on the solid region. We consider noslip boundary conditions for the velocity on all surfaces

$$\boldsymbol{u}^f = 0 \text{ on } \partial\Omega, \qquad \boldsymbol{u}^f[t] = \boldsymbol{u}^s[t] \text{ on } \partial S[t],\ \forall t \in I. \tag{1.41}$$

For the temperature, there are natural Neumann boundary conditions

$$\boldsymbol{q}^f \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega \qquad \boldsymbol{q}^f \cdot \boldsymbol{n}[t] = \boldsymbol{q}^s \cdot \boldsymbol{n}[t] \text{ on } \partial S[t],\ \forall t \in I. \tag{1.42}$$

Another boundary conditions have appeared already in the boundary terms in (1.12) and (1.13). To give a sense to all these boundary conditions, we have to assume that all quantities in question, namely $\boldsymbol{u}^f$, $\boldsymbol{q}^f$, $\mathbb{T}$, and $\boldsymbol{u}^s$, $\boldsymbol{q}^s$, are continuous up to the boundary $\partial S \cup \partial\Omega$.

Chapter 2

Mathematical apparatus, Notation

We denote by $\mathbb{N}$ a set of positive integers and by $\mathbf{R}$ the real numbers. An N-dimensional Euclidean space will be denoted by $\mathbf{R}^N$. Elements of $\mathbf{R}^N$ are (column) vectors and we denote them by bold letters. By a doubled typeface, e.g. $\mathbb{A}, \mathbb{B}, \ldots$, we will denote 2-tensors on $\mathbf{R}^N$, which can be understood as linear mappings from $\mathbf{R}^N$ to $\mathbf{R}^N$ as well as $N \times N$ matrices.

An open set $\Omega \subset \mathbf{R}^N$ is called *connected* if every two points in Ω can be connected be a piecewise linear curve in Ω. By Ω we usually denote a *domain*, i.e. an open and connected set. If the boundary $\partial\Omega$ of the domain $\Omega \subset \mathbf{R}^N$ can be locally described by a finite system of Lipschitz continuous mappings $f_j : \mathbf{R}^{N-1} \to \mathbf{R}$, we speak about Lipschitz domain. If the mappings f_j are from $C^{k,\mu}(R^{N-1})$, see definition below, we speak about domain of class $C^{k,\mu}$ and we write $\partial\Omega \in C^{k,\mu}$.

Next, we denote $B_r(\boldsymbol{x}) := \{\boldsymbol{y} \in \mathbf{R}^N \,|\, |\boldsymbol{x} - \boldsymbol{y}| \leq r\}$ the ball of radius r at the point $\boldsymbol{x}$. A ball at origin, $B_r(0)$, we denote simply B_r. For a set M, we denote by 1_M its *characteristic function*.

2.1 Spaces

A vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N)$ of non-negative integers is called a *multiindex* of dimension N. The *length of multiindex* α is a number $|\alpha| = \sum_{i=1}^{N} \alpha_i$. With help of multiindexes, we can write

$$D^{\boldsymbol{\alpha}} f := \frac{\partial_1^{\alpha} \ldots \partial_N^{\alpha} f}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}}$$

for multiple partial derivatives of a function f of N variables.

For a domain Ω, we introduce following linear spaces: a space $C(\Omega)$ of functions continuous on Ω and $C_0(\Omega) \subset C(\Omega)$ a space of functions with compact support in Ω. For $k \in \mathbb{N}$, we denote by $C^k(\Omega)$ a space of functions with continuous partial derivatives up to the order k on Ω and by $C^k(\overline{\Omega})$ a space of functions, which derivatives up to the order k can be

continuously extended up to the boundary. In particular, we have $C^0(\overline{\Omega}) = C(\overline{\Omega})$. Finally, we shall denote $AC(\Omega)$ absolutely continuous functions on the domain Ω. Let us recall that the first derivative of an absolutely continuous function belongs to the space $L^1(\Omega)$. The space $C^k(\overline{\Omega})$ and the spaces $C_0(\Omega), AC(\overline{\Omega}) \subset C^k(\overline{\Omega})$ endowed with norm

$$\|f\|_{C^k(\Omega)} := \sum_{|\boldsymbol{\alpha}\leq k|} \sup_{\boldsymbol{x}\in\Omega} |D^{\boldsymbol{\alpha}} f|.$$

are Banach spaces. Moreover, we set

$$C^\infty(\Omega) = \cap_{k=1}^\infty C^k(\Omega), \quad C^\infty(\overline{\Omega}) = \cap_{k=1}^\infty C^k(\overline{\Omega}), \quad \text{and} \quad \mathcal{D}(\Omega) = C_0^\infty(\Omega) = C_0(\Omega) \cap C^\infty(\Omega).$$

The space $\mathcal{D}(\Omega)$ is endowed with the topology of local uniform convergence, i.e. $v_n \to v$ in $\mathcal{D}(\Omega)$, iff there exists a compact K such that

$$\operatorname{supp} v_n \subset K, \text{ and } D^\alpha v_n \to D^\alpha v, \text{ in } C(K) \text{ for all multiindices } \alpha.$$

A function $f \in C(\overline{\Omega})$ is *Hölder continuous*, if there exist constants $\mu \in (0,1]$, $L > 0$ such that

$$|f(\boldsymbol{x}) - f(\boldsymbol{y})| \leq L|\boldsymbol{x} - \boldsymbol{y}|^\mu, \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \Omega.$$

If $\mu = 1$, we speak of *Lipschitz functions*. We define a Banach space $C^{k,\mu}(\overline{\Omega})$, $k \in 0, 1, \ldots$ of the functions with Hölder continuous derivatives up to order k endowned with norm

$$\|f\|_{C^{k,\mu}(\Omega)} := \|f\|_{k-1,\mu} + \sum_{|\boldsymbol{\alpha}|=k} \sup_{\substack{\boldsymbol{x},\boldsymbol{y}\in\Omega \\ \boldsymbol{x}\neq\boldsymbol{y}}} \frac{|D^{\boldsymbol{\alpha}} f(\boldsymbol{x}) - D^{\boldsymbol{\alpha}} f(\boldsymbol{y})|}{|\boldsymbol{x} - \boldsymbol{y}|^\mu}.$$

If X is a metric space, one can define a Banach space $C(\overline{\Omega}; X)$ of continuous functions on Ω with values in the space X. Similarly one can define Banach spaces $C^k(\overline{\Omega}; X)$ and $C^{k,\mu}(\overline{\Omega}; X)$.

We shall use standard the Lebesgue spaces $L^p(\Omega)$, $1 \leq p \leq \infty$ and the Lebesgue spaces of functions with zero average

$$\overline{L^p(\Omega)} := \{f \in L^p(\Omega) \mid \int_\Omega f \, \mathrm{d}\boldsymbol{x} = 0\}.$$

Similarly, the standard notation $W^{k,p}(\Omega)$ is used for the Sobolev spaces with norm

$$\|f\|_{W^{k,p}(\Omega)} = \|f\|_{k,p} := \Big(\sum_{|\boldsymbol{\alpha}|=0}^{k} \|D^{\boldsymbol{\alpha}} f(\boldsymbol{x})\|_{L^p(\Omega)}^p \Big)^{\frac{1}{p}}.$$

In particular, $W^{0,p}(\Omega) = L^p(\Omega)$ and $\|f\|_{L^p(\Omega)} = \|f\|_p$. The space $W_0^{k,p}(\Omega)$ is closure of $\mathcal{D}(\Omega)$ in (k,p)-norm. We denote $W^{-k,p'}(\Omega)$ the dual space of $W_0^{k,p}(\Omega)$, where p' is dual exponent $1 = \frac{1}{p} + \frac{1}{p'}$. A natural extension are the spaces of vector valued functions $W^{k,p}(\Omega; \mathbf{R}^N)$, $W_0^{k,p}(\Omega; \mathbf{R}^N)$. Developing further this idea, one obtains the *Bochner spaces* $L^p(I; X)$ of L^p-integrable functions with values in some Banach space X, where the norm is defined as

$$\|f\|_{L^p(I;X)} = \| \|f(t)\|_X \|_{L^p(I)}.$$

For an unbounded domain Ω, we denote $C_{loc}^{k,\mu}(\Omega)$, $W_{loc}^{k,p}(\Omega)$ the functions that are in $C^{k,\mu}(K)$, $W^{k,p}(K)$ for any compact $K \subset \Omega$.

The Sobolev spaces have number of important and well known properties, in particular there exist extension and trace operators for the Lipschitz domains. One can find for example in monographs [2], [46], the corresponding theorems and proofs. Here, we cite only the most important for us, the *Rellich-Kondrachov compactness theorem*:

Theorem 2.1.1. *Let $\Omega \subset \mathbf{R}^N$ be a bounded domain and $p \geq 1$, $k > 0$. Then*

- *if $kp < N$, the space $W_0^{k,p}(\Omega)$ is continuously imbedded in $L^q(\Omega)$ for any*
$$1 \leq q \leq p^* = \frac{Np}{N-kp}.$$
Moreover, the imbedding is compact if $q < p*$.
- *If $kp = N$, the space $W_0^{k,p}(\Omega)$ is compactly imbedded in $L^q(\Omega)$ for any $1 \leq q < \infty$.*
- *If $kp > N + \mu$, $\mu > 0$, the space $W_0^{k,p}(\Omega)$ is compactly imbedded in $C^{0,\mu}(\overline{\Omega})$.*

Using the extension operator, the conclusion of the theorem remains valid even for the space $W^{k,p}(\Omega)$, provided Ω is a Lipschitz domain.

Next, we introduce smoothing kernels

$$\omega_\delta(\boldsymbol{x}) = \frac{\tilde{\omega}_\delta(\boldsymbol{x})}{\big\|\tilde{\omega}_\delta\big\|_{L^1(\mathbf{R}^N)}}, \quad \tilde{\omega}_\delta(\boldsymbol{x}) = \omega\Big(\frac{|\boldsymbol{x}|}{\delta}\Big), \tag{2.1}$$

where $\omega \in \mathcal{D}(\mathbf{R})$ is an even, non-negative function with support $(-1, 1)$. The smoothing kernels can be used to construct smooth approximations of distributions. Let $v \in \mathcal{D}'(\Omega)$, then $\omega_\delta \star v \in \mathcal{D}(\mathbf{R}^N)$ and if $v \in X(\Omega)$, where X stands for L^p, $W^{k,p}$, or $C^{k.\mu}$ space, then

$$\omega_\delta \star v \to v \text{ strongly in } X(\Omega).$$

For a non-decreasing, concave function

$$T \in C^\infty([0,\infty)),\ T(t) = \begin{cases} t & \text{on } [0,1], \\ 2 & \text{on } [3,\infty), \end{cases}$$

we define *cut-off functions*.

$$T_k(t) := kT\Big(\frac{t}{k}\Big).$$

If $f \in L^p(\Omega)$, then $T_k(f)$ belongs in $L^\infty(\Omega)$ and by virtue of Levi's theorem,

$$T_k(f) \to f \quad \text{strongly in } L^p(\Omega) \text{ as } k \to \infty.$$

Poincaré inequalities allow to control $W^{1,p}$-norm by the L^p-norm of the first derivatives and L^1-norm of the function. The following lemma is a version of such an inequality inspired by Lemma 3.2 in [14].

Lemma 2.1.2. *Let $\Omega \subset \mathbf{R}^N$ be a bounded domain, and let $0 < \lambda < p$, $p > 1$ be given. Let $v \in W^{1,p}(\Omega)$, and let ϱ be a non-negative function such that $0 < m \leq \big\|\varrho\big\|_1$ and $\big\|\varrho\big\|_\gamma \leq M$, $\gamma > 1$. Then*

$$\big\|v\big\|_{L^p(\Omega)} \leq c(p,\lambda,m,M)\left(\big\|\nabla v\big\|_{L^p(\Omega;\mathbf{R}^N)} + \Big(\int_\Omega \varrho|v|^\lambda \,\mathrm{d}\boldsymbol{x}\Big)^{\frac{1}{\lambda}}\right). \tag{2.2}$$

Proof. Let assume by contradiction that the statement does not hold. Then for any $n \in \mathbb{N}$, we can find v_n, ϱ_n such that

$$\|v_n\|_{L^p(\Omega)} \geq n \left(\|\nabla v_n\|_{L^p(\Omega;\mathbf{R}^N)} + \Big(\int_\Omega \varrho_n |v_n|^\lambda \, \mathrm{d}\boldsymbol{x} \Big)^{\frac{1}{\lambda}}. \right) \tag{2.3}$$

Consequently, for $w_n = v_n \|v_n\|^{-1}_{L^p(\Omega)}$, we have

$$\|\nabla w_n\|_{L^p(\Omega)} \to 0 \quad \text{and} \quad \|w_n\| \to w \text{ in } L^p(\Omega)$$

because of compact imbedding, where $w \equiv |\Omega|^{-\frac{1}{p}}$. This, in fact, yields strong convergence

$$w_n \to w \text{ in } W^{1,p}(\Omega).$$

On the other hand, as $\|\varrho_n\|_\gamma \leq M$, we can find k such that

$$T_k(\varrho_n) \to \overline{T_k(\varrho)} \text{ weakly in } L^\beta(\Omega) \text{ for every } \beta \geq 1,$$

where

$$\int_\Omega \overline{T_k(\varrho)} \, \mathrm{d}\boldsymbol{x} \geq \frac{m}{2}. \tag{2.4}$$

Here, we introduce a notation that $\overline{f}$ denotes the weak L^1-limit of the sequence f_n. The former fact together with (2.3) leads to

$$0 = \lim_{n\to\infty} \int_\Omega \varrho_n |w_n|^\lambda \, \mathrm{d}\boldsymbol{x} \geq \lim_{n\to\infty} \int_\Omega T_k(\varrho_n) |w_n|^\lambda \, \mathrm{d}\boldsymbol{x} = |\Omega|^{-\frac{\lambda}{p}} \int_\Omega \overline{T_k(\varrho)} \, \mathrm{d}\boldsymbol{x},$$

which is in contrast with (2.4). □

We enclose this section by several auxiliary resutls. In the following lemma, we introduce the *Bogovskii operator* $\mathcal{B}$, which is a particular inverse operator of the divergence on a Lipschitz domain. The full proof of the lemma as well as references to the original works, one can find in Section 3.3 of [37].

Lemma 2.1.3. *Let Ω be a bounded Lipschitz domain. Then there exists a linear operator*

$$\mathcal{B} : \overline{L^p}(\Omega) \to W^{1,p}_0(\Omega;\mathbf{R}^N), \quad 1 < p < \infty,$$

such that

$$\mathrm{div}\mathcal{B}(f) = f \quad \textit{a.e. in } \Omega \textit{ for all } f \in \overline{L^p(\Omega)}.$$

This operator is continuous,

$$\|\mathcal{B}(f)\|_{1,p} \leq c(p,\Omega)\|f\|_p, \tag{2.5}$$

and if $f = \operatorname{div} \boldsymbol{g}$ for some $\boldsymbol{g} \in L^q(\Omega;\mathbf{R}^N)$ (with $\operatorname{div} \boldsymbol{g} \in L^p(\Omega)$), then

$$\|\mathcal{B}(f))\|_q \leq c(q,\Omega)\|\boldsymbol{g}\|_q. \tag{2.6}$$

Remark 2.1.4. *The operator itself does not depend on p but only on the domain Ω. Furthermore, the constants in* (2.5) *and* (2.6) *in fact depends only on the Lipschitz constant of the boundary $\partial\Omega$.*

Next, we recall the *Gronwall lemma*, which is essentially used in the energy estimates of evolution equations.

Lemma 2.1.5. *[26, Lemma 4.3.1] Let $h \in L^\infty(0,T)$, $h \geq 0$, $a \in \mathbf{R}$, and $b \in L^1(0,T)$, $b \geq 0$ satisfy the inequality*

$$h(t) \leq a + \int_0^t b(s)h(s)\,\mathrm{d}s, \quad \textit{for all } t \in [0,T].$$

Then

$$h(t) \leq a \exp\Big(\int_0^t b(s)\,\mathrm{d}s\Big) \quad \textit{for a.e.t} \in [a,b].$$

We end with the *Shauder fixed point theorem*:

Theorem 2.1.6 (Theorem 2.1.2 in [33])**.** *Let A be a closed bounded convex subset of a Banach space X and $f : A \to A$ a compact mapping, i.e. continuous mapping that maps bounded sets on the compats. Then, there exists $v \in A$ such that $f(v) = v$. Such v is called a fixed point of f.*

2.2 Compactness tools

We say that a sequence v_n in a Banach space X *converges weakly* to $v \in X$, iff

$$\langle f, v_n\rangle \to \langle f, v\rangle \quad \text{for every } f \in X^*,$$

where X^* denotes the dual of X. Similarly, a sequence f_n in X^* converges weakly-$*$ to $f \in X^*$, iff

$$\langle f_n, v\rangle \to \langle f, v\rangle \quad \text{for every } v \in X.$$

For reflexive spaces the weak and weak-$*$ topology are equivalent. We shall denote X_{weak} a linear space X with the weak topology, A crucial result in the theory of partial differential equations is weak-$*$ compactness of bounded sets stated in the Alaoglu-Bourbaki theorem:

Theorem 2.2.1. *Let X^* be the dual space of a Banach space X and M a bounded subset of X^*. Then M is precompact in weak-$*$ topology of X^*. In particular, from any sequence $v_n \in M$, one can take a subsequence (not relabeled) such that*

$$v_n \to v \quad \textit{weakly-}* \textit{ in } X^*.$$

If the space X is separable, the weak-$*$ topology is metrizable on bounded sets of X^*. Then one can introduce the space $C(\overline{\Omega}; X^*_{weak})$. For every its member f, the function $\langle f(t), v\rangle$ is continuous uniformly with respect to the choice of $v \in X$. Very usefull characterization of compact sets in $C(\overline{\Omega}; X)$ is the following abstract *Arzelà-Ascoli theorem:*

Theorem 2.2.2. *[25, Chapter 7, Theorem 17] Let $K \subset \mathbf{R}^N$ be compact and X a compact metric space endowed with a metric d_X. Let $\{v_n\}_{n=1}^\infty$ be a sequence of functions in $C(K;X)$ which is equi-continuous, that is for any $\varepsilon > 0$ there is $\delta > 0$ such that*

$$|\boldsymbol{y} - \boldsymbol{z}| < \delta \quad \Longrightarrow \quad d_X\big(v_n(\boldsymbol{y}), v_n(\boldsymbol{z})\big) \leq \varepsilon, \quad \textit{independently of } n.$$

Then $\{v_n\}_{i=1}^{\infty}$ is precompact in $C(K;X)$. In particular, there exists a subsequence (not relabeled) and a function $v \in C(K;X)$ such that

$$\sup_{\boldsymbol{y}\in K} d_X\big(v_n(\boldsymbol{y}), v(\boldsymbol{y})\big) \to 0 \text{ as } n \to \infty.$$

In the next theorem, we have summarized some usefull properties of convex functions.

Theorem 2.2.3. *[14, Theorem 2.11 and its Corollary 2.2] Let $O \subset \mathbf{R}^N$ be a bounded measurable set and $\Phi : \mathbf{R}^M \to (-\infty, +\infty]$ be a lower semi-continuous convex function. Let $\{\boldsymbol{v}_n\}_{n=1}^{\infty}$ be a sequence of functions such that*

$$\boldsymbol{v}_n \to \boldsymbol{v} \text{ weakly in } L^1(O;\mathbf{R}^M)$$

and

$$\Phi(\boldsymbol{v}_n) \to \overline{\Phi(\boldsymbol{v})} \text{ weakly in } L^1(O).$$

Then

$$\Phi(\boldsymbol{v}) \le \overline{\Phi(\boldsymbol{v})} \text{ a.e. on } O$$

and

$$\int_O \Phi(\boldsymbol{v})\,\mathrm{d}\boldsymbol{y} \le \liminf_{n\to\infty} \int_O \Phi(\boldsymbol{v}_n)\,\mathrm{d}\boldsymbol{y}.$$

If, moreover, Φ is strictly convex on an open convex set $U \subset \mathbf{R}^M$, and

$$\Phi(\boldsymbol{v}) = \overline{\Phi(\boldsymbol{v})} \text{ a.e. on } O,$$

then

$$\boldsymbol{v}_n(\boldsymbol{y}) \to \boldsymbol{v}(\boldsymbol{y}) \text{ for a.e. } \boldsymbol{y} \in O \cap \{\boldsymbol{v} \in U\},$$

extracting a subsequence as the case may be.

In general, the weak limits does not commute with multiplication, i.e. $\overline{w\,v} \neq \overline{w}\,\overline{v}$. However, for certain products, the celebrated Div-Curl lemma due to Tartar [44] can be used.

Lemma 2.2.4. *Let $\Omega \subset \mathbf{R}^N$ be a bounded domain. Let*

$$\boldsymbol{v}_n \rightharpoonup \boldsymbol{v} \text{ weakly in } L^p(\Omega;\mathbf{R}^N), \quad \boldsymbol{w}_n \rightharpoonup \boldsymbol{w} \text{ weakly in } L^q(\Omega;\mathbf{R}^N),$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1$, $1 < p, q < \infty$ and let

$$\mathrm{div}\boldsymbol{v}_n \text{ and } \mathrm{curl}\,\boldsymbol{w}_n \text{ be precompact in } W^{-1,s}(\Omega),\ W^{-1,s}(\Omega;\mathbf{R}^{N\times N})$$

respectively, for some $s > 1$. Then

$$\boldsymbol{v}_n \cdot \boldsymbol{w}_n \to \boldsymbol{v} \cdot \boldsymbol{w} \quad \text{weakly in } L^r(\mathbf{R}^N).$$

We conclude with two results concerning the R-operator in $\mathbf{R}^3$, which is defined via Fourier transform by the formula

$$\mathcal{R}_{i,j}[v] := \mathcal{F}^{-1}\big(-\boldsymbol{\xi}_i\boldsymbol{\xi}_j\,|\boldsymbol{\xi}|^{-2}\,\mathcal{F}(v)\big) = \nabla_i\nabla_j\Delta^{-1}v, \tag{2.7}$$

where

$$\Delta^{-1}v[\boldsymbol{x}] = \mathcal{F}^{-1}\big(-|\boldsymbol{\xi}|^{-2}\mathcal{F}(v)\big) = \int_{\mathbf{R}^3} v(\boldsymbol{y})|\boldsymbol{x}-\boldsymbol{y}|^{-1}\,\mathrm{d}\boldsymbol{x}. \tag{2.8}$$

It is a continuous operator from $L^p(\mathbf{R}^3)$ to $L^p(\mathbf{R}^3)$, $1<p<\infty$ and it holds

$$\mathcal{R}_{i,j} = \mathcal{R}_{j,i}, \quad \int_{\mathbf{R}^3} \mathcal{R}_{i,j}[v]w\,\mathrm{d}\boldsymbol{x} = \int_{\mathbf{R}^3} v\mathcal{R}_{i,j}[w]\,\mathrm{d}\boldsymbol{x}.$$

As a corollary of Lemma 2.2.4, we get the following commutator lemma (see [14, Corollary 6.1])

Corollary 2.2.5. *Let $1<p,q<\infty$, $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}<1$ and*

$$\begin{aligned} f_n &\to f \quad \textit{weakly in } L^p(\mathbf{R}^3),\\ g_n &\to g \quad \textit{weakly in } L^q(\mathbf{R}^3). \end{aligned}$$

Then

$$f_n\mathcal{R}_{i,j}[g_n] - g_n\mathcal{R}_{i,j}[f_n] \longrightarrow f\mathcal{R}_{i,j}[g] - g\mathcal{R}_{i,j}[f] \quad \textit{weakly in } L^r(\mathbf{R}^3). \tag{2.9}$$

Furthermore, we cite the result due to Feireisl [12] in spirit of Coifman and Meyer [6].

Lemma 2.2.6. *Let $\boldsymbol{V}\in L^2(\mathbf{R}^3;\mathbf{R}^3)$ and $w\in W^{1,r}(\mathbf{R}^3)$, $r>\frac{6}{5}$. Then there exist constants $c(r)>0$, $\omega(r)>0$, and $p(r)>1$ such that*

$$\big\|\mathcal{R}_{i,j}[wV_j] - w\mathcal{R}_{i,j}[V_j]\big\|_{W^{\omega,p}(\mathbf{R}^3,\mathbf{R}^3)} \le c\big\|w\big\|_{W^{1,r}(\mathbf{R}^3)}\big\|\boldsymbol{V}\big\|_{L^2(\mathbf{R}^3,\mathbf{R}^3)}.$$

2.2.1 Young measures

Let $\Omega\subset\mathbf{R}^N$ be a domain. We say that $\psi(\boldsymbol{x},\boldsymbol{y})$ is a *Carahheodory function* on $Q\times\mathbf{R}^m$ if

$$\left.\begin{aligned} &\text{the function } \psi(\boldsymbol{x},\,\cdot\,) \text{ is continuous on } \mathbf{R}^m \text{ for a.a. } x\in Q\\ &\text{the function } \psi(\,\cdot\,,\boldsymbol{y}) \text{ is measurable on } Q \text{ for all } \boldsymbol{y}\in\mathbf{R}^m. \end{aligned}\right\} \tag{2.10}$$

Let consider a family probablity measures $\{\nu_{\boldsymbol{x}}\}$ parametrized by points $\boldsymbol{x}\in Q$. This family we call *Young measure* if for every $\varphi\in C(\mathbf{R}^m)\cap L^\infty(\mathbf{R}^m)$ the function

$$\boldsymbol{x}\to\int_{\mathbf{R}^m}\varphi(\boldsymbol{y})\,\mathrm{d}\nu_{\boldsymbol{x}}(\boldsymbol{y}) \equiv \langle\nu_{\boldsymbol{x}},\varphi\rangle$$

is meeasurable on Q. Young measures can be used to represent a limit of a weakly converging sequence composed with a nonlinear function, which is statement of the following theorem.

Theorem 2.2.7. *[38, Theorem 6.2] Let $\Omega\subset\mathbf{R}^N$ be a domain and $f_n: Q\to\mathbf{R}^m$ a sequence of functions converging weakly to f in $L^p(\Omega;\mathbf{R}^m)$ for some $p>1$. Then there exists a Young measure $\nu_{\boldsymbol{x}}$ such that*

$$\overline{\psi}(\boldsymbol{x}) = \int_{\mathbf{R}^m}\psi(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\nu_{\boldsymbol{x}}(\boldsymbol{y})$$

whenever ψ is a Caratheodory function on $Q\times\mathbf{R}^m$ and the sequence $\psi(\,\cdot\,,f_n(\,\cdot\,))$ admits a weak L^1-limit $\overline{\psi}$.

Chapter 3

Existence theory for a non-homogeneous fluid

In this chapter, we shall treat the existence problem for a gas with heterogeneous constitutive laws, which we will use later in Chapter 4 as an approximation for the problem of rigid bodies. As a matter of fact, the variational formulation based on the entropy inequality does not cope well with fully heterogeneous constitutive equation for the internal energy and the entropy. The problem is explained in Section 4.9. Therefore we consider only the transport coefficients, namely the viscosities μ, λ and the heat conductivity coefficient κ, to be space and/or time dependent. More specifically, we shall assume that they are transported by some smooth artificial velocity field $\boldsymbol{U}$.

The first existence result for global-in-time solutions to the incompressible Navier-Stokes system was achieved by Leray [28] in 1934. In this pioneering work he also introduced the concept of weak solutions. A similar result for a compressible fluid in isentropic regime was proved by P.-L. Lions in [29] (1998). Further essential contribution and extension of the existence theory for the complete system with the temperature is mainly due to Feireisl. In [13] he introduced a concept of oscillation defect measure for the density, which allows to treat an isentropic flow with realistic adiabatic constants in three dimensions. In the book [14] he has presented complete existence theory for a compressible heat conducting fluid with quite general constitutive laws. Later improvements cope, among other things, with temperature dependent viscosities [12], the total energy equality and entropy formulation [9], and the general constitutive equation for the ideal gas [17].

The existence theory presented in this chapter is based on papers [9], [17] with only slight modifications in order to accommodate heterogeneous constitutive equations for the transport coefficients. In Section 3.1, we define the variational solution and we state the main result. Its proof, which is performed in subsequent sections, consists of the standard chain of approximations. First, in Section 3.2, we construct local-in-time solution to the Faedo-Galerkin approximation of modified system. Then, we gain the estimates independent of time and extend the solution on an arbitrary time interval. In Section 3.4, we pass to the limit in the sequence of Faedo-Galerkin approximations and we obtain a solution of the system with several regularizing terms. The aim of the last two sections 3.5, 3.6 is to remove the additional terms letting their coefficients tend to zero. These last two limits share a lot of

features with the high viscosity limit in Chapter 4, where the proof is explained in full detail, therefore in actual chapter we present finely only the different steps.

3.1 Problem Formulation

As usual, we denote ϱ, $\boldsymbol{u}$, ϑ the density, the velocity, and the temperature, respectively. Following [14], we introduce a concept of *variational solutions* to the system of equations (1.7), (1.10), and (1.17).

Definition 3.1.1. *We shall say that functions*

$$\varrho \in L^\infty(I; L^\gamma(\Omega)),\ \boldsymbol{u} \in L^2(I; W_0^{1,2}(\Omega)),\ \vartheta \in L^2(I; W^{1,2}(\Omega)) \tag{3.1}$$

form a **variational solution** *of* **problem (F)** *if*

- *The density ϱ is non-negative function a.e. on $I \times \Omega$ and $\vartheta > 0$ a.e. on $I \times \Omega$)).*
- *The continuity equation* (1.7) *is satisfied in the sense of distributions,*

$$\int_I \int_\Omega \varrho \partial_t \varphi + \varrho \boldsymbol{u} \cdot \nabla \varphi \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t = 0, \quad \varphi \in \mathcal{D}(I \times \overline{\Omega}). \tag{3.2}$$

- *The momentum equation* (1.10) *is satisfied in the sense of distributions as well,*

$$\int_I \int_\Omega \varrho \boldsymbol{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho \boldsymbol{u} \otimes \boldsymbol{u} : \nabla \boldsymbol{\varphi} + p \mathrm{div} \boldsymbol{\varphi} - \mathbb{S} : \nabla \boldsymbol{\varphi} + \varrho \boldsymbol{f} \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t = 0 \tag{3.3}$$

  for any $\varphi \in \mathcal{D}(I \times \Omega; \mathbf{R}^3)$.

- *The specific entropy satisfies an inequality*

$$\int_I \int_\Omega \varrho s \partial_t \varphi + \varrho s \boldsymbol{u} \cdot \nabla \varphi - \frac{\kappa \nabla \vartheta \cdot \nabla \varphi}{\vartheta} + \Big(\frac{\mathbb{S} : \nabla \boldsymbol{u}}{\vartheta} + \frac{\kappa |\nabla \vartheta|^2}{\vartheta^2} \Big) \varphi \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \leq 0 \tag{3.4}$$

  for any $\varphi \in \mathcal{D}(I \times \overline{\Omega})$, $\varphi \geq 0$.

- *At last, the total energy balance over the whole domain holds:*

$$\int_\Omega E[t_2] - E[t_1] \, \mathrm{d}\boldsymbol{x} = \int_{t_1}^{t_2} \int_\Omega \varrho \boldsymbol{f} \cdot \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t, \quad \text{for a.e. } t_1,\ t_2 \in I. \tag{3.5}$$

In Definition 3.1.1, we have replaced the equation for the temperature by the entropy inequality (3.4) and the energy balance (3.5). The idea is to get rid of the term $\mathbb{T} : \nabla \boldsymbol{u}$ in the internal energy equation (1.17). This term, as well as its counterparts in the equivalent equations (1.19) and (1.15), is known to be only L^1-integrable in time. Using the entropy inequality is convenient, since one part of $\mathbb{T} : \nabla \boldsymbol{u}$, namely $\mathbb{S} : \nabla \boldsymbol{u}$ is convex and can be treated via weak lower semi-continuity, while the other part, $p\mathrm{div}\boldsymbol{u}$, disappears during the calculation (1.18). The total energy balance has to be included into formulation, in order to keep formal equivalence with the original problem.

Indeed, if the solution $(\varrho, \boldsymbol{u}, \vartheta)$ is smooth, we can take $\vartheta\varphi$ as a test function in (3.4) and perform a reversed calculation (1.18) to deduce

$$\int_0^T \int_\Omega \varrho e \varphi_t + \varrho e \boldsymbol{u} \cdot \nabla \varphi - \kappa_\delta \nabla \vartheta \cdot \nabla \varphi + (\mathbb{S} : \nabla \boldsymbol{u} - p \,\mathrm{div} \boldsymbol{u})\varphi \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}t \leq 0 \tag{3.6}$$

for any $\varphi \in \mathcal{D}(I \times \overline{\Omega})$, $\varphi \geq 0$. On the other hand for the smooth solution we easily obtain the strong momentum equation (1.10) and the equation for the kinetic energy (1.16). The later one can be integrated over Ω and subtracted from (3.5) to get

$$\int_\Omega (\varrho e)[t_2] \,\mathrm{d}\boldsymbol{x} - \int_\Omega (\varrho e)[t_1] \,\mathrm{d}\boldsymbol{x} = \int_{t_1}^{t_2} \int_\Omega p \,\mathrm{div} \boldsymbol{u} - \mathbb{S} : \nabla \boldsymbol{u} \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}t,$$

which excludes strict inequality in (3.6).

3.1.1 Renormalized Continuity Equation

In addition to the declarations of Definition (3.1.1), the solution we are going to construct will satisfy the so-called *renormalized continuity equation*. The idea of renormalization for the equations of hyperbolic type is due to DiPerna, Lions [8]. For regular solutions, one can multiply (1.7) by $B'(\varrho)$, where

$$B \in C^1[0, \infty), \ B(r) \equiv C_M \text{ for } r > M, \tag{3.7}$$

and obtain

$$\partial_t B(\varrho) + \mathrm{div}(B(\varrho)\boldsymbol{u}) + b(\varrho)\mathrm{div}\boldsymbol{u} = 0, \quad b(r) = B'(r)r - B(r). \tag{3.8}$$

Similar renormalization procedure can be done even for the weak solutions provided the density is square integrable.

Proposition 3.1.2. *[14, Proposition 4.2] Let $\Omega \subset \mathbf{R}^3$ be a domain and*

$$\varrho \in L^2(I; L^2(\Omega)), \ \boldsymbol{u} \in L^2(I; W^{1,2}(\Omega; \mathbf{R}^3)), \ \text{and } h \in L^1(I \times \Omega)$$

satisfy

$$\partial_t \varrho + \mathrm{div}(\varrho \boldsymbol{u}) = h \quad \text{in } \mathcal{D}'(I \times \Omega).$$

Then

$$\int_0^T \int_\Omega B(\varrho)\partial_t \varphi + B(\varrho)\boldsymbol{u} \cdot \nabla \varphi - \big(b(\varrho)\mathrm{div}\boldsymbol{u} + B'(\varrho)h\big)\varphi \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}t = 0, \quad \varphi \in \mathcal{D}(I \times \Omega) \tag{3.9}$$

for any B satisfying (3.7).

In fact, equation (3.9) holds even for a larger family of functions B according to available estimates of the density.

Proposition 3.1.3. *Let $\Omega \subset \mathbf{R}^3$ be a domain and*

$$\varrho \in L^p(I; L^p(\Omega)), \ \boldsymbol{u} \in L^2(I; W^{1,2}(\Omega; \mathbf{R}^3))$$

satisfy (3.9) *for every B from the class* (3.7). *Then the same equation holds even for every*

$$B \in W^{1,\infty}_{loc}[0, \infty), \ |B'(r)| \leq C(1 + r^{\frac{p}{2}-1}). \tag{3.10}$$

Proof. The functions (3.7) are dense in $W^{1,\infty}_{loc}[0,\infty)$, thus we find among them a sequence B_n approximating a function B from (3.10). Further, according to (3.10), we can assume

$$|B_n(\varrho)|,\ |B_n'(\varrho)\varrho - B_n(\varrho)| \leq C(1+\varrho^{\frac{p}{2}}).$$

Consequently, we can find an integrable majorant for every term in the equation (3.9) and use Lebesgue convergence theorem to pass to the limit in equation (3.9) as $n \to \infty$. □

Next, we report the validity of the renormalized continuity equation even on the whole space, provided the renormalized density is square-integrable.

Proposition 3.1.4. *[14, see Proposition 4.1] Let $\Omega \subset \mathbf{R}^3$ be a bounded Lipschitz domain and B function such that*

$$B(\varrho) \in L^2(I; L^2(\Omega)),\ \boldsymbol{u} \in L^2(I; W^{1,2}_0(\Omega; \mathbf{R}^3)).$$

satisfy

$$\partial_t B(\varrho) + \mathrm{div}(B(\varrho)\boldsymbol{u}) + b(\varrho)\mathrm{div}\boldsymbol{u} = 0, \quad \text{in } \mathcal{D}'(I\times\Omega). \tag{3.11}$$

Then (3.11) *holds even in $\mathcal{D}'(I\times\mathbf{R}^3)$, assuming $(\varrho,\boldsymbol{u})$ extended by zero outside Ω.*

Applying the smoothing kernels $\omega_\varepsilon(|\boldsymbol{x}-\boldsymbol{y}|)$, introduced in (2.1), on the both sides of (3.11), we get

$$\partial_t(\omega_\varepsilon \star B(\varrho)) + \mathrm{div}((\omega_\varepsilon \star B(\varrho))\boldsymbol{u}) + \omega_\varepsilon \star (b(\varrho)\mathrm{div}\boldsymbol{u}) = r_\varepsilon \quad \text{a.e. in } I\times\mathbf{R}^3, \tag{3.12}$$

where by virtue of Lemma 4.3 in [14]

$$r_\epsilon = \omega_\varepsilon \star \mathrm{div}(B(\varrho)\boldsymbol{u}) - \mathrm{div}((\omega_\varepsilon \star B(\varrho))\boldsymbol{u}) \to 0 \text{ in } L^r(\mathbf{R}^3)$$

as $\varepsilon \to 0$, provided $B(\varrho) \in L^p(\Omega)$, $\boldsymbol{u} \in W^{1,2}(\Omega;\mathbf{R}^3)$, and $\frac{1}{r} \geq \frac{1}{2} + \frac{1}{p}$.

For a renormalized solution $\varrho \in L^\infty(I; L^\gamma(\Omega))$, one have

$$B(\varrho) \in C(I; L^\gamma_{weak}(\Omega)) \quad \text{and} \quad \varrho \in C(I; L^\gamma_{weak}(\Omega)),$$

see the discusion before (4.59). Then the question is, whether or not the identity

$$B(\varrho)[t] = B(\varrho[t]).$$

holds. The last result of this section gives a positive answer.

Proposition 3.1.5. *[14, Proposition 4.3] Let $\Omega \subset \mathbf{R}^3$ be a bounded domain. Let $\varrho \geq 0$,*

$$\varrho \in L^\infty(I; L^\gamma(\Omega)),\ \boldsymbol{u} \in L^2(I; W^{1,2}(\Omega;\mathbf{R}^3)),\ \gamma > \frac{6}{5},$$

be a solution of the renormalized continuity equation (3.9)*. Then*

$$\varrho \in C(I; L^1(\Omega)). \tag{3.13}$$

3.1.2 Heterogeneous Constitutive equations

We suppose that the thermodynamic quantities p, e, s are functions of the state variables ϑ, ϱ according to the discussion in Section 1.2.1. More specifically, we consider the pressure

$$\left\{\begin{array}{c} p(\varrho,\vartheta)=p_G(\varrho,\vartheta)+p_R(\vartheta) \\ p_G(\varrho,\vartheta)=\vartheta^{\frac{5}{2}}P\big(\varrho\vartheta^{-\frac{3}{2}}\big),\ p_R(\vartheta)=\dfrac{a}{3}\vartheta^4,\ a>0, \end{array}\right\} \tag{3.14}$$

where the function P meets hypothesis

$$\left\{\begin{array}{c} P\in C^1[0,\infty),\ P(0)=0,\ \text{non-decreasing}, \\ 0<\dfrac{5}{3}P(Y)-P'(Y)Y\le c(1+Y^\alpha),\ 0<\alpha<\dfrac{50}{39}, \\ \lim\limits_{Y\to\infty}P(Y)Y^{-\frac{5}{3}}\ge P_\infty>0. \end{array}\right\} \tag{3.15}$$

The growth condition on the second line is only technical and used only in the proof of strong convergence of the density. As a direct consequence, we get

$$P(Y)\le C(Y+Y^{\frac{5}{3}})\quad\text{and}\quad P'(Y)\le C(1+Y^{\frac{2}{3}}). \tag{3.16}$$

The internal energy is determined by (3.14) as

$$\left\{\begin{array}{c} e(\varrho,\vartheta)=e_G(\varrho,\vartheta)+e_R(\vartheta), \\ e_G=\dfrac{3}{2}p_G(\varrho,\vartheta)\varrho^{-1},\ e_R=a\vartheta^4\varrho^{-1}. \end{array}\right\} \tag{3.17}$$

In view of (1.27) and (1.28) the corresponding specific entropy reads

$$\left\{\begin{array}{c} s(\varrho,\vartheta)=s_G(\varrho,\vartheta)+s_R(\vartheta), \\ s_G=S\big(\varrho\vartheta^{-\frac{3}{2}}\big),\ s_R=\dfrac{4}{3}a\vartheta^3\varrho^{-1}, \end{array}\right\} \tag{3.18}$$

where S is a C^1-function interrelated with P through the relation (1.38). Obviously S is non-increasing in Y and since $c_v(Y)$ is positive, we have

$$s_G=S(Y)\le-\frac{2}{3}C_V\log Y,\ C_V=\max_{y\in[0,M]}c_v(y),\quad\text{for } 0<Y<M. \tag{3.19}$$

The Cauchy stress tensor $\mathbb{T}$, the viscous stress tensor $\mathbb{S}$, and the heat flux $\boldsymbol{q}$ are given by (1.21), (1.22), and (1.23) respectively. The corresponding transport coefficients are allowed to be also function of time and space, moreover they depends on $\boldsymbol{u}$. More specificaly, the constitutive functions of transport coeficients are given by a mapping

$$\mathcal{Z}:\boldsymbol{u}\in L^2(Q)\longrightarrow(\mu,\zeta,\kappa_G,\kappa_R), \tag{3.20}$$

with following properties. For a fixed $\boldsymbol{u}\in L^2(Q)$ the viscosities $\mu[\boldsymbol{u}](t,\boldsymbol{x};\vartheta)$ and $\zeta[\boldsymbol{u}](t,\boldsymbol{x};\vartheta)$ are C^1-functions of ϑ and measurable in t and $\boldsymbol{x}$. Further, they obey growth conditions

$$\left\{\begin{array}{l} 0<\underline{\mu}(1+\vartheta)\le\mu(t,\boldsymbol{x};\vartheta);\ |\partial_\vartheta\mu|\le\overline{\mu}, \\ 0\le\underline{\zeta}(1+\vartheta)\le\zeta(t,\boldsymbol{x};\vartheta);\ |\partial_\vartheta\zeta|\le\overline{\mu}, \end{array}\right\} \tag{3.21}$$

where constants $\underline{\mu}$, $\overline{\mu}$, $\underline{\zeta}$, $\overline{\zeta}$ are independent of $\boldsymbol{u}$.

Having $\boldsymbol{u}$ still fixed, the heat conductivity κ consists of the perfect gas part κ_G and the radiation part κ_R, which are $W^{1,2}$-functions of t, C^2-functions of $\boldsymbol{x}$ and C^1-functions of ϑ. Moreover they satisfy

$$\left.\begin{array}{c} 0 < \underline{\kappa} \leq \kappa^G(t,\boldsymbol{x};\vartheta) \leq \overline{\kappa}\vartheta, \\ 0 < \underline{\kappa}\vartheta^3 \leq \kappa^R(t,\boldsymbol{x};\vartheta) \leq \overline{\kappa}\vartheta^3, \\ |\partial_x\partial_x(\kappa_G+\kappa_R)| \leq \overline{K_x}\vartheta^3, \\ |\partial_t(\kappa_G+\kappa_R)(t,\boldsymbol{x};\vartheta)| \leq \overline{K_t}(t,\boldsymbol{x})\vartheta^3. \end{array}\right\} \tag{3.22}$$

Here the constants $\underline{\kappa}$, $\overline{\kappa}$ are independent of $\boldsymbol{u}$, while the constant $\overline{K_x}$ and the function $\overline{K_t}$ from $L^2(I;L^\infty(\Omega))$ depends on $\|\boldsymbol{u}\|_{L^2(Q)}$. The last two lines in (3.22) are only technical conditions, which are used only at the very beginning in construction of the first approximation and they can probably be eliminated.

Finally, we assume that the mapping $\mathcal{Z}$ satisfies a following compact property. Whenever

$$\boldsymbol{u}_n \rightharpoonup \boldsymbol{u} \quad \text{weakly in } L^2(I\times\Omega;\mathbf{R}^3),$$

there exists a subsequence (not relabeled) such that

$$\mathcal{Z}(\boldsymbol{u}_n) \to \mathcal{Z}(\boldsymbol{u}) \text{ a.e. on } I\times\Omega\times\mathbf{R}. \tag{3.23}$$

Now, we are ready to state the main result about the existence of global-in-time variational solutions to **problem (F)**

Theorem 3.1.6. *Let $\Omega \subset \mathbf{R}^3$ be a bounded domain of class $C^{2+\nu}$, $\nu > 0$. Assume p, e, s are given through* (3.14), (3.17), (3.18), *respectively. Let μ, λ, κ obey* (3.21) *and* (3.22). *Then for any force $\boldsymbol{f} \in L^\infty(\Omega;\mathbf{R}^3)$ and initial data $\varrho_0, \vartheta_0 \in L^\infty(\Omega)$, $\boldsymbol{u}_0 \in L^\infty(\Omega;\mathbf{R}^3)$, there exists at least one variational solution of* **problem (F)** *in the sense of Definition 3.1.1, which satisfies the initial conditions*

$$\left.\begin{array}{c} \varrho[t] \to \varrho[0] = \varrho_0 \text{ in } L^1(\Omega), \\ (\varrho\boldsymbol{u})[t] \to (\varrho\boldsymbol{u})[0] = \varrho_0\boldsymbol{u}_0 \text{ weakly in } L^1(\Omega;\mathbf{R}^3), \\ E[t] \to E[0] = E_0 = E(\varrho_0,\boldsymbol{u}_0,\vartheta_0) \end{array}\right\} \text{as } t \to 0+$$

and

$$\operatorname*{esslim}_{t\to 0+}\int_\Omega (\varrho s(\varrho,\vartheta))[t]\varphi \,\mathrm{d}\boldsymbol{x} \geq \lim_{\tau\to 0+}\int_0^\tau\int_\Omega (\varrho s(\varrho,\vartheta))\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_\Omega \varrho_0 s(\varrho_0,\vartheta_0)\varphi\,\mathrm{d}\boldsymbol{x}$$

for any $\varphi \in \mathcal{D}(\overline{\Omega})$, $\varphi \geq 0$.

Rest of the chapter is devoted to the proof, which consist of the following steps

- The continuity equation is equipped with an artificial viscosity term, the entropy inequality and the energy balance are replaced by the equation for the internal energy. Several terms are added in order to improve the estimates. The Faedo-Galerkin approximation of the momentum equation is considered as a fixed point problem, which is solved by means of Schauder fixed point theorem on a short time interval. Then the solution is extended on the whole interval $(0,T)$.

- Passing to the limit in a sequence of solutions, where the velocity lives in an n-dimensional approximation of the target space $W_0^{1,2}$, we obtain a weak solution of the modified system.
- Next, we perform a vanishing viscosity limit, letting all the artificial terms but the artificial pressure go to zero.
- We finish the proof, letting the artificial pressure go to zero.

3.2 Faedo-Galerkin approximation

At the first approximation level, the system consists of the continuity equation augmented by the artificial viscosity, the internal energy equation, and the Galerkin approximation of the momentum equation. Our strategy is to construct solving operators of the first two equations, then plug the solution into the momentum equation, and find a finite dimensional approximation of the velocity field as the fixed point of a suitable non-linear operator. All this on some small time interval $J = (0, T^*)$.

3.2.1 Continuity Equation

We endow the continuity equation (3.2) by a parabolic perturbation, a homogeneous Neumann boundary condition, and smoothed initial $\varrho_{0,\varepsilon}$ such that

$$\varrho_{0,\varepsilon} \in C^{2+\nu}(\overline{\Omega}),\ \inf_{\Omega} \varrho_{0,\varepsilon} > 0,\ \nabla \varrho_{0,\varepsilon} \cdot \boldsymbol{n}|_{\partial\Omega} = 0, \tag{3.24}$$

and $\varrho_{0,\varepsilon}$ tends to ϱ_0 in $L^\beta(\Omega)$ as $\varepsilon \to 0$. After these modifications, we obtain a Neumann problem

$$\left.\begin{aligned} \partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) &= \varepsilon \Delta \varrho && \text{on } J \times \Omega, \\ \nabla \varrho_\varepsilon \cdot \boldsymbol{n} &= 0 && \text{on } J \times \partial\Omega, \\ \varrho[0] &= \varrho_{0,\varepsilon} && \text{on } \Omega. \end{aligned}\right\} \tag{3.25}$$

For this problem, we report the following result.

Lemma 3.2.1. *Let $\Omega \subset \mathbf{R}^3$ be a bounded domain of class $C^{2+\nu}$, $\nu > 0$. Let $\boldsymbol{u}$ be a given vector field from the space $C(\overline{J}; C_0^2(\overline{\Omega}; \mathbf{R}^3))$. Then for any $\varrho_{0,\varepsilon}$ satisfying* (3.24)*, there exists a unique solution of the problem* (3.25) *from the space*

$$X_\varrho = C(\overline{J}; C^{2+\nu}(\overline{\Omega})) \cap C^1(\overline{J}; C^\nu(\overline{\Omega})).$$

This solution also satisfy the maximal principle

$$\underline{\varrho}_0 e^{-U(\tau)} \le \varrho(\tau, \boldsymbol{x}) \le \overline{\varrho}_0 e^{+U(\tau)} \text{ for a.a. } \tau \in J,\ \boldsymbol{x} \in \Omega, \tag{3.26}$$

$$U(\tau) = \int_0^\tau \left\| \operatorname{div} \boldsymbol{u} \right\|_\infty \,\mathrm{d}t,\ \underline{\varrho}_0 = \operatorname{ess\,inf}_{\Omega} \varrho_{0,\varepsilon},\ \overline{\varrho}_0 = \operatorname{ess\,sup}_{\Omega} \varrho_{0,\varepsilon}.$$

Finally, the solution operator $\boldsymbol{u} \to \varrho[\boldsymbol{u}]$ maps bounded sets in $C(\overline{J}; C_0^2(\overline{\Omega}; \mathbf{R}^3))$ on the bounded sets in X_ϱ and it is continuous mapping at least into the space $C^1(\overline{J} \times \overline{\Omega})$.

The proofs of these statements one can find in [14, Chapter 7] and its references.

3.2.2 Internal energy equation

Instead of the entropy inequality (3.4) and the total energy balance (3.5), we use the equation for the internal energy. We add some regularizing terms, in particular modify the internal energy and the heat conductivity as follows

$$e_\delta(\varrho,\vartheta) = e(\varrho,\vartheta) + \delta\vartheta, \tag{3.27}$$

$$\kappa_\varepsilon(t,\boldsymbol{x};\vartheta) = \kappa(t,\boldsymbol{x};\vartheta) + \delta\vartheta^\beta + \sqrt{\varepsilon}\vartheta^{-1}. \tag{3.28}$$

We assume also smooth approximation $\vartheta_{0,\varepsilon}$ of the initial condition, which satisfies

$$\vartheta_{0,\varepsilon} \in L^\infty(\Omega)\cap W^{1,2}(\Omega),\ \operatorname{ess}\inf_\Omega \vartheta_{0,\varepsilon}(\boldsymbol{x}) > 0,\ \text{and } \vartheta_{0,\varepsilon}\to\vartheta_0 \text{ in } L^1(\Omega). \tag{3.29}$$

Finally, the equation for the temperature is represented by the following semi-linear parabolic problem

$$\left.\begin{aligned} &\partial_t(\varrho e_\delta(\varrho,\vartheta))) + \operatorname{div}(\varrho e_\delta(\varrho,\vartheta)\boldsymbol{u}) - \operatorname{div}(\kappa_\varepsilon(t,\boldsymbol{x};\vartheta)\nabla\vartheta) \\ &\qquad = \varepsilon\delta|\nabla\varrho|^2(\beta\varrho^{\beta-2}+2) - p_\delta(\varrho,\vartheta)\operatorname{div}\boldsymbol{u} + \mathbb{S}:\nabla\boldsymbol{u} + \varepsilon(\vartheta^{-\beta}-\vartheta^\beta) && \text{on } J\times\Omega,\\ &\nabla\vartheta\cdot\boldsymbol{n} = 0 && \text{on } J\times\partial\Omega,\\ &\vartheta[0] = \vartheta_{0,\varepsilon} && \text{on } \Omega. \end{aligned}\right\} \tag{3.30}$$

Concerning the problem (3.30), we claim the following result

Lemma 3.2.2. *Let $\Omega\subset\mathbf{R}^3$ be a bounded domain of class $C^{2+\nu}$. Let $\boldsymbol{u}\in C(\overline{J};C_0^2(\overline{\Omega};\mathbf{R}^3))$ be a given velocity field and $\varrho = \varrho[\boldsymbol{u}]\in X_\varrho$ be the density according to Lemma 3.2.1, in particular $\varrho(t,\boldsymbol{x})>\underline{\varrho}$. Then for $\vartheta_{0,\varepsilon}$ satisfying* (3.29)*, there exists a unique ϑ in a class*

$$X_\vartheta\ \begin{cases} \vartheta\in C(\overline{J};W^{1,2}(\Omega)),\ \partial_t\vartheta\in L^2(J\times\Omega)\\ \operatorname{div}\big(\kappa_\varepsilon(t,\boldsymbol{x};\vartheta)\nabla\vartheta\big)\in L^2(J\times\Omega)\end{cases} \tag{3.31}$$

such that (3.30) *is satisfied a.e. on $J\times\Omega$. For this solution, there exist constants $\underline{\vartheta}$, $\overline{\vartheta}$ depending solely on*

$$\big\|\boldsymbol{u}\big\|_{C(J;X_n)} \quad \textit{and} \quad \big\|\varrho[\boldsymbol{u}]\big\|_{C^1(\overline{J}\times\Omega)},$$

such that

$$\underline{\vartheta}\le\vartheta(\tau,\boldsymbol{x})\le\overline{\vartheta}. \tag{3.32}$$

Moreover, there exists a continuous solution operator $\vartheta[\boldsymbol{u}]$ from $C(\overline{J};C_0^2(\overline{\Omega}))$ to X_ϑ.

Proof. Step 1 Maximum principle. Let $\underline{\vartheta}$ be a subsolution and $\overline{\vartheta}$ a supersolution of (3.30). Then in accordance with assumptions about $\boldsymbol{u}$ and $\varrho[\boldsymbol{u}]$, using hypotheses (3.14), (3.21), and due to presence of the term ϑ^β for $\beta>4$, we infer

$$\big[\partial_t w + \nabla w\cdot\boldsymbol{u}\big]\operatorname{sgn}^+ w - \operatorname{div}\big(\kappa(t,\boldsymbol{x};\underline{\vartheta})\nabla\underline{\vartheta} - \kappa(t,\boldsymbol{x};\overline{\vartheta})\nabla\overline{\vartheta}\big)\operatorname{sgn}^+ w \le C(\underline{\vartheta}-\overline{\vartheta})\operatorname{sgn}^+ w \tag{3.33}$$

for the difference

$$w = \varrho e_\delta(\varrho,\underline{\vartheta}) - \varrho e_\delta(\varrho,\overline{\vartheta}).$$

In (3.33) we have denoted sgn^+ the positive part of the standard signum function. For any $v\in W^{1,2}(J\times\Omega)$, in particular for $v=w$, we have

$$\partial_t|v|^+ = \operatorname{sgn}^+(v)\partial_t v,\quad \nabla|v|^+ = \operatorname{sgn}^+(v)\nabla v$$

(see [46]), which can be used in the first term of (3.33). Conserning the second term, as e_δ is an increasing function of ϑ, we can replace $\mathrm{sgn}^+ w$ by $\mathrm{sgn}^+(\underline{\vartheta}-\overline{\vartheta})$. Further, we approximate function sgn^+ by the sequence

$$\mathrm{sg}_n(t) = \begin{cases} nt & \text{on } (0,\frac{1}{n}) \\ \mathrm{sgn}^+(t) & \text{elsewhere.} \end{cases}$$

Then we integrate by parts to get

$$\begin{aligned}\int_\Omega \mathrm{div}\big(\kappa(t,\boldsymbol{x};\underline{\vartheta})\nabla\underline{\vartheta} - \kappa(t,\boldsymbol{x};\overline{\vartheta})\nabla\overline{\vartheta}\big)\mathrm{sgn}^+ w \,\mathrm{d}\boldsymbol{x} = \\ -\lim_{n\to\infty}\int_{\{0<\underline{\vartheta}-\overline{\vartheta}<\frac{1}{n}\}} \big[\big(\kappa(t,x;\underline{\vartheta})-\kappa(t,x;\overline{\vartheta})\big)\nabla\underline{\vartheta} + \kappa(t,x;\overline{\vartheta})\nabla(\underline{\vartheta}-\overline{\vartheta})\big] n\nabla(\underline{\vartheta}-\overline{\vartheta}) \,\mathrm{d}\boldsymbol{x} \le \\ \lim_{n\to\infty}\int_{\{0<\underline{\vartheta}-\overline{\vartheta}<\frac{1}{n}\}} \big\|\partial_\vartheta\kappa\big\|_\infty |\nabla\underline{\vartheta}||\nabla(\underline{\vartheta}-\overline{\vartheta})| \,\mathrm{d}\boldsymbol{x}. \quad (3.34)\end{aligned}$$

If $\underline{\vartheta}$ is homogeneous function in space, the right hand side is equal to zero. Using the oposite spliting, we get the same inequality for the spatially homogeneous $\overline{\vartheta}$. Consequently, integrating (3.33) over the time interval $(0,\tau)$, we arrive at

$$\int_\Omega |w|^+(\tau) \,\mathrm{d}\boldsymbol{x} \le C\int_0^\tau\int_\Omega (1+|\mathrm{div}\boldsymbol{u}|)|w|^+ \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

Then an application of the Gronwall lemma 2.1.5 yields $\underline{\vartheta}\le\overline{\vartheta}$ a.a. on $J\times\Omega$. It is easy to check, that because of the term $\varepsilon(\vartheta^{-\beta}-\vartheta^\beta)$ one can find constants $\underline{\vartheta}$, $\overline{\vartheta}$, which are subsolution and supersolution of (3.30), respectively. Then (3.32) follows for any solution ϑ of (3.30).

Next, we take two (possibly) different solutions as $\underline{\vartheta}$ and $\overline{\vartheta}$, both from the regularity class X_ϑ. We already know, they are uniformly bounded on $J\times\Omega$. Then we can improve spatial regularity of $\underline{\vartheta}$ using the L^p theory for the Laplace equation and hypothesis (3.22). We estimate

$$\begin{aligned}\underline{\kappa}^2\big\|\underline{\vartheta}\big\|^2_{2,2,\Omega} \le \int_\Omega \big|(\kappa(t,\boldsymbol{x};\underline{\vartheta})\nabla\underline{\vartheta}\big|^2 \,\mathrm{d}\boldsymbol{x} \le \\ C\int_\Omega \big|\mathrm{div}(\kappa(t,\boldsymbol{x};\underline{\vartheta})\nabla\underline{\vartheta})\big|^2 + |\partial_x\kappa(t,x;\underline{\vartheta})|^2|\nabla\underline{\vartheta}|^2 + |\partial_\vartheta\kappa(t,\boldsymbol{x};\underline{\vartheta})|^2|\nabla\underline{\vartheta}|^2 \,\mathrm{d}\boldsymbol{x}. \quad (3.35)\end{aligned}$$

Using conveniently the Young inequality on the right-hand side, we get

$$\underline{\vartheta} \text{ bounded in } L^2(I;W^{2,2}(\Omega)),$$

since $\partial_{\boldsymbol{x}}\kappa$ and $\partial_\vartheta\kappa$ are bounded on bounded sets. Now we can use Hölder inequality to continue with calculation (3.34):

$$\begin{aligned}\int_{\{0<\underline{\vartheta}-\overline{\vartheta}<\frac{1}{n}\}} \partial_\vartheta|\kappa(t,\boldsymbol{x};\tilde{\vartheta})||\nabla\underline{\vartheta}||\nabla(\underline{\vartheta}-\nabla\overline{\vartheta})| \,\mathrm{d}\boldsymbol{x} \le \\ |\{0<\underline{\vartheta}-\overline{\vartheta}<\tfrac{1}{n}\}|^{\frac{1}{3}}\big\|\underline{\vartheta}\big\|_{2,2,\Omega}\big\|\underline{\vartheta}-\overline{\vartheta}\big\|_{1,2,\Omega} \to 0. \quad (3.36)\end{aligned}$$

Finally, we use the Gronwall lemma similarly as in the previous paragraph to deduce $\underline{\vartheta}=\overline{\vartheta}$ a.e. on $J\times\Omega$.

Step 2. A priori estimates. Multiplying equation (3.30) by $\partial_t \mathcal{K}(t,\boldsymbol{x};\vartheta)$, where

$$\mathcal{K}(t,\boldsymbol{x};\vartheta) = \int_1^\vartheta \kappa(t,\boldsymbol{x};\theta)\,\mathrm{d}\theta,$$
$$\nabla\mathcal{K}(t,\boldsymbol{x};\vartheta) = \kappa(t,\boldsymbol{x};\vartheta)\nabla\vartheta + \big[\partial_{\boldsymbol{x}} K\big], \quad \big[\partial_{\boldsymbol{x}} K\big](t,\boldsymbol{x};\vartheta) = \int_1^\vartheta \partial_{\boldsymbol{x}}\kappa(t,\boldsymbol{x};\theta)\,\mathrm{d}\theta,$$
$$\partial_t\mathcal{K}(t,\boldsymbol{x};\vartheta) = \kappa(t,\boldsymbol{x};\vartheta)\partial_t\vartheta + \big[\partial_t K\big], \quad \big[\partial_t K\big](t,\boldsymbol{x};\vartheta) = \int_1^\vartheta \partial_t\kappa(t,\boldsymbol{x};\theta)\,\mathrm{d}\theta.$$

We get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}|\nabla\mathcal{K}|^2\,\mathrm{d}\boldsymbol{x} + \int_\Omega \frac{\varrho}{\kappa}\frac{\partial e_\delta}{\partial\vartheta}|\partial_t\mathcal{K}|^2\,\mathrm{d}\boldsymbol{x} = \int_\Omega \big[\partial_{\boldsymbol{x}} K\big]\cdot\nabla\partial_t\mathcal{K}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \frac{\varrho}{\kappa}\frac{\partial e_\delta}{\partial\vartheta}\big[\partial_t K\big]\partial_t\mathcal{K} - \varrho\boldsymbol{u}\cdot\frac{\partial e_\delta}{\partial\vartheta}\nabla\vartheta\partial_t\mathcal{K} + G(t,\boldsymbol{x})\partial_t\mathcal{K}\,\mathrm{d}\boldsymbol{x}, \quad (3.37)$$

where, in view of (3.32),

$$G(t,\boldsymbol{x}) = -\frac{\partial(\varrho e_\delta)}{\partial\varrho}(\partial_t\varrho + \nabla\varrho\cdot\boldsymbol{u}) - \varrho e_\delta \mathrm{div}\boldsymbol{u} + \mathbb{S}:\nabla\boldsymbol{u} - p_\delta\mathrm{div}\boldsymbol{u} + \varepsilon\delta|\nabla\varrho|^2(\beta\varrho^{\beta-2}+2) + \varepsilon(\vartheta^{-\beta}-\vartheta^\beta) \quad (3.38)$$

is bounded in $L^\infty(J\times\Omega)$ while $\frac{\varrho}{\kappa}\frac{\partial e_\delta}{\partial\vartheta}$ is uniformaly greater than zero and bounded.

Further, we can integrate by parts in the first and the most delicate term on the right-hand side to estimate

$$\int_\Omega \big[\partial_{\boldsymbol{x}} K\big](t,\boldsymbol{x};\vartheta)\cdot\nabla\partial_t\mathcal{K}(t,\boldsymbol{x};\vartheta)\,\mathrm{d}\boldsymbol{x} \le \big(\big\|\partial_{\boldsymbol{x}}\kappa\big\|_\infty\big\|\nabla\vartheta\big\|_2 + \big\|\big[\partial_x\partial_{\boldsymbol{x}} K\big]\big\|_2\big)\big\|\partial_t\mathcal{K}\big\|_2,$$

where

$$\big[\partial_x\partial_{\boldsymbol{x}} K\big](t,\boldsymbol{x};\vartheta) = \int_1^\vartheta \partial_{\boldsymbol{x}}\partial_{\boldsymbol{x}}\kappa(t,\boldsymbol{x};\theta)\,\mathrm{d}\theta.$$

The norm $\big\|\nabla\vartheta\big\|_2$ is dominated by $\big\|\nabla\mathcal{K}\big\|_2$ as follows

$$\underline{\kappa}^2\big\|\nabla\vartheta\big\|_2^2 \le \int_\Omega|\kappa\nabla\vartheta|^2\,\mathrm{d}\boldsymbol{x} \le C(\big\|\nabla\mathcal{K}\big\|_2^2 + \big\|\big[\partial_x K\big]\big\|_2^2).$$

Similarly the norm $\big\|\partial_t\vartheta\big\|_2$ is dominated by $\big\|\partial_t\mathcal{K}\big\|_2$. In accordance with hypotehesis (3.22), we have

$$\big[\partial_{\boldsymbol{x}} K\big],\ [\partial_{\boldsymbol{x}}\kappa],\ \big[\partial_{\boldsymbol{x}}\partial_{\boldsymbol{x}} K\big] \in L^\infty(J\times\Omega) \text{ and } \big[\partial_t K\big] \in L^2(J;L^\infty(\Omega)).$$

Then a direct application of the Gronwall lemma yields

$$\nabla\vartheta,\ \nabla\mathcal{K} \in C(\overline{J};L^2(\Omega)); \quad \partial_t\vartheta,\ \partial_t\mathcal{K} \in L^2(I\times\Omega).$$

Finally, using the equation again, we get also

$$\mathrm{div}(\kappa(t,\boldsymbol{x};\vartheta)\nabla\vartheta) \in L^2(I\times\Omega).$$

Step 3 Existence. To begin with, we rewrite the equation into the variable $\mathcal{K} = \mathcal{K}(t, \boldsymbol{x}; \vartheta)$

$$\partial_t \mathcal{K} - A(t, \boldsymbol{x}; \vartheta)\Delta \mathcal{K} = A(t, \boldsymbol{x}; \vartheta)\mathrm{div}\big[\partial_x K\big] + \big[\partial_t K\big] \\ - \varrho \boldsymbol{u} \cdot (\nabla \mathcal{K} - \big[\partial_x K\big]) + A(t, \boldsymbol{x}; \vartheta) G(t, x; \vartheta), \quad (3.39)$$

where $G(t, x; \vartheta)$ is the same as in (3.38) and

$$A(t, \boldsymbol{x}; \vartheta) = \kappa(t, \boldsymbol{x}; \vartheta)\Big[\varrho(t, \boldsymbol{x}) \frac{\partial e_G}{\partial \vartheta}(\varrho(t, \boldsymbol{x}), \vartheta) + a\vartheta^3 + \delta\varrho(t, \boldsymbol{x})\Big]^{-1}.$$

Next, we continuously extend $\boldsymbol{u}$, $\varrho[\boldsymbol{u}]$ in time, in such a way that they are defined on the whole $\mathbf{R}$, but still have a compact support. Then also A, G are defined for all $t \in \mathbf{R}$. Consequently, we can use mollifiers (2.1) with parameter ω to smooth out A, and G in the time variable. Moreover, we replace ϑ by

$$\theta_\omega = \frac{\sqrt{\vartheta^2 + \omega^2}}{1 + \omega\sqrt{\vartheta^2 + \omega^2}}$$

in

$$A(t, \boldsymbol{x}; \vartheta),\ G(t, \boldsymbol{x}; \vartheta),\ \big[\partial_x K\big](t, \boldsymbol{x}; \vartheta)\ \big[\partial_t K\big](t, \boldsymbol{x}; \vartheta).$$

For the smoothed equation one can apply Theorem 8.1 in Chapter V of [27] to obtain the unique classical solution $\mathcal{K}_\omega$ and thus also the temperature ϑ_ω. Then performing similar estimates as in Step 1 and Step 2, one can pass to the limit as $\omega \to 0$ and find the solution of the equation (3.30) □

3.2.3 Momentum equation and existence of fixed point

The viscosity term in (3.25) involves a new term, $\varepsilon \nabla \boldsymbol{u} \nabla \varrho$, in the momentum equation in order to preserve the total energy balance. Next,we enhance the pressure by an artificial pressure term setting

$$p_\delta(\varrho, \vartheta) = p(\varrho, \vartheta) + \delta\varrho^\beta + \delta\varrho^2$$

Finally, we project the resulting equation onto the n-dimensional space

$$X_n \subset C^\infty(\Omega; \mathbf{R}^3) \cap C_0(\Omega; \mathbf{R}^3) \subset L^2(\Omega; \mathbf{R}^3),$$

endowed by the structure of the Hilbert space $L^2(\Omega; \mathbf{R}^3)$. More specifically, we look for $\boldsymbol{u} \in X_n$, which satisfies

$$\int_\Omega \varrho \boldsymbol{u}(t) \cdot \boldsymbol{\varphi}\ \mathrm{d}\boldsymbol{x} - \int_\Omega \varrho_0 \boldsymbol{u}_0 \cdot \boldsymbol{\varphi}\ \mathrm{d}\boldsymbol{x} = \\ \int_0^t \int_\Omega [\varrho \boldsymbol{u} \otimes \boldsymbol{u} - \mathbb{S}] : \nabla \boldsymbol{\varphi} + p_\delta(\varrho, \vartheta)\mathrm{div}\boldsymbol{\varphi} + [\varrho \boldsymbol{f} - \varepsilon \nabla \boldsymbol{u} \nabla \varrho] \cdot \boldsymbol{\varphi}\ \mathrm{d}\boldsymbol{x}\,\mathrm{d}s \quad (3.40)$$

for any $\boldsymbol{\varphi} \in X_n$.

We shall solve this equation on a short time interval J by means of the Schauder fixed point theorem. In order to reformulate it as a fixed point problem on the Banach space $C(\overline{J}; X_n)$, we have to set up a convenient notation. For a fixed $\varrho \in L^1(\Omega)$, we introduce an operator

$$M[\varrho] : X_n \to X_n^* \equiv X_n; \quad \langle M[\varrho]\boldsymbol{v}, \boldsymbol{w}\rangle := \int_\Omega \varrho \boldsymbol{v} \cdot \boldsymbol{w}\ \mathrm{d}\boldsymbol{x}.$$

This is nothing else, then projection of an L^1-function $\varrho\boldsymbol{v}$ on the space X_n^*. Therefore, $M[\varrho]$ is invertible provided $\varrho > 0$. Observing that

$$\inf_{\boldsymbol{x}\in\Omega} \varrho(\boldsymbol{x})\langle \boldsymbol{v}, \boldsymbol{w}\rangle \le \int_\Omega \varrho\boldsymbol{v}\cdot\boldsymbol{w}\ \mathrm{d}\boldsymbol{x} \le \sup_{\boldsymbol{x}\in\Omega} \varrho(\boldsymbol{x})\langle \boldsymbol{v}, \boldsymbol{w}\rangle,$$

we deduce

$$\big\|M[\varrho]\big\|_{\mathcal{L}(X_n,X_n^*)} \le \big\|\varrho\big\|_{C(\Omega)}, \quad \big\|M[\varrho]^{-1}\big\|_{\mathcal{L}(X_n^*,X_n)} \le \big\|\varrho^{-1}\big\|_{C(\Omega)} = (\inf_{\boldsymbol{x}\in\Omega} \varrho(\boldsymbol{x}))^{-1}. \tag{3.41}$$

Clearly, $M[\,\cdot\,]$ is Lipschitz continuous mapping form $L^1(\Omega)$ to $\mathcal{L}(X_n, X_n)$, but the same is true for $M[\,\cdot\,]^{-1}$, at least on the sets $\{\underline{\varrho} < \varrho(\,\cdot\,) < \overline{\varrho}\}$. Indeed, a simple calculation yields

$$\big\|M[\varrho_1]^{-1} - M[\varrho_2]^{-1}\big\| = \big\|M[\varrho_2]^{-1}(M[\varrho_2] - M[\varrho_1])M[\varrho_1]^{-1}\big\| \le \underline{\varrho}^{-2}\overline{\varrho}\big\|\varrho_1 - \varrho_2\big\|_{L^1(\Omega)}. \tag{3.42}$$

Finally, we denote

$$\boldsymbol{u}_{n,0} = M^{-1}[\varrho_{\varepsilon,0}](\varrho_{0,\varepsilon}\boldsymbol{u}_0)$$

Now, we are ready to define a mapping

$$\mathcal{T} : \mathcal{B} \to C(\overline{J}; X_n); \quad \mathcal{B} := \{\boldsymbol{v} \in C(\overline{J}; X_n) \,|\, \big\|\boldsymbol{v} - \boldsymbol{u}_{n,0}\big\|_{C(J;X_n)} \le 1\}$$

given by formula

$$\mathcal{T}[\boldsymbol{u}](t) := M[\varrho[\boldsymbol{u}](t)]^{-1}\Big((\varrho\boldsymbol{u})_{n,0} + \int_0^t F(\varrho[\boldsymbol{u}], \boldsymbol{u}, \vartheta[\boldsymbol{u}])(\tau)\,\mathrm{d}\tau\Big), \tag{3.43}$$

where for $\boldsymbol{\varphi} \in X_n$, we denote

$$\langle F(\varrho, \boldsymbol{u}, \vartheta), \boldsymbol{\varphi}\rangle = \int_\Omega \big(\varrho\boldsymbol{u}\otimes\boldsymbol{u} - \mathbb{S}(\boldsymbol{u}, \vartheta)\big) : \nabla\boldsymbol{\varphi} + p_\delta(\varrho, \vartheta)\mathrm{div}\,\boldsymbol{\varphi} + \big(\varrho\boldsymbol{f} - \varepsilon(\nabla\varrho\cdot\nabla)\boldsymbol{u}\big)\cdot\boldsymbol{\varphi}\ \mathrm{d}\boldsymbol{x},$$

$$\langle (\varrho\boldsymbol{u})_{n,0}, \boldsymbol{\varphi}\rangle = \int_\Omega \varrho_{0,\varepsilon}\boldsymbol{u}_0\cdot\boldsymbol{\varphi}\ \mathrm{d}\boldsymbol{x}.$$

The solution operators $\varrho[\boldsymbol{u}]$ and $\vartheta[\boldsymbol{u}]$ of the equations (3.25) and (3.30) are given by Lemmas 3.2.1, 3.2.2 respectively. With help of the mapping $\mathcal{T}$, the equation (3.40) is equivalent to

$$\mathcal{T}[\boldsymbol{u}] = \boldsymbol{u}.$$

Using Lemma 3.2.1, we can find $\underline{\varrho}$ and $\overline{\varrho}$ such that

$$\underline{\varrho} \le \varrho_{0,\varepsilon}(\boldsymbol{x}) \le \overline{\varrho}, \quad \text{on } \overline{\Omega},$$
$$\underline{\varrho} \le \varrho[\boldsymbol{u}](t,\boldsymbol{x}) \le \overline{\varrho}, \quad \text{on } \overline{J}\times\overline{\Omega} \text{ for any } \boldsymbol{u}\in\mathcal{B}.$$

Further, one can use Lemmas 3.2.1 and 3.2.2 to get

$$\big\|F(\varrho[\boldsymbol{u}], \boldsymbol{u}, \vartheta[\boldsymbol{u}]\big\|_{L^p(J)} \le C\big\|1 + \varrho^\beta + |\nabla\varrho| + \vartheta^4\big\|_{L^p(J\times\Omega)} \le C\big\|\boldsymbol{u}\big\|_{C(J;X_n)},$$

for any $p > 1$. Let us note that once this bound holds for some $J = J_0$, it remains valid even for shorter time interval, with the same constant. Then, for any $\boldsymbol{u}\in\mathcal{B}$, we have

$$\big\|\mathcal{T}[\boldsymbol{u}] - \boldsymbol{u}_{n,0}\big\|_{C(J;X_n)} \le \underline{\varrho}^{-1}\int_J |F(\varrho[\boldsymbol{u}], \boldsymbol{u}, \vartheta[\boldsymbol{u}])|\ \mathrm{d}t$$
$$+ \underline{\varrho}^{-2}\overline{\varrho}\sup_{t\in J}\big\|\varrho[\boldsymbol{u}](t) - \varrho_{\varepsilon,0}\big\|_{L^1}\big\|\varrho_{0,\varepsilon}\boldsymbol{u}_0\big\|_{L^1} \le \Psi(|J|),$$

where Ψ is monotone function, which approach zero as $T^* = |J|$ goes to zero. Hence, there exists a time interval J such that $\mathcal{T}$ is self-mapping on the set $\mathcal{B}$. Similar calculation,

$$\big\|\mathcal{T}[\boldsymbol{u}](t_1) - \mathcal{T}[\boldsymbol{u}](t_2)\big\|_{X_n} \le \underline{\varrho}^{-1} \int_{t_1}^{t_2} |F(\varrho[\boldsymbol{u}], \boldsymbol{u}, \vartheta[\boldsymbol{u}])| \,\mathrm{d}t,$$

yields equi-continuity of the set $\mathcal{T}[\mathcal{B}]$. Consequently, by virtue of the Arzelà-Ascoli theorem 2.2.2, the mapping $\mathcal{T}$ is compact on $\mathcal{B}$. In order to apply the Schauder theorem it remains to verify a continuity of $\mathcal{T}$. To this end we write

$$\begin{aligned}\big\|\mathcal{T}[\boldsymbol{u}_1]-\mathcal{T}[\boldsymbol{u}_2]\big\|_{C(J;X_n)} \le \underline{\varrho}^{-2}\overline{\varrho}\big\|\varrho[\boldsymbol{u}_1]-\varrho[\boldsymbol{u}_2]\big\|_{C(J;L^1(\Omega))}\big(\big\|\varrho_{0,\varepsilon}\boldsymbol{u}_0\big\|_{L^1(\Omega)}+\big\|F[\boldsymbol{u}_1]\big\|_{L^1(J;X_n^*)}\big)\\ + \underline{\varrho}^{-1}\big\|F[\boldsymbol{u}_1] - F[\boldsymbol{u}_2]\big\|_{L^1(J;X_n^*)}.\end{aligned}$$

Further for the last term, we have

$$\begin{aligned}&\big\|F[\boldsymbol{u}_1] - F[\boldsymbol{u}_2]\big\|_{L^1(J;X_n^*)} \le C\big[|\partial_\varrho p_\delta|, |\partial_\vartheta p_\delta|, |\partial_\vartheta(\mu,\zeta)|\big]\Big(\big\|\varrho[\boldsymbol{u}_1] - \varrho[\boldsymbol{u}_2]\big\|_{L^1(J\times\Omega)}\\ &+\big\|\boldsymbol{u}_1-\boldsymbol{u}_2\big\|_{C(J;X_n^*)}+\big\|\vartheta[\boldsymbol{u}_1]-\vartheta[\boldsymbol{u}_2]\big\|_{L^1(J\times\Omega)}\Big)+C\big\|(\mu,\zeta)[\boldsymbol{u}_1](\vartheta_2)-(\mu,\zeta)[\boldsymbol{u}_2](\vartheta_2)\big\|_{L^1(J\times\Omega)}.\end{aligned}$$

Thus according to Lemmas 3.2.1, 3.2.2 and due to (3.23) the mapping $\mathcal{T}$ is continuous. Then on can apply the Schauder theorem 2.1.6 to obtain the solution $\varrho = \varrho[\boldsymbol{u}]$, $\boldsymbol{u}$, $\vartheta = \vartheta[\boldsymbol{u}]$ of the equations (3.25), (3.40), (3.30) on some short time interval J.

3.3 Time independent estimates

In the previous step, we have constructed the Galerkin approximation $(\varrho, \boldsymbol{u}, \vartheta) = (\varrho_n, \boldsymbol{u}_n, \vartheta_n)$ on a short time interval and for a fixed n. Our next aim is to derive time independent estimates and prolongate the solution on the whole time interval. In fact these estimates will be also independent of the dimension n.

In order to enter the framework of variational solutions the equation (3.30) will be replaced by the entropy equation and the balance of the energy. Since our solution is regular and the temperature is strictly positive, a calculation similar to (1.18) can be performed to get the entropy equality. Nevertheless, there are some extra terms because of the modification in the continuity equation. We compute

$$\begin{aligned}&\frac{1}{\vartheta}\big(\partial_t(\varrho e_G) + \mathrm{div}(\varrho\boldsymbol{u}e_G) + p_G\mathrm{div}\boldsymbol{u}\big) = \varepsilon\Delta\varrho\frac{e_G}{\vartheta}+\\ &\quad\varrho\Big(\frac{1}{\vartheta}\frac{\partial e_G}{\partial\vartheta}(\partial_t\vartheta + \boldsymbol{u}\cdot\nabla\vartheta) - \frac{2}{3\varrho}\frac{\partial e_G}{\partial\vartheta}(\partial_t\varrho + \boldsymbol{u}\cdot\nabla\varrho)\Big) + \frac{p_G}{\vartheta\varrho}(\partial_t\varrho + \boldsymbol{u}\cdot\nabla\varrho + \varrho\mathrm{div}\boldsymbol{u}) =\\ &\qquad\qquad\partial_t(\varrho s_G) + \mathrm{div}(\varrho\boldsymbol{u}s_G) + \varepsilon\Delta\varrho\Big(\frac{e_G}{\vartheta} + \frac{p_G}{\vartheta\varrho} - s_G\Big). \quad (3.44)\end{aligned}$$

Applying a similar procedure for e_R, and $\delta\vartheta$ one gets

$$\frac{1}{\vartheta}\big(\partial_t(\varrho e_\delta) + \mathrm{div}(\varrho\boldsymbol{u}e_\delta) + p_\delta\mathrm{div}\boldsymbol{u} = \partial_t(\varrho s_\delta) + \mathrm{div}(\varrho\boldsymbol{u}s_\delta) + \varepsilon\Delta\varrho A_\varepsilon$$

where

$$A_\varepsilon[\boldsymbol{u}](t,\boldsymbol{x};\vartheta,\varrho) = \frac{e_G}{\vartheta} + \frac{p_G}{\vartheta\varrho} - s_G + \delta(1-\log\vartheta).$$

Using the Green theorem one has

$$\varepsilon \int_\Omega \Delta \varrho A_\varepsilon \varphi \,\mathrm{d}\boldsymbol{x} = -\varepsilon \int_\Omega \nabla \varrho \cdot \nabla \varrho \,\partial_\varrho A_\varepsilon + \nabla \varrho \cdot \nabla \vartheta \,\partial_\vartheta A_\varepsilon + \nabla \varrho \cdot \nabla \varphi \, A_\varepsilon \,\mathrm{d}\boldsymbol{x},$$

where, due to Maxwell's relations (1.27), (1.28),

$$\partial_\varrho A_\varepsilon = \frac{1}{\vartheta \varrho} \partial_\varrho p_G, \quad \partial_\vartheta A_\varepsilon = -\frac{1}{\vartheta^2}(e_G + \varrho \partial_\varrho e_G + \delta\vartheta).$$

Then a weak form of the entropy equality reads

$$\int_J \int_\Omega \varrho s_\delta \partial_t \varphi + \varrho \boldsymbol{u} s_\delta \cdot \nabla \varphi - \frac{\kappa_\varepsilon[\boldsymbol{u}] \nabla \vartheta \cdot \nabla \varphi}{\vartheta} + \sigma_\varepsilon \varphi - \varepsilon \vartheta^{\beta-1} + \varepsilon A_\varepsilon \nabla \varrho \cdot \nabla \varphi \\ - \varepsilon \frac{1}{\vartheta^2}(e_G + \varrho \partial_\varrho e_G + \delta\vartheta) \nabla \varrho \cdot \nabla \vartheta \, \varphi \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}t = 0 \quad (3.45)$$

for any $\varphi \in \mathcal{D}(J \times \overline{\Omega})$. The approximate entropy is

$$s_\delta = s_G + s_R + \delta \log \vartheta$$

and the modified entropy production reads

$$\sigma_\varepsilon = \frac{\kappa_\varepsilon |\nabla \vartheta|^2}{\vartheta^2} + \frac{\mathbb{S} : \nabla \boldsymbol{u}}{\vartheta} + \varepsilon \vartheta^{-\beta-1} + \varepsilon \frac{\partial_\varrho p_G}{\vartheta \varrho} |\nabla \varrho|^2 + \delta\varepsilon(\beta \varrho^{\beta-2} + 2) \frac{|\nabla \varrho|^2}{\vartheta}.$$

In accordance with hypothesis (3.15), we have $\partial_\varrho p_G \geq 0$, thus σ_ε has a sign. Further, we can test (3.40) by $\boldsymbol{u}$ and obtain the balance of kinetic energy. This added to the internal energy equation (3.30) integrated over Ω gives rise to the balance of total energy:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \Big(\frac{1}{2} \varrho |\boldsymbol{u}|^2 + \varrho e_\delta(\varrho, \vartheta) + \delta\big(\frac{\varrho^\beta}{\beta - 1} + \varrho^2 \big) \Big)[t] \,\mathrm{d}\boldsymbol{x} = \int_\Omega \varrho \boldsymbol{f} \cdot \boldsymbol{u} + \varepsilon(\vartheta^{-\beta} - \vartheta^\beta) \,\mathrm{d}\boldsymbol{x} \quad (3.46)$$

for a.e. $t \in J$.

Now, we are ready to accomplish the estimates independent of time. We take a sequence of spatially homogeneous functons, which approximates $\varphi = 1_{(0,t)}$, as the test functions in (3.45). Then we add up the result and (3.46) together. Integrating by parts in time, we arrive at

$$\int_\Omega E_n[t] \,\mathrm{d}\boldsymbol{x} + \int_0^t \int_\Omega \sigma_\varepsilon \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}\tau \leq \int_\Omega E_n[0] - (\varrho s_\varepsilon)[0] + (\varrho s_\delta)[t] \,\mathrm{d}\boldsymbol{x} + \sum_{j=1}^3 I_j \quad (3.47)$$

for a.e. $t \in J$, where

$$E_n[t] = E_\delta(\varrho, \boldsymbol{u}, \vartheta) = (\frac{1}{2} \varrho |\boldsymbol{u}|^2 + \varrho e_\delta + \frac{\delta}{\beta - 1} \varrho^\beta + \delta \varrho^2)[t],$$

$$\sum_{j=1}^3 I_j = \int_0^t \int_\Omega \varrho \boldsymbol{f} \cdot \boldsymbol{u} + \varepsilon(\vartheta^{-\beta} - \vartheta^\beta + \vartheta^{\beta-1}) + \frac{\varepsilon}{\vartheta^2}(e_G + \varrho \partial_\varrho e_G + \delta\vartheta)) \nabla \varrho \cdot \nabla \vartheta \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}\tau$$

In view of hypotheses (3.21), (3.22) and the Korn inequality, the entropy production rate σ_ε dominates following quantities

$$C\sigma_\varepsilon \geq \tilde{\sigma} = |\nabla \log \vartheta|^2 + |\nabla(\vartheta^{\frac{3}{2}})|^2 + |\nabla \boldsymbol{u}|^2 + \delta \vartheta^{\beta-2} |\nabla \vartheta|^2 \\ + \sqrt{\varepsilon} \vartheta^{-3} |\nabla \vartheta|^2 + \varepsilon \vartheta^{-(\beta+1)} + \delta\varepsilon(\beta \varrho^{\beta-2} + 2) \frac{|\nabla \varrho|^2}{\vartheta} \quad (3.48)$$

where the constant C is independent of T, n, ε, δ.

In accordance with assumptions about $\varrho_{\varepsilon,0}$ and $\vartheta_{\varepsilon,0}$ in (3.25), (3.30), respectively, the zero time terms on the right-hand side of (3.47) are bounded uniformly with respect to T, n, ε, and δ. On the other hand, for the entropy term one can use (3.19) to infer

$$\int_\Omega \varrho s_\delta \le S(1)\int_{\varrho\vartheta^{-\frac{3}{2}}\ge 1}\varrho\,\mathrm{d}\boldsymbol{x} + C\int_{\varrho\vartheta^{-\frac{3}{2}}<1}\frac{3}{2}\varrho\log\vartheta - \varrho\log\varrho\,\mathrm{d}\boldsymbol{x}$$
$$+\delta\int_\Omega \varrho\log\vartheta + \frac{4}{3}a\vartheta^3\,\mathrm{d}\boldsymbol{x} \le C\int_\Omega \vartheta^4 + \varrho^{\frac{5}{3}}\,\mathrm{d}\boldsymbol{x} - \delta\int_\Omega \varrho|\log\vartheta|\,\mathrm{d}\boldsymbol{x},$$

where the last term gives an estimate on the left-hand side and the rest is dominated by $CE_n[t]$, in particular by the internal energy. Again the constant C is independent of T, n, ε, and δ.

In order to apply the Gronwall lemma, it remains to bound the terms I_j by an integral

$$\int_0^t \omega\big\|\tilde\sigma\big\|_{1,\Omega}[\tau] + C(\omega)E_n[\tau]\,\mathrm{d}\tau,$$

where ω is a suitable small parameter and $C(\omega)$ is a possibly large constant independent of T, n, ε, and δ. For the first term, we have

$$|I_1| \le \big\|f\big\|_{L^\infty(Q)}\int_0^t\int_\Omega \frac{1}{2}\varrho|\boldsymbol{u}|^2 + \frac{1}{2}\varrho\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\tau.$$

For the second one, we use the interpolation (3.54) and Lemma 2.1.2 to get

$$|I_2| \le C(\omega)\int_0^t \big\|\vartheta\big\|_4^4\,\mathrm{d}\tau + \omega\int_0^t \varepsilon\big\|\vartheta^{-(\beta+1)}\big\|_1 + \delta\big\|\nabla(\vartheta^{\frac{\beta}{2}})\big\|_2^2\,\mathrm{d}\tau.$$

The third term I_3 is more tricky. First, we observe that because of (1.26), (3.17), and (3.16) one have

$$|e_G + \varrho\partial_\varrho e_G| = |\tfrac{5}{3}e_G - \tfrac{2}{3}\vartheta c_v| \le C\vartheta(|P(Y)Y^{-1}| + |P'(Y)|) \le C(\varrho^{\frac{2}{3}} + \vartheta).$$

Further, taking $\varphi = \varrho$ as a test function in (3.25), we arrive at the "energy" equality for the density

$$\big\|\varrho\big\|_2^2[t] + 2\varepsilon\int_0^t \big\|\nabla\varrho\big\|_2^2\,\mathrm{d}\tau = \big\|\varrho_{0,\varepsilon}\big\|_2^2 - \int_0^t\int_\Omega \varrho^2\mathrm{div}\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\tau. \tag{3.49}$$

In particular,

$$\varepsilon\int_0^t\int_\Omega |\nabla\varrho|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\tau \le \int_0^t \frac{1}{2}\big\|\boldsymbol{u}\big\|_{1,2}^2 + \big\|\varrho\big\|_4^2\,\mathrm{d}\tau.$$

Then, taking ε small enough, especially $\varepsilon < \delta$, one can estimate I_3 as follows

$$|I_3| \le C(\delta)\int_0^t\int_\Omega \varepsilon\frac{\varrho^{\frac{2}{3}} + \vartheta}{\vartheta^2}|\nabla\varrho||\nabla\vartheta|\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\tau \le$$
$$\int_0^t\int_\Omega \omega\delta\varepsilon(\beta\varrho^{\beta-2}+2)\frac{|\nabla\varrho|^2}{\vartheta} + C(\omega,\delta,\beta)\,\varepsilon\vartheta^{-3}|\nabla\vartheta|^2$$
$$+ C(\omega)\varepsilon^2|\nabla\varrho|^2 + \omega|\nabla\log\vartheta|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\tau \le C(\omega)\int_0^t E[t] + \omega\int_\Omega \tilde\sigma\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\tau. \tag{3.50}$$

After we have overcome all the terms on the right-hand side of (3.47), we can use the Gronwall lemma to get estimates independent of the time, n, ε, and δ. In particular for the velocity, we get

$$\boldsymbol{u} \text{ bounded in } L^2(J; W^{1,2}(\Omega; \mathbf{R}^3)).$$

Since all norms are equivalent on X_n, the velocity is bounded also in $L^2(J; W^{1,\infty}(\Omega))$. Then by virtue of (3.26), the density is strictly positive and bounded in $L^\infty(J \times \Omega)$. Seeing that also

$$\varrho|\boldsymbol{u}|^2 \text{ is bounded in } L^\infty(J; L^1(\Omega)),$$

we conclude that the norm of the velocity $\|\boldsymbol{u}\|_{X_n}[t]$ is uniformly bounded on the interval $\overline{J} = [0, T^*]$ and therefore $\boldsymbol{u}[T^*] \in X_n$. Lemma 3.2.2 ensures that the temperature $\vartheta[T^*]$ is strictly positive and belongs in $W^{1,2}(\Omega) \cap L^\infty(\Omega)$. Repeating this procedure one can prolongate the solution up to any finite interval $I = (0, T)$. Consequently, the estimates we have derived holds on the whole time interval I, namely

$$\varrho|\boldsymbol{u}|^2,\ \varrho^{\frac{5}{3}},\ \delta\varrho^\beta,\ \vartheta^4,\ \delta\varrho|\log\vartheta| \quad \text{are bounded in } L^\infty(I; L^1(\Omega)). \tag{3.51}$$

Further from (3.48),

$$|\nabla\boldsymbol{u}|^2,\ |\nabla\vartheta^{\frac{3}{2}}|^2,\ |\nabla\log\vartheta|^2,\ \delta|\nabla\vartheta^{\frac{\beta}{2}}|^2,$$
$$\sqrt{\varepsilon}|\nabla\vartheta^{-\frac{1}{2}}|^2,\ \varepsilon\vartheta^{-(\beta+1)},\ \varepsilon\delta\frac{|\nabla\varrho^{\frac{\beta}{2}}|^2}{\vartheta} \quad \text{are bounded in } L^1(Q),\ Q := I \times \Omega. \tag{3.52}$$

Appling Lemma 2.1.2, we get also

$$\boldsymbol{u},\ \vartheta^{\frac{3}{2}},\ \sqrt{\delta}\log\vartheta,\ \sqrt{\delta}\vartheta^{\frac{\beta}{2}} \quad \text{bounded in } L^2(I; W^{1,2}(\Omega)). \tag{3.53}$$

Using an interpolation and the Sobolev imbedding, the temperature can be bounded also in homogeneous spaces

$$\|\vartheta\|_{L^q(Q)} \le C\delta^{-\frac{1}{2}\frac{3\beta}{3\beta+8}},\ q = \beta + \frac{8}{3}, \quad \|\vartheta\|_{L^p(Q)} \le C,\ p = \frac{17}{3}. \tag{3.54}$$

Then a similar estimate follows for the density, namely

$$\|\nabla\varrho^{\frac{\beta}{2}}\|_{L^p(Q)} \le \|\vartheta^{-1}|\nabla\varrho^{\frac{\beta}{2}}|^2\|_{\frac{2}{p}}\|\vartheta\|_{\frac{2}{qp}} \le C(\delta, \varepsilon)$$

for $\frac{p}{2}(1 + \frac{1}{q}) \le 1$, $q = \beta + \frac{8}{3}$. Hence for a fixed ε the density ϱ is bounded in $L^r(I; L^{3r}(\Omega))$ for some $r > 1$, which can be made arbitrary large taking an appropriate β.

3.4 Limit in Galerkin approximation

At this stage, we take a sequence of spaces $X_n \subset C_0^\infty(\Omega; \mathbf{R}^3)$ such that

$$\overline{\bigcup_{n=1}^{\infty} X_n} = W_0^{1,2}(\Omega; \mathbf{R}^3).$$

This is possible as $W_0^{1,2}(\Omega;\mathbf{R}^3)$ is a separable space. For any n, we can perform the construction procedure described above to get solution $(\varrho,\boldsymbol{u},\vartheta)=(\varrho_n,\boldsymbol{u}_n.\vartheta_n)$ to the system of equations

$$\left.\begin{aligned}\partial_t\varrho_n+\operatorname{div}(\varrho_n\boldsymbol{u}_n)&=\varepsilon\Delta\varrho_n, && \text{on } I\times\Omega,\\ \nabla\varrho_n\cdot\boldsymbol{n}&=0 && \text{on } I\times\partial\Omega,\\ \varrho_n[0]&=\varrho_{0,\varepsilon} && \text{on } \Omega,\end{aligned}\right\} \tag{3.55}$$

$$\int_I\int_\Omega \varrho_n\boldsymbol{u}_n\cdot\partial_t\boldsymbol{\varphi}+\varrho_n\boldsymbol{u}_n\otimes\boldsymbol{u}_n:\nabla\boldsymbol{\varphi}+p_\delta(\varrho_n,\vartheta_n)\operatorname{div}\boldsymbol{\varphi}$$
$$-\mathbb{S}_n[\boldsymbol{u}_n]:\nabla\boldsymbol{\varphi}+(\varrho_n\boldsymbol{f}-\varepsilon\nabla\boldsymbol{u}_n\nabla\varrho_n)\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t=0, \tag{3.56}$$

for any $\boldsymbol{\varphi}\in C_0^1(I;X_n)$.

$$\int_I\int_\Omega \varrho_n s_{\delta,n}\partial_t\varphi+\varrho_n\boldsymbol{u}_n s_{\delta,n}\cdot\nabla\varphi-\frac{\kappa_{\varepsilon,n}[\boldsymbol{u}_n]\nabla\vartheta_n\cdot\nabla\varphi}{\vartheta}+\sigma_{\varepsilon,n}\varphi+\varepsilon A_{\varepsilon,n}\nabla\varrho_n\cdot\nabla\varphi$$
$$-\varepsilon\frac{1}{\vartheta_n^2}(e_{G,n}+\varrho_n\partial_\varrho e_{G,n}+\delta\vartheta_n)\nabla\varrho_n\cdot\nabla\vartheta_n\,\varphi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t=0 \tag{3.57}$$

for any $\varphi\in\mathcal{D}(I\times\overline{\Omega})$.

$$\int_\Omega E_n[t_2]-E_n[t_1]\,\mathrm{d}\boldsymbol{x}=\int_{t_1}^{t_2}\int_\Omega \varrho_n\boldsymbol{f}\cdot\boldsymbol{u}_n+\varepsilon(\vartheta_n^{-\beta}-\vartheta^\beta)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \tag{3.58}$$

for a.e. t_1, $t_2\in I$, where

$$E_n=E_\delta(\varrho_n,\boldsymbol{u}_n,\vartheta_n)=\frac{1}{2}\varrho_n|\boldsymbol{u}_n|^2+\varrho_n e_{\delta,n}+\delta\big(\frac{\varrho_n^\beta}{\beta-1}+\varrho_n^2\big).$$

In addition, ϱ_n, $\boldsymbol{u}_n$, ϑ_n satisfy the initial conditions

$$\varrho_n[0]\boldsymbol{u}_n[0]=\varrho_0\boldsymbol{u}_0 \text{ in } X_n^*, \tag{3.59}$$
$$\varrho_n[0]s_\varepsilon(\varrho_n[0],\vartheta_n[0])=\varrho_{0,\varepsilon}s_\varepsilon(\varrho_{0,\varepsilon},\vartheta_{0,\varepsilon}) \text{ on } \Omega, \tag{3.60}$$
$$E_n[0]=E_n(\varrho_{0,\varepsilon},\boldsymbol{u}_0,\vartheta_{0,\varepsilon}). \tag{3.61}$$

The sequence $(\varrho_n,\boldsymbol{u}_n,\vartheta_n)$ complies with the n-independent estimates (3.51), (3.53). Consequently, we can identify the limits using Theorem 2.2.1,

$$\varrho_n\to\varrho \qquad \text{weakly-}* \text{ in } L^\infty(I;L^\beta(\Omega)), \tag{3.62}$$
$$\boldsymbol{u}_n\to\boldsymbol{u} \qquad \text{weakly in } L^2(I;W_0^{1,2}(\Omega;\mathbf{R}^3)), \tag{3.63}$$
$$\vartheta_n\to\vartheta \qquad \text{weakly-}* \text{ in } L^\infty(I;L^4(\Omega)). \tag{3.64}$$

Our next task is pass to the limit in the equations (3.55), (3.56), (3.57), and (3.58).

3.4.1 Limit in the continuity equation

To begin with, we shall need an equi-integrability of the terms $\partial_t\varrho_n$, $\Delta\varrho_n$. This is of course a direct consequence of the L^p theory for parabolic equations, provided we can control the term

$$\operatorname{div}(\varrho_n\boldsymbol{u}_n)=\nabla\varrho_n\cdot\boldsymbol{u}_n+\varrho_n\operatorname{div}\boldsymbol{u}_n.$$

in $L^p(I\times\Omega)$, $p>1$. Unfortunately, in view of (3.51), (3.53), the first term is merely integrable with respect to time, so we need to gain better estimates. To this end, one can test the continuity equation (3.55) by $\log\varrho_n+1=(\varrho_n\log\varrho_n)'$, to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho_n\log\varrho_n\ \mathrm{d}\boldsymbol{x}+\varepsilon\int_\Omega\frac{|\nabla\varrho_n|^2}{\varrho_n}\ \mathrm{d}\boldsymbol{x}=\int_\Omega\varrho_n\mathrm{div}\boldsymbol{u}_n\ \mathrm{d}\boldsymbol{x}.$$

Hence

$$\varepsilon\int_\Omega\frac{|\nabla\varrho_n|^2}{\varrho_n}\ \mathrm{d}\boldsymbol{x}\le C. \tag{3.65}$$

Further, we can use the estimate for the kinetic energy to infer

$$\big\|\nabla\varrho_n\cdot\boldsymbol{u}_n\big\|_{L^2(I;L^1(\Omega))}\le\Big\|\frac{\nabla\varrho_n}{\sqrt{\varrho_n}}\Big\|_{L^2(I\times\Omega)}\big\|\sqrt{\varrho_n}\boldsymbol{u}_n\big\|_{L^\infty(I;L^2(\Omega))}\le C(\varepsilon).$$

On the other hand, $\nabla\varrho_n\cdot\boldsymbol{u}_n$ is bounded also in $L^1(I;L^{\frac{3}{2}}(\Omega))$ and $\varrho_n\mathrm{div}\boldsymbol{u}_n$ in some $L^r(I\times\Omega)$, $1<r(\beta)<2$. Then an interpolation argument keeps the term $\mathrm{div}(\varrho_n\boldsymbol{u}_n)$ bounded in $L^p(I\times\Omega)$ for some $p>1$ and we conclude

$$\partial_t\varrho_n,\ \Delta\varrho_n \text{ bounded in } L^p(I\times\Omega),\ p>1. \tag{3.66}$$

In particular, ϱ_n is bounded in $W^{1,p}(I\times\Omega)$ for some $p>1$, whence the compact imbedding yields the strong convergence

$$\varrho_n\to\varrho \text{ a.e. on } I\times\Omega. \tag{3.67}$$

Consequently, one can pass to the limit in the continuity equation (3.55) and obtain

$$\partial_t\varrho+\mathrm{div}(\varrho\boldsymbol{u})-\varepsilon\Delta\varrho=0\quad\text{a.e. in } I\times\Omega, \tag{3.68}$$

where ϱ is non-negative and it satisfies

$$\nabla\varrho\cdot\boldsymbol{n}|_{\partial\Omega}=0,\quad \varrho[0]=\varrho_{0,\varepsilon}. \tag{3.69}$$

Multiplying by a smooth function φ and integrating by parts we get also weak formulation

$$\int_I\int_\Omega\varrho\partial_t\varphi+\varrho\boldsymbol{u}\cdot\nabla\varphi-\varepsilon\nabla\varrho\cdot\nabla\varphi\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t=0,\quad\text{for any }\varphi\in\mathcal{D}(I\times\overline{\Omega}). \tag{3.70}$$

Then we take $\varphi=\varrho$ and we arrive at the "energy" equality for ϱ, which subtraced from (3.49), yields

$$2\varepsilon\int_0^T\int_\Omega|\nabla\varrho_n|^2-|\nabla\varrho|^2\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t=\int_\Omega(\varrho_n^2-\varrho^2)[T]\ \mathrm{d}\boldsymbol{x}-\int_0^T\int_\Omega\varrho_n^2\mathrm{div}\boldsymbol{u}_n-\varrho^2\mathrm{div}\boldsymbol{u}\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t.$$

Since the right-hand side tends to zero, we have proved the strong convergence of gradients

$$\nabla\varrho_n\to\nabla\varrho \text{ in } L^2(I\times\Omega).$$

3.4.2 Limit in the entropy equation

Our next goal is to establish the strong convergence of the temperature field and pass to the limit in the entropy equation and the balance of the total energy. Following Section 4.6, we shall apply Lemma 4.6.1 on the entropy equality (3.57). To this end, we need

$$\varrho_n s_\delta(\varrho_n,\vartheta_n) \text{ bounded in } L^2(I;L^p(\Omega))\cap L^\infty(I;L^1(\Omega)),\ p>\frac{6}{5}. \tag{3.71}$$

From the hypothesis (3.18) we deduce

$$|\varrho s_\delta(\varrho,\vartheta)| \le C(\vartheta^3 + \varrho|\log\varrho| + \varrho|\log\vartheta| + 1).$$

Then (3.71) is a straightforward consequence of estimates (3.51), (3.52). In fact (3.71) is uniform with respect to ε. Further, we need to bound all other terms in the entropy equation (3.57) at least in some $L^1(I;W^{-n,p}(\Omega))$, $p>1$. The entropy production σ_ε is uniformly bounded in $L^1(I)$. For the other terms, we need also L^1-estimates, nevertheless with future application in mind, we shall gain even estimates in $L^p(I\times\Omega)$ for some $p>1$ uniform with respect to ε.

Since $\varrho_n s_\delta(\varrho_n,\vartheta_n)$ can be interpolated between the spaces in (3.71) and because the velocity is bounded in $L^2(I;L^6(\Omega;\mathbf{R}^3))$ we easily deduce

$$\big\|\varrho_n s_\delta(\varrho_n,\vartheta_n)\boldsymbol{u}_n\big\|_{L^p(I\times\Omega)} \le C(\delta).$$

Next in view of hypothesis (3.22), we observe that

$$\left|\frac{\kappa_\delta(t,\boldsymbol{x};\vartheta_n)\nabla\vartheta_n}{\vartheta_n}\right| \le C(\delta)\big(\nabla\vartheta_n + \vartheta_n^{\frac{\beta}{2}}|\nabla(\vartheta_n^{\frac{3}{2}})|\big)$$

where, by virtue of (3.52) and (3.54), the right-hand side is bounded in $L^p(I\times\Omega)$. Similarly the remaining epsilon part of the modified κ can be estimated this way

$$\left|\frac{(\sqrt{\varepsilon}\vartheta_n^{-1})\nabla\vartheta_n}{\vartheta_n}\right| \le \varepsilon^{\frac14-\frac{1}{\beta+1}}\big(\varepsilon^{\frac{1}{\beta+1}}|\vartheta^{-\frac12}|\big)\big(\varepsilon^{\frac14}|\nabla(\vartheta^{-\frac12})|\big).$$

The L^p-norm of the right-hand side of both inequalities is dominated by $\varepsilon^r C$ for some $r>0$. The term $\varepsilon A_{\varepsilon,n}\nabla\varrho_n$ is more delicate. In accordance with hypotheses (3.14), (3.17), and (3.18), we have

$$\varepsilon|A_{\varepsilon,n}||\nabla\varrho_n| \le \varepsilon C(\delta)\big(1+\frac{\varrho_n^{\frac23}}{\vartheta_n} + |\log\varrho_n| + |\log\vartheta_n|\big)|\nabla\varrho_n|.$$

Omitting the index n, the terms on the right-hand side can be estimated as follows

$$\varepsilon\big\|\nabla\varrho\big\|_p \le \sqrt{\varepsilon}C(\delta),$$

$$\varepsilon\big\|\varrho^{\frac23}\vartheta^{-1}\nabla\varrho\big\|_p \le \varepsilon^{\frac12-\frac{1}{\beta+1}}C(\delta)\big\|\varrho\big\|_\beta^{\,\varepsilon^{\frac{1}{\beta+1}}}\big\|\vartheta^{-1}\big\|_{\beta+1}\sqrt{\varepsilon}\big\|\nabla\varrho\big\|_2 \le \varepsilon^{r(\beta)}C(\delta),$$

$$\varepsilon\big\|\log\varrho\nabla\varrho\big\|_p \le \sqrt{\varepsilon}C(\delta)(\big\|\varrho\big\|_\beta^2 + \varepsilon\big\|\frac{|\nabla\varrho|}{\varrho}\big\|_2^2) \le \sqrt{\varepsilon}C(\delta),$$

and

$$\begin{aligned}\varepsilon\big\|\log\vartheta\nabla\varrho\big\|_p &= \varepsilon\big\|(\varrho|\log\vartheta|)^{\frac1\alpha}\,|\log\vartheta|^{1-\frac1\alpha}\,\frac{|\nabla\varrho|}{\sqrt{\varrho}}\,\varrho^{\frac12-\frac1\alpha}\big\|_{L^p(I\times\Omega)}\\ &\le \sqrt{\varepsilon}\big\|\varrho\log\vartheta\big\|_{L^\infty(I;L^1(\Omega))}^{\alpha}\big\|\log\vartheta\big\|_{L^2(I;L^6(\Omega))}^{1-\frac1\alpha}\sqrt{\varepsilon}\big\|\varrho^{-1}|\nabla\varrho|^2\big\|_1\big\|\varrho\big\|_\beta \le \sqrt{\varepsilon}C(\delta)\end{aligned} \tag{3.72}$$

for some $p>1$, provided β is large enough and $\alpha>\frac52$.

Concerning the last term in (3.57) we have at our disposal only the L^1-estimate (3.50), namely

$$\big\|\varepsilon(e_G + \varrho\partial_\varrho e_G + \delta\vartheta)\frac{\nabla\varrho\cdot\nabla\vartheta}{\vartheta^2}\big\|_1 \leq \sqrt[4]{\varepsilon}C(\delta).$$

Nevertheless, this is enough to meet the assumptions of Lemma 4.6.1 and conclude

$$\frac{4}{3}a(\vartheta\overline{\vartheta^3} - \overline{\vartheta\vartheta^3}) = (\overline{\vartheta\varrho s_G} - \vartheta\overline{\varrho s_G}) + \delta(\overline{\vartheta\varrho\log\vartheta} - \vartheta\overline{\varrho\log\vartheta}). \tag{3.73}$$

The right-hand side integrated over a ball $B \subset I\times\Omega$ can be split,

$$\lim_{n\to\infty}\int_B \varrho_n(s_G(\varrho_n,\vartheta_n) + \delta\log\vartheta_n)(\vartheta_n - \vartheta)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t =$$
$$\lim_{n\to\infty}\int_B \varrho_n\big(s_G(\varrho_n,\vartheta_n) - s_G(\varrho_n,\vartheta)\big)(\vartheta_n-\vartheta) + \delta\varrho_n\big(\log\vartheta_n - \log\vartheta)(\vartheta_n-\vartheta)$$
$$+ \varrho_n\big(s_G(\varrho_n,\vartheta) + \delta\log\vartheta\big)(\vartheta_n-\vartheta)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,$$

whence, in accordance with hypothesis (3.18), the first two terms on the right-hand side are non-negative, while the last one tends to zero, because of the strong convergence of the density. Then (3.73) yields

$$\overline{\vartheta^3}\vartheta \geq \overline{\vartheta^3}\,\overline{\vartheta} \qquad \text{a.e. on } I\times\Omega. \tag{3.74}$$

Then using the Minty trick, we conclude

$$\vartheta_n \to \vartheta \text{ in } L^4(I\times\Omega). \tag{3.75}$$

The point-wise convergence of the temperature and the compactness property (3.23) of the mapping $\mathcal{Z}$ imply also the point-wise convergence of the quantities κ_G, κ_R, μ, and ζ, which are mainly functions of t, $\boldsymbol{x}$, and ϑ, but depends also on the velocity through the non-local mapping $\mathcal{Z}$. Then the terms

$$\frac{\kappa[\boldsymbol{u}_n](t,\boldsymbol{x};\vartheta_n)\nabla\vartheta_n}{\vartheta_n} \text{ and } \mathbb{S}[\boldsymbol{u}_n](t,\boldsymbol{x};\vartheta_n)$$

tends to their counterparts as they are bounded in some $L^p(I\times\Omega)$, $p>1$. The same argument is tacitly used in all succeeding limit processes.

The strong convergence of the temperature allows us to pass to the limit in (3.57) and (3.58). In view of the L^p-estimates from the beginning of this section, the majority of the terms in (3.57) tends to their counterparts. based on the limit quantities ϱ, $\boldsymbol{u}$, ϑ. On the other hand the L^1-terms tend to the Radon measures:

$$\varepsilon(e_G + \varrho_n\partial_\varrho e_{G,n} + \delta\vartheta_n)\frac{\nabla\varrho_n\cdot\nabla\vartheta_n}{\vartheta_n^2} \to \Gamma_\varepsilon, \text{ where } \big\|\Gamma\big\|_{\mathcal{M}(I\times\Omega)} \leq \sqrt[4]{\varepsilon}C(\delta),$$

and

$$\sigma_{\varepsilon,n} \to \Sigma_\varepsilon \geq \sigma_\varepsilon = \frac{\kappa_\varepsilon[\boldsymbol{u}]|\nabla\vartheta|^2}{\vartheta^2} + \frac{\mathbb{S}:\nabla\boldsymbol{u}}{\vartheta} + \varepsilon\vartheta^{-\beta-1} + \varepsilon\frac{\partial_\varrho p_G}{\vartheta\varrho}|\nabla\varrho|^2 + \delta\varepsilon(\beta\varrho^{\beta-2}+2)\frac{|\nabla\varrho|^2}{\vartheta},$$

where $\Sigma_\varepsilon \geq \sigma_\varepsilon$ (in sense of distributions) is due to weak lower semi-continuity of the convex term $\sigma_{\varepsilon,n}$, cf. Section 4.8. Thus the limit equation reads

$$\int_I\int_\Omega \varrho s_\delta\partial_t\varphi + \varrho\boldsymbol{u}s_\delta\cdot\nabla\varphi - \frac{\kappa_\varepsilon[\boldsymbol{u}]\nabla\vartheta\cdot\nabla\varphi}{\vartheta} + \varepsilon A_\varepsilon\nabla\varrho\cdot\nabla\varphi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$+ \langle\Sigma_\varepsilon,\varphi\rangle_{I\times\Omega} - \varepsilon\vartheta^{\beta-1} - \langle\Gamma_\varepsilon,\varphi\rangle_{I\times\Omega} = 0, \quad \text{for any } \varphi\in\mathcal{D}(I\times\overline{\Omega}). \tag{3.76}$$

Concerning the initial condition, we use (3.60) and we denote $s_{0,\varepsilon} = s_\varepsilon(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon})$. Then we get

$$\int_\Omega \varrho_{0,\varepsilon} s_{0,\varepsilon}\varphi \,\mathrm{d}\boldsymbol{x} = \int_\Omega \varrho_n[0] s_\varepsilon(\varrho_n[0], \vartheta_n[0])\varphi \,\mathrm{d}\boldsymbol{x} \\ = \lim_{\tau\to 0+} \frac{1}{\tau}\int_0^\tau \int_\Omega \varrho_n s_\varepsilon(\varrho_n, \vartheta_n)\varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t, \quad (3.77)$$

for any $\varphi \in \mathcal{D}(\overline{\Omega})$ and every n. . Having $\varrho_n s_{\varepsilon,n}$ bounded in $L^p(I\times\Omega)$, $p>1$, one can pass to the limit in this equality. Next, we test (3.76) by $\psi(t)\varphi(\boldsymbol{x})$, where ψ is an aproximation of the characteristic function of the interval $[0,t]$ and $\varphi \geq 0$. Letting t go to zero we arrive at

$$\operatorname*{esslim}_{t\to 0+} \int_\Omega \varrho s_\varepsilon(\varrho,\vartheta)[t]\varphi \,\mathrm{d}\boldsymbol{x} \geq \lim_{\tau\to 0+}\frac{1}{\tau}\int_0^\tau\int_\Omega \varrho s_\varepsilon\varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_\Omega \varrho_{0,\varepsilon}s_{0,\varepsilon}\varphi \,\mathrm{d}\boldsymbol{x}. \quad (3.78)$$

In order to pass to the limit in the total energy balance we rewrite it in the weak form:

$$\int_I \partial_t\psi(t)\int_\Omega \frac{1}{2}\varrho_n|\boldsymbol{u}_n|^2 + \varrho_n e_{\varepsilon,n} + \delta\big(\frac{\varrho_n^\beta}{\beta-1} + \varrho_n^2\big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_I \psi(t)\int_\Omega \varrho_n \boldsymbol{f}\cdot\boldsymbol{u}_n + \varepsilon\vartheta_n^{-\beta}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \quad (3.79)$$

for any $\psi \in \mathcal{D}(I)$. Here all the terms are equi-integrable, thus we can pass to the limit and use a sequence of test functions approaching the characteristic function of the interval $[t_1,t_2]$. Finally, we get

$$\int_\Omega E_\varepsilon[t_2] - E_\varepsilon[t_1]\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_{t_1}^{t_2}\int_\Omega \varrho\boldsymbol{f}\cdot\boldsymbol{u} + \varepsilon(\vartheta^{-\beta} - \vartheta^\beta)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \quad \text{for a.e. } t_1,\ t_2 \in I, \quad (3.80)$$

$$E_\varepsilon = \frac{1}{2}\varrho|\boldsymbol{u}|^2 + \varrho e_\delta(\varrho,\vartheta) + \delta\big(\frac{\varrho^\beta}{\beta-1} + \varrho^2\big).$$

Finally, from (3.61) and (3.80) it follows

$$E_\varepsilon[t] \to E_\varepsilon[0] = E_\delta(\varrho_{0,\varepsilon}, \boldsymbol{u}_0, \vartheta_{0,\varepsilon}). \quad (3.81)$$

3.4.3 Limit in the momentum equation

In view of the available estimates, one can use the momentum equation and the Arzelà-Ascoli theorem similarly as in (4.65) in order to get

$$\varrho_n\boldsymbol{u}_n \to \varrho\boldsymbol{u} \text{ in } C([0,T]; L^{\frac{5}{4}}_{weak}(\Omega;\mathbf{R}^3)). \quad (3.82)$$

This space is compactly imbedded into $C([0,T]; W^{-1,2}(\Omega;\mathbf{R}^3))$, thus taking into account (3.53), we deduce

$$\varrho_n\boldsymbol{u}_n\otimes\boldsymbol{u}_n \to \varrho\boldsymbol{u}\otimes\boldsymbol{u} \text{ in } L^2(I; L^p(\Omega;\mathbf{R}^3\times\mathbf{R}^3)),\ p>1.$$

Because of the strong convergence of the density gradients (3.67), we infer

$$\nabla\varrho_n\nabla\boldsymbol{u}_n \to \nabla\varrho\nabla\boldsymbol{u} \text{ in } \mathcal{D}'(I\times\Omega;\mathbf{R}^3).$$

The limit in the other terms is a straightforward consequence of the available estimates and the strong convergence of the density and the temperature. Consequently we obtain the limit of the momentum equation (3.56):

$$\int_I \int_\Omega \varrho \boldsymbol{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho \boldsymbol{u} \otimes \boldsymbol{u} : \nabla \boldsymbol{\varphi} + p_\delta \mathrm{div} \boldsymbol{\varphi} - \mathbb{S} : \nabla \boldsymbol{\varphi} + \varrho \boldsymbol{f} \cdot \varphi \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t = 0 \tag{3.83}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}(I \times \Omega; \mathbf{R}^3)$. Because of (3.82), we can pass to the limit in (3.59) as well and get

$$(\varrho \boldsymbol{u})[t] \to (\varrho \boldsymbol{u})[0] = \varrho_{0,\varepsilon} \boldsymbol{u}_0 \text{ weakly in } L^1(\Omega). \tag{3.84}$$

3.5 Vanishing viscosity limit

In the previous section, we have constructed a solution $(\varrho, \boldsymbol{u}, \vartheta) = (\varrho_\varepsilon, \boldsymbol{u}_\varepsilon, \vartheta_\varepsilon)$ of the equations (3.70), (3.83), (3.76), and (3.80) for any fixed ε. Our next goal is to let ε tend to zero and pass to the limit in our system. We can take advantage of estimates (3.51), (3.52), and (3.53), which are uniform with respect to ε. In particular, all the terms containing ε vanish in the limit, since

$$\left\| \varepsilon \nabla \varrho_\varepsilon \right\|_{L^1(Q)} + \left\| \varepsilon \nabla \varrho_\varepsilon \nabla \boldsymbol{u}_\varepsilon \right\|_{L^1(Q)} \leq \sqrt{\varepsilon} C(\delta)$$

and

$$\left\| \varepsilon \vartheta^{-\beta} \right\|_{L^1(Q)} \leq \varepsilon^{\frac{\beta}{\beta+1}} C, \quad \left\| \varepsilon \vartheta^{\beta} \right\|_{L^1(Q)} \leq \varepsilon C.$$

while the epsilon terms in the entropy inequality (3.76) either have sign and can be forgotten or are dominated by $\varepsilon^r C(\delta)$, $r(\beta) > 0$ as was discussed in Section 3.4.2. In view of the available estimates, we can choose a subsequence such that

$$\left.\begin{aligned} \varrho_\varepsilon &\to \varrho \quad \text{in } C(I; L^{\frac{5}{3}}_{weak}(\Omega)), \\ \boldsymbol{u}_\varepsilon &\to \boldsymbol{u} \quad \text{weakly in } L^2(I; W^{1,2}(\Omega; \mathbf{R}^3)), \\ \vartheta_\varepsilon &\to \vartheta \quad \text{weakly in } L^2(I; W^{1,2}(\Omega)). \end{aligned}\right\} \tag{3.85}$$

3.5.1 Refined pressure estimates

According to the estimates $(3.51 - 3.53)$, the artificial pressure as well as the corresponding term in the energy balance are known to be bounded only in $L^1(Q)$, which is not enough to exclude possible concentrations in the limit. Nevertheless, an additional estimate can be gained from the momentum equation. Following [18], we test (3.83) by

$$\boldsymbol{\varphi} = \psi(t) \mathcal{B}[\pi], \ \pi = \varrho - \fint_\Omega \varrho \, \mathrm{d}\boldsymbol{x},$$

where $\mathcal{B}$ is the Bogovskii operator on the domain Ω introduced in Lemma 2.1.3. A straightforward calculation yields

$$\int_0^T \int_\Omega \psi p_\delta \varrho \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \sum_{j=1}^4 I_j = \int_0^T \psi \int_\Omega p \,\mathrm{d}\boldsymbol{x} \int_\Omega \varrho \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \tag{3.86}$$

$$\begin{aligned} -\int_0^T \int_\Omega \psi\varrho\boldsymbol{f}\cdot\mathcal{B}[\pi] + \psi\varrho\boldsymbol{u}\otimes\boldsymbol{u} : \nabla\mathcal{B}[\pi] + \partial_t\psi\varrho\boldsymbol{u}\cdot\mathcal{B}[\pi]\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ + \int_0^T \psi \int_\Omega \big(2\mu^f\mathbb{D}\boldsymbol{u} + \big(\zeta^f - \frac{2}{3}\mu^f\big)\mathrm{div}\,\boldsymbol{u}\mathbb{I}\big) : \nabla\mathcal{B}[\pi]\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ - \int_0^T \psi \int_\Omega \varrho\boldsymbol{u}\cdot\partial_t\mathcal{B}[\pi]\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t. \end{aligned}$$

The first three terms I_1, I_2, I_3 are bounded by virtue of estimates (3.51), (3.53). In the last term I_4, we use essentially the strong continuity equation (3.68) to deduce

$$\big\|\partial_t\mathcal{B}[\pi]\big\|_{L^2(I\times\Omega)} = \big\|\mathcal{B}[\varepsilon\Delta\varrho - \mathrm{div}(\varrho\boldsymbol{u})]\big\|_{L^2(I\times\Omega)} \le C\big(\big\|\varrho\boldsymbol{u}\big\|_2 + \varepsilon\big\|\nabla\varrho\big\|_2\big),$$

where the right hand side is bounded provided $\beta > 3$. Then I_4 is bounded as well and it follows

$$\big\|p_G(\varrho,\vartheta)\varrho + \delta(\varrho^{\beta+1} + \varrho^3)\big\|_{L^1(Q)} \le C. \tag{3.87}$$

3.5.2 Limit passage

Using (3.85), one can pass to the limit in the continuity equation and get

$$\int_I\int_{\mathbf{R}^3} \varrho\partial_t\varphi + \varrho\boldsymbol{u}\cdot\nabla\varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = 0 \quad \text{for all } \varphi\in\mathcal{D}(I\times\mathbf{R}^3). \tag{3.88}$$

Moreover, thanks to the artificial pressure $\delta\varrho^\beta$, the density is square integrable and Lemma 3.1.2 can be used to obtain also the renormalized equation.

Next, we turn our attention to the momentum equation. Employing the time term, it follows

$$\varrho_\varepsilon\boldsymbol{u}_\varepsilon \to \varrho\boldsymbol{u} \text{ in } C([0,T];L^{\frac{5}{4}}_{weak}(\Omega;\mathbf{R}^3)). \tag{3.89}$$

Hence, thanks to the compact imbedding $L^{\frac{5}{4}}_{weak} \hookrightarrow W^{-1,2}$, we infer

$$\varrho_\varepsilon\boldsymbol{u}_\varepsilon\otimes\boldsymbol{u}_\varepsilon \to \varrho\boldsymbol{u}\otimes\boldsymbol{u} \text{ weakly in } L^p(Q) \tag{3.90}$$

for some $p > 1$. Moreover, in view of (3.87), the terms $\mathbb{S}$, p_δ are also bounded in $L^p(Q)$, $p > 1$. Then one can pass to the limit in the momentum equation (3.83) and obtain

$$\int_I\int_\Omega \varrho\boldsymbol{u}\cdot\partial_t\boldsymbol{\varphi} + \varrho\boldsymbol{u}\otimes\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + \overline{p_\delta}\mathrm{div}\boldsymbol{\varphi} - \overline{\mathbb{S}} : \mathbb{D}\boldsymbol{\varphi} + \varrho\boldsymbol{f}\cdot\varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = 0 \tag{3.91}$$

for any $\boldsymbol{\varphi}\in\mathcal{D}(I\times\Omega;\mathbf{R}^3)$.

3.5.3 Strong convergence of the temperature

Our next goal is to establish the strong convergence of the temperature field. We present a method based on the Div-Curl lemma 2.2.4, which was probably first used in [16]. We rewrite the entropy equation (3.76) as

$$\mathrm{Div}_{t,\boldsymbol{x}} \boldsymbol{U}_\varepsilon = \Sigma_\varepsilon - \Gamma_\varepsilon$$

where

$$\boldsymbol{U}_\varepsilon = \Big[\varrho_\varepsilon s_\delta(\varrho_\varepsilon, \vartheta_\varepsilon), \varrho_\varepsilon s_\delta(\varrho_e ps, \vartheta_\varepsilon) + \frac{\kappa_\varepsilon \nabla \vartheta_\varepsilon}{\vartheta_\varepsilon} + \varepsilon A_\varepsilon \nabla \varrho_\varepsilon\Big]$$

and $\mathrm{Div}_{t,\boldsymbol{x}}$ is the divergence operator on the four-dimensional space-time. It follows that $\mathrm{Div}_{t.\boldsymbol{x}} \boldsymbol{U}_\varepsilon$ is relatively compact in $W^{-1,s}(I \times \Omega)$ for some $s > 1$. Moreover, in view of the estimates collected in Section 3.4.2, all terms in $\boldsymbol{U}_\varepsilon$ are bounded in $L^p(I \times \Omega)$ for some $p > 1$. On the other hand, the field

$$\boldsymbol{V}_\varepsilon = (\vartheta_\varepsilon, 0, 0, 0),$$

is bounded in any $L^q(I \times \Omega; \mathbf{R}^4)$, $1 < q < \infty$, provided β is large enough, and $\mathrm{Curl}_{t,\boldsymbol{x}} \boldsymbol{V}_\varepsilon$ is relatively compact in $W^{-1,s}(I \times \Omega; \mathbf{R}^4)$ for $s > 1$. Then a direct application of Lemma 2.2.4 yields

$$\frac{4}{3} a(\vartheta \overline{\vartheta^3} - \overline{\vartheta \vartheta^3}) = (\overline{\vartheta \varrho s_G} - \vartheta \overline{\varrho s_G}). \tag{3.92}$$

The right-hand side integrated over a ball $B \subset I \times \Omega$, can be split

$$\lim_{\varepsilon \to 0} \int_B \varrho_\varepsilon s_G(\varrho_\varepsilon, \vartheta_\varepsilon)(\vartheta_\varepsilon - \vartheta) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t =$$
$$\lim_{\varepsilon \to 0} \int_B \varrho_\varepsilon \big(s_G(\varrho_\varepsilon, \vartheta_\varepsilon) - s_G(\varrho_\varepsilon, \vartheta)\big)(\vartheta_\varepsilon - \vartheta) + \varrho_\varepsilon s_G(\varrho_\varepsilon, \vartheta)(\vartheta_\varepsilon - \vartheta) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t,$$

where the former term is non-negative, while the later one tends to zero, by virtue of calculation (4.71) if we prove

$$\overline{B(\varrho) G(\vartheta)} = \overline{B(\varrho)} \, \overline{G(\vartheta)}. \tag{3.93}$$

Using the renormalized continuity equation, one can show similarly as in (4.59) that

$$B(\varrho_\varepsilon) \to \overline{B(\varrho)} \text{ in } C([0,T]; L^p_{weak}(\Omega)). \tag{3.94}$$

for any $B \in W^{1,\infty}(\mathbf{R})$. On the other hand in view of the estimate (3.53), we have

$$G(\vartheta_\varepsilon) \to \overline{G(\vartheta)} \quad \text{weakly in } L^2(I; W^{1,2}(\Omega))$$

for any $G \in W^{1,\infty}(\mathbf{R})$. Then (3.93) is a consequence of the compact imbedding $L^p(\Omega) \hookrightarrow W^{-1,2}(\Omega)$ for $p > \frac{6}{5}$. Finally, (3.92) yields

$$\overline{\vartheta^3} \vartheta \geq \overline{\vartheta^3 \vartheta} \qquad \text{a.e. on } I \times \Omega \tag{3.95}$$

and using the Minty trick, we conclude

$$\vartheta_n \to \vartheta \text{ in } L^4(I \times \Omega). \tag{3.96}$$

3.5.4 Strong convergence of the density field

In order to identify weak limits of the nonlinear terms, like $\overline{p_\delta}(\varrho_\varepsilon, \vartheta_\varepsilon)$, we have to prove the strong convergence of the density field. Following Section 4.7, we begin with compactness properties of the effective viscous pressure. Testing (3.83) by $\psi\eta\boldsymbol{\varphi}_\varepsilon = \psi(t)\eta(\boldsymbol{x})\nabla\Delta^{-1}[\xi(\boldsymbol{x})\varrho_\varepsilon)]$ and using the continuity equation (3.70), we get

$$\int_Q \psi\xi\big(\eta(p_{\delta,\varepsilon} - \mathcal{R} : [\eta\mathbb{S}_\varepsilon])\big)\,\varrho_\varepsilon \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \tag{3.97}$$

$$\int_Q \psi(\mathbb{S}_\varepsilon - \varrho_\varepsilon \boldsymbol{u}_\varepsilon \otimes \boldsymbol{u}_\varepsilon) : \nabla\eta \otimes \boldsymbol{\varphi}_\varepsilon - \psi p_{\delta,\varepsilon}\nabla\eta\cdot\boldsymbol{\varphi}_\varepsilon)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

$$-\int_Q \eta\varrho_\varepsilon\boldsymbol{u}_\varepsilon\cdot\big(\partial_t\psi\boldsymbol{\varphi}_\varepsilon + \psi\nabla\Delta^{-1}[\nabla\xi\cdot\varrho_\varepsilon\boldsymbol{u}_\varepsilon]\big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_Q \psi\eta\boldsymbol{\varphi}_\varepsilon\cdot(\varrho_\varepsilon\boldsymbol{f})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

$$+\varepsilon\int_Q (\nabla\varrho_\varepsilon\cdot\nabla)\boldsymbol{u}_\varepsilon\cdot\nabla(\psi\eta\boldsymbol{\varphi}_\varepsilon) + \psi\eta\varrho_\varepsilon\boldsymbol{u}_\varepsilon\cdot\mathcal{R}\cdot[\xi\nabla\varrho_\varepsilon] - \psi\eta\nabla\Delta^{-1}[\nabla\eta\cdot\nabla\varrho_\varepsilon]\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

$$+\int_Q \psi\boldsymbol{u}_\varepsilon\cdot\big(\mathcal{R}\cdot[\eta\varrho_\varepsilon\boldsymbol{u}_\varepsilon]\xi\varrho_\varepsilon - \eta\varrho_\varepsilon\boldsymbol{u}_\varepsilon\cdot\mathcal{R}[\xi\varrho_\varepsilon]\big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

On the other hand, (3.91) tested by $\psi\eta\overline{\boldsymbol{\varphi}} = \psi(t)\eta(\boldsymbol{x})\nabla\Delta^{-1}[\xi(\boldsymbol{x})\varrho]$ together with the limit of the continuity equation (3.88) gives

$$\int_Q \psi\xi\big(\overline{\eta p_\delta - \mathcal{R} : [\eta\mathbb{S}]}\big)\,\varrho\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \tag{3.98}$$

$$\int_Q \psi(\overline{\mathbb{S}} - \varrho\boldsymbol{u}\otimes\boldsymbol{u}) : \nabla\eta\otimes\overline{\boldsymbol{\varphi}} - \psi\overline{p_\delta}\nabla\eta\cdot\overline{\boldsymbol{\varphi}}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

$$-\int_Q \eta\varrho\boldsymbol{u}\cdot\big(\partial_t\psi\overline{\boldsymbol{\varphi}} + \psi\nabla\Delta^{-1}[\nabla\xi\cdot\varrho\boldsymbol{u}]\big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_Q \psi\eta\overline{\boldsymbol{\varphi}}\cdot(\varrho\boldsymbol{f})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

$$+\int_Q \psi\boldsymbol{u}\cdot\big(\mathcal{R}\cdot[\eta\varrho\boldsymbol{u}]\xi\varrho - \eta\varrho\boldsymbol{u}\cdot\mathcal{R}[\xi\varrho]\big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

The right-hand side of (3.97) tends to the right-hand side of (3.98), in particular for the very last terms one can apply Corollary 2.2.5. Then the left-hand sides imply

$$\lim_{\varepsilon\to0}\int_Q \psi\eta\xi\big(p_G(\varrho_\varepsilon,\vartheta_\varepsilon) + p_R(\vartheta_\varepsilon) + \delta(\varrho_\varepsilon^\beta + \varrho_\varepsilon^2) - (\zeta_\varepsilon + \tfrac{4}{3}\mu_\varepsilon)\mathrm{div}\,\boldsymbol{u}_\varepsilon\big)(\varrho_\varepsilon - \varrho)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

$$= \lim_{\varepsilon\to0}\int_Q 2\psi\xi\Big(\mathcal{R} : [\eta\mu_\varepsilon\nabla\boldsymbol{u}_\varepsilon] - \eta\mu_\varepsilon\mathcal{R} : [\nabla\boldsymbol{u}_\varepsilon]\Big)(\varrho_\varepsilon - \varrho)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t. \tag{3.99}$$

The right-hand side tends to zero by means of Lemma 2.2.6, the interpolation, and strong convergence of the temperature. Then (3.99) reduces to

$$\lim_{\varepsilon\to0}\int_Q \varphi(\zeta(\vartheta) + \frac{4}{3}\mu(\vartheta))\mathrm{div}\boldsymbol{u}_\varepsilon(\varrho_\varepsilon - \varrho)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

$$= \lim_{\varepsilon\to0}\int_Q \varphi\big(p_G(\varrho_\varepsilon,\vartheta_\varepsilon) + \delta(\varrho_\varepsilon^\beta + \varrho_\varepsilon^2)\big)(\varrho_\varepsilon - \varrho)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \quad \text{for any } \varphi\in\mathcal{D}(Q). \tag{3.100}$$

Since $p_G(\varrho,\vartheta) + \delta(\varrho^\beta + \varrho^2)$ is a non-decreasing function of ϱ, one have

$$\big(p_G(\varrho_\varepsilon,\vartheta_\varepsilon) + \delta\varrho_\varepsilon^\beta - p_G(\varrho,\vartheta_\varepsilon) - \delta\varrho^\beta\big)(\varrho_\varepsilon - \varrho) \geq 0.$$

Then because of the strong convergence of the temperature the right-hand side of (3.100) is greater than zero and one gets

$$\overline{\varrho \mathrm{div} \boldsymbol{u}} - \varrho \mathrm{div} \boldsymbol{u} \geq 0. \tag{3.101}$$

Using Lemma 3.1.2, Lemma 3.1.3 and Lemma 3.1.4, one can check that ϱ, $\boldsymbol{u}$ satisfy

$$\partial_t(\varrho \log(\varrho)) + \mathrm{div}(\varrho \log(\varrho)\boldsymbol{u}) + \varrho \mathrm{div} \varrho = 0 \quad \text{in } \mathcal{D}'(I \times \mathbf{R}^3), \tag{3.102}$$

provided ϱ, $\boldsymbol{u}$ are extended by zero outside of Ω.

On the other hand for a $B \in C^2[0, \infty)$, one can renormalize the equation (3.68) and deduce

$$\partial_t B(\varrho_\varepsilon) + \mathrm{div}(B(\varrho_\varepsilon)\boldsymbol{u}_\varepsilon) + \big(B'(\varrho_\varepsilon)\varrho_\varepsilon - B(\varrho_\varepsilon ps)\big)\mathrm{div}\boldsymbol{u}_\varepsilon = \\ \varepsilon \mathrm{div}(1_\Omega \nabla B(\varrho_\varepsilon)) - \varepsilon 1_\Omega B''(\varrho_\varepsilon)|\nabla \varrho_\varepsilon|^2 \quad \text{in } \mathcal{D}'(I \times \mathbf{R}^3).$$

Taking a convex function $B(\varrho) = \varrho \log \varrho$, integrating over Ω and letting $\varepsilon \to 0$, we infer

$$\int_0^\tau \int_\Omega \overline{\varrho \mathrm{div} \boldsymbol{u}} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \leq \int_\Omega (\varrho \log \varrho)[0] - (\overline{\varrho \log \varrho})[\tau],$$

which combined with (3.102) and (3.101) yields

$$\int_\Omega (\varrho \log \varrho - \overline{\varrho \log \varrho})[\tau] \, \mathrm{d}\boldsymbol{x} \geq \int_0^\tau \int_\Omega \overline{\varrho \mathrm{div} \boldsymbol{u}} - \varrho \mathrm{div} \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \geq 0.$$

Then applying Theorem 2.2.3 we conclude

$$\varrho_\varepsilon \to \varrho \quad \text{in } L^1(I \times \Omega).$$

3.5.5 Limit passage - continued

With the strong convergence at hand, we can finish the passage to the limit in equations. First, we can identify $\overline{p_\delta}$, $\overline{\mathbb{S}}$ in the momentum equation with p_δ, $\mathbb{S}$, respectively. Next, in the entropy equation the epsilon terms vanish, while the other terms are equi-integrable except the measure Σ_ε. Nevertheless, Σ_ε is uniformly bounded in the space of Radon measures $\mathcal{M}(I \times \Omega)$, so up to the subsequence we have

$$\langle \Sigma_\varepsilon, \varphi \rangle \to \langle \Sigma_\delta, \varphi \rangle \quad \text{as } \varepsilon \to 0$$

for any $\varphi \in \mathcal{D}(I \times \Omega)$. Moreover,

$$\Sigma_\delta \geq \sigma_\delta = \frac{\kappa[\boldsymbol{u}]|\nabla \vartheta|^2}{\vartheta^2} + \delta \vartheta^{\beta-2}|\nabla \vartheta|, + \frac{\mathbb{S} : \nabla \boldsymbol{u}}{\vartheta}$$

in $\mathcal{M}(I \times \Omega)$ because of the lower semi-continuity of the convex terms in σ_ε.

In the energy equality, one can use a similar argument as for the convective term to pass with the kinetic energy term $\varrho|\boldsymbol{u}|^2$, the artificial pressure is equi-integrable due to (3.87) and the term $\varepsilon(\vartheta^{-\beta} - \vartheta^\beta)$ tends to zero.

Finally, we obtain a solution $(\varrho, \boldsymbol{u}, \vartheta)$ of the limit system

$$\int_I \int_\Omega B(\varrho)\partial_t \varphi + B(\varrho)\boldsymbol{u} \cdot \nabla \varphi - b(\varrho)\mathrm{div}\boldsymbol{u}\varphi \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t = 0, \quad \text{for } \varphi \in \mathcal{D}(I \times \overline{\Omega}) \tag{3.103}$$

and any B satisfying (3.7),

$$\int_I \int_\Omega \varrho \boldsymbol{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho \boldsymbol{u} \otimes \boldsymbol{u} : \nabla \boldsymbol{\varphi} + p_\delta \mathrm{div}\, \boldsymbol{\varphi} - \mathbb{S}[\boldsymbol{u}] : \nabla \boldsymbol{\varphi} + \varrho \boldsymbol{f} \cdot \boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = 0,$$
$$\text{for } \boldsymbol{\varphi} \in \mathcal{D}(I \times \Omega; \mathbf{R}^3), \quad (3.104)$$

$$\int_I \int_\Omega (\varrho s_\delta)\partial_t \varphi + (\varrho s_\delta)\boldsymbol{u} \cdot \nabla \varphi - \frac{\kappa_\delta[\boldsymbol{u}]\nabla\vartheta \cdot \nabla\varphi}{\vartheta} + \langle \Sigma_\delta, \varphi \rangle \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = 0,$$
$$\text{for } \varphi \geq 0,\ \varphi \in \mathcal{D}(I \times \overline{\Omega}), \quad (3.105)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \big(\frac{1}{2}\varrho|\boldsymbol{u}|^2 + \varrho e + \delta\big(\frac{\varrho^\beta}{\beta - 1} + \varrho^2\big)\big)[t] \,\mathrm{d}\boldsymbol{x} = \int_\Omega \varrho \boldsymbol{f} \cdot \boldsymbol{u} \,\mathrm{d}\boldsymbol{x} \quad \text{for a.e. } t \in I. \quad (3.106)$$

It remains to clarify initial conditions. In view of (3.24) and the strong convergence of the density, one can pass to the limit in the initial condition (3.69) to deduce

$$\varrho[t] \to \varrho[0] = \varrho_0 \text{ weakly in } L^1(\Omega).$$

Moreover, applying Lemma 3.1.5, one obtains even the strong convergence.

Similarly we can treat the other initial conditions (3.84), (3.78), (3.81) using the limit equations and assumptions (3.24), (3.29). We conclude

$$\left.\begin{aligned} \varrho[t] &\to \varrho[0] = \varrho_0 \quad \text{in } L^1(\Omega), \\ (\varrho\boldsymbol{u})[t] &\to (\varrho\boldsymbol{u})[0] = \varrho_0\boldsymbol{u}_0 \quad \text{weakly in } L^1(\Omega; \mathbf{R}^3), \\ E_\delta[t] &\to E_\delta[0] = E_\delta(\varrho_0, \boldsymbol{u}_0, \vartheta_0), \end{aligned}\right\} \quad (3.107)$$

and

$$\operatorname*{esslim}_{t \to 0+} \int_\Omega (\varrho s_\delta(\varrho, \vartheta))[t]\varphi \,\mathrm{d}\boldsymbol{x} \geq \lim_{\tau \to 0+} \int_0^\tau \int_\Omega (\varrho s_\delta(\varrho, \vartheta)) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_\Omega \varrho_0 s_\delta(\varrho_0, \vartheta_0)\varphi \,\mathrm{d}\boldsymbol{x} \quad (3.108)$$

for any $\varphi \in \mathcal{D}(\overline{\Omega})$, $\varphi \geq 0$.

3.6 Vanishing artificial pressure

Our ultimate goal is pass to the limit in the system (3.103 – 3.106) as $\delta \to 0$ and remove remaining artificial terms.

3.6.1 The estimates revisited

One can perform the same estimates as in Section 3.3. Integrating the total energy balance (3.106) over the time interval $[0, t]$ and testing the entropy equation (3.105) by an approximating sequence of $\varphi = 1_{[0,t]}$, we get

$$\varrho|\boldsymbol{u}|^2,\ \varrho^{\frac{5}{3}},\ \delta\varrho^\beta,\ \vartheta^4,\ \varrho s_G,\ \delta\varrho|\log\vartheta| \quad \text{bounded in } L^\infty(I; L^1(\Omega)), \quad (3.109)$$

and

$$|\nabla \boldsymbol{u}|^2,\ |\nabla \vartheta^{\frac{3}{2}}|^2,\ |\nabla \log \vartheta|^2,\ \delta|\nabla \vartheta^{\frac{\beta}{2}}|^2 \quad \text{bounded in } L^1(I\times\Omega) \tag{3.110}$$

uniformly with respect to δ.

In contrast to the previous limits, we have no uniform estimate for $\varrho \log \vartheta$. However, using the entropy inequality once again, one can proceed similarly as in (4.51), namely one can find M, $\delta^* > 0$, independent of δ, such that

$$|\{\vartheta[\tau] \geq \delta^*\}| \geq M > 0, \text{ while } \int_{\{\vartheta[\tau]\geq\delta^*\}} \log \vartheta[\tau]\ \mathrm{d}\boldsymbol{x} \leq C$$

uniformly with respect to $\tau \in I$ and δ. Then the Poincaré inequality 2.1.2 leads to

$$\boldsymbol{u},\ \vartheta^{\frac{3}{2}},\ \log\vartheta,\ \sqrt{\delta}\vartheta^{\frac{\beta}{2}} \quad \text{bounded in } L^2(I;W^{1,2}(\Omega)). \tag{3.111}$$

Using these estimates, we have also ϑ bounded in $L^{\frac{17}{3}}(I\times\Omega)$ and $\mathbb{S}$ bounded in $L^p(I\times\Omega)$ for some $p>1$.

3.6.2 Modification of pressure estimate

Similarly to Section 3.5.1, we have to derive better than L^1-estimate for the pressure. However, several modifications have to be made because of weaker uniform estimates for the density. On the other hand, we can take advantage of the renormalized continuity equation (3.103) without the inconvenient elliptic term.

We use

$$\boldsymbol{\varphi} = \psi(t)\mathcal{B}[\pi_\varepsilon^\delta],\ \pi_\varepsilon^\delta = \omega_\varepsilon \star \varrho_\delta^\nu - \fint_\Omega \omega_\varepsilon \star \varrho_\delta^\nu\ \mathrm{d}\boldsymbol{x}$$

as a test function in (3.104). For the time derivative $\partial_t\boldsymbol{\varphi}$, we shall use the mollified version of the renormalized continuity equation (3.12). Provided $\nu>0$ is sufficiently small, specifically $\nu < \frac{1}{18}$, we deduce

$$\begin{aligned}\big\|\partial_t\mathcal{B}[\pi_\varepsilon]\big\|_{L^1(I;L^5(\Omega;\mathbf{R}^3))} &\leq C\Big(\big\|(\omega_\varepsilon\star\varrho_\delta^\nu)\boldsymbol{u}_\delta\big\|_{L^1(I;L^5(\Omega;\mathbf{R}^3))} + \big\|\omega_\varepsilon\star(\varrho_\delta^\nu \mathrm{div}\boldsymbol{u}_\delta) + r_\varepsilon\big\|_{L^1(I;L^{\frac{15}{8}}(\Omega))}\\ &\quad + \int_I\int_\Omega \omega_\varepsilon\star(\varrho_\delta^\nu\mathrm{div}\boldsymbol{u}_\delta) + r_\varepsilon\ \mathrm{d}\boldsymbol{x}\Big) \leq C(\big\|\varrho_\delta^\nu\boldsymbol{u}_\delta\big\|_{L^1(L^5)} + \big\|\varrho_\delta^\nu\mathrm{div}\boldsymbol{u}_\delta\big\|_{L^1(L^{\frac{15}{8}})} + 1),\end{aligned}$$

where the right-hand side is bounded independently of δ and ε. Now, testing (3.104) by $\boldsymbol{\varphi}$, a straightforward calculation yields

$$\begin{aligned}\int_0^T\int_\Omega \psi p_\delta(\omega_\varepsilon\star\varrho_\delta^\nu)\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t = \sum_{j=1}^4 I_j &= \int_0^T \psi\int_\Omega p_\delta\ \mathrm{d}\boldsymbol{x}\fint_\Omega \omega_\varepsilon\star\varrho_\delta^\nu\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t \\ &- \int_0^T\int_\Omega \psi\varrho_\delta\boldsymbol{f}\cdot\mathcal{B}[\pi_\varepsilon^\delta] + \psi\varrho_\delta\boldsymbol{u}_\delta\otimes\boldsymbol{u}_\delta : \nabla\mathcal{B}[\pi_\varepsilon^\delta] + \partial_t\psi\varrho_\delta\boldsymbol{u}_\delta\cdot\mathcal{B}[\pi_\varepsilon^\delta]\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t \\ &+ \int_0^T\psi\int_\Omega \big(2\mu^f\mathbb{D}\boldsymbol{u}_\delta + \big(\zeta^f - \frac{2}{3}\mu^f\big)\mathrm{div}\,\boldsymbol{u}_\delta\mathbb{I}\big) : \nabla\mathcal{B}[\pi_\varepsilon^\delta]\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t \\ &- \int_0^T\psi\int_\Omega \varrho_\delta\boldsymbol{u}_\delta\cdot\partial_t\mathcal{B}[\pi_\varepsilon^\delta]\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t.\end{aligned} \tag{3.112}$$

By virtue of estimates (3.109), (3.111) the first three integrals are bounded independently of δ and ε, where the most restrictive convective term leads to the condition $\nu \leq \frac{5}{9}$. For the last integral, we have

$$I_4 \leq \left\|\varrho \boldsymbol{u}\right\|_{L^\infty(I;L^{\frac{5}{4}}(\Omega))} \left\|\partial_t \mathcal{B}[\pi_\varepsilon]\right\|_{L^1(I;L^5(\Omega)} \leq C$$

Finally, using Fatou's lemma, we get

$$\int_0^T \int_\Omega p_\delta(\varrho_\delta, \vartheta_\delta)\varrho_\delta^\nu \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \leq \liminf_{\varepsilon\to 0} \int_0^T \int_\Omega p_\delta(\varrho_\delta, \vartheta_\delta)\omega_\varepsilon \star \varrho_\delta^\nu \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \leq C, \tag{3.113}$$

whence in view of the hypothesis (3.15), we conclude

$$\left\|p_G(\varrho_\delta, \vartheta_\delta)\right\|_{L^p(I\times\Omega)} \leq C \text{ for some } p > 1 \quad \text{and} \quad \left\|\delta(\varrho^\beta + \varrho^2)\right\|_{L^1(I\times\Omega)} \leq \delta^{\frac{\nu}{\beta+\nu}} C.$$

3.6.3 Limit passage

In view of the uniform estimates (3.109 – 3.111), we can use the Alaoglu-Bourbaki theorem 2.2.1 to identify limits

$$\varrho_\delta \to \varrho \qquad \text{weakly-}* \text{ in } L^\infty(I; L^{\frac{5}{3}}(\Omega)), \tag{3.114}$$

$$\boldsymbol{u}_\delta \to \boldsymbol{u} \qquad \text{weakly in } L^2(I; W_0^{1,2}(\Omega; \mathbf{R}^3)), \tag{3.115}$$

$$\vartheta_\delta \to \vartheta \qquad \text{weakly-}* \text{ in } L^\infty(I; L^4(\Omega)). \tag{3.116}$$

as $\delta \to 0$, passing to the subsequence as the case may be. Moreover, since ϱ_δ satisfies (3.103) and $\varrho_\delta \boldsymbol{u}_\delta$ satisfies (3.104), we have even

$$\varrho_\delta \to \varrho \qquad \text{in } C([0,T]; L^{\frac{5}{3}}_{weak}(\Omega)), \tag{3.117}$$

$$B(\varrho_\delta) \to \overline{B(\varrho)} \qquad \text{in } C([0,T]; L^{p}_{weak}(\Omega)), \tag{3.118}$$

$$\varrho_\delta \boldsymbol{u}_\delta \to \varrho \boldsymbol{u} \qquad \text{in } C([0,T]; L^{\frac{5}{4}}_{weak}(\Omega; \mathbf{R}^3)) \tag{3.119}$$

for any

$$B(z) \in C^1[0,\infty),\ |B(z)| \leq 1 + z^\lambda,\ \lambda \in (0, \frac{5}{3}),$$

and $p \in (1, \frac{5}{3\lambda})$.

Then one can pass to the limit in the continuity equation, i.e. (3.103) with $B(z) = z$, and get

$$\int_I \int_{\mathbf{R}^3} \varrho \partial_t \varphi + \varrho \boldsymbol{u} \cdot \nabla \varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = 0 \quad \text{for any} \varphi \in \mathcal{D}(I \times \mathbf{R}^3). \tag{3.120}$$

Further, as $L^{\frac{5}{4}}(\Omega; \mathbf{R}^3)$ is compactly imbedded into $W^{-1,2}(\Omega; \mathbf{R}^3)$, the sequence $\varrho_\delta \boldsymbol{u}_\delta$ converge strongly in $C(I; W^{-1,2}(\Omega))$ and therefore

$$\varrho_\delta \boldsymbol{u}_\delta \otimes \boldsymbol{u}_\delta \to \varrho \boldsymbol{u} \otimes \boldsymbol{u} \text{ weakly in } L^2(I; L^{\frac{30}{29}}(\Omega; \mathbf{R}^3 \times \mathbf{R}^3)).$$

Using the equi-integrability of $\mathbb{S}$ and p_δ, the artificial pressure vanishes as we pass to the limit in the momentum equation and we get

$$\int_I \int_\Omega \varrho \boldsymbol{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho \boldsymbol{u} \otimes \boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + \overline{p}\mathrm{div}\boldsymbol{\varphi} - \overline{\mathbb{S}} : \mathbb{D}\boldsymbol{\varphi} + \varrho \boldsymbol{f} \cdot \varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = 0 \tag{3.121}$$

for any $\boldsymbol{\varphi} \in \mathcal{D}(I \times \Omega; \mathbf{R}^3)$.

If we succeed in proving the strong convergence of the temperature and the density, we can finish the limit process. We can pass to the limit in the renormalized continuity equation. In the momentum equation (3.121), we identify $\overline{p}$, $\overline{\mathbb{S}}$ with p, $\mathbb{S}$, respectively. In the entropy inequality (3.105), we use the weak lower semi-continuity of the convex term σ_δ and the equi-integrability of the other terms (see estimates in Section 3.6.4 later on). Finally, rewriting the total energy balance int the weak form (3.79), we obtain the last equation in the target system $3.2 - 3.5$.

3.6.4 Pointwise convergence of the temperature

In order to prove the strong convergence of the temperature, we use again Proposition 4.6.1. Direct use of the estimates (3.109) and (3.111) imply

$$\varrho_\delta s_\delta(\varrho_\delta, \vartheta_\delta) \text{ bounded in } L^2(I; L^p(\Omega)) \cap L^\infty(I; L^1(\Omega)),\ p = \frac{30}{23} > \frac{6}{5}, \tag{3.122}$$

whence by the interpolation, $\varrho_\delta s_\delta \boldsymbol{u}_\delta$ is bounded in some $L^p(I \times \Omega)$, $p > 1$. Further, in accordance with hypothesis (3.22)

$$\left\| \kappa(\vartheta_\delta) \vartheta_\delta^{-1} \nabla \vartheta_\delta \right\|_{L^2(I \times \Omega)} \leq C \left\| \nabla \vartheta_\delta \right\|_2 \leq C$$

and by virtue of (3.54)

$$\delta \left\| \vartheta_\delta^{\beta - 1} \nabla \vartheta_\delta \right\|_p \leq \delta^{\frac{1}{2}} \left\| \nabla (\vartheta_\delta^{\frac{\beta}{2}}) \right\|_2 \delta^{\frac{1}{2} \frac{3\beta}{3\beta + 8}} \left\| \vartheta_\delta \right\|_{\beta + \frac{8}{3}}^{\frac{3\beta + 12}{3\beta + 8}} \leq C \delta^{\frac{3\beta + 12}{3\beta + 8}} \to 0.$$

Finally, σ_δ is uniformly bounded in $L^1(I \times \Omega)$. Then Proposition 4.6.1 applied on the entropy inequality (3.105), using $\Sigma_\delta \geq \sigma_\delta$, yields

$$\varrho_\delta s(\varrho_\delta, \vartheta_\delta) \to \overline{\varrho s(\varrho, \vartheta)} \text{ in } L^2(I; W^{-1,2}(\Omega)),$$

while

$$\delta \varrho_\delta \log \vartheta_\delta \to 0 \text{ at least in } L^1(I \times \Omega).$$

Next, we use the structure of the entropy and the weak convergence of the temperature in the space $L^2(I; W^{1,2}(\Omega))$ to conclude

$$\frac{4}{3} a(\vartheta \overline{\vartheta^3} - \overline{\vartheta \vartheta^3}) = (\overline{\vartheta \varrho s_G} - \vartheta \overline{\varrho s_G}). \tag{3.123}$$

Then, exactly as in Section 3.4.2, one can deduce the strong convergence of the temperature

$$\vartheta_\delta \to \vartheta \text{ in } L^4(I \times \Omega). \tag{3.124}$$

3.6.5 Strong convergence of the density field

Compactness of the density shall be proved in the very same way as in Section 4.7. In fact it is simpler as one can work on the whole domain Ω.

Testing (3.104) by $\psi\eta\boldsymbol{\varphi}_\delta = \psi(t)\eta(\boldsymbol{x})\nabla\Delta^{-1}[\xi(\boldsymbol{x})T_k^\nu(\varrho_\delta)]$, $0<\nu\leq 1$, and using the renormalized continuity equation

$$\partial_t\big(T_k^\nu(\varrho_\delta)\big)+\mathrm{div}\big(T_k^\nu(\varrho_\delta)\boldsymbol{u}_\delta\big)+\big[(T_k^\nu)'(\varrho_\delta)\varrho_\delta-T_k^\nu(\varrho_\delta)\big]\mathrm{div}\boldsymbol{u}_\delta=0\quad\text{in }\mathcal{D}'(I\times\Omega),$$

we get

$$\begin{aligned}\int_Q\psi\xi\big(\eta p_\delta-\mathcal{R}:[\eta\mathbb{S}_\delta]\big)\big)\,T_k^\nu(\varrho_\delta)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t=\qquad(3.125)\\ \int_Q\psi(\mathbb{S}_\delta-\varrho_\delta\boldsymbol{u}_\delta\otimes\boldsymbol{u}_\delta):\nabla\eta\otimes\boldsymbol{\varphi}_\delta-\psi p_\delta\nabla\eta\cdot\boldsymbol{\varphi}_\delta)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ -\int_Q\eta\varrho_\delta\boldsymbol{u}_\delta\cdot\big(\partial_t\psi\boldsymbol{\varphi}_\delta+\psi\nabla\Delta^{-1}[\nabla\xi\cdot T_k^\nu(\varrho_\delta)\boldsymbol{u}_\delta]\big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ -\int_Q\psi\eta\varrho_\delta\boldsymbol{u}_\delta\cdot\nabla\Delta^{-1}[\xi((T_k^\nu)'(\varrho_\delta)\varrho_\delta-T_k^\nu(\varrho_\delta))\mathrm{div}\,\boldsymbol{u}_\delta]+\psi\eta\boldsymbol{\varphi}_\delta\cdot(\varrho_\delta\boldsymbol{f})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ +\int_Q\psi\boldsymbol{u}_\delta\cdot\big(\mathcal{R}\cdot[\eta\varrho_\delta\boldsymbol{u}_\delta]\xi T_k^\nu(\varrho_\delta)-\eta\varrho_\delta\boldsymbol{u}_\delta\cdot\mathcal{R}[\xi T_k^\nu(\varrho_\delta)]\big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.\end{aligned}$$

On the other hand, testing (3.121) by $\psi\eta\overline{\boldsymbol{\varphi}}=\psi(t)\eta(\boldsymbol{x})\nabla\Delta^{-1}[\xi(\boldsymbol{x})\overline{T_k^\nu(\varrho)}]$, $0<\nu\leq 1$ and using L^1-limit of the renormalized continuity equation

$$\partial_t\big(\overline{T_k^\nu(\varrho)}\big)+\mathrm{div}\big(\overline{T_k^\nu(\varrho)}\boldsymbol{u}\big)+\overline{\big[(T_k^\nu)'(\varrho)\varrho-T_k^\nu(\varrho)\big]\mathrm{div}\boldsymbol{u}}=0\quad\text{in }\mathcal{D}(I\times\Omega),$$

we get

$$\begin{aligned}\int_Q\psi\xi\big(\overline{\eta p-\mathcal{R}:[\eta\mathbb{S}]}\big)\,\overline{T_k^\nu(\varrho)}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t=\qquad(3.126)\\ \int_Q\psi(\overline{\mathbb{S}}-\varrho\boldsymbol{u}\otimes\boldsymbol{u}):\nabla\eta\otimes\overline{\boldsymbol{\varphi}}-\psi\overline{p}\nabla\eta\cdot\overline{\boldsymbol{\varphi}})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ -\int_Q\eta\varrho\boldsymbol{u}\cdot\big(\partial_t\psi\overline{\boldsymbol{\varphi}}+\psi\nabla\Delta^{-1}[\nabla\xi\cdot\overline{T_k^\nu(\varrho)}\boldsymbol{u}]\big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ -\int_Q\psi\eta\varrho\boldsymbol{u}\cdot\nabla\Delta^{-1}[\xi\overline{((T_k^\nu)'(\varrho)\varrho-T_k^\nu(\varrho))\mathrm{div}\,\boldsymbol{u}}]+\psi\eta\overline{\boldsymbol{\varphi}}\cdot(\varrho\boldsymbol{f})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ +\int_Q\psi\boldsymbol{u}\cdot\big(\mathcal{R}\cdot[\eta\varrho\boldsymbol{u}]\xi\overline{T_k^\nu(\varrho)}-\eta\varrho\boldsymbol{u}\cdot\mathcal{R}[\xi\overline{T_k^\nu(\varrho)}]\big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.\end{aligned}$$

In view of the available estimates and using Corollary 2.2.5, the right-hand side of (3.125) tends to the right-hand side of (3.126) and the left-hand sides yields

$$\begin{aligned}\lim_{\delta\to0}\int_Q\psi\eta\xi\big(p_G(\varrho_\delta,\vartheta_\delta)+p_R(\vartheta_\delta)-(\zeta_\delta+\tfrac{4}{3}\mu_\delta)\mathrm{div}\,\boldsymbol{u}_\delta\big)\big(T_k^\nu(\varrho_\delta)-\overline{T_k^\nu(\varrho)}\big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ =\lim_{\delta\to0}\int_Q\psi\xi\Big(\mathcal{R}:[\eta\mu_\delta\nabla\boldsymbol{u}_\delta]-\eta\mu_\delta\mathcal{R}:[\nabla\boldsymbol{u}_\delta]\Big)\big(T_k^\nu(\varrho_\delta)-\overline{T_k^\nu(\varrho)}\big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.\qquad(3.127)\end{aligned}$$

The right-hand side tends to zero by means of Lemma 2.2.6, the interpolation, and strong convergence of the temperature, thus (3.127) reduces to

$$\begin{aligned}\lim_{\delta\to0}\int_Q\varphi(\zeta(\vartheta)+\frac{4}{3}\mu(\vartheta))\mathrm{div}\boldsymbol{u}_\delta\big(T_k^\nu(\varrho_\delta)-\overline{T_k^\nu(\varrho)}\big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ =\lim_{\delta\to0}\int_Q\varphi\big(p_G(\varrho_\delta,\vartheta_\delta)\big)\big(T_k^\nu(\varrho_\delta)-\overline{T_k^\nu(\varrho)}\big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\quad\text{for any }\varphi\in\mathcal{D}(Q).\qquad(3.128)\end{aligned}$$

Using essentially this relation and hypothesis (3.14), we can proceed exactly as in Section 4.7.2 and deduce the bound for the oscillation defect measure

$$\mathrm{osc}_r[\varrho_\delta \to \varrho] := \sup_{k\geq 1} \limsup_{\delta\to 0} \int_Q |T_k(\varrho_\delta) - T_k(\varrho)|^r \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \leq C \quad \text{for some } r > 2. \tag{3.129}$$

In the next step, we prove that ϱ, $\boldsymbol{u}$ solve the renormalized continuity equation. Taking $B(z) = T_k(z)$ in (3.103), then passing to the limit and applying Proposition 3.1.2, we arrive at

$$\partial_t(B(\overline{T_k(\varrho)})) + \mathrm{div}(B(\overline{T_k})\boldsymbol{u}) + b(\overline{T_k(\varrho)})\mathrm{div}\boldsymbol{u} = B'(\overline{T_k(\varrho)})\overline{\big[T_k(\varrho) - T_k'(\varrho)\varrho\big]\mathrm{div}\boldsymbol{u}} \text{ in } \mathcal{D}'(I\times\Omega) \tag{3.130}$$

for any $B(z)$ satisfying (3.7). The weak lower semicontinuity of the norm yields

$$\big\|\overline{T_k(\varrho)} - \varrho\big\|_{L^1(Q)} \leq \liminf_{\delta\to 0}\big\|T_k(\varrho_\delta) - \varrho_\delta\big\|_{L^1(Q)} \leq \sup_\delta \big\|\varrho_\delta\big\|_{L^1(\{\varrho_\delta\geq k\})} \leq k^{-\frac{2}{5}} \sup_\delta \big\|\varrho_\delta\big\|_{L^{\frac{5}{3}}(Q)}.$$

Hence we have

$$B(\overline{T_k(\varrho)}) \to B(\varrho),\ b(\overline{T_k(\varrho)}) \to b(\varrho) \text{ in any } L^p(Q),\ p\geq 1, \tag{3.131}$$

as $k\to\infty$. Then it remains to prove that the right-hand side of (3.130) tends to zero. To this end, we estimate

$$\begin{aligned}&\big\|B'(\overline{T_k(\varrho)})\overline{\big[T_k(\varrho) - T_k'(\varrho)\varrho\big]\mathrm{div}\boldsymbol{u}}\big\|_{L^1(Q)}\\ &\leq \max_{0\leq z\leq M}|B'(z)|\ \sup_\delta\big\|\mathrm{div}\boldsymbol{u}_\delta\big\|_{L^2(Q)}\ \liminf_{\delta\to 0}\big\|T_k(\varrho_\delta) - T_k'(\varrho_\delta)\varrho_\delta\big\|_{L^2(Q_M)},\end{aligned}$$

where $Q_M = \{\overline{T_k(\varrho)} \leq M\}$. We shall continue by interpolation of the last term

$$\big\|T_k(\varrho_\delta) - T_k'(\varrho_\delta)\varrho_\delta\big\|_{L^2(Q_M)} \leq \big\|T_k(\varrho_\delta) - T_k'(\varrho_\delta)\varrho_\delta\big\|^{\lambda}_{L^1(Q)}\big\|T_k(\varrho_\delta)\big\|^{1-\lambda}_{L^p(Q_M)} \tag{3.132}$$

for certain $p > 2$ and $\lambda\in(0,1)$. Now the first norm tends to zero, since

$$\big\|T_k(\varrho_\delta) - T_k'(\varrho_\delta)\varrho_\delta\big\|_{L^1(Q)} \leq \int_{\{\varrho_\delta\geq k\}} \varrho_\delta\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \leq k^{-\frac{2}{5}}\sup_\delta\big\|\varrho_\delta\big\|_{L^{\frac{5}{3}}(Q)},$$

while the second is bounded

$$\begin{aligned}\limsup_{n\to\infty}\big\|T_k(\varrho_\delta)\big\|_{L^p(Q_M)} &\leq \limsup_{n\to\infty}\big\|T_k(\varrho_\delta) - T_k(\varrho)\big\|_{L^p(Q)} + \big\|T_k(\varrho) - \overline{T_k(\varrho)}\big\|_{L^p(Q)}\\ &\quad + \big\|\overline{T_k(\varrho)}\big\|_{L^p(Q_M)} \leq 2\,\mathrm{osc}_p[\varrho_\delta\to\varrho](Q) + M|Q|^{\frac{1}{p}}.\end{aligned}$$

Finally, we can use Proposition 3.1.3 to extend the class of valid B-functions.

In particular, the function

$$B(z) = L_k(z)z,\ L_k := \int_1^z \frac{T_k(s)}{s^2}\,\mathrm{d}s,$$

can be used in the renormalized equation (3.2). Taking a difference of (3.2) and the weak limit of the equation (3.103) for ϱ_δ, $\boldsymbol{u}_\delta$, we obtain

$$\int_\Omega \big[\overline{\varrho L_k(\varrho)} - \varrho L_k(\varrho)\big](\tau)\ \mathrm{d}\boldsymbol{x} = \int_\Omega \big[\overline{\varrho L_k(\varrho)} - \varrho L_k(\varrho)\big](0)\ \mathrm{d}\boldsymbol{x} + \int_0^\tau \int_\Omega \big(T_k(\varrho) - \overline{T_k(\varrho)}\big)\mathrm{div}\boldsymbol{u}\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t + \int_0^\tau \int_\Omega \overline{T_k(\varrho)}\mathrm{div}\boldsymbol{u} - \overline{T_k(\varrho)\mathrm{div}\boldsymbol{u}}\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t. \quad (3.133)$$

The first term on the right-hand side is in fact zero as $\varrho_\delta[0] = \varrho_0$, the second term tends to zero as $k \to \infty$ by the same argument as above, namely

$$\big\|T_k(\varrho) - \overline{T_k(\varrho)}\big\|_{L^2(Q)} \leq \liminf_{\delta\to 0}\big\|T_k(\varrho) - T_k(\varrho_\delta)\big\|_1^\lambda (\mathrm{osc}_p[\varrho_\delta \to \varrho](Q))^{1-\lambda} \leq Ck^{-\frac{2}{5}}$$

for suitable $\lambda \in (0,1)$, $p > 2$. For the third term, we use the monotonicity of pressure with respect to the density and (3.128) with $\nu = 1$ to deduce

$$\overline{T_k(\varrho)}\mathrm{div}\boldsymbol{u} - \overline{T_k(\varrho)\mathrm{div}\boldsymbol{u}} \leq 0 \quad \text{a.e. on } Q.$$

Then, passing to the limit in (3.133) as $k \to \infty$, we get

$$\int_\Omega (\overline{\varrho \log \varrho} - \varrho \log \varrho)(\tau)\ \mathrm{d}\boldsymbol{x} \leq 0,$$

which implies the pointwise convergence of the density,

$$\varrho_n \to \varrho \quad \text{a.e. on} Q. \quad (3.134)$$

Chapter 4

Evolution of solid-fluid system

This chapter is devoted to the existence theory for the problem of rigid bodies drifted in a compressible fluid. More specifically, we will show that one can get variational solutions to this problem as a limit of solutions to the Navier-Stokes-Fourier system with spatially dependent viscosities approaching infinity on the regions corresponding to the bodies. This clever penalization method has been used by Conca, San Martin, Tucsnak [7] and San Martin, Starovoitov,Tucsnak [30] to treat a similar problem for an incompressible fluid. Later, the same method was used by Feireisl [15] for bodies in a compressible fluid in the isentropic regime. Our aim is to extend this result to the case of a general heat conducting gas in the spirit of the theory discussed in Chapter 3.

In the first section, we derive the definition of variational solutions and we state the main existence result. Then in Section 4.2, we use Theorem 3.1.6 with suitably chosen transport coefficients to construct a sequence of approximate solutions. The displacement mappings, which describe the motion of approximate bodies, are constructed in Section 4.3. Then we derive necessary estimates in Section 4.4 and prove the strong convergence of the temperature, Section 4.6, and the density, Section 4.7. This allows us to pass to the limit in the equations within Sections 4.5 and 4.8 and finish the proof of the main result.

During the work on my thesis, it appears that performing the high viscosity limit as the last step is probably not optimal. It seems that one can obtain a better result inserting the high viscosity limit before the vanishing viscosity limit similarly as in [15]. Some ideas in this direction are presented in the last section 4.9.

4.1 Variational formulation

In the classical formulation, which was outlined in Chapter 1, there are separated equations and state quantities for the fluid and for the solid region. The fluid-solid interaction was represented by the continuity of the velocity and the temperature over the (smooth) boundary of the bodies and also by the continuity of the stress and the heatflux in the normal direction. The weak solutions need not to be continuous, therefore the interaction has to be treated differently. We shall join the integral formulations of the equations for the fluid and for the

solid part. Because of the continuity conditions on the boundary of the bodies, the boundary integrals mutually vanish and we obtain unified weak formulation of the balance laws on the whole domain Ω.

Description of the bodies and their motion is similar to the classical formulation. We shall call the *displacement (mapping)* a family of diffeomorphisms $\boldsymbol{\eta}[t]$, $\boldsymbol{\eta} \in AC(I; C_{loc}(\mathbf{R}^3))$, which describes motion of particles, cf. (1.1). Further, we shall say that $\boldsymbol{\eta}$ is the *rigid displacement (mapping)* if it is an affine isometry (1.3).

For the weak solution the velocity need not be continuous. Consequently the relation (1.4) has to be revisited. We shall say that a (velocity) field $\boldsymbol{u}$ is *compatible* with $\{\boldsymbol{\eta}, \overline{S}\}$, if $\boldsymbol{\eta}$ is a displacement, $\overline{S}$ is an evolution of the compact set (the body) $\overline{S}[t] = \boldsymbol{\eta}[t](\overline{S}_0)$, and it holds

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\eta}[t]\big(\boldsymbol{\eta}[-t](\,\cdot\,)\big) = \boldsymbol{u}(t,\,\cdot\,) \quad \text{on } \overline{S}[t]. \tag{4.1}$$

The equality in (4.1) has to be understood in the sense of corresponding spaces. More specifically, as the velocity lives in the space $W^{1,2}(\Omega; \mathbf{R}^3)$ the equality $\boldsymbol{v} = \boldsymbol{u}$ on the compact set S means $\boldsymbol{u} - \boldsymbol{v} \in W^{1,2}_0(\mathbf{R}^3 \setminus S; \mathbf{R}^3)$. We shall also say that $\boldsymbol{u}$ is compatible with $\boldsymbol{\eta}$ if (4.1) holds on the whole domain Ω.

In what follows, we consider the motion of fluid and rigid bodies $\overline{S}^i$, $i = 1, \dots, N$ in the domain $\Omega \subset \mathbf{R}^3$ during the time interval $I = (0, T)$. The motion of the bodies is given by the rigid displacements $\boldsymbol{\eta}^i$ through the formula $\overline{S}^i[t] := \boldsymbol{\eta}^i[t](\overline{S}^i_0)$, where $\overline{S}^i_0$ is the body in the initial position. The bodies $\overline{S}^i[t]$ are compact connected sets with non-empty interior and boundary of zero measure for all times $t \in I$. In terms of $\boldsymbol{\eta}^i$ we introduce domains

$$\left.\begin{aligned} Q &:= I \times \Omega, & Q^i &:= \{(t, \boldsymbol{x}) \,|\, t \in I, \boldsymbol{x} \in \overline{S}^i(t)\}, \\ Q^s &:= \bigcup_{i=1}^N Q^i, & Q^f &:= Q \setminus Q^s, \\ \Omega^s[t] &:= \bigcup_{i=1}^N S^i[t], & \Omega^f[t] &:= \Omega \setminus \overline{\Omega}^s[t]. \end{aligned}\right\} \tag{4.2}$$

We merge the state quantities as follows

$$\varrho(t, \boldsymbol{x}) = \begin{cases} \varrho^f(t, \boldsymbol{x}) & \text{on } Q^f, \\ \varrho^{S^i}(\boldsymbol{\eta}^i[-t](\boldsymbol{x})) & \text{on } Q^i, \\ 0 & \text{on } \mathbf{R}^3 \setminus \Omega, \end{cases}$$

$$\boldsymbol{u}(t, \boldsymbol{x}) = \begin{cases} \boldsymbol{u}^f(t, \boldsymbol{x}) & \text{on } Q^f, \\ \boldsymbol{u}^{S^i}(t, \boldsymbol{x}) = \boldsymbol{V}(t) + \boldsymbol{\omega}(t) \times (\boldsymbol{x} - \boldsymbol{X}_{S^i}(t)) & \text{on } Q^i, \\ 0 & \text{on } \mathbf{R}^3 \setminus \Omega, \end{cases}$$

$$\vartheta(t, \boldsymbol{x}) = \begin{cases} \vartheta^f(t, \boldsymbol{x}) & \text{on } Q^f, \\ \vartheta^i(t, \boldsymbol{r}),\ \boldsymbol{r} = \boldsymbol{\eta}^i[-t](\boldsymbol{x}) - X^i(0) & \text{on } Q^i. \end{cases}$$

To keep the consistency between the global velocity $\boldsymbol{u}$ and the bodies $\overline{S}[t]$, we require that $\boldsymbol{u}$ is compatible with $\{\boldsymbol{\eta}^i, \overline{S}^i\}$ for all $i = 1, \dots N$. This condition, which can also be viewed as an expresion of the boundary conditions (1.41), is enough to ensure the impermeability of the bodies even in the case of weak solution.

Lemma 4.1.1. *[15, Lemma 3.1] Let the velocity $\boldsymbol{u}$ be compatible with $\{\boldsymbol{\eta}^i, \overline{S}^i\}$ for $i = 1, 2$. Define $S^i := \left(\overline{S}^i\right)^\circ$, $i = 1, 2$. Then either $S^1[t] \cap S^2[t] = \emptyset$ for all $t \in [0, T]$ or $S^1[t] \cap S^2[t] \neq \emptyset$ and $\boldsymbol{\eta}^1[t] = \boldsymbol{\eta}^2[t]$ for all $t \in [0, T]$.*

Proof. Assume that there exists $\boldsymbol{y} \in S^1[\tau] \cap S^2[\tau]$ for some $\tau \in [0, T]$. Since $\boldsymbol{\eta}^i$ are continuous and $S^i[t]$ are open, we can find ε such that

$$B_\varepsilon(\boldsymbol{y}) \subset S^1[t] \cap S^2[t] \quad \forall\, t \in B_\varepsilon(\tau).$$

Further, $\boldsymbol{\eta}^i$ are both compatible with the same velocity and they are rigid displacements, thus they obey (1.4). Consequently it holds

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{X}^1(t) + \mathbb{Q}^1[t](\boldsymbol{x} - \boldsymbol{X}^1(t)) = \tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{X}^2(t) + \mathbb{Q}^2[t](\boldsymbol{x} - \boldsymbol{X}^2(t)) \quad \forall\, t \in B_\varepsilon(\tau),\ \boldsymbol{x} \in B_\varepsilon(\boldsymbol{y}).$$

Then for every $t \in B_\varepsilon(\tau)$ it must be $\mathbb{Q}^1[t] = \mathbb{Q}^2[t] = \mathbb{Q}[t]$ while $\boldsymbol{X}^i$ satisfy

$$\tfrac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{X}^1(t) - \boldsymbol{X}^2(t)) = \mathbb{Q}[t](\boldsymbol{X}^1(t) - \boldsymbol{X}^2(t)). \tag{4.3}$$

In accordance with (1.3) the displacement between times s and t have a form

$$\boldsymbol{\eta}[s \to t](\boldsymbol{x}) := \boldsymbol{\eta}[t]\big(\boldsymbol{\eta}[-s](\boldsymbol{x})\big) = \boldsymbol{X}(t) + \mathbb{O}[s \to t](\boldsymbol{x} - \boldsymbol{X}(s))$$

where $\mathbb{O}[s \to t] := \mathbb{O}[t]\mathbb{O}^{-1}[s]$. As (1.3) is solution of (1.4) the solution of (4.3) reads

$$(\boldsymbol{X}^1(t) - \boldsymbol{X}^2(t)) = \mathbb{O}[\tau \to t](\boldsymbol{X}^1(\tau) - \boldsymbol{X}^2(\tau))$$

Now it is easy to see that $\boldsymbol{\eta}^1[\tau \to t] = \boldsymbol{\eta}^2[\tau \to t]$ for every $t \in B_\varepsilon(\tau)$. Finally, using the continuity of $\boldsymbol{\eta}^i$, we can extend this equality to the whole interval $[0, T]$. □

4.1.1 Continuity equation

In order to derive unified continuity equation, we apply the transport theorem 1.1.1 on a the quantity $\varrho\varphi$, $\varphi \in \mathcal{D}(I \times \mathbf{R}^3)$ considering successively the domains $\Omega^f[t]$ and $\Omega^s[t]$. Since the function φ is compactly supported in $(0, T)$ we get

$$\begin{aligned}
0 &= \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega^f(t)} \varrho\varphi \,\mathrm{d}\boldsymbol{x} + \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega^s(t)} \varrho\varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&= \int_0^T \int_{\Omega^f(t)} \varrho\partial_t\varphi + \varrho\boldsymbol{u}\cdot\nabla\varphi + [\partial_t\varrho + \operatorname{div}(\varrho\boldsymbol{u})]\varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&+ \int_0^T \int_{\Omega^s(t)} \varrho\partial_t\varphi + \varrho\boldsymbol{u}\cdot\nabla\varphi + [\partial_t\varrho + \operatorname{div}(\varrho\boldsymbol{u})]\varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.
\end{aligned}$$

By virtue of (1.7) and (1.8), the square brackets are equal to zero and the unified equation follows:

$$\int_0^T \int_{\mathbf{R}^3} \varrho\partial_t\varphi + \varrho\boldsymbol{u}\nabla\varphi \;\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = 0 \qquad \varphi \in \mathcal{D}(I \times \mathbf{R}^3). \tag{4.4}$$

Conversely, provided ϱ, $\boldsymbol{u}$ are solutions of (4.4) smooth in Q^f, one can test this equation by any φ supported in Q^f to get (1.7). On the other hand, the density on the solid region is perfectly propagated even in the case of weak solution, in particular we claim:

Lemma 4.1.2. *[15, Lemma 3.2] Let ϱ, $\boldsymbol{u}$ satisfy the continuity equation* (4.4)*,*

$$\varrho \in L^\infty(I; L^\gamma(\Omega)), \quad \boldsymbol{u} \in L^2(I; W^{1,2}_0(\Omega)), \quad \gamma > 1, \tag{4.5}$$

and let $\boldsymbol{u}$ be compatible with $\{\boldsymbol{\eta}, \overline{S}\}$, where $\boldsymbol{\eta}$ is a rigid displacement. Then

$$\varrho(t, \boldsymbol{\eta}[t](\boldsymbol{x})) = \varrho(0, \boldsymbol{x}) \quad \textit{for a.e. } \boldsymbol{x} \in \big(\overline{S}(0)\big)^\circ \textit{ and any } t \in [0, T].$$

Proof. To begin with, let us recall that

$$\boldsymbol{u}(t, \boldsymbol{x}) = \boldsymbol{u}^i(t, \boldsymbol{x}) = \boldsymbol{V}(t) + \mathbb{Q}(t)(\boldsymbol{x} - \boldsymbol{X}(t)) \text{ on every } S^i[t], \tag{4.6}$$

where obviously $\boldsymbol{u}^i \in L^2(I; W^{1,\infty}(\mathbf{R}^3))$ and $\mathbb{D}\boldsymbol{u}^i = 0$. Next, we use the regularizing kernels $\omega_\delta(\boldsymbol{y} - \boldsymbol{x})$ as a test functions in (4.4). For the fixed time $t \in I$, we get

$$\begin{aligned}\big(\partial_t(\omega_\delta \star \varrho) + \nabla(\omega_\delta \star \varrho) \cdot \boldsymbol{u}\big)[\boldsymbol{y}] &= \int_{\mathbf{R}^3} \varrho(\boldsymbol{x})(\boldsymbol{u}(\boldsymbol{y}) - \boldsymbol{u}(\boldsymbol{x})) \cdot \nabla\omega_\delta(\boldsymbol{y} - \boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \\ &= \int_{\mathbf{R}^3} \varrho(\boldsymbol{x} - \boldsymbol{z}) \frac{\boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{u}(\boldsymbol{x} - \boldsymbol{z})}{|\boldsymbol{z}|} \cdot \nabla\omega_\delta(\boldsymbol{z}) |\boldsymbol{z}| \,\mathrm{d}z \end{aligned} \tag{4.7}$$

for any $\boldsymbol{y} \in K_\delta[t]$, $\operatorname{dist}(K_\delta[t], \Omega^f[t]) > \delta$. Now, for any ball $\overline{B[0]} \subset \Omega^s[0]$ we can find δ such that $B[t] \subset K_\delta[t]$ for all times $t \in I$. Thus (4.7) implies

$$\left| \int_{B[t]} \omega_\delta \star \varrho[t] \,\mathrm{d}y - \int_{B[0]} \omega_\delta \star \varrho[0] \,\mathrm{d}y \right| \le \int_0^t \int_{B[s]} |B_\delta(\boldsymbol{y})|^{\frac{1}{\gamma}} \big\|\varrho[s]\big\|_\gamma \big\|\boldsymbol{u}[s]\big\|_{1,\infty} \,\mathrm{d}y \,\mathrm{d}s. \tag{4.8}$$

Passing to the limit for $\delta \to 0$, the right-hand side tends to zero because of (4.5) and (4.6), while the left-hand side yields

$$\int_{B[t]} \varrho[t] \,\mathrm{d}\boldsymbol{x} = \int_{B[0]} \varrho[0] \,\mathrm{d}\boldsymbol{x}.$$

Then, using the Lebesgue point property, we conclude

$$\varrho(t, \eta^i[t](\boldsymbol{x})) = \varrho(0, \boldsymbol{x}) \quad \text{for a.e. } \boldsymbol{x} \in S^i(0).$$

□

4.1.2 Momentum equation

Taking the scalar product of (1.12) with the fixed vector $\boldsymbol{a}$ one gets

$$\begin{aligned}\boldsymbol{a} \cdot \Big(\frac{\mathrm{d}}{\mathrm{d}t} \int_{S(t)} \varrho \underbrace{(\boldsymbol{V} + \boldsymbol{r} \times \boldsymbol{\omega})}_{\boldsymbol{u}} \,\mathrm{d}\boldsymbol{r} \Big)[\tau] &= \int_{S(\tau)} \boldsymbol{a} \cdot \Big[\partial_t(\varrho\boldsymbol{u}) + \operatorname{div}(\varrho\boldsymbol{u} \otimes \boldsymbol{u}) \Big] \,\mathrm{d}x = \\ &= \boldsymbol{a} \cdot \int_{\partial S(\tau)} \mathbb{T}\boldsymbol{n} \,\mathrm{d}\sigma + \boldsymbol{a} \cdot \int_{S(\tau)} \varrho\boldsymbol{f} \,\mathrm{d}r.\end{aligned}$$

Similarly (1.13) multiplied by the vector $\boldsymbol{b}$ yields

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t}\int_{S(t)}\varrho\big(\boldsymbol{r}\times\underbrace{(\boldsymbol{V}+\boldsymbol{r}\times\boldsymbol{\omega})}_{\boldsymbol{u}}\big)\cdot\boldsymbol{b}\,\mathrm{d}\boldsymbol{r}\Big)[\tau]=\Big(\frac{\mathrm{d}}{\mathrm{d}t}\int_{S(t)}(\boldsymbol{b}\times\boldsymbol{r})\cdot\varrho\boldsymbol{u}\,\mathrm{d}\boldsymbol{r}\Big)[\tau]=$$
$$=\int_{S(\tau)}(\boldsymbol{b}\times\boldsymbol{r})\cdot\big[\partial_t(\varrho\boldsymbol{u})+\operatorname{div}(\varrho\boldsymbol{u}\otimes\boldsymbol{u})\big]\underbrace{\nabla(\boldsymbol{b}\times\boldsymbol{r}):\varrho\boldsymbol{u}\otimes\boldsymbol{u}}_{0}\,\mathrm{d}\boldsymbol{r}=$$
$$=\int_{\partial S(\tau)}(\boldsymbol{b}\times\boldsymbol{r})\cdot\mathbb{T}\boldsymbol{n}\,\mathrm{d}\sigma+\int_{S(\tau)}(\boldsymbol{b}\times\boldsymbol{r})\cdot\varrho\boldsymbol{f}\,\mathrm{d}r,$$

where we have used the fact that the scalar product of the symmetric tensor $\varrho\boldsymbol{u}\otimes\boldsymbol{u}$ and the antisymmetric tensor $\nabla(\boldsymbol{b}\times\boldsymbol{r})$ is zero. Adding up both equations and using substitution $\boldsymbol{r}=\boldsymbol{x}-\boldsymbol{X}_S(\tau)$, we obtain

$$\int_{S(\tau)}\big[\partial_t(\varrho\boldsymbol{u})^s+\operatorname{div}(\varrho\boldsymbol{u}\otimes\boldsymbol{u})^s\big]\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}=\int_{\partial S(\tau)}\mathbb{T}^f\boldsymbol{n}\cdot\boldsymbol{\varphi}+\int_{S(\tau)}\varrho^s\boldsymbol{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}\tag{4.9}$$

for every $\tau\in I$ and $\boldsymbol{\varphi}=\boldsymbol{a}+\boldsymbol{b}\times(\boldsymbol{x}-\boldsymbol{X}_S(\tau))$. Equivalently, one can take any $\boldsymbol{\varphi}\in\mathcal{D}(\mathbf{R}^3)$, $\mathbb{D}\boldsymbol{\varphi}=0$ on $S(\tau)$, which is the statement of the following lemma.

Lemma 4.1.3. *Let M be a domain in $\mathbf{R}^3$ then a set*

$$A:=\{\boldsymbol{f}\in W^{1,1}(M)\,|\,\mathbb{D}\boldsymbol{f}=0\ \textit{for a.e.}\ \boldsymbol{x}\in M\}$$

coincide with a set

$$B:=\{f(\boldsymbol{x})=\boldsymbol{a}+\boldsymbol{x}\times\boldsymbol{b}\,|\,\boldsymbol{a},\boldsymbol{b}\in\mathbf{R}^3\}.$$

Proof. Apparently $B\subset A$. On the other hand every $\boldsymbol{f}$ in A have an antisymmetric gradient. Hence taking second derivatives in the sense of distribution we observe

$$0=\partial_j\partial_i\boldsymbol{f}_i=\partial_i\partial_j\boldsymbol{f}_i=-\partial_i\partial_i\boldsymbol{f}_j$$

and

$$\partial_i\partial_j\boldsymbol{f}_k=-\partial_k\partial_i\boldsymbol{f}_j=\partial_j\partial_k f_i=-\partial_i\partial_j\boldsymbol{f}_k.$$

Thus $\nabla\nabla\boldsymbol{f}=0$. Consequently, $\nabla\boldsymbol{f}$ is a constant antisymmetric tensor and $\boldsymbol{f}$ has a form $\boldsymbol{f}(\boldsymbol{x})=\boldsymbol{a}+\mathbb{Q}\boldsymbol{x}$ for some constant vector $\boldsymbol{a}$ and antisymmetric tensor $\mathbb{Q}$. Since the antisymmetric tensors operate like the vector product, we conclude $\boldsymbol{f}\in B$. □

Similarly as in the previous section, we apply the transport theorem on the quantity $\varrho\boldsymbol{u}\cdot\boldsymbol{\varphi}$, successively on the domains $\Omega^f[t]$ and $\Omega^s[t]$. In view of Lemma 4.1.3, we choose $\boldsymbol{\varphi}\in\mathcal{T}$, where

$$\mathcal{T}:=\{\boldsymbol{\varphi}\in\mathcal{D}(Q)\,|\,\mathbb{D}\boldsymbol{\varphi}=0\text{ on some open neighborhood of }Q^s\}.\tag{4.10}$$

Using (1.10), (4.9), and the Green theorem we compute

$$\begin{aligned}0=&\int_0^T\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^f(t)\cap\Omega^s(t)}\varrho\boldsymbol{u}\cdot\boldsymbol{\varphi}\;\mathrm{d}\boldsymbol{x}\;\mathrm{d}t=\int_0^T\int_{\Omega^f(t)}(\varrho\boldsymbol{u})^f\partial_t\boldsymbol{\varphi}+(\varrho\boldsymbol{u}\otimes\boldsymbol{u})^f:\nabla\boldsymbol{\varphi}\;\mathrm{d}\boldsymbol{x}\;\mathrm{d}t\\&+\int_0^T\int_{\Omega^f(t)}-\nabla\boldsymbol{\varphi}:\mathbb{T}^f+\boldsymbol{\varphi}\cdot\varrho^f\boldsymbol{f}\;\mathrm{d}\boldsymbol{x}+\int_{\partial\Omega}\boldsymbol{\varphi}(\mathbb{T}^f\boldsymbol{n})\,\mathrm{d}\sigma-\int_{\partial\Omega^s(t)}\boldsymbol{\varphi}(\mathbb{T}^f\boldsymbol{n})\,\mathrm{d}\sigma\;\mathrm{d}t\\&+\int_0^T\int_{\Omega^s(t)}(\varrho\boldsymbol{u})^s\partial_t\boldsymbol{\varphi}+(\varrho\boldsymbol{u}\otimes\boldsymbol{u})^s:\nabla\boldsymbol{\varphi}+\boldsymbol{\varphi}\cdot\varrho^s\boldsymbol{f}\;\mathrm{d}\boldsymbol{x}\;\mathrm{d}t+\int_{\partial\Omega^s(t)}\boldsymbol{\varphi}(\mathbb{T}^f\boldsymbol{n})\;\mathrm{d}\sigma.\end{aligned}\tag{4.11}$$

Since the scalar product of the symmetric and the anti-symmetric tensor is zero, we conclude

$$\int_Q (\varrho \boldsymbol{u}) \cdot \partial_t \boldsymbol{\varphi} + (\varrho \boldsymbol{u} \otimes \boldsymbol{u}) : \mathbb{D}\boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t = \int_Q \mathbb{T} : \mathbb{D}\boldsymbol{\varphi} - \varrho \boldsymbol{f} \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \qquad \boldsymbol{\varphi} \in \mathcal{T}. \tag{4.12}$$

Remark 4.1.4. *Later, when we derive suitable estimates, one can use a density argument to extend the set of admissible test function up to*

$$\tilde{\mathcal{T}} = \{\boldsymbol{\varphi}_1 + \boldsymbol{\varphi}_2 \,|\, \boldsymbol{\varphi}_1 \in \mathcal{T}, \boldsymbol{\varphi}_2 \in W_0^{1,p}(Q^f)\} \tag{4.13}$$

for some p large enough.

Conversely, if $(\varrho,\ \boldsymbol{u})$ is a regular solution of (4.12), we can recover (1.10) on Q^f taking the test functions $\boldsymbol{\varphi}$ with $\operatorname{supp} \boldsymbol{\varphi} \subset Q^f$ in the equation (4.11). Similarly, one can take a sequence $\boldsymbol{\varphi}_n \in \mathcal{T}$, $|\operatorname{supp} \boldsymbol{\varphi}_n \cap Q^f| \to 0$ to get (4.9) and then (1.12) and (1.13). Unfortunately, the latter localizing procedure is applicable only in the times where no collision occurs, i.e. $\partial S_i \cap \partial S_j = \emptyset$ and $\partial S_i \cap \partial\Omega = \emptyset$ for $i \neq j$.

4.1.3 Thermal inequalities

Like in the previous cases one can use the transport theorem and the entropy equation (1.19) separately for the solid and the fluid part to get

$$\int_0^T \int_\Omega \varrho s \partial_t \varphi + \varrho s \boldsymbol{u} \cdot \nabla \varphi + \frac{\boldsymbol{q} \cdot \nabla \varphi}{\vartheta} + \sigma \varphi \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \leq 0 \tag{4.14}$$

for any $\varphi \in \mathcal{D}(I \times \overline{\Omega})$, $\varphi \geq 0$. Similarly, we get the global energy inequality

$$\int_\Omega E[t_2] - E[t_1] \, \mathrm{d}\boldsymbol{x} \leq \int_{t_1}^{t_2} \int_\Omega \varrho \boldsymbol{f} \cdot \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t. \tag{4.15}$$

where $E = \frac{1}{2}\varrho |\boldsymbol{u}|^2 + \varrho e$.

For a regular solution, indeed one can expect at most continuity of $\boldsymbol{u}$ and ϑ over $\partial\Omega^s(t)$ and ϱ even with the jump, nevertheless we can use the transport theorem once again and integrate by parts in (4.14). Using $\vartheta\varphi$ as the test function and performing the calculation (1.18) separately on the fluid and the solid part, we get

$$\begin{aligned} \int_0^T \int_{\Omega^f[t]} [\partial_t(\varrho e) + \operatorname{div}(\varrho e \boldsymbol{u}) - \operatorname{div}\boldsymbol{q} + \mathbb{S} : \nabla \boldsymbol{u}]^f \varphi \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\ \int_0^T \int_{\Omega^s[t]} [\partial_t(\varrho e) + \operatorname{div}(\varrho e \boldsymbol{u}) - \operatorname{div}\boldsymbol{q} + \mathbb{S} : \nabla \boldsymbol{u}]^s \varphi \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\ + \int_0^T \int_{\partial\Omega} \boldsymbol{q} \cdot \boldsymbol{n} \varphi \, \mathrm{d}\sigma \, \mathrm{d}t + \int_0^T \int_{\partial\Omega^s(t)} [\boldsymbol{q}^s - \boldsymbol{q}^f] \cdot \boldsymbol{n} \varphi \, \mathrm{d}\sigma \, \mathrm{d}t \geq 0. \end{aligned} \tag{4.16}$$

If we take $\varphi = 1_{[0,t]}$ and add the kinetic energy equation (1.16), it became obvious that the strict inequality in (4.16) is in contradiction with (4.15), so we recover the internal energy equation (1.17).

Now we are ready to introduce the *variational solutions* of our system.

Definition 4.1.5. *We shall say that functions*

$$\varrho \in L^\infty(I; L^\gamma(\Omega)),\ \boldsymbol{u} \in L^2(I; W_0^{1,2}(\Omega)),\ \vartheta \in L^2(I; W^{1,2}(\Omega)) \tag{4.17}$$

and rigid displacement mappings $\{\boldsymbol{\eta}^i\}_{i=1}^N$ *with bodies* $\{S^i\}_{i=1}^N$ *form the* **variational solution** *of* **problem (B)** *if*

- *The density* ϱ *is non-negative a.e. in* Q*, the temperature is positive a.e. in* Q*, and the velocity* $\boldsymbol{u}$ *is compatible with* $\{\boldsymbol{\eta}^i, \overline{S}^i\}$ *for every* $i = 1 \dots N$*.*
- *The weak formulation (4.4) of the continuity equation holds, provided the density and the velocity are extended by zero outside of the domain* Ω*.*
- *The momentum equation holds in the sense of distributions, namely* (4.12)*.*
- *The inequality* (4.14) *for the specific entropy and the opposite inequality* (4.15) *for the total energy are satisfied.*

Besides the properties specified in Definition 4.1.5, the solution we are going to construct will satisfy also the renormalized continuity equation (3.9).

4.1.4 Constitutive equations - hypotheses

We assume that the pressure, as well as the internal energy and the entropy, are given by the same functions on the solid and on the fluid region. Although the original aim was to allow different pressure for the fluid and for the bodies, there appears a problem the formulation based on the entropy inequality. We discuss it at the end of the chapter. However, we allow the heat conductivity coefficient to be different.

Following discussion about the constitutive equations in Section 1.2.1, we assume the pressure in the form

$$\left\{\begin{array}{c} p = p_G(\varrho, \vartheta) + p_R(\vartheta), \\ p_G = \vartheta^{\frac{5}{2}} P(\varrho \vartheta^{-\frac{3}{2}}),\ p_R = \dfrac{a}{3}\vartheta^4, a > 0, \end{array}\right\} \tag{4.18}$$

where P meets hypotheses

$$\left\{\begin{array}{c} P \in C^1[0,\infty), P(0) = 0, P'(Y) \geq 0 \text{ on } (0,\infty), \\ 0 < \dfrac{5}{3}P(Y) - P'(Y)Y \leq c(1 + Y^\alpha), 0 < \alpha < \dfrac{50}{39}, \\ \lim\limits_{Y\to\infty} P(Y)Y^{-\frac{5}{3}} = P_\infty > 0. \end{array}\right\} \tag{4.19}$$

Further, we assume the internal energy interrelated to the pressure by the Gibbs law (1.24). In particular we assume

$$\left\{\begin{array}{c} e = e_G(\varrho, \vartheta) + e_R(\vartheta) \\ e_G = \dfrac{3}{2}\varrho^{-1} p_G,\ e_R = a\varrho^{-1}\vartheta^4. \end{array}\right\} \tag{4.20}$$

Finally, according to (1.27) and (1.28), we assume the entropy

$$\left\{\begin{array}{c} s = s_G(\varrho, \vartheta) + s_R(\vartheta), \\ s_G = S(\varrho \vartheta^{-\frac{3}{2}}),\ s_R = \dfrac{4a}{3}\varrho^{-1}\vartheta^3, \end{array}\right\} \tag{4.21}$$

where S is interrelated with P through the relation (1.38).

Besides these structural assumptions on the thermodynamical quantities, we assume that the fluid viscosities μ^f, ζ^f are C^1-functions of temperature with linear growth:

$$\left\{\begin{array}{c} \mu = \mu^f(\vartheta),\ \zeta = \zeta^f(\vartheta) \text{ on } Q^f \\ 0 < \underline{\mu}(1+\vartheta) \le \mu^f(\vartheta);\ |(\mu^f)'| \le \overline{\mu} \\ 0 \le \underline{\mu}(1+\vartheta) \le \zeta^f(\vartheta);\ |(\zeta^f)'| \le \overline{\mu}. \end{array}\right\} \tag{4.22}$$

On the other hand, the symmetric part of the velocity gradient is zero on the solid region and thus the term $\mathbb{S} : \nabla \boldsymbol{u}$ is zero independently of the viscosities. The same holds for the term $\mathbb{S} : \nabla\varphi$ in the momentum equation, since $\varphi \in \mathcal{T}$.

The heat conductivity coefficient consists of the part due to the motion of the particles and the part caused by the radiation. The former part should have a linear growth in accordance with linear grow of the viscosities, while the latter one should behave like ϑ^3. We assume different relations for the fluid and for the solid region, namely

$$\left\{\begin{array}{c} \kappa = \kappa^f = \kappa^f_G(\vartheta) + \kappa^f_R(\vartheta) \quad \text{on } Q^f, \\ \kappa = \kappa^s = \kappa^s_G(\vartheta) + \kappa^s_R(\vartheta) \quad \text{on } Q^s, \\ 0 < \underline{\kappa} \le\ (\kappa^f_G(\vartheta), \kappa^s_G(\vartheta))\ \le \overline{\kappa}\vartheta, \\ 0 < \underline{\kappa}\vartheta^3 \le\ (\kappa^f_R(\vartheta), \kappa^s_R(\vartheta))\ \le \overline{\kappa}\vartheta^3, \end{array}\right\} \tag{4.23}$$

where κ^f_G, κ^s_G, κ^f_R, κ^s_R are C^1-functions of the temperature.

Now we are ready to state the main result about the existence of global-in-time the variational solutions to **problem (B)**

Theorem 4.1.6. *Let $\Omega \subset \mathbf{R}^3$ be a bounded domain with $C^{2+\nu}$ boundary ($\nu > 0$). Assume p, e, s are given through* (4.18), (4.20), (4.21) *respectively. Let μ, λ, κ obey* (4.22), (4.23). *Finally, let the external force $\boldsymbol{f} \in L^\infty(\Omega; \mathbf{R}^3)$ and the initial data $\varrho_0, \vartheta_0 \in L^\infty(\Omega)$, $\boldsymbol{u}_0 \in L^\infty(\Omega; \mathbf{R}^3)$ be given as well as the initial position of the bodies $\{S^i_0\}_{i=1}^N$, where S^i_0 are open, connected sets with a C^2-boundary. Then there exists at least one solution ϱ, $\boldsymbol{u}$, ϑ, $\{\boldsymbol{\eta}^i, S^i\}_{i=1}^N$ of* **problem (B)** *in the sense of Definition 3.1.1, which satisfies the initial conditions*

$$\begin{gathered} \varrho[t] \to \varrho_0 \text{ in } L^1(\Omega), \\ (\varrho\boldsymbol{u})[t] \to \varrho_0\boldsymbol{u}_0 \text{ weakly in } L^1(\Omega; \mathbf{R}^3), \\ \operatorname*{esslim}_{t\to 0+} \int_\Omega (\varrho s)[t]\varphi \,\mathrm{d}\boldsymbol{x} \ge \int_\Omega \varrho_0 s_0 \varphi \,\mathrm{d}\boldsymbol{x}, \quad \forall\, \varphi \in \mathcal{D}(\Omega),\ \varphi \ge 0, \\ E[t] \to E_0 = E(\varrho_0, \boldsymbol{u}_0, \vartheta_0).\boldsymbol{\eta}^i[0](\boldsymbol{x}) = \boldsymbol{x},\ S^i[0] = S^i_0, \end{gathered}$$

where $s_0(\boldsymbol{x}) = s(\varrho_0(\boldsymbol{x}), \vartheta_0(\boldsymbol{x}))$.

4.2 Approximating problems

As usual, the solution will be constructed via the sequence of approximate solutions. We use a sequence of the solutions $(\varrho_n, \boldsymbol{u}_n, \vartheta_n)$ to **Problem (F)** from Chapter 3 with suitably

chosen constitutive relations. In particular, we take viscosities approaching infinity on the solid region given by

$$\mu_n(\vartheta_n;t,\boldsymbol{x})=\mu^f(\vartheta_n)+n\chi_n(t,\boldsymbol{x}),\quad \zeta_n(t,\boldsymbol{x};\vartheta_n)=\zeta^f(\vartheta_n)+n\chi_n(t,\boldsymbol{x}). \tag{4.24}$$

The heat conductivity is just a smooth approximation of the discontinuous relation (4.23), namely

$$\kappa_n(t,\boldsymbol{x};\vartheta_n)=(1-\chi_n(t,\boldsymbol{x}))\kappa^f(\vartheta_n)+\chi_n(t,\boldsymbol{x})\kappa^s(\varrho_n,\vartheta_n), \tag{4.25}$$

where χ_n is an approximation of the characteristic function of the solid region Q^s, which will be specified in the following section. For this setting, we can use Theorem 3.1.6 to get

$$\varrho_n\in L^\infty(I;L^{\frac{5}{3}}(\Omega)),\ \boldsymbol{u}_n\in L^2(I;W^{1,2}_0(\Omega;\mathbf{R}^3)),\ \vartheta_n\in L^2(I;W^{1,2}(\Omega))\cap L^\infty(I;L^4(\Omega)),$$

which satisfy

$$\int_0^T\int_\Omega B(\varrho_n)\partial_t\varphi+B(\varrho_n)\boldsymbol{u}_n\cdot\nabla\varphi-b(\varrho_n)\,\mathrm{div}\boldsymbol{u}_n\,\varphi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t=0,\quad \varphi\in\mathcal{D}(I\times\mathbf{R}^3), \tag{4.26}$$

$$\begin{aligned}\int_0^T\int_\Omega &\varrho_n\boldsymbol{u}_n\cdot\partial_t\boldsymbol{\varphi}+\varrho_n\boldsymbol{u}_n\otimes\boldsymbol{u}_n:\nabla\boldsymbol{\varphi}+p_n\mathrm{div}\boldsymbol{\varphi}-2\mu_n\mathbb{D}\boldsymbol{u}:\nabla\boldsymbol{\varphi}\\ &-\big(\zeta_n-\frac{2}{3}\mu_n\big)\mathrm{div}\boldsymbol{u}\,\mathrm{div}\boldsymbol{\varphi}+\varrho_n\boldsymbol{f}\cdot\varphi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t=0,\quad \boldsymbol{\varphi}\in\mathcal{D}(I\times\Omega;\mathbf{R}^3),\end{aligned} \tag{4.27}$$

$$\begin{aligned}\int_0^T\int_\Omega &\varrho_n s_n\partial_t\varphi+\varrho_n s_n\boldsymbol{u}_n\cdot\nabla\varphi-\frac{\kappa_n\nabla\vartheta_n\cdot\nabla\varphi}{\vartheta_n}\\ &+\Big(\frac{\mathbb{S}_n:\nabla\boldsymbol{u}_n}{\vartheta_n}+\frac{\kappa_n\,|\vartheta_n|^2}{\vartheta_n^2}\Big)\varphi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\le 0,\ \varphi\in\mathcal{D}(I\times\overline{\Omega}),\ \varphi\ge 0,\end{aligned} \tag{4.28}$$

$$\int_0^T\psi(t)E_n[t]\,\mathrm{d}t=\int_0^T\psi\int_\Omega\varrho_n\boldsymbol{f}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,\quad \psi\in\mathcal{D}(I). \tag{4.29}$$

Moreover

$$\varrho_n[t]\to\varrho_0\text{ in }L^1(\Omega),\ (\varrho_n\boldsymbol{u}_n)[t]\to(\varrho\boldsymbol{u})_0\text{ weakly in }L^1(\Omega),\ E[t]\to E_0\quad\text{as } t\to 0, \tag{4.30}$$

and

$$\operatorname{ess}\lim_{t\to 0+}\int_\Omega(\varrho_n s_n)[t]\varphi\,\mathrm{d}\boldsymbol{x}\ge\int_\Omega\varrho_0 s_0\varphi\,\mathrm{d}\boldsymbol{x}\quad\forall\varphi\in\mathcal{D}(\overline{\Omega}),\ \varphi\ge 0. \tag{4.31}$$

4.3 Construction of displacement mappings

For a nonvoid compact set $K\subset\mathbf{R}^3$, we define the distance function

$$\mathsf{d}_K(\boldsymbol{x})=\min_{\boldsymbol{y}\in K}|\boldsymbol{x}-\boldsymbol{y}|.$$

It is easy to see that d_K is a Lipschitz continuous function with the amplitude of the gradient equal to 1 a.e. on $\mathbf{R}^3\setminus K$. Further, for any set $S\subset\mathbf{R}^3$, we define a signed distance function from the boundary ∂S as

$$\mathsf{db}_S(\boldsymbol{x})=d_{\overline{\mathbf{R}^3\setminus S}}(\boldsymbol{x})-d_{\overline{S}}(\boldsymbol{x}).$$

We shall say that a sequence of sets $S_n\subset\mathbf{R}^3$ converges to $S\subset\mathbf{R}^3$ in the sense of boundaries, $S_n\xrightarrow{\mathsf{db}}S$, if

$$\mathsf{db}_{S_n}\to\mathsf{db}_S\text{ in }C_{loc}(\mathbf{R}^3).$$

The following lemma, we shall find useful for the investigation of db-convergence.

Lemma 4.3.1. *Let $f, g \in C^{0,\alpha}(\mathbf{R}^3;\mathbf{R}^3)$ be homeomorphisms on $\mathbf{R}^3$ and $A, B \subset \mathbf{R}^3$. Then it holds*

$$\left\|\mathsf{db}_{f(A)} - \mathsf{db}_{g(B)}\right\|_{C(\mathbf{R}^3)} \leq \left(\|f\|_{C^{0,\alpha}(\mathbf{R}^3)} + \|g\|_{C^{0,\alpha}(\mathbf{R}^3)}\right) \left\|\mathsf{db}_A - \mathsf{db}_B\right\|^{\alpha}_{C(\mathbf{R}^3)} + \|f-g\|_{C(\mathbf{R}^3)}.$$

Proof. The continuous properties of f imply $\mathbf{R}^3 \setminus f(K) = f(\mathbf{R}^3 \setminus K)$, $\overline{f(K)} = f(\overline{K})$ for any $K \subset \mathbf{R}^3$ and the same holds for g. Then, from the definition of db distance function, it follows

$$\left\|\mathsf{db}_{f(A)} - \mathsf{db}_{g(B)}\right\|_{C(\mathbf{R}^3)} \leq \left\|\mathsf{d}_{\overline{f(A)}} - \mathsf{d}_{\overline{g(B)}}\right\|_{C(\mathbf{R}^3)} + \left\|\mathsf{d}_{\overline{f(\mathbf{R}^3\setminus A)}} - \mathsf{d}_{g(\mathbf{R}^3\setminus B)}\right\|_{C(\mathbf{R}^3)}.$$

Both terms on the right-hand side can be further estimated this way

$$\left\|\mathsf{d}_{\overline{f(A)}} - \mathsf{d}_{\overline{g(B)}}\right\|_{C(\mathbf{R}^3)} \leq \max\left\{\sup_{\boldsymbol{x}\in\overline{f(A)}} \mathsf{d}_{\overline{g(B)}}(\boldsymbol{x}), \sup_{\boldsymbol{x}\in\overline{f(B)}} \mathsf{d}_{\overline{g(A)}}(\boldsymbol{x})\right\}.$$

Again, it is enough to explore the first supremum. For any $\boldsymbol{x} \in \overline{f(A)}$, there is $\boldsymbol{x}_0 \in \overline{A}$ such that $f(\boldsymbol{x}_0) = \boldsymbol{x}$ and there is $\boldsymbol{y}_0 \in \overline{B}$ such that $\mathsf{d}_{\overline{B}}(\boldsymbol{x}_0) = |\boldsymbol{x}_0 - \boldsymbol{y}_0|$. Following estimate then finish the proof

$$\begin{aligned}\mathsf{d}_{\overline{g(B)}}(\boldsymbol{x}) &= \mathsf{d}_{g(\overline{B})}(f(\boldsymbol{x}_0)) \leq |f(\boldsymbol{x}_0) - g(\boldsymbol{y}_0)| \\ &\leq |f(\boldsymbol{x}_0) - f(\boldsymbol{y}_0)| + |f(\boldsymbol{y}_0) - g(\boldsymbol{y}_0)| \leq \|f\|_{C^{0,\alpha}(\mathbf{R}^3)}|\boldsymbol{x}_0 - \boldsymbol{y}_0|^{\alpha} + \|f-g\|_{C(\mathbf{R}^3)}.\end{aligned}$$

□

Finally, we introduce a notation of δ-stretch and δ-shrink of the set M as

$$\oplus_\delta M := \bigcup_{\boldsymbol{x}\in M} B_\delta(\boldsymbol{x}), \quad \ominus_\delta M := \{\boldsymbol{x} \,|\, B_\delta(\boldsymbol{x}) \subset M\}, \quad \text{respectively.}$$

Having collected the preliminary material, we introduce displacement mappings $\boldsymbol{\eta}_n$ for the approximating problems. It can not be done directly through (4.1), since the velocities $\boldsymbol{u}_n$ are not regular enough. Instead, we use the velocities smoothed over some δ-neighborhood:

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\eta}_n[t](\boldsymbol{x}_0) = (\omega_\delta \star \boldsymbol{u}_n)(t, \boldsymbol{\eta}_n[t](\boldsymbol{x}_0)), \quad \boldsymbol{\eta}_n[0](\boldsymbol{x}_0) = \boldsymbol{x}_0, \tag{4.32}$$

where ω_δ are smoothing kernels introduced in (2.1). Although $\boldsymbol{\eta}_n$ are not compatible with $\boldsymbol{u}_n$, we hope to get the limit displacement $\boldsymbol{\eta}$ and the limit velocity $\boldsymbol{u}$ such that $\mathbb{D}\boldsymbol{u} = 0$ on the set M, where the viscosity penalization takes effect (roughly speaking it is the limit of $\mathrm{supp}\chi_n$). In that case the limit velocity $\boldsymbol{u}$ should coincide with $\omega_\delta \star \boldsymbol{u}$ on $\ominus_\delta M$. On the same set, $\boldsymbol{\eta}$ should be the rigid displacement compatible with $\boldsymbol{u}$. This deliberation motivates following definition of χ_n. First, we denote

$$O := \bigcup_{i=1}^{N} \ominus_\delta S_0^i, \quad O_n[t] := \boldsymbol{\eta}_n[t](O). \tag{4.33}$$

The sets O_n are bounded, open and non-empty since we assume bodies S_0^i with C^2-boundary. By the same token, we can choose $\delta > 0$ such that $\oplus_\delta O = \cup S_0^i$. At last, we define

$$\chi_n(t, \boldsymbol{x}) := H(n(\delta + \mathsf{db}_{O_n[t]}(\boldsymbol{x}))), \tag{4.34}$$

where $H \in C^\infty(\mathbf{R})$ is a non-decreasing function such that $H = 0$ on $(-\infty, 0]$, and $H = 1$ on $(1, +\infty)$.

Let us show briefly that the constitutive relations (4.24), (4.25) completed with definition (4.34) meet assumptions of Theorem 3.1.6, in particular hypotheses (3.21), (3.22), and (3.23). The first two are satisfied because of (4.22) and (4.23). However, we have to check bounds for $\partial_{\boldsymbol{x}}\partial_{\boldsymbol{x}}\kappa$ and $\partial_t\kappa$. The former is a direct consequence of (4.32) and assumption of C^2-boundary of the bodies. For the later bound we have

$$|\partial_t \kappa_G| \le |\partial_t \chi_n| C\vartheta, \quad |\partial_t \kappa_R| \le |\partial_t \chi_n| C\vartheta^2$$

and

$$\begin{aligned}|\partial_t \chi_n(t,\boldsymbol{x})| &\le C(n)|\partial_t \mathsf{db}_{O_n[t]}(\boldsymbol{x})| \le C(n)|\nabla \mathsf{db}_{O_n[0]}| \left|\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\eta}_n^{-1}[t](\boldsymbol{x})\right| \\ &\le C(n)|\nabla \boldsymbol{\eta}_n[t](\boldsymbol{x}_0)| \left|\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\eta}_n[t](\boldsymbol{x}_0)\right| \le C\big(n, \|\boldsymbol{u}_n\|_{L^2(J\times\Omega)}\big)|\omega_\delta \star \boldsymbol{u}_n(\boldsymbol{x})|.\end{aligned}$$

Next, we have to prove the compactness property (3.23). Assuming $\boldsymbol{u}_m \to \boldsymbol{u}$ weakly in $L^2(I\times\Omega;\mathbf{R}^3)$ and using Lemma 4.3.1, we have

$$\begin{aligned}|\mu[\boldsymbol{u}_m](t,\boldsymbol{x};\vartheta) - \mu[\boldsymbol{u}](t,\boldsymbol{x};\vartheta)| &\le C|\chi[\boldsymbol{u}_m](t,\boldsymbol{x}) - \chi[\boldsymbol{u}](t,\boldsymbol{x})| \\ &\le C|\mathsf{db}_{O_m[t]}(\boldsymbol{x}) - \mathsf{db}_{O[t]}(\boldsymbol{x})| \le C|\boldsymbol{\eta}_m[t] - \boldsymbol{\eta}[t]|_{C(\Omega)},\end{aligned}$$

where $\boldsymbol{\eta}_m$, $\boldsymbol{\eta}$ are displacements compatible with $\boldsymbol{u}_m$, $\boldsymbol{u}$ respectively. The same holds for ζ, while for the heat conductivity, we get

$$\begin{aligned}|\kappa[\boldsymbol{u}_m](t,\boldsymbol{x};\vartheta) - \kappa[\boldsymbol{u}](t,\boldsymbol{x};\vartheta)| &\le |\chi[\boldsymbol{u}_m](t,\boldsymbol{x}) - \chi[\boldsymbol{u}](t,\boldsymbol{x})||\kappa^s - \kappa^f| \\ &\le C(\vartheta)|\mathsf{db}_{O_m[t]}(\boldsymbol{x}) - \mathsf{db}_{O[t]}(\boldsymbol{x})| \le C(\vartheta)|\boldsymbol{\eta}_m[t] - \boldsymbol{\eta}[t]|_{C(\Omega)}.\end{aligned}$$

To finish the proof, it suffices to apply the following compactness result for the displacement mappings.

Lemma 4.3.2. *Let $\boldsymbol{v}_n$ be a sequence of (velocity) fields,*

$$\boldsymbol{v}_n \text{ uniformly bounded in } L^p(I; W^{1,q}(\mathbf{R}^3;\mathbf{R}^3)),\ p>1,\ q>3.$$

Let $\boldsymbol{\eta}_n$ be a sequence of compatible (displacement) mappings, i.e. given through

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\eta}_n[t](\boldsymbol{x}_0) = \boldsymbol{v}_n(t, \boldsymbol{\eta}_n[t](\boldsymbol{x}_0)),\ \boldsymbol{\eta}_n[0](\boldsymbol{x}_0) = \boldsymbol{x}_0. \tag{4.35}$$

Finally, let $S_n \subset \mathbf{R}^3$ be a sequence of sets, which converge to the set S in the sense of boundaries. Then

$$\boldsymbol{v}_n \rightharpoonup \boldsymbol{v} \text{ weakly in } L^p(I; W^{1,q}(\mathbf{R}^3;\mathbf{R}^3)) \tag{4.36}$$

passing to the subsequence as the case may be, and

$$\boldsymbol{\eta}_n[t] \to \boldsymbol{\eta}[t] \text{ in } C_{loc}(\mathbf{R}^3) \text{ uniformly in } t \in [0,T], \tag{4.37}$$

where $\boldsymbol{\eta}[t]$ is compatible with $\boldsymbol{v}$. Moreover

$$S_n(t) \xrightarrow{\mathsf{db}} S(t) \tag{4.38}$$

uniformly in $t \in [0,T]$, where $S_n(t) := \boldsymbol{\eta}_n[t](S_n)$ and $S(t) := \boldsymbol{\eta}[t](S)$.

Proof. Applying the Alaoglu-Bourbaki theorem 2.2.1 on the sequence $\boldsymbol{v}_n$ bounded in the reflexive space $L^p(I;W^{1,q}(\mathbf{R}^3;\mathbf{R}^3))$, we get (4.36). Using the Sobolev imbedding, we have

$$\boldsymbol{v}_n, \boldsymbol{v} \ \in L^p(I;C^{0,\alpha}_{loc}(\mathbf{R}^3;\mathbf{R}^3)) \quad \text{for some } 0<\alpha<1, \tag{4.39}$$

which justify (4.35). Then (4.35) can be used to deduce

$$\big\|\boldsymbol{\eta}_n[t]-\boldsymbol{\eta}_n[s]\big\|_C = \sup_{\boldsymbol{x}\in\mathbf{R}^3}\left|\int_s^t \boldsymbol{v}_n(\tau,\boldsymbol{\eta}_n(t,\boldsymbol{x}))\,\mathrm{d}\tau\right| \le |t-s|^{1/p'}\big\|\boldsymbol{v}_n\big\|_{L^p(I;C^{0,\alpha}(\mathbf{R}^3;\mathbf{R}^3))} \tag{4.40}$$

which means that $\boldsymbol{\eta}_n$ are equi-continuous. Employing the abstract Arzelà-Ascoli theorem 2.2.2 we conclude (4.37). Furthermore, it follows from (4.39) and (4.35) that $\boldsymbol{\eta}_n$ as well as $\boldsymbol{\eta}$ belong to $C(I;C^{0,\alpha}_{loc}(\mathbf{R}^3;\mathbf{R}^3))$. Thus one can use Lemma 4.3.1 with $f=\boldsymbol{\eta}_n[t]$, $g=\boldsymbol{\eta}[t]$ and $A=S_n(0)$, $B=S(0)$ to verify (4.38). □

4.4 Uniform estimates

4.4.1 Energy estimate

Direct use of the total energy inequality (4.29) and the constitutive law for internal energy (4.29) yields

$$\begin{aligned} E[\tau] &= \left(\int_\Omega \frac{1}{2}\varrho_n|\boldsymbol{u}_n|^2+(\varrho_n e_G + a\vartheta_n^4)\,\mathrm{d}\boldsymbol{x}\right)[\tau] \le \\ &\le E[0]+\int_0^\tau\int_\Omega \varrho_n\boldsymbol{u}_n\cdot\boldsymbol{f}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \le E[0]+\int_0^\tau \big\|\boldsymbol{f}\big\|_\infty\big\|\varrho_n\boldsymbol{u}_n^2\big\|_1^{\frac12}\big\|\varrho_n\big\|_1^{\frac12}\,\mathrm{d}t \end{aligned} \tag{4.41}$$

Thanks to the particular form of e_G, and due to the hypotheses about positive cold pressure (1.36), (4.19), the left-hand side provides an estimate of the norms $\big\|\varrho_n\boldsymbol{u}_n^2\big\|_1$, $\big\|\varrho_n\big\|_1$. Thus we can apply the Gronwall lemma to infer

$$\sqrt{\varrho_n}\boldsymbol{u}_n \text{ bounded in } L^\infty(I;L^2(\Omega;\mathbf{R}^3)), \tag{4.42}$$

$$\varrho_n e_n,\ p_n \text{ bounded in } L^\infty(I;L^1(\Omega)), \tag{4.43}$$

$$\varrho_n \text{ bounded in } L^\infty(I;L^{\frac53}(\Omega)), \tag{4.44}$$

$$\vartheta_n \text{ bounded in } L^\infty(I;L^4(\Omega)). \tag{4.45}$$

4.4.2 Entropy estimate

Using a spatially homogeneous test function in the entropy inequality (4.28), initial condition (4.31), and hypothesis (4.21) about the entropy, we arrive at

$$\int_0^\tau\int_\Omega \frac{\kappa_n|\nabla\vartheta_n|^2}{\vartheta_n^2}+\frac{\mathbb{S}_n:\nabla\boldsymbol{u}_n}{\vartheta_n}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \le \int_\Omega \big(\varrho_n s_G(\varrho_n,\vartheta_n)+\frac43 a\vartheta_n^3\big)[\tau]\,\mathrm{d}\boldsymbol{x}-\int_\Omega \varrho_0 s_0\,\mathrm{d}\boldsymbol{x}. \tag{4.46}$$

Since $P\in C^1[0,\infty)$, there exists a constant C_V such that

$$c_v(Y)\le C_V \quad \text{for any } 0<Y<1,$$

c.f. (1.34). Hence we have

$$\varrho_n s_G(\varrho_n,\vartheta_n)=\varrho_n S\big(\varrho_n\vartheta_n^{-\frac{3}{2}}\big)\le C_V(-\varrho_n\log\varrho_n+\frac{3}{2}\varrho_n\log\vartheta_n)\le C(\varrho^{\frac{5}{3}}+\varrho\vartheta). \tag{4.47}$$

Furthermore, the function S is decreasing, thus

$$\varrho_n s_G(\varrho_n,\vartheta_n)=\varrho_n S\big(\varrho_n\vartheta^{-\frac{3}{2}}\big)\le \varrho_n S(1) \text{ for } \varrho\vartheta^{-\frac{3}{2}}=Y\ge 1.$$

Consequently, the right-hand side of (4.46) is bounded. Using the hypotheses (4.23) and (4.22), the left-hand side yields estimates

$$\nabla\log\vartheta_n,\ \nabla(\vartheta_n^{\frac{3}{2}}),\quad \text{bounded in } L^2(Q), \tag{4.48}$$

$$\boldsymbol{u}_n\quad \text{bounded in } L^2(W^{1,2}(I;\Omega;\mathbf{R}^3)), \tag{4.49}$$

and

$$n\int_0^T\int_\Omega \chi_n|\mathbb{D}\boldsymbol{u}_n|^2\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t\quad \text{is bounded.} \tag{4.50}$$

In order to keep the positivity of the temperature, we use (4.46) once again. We forget the non-negative left-hand side and rearrange the remaining terms to get

$$\begin{aligned}\int_\Omega \varrho_0 s_0\ \mathrm{d}\boldsymbol{x}-S(N)\int_\Omega\varrho\ \mathrm{d}\boldsymbol{x}\le\int_\Omega\frac{4a}{3}\vartheta^3\ \mathrm{d}\boldsymbol{x}+\int_{\{\varrho\vartheta^{-\frac{3}{2}}\le 1\}}\varrho s_G\ \mathrm{d}\boldsymbol{x}\\ +\int_{\{1\le\varrho\vartheta^{-\frac{3}{2}}\le N\}}\varrho s_G\ \mathrm{d}\boldsymbol{x}+|S(N)|\int_{\{\varrho\vartheta^{-\frac{3}{2}}\le N\}}\varrho\ \mathrm{d}\boldsymbol{x},\end{aligned} \tag{4.51}$$

where we have denoted $\varrho=\varrho_n[\tau]$, $\vartheta=\vartheta_n[\tau]$. Taking N large enough, we can make left-hand side strictly positive, greater than some $S_0>0$. Similarly as in (4.47) one gets

$$\int_{\{\varrho\vartheta^{-\frac{3}{2}}\le 1\}}\varrho s_G\ \mathrm{d}\boldsymbol{x}\le C\int_{\{\varrho\vartheta^{-\frac{3}{2}}\le 1\}}(-\varrho\log\varrho+\varrho\log\vartheta)\ \mathrm{d}\boldsymbol{x}\le C\int_\Omega\vartheta^3\ \mathrm{d}\boldsymbol{x},$$

while for the other terms, we have

$$\int_{\{1\le\varrho\vartheta^{-\frac{3}{2}}\le N\}}\varrho s_G\ \mathrm{d}\boldsymbol{x}\le NS(1)\int_\Omega\vartheta^{\frac{3}{2}}\ \mathrm{d}\boldsymbol{x},\quad \int_{\{\varrho\vartheta^{-\frac{3}{2}}\le N\}}\varrho\ \mathrm{d}\boldsymbol{x}\le N\int_\Omega\vartheta^{\frac{3}{2}}\ \mathrm{d}\boldsymbol{x}.$$

Consequently,

$$0<S_0\le C\int_\Omega\vartheta_n[\tau]^3+\vartheta_n[\tau]^{\frac{3}{2}}\ \mathrm{d}\boldsymbol{x}$$

for a.a. $\tau\in I$ and uniformly in n. Then, by virtue of the Hölder inequality,

$$S_0-|\Omega|(\delta^3+\delta^{\frac{3}{2}})\le|\{\vartheta_n\ge\delta\}|^{\frac{1}{4}}\big\|\vartheta_n\big\|^3_{L^4(\Omega)}|\{\vartheta_n\ge\delta\}|^{\frac{5}{8}}\big\|\vartheta_n\big\|^{\frac{3}{2}}_{L^4(\Omega)}.$$

Thus there exist constants M, $\delta>0$, independent of n such that

$$|\{\vartheta_n[\tau]\ge\delta\}|\ge M>0 \text{ for a.a. } t\in I$$

while

$$\int_{\vartheta_n[\tau]\ge\delta}\log\vartheta_n[\tau]\ \mathrm{d}\boldsymbol{x}\le C \text{ uniformly in } \tau,\ n.$$

Now, one can apply Poincaré inequality formulated in Proposition 2.1.2, to get

$$\log \vartheta_n,\ \vartheta_n^{\frac{3}{2}} \text{ bounded in } L^2(I;W^{1,2}(\Omega)). \tag{4.52}$$

From the latter estimate we have ϑ_n bounded in $L^3(I;L^9(\Omega))$ due to the Sobolev imbedding. Interpolating, we obtain ϑ_n bounded in $L^{\frac{17}{3}}(Q)$ and consequently, using hypothesis (4.22) and estimate (4.49), we get

$$\mathbb{S}_n \text{ bounded in } L^p(Q) \text{ for some } p>1. \tag{4.53}$$

4.4.3 Refined pressure estimate

In view of the estimates derived up to now, the pressure and the internal energy are known to be bounded only in the non-reflexive space $L^\infty(I;L^1(\Omega))$. This is not enough to pass to the limit neither with the pressure in the momentum equation nor with the internal energy in the total energy balance. In contrast to Section 3.6.2, we can not test the momentum equation by a function with non-zero support on Q_s, since there we can not control the penalizing term $n\chi_n\mathbb{D}\boldsymbol{u}_n$. Fortunately, in our formulation the test functions in (4.12) have a compact support in Q^f, thus for the pressure, we need only the local estimate in Q^f. On the other hand, the L^1-estimates for the internal energy are sufficient to get the total energy inequality.

Let $J\subset I$ be an open time interval and $U\subset\Omega$ be an open ball such that

$$K=\overline{J\times U}\subset Q^f.$$

We use

$$\boldsymbol{\varphi}=\psi(t)\mathcal{B}_U[\pi_\varepsilon^n],\ \pi_\varepsilon^n=\omega_\varepsilon\star\varrho_n^\nu,\ \psi\in\mathcal{D}(J)-\fint_K\omega_\varepsilon\star\varrho_n^\nu\,\mathrm{d}\boldsymbol{x}$$

as the test function in (4.27). Then, exactly as in Section 3.6.2, we derive the estimate

$$\int_K p(\varrho_n,\vartheta_n)\varrho_n^\nu\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\le C.$$

Since, $p_G\le C(\varrho^{\frac{5}{3}}+\varrho\vartheta)$ because of the hypothesis (4.18), we conclude

$$\big\|p(\varrho_n,\vartheta_n)\big\|_{L^p(\mathcal{K}^f)}\le C\quad\text{for some } p>1 \text{ and any compact } \mathcal{K}^f\subset Q^f.$$

4.5 Limit passage

In view of the estimates (4.42 – 4.45) and (4.48 – 4.49), we can use Alaoglu-Bourbaki theorem 2.2.1. Passing to the subsequence as the case may be, one gets

$$\varrho_n\to\varrho \qquad \text{weakly-}*\text{r in } L^\infty(I;L^{\frac{5}{3}}(\Omega)), \tag{4.54}$$

$$\boldsymbol{u}_n\to\boldsymbol{u} \qquad \text{weakly in } L^2(I;W_0^{1,2}(\Omega;\mathbf{R}^3)), \tag{4.55}$$

$$\vartheta_n\to\vartheta \qquad \text{weakly-}* \text{ in } L^\infty(I;L^4(\Omega)). \tag{4.56}$$

Due to (4.49) we have $\omega_\delta\star\boldsymbol{u}_n$ bounded in $L^2(I;W^{1,\infty}(\Omega;\mathbf{R}^3))$, thus Lemma 4.3.2 yields

$$\boldsymbol{\eta}_n\to\boldsymbol{\eta} \text{ in } C(I;C_{loc}(\mathbf{R}^3));\quad O_n[t]\xrightarrow{\mathsf{db}}O[t] \text{ uniformly for } t\in[0,T].$$

Moreover, it holds $\chi_n[t] \to \chi[t] = 1_{\oplus_\delta O[t]}$ in $C(I; L^p_{loc}(\mathbf{R}^3))$ for any $p \geq 1$. Indeed, we can compute

$$\big\|\chi_n[t] - \chi[t]\big\|_p^p \leq \int_{\mathbf{R}^3} |H_n(\mathsf{db}_{\oplus_\delta O_n[t]}) - 1_{\oplus_\delta O_n[t]}|^p + |1_{\oplus_\delta O_n[t]} - 1_{\oplus_\delta O[t]}|^p \,\mathrm{d}\boldsymbol{x}$$
$$\leq C\frac{1}{n}|\partial(\oplus_\delta O_n[t])| + C\big\|\mathsf{db}_{O_n[t]} - \mathsf{db}_{O[t]}\big\|_C\,|\partial(\oplus_\delta O[t])|. \quad (4.57)$$

Further, we can pass to the limit in (4.50) to get $\mathbb{D}\boldsymbol{u} = 0$ a.e. on $\oplus_\delta O[t]$ for a.e. $t \in I$. Consequently $\omega_\delta \star \boldsymbol{u}$ coincide with $\boldsymbol{u}$ on $O[t]$ and $\boldsymbol{u}$ is compatible with $\{\boldsymbol{\eta}, \ominus_\delta S_0^i\}$ for $i = 1 \dots N$, cf. definition (4.33). Then it is natural to assign to each body S^i the corresponding rigid displacement $\boldsymbol{\eta}^i$, setting

$$\boldsymbol{\eta}^i[t] = \boldsymbol{\eta}[t] \text{ on } \boldsymbol{\eta}[t](\ominus_\delta S_0^i).$$

Since $\boldsymbol{\eta}^i$ is defined on the whole $\mathbf{R}^3$, the velocity $\boldsymbol{u}$ is compatible with $\{\boldsymbol{\eta}^i, S_0^i\}$, $i = 1 \dots N$.

We consider the renormalized continuity equation (4.26) with some $B(z)$ according to (3.7). Testing it by some fixed $\varphi \in \mathcal{D}(\Omega)$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega B(\varrho_n)\varphi \,\mathrm{d}\boldsymbol{x}\Big)[t] = g_{n,\varphi}[t] := -\int_\Omega (B(\varrho_n)\boldsymbol{u}_n)[t]\cdot\nabla\varphi + (b(\varrho_n)\mathrm{div}\,\boldsymbol{u}_n)[t]\varphi \,\mathrm{d}\boldsymbol{x} \quad (4.58)$$

where functions $g_{n,\varphi}$ are bounded in $L^2(I)$ independently of n. Moreover, in view of (3.7), we have

$$\operatorname*{ess\,sup}_{t\in I}\big\|B(\varrho_n)\big\|_{L^p(\Omega)} \leq C_B.$$

for any $1 < p < \infty$. Then taking any fixed $\tilde\varphi \in L^{p'}(\Omega)$, we estimate

$$\Big|\int_\Omega (B(\varrho_n)[t] - B(\varrho_n)[s])\tilde\varphi \,\mathrm{d}\boldsymbol{x}\Big| \leq C_B\big\|\tilde\varphi - \varphi\big\|_{L^{p'}(\Omega)} + \Big|\int_\Omega (B(\varrho_n)[t] - B(\varrho_n)[s])\varphi \,\mathrm{d}\boldsymbol{x}\Big|,$$

for any $\varphi \in \mathcal{D}(\Omega)$. This together with (4.58) yields equi-continuity of the functions

$$t \to \int_\Omega B(\varrho_n)[t]\tilde\varphi \,\mathrm{d}\boldsymbol{x}$$

Then the Arzelà-Ascoli theorem 2.2.2 can be applied on the ball $B_{C_B}(0)$ in the reflexive space $L^p(\Omega)$, where the weak topology is metrizable, to conclude

$$B(\varrho_n) \to \overline{B(\varrho)} \text{ in } C(\overline{I}; L^p_{weak}(\Omega)), \quad (4.59)$$

for any $1 < p < \infty$. Using the density argument similarly as in Proposition 3.1.3, the same is true for any

$$B(z) \in C^1[0,\infty),\ |B(z)| \leq 1 + z^\lambda,\ \lambda \in (0, \frac{5}{3}),$$

but only for $p \in (1, \frac{5}{3\lambda})$. In particular for $B(z) = z$ we get

$$\varrho_n \to \varrho \text{ in } C(\overline{I}; L^{\frac{5}{3}}_{weak}(\Omega)). \quad (4.60)$$

Since $L^{\frac{5}{3}}(\Omega)$ is compactly imbedded into $W^{-1,2}(\Omega)$, taking into account (4.55), we deduce

$$\varrho_n\boldsymbol{u}_n \to \varrho\boldsymbol{u} \text{ weakly-star in } L^\infty(I; L^{\frac{5}{4}}(\Omega)), \quad (4.61)$$

where we have also used (4.44), (4.42) in order to estimate $\varrho_n \boldsymbol{u}_n$. Consequently, we can pass to the limit in the continuity equation and get

$$\int_I \int_{\mathbf{R}^3} \varrho \partial_t \varphi + \varrho \boldsymbol{u} \cdot \nabla \varphi \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t = 0 \quad \text{for all } \varphi \in \mathcal{D}(I \times \mathbf{R}^3). \tag{4.62}$$

By the same token, we have

$$\varrho_n \vartheta_n \to \varrho \vartheta \text{ weakly in } L^2(I; L^p(\Omega)), \text{ where } \frac{1}{p} = \frac{23}{30} = \frac{3}{5} + \frac{1}{6}. \tag{4.63}$$

Due to (4.55) and (4.61), we have

$$\varrho_n \boldsymbol{u}_n \otimes \boldsymbol{u}_n \to \overline{\varrho \boldsymbol{u} \otimes \boldsymbol{u}} \text{ weakly in } L^p(Q) \tag{4.64}$$

for some $p > 1$. Now, consider $(t, \boldsymbol{x}) \in Q^f$ and its open neighborhood $J \times U$ such that $t \in J \subset I$, $U = B_\varepsilon(\boldsymbol{x})$ and $\mathcal{K}^f = \overline{J \times U} \subset Q^f$. Using the momentum equation (4.27) and (4.61), we establish equi-continuity of functions

$$t \to \int_U \varrho_n \boldsymbol{u}_n \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x}$$

for any fixed $\boldsymbol{\varphi} \in L^5(U)$ in the similar way as in the previous paragraph. Then, we can apply the Arzelà-Ascoli theorem, to get

$$\varrho_n \boldsymbol{u}_n \to \varrho \boldsymbol{u} \text{ in } C(J; L^{\frac{5}{4}}_{weak}(U)). \tag{4.65}$$

Employing the compact imbedding $L^{\frac{5}{4}}(U) \subset W^{-1,2}(U)$, we obtain

$$\varrho_n \boldsymbol{u}_n \otimes \boldsymbol{u}_n \to \varrho \boldsymbol{u} \otimes \boldsymbol{u} \text{ weakly in } L^p(J \times U). \tag{4.66}$$

Thus, with regard to (4.64), it must be

$$\overline{\varrho \boldsymbol{u} \otimes \boldsymbol{u}} = \varrho \boldsymbol{u} \otimes \boldsymbol{u} \text{ a.e. on } Q^f.$$

This together with estimates (4.53) and (3.113) allow us to pass to the limit in the momentum equation (4.27) and obtain

$$\int_I \int_\Omega \varrho \boldsymbol{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho \boldsymbol{u} \otimes \boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + \overline{p} \mathrm{div} \boldsymbol{\varphi} - \overline{\mathbb{S}} : \mathbb{D}\boldsymbol{\varphi} + \varrho \boldsymbol{f} \cdot \varphi \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t = 0 \tag{4.67}$$

for any $\boldsymbol{\varphi} \in \mathcal{T}$.

4.6 Pointwise convergence of the temperature

Let us start with following version of the Aubin-Lions lemma taken from [14] (Lemma 6.3)

Lemma 4.6.1. *Let $\{v_n\}_{n=1}^\infty$ be a sequence of functions bounded in*

$$L^2(I; L^q(\Omega)) \cap L^\infty(I; L^1(\Omega)), \quad \textit{for some } q > \frac{2N}{N+2}. \tag{4.68}$$

Furthermore, we assume that

$$\partial_t v_n \geq g_n \text{ in } \mathcal{D}'(I \times \Omega),$$

where g_n are bounded in $L^1(I; W^{-m,r}(\Omega))$ for certain $m \geq 1$, $r > 1$. Then we have

$$v_n \to v \quad \text{in } L^2(I; W^{-1,2}(\Omega)) \tag{4.69}$$

passing to the subsequence as the case may be.

In view of estimates (4.42 – 4.45), (4.52), we can apply Lemma 4.6.1 on the entropy inequality. In particular, we have $\varrho_n s_n$ already bounded in $L^\infty(I; L^1(\Omega))$ from the entropy estimate. On the other hand, since

$$\varrho_n s_n = C\vartheta_n^3 + \varrho_n S\big(\varrho_n \vartheta_n^{-\frac{3}{2}}\big),$$

we can estimate $\varrho_n s_n$ uniformly in $L^2(I; L^q(\Omega))$ for some $q > \frac{2N}{N+2} = \frac{6}{5}$ as follows

$$\vartheta_n^3 \text{ bounded in } L^\infty(I; L^{\frac{4}{3}}(\Omega)),\ \frac{4}{3} > \frac{6}{5},$$
$$\varrho_n \log \vartheta_n \text{ bounded in } L^2(I; L^p(\Omega)),\ p = \frac{30}{23} > \frac{6}{5},$$
$$\varrho_n S\big(\varrho_n \vartheta_n^{-\frac{3}{2}}\big) \leq C(\varrho_n \log \varrho_n + \varrho_n \log \vartheta_n) \text{ bounded in } L^2(I; L^p(\Omega)),\ \text{cf. (4.47)}.$$

Then Lemma 4.6.1 yields

$$\varrho_n s_n \to \overline{\varrho s} \qquad \text{in } L^2(I; W^{-1,2}(\Omega)),$$
$$\overline{\varrho s} = \overline{\varrho s_G} + \frac{4}{3} a \overline{\vartheta^3},$$

which together with (4.52) implies

$$\frac{4}{3} a(\vartheta \overline{\vartheta^3} - \overline{\vartheta}\,\overline{\vartheta^3}) = (\overline{\vartheta \varrho s_G} - \vartheta \overline{\varrho s_G}) \quad \text{a.e. on } Q. \tag{4.70}$$

The right-hand side integrated over a ball $B \subset Q$ can be split this way

$$\lim_{n\to\infty} \int_B \varrho_n s_G(\varrho_n, \vartheta_n)(\vartheta_n - \vartheta)\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t =$$
$$\lim_{n\to\infty} \int_B \varrho_n \big(s_G(\varrho_n, \vartheta_n) - s_G(\varrho_n, \vartheta)\big)(\vartheta_n - \vartheta) + \varrho_n s_G(\varrho_n, \vartheta)(\vartheta_n - \vartheta)\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t,$$

where the first term is non-negative by virtue of (1.38) and (1.34). In the sequel, we shall prove that the second term tends to zero. We use Theorem 2.2.7 to introduce Young measures $\nu^{(\varrho,\vartheta)}_{(t,\boldsymbol{x})}$, $\nu^{\varrho}_{(t,\boldsymbol{x})}$, and $\nu^{\vartheta}_{(t,\boldsymbol{x})}$ corresponding to the sequences (ϱ_n, ϑ_n), ϱ_n, and ϑ_n respectively. By virtue of (4.59) and compact imbedding one deduces

$$B(\varrho_n) \to \overline{B(\varrho)} \text{ strongly in } L^2(I; W^{-1,2}(\Omega))$$

for any $B \in W^{1,\infty}(\mathbf{R})$. On the other hand (4.52) implies

$$G(\vartheta_n) \to \overline{G(\vartheta)} \text{ weakly in } L^2(I; W^{1,2}(\Omega))$$

for $G \in W^{1,\infty}(\mathbf{R})$ at least for a subsequence. Then we have $\overline{B(\varrho)G(\vartheta)} = \overline{B(\varrho)}\,\overline{G(\vartheta)}$, which implies

$$\nu^{(\varrho,\vartheta)}_{(t,\boldsymbol{x})} = \nu^{\varrho}_{(t,\boldsymbol{x})} \otimes \nu^{\vartheta}_{(t,\boldsymbol{x})}.$$

Using this fact and Theorem 2.2.7, we can compute

$$\lim_{n\to\infty} \int_B \varrho_n s_g(\varrho_n,\vartheta)(\vartheta_n - \vartheta)\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t = \int_B \int_{\mathbf{R}^2} \lambda s_G(\lambda,\vartheta(\boldsymbol{x}))(\mu - \vartheta(\boldsymbol{x}))\,\mathrm{d}\nu^{(\varrho,\vartheta)}_{(t,\boldsymbol{x})}(\lambda,\mu)\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t =$$
$$\int_B \int_{\mathbf{R}} \lambda s_G(\lambda,\vartheta(\boldsymbol{x}))\,\mathrm{d}\nu^{\varrho}_{(t,\boldsymbol{x})}(\lambda) \int_{\mathbf{R}} \mu - \vartheta(\boldsymbol{x})\,\mathrm{d}\nu^{\vartheta}_{(t,\boldsymbol{x})}(\mu)\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t = 0 \quad (4.71)$$

Consequently, (4.70) yields

$$\overline{\vartheta^3}\vartheta \geq \overline{\vartheta^3}\,\overline{\vartheta} \qquad \text{a.e. on } Q. \quad (4.72)$$

To conclude the strong convergence we shall parform the so-called *Minty trick*. For any any $w \in \mathcal{D}(Q)$, we observe

$$0 \leq (\vartheta_n^3 - (\vartheta + \alpha w)^3)(\vartheta_n - (\vartheta + \alpha w)) \to \overline{\vartheta^4} - \overline{\vartheta^3}\vartheta - \alpha w(\overline{\vartheta^3} - (\vartheta + \alpha w)^3).$$

Then, using (4.72) and passing to the limit as $\alpha \to 0$, we get

$$0 \leq \int_Q w(\overline{\vartheta^3} - \vartheta^3)\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t.$$

Hence $\overline{\vartheta^3} = \vartheta^3$ and Theorem 2.2.3 yields

$$\vartheta_n \to \vartheta \text{ in } L^4(Q). \quad (4.73)$$

4.7 Strong convergence of the density field

In order to finish the limit passage, we have to show the strong compactness of the density. The proof is based on compactness properties of the quantity

$$P = p - (\lambda - 2\mu)\mathrm{div}\boldsymbol{u}$$

usually called the *effective viscous pressure*. Importance of this quantity in the theory of compressible fluids was first observed by Serre [42] and some regularity properties of P were discovered by Hoff [24]. Later on, P.-L. Lions has proved the celebrated relation

$$\overline{P}\,\overline{B(\varrho)} = \overline{P\,B(\varrho)}. \quad (4.74)$$

This result, together with the theory of renormalized solutions developed with DiPerna in [8], forms the corner stone of his existence theory for barotropic flows presented in [29].

Further significant improvements are due to Feireisl. His concept of oscillations defect measure allows to precisely analyse the transport of the density oscillations using the renormalized continuity equation, see Chapter 6 of [14]. Besides the better theory for barotropic flows, this precise analysis can handle also the pressure and the viscosities dependent on the temperature. In this section, we present Feireisl's method with a little modification of the estimates for the oscillations defect measure in Section 4.7.2. It seems that this approach could be less restrictive for the growth of transport coefficients, but we rather do not present the complete analysis in this direction in order to keep the complexity at a reasonable level.

4.7.1 Compactness of the effective viscous pressure

We consider a time interval $J \subset I$ and a ball $U \subset \Omega$ such that

$$\overline{J \times U} \subset Q_n^f, \quad \forall n \in \mathbb{N}.$$

Then we take $\psi \in \mathcal{D}(J)$ and $\eta, \xi \in \mathcal{D}(U)$ and we use the function

$$\psi\eta\boldsymbol{\varphi}_n = \psi(t)\eta(\boldsymbol{x})\nabla\Delta^{-1}[\xi(\boldsymbol{x})T_k^\nu(\varrho_n)], \quad 0 < \nu \leq 1,$$

as a test function in the momentum equation (4.12). Using the renormalized continuity equation

$$\partial_t\big(T_k^\nu(\varrho_n)\big) + \operatorname{div}\big(T_k^\nu(\varrho_n)\boldsymbol{u}_n\big) + \big[(T_k^\nu)'(\varrho_n)\varrho_n - T_k^\nu(\varrho_n)\big]\operatorname{div}\boldsymbol{u}_n = 0 \quad \text{in } \mathcal{D}'(I\times\Omega),$$

for the time term, we deduce

$$\begin{aligned}
\int_Q \psi\xi\big(\eta p_n - \mathcal{R} : [\eta\mathbb{S}_n]\big)\, T_k^\nu(\varrho_n)\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t = \qquad (4.75)\\
\int_Q \psi(\mathbb{S}_n - \varrho_n\boldsymbol{u}_n\otimes\boldsymbol{u}_n) : \nabla\eta\otimes\boldsymbol{\varphi}_n - \psi p_n\nabla\eta\cdot\boldsymbol{\varphi}_n)\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t\\
-\int_Q \eta\varrho_n\boldsymbol{u}_n\cdot\big(\partial_t\psi\boldsymbol{\varphi}_n + \psi\nabla\Delta^{-1}[\nabla\xi\cdot T_k^\nu(\varrho_n)\boldsymbol{u}_n]\big)\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t\\
-\int_Q \psi\eta\varrho_n\boldsymbol{u}_n\cdot\nabla\Delta^{-1}[\xi((T_k^\nu)'(\varrho_n)\varrho_n - T_k^\nu(\varrho_n))\operatorname{div}\boldsymbol{u}_n] + \psi\eta\boldsymbol{\varphi}_n\cdot(\varrho_n\boldsymbol{f})\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t\\
+\int_Q \psi\boldsymbol{u}_n\cdot\big(\mathcal{R}\cdot[\eta\varrho_n\boldsymbol{u}_n]\xi T_k^\nu(\varrho_n) - \eta\varrho_n\boldsymbol{u}_n\cdot\mathcal{R}[\xi T_k^\nu(\varrho_n)]\big)\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t.
\end{aligned}$$

On the other hand, testing the limit momentum equation (4.67) by the function

$$\psi\eta\overline{\boldsymbol{\varphi}} = \psi(t)\eta(\boldsymbol{x})\nabla\Delta^{-1}[\xi(\boldsymbol{x})\overline{T_k^\nu(\varrho)}], \quad 0 < \nu \leq 1,$$

using the L^1-limit of the renormalized continuity equation:

$$\partial_t\big(\overline{T_k^\nu(\varrho)}\big) + \operatorname{div}\big(\overline{T_k^\nu(\varrho)}\boldsymbol{u}\big) + \overline{\big[(T_k^\nu)'(\varrho)\varrho - T_k^\nu(\varrho)\big]\operatorname{div}\boldsymbol{u}} = 0 \quad \text{in } \mathcal{D}(I\times\Omega), \qquad (4.76)$$

we get

$$\begin{aligned}
\int_Q \psi\xi\big(\overline{\eta p - \mathcal{R} : [\eta\mathbb{S}]}\big)\,\overline{T_k^\nu(\varrho)}\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t = \qquad (4.77)\\
\int_Q \psi(\overline{\mathbb{S}} - \varrho\boldsymbol{u}\otimes\boldsymbol{u}) : \nabla\eta\otimes\overline{\boldsymbol{\varphi}} - \psi\overline{p}\nabla\eta\cdot\overline{\boldsymbol{\varphi}})\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t\\
-\int_Q \eta\varrho\boldsymbol{u}\cdot\big(\partial_t\psi\overline{\boldsymbol{\varphi}} + \psi\nabla\Delta^{-1}[\nabla\xi\cdot\overline{T_k^\nu(\varrho)}\boldsymbol{u}]\big)\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t\\
-\int_Q \psi\eta\varrho\boldsymbol{u}\cdot\nabla\Delta^{-1}[\xi\overline{\big((T_k^\nu)'(\varrho)\varrho - T_k^\nu(\varrho)\big)\operatorname{div}\boldsymbol{u}}] + \psi\eta\overline{\boldsymbol{\varphi}}\cdot(\varrho\boldsymbol{f})\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t\\
+\int_Q \psi\boldsymbol{u}\cdot\big(\mathcal{R}\cdot[\eta\varrho\boldsymbol{u}]\xi\overline{T_k^\nu(\varrho)} - \eta\varrho\boldsymbol{u}\cdot\mathcal{R}[\xi\overline{T_k^\nu(\varrho)}]\big)\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t.
\end{aligned}$$

In view of the available estimates, the right-hand side of (4.75) tends to the right-hand side of (4.77). In particular for the last terms, one can employ Corollary 2.2.5, using (4.59), (4.65), and a compact imbedding $L^q \subset W^{-1,2}$ for some $q > \frac{5}{4} > \frac{6}{5}$ to get

$$\mathcal{R}\cdot[\eta\varrho_n\boldsymbol{u}_n]\xi T_k^\nu(\varrho_n) - \eta\varrho_n\boldsymbol{u}_n\cdot\mathcal{R}[\xi T_k^\nu(\varrho_n)] \longrightarrow \mathcal{R}\cdot[\eta\varrho\boldsymbol{u}]\xi\overline{T_k^\nu(\varrho)} - \eta\varrho\boldsymbol{u}\cdot\mathcal{R}[\xi\overline{T_k^\nu(\varrho)}]$$

strongly in $C(I;W^{-1,2}(\Omega))$. Then the left-hand sides of (4.75), (4.77) yields an equality

$$\begin{aligned}\lim_{n\to\infty}\int_Q \psi\eta\xi\big(p_G(\varrho_n,\vartheta_n)+p_R^f(\vartheta_n)-(\zeta_n+\tfrac{4}{3}\mu_n)\mathrm{div}\,\boldsymbol{u}_n\big)\big(T_k^\nu(\varrho_n)-\overline{T_k^\nu(\varrho)}\big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ =\lim_{n\to\infty}\int_Q 2\psi\xi\Big(\mathcal{R}:[\eta\mu_n\nabla\boldsymbol{u}_n]-\eta\mu_n\mathcal{R}:[\nabla\boldsymbol{u}_n]\Big)\big(T_k^\nu(\varrho_n)-\overline{T_k^\nu(\varrho)}\big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \quad (4.78)\end{aligned}$$

Our next aim is to show that the right-hand side tend to zero. We apply Lemma 2.2.6 on the components of $\nabla\boldsymbol{u}_n \in L^2(Q;\mathbf{R}^3\times\mathbf{R}^3)$ and on the viscosity $\mu_n = \mu(\vartheta_n)$, which is bounded in $L^2(I;W^{1,2}(\Omega))$. As a consequence, we have

$$R_n := \mathcal{R}:[\eta\mu_n\nabla\boldsymbol{u}_n]-\eta\mu_n\mathcal{R}:[\nabla\boldsymbol{u}_n] \quad \text{bounded in } L^1(I;W^{\lambda,q}(\Omega))$$

for some $\lambda>0$, $q>1$. On the other hand, the same sequence belongs also to $L^2(I;L^{\frac{4}{3}}(\Omega))$ because of (4.45) and (4.49). Thus by an interplation argument we have R_n bounded in $L^p(I;W^{\lambda',p}(\Omega)$ for certain $\lambda'>0$, $p>1$. Further, taking in to account (4.59), the right-hand side of (4.78) tends to zero. Using the strong convergence of the temperature, the left-hand side of (4.78) implies

$$\begin{aligned}\lim_{n\to\infty}\int_Q \varphi(\zeta(\vartheta)+\frac{4}{3}\mu(\vartheta))\mathrm{div}\boldsymbol{u}_n(T_k^\nu(\varrho_n)-\overline{T_k^\nu(\varrho)})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ =\lim_{n\to\infty}\int_Q \varphi p_G(\varrho_n,\vartheta_n)(T_k^\nu(\varrho_n)-\overline{T_k^\nu(\varrho)})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \quad \text{for any } \varphi\in\mathcal{D}(Q^f). \quad (4.79)\end{aligned}$$

4.7.2 The density oscillations bounded

Our next task is to control the oscillations of the density estimating the quantity

$$\mathrm{osc}_p[\varrho_n\to\varrho] := \sup_{k\ge1}\limsup_{n\to\infty}\int_{Q^f}|T_k(\varrho_n)-T_k(\varrho)|^p\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

called *oscillations defect measure*. To this end we shall use few simple algebraic inequalities. Let $0\le b\le a<\infty$, then

$$a^\gamma-b^\gamma=\int_b^a\gamma t^{\gamma-1}\,\mathrm{d}t\ge\int_a^b\gamma(t-b)^{\gamma-1}\,\mathrm{d}t=(a-b)^\gamma,\ \text{for } \gamma\ge1, \quad (4.80)$$

while, by the same token,

$$(a-b)^\gamma\ge a^\gamma-b^\gamma,\ \text{for } \gamma\le1. \quad (4.81)$$

Further, assuming $\gamma\ge1$, $\nu\le1$, $\gamma+\nu\ge2$, we observe that

$$\nu|1-x|^{\gamma+\nu}\le\nu(1-x)^2\le(1-x)(\nu-\nu x)\le(1-x^\gamma)(1-x^\nu),\ \text{for } 0\le x\le1.$$

Then, taking $x = \frac{b}{a}$, we obtain

$$|a-b|^{\gamma+\nu} \leq \frac{1}{\nu}(a^\gamma - b^\gamma)(a^\nu - b^\nu). \tag{4.82}$$

With a help of these inequalities, taking $c = \nu p_c(1)$, $1 \geq \nu \geq \frac{1}{3}$, we can estimate the oscillations defect measure as follows

$$\begin{aligned} c\limsup_{n\to\infty}\int_{Q^f} |T_k(\varrho_n) - T_k(\varrho)|^{\frac{5}{3}+\nu}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t &\leq c\limsup_{n\to\infty}\int_{Q^f}(\varrho_n^{\frac{5}{3}} - \varrho^{\frac{5}{3}})(T_k^\nu(\varrho_n) - T_k^\nu(\varrho))\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ = \limsup_{n\to\infty}\int_{Q^f} p_c(\varrho_n)\big(T_k^\nu(\varrho_n) - \overline{T_k^\nu(\varrho)}\big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_{Q^f}\underbrace{(\overline{p_c(\varrho)} - p_c(\varrho))(T_k^\nu(\varrho) - \overline{T_k^\nu(\varrho)})}_{\triangle}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ \leq \limsup_{n\to\infty}\int_{Q^f}|p_c(\varrho_n) - p_G(\varrho_n,\vartheta_n)||T_k^\nu(\varrho_n) - \overline{T_k^\nu(\varrho)}|\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ + \limsup_{n\to\infty}\int_{Q^f}(\zeta(\vartheta) + \frac{4}{3}\mu(\vartheta))\mathrm{div}\boldsymbol{u}_n(T_k^\nu(\varrho_n) - \overline{T_k^\nu(\varrho)})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t. \end{aligned} \tag{4.83}$$

The latter inequality, $\triangle \geq 0$, is due to the convexity of the functions $\varrho \to \varrho^\gamma$ and $\varrho \to -T_k(\varrho)$, while in the very last term we have used (4.79). We continue by the estimate of the right-hand side. In accordance with the hypothesis (4.18), we have

$$\begin{aligned} 0 \leq p_G(\varrho,\vartheta) - p_c(\varrho) = \int_0^\vartheta \partial_\vartheta p_G(\varrho,s)\,\mathrm{d}s = \int_0^\vartheta \frac{3}{2}\frac{\varrho}{Y}\big(\frac{5}{3}P(Y) - P'(Y)Y\big)\,\mathrm{d}s \\ \leq C(\vartheta^{\frac{3}{2}} + \varrho^\alpha\vartheta^{\frac{5-3\alpha}{2}}). \end{aligned} \tag{4.84}$$

Hence, by virtue of the estimates (4.44), (4.52), we conclude

$$\big\|p_g(\varrho_n,\vartheta_n) - p_c(\varrho_n)\big\|_{L^q(Q)} \leq C, \text{ for certain } q > \frac{6}{5},$$

provided

$$\frac{\alpha}{\frac{5}{3}} + \frac{\frac{5-3\alpha}{2}}{9} < \frac{5}{6},$$

which is equivalent to the condition for alpha in (4.18). On the other hand, due to (4.22), for any $q > \frac{6}{5}$, we get

$$\big\|(\zeta(\vartheta) + \frac{4}{3}\mu(\vartheta))\mathrm{div}\boldsymbol{u}_n\big\|_{L^q(\Omega)} \leq \big\|\vartheta_n\big\|_{L^{\frac{17}{3}}(Q)}\big\|\mathrm{div}\,\boldsymbol{u}\big\|_{L^2(Q)} \leq C.$$

In order to estimate the right-hand side of (4.83), we use weak lower semi-continuity of the norm and inequality (4.81) to infer

$$\begin{aligned} \limsup_{n\to\infty}\int_{Q^f}|T_k^\nu(\varrho_n) - \overline{T_k^\nu(\varrho)}|^p\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \leq \limsup_{n\to\infty}\big\|T_k(\varrho_n) - T_k(\varrho)\big\|^{p\nu}_{L^{p\nu}(Q^f)} \\ + \big\|T_k^\nu(\varrho) - \overline{T_k^\nu(\varrho)}\big\|^p_{L^p(Q^f)} \leq 2\limsup_{n\to\infty}\big\|T_k(\varrho_n) - T_k(\varrho)\big\|^{p\nu}_{L^{p\nu}(Q^f)}, \end{aligned}$$

for any $p \geq \nu^{-1}$. If we assume $p = q'$ and $p\nu = \frac{5}{3} + \nu$, we can estimate the right-hand side of (4.83) and get

$$\limsup_{n\to\infty} \int_{Q^f} |T_k(\varrho_n) - T_k(\varrho)|^{\frac{5}{3}+\nu} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \leq C \limsup_{n\to\infty} \big\|T_k(\varrho_n) - T_k(\varrho)\big\|_{L^{\frac{5}{3}+\nu}(Q^f)},$$

while the condition $q > \frac{6}{5}$ is equivalent with $\nu > \frac{1}{3}$. Then, using conveniently Young's inequality, we conclude

$$\mathrm{osc}_r[\varrho_n \to \varrho] \leq C \quad \text{for some } r > 2. \tag{4.85}$$

4.7.3 Limit in the renormalized continuity equation

As a next step, we will proof that ϱ, $\boldsymbol{u}$ solve the renormalized continuity equation (3.9). Following the proof in [14, Proposition 6.3], we apply the renormalization procedure of Lemma 3.1.2 on the equation (4.76) with $\nu = 1$ in order to get

$$\partial_t(B(\overline{T_k(\varrho)})) + \mathrm{div}(B(\overline{T_k})\boldsymbol{u}) + b(\overline{T_k(\varrho)})\mathrm{div}\boldsymbol{u} = B'(\overline{T_k(\varrho)})\overline{\big[T_k(\varrho) - T_k'(\varrho)\varrho\big]\mathrm{div}\boldsymbol{u}} \text{ in } \mathcal{D}'(I\times\Omega) \tag{4.86}$$

for any $B(z)$ satisfying (3.7). In particular, we have $B(z) = C_M$ for $z \geq M$. Due to the weak lower semi-continuity of the norm we get an estimate

$$\big\|\overline{T_k(\varrho)} - \varrho\big\|_{L^1(Q)} \leq \liminf_{n\to\infty} \big\|T_k(\varrho_n) - \varrho_n\big\|_{L^1(Q)} \leq \sup_{n\geq 1} \big\|\varrho_n\big\|_{L^1(\{\varrho_n \geq k\})} \leq k^{-\frac{2}{5}} \sup_{n\geq 1} \big\|\varrho_n\big\|_{\frac{5}{3},Q}$$

and consequently

$$B(\overline{T_k(\varrho)}) \to B(\varrho) \text{ and } b(\overline{T_k(\varrho)}) \to b(\varrho) \text{ in any } L^p(Q),\ p \geq 1,\ \text{as } k \to \infty. \tag{4.87}$$

It remains to show that the right-hand side of (4.86) tends to zero. To this end, we estimate

$$\big\|B'(\overline{T_k(\varrho)})\overline{\big[T_k(\varrho) - T_k'(\varrho)\varrho\big]\mathrm{div}\boldsymbol{u}}\big\|_{L^1(Q)} \leq \max_{0\leq z\leq M} |B'(z)| \sup_{n\geq 1} \big\|\mathrm{div}\boldsymbol{u}_n\big\|_{L^2(Q)} \liminf_{n\to\infty} \big\|T_k(\varrho_n) - T_k'(\varrho_n)\varrho_n\big\|_{L^2(Q_M^f)},$$
$$Q_M^f = \{\overline{T_k(\varrho)} \leq M\} \cap Q^f,$$

where we have used fact that $\mathrm{div}\boldsymbol{u}_n$ tends to zero on Q^s as a consequence of (4.50). We shall continue by interpolation of the last term

$$\big\|T_k(\varrho_n) - T_k'(\varrho_n)\varrho_n\big\|_{L^2(Q_M^f)} \leq \big\|T_k(\varrho_n) - T_k'(\varrho_n)\varrho_n\big\|_{L^1(Q^f)}^{\lambda} 2\big\|T_k(\varrho_n)\big\|_{L^p(Q_M^f)}^{1-\lambda} \tag{4.88}$$

for certain $p > 2$ and $\lambda \in (0,1)$. Now the first norm tends to zero, since

$$\big\|T_k(\varrho_n) - T_k'(\varrho_n)\varrho_n\big\|_{L^1(Q^f)} \leq \int_{\{\varrho_n \geq k\}} \varrho_n \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \leq k^{-\frac{2}{5}} \sup_{n\geq 1} \big\|\varrho_n\big\|_{L^{\frac{5}{3}}(Q)},$$

while the second is bounded due to (4.85):

$$\limsup_{n\to\infty} \big\|T_k(\varrho_n)\big\|_{L^p(Q_M^f)} \leq \limsup_{n\to\infty} \big\|T_k(\varrho_n) - T_k(\varrho)\big\|_{L^p(Q^f)} + \big\|T_k(\varrho) - \overline{T_k(\varrho)}\big\|_{L^p(Q^f)} + \big\|\overline{T_k(\varrho)}\big\|_{L^p(Q_M^f)} \leq 2\,\mathrm{osc}_p[\varrho_n \to \varrho](Q^f) + M|Q^f|^{\frac{1}{p}}.$$

4.7.4 Strong convergence

Having the renormalized continuity equation (3.9) satisfied by the limit fields ϱ, $\boldsymbol{u}$, we can take its difference with L^1-limit of (4.26). Using

$$B(z) := L_k(z)z,\ L_k := \int_1^z \frac{T_k(s)}{s^2}\,\mathrm{d}s,$$

which is a valid B-function in view of Proposition 3.1.3, we have

$$\int_\Omega \big(\overline{\varrho L_k(\varrho)} - \varrho L_k(\varrho)\big)[\tau]\ \mathrm{d}\boldsymbol{x} = \int_\Omega \big(\overline{\varrho L_k(\varrho)} - \varrho L_k(\varrho)\big)[0]\ \mathrm{d}\boldsymbol{x} + \int_0^\tau \int_\Omega \big(T_k(\varrho) - \overline{T_k(\varrho)}\big)\mathrm{div}\boldsymbol{u}\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t + \int_0^\tau \int_\Omega \overline{T_k(\varrho)}\mathrm{div}\boldsymbol{u} - \overline{T_k(\varrho)\mathrm{div}\boldsymbol{u}}\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t. \quad (4.89)$$

The first term on the right-hand side is in fact zero as $\varrho_n[0] = \varrho_0$. The second term tends to zero as $k \to \infty$ by the same argument as above, namely

$$\big\|T_k(\varrho) - \overline{T_k(\varrho)}\big\|_{2,Q^f} \le \liminf_{n\to\infty} \big\|T_k(\varrho) - T_k(\varrho_n)\big\|_1^\lambda (\mathrm{osc}_p[\varrho_n \to \varrho](Q^f))^{1-\lambda} \le Ck^{-\frac{2}{5}}$$

for suitable $\lambda \in (0,1)$, $p > 2$, while $\mathrm{div}\boldsymbol{u} = 0$ on Q^s. It remains to manage the third term. Since the pressure $p_G(\varrho,\vartheta)$ is a monotone function of the density, we have

$$\big(p_G(\varrho_n,\vartheta_n) - p_G(\varrho,\vartheta_n)\big)\big(\overline{T_k(\varrho)} - T_k(\varrho_n)\big) \le 0,$$

whereas, due to the strong convergence of the temperature,

$$p_G(\varrho,\vartheta_n)\big(\overline{T_k(\varrho)} - T_k(\varrho_n)\big) \to 0 \quad \text{as } n \to \infty.$$

Then (4.79) with $\nu = 1$ yields

$$\overline{T_k(\varrho)}\mathrm{div}\boldsymbol{u} - \overline{T_k(\varrho)\mathrm{div}\boldsymbol{u}} \le 0 \quad \text{on } Q^f,$$

while on the solid region Q^s, we have

$$\overline{T_k(\varrho)}\mathrm{div}\boldsymbol{u} - \overline{T_k(\varrho)\mathrm{div}\boldsymbol{u}} = 0$$

because of (4.50).

Passing to the limit in (4.89) as $k \to \infty$, we conclude

$$\int_\Omega \big(\overline{\varrho\log\varrho} - \varrho\log\varrho\big)(\tau)\ \mathrm{d}\boldsymbol{x} \le 0.$$

Hence by virtue of Theorem 2.2.3 we obtain the pointwise convergence of the density,

$$\varrho_n \to \varrho \quad \text{a.e. on } Q. \quad (4.90)$$

4.8 Limit passage - continued

After we have established the strong convergence of the temperature and the density, we can finish the limit passage. The continuity equation was well as its renormalized version are

already verified. In the momentum equation (4.67), we identify $\overline{p}$, $\overline{\mathbb{S}}$ with p and $\mathbb{S}$, respectively and derive (4.12). Concerning the entropy inequality, the first three terms in (4.28), namely $\varrho_n s_n$, $\varrho_n s_n \boldsymbol{u}_n$, and $\boldsymbol{q}_n \vartheta_n^{-1} = -\kappa[\boldsymbol{u}_n](\vartheta_n)\nabla \log \vartheta_n$, are equi-integrable especially due to the estimate (4.68) for the entropy $\varrho_n s_n$. Consequently, these terms converge to its counterparts in (4.14). Next, as the entropy production rate is a convex function, which is weakly lower semi-continuous, we have

$$\liminf_{n\to\infty} \int_Q \sigma_n \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \liminf_{n\to\infty} \int_Q |A_n|^2 + |B_n|^2 + |C_n|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \geq \int_Q |\overline{A}|^2 + |\overline{B}|^2 + |\overline{C}|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

where

$$A_n = \sqrt{\frac{\mu(\vartheta_n)}{\vartheta_n}}\big(2\mathbb{D}\boldsymbol{u}_n - \frac{1}{3}\mathrm{div}\,\boldsymbol{u}_n\mathbb{I}\big), \quad B_n = \sqrt{\frac{\zeta(\vartheta_n)}{\vartheta_n}}\mathrm{div}\,\boldsymbol{u}_n, \quad C_n = \frac{\sqrt{[\boldsymbol{u}_n](\vartheta_n)}}{\vartheta_n}\nabla\vartheta_n.$$

Using the strong convergence of the temperature, we observe that $\overline{A}^2 + \overline{B}^2 + \overline{C}^2 = \sigma$, which finish the proof of (4.14).

Finally, we have to deal with the energy inequality. As was mentioned above in Section 4.4.3 the internal energy is not equi-integrable on the solid region. Fortunately, it can be split into the convex and the bounded part as follows

$$\varrho_n e_n = \varrho^{\frac{5}{3}} + \frac{3}{2}\big(p_G(\varrho_n,\vartheta_n) - p_c(\varrho_n)\big) + a\vartheta_n^4.$$

The first term is convex and thus weakly lower semi-continuous, while the other terms are equi-integrable, in particular the second term can be treated like in (4.84). Since the other terms in (4.29) are equi-integrable, we can pass to the limit and obtain (4.15). The initial conditions are the direct consequence of the initial conditions specified in Theorem (3.1.6).

4.9 Partial results and open problems

I have started the work on my thesis with belief that the existence problem for a compressible fluid with rigid bodies can be solved by a direct penalization of the corresponding problem for a fluid with heterogeneous constitutive laws. We have proved that this can be done assuming homogeneous constitutive equations for the pressure, the internal energy, and the entropy. A distinct internal energy for the fluid and for the bodies can not be used, when the high viscosity limit takes the last place in the chain of approximations. The reason is that the variational formulation is based on the entropy inequality together with the total energy balance, where a calculation similar to (1.18) is crucial for the formal representation as well as for the construction of solutions. However, in the case of the heterogeneous internal energy there appears a new term in this calculation:

$$\begin{aligned}\frac{1}{\vartheta}\big(\partial_t(\varrho e) + \mathrm{div}(\varrho\boldsymbol{u}e) + p\mathrm{div}\boldsymbol{u}\big) &= \varrho(\boldsymbol{u}-\boldsymbol{U})\cdot(\vartheta^{-1}\partial_{\boldsymbol{x}}e) + \\ \varrho\Big(\frac{1}{\vartheta}\frac{\partial e}{\partial\vartheta}(\partial_t\vartheta + \boldsymbol{u}\cdot\nabla\vartheta) - \frac{2}{3\varrho}\frac{\partial e}{\partial\vartheta}(\partial_t\varrho + \boldsymbol{u}\cdot\nabla\varrho)\Big) + \frac{p}{\vartheta\varrho}\partial_t\varrho + \frac{p}{\vartheta\varrho}\boldsymbol{u}\cdot\nabla\varrho + \frac{p}{\vartheta\varrho}\varrho\mathrm{div}\boldsymbol{u} &= \\ \partial_t(\varrho s) + \mathrm{div}(\varrho\boldsymbol{u}s) + \varrho(\boldsymbol{u}-\boldsymbol{U})\cdot(\vartheta^{-1}\partial_{\boldsymbol{x}}e - \partial_{\boldsymbol{x}}s),\end{aligned}$$

where $\boldsymbol{U}$ is some velocity field, which transports the heterogeneous constitutive equations, namely if this field is compatible with some displacement $\boldsymbol{\eta}$, it holds

$$e(t, \boldsymbol{x}; \varrho, \vartheta) = e(0, \boldsymbol{\eta}[-t](\boldsymbol{x}); \varrho, \vartheta).$$

Now it is clear that the term $\varrho(\boldsymbol{u} - \boldsymbol{U}) \cdot \partial_{\boldsymbol{x}} D$, $D = e_{t,\boldsymbol{x}} - \vartheta s_{t,\boldsymbol{x}}$ has to be added into the total energy balance. After the high viscosity limit this term should disappear, since $\partial_{\boldsymbol{x}} D = 0$ on the fluid part and $\boldsymbol{u} = \boldsymbol{U}$ on the bodies, but the problem is how to prove it. The problem is even how to bound this term, since a rough calculation

$$\partial_{\boldsymbol{x}} e(t, \boldsymbol{x}) = \partial_{\boldsymbol{x}} e(0, \boldsymbol{\eta}[-t](\boldsymbol{x})) = \partial_{\boldsymbol{x}_0}(0, \boldsymbol{x}_0) \cdot \nabla \boldsymbol{\eta}[-t](\boldsymbol{x}),$$
$$\frac{\mathrm{d}}{\mathrm{d}t} \nabla \boldsymbol{\eta}[t](\boldsymbol{x}_0) = \nabla \boldsymbol{U}(t, \boldsymbol{\eta}[t](\boldsymbol{x}_0)) \nabla \boldsymbol{\eta}[t](\boldsymbol{x}_0)$$

indicates that $\partial_{\boldsymbol{x}} D$ could have an exponential grow in time.

4.9.1 Total energy equality

One of the considerable achievements in the theory of the full Navier-Stokes-Fourier system is the possibility to construct weak solutions, which preserve the total energy. In particular (4.15) holds with the equality sign. It is natural to ask whether or not the system fluid-bodies enjoy the same property. After the first contact it is unlikely to be true, as the weak solutions does not cope well with instantenous contacts of the rigid bodies. On the other hand, up to the first contact it is probably true. In this section we shall outline some ideas of the proof.

The reason, why we were not able to prove the total energy balance with the equality sign, is that we lack an L^p-estimate for the pressure and/or the internal energy on the whole domain Ω. To overcome this obstacle, we can perform the high viscosity limit before the vanishing viscosity limit. There one can use the term $\vartheta^{-1}|\varrho^{\frac{\beta}{2}}|^2$ to control ϱ^β, the dominant part of the pressure at this approximation level. After the high viscosity limit, the density is perfectly transported on the bodies, thus we need an L^p-estimate for the pressure only on Q^f up to the boundary. This problem, we shall discuss for the last step: vanishing artificial pressure limit.

The idea is still the same as in [18] namely to test the momentum equation by $\mathrm{div}^{-1}\varrho^\nu$. Unfortunately it seems that one can not use the Bogovskii operator, since it is composed in the nonconstructive way from operators on star-shaped subdomains and therefore we don't know how to identify its time derivative, namely $\partial_t \mathcal{B}_{\Omega[t]}(\varrho^\nu)$. Therefore, we better follow Geissert, Heck, Hieber [23] and use the Stokes problem to construct a sort of div^{-1} operator on the time dependent domain $\Omega[t]$ as well as its time derivative.

Suitable inverse of the divergence

The classical result concerning the Stokes problem reads:

Proposition 4.9.1. *Let $1 < q < \infty$ and Ω be a bounded domain with C^2 boundary. Then for any $\boldsymbol{f} \in W^{-1,q}(\Omega; \mathbf{R}^N)$ and $g \in \overline{L^q}(\Omega)$, there exists one and only one solution*

$$(\boldsymbol{v}, p) \in W_0^{1,q}(\Omega; \mathbf{R}^N) \times \overline{L^q}(\Omega)$$

of the system

$$-\Delta \boldsymbol{v} + \nabla p = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{v} = g \qquad \text{on } \Omega, \tag{4.91}$$

which satisfies the estimate

$$\|\nabla \boldsymbol{v}\|_{L^q(\Omega)} + \|p\|_{L^q(\Omega)} \le C(\Omega, q, N)(\|\boldsymbol{f}\|_{W^{-1,q}(\Omega)} + \|g\|_{L^q(\Omega)}. \tag{4.92}$$

Moreover, for $\boldsymbol{f} \in L^q(\Omega)$, $g \in W^{1,q}(\Omega)$ *we have*

$$\|\nabla \boldsymbol{v}\|_{W^{1,q}(\Omega)} + \|p\|_{W^{1,q}(\Omega)} \le C(\Omega, q, N)(\|\boldsymbol{f}\|_{L^q(\Omega)} + \|g\|_{W^{1,q}(\Omega)}). \tag{4.93}$$

This version is taken from [23], but the original works are due to Catabriga [5] and Galdi, Simander [21], [20] and others. In addition to the estimates (4.92), (4.93), it holds

$$\|\boldsymbol{v}\|_{L^q(\Omega)} \le C(\Omega, q, N)\Big(\|\boldsymbol{f}\|_{W^{-2,q}(\Omega;\mathbf{R}^N)} + \|g\|_{W^{-1,q}(\Omega)}\Big). \tag{4.94}$$

Indeed, taking an arbitrary $\tilde{\boldsymbol{\varphi}} \in L^{q'}(\Omega; \mathbf{R}^N)$, we can solve the Stokes problem

$$-\Delta \boldsymbol{\varphi} - \nabla \psi = \tilde{\boldsymbol{\varphi}}, \quad \operatorname{div} \boldsymbol{\varphi} = 0 \qquad \text{on } \Omega,$$

to get $\boldsymbol{\varphi} \in W_0^{2,q'}(\Omega; \mathbf{R}^N)$ and $\psi \in W^{1,q'}(\Omega)$ satisfying

$$\|\boldsymbol{\varphi}\|_{W^{2,q'}(\Omega)} + \|\psi\|_{W^{1,q'}(\Omega)} \le C\|\tilde{\boldsymbol{\varphi}}\|_{L^{q'}(\Omega)}.$$

Now, if we test equations (4.91) by $\boldsymbol{\varphi}$, ψ respectively, we get

$$\int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} = \int_\Omega -\boldsymbol{v} \cdot \Delta \boldsymbol{\varphi} - p \operatorname{div} \boldsymbol{\varphi} = \int_\Omega \boldsymbol{v} \cdot \tilde{\boldsymbol{\varphi}} + \boldsymbol{v} \cdot \nabla \psi \, \mathrm{d}\boldsymbol{x}, \tag{4.95}$$

$$\int_\Omega g\psi \, \mathrm{d}\boldsymbol{x} = -\int_\Omega \boldsymbol{v} \cdot \nabla \psi \, \mathrm{d}\boldsymbol{x}. \tag{4.96}$$

Then we can estimate the L^q-norm of $\boldsymbol{v}$ as follows

$$\sup_{\tilde{\boldsymbol{\varphi}} \in L^{q'}} \int_\Omega \boldsymbol{v} \cdot \tilde{\boldsymbol{\varphi}} \, \mathrm{d}\boldsymbol{x} = \sup \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} + g\psi \, \mathrm{d}\boldsymbol{x} \le C(\|f\|_{W^{-2,q}} + \|g\|_{W^{-1,q}})\|\tilde{\boldsymbol{\varphi}}\|_{L^{q'}}. \tag{4.97}$$

Considering the problem (4.91) with $\boldsymbol{f} = 0$, Proposition 4.9.1 yields the linear bounded operator

$$\mathcal{H} : \overline{L^q}(\Omega) \to W_0^{1,q}(\Omega; \mathbf{R}^N),$$

defined by the formula $\mathcal{H}[g] := \boldsymbol{v}$ and satisfying

$$\operatorname{div} \mathcal{H}[g] = g.$$

Our next aim is to identify the time derivative $\partial_t \mathcal{H}[g]$ for an operator $\mathcal{H}$ on the time dependent domain $\Omega[t]$ and with time dependent $g = g[t]$. The proof of the following proposition is inspirated by paper [43] due to Simon, which deals with the related shape optimization problem.

Proposition 4.9.2. *Let Ω_0 be a bounded domain in $\mathbf{R}^3$ and $J=(0,\tau)$ a short time interval. Let $\boldsymbol{U}[t]\equiv\boldsymbol{U}\in W^{1,\infty}(\mathbf{R}^3;\mathbf{R}^3)$, be some velocity field and $\boldsymbol{\eta}$ the displacement mapping compatible with it on J. Let Ω_0 as well as $\Omega[t]:=\boldsymbol{\eta}[t](\Omega_0)$ be domains with C^2-boundary. Assume $g[t]$ from $\overline{L^p}(\Omega)\cap W^{1,q}(\Omega)$ for a.e. $t\in J$ and denote $\boldsymbol{v}[t]=\mathcal{H}_{\Omega[t]}(g[t])$. Then if $\partial_t g[0]$ belongs to $L^q(\Omega)$, the time derivative $\partial_t\boldsymbol{v}=\partial_t\mathcal{H}_{\Omega[t]}[g[t]]|_{t=0}$ is the (unique) solution of the non-homogeneous Stokes problem*

$$\left.\begin{aligned} -\Delta\partial_t\boldsymbol{v}+\nabla\partial_t p=0,\quad \operatorname{div}\partial_t\boldsymbol{v}=\partial_t g[0] \qquad & \text{on } \Omega_0,\\ \partial_t\boldsymbol{v}=-(\boldsymbol{U}\cdot\boldsymbol{n})\partial_{\boldsymbol{n}}\boldsymbol{v} \qquad & \text{on } \partial\Omega_0,\\ \int_{\Omega_0}\partial_t p\,\mathrm{d}\boldsymbol{x}=-\int_{\partial\Omega_0}(\boldsymbol{U}\cdot\boldsymbol{n})p[0]\,\mathrm{d}\sigma, & \end{aligned}\right\} \tag{4.98}$$

and following estimates hold

$$\begin{aligned} \|\partial_t\boldsymbol{v}\|_{W^{1,q}(\Omega_0;\mathbf{R}^3)}&\le C(\|g\|_{W^{1,q}(\Omega_0)}\|\boldsymbol{U}\|_{W^{1,\infty}(\Omega_0;\mathbf{R}^3)}+\|\partial_t g\|_{L^q(\Omega_0)}),\\ \|\partial_t\boldsymbol{v}\|_{L^q(\Omega_0)}&\le C(\|g\|_{L^q(\Omega_0)}\|\boldsymbol{U}\|_{L^\infty(\Omega_0)}+\|\partial_t g\|_{W^{-1,q}(\Omega_0)}). \end{aligned} \tag{4.99}$$

Proof. By virtue of Proposition 4.9.1, the problem (4.91) with $\boldsymbol{f}=0$ and on the domain $\Omega[t]$ has the solution $(\boldsymbol{v}[t],p[t])$ unique in class $W^{1,q}_0(\Omega;\mathbf{R}^3)\times\overline{L^q}(\Omega)$ for a.e. $t\in J$. Since the right-hand sides of equations (4.91) are differentiable, we can differentiate both equations. Further, we differentiate the condition $\boldsymbol{v}=0$ on $\partial\Omega[t]$. For every $\boldsymbol{x}=\boldsymbol{\eta}[t](\boldsymbol{x}_0)$ from $\partial\Omega[t]$, we have

$$0=\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}(\boldsymbol{\eta}[t](\boldsymbol{x}_0))|_{t=0}=\partial_t\boldsymbol{v}(0,\boldsymbol{x}_0)+(\boldsymbol{U}(\boldsymbol{x}_0)\cdot\nabla)\boldsymbol{v}(0,\boldsymbol{x}_0),\quad \boldsymbol{x}_0\in\partial\Omega_0.$$

Moreover, on $\partial\Omega[t]$, the function v is constant in directions tangential to the boundary, thus $(\boldsymbol{U}\cdot\nabla)\boldsymbol{v}=(\boldsymbol{U}\cdot\boldsymbol{n})\partial_{\boldsymbol{n}}\boldsymbol{v}$, noting that $\boldsymbol{v}\in W^{2,q}(\Omega;\mathbf{R}^3)$ since $g\in W^{1,q}(\Omega)$. Differentiating also the condition $0=\int_\Omega[t]p[t]\,\mathrm{d}\boldsymbol{x}$, we obtain the system (4.98). Next, we transform this non-homogeneous system to the homogeneous one in order to apply Proposition 4.9.1. Setting

$$\boldsymbol{v}_0=(\boldsymbol{U}\cdot\nabla)\boldsymbol{v}\quad\text{and}\quad p_0=-\int_{\partial\Omega[t]}(\boldsymbol{U}[t]\cdot\boldsymbol{n})p[t]\,\mathrm{d}\sigma,$$

the functions $\partial_t\boldsymbol{v}+\boldsymbol{v}_0$, $\partial_t p+p_0$ form the solution of a homogeneous Stokes problem with the right-hand side $\tilde{\boldsymbol{f}}=\Delta\boldsymbol{v}_0$, $\tilde g=\partial_t g+\operatorname{div}\boldsymbol{v}_0$. In accordance with our assumptions, we have $\tilde{\boldsymbol{f}}\in W^{-1,2}(\Omega_0,\mathbf{R}^3)$ and $\tilde g\in L^q(\Omega_0)$. It remains to show that $\tilde g$ is even in $\overline{L^q}(\Omega)$. First, we compute

$$\int_{\Omega[t]}\operatorname{div}\boldsymbol{v}_0\,\mathrm{d}\boldsymbol{x}=\int_{\partial\Omega[t]}(\boldsymbol{U}\cdot\boldsymbol{n})(n\cdot\partial_{\boldsymbol{n}}\boldsymbol{v})\,\mathrm{d}\sigma=\int_{\partial\Omega[t]}(\operatorname{div}\boldsymbol{v}\boldsymbol{U})\boldsymbol{n}\,\mathrm{d}\sigma=\int_{\Omega[t]}\operatorname{div}(g\boldsymbol{U}),$$

where in the first two equalities we have used once more that $\boldsymbol{v}$ is the constant zero in the directions tangential to the boundary. The third one is because of the fact, that $\operatorname{div}\boldsymbol{v}=g\in W^{1,q}(\Omega_0)$ has a trace on $\partial\Omega_0$. Then

$$\int_\Omega\tilde g\,\mathrm{d}\boldsymbol{x}=\int_\Omega\partial_t g+\operatorname{div}(g\boldsymbol{U})\,\mathrm{d}\boldsymbol{x}=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega[t]}g\,\mathrm{d}\boldsymbol{x}=0.$$

Now, we can apply Proposition 4.9.1 to obtain the solution $(\tilde{\boldsymbol{v}},\tilde p)$ of the homogeneous problem and also the solution of the original non-homogeneous problem:

$$\partial_t\boldsymbol{v}=\tilde{\boldsymbol{v}}-\boldsymbol{v}_0,\quad \partial_t p=\tilde p-\boldsymbol{p}_0,$$

which is unique in the class $W^{1,q}(\Omega;\mathbf{R}^3)\times\overline{L^q}(\Omega)$. Finally, by virtue of estimates (4.92), (4.94), we infer

$$\begin{aligned}\big\|\partial_t\boldsymbol{v}\big\|_{1,q}&\le C\Big(\big\|\boldsymbol{v}_0\big\|_{1,q}+\big\|\partial_t g\big\|_q\Big)\le C\Big(\big\|g\big\|_{1,q}\big\|\boldsymbol{U}\big\|_{1,\infty}+\big\|\partial_t g\big\|_q\Big),\\ \big\|\partial_t\boldsymbol{v}\big\|_{q}&\le C\Big(\big\|\boldsymbol{v}_0\big\|_{q}+\big\|\partial_t g\big\|_{-1,q}\Big)\le C\Big(\big\|g\big\|_{q}\big\|\boldsymbol{U}\big\|_{\infty}+\big\|\partial_t g\big\|_{-1,q}\Big),\end{aligned}$$

where all norms are considered on Ω_0. □

Refined pressure estimate on the whole fluid region

Since, we work on the time interval I up to the first contact, there exists an artificial velocity field $\boldsymbol{U}\in L^2(I;W^{3,\infty}(\Omega;\mathbf{R}^3))$, which coincides with the actual velocity $\boldsymbol{u}$ on the bodies. There exists also displacement mapping $\boldsymbol{\eta}$, which coincide with $\boldsymbol{\eta}^i$ on the bodies and which is compatible with $\boldsymbol{U}$. We consider the operator $\mathcal{H}$ on the domain

$$F[t]=\Omega^f[t]=\boldsymbol{\eta}[t](F_0),\quad F_0=\Omega\setminus\bigcup_{i=1}^N S^i.$$

We assume that F_0 is bounded domain of the class C^2. Then the same holds for domains $F[t]$ because of the regular velocity field $\boldsymbol{U}$.

Next, taking a fixed $\epsilon>0$, and $t\in I$, we shall use the following test function in the momentum equation:

$$\boldsymbol{\varphi}=\mathcal{H}[\pi_\epsilon]=\mathcal{H}_{F[t]}[\pi_\epsilon[t]],\ \pi_\epsilon[t]=\omega_\epsilon\star\varrho^\nu[t]-\fint_{F[t]}\omega_\epsilon\star\varrho^\nu[t],\quad \nu>0,$$

where ω_ϵ are smoothing kernels (2.1) acting in the space variable. Obviously, $\pi_\epsilon[t]$ belongs to $W^{1,q}(\mathbf{R}^3)\cap\overline{L^q}(F[t])$ for any $q>1$ and for a.e. $t\in I$. On the other hand, using the smoothed renormalized continuity equation (3.12) with $B(\varrho)=\varrho^\nu$, we get

$$\partial_t(\omega_\epsilon\star\varrho^\nu)+\mathrm{div}((\omega_\epsilon\star\varrho^\nu)\boldsymbol{u})+(\nu-1)\omega_\epsilon\star(\varrho^\nu\mathrm{div}\boldsymbol{u})=r_\epsilon\ \text{a.e. on}\ I\times\mathbf{R}^3,\tag{4.100}$$

where $r_\epsilon\to0$ in $L^p(I\times\mathbf{R}^3)$, $\frac{1}{p}\ge\frac{3\nu}{5}+\frac12$. Consequently, $\partial_t\pi_\epsilon[t]$ is bounded in $L^p(\mathbf{R}^3)$ for a.e. $t\in I$. Then Propositions 4.9.1, 4.9.2 ensure that $\boldsymbol{\varphi}$ is a valid test function for the momentum equation, which yields

$$\begin{aligned}\int_0^T\int_F p(\varrho,\vartheta)\omega_\epsilon\star\varrho^\nu\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t&=\int_0^T\int_F p(\varrho,\vartheta)\,\mathrm{d}\boldsymbol{x}\fint_F\omega_\epsilon\star\varrho^\nu\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ &-\int_0^T\int_F\varrho\boldsymbol{f}\cdot\mathcal{H}(\pi_\epsilon)+\varrho\boldsymbol{u}\otimes\boldsymbol{u}:\nabla\mathcal{H}(\pi_\epsilon)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ &+\int_0^T\int_F\big(2\mu^f(\vartheta)\mathbb{D}\boldsymbol{u}+\big(\zeta^f(\vartheta)-\frac23\mu^f(\vartheta)\big)\mathrm{div}\,\boldsymbol{u}\mathbb{I}\big):\nabla\mathcal{H}(\pi_\epsilon)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ &-\int_0^T\int_F\varrho\boldsymbol{u}\cdot\partial_t\mathcal{H}[\pi_\epsilon]\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\end{aligned}\tag{4.101}$$

By virtue of the energy and entropy estimates, the first three integrals are bounded independently of ϵ, where the most restrictive convective term leads to the condition $\nu\le\frac59$. For the

last one, we employ (4.99) to get

$$I_4 = \int_0^T \int_F \varrho \boldsymbol{u} \cdot \partial_t \mathcal{H}[\pi_\epsilon] \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}t$$
$$\leq C \int_0^T \big\|(\varrho \boldsymbol{u})[t]\big\|_{L^{\frac{5}{4}}(\Omega)} \big(\big\|\pi_\epsilon[t]\big\|_{L^5(\Omega)} \big\|\boldsymbol{U}[t]\big\|_{L^\infty(\Omega)} + \big\|\partial_t \pi_\epsilon[t]\big\|_{W^{-1,5}(F[t])} \big),$$

The terms on the right-hand side can be further estimated using the properties of the smoothing kernels, equation (4.100), and the Sobolev imbedding as follows

$$\big\|\pi_\epsilon[t]\big\|_5 \leq C \big\|\varrho^{5\nu}[t]\big\|_1,$$

$$\big\|\partial_t \pi_\epsilon[t]\big\|_{W^{-1,5}(F[t])} \leq C \Big(\big\|(\varrho^\nu \boldsymbol{u})[t]\big\|_{L^5(\Omega)} + \big\|(\varrho^\nu \mathrm{div} \boldsymbol{u})[t]\big\|_{L^1(\Omega)} + \big\|r_\varepsilon\big\|_{L^1(\Omega)} + |I_5| \Big),$$

where

$$I_5 = \frac{\mathrm{d}}{\mathrm{d}t} \int_{F[t]} \varrho^\nu[t] \star \omega_\epsilon \,\mathrm{d}\boldsymbol{x}.$$

Taking ν small enough, namely $\nu \leq \frac{1}{18}$, the integral I_4 will be bounded independently of ϵ as soon as the term I_5 is integrable in time uniformly with respect to ϵ.

And here an open problem arise, since we know that $|F[t]| = |F_0|$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_F [t] \varrho^\nu[t] \,\mathrm{d}\boldsymbol{x} = 0,$$

but it doesn't need to be true for I_5, because of smoothing over the boundary $\partial F[t]$. So, the question is whether one can find a regularization of the density field, which behaves better on the boundary.

Chapter 5

Steady barotropic flow for monatomic gas

In this chapter we shall deal with the existence of steady (i.e. time independent) solutions $(\varrho, \boldsymbol{u})$ to the system of equations for the isentropic flow of the Newtonian fluid. As was explained in the Chapter 1, especialy in Section 1.2.2, for such a flow the Navier-Stokes-Fourier system reduces to

$$\operatorname{div}(\varrho \boldsymbol{u}) = 0 \tag{5.1}$$

$$\operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) - \mu \Delta \boldsymbol{u} - (\mu + \lambda) \nabla \operatorname{div} \boldsymbol{u} + \nabla p(\varrho) = \varrho \boldsymbol{f} + \boldsymbol{g}. \tag{5.2}$$

Without loss of generality, we can set $a = 1$ in (1.40) and assume pressure $p(\varrho) = \varrho^\gamma$.

The first existence result for the system (5.1 − 5.2) is due to the pioneering work of Lions [29]. There he assumes $\gamma > \frac{5}{3}$. Later, Novotny, Novo [34] have adapted a method of Feireisl [13] to prove existence in the case of the potential $\boldsymbol{f}$ (and arbitrary $\boldsymbol{g}$) with $\gamma > \frac{3}{2}$, see also [37]. Recently, Frehse, Goj, Steinhauer [19] and Plotnikov, Sokolowski [39] have independently obtained new L^∞ estimates for the quantity $\Delta^{-1} p$ and have proposed several methods to improve estimates of the density. Both works however assume a priori bound for L^1 norm of $\varrho \boldsymbol{u}^2$ which is not available for the general system (5.1 − 5.2). Before this these was finished, there appeared a new result of Plotnikov, Sokolowski [40] in the same spirit as ours.

The main goals of this chapter are:

- To put the Frehse, Goj, Steinhauer [19] and the Plotnikov, Sokolowski [39] estimates into the context of the modern potential theory (see Adams, Hedberg [1]).
- To show how the L^∞ estimate of $\Delta^{-1} p$ can be combined with the standard energy and density bounds even without the a priori L^1 bound for $\varrho \boldsymbol{u}^2$.
- To use these observations to prove existence of solutions for small values of γ, namely $\gamma > \frac{1}{3}(1 + \sqrt{13}) \approx 1.53$ in the case of three dimensional flows and arbitrary $\boldsymbol{f}$, and $\gamma > \frac{1}{8}(3 + \sqrt{41}) \approx 1.175$, if $\boldsymbol{f}$ is potential.

The condition for the general $\boldsymbol{f}$ allows to treat at least the monatomic gas. As the estimate of $\Delta^{-1}p$ is essentially of the local character we limit ourselves to the periodic boundary conditions and periodic domain. In order to guarantee existence of space periodic solutions, we assume $\boldsymbol{f}$ and $\boldsymbol{g}$ with certain symmetries.

5.1 Formulation of the problem and main results

We consider equations (5.1 – 5.2) on a periodic cell

$$\Omega = \Big([-\pi,+\pi]\big|_{\{-\pi,\pi\}}\Big)^3 \tag{5.3}$$

with the periodic boundary conditions and $\boldsymbol{f}$, $\boldsymbol{g}$ with symmetry

$$\begin{aligned} &\boldsymbol{f}_i(\boldsymbol{x}) = -\boldsymbol{f}_i(\boldsymbol{Y}_i(\boldsymbol{x})), \quad \boldsymbol{f}_i(\boldsymbol{x}) = \boldsymbol{f}_i(\boldsymbol{Y}_j(\boldsymbol{x})) \qquad \text{and} \\ &\boldsymbol{g}_i(\boldsymbol{x}) = -\boldsymbol{g}_i(\boldsymbol{Y}_i(\boldsymbol{x})), \quad \boldsymbol{g}_i(\boldsymbol{x}) = \boldsymbol{g}_i(\boldsymbol{Y}_j(\boldsymbol{x})) \qquad \text{for } i\neq j,\ i,j\in\{1,2,3\}, \end{aligned} \tag{5.4}$$

where

$$\boldsymbol{Y}_i(\dots,x_i,\dots) = (\dots,-x_i,\dots).$$

This implies the same symmetry of $\boldsymbol{u}$, and ϱ with the symmetry

$$\varrho(\boldsymbol{x}) = \varrho(\boldsymbol{Y}_i(\boldsymbol{x})) \quad \text{for } i=1,2,3. \tag{5.5}$$

Consequently the investigated problem can be viewed also as the problem on the cube $(0,\pi)^3$ with slip boundary conditions

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0, \quad \boldsymbol{n}\,\mathbb{S}\,\boldsymbol{\tau} = 0 \quad \text{both on } \partial\,(0,\pi)^3,$$

see Ebin [10].

In addition to the notation introduced in Chapter 2, we shall use spaces of symmetric functions: for example, $W^{1,2}_{sym}(\Omega;\mathbf{R}^3)$ stands for the (vector valued) functions from $W^{1,2}(\Omega;\mathbf{R}^3)$ that enjoy symmetric property (5.4) and $L^p_{sym}(\Omega)$ denotes (scalar) functions from $L^p(\Omega)$ that satisfy symmetry (5.5). Another rather uncommon notation is that a set as an index of a measure (or a function) means the measure restricted to the set, e.g. p_Ω is the measure $p_\Omega(M) = p(\Omega\cap M) = \int_{\Omega\cap M} p$.

Suppose for a moment that $(\varrho,\boldsymbol{u})$ is a classical solution to (5.1 – 5.2) and let $b\in C^1(0,\infty)$. Multiplying continuity equation (5.1) by $b'(\varrho)$, we obtain the renormalized continuity equation

$$\operatorname{div}(b(\varrho)\boldsymbol{u}) + \big(\varrho b'(\varrho) - b(\varrho)\big)\operatorname{div}\boldsymbol{u} = 0. \tag{5.6}$$

To keep this equation valid even for a weak solution $\varrho\in L^\gamma(\Omega)$ and $\boldsymbol{u}\in W^{1,2}(\Omega;\mathbf{R}^3)$ (see Definition 5.1.1 later on) we require that (5.6) is satisfied in the sense of distributions $\mathcal{D}'(\Omega)$ for any

$$\left.\begin{aligned} &b\in C\big([0,\infty)\big)\cap C^1\big((0,\infty)\big) \\ &\sup_{t\in(0,1)}\big|t^\alpha b'(t)\big| < \infty, \qquad \text{for some } \alpha\in[0,1), \\ &\sup_{t\in(1,\infty)}\big|t^{-\alpha} b'(t)\big| < \infty, \qquad \text{for some } \alpha\le\frac{\gamma}{2}-1. \end{aligned}\right\} \tag{5.7}$$

Similarly, we take a scalar product of momentum equation (5.2) with $\boldsymbol{u}$ and we integrate over Ω. Using continuity equation (5.1) and taking advantage of the periodicity of solutions, after several integrations by parts, we obtain the energy equality

$$\int_\Omega \mu|\nabla \boldsymbol{u}|^2 + (\mu+\lambda)|\mathrm{div}\boldsymbol{u}|^2 \,\mathrm{d}\boldsymbol{x} = \int_\Omega \varrho \boldsymbol{f}\cdot\boldsymbol{u} + \boldsymbol{g}\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}. \tag{5.8}$$

Of course, due to the presence of the weakly lower semi-continuous functionals

$$\nabla \boldsymbol{u} \to \int_\Omega |\nabla \boldsymbol{u}|^2\,\mathrm{d}\boldsymbol{x}, \quad \nabla \boldsymbol{u} \to \int_\Omega |\mathrm{div}\boldsymbol{u}|^2\,\mathrm{d}\boldsymbol{x},$$

on $L^2(\Omega;\mathbf{R}^3)$, for weak solutions, we can expect only the energy inequality

$$\int_\Omega \mu|\nabla \boldsymbol{u}|^2 + (\mu+\lambda)|\mathrm{div}\boldsymbol{u}|^2 \,\mathrm{d}\boldsymbol{x} \leq \int_\Omega \varrho \boldsymbol{f}\cdot\boldsymbol{u} + \boldsymbol{g}\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}. \tag{5.9}$$

Last but not least, integrating momentum equation (5.2) over the periodic cell Ω, in accordance with the periodicity of solutions, we obtain the compatibility relation

$$\int_\Omega \varrho\boldsymbol{f} + \boldsymbol{g}\,\mathrm{d}\boldsymbol{x} = 0. \tag{5.10}$$

This condition is automatically satisfied by any solution induced by $\boldsymbol{f}$ and $\boldsymbol{g}$ with symmetry (5.4). Finally, we denote by $m > 0$ the total mass of the gas in the volume Ω.

Following the terminology of [37] we define a *renormalized bounded energy weak solution* of the periodic problem (5.1 – 5.2) on the domain Ω as follows:

Definition 5.1.1. *Let the viscosity coefficients μ, λ satisfy $\mu > 0$, $2\mu + 3\lambda > 0$. Suppose that $\gamma > 1$ and $m > 0$ are given constants and assume that both $\boldsymbol{f}, \boldsymbol{g} \in L^\infty(\Omega)$ satisfy (5.4). We say that a couple $(\varrho, \boldsymbol{u})$ is a renormalized bounded energy weak solution of the periodic problem* (5.1 – 5.2) *on the periodic cell Ω if*

$$\varrho \in L^\gamma_{sym}(\Omega),\ \boldsymbol{u} \in W^{1,2}_{sym}(\Omega;\mathbf{R}^3), \tag{5.11}$$

$$\int_\Omega \varrho\,\mathrm{d}\boldsymbol{x} = m, \tag{5.12}$$

the renormalized continuity equation (5.6) is valid for any b satisfying (5.7), the momentum equation (5.2) holds in $\mathcal{D}'(\Omega)$, and (5.9) is satisfied.

Remark 5.1.2. *In view of (5.11) the simple density argument can be used to see that (5.2) holds even in $(W^{1,q}(\Omega;\mathbf{R}^3))'$ for any $q \geq \max(2, \frac{3\gamma}{2\gamma-3})$.*

Now we are ready to state the main result.

Theorem 5.1.3. *Let Ω, m, μ, λ, $\boldsymbol{f}$, $\boldsymbol{g}$ satisfy hypothesis of Definition 5.1.1. Let*

$$\gamma > \gamma_{gen.} := \frac{1}{3}(1+\sqrt{13}) \approx 1.53 \tag{5.13}$$

or let $\boldsymbol{f}$ be potential and

$$\gamma > \gamma_{pot.} := \frac{1}{8}(3+\sqrt{41}) \approx 1.175. \tag{5.14}$$

Then there exists a renormalized bounded energy weak solution $(\varrho, \boldsymbol{u})$ of the periodic problem (5.1 – 5.2) *which satisfies*

$$\varrho \in L^{\gamma q}(\Omega), \qquad q = \frac{3\gamma}{2+\gamma}. \tag{5.15}$$

The rest of the chapter is devoted to the proof, In Section 3 we derive L^∞ estimates for $\Delta^{-1}p$. Then in Section 4 we use the nonlinear potential theory due to Adams, Hedberg [1] to find a convenient L^1 bound for the quantity $p\boldsymbol{u}^2$. In Section 5 we use this estimate together with standard energy and density bounds to estimate the density in the space $L^{\gamma q}$, $q = \frac{3\gamma}{\gamma+2}$. Finally, in Section 6, we combine this piece of information with the recently discovered compactness properties of the so called effective viscous flux and with the notion of the renormalized solutions to the continuity equation (cf. P.L. Lions' results [29] and [14], [37]). This makes possible to prove compactness of the set of weak solutions as well as to construct weak solutions via a several level approximation scheme. The limit passage from one level to another is standard, see e.g. [37]. Nevertheless, the necessary modifications in the construction of approximations to accommodate the periodic boundary conditions, as well as the last (and the most delicate) limit process are performed in all details in Sections 6.1 and 6.2.

5.2 Potential estimate

Let $(\varrho_\delta, \boldsymbol{u}_\delta)$ be a sequence of renormalized bounded energy weak solutions to the problem (5.1 − 5.2), where, as well as in sequel, p stands for p_δ. Our aim is to derive for ϱ_δ sufficiently strong estimates independent of $\delta > 0$ in terms of the external data $\|\boldsymbol{f}\|_\infty$, $\|\boldsymbol{g}\|_\infty$ (and, of course, of the coefficients μ, λ).

Choose $\boldsymbol{y} \in \Omega$. Since the periodic problem is invariant with respect to the translation of the periodic cell, we can assume $y = 0$. As in [19] and [39], the main estimate of this section can be obtained testing formally the momentum equation (5.2) by $\varphi(x) = (x-y)|x-y|^{-1}$. Since this is not an admissible test function in the sense of Remark 5.1.2, we shall truncate it as follows:

$$\varphi = (x-y)\eta(|x-y|),$$
$$\eta(t) = \begin{cases} \frac{1}{r} - \frac{1}{R} & \text{on } [0,r) \\ \frac{1}{t} - \frac{1}{R} & \text{on } [r,R) \\ 0 & \text{on } [R,\infty) \end{cases}$$

where $0 < r < \frac{\pi}{2} < R < \pi$. Denoting $\mathbb{P} = \varrho\boldsymbol{u}\otimes\boldsymbol{u} + p\,\mathbb{I}$ and $\boldsymbol{n} = \frac{(x-y)}{|x-y|}$, a short calculation yields

$$\frac{1}{r}\int_{B_r} \mathrm{Tr}(\mathbb{P}-\mathbb{S}) + (\varrho\boldsymbol{f}+\boldsymbol{g})\cdot(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}\boldsymbol{x} - \frac{1}{R}\int_{B_R} \mathrm{Tr}(\mathbb{P}-\mathbb{S}) + (\varrho\boldsymbol{f}+\boldsymbol{g})\cdot(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}$$
$$+ \int_{B_R\setminus B_r} \frac{\mathrm{Tr}(\mathbb{P}-\mathbb{S}) - (\mathbb{P}-\mathbb{S}):\boldsymbol{n}\otimes\boldsymbol{n}}{|\boldsymbol{x}-\boldsymbol{y}|} + (\varrho\boldsymbol{f}+\boldsymbol{g})\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{x} = 0, \quad (5.16)$$

where $B_s = \{\boldsymbol{x} : |\boldsymbol{x}-\boldsymbol{y}| < s\}$. Since $\varrho \in L^\beta(\Omega)$ for a fixed δ, we realize that the term $Q := \big(\mathrm{Tr}(\mathbb{P}-\mathbb{S}) + (\varrho\boldsymbol{f}+\boldsymbol{g})\cdot(\boldsymbol{x}-\boldsymbol{y})\big)$ belongs in $L^1(\Omega)$. Thus the Lebesgue point property implies

$$\frac{1}{r}\int_{B_r} Q\,\mathrm{d}\boldsymbol{x} = \frac{4\pi\, r^2}{|B_r|}\int_{B_r} Q\,\mathrm{d}\boldsymbol{x} \to 0 \quad \text{as } r \to 0.$$

Rearranging the remaining terms in (5.16) and estimating the resulting right-hand side, we

obtain

$$\sup_{r>0}\int_{B_R\setminus B_r}\frac{\operatorname{Tr}\mathbb{P}-\mathbb{P}:\boldsymbol{n}\otimes\boldsymbol{n}}{|\boldsymbol{y}-\boldsymbol{x}|}\,\mathrm{d}\boldsymbol{x}\le\frac{1}{R}\int_{B_R}\operatorname{Tr}(\mathbb{P}-\mathbb{S})+(\varrho\boldsymbol{f}+\boldsymbol{g})\cdot(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}$$
$$+\int_{B_R}\frac{2|\mathbb{S}|}{|\boldsymbol{y}-\boldsymbol{x}|}+|\varrho\boldsymbol{f}+\boldsymbol{g}|\,\mathrm{d}\boldsymbol{x}\le C(1+\|\mathbb{P}\|_{1,\Omega}+\|\mathbb{S}\|_{2,\Omega}+\|\varrho\|_{1,\Omega}).$$

Here and in the sequel, C is a generic positive constant independent of δ. Next, we observe that

$$\operatorname{Tr}\mathbb{P}-\mathbb{P}:\boldsymbol{n}\otimes\boldsymbol{n}=\varrho\boldsymbol{u}^2+3p-(\varrho(\boldsymbol{u}\cdot\boldsymbol{n})^2+p)\ge 2p\ge 0.$$

Thus, recalling the structure of $\mathbb{S}$, we get

$$\int_{B_R}\frac{2p}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{x}\le C\big(1+\|\varrho\boldsymbol{u}^2\|_{1,\Omega}+\|p\|_{1,\Omega}+\|\boldsymbol{u}\|_{1,2,\Omega}\big).\tag{5.17}$$

Finally, denoting the periodic extension of p from $L^1(\Omega)$ to $L^1_{loc}(\mathbf{R}^3)$ again by p and extending the integral at the left-hand side of (5.17) to the whole $\mathbf{R}^3$, we arrive at

$$(\Delta^{-1}p_\Omega)[\boldsymbol{y}]:=\int_{\mathbf{R}^3}\frac{p_\Omega(\boldsymbol{x})}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{x}\le\int_{B_R}\frac{p}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{x}+\frac{1}{R}\int_\Omega p\,\mathrm{d}\boldsymbol{x}$$
$$\le C(1+\|\varrho\boldsymbol{u}^2\|_{1,\Omega}+\|p\|_{1,\Omega}+\|\boldsymbol{u}\|_{1,2,\Omega}).\tag{5.18}$$

5.3 An application of the potential theory

In this part we will apply the general potential theory developed by Adams, Hedberg [1] to obtain a convenient estimate for $p\boldsymbol{u}^2$. Similar estimate has been proved in [39], in a direct way. Slightly weaker one, for the quantity $p|\boldsymbol{u}|$, was derived in [19] via the theory of Morrey spaces. The main advantage of our approach are accurate expressions for the best constants (see (5.27)) of estimates, which will be crucial for the bootstrapping argument in Section 5.

We shall say that a function g on $\mathbf{R}^N$ is a *radially decreasing convolution kernel* if $g(\boldsymbol{x})=g_0(|\boldsymbol{x}|)$, for some non-negative, lower semi-continuous, non-increasing function g_0 on $\mathbf{R}^+$ for which $\int_0^1 g_0(t)t^{N-1}\mathrm{d}t<\infty$. The key ingredient of our proof is the following theorem.

Theorem 5.3.1. *[1, Theorem 7.2.1] Let g be a radially decreasing convolution kernel, and let $\mu\in\mathcal{M}^+(\mathbf{R}^N)$ be a positive Radon measure. Then for $1<p\le q<\infty$ the following properties of μ are equivalent:*

(a) There is a constant A_1 such that

$$\left(\int_{\mathbf{R}^N}|g\star f|^q\,\mathrm{d}\mu\right)^{1/q}\le A_1\|f\|_p\tag{5.19}$$

for all $f\in L^p(\mathbf{R}^N)$.

(b) There is a constant A_2 such that

$$\|g\star\mu_K\|_{p'}\le A_2\,\mu(K)^{1/q'}\tag{5.20}$$

for all compact sets $K\subset\mathbf{R}^N$.

Moreover, the least possible values of A_1 and A_2 are the same. As a matter of fact one can take $A_1 = A_2$.

The following preliminary material is taken again from [1, Chapter 1]. We shall be concerned with the *Bessel kernels* G_α, which are defined for any real (or even complex) index α via the Fourier transform by the formula

$$G_\alpha := \mathcal{F}^{-1}\big((1+|\boldsymbol{\xi}|^2)^{-\frac{\alpha}{2}}\big). \tag{5.21}$$

The Bessel kernel G_α is radially decreasing convolution kernel, in particular it is real and positive. It has exponential decay at infinity and the following asymptotics at zero

$$G_\alpha(\boldsymbol{x}) \le C(\alpha,N)|\boldsymbol{x}|^{\alpha-N} \quad \text{as } |\boldsymbol{x}|\to 0, \quad \text{for } 0<\alpha<N. \tag{5.22}$$

Due to the definition (5.21) it is easy to see that the kernels G_α form a group, namely

$$G_\alpha \star G_\beta = G_{\alpha+\beta}. \tag{5.23}$$

For the kernel G_α one can define the *Bessel potential space*

$$L^{\alpha,p}(\mathbf{R}^N) := \{\varphi = G_\alpha \star f \,|\, f\in L^p(\mathbf{R}^N)\},$$

with the norm

$$\big\|G_\alpha\star f\big\|_{L^{\alpha,p}(\mathbf{R}^N)} := \big\|f\big\|_{L^p(\mathbf{R}^N)}.$$

The fundamental theorem of A. P. Calderon [4] identifies these spaces with the Sobolev spaces.

Theorem 5.3.2. *[1, Theorem 1.2.3] For $\alpha\in\mathbb{N}$, $W^{\alpha,p}(\mathbf{R}^N) = L^{\alpha,p}(\mathbf{R}^N)$, $1<p<\infty$, with equivalence of norms. In particular, for all $\varphi\in W^{\alpha,p}(\mathbf{R}^N)$ there exists a unique $f\in L^p(\mathbf{R}^N)$ such that $\varphi = G_\alpha\star f$, and there is a constant A such that*

$$A^{-1}\big\|\varphi\big\|_{L^{\alpha,p}(\mathbf{R}^N)} \le \big\|\varphi\big\|_{W^{\alpha,p}(\mathbf{R}^N)} \le A\big\|\varphi\big\|_{L^{\alpha,p}(\mathbf{R}^N)}.$$

Due to Theorem 5.3.2, for any $u^i\in W^{1,2}(\Omega)$ there exists a unique $f\in L^2(\Omega)$ such that $E(u^i) = G_1\star f$, where $E: W^{1,2}(\Omega)\to W^{1,2}(\mathbf{R}^3)$ is a continuous extension operator. Now, we are in the position to use Theorem 5.3.1 with $N=3$, $p=q=2$, $\mu = p_\Omega\,\mathrm{d}\boldsymbol{x}$, $g=G_1$ and f. First we apply Fubini's theorem to check the condition (b) of Theorem 5.3.1

$$\big\|G_1\star p_{\Omega\cap K}\big\|_2^2 = \int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3} G_1(\boldsymbol{y}-\boldsymbol{x})\,p_{\Omega\cap K}(\boldsymbol{y})\;G_1(\boldsymbol{z}-\boldsymbol{x})\,p_{\Omega\cap K}(\boldsymbol{z})\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{z}\,\mathrm{d}\boldsymbol{x} \tag{5.24}$$

$$= \int_{\mathbf{R}^3}\big((G_1\star G_1)\star p_{\Omega\cap K}\big)(\boldsymbol{z})\,p_{\Omega\cap K}(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z} \tag{5.25}$$

$$\le \big\|G_2\star p_\Omega\big\|_\infty\; p_\Omega(K) \le C\big\|\Delta^{-1}p_\Omega\big\|_\infty\; p_\Omega(K) \le A_2^2\,p_\Omega(K), \tag{5.26}$$

where on the last line we have used (5.23), (5.22), (5.18), and we have put

$$A_2^2 = C(1+\big\|\varrho\boldsymbol{u}^2\big\|_1 + \big\|p\big\|_1 + \big\|\boldsymbol{u}\big\|_{1,2}). \tag{5.27}$$

Finally, using the statement (a) of Theorem 5.3.1 and Theorem 5.3.2 we conclude that

$$\big\|p\boldsymbol{u}^2\big\|_{L^1(\Omega)} = \sum_{i=1}^3\int_{\mathbf{R}^3} E(u^i)^2\,p_\Omega\,\mathrm{d}\boldsymbol{x} \le \sum_{i=1}^3 A_1^2\big\|E(u^i)\big\|^2_{L^{1,2}(\mathbf{R}^3)} \le C\,A_2^2\big\|\boldsymbol{u}\big\|^2_{W^{1,2}(\Omega)}\,. \tag{5.28}$$

5.4 Bootstrapping argument

There are two standard estimates for the renormalized bounded energy weak solutions we have not yet exploited. First, if we use the energy inequality (5.9), Korn's inequality, the Young inequality, and the Sobolev imbedings we arrive at the estimate

$$\|\boldsymbol{u}\|_{1,2} \leq C(\Omega)\|\boldsymbol{f}\|_{\infty}\|\varrho\|_{\frac{6}{5}}. \tag{5.29}$$

Second, we introduce the so called *Bogovskii operator*, which is a particular solving operator

$$\mathcal{B}:\ \varphi \in L^q(\Omega) \to \boldsymbol{v} \in W^{1,q}(\Omega;\mathbf{R}^3), \quad 1 < q < \infty \tag{5.30}$$

of the problem

$$\left\{ \begin{array}{ll} \operatorname{div} \boldsymbol{v} = \varphi - \fint_{\Omega} \varphi \,\mathrm{d}\boldsymbol{x} & \text{in } (-\pi,\pi)^3, \\ \boldsymbol{v} = 0 & \text{on } \partial(-\pi,\pi)^3. \end{array} \right\} \tag{5.31}$$

The operator $\mathcal{B}$ is continuous, namely $\|v\|_{1,q} \leq C\|\varphi\|_q$. For details see [37, Section 3] and references quoted there. In view of Remark 5.1.2 we can test (5.2) by the function $\mathcal{B}[\varphi]$, where $\varphi \in L^{q'}(\Omega)$, $1 < q \leq 2$, to get

$$\begin{aligned} \int_{\Omega} p \operatorname{div}(\mathcal{B}[\varphi]) \,\mathrm{d}\boldsymbol{x} &= \int_{\Omega} (\mathbb{S} - \varrho \boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \mathcal{B}[\varphi] - (\varrho \boldsymbol{f} + \boldsymbol{g}) \cdot \mathcal{B}[\varphi] \,\mathrm{d}\boldsymbol{x} \\ &\leq C\big(\|\boldsymbol{u}\|_{1,2} + \|\varrho \boldsymbol{u}^2\|_q + \|\varrho\|_{\frac{6}{5}}\|\boldsymbol{f}\|_{\infty} + \|\boldsymbol{g}\|_{\infty}\big)\|\varphi\|_{q'}\,. \end{aligned} \tag{5.32}$$

For $\gamma q > \frac{6}{5}$, the Young inequality together with (5.29) yields

$$\|p\|_q = \sup_{\varphi \in L^{q'}(\Omega)} \|\varphi\|_{q'}^{-1} \int_{\Omega} p \left(\operatorname{div} \mathcal{B}[\varphi] + \int_{\Omega} \varphi \,\mathrm{d}\boldsymbol{x} \right) \mathrm{d}\boldsymbol{x} \leq C\big(1 + \|\varrho \boldsymbol{u}^2\|_q\big). \tag{5.33}$$

Next, we split the right-hand side,

$$\|\varrho \boldsymbol{u}^2\|_q^q = \int_{\Omega} (\varrho^{\gamma} \boldsymbol{u}^2)^b \boldsymbol{u}^c \,\mathrm{d}\boldsymbol{x}, \quad q = \gamma b,\ 2q = 2b + c,$$

and apply the Hölder inequality to get

$$\|\varrho \boldsymbol{u}^2\|_q^q \leq \|\varrho^{\gamma} \boldsymbol{u}^2\|_1^b \|\boldsymbol{u}\|_6^c, \tag{5.34}$$

provided

$$b + \frac{c}{6} \leq 1 \text{ or equivalently } q \leq \frac{3\gamma}{\gamma + 2}\,. \tag{5.35}$$

With help of estimates (5.29), (5.33) we can rewrite (5.28) as

$$\|p\boldsymbol{u}^2\|_1 \leq C(1 + \|\varrho \boldsymbol{u}^2\|_{1+\varepsilon})\|\varrho\|_{\frac{6}{5}}^2. \tag{5.36}$$

Further application of the Hölder inequality together with the imbeding $L^6(\Omega) \hookrightarrow W^{1,2}(\Omega)$ and with (5.29) yields

$$\|p\boldsymbol{u}^2\|_1 \leq C(1 + \|\varrho\|_{\frac{3}{2}+\varepsilon}\|\varrho\|_{\frac{6}{5}}^2)\|\varrho\|_{\frac{6}{5}}^2. \tag{5.37}$$

In (5.36), (5.37), ε can be chosen arbitrary from the interval $(0, \varepsilon_0)$ where ε_0 is sufficiently small and C depends on ε_0 but is independent of ε. Taking into account (5.33), (5.34), and (5.37) we arrive at

$$\big\|p\big\|_q^q = \big\|\varrho^\gamma + \delta\varrho^\beta\big\|_q^q \leq C(1 + \big\|\varrho\big\|_{\frac{3}{2}+\varepsilon}^b \big\|\varrho\big\|_{\frac{6}{5}}^{2b+2q}). \tag{5.38}$$

In the next step we shall interpolate the norms at the right-hand side of (5.38) between $L^1(\Omega)$ and $L^{\gamma q}(\Omega)$ as follows

$$\big\|\varrho\big\|_r \leq \big\|\varrho\big\|_1^x \big\|\varrho\big\|_{\gamma q}^y = C\big\|\varrho\big\|_{\gamma q}^y,\ y = \frac{\gamma q}{(\gamma q - 1)} \frac{(r-1)}{r}. \tag{5.39}$$

Applying (5.39) to (5.38) with r successively equal to $\frac{3}{2} + \varepsilon$ and $\frac{6}{5}$, under the necessary conditions $\gamma q > \frac{3}{2}$ and $\gamma q \geq \frac{6}{5}$ respectively, and noticing that $\big\|\varrho\big\|_1 = m$, we get

$$\big\|\varrho^\gamma + \delta\varrho^\beta\big\|_q^q \leq C(1 + \big\|\varrho\big\|_{\gamma q}^{z+\hat{C}\varepsilon}), \quad z = \frac{\gamma q}{\gamma q - 1}\left(\frac{b}{3} + \frac{2b+2q}{6}\right),\ \hat{C} \leq \frac{\gamma q}{\gamma q - 1}\frac{2}{3}b \tag{5.40}$$

This formula yields

$$\big\|\varrho^\gamma + \delta\varrho^\beta\big\|_q \leq C(\Omega, m, \boldsymbol{f}, \boldsymbol{g})$$

provided

$$\gamma q > z = \frac{\gamma q}{\gamma q - 1}\,\frac{\gamma + 2}{3\gamma}\,q. \tag{5.41}$$

The expression $\frac{\gamma q}{\gamma q - 1}$ is a decreasing function of q, consequently (5.41) can be understood as an inequality to determine the lower bound for q. Thus, in accordance with (5.35), $q = \frac{3\gamma}{\gamma+2}$ represents the optimal choice of q. Then (5.41) reduce to $\gamma q > 2$ or equivalently $3\gamma^2 - 2\gamma - 4 > 0$. The latter inequality leads directly to the condition $\gamma > \gamma_{gen.}$ (5.13).

If the volume force $\boldsymbol{f}$ is potential, the term $\int_\Omega \varrho\boldsymbol{f}\cdot\boldsymbol{u}$ on the right-hand side of (5.9) is zero thanks to (5.1). Thus we obtain, instead of (5.29), a priori bound for $\big\|\boldsymbol{u}\big\|_{1,2}$. Consequently (5.37) takes the form

$$\big\|p\boldsymbol{u}^2\big\|_1 \leq C(1 + \big\|\varrho\boldsymbol{u}^2\big\|_{1+\varepsilon}) \tag{5.42}$$

and interpolation (5.34) yields

$$\big\|\varrho\boldsymbol{u}^2\big\|_q^q \leq C\big\|\varrho^\gamma\boldsymbol{u}^2\big\|_1^b \big\|\boldsymbol{u}\big\|_6^c \leq C(1 + \big\|\varrho\boldsymbol{u}^2\big\|_{1+\varepsilon}^b). \tag{5.43}$$

As $b < q$, we get estimate for $\big\|\varrho\boldsymbol{u}^2\big\|_q^q$. Using (5.33), we arrive at

$$\big\|\varrho^\gamma + \delta\varrho^\beta\big\|_q^q \leq C(1 + \big\|\varrho\boldsymbol{u}^2\big\|_q^q) \leq C(\Omega, m, \boldsymbol{f}, \boldsymbol{g}) \tag{5.44}$$

for every $1 < q \leq \frac{3\gamma}{\gamma+2}$ and for all $\gamma > 1$.

Summarizing all estimates, we have

$$\begin{aligned} \delta^{1/\beta}\varrho_\delta &\quad \text{bounded in } L^{\beta q}(\Omega), & \varrho_\delta &\quad \text{bounded in } L^{\gamma q}(\Omega), \\ \varrho_\delta\boldsymbol{u}_\delta^2 &\quad \text{bounded in } L^q(\Omega), & \boldsymbol{u}_\delta &\quad \text{bounded in } W^{1,2}(\Omega;\mathbf{R}^3) \end{aligned} \tag{5.45}$$

uniformly with respect to δ, provided $\gamma > \gamma_{gen.}$, or provided $\gamma > 1$ and $\boldsymbol{f}$ is potential. To prove strong convergence of the density, we shall also need the estimate

$$\big\|\varrho_\delta\boldsymbol{u}_\delta\big\|_r \leq \big\|\varrho_\delta\big\|_{\gamma q}^{\frac{1}{2}} \big\|\varrho_\delta\boldsymbol{u}_\delta^2\big\|_q^{\frac{1}{2}} \leq C \quad \text{with some } r > \tfrac{6}{5}. \tag{5.46}$$

This is true provided $\frac{5}{6} > \frac{1}{2q}(1 + \frac{1}{\gamma})$ which is equivalent to condition (5.14).

5.5 Existence of a solution

The first part of this section is devoted to the construction of the bounded energy weak solutions to problem (5.1 – 5.2) by using several level approximation scheme. We also explain (referring to the second part) how to pass to the limit between the levels. In the second part we combine the estimates of Section 5 with the compactness properties of the effective viscous flux and with the convenient estimates of oscillations to the density sequence to carry out the last limit process $\delta \to 0+$.

5.5.1 Approximations

In this section we explain how to construct the renormalized bounded energy weak solutions to problem (5.1 – 5.2) on the periodic cell (5.3). We adopt the same chain of approximations as described in Chapter 4 of [37], where a similar problem is treated for larger values of the adiabatic constant and the homogeneous Dirichlet boundary conditions for the velocity. The problem of density estimates for the small adiabatic constants was already treated in Section 5. Due to this fact, we shall concentrate in this part essentially to the changes which are necessary to be operated in order to accommodate the periodic boundary conditions and the symmetries (5.4), (5.5).

To this end, we consider an approximating problem with positive parameters α, ε, and δ:

$$\alpha(\varrho - h) + \operatorname{div}(\varrho \boldsymbol{u}) - \varepsilon \Delta \varrho = 0, \tag{5.47}$$

$$\alpha(h + \varrho)\boldsymbol{u} + \frac{1}{2}\big(\operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \varrho \boldsymbol{u} \nabla \boldsymbol{u}\big) + \nabla(\varrho^\gamma + \delta \varrho^\beta) - \operatorname{div}\mathbb{S} = \varrho \boldsymbol{f} + \boldsymbol{g}, \tag{5.48}$$

on the periodic cell Ω. Here h is a smooth periodic function with the symmetry (5.5) satisfying $\int_\Omega h = m$. Further, ρ and $\boldsymbol{u}$ are unknowns which has to obey symmetries (5.4) and (5.5), respectively. Notice that in this case $\boldsymbol{u} \cdot n$ and $\partial_n \varrho$ necessarily vanish on $\partial(-\pi, \pi)^3$. In order to solve this system we employ the Leray-Schauder fixed point theorem

Theorem 5.5.1 (see [37] Section 1.4.11.8)**.** *Let X be a Banach space and $D \subset X$ bounded open set. Let $H : \overline{D} \times [0,1] \to X$ be a homotopy of compact transformations, which means that H is a compact mapping for every $t \in [0,1]$ and that it is uniformly continuous in t on any bounded set $B \subset \overline{D}$. Let*

$$\omega - H(\omega, t) \neq 0, \quad \forall t \in [0,1], \quad \forall \omega \in \partial D. \tag{5.49}$$

If there exists $\omega_0 \in D$ such that $H(\omega_0, 0) = \omega_0$, then, for any $t \in [0,1]$, there exists $\omega_t \in D$, satisfying $H(\omega_t, t) = u_t$ as well.

We take $\boldsymbol{v} \in W^{1,\infty}_{sym}(\Omega; \mathbf{R}^3)$ such that $\|\boldsymbol{v}\|_{1,\infty} \leq K$ for some $K > 0$. Using the standard theory of elliptic operators, see e.g. Nečas [32], we can construct solving operators

$$\Pi_t : \xi \in W^{1,p}_{sym}(\Omega) \cap \big\{\textstyle\int_\Omega \xi = m\big\} \to \varrho_t \in \big(W^{2,p}_{sym}(\Omega) \cap \big\{\textstyle\int_\Omega \varrho = m\big\}\big)$$

to the problems

$$-\varepsilon \Delta \varrho_t = -t\big(\alpha(\xi - h) + \operatorname{div}(\xi \boldsymbol{v})\big) \text{ in } \Omega, \quad \int_\Omega \varrho_t \, \mathrm{d}\boldsymbol{x} = m, \quad t \in [0,1], \tag{5.50}$$

which, for any $1 < p < \infty$, forms a homotopy of compact transformations by virtue of the compact imbedding $W^{2,p}_{sym}(\Omega) \hookrightarrow\hookrightarrow W^{1,p}_{sym}(\Omega)$. Testing

$$\alpha(\varrho - h) + \mathrm{div}(\varrho \boldsymbol{v}) - \varepsilon \Delta \varrho = 0 \tag{5.51}$$

(compare with (5.47)) by ϱ and using conveniently a bootstrapping argument we realize that any fixed point $\varrho_t \in W^{1,p}_{sym}(\Omega) \cap \{\int_\Omega \varrho = m\}$ of Π_t satisfies

$$\big\|\varrho_t\big\|_{1,p} \leq C_S(K, p, \varepsilon, \alpha, h), \tag{5.52}$$

where C_S is a positive constant independent of t. As a consequence the domain

$$D = \{\xi \in W^{1,p}_{sym}(\Omega) \,|\, \big\|\xi\big\|_{1,p} \leq 2C_S, \ \fint_\Omega \varrho = m\}$$

verifies (5.49) with the homotopy $H(\,\cdot\,, t) = \Pi_t(\,\cdot\,)$. We can therefore employ Theorem 5.5.1, taking $X = W^{1,p}_{sym}(\Omega) \cap \{\int_\Omega \varrho = m\}$, to construct the operator S

$$S : \boldsymbol{v} \in W^{1,\infty}_{sym}(\Omega; \mathbf{R}^3) \to (\varrho = \Pi_1(\varrho)) \in W^{1,p}_{sym}(\Omega) \tag{5.53}$$

such that $\varrho = S(\boldsymbol{v})$ solves equation (5.47).

Similarly we define operators $T_t : \boldsymbol{v} \to \boldsymbol{u}_t$, $t \in [0, 1]$ as the solving operators to the problems

$$-\mu \Delta \boldsymbol{u} - (\mu + \lambda) \nabla \mathrm{div} \boldsymbol{u} = -t \boldsymbol{F}(S(\boldsymbol{v}), \boldsymbol{v}), \tag{5.54}$$

on the periodic cell Ω, where

$$\boldsymbol{F}(\varrho, \boldsymbol{v}) := \alpha(h + \varrho)\boldsymbol{v} + \frac{1}{2}\mathrm{div}(\varrho \boldsymbol{v} \otimes \boldsymbol{v}) + \frac{1}{2}\varrho \boldsymbol{v} \nabla \boldsymbol{v} + \nabla(\varrho^\gamma + \delta \varrho^\beta) - \varrho \boldsymbol{f} - \boldsymbol{g}. \tag{5.55}$$

The necessary condition to guarantee the existence of solutions to this system is $\int_\Omega \boldsymbol{F} = 0$. This condition is always satisfied provided $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{v}$ and ϱ, h posses symmetries (5.4) and (5.5), respectively.

Referring to the standard results of the regularity to the elliptic systems, see again [32], we conclude that

$$T_t : \boldsymbol{v} \in W^{1,\infty}_{sym}(\Omega; \mathbf{R}^3) \to \boldsymbol{u}_t \in W^{2,p}_{sym}(\Omega; \mathbf{R}^3) \hookrightarrow\hookrightarrow W^{1,\infty}_{sym}(\Omega; \mathbf{R}^3)$$

for any $p > 3$.

We test (5.48) by $\boldsymbol{u}$, where (5.48) can be viewed as the Lamé type system (5.54) with $\boldsymbol{v} = \boldsymbol{u}$. After a long but standard calculation, employing among others (5.47), we get

$$\int_\Omega \mu |\nabla \boldsymbol{u}|^2 + (\mu + \lambda)|\mathrm{div} \boldsymbol{u}|^2 \,\mathrm{d}\boldsymbol{x} + \varepsilon\delta \big\|\nabla(\varrho^{\beta/2})\big\|^2_{0,2} \leq \int_\Omega (\varrho \boldsymbol{f} + \boldsymbol{g}) \cdot \boldsymbol{u} \,\mathrm{d}\boldsymbol{x} + \alpha C(h), \tag{5.56}$$

where $C(h)$ is a positive constant dependent on h. Taking advantage of the symmetries of $\boldsymbol{u}$ and of the fact that $\int_\Omega (\varrho - h) = 0$, one can use the Sobolev and Poincaré type inequalities as well as a bootstrapping via $\boldsymbol{F}(S(\boldsymbol{u}), \boldsymbol{u})$ and the elliptic regularity of (5.54) to conclude that

$$\big\|\boldsymbol{u}\big\|_{2,6} + \big\|\varrho\big\|_{0,3\beta} \leq C_T(\alpha, \delta, \varepsilon, \boldsymbol{f}, \boldsymbol{g}, h). \tag{5.57}$$

Now we shall take $K = 2C_T$ in the definition of C_S (see (5.52)) in order to have the operator S well defined.

The domain $D = \{\boldsymbol{v} \in W^{1,\infty}_{sym}(\Omega, \mathbf{R}^3) \,|\, \|\boldsymbol{v}\|_{1,\infty} \leq 2C_T\}$, verifies (5.49) with $H(\cdot, t) = T_t$. Once again, we can use Theorem 5.5.1 with $X = W^{1,\infty}_{sym}(\Omega)$, to guarantee existence of a fixed point $\boldsymbol{u}_\varepsilon = T_1(\boldsymbol{u}_\varepsilon)$ and then we set $\varrho_\varepsilon = S(\boldsymbol{u}_\varepsilon)$. Evidently, the couple $(\varrho_\varepsilon, \boldsymbol{u}_\varepsilon)$ solves $(5.47 - 5.48)$.

To pass to the limit $\varepsilon \to 0+$, we have on our disposal estimate (5.56) and another estimate

$$\|\varrho\|_{0,2\beta} \leq C(\delta, \boldsymbol{f}, \boldsymbol{g}, h).$$

It can be obtained by testing the momentum equation (5.48) by the Bogovskii operator $\mathcal{B}[\varphi]$, see (5.30), (5.31), using the known bound (5.56), and applying conveniently the Sobolev imbeddings and the Hölder inequality in a way similar to (5.32). Both estimates provide uniform bounds for $\|\boldsymbol{u}_\varepsilon\|_{1,2}$ and $\|\varrho_\varepsilon\|_{0,2\beta}$ independent of ε.

These estimates are sufficient to pass to the limit in the continuity equation (5.47), the energy inequality (5.56), and in all terms of the momentum equation (5.48) except the pressure term $p_\delta(\varrho_\varepsilon)$.

To pass to the limit in $p_\delta(\varrho_\varepsilon)$, one needs to show that the weak limits $\boldsymbol{u}$ and ϱ of the sequences $\boldsymbol{u}_\varepsilon$ and ϱ_ε satisfy also the renormalized continuity equation similar to (5.6), namely

$$\begin{gathered}\alpha\varrho b'(\varrho) + \mathrm{div}(b(\varrho)\boldsymbol{u}) + (\varrho b'(\varrho) - b(\varrho))\mathrm{div}\boldsymbol{u} = \\ \alpha h b'(\varrho) + \varepsilon \mathrm{div}(b'(\varrho)\nabla\varrho)) - \varepsilon b''(\varrho)|\nabla\varrho|^2\end{gathered} \tag{5.58}$$

with a convenient function $b \in C^2(0, \infty)$. This equation can be obtained via multiplying equation (5.51) by $b'(\varrho)$. Further, one needs to prove that the quantity

$$P_\delta(\varrho) = p_\delta(\varrho) - (2\mu + \lambda)\mathrm{div}\boldsymbol{u}, \tag{5.59}$$

called *effective viscous pressure*, satisfies the identity

$$\overline{P_\delta(\varrho)b(\varrho)} - \overline{P_\delta(\varrho)}\,\overline{b(\varrho)} = (2\mu + \lambda)\Big(\overline{b(\varrho)\mathrm{div}\boldsymbol{u}} - \overline{b(\varrho)}\,\mathrm{div}\boldsymbol{u}\Big) \tag{5.60}$$

with another convenient function b. Here and in what follows the overlined quantities denote corresponding weak limits in $\mathcal{D}'(\Omega)$.

The same holds for the passage $\alpha \to 0+$, but now, (5.58) is replaced by the renormalized continuity equation (5.6).

Importance of the effective viscous pressure (5.59) and some of their properties was recovered in various contexts by several authors Lions [29], Serre [42], Hoff [24], Novotný, Padula [36] and [35]. Finally it was successfully used in existence theory by Lions [29]. Its rigorous mathematical realization is deeply related to the quality of density estimates and therefore to the value of γ (resp. β, in the case of limits $\varepsilon \to 0+$ and $\alpha \to 0+$). In fact, the difficulty of the underlying mathematical analysis increases with decreasing values of adiabatic constant. Intimately related to the DiPerna-Lions transport theory and to the Friedrich's lemma about commutators [8], the Lions method is applicable provided ϱ is square integrable. Thus, for general $\boldsymbol{f}$, it could be used without additional restriction as the condition $\gamma > \gamma_{gen.}$ is equivalent to $\gamma q > 2$ (cf. discussion after (5.41)). To treat also the case of potential $\boldsymbol{f}$ we shall rather apply another method proposed by Feireisl [13] (see also [18]) which is better adapted to investigate small adiabatic constants. We shall describe all details of this approach in the next section.

To conclude, both previous limit procedures, namely $\varepsilon \to 0+$ and $\alpha \to 0+$ have common features with the limit passage $\delta \to 0+$. The latter (most difficult) limit contains all of essential mathematical aspects of limits $\varepsilon \to 0+$, $\alpha \to 0+$. Consequently, the reader can, by himself, adapt the arguments of Section 6.2 to these situations.

5.5.2 Vanishing artificial pressure

Let $\varrho_\delta \in L^\beta_{sym}(\Omega)$, $\boldsymbol{u}_\delta \in W^{1,2}_{sym}(\Omega;\mathbf{R}^3)$ be sequence of bounded energy renormalized weak solutions to the problem

$$\operatorname{div}\big(b(\varrho_\delta)\boldsymbol{u}_\delta\big) + \big(\varrho_\delta b'(\varrho_\delta - b(\varrho_\delta)\big)\operatorname{div}\boldsymbol{u}_\delta = 0 \qquad \text{in } \mathcal{D}'(\Omega), \tag{5.61}$$

$$\operatorname{div}(\varrho_\delta \boldsymbol{u}_\delta \otimes \boldsymbol{u}_\delta) - \mu\Delta\boldsymbol{u}_\delta - (\mu+\lambda)\nabla\operatorname{div}\boldsymbol{u}_\delta + \nabla(\varrho_\delta^\gamma + \delta\varrho_\delta^\beta) = \varrho_\delta \boldsymbol{f} + \boldsymbol{g} \quad \text{in } \mathcal{D}'(\Omega;\mathbf{R}^3), \tag{5.62}$$

$$\int_\Omega \mu|\nabla\boldsymbol{u}_\delta|^2 + (\mu+\lambda)|\operatorname{div}\boldsymbol{u}_\delta|^2\,\mathrm{d}\boldsymbol{x} \le \int_\Omega (\varrho_\delta \boldsymbol{f} + \boldsymbol{g})\cdot\boldsymbol{u}_\delta\,\mathrm{d}\boldsymbol{x}, \tag{5.63}$$

where b is the same as in (5.6). By virtue of the estimates (5.45), (5.46), and the compact imbedding $W^{1,2}(\Omega;\mathbf{R}^3) \hookrightarrow\hookrightarrow L^p(\Omega;\mathbf{R}^3)$, $1 \le p < 6$ we obtain following limits

$$\left.\begin{aligned} \delta\varrho^\beta &\to 0 && \text{in } \mathcal{D}'(\Omega),\\ \varrho_\delta &\rightharpoonup \varrho && \text{weakly in } L^{q\gamma}(\Omega),\\ \boldsymbol{u}_\delta &\rightharpoonup \boldsymbol{u} && \text{weakly in } W^{1,2}(\Omega;\mathbf{R}^3),\\ \boldsymbol{u}_\delta &\to \boldsymbol{u} && \text{in } L^p(\Omega;\mathbf{R}^3),\ 1 \le p < 6, \end{aligned}\right\} \tag{5.64}$$

$$\left.\begin{aligned} \varrho_\delta\boldsymbol{u}_\delta &\rightharpoonup \varrho\boldsymbol{u} && \text{weakly in } L^r(\Omega), \text{ for some } r > 6/5,\\ \varrho_\delta\boldsymbol{u}_\delta\otimes\boldsymbol{u}_\delta &\rightharpoonup \varrho\boldsymbol{u}\otimes\boldsymbol{u} && \text{weakly in } L^q(\Omega). \end{aligned}\right\} \tag{5.65}$$

at least for a chosen subsequence.

Using these facts and the weak lower semi-continuity of the left hand side of (5.63) we can pass to the limit in (5.61 − 5.63) and we get

$$\operatorname{div}(\varrho\boldsymbol{u}) = 0 \qquad \text{in } \mathcal{D}'(\Omega) \tag{5.66}$$

$$\operatorname{div}\big(\overline{b(\varrho)}\boldsymbol{u}\big) + \overline{\big(\varrho b'(\varrho - b(\varrho)\big)\operatorname{div}\boldsymbol{u}} = 0 \qquad \text{in } \mathcal{D}'(\Omega), \tag{5.67}$$

$$\operatorname{div}(\varrho\boldsymbol{u}\otimes\boldsymbol{u}) - \mu\Delta\boldsymbol{u} - (\mu+\lambda)\nabla\operatorname{div}\boldsymbol{u} + \nabla\overline{\varrho^\gamma} = \varrho\boldsymbol{f} + \boldsymbol{g} \qquad \text{in } \mathcal{D}'(\Omega;\mathbf{R}^3), \tag{5.68}$$

$$\int_\Omega \mu|\nabla\boldsymbol{u}|^2 + (\mu+\lambda)|\operatorname{div}\boldsymbol{u}|^2\,\mathrm{d}\boldsymbol{x} \le \int_\Omega (\varrho\boldsymbol{f} + \boldsymbol{g})\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}. \tag{5.69}$$

The proof will be complete provided we show the strong convergence of ϱ_δ in $L^1(\Omega)$. This will be done in several steps following [37]. In the first step we shall prove identity (5.60) with $b = T_k$, $k > 0$, where

$$T_k(z) = kT\Big(\frac{z}{k}\Big);\ T \in C^\infty(\mathbf{R}^+),\ \text{concave};\ T(z) = z \text{ for } z \le 1;\ T(z) = 2 \text{ for } z \ge 3. \tag{5.70}$$

In the second step, we deduce from (5.60) an estimate measuring oscillations of the sequence of densities ρ_δ (see formula (5.77)). This information is used in the third step to prove that the couple $(\varrho, \boldsymbol{u})$ satisfies the renormalized continuity equation (see Lemma 5.5.2). The last

fourth step consists in comparing the weak limit of the renormalized continuity equation for $(\varrho_\delta, \boldsymbol{u}_\delta)$ with the renormalized continuity equation for the weak limit $(\varrho, \boldsymbol{u})$.

Step 1: Compactness properties of the effective viscous pressure (5.59). Testing (5.62) by $\eta\boldsymbol{\varphi}_\delta = \eta\nabla\Delta^{-1}(\xi T_k(\varrho_\delta))$ with η, $\xi \in \mathcal{D}(\Omega)$ we obtain

$$\int_\Omega \eta\xi\big(\varrho_\delta^\gamma - (2\mu+\lambda)\,\mathrm{div}\boldsymbol{u}_\delta\big)T_k(\varrho_\delta)\,\mathrm{d}\boldsymbol{x} = \mathrm{GoodTerms}_\delta + \int_\Omega \eta\,\underbrace{\mathcal{R}_{i,j}(\xi T_k(\varrho_\delta)\boldsymbol{u}_\delta^j)\varrho_\delta\boldsymbol{u}_\delta^i}_{\mathrm{DivCurl}_\delta}\,\mathrm{d}\boldsymbol{x}$$
$$+ \int_\Omega \boldsymbol{u}_\delta^j \cdot \underbrace{[\xi T_k(\varrho_\delta)\mathcal{R}_{i,j}(\eta\varrho_\delta\boldsymbol{u}_\delta^i) - \eta\varrho_\delta\boldsymbol{u}_\delta^i\mathcal{R}_{i,j}(\xi T_k(\varrho_\delta))]}_{\mathrm{Commutator}_\delta}\,\mathrm{d}\boldsymbol{x}, \quad (5.71)$$

$$\mathrm{GoodTerms}_\delta = \int_\Omega \big((\mu+\lambda)\mathrm{div}\boldsymbol{u}_\delta - \varrho_\delta^\gamma\big)\nabla\eta\cdot\boldsymbol{\varphi}_\delta - \delta\varrho_\delta^\beta\mathrm{div}(\eta\boldsymbol{\varphi}_\delta) + \big(\mu\nabla\boldsymbol{u}_\delta - \varrho_\delta\boldsymbol{u}_\delta\otimes\boldsymbol{u}_\delta\big)\nabla\eta\otimes\boldsymbol{\varphi}_\delta$$
$$- \mu\nabla\eta\otimes\boldsymbol{u}_\delta : \nabla\boldsymbol{\varphi}_\delta + \mu\boldsymbol{u}_\delta\cdot\nabla\eta(\xi T_k(\varrho_\delta)) - (f\varrho_\delta + g)\eta\boldsymbol{\varphi}_\delta\,\mathrm{d}\boldsymbol{x}. \quad (5.72)$$

Similarly we can test (5.68) by $\eta\boldsymbol{\varphi} = \eta\nabla\Delta^{-1}(\xi\overline{T_k(\varrho)})$ to get

$$\int_\Omega \eta\xi\big(\overline{\varrho^\gamma} - (2\mu+\lambda)\,\mathrm{div}\boldsymbol{u}\big)\overline{T_k(\varrho)}\,\mathrm{d}\boldsymbol{x} = \mathrm{GoodTerms} + \int_\Omega \eta\,\underbrace{\mathcal{R}_{i,j}(\xi\overline{T_k(\varrho)}\boldsymbol{u}^j)\varrho\boldsymbol{u}^i}_{\mathrm{DivCurl}}\,\mathrm{d}\boldsymbol{x}$$
$$+ \int_\Omega \boldsymbol{u}^j \cdot \underbrace{[\xi\overline{T_k(\varrho)}\mathcal{R}_{i,j}(\eta\varrho\boldsymbol{u}^i) - \eta\varrho\boldsymbol{u}^i\mathcal{R}_{i,j}(\xi\overline{T_k(\varrho)})]}_{\mathrm{Commutator}}\,\mathrm{d}\boldsymbol{x}, \quad (5.73)$$

$$\mathrm{GoodTerms} = \int_\Omega \big((\mu+\lambda)\mathrm{div}\boldsymbol{u} - \overline{\varrho^\gamma}\big)\nabla\eta\cdot\boldsymbol{\varphi} + \big(\mu\nabla\boldsymbol{u} - \varrho\boldsymbol{u}\otimes\boldsymbol{u}\big)\nabla\eta\otimes\boldsymbol{\varphi}$$
$$- \mu\nabla\eta\otimes\boldsymbol{u} : \nabla\boldsymbol{\varphi} + \mu\boldsymbol{u}\cdot\nabla\eta(\xi\overline{T_k(\varrho)}) - (f\varrho + g)\eta\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}. \quad (5.74)$$

Next we shall pass to the limit in (5.71) as $\delta \to 0+$. Realizing that $\boldsymbol{\varphi}_\delta \to \boldsymbol{\varphi}$ in any $L^p(\Omega;\mathbf{R}^3)$, $p > 1$ and taking into account limits (5.64), (5.65) it is straightforward to show that $(\mathrm{GoodTerms}_\delta) \to (\mathrm{GoodTerms})$. Furthermore, applying Lemma 2.2.4 and Lemma 2.2.5 we easily verify that $(\mathrm{DivCurl}_\delta) \rightharpoonup (\mathrm{DivCurl})$ weakly in $\mathcal{D}'(\Omega)$ and $(\mathrm{Commutator}_\delta) \rightharpoonup (\mathrm{Commutator})$ weakly in $L^r(\Omega)$, respectively. This is the only place where we need quite restrictive estimate (5.46).

Finally, subtracting (5.73) and the limit of (5.71) as $\delta \to 0+$, we obtain the famous identity for the effective viscous pressure, cf. (5.59), namely

$$\overline{\varrho^\gamma T_k(\varrho)} - \overline{\varrho^\gamma}\,\overline{T_k(\varrho)} = -(2\mu+\lambda)\big(\overline{T_k(\varrho)\mathrm{div}\boldsymbol{u}} - \overline{T_k(\varrho)}\mathrm{div}\boldsymbol{u}\big) \quad \text{a.e. in } \Omega. \quad (5.75)$$

Step 2: Defect measure of oscillations. Using in succesive steps the elementary algebraic inequality $(a-b)^\gamma \le a^\gamma - b^\gamma$, $a \ge b \ge 0$, weak lower semi-continuity of convex functionals $\varrho \to \int_\Omega \varrho^\gamma$, $\varrho \to -\int_\Omega T_k(\varrho)$, and (5.75) we succeed to control oscillations of the density sequence ϱ_δ in the following way

$$\limsup_{\delta\to 0+}\int_\Omega |T_k(\varrho) - T_k(\varrho_\delta)|^{\gamma+1}\,\mathrm{d}\boldsymbol{x} \le \limsup_{\delta\to 0}\int_\Omega (\varrho^\gamma - \varrho_\delta^\gamma)\big(T_k(\varrho) - T_k(\varrho_\delta)\big)\,\mathrm{d}\boldsymbol{x}$$
$$\le \int_\Omega \overline{\varrho^\gamma T_k(\varrho)} - \overline{\varrho^\gamma}\,\overline{T_k(\varrho)}\,\mathrm{d}\boldsymbol{x} \le C\big\|\mathrm{div}\boldsymbol{u}_\delta\big\|_2 \limsup_{\delta\to 0+}\big\|T_k(\varrho) - T_k(\varrho_\delta)\big\|_2. \quad (5.76)$$

Hence, thanks to (5.45),

$$\sup_{k>0} \limsup_{\delta\to 0+} \big\|T_k(\varrho) - T_k(\varrho_\delta)\big\|_{\gamma+1} \leq C. \tag{5.77}$$

Step 3: Renormalized continuity equation. The control of the density oscillations allows us to keep the renormalized continuity equation (5.6) valid for the limits ϱ, $\boldsymbol{u}$ even if the density is not known to be square integrable. More precisely we claim (see e.g. [37, Lemma 4.50]):

Lemma 5.5.2. *Let b belong to (5.7), $\boldsymbol{u}_\delta \rightharpoonup \boldsymbol{u}$ weakly in $W^{1,2}(\Omega;\mathbf{R}^3)$ and $\varrho_\delta \rightharpoonup \varrho$ weakly in $L^s(\Omega)$, $s > 1$ and suppose that (5.61), (5.67) and (5.77) hold. Then $(\varrho, \boldsymbol{u})$ satisfies renormalized continuity equation (5.6) in $\mathcal{D}'(\Omega)$.*

If $s \geq 2$, Lemma 5.5.2 is a particular case of the DiPerna-Lions transport theory, which is, in this case, a direct consequence of (5.66) and the Fridrichs' lemma about commutators [8].

If $s \in (1,2)$ one may adapt to the steady situation the "nonsteady" aproach of Feireisl [13] (see also [18]). Since $\overline{T_k(\varrho)}$ belongs, in particular, to $L^2(\Omega)$, one can apply the Di-Perna, Lions transport theory to (5.67) with $b = T_k$ to conclude that

$$\mathrm{div}\big(b(\overline{T_k(\varrho)})\boldsymbol{u}\big) + \Big(\overline{T_k(\varrho)}b'\big(\overline{T_k(\varrho)}\big) - b\big(\overline{T_k(\varrho)}\big)\Big)\,\mathrm{div}\boldsymbol{u} = b'\big(\overline{T_k(\varrho)}\big)\overline{\big(\varrho T_k(\varrho) - T_k(\varrho)\big)\mathrm{div}\boldsymbol{u}}, \tag{5.78}$$

e.g. for any $b \in C^1([0,\infty)) \cap C_0([0,\infty))$. As the consequence of the weak lower semi-continuity of norms we get

$$\big\|\overline{T_k(\varrho)} - \varrho\big\|_1 \leq Ck^{1-p}, \quad \big\|T_k(\varrho) - \varrho\big\| \leq Ck^{1-p}, \qquad \text{for } 1 \leq p < \gamma q. \tag{5.79}$$

Using this fact and (5.77) one verifies that

$$b'(\overline{T_k(\varrho)})\overline{(\varrho T_k(\varrho) - T_k(\varrho))\mathrm{div}\boldsymbol{u}} \to 0 \quad \text{in } L^1(\Omega).$$

Consequently (5.78) yields (5.6) for a compactly supported b. The passage to general b given by (5.7) can be performed via the Lebesgue dominated convergence theorem.

Step 4: Strong convergence of ρ_δ. Finally we use (5.6) to prove the strong convergence of ϱ_δ in $L^1(\Omega)$. We introduce functions $L_k(z) \approx z\log(z)$ by the equation $tL_k'(t) - L_k(t) = T_k(t)$. Using L_k as b in (5.6) and (5.66) leads to $\int_\Omega T_k(\varrho)\mathrm{div}\boldsymbol{u} = 0$ and $\int_\Omega \overline{T_k}\mathrm{div}\boldsymbol{u} = 0$, respectively. With this information at hand, the revisited proof of formula (5.76) yields

$$\begin{aligned}\limsup_{\delta\to 0+}\big\|T_k(\varrho) - T_k(\varrho_\delta)\big\|_{\gamma+1}^{\gamma+1} &\leq C\int_\Omega \mathrm{div}\boldsymbol{u}(T_k(\varrho) - \overline{T_k(\varrho)})\,\mathrm{d}\boldsymbol{x}\\ &\leq C\,\big\|T_k(\varrho) - \overline{T_k(\varrho)}\big\|_1^{\frac{\gamma-1}{2\gamma}} \limsup_{\delta\to 0+}\big\|T_k(\varrho) - T_k(\varrho_\delta)\big\|_{\gamma+1}^{\frac{\gamma+2}{2\gamma}}.\end{aligned} \tag{5.80}$$

Recalling (5.79), the right-hand side of (5.80) tends to zero with k. Now, we write

$$\limsup_{\delta\to 0+}\big\|\varrho_\delta - \varrho\big\|_1 \leq \big\|\varrho_\delta - T_k(\varrho_\delta)\big\|_1 + \limsup_{\delta\to 0+}\big\|T_k(\varrho_\delta) - T_k(\varrho)\big\|_1 + \big\|T_k(\varrho) - \varrho\big\|_1.$$

By virtue of (5.79) and (5.80), the right hand side of the above formula tends to zero. Consequently, the sequence ρ_δ converges strongly in $L^s(\Omega)$, for all $1 \leq s < \gamma q$ and $\overline{\rho^\gamma}$ in equation (5.68) is equal to ρ^γ. This completes the proof of Theorem 5.1.3.

Bibliography

[1] D. R. Adams and L. I. Hedberg. *Function spaces and potential theory.* Springer-Verlag, Berlin, 1996.

[2] R. A. Adams and J. J. F. Frounier. *Sobolev Spaces.* Elsevier, 2003.

[3] C. Buet and B Després. Asymptotic analysis of fluid models for the coupling of radiation and hydrodynamics. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 85:385–418, 2004.

[4] A. P. Calderon. Lebesgue spaces of differentiable functions and distributions. In *Partial Differential Equations, Proc. Sympos. Pure Math.*, number 4, pages 33–49. Amer. Math. Soc., Providence, Rhode Island, 1961.

[5] L. Cattabriga. Su un problema al contorno relativo al sistema di equazioni di stokes. *Rend. Sem. Mat. Univ. Padova*, 31:308–340, 1961.

[6] R. Coifman and Y. Meyer. On commutators of singular integrals and bilinear singular integrals. *Trans. Amer. Math. Soc.*, 212:315–331, 1975.

[7] C. Conca, J. H. San Martin, and M. Tucsnak. Motion of a rigid body in a viscous fluid. 328(6):473–478, 1999.

[8] R. J. DiPerna and P.-L. Lions. Ordinary differential equations, transport theory and sobolev spaces. *Invent. Math.*, 98:511–547, 1989.

[9] B. Ducomet and E. Feireisl. On the dynamics of gaseous stars. *Arch. Rational Mech. Anal.*, 174(2):221–266, 2004.

[10] D. B. Ebin. Viscous fluids in a domain with frictionless boundary. In H. Kurke, J. Mecke, H. Triebel, and R.Thiele, editors, *Global Analysis - Analysis on Manifolds*, pages 93–110. Teubner, Leipzig, 1983.

[11] S. Eliezer, A. Ghatak, and H. Hora. *An introduction to equations of states, theory and applications.* Cambridge University Press, Cambridge, 1998.

[12] E. Feireisl. On the motion of a viscous, compressible, and heat conducting fluid. *Indiana Univ. Math. J.*, 53:1705–1738, 2000.

[13] E. Feireisl. On compactness of solutions to the compressible isentropic navier-stokes equations when the density is not square integrable. *Comment. Math. Univ. Carolinae*, 42(1):83–98, 2001.

[14] E. Feireisl. *Dynamics of Viscous Compressible Fluids*. Oxford University Press, Oxford, 2003.

[15] E. Feireisl. On the motion of rigid bodies in a viscous compressible fluid. *Arch. Rational Mech. Anal.*, 167:281–308, 2003.

[16] E. Feireisl. Mathematical theory of compressible, viscous, and heat conductiong fluids. *Comp. and Math. with Applications*, 53:461–490, 2007.

[17] E. Feireisl and A. Novotný. On a simple model of reacting compressible flows arising in astrophysics. *Proceedings Section A: Mathematics -Royal Society of Edimburgh*, 135(6):1169–1194, 2005.

[18] E. Feireisl, A. Novotný, and H. Petzeltová. On the existence of globally defined weak solutions to the navier-stokes equations of compressible isentropic fluids. *J. Math. Fluid Dynamics*, (3):358–392, 2001.

[19] J. Frehse, S. Goj, and M. Steinhauer. L^p - estimates for the navier-stokes equations for steady compressible flow. *Manuscripta Math.*, 116(3):265–275, 2005.

[20] G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I, Linearized steady problems*. Springer, 1994.

[21] G. P. Galdi and C. G. Simander. Existence, uniqueness and l^q-estimates for the stokes problem in an exterior domain. *Arch. Ration. Mech. Anal.*, 112(4):291–318, 1990.

[22] G. Gallavotti. *Statistical mechanics: A short treatise*. 1999.

[23] M. Geiert, H. Heck, and M. Hieber. On the equation div u = g and bogovskii's operator in sobolev spaces of negative order. *Operator Theory, Advances and Applications*, 168:113–121, 2006.

[24] D. Hoff. Strong convergence to global solutions for multidimensional flows of compressible, viscous fluids with polytropic equations of state and discontinuous initial data. *Arch. Rational Mech. Anal.*, 132:1–14, 1995.

[25] J. L. Kelley. *General topology*. Van Nostrand, Inc. Princeton, 1957.

[26] J. Kurzweil. *Ordinary differential equations*. Elsevier, Amsterdam, 1986.

[27] O. A. Ladhyzenskaya, V. A. Solonikov, and N. N. Uraltseva. *Linear and quasi-linear equations of parabolic type*, volume 23 of *Translations of Mathematical Monographs*. Providence, RI: AMS, 1988.

[28] J. Leray. Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Math.*, 63:193–248, 1934.

[29] P.-L. Lions. *Compressible models*, volume 2 of *Mathematical topics in fluid dynamics*. Oxford Science Publication, Oxford, 1998.

[30] J. A. San Martin, V. Starovoitov, and M. Tucsnak. Global weak solutions for the two dimensional motion of several rigid bodies in an incomressible viscous fluid. 161:93–112, 2002.

[31] I. Müller and T. Ruggeri. *Rational extended thermodynamics*, volume 37 of *Springer Tracts in Natural Philosophy*. Springer-Verlag, Heidelberg, 1998.

[32] J. Nečas. *Les Methodes Directes en théorie des Équations Elliptiques*. Masson & CIE, Éditeurs, Paris, 1967.

[33] L. Nirenberg. *Topics in nonlinear functional analysis*. Courant Intstitute, New York, 1974.

[34] S. Novo and A. Novotný. On the existence of weak solutions to steady compressible navier-stokes equations when the density is not square integrable. *J. Math. Kyoto Univ.*, 42(3):531–550, 2002.

[35] A. Novotný. Some remarks to the compactness of steady compressible isentropic navier-stokes equations via decomposition method. *Comment. Math. Univ. Carolinae*, 37(2):305–342, 1996.

[36] A. Novotný and M. Padula. Existence and uniquaness of stationary solutions for viscous compressible heat-conductive fluid with large potential and small nonpotential external forces. *Sib. Math. J.*, 34:120–146, 1991.

[37] A. Novotný and I. Straškraba. *Introduction to the Mathematical Theory of Compressible Flow*. Oxford University Press, Oxford, 2004.

[38] P. Pedregal. *Parametrized measures and variational principles*. Birkhauser, Basel, 1997.

[39] P. I. Plotnikov and J. Sokolowski. Concentrations of stationary solutions to compressible navier-stokes equations. *Comm. Math. Phys.*, 258(3), 2005.

[40] P. I. Plotnikov and J. Sokolowski. Stationary solutions of navier-stokes equations for diatomic gases. *Uspekhi Mat. Nauk*, 62(3), 2007.

[41] T. Ruggeri and M. Trovato. Hyperbolicity in extended thermodynamics of fermi and bose gases. *Continuum Mech. Thermodyn.*, 2004. Published online.

[42] D. Serre. Variation de grande amplitude pour la densité d'un fluid viscueux compressible. *Physica D.*, 48:113–128, 1991.

[43] J. Simon. Domain variation for drag in stokes flow. In *Control theory of distributed parameter systems and applications (Shanghai, 1990)*, number 159 in Lecture Notes in Control and Inform. Sci., pages 28–42. Springer, Berlin, 1991.

[44] L. Tartar. Compensated compactness and applications to partial differential equations. In L. J. Knopps, editor, *Nonlinear. Anal. and Mech.*, number 39 in Research Notes in Math, pages 138–211. Heriot-Watt Sympos., Pitman, Boston, 1975.

[45] C. Truesdell and K. R. Rajagopal. *An introduction to the mechanics of fluids*. Birkhäuser, Boston, 2000.

[46] W. P. Ziemer. *Weakly differentiable functions*. Springer-Verlag, New York, 1989.