Interacting particle systems: stochastic order, attractiveness and random walks on small world graphs.

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INTERACTING PARTICLE SYSTEMS: STOCHASTIC ORDER, ATTRACTIVENESS AND RANDOM WALKS ON SMALL WORLD GRAPHS

Ph.D. Thesis

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The main subject of the thesis is concerned with interacting particle systems, which are classes of spatio-temporal stochastic processes describing the evolution of particles in interaction with each other.
The particles move on a finite or infinite discrete space and on each element of this space the state of the configuration is integer valued. Configurations of particles evolve in continuous time according to a Markov process. Here the space is either the infinite deterministic d-dimensional lattice or a random graph given by the finite d-dimensional torus with random matchings.

In Part I we investigate the stochastic order in a particle system with multiple births, deaths and jumps on the d-dimensional lattice: stochastic order is a key tool to understand the ergodic properties of a system. We give applications on biological models of spread of epidemics and metapopulation dynamics systems.

In Part II we analyse the coalescing random walk in a class of finite random graphs modeling social networks, the small world graphs. We derive the law of the meeting time of two random walks on small world graphs and we use this result to understand the role of random connections in meeting time of random walks and to investigate the behavior of coalescing random walks.

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Part I

Stochastic Order, Attractiveness and Applications to Biological Models

Chapter 1

Introduction

1.1 Introduction (français)

1.1.1 Systèmes de particules en interaction

Les systèmes de particules en interaction sont des processus de Markov $(\eta_t)_{t\geq 0}$ avec $\eta_t : S \to X$, où S est un ensemble de sites et X est un ensemble dénombrable d'états possibles sur chaque site. Nous nous concentrons sur le cas $S = \mathbb{Z}^d$ et $X \subseteq \mathbb{N}$. Pour tout $x \in S$, soit $(X, \rho_x, \mathcal{X}_x)$ un espace métrique séparable complet, où ρ_x est une métrique sur X et $\mathcal{X}_x$ est la tribu engendrée (par les ouverts de la topologie engendrée) par ρ_x. Soit $\Omega = X^S$ *l'espace d'états* du système de particules et $\mathcal{X} = \prod_{x\in S} \mathcal{X}_x$ la tribu produit sur Ω. La valeur $\eta_t(x)$ est l'état du site $x \in S$ à l'instant $t \geq 0$ et η_t est la *configuration* du système à l'instant $t \geq 0$. Ces processus de Markov décrivent l'évolution d'un système de particules sur S en interaction les unes avec les autres, et ils sont utilisés pour modéliser des phénomènes d'épidémies, de dynamiques des populations ou des processus chimiques.

La première question est l'existence de tels processus. L'évolution temporelle de ces systèmes est décrite par des taux de transitions. Ces taux décrivent comment un processus passe d'un état à un autre. Pour $\eta \in \Omega$ et $\xi \in \Omega$, $c(\eta, \xi)$ est le taux de transition de la configuration η à la configuration ξ, c'est-à-dire

$$\lim_{h\to 0} \frac{P^\eta(\eta_h = \xi)}{h} = c(\eta, \xi)$$

où P^η dénote la loi du processus partant de η. Le *générateur infinitésimal* $\mathcal{L}$ est un opérateur déterminé a priori par ses valeurs sur l'ensemble des fonctions f sur Ω qui ne dépendent que d'un nombre fini de coordonnées, défini par

$$\mathcal{L}f(\eta) = \sum_{\xi\in\Omega} c(\eta, \xi)[f(\xi) - f(\eta)].$$

Afin d'assurer que $\mathcal{L}$ est un opérateur bien défini, nous travaillons avec *des interactions locales*: on supposera qu'il existe K tel que au plus $l \leq K$ sites changent d'état à chaque instant t et que les taux de transition dépendent uniquement de l'état d'un nombre fini de sites voisins. Nous supposons aussi $c(\eta, \xi)$ invariant par translation, c'est-à-dire

$$c(\eta, \xi) = c(\sigma_x\eta, \sigma_x\xi)$$

pour tous $x \in S$, où $(\sigma_x\eta)(z) = \eta(x + z)$ pour $z \in S$.
Si $X = \{0, 1, \ldots, N\}$ est fini, l'existence et l'unicité du processus ont été démontrées

par Liggett [33] sous des conditions générales; une construction graphique alternative a été démontrée par Harris [29] et reformulée par Durrett [22]. Si X est un ensemble dénombrable fini ou infini, des conditions suffisantes d'existence et d'unicité sont données par Chen [12, Chapitre 13].
En ce cas le processus a un espace d'états plus petit $\Omega_0 \subset \Omega$. Nous posons $\Omega_0 = \Omega$ si X est fini. Plus précisément, étant donnée une suite $(k_x)_{x\in\mathbb{Z}^d}$ telle que $\sum_{x\in\mathbb{Z}^d} k_x < \infty$, nous définissons

$$\Omega_0 = \{\eta \in \Omega : \sum_{x\in\mathbb{Z}^d} \rho_x(0, \eta(x))k_x < \infty\}. \tag{1.1}$$

Pour $\eta \in \Omega_0$, soit E^η l'espérance correspondant à P^η. Pour $f \in C(\Omega_0)$, l'ensemble des fonctions continues sur Ω_0, nous définissons

$$T(t)f(\eta) := E^\eta f(\eta_t).$$

Pour chaque probabilité μ sur $(\Omega_0, \mathcal{X})$, $\mu T(t)$, la probabilité sur $(\Omega_0, \mathcal{X})$ définie par

$$\int f[d\mu T(t)] = \int [T(t)f]d\mu$$

pour tous $f \in C(\Omega_0)$ (voir [33, Chapitre I]), est la loi à l'instant t du processus de distribution initiale μ.

Une des questions principales concernant un système de particules en interaction est l'existence d'une *mesure invariante* non triviale, soit une mesure μ telle que $P_\mu(\eta_t \in A) = \mu(A)$ pour tous $t \geq 0$, $A \in \mathcal{X}$, où P_μ est la loi du processus de distribution initiale μ. Le terme triviale signifie que la mesure est concentrée sur un état absorbant, s'il en existe un. Soit $\mathcal{I}$ l'ensemble de toutes les mesures invariantes du processus. Celui-ci est *ergodique* s'il y a une unique mesure invariante et si le processus converge vers cette mesure pour toute distribution initiale (voir [33, Définition 1.9]).

Afin d'étudier l'ergodicité d'un système, on peut le comparer avec d'autres processus de comportements mieux connus. Le *couplage* est une technique qui consiste, à partir de différents processus $(\eta_t)_{t\geq 0}$ et $(\xi_t)_{t\geq 0}$, à construire un processus *couplé* $(\eta_t, \xi_t)_{t\geq 0}$, qui est un processus de Markov avec espace d'états $\Omega_0 \times \Omega_0$, tel que chaque marginale est une copie du processus original.
Un outil fondamental pour les systèmes de particules en interaction est l'*attractivité*, une propriété concernant la distribution au temps t de deux processus *avec le même générateur* partant de distributions initiales différentes (voir la Section 2.1). L'attractivité correspond à l'existence d'un processus de Markov couplé $(\xi_t, \eta_t)_{t\geq 0}$ tel que si $\xi_0(x) \leq \eta_0(x)$ (pour tout $x \in \mathbb{Z}^d$) alors, pour tout $t \geq 0$, $\xi_t(x) \leq \eta_t(x)$ presque sûrement.

Dans les applications biologiques, une particule représente un individu d'une espèce sur un site x et la configuration identiquement égale à 0 (notée $\underline{0}$) correspond à l'extinction de l'espèce; c'est souvent un état absorbant: un problème important dans ce cadre est de trouver des conditions sur les taux qui induisent soit une mesure invariante non triviale, c'est-à-dire la survie de l'espèce pour tous $t \geq 0$, soit l'ergodicité, c'est-à dire l'extinction presque sûre.
Il est donc utile de comprendre quand la population d'un système est toujours supérieure (ou inférieure) au nombre d'individus d'une autre. Le résultat principal de la première partie, le Théorème 1.4 introduit dans la Section 1.1.3, donne des conditions nécessaires et

suffisantes sur les taux de transition de deux systèmes qui garantissent que sur chaque site le nombre de particules d'un système est toujours plus grand que le nombre de particules de l'autre.
Pour une large classe de systèmes de particules en interaction nous obtenons des conditions nécessaires et suffisantes sur les taux qui assurent l'attractivité comme corollaire du Théorème 1.4. L'attractivité est un instrument essentiel pour déterminer l'ensemble des probabilités invariantes. Soivent $\overline{\nu}$ et $\underline{\nu}$ les distributions limites respectives (quand t tend vers l'infini) du processus partant de la plus grande configuration initiale et de la plus petite, si l'espace d'états est compact; si le processus est attractif et $\overline{\nu} = \underline{\nu}$, alors le système est ergodique: c'est l'idée principale du *critère d'ergodicité* expliqué dans la Section 1.1.2. Nous utilisons cette technique pour améliorer les conditions d'ergodicité pour un modèle d'épidémie déjà bien étudié (voir la Section 1.1.3).

Dans la Section 1.1.2, nous expliquons la *technique de comparaison avec la percolation orientée*. Dans la dernière section de la première partie de la thèse nous étudions plusieurs modèles de dynamiques des populations, et nous utilisons les arguments de comparaison pour démontrer l'existence de mesures invariantes non triviales.

Cette introduction est organisée comme suit. La Section 1.1.2 décrit les principales techniques que nous utilisons, à savoir une brève définition du couplage, le critère d'ergodicité et la comparaison avec la percolation orientée. Dans la Section 1.1.3 nous donnons les conditions nécessaires et suffisantes qui garantissent l'ordre stochastique et l'attractivité. Les Sections 1.1.4 et 1.1.5 contiennent des applications à l'ergodicité et à l'existence de mesures invariantes non triviales dans des modèles biologiques.

1.1.2 Techniques

Couplage

Cette technique très puissante est utilisée dans de nombreux domaines de la théorie des probabilités.
Étant donnés deux systèmes de particules en interaction $(\eta_t)_{t\geq 0}$ et $(\xi_t)_{t\geq 0}$, respectivement sur $(\Omega, \mathcal{X}_1, \mathbb{P}_1)$ et $(\Omega, \mathcal{X}_2, \mathbb{P}_2)$, de générateurs $\mathcal{L}_1$ et $\mathcal{L}_2$, *le processus couplé* $(\eta_t, \xi_t)_{t\geq 0}$ est un processus de Markov sur $(\Omega\times\Omega, \mathcal{X}_1\times\mathcal{X}_2, \widetilde{\mathbb{P}})$ de générateur $\widetilde{\mathcal{L}}$ tel que chaque marginale est une copie du processus original. En d'autres termes, pour tous $A \in \mathcal{X}_1$, $B \in \mathcal{X}_2$,

$$\begin{aligned}\widetilde{\mathbb{P}}\Big((\eta_t, \xi_t) \in (A \times \Omega)\Big) &= \mathbb{P}_1(\eta_t \in A)\\ \widetilde{\mathbb{P}}\Big((\eta_t, \xi_t) \in (\Omega \times B)\Big) &= \mathbb{P}_2(\xi_t \in B)\end{aligned}$$

et $\widetilde{\mathbb{P}}$ est une *mesure de probabilité couplée*.
Une définition équivalente est donnée à partir du générateur infinitésimal $\widetilde{\mathcal{L}}$: pour tous $A \in \mathcal{X}_1$ et $B \in \mathcal{X}_2$,

$$\begin{aligned}\widetilde{\mathcal{L}}\mathbb{1}_{A\times\Omega}(\eta,\xi) &= \mathcal{L}_1\mathbb{1}_A(\eta)\\ \widetilde{\mathcal{L}}\mathbb{1}_{\Omega\times B}(\eta,\xi) &= \mathcal{L}_2\mathbb{1}_B(\xi).\end{aligned}$$

Il y a plusieurs manières de construire un processus couplé: nous présentons brièvement trois des couplages principaux utilisés.
Couplage indépendant

$$\widetilde{\mathcal{L}}^i f(\eta,\xi) = [\mathcal{L}_1 f(\cdot,\xi)](\eta) + [\mathcal{L}_2 f(\eta,\cdot)](\xi)$$

les deux processus se déplacent de manière indépendante via des taux non couplés.
Couplage classique
Supposons $\mathcal{L}_1 = \mathcal{L}_2$, on note $g(\eta) = f(\eta, \eta)$ et $\Delta = \{(\eta, \xi) \in \Omega^2 : \eta = \xi\}$.

$$\widetilde{\mathcal{L}}^c f(\eta, \xi) = 1\!\!1_{\Delta^c}(\eta, \xi)\widetilde{\mathcal{L}}^i f(\eta, \xi) + 1\!\!1_{\Delta}(\eta, \xi)\mathcal{L}_1 g(\eta)$$

les processus se déplacent de manière indépendante jusqu'à ce qu'ils se rencontrent, et après ils se déplacent ensemble.
Couplage de base
Pour η, ξ dans Ω, soit $I(\eta, \xi) = \{\zeta \in \Omega : \eta+\zeta \in \Omega_0, \xi+\zeta \in \Omega_0\}$ où $\eta+\zeta$ est la configuration définie par $(\eta + \zeta)(x) = \eta(x) + \zeta(x)$ pour tous $x \in S$

$$\begin{aligned}\widetilde{\mathcal{L}}^b f(\eta, \xi) = &\sum_{\zeta\in I(\eta,\xi)} [c_1(\eta, \eta + \zeta) \wedge c_2(\xi, \xi + \zeta)]\big(f(\eta + \zeta, \xi + \zeta) - f(\eta, \xi)\big)\\ &+ \sum_{\zeta\in I(\eta,\xi)} [c_1(\eta, \eta + \zeta) - c_2(\xi, \xi + \zeta)]^+\big(f(\eta + \zeta, \xi) - f(\eta, \xi)\big)\\ &+ \sum_{\zeta\in I(\eta,\xi)} [c_2(\eta, \eta + \zeta) - c_1(\xi, \xi + \zeta)]^+\big(f(\eta, \xi + \zeta) - f(\eta, \xi)\big)\end{aligned}$$

en ce cas les processus se déplacent ensemble autant que possible.

Comparaison avec la percolation orientée

Nous reprenons la présentation faite dans [22] du *théorème de comparaison* afin de prouver l'existence de mesures invariantes non triviales pour des systèmes de particules d'espace d'états non compact Ω. L'idée consiste à renormaliser le processus en espace et en temps, et à faire une comparaison avec un modèle de percolation orientée. Soit

$$\mathcal{N} = \{(x, n) \in \mathbb{Z}^2 : x + n \text{ is even}, n \geq 0\}.$$

Pour tout $(x, n) \in \mathbb{Z}^2$, on place une arête orientée de (x, n) vers $(x + 1, n + 1)$ et de (x, n) vers $(x - 1, n - 1)$. Soit $w(x, n)$ une variable aléatoire qui indique si le site est ouvert (1) ou fermé (0). On dit qu'un site (y, n) peut être atteint à partir de (x, m) et on écrit

$$(x, m) \to (y, n)$$

s'il existe une suite de points $x_m = x, \ldots, x_n = y$ tels que $|x_k - x_{k-1}| = 1$ pour $m < k \leq n$ et $w(x_k, k) = 1$ pour $m \leq k \leq n$, c'est-à-dire une trajectoire de sites ouverts.
Nous disons que les $w(x, n)$ sont M-dépendants de densité $1-\gamma$, si pour toute suite (x_i, n_i), $1 \leq i \leq I$ telle que $\|(x_i, n_i) - x_j, n_j)\|_\infty > M$, pour $i \neq j$

$$\mathbb{P}(w(x_i, n_i) = 0 \text{ for } 1 \leq i \leq I) \leq \gamma^I.$$

On se donne une condition initiale $W_0 \subset 2\mathbb{Z} = \{x \in \mathbb{Z} : (x, 0) \in \mathcal{N}\}$ et on définit le processus de *percolation orientée* par

$$W_n = \{y \in \mathbb{Z} : (x, 0) \to (y, n) \text{ pour un } x \in W_0\}$$

autrement dit W_n est l'ensemble des sites ouverts au temps n. Dans le cas où $W_0^0 = \{0\}$, nous notons

$$\mathcal{C}_0 = \{(y, n) \in \mathcal{N} : (0, 0) \to (y, n)\}$$

l'ensemble des sites qui peuvent être atteints par un chemin à partir de $(0, 0)$. Lorsque $P(|C_0| = \infty) > 0$ on dit que la *percolation se réalise*. Le résultat suivant indique que, si la densité de sites ouverts est assez grande, il y a percolation:

Théorème 1.1 *[22, Théorème 4.1] Si $\gamma \leq 6^{-4(2m+1)^2}$, alors*

$$\mathbb{P}(|\mathcal{C}_0| < \infty) \leq 1/20.$$

Afin de prouver l'existence d'une mesure invariante non triviale, on a besoin de montrer que, si on part d'une "bonne configuration", on peut obtenir une densité positive de sites ouverts. Soit W_0^p une configuration initiale définie par: les sites de $\{x \in 2\mathbb{Z}, x \in W_0^p\}$, sont ouverts indépendamment avec probabilité p. Alors

Théorème 1.2 *[22, Théorème 4.2] Si $p > 0$ et $\gamma \leq 6^{-4(2M+1)^2}$, alors*

$$\liminf_{n\to\infty} \mathbb{P}(0 \in W_{2n}^p) \geq 19/20.$$

La relation entre le système de particules et le modèle de percolation est donnée par le théorème de comparaison suivant. L'idée est de partitionner l'espace-temps en boîtes spatio-temporelles et d'associer à chaque boîte un *bon évènement* $G_{m,n}$, $(m,n) \in \mathcal{N}$. Chaque fois que l'évènement est réalisé on place une arête orientée du site (m,n) vers les sites $(m+1,n+1)$ et $(m-1,n+1)$.
Soit $(\eta_t)_{t\geq 0}$ un processus invariant par translation et à portée finie construit à partir d'une représentation graphique tel que $\eta_t : S \to X = \{0,1,\ldots,N\}$. On fixe $T > 0$, L, k_0 et j_0 des entiers positifs et $M = \max\{k_0, j_0\}$. Pour $(m,n) \in \mathcal{N}$, nous définissons la boîte spatio-temporelle

$$\mathcal{R}_{m,n} = (2mLe_1, nT) + \{[-k_0L, k_0L]^d \times [0, j_0T]\},$$

où $e_1 = (1,0,\ldots,0)$.
Soit H un ensemble de configurations déterminé par les valeurs de ξ sur $[-L,L]^d$. On dira que (m,n) est occupé si $\eta_{nT} \in \sigma_{2mLe_1}H$.
Condition de comparaison. *On suppose que pour tout $(m,n) \in \mathcal{N}$, il existe un évènement $G_{m,n}$ ne dépendant que de l'état du système dans la boîte $\mathcal{R}_{m,n}$, tel que $P(G_{m,n}) \geq 1-\gamma$ et tel que si (m,n) est occupé alors sur l'évènement $G_{m,n}$ les sites $(m+1,n+1)$ et $(m-1,n+1)$ sont aussi occupés, i.e.*

$$\eta_{(n+1)T} \in \sigma_{2(m+1)Le_1}H \ \text{ et } \eta_{(n+1)T} \in \sigma_{2(m-1)Le_1}H.$$

Soit $X_n = \{m : (m,n) \in \mathcal{N}, \eta_{nT}) \in \sigma_{2mLe_1}H\}$ l'ensemble des sites occupés au niveau n. La relation finale entre le système de particules et le modèle de percolation est donnée par

Théorème 1.3 *[22, Théorème 4.3] Si la condition de comparaison est satisfaite, alors on peut trouver des variables aléatoires $w(x,n)$ telles que pour tout $n \geq 0$, X_n domine l'ensemble des sites occupés W_n d'un modèle de percolation orientée M-dépendant de densité $1-\gamma$, de configuration initiale $W_0 = X_0$, i.e. $X_n \subset W_n$ pour tout $n \geq 0$.*

Soit $(\eta_t)_{t\geq 0}$ un processus attractif de taux de transition invariants par translation. Soit $H = \{\xi : \xi(x) = 1 \text{ pour un } x \in [-L,L]^d\}$, et on suppose la condition de comparaison vérifiée. Soit une configuration initiale η_0 telle que $\eta_0(y) = 1\!\!1_{\{y=x\}}(y)$ pour un $x \in \mathbb{Z}^d$, par invariance par translation nous pouvons supposer $x = 0$. Par les Théorèmes 1.3 et 1.1, si γ est suffisamment petit et δ_η est la mesure de Dirac concentrée sur $\eta \in \Omega$,

$$\liminf_{t\to\infty} \delta_{\eta_0} T(t)\big\{\eta : \eta(x) > 0 \text{ pour un } x \in \mathbb{Z}^d\big\} \geq \mathbb{P}(|\mathcal{C}_0| = \infty) > 0 \tag{1.2}$$

c'est-à-dire l'existence d'un chemin infini de sites ouverts dans la percolation orientée implique l'existence d'un chemin infini d'individus dans le système de particules.
Afin de construire une mesure invariante non triviale, on suppose $\eta_0(x) = N$ pour tout $x \in \mathbb{Z}^d$. Par l'attractivité

$$\lim_{t\to\infty} \delta_{\eta_0} T(t) = \overline{\nu}$$

qui est une measure invariante.
Si la condition de comparaison est vérifiée, par le Théorème 1.3, X_n domine un processus de percolation orientée W_n^p tel que un site $x \in 2\mathbb{Z}$ est ouvert à l'instant 0 s'il existe $y \in x + \sigma_{2mLe_1} H$ tel que $\eta_0(y) \geq 1$. Puisque $\eta_0(x) = N$ pour tout $x \in \mathbb{Z}^d$, dans la configuration initiale W_0^p tous les sites sont ouverts, c'est-à-dire $p = 1$. Par le Théorème 1.2

$$\begin{aligned} \nu\{\eta : \eta(x) > 0 \text{ pour } x \in 2mLe_1 + [-L, L]^d\} &\geq \liminf_{n\to\infty} \delta_{\eta_0} T(n)\{(m,n) \text{ est occupé}\} \\ &\geq \liminf_{n\to\infty} \mathbb{P}(0 \in W_{2n}^1) > 0 \end{aligned} \tag{1.3}$$

ce qui implique que ν se concentre sur des configurations avec une infinité d'individus.

Critère d'ergodicité

Cette technique donne des conditions suffisantes sur les taux de transition pour l'ergodicité d'un processus attractif et invariant par translation. Elle a été utilisée par plusieurs auteurs (voir [12], [13], [35]) pour des processus de réaction-diffusion.
Soit $(\eta_t)_{t\geq 0}$ un système attractif tel que $\eta_t : \mathbb{Z}^d \to X = \{0, 1, \ldots, N\} \subset \mathbb{N}$, et pour chaque $x \in S$ soit $F_x : X \times X \to \mathbb{R}$ une métrique sur X. Nous supposons que F_x ne dépend pas de x, c'est-à-dire $F_x = F$ pour tout $x \in \mathbb{Z}^d$. Nous introduisons un ordre partiel sur $\Omega = X^{\mathbb{Z}^d}$. Brièvement, étant données deux configurations $\eta \in \Omega$ et $\xi \in \Omega$, $\eta \leq \xi$ si $\eta(x) \leq \xi(x)$ pour tout $x \in \Omega$ (voir la Section 2.1 pour plus de détails).
La métrique sur X induit une métrique sur Ω. À savoir, pour tous η et ξ dans Ω, nous définissons

$$\rho_\alpha(\eta, \xi) := \sum_{x\in\mathbb{Z}^d} F_x(\eta(x), \xi(x))\alpha(x) \tag{1.4}$$

où $\{\alpha(x)\}_{x\in\mathbb{Z}^d}$ est une suite telle que $\alpha(x) \in \mathbb{R}$, $\alpha(x) > 0$ pour tout $x \in \mathbb{Z}^d$ et

$$\sum_{x\in\mathbb{Z}^d} \alpha(x) < \infty. \tag{1.5}$$

Si $X = \mathbb{N}$ (c'est-à-dire "$N = \infty$"), le processus doit être construit sur un espace d'états plus petit (voir [13])

$$\Omega_0 = \{\eta \in \Omega : \sum_{x\in\mathbb{Z}^d} \eta(x)\alpha(x) < \infty\}.$$

Étant données deux probabilités μ_1 et μ_2 sur $(\Omega, \mathcal{X})$, nous définissons la 1-distance de Kantorovich

$$W(\mu_1, \mu_2) = \inf_{\widetilde{P}} \Big\{ \int \rho_\alpha(\eta, \xi) \widetilde{P}(d\eta, d\xi) \Big\}$$

où $\widetilde{P}$ est dans l'ensemble de toutes les probabilités de couplage de marginales μ_1 et μ_2.
Soit $\widetilde{\mathbb{P}}$ une probabilité de couplage sur $(\Omega \times \Omega, \mathcal{X}_1 \times \mathcal{X}_2)$ et $\widetilde{\mathbb{E}}$ l'espérance par rapport à $\widetilde{\mathbb{P}}$.
On note $E^n = \{\eta \in \mathbb{Z}^d : \eta(x) = n \text{ pour tout } x \in \mathbb{Z}^d\}$, $n \in X$. Soit η_t^k le processus tel que

$\eta_0^k \in E^k$, $k \in X$. L'attractivité et l'invariance par translation impliquent l'existence de deux mesures invariantes $\bar{\nu}$ et $\underline{\nu}$ tels que

$$\lim_{t\to\infty} \delta_{\eta^N} T(t) = \bar{\nu}; \qquad \lim_{t\to\infty} \delta_{\eta^0} T(t) = \underline{\nu}.$$

Encore par attractivité, si $\eta_0^0 \le \xi_0 \le \eta_0^N$ on obtient

$$\lim_{t\to\infty} \eta_t^0(x) \le \lim_{t\to\infty} \xi_t(x) \le \lim_{t\to\infty} \eta_t^N(x)$$

pour tout $x \in \mathbb{Z}^d$, $\widetilde{\mathbb{P}}$ a.s. Si $N = \infty$ une étape supplémentaire consiste à prouver l'existence de $\lim_{n\to\infty} \eta_t^n$ pour tous $t \ge 0$. Enfin, si on démontre que $\bar{\nu} = \underline{\nu}$ l'ergodicité suit.
L'idée principale consiste à choisir une bonne "métrique" F sur X, à évaluer la distance entre η_t^n et η_t^0 à chaque instant $t \ge 0$ et à chercher des conditions sur les taux telles que l'éspérance de la distance entre η_t^n et η_t^0 converge vers zéro uniformément par rapport à $x \in S$ si t tend vers l'infini. Si on démontre que

$$\widetilde{\mathbb{E}}\Big(\lim_{t\to\infty} F(\eta_t^0(x), \eta_t^N(x))\Big) = 0 \tag{1.6}$$

uniformément par rapport à $x \in S$, alors

$$\begin{aligned} W(\underline{\nu}, \bar{\nu}) =& \widetilde{\mathbb{E}}\Big(\lim_{t\to\infty} \rho_\alpha(\eta_t^0, \eta_t^N)\Big) = \widetilde{\mathbb{E}}\Big(\lim_{t\to\infty} \sum_{x\in\mathbb{Z}^d} F\big(\eta_t^0(x), \eta_t^N(x)\big)\alpha(x)\Big) \\ \le& \sum_{x\in\mathbb{Z}^d} \alpha(x) \widetilde{\mathbb{E}}\Big(\lim_{t\to\infty} \sup_{x\in\mathbb{Z}^d} F\big(\eta_t^0(x), \eta_t^N(x)\big)\Big) = 0 \end{aligned} \tag{1.7}$$

et l'ergodicité suit.
Nous allons utiliser les propriétés du générateur et le Lemme du Gronwall (voir la Section 3.1.1) pour prouver que s'il existe une métrique F et $\epsilon > 0$ tels que pour tout $x \in \mathbb{Z}^d$

$$\widetilde{\mathbb{E}}\Big(\mathcal{L}F(\eta_t^0(x), \eta_t^N(x))\Big) \le -\epsilon \widetilde{\mathbb{E}}\Big(F(\eta_t^0(x), \eta_t^N(x))\Big) \tag{1.8}$$

alors (1.6) est vérifiée.
Dans [13, Théorème 14.10] l'auteur utilise la métrique

$$F(k,l) = \left| \sum_{j<k} u_j - \sum_{j<l} u_j \right|, \qquad k,l \in X = \mathbb{N}$$

où $\{u_k\}_{k\in\mathbb{N}}$ est une suite de réels positifs et on pose $\sum_{0\le j<0} u_j = 0$. La métrique

$$F(k,l) = 1\!\!1_{\{k\neq l\}} \sum_{j=0}^{|k-l|-1} u_j \qquad k,l \in X = \mathbb{N} \tag{1.9}$$

a été utilisée dans [35] si $1 \ge u_j \ge \gamma > 0$ et dans [12] si $1 \ge u_j > 0$ pour obtenir des conditions suffisantes d'ergodicité pour les processus de réaction diffusion.
Comme nous travaillons avec deux copies du même processus, nous pouvons les coupler et évaluer (1.8) par des couplages différents afin d'obtenir des conditions d'ergodicité différentes: dans [12] l'auteur utilise plusieurs couplages et il donne le meilleur (par rapport à un critère donné) pour l'ergodicité dans une classe de processus de réaction diffusion.
Si $\underline{\nu} \neq \delta_{\underline{0}}$, comme dans de nombreux processus de réaction-diffusion, on peut travailler

avec des couplages différents (voir [12]); sinon $\eta_t^0 = \eta_0^0$ pour tout $t \geq 0$ et on utilise le couplage indépendant.
Après avoir choisi une métrique (et un couplage), nous évaluons $\widetilde{\mathbb{E}}\big(\mathcal{L}F(\eta_t^0(x), \eta_t^N(x))\big)$ et nous cherchons des conditions sur les taux de transition qui assurent l'existence de $\{u_j\}_{j\in X}$ qui satisfait (1.8). Finalement, le problème se réduit à l'existence d'une suite de réels qui satisfont une condition sur les taux de transition, appelée *u-critère*.

1.1.3 Théorème de comparaison et attractivité

Le résultat principal est un théorème de comparaison entre différents systèmes de particules en interaction, qui nous donne des conditions nécessaires et suffisantes d'attractivité comme corollaire. Grosso modo, on dit qu'un processus est *stochastiquement plus grand* qu'un autre si nous pouvons coupler les deux processus de manière à ce que le plus grand au temps 0 reste toujours le plus grand, voir la Section 2.2 pour une définition plus rigoureuse: nous donnons des conditions nécessaires et suffisantes sur les taux de transition d'un processus qui garantissent qu'un système est stochastiquement plus grand (ou plus petit) qu'un autre.

On considère des systèmes avec des naissances, des morts et des sauts de plus d'une particule à la fois qui dépendent de la configuration de manière très générale. Soit $X \subseteq \mathbb{N}$, l'espace d'états du processus $(\eta_t)_{t\geq 0}$ est $\Omega = X^{\mathbb{Z}^d}$. Le générateur infinitésimal $\mathcal{L}$ est donné, pour une fonction locale f (c'est-à-dire une fonction qui ne dépend que d'un nombre fini de coordonnées) par

$$\begin{aligned}\mathcal{L}f(\eta) = \sum_{x,y\in S}\sum_{\alpha,\beta\in X}\chi_{\alpha,\beta}^{x,y}(\eta)p(x,y)\sum_{k>0}\Big(\Gamma_{\alpha,\beta}^{k}(f(S_{x,y}^{-k,k}\eta)-f(\eta))+\\ +(R_{\alpha,\beta}^{0,k}+P_{\beta}^{k})(f(S_{y}^{k}\eta)-f(\eta))+(R_{\alpha,\beta}^{-k,0}+P_{\alpha}^{-k})(f(S_{x}^{-k}\eta)-f(\eta))\Big)\end{aligned}\tag{1.10}$$

où $\chi_{\alpha,\beta}^{x,y}$ est l'indicateur des configurations de valeurs (α,β) sur (x,y), c'est-à-dire

$$\chi_{\alpha,\beta}^{x,y}(\eta) = \begin{cases} 1 & \text{si } \eta(x)=\alpha \text{ et } \eta(y)=\beta \\ 0 & \text{autrement.}\end{cases}$$

$S_{x,y}^{-k,k}$, S_y^k et S_y^{-k}, où $k > 0$, sont des opérateurs locaux exécutant les transformations chaque fois que c'est possible,

$$(S_{x,y}^{-k,k}\eta)(z) = \begin{cases} \eta(x)-k & \text{si } z=x \text{ et } \eta(x)-k\in X, \eta(y)+k\in X \\ \eta(y)+k & \text{si } z=y \text{ et } \eta(x)-k\in X, \eta(y)+k\in X \\ \eta(z) & \text{autrement}\end{cases}\tag{1.11}$$

$$(S_y^{k}\eta)(z) = \begin{cases} \eta(y)+k & \text{si } z=y \text{ et } \eta(y)+k\in X \\ \eta(z) & \text{autrement}\end{cases}\tag{1.12}$$

$$(S_y^{-k}\eta)(z) = \begin{cases} \eta(y)-k & \text{si } z=y \text{ et } \eta(y)-k\in X \\ \eta(z) & \text{autrement}\end{cases}\tag{1.13}$$

et $p(x,y)$ est une probabilité symétrique et invariante par translation sur $\mathbb{Z}^d$.
Étant donnée une configuration $\eta \in \Omega$, les taux de transition ont la signification suivante: sur chaque site x nous pouvons avoir une *naissance* (une *mort*) de k individus qui dépend

de l'état de la configuration sur le même site $\eta(x)$ avec taux $P^k_{\eta(x)}$ et qui dépend aussi de l'état sur les autres sites $y \neq x$ avec taux $R^{0,k}_{\eta(y),\eta(x)}p(y,x)$. Cela représente une règle d'interaction différente entre individus de la même population et individus de populations différentes. De plus nous pouvons avoir un saut de k particules de y à x avec taux $\Gamma^k_{\eta(y),\eta(x)}p(y,x)$, qui représente la migration d'un groupe de particules (voir la Section 2.2). On note $\Pi^{0,k}_{\alpha,\beta} = R^{0,k}_{\alpha,\beta} + P^k_\beta$ pour tous $(\alpha,\beta) \in X^2$.
Notons par $\mathcal{S}$ et $\widetilde{\mathcal{S}}$ deux systèmes de générateurs respectifs $\mathcal{L}$ et $\widetilde{\mathcal{L}}$ et de taux $\{\Pi^{\cdot,\cdot}_{\alpha,\beta}, \Gamma^{\cdot}_{\alpha,\beta}\}$ et $\{\widetilde{\Pi}^{\cdot,\cdot}_{\cdot,\cdot}, \widetilde{\Gamma}^{\cdot}_{\cdot,\cdot}\}$,

Théorème 1.4 *(Théorème 2.15, Section 2.2) Un $\mathcal{S}$ système de particules $(\eta_t)_{t\geq 0}$ est* stochastiquement plus grand *qu'un $\widetilde{\mathcal{S}}$ système de particules $(\xi_t)_{t\geq 0}$ si et seulement si pour tous $(\alpha,\beta),(\gamma,\delta) \in X^2 \times X^2$ avec $(\alpha,\beta) \leq (\gamma,\delta)$,*

$$\sum_{k>\delta-\beta+l_1} \widetilde{\Pi}^{0,k}_{\alpha,\beta} + \sum_{k\in I_a} \widetilde{\Gamma}^k_{\alpha,\beta} \leq \sum_{l>l_1} \Pi^{0,l}_{\gamma,\delta} + \sum_{l\in I_b} \Gamma^l_{\gamma,\delta} \tag{1.14}$$

$$\sum_{k>k_1} \widetilde{\Pi}^{-k,0}_{\alpha,\beta} + \sum_{k\in I_d} \widetilde{\Gamma}^k_{\alpha,\beta} \geq \sum_{l>\gamma-\alpha+k_1} \Pi^{-l,0}_{\gamma,\delta} + \sum_{l\in I_c} \Gamma^l_{\gamma,\delta} \tag{1.15}$$

pour tous choix de K, I_a, I_b, I_c, I_d qui dépendent de K, où I_a, I_b, I_c et I_d sont des ensembles d'indices donnés par la Définition 2.14.

En d'autres termes, nous obtenons des conditions sur les taux de transition qui caractérisent l'ordre stochastique dans la classe des systèmes avec lesquels nous travaillons. Si $\mathcal{S} = \widetilde{\mathcal{S}}$ l'énoncé du Théorème 1.4 donne les conditions nécessaires et suffisantes d'attractivité.

L'analyse des systèmes de particules en interaction a commencé avec les Systèmes de Spins, qui sont des processus d'espace d'états $\{0,1\}^{\mathbb{Z}^d}$. Nous nous référons à [33] et [34] pour la construction et les résultats principaux. Les exemples les plus célèbres sont le modèle d'Ising, le processus de contact et le modèle du votant. Ces processus ont été largement étudiés, en particulier leur attractivité (voir [33, Chapitre III, Section II].
De nombreux autres modèles avec plus d'1 particule par site ont été étudiés. Les processus de réaction-diffusion, par exemple, sont des processus d'espace d'états $\mathbb{N}^{\mathbb{Z}^d}$ (donc non-compact), utilisés pour modéliser des réactions chimiques. Nous référons à [12, Chapitre 15] pour une introduction générale et pour la construction de ces processus. Schinazi a analysé un grand nombre de modèles pour des applications biologiques (de [39] à [44]) sur $X^{\mathbb{Z}^d}$, où $X = \{0,1,\dots N\}$. Dans ces processus de réaction-diffusion et modèles biologiques, seulement des naissances, morts ou sauts d'au plus une particule à la fois sont permis: dans ce cas on obtient une caractérisation de l'ordre partiel et de l'attractivité de les systèmes comme corollaire de notre résultat principal, voir la Section 2.3.5.
Mais parfois, le modèle nécessite des naissances, des morts ou des sauts de plus d'une particule à la fois. C'est le cas de certains systèmes biologiques d'extinction de masse ([41], [43] ou processus de contact "multitype" ([23], [22], [36]). Une compréhension partielle des propriétés d'attractivité se trouve dans [48] pour le processus de contact multitype. Dans [27, Theorem 2.21] les auteurs ont trouvé des conditions nécessaires et suffisantes d'attractivité pour un système de particules *conservatif* avec des sauts multiples sur $\mathbb{N}^{\mathbb{Z}^d}$: en d'autres termes ils travaillent avec un système avec des sauts de plus de 1 particule, mais des naissances ou des morts ne sont pas possibles.

Les modèles biologiques sont souvent très compliqués: de nombreux modèles de dynamiques des populations sont réglés par des naissances, des morts et des migrations de plus d'un individu à la fois (voir [28]).
Afin de comprendre les propriétés ergodiques d'un système, il est parfois utile de le comparer avec un autre système plus facile à étudier: donc une analyse de l'ordre stochastique et de l'attractivité qui comprend tous ces systèmes peut être utile. Le Théorème 1.4 donne les conditions nécessaires et suffisantes pour comparer des systèmes différents.

Dans la Section 2.1, nous rappelons quelques définitions classiques et propositions bien connues dont on aura besoin dans la suite. Le système de particules est introduit avec plus de détails dans la Section 2.2, où nous expliquons les résultats principaux et nous définissons le couplage utilisé pour démontrer le Théorème 1.4. Une analyse détaillée du mécanisme du couplage se trouve dans la Section 2.4.3. Dans la Section 2.3, nous explicitons les conditions sur plusieurs exemples. Finalement, la Section 2.4 est dédiée aux preuves.

1.1.4 Un résultat d'ergodicité pour un modèle d'épidémie

Le système de particules en interaction le plus étudié pour modéliser la propagation d'une maladie contagieuse est le processus de contact, introduit par Harris [30]. Il s'agit d'un système de spins réglé par les transitions

$$\begin{aligned} \eta_t(x) &= 0 \to 1 \text{ avec taux } \sum_{y\sim x} \eta_t(y) \\ \eta_t(x) &= 1 \to 0 \text{ avec taux } 1 \end{aligned}$$

où $y \sim x$ est un des $2d$ voisins du site x (voir [33] et [34] pour une analyse exhaustive de ce modèle).
Afin d'étudier le rôle des regroupements sociaux dans la propagation d'une épidémie, Schinazi [40] a introduit une généralisation du processus de contact. Après, Belhadji [5] a étudié certaines généralisations du modèle. Sur chaque site dans $\mathbb{Z}^d$ il y a un groupe de $N \leq \infty$ individus: chaque individu peut être *sain* ou *infecté*. Un groupe est infecté s'il contient au moins une personne infectée, sinon il est sain. La maladie se déplace d'une personne infectée à une personne saine avec taux ϕ si elles sont dans le même groupe. Le taux d'infection entre groupes différents est différent: l'épidémie se déplace à partir d'un individu infecté dans un groupe en y à un individu dans un groupe voisin en x avec le taux λ si x est sain et avec le taux β si x est infecté.
Plus précisément, nous avons un processus de Markov $(\eta_t)_{t\geq 0}$ tel que $\eta_t : \mathbb{Z}^d \to \{0, 1, \ldots, N\}$, où $\eta_t(x)$ est le nombre d'individus infectés sur le site $x \in \mathbb{Z}^d$ et N est un nombre maximal fixé d'individus par site. Les transitions sont

$$\begin{array}{lll} \eta_t(x) = 0 \to 1 & \text{au taux} & \lambda \sum_{y\sim x} \eta(y) + \gamma, \\ \eta_t(x) = i \to i+1 & \text{au taux} & \beta \sum_{y\sim x} \eta(y) + i\phi, \quad i = 1, 2, \ldots, N-1, \\ \eta_t(x) = i \to i-1 & \text{au taux} & i, \qquad\qquad\qquad\quad i = 1, 2, \ldots, N. \end{array} \tag{1.16}$$

Puisque chaque individu guérit au taux 1, le processus est appelé *modèle à guérisons individuelles*. Le taux γ représente une naissance pure de la maladie: dans le modèle classique à guérisons individuelles, on a $\gamma = 0$.
Soit λ_c la valeur critique du processus de contact; Belhadji a prouvé que

Théorème 1.5 *[5, Théorème 14]*
Pour tous $\phi \geq 0$ et $\beta \geq 0$, si $\lambda > \lambda_c$ une épidémie survient avec probabilité positive. Pour tous λ et β tels que $\beta \leq \lambda \leq 1/2d$, il existe $\phi_c(\lambda, \beta) \in (0, \infty)$ tel que si $\phi < \phi_c(\lambda, \beta)$ aucune épidémie n'est possible, tandis que si $\phi > \phi_c(\lambda, \beta)$ une épidemie peut survenir avec probabilité positive.

Des conditions suffisantes d'ergodicité sont obtenues comme corollaire de [12, Théorème 14.3]

Théorème 1.6 *[5, Théorème 15]*
Si

$$\lambda \vee \beta < \frac{1-\phi}{2d},$$

aucune épidémie n'est possible pour le processus, quelle que soit la taille N des regroupements.

En utilisant l'attractivité du modèle et le u-critère expliqué dans la Section 1.1.2, nous améliorons cette condition d'ergodicité. Notons qu'une dépendance en la taille N des regroupements apparaît.

Théorème 1.7 (Théorème 3.1.6, Section 3.1) *Supposons*

$$\lambda \vee \beta < \frac{1-\phi}{2d(1-\phi^N)} \tag{1.17}$$

avec $\phi < 1$ et: soit i) $\gamma = 0$, soit ii) $\gamma > 0$ et $\beta - \lambda \leq \gamma/(2d)$.
Alors le système est ergodique.

Si $N = 1$ et $\gamma = 0$, le processus est le processus de contact et le résultat est bien connu (et déjà amélioré, voir par exemple [33, Corollaire 4, 4, chapitre VI]); comme corollaire, si N tend vers infini, nous obtenons le résultat d'ergodicité dans le cas non compact.

1.1.5 Dynamiques de métapopulations

Nous utilisons la technique de comparaison avec la percolation orientée pour analyser la survie ou l'extinction de plusieurs modèles de dynamiques de métapopulations. Un modèle de métapopulations se réfère à de nombreuses populations locales liées par des migrations dans un environnement fragmenté. Chaque population locale peut augmenter ou diminuer, survivre, s'éteindre ou migrer à partir de son site de manière très compliquée (voir Hanski [28] pour en savoir plus sur les métapopulations).
Il y a plusieurs facteurs qui influencent le taux de croissance d'une population, c'est-à-dire la différence entre le taux de naissance et le taux de mort. L'hypothèse classique considère une croissance dépendante de la densité. Si la taille de la population croît le taux de naissance et le taux de mort augmentent car il y a un nombre plus grand d'individus qui peuvent, respectivement, se reproduire et mourir. De plus, les taux peuvent croître de manières différentes. Si le taux de naissance est toujours plus grand que le taux de mort, la population va grandir indéfiniment. Si le taux de naissance est toujours plus petit, la population va disparaître. La situation la plus intéressante est donnée par un taux de naissance plus petit que le taux de mortalité au-dessus d'une taille particulière N de la population, et plus grand au-dessous. D'un point de vue biologique, cela signifie que si la densité est trop grande les ressources ne sont pas suffisantes pour tous les individus et le

taux de croissance diminue, puisque le taux de mort augmente. Dans les environnements réels ce phénomène est progressif, c'est-à-dire que le taux de croissance diminue au-dessus de N si la densité de la population augmente. Dans certaines de nos applications, nous supposons qu'au-dessus de N le taux de croissance est nul. En ce cas, N est la *capacité* du site, c'est-à-dire le nombre maximal d'individus qu'il peut contenir.
Une des stratégies les plus importantes pour augmenter la probabilité de survie d'une espèce est la migration de groupes d'individus. Lorsque la taille de la population est grande, et donc les ressources sont rares, un ou plusieurs individus quittent leur population pour chercher de nouvelles ressources dans d'autres sites avec des populations différentes.
Dans certains cas, un autre facteur peut faciliter l'extinction d'une espèce: *l'effet Allee*. Si une population est soumise à l'effet Allee, si la densité d'individus est petite le taux de mortalité augmente. La raison est que si la densité est petite certains facteurs comme la difficulté à trouver des compagnons causent une diminution de la fécondité et une augmentation de la mortalité (voir [2], [15], [47], [45]).
Pour résumer, les dynamiques de populations sont gouvernées par de nombreux paramètres. Nous simplifions la structure réelle et nous traitons 4 modèles de métapopulations d'un point de vue mathématique.

Le modèle mathématique est un système de particules en interaction $(\eta_t)_{t\geq 0}$ sur $\Omega = X^{\mathbb{Z}^d}$, où $X = \{0, 1, \ldots, N\} \subseteq \mathbb{N}$ et N désigne la taille commune (capacité) des regroupements. La valeur de $\eta_t(x)$, $x \in \mathbb{Z}^d$ est le nombre d'individus présents dans le groupe sur le site x à l'instant $t \geq 0$. Notons que $N = \infty$ dans le Modèle IV.
Pour tous $x, y \in \mathbb{Z}^d$, nous écrivons $y \sim x$ si y est un des $2d$ voisins du site x. Le générateur infinitésimal $\mathcal{L}$ du processus est donné (pour une fonction locale f et un élément $\eta \in \Omega$) par

$$\begin{aligned}\mathcal{L}f(\eta) = \sum_{x\in X} \Big\{ & B(\eta(x))\big(f(S_x^1\eta) - f(\eta)\big) + D(\eta(x))\big(f(S_x^{-1}\eta(x)) - f(\eta)\big) \\ & + \sum_{y\sim x}\sum_{k\in X} J(\eta(x), \eta(y), k)\big(f(S_{x,y}^{-k,k}\eta) - f(\eta)\big)\Big\} \end{aligned} \tag{1.18}$$

où $S_x^{\pm 1}$ et $S_{x,y}^{-k,k}$ sont définis dans (1.11), (1.12) et (1.12 et $B(\cdot)$, $D(\cdot)$ sont des fonctions positives de X dans $\mathbb{R}$. Les rôles de $B(\eta(x))$ et $D(\eta(x))$ sont respectivement le taux de naissance et le taux de mort sur x, qui ne dépendent que de la configuration sur le site x. Nous n'autorisons pas plus d'une naissance ou d'une mort à la fois.
On suppose que $B(0) = 0$, c'est-à-dire que $\delta_{\underline{0}}$ est une mesure invariante.
La fonction $J : X^3 \to \mathbb{R}^+$ est telle que $J(\eta(x), \eta(y), k)$ correspond à une migration (saut) et elle dépend de la configuration sur x et y. Notons qu'un saut de plus d'une particule à la fois est possible. Nous appelons *émigration* de x un saut qui réduit le nombre de particules sur x et *immigration* un saut qui l'augmente.
Lorsque X est fini, ce qui est le cas des Modèles I, II et III, nous nous référons à la construction dans [33]; lorsque X est infini, dans le Modèle IV, nous nous référons à [12] et on exige des conditions supplémentaires sur les taux de transition (voir la Section 3.2.4).
On dit que la survie est possible si

$$\mathbb{P}(|\eta_t| \geq 1 \textit{ for all } t \geq 0) > 0 \tag{1.19}$$

où $|\eta_0|$ est fini et $|\eta_t|$ est le nombre d'individus de $(\eta_t)_{t\geq 0}$ à l'instant t.
On dit qu'il y a extinction de l'espèce si pout toute configuration initiale finie ou infinie la

loi du processus tend vers $\delta_{\underline{0}}$.

Modèle I: le modèle de base.

Dans [43] Schinazi introduit le modèle suivant pour analyser le rôle de l'agrégation sociale dans l'extinction d'une espèce: étant donné $N < \infty$, sur chaque site de $\mathbb{Z}^d$, on peut avoir $0, 1, \dots, N$ individus. Les taux de transition du processus de Markov $(\eta_t)_{t\geq 0}$ sont

$$\begin{array}{lll} \eta_t(x) \to \eta_t(x)+1 & \text{au taux } \eta_t(x)\phi + \lambda n_N(x,\eta_t) & \text{si } 0 \leq \eta_t(x) \leq N-1, \\ \eta_t(x) \to \eta_t(x)-1 & \text{au taux } 1 & \text{si } 1 \leq \eta_t(x) \leq N, \end{array}$$

où $n_N(x, \eta_t)$ est le nombre de voisins y de x tels que $\eta_t(y) = N$. Autrement dit, chaque individu cause une naissance sur le même site avec taux ϕ et il meurt avec taux 1. Un individu sur x cause une naissance d'un nouvel individu sur un site $y \sim x$ seulement si la population sur x a atteint la capacité N. Il y a transition de phase sur la capacité N.

Théorème 1.8 *[43, Théorème 2]*
Supposons que $d \geq 2$, $\lambda > 0$ et $\phi > 0$. Il existe une valeur critique $N_c(\lambda, \phi)$ telle que si $N > N_c(\lambda, \phi)$, alors, si le processus a initialement un nombre fini d'individus, la population a une probabilité strictement positive de survivre.

L'auteur compare ce résultat avec la transition de phase opposée qui apparaît quand, au lieu de la mort d'une seule personne avec taux 1, on considère la mort de tous les individus d'un regroupement avec taux 1, qu'il appelle modèle en *temps catastrophiques*. Dans les temps catastrophiques, voir [43, Théorème 1], il existe $N_c(\lambda, \phi)$ tel que si $N > N_c(\lambda, \phi)$ la population s'éteint.
La conséquence est que l'agrégation spatiale peut être positive pour la survie d'une espèce en temps non catastrophiques. Dans ce cas, si N est petit, par exemple si $N = 1$, et $\lambda < \lambda_c$ (où λ_c est la valeur critique du processus de contact), l'espèce s'éteint: comme déjà remarqué, cela est cohérent avec l'effet Allee, puisque pour une petite densité la probabilité de survivre devrait être plus petite.
Nous voulons étudier le rôle des migrations dans la survie d'une espèce, à partir de ce modèle. Nous commençons par un système très proche du modèle de Schinazi en temps non catastrophiques: la seule différence est qu'au lieu d'une naissance d'un nouvel individu on considère une migration d'un individu sur un site voisin. Il s'agit d'une petite différence et on peut imaginer que cela ne change pas le comportement du modèle, si N est grand. De toute façon, si $N = 1$, de nouvelles naissances ne sont pas possibles et le processus s'éteint pour tout λ: ce n'est certainement pas le cas du modèle en temps non catastrophiques avec $N = 1$, qui est le processus de contact.
Nous introduisons le Modèle I. Nous avons choisi de fixer un taux de naissance 1 et d'associer deux paramètres aux taux de mort et de migration. Étant donnés δ et λ des nombres réels positifs, les taux de transition de naissance et de mort sont, pour tout $x \in S$

$$\begin{array}{ll} \eta_t(x) \to \eta_t(x)+1 & \text{au taux } \eta_t(x)\mathbb{1}_{\{0\leq\eta_t(x)<N\}}, \\ \eta_t(x) \to \eta_t(x)-1 & \text{au taux } \delta\eta_t(x)\mathbb{1}_{\{0\leq\eta_t(x)\leq N\}}. \end{array}$$

Pour tout $x \sim y$ le taux de migration est

$$(\eta_t(x), \eta_t(y)) \to (\eta_t(x)-1, \eta_t(y)+1) \qquad \text{au taux } \mathbb{1}_{\{\eta_t(x)=N\}}\lambda M(\eta_t(y))$$

où $M : X \to \mathbb{R}^+$. Nous supposons $M(l)$ décroissante en l, c'est-à-dire que les individus essaient de se déplacer vers les sites avec moins individus (et plus de ressources). Nous

supposons $M(N) = 0$: une migration de x à y n'est pas possible si $\eta(y) = N$.
Le premier résultat correspond au Théorème 1.8 pour le modèle en temps non catastrophiques et il est prouvé de manière similaire.

Théorème 1.9 ***(Théorème 3.14, Section 3.2.1)***
Supposons $d \geq 2$, $\lambda > 0$ et $\delta < 1$. Il existe une valeur critique $N_c(\lambda, \delta)$ telle que si $N > N_c(\lambda, \delta)$, alors, à partir de $\eta_0 \in \Omega$ tel que $|\eta_0| \geq 1$ l'espèce a une probabilité positive de survivre. De plus, si $\eta_0 \in E^N$, le processus converge vers une mesure invariante non triviale avec probabilité positive.

Si on fixe la capacité N, nous prouvons qu'il existe une transition de phase aussi par rapport au taux de mort δ. Une preuve similaire a été utilisée dans [42] pour démontrer une transition de phase pour un modèle stochastique pour la dynamique des virus.

Théorème 1.10 ***(Théorème 3.18, Section 3.2.1)***

i) Pour tout $\lambda > 0$, $1 < N < \infty$, il existe $\delta_c^1(\lambda, N) > 0$ tel que si $\delta < \delta_c^1(\lambda, N)$ le processus a une probabilité positive de survivre.

ii) Il existe $\delta_c^2 > 0$ tel que si $\delta > \delta_c^2$ le processus s'éteint pour tous λ, N.

iii) Pour tous $\lambda > 0$, $1 < N < \infty$, il existe $\delta_c(\lambda, N) > 0$ tel que, si $\delta < \delta_c(\lambda, N)$ le processus avec configuration initiale η_0 avec $|\eta_0| \geq 1$ a une probabilité positive de survivre et si $\delta > \delta_c(\lambda, N)$ le processus s'éteint. Si $\delta < \delta_c(\lambda, N)$ et $\eta_0 \in E^N$ le processus converge vers une mesure invariante non triviale.

Modèle II: l'effet Allee.

On rajoute l'effet Allee dans le Modèle I. De Nombreuses études biologiques conviennent de l'existence de l'effet Allee dans les systèmes biologiques (voir [2], [15], [47], [45]). Dans une population soumise à l'effet Allee, si la densité des individus est petite le taux de mort augmente.
Plusieurs techniques différentes ont déjà été utilisées pour mettre en lumière l'effet Allee comme les équations différentielles stochastiques (voir [20]), les chaînes de Markov à temps discret (voir [3]) ou les processus de réaction-diffusion (voir [21]), mais aucun de ces modèles n'a une structure spatiale.

Nous introduisons pour la première fois un modèle avec une structure spatiale pour expliquer l'effet Allee.
Nous traduisons l'effet Allee en termes mathématiques dans un modèle de métapopulations. Comme dans le Modèle I, on fixe une capacité N pour tous les sites, mais nous supposons le taux de mort plus grand que le taux de naissance si la densité est petite.
À savoir, on fixe $N_A \leq N$ un entier positif et δ_A, δ, λ des nombres réels positifs, les taux de naissance et mort sont

$$\begin{aligned}
\eta_t(x) \to& \eta_t(x) + 1 \qquad \text{au taux } \mathbb{1}_{\{\eta_t(x) \leq N-1\}} \eta_t(x), \\
\eta_t(x) \to& \eta_t(x) - 1 \qquad \text{au taux } \delta_A \eta_t(x) \mathbb{1}_{\{\eta_t(x) \leq N_A\}} + \delta \eta_t(x) \mathbb{1}_{\{N_A < \eta_t(x) \leq N\}}.
\end{aligned}$$

Pour tout $x \sim y$ le taux de migration est

$$(\eta_t(x), \eta_t(y)) \to (\eta_t(x) - 1, \eta_t(y) + 1) \qquad \text{au taux } \mathbb{1}_{\{\eta_t(x) = N\}} \lambda M(\eta_t(y)),$$

où $M : X \to \mathbb{R}^+$ est une fonction décroissante telle que $M(N) = 0$ comme dans le Modèle I.
Nous supposons $\delta_A \geq 1$ et $\delta_A \geq \delta$, autrement dit si $\eta_t(x) \leq N_A$, alors le taux de mort $\delta_A \eta_t(x)$ est supérieur ou égal au taux de naissance $\eta_t(x)$ à cause de l'effet Allee.
Si $\eta_t(x) \geq N_A$, la situation la plus intéressante est donnée par un taux de mort $\delta\eta_t(x)$ plus petit que le taux de naissance $\eta_t(x)$, soit $\delta < 1$. Si $\delta > 1$, le taux de mort est toujours plus grand que le taux de naissance et l'espèce s'éteint comme on l'a prouvé dans le Théorème 1.10 *ii)*.
Si $N^A = 0$ (pas d'effet Allee) ou $N^A = N$ (taux de mort toujours plus grand que le taux de naissance), il y a un seul taux de mortalité et nous nous référons aux résultats du Modèle I.

Théorème 1.11 ***(Théorème 3.20, Section 3.2.2)***
Supposons $\delta_A > 1$.

i) Pour tous $\delta < 1$, $\lambda > 0$, $0 < N < \infty$, $0 \leq N_A \leq N$ il existe $\delta_c^A(\delta, \lambda, N, N_A)$ tel que si $\delta_A > \delta_c^A(\delta, \lambda, N, N_A)$ le processus converge vers la mesure invariante triviale $\delta_{\underline{0}}$ pour chaque configuration initiale $\eta_0 \in \Omega$.

ii) Si $\delta > 1$ le processus s'éteint pour tous $\delta_A > 1$, N, N_A, λ et pour toute configuration initiale.

Autrement dit même si on prend une population avec taux de migration et capacité très grands, il y a un effet Allee assex grand pour que l'espèce s'éteint.

Afin de modéliser l'effet Allee on a requis $\delta_A > 1$ et $\delta < 1$. D'un point de vue mathématique, il est intéressant d'étudier un modèle avec $\delta_A < 1$ et $\delta > 1$. On fixe N, N_A et λ et nous démontrons que nous ne pouvons choisir ni $\delta_A < 1$ suffisamment petit pour la survie de l'espèce pour tous δ, ni $\delta > 1$ suffisamment grand pour l'extinction de l'espèce pour tous δ_A.

Théorème 1.12 ***(Théorème 3.21, Section 3.2.2)***
Supposons $\delta_A < 1$ et $\delta > 1$.

i) Pour tous $1 < N_A < N$, $\lambda > 0$, $\delta > 1$, il existe $\delta_c^A(\lambda, N_A, N, \delta)$ tel que, si $\delta_A < \delta_c^A(\lambda, N_A, N, \delta)$, le processus a une probabilité positive de survivre pour chaque configuration initiale η_0 avec $|\eta_0| \geq 1$.

ii) Pour tous $1 < N_A < N$, $\lambda > 0$, $\delta_A < 1$, il existe $\delta_c(\lambda, N_A, N, \delta_A)$ tel que, si $\delta > \delta_c(\lambda, N_A, N, \delta_A)$, le processus s'éteint quelle que soit la distribution initiale.

Modèle III: migration de masse.

Nous introduisons la migration de masse comme solution de l'effet Allee. La migration est une des stratégies les plus importantes qu'une espèce peut adopter pour améliorer ses chances de survie (voir [47]). Nous avons déjà observé dans le Modèle I que la migration d'un seul individu est positive en l'absence de l'effet Allee. Dans le Modèle II, au contraire, la migration d'un seul individu ne suffit pas si l'effet Allee est trop fort.
Nous démontrons que, au moins en théorie, *des migrations de grands groupes d'individus améliorent la probabilité de survie d'une espèce pour tous les effets Allee.* Une migration de nombreux individus évite une densité petite dans un nouvel environnement (qui n'est

pas favorable à la survie de l'espèce).

Nous introduisons des paramètres positifs δ_A, δ, N_A, N tels que $0 \leq N_A \leq N$, $\delta_A > 1$, $\delta > 0$ et nous prenons les mêmes taux de naissance et de mort que dans le Modèle II,

$$\begin{aligned}
\eta_t(x) \to& \eta_t(x) + 1 && \text{au taux } \mathbb{1}_{\{\eta_t(x) \leq N-1\}} \eta_t(x), \\
\eta_t(x) \to& \eta_t(x) - 1 && \text{au taux } \delta_A \eta_t(x) \mathbb{1}_{\{\eta_t(x) \leq N_A\}} + \delta \eta_t(x) \mathbb{1}_{\{N_A < \eta_t(x) \leq N\}}.
\end{aligned}$$

Étant donnés $0 < M \leq N$, $M \in \mathbb{N}$, pour tout $y \sim x$, nous introduisons le taux de migration:

$$(\eta_t(x), \eta_t(y)) \to (\eta_t(x) - k, \eta_t(y) + k) \qquad \text{au taux } J(\eta_t(x), \eta_t(y), k) \tag{1.20}$$

où pour tous $\alpha \in X$, $\beta \in X$, $k \in \{1, \dots, M\}$

$$J(\alpha, \beta, k) := \begin{cases} \lambda & \text{si } \alpha - k \geq N - M \text{ et } \beta + k \leq N \\ 0 & \text{autrement} \end{cases} \tag{1.21}$$

c'est-à-dire que si k individus tentent de migrer de x à y et $\eta_t(y) + k > N$, la migration ne se produit pas.
Notons que si $\eta_t(x) < N - M$ le taux de migration est nul. Cela signifie que des *individus essayent de migrer seulement s'il y a plus de $N - M$ individus sur un site.* D'un point de vue biologique, cela signifie que s'il y a peu d'individus les ressources sont suffisantes pour tous et il n'y a pas de raisons de migrer.
Lorsque $\alpha \geq N - M$ il y a une probabilité positive de migrer et le nombre d'individus qui migrent est croissante en la taille de la population. Si $\alpha = N - M + 1$, nous permettons une migration d'au plus 1 individu de x vers un site voisin, si $\alpha = N - M + 2$, nous permettons une migration d'au plus 2 individus et cetera. Si $\alpha = N = (N - M) + M$, nous permettons une migration du plus grand groupe de M particules. Le taux de migration est donné par λ.
Nous prouvons que, pour tous les effets Allee nous pouvons prendre une taille de population N et une taille maximale de migration de masse M suffisamment grande pour avoir la survie de l'espèce.

Théorème 1.13 ***(Théorème 3.23, Section 3.2.3)***
Soit $d \geq 2$.

i) Pour tous $\delta < 1$, $\lambda > 0$, $N_A \geq 0$, il existe $N_c(\delta, \lambda, N_A)$ tel que pour tout $N > N_c(\delta, \lambda, N_A)$, il existe $M(N_A)$ tel que le processus de configuration initiale η_0 avec $|\eta_0| \geq 1$ ait une probabilité positive de survie pour tous $\delta_A < \infty$. De plus, si $\eta_0 \in E^N$ le processus converge vers une mesure invariante non-triviale pour tous $\delta_A < \infty$.

ii) Si $\delta > 1$, pour tous N, N_A, λ, δ_A, M le processus s'éteint pour chaque configuration initiale.

En d'autres termes, étant donnés N_A, λ et δ, même si δ_A est le plus grand possible, *si l'espèce vit dans des populations assez grandes et migre en groupes assez grands la survie est possible.*

Dans le Modèle II on a montré qu'un effet Allee très fort cause l'extinction même d'une population très nombreuse. Le Théorème 1.13 affirme que la stratégie qu'une espèce doit utiliser n'est pas l'augmentation du taux de migration, mais l'augmentation du nombre d'individus qui migrent à la fois. De plus, la capacité de la population est importante et l'agrégation spatiale est positive pour la survie de l'espèce.

Modèle IV: équilibre écologique.

L'environnement réel n'a pas une limitation a priori sur la taille de la population, mais il y a une sorte de *mécanisme d'auto-régulation* qui ne permet pas l'explosion du nombre d'individus par site. *L'équilibre écologique* a été introduit dans [6] pour la "marche aléatoire de branchement restrained": certaines restrictions sur les taux de naissance donnent la survie sans explosion du nombre d'individus. Nous montrons qu'on peut avoir la même situation dans notre système avec un mécanisme différent.

Nous supposons que dans notre environnement il n'y a pas une taille maximale N comme dans les modèles précédents, mais le taux de naissance est toujours positif. Nous supposons également que, si la taille de la population est plus grande qu'une valeur fixée N, le taux de croissance est négatif.
Nous pouvons changer chacun des modèles précédents de la façon suivante lorsque le nombre d'individus par site $\eta_t(x)$ est plus grand que N:

$$\begin{aligned} \eta_t(x) \to& \eta_t(x)+1 & &\text{au taux } \eta(x)1\!\!1_{\{\eta_t(x)\geq N\}}, \\ \eta_t(x) \to& \eta_t(x)-1 & &\text{au taux } \widetilde{\delta}\eta(x)1\!\!1_{\{\eta_t(x)\geq N+1\}}, \end{aligned}$$

où $\widetilde{\delta} > 1$, c'est-à-dire lorsque la densité est grande le taux de mort prévaut sur le taux de naissance.
Afin de simplifier la notation et les preuves, nous travaillons sur le Modèle I. Nous pouvons obtenir des résultats similaires sur le Modèle III. À savoir, nous prenons les taux de naissance et de mort suivants

$$\begin{aligned} \eta_t(x) \to& \eta_t(x)+1 & &\text{au taux } \eta_t(x), \\ \eta_t(x) \to& \eta_t(x)-1 & &\text{au taux } \delta\eta_t(x)1\!\!1_{\{N_A\leq\eta_t(x)\leq N\}} + \widetilde{\delta}\eta(x)1\!\!1_{\{\eta(x)>N\}}, \end{aligned}$$

et pour tout $x \sim y$ le taux de migration est

$$(\eta_t(x),\eta_t(y)) \to (\eta_t(x)-1,\eta_t(y)+1) \qquad \text{au taux } 1\!\!1_{\{y\sim x\}}\lambda M(\eta_t(y))1\!\!1_{\{\eta_t(x)\geq N\}}$$

où $M : X \to \mathbb{R}^+$ est une fonction décroissante telle que $M(l) = 0$ pour tous $l \geq N$.
Cela signifie que si la taille de la population $\eta_t(x)$ est plus grande que N le taux de mort $\eta_t(x)\widetilde{\delta}$ est plus grand que le taux de naissance $\eta_t(x)$. Une migration est accordée à partir d'un site avec plus de N individus vers un site avec moins de N individus. Nous travaillons sans aucune capacité a priori, donc nous nous référons à la construction dans les cas non compacts et l'espace d'états est $\Omega_0 \subset \Omega$ (voir [13, Chapitre 13]). On peut démontrer que les conditions suffisantes pour l'existence du processus sont vérifiées (voir [13, Théorème 13.1]).
Nous prouvons que la survie de l'espèce est possible et la population sur chaque site n'explose pas; dans les autres cas l'espèce s'éteint.

Théorème 1.14 ***(Théorème 3.25, Section 3.2.4)***
Supposons que $|\eta_0| \geq 1$ *et qu'il existe* $n < \infty$ *tel que* $\eta_0(x) \leq n$ *pour tout* $x \in \mathbb{Z}^d$.

i) Pour tous $\lambda > 0$, $1 < N < \infty$, $\epsilon > 0$ *il existe* $\delta_c^1(\lambda, N) > 0$ *et* $K < \infty$ *tels que si* $\delta < \delta_c^1(\lambda, N)$, *le processus a une probabilité positive de survivre pour chaque* $\widetilde{\delta} > 1$ *et* $\lim_{t\to\infty} \mathbb{E}(\eta_t(x)) \leq K$ *pour tous* $x \in \mathbb{Z}^d$ *et* $\widetilde{\delta} \geq 1 + \epsilon$.

ii) Il existe $\delta_c^2 > 0$ *tel que si* $\delta > \delta_c^2$, *le processus s'éteint pour tous* λ, N, $\widetilde{\delta} > 1$ *et chaque configuration initiale.*

iii) Pour tous $\lambda > 0$, $1 < N < \infty$, $\widetilde{\delta}$, *il existe* $\delta_c(\lambda, N, \widetilde{\delta}) > 0$ *tel que si* $\delta < \delta_c(\lambda, N, \widetilde{\delta})$ *le processus survit et si* $\delta > \delta_c(\lambda, N, \widetilde{\delta})$ *le processus s'éteint.*

1.1 Introduction

1.1.1 Interacting particle systems

Interacting particle systems are Markov processes $(\eta_t)_{t\geq 0}$, with $\eta_t : S \to X$, where S is a set of sites and X is a countable set of possible states on each site. We focus on cases where $S = \mathbb{Z}^d$ and $X \subseteq \mathbb{N}$. For each $x \in S$, let $(X, \mathcal{X}_x, \rho_x)$ be a separable metric space, where ρ_x is a metric on X and $\mathcal{X}_x$ is the σ-algebra induced (by the open sets in the topology induced) by ρ_x. Let $\Omega = X^S$ be the *state space* of the particle system and $\mathcal{X} = \prod_{x\in S}\mathcal{X}_x$ the product σ-algebra on Ω. The value $\eta_t(x)$ is the state of site $x \in S$ at time $t \geq 0$ and η_t is the *configuration* of the system at time t. These Markov processes describe the evolution of particles on S which interact with each other and which are used to model many real systems involving spreads of diseases, population dynamics and chemical processes.

The first question is the existence of such processes. The evolution of the process is ruled by infinitesimal transition rates, under which the system moves from a state to another one. For $\eta \in \Omega$ and $\xi \in \Omega$, $c(\eta, \xi)$ is the transition rate from configuration η to configuration ξ, that is

$$\lim_{h\to 0} \frac{P^\eta(\eta_h = \xi)}{h} = c(\eta, \xi)$$

where P^η denotes the law of the process starting at η. Then the *infinitesimal generator* $\mathcal{L}$ is an operator, defined a priori on functions f on Ω depending only on a finite number of coordinates, given by

$$\mathcal{L}f(\eta) = \sum_{\xi\in\Omega} c(\eta, \xi)[f(\xi) - f(\eta)].$$

In order to ensure that $\mathcal{L}$ is a well defined operator, we work with *local interactions*: we assume that there exists K such that at each time at most $l \leq K$ sites may change state and that the transition rates depend only on the state of a finite number of neighbors of these sites. We also suppose $c(\eta, \xi)$ translation invariant, that is

$$c(\eta, \xi) = c(\sigma_x\eta, \sigma_x\xi)$$

for each $x \in S$, where $(\sigma_x\eta)(z) = \eta(x + z)$ for $z \in S$.
If $X = \{0, 1, \dots, N\}$ is finite, existence and unicity of the process have been proved by Liggett [33] under general conditions; an alternative graphical construction has been proved by Harris [29] and reformulated by Durrett [22]. If X is a countable finite or infinite set, sufficient conditions for existence and unicity are given by Chen [12, Chapter 13]. In this case the process takes place in a smaller state space $\Omega_0 \subset \Omega$. We set $\Omega_0 = \Omega$ if X is finite. Namely, given a sequence $(k_x)_{x\in\mathbb{Z}^d}$ such that $\sum_{x\in\mathbb{Z}^d} k_x < \infty$, we define

$$\Omega_0 = \{\eta \in \Omega : \sum_{x\in\mathbb{Z}^d} \rho_x(0, \eta(x))k_x < \infty\}. \tag{1.1}$$

For $\eta \in \Omega_0$, let E^η be the expectation corresponding to P^η. For $f \in C(\Omega_0)$, the set of continuous functions on Ω_0, we define

$$T(t)f(\eta) := E^\eta f(\eta_t).$$

For each probability measure μ on $(\Omega_0, \mathcal{X})$, $\mu T(t)$, the probability measure on $(\Omega_0, \mathcal{X})$ defined by

$$\int f[d\mu T(t)] = \int [T(t)f]d\mu$$

for all $f \in C(\Omega_0)$ (see [33, Chapter I]), is the distribution at time t of the process with initial distribution μ.

One of the main questions about an interacting particle system is the existence of a non-trivial *invariant measure*, that is μ such that $P_\mu(\eta_t \in A) = \mu(A)$ for each $t \geq 0$, $A \in \mathcal{X}$, where P_μ is the law of the process starting from the initial distribution μ. An invariant measure is trivial if it is concentrated on an absorbing state when there exists any.
Let $\mathcal{I}$ be the set of all the invariant measures of the process. The latter is *ergodic* if there is a unique invariant measure to which the process converges starting from each initial distribution (see [33, Definition 1.9]).

In order to investigate the ergodic properties of a system, one can compare it with different processes with well known behaviours. *Coupling* is a technique which consists, starting from different processes $(\eta_t)_{t\geq 0}$ and $(\xi_t)_{t\geq 0}$, in constructing a *Markovian coupled process* $(\eta_t, \xi_t)_{t\geq 0}$, that is a Markov process with state space $\Omega_0 \times \Omega_0$, such that each marginal is a copy of the original process.
A fundamental tool in interacting particle systems is *attractiveness*, which is a property concerning the distribution at time t of two processes *with the same generator* which start with different initial distributions (see Section 2.1). Attractiveness corresponds to the existence of a Markovian coupled process $(\xi_t, \eta_t)_{t\geq 0}$ such that if $\xi_0 \leq \eta_0$ (coordinate-wise) then, for all $t \geq 0$, $\xi_t \leq \eta_t$ a.s.

In biological applications a particle represents an individual from a species and the empty configuration corresponds to the extinction of that species; it is often an absorbing state: an important problem in this case is to find conditions on rates which give either a non-trivial invariant measure, that is the survival of the species for each time, or ergodicity, that is almost sure extinction. Therefore it is useful to understand when the population in a system is always larger (or smaller) than the number of individuals of another one. The main result of the first part, Theorem 1.4 introduced in Section 1.1.3, gives necessary and sufficient conditions on transition rates of two systems which ensure that on each site the number of particles of one system is always larger than the number of particles of the other one.
For a large class of interacting particle systems we get necessary and sufficient conditions on the rates to insure attractiveness as a corollary of Theorem 1.4. Attractiveness is a key tool to determine the set of invariant probability measures. Let $\overline{\nu}$ and $\underline{\nu}$ be the limiting distributions of the process as t goes to infinity starting respectively from the larger initial configuration and from the smaller one in the case of a compact state space; if the process is attractive and $\overline{\nu} = \underline{\nu}$, then the process is ergodic: this is the basic idea for the *ergodicity criterion* explained in Section 1.1.2. We use this technique to improve ergodicity conditions on a well studied epidemic model, see Section 1.1.4.

In Section 1.1.2 we explain a *comparison with oriented percolation technique.* To show the existence of non-trivial invariant measures, in the last section of the first part of the thesis we focus on several population dynamics models to which we apply these comparison arguments.

The rest of this introduction is organized in the following way. Section 1.1.2 describes the main techniques we use, that is a brief definition of coupling, the ergodicity criterion and comparison with oriented percolation. In Section 1.1.3 we state necessary and sufficient conditions for stochastic order and attractiveness. Sections 1.1.4 and 1.1.5 consist in some applications to determine ergodicity and existence of non-trivial invariant measures in biological models.

1.1.2 Techniques

Coupling

This very powerful technique is used in many different areas of probability theory.
Given two interacting particle systems $(\eta_t)_{t\geq 0}$ and $(\xi_t)_{t\geq 0}$ respectively on $(\Omega, \mathcal{X}_1, \mathbb{P}_1)$ and $(\Omega, \mathcal{X}_2, \mathbb{P}_2)$ with generators $\mathcal{L}_1$ and $\mathcal{L}_2$ and transition rates $c_1(\cdot,\cdot)$ and $c_2(\cdot,\cdot)$, *the coupled process* $(\eta_t, \xi_t)_{t\geq 0}$ is a Markov process on $(\Omega\times\Omega, \mathcal{X}_1\times\mathcal{X}_2, \widetilde{\mathbb{P}})$ with generator $\widetilde{\mathcal{L}}$ such that each marginal is a copy of the original process. In other words, for each $A\in\mathcal{X}_1$, $B\in\mathcal{X}_2$

$$\begin{aligned}\widetilde{\mathbb{P}}\Big((\eta_t,\xi_t)\in(A\times\Omega)\Big) =&\mathbb{P}_1(\eta_t\in A)\\ \widetilde{\mathbb{P}}\Big((\eta_t,\xi_t)\in(\Omega\times B)\Big) =&\mathbb{P}_2(\xi_t\in B)\end{aligned}$$

and $\widetilde{\mathbb{P}}$ is a *coupling probability measure.*
An equivalent definition can be stated starting from the infinitesimal general $\widetilde{\mathcal{L}}$: for each $A\in\mathcal{X}_1$ and $B\in\mathcal{X}_2$,

$$\begin{aligned}\widetilde{\mathcal{L}}1\!\!1_{A\times\Omega}(\eta,\xi) &= \mathcal{L}_1 1\!\!1_A(\eta)\\ \widetilde{\mathcal{L}}1\!\!1_{\Omega\times B}(\eta,\xi) &= \mathcal{L}_2 1\!\!1_B(\xi).\end{aligned}$$

A coupled process can be constructed in several ways: we briefly introduce three of the main couplings used.
Independent coupling

$$\widetilde{\mathcal{L}}^i f(\eta,\xi) = [\mathcal{L}_1 f(\cdot,\xi)](\eta) + [\mathcal{L}_2 f(\eta,\cdot)](\xi)$$

the processes move independently through uncoupled rates.
Classical coupling
Suppose $\mathcal{L}_1=\mathcal{L}_2$, denote by $g(\eta)=f(\eta,\eta)$ and by $\Delta=\{(\eta,\xi)\in\Omega^2:\eta=\xi\}$. Let

$$\widetilde{\mathcal{L}}^c f(\eta,\xi) = 1\!\!1_{\Delta^c}(\eta,\xi)\widetilde{\mathcal{L}}^i f(\eta,\xi) + 1\!\!1_{\Delta}(\eta,\xi)\mathcal{L}_1 g(\eta)$$

that is the processes move independently until they meet, and then they move together.
Basic coupling
For $\eta,\xi\in\Omega$, let $I(\eta,\xi)=\{\zeta\in\Omega:\eta+\zeta\in\Omega_0,\xi+\zeta\in\Omega_0\}$, where $\eta+\zeta$ denotes the configuration defined by $(\eta+\zeta)(x)=\eta(x)+\zeta(x)$ for all $x\in S$

$$\begin{aligned}\widetilde{\mathcal{L}}^b f(\eta,\xi) =& \sum_{\zeta\in I(\eta,\xi)}[c_1(\eta,\eta+\zeta)\wedge c_2(\xi,\xi+\zeta)]\big(f(\eta+\zeta,\xi+\zeta)-f(\eta,\xi)\big)\\ &+\sum_{\zeta\in I(\eta,\xi)}[c_1(\eta,\eta+\zeta)-c_2(\xi,\xi+\zeta)]^+\big(f(\eta+\zeta,\xi)-f(\eta,\xi)\big)\\ &+\sum_{\zeta\in I(\eta,\xi)}[c_2(\eta,\eta+\zeta)-c_1(\xi,\xi+\zeta)]^+\big(f(\eta,\xi+\zeta)-f(\eta,\xi)\big)\end{aligned}$$

that is the processes move together as much as possible.

Comparison with oriented percolation

We follow the presentation in [22] of the *comparison theorem* (introduced in [10]) for proving the existence of non-trivial invariant measures for particle systems with compact state space Ω. The idea consists in rescaling the process in time and space and then making a comparison with an oriented percolation model. Let

$$\mathcal{N} = \{(x,n) \in \mathbb{Z}^2 : x+n \text{ is even}, n \geq 0\}.$$

For each $(x,n) \in \mathbb{Z}^2$, we draw an oriented edge from (x,n) to $(x+1,n+1)$ and from (x,n) to $(x-1,n-1)$. Let $w(x,n)$ be a random variable that indicates whether the site (x,n) is open (1) or closed (0). We say that (x,m) can reach (y,n) and write

$$(x,m) \to (y,n)$$

if there exists a sequence of points $x_m = x, \ldots, x_n = y$ such that $|x_k - x_{k-1}| = 1$ for $m < k \leq n$ and $w(x_k,k) = 1$ for $m \leq k \leq n$, that is a path of open sites.
We say that the $w(x,n)$ are M dependent with density at least $1-\gamma$ if whenever (x_i,n_i), $1 \leq i \leq I$ is a sequence such that $\|(x_i,n_i)-(x_j,n_j)\|_\infty > M$, if $i \neq j$

$$\mathbb{P}(w(x_i,n_i) = 0 \text{ for } 1 \leq i \leq I) \leq \gamma^I$$

We take as initial condition $W_0 \subset 2\mathbb{Z} = \{x \in \mathbb{Z} : (x,0) \in \mathcal{N}\}$ and we define an *oriented percolation* process by

$$W_n = \{y \in \mathbb{Z} : (x,0) \to (y,n) \text{ for some } x \in W_0\}$$

that is W_n are sites which are open at time n. If $W_0 = \{0\}$, we denote by

$$\mathcal{C}_0 = \{(y,n) \in \mathcal{N} : (0,0) \to (y,n)\}$$

the set of all space-time points that can be reached by an open path from $(0,0)$. If $P(|C_0| = \infty) > 0$ *percolation occurs.* The following result states that if the density of open sites is large enough we have percolation:

Theorem 1.1 *[22, Theorem 4.1] If* $\gamma \leq 6^{-4(2M+1)^2}$ *then*

$$\mathbb{P}(|\mathcal{C}_0| < \infty) \leq 1/20.$$

In order to prove the existence of a non-trivial invariant measure, we have to prove that if we start from a particular configuration we can get a positive density of open sites. Let W_0^p be the initial condition defined by: sites of $\{x \in 2\mathbb{Z} : x \in W_0^p\}$ are open independently with probability p. Then

Theorem 1.2 *[22, Theorem 4.2] If* $p > 0$ *and* $\gamma \leq 6^{-4(2M+1)^2}$ *then*

$$\liminf_{n\to\infty} \mathbb{P}(0 \in W_{2n}^p) \geq 19/20.$$

The relation between the particle system and the percolation model is stated by the following comparison theorem. The idea consists in constructing space-time blocks and to associate to each of them a *good event* $G_{m,n}$, $(m,n) \in \mathcal{N}$. Each time the good event happens, we put an oriented edge from (m,n) to $(m+1,n+1)$ and to $(m-1,n+1)$.
Let $(\eta_t)_{t\geq 0}$ be a translation invariant finite range process that is constructed from the graphical representation such that $\eta_t : S \to X = \{0,1,\dots,N\}$. Let T, L, k_0 and j_0 be positive integers and $M = \max\{k_0, j_0\}$. Given $(m,n) \in \mathcal{N}$, the corresponding space time cell is

$$\mathcal{R}_{m,n} = (2mLe_1, nT) + \{[-k_0L, k_0L]^d \times [0, j_0T]\},$$

where $e_1 = (1,0,\dots,0)$.
Let H be a set of configurations determined by the values of η_0 on $[-L,L]^d$. We say that (m,n) is occupied if $\eta_{nT} \in \sigma_{2mLe_1}H$.
Comparison assumption. *We suppose that for each $(m,n) \in \mathcal{N}$ there exists an event $G_{m,n}$ depending only on the configuration in $\mathcal{R}_{m,n}$, so that $P(G_{m,n}) \geq 1-\gamma$ and if (m,n) is occupied then on $G_{m,n}$ the sites $(m+1,n+1)$ and $(m-1,n+1)$ are also occupied, that is*

$$\eta_{(n+1)T} \in \sigma_{2(m+1)Le_1}H \text{ and } \eta_{(n+1)T} \in \sigma_{2(m-1)Le_1}H.$$

Let $X_n = \{m \in \mathbb{Z} : (m,n) \in \mathcal{N}, \xi_{nT} \in \sigma_{2mLe_1}H\}$ be the set of occupied sites at level n. The final relation between particle systems and percolation is given by

Theorem 1.3 *[22, Theorem 4.3] If the comparison assumption holds then we can define random variables $w(x,n)$ so that X_n dominates an M dependent oriented percolation process with initial configuration $W_0 = X_0$ and density at least $1-\gamma$, i.e. $X_n \supset W_n$ for all n.*

Let $(\eta_t)_{t\geq 0}$ be an attractive process with translation invariant transition rates.
Let $H = \{\xi : \xi(x) = 1 \text{ for some } x \in [-L,L]^d\}$ and assume that the comparison assumption holds.
Given an initial configuration η_0 such that $\eta_0(y) = \mathbb{1}_{\{y=x\}}(y)$ for some $x \in \mathbb{Z}^d$, by translation invariance we can suppose $x = 0$. By Theorems 1.1 and 1.3, if γ is sufficiently small and δ_η is the Dirac measure concentrated on $\eta \in \Omega$

$$\liminf_{t\to\infty} \delta_{\eta_0} T(t)\{\eta : \eta(x) > 0 \text{ for some } x \in \mathbb{Z}^d\} \geq \mathbb{P}(|\mathcal{C}_0| = \infty) > 0 \qquad (1.2)$$

that is the existence of an infinite path of open sites in the oriented percolation implies the existence of an infinite path of individuals in the particle system.
To construct a non-trivial invariant measure, we assume $\eta_0(x) = N$ for each $x \in \mathbb{Z}^d$. By attractiveness,

$$\lim_{t\to\infty} \delta_{\eta_0} T(t) = \overline{\nu}$$

which is a invariant measure.
If the comparison assumption holds, by Theorem 1.3, X_n dominates an oriented percolation process W_n^p such that a site $x \in 2\mathbb{Z}$ is open at time 0 if there exists $y \in x + \sigma_{2mLe_1}H$ such that $\eta_0(y) \geq 1$. Since $\eta_0(x) = N$ for each $x \in \mathbb{Z}^d$, then in the initial configuration W_0^p all sites are open, that is $p = 1$. By Theorem 1.2

$$\begin{aligned}\overline{\nu}\{\eta : \eta(x) > 0 \text{ for some } x \in 2mLe_1 + [-L,L]^d\} &\geq \liminf_{n\to\infty} \delta_{\eta_0} T(n)\{(m,n) \text{ is occupied}\} \\ &\geq \liminf_{n\to\infty} \mathbb{P}(0 \in W_{2n}^1) > 0 \qquad (1.3)\end{aligned}$$

which implies that $\overline{\nu}$ concentrates on configurations with infinitely many individuals.

Ergodicity criterion

This technique gives sufficient conditions on transition rates which yield ergodicity of an attractive translation invariant process. It has been used by several authors (see [12], [13], [35]) for reaction-diffusion processes.
Let $(\eta_t)_{t\geq 0}$ be a translation invariant attractive process such that $\eta_t : \mathbb{Z}^d \to X = \{0, 1, \dots, N\} \subseteq \mathbb{N}$ and for each $x \in S$ let $F_x : X \times X \to \mathbb{R}$ be a metric on X. We suppose F_x does not depend on x, that is $F_x = F$ for each $x \in \mathbb{Z}^d$. We introduce a partial order on $\Omega = X^{\mathbb{Z}^d}$. Briefly, given two configurations $\eta \in \Omega$ and $\xi \in \Omega$, $\eta \leq \xi$ if $\eta(x) \leq \xi(x)$ for each $x \in S$ (see Section 2.1 for more details).
A metric on X induces in a natural way a metric on Ω. Namely, for each η and ξ in Ω we define

$$\rho_\alpha(\eta, \xi) := \sum_{x\in\mathbb{Z}^d} F_x(\eta(x), \xi(x))\alpha(x) \tag{1.4}$$

where $\{\alpha(x)\}_{x\in\mathbb{Z}^d}$ is a sequence such that $\alpha(x) \in \mathbb{R}$, $\alpha(x) > 0$ for each $x \in \mathbb{Z}^d$ and

$$\sum_{x\in\mathbb{Z}^d} \alpha(x) < \infty. \tag{1.5}$$

If $X = \mathbb{N}$ (that is "$N = \infty$"), the process has to be constructed on a smaller state space (see [13])

$$\Omega_0 = \{\eta \in \Omega : \sum_{x\in\mathbb{Z}^d} \eta(x)\alpha(x) < \infty\}.$$

We define the Kantorovich 1-distance between probability measures μ_1 and μ_2 on $(\Omega, \mathcal{X})$:

$$W(\mu_1, \mu_2) = \inf_{\widetilde{P}} \Big\{ \int \rho_\alpha(\eta, \xi)\widetilde{P}(d\eta, d\xi)\Big\}$$

where $\widetilde{P}$ varies over all coupling measures with marginals μ_1 and μ_2.
Let $\widetilde{\mathbb{P}}$ be a coupling probability measure on $(\Omega \times \Omega, \mathcal{X}_1 \times \mathcal{X}_2)$ and $\widetilde{\mathbb{E}}$ the expected value with respect to $\widetilde{\mathbb{P}}$.
Denote by $E^n = \{\eta \in \mathbb{Z}^d : \eta(x) = n \text{ for each } x \in \mathbb{Z}^d\}$, $n \in X$. Let η_t^k be the process starting from $\eta_0^k \in E^k$, $k \in X$. Attractiveness and translation invariance imply the existence of translation invariant measures $\bar{\nu}$ and $\underline{\nu}$ such that

$$\lim_{t\to\infty} \delta_{\eta^N} T(t) = \bar{\nu}, \qquad \lim_{t\to\infty} \delta_{\eta^0} T(t) = \underline{\nu}.$$

Again by attractiveness, if $\eta_0^0 \leq \xi_0 \leq \eta_0^N$ we get

$$\lim_{t\to\infty} \eta_t^0(x) \leq \lim_{t\to\infty} \xi_t(x) \leq \lim_{t\to\infty} \eta_t^N(x)$$

for each $x \in \mathbb{Z}^d$, $\widetilde{\mathbb{P}}$-a.s. Notice that if $N = \infty$ an additional step consists in proving the existence of $\lim_{n\to\infty} \eta_t^n$ for each $t \geq 0$. Finally if we prove that $\bar{\nu} = \underline{\nu}$ ergodicity follows.
The key idea consists in taking a "good metric" F on X to evaluate the distance between η_t^N and η_t^0 at each time $t \geq 0$ and to look for conditions on the rates so that the expected value of the distance between η_t^N and η_t^0 converges to zero as t goes to infinity uniformly with respect to $x \in S$. If we prove that

$$\widetilde{\mathbb{E}}\Big(\lim_{t\to\infty} F(\eta_t^0(x), \eta_t^N(x))\Big) = 0 \tag{1.6}$$

uniformly with respect to $x \in S$, then

$$\begin{aligned} W(\underline{\nu},\overline{\nu}) \leq & \widetilde{\mathbb{E}}\Big(\lim_{t\to\infty}\rho_\alpha(\eta_t^0,\eta_t^N)\Big) = \widetilde{\mathbb{E}}\Big(\lim_{t\to\infty}\sum_{x\in\mathbb{Z}^d}F\big(\eta_t^0(x),\eta_t^N(x)\big)\alpha(x)\Big) \\ & \leq \sum_{x\in\mathbb{Z}^d}\alpha(x)\widetilde{\mathbb{E}}\Big(\lim_{t\to\infty}\sup_{x\in\mathbb{Z}^d}F\big(\eta_t^0(x),\eta_t^N(x)\big)\Big) = 0 \end{aligned} \tag{1.7}$$

and ergodicity follows.
We will use generator properties and Gronwall Lemma (See Section 3.1.1) to prove that if there exists a metric F and $\epsilon > 0$ such that for each $x \in \mathbb{Z}^d$

$$\widetilde{\mathbb{E}}\Big(\mathcal{L}F(\eta_t^0(x),\eta_t^N(x))\Big) \leq -\epsilon\widetilde{\mathbb{E}}\Big(F(\eta_t^0(x),\eta_t^N(x))\Big) \tag{1.8}$$

then (1.6) holds.
In [13, Theorem 14.10] the author uses the metric

$$F(k,l) = \left|\sum_{0\leq j<k}u_j - \sum_{0\leq j<l}u_j\right|, \qquad k,l\in X=\mathbb{N}$$

for a given sequence $\{u_k\}_{k\in\mathbb{N}}$ of real positive numbers and we set $\sum_{0\leq j<0}u_j = 0$. The metric

$$F(k,l) = 1\!\!1_{\{k\neq l\}}\sum_{j=0}^{|k-l|-1}u_j \qquad k,l\in X=\mathbb{N} \tag{1.9}$$

was used in [35] if $1 \geq u_j \geq \gamma > 0$ and in [12] if $1 \geq u_j > 0$ to get sufficient ergodicity conditions on reaction diffusion processes.
Since we are working with two copies of the same process, we can couple them and evaluate (1.8) with different couplings in order to get different ergodicity conditions: in [12] the author works with several couplings and he gets the best one (in some sense) for ergodicity for a class of reaction diffusion processes.
When $\underline{\nu} \neq \delta_{\underline{0}}$ (where $\delta_{\underline{0}}$ is the Dirac measure concentrated on the empty configuration), as in many reaction diffusion processes, one can work with different couplings (see [12]); otherwise $\eta_t^0 = \eta_0^0$ for each $t \geq 0$ and we use the independent coupling.
After choosing a metric (and a coupling), we evaluate $\widetilde{\mathbb{E}}\Big(\mathcal{L}F(\eta_t^0(x),\eta_t^N(x))\Big)$ and we look for conditions on the transition rates which ensure the existence of $\{u_j\}_{j\in X}$ which satisfies (1.8). Finally the problem reduces to the existence of a sequence of real numbers which satisfy a condition involving transition rates, called *u-criterion*.

1.1.3 Comparison theorem and attractiveness

The main result is a comparison theorem between different interacting particle systems in a large class of processes, which gives us necessary and sufficient conditions for attractiveness as a corollary. Roughly speaking, we say that a process is *stochastically larger* than another one if we can couple the two processes in such a way that the initial larger one stays always the larger, see Section 2.2 for a more rigourous definition: we give necessary and sufficient conditions on transition rates of those processes which ensure that a system is stochastically larger (or smaller) than the other one.

We consider systems with births, deaths and jumps of more than one particle per time depending on the configuration in a very general way. Let $X \subseteq \mathbb{N}$, the state space of the process $(\eta_t)_{t\geq 0}$ is $\Omega = X^{\mathbb{Z}^d}$. The infinitesimal generator $\mathcal{L}$ of the process is given, for a local function f, that is a function depending only on a finite number of coordinates, and for $\eta \in \Omega$, by

$$\begin{aligned}\mathcal{L}f(\eta) = \sum_{x,y\in S}\sum_{\alpha,\beta\in X}\chi^{x,y}_{\alpha,\beta}(\eta)p(x,y)\sum_{k>0}\Big(\Gamma^k_{\alpha,\beta}(f(S^{-k,k}_{x,y}\eta)-f(\eta))+\\ +(R^{0,k}_{\alpha,\beta}+P^k_\beta)(f(S^k_y\eta)-f(\eta))+(R^{-k,0}_{\alpha,\beta}+P^{-k}_\alpha)(f(S^{-k}_x\eta)-f(\eta))\Big)\end{aligned} \tag{1.10}$$

where $\chi^{x,y}_{\alpha,\beta}$ is the indicator of configurations with values (α,β) on (x,y), that is

$$\chi^{x,y}_{\alpha,\beta}(\eta) = \begin{cases} 1 & \text{if } \eta(x)=\alpha \text{ and } \eta(y)=\beta \\ 0 & \text{otherwise}\end{cases}$$

$S^{-k,k}_{x,y}$, S^k_y and S^{-k}_y, where $k>0$, are local operators performing the transformations whenever possible

$$(S^{-k,k}_{x,y}\eta)(z) = \begin{cases} \eta(x)-k & \text{if } z=x \text{ and } \eta(x)-k\in X, \eta(y)+k\in X\\ \eta(y)+k & \text{if } z=y \text{ and } \eta(x)-k\in X, \eta(y)+k\in X\\ \eta(z) & \text{otherwise}\end{cases} \tag{1.11}$$

$$(S^k_y\eta)(z) = \begin{cases} \eta(y)+k & \text{if } z=y \text{ and } \eta(y)+k\in X\\ \eta(z) & \text{otherwise}\end{cases} \tag{1.12}$$

$$(S^{-k}_y\eta)(z) = \begin{cases} \eta(y)-k & \text{if } z=y \text{ and } \eta(y)-k\in X\\ \eta(z) & \text{otherwise}\end{cases} \tag{1.13}$$

and $p(x,y)$ is a symmetric translation invariant probability distribution on $\mathbb{Z}^d$.
Given a configuration $\eta\in\Omega$, the transition rates have the following meaning: on each site x we can have a *birth* (*death*) of k individuals depending on the configuration state $\eta(x)$ on the same site with rate $P^k_{\eta(x)}$ and depending also on a state on another site $y\neq x$ with rate $R^{0,k}_{\eta(y),\eta(x)}p(y,x)$. This represents a possible different interaction rule between individuals of the same population and individuals from different populations. Moreover we can have a jump of k particles from y to x with rate $\Gamma^k_{\eta(y),\eta(x)}p(y,x)$, which represents a migration of a flock of individuals (see Section 2.2). Let $\Pi^{0,k}_{\alpha,\beta} = R^{0,k}_{\alpha,\beta}+P^k_\beta$ for each $(\alpha,\beta)\in X^2$.
Denote by $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ two systems with respective generators $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ and rates $\{\Pi^{\cdot,\cdot}_{\alpha,\beta},\Gamma^{\cdot}_{\alpha,\beta}\}$ and $\{\widetilde{\Pi}^{\cdot,\cdot}_{\cdot,\cdot},\widetilde{\Gamma}^{\cdot}_{\cdot,\cdot}\}$,

Theorem 1.4 ***(Theorem* 2.15*, Section* 2.2*).***
A $\mathcal{S}$ particle system $(\eta_t)_{t\geq 0}$ is stochastically larger *than a $\widetilde{\mathcal{S}}$ particle system $(\xi_t)_{t\geq 0}$ if and only if for all $(\alpha,\beta),(\gamma,\delta)\in X^2\times X^2$ with $(\alpha,\beta)\leq(\gamma,\delta)$,*

$$\sum_{k>\delta-\beta+l_1}\widetilde{\Pi}^{0,k}_{\alpha,\beta}+\sum_{k\in I_a}\widetilde{\Gamma}^k_{\alpha,\beta}\leq\sum_{l>l_1}\Pi^{0,l}_{\gamma,\delta}+\sum_{l\in I_b}\Gamma^l_{\gamma,\delta} \tag{1.14}$$

$$\sum_{k>k_1}\widetilde{\Pi}^{-k,0}_{\alpha,\beta}+\sum_{k\in I_d}\widetilde{\Gamma}^k_{\alpha,\beta}\geq\sum_{l>\gamma-\alpha+k_1}\Pi^{-l,0}_{\gamma,\delta}+\sum_{l\in I_c}\Gamma^l_{\gamma,\delta} \tag{1.15}$$

for each choice of K, I_a, I_b, I_c, I_d depending on K, where I_a, I_b, I_c and I_d are sets of indexes given in Definition 2.14.

In other words we get conditions on transition rates that characterize the stochastic order in the class of systems we work with. If $\mathcal{S}=\widetilde{\mathcal{S}}$ the claim gives necessary and sufficient conditions for attractiveness.

The analysis of interacting particle systems began with Spin Systems, that are processes with state space $\{0,1\}^{\mathbb{Z}^d}$. We refer to [33] and [34] for construction and main results. The most famous examples are Ising model, contact process and voter model. These processes have been largely investigated, in particular their attractiveness (see [33, Chapter III, Section II].
Many other models with more than 1 particle per site have been studied. Reaction diffusion processes, for example, are processes with state space $\mathbb{N}^{\mathbb{Z}^d}$ (hence non compact), used to model chemical reactions. We refer to [12, Chapter 13] for a general introduction and construction of these processes. Schinazi analyzed many models for biological applications ([39] to [44]) taking place on $X^{\mathbb{Z}^d}$, where $X=\{0,1,\dots N\}$. In such reaction diffusion processes and biological models a birth, death or jump of at most one particle per time is allowed: in this case we get a characterization of the partial order and attractiveness of the systems as a corollary of our main result, see Section 2.3.5.
But sometimes the model requires births or deaths of more than one particle per time. This is the case of some biological systems with mass extinction ([41], [43]) or multitype contact process ([23], [22], [36]). A partial understanding of attractiveness properties can be found in [48] for the multitype contact process.
In [27, Theorem 2.21] the authors found necessary and sufficient conditions for attractiveness for a *conservative* particle system with multiple jumps on $\mathbb{N}^{\mathbb{Z}^d}$: in other words they work with a system with jumps of more than 1 particle, but no births or deaths.

Biological models are often very complex: many population dynamics are ruled by births, deaths and migrations of more than one individual per time (see [28]).
In order to understand the ergodic properties of a system one may compare it with another one easier to study: therefore an analysis of the stochastic order and attractiveness behaviour which involve all these systems can be useful. Theorem 1.4 gives conditions on transition rates to compare different particles systems.

In Section 2.1 we recall some classical definitions and well known propositions needed in the sequel. The particle system is introduced with more details in Section 2.2, where we also explain the main results and we define the coupling used to prove Theorem 1.4. A detailed analysis of coupling mechanisms can be found in Section 2.4.3. In Section 2.3 we explicit the conditions on several examples. Finally Section 2.4 is devoted to proofs.

1.1.4 Ergodicity result for an epidemic model

The most investigated interacting particle system that models the spread of epidemics is the contact process, introduced by Harris [30]. It is a spin system $(\eta_t)_{t\geq 0}$ on $\{0,1\}^{\mathbb{Z}^d}$ ruled by the transitions

$$\begin{aligned} \eta_t(x) = 0 \to 1 &\text{ at rate } \sum_{y\sim x} \eta_t(y) \\ \eta_t(x) = 1 \to 0 &\text{ at rate } 1 \end{aligned}$$

where $y \sim x$ is one of the $2d$ nearest neighbours of site x. See [33] and [34] for an exhaustive analysis of this model.
In order to understand the role of social clusters in the spread of epidemics, Schinazi [40] introduced a generalization of the contact process. Then, Belhadji [5] investigated some generalizations of this model: on each site in $\mathbb{Z}^d$ there is a cluster of $N \leq \infty$ individuals, where each individual can be *healthy* or *infected*. A cluster is infected if there is at least one infected individual, otherwise it is healthy. The illness moves from an infected individual to a healthy one with rate ϕ if they are in the same cluster. The infection rate between different clusters is different: the epidemics moves from an infected individual in a cluster y to an individual in a neighboring cluster x with rate λ if x is healthy, and with rate β if x is infected.
More precisely we have a Markov process $(\eta_t)_{t\geq 0}$ such that $\eta_t : \mathbb{Z}^d \to \{0,1,\ldots,N\}$, where $\eta_t(x)$ is the number of infected individuals at site $x \in \mathbb{Z}^d$ and N is a fixed number of individuals on each site. The transitions are

$$\begin{aligned} \eta_t(x) = 0 \to 1 \quad &\text{at rate} \quad \lambda \sum_{y\sim x} \eta_t(y) + \gamma, \\ \eta_t(x) = i \to i+1 \quad &\text{at rate} \quad \beta \sum_{y\sim x} \eta_t(y) + i\phi, \quad i = 1,2,\ldots,N-1, \\ \eta_t(x) = i \to i-1 \quad &\text{at rate} \quad i, \qquad\qquad\qquad\quad i = 1,2,\ldots,N. \end{aligned} \tag{1.16}$$

Since each individual recovers with rate 1, the process is called a *local recovery model.* The rate γ represents a positive "pure birth" of the illness: in the classical individual recovery model we set $\gamma = 0$.
Let λ_c be the critical value of the contact process; Belhadji proved that

Theorem 1.5 *[5, Theorem 14]*
For each $\phi \geq 0$ and $\beta \geq 0$, if $\lambda > \lambda_c$ an epidemic occurs with positive probability. For each λ and β such that $\beta \leq \lambda \leq 1/2d$, there exists a critical $\phi_c(\lambda,\beta) \in (0,\infty)$ such that if $\phi > \phi_c(\lambda,\beta)$ an epidemic is possible and if $\phi < \phi_c(\lambda,\beta)$ we get extinction of the disease almost surely.

Sufficient conditions for ergodicity are done as a corollary of [12, Theorem 14.3]

Theorem 1.6 *[5, Theorem 15]*
If

$$\lambda \vee \beta < \frac{1-\phi}{2d}, \tag{1.17}$$

the disease dies out for each cluster size N.

By using attractiveness of the model and the u-criterion explained in Section 1.1.2, we improve this ergodicity condition. Notice that a dependence on the cluster size N appears.

Theorem 1.7 ***(Theorem 3.2, Section 3.1)***
Suppose

$$\lambda \vee \beta < \frac{1-\phi}{2d(1-\phi^N)},$$

with $\phi < 1$ and either i) $\gamma = 0$, or ii) $\gamma > 0$ and $\beta - \lambda \leq \gamma/(2d)$.
Then the system is ergodic.

If $N = 1$ and $\gamma = 0$ the process reduces to the contact process and the result is a well known (and already improved) ergodic result (see for instance [33, Corollary 4.4, Chapter VI]); as a corollary we get the ergodicity result in the non compact case as N goes to infinity.

1.1.5 Metapopulation dynamics models

We use comparison arguments, with oriented percolation and with different processes, to investigate survival and extinction in several models of metapopulation dynamics. A metapopulation model refers to many small local populations connected via migrations in fragmented environment. Each local population can increase or decrease, survive, get extinct or migrate from its site in a very complicated way (see Hanski [28], for more about metapopulations).
Many factors influence the growth rate, that is the difference between birth and death rates, of a (local) population. The classical hypothesis considers a density dependent growth rate. As the population size increases both the birth and the death rates increase because there are more individuals that can respectively reproduce themselves and die. By the way, the number of individuals can increase in different ways. If the birth rate is always larger than the death rate the population will increase indefinitely. If it is always smaller the population will get extinct. A more interesting situation is given by a birth rate larger than the death rate under a particular population size N and smaller over that. From a biological point of view this means that when the population density is too large resources are not enough for all individuals and the growth rate decreases, since the death rate increases.
In real environment this process is gradual, that is the growth rate decreases over N as population density increases. In some of our applications we suppose that over N the growth rate is null. In this case N is the *capacity* of the site, that is the maximal number of individuals that it can hold.
One of the most important strategies to ensure the survival of a species is the migration of flocks of individuals. When the population size is large, that is resources are few, one or more individuals leave their population, that is the site where they are located, and look for new resources in other sites with different populations.
In some cases another factor can favor the extinction of a species: *the Allee effect.* When a population is subjected to the Allee effect, if the density of individuals is small the death rate increases. The reason is that at low density many factors (as difficulties in finding mates) cause a decrease of fecundity and an increase of mortality (see [2], [15], [45], [47]).
To summarize, population dynamics is ruled by many parameters. We simplify the real structure and we treat 4 metapopulation models from a mathematical point of view.

The mathematical model is an interacting particle system $(\eta_t)_{t\geq 0}$ on $\Omega = X^{\mathbb{Z}^d}$, where $X = \{0, 1, \ldots, N\} \subseteq \mathbb{N}$ and N denotes the common size (capacity) of the clusters, if finite. The value $\eta_t(x)$, $x \in \mathbb{Z}^d$ is the number of individuals present in cluster x at time $t \geq 0$.

We will take $N = \infty$ in Model IV.
For any $x, y \in \mathbb{Z}^d$, we write $y \sim x$ if y is one of the $2d$ nearest neighbors of site x. The infinitesimal generator $\mathcal{L}$ of the process is given by

$$\begin{aligned}\mathcal{L}f(\eta) = \sum_{x\in X} \Big\{ & B(\eta(x))\big(f(S_x^1\eta) - f(\eta)\big) + D(\eta(x))\big(f(S_x^{-1}\eta(x)) - f(\eta)\big) \\ & + \sum_{y\sim x}\sum_{k\in X} J(\eta(x),\eta(y),k)\big(f(S_{x,y}^{-k,k}\eta) - f(\eta)\big)\Big\} \qquad (1.18)\end{aligned}$$

where f is a local function, $\eta \in \Omega$, $S_x^{\pm 1}$ and $S_{x,y}^{-k,k}$ are defined in (1.11), (1.12) and (1.13), and $B(\cdot)$, $D(\cdot)$ are positive functions from X to $\mathbb{R}$. The respective birth and death rates on x, $B(\eta(x))$ and $D(\eta(x))$, depend only on the configuration on site x. We allow at most one birth or death per time.
We assume $B(0) = 0$, that is $\delta_{\underline{0}}$ is a trivial invariant measure.
The function $J : X^3 \to \mathbb{R}^+$ is such that $J(\eta(x), \eta(y), k)$ represents the migration (jump) rate and it depends on the configurations on x and y. Notice that a jump of more than one particle per time is possible. We call *emigration* from x a jump that reduces the number of particles on x and *immigration* a jump that increases it.
When X is finite, which is the case in Models I, II and III, we refer to the construction in [33]; when X is infinite, that is in Model IV, we refer to [12] and we require additional conditions on the transition rates (see section 3.2.4).

We say that there is survival of the species if

$$\mathbb{P}(|\eta_t| \geq 1 \textit{ for all } t \geq 0) > 0 \qquad (1.19)$$

where $|\eta_0|$ is finite and $|\eta_t|$ is the number of individuals of $(\eta_t)_{t\geq 0}$ at time t.

The species gets extinct if for each finite or infinite initial configuration the process converges to $\delta_{\underline{0}}$.

Model I: the basic model.

In [43] Schinazi introduced the following model to investigate the role of social aggregation in the extinction of a species: for a fixed $N < \infty$, on each site on $\mathbb{Z}^d$ we may have $0, 1, \ldots N$ individuals. The transition rates of the Markov process $(\eta_t)_{t\geq 0}$ are

$$\begin{array}{lll} \eta_t(x) \to \eta_t(x) + 1 & \text{at rate } \eta_t(x)\phi + \lambda n_N(x, \eta_t) & \text{for } 0 \leq \eta_t(x) \leq N-1, \\ \eta_t(x) \to \eta_t(x) - 1 & \text{at rate } 1 & \text{for } 1 \leq \eta_t(x) \leq N, \end{array}$$

where $n_N(x, \eta_t)$ is the number of neighbors y of x such that $\eta_t(y) = N$. In other words each individual gives birth to another one on the same site with rate ϕ and dies with rate 1. An individual on site x gives birth to a new individual in a site $y \sim x$ only when the population at x has reached the maximal size N. There is a phase transition on the capacity N of sites:

Theorem 1.8 *[43, Theorem 2]*
Assume that $d \geq 2$, $\lambda > 0$ and $\phi > 0$. There is a critical value $N_c(\lambda, \phi)$ such that if $N > N_c(\lambda, \phi)$, then starting from any finite number of individuals, the population has a strictly positive probability of surviving.

The author compares this result with the opposite phase transition that appears when instead of deaths of single individuals with rate 1 we consider the death with rate 1 of all individuals in a cluster, that he calls *catastrophic times.* In catastrophic times, see [43, Theorem 1], there exists $N_c(\lambda, \phi)$ such that if $N > N_c(\lambda, \phi)$ the population dies out.
The consequence is that cluster aggregation may be good in non catastrophic times. In this case, if N is small, for example if $N = 1$ and $\lambda < \lambda_c$ (where λ_c is the critical value for the contact process), the species dies out: this is consistent with the Allee effect, since for a small density the probability of surviving should be smaller.
We want to investigate the role of migration in the survival of a species, starting from this model. We begin from a system very similar to Schinazi's model in non catastrophic times: the only difference is that instead of a birth of a new individual we take a migration of one individual to a neighboring site. We could imagine that this small difference does not change the behaviour of the model, if N is large. Whatever, if $N = 1$ no new births are possible and the process gets extinct for any λ: this is definitely not the case for the non catastrophic times model for $N = 1$, which is the contact process.

We introduce Model I. We choose to fix a birth rate equal to 1 and to associate two parameters to death and migration rates. Given δ and λ positive real numbers, the birth and death transition rates are, for each $x \in S$

$$\begin{aligned} \eta_t(x) \to & \eta_t(x) + 1 & \text{at rate } \eta_t(x) \mathbb{1}_{\{\eta_t(x) < N\}}, \\ \eta_t(x) \to & \eta_t(x) - 1 & \text{at rate } \delta \eta_t(x). \end{aligned}$$

For each $x \sim y$ the migration rate is

$$(\eta_t(x), \eta_t(y)) \to (\eta_t(x) - 1, \eta_t(y) + 1) \qquad \text{at rate } \mathbb{1}_{\{\eta_t(x) = N\}} \lambda M(\eta_t(y))$$

where $M : X \to \mathbb{R}^+$ is such that $M(l)$ is non increasing in l, that is individuals try to move towards sites with less individuals (and more resources). We assume $M(N) = 0$, that is a migration from x to y is not possible if $\eta(y) = N$.
The first result corresponds to Theorem 1.8 for non catastrophic times model and it is proved in a similar way.

Theorem 1.9 ***(Theorem* 3.14*, Section* 3.2.1*)***
Suppose $d \geq 2$, $\lambda > 0$ and $\delta < 1$. There exists a critical value $N_c(\lambda, \delta)$ such that if $N > N_c(\lambda, \delta)$, then starting from $\eta_0 \in \Omega$ such that $|\eta_0| \geq 1$ the process has a positive probability of survival. Moreover if $\eta_0 \in E^N$ the process converges to a non-trivial invariant measure with positive probability.

If we fix the capacity N, we prove that there is a phase transition also with respect to the death rate δ. A similar proof has been used in [42] to show a phase transition for a stochastic model for virus dynamics.

Theorem 1.10 ***(Theorem* 3.18*, Section* 3.2.1*)***

i) For all $\lambda > 0$, $1 < N < \infty$, there exists $\delta_c^1(\lambda, N) > 0$ such that if $\delta < \delta_c^1(\lambda, N)$ the process has a positive probability of survival.

ii) There exists $\delta_c^2 > 0$ such that if $\delta > \delta_c^2$ the process dies out for all λ, N.

iii) For all $\lambda > 0$, $1 < N < \infty$, there exists $\delta_c(\lambda, N) > 0$ such that, if $\delta < \delta_c(\lambda, N)$ the process starting from η_0 with $|\eta_0| \geq 1$ has a positive probability of survival and if $\delta > \delta_c(\lambda, N)$ the process dies out. Moreover if $\delta < \delta_c(\lambda, N)$ and $\eta_0 \in E^N$ the process converges to a non-trivial invariant measure with positive probability.

Model II: the Allee effect.

We add the Allee effect to Model I. Many biological investigations agree with the existence of the Allee effect in some biological systems (see [2], [15], [45], [47]). In a population subjected to the Allee effect, if the density of individuals is small the death rate increases. Different stochastic techniques have been already used to investigate the Allee effect like stochastic differential equations (see [20]), discrete-time Markov chains (see [3]) or diffusion processes (see [21]), but none of these models has a spatial structure.

We introduce for the first time a model with spatial structure for the Allee effect. We translate the Allee effect into mathematical terms for a metapopulation model. As in Model I, we fix a capacity N for all sites, but we assume that the death rate is larger than the birth rate when the density is small. Namely, fix $N_A \leq N$ a positive integer and δ_A, δ, λ positive real numbers; birth and death transitions are

$$\begin{aligned}
\eta_t(x) \to &\eta_t(x) + 1 \qquad && \text{at rate } \eta_t(x)1\!\!1_{\{\eta_t(x)\leq N-1\}},\\
\eta_t(x) \to &\eta_t(x) - 1 \qquad && \text{at rate } \eta_t(x)\big(\delta_A 1\!\!1_{\{\eta_t(x)\leq N_A\}} + \delta 1\!\!1_{\{N_A<\eta_t(x)\}}\big).
\end{aligned}$$

For each $x \sim y$, migrations are given by

$$(\eta_t(x), \eta_t(y)) \to (\eta_t(x) - 1, \eta_t(y) + 1) \qquad \text{at rate } 1\!\!1_{\{\eta_t(x)=N\}}\lambda M(\eta_t(y)),$$

where $M : X \to \mathbb{R}^+$ is a non increasing function such that $M(N) = 0$ as in Model I.
We assume $\delta_A > 1$ and $\delta_A \geq \delta$; in other words if $\eta_t(x) \leq N_A$ then the death rate $\delta_A\eta_t(x)$ is larger (or equal) than the birth rate $\eta_t(x)$ because of the Allee effect.
If $\eta_t(x) \geq N_A$, the most interesting situation is given by a death rate $\delta\eta_t(x)$ smaller than the birth rate $\eta_t(x)$, that is $\delta < 1$. If $\delta > 1$ the death rate is always larger than the birth rate and the species gets extinct as proved in Theorem 1.10 *ii)*.
If $N^A = 0$ (no Allee effect) or $N^A = N$ (death rate always larger than birth rate) there is only one death rate and we are back to Model I.

We prove that the Allee effect changes the behaviour of the system: for any possible capacity N and migration rates there exists an Allee effect large enough for the species to get extinct.

Theorem 1.11 ***(Theorem 3.20, Section 3.2.2)***
Assume $\delta_A > 1$.

i) For all $\delta < 1$, $\lambda > 0$, $0 < N < \infty$, $0 \leq N_A \leq N$ there exists $\delta_c^A(\delta, \lambda, N, N_A)$ such that if $\delta_A > \delta_c^A(\delta, \lambda, N, N_A)$ the process converges to the trivial invariant measure $\delta_{\underline{0}}$ for any initial configuration $\eta_0 \in \Omega$.

ii) If $\delta > 1$ the process dies out for all $\delta_A > 1$, N, N_A, λ and any initial configuration.

In other words as small as we take the death rate and as large as we take the population size and the migration rate, there exists an Allee effect strong enough so that the species gets extinct.

In order to model the Allee effect, we need $\delta_A > 1$ and $\delta < 1$. From a mathematical point of view, it would be interesting to investigate a model where δ and δ_A play symmetric roles, that is $\delta_A < 1$ and $\delta > 1$.

For fixed N, N_A and λ we prove that we can choose neither $\delta_A < 1$ small enough for the species to survive for each δ nor $\delta > 1$ large enough for the species to get extinct for each δ_A.

Theorem 1.12 ***(Theorem 3.21, Section 3.2.2)***
Assume $\delta_A < 1$ and $\delta > 1$.

i) For all $1 < N_A < N$, $\lambda > 0$, $\delta > 1$ there exists $\delta_c^A(\lambda, N_A, N, \delta)$ such that, if $\delta_A < \delta_c^A(\lambda, N_A, N, \delta)$, the process survives for any initial configuration η_0 such that $|\eta_0| \geq 1$ with positive probability.

ii) For all $1 < N_A < N$, $\lambda > 0$, $\delta_A < 1$ there exists $\delta_c(\lambda, N_A, N, \delta_A)$ such that, if $\delta > \delta_c(\lambda, N_A, N, \delta_A)$, the process dies out for any initial distribution.

Model III: mass migration.

We introduce mass migration as Allee effect solution. Migration is one of the most important strategies that a species can adopt to improve its probability of survival ([11], [47]).
We have already observed in Model I that a migration of a single individual is good in absence of the Allee effect. The model without migrations dies out, but if we add a possible migration of one individual there is a positive probability of survival. In Model II, anyhow, a single individual migration may not be enough: even in the supercritical region of δ in Model I there exists an Allee effect strong enough so that the species gets extinct. Which strategy may a species adopt to reduce the Allee effect?
We show that, at least in theory, *migrations of large flocks of individuals improve the probability of survival for any Allee effect.* A migration of many individuals in a new environment improves the probability of a successful colonization avoiding a small density in that new environment which is influenced by the Allee effect.

We introduce positive parameters δ_A, δ, N_A, N such that $0 \leq N_A \leq N$, $\delta_A > 1$, $\delta > 0$ and we take birth and death rates as in Model II, that is

$$\begin{aligned} \eta_t(x) \to & \eta_t(x) + 1 && \text{at rate } \eta_t(x) \mathbb{1}_{\{\eta_t(x) \leq N-1\}}, \\ \eta_t(x) \to & \eta_t(x) - 1 && \text{at rate } \eta_t(x)\big(\delta_A \mathbb{1}_{\{\eta_t(x) \leq N_A\}} + \delta \mathbb{1}_{\{N_A < \eta_t(x)\}}\big). \end{aligned}$$

Given $M \in \mathbb{N}$, $0 < M \leq N$, for each $y \sim x$ we introduce the migration rate

$$(\eta_t(x), \eta_t(y)) \to (\eta_t(x) - k, \eta_t(y) + k) \qquad \text{at rate } J(\eta_t(x), \eta_t(y), k) \tag{1.20}$$

where for each $l \in X$, $m \in X$, $k \in \{1, \ldots, M\}$

$$J(l, m, k) := \begin{cases} \lambda & \text{if } l - k \geq N - M \text{ and } m + k \leq N \\ 0 & \text{otherwise} \end{cases} \tag{1.21}$$

that is if k individuals try to migrate from x to y but $\eta_t(y) + k > N$, the migration does not happen.
Notice that if $\eta_t(x) < N - M$ the migration rate is null. This means that *individuals try to migrate only when there are more than $N - M$ individuals on a site.* From a biological point of view, this means that when there are few individuals resources are enough for all

and there are no reasons to migrate.
When $l \geq N-M$ there is a positive probability of migration and the number of individuals that migrate is increasing with the population size. If $l = N-M+1$ we allow a migration of at most 1 individual from x to a nearest neighbor site, when $l = N-M+2$ we allow a migration of at most 2 individuals and so on. If $l = N = (N-M)+M$ we allow a migration of the largest flock of M individuals. The migration rate is given by λ.
We prove that for any Allee effect we can take a population size N and a maximal migration flock size M large enough for the species to survive.

Theorem 1.13 ***(Theorem 3.23, Section 3.2.3)***
Let $d \geq 2$.

i) For all $\delta < 1$, $\lambda > 0$, $N_A \geq 0$, there exists $N_c(\delta, \lambda, N_A)$ such that for each $N > N_c(\delta, \lambda, N_A)$, there exists $M(N_A)$ so that the process starting from η_0 with $|\eta_0| \geq 1$ has a positive probability of survival for each $\delta_A < \infty$. Moreover if $\eta_0 \in E^N$ the process converges to a non-trivial invariant measure for each $\delta_A < \infty$.

ii) If $\delta > 1$, for all N, N_A, λ, δ_A, M the process dies out for any initial configuration.

In other words, for fixed N_A, λ, δ, even if δ_A is the strongest one, *if the species lives in large populations and migrates in large flocks, survival is possible.*

In Model II we showed that a strong Allee effect dooms even a very large population with a large migration rate. Theorem 1.13 states that the strategy that the species may adopt to reduce the Allee effect is not an increase of the migration rate, but an increase of the number of individuals which migrate. Moreover the population size is important and spatial aggregation is good for the survival of the species.

Model IV: ecological equilibrium.

Real natural environments do not have any a priori bound on the population size, but there is a kind of *self-regulating mechanism* that does not allow an "explosion" of the number of individuals per site. *Ecological equilibrium* has been introduced in [6] for restrained branching random walks: some restrictions on branching random walks birth rates provides a survival through nonexploding populations. We show that this is also possible in a system with a different mechanism.

We suppose that in our environment there is no maximal population size as in previous models, and the birth rate is always positive. We also assume that when the population size is larger than some value N the growth rate is negative.
We change each of the previous models in the following way when the number of individuals per site $\eta(x)$ is larger than N:

$$\begin{aligned} \eta_t(x) \to& \eta_t(x)+1 && \text{at rate } \eta_t(x)\mathbb{1}_{\{N \leq \eta_t(x)\}}, \\ \eta_t(x) \to& \eta_t(x)-1 && \text{at rate } \widetilde{\delta}\eta_t(x)\mathbb{1}_{\{N+1 \leq \eta_t(x)\}}, \end{aligned}$$

where $\widetilde{\delta} > 1$, that is when the density is large the death rate prevales over the birth rate. In order to simplify notation and proofs, we work on Model I. We can get a similar result on Model III. Namely we take the following birth and death transition rates

$$\begin{aligned} \eta_t(x) \to& \eta_t(x)+1 && \text{at rate } \eta_t(x), \\ \eta_t(x) \to& \eta_t(x)-1 && \text{at rate } \eta_t(x)\big(\delta\mathbb{1}_{\{\eta_t(x) \leq N\}} + \widetilde{\delta}\mathbb{1}_{\{N < \eta_t(x)\}}\big), \end{aligned}$$

and for each $x \sim y$ the migration rate is

$$(\eta_t(x), \eta_t(y)) \rightarrow (\eta_t(x) - 1, \eta_t(y) + 1) \qquad \text{at rate } \mathbb{1}_{\{y \sim x\}} \lambda M(\eta_t(y)) \mathbb{1}_{\{\eta_t(x) \geq N\}}$$

where $M : X \rightarrow \mathbb{R}^+$ is a non increasing function such that $M(l) = 0$ for each $l \geq N$.
This means that when the population size $\eta_t(x)$ is larger than N the death rate $\widetilde{\delta}\eta_t(x)$ is larger than the birth rate $\eta_t(x)$. A migration is allowed from a site with more than N individuals to a site with less than N individuals. Since we are working without any a priori bound, we refer to construction techniques in non compact cases and we restrict the state space to $\Omega_0 \subseteq \Omega$ (See [13, Chapter 13]). One can check that sufficient conditions for the existence of the process (see [13, Theorem 13.1]) are satisfied.
We prove that we can have survival of the species but on each site the population size does not explode. In other cases the species gets extinct.

Theorem 1.14 ***(Theorem 3.25, Section 3.2.4)***
Suppose $|\eta_0| \geq 1$ and there exists $n < \infty$ such that $\eta_0(x) \leq n$ for each $x \in \mathbb{Z}^d$.

i) For each $\lambda > 0$, $1 < N < \infty$, $\epsilon > 0$ there exists $\delta_c^1(\lambda, N) > 0$ and $K < \infty$ such that if $\delta < \delta_c^1(\lambda, N)$, the process has a positive probability of survival for each $\widetilde{\delta} > 1$ and $\lim_{t \rightarrow \infty} \mathbb{E}(\eta_t(x)) \leq K$ for each $x \in \mathbb{Z}^d$ and $\widetilde{\delta} \geq 1 + \epsilon$.

ii) There exists $\delta_c^2 > 0$ such that if $\delta > \delta_c^2$, the process dies out for each λ, N, $\widetilde{\delta} > 1$ and any initial configuration.

iii) For each $\lambda > 0$, $1 < N < \infty$, $\widetilde{\delta}$ there exists $\delta_c(\lambda, N, \widetilde{\delta}) > 0$ such that, if $\delta < \delta_c(\lambda, N, \widetilde{\delta})$ the process survives and if $\delta > \delta_c(\lambda, N, \widetilde{\delta})$ the process dies out.

Chapter 2

Stochastic order and attractiveness

2.1 Background

We recall the definitions and results about attractiveness and couplings that we use.

Definition 2.1 *Let W be a set endowed with a partial order relation.*
A set $V \subset W$ is increasing if

$$\forall l \in V, m \in W, \qquad l \leq m \Rightarrow m \in V.$$

A set $V \subset W$ is decreasing if

$$\forall l \in V, m \in W, \qquad l \geq m \Rightarrow m \in V.$$

A function $f : W \to \mathbb{R}$ is monotone if

$$\forall l, m \in W, \qquad l \leq m \Rightarrow f(l) \leq f(m).$$

For instance if $l \in W$, then $I_l = \{m \in W : l \leq m\}$ is an increasing set, and $D_l = \{m \in W : l \geq m\}$ a decreasing one.

Remark 2.2 *Let J be a set of indexes. If $\{I_j\}_{j\in J}$ is a family of increasing sets on W, then $\bigcup_{j\in J} I_j$ is an increasing set. If $\{D_j\}_{j\in J}$ is a family of decreasing sets on W, then $\bigcup_{j\in J} D_j$ is a decreasing set.*

Remark 2.3 *For any subset $V \in W$,*

$$V \textit{ is increasing} \Leftrightarrow W \backslash V \textit{ is decreasing} \Leftrightarrow 1\!\!1_V \textit{ is monotone.}$$

Now we introduce the interacting particle system. Denote by $S = \mathbb{Z}^d$ and let $X \subseteq \mathbb{N}$. Let $(\eta_t)_{t\geq 0}$ be a Markov process on $\Omega = X^S$ with semi-group $T(t)$ and infinitesimal generator $\mathcal{L}$. The set Ω is the state space, S is the set of sites and X is the set of possible states on each site. We refer to [33] for the classical construction in a compact state space. Since we are interested also in non compact cases, we assume that $(\eta_t)_{t\geq 0}$ is a well defined Markov process on a subset $\Omega_0 \subset \Omega$, such that for any bounded local function f on Ω_0,

$$\forall \eta \in \Omega_0, \quad \lim_{t\to 0} \frac{T(t)f(\eta) - f(\eta)}{t} = \mathcal{L}f(\eta) < \infty. \tag{2.1.1}$$

Given two processes $(\xi_t)_{t\geq 0}$ and $(\zeta_t)_{t\geq 0}$, a coupled process $(\xi_t,\zeta_t)_{t\geq 0}$ is a Markov process with state space $\Omega_0\times\Omega_0$ such that each marginal is a copy of the original process.
We define a partial order on the state space:

$$\forall \xi,\zeta\in\Omega, \qquad \xi\leq\zeta \Leftrightarrow (\forall x\in S, \xi(x)\leq\zeta(x)). \tag{2.1.2}$$

We denote by $\mathcal{M}$ the set of all bounded, monotone continuous functions on Ω. The partial order on Ω induces a stochastic order on the set $\mathcal{P}$ of probability measures on Ω endowed with the weak topology:

$$\forall \nu,\nu'\in\mathcal{P}, \qquad \nu\leq\nu' \Leftrightarrow (\forall f\in\mathcal{M}, \nu(f)\leq\nu'(f)). \tag{2.1.3}$$

Remark 2.4 *$\nu\leq\nu'$ is equivalent to $V\subset\Omega$, $\nu(V)\leq\nu'(V)$ for all increasing sets.*

Indeed for any non-negative $f\in\mathcal{M}$ and all $s\geq 0$,

$$\{f\geq s\}=\{f(x)\geq s,\forall x\}=\bigcap_{x\in\Omega}\{f(x)\geq s\}=\Big(\bigcup_{x\in\Omega}\{f(x)<s\}\Big)^c$$

is an increasing set because it is the complement of a decreasing one, by Remark 2.2 and Remark 2.3, and $\nu(f)=\int_0^\infty \nu\{f\geq s\}ds$. On the other hand use Remark 2.3.

Theorem 2.5 *[33, Theorem II.2.2] For the particle system $(\eta_t)_{t\geq 0}$, the following two statements are equivalent:*

(a) $f\in\mathcal{M}$ implies $T(t)f\in\mathcal{M}$ for all $t\geq 0$.

(b) For $\nu,\nu'\in\mathcal{P},\nu\leq\nu'$ implies $\nu T(t)\leq\nu' T(t)$ for all $t\geq 0$.

Definition 2.6 *[33, Definition II.2.3] The particle system $(\eta_t)_{t\geq 0}$ is attractive if the equivalent conditions of Theorem* 2.5 *are satisfied.*

Attractiveness is a property concerning the distribution at time t of two processes *with the same generator* which start with different initial distributions. We are also interested in comparing distributions of two processes with *different generators* starting with different initial distributions. We slightly modify Theorem 2.5 and Definition 2.6 in order to get a more general setting.

Theorem 2.7 *Let $(\xi_t)_{t\geq 0}$ and $(\zeta_t)_{t\geq 0}$ be two processes with generators $\widetilde{\mathcal{L}}$ and $\mathcal{L}$ and semigroups $\widetilde{T}(t)$ and $T(t)$ respectively. The following two statements are equivalent:*

(a) $f\in\mathcal{M}$ and $\xi_0\leq\zeta_0$ implies $\widetilde{T}(t)f(\xi_0)\leq T(t)f(\zeta_0)$ for all $t\geq 0$.

(b) For $\nu,\nu'\in\mathcal{P},\nu\leq\nu'$ implies $\nu\widetilde{T}(t)\leq\nu' T(t)$ for all $t\geq 0$.

Definition 2.8 *A process $(\zeta_t)_{t\geq 0}$ is* stochastically larger *than a process $(\xi_t)_{t\geq 0}$ if the equivalent conditions of Theorem* 2.7 *are satisfied. In this case the process $(\xi_t)_{t\geq 0}$ is stochastically smaller than $(\zeta_t)_{t\geq 0}$ and the pair $(\xi_t,\zeta_t)_{t\geq 0}$ is stochastically ordered.*

Notice that if $\widetilde{T}=T$ we get Theorem 2.5 and Definition 2.6.

Theorem 2.9 *[33, Theorem II.2.4]. For $\nu, \nu' \in \mathcal{P}$, a necessary and sufficient condition for $\nu \leq \nu'$ is the existence of a probability measure μ on $\Omega \times \Omega$, called a coupled measure, which satisfies*

(a) $\mu\{(\xi,\zeta) : \xi \in A\} = \nu(A), \mu\{(\xi,\zeta) : \zeta \in A\} = \nu'(A)$, for all Borel sets $A \in \Omega$,
(b) $\mu\{(\xi,\zeta) : \xi \leq \zeta\} = 1$.

In order to characterize the stochastic ordering of two processes, first of all we find necessary conditions on the transition rates of the model. Then we construct a Markovian increasing (we also say *attractive*) coupling, that is a coupled process $(\xi_t, \zeta_t)_{t\geq 0}$ which has the property that $\xi_0 \leq \zeta_0$ implies

$$P^{(\xi_0,\zeta_0)}\{\xi_t \leq \zeta_t\} = 1,$$

for all $t \geq 0$. Here $P^{(\xi_0,\zeta_0)}$ denotes the distribution of $(\xi_t, \zeta_t)_{t\geq 0}$ with initial state (ξ_0, ζ_0). A characterization of attractiveness follows.

2.2 Main results

We look for necessary and sufficient conditions on the transition rates that yield stochastic order or attractiveness of the following class of interacting particle systems with births, deaths and jumps. Referring to notations in Section 1.1.3, the infinitesimal generator $\mathcal{L}$ of the process $(\eta_t)_{t\geq 0}$ on $\Omega = X^S$ is given, for a local function f, by

$$\begin{aligned} \mathcal{L}f(\eta) = \sum_{x,y\in S}\sum_{\alpha,\beta\in X} \chi^{x,y}_{\alpha,\beta}(\eta)p(x,y)\sum_{k>0}\Big(\Gamma^k_{\alpha,\beta}(f(S^{-k,k}_{x,y}\eta) - f(\eta)) + \\ + (R^{0,k}_{\alpha,\beta} + P^k_\beta)(f(S^k_y\eta) - f(\eta)) + (R^{-k,0}_{\alpha,\beta} + P^{-k}_\alpha)(f(S^{-k}_x\eta) - f(\eta))\Big) \end{aligned} \tag{2.2.1}$$

where the rates have the following meaning:

- $p(x,y)$ is a symmetric translation invariant probability distribution on $\mathbb{Z}^d$;
- $\Gamma^k_{\alpha,\beta}p(x,y)$ is the jump rate of k particles from x, where $\eta(x) = \alpha$, to y, where $\eta(y) = \beta$;
- $R^{0,k}_{\alpha,\beta}p(x,y)$ is the part of the birth rate of k particles in y such that $\eta(y) = \beta$ dependent on the value of η in x (that is α);
- $R^{-k,0}_{\alpha,\beta}p(x,y)$ is the part of the death rate of k particles in x such that $\eta(x) = \alpha$ dependent on the value of η in y (that is β);
- $P^{\pm k}_\beta$ is the birth/death rate of k particles in $\eta(y) = \beta$ dependent only on the value of η in y: we call it an *independent* birth/death rate.

We call *addition* on y (*subtraction* from x) of k particles the birth on y (death on x) or jump from x to y of k particles. The correspondent rate is called *addition* (*subtraction*) *rate*.
By convention we take births on the right subscript and deaths on the left one: formula (2.2.1) involves births upon β, deaths from α and a fixed direction, from α to β, for jumps of particles. We define, for notational convenience

$$\Pi^{0,k}_{\alpha,\beta} := R^{0,k}_{\alpha,\beta} + P^k_\beta; \qquad \Pi^{-k,0}_{\alpha,\beta} := R^{-k,0}_{\alpha,\beta} + P^{-k}_\alpha. \tag{2.2.2}$$

We assume (2.1.1) and we will be more precise on the induced conditions on transition rates in the applications. We state here only a common necessary condition on the rates.

Hypothesis 2.2.1 *We assume that for each* $(\alpha, \beta) \in X^2$

$$W(\alpha,\beta) := \sup\{n : \Gamma^n_{\alpha,\beta} + \Pi^{0,n}_{\alpha,\beta} + \Pi^{-n,0}_{\alpha,\beta} > 0\} < \infty,$$
$$\sum_{k>0} \Gamma^k_{\alpha,\beta} < \infty; \qquad \sum_{k>0} \Pi^{0,k}_{\alpha,\beta} < \infty; \qquad \sum_{k>0} \Pi^{-k,0}_{\alpha,\beta} < \infty.$$

In other words, for each α, β there exists a maximal number of particles involved in birth, death and jump rates. Notice that $W(\alpha, \beta)$ is not necessarily equal to $W(\beta, \alpha)$, which involves deaths from β, births upon α and jumps from β to α.

Definition 2.10 *Given* $(\alpha, \beta), (\gamma, \delta) \in X^2 \times X^2$*, the notation* $(\alpha, \beta) \leq (\gamma, \delta)$ *is equivalent to* $\alpha \leq \gamma$ *and* $\beta \leq \delta$*;* (α, β)*,* (γ, δ) *are* ordered *if* $(\alpha, \beta) \leq (\gamma, \delta)$ *or* $(\alpha, \beta) \geq (\gamma, \delta)$*; they are* not ordered *otherwise, that is when* ($\alpha < \gamma$ *and* $\beta > \delta$) *or* ($\alpha > \gamma$ *and* $\beta < \delta$)*.*

Definition 2.11 *Let* $(\alpha, \beta), (\gamma, \delta) \in X^2 \times X^2$ *be such that* $(\alpha, \beta) \leq (\gamma, \delta)$*. There is a* lower attractiveness problem *on* $(\alpha, \beta), (\gamma, \delta)$ *if there exists* k *such that* $\beta + k > \delta$ *and* $\Pi^{0,k}_{\alpha,\beta} + \Gamma^k_{\alpha,\beta} > 0$*;* (β, δ) *is a* k-bad pair *(of states) and* k *is a* low-bad value *(with respect to* (β, δ)*). There is a* higher attractiveness problem *on* $(\alpha, \beta), (\gamma, \delta)$ *if there exists* k *such that* $\gamma - k < \alpha$ *and* $\Pi^{-k,0}_{\gamma,\delta} + \Gamma^k_{\gamma,\delta} > 0$*;* (α, γ) *is a* k-bad pair *(of states) and* k *is a* high-bad value *(with respect to* (α, γ)*). Otherwise* (β, δ) *(resp.* (α, γ)*) is a* k*-good pair (of states) and* k *is a* low-good (resp. high-good) value*.*
If either (β, δ) *or* (α, γ) *is a bad pair there is an* attractiveness problem *on* $(\alpha, \beta), (\gamma, \delta)$*.*

In other words we distinguish bad situations, where an addition of particles allows lower states to go over upper ones (or upper ones to go under lower ones) from good ones, where it cannot happen.
Notice that Definition 2.11 involves addition of particles upon β and subtraction of particles from γ. If we are interested in attractiveness problems coming from addition upon α and subtraction from δ we refer to $(\beta, \alpha), (\delta, \gamma)$.

Definition 2.12 *Given* $(\alpha, \beta), (\gamma, \delta) \in X^2 \times X^2$ *such that* $(\alpha, \beta) \leq (\gamma, \delta)$*, denote by*

$$A := \{n \in \mathbb{N} : \beta + n > \delta \text{ and } \Gamma^n_{\alpha,\beta} + \Pi^{0,n}_{\alpha,\beta} > 0\} \tag{2.2.3}$$
$$B := \{n \in \mathbb{N} : \gamma - n < \alpha \text{ and } \Gamma^n_{\gamma,\delta} + \Pi^{-n,0}_{\gamma,\delta} > 0\} \tag{2.2.4}$$
$$L(\alpha,\beta,\gamma,\delta) := \begin{cases} \sup A & \text{if } A \neq \emptyset \\ \delta - \beta & \text{if } A = \emptyset \end{cases} \tag{2.2.5}$$
$$U(\alpha,\beta,\gamma,\delta) := \begin{cases} \sup B & \text{if } B \neq \emptyset \\ \gamma - \alpha & \text{if } B = \emptyset \end{cases} \tag{2.2.6}$$
$$N(\alpha,\beta,\gamma,\delta) := L(\alpha,\beta,\gamma,\delta) \vee U(\alpha,\beta,\gamma,\delta) \tag{2.2.7}$$
$$M(\alpha,\beta,\gamma,\delta) := W(\alpha,\beta) \vee W(\gamma,\delta). \tag{2.2.8}$$

The notation L *means Lower (configuration) and* U *means Upper (configuration).*

Notice that $L(\alpha, \beta, \gamma, \delta)$ (resp. $U(\alpha, \beta, \gamma, \delta)$) involves only rates that cause a lower (resp. higher) attractiveness problem on $(\alpha, \beta), (\gamma, \delta)$. Therefore if $L > \delta - \beta$ and $U > \gamma - \alpha$ then there are both a lower and a higher attractiveness problem, and $N(\alpha, \beta, \gamma, \delta)$ is the maximal bad value with respect to (α, γ), (β, δ), while $M(\alpha, \beta, \gamma, \delta)$ is the maximal change of particles (good or bad value) involving all rates, even births on the higher configuration ($\Pi^{0,n}_{\gamma,\delta}$) and deaths on the lower one ($\Pi^{-n,0}_{\alpha,\beta}$), so that $M(\alpha, \beta, \gamma, \delta) \geq N(\alpha, \beta, \gamma, \delta)$.

Remark 2.13 *Hypothesis (2.2.3) implies that (2.2.5), (2.2.6), (2.2.7) and (2.2.8) are finite.*

Definition 2.14 *Given $K \in \mathbb{N}$, $l := \{l_i\}_{i \le K}$, $l_i \in \mathbb{N}$, $m := \{m_i\}_{i \le K}$, $m_i \in \mathbb{N}$, $k := \{k_i\}_{i \le K}$, $k_i \in \mathbb{N}$ such that for each $i < j$, $l_i \le l_j$, $m_i \le m_j$, $k_i \le k_j$ let*

$$I_a := I_a^K(l,m) = \bigcup_{i=1}^K \{k : m_i \ge k > \delta - \beta + l_i\} \tag{2.2.9}$$

$$I_b := I_b^K(l,m) = \bigcup_{i=1}^K \{l : \gamma - \alpha + m_i \ge l > l_i\} \tag{2.2.10}$$

$$I_c := I_c^K(k,m) = \bigcup_{i=1}^K \{l : m_i \ge l > \gamma - \alpha + k_i\} \tag{2.2.11}$$

$$I_d := I_d^K(k,m) = \bigcup_{i=1}^K \{k : \delta - \beta + m_i \ge k > k_i\} \tag{2.2.12}$$

We denote by $\mathcal{S} = \mathcal{S}(\Gamma, R, P, p) = \{\Gamma^l_{\alpha,\beta}, R^{0,k}_{\alpha,\beta}, R^{-k,0}_{\alpha,\beta}, P^{\pm k}_{\beta}, p(x,y)\}_{\{\alpha,\beta \in X, l>0, k>0, (x,y)\in S^2\}}$, the set of parameters that characterize the generator (2.2.1) of the system, and we will identify the system with this set $\mathcal{S}$. An *$\mathcal{S}$ particle system* is a process $(\eta_t)_{t\ge 0}$ whose law is given by generator (2.2.1) and we write $\eta_t \sim \mathcal{S}$.
Let $\mathcal{S} = \mathcal{S}(\Gamma, R, P, p)$ and $\widetilde{\mathcal{S}} = \mathcal{S}(\widetilde{\Gamma}, \widetilde{R}, \widetilde{P}, p)$ be two processes with different transition rates (but the same p). The main result is

Theorem 2.15 *A $\mathcal{S}$ particle system $(\eta_t)_{t\ge 0}$ is* stochastically larger *than a $\widetilde{\mathcal{S}}$ particle system $(\xi_t)_{t\ge 0}$ if and only if for all $(\alpha,\beta),(\gamma,\delta) \in X^2 \times X^2$ with $(\alpha,\beta) \le (\gamma,\delta)$,*

$$\sum_{k>\delta-\beta+l_1} \widetilde{\Pi}^{0,k}_{\alpha,\beta} + \sum_{k\in I_a} \widetilde{\Gamma}^k_{\alpha,\beta} \le \sum_{l>l_1} \Pi^{0,l}_{\gamma,\delta} + \sum_{l \in I_b} \Gamma^l_{\gamma,\delta} \tag{2.2.13}$$

$$\sum_{k>k_1} \widetilde{\Pi}^{-k,0}_{\alpha,\beta} + \sum_{k\in I_d} \widetilde{\Gamma}^k_{\alpha,\beta} \ge \sum_{l>\gamma-\alpha+k_1} \Pi^{-l,0}_{\gamma,\delta} + \sum_{l\in I_c} \Gamma^l_{\gamma,\delta} \tag{2.2.14}$$

for all choices of K, I_a, I_b, I_c, I_d in Definition 2.14.

Remark 2.16 *If $\mathcal{S} = \widetilde{\mathcal{S}}$, Theorem* 2.15 *states necessary and sufficient conditions for attractiveness of $\mathcal{S}$.*

We split the proof of Theorem 2.15 in two parts:

Proposition 2.17 *If the particle system $(\eta_t)_{t\ge 0}$ is stochastically larger than $(\xi_t)_{t\ge 0}$, then for all $(\alpha,\beta),(\gamma,\delta) \in X^2 \times X^2$ with $(\alpha,\beta) \le (\gamma,\delta)$, for all $(x,y) \in S^2$, Conditions* (2.2.13), (2.2.14) *hold.*

The other part of the proof of Theorem 2.15 is harder and is obtained by showing the existence of a Markovian coupling, which appears to be increasing under Conditions (2.2.13), (2.2.14).

We fix two configurations ξ and η such that $\xi \le \eta$ and $\xi_t \sim \widetilde{\mathcal{S}}$, $\eta_t \sim \mathcal{S}$. The first step consists in proving that instead of taking all possible sites, it is enough to consider an ordered pair of sites (x, y) and to construct an increasing coupling concerning some of the rates depending on $\eta(x), \eta(y)$ and $\xi(x), \xi(y)$ (remember that we chose to take births on y, deaths on x and jumps from x to y) and a small part of the independent rates (by this, we mean deaths from x with a rate depending only on $\eta(x)$ and $\xi(x)$ and births upon y with a rate depending only on $\eta(y)$ and $\xi(y)$). We do not have to combine any "dependent reaction" $\widetilde{R}^{\cdot,\cdot}_{\xi(x),\xi(y)}$ or jumps rate $\widetilde{\Gamma}^{\cdot}_{\xi(x),\xi(y)}$ on y with any rate $R^{\cdot,\cdot}_{\eta(z),\eta(y)}$ or $\Gamma^{\cdot}_{\eta(z),\eta(y)}$ if z is different from x.

Definition 2.18 *An ordered pair of sites (x, y) is an attractive pair for $(\widetilde{\mathcal{S}}, \mathcal{S})$ if there exists an increasing coupling for $(\xi_t, \eta_t)_{t\geq 0}$ where $\xi_t \sim \widetilde{\mathcal{S}}(\widetilde{\Gamma}, \widetilde{R}, h(\widetilde{P}), q)$, $\eta_t \sim \mathcal{S}(\Gamma, R, h(P), q)$ and*

$$q(z, w) = \begin{cases} p(x, y) & \text{if } z = x, w = y, \\ 0 & \text{otherwise;} \end{cases}$$

and, for each $\eta \in \Omega$, $k \in \mathbb{N}$,

$$h(P^k_{\eta(z)}) = \begin{cases} P^k_{\eta(y)} p(x, y) & z = y, \\ 0 & \text{otherwise;} \end{cases} \qquad h(P^{-k}_{\eta(z)}) = \begin{cases} P^{-k}_{\eta(x)} p(x, y) & z = x, \\ 0 & \text{otherwise.} \end{cases}$$

For notational convenience we simply call these new systems $\mathcal{S}_{(x,y)}$ and $\widetilde{\mathcal{S}}_{(x,y)}$.

Notice that $\mathcal{S}_{(x,y)} \neq \mathcal{S}_{(y,x)}$, because we take into account births on the second site and deaths on the first one. In the same way we only take particles' jumps from the first site to the second one. However by symmetry and translation invariance the conditions on $\mathcal{S}_{(x,y)}$ and $\mathcal{S}_{(y,x)}$ will be the same. The same remark holds for $\widetilde{\mathcal{S}}_{(x,y)}$.
In other words in order to see if a pair is attractive, we reduce ourselves to a system where we only take part of the rates depending on two sites, and a part of the independent rates depending on $p(x, y)$ ($P^k_{\eta(y)} p(x, y)$ and $P^{-k}_{\eta(x)} p(x, y)$).

Proposition 2.19 *The process $\eta_t \sim \mathcal{S}$ is stochastically larger than $\xi_t \sim \widetilde{\mathcal{S}}$ if and only if all its pairs are attractive pairs for $(\widetilde{\mathcal{S}}, \mathcal{S})$.*

The harder direction consists in proving that if two systems are ordered we are able to construct an increasing coupling for each pair of sites. The final coupling will be formed by superposing couplings for all possible pairs of sites.
From now on, we will work on a fixed pair of sites $(x, y) \in S^2$ and on two configurations $(\eta_t, \xi_t) \in \Omega^2$ for a fixed $t \geq 0$, $\eta_t \sim \mathcal{S}$, $\xi_t \sim \widetilde{\mathcal{S}}$ such that $\xi_t \leq \eta_t$ and

$$\xi_t(x) = \alpha, \quad \xi_t(y) = \beta, \quad \eta_t(x) = \gamma, \quad \eta_t(y) = \delta. \tag{2.2.15}$$

Remark 2.20 *With a slight abuse of notation, since (α, β) involves the lower system and (γ, δ) the upper one, we denote by*

$$\widetilde{\Pi}^{\cdot,\cdot}_{\alpha,\beta} = \Pi^{\cdot,\cdot}_{\alpha,\beta} \qquad \widetilde{\Gamma}^{\cdot}_{\alpha,\beta} = \Gamma^{\cdot}_{\alpha,\beta}$$

and $\mathcal{S} := (\widetilde{\mathcal{S}}, \mathcal{S})$. Indeed, since rates on (α, β) involve $\widetilde{\mathcal{S}}$ and rates on (γ, δ) involve $\mathcal{S}$ we omit the superscript $\sim$ on the lower system rates.

Therefore, when we refer to a coupling, we mean a coupling for this fixed pair (x, y) and with fixed $(\alpha, \beta), (\gamma, \delta)$ defined above. For this reason we write β (or γ) k-bad instead of (β, δ) (or (α, γ)) k-bad and (see Definition 2.12) we call $L := L(\alpha, \beta, \gamma, \delta)$, $U := U(\alpha, \beta, \gamma, \delta)$, $N := N(\alpha, \beta, \gamma, \delta)$ and $M := M(\alpha, \beta, \gamma, \delta)$. We drop in $p(x, y)$ the dependence on (x, y), and simply write $p(x, y) =: p$.

For a given configuration, we call "higher birth rate" the positive birth rate giving the maximal number of births, "higher death rate" the positive death rate giving the maximal number of deaths, "higher jump rate" the positive jump rate giving the maximal number of jumps. We speak of "higher rate", if we are not interested in making any distinction between births/deaths or jumps.
Our purpose is to construct an increasing coupling under Conditions (2.2.13) – (2.2.14).

We define a coupling rate that moves both processes only if we are dealing with an attractiveness problem, otherwise we let the two processes move independently through uncoupled rates. Since $W(\alpha,\beta)$ is finite, we can construct the coupling by a downwards recursion on k, starting from M on rates that do not give attractiveness problems and from N otherwise.

Suppose $L = U = N$ and that there is a lower attractiveness problem on β, then an addition of N particles upon β breaks the partial order, that is $\beta + N > \delta$. Such a problem could come both from births ($\Pi^{0,N}_{\alpha,\beta}p$) and from jumps ($\Gamma^{N}_{\alpha,\beta}p$) rates. The coupling construction takes place in three main steps. We just give an idea of these steps and we will investigate them better going on.

Step 1) We begin from jump rates: in order to "solve the problem", we could couple $\Gamma^{N}_{\alpha,\beta}p$ with either a jump or a birth rate of the upper configuration. *We give priority to jump rates.* We first couple it with $\Gamma^{N}_{\gamma,\delta}p$, because $\alpha \leq \gamma$, $\alpha - N \leq \gamma - N$ and the final pairs of values are $(\alpha - N, \beta + N) \leq (\gamma - N, \delta + N)$.
Then while the attractiveness problem is not solved, we go on by coupling the remainder of the lower configuration jump rate with the upper configuration jump rate with a decreasing change of particles $l \leq N - 1$, that is $\Gamma^{l}_{\gamma,\delta}p$ at the l^{th} step. The final pairs of values we reach are $(\alpha - N, \beta + N), (\gamma - l, \delta + l)$, which preserves the partial order on x since $\alpha - N \leq \gamma - l$ when $l \leq N$, but not necessarily on y, since $\beta + N \leq \delta + l$ only if $l \geq N - \delta + \beta$. For this reason, we stop the coupling between jumps at the $(N - \delta + \beta)^{th}$ step. We will come back to jump rates later on, on Step 3. Notice that after Step 1 we are left with only remaining parts of the involved rates.
If γ is N-bad in order to solve the higher attractiveness problem given by $\Gamma^{N}_{\gamma,\delta}p$ we construct the coupling rates involving it in a symmetric way.

Step 2) We couple birth and death rates:

Step 2a) Another rate that causes an attractiveness problem is $\Pi^{0,N}_{\alpha,\beta}p$. We cannot couple it with the remaining part of a jump rate $\Gamma^{l}_{\gamma,\delta}p$ with $l > \gamma - \alpha$ from Step 1, otherwise we would reach $(\alpha, \beta + N), (\gamma - l, \delta + l)$ such that $\gamma - l < \alpha$, which does not preserve the partial order. So *we choose* to couple it with the births rate of the upper configuration with larger change of particles, $\Pi^{0,M}_{\gamma,\delta}p$ and going down we couple it with $\Pi^{0,l}_{\gamma,\delta}p$ at l^{th}-step, until $l = \gamma - \alpha + 1$.
We do not impose restrictions as in Step 1, that is we do not stop when $l = N - \delta + \beta$. Thanks to attractiveness (which will follow from Conditions (2.2.13) – (2.2.14)), we will see that we cannot reach pairs of values breaking the partial order $\beta + N \leq \delta + l$.

Step 2b) When $l \leq \gamma - \alpha$, we can use both $\Pi^{0,l}_{\gamma,\delta}p$ and the remainder of $\Gamma^{l}_{\gamma,\delta}p$ from Step 1. *We give priority to l-births rate, but after that we couple immediately with the l-jumps rate*, that is first $\Pi^{0,\gamma-\alpha}_{\gamma,\delta}p$, then the remainder of $\Gamma^{\gamma-\alpha}_{\gamma,\delta}p$ from Step 1, then $\Pi^{0,\gamma-\alpha-1}_{\gamma,\delta}p$, then

the remainder of $\Gamma^{\gamma-\alpha-1}_{\gamma,\delta}p$ and so on.
If γ is N-bad we construct the coupling terms involving $\Pi^{-N,0}_{\gamma,\delta}p$ in order to solve the higher attractiveness problem given by this rate in a symmetric way.

Step 3) We come back to Step 1, where even if the remaining part of $\Gamma^{N}_{\alpha,\beta}p$ was still positive at step $N-\delta+\beta$, we decided to stop. We use the upper configuration reaction rate remaining from Step 2 in order to solve the attractiveness problem: we couple the remainder from Step 1 of $\Gamma^{N}_{\alpha,\beta}p$ with the remainder from Step 2 of $\Pi^{0,M}_{\gamma,\delta}p$, then with the remainder of $\Pi^{0,M-1}_{\gamma,\delta}p$ and so on downwards in l with $\Pi^{0,l}_{\gamma,\delta}p$. As in Step 2, the pairs of values $(\alpha-N,\beta+N),(\gamma,\delta+l)$ breaking the partial order (that is such that $\beta+N>\delta+l$) that we could reach a priori will be avoided by attractiveness.
If γ is N-bad and we have a remainder of $\Gamma^{N}_{\gamma,\delta}p$ from Step 1, we proceed in a symmetric way.

Once we constructed all the coupling rates involving $\Gamma^{N}_{\alpha,\beta}p$, $\Pi^{0,N}_{\alpha,\beta}p$, $\Gamma^{N}_{\gamma,\delta}p$ and $\Pi^{-N,0}_{\gamma,\delta}p$ we move to rates involving less than N particles. If β is $(N-1)$-bad (and/or γ is $(N-1)$-bad) in order to solve the lower (higher) attractiveness problems given by $\Gamma^{N-1}_{\alpha,\beta}p$ and $\Pi^{0,N-1}_{\alpha,\beta}p$ ($\Gamma^{N-1}_{\gamma,\delta}p$ and $\Pi^{-(N-1),0}_{\gamma,\delta}p$) (or their remainders from previous steps) we repeat Step 1, Step 2 and Step 3 for such terms. We proceed this way with the remainders of $\Gamma^{k}_{\alpha,\beta}p$ and $\Pi^{0,k}_{\alpha,\beta}p$ ($\Gamma^{l}_{\gamma,\delta}p$ and $\Pi^{-l,0}_{\gamma,\delta}p$) going downwards with respect to k (l) until (β,δ) is k-good ((α,γ) is l-good). We call the part of Step 1 (resp. 2 or 3) involving rates $\Gamma^{k}_{\alpha,\beta}p$ or $\Pi^{0,k}_{\alpha,\beta}p$ when (β,δ) is k-bad for $k\le M$ *Lower Step* 1 (resp. 2 and 3) and the symmetric one when (α,γ) is l-bad *Upper Step* 1 (resp. 2 or 3).
If $N=L>U$ (resp $N=U>L$) we begin the coupling construction from Lower Step 1 by $\Gamma^{L}_{\alpha,\beta}p$ (resp. Upper Step 1 by $\Gamma^{U}_{\gamma,\delta}p$). Then since $\Gamma^{L}_{\gamma,\delta}p=0$ we skip Upper Step 1 and we pass to Lower Step 2. Since $\Pi^{-L,0}_{\gamma,\delta}p=0$ we skip Upper Step 2 and we pass to Lower Step 3 and so on until we reach the first positive rate giving a higher attractiveness problem.

We couple an upper configuration jump rate $\Gamma^{l}_{\gamma,\delta}p$ with a lower configuration birth rate $\Pi^{0,k}_{\alpha,\beta}p$ if $\beta+k>l$ and $\alpha\le\gamma-l$ in order to solve a lower attractiveness problem, and with a lower configuration death rate $\Pi^{-k,0}_{\alpha,\beta}p$ if $\alpha>\gamma-l$ in order to solve a higher attractiveness problem: in other words we cannot couple the same higher jump rate both with lower birth and death rates. A symmetric remark holds for lower jump rates.

Our purpose now is to describe a coupling for $(\widetilde{\mathcal{S}}_{(x,y)},\mathcal{S}_{(x,y)})$, that we denote by $\mathcal{H}(x,y)$ (or simply $\mathcal{H}$), which will be increasing under conditions (2.2.13)−(2.2.14). Coupling rates are given below by downwards recursive relations

Definition 2.21

•*If $k\le L$ and $l\le U$,*

$$J^{k,l}_{\alpha,\beta} = \Gamma^{k}_{\alpha,\beta}p-\sum_{l'>l}H^{k,k,l',l'}_{\alpha,\beta,\gamma,\delta}-\sum_{l'>l}H^{k,k,-l',0}_{\alpha,\beta,\gamma,\delta} \tag{2.2.16}$$

$$J^{k,l}_{\gamma,\delta} = \Gamma^{l}_{\gamma,\delta}p-\sum_{k'>k}H^{k',k',l,l}_{\alpha,\beta,\gamma,\delta}-\sum_{k'>k}H^{0,k',l,l}_{\alpha,\beta,\gamma,\delta} \tag{2.2.17}$$

$$H^{k,k,l,l}_{\alpha,\beta,\gamma,\delta} = \begin{cases} J^{k,l}_{\alpha,\beta} \wedge J^{k,l}_{\gamma,\delta} & \textit{if } (\alpha > \gamma - l \textit{ or } \beta + k > \delta) \\ & \textit{and } (\beta + k \leq \delta + l) \textit{ and } (\alpha - k \leq \gamma - l) \\ 0 & \textit{otherwise.} \end{cases} \tag{2.2.18}$$

•*If* $k \leq L$ *and* $l \leq M$,

$$B^{k,l}_{\alpha,\beta} = \Pi^{0,k}_{\alpha,\beta} p - \sum_{l'>l} H^{0,k,0,l'}_{\alpha,\beta,\gamma,\delta} - \sum_{U\wedge(\gamma-\alpha)\geq l'>l} H^{0,k,l',l'}_{\alpha,\beta,\gamma,\delta} \tag{2.2.19}$$

$$B^{k,l}_{\gamma,\delta} = \Pi^{0,l}_{\gamma,\delta} p - \sum_{k'>k} H^{0,k',0,l}_{\alpha,\beta,\gamma,\delta} - \sum_{k'>k} H^{k',k',0,l}_{\alpha,\beta,\gamma,\delta} \tag{2.2.20}$$

$$H^{0,k,0,l}_{\alpha,\beta,\gamma,\delta} = \begin{cases} B^{k,l}_{\alpha,\beta} \wedge B^{k,l}_{\gamma,\delta} & \textit{if } \beta + k > \delta \\ 0 & \textit{otherwise} \end{cases} \tag{2.2.21}$$

$$H^{0,k,l,l}_{\alpha,\beta,\gamma,\delta} = \begin{cases} [B^{k,l}_{\alpha,\beta} - B^{k,l}_{\alpha,\beta} \wedge B^{k,l}_{\gamma,\delta}] \wedge [J^{k,l}_{\gamma,\delta} - J^{k,l}_{\alpha,\beta} \wedge J^{k,l}_{\gamma,\delta}] & \textit{if } \beta + k > \delta \\ & \textit{and } \alpha \leq \gamma - l \\ 0 & \textit{otherwise.} \end{cases} \tag{2.2.22}$$

•*If* $k \leq M$ *and* $l \leq U$,

$$D^{k,l}_{\alpha,\beta} = \Pi^{-k,0}_{\alpha,\beta} p - \sum_{l'>l} H^{-k,0,-l',0}_{\alpha,\beta,\gamma,\delta} - \sum_{l'>l} H^{-k,0,l',l'}_{\alpha,\beta,\gamma,\delta} \tag{2.2.23}$$

$$D^{k,l}_{\gamma,\delta} = \Pi^{-l,0}_{\gamma,\delta} p - \sum_{k'>k} H^{-k',0,-l,0}_{\alpha,\beta,\gamma,\delta} - \sum_{(\delta-\beta)\wedge L\geq k'>k} H^{k',k',-l,0}_{\alpha,\beta,\gamma,\delta} \tag{2.2.24}$$

$$H^{-k,0,-l,0}_{\alpha,\beta,\gamma,\delta} = \begin{cases} D^{k,l}_{\alpha,\beta} \wedge D^{k,l}_{\gamma,\delta} & \textit{if } \alpha > \gamma - l \\ 0 & \textit{otherwise} \end{cases} \tag{2.2.25}$$

$$H^{k,k,-l,0}_{\alpha,\beta,\gamma,\delta} = \begin{cases} [J^{k,l}_{\alpha,\beta} - J^{k,l}_{\alpha,\beta} \wedge J^{k,l}_{\gamma,\delta}] \wedge [D^{k,l}_{\gamma,\delta} p - D^{k,l}_{\alpha,\beta} \wedge D^{k,l}_{\gamma,\delta}] & \textit{if } \alpha > \gamma - l \\ & \textit{and } \beta + k \leq \delta \\ 0 & \textit{otherwise.} \end{cases} \tag{2.2.26}$$

•*If* $k \leq L$ *and* $l \leq M$,

$$T^{k,l}_{\alpha,\beta} = \Gamma^{k}_{\alpha,\beta} p - \sum_{l'\geq k-\delta+\beta} H^{k,k,l',l'}_{\alpha,\beta,\gamma,\delta} - \sum_{l'>l} H^{k,k,0,l'}_{\alpha,\beta,\gamma,\delta} \tag{2.2.27}$$

$$H^{k,k,0,l}_{\alpha,\beta,\gamma,\delta} = \begin{cases} T^{k,l}_{\alpha,\beta} \wedge [B^{k,l}_{\gamma,\delta} - B^{k,l}_{\alpha,\beta} \wedge B^{k,l}_{\gamma,\delta}] & \textit{if } \beta + k > l \\ 0 & \textit{otherwise.} \end{cases} \tag{2.2.28}$$

•*If* $k \leq M$ *and* $l \leq U$,

$$T^{k,l}_{\gamma,\delta} = \Gamma^{l}_{\gamma,\delta} p - \sum_{k'\geq l-\gamma+\alpha} H^{k',k',l,l}_{\alpha,\beta,\gamma,\delta} - \sum_{k'>k} H^{-k',0,l,l}_{\alpha,\beta,\gamma,\delta} \tag{2.2.29}$$

$$H^{-k,0,l,l}_{\alpha,\beta,\gamma,\delta} = \begin{cases} [D^{k,l}_{\alpha,\beta} - D^{k,l}_{\alpha,\beta} \wedge D^{k,l}_{\gamma,\delta}] \wedge T^{k,l}_{\gamma,\delta} & \textit{if } \alpha > \gamma - l \\ 0 & \textit{otherwise.} \end{cases} \tag{2.2.30}$$

The uncoupled terms are

$$i)\quad H^{k,k,0,0}_{\alpha,\beta,\gamma,\delta} = \begin{cases} T^{k,0}_{\alpha,\beta} & \text{if } \beta+k>\delta,\\ J^{k,0}_{\alpha,\beta} & \text{otherwise;}\end{cases} \qquad ii)\quad H^{0,0,l,l}_{\alpha,\beta,\gamma,\delta} = \begin{cases} T^{0,l}_{\gamma,\delta} & \text{if } \gamma-l<\alpha,\\ J^{0,l}_{\gamma,\delta} & \text{otherwise;}\end{cases} \tag{2.2.31}$$

$$i)\quad H^{0,k,0,0}_{\alpha,\beta,\gamma,\delta} = B^{k,0}_{\alpha,\beta}; \qquad ii)\quad H^{0,0,0,l}_{\alpha,\beta,\gamma,\delta} = B^{0,l}_{\gamma,\delta}; \tag{2.2.32}$$

$$i)\quad H^{-k,0,0,0}_{\alpha,\beta,\gamma,\delta} = D^{-k,0}_{\alpha,\beta}; \qquad ii)\quad H^{0,0,-l,0}_{\alpha,\beta,\gamma,\delta} = D^{-l,0}_{\gamma,\delta}. \tag{2.2.33}$$

The notation J means jump, B birth, D death and T third, since this term is positive only if we arrive to Step 3 of the coupling construction.

Remark 2.22 *By Definition* (2.2.18) *coupling terms between jumps such that the final pairs of states break the partial order are null.*

$$H^{k,k,l,l}_{\alpha,\beta,\gamma,\delta} = 0 \qquad \text{if } (\beta+k>\delta+l) \text{ or } (\alpha-k>\gamma-l). \tag{2.2.34}$$

By Definition (2.2.22) *(resp.* (2.2.26)*) coupling terms between lower births (resp. upper deaths) and upper (resp. lower) jumps such that the final pairs of states break the partial order are null.*

$$H^{0,k,l,l}_{\alpha,\beta,\gamma,\delta} = 0 \qquad \text{if } \alpha>\gamma-l; \tag{2.2.35}$$

$$H^{k,k,-l,0}_{\alpha,\beta,\gamma,\delta} = 0 \qquad \text{if } \beta+k>\delta. \tag{2.2.36}$$

We will explain how this coupling construction works at the end of this section; see also in Example 2.3.5 the easier case $M=1$. We complete the proof of Theorem 2.15 by

Proposition 2.23 *Under Conditions (*2.2.13*)–(*2.2.14*), $\mathcal{H}$ is increasing.*

In order to prove Proposition 2.23, we define a new system $\overline{\mathcal{S}}$ (in fact a new pair of systems $\overline{\mathcal{S}} := (\widetilde{\overline{\mathcal{S}}}, \overline{\mathcal{S}})$ by Remark 2.20), depending on $\mathcal{S}$ and $\mathcal{H}$, whose rates are those of $\mathcal{S}$ to whom we subtract the coupled rates of $\mathcal{H}$ involving the higher addition or subtraction of L or U particles.

Definition 2.24 *Given $\mathcal{S}$, we define a new pair of systems $\overline{\mathcal{S}}$, through the transition rates*

$$\overline{\Gamma}^k_{\alpha,\beta}p = \begin{cases} \Gamma^L_{\alpha,\beta}p - \sum_{l>0} H^{L,L,l,l}_{\alpha,\beta,\gamma,\delta} - \sum_{l>0} H^{L,L,0,l}_{\alpha,\beta,\gamma,\delta} & k=L \text{ and } \beta \text{ is } L\text{-bad},\\ \Gamma^k_{\alpha,\beta}p - H^{k,k,U,U}_{\alpha,\beta,\gamma,\delta} - H^{k,k,-U,0}_{\alpha,\beta,\gamma,\delta} & (k<L \text{ and } \gamma \text{ is } U\text{-bad})\\ & \text{ or } (k=L \text{ and } \beta \text{ is } L\text{-good}),\\ \Gamma^k_{\alpha,\beta}p & \text{otherwise;}\end{cases}$$

$$\overline{\Gamma}^l_{\gamma,\delta}p = \begin{cases} \Gamma^U_{\gamma,\delta}p - \sum_{k>0} H^{k,k,U,U}_{\alpha,\beta,\gamma,\delta} - \sum_{k>0} H^{-k,0,U,U}_{\alpha,\beta,\gamma,\delta} & l=U \text{ and } \gamma \text{ is } U\text{-bad},\\ \Gamma^l_{\gamma,\delta}p - H^{L,L,l,l}_{\alpha,\beta,\gamma,\delta} - H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta} & (l<U \text{ and } \beta \text{ is } L\text{-bad})\\ & \text{ or } (l=U \text{ and } \gamma \text{ is } U\text{-good}),\\ \Gamma^l_{\gamma,\delta}p & \text{otherwise;}\end{cases}$$

$$\overline{\Pi}^{0,k}_{\alpha,\beta}p = \begin{cases} \Pi^{0,L}_{\alpha,\beta}p - \sum_{l>0} H^{0,L,0,l}_{\alpha,\beta,\gamma,\delta} - \sum_{l>0} H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta} & k = L, \\ \Pi^{0,k}_{\alpha,\beta}p & \textit{otherwise;} \end{cases}$$

$$\overline{\Pi}^{0,l}_{\gamma,\delta}p = \Pi^{0,l}_{\gamma,\delta}p - H^{0,L,0,l}_{\alpha,\beta,\gamma,\delta} - H^{L,L,0,l}_{\alpha,\beta,\gamma,\delta} \qquad \textit{for each } l;$$

$$\overline{\Pi}^{-l,0}_{\gamma,\delta}p = \begin{cases} \Pi^{-U,0}_{\gamma,\delta}p - \sum_{k>0} H^{-k,0,-U,0}_{\alpha,\beta,\gamma,\delta} - \sum_{k>0} H^{k,k,-U,0}_{\alpha,\beta,\gamma,\delta} & l = U, \\ \Pi^{-l,0}_{\gamma,\delta}p & \textit{otherwise;} \end{cases}$$

$$\overline{\Pi}^{-k,0}_{\alpha,\beta}p = \Pi^{-k,0}_{\alpha,\beta}p - H^{-k,0,-U,0}_{\alpha,\beta,\gamma,\delta} - H^{-k,0,U,U}_{\alpha,\beta,\gamma,\delta} \qquad \textit{for each } k.$$

Our plan consists in working by induction on N, L or U according to the situation. Denote by

$$\overline{L} = L(\alpha,\beta,\gamma,\delta,\overline{\mathcal{S}}); \qquad \overline{U} = U(\alpha,\beta,\gamma,\delta,\overline{\mathcal{S}});$$
$$\overline{N} = N(\alpha,\beta,\gamma,\delta,\overline{\mathcal{S}}); \qquad \overline{M} = M(\alpha,\beta,\gamma,\delta,\overline{\mathcal{S}}).$$

In order to prove that Proposition 2.23 holds for a system $\mathcal{S}$ with $N = n$ (resp. $L = n$ or $U = n$), we assume that Conditions (2.2.13)–(2.2.14) are satisfied and that Proposition 2.23 holds for each system $\widetilde{\mathcal{S}}$ such that $N(\widetilde{\mathcal{S}}) \leq n-1$. We prove that if Conditions (2.2.13)–(2.2.14) are satisfied on $\mathcal{S}$ then they hold on $\overline{\mathcal{S}}$: by induction hypothesis, if $\overline{N} \leq n-1$, we get that $\mathcal{H}(\overline{\mathcal{S}})$ is increasing from which we will deduce that $\mathcal{H}(\mathcal{S})$ is also increasing. In other words we work by induction on the higher rate which causes an attractiveness problem by defining at each step a new system and, if the induction basis is satisfied, we get the result by the induction step as explained.
Since we need that $\overline{N} \leq n-1$ (resp. $\overline{L} \leq n-1$ or $\overline{U} \leq n-1$) the following propositions allow us to work by induction.

Proposition 2.25 *i) If β is L-bad, then $\overline{L} \leq L-1$.*
ii) If γ is U-bad, then $\overline{U} \leq U-1$.

The harder part is:

Proposition 2.26 *If $\mathcal{S}$ satisfies Conditions* (2.2.13)–(2.2.14)*, then the system $\overline{\mathcal{S}}$ satisfies Conditions* (2.2.13)–(2.2.14)*.*

Proposition 2.27 *If $\overline{\mathcal{H}} := \mathcal{H}(\overline{\mathcal{S}})$ is increasing, then $\mathcal{H}(\mathcal{S})$ is increasing.*

Finally, the first step of the induction is given by Lemmas 2.28, 2.29 below.

• By Definition 2.12, if both $L = \delta - \beta$ and $U = \gamma - \alpha$, then there is no attractiveness problem; the coupling $\mathcal{H}$ is the *independent* coupling, it consists in uncoupled terms and it is increasing.
• If $U = \gamma - \alpha$ (that is γ is U-good, which implies that γ is $(U-1)$-good) but $L > \delta - \beta$, then there is only a lower attractiveness problem, $\overline{L} \leq L-1$ by Proposition 2.25 *i)* and we work by induction on L: by Proposition 2.26 and the induction hypothesis $\overline{\mathcal{H}}$ is increasing

and the claim follows from Proposition 2.27. The induction basis consists in proving the result for a system with $L_0 = \delta - \beta + 1$ (that is $\beta + L_0 = \delta + 1$, the minimal lower attractiveness problem) and no higher problem (see Lemma 2.28 *ii*)).
• If $L = \delta - \beta$ (that is β is L-good, which implies that β is $(L-1)$-good) but $U > \gamma - \alpha$, then there is only a higher attractiveness problem, $\overline{U} \le U - 1$ by Proposition 2.25 *ii*) and we work by induction on U as in previous case by using Propositions 2.26, 2.27 and the induction hypothesis. The induction basis consists in proving the result for a system with $U_0 = \gamma - \alpha + 1$ (that is $\gamma - U_0 = \alpha - 1$, the minimal higher attractiveness problem) and no lower problem (see Lemma 2.28 *iii*)).
• Suppose $L > \delta - \beta$ and $U > \gamma - \alpha$: we have both a lower and a higher attractiveness problem, $\overline{N} = \max\{\overline{L}, \overline{U}\} \le \max\{L-1, U-1\} = N - 1$ by Proposition 2.25 and we work by induction on N. We use the induction hypothesis as in previous cases, and we proceed downwards with respect to N by defining new attractive systems until we reach one of the following cases in the final system $\mathcal{S}_0$:

- If $L_0 = \delta - \beta + 1$ and $U_0 = \gamma - \alpha + 1$, then $N_0 = (\delta - \beta + 1) \vee (\gamma - \alpha + 1)$ and the induction basis is given by Lemma 2.28 *i*);
- If $L_0 > \delta - \beta$ and $U_0 = \gamma - \alpha$, that is we reach a system without higher attractiveness problem, then the induction basis is given by Lemma 2.29 *i*);
- If $L_0 = \delta - \beta$ and $U_0 > \gamma - \alpha$, that is we reach a system without lower attractiveness problem, then the induction basis is given by Lemma 2.29 *ii*).

Lemma 2.28 *i) If $L = \delta - \beta + 1$ and $U = \gamma - \alpha + 1$, Conditions* (2.2.13), (2.2.14) *reduce to*

$$\Pi^{0,L}_{\alpha,\beta} + \Gamma^L_{\alpha,\beta}1\!\!1_{\{L\in I_a\}} \le \sum_{l>0} \Pi^{0,l}_{\gamma,\delta} + \sum_{\gamma-\alpha+m_1\ge l>0} \Gamma^l_{\gamma,\delta} \tag{2.2.37}$$

$$\sum_{k>0} \Pi^{-k,0}_{\alpha,\beta} + \sum_{\delta-\beta+m_1\ge k>0} \Gamma^k_{\alpha,\beta} \ge \Pi^{-U,0}_{\gamma,\delta} + \Gamma^U_{\gamma,\delta}1\!\!1_{\{U\in I_c\}} \tag{2.2.38}$$

for each m_1. Under conditions (2.2.37)–(2.2.38)*, $\mathcal{H}$ is an increasing coupling.*
ii) If $L = \delta - \beta + 1$, and $\alpha \le \gamma - U$, Conditions (2.2.13)–(2.2.14) *reduce to*

$$\Pi^{0,L}_{\alpha,\beta} + \Gamma^L_{\alpha,\beta} \le \sum_{l>0} \Pi^{0,l}_{\gamma,\delta} + \sum_{l>0} \Gamma^l_{\gamma,\delta} \tag{2.2.39}$$

*Under condition (*2.2.39*), $\mathcal{H}$ is an increasing coupling.*
iii) If $U = \gamma - \alpha + 1$, and $\beta + L \le \delta$, Conditions (2.2.13)–(2.2.14) *reduce to*

$$\sum_{k>0} \Pi^{-k,0}_{\alpha,\beta} + \sum_{k>0} \Gamma^k_{\alpha,\beta} \ge \Pi^{-U,0}_{\gamma,\delta} + \Gamma^U_{\gamma,\delta} \tag{2.2.40}$$

*Under condition (*2.2.40*), $\mathcal{H}$ is an increasing coupling.*

Lemma 2.29 *i) If $\beta + L > \delta$ and $\alpha \le \gamma - U$, under Condition* (2.2.13) *$\mathcal{H}$ is an increasing coupling.*
ii) If $\beta + L \le \delta$ and $\alpha > \gamma - U$, under Condition (2.2.14) *$\mathcal{H}$ is an increasing coupling.*

In the last part of this section we explain how the coupling in Definition 2.21 works. We begin by writing down explicitely the terms concerning the maximal sizes M and N for change. These terms will be important in order to prove stochastic order and attractiveness. We suppose $L = U = N$. Notice that by definition of M, all coupling terms concerning addition or subtraction of a number of particles bigger than M are null.

Step 1) If $k \le N$, $l \le N$, by (2.2.16), (2.2.17) and (2.2.18)

$$H^{N,N,N,N}_{\alpha,\beta,\gamma,\delta} = (\Gamma^N_{\alpha,\beta} \wedge \Gamma^N_{\gamma,\delta})p \qquad \text{if } \beta + N > \delta \text{ or } \gamma - N < \alpha \tag{2.2.41}$$

$$H^{N,N,l,l}_{\alpha,\beta,\gamma,\delta} = [\Gamma^N_{\alpha,\beta}p - \sum_{l'>l} H^{N,N,l',l'}_{\alpha,\beta,\gamma,\delta}] \wedge [\Gamma^l_{\gamma,\delta}p] \qquad \text{if } \delta < \beta + N \le \delta + l \text{ or } \gamma - l < \alpha \tag{2.2.42}$$

$$H^{k,k,N,N}_{\alpha,\beta,\gamma,\delta} = [\Gamma^k_{\alpha,\beta}p] \wedge [\Gamma^N_{\gamma,\delta}p - \sum_{k'>k} H^{k',k',N,N}_{\alpha,\beta,\gamma,\delta}] \qquad \text{if } \alpha - k \le \gamma - N < \alpha \text{ or } \beta + k > \delta. \tag{2.2.43}$$

Otherwise $H^{N,N,l,l}_{\alpha,\beta,\gamma,\delta} = H^{k,k,N,N}_{\alpha,\beta,\gamma,\delta} = 0$. Notice that in (2.2.41) conditions $\alpha - N \le \gamma - N$ and $\beta + N \le \delta + N$ are always satisfied since $(\alpha, \beta) \le (\gamma, \delta)$; in (2.2.42) condition $\alpha - N \le \gamma - l$ is always satisfied since $l \le N$; in (2.2.43) condition $\beta + k \le \delta + N$ is always satisfied since $k \le N$. Such conditions are not always true in case $L \ne U$ (see Remark 2.31).
If β is N-bad, we couple the N-particles jump rates by taking the minimum between $\Gamma^N_{\alpha,\beta}p$ and $\Gamma^N_{\gamma,\delta}p$ (Formula (2.2.41)). After that, if the remainder of $\Gamma^N_{\alpha,\beta}p$ is positive, we proceed recursively downwards by coupling it with $\Gamma^l_{\gamma,\delta}p$ when $\beta + N \le \delta + l$, that is when the final pair does not break the partial order (Formula (2.2.42)).
More precisely, looking for instance at (2.2.41), (2.2.42) when $\Gamma^N_{\alpha,\beta} > \Gamma^N_{\gamma,\delta}$ and $\beta + N \le \delta + N - 1$, we get the first coupling rates

$$H^{N,N,N,N}_{\alpha,\beta,\gamma,\delta} = \Gamma^N_{\gamma,\delta}p; \qquad H^{N,N,N-1,N-1}_{\alpha,\beta,\gamma,\delta} = (\Gamma^N_{\alpha,\beta} - \Gamma^N_{\gamma,\delta})p \wedge \Gamma^{N-1}_{\gamma,\delta}p.$$

Then either $H^{N,N,N-1,N-1}_{\alpha,\beta,\gamma,\delta} = (\Gamma^N_{\alpha,\beta} - \Gamma^N_{\gamma,\delta})p$, and with the two steps $l = N$ and $l = N - 1$ we have no remaining part of $\Gamma^N_{\alpha,\beta}p$, or $H^{N,N,N-1,N-1}_{\alpha,\beta,\gamma,\delta} = \Gamma^{N-1}_{\gamma,\delta}p$ and we are left with a remainder $(\Gamma^N_{\alpha,\beta} - \Gamma^N_{\gamma,\delta} - \Gamma^{N-1}_{\gamma,\delta})p$, to go on with $l = N - 2, \ldots$ until either we have no remainder of $\Gamma^N_{\alpha,\beta}p$ or we have reached $l = \beta + N - \delta$ with the remainder $\Gamma^N_{\alpha,\beta}p - \sum_{l' \ge N-\delta+\beta} \Gamma^l_{\gamma,\delta}p$.
Suppose γ is N-bad. Then in a symmetric way we couple the jump rate $\Gamma^N_{\gamma,\delta}p$ with $\Gamma^N_{\alpha,\beta}p$ if we have not still done it (that is if β is not N-bad) and we proceed downwards in k coupling the remainder of $\Gamma^N_{\gamma,\delta}p$ with $\Gamma^k_{\alpha,\beta}p$ until $k \le N - \gamma + \alpha$ (Formulas (2.2.41) and (2.2.43)).

Remark 2.30 *The coupling rates* (2.2.42) *and* (2.2.43) *are minima of expressions involving only sums of rates* $\Gamma^{\cdot}_{\cdot,\cdot}p$.

Remark 2.31 *If* $N = L > U$, *we proceed in the same way by following Definition* 2.21. *We get* $\Gamma^N_{\alpha,\beta}p \wedge \Gamma^N_{\gamma,\delta}p = \Gamma^L_{\alpha,\beta}p \wedge 0 = 0$. *We have* $\alpha - L \le \gamma - U$ *because* $L > U$.
• *If* $\beta + L \le \gamma + U$, *the first positive coupling term* $H^{L,L,l,l}_{\alpha,\beta,\gamma,\delta}$ *is* $\Gamma^L_{\alpha,\beta}p \wedge \Gamma^U_{\gamma,\delta}p$, *it is between jumps and involves the higher jump rates on the lower and on the upper configuration.*
• *If* $\beta + L > \delta + U$, *we cannot couple a jump of* L *particles upon* β *with a jump of* U *or less particles upon* δ, *and coupling terms involving* $\Gamma^L_{\alpha,\beta}p$ *appear only in Step* 3; *if we look at* $\Gamma^U_{\gamma,\delta}p$, *since* $\beta + L > \delta + U$ *by Definition* 2.21 *Formulas* (2.2.16), (2.2.17) *and* (2.2.18)

the terms $H^{k,k,U,U}_{\alpha,\beta,\gamma,\delta}$ are null until $\beta+k\le\delta+U$. The first positive coupling term in Upper Step 1 is $H^{k,k,U,U}_{\alpha,\beta,\gamma,\delta}$, where $k=U+\delta-\beta$. To summarize, let

$$\bar{k}:=L\wedge(U+\delta-\beta) \tag{2.2.44}$$

then $H^{\bar{k},\bar{k},U,U}_{\alpha,\beta,\gamma,\delta}$ is the first positive coupling term between jumps involving $\Gamma^{U}_{\gamma,\delta}p$.
Symmetric arguments hold in case $L<U$. Let

$$\bar{l}:=U\wedge(L+\gamma-\alpha) \tag{2.2.45}$$

then $H^{L,L,\bar{l},\bar{l}}_{\alpha,\beta,\gamma,\delta}$ is the first positive coupling term between jumps involving $\Gamma^{L}_{\alpha,\beta}p$.
In any case the construction starting from N holds and implies these remarks.

Step 2) (Birth rates). If $\beta+N>\delta$, by (2.2.19), (2.2.20) and (2.2.21)

$$H^{0,N,0,M}_{\alpha,\beta,\gamma,\delta}=(\Pi^{0,N}_{\alpha,\beta}\wedge\Pi^{0,M}_{\gamma,\delta})p \tag{2.2.46}$$

$$H^{0,N,0,l}_{\alpha,\beta,\gamma,\delta}=[\Pi^{0,N}_{\alpha,\beta}p-\sum_{l'>l}H^{0,N,0,l'}_{\alpha,\beta,\gamma,\delta}-\sum_{\gamma-\alpha\ge l'>l}H^{0,N,l',l'}_{\alpha,\beta,\gamma,\delta}]\wedge[\Pi^{0,l}_{\gamma,\delta}p]. \tag{2.2.47}$$

Otherwise $H^{0,N,0,l}_{\alpha,\beta,\gamma,\delta}=0$. If $\beta+N>\delta$, by (2.2.19), (2.2.20) and (2.2.22)

$$H^{0,N,N,N}_{\alpha,\beta,\gamma,\delta}=[\Pi^{0,N}_{\alpha,\beta}p-\sum_{l'\ge N}H^{0,N,0,l'}_{\alpha,\beta,\gamma,\delta}]\wedge[\Gamma^{N}_{\gamma,\delta}p-H^{N,N,N,N}_{\alpha,\beta,\gamma,\delta}]\qquad\text{if }\gamma-N\ge\alpha \tag{2.2.48}$$

$$H^{0,N,l,l}_{\alpha,\beta,\gamma,\delta}=[\Pi^{0,N}_{\alpha,\beta}p-\sum_{l'\ge l}H^{0,N,0,l'}_{\alpha,\beta,\gamma,\delta}-\sum_{\gamma-\alpha\ge l'>l}H^{0,N,l',l'}_{\alpha,\beta,\gamma,\delta}]\wedge[\Gamma^{l}_{\gamma,\delta}p-H^{N,N,l,l}_{\alpha,\beta,\gamma,\delta}]\qquad\text{if }\gamma-l\ge\alpha. \tag{2.2.49}$$

In other cases, $H^{0,N,l,l}_{\alpha,\beta,\gamma,\delta}=0$.

Step 2a) (Birth rates). If β is N-bad, we couple the N-births rate $\Pi^{0,N}_{\alpha,\beta}p$ with the M-births rate $\Pi^{0,M}_{\gamma,\delta}p$ through (2.2.46). If $\Pi^{0,M}_{\gamma,\delta}p<\Pi^{0,N}_{\alpha,\beta}p$, we have to couple with another rate. If $N<M$ there is no jump rate involving M particles (see Definitions (2.2.7) and (2.2.8)), hence we proceed downwards by coupling with $\Pi^{0,l}_{\gamma,\delta}p$ with $l\le M$ until either the problem is solved or we have coupled with $\Pi^{0,l}_{\gamma,\delta}p$ with $l=N$ (Formula (2.2.47) when $l\ge N$). If after coupling with $\Pi^{0,N}_{\gamma,\delta}p$ the remaining birth rate $\Pi^{0,N}_{\alpha,\beta}p-\sum_{l'\ge N}H^{0,N,0,l'}_{\alpha,\beta,\gamma,\delta}$ is still positive, we could also couple it with the remainder of the jump rate $\Gamma^{N}_{\gamma,\delta}p$, but only if the final pairs of states do not break the partial order, that is if $\gamma-N\ge\alpha$ (Formula (2.2.48)). Indeed if $\gamma-N<\alpha$ we would reach a configuration such that $\eta(x)=\gamma-N<\alpha=\xi(x)$: hence we do not couple with $\Gamma^{N}_{\gamma,\delta}p$ and the next rate we use is $\Pi^{0,N-1}_{\gamma,\delta}p$. We proceed downwards in l by coupling with $\Pi^{0,l}_{\gamma,\delta}p$ when $l\le N-1$ until $\gamma-l=\alpha$ (Formula (2.2.47) when $l>\gamma-\alpha$).

Step 2b)(Birth rates). If $l=\gamma-\alpha$, we take the minimum between $\Pi^{0,\gamma-\alpha}_{\gamma,\delta}p$ and $\Pi^{0,N}_{\alpha,\beta}p-\sum_{l>\gamma-\alpha}H^{0,N,0,l}_{\alpha,\beta,\gamma,\delta}$. After that, we couple the remainder of $\Pi^{0,N}_{\alpha,\beta}p$ with the remainder of $\Gamma^{\gamma-\alpha}_{\gamma,\delta}p$, that is $\Gamma^{\gamma-\alpha}_{\gamma,\delta}p-H^{N,N,\gamma-\alpha,\gamma-\alpha}_{\alpha,\beta,\gamma,\delta}$ (Formula (2.2.49) when $l=\gamma-\alpha$). If the problem persists, we couple the remainder of $\Pi^{0,N}_{\alpha,\beta}p$ with $\Pi^{0,\gamma-\alpha-1}_{\gamma,\delta}p$, then with the remainder of $\Gamma^{\gamma-\alpha-1}_{\gamma,\delta}p$ and so on. In other words we couple lower configuration's N-births only with upper configuration's l-births while $l>\gamma-\alpha$ (Step 2a), and we alternate a coupling with the upper configuration's l-births rate and l-jumps rate if $l\le\gamma-\alpha$ (Step 2b), giving priority to births. Following this idea we get the recursive Formulas (2.2.47) and (2.2.49).

Remark 2.32 *We proceed downwards by coupling $\Pi^{0,N}_{\alpha,\beta}p$ with upper configuration's birth and jump rates until $l = 0$, hence we do not stop when $\beta + N > \delta + l$, that is when $l = N - \delta + \beta$. This means that a priori we could break the order between configurations, but this will be not the case as consequence of Conditions* (2.2.13)*–*(2.2.14)*.*

Step 2) (Death rates). If $\gamma - N < \alpha$, by (2.2.23), (2.2.24) and (2.2.25)

$$H^{-M,0,-N,0}_{\alpha,\beta,\gamma,\delta} =(\Pi^{-N,0}_{\alpha,\beta} \wedge \Pi^{-M,0}_{\gamma,\delta})p \tag{2.2.50}$$

$$H^{-k,0,-N,0}_{\alpha,\beta,\gamma,\delta} =[\Pi^{-k,0}_{\alpha,\beta}p] \wedge [\Pi^{-N,0}_{\gamma,\delta}p - \sum_{k'>k} H^{-k',0,-N,0}_{\alpha,\beta,\gamma,\delta} - \sum_{\delta-\beta\geq k'>k} H^{k',k',-N,0}_{\alpha,\beta,\gamma,\delta}]. \tag{2.2.51}$$

In other cases $H^{-k,0,-N,0}_{\alpha,\beta,\gamma,\delta} = 0$. If $\gamma - N < \alpha$ by (2.2.23), (2.2.24) and (2.2.26)

$$H^{N,N,-N,0}_{\alpha,\beta,\gamma,\delta} =[\Gamma^N_{\alpha,\beta}p - H^{N,N,N,N}_{\alpha,\beta,\gamma,\delta}] \wedge [\Pi^{-N,0}_{\gamma,\delta}p - \sum_{k'\geq N} H^{-k',0,-N,0}_{\alpha,\beta,\gamma,\delta}] \qquad \text{if } \beta + N \leq \delta \tag{2.2.52}$$

$$\begin{aligned} H^{k,k,-N,0}_{\alpha,\beta,\gamma,\delta} =&[\Gamma^k_{\alpha,\beta}p - H^{k,k,N,N}_{\alpha,\beta,\gamma,\delta}] \\ &\wedge [\Pi^{-N,0}_{\gamma,\delta}p - \sum_{k'\geq k} H^{-k',0,-N,0}_{\alpha,\beta,\gamma,\delta} - \sum_{\delta-\beta\geq k'>k} H^{k',k',-N,0}_{\alpha,\beta,\gamma,\delta}] \qquad \text{if } \beta + k \leq \delta. \end{aligned} \tag{2.2.53}$$

Otherwise $H^{k,k,-N,0}_{\alpha,\beta,\gamma,\delta} = 0$.

Step 2a) (Death rates). If γ is N-bad we proceed in a symmetric way with respect to birth rates. We couple the death rate $\Pi^{-N,0}_{\gamma,\delta}p$ with $\Pi^{-M,0}_{\alpha,\beta}p$ (Formula (2.2.50)) and we proceed downwards in k with $\Pi^{-k,0}_{\gamma,\delta}p$, when $k \leq M - 1$ until $\beta + k > \delta$ (Formula (2.2.51)).

Step 2b) (Death rates). When $\beta + k \leq \delta$ we couple the remaining death rate $\Pi^{-N,0}_{\gamma,\delta}p - \sum_{k'\geq\delta-\beta} H^{-k',0,-N,0}_{\alpha,\beta,\gamma,\delta}$ alternatively with the upper death rate $\Pi^{-k,0}_{\alpha,\beta}p$ and with the remaining jump rate $\Gamma^k_{\alpha,\beta}p - H^{k,k,N,N}_{\alpha,\beta,\gamma,\delta}$ (Formula (2.2.53)). Remark 2.32 holds in a symmetric way for death rates.

Step 3) If $\beta + N > \delta$ and $l \leq M$, by (2.2.27) and (2.2.28)

$$H^{N,N,0,l}_{\alpha,\beta,\gamma,\delta} = [\Gamma^N_{\alpha,\beta}p - \sum_{l'\geq N-\delta+\beta} H^{N,N,l',l'}_{\alpha,\beta,\gamma,\delta} - \sum_{l'>l} H^{N,N,0,l'}_{\alpha,\beta,\gamma,\delta}] \wedge [\Pi^{0,l}_{\gamma,\delta}p - H^{0,N,0,l}_{\alpha,\beta,\gamma,\delta}]. \tag{2.2.54}$$

Otherwise $H^{N,N,0,l}_{\alpha,\beta,\gamma,\delta} = 0$. If $\gamma - N < \alpha$ and $k \leq M$, by (2.2.29) and (2.2.30)

$$H^{-k,0,N,N}_{\alpha,\beta,\gamma,\delta} = [\Pi^{-k,0}_{\alpha,\beta}p - H^{-k,0,-N,0}_{\alpha,\beta,\gamma,\delta}] \wedge [\Gamma^N_{\gamma,\delta}p - \sum_{k'\geq N-\gamma+\alpha} H^{k',k',N,N}_{\alpha,\beta,\gamma,\delta} - \sum_{k'>k} H^{-k,0,N,N}_{\alpha,\beta,\gamma,\delta}]. \tag{2.2.55}$$

Otherwise $H^{-k,0,N,N}_{\alpha,\beta,\gamma,\delta} = 0$.
Suppose we still have from Step 1 a remainder $\Gamma^N_{\alpha,\beta} - \sum_{l'\geq N-\delta+\beta} \Gamma^{l'}_{\gamma,\delta} > 0$, we couple it with the remaining birth rate $\Pi^{0,l}_{\gamma,\delta}p - H^{0,N,0,l}_{\alpha,\beta,\gamma,\delta}$ starting from $l = M$ and going downwards (recursive formula (2.2.54)). In a symmetric way we get (2.2.55) in case $\Gamma^N_{\gamma,\delta} - \sum_{k'\geq N-\gamma+\alpha} \Gamma^{k'}_{\alpha,\beta} > 0$, that is if Step 1 is not enough to solve the higher attractiveness given by $\Gamma^N_{\gamma,\delta}p$. Remark 2.32 holds in a similar way for Step 3.

We focused on coupling terms concerning some particular rates, $\Gamma^N_{\alpha,\beta}$ and $\Pi^{0,N}_{\alpha,\beta}$ if $\beta+N>\delta$ and $\Gamma^N_{\gamma,\delta}$ and $\Pi^{-N,0}_{\gamma,\delta}$ if $\gamma-N<\alpha$. These are the terms such that an addition or a subtraction of N particles gives an attractiveness problem. The coupling construction in Definition 2.21 works exactly in the same way for other terms. Roughly speaking, we couple only when we have an attractiveness problem: if the problem comes from a jump rate of the lower configuration, we couple as much as possible with upper configuration's jump rates (Step 1) and, only if necessary, with births or deaths rates (Step 3). If the problem comes from a birth (death) rate, we alternate a coupling with the upper (lower) configuration's maximal birth (death) rate with a coupling with the upper (lower) configuration's maximal jump rate if it does not give any attractiveness problem (Step 2).

2.3 Examples

In the following examples we suppose $\widetilde{\mathcal{S}}=\mathcal{S}$, that is we consider necessary and sufficient conditions for attractiveness.

2.3.1 Conservative dynamics

If $\Pi^{0,k}_{\alpha,\beta}=0$, for all $(\alpha,\beta)\in X^2, k\in\mathbb{N}$, we get a particular case of the model introduced by Gobron, Saada (see [27]), for which no particles births or deaths are allowed and the particle system is conservative.

Suppose that in this model the number k of particles which make a jump together is bounded by a constant and their rate $\Gamma^k_{\alpha,\beta}(y-x)$ has the form $\Gamma^k_{\alpha,\beta}p(y-x)$ that we introduced in (2.2.1). Denote by

$$M:=\max\{k:\Gamma^k_{\alpha,\beta}(y-x)>0 \text{ for each } \alpha,\beta,x,y\}.$$

Necessary and sufficient conditions for attractiveness for this model are given by [27, Theorem 2.21]:

$$\sum_{k'>\delta-\beta+l}\Gamma^{k'}_{\alpha,\beta}p(y-x)\le\sum_{l'>l}\Gamma^{l'}_{\gamma,\delta}p(y-x) \qquad \text{for each } l\ge 0 \tag{2.3.1}$$

$$\sum_{k'>k}\Gamma^{k'}_{\alpha,\beta}p(y-x)\ge\sum_{l'>\gamma-\alpha+k}\Gamma^{l'}_{\gamma,\delta}p(y-x) \qquad \text{for each } k\ge 0 \tag{2.3.2}$$

while (2.2.13) $-$ (2.2.14) become

$$\sum_{k'\in I_a}\Gamma^{k'}_{\alpha,\beta}p(y-x)\le\sum_{l'\in I_b}\Gamma^{l'}_{\gamma,\delta}p(y-x) \tag{2.3.3}$$

$$\sum_{k'\in I_d}\Gamma^{k'}_{\alpha,\beta}p(y-x)\ge\sum_{l'\in I_c}\Gamma^{l'}_{\gamma,\delta}p(y-x) \tag{2.3.4}$$

for all I_a, I_b, I_c and I_d given in Definition 2.14.

Proposition 2.33 *Conditions* (2.3.1)*–*(2.3.2) *and Conditions* (2.3.3)*–*(2.3.4) *are equivalent.*

Proof .
Suppose (2.3.3) and (2.3.4) hold. By choosing $m_i=m$, $l_i=l$ for each i, m larger than M, $I_a=\{M\ge k'>\delta-\beta+l\}$ and $I_b=\{M\ge l'>l\}$, therefore

$$\sum_{k'>\delta-\beta+l}\Gamma^{k'}_{\alpha,\beta}p(y-x)=\sum_{k'\in I_a}\Gamma^{k'}_{\alpha,\beta}p(y-x)\le\sum_{l'\in I_b}\Gamma^{l'}_{\gamma,\delta}p(y-x)=\sum_{l'>l}\Gamma^{l'}_{\gamma,\delta}p(y-x)$$

that is (2.3.1). By choosing $k_i = k$ for each i and again $m \geq M$, $I_c = \{M \geq k' > \gamma-\alpha+k\}$ and $I_d = \{M \geq l' > k\}$. Then

$$\sum_{l'>\gamma-\alpha+k} \Gamma^{l'}_{\gamma,\delta} p(y-x) = \sum_{l'\in I_c} \Gamma^{l'}_{\gamma,\delta} p(y-x) \leq \sum_{k'\in I_d} \Gamma^{k'}_{\alpha,\beta} p(y-x) = \sum_{k'>k} \Gamma^{k'}_{\alpha,\beta} p(y-x)$$

that is (2.3.2).
Now suppose (2.3.1)–(2.3.2) hold. If we take $k > \delta - \beta + l$ in (2.3.2) and we call $k := m$, by subtracting (2.3.2) to (2.3.1) we get

$$\sum_{k'>\delta-\beta+l} \Gamma^{k'}_{\alpha,\beta} p(y-x) - \sum_{k'>m} \Gamma^{k'}_{\alpha,\beta} p(y-x) \leq \sum_{l'>l} \Gamma^{l'}_{\gamma,\delta} p(y-x) - \sum_{l'>\gamma-\alpha+m} \Gamma^{l'}_{\gamma,\delta} p(y-x)$$

that is for each choice of l and m

$$\sum_{m\geq k'>\delta-\beta+l} \Gamma^{k'}_{\alpha,\beta} p(y-x) \leq \sum_{\gamma-\alpha+m\geq l'>l} \Gamma^{l'}_{\gamma,\delta} p(y-x). \tag{2.3.5}$$

If we take two pairs of values m_1, l_1 and m_2, l_2 in (2.3.5) and add the correspondent inequalities, we get

$$\begin{aligned} \sum_{m_2\geq k'>\delta-\beta+l_2} \Gamma^{k'}_{\alpha,\beta} p(y-x) + & \sum_{m_1\geq k'>\delta-\beta+l_1} \Gamma^{k'}_{\alpha,\beta} p(y-x) \\ & \leq \sum_{\gamma-\alpha+m_2\geq l'>l_2} \Gamma^{l'}_{\gamma,\delta} p(y-x) + \sum_{\gamma-\alpha+m_1\geq l'>l_1} \Gamma^{l'}_{\gamma,\delta} p(y-x). \end{aligned} \tag{2.3.6}$$

Since

$$\begin{aligned} I_a^2 =& \{m_2 \geq k' > \delta - \beta + l_2\} \cup \{m_1 \geq k' > \delta - \beta + l_1\} \\ I_b^2 =& \{\gamma - \alpha + m_2 \geq k' > l_2\} \cup \{\gamma - \alpha + m_1 \geq k' > l_1\} \end{aligned}$$

then

$$\sum_{k'\in I_a^2} \Gamma^{k'}_{\alpha,\beta} p(y-x) \leq \sum_{m_2\geq k'>\delta-\beta+l_2} \Gamma^{k'}_{\alpha,\beta} p(y-x) + \sum_{m_1\geq k'>\delta-\beta+l_1} \Gamma^{k'}_{\alpha,\beta} p(y-x). \tag{2.3.7}$$

If $l_2 \geq \gamma - \alpha + m_1$, for I_a^2 and I_b^2

$$\sum_{\gamma-\alpha+m_2\geq l'>l_2} \Gamma^{l'}_{\gamma,\delta} p(y-x) + \sum_{\gamma-\alpha+m_1\geq l'>l_1} \Gamma^{l'}_{\gamma,\delta} p(y-x) = \sum_{l'\in I_b^2} \Gamma^{l'}_{\gamma,\delta} p(y-x) \tag{2.3.8}$$

hence by using (2.3.6), (2.3.7) and (2.3.8) we get (2.3.3).
If $l_2 < \gamma - \alpha + m_1$, then $I_b^2 = \{\gamma - \alpha + m_2 \geq l' > l_1\}$ and $I_a^2 \subseteq \{m_2 \geq k' > \delta - \beta + l_1\}$. By using that and (2.3.5) with $l = l_1$ and $m = m_2$,

$$\sum_{k'\in I_a^2} \Gamma^{k'}_{\alpha,\beta} p(y-x) \leq \sum_{m_2\geq k'>\delta-\beta+l_1} \Gamma^{k'}_{\alpha,\beta} p(y-x) = \sum_{l'\in I_b^2} \Gamma^{l'}_{\gamma,\delta} p(y-x).$$

We can repeat the same passages and prove the result for I_a^K and I_b^K with $K > 2$. Hence condition (2.3.3) holds. We prove (2.3.4) in a similar way. $\square$

2.3.2 Multitype contact processes

If $\Gamma^k_{\alpha,\beta} = 0$, for all $(\alpha, \beta) \in X^2, k \geq 0$, that is when jumps of particles are not present, we get a generalization of the multitype contact process studied by Stover (see [48]). In this multitype contact process, for a site x, $\eta(x) = \alpha$ causes on $y \sim x$ with $\eta(y) = \beta$ one possibility of birth or death of one or more individuals. This means $p(x, y) = (2d)^{-1}1\!\!1_{\{y\sim x\}}$ and that there exists at most one value $k > 0$ such that $\Pi^{0,k}_{\alpha,\beta} > 0$ and one value $l > 0$ such that $\Pi^{0,-l}_{\alpha,\beta} > 0$.
Following Stover's notations, let $\mathcal{U}_\beta$ be the set of values of $\eta(x)$ that induce a growth on $\eta(y) = \beta$, and $\mathcal{D}_\beta$ be the set of values that induce a decrease on $\eta(y) = \beta$: if $\alpha \in \mathcal{U}_\beta$ there exists at most one $k_u = k_u(\alpha, \beta)$ such that $\Pi^{0,k_u}_{\alpha,\beta} > 0$ and if $\alpha \in \mathcal{D}_\beta$ there exists at most one $k_d = k_d(\alpha, \beta)$ such that $\Pi^{0,-k_d}_{\alpha,\beta} > 0$.
Then [48, Definition 19] and [48, Theorem 19] state that the system is monotone if and only if:
for each α, β, γ, $\delta \in X$,

- if $\beta + k_u(\alpha, \beta) > \delta$, then $k_u(\gamma, \delta)$ is such that $\beta + k_u(\alpha, \beta) \leq \delta + k_u(\gamma, \delta)$ and
$$\Pi^{0,k_u(\alpha,\beta)}_{\alpha,\beta} \leq \Pi^{0,k_u(\gamma,\delta)}_{\gamma,\delta}; \tag{2.3.9}$$
- if $\delta - k_d(\gamma, \delta) < \beta$, then $k_d(\alpha, \beta)$ is such that $\beta - k_d(\alpha, \beta) \leq \delta - k_d(\gamma, \delta)$ and
$$\Pi^{0,-k_d(\alpha,\beta)}_{\alpha,\beta} \geq \Pi^{0,-k_d(\gamma,\delta)}_{\gamma,\delta}; \tag{2.3.10}$$

Proposition 2.34 *Conditions* (2.3.9)*–*(2.3.10) *are equivalent to* (2.2.13)*–*(2.2.14)*.*

Proof .
For the multitype contact process, which has no jumps, (2.2.13)–(2.2.14) become, for all $(\alpha, \beta) \in X^2$, $(\gamma, \delta) \in X^2$, $k_1 \geq 0$, $l_1 \geq 0$,

$$\sum_{k'>\delta-\beta+l_1} \Pi^{0,k'}_{\alpha,\beta} \leq \sum_{l'>l_1} \Pi^{0,l'}_{\gamma,\delta} \tag{2.3.11}$$
$$\sum_{k'>k_1} \Pi^{0,-k'}_{\alpha,\beta} \geq \sum_{l'>\delta-\beta+k_1} \Pi^{0,-l'}_{\gamma,\delta} \tag{2.3.12}$$

There exists at most one possible positive birth (and death) rate involving a birth of $k_u(\alpha, \beta)$ and $k_u(\gamma, \delta)$ particles (respectively $k_d(\alpha, \beta)$ and $k_d(\gamma, \delta)$ particles). Hence (2.3.11)–(2.3.12) become, for all l_1 and k_1 positive integers,

$$1\!\!1_{\{k_u(\alpha,\beta)>\delta-\beta+l_1\}}\Pi^{0,k_u(\alpha,\beta)}_{\alpha,\beta} \leq 1\!\!1_{\{k_u(\gamma,\delta)>l_1\}}\Pi^{0,k_u(\gamma,\delta)}_{\gamma,\delta} \tag{2.3.13}$$
$$1\!\!1_{\{k_d(\alpha,\beta)>k_1\}}\Pi^{0,-k_d(\alpha,\beta)}_{\alpha,\beta} \geq 1\!\!1_{\{k_d(\gamma,\delta)>\delta-\beta+k_1\}}\Pi^{0,-k_d(\gamma,\delta)}_{\gamma,\delta} \tag{2.3.14}$$

which are equivalent to Conditions (2.3.9)–(2.3.10). Indeed we prove the equivalence of first conditions, the other ones can be proved in a similar way. If (2.3.9) holds, then if we take l_1 such that $k_u(\alpha, \beta) \leq \delta - \beta + l_1$, Condition (2.3.13) is trivially satisfied since the left hand side is null. Suppose $k_u(\alpha, \beta) > \delta - \beta + l_1$, by using (2.3.9) we get that $k_u(\gamma, \delta) \geq k_u(\alpha, \beta) - \delta + \beta > l_1$, and (2.3.13) holds. Conversely, if (2.3.13) holds, then by taking $l_1 = k_u(\gamma, \delta)$ we get

$$1\!\!1_{\{k_u(\alpha,\beta)>\delta-\beta+k_u(\gamma,\delta)\}}\Pi^{0,k_u(\alpha,\beta)}_{\alpha,\beta} \leq 1\!\!1_{\{k_u(\gamma,\delta)>k_u(\gamma,\delta)\}}\Pi^{0,k_u(\gamma,\delta)}_{\gamma,\delta} = 0.$$

That is $\beta + k_u(\alpha, \beta) \leq \delta + k_u(\gamma, \delta)$. By taking $l_1 = 0$ we get (2.3.9) and the equivalence is proved. $\square$

In [48, Theorem 21.1] the author gives only sufficiency conditions for attractiveness in our more general setup. In this case necessary and sufficient conditions for attractiveness are given by (2.3.11)–(2.3.12).

2.3.3 Reaction diffusion processes

If we consider births, deaths and jumps of at most one particle and $R^{0,k}_{\alpha,\beta} = 0$ for each α, β, k, that is no dependent reaction rates, the model is the reaction diffusion process studied by Chen (see [12]). Necessary and sufficient conditions for its attractiveness are given in Section 2.3.5 below, where a general example with $M = 1$ is treated. When $R^{0,k}_{\alpha,\beta} = 0$, Conditions (2.2.13)–(2.2.14) reduce to

$$\begin{aligned} P^{0,1}_{\beta} + \Gamma^1_{\alpha,\beta} \leq& P^{0,1}_{\delta} + \Gamma^1_{\gamma,\delta} && \text{if } \beta = \delta \text{ and } \gamma > \alpha \\ P^{-1,0}_{\alpha} + \Gamma^1_{\alpha,\beta} \geq& P^{-1,0}_{\gamma} + \Gamma^1_{\gamma,\delta} && \text{if } \gamma = \alpha \text{ and } \delta > \beta \end{aligned}$$

that is

$$\begin{aligned} \Gamma^1_{\alpha,\beta} \leq& \Gamma^1_{\gamma,\beta} && \text{if } \gamma > \alpha \\ \Gamma^1_{\alpha,\delta} \geq& \Gamma^1_{\alpha,\beta} && \text{if } \delta > \beta. \end{aligned}$$

In other words we need $\Gamma^1_{\alpha,\beta}$ to be non decreasing with respect to α for each fixed β and non increasing with respect to β for each fixed α.
In [12], the author introduces several couplings in order to find ergodicity conditions of reaction diffusion processes. All these couplings are identical to $\mathcal{H}$ on configurations with attractiveness problems but differ from $\mathcal{H}$ on configurations where the two processes move independently.

2.3.4 Minimal increasing coupling

We constructed a coupling on configurations for which a simultaneous change of the two processes is necessary. For instance, if $\beta + N \leq \delta$ and $\alpha \leq \gamma - N$, we do not have any attractiveness problem and the two processes move independently. Hence, in some sense, that we will specify later, this coupling characterizes attractiveness. Let us consider the model introduced in [27]. If $M < \infty$, the coupling they obtain (notice that in this case the reaction is always null) is $\mathcal{H}$ in Definition 2.21 extended to the case $\beta + k \leq \delta$. But the definition of coupling terms in these configurations is useless for our purposes, indeed non-trivial coupling terms in Definition 2.21 are null.
If we consider the reaction-diffusion process studied by Chen (see [12]), in order to prove sufficient conditions for ergodicity, he uses different couplings, but in bad pairs of values all of them coincide with the coupling given by Definition 2.21, that is basic coupling for $M = 1$ and null dependent reaction rates. In good pairs, where we use independent coupling, he coupled in different ways and, inside the set of increasing couplings, he found the best one (in some sense) for ergodicity.
Now we introduce the notion of *minimal increasing coupling*:

Definition 2.35 *A coupling is a* minimal increasing coupling *for a system $\mathcal{S}$ if the following properties hold:*

i) It is an increasing coupling for each possible probability distribution $p(x, y)$, maximum jump rate $M \in \mathbb{N}$ and rates $\{\Gamma^k_{\alpha,\beta}, \Pi^{0,k}_{\alpha,\beta}, \Pi^{-k,0}_{\alpha,\beta}\}_{\{(\alpha,\beta)\in X^2\}}$.

ii) If we replace some coupling rate by uncoupled moves for the two configurations, we get a coupling that does not satisfy property i).

By construction, Definition 2.21 gives a minimal increasing coupling.

2.3.5 The case M=1

Conditions (2.2.13) and (2.2.14) become

$$\begin{aligned}
\Pi^{0,1}_{\alpha,\beta}+\Gamma^{1}_{\alpha,\beta} &\leq \Pi^{0,1}_{\gamma,\delta}+\Gamma^{1}_{\gamma,\delta} && \text{if } \beta=\delta \text{ and } \gamma>\alpha && (2.3.15)\\
\Pi^{-1,0}_{\alpha,\beta}+\Gamma^{1}_{\alpha,\beta} &\geq \Pi^{-1,0}_{\gamma,\delta}+\Gamma^{1}_{\gamma,\delta} && \text{if } \gamma=\alpha \text{ and } \delta>\beta. && (2.3.16)
\end{aligned}$$

Indeed, since $M=1$, if $\delta>\beta$, the left hand side of (2.2.13) is null; if $\delta=\beta$ the only case for which left hand side of (2.2.13) is not null is $K=1$ and $l_1=0$, which gives

$$\sum_{k'>0}\Pi^{0,k'}_{\alpha,\beta}+\sum_{k'\in I_a}\Gamma^{k'}_{\alpha,\beta}\leq\sum_{l'>0}\Pi^{0,l'}_{\gamma,\beta}+\sum_{l'\in I_b}\Gamma^{l'}_{\gamma,\beta}$$

where $M=1$, $K=1$, $I_a=\{k':m_1\geq k'>0\}$ and $I_b=\{1\}$ since we can suppose $\gamma>\alpha$. If $m_1>0$, we get (2.3.15). We prove (2.3.16) in a similar way.
Notice that $\beta+1>\delta$ only if $\beta=\delta$, and $\gamma-1<\alpha$ only if $\gamma=\alpha$. Definition 2.21 becomes

$$H^{1,1,1,1}_{\alpha,\beta,\gamma,\delta}=\begin{cases}(\Gamma^{1}_{\alpha,\beta}\wedge\Gamma^{1}_{\gamma,\delta})p & \text{if } \beta=\delta \text{ or } \alpha=\gamma\\ 0 & \text{otherwise;}\end{cases}$$

$$H^{0,1,0,1}_{\alpha,\beta,\gamma,\delta}=\begin{cases}(\Pi^{0,1}_{\alpha,\beta}\wedge\Pi^{0,1}_{\gamma,\delta})p & \text{if } \beta=\delta\\ 0 & \text{otherwise;}\end{cases}$$

$$H^{0,1,1,1}_{\alpha,\beta,\gamma,\delta}=\begin{cases}[\Pi^{0,1}_{\alpha,\beta}p-H^{0,1,0,1}_{\alpha,\beta,\gamma,\delta}]\wedge[\Gamma^{1}_{\gamma,\delta}p-H^{1,1,1,1}_{\alpha,\beta,\gamma,\delta}] & \text{if } \beta=\delta \text{ and } \gamma>\alpha\\ 0 & \text{otherwise;}\end{cases}$$

$$H^{-1,0,-1,0}_{\alpha,\beta,\gamma,\delta}=\begin{cases}(\Pi^{-1,0}_{\alpha,\beta}\wedge\Pi^{-1,0}_{\gamma,\delta})p & \text{if } \alpha=\gamma\\ 0 & \text{otherwise;}\end{cases}$$

$$H^{1,1,-1,0}_{\alpha,\beta,\gamma,\delta}=\begin{cases}[\Gamma^{1}_{\alpha,\beta}p-H^{1,1,1,1}_{\alpha,\beta,\gamma,\delta}]\wedge[\Pi^{-1,0}_{\gamma,\delta}p-H^{-1,0,-1,0}_{\alpha,\beta,\gamma,\delta}] & \text{if } \alpha=\gamma \text{ and } \beta<\delta\\ 0 & \text{otherwise;}\end{cases}$$

$$H^{1,1,0,1}_{\alpha,\beta,\gamma,\delta}=\begin{cases}[\Gamma^{1}_{\alpha,\beta}p-H^{1,1,1,1}_{\alpha,\beta,\gamma,\delta}]\wedge[\Pi^{0,1}_{\gamma,\delta}p-H^{0,1,0,1}_{\alpha,\beta,\gamma,\delta}] & \text{if } \beta=\delta\\ 0 & \text{otherwise;}\end{cases}$$

$$H^{-1,0,1,1}_{\alpha,\beta,\gamma,\delta}=\begin{cases}[\Pi^{-1,0}_{\alpha,\beta}p-H^{-1,0,-1,0}_{\alpha,\beta,\gamma,\delta}]^{+}\wedge[\Gamma^{1}_{\alpha,\beta}p-H^{1,1,1,1}_{\alpha,\beta,\gamma,\delta}] & \text{if } \alpha=\gamma\\ 0 & \text{otherwise;}\end{cases}$$

$$\begin{aligned}
H^{0,1,0,0}_{\alpha,\beta,\gamma,\delta} &=\Pi^{0,1}_{\alpha,\beta}p-H^{0,1,0,1}_{\alpha,\beta,\gamma,\delta}-H^{0,1,1,1}_{\alpha,\beta,\gamma,\delta}; & H^{1,1,0,0}_{\alpha,\beta,\gamma,\delta} &=\Gamma^{1}_{\alpha,\beta}p-H^{1,1,1,1}_{\alpha,\beta,\gamma,\delta}-H^{1,1,-1,0}_{\alpha,\beta,\gamma,\delta}-H^{1,1,0,1}_{\alpha,\beta,\gamma,\delta};\\
H^{0,0,0,1}_{\alpha,\beta,\gamma,\delta} &=\Pi^{0,1}_{\gamma,\delta}p-H^{0,1,0,1}_{\alpha,\beta,\gamma,\delta}-H^{1,1,0,1}_{\alpha,\beta,\gamma,\delta}; & H^{0,0,1,1}_{\alpha,\beta,\gamma,\delta} &=\Gamma^{1}_{\gamma,\delta}p-H^{1,1,1,1}_{\alpha,\beta,\gamma,\delta}-H^{0,1,1,1}_{\alpha,\beta,\gamma,\delta}-H^{-1,0,1,1}_{\alpha,\beta,\gamma,\delta};\\
H^{-1,0,0,0}_{\alpha,\beta,\gamma,\delta} &=\Pi^{-1,0}_{\alpha,\beta}p-H^{-1,0,-1,0}_{\alpha,\beta,\gamma,\delta}-H^{-1,0,1,1}_{\alpha,\beta,\gamma,\delta}; & H^{0,0,-1,0}_{\alpha,\beta,\gamma,\delta} &=\Pi^{-1,0}_{\gamma,\delta}p-H^{-1,0,-1,0}_{\alpha,\beta,\gamma,\delta}-H^{1,1,-1,0}_{\alpha,\beta,\gamma,\delta}.
\end{aligned}$$

The definition of uncoupled terms ensures this is a coupling. We have to prove attractiveness.
Case a) $\beta=\delta$ and $\gamma>\alpha$.
In this case $H^{-1,0,1,1}_{\alpha,\beta,\gamma,\beta}=H^{-1,0,-1,0}_{\alpha,\beta,\gamma,\beta}=H^{1,1,-1,0}_{\alpha,\beta,\gamma,\beta}=0$.

• Suppose $\Gamma^1_{\alpha,\beta} \geq \Gamma^1_{\gamma,\delta}$. By Condition (2.3.15) we must have $\Pi^{0,1}_{\alpha,\beta}p \leq \Pi^{0,1}_{\gamma,\beta}p$, then $H^{0,1,0,0}_{\alpha,\beta,\gamma,\beta} = 0$.

$$H^{1,1,1,1}_{\alpha,\beta,\gamma,\beta} = \Gamma^1_{\gamma,\beta}p, \qquad H^{0,1,0,1}_{\alpha,\beta,\gamma,\beta} = \Pi^{0,1}_{\alpha,\beta}p, \qquad H^{0,1,1,1}_{\alpha,\beta,\gamma,\beta} = 0$$

$$H^{1,1,0,1}_{\alpha,\beta,\gamma,\delta} = [(\Gamma^1_{\alpha,\beta} - \Gamma^1_{\gamma,\beta})p] \wedge [(\Pi^{0,1}_{\gamma,\delta} - \Pi^{0,1}_{\alpha,\beta})p] = (\Gamma^1_{\alpha,\beta} - \Gamma^1_{\gamma,\beta})p$$

by Condition (2.3.15). Finally

$$H^{1,1,0,0}_{\alpha,\beta,\gamma,\beta} = \Gamma^1_{\alpha,\beta}p - \Gamma^1_{\gamma,\beta}p - \Gamma^1_{\alpha,\beta} + \Gamma^1_{\gamma,\beta}p = 0.$$

• Suppose $\Gamma^1_{\alpha,\beta} < \Gamma^1_{\gamma,\delta}$. Then $H^{1,1,1,1}_{\alpha,\beta,\gamma,\beta} = \Gamma^1_{\alpha,\beta}p$ and we get $H^{1,1,0,1}_{\alpha,\beta,\gamma,\beta} = H^{1,1,0,0}_{\alpha,\beta,\gamma,\delta} = 0$. We have to prove that $H^{0,1,0,0}_{\alpha,\beta,\gamma,\beta} = 0$.

$$H^{0,1,0,0}_{\alpha,\beta,\gamma,\beta} = \Pi^{0,1}_{\alpha,\beta}p - \Pi^{0,1}_{\alpha,\beta}p \wedge \Pi^{0,1}_{\gamma,\beta}p - [\Pi^{0,1}_{\alpha,\beta}p - \Pi^{0,1}_{\alpha,\beta}p \wedge \Pi^{0,1}_{\gamma,\beta}p] \wedge [\Gamma^1_{\gamma,\beta}p - \Gamma^1_{\alpha,\beta}p]$$

By using the relation

$$a \wedge (c - c \wedge b) = (a + b) \wedge c - b \wedge c \qquad \text{for } a, b, c \geq 0$$

with $a = \Gamma^1_{\gamma,\beta}p - \Gamma^1_{\alpha,\beta}p$, $b = \Pi^{0,1}_{\gamma,\beta}p$ and $c = \Pi^{0,1}_{\alpha,\beta}p$, we get by Condition (2.3.15),

$$\begin{aligned} H^{0,1,0,0}_{\alpha,\beta,\gamma,\beta} =& \Pi^{0,1}_{\alpha,\beta}p - \Pi^{0,1}_{\alpha,\beta}p \wedge \Pi^{0,1}_{\gamma,\beta}p - [\Gamma^1_{\gamma,\beta}p - \Gamma^1_{\alpha,\beta}p + \Pi^{0,1}_{\gamma,\beta}p] \wedge \Pi^{0,1}_{\alpha,\beta}p - \Pi^{0,1}_{\alpha,\beta}p \wedge \Pi^{0,1}_{\gamma,\beta}p \\ =& \Pi^{0,1}_{\alpha,\beta}p - \Pi^{0,1}_{\alpha,\beta}p = 0 \end{aligned}$$

Case b) $\alpha = \gamma$ and $\delta > \beta$: we check that the coupling is increasing in the same way.

2.4 Proofs

2.4.1 Proof of Proposition 2.17

Let $(\xi, \eta) \in \Omega \times \Omega$ be two configurations such that $\xi \leq \eta$. Let $V \subset \Omega$ be an increasing cylinder set. If $\xi \in V$ or $\eta \notin V$,

$$\mathbb{1}_V(\xi) = \mathbb{1}_V(\eta). \tag{2.4.1}$$

Since η_t is stochastically larger than ξ_t, for all $t \geq 0$,

$$(\widetilde{T}(t)\mathbb{1}_V)(\xi) \leq (T(t)\mathbb{1}_V)(\eta)$$

by using Remark 2.4 and Theorem 2.7 (or Theorem 2.5 if we are interested in attractiveness). Combining this with (2.4.1),

$$t^{-1}[(\widetilde{T}(t)\mathbb{1}_V)(\xi) - \mathbb{1}_V(\xi)] \leq t^{-1}[(T(t)\mathbb{1}_V)(\eta) - \mathbb{1}_V(\eta)],$$

which gives, by Assumption (2.1.1),

$$(\widetilde{\mathcal{L}}\mathbb{1}_V)(\xi) \leq (\mathcal{L}\mathbb{1}_V)(\eta). \tag{2.4.2}$$

We have, by using (2.2.1),

$$\begin{aligned}(\widetilde{\mathcal{L}}1\!\!1_V)(\xi) =& \sum_{x,y\in S} p(x,y) \sum_{\alpha,\beta\in X} \chi^{x,y}_{\alpha,\beta}(\xi) \sum_{k>0} \Big(\widetilde{\Gamma}^k_{\alpha,\beta}(1\!\!1_V(S^{-k,k}_{x,y}\xi) - 1\!\!1_V(\xi)) \\ &+ \widetilde{\Pi}^{0,k}_{\alpha,\beta}(1\!\!1_V(S^{0,k}_{x,y}\xi) - 1\!\!1_V(\xi)) + \widetilde{\Pi}^{-k,0}_{\alpha,\beta}(1\!\!1_V(S^{0,-k}_{x,y}\xi) - 1\!\!1_V(\xi))\Big) \\ =& - 1\!\!1_V(\xi) \sum_{x,y\in S} p(x,y) \sum_{\alpha,\beta\in X} \chi^{x,y}_{\alpha,\beta}(\xi) \sum_{k\geq 0} \Big(\widetilde{\Gamma}^k_{\alpha,\beta} 1\!\!1_{\Omega\setminus V}(S^{-k,k}_{x,y}\xi) \\ &+ \widetilde{\Pi}^{0,k}_{\alpha,\beta} 1\!\!1_{\Omega\setminus V}(S^{0,k}_{x,y}\xi) + \widetilde{\Pi}^{-k,0}_{\alpha,\beta} 1\!\!1_{\Omega\setminus V}(S^{0,-k}_{x,y}\xi)\Big) \\ &+ 1\!\!1_{\Omega\setminus V}(\xi) \sum_{x,y\in S} p(x,y) \sum_{\alpha,\beta\in X} \chi^{x,y}_{\alpha,\beta}(\xi) \sum_{k\geq 0} \Big(\widetilde{\Gamma}^k_{\alpha,\beta} 1\!\!1_V(S^{-k,k}_{x,y}\xi) \\ &+ \widetilde{\Pi}^{0,k}_{\alpha,\beta} 1\!\!1_V(S^{0,k}_{x,y}\xi) + \widetilde{\Pi}^{-k,0}_{\alpha,\beta} 1\!\!1_V(S^{0,-k}_{x,y}\xi)\Big). \end{aligned} \tag{2.4.3}$$

We write $(\mathcal{L}1\!\!1_V)(\eta)$ as in (2.4.3) by using the corresponding rates of $\mathcal{S}$.
We fix $y \in S$, $(\alpha_z, \gamma_z, \beta, \delta) \in X^4$ with $(\alpha_z, \beta) \leq (\gamma_z, \delta)$ for all $z \in S$, $z \neq y$, and two configurations $(\xi, \eta) \in \Omega \times \Omega$ such that $\xi(z) = \alpha_z$, $\eta(z) = \gamma_z$, for all $z \in S$, $z \neq y$, $\xi(y) = \beta$, $\eta(y) = \delta$. Thus $\xi \leq \eta$.
We define the set C^+_y of sites which interact with y with an increase of the configuration on y,

$$C^+_y = \Big\{z \in S : p(z,y) > 0 \text{ and } \sum_{k>0} \big(\widetilde{\Gamma}^k_{\alpha_z,\beta} + \Gamma^k_{\gamma_z,\delta} + \widetilde{\Pi}^{0,k}_{\alpha_z,\beta} + \Pi^{0,k}_{\gamma_z,\delta}\big) > 0\Big\}. \tag{2.4.4}$$

Denote by $x = (x_1, \ldots, x_d)$ the coordinates of each $x \in S$. We define, for each $n \in \mathbb{N}$, $C^+_y(n) = C^+_y \cap \{z \in S : \sum_{i=1}^d |z_i - y_i| \leq n\}$.
We may suppose $C^+_y \neq \emptyset$, since otherwise (2.2.13)–(2.2.14) would be trivially satisfied. Given $K \in \mathbb{N}$, we fix $\{p^i_z\}_{i\leq K, z\in C^+_y}$ such that for each for each i, z, $p^i_z \in X$ and $p^i_z \leq \xi(z)$. Moreover we fix $\{p^i_y\}_{i\leq K}$ such that $p^i_y > \delta$, for each i. For $n \in \mathbb{N}$, let

$$I_y(n) = \bigcup_{i=1}^K \big\{\zeta \in \Omega : \zeta(y) \geq p^i_y \text{ and } \zeta(z) \geq p^i_z, \text{ for all } z \in C^+_y(n)\big\}.$$

By Example 2.2, $I_y(n)$ is an increasing cylinder set, to which neither ξ nor η belong. We compute, using (2.4.3),

$$\begin{aligned}(\widetilde{\mathcal{L}}I_y(n))(\xi) =& \sum_{z\in C^+_y(n)} p(z,y) \sum_{k>0} \big(1\!\!1_{\bigcup_{i=1}^K \{\alpha_z - k \geq p^i_z, \beta+k\geq p^i_y\}} \widetilde{\Gamma}^k_{\alpha_z,\beta} + 1\!\!1_{\bigcup_{i=1}^K \{\beta+k\geq p^i_y\}} \widetilde{\Pi}^{0,k}_{\alpha_z,\beta}\big) \\ &+ \sum_{z\notin C^+_y(n)} p(z,y) \sum_{k>0} 1\!\!1_{\bigcup_{i=1}^K \{\beta+k\geq p^i_y\}} \big(\widetilde{\Gamma}^k_{\alpha_z,\beta} + \widetilde{\Pi}^{0,k}_{\alpha,\beta}\big) \\ (\mathcal{L}I_y(n))(\eta) =& \sum_{z\in C^+_y(n)} p(z,y) \sum_{l>0} \big(1\!\!1_{\bigcup_{i=1}^K \{\gamma_z - l \geq p^i_z, \delta+l\geq p^i_y\}} \Gamma^l_{\gamma_z,\delta} + 1\!\!1_{\bigcup_{i=1}^K \{\delta+l\geq p^i_y\}} \Pi^{0,l}_{\gamma_z,\delta}\big) \\ &+ \sum_{z\notin C^+_y(n)} p(z,y) \sum_{l>0} 1\!\!1_{\bigcup_{i=1}^K \{\delta+l\geq p^i_y\}} \big(\Gamma^l_{\gamma_z,\delta} + \Pi^{0,l}_{\gamma,\delta}\big).\end{aligned}$$

So, by setting

$$J(p_z, a, b) := J(\{p^i_z\}_{i\leq K}, a, b) = \bigcup_{i=1}^K \{l : a - l \geq p^i_z, b + l \geq p^i_y\},$$

and by (2.4.2), if $p^1_y := \min_i\{p^i_y\}$, we get

$$\sum_{z\in C^+_y(n)} p(z,y)\Big(\sum_{k\in J(p_z,\alpha_z,\beta)} \widetilde{\Gamma}^k_{\alpha_z,\beta} + \sum_{k\geq p^1_y-\beta} \widetilde{\Pi}^{0,k}_{\alpha_z,\beta}\Big) + \sum_{z\notin C^+_y(n)} p(z,y) \sum_{k\geq p^1_y-\beta} \big(\widetilde{\Pi}^{0,k}_{\alpha_z,\beta} + \widetilde{\Gamma}^k_{\alpha_z,\beta}\big)$$
$$\leq \sum_{z\in C^+_y(n)} p(z,y)\Big(\sum_{l\in J(p_z,\gamma_z,\delta)} \Gamma^l_{\gamma_z,\delta} + \sum_{l\geq p^1_y-\delta} \Pi^{0,l}_{\gamma_z,\delta}\Big) + \sum_{z\notin C^+_y(n)} p(z,y) \sum_{l\geq p^1_y-\delta} \big(\Pi^{0,l}_{\gamma_z,\delta} + \Gamma^l_{\gamma_z,\delta}\big)$$

Taking the monotone limit $n \to \infty$ gives

$$\sum_{z\in S}\Big(\sum_{k\in J(p_z,\alpha_z,\beta)} \widetilde{\Gamma}^k_{\alpha_z,\beta} + \sum_{k\geq p^1_y-\beta} \widetilde{\Pi}^{0,k}_{\alpha_z,\beta}\Big)p(z,y) \leq \sum_{z\in S}\Big(\sum_{l\in J(p_z,\gamma_z,\beta)} \Gamma^l_{\gamma_z,\delta} + \sum_{l\geq p^1_y-\delta} \Pi^{0,l}_{\gamma_z,\delta}\Big)p(z,y)$$

By choosing the values $\alpha_z \equiv \alpha$, $\gamma_z \equiv \gamma$, $p^i_z \equiv p^i_\alpha$,

$$\sum_{k\in J(p_\alpha,\alpha,\beta)} \widetilde{\Gamma}^k_{\alpha,\beta} + \sum_{k\geq p^1_y-\beta} \widetilde{\Pi}^{0,k}_{\alpha,\beta} \leq \sum_{l\in J(p_\alpha,\gamma,\delta)} \Gamma^l_{\gamma,\delta} + \sum_{l\geq p^1_y-\delta} \Pi^{0,l}_{\gamma,\delta}. \tag{2.4.5}$$

A similar argument with

$$C^-_x = \{z \in S : p(z,y) > 0 \text{ and } \sum_{k>0} \big(\widetilde{\Gamma}^k_{\alpha,\gamma(z)} + \Gamma^k_{\gamma,\delta_z} + \widetilde{\Pi}^{-k,0}_{\alpha,\beta_z} + \Pi^{-k,0}_{\gamma,\delta_z}\big) > 0\}$$

subsets $C^-_x(n), n \in \mathbb{N}$, $\{p^i_z\}_{z\in C^-_x(n), i\leq K} \in X$, $\{p^i_x\}_{i\leq K} \in X$ with $p^i_x < \alpha$, for each i, $p^i_z \geq \delta_z$, for all $z \in C^-_y(n)$, $i \leq K$, decreasing cylinder sets

$$D_x(n) = \bigcup_{i=1}^K \big\{\zeta \in \Omega : \zeta(x) \leq p^i_x \text{ and } \zeta(z) \leq p^i_z, \text{ for all } z \in C^-_y(n)\big\}$$

to the complement of which ξ and η belong, and the application of inequality (2.4.2) to $\xi, \eta, \Omega\backslash D_x(n)$ (which is increasing by Remark 2.3) leads to

$$\sum_{k\in J^-(p_\gamma,\alpha,\beta)} \widetilde{\Gamma}^k_{\alpha,\beta} + \sum_{\alpha-k\leq p^1_x} \widetilde{\Pi}^{0,k}_{\alpha,\beta} \geq \sum_{l\in J^-(p_\gamma,\gamma,\delta)} \Gamma^l_{\gamma,\delta} + \sum_{\gamma-l\leq p^1_x} \Pi^{0,l}_{\gamma,\delta} \tag{2.4.6}$$

where $p^1_x = \max_i\{p^i_x\}$,

$$J^-(p_\gamma, a, b) := J^-(\{p^i_\gamma\}_{i\leq K}, a, b) = \bigcup_{i=1}^K \{l : a - l \leq p^i_x, b + l \leq p^i_\gamma\}.$$

Finally, taking $p^i_y = \delta + l_i + 1$, $p^i_\alpha = \alpha - m_i$ in (2.4.5), $p^i_x = \alpha - k_i - 1$, $p^i_\gamma = \delta + m_i$ in (2.4.6) gives (2.2.13)–(2.2.14). $\square$

2.4.2 Proof of Proposition 2.19

Suppose that for each pair (x, y) we are able to construct an increasing coupling for $(\widetilde{\mathcal{S}}_{(x,y)}, \mathcal{S}_{(x,y)})$ (cf. Definition 2.18). We fix $\xi \leq \eta$ and we denote by $\mathcal{H}_{(x,y)}(\widetilde{\mathcal{S}}, \mathcal{S})$ such a coupling on these configurations. We define a coupling for $(\widetilde{\mathcal{S}}, \mathcal{S})$ simply by taking every single coupling for each pair (x, y) that is $\mathcal{H}(\widetilde{\mathcal{S}}, \mathcal{S}) = \{\mathcal{H}_{(x,y)}(\widetilde{\mathcal{S}}, \mathcal{S})\}_{(x,y)\in X^2}$. By attractiveness of each coupling we get the attractiveness of the final one. We have to prove that we actually get a coupling, by summing each marginal.
Consider a pair of systems $(\widetilde{\mathcal{S}}_{(x,y)}, \mathcal{S}_{(x,y)})$ and the jump rates for such a system. By

Definition 2.18, all jump rates but $\widetilde{\Gamma}^k_{\xi(x),\xi(y)}p(x,y)$ and $\Gamma^l_{\eta(x),\eta(y)}p(x,y)$ are null. Notice that even jumps from y to x do not appear because $q(y,x)=0$.
Since $\mathcal{H}_{x,y}(\widetilde{\mathcal{S}},\mathcal{S})$ is a coupling for $(\widetilde{\mathcal{S}}_{(x,y)},\mathcal{S}_{(x,y)})$, by summing all the marginals we get

$$\textstyle\sum_l \mathcal{H}_{x,y}(\widetilde{\Gamma}^k_{\xi(x),\xi(y)}, R^{0,l}_{\eta(x),\eta(y)}, R^{-l,0}_{\eta(x),\eta(y)}, P^l_{\eta(y)}, P^{-l}_{\eta(x)}, \Gamma^l_{\eta(x),\eta(y)}) = \widetilde{\Gamma}^k_{\xi(x),\xi(y)}p(x,y)$$

$$\textstyle\sum_k \mathcal{H}_{x,y}(\Gamma^l_{\eta(x),\eta(y)}, \widetilde{R}^{0,k}_{\xi(x),\xi(y)}, \widetilde{R}^{-k,0}_{\xi(x),\xi(y)}, \widetilde{P}^k_{\xi(y)}, \widetilde{P}^{-k}_{\xi(x)}, \widetilde{\Gamma}^k_{\xi(x),\xi(y)}) = \Gamma^l_{\eta(x),\eta(y)}p(x,y).$$

If we consider $(\widetilde{\mathcal{S}},\mathcal{S})$, a jump rate depending on configurations on (x,y) gives coupling terms only on $\mathcal{H}_{(x,y)}(\widetilde{\mathcal{S}},\mathcal{S})$, because $q(w,z)=0$ for each $(\widetilde{\mathcal{S}}_{(w,z)},\mathcal{S}_{(w,z)})$ such that $x\neq w$ or $y\neq z$. Therefore

$$\textstyle\sum_l \mathcal{H}(\widetilde{\Gamma}^k_{\xi(x),\xi(y)},\cdots) = \sum_l \mathcal{H}_{x,y}(\widetilde{\Gamma}^k_{\xi(x),\xi(y)},\cdots) = \widetilde{\Gamma}^k_{\xi(x),\xi(y)}p(x,y)$$

$$\textstyle\sum_k \mathcal{H}(\Gamma^l_{\eta(x),\eta(y)},\cdots) = \sum_k \mathcal{H}_{x,y}(\Gamma^l_{\eta(x),\eta(y)},\cdots) = \Gamma^l_{\eta(x),\eta(y)}p(x,y).$$

The same arguments hold for the dependent birth or death rates.
On the contrary the independent reaction rates take a small contribution for each pair. On $(\widetilde{\mathcal{S}}_{(x,y)},\mathcal{S}_{(x,y)})$, by Definition 2.18, all birth rates but $\widetilde{P}^k_{\xi(y)}p(x,y)$ and $P^l_{\eta(y)}p(x,y)$ for each $k>0$, $l>0$ are null, even $\widetilde{P}^k_{\xi(x)}p(x,y) = P^l_{\eta(x)}p(x,y) = 0$, and all death rates but $\widetilde{P}^{-k}_{\xi(x)}p(x,y)$ and $P^{-l}_{\eta(y)}p(x,y)$ are null, even $\widetilde{P}^{-k}_{\xi(y)}p(x,y) = P^{-l}_{\eta(y)}p(x,y) = 0$. Therefore, for each $k>0$ and for each $l>0$,

$$\textstyle\sum_l \mathcal{H}(\widetilde{P}^k_{\xi(y)},\cdots) = \sum_{x\in X}\sum_l \mathcal{H}_{x,y}(\widetilde{P}^k_{\xi(y)},\cdots) = \sum_{x\in X}\widetilde{P}^k_{\xi(y)}p(x,y) = \widetilde{P}^k_{\xi(y)}$$

$$\textstyle\sum_k \mathcal{H}(P^l_{\eta(y)},\cdots) = \sum_{x\in X}\sum_k \mathcal{H}_{x,y}(P^l_{\eta(y)},\cdots) = \sum_{x\in X}P^l_{\eta(y)}p(x,y) = P^l_{\eta(y)}.$$

The same result holds for the death rates,

$$\textstyle\sum_l \mathcal{H}(\widetilde{P}^{-k}_{\xi(x)},\cdots) = \sum_{y\in X}\sum_l \mathcal{H}_{x,y}(\widetilde{P}^{-k}_{\xi(x)},\cdots) = \sum_{y\in X}\widetilde{P}^{-k}_{\xi(x)}p(x,y) = \widetilde{P}^{-k}_{\xi(x)}$$

$$\textstyle\sum_k \mathcal{H}(P^{-l}_{\eta(x)},\cdots) = \sum_{y\in X}\sum_k \mathcal{H}_{x,y}(P^{-l}_{\eta(x)},\cdots) = \sum_{y\in X}P^{-l}_{\eta(x)}p(x,y) = P^{-l}_{\eta(x)}.$$

□

2.4.3 Coupling properties

Now we focus on rates involving L or U particles (Definition 2.12) which cause an attractiveness problem involving the maximal possible number of particles.

Definition 2.36 *Let*

$$\widehat{L} := L-\delta+\beta; \qquad \widehat{U} := U-\gamma+\alpha; \tag{2.4.7}$$

and

$$\mathcal{R}^L_{\alpha,\beta} = [\Gamma^L_{\alpha,\beta}p - \sum_{\bar{l}\geq l\geq \widehat{L}} \Gamma^l_{\gamma,\delta}p]^+; \tag{2.4.8}$$

$$\mathcal{R}^U_{\gamma,\delta} = [\Gamma^U_{\gamma,\delta}p - \sum_{\bar{k}\geq k\geq \widehat{U}} \Gamma^k_{\alpha,\beta}p]^+; \tag{2.4.9}$$

where $\bar{l}$ *and* $\bar{k}$ *were defined in* (2.2.45) *and* (2.2.44).

Hence β is L-bad (resp. γ is U-bad) is equivalent to $\widehat{L} > 0$ (resp. $\widehat{U} > 0$).
Formula (2.4.8) (resp. (2.4.9)) gives the remainder of Lower (resp. Upper) Step 1.
If $L \geq U$ then $\mathcal{R}^L_{\alpha,\beta} = [\Gamma^L_{\alpha,\beta}p - \sum_{U \geq l \geq \widehat{L}} \Gamma^l_{\gamma,\delta}p]^+$ since $\bar{l} = U$, by Remark 2.31. In the symmetric case, that is if $U \geq L$, we get $\mathcal{R}^U_{\gamma,\delta} = [\Gamma^U_{\gamma,\delta}p - \sum_{L \geq k \geq \widehat{U}} \Gamma^k_{\alpha,\beta}p]^+$ since $\bar{k} = L$.

The aim of this section is to prove

Proposition 2.37 *i) If β is L-bad and $l < \widehat{L}$ then under Condition (2.2.13)*

$$H^{L,L,l,l}_{\alpha,\beta,\gamma,\delta} = H^{0,L,0,l}_{\alpha,\beta,\gamma,\delta} = H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta} = H^{L,L,0,l}_{\alpha,\beta,\gamma,\delta} = 0. \tag{2.4.10}$$

ii) If γ is U-bad and $k < \widehat{U}$, then under Condition (2.2.14)

$$H^{k,k,U,U}_{\alpha,\beta,\gamma,\delta} = H^{-k,0,-U,0}_{\alpha,\beta,\gamma,\delta} = H^{k,k,-U,0}_{\alpha,\beta,\gamma,\delta} = H^{-k,0,U,U}_{\alpha,\beta,\gamma,\delta} = 0. \tag{2.4.11}$$

This is the most important proposition and it permits to prove that coupling $\mathcal{H}$ is increasing for jumps, births and deaths involving L or U particles, under Conditions (2.2.13) and (2.2.14) if β or/and γ are bad values. Indeed Proposition 2.37 states that all coupling terms that would break the order of configurations are equal to 0.
Going downwards from N (resp. L, U), to exhibit the indexes from which the attractiveness problem is solved, we introduce

Definition 2.38

$$N^{d+} = \begin{cases} \max\{l : H^{L,L,l,l}_{\alpha,\beta,\gamma,\delta} = J^{L,l}_{\alpha,\beta}\} & \text{if } \widehat{L} > 0 \text{ and } \mathcal{R}^L_{\alpha,\beta} = 0 \\ \widehat{L} - 1 & \text{if } \bar{l} \geq \widehat{L} > 0 \text{ and } \mathcal{R}^L_{\alpha,\beta} > 0 \\ M+1 & \text{otherwise;} \end{cases} \tag{2.4.12}$$

$$N^{d-} = \begin{cases} \max\{k : H^{k,k,U,U}_{\alpha,\beta,\gamma,\delta} = J^{k,U}_{\gamma,\delta}\} & \text{if } \widehat{U} > 0 \text{ and } \mathcal{R}^U_{\gamma,\delta} = 0 \\ \widehat{U} - 1 & \text{if } \bar{k} \geq \widehat{U} > 0 \text{ and } \mathcal{R}^U_{\gamma,\delta} > 0 \\ M+1 & \text{otherwise;} \end{cases} \tag{2.4.13}$$

$$N^{B} = \begin{cases} \max\{l : H^{0,L,0,l}_{\alpha,\beta,\gamma,\delta} = B^{L,l}_{\alpha,\beta}\} & \text{if } \widehat{L} > 0 \\ M+1 & \text{otherwise;} \end{cases} \tag{2.4.14}$$

$$N^{Bd} = \begin{cases} \max\{l : H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta} = B^{L,l}_{\alpha,\beta} - B^{L,l}_{\alpha,\beta} \wedge B^{L,l}_{\gamma,\delta}\} & \text{if } \widehat{L} > 0 \text{ and } N^B < (\gamma-\alpha) \wedge U \\ M+1 & \text{otherwise;} \end{cases} \tag{2.4.15}$$

$$N^{D} = \begin{cases} \max\{k : H^{-k,0,-U,0}_{\alpha,\beta,\gamma,\delta} = D^{k,U}_{\gamma,\delta}\} & \text{if } \widehat{U} > 0 \\ M+1 & \text{otherwise;} \end{cases} \tag{2.4.16}$$

$$N^{Dd} = \begin{cases} \max\{k : H^{-k,k,-U,0}_{\alpha,\beta,\gamma,\delta} = D^{k,U}_{\alpha,\beta} - D^{k,U}_{\gamma,\delta} \wedge D^{k,U}_{\gamma,\delta}\} & \text{if } \widehat{U} > 0 \text{ and } N^D < (\delta-\beta) \wedge L \\ M+1 & \text{otherwise;} \end{cases} \tag{2.4.17}$$

$$N^{dB} = \begin{cases} \max\{l : H^{L,L,0,l}_{\alpha,\beta,\gamma,\delta} = T^{L,l}_{\alpha,\beta}\} & \text{if } \widehat{L} > 0 \text{ and } \mathcal{R}^L_{\alpha,\beta} > 0 \\ M+1 & \text{otherwise;} \end{cases} \tag{2.4.18}$$

$$N^{dD} = \begin{cases} \max\{k : H^{-k,0,U,U}_{\alpha,\beta,\gamma,\delta} = T^{k,U}_{\gamma,\delta}\} & \text{if } \widehat{U} > 0 \text{ and } \mathcal{R}^U_{\gamma,\delta} > 0 \\ M+1 & \text{otherwise.} \end{cases} \tag{2.4.19}$$

The superscript B means birth, D death, d^+ diffusion (jump) on the lower configuration, d^- diffusion (jump) on the upper configuration, Bd birth and diffusion, dB diffusion and birth, Dd death and diffusion, dD diffusion and death.

Notice that Definition (2.2.32) of $H^{0,L,0,0}_{\alpha,\beta,\gamma,\delta}$ ensures that N^B is well defined if β is L-bad, since $0 \in \{l : H^{0,L,0,l}_{\alpha,\beta,\gamma,\delta} = B^{L,0}_{\alpha,\beta}\}$. The same holds for all variables in Definition 2.38 by using uncoupled terms in Definition 2.21. Each variable depends on the system, that is $N^\cdot = N^\cdot(\mathcal{S})$, in particular on $N(\mathcal{S})$, $M(\mathcal{S})$, and $L(\mathcal{S})$. We drop when not necessary the dependence on $\mathcal{S}$.
A first step towards the attractiveness of $\mathcal{H}$ is

Proposition 2.39 *Under conditions* (2.2.13), (2.2.14) *the following inequalities hold:*
i) if β is L-bad and $\mathcal{R}^L_{\alpha,\beta} = 0$, then $N^{d+} \geq \widehat{L}$;
ii) if γ is U-bad and $\mathcal{R}^U_{\gamma,\delta} = 0$, then $N^{d-} \geq \widehat{U}$;
iii) if β is L-bad and $N^B \geq (\gamma - \alpha) \wedge U$, then $N^B \geq \widehat{L}$;
iv) if β is L-bad and $N^B < (\gamma - \alpha) \wedge U$, then $N^B \vee N^{Bd} \geq \widehat{L}$;
v) if γ is U-bad and $N^D \geq (\delta - \beta) \wedge L$, then $N^D \geq \widehat{U}$;
vi) if γ is U-bad and $N^D < (\delta - \beta) \wedge L$, then $N^D \vee N^{Dd} \geq \widehat{U}$;
vii) if β is L-bad and $\mathcal{R}^L_{\alpha,\beta} > 0$, then $N^{dB} \geq \widehat{L}$;
viii) if γ is U-bad and $\mathcal{R}^U_{\gamma,\delta} > 0$, then $N^{dD} \geq \widehat{U}$.

Proposition 2.39 ensures that we are able to solve all the attractiveness problems coming from rates involving L or U particles by using coupling $\mathcal{H}$. To prove Propositions 2.39 and 2.37, we detail how variables in Definition 2.38 are related to the coupling construction, by following the main three steps.

Step 1) Suppose $\beta + L > \delta$.

• We couple the lower configuration L-jump rate with the upper configuration U-jump rate if $U \geq \widehat{L}$ and $L \geq \widehat{U}$ (hence $\bar{l} = U$ and $\bar{k} = L$ by (2.2.45) and (2.2.44))

$$H^{L,L,U,U}_{\alpha,\beta,\gamma,\delta} = J^{L,U}_{\alpha,\beta} \wedge J^{L,U}_{\gamma,\delta} = (\Gamma^L_{\alpha,\beta} \wedge \Gamma^U_{\gamma,\delta})p. \tag{2.4.20}$$

• If $U < \widehat{L}$ ($\beta + L > \delta + U$) hence $L > U$ (and $U = \bar{l}$ by (2.2.45)), $\beta + L > \delta + l$ for each $l \leq U$ and

$$H^{L,L,l,l}_{\alpha,\beta,\gamma,\delta} = 0 \qquad \text{for all } l \tag{2.4.21}$$

(see (2.2.34)). Since $\bar{l} = U < \widehat{L}$, by Definition 2.36,

$$\mathcal{R}^L_{\alpha,\beta} = \Gamma^L_{\alpha,\beta} p > 0$$

and we will come back to $\Gamma^L_{\alpha,\beta} p$ only in Step 3. In this case since $\mathcal{R}^L_{\alpha,\beta} > 0$ and $\bar{l} < \widehat{L}$ then $N^{d+} = M + 1$ by Definition (2.4.12).
• If $L < \widehat{U}$ ($\alpha - L > \gamma - U$), then $\bar{l} = L + \gamma - \alpha$ by (2.2.45), $L < U$, γ is also U-bad and the first positive coupling term, such that $\gamma - l \geq \alpha - L$ by (2.2.34) is, by Definition (2.2.18)

$$H^{L,L,\bar{l},\bar{l}}_{\alpha,\beta,\gamma,\delta} = J^{L,\bar{l}}_{\alpha,\beta} \wedge J^{L,\bar{l}}_{\gamma,\delta} \tag{2.4.22}$$

Therefore if $\bar{l} < \widehat{L}$ we will come back to $\Gamma^L_{\alpha,\beta} p$ in Step 3 and if $\bar{l} \geq \widehat{L}$ we proceed downwards in l from $\bar{l}$ with

$$H^{L,L,l,l}_{\alpha,\beta,\gamma,\delta} = J^{L,l}_{\alpha,\beta} \wedge J^{L,l}_{\gamma,\delta} \qquad \text{if } l \geq \widehat{L}$$

where since births and deaths are introduced only in Step 2,

$$J^{L,l}_{\gamma,\delta} = \Gamma^l_{\gamma,\delta}p - \sum_{k'>L} H^{k',k',l,l}_{\alpha,\beta,\gamma,\delta} - \sum_{k'>L} H^{0,k',l,l}_{\alpha,\beta,\gamma,\delta} = \Gamma^l_{\gamma,\delta}p \tag{2.4.23}$$

$$\begin{aligned} J^{L,l}_{\alpha,\beta} &= \Gamma^L_{\alpha,\beta}p - \sum_{l'>l} H^{L,L,l',l'}_{\alpha,\beta,\gamma,\delta} - \sum_{l'>l} H^{L,L,-l',0}_{\alpha,\beta,\gamma,\delta} = \Gamma^L_{\alpha,\beta}p - \sum_{l'>l} H^{L,L,l',l'}_{\alpha,\beta,\gamma,\delta} \\ &= \Gamma^L_{\alpha,\beta}p - \sum_{l'>l\vee N^{d+}} \Gamma^{l'}_{\gamma,\delta}p \end{aligned} \tag{2.4.24}$$

by (2.4.23) and Definition (2.4.12).

Suppose we begin from (2.4.20) and $H^{L,L,U,U}_{\alpha,\beta,\gamma,\delta} = \Gamma^U_{\gamma,\delta}p$. It implies that going downwards there is no remaining part of $\Gamma^U_{\gamma,\delta}p$ and $H^{k,k,U,U}_{\alpha,\beta,\gamma,\delta} = 0$ for each $k < U$. Since N^{d-} denotes the first value of k such that $H^{k,k,U,U}_{\alpha,\beta,\gamma,\delta} = J^{k,U}_{\alpha,\beta} \wedge J^{k,U}_{\gamma,\delta} = J^{k,U}_{\gamma,\delta}$, $N^{d-} = L$. Similarly, if $H^{L,L,U,U}_{\alpha,\beta,\gamma,\delta} = \Gamma^L_{\alpha,\beta}p$ then $N^{d+} = U$.

Remark 2.40 *Therefore if we start from* (2.4.20) *at least one term between* N^{d+} *and* N^{d-} *is equal to its maximal possible value and, if* $L \neq U$*, it cannot be both.*

This construction is detailed in Tables 2.1 and 2.2. While the minimum defining $H^{L,L,l,l}_{\alpha,\beta,\gamma,\delta}$

Table 2.1: L jump rate, $N^{d+} \neq M+1$, $\mathcal{R}^L_{\alpha,\beta} = 0$, $\bar{l} \geq \widehat{L}$

l	$J^{L,l}_{\alpha,\beta}$	$\wedge$	$J^{L,l}_{\gamma,\delta}$	$=$ $H^{L,L,l,l}_{\alpha,\beta,\gamma,\delta}$
$\bar{l}$			$\times$	
$\vdots$			$\vdots$	$J^{L,l}_{\gamma,\delta} = \Gamma^l_{\gamma,\delta}p$
$N^{d+}+1$			$\times$	
N^{d+}	$\times$			$J^{L,l}_{\alpha,\beta} = \Gamma^L_{\alpha,\beta}p - \sum_{\bar{l}\geq l'>N^{d+}} \Gamma^{l'}_{\gamma,\delta}p \geq 0$
$N^{d+}-1$	$\times$			
$\vdots$	$\vdots$			$J^{L,l}_{\alpha,\beta} = 0$
$\widehat{L}$	$\times$			

is the second term, we still have a lower attractiveness problem and we go on downwards in l. As soon as the minimum is the first term, we have no remainder of $\Gamma^L_{\alpha,\beta}p$, so the problem is solved, and N^{d+} corresponds to the first such l. Then for l smaller $J^{L,l}_{\alpha,\beta} = 0$. In other words when $l > N^{d+}$, the coupling term is the upper configuration l-jump rate, when $l = N^{d+}$ it is the remainder of the lower configuration L-jump rate, and when $l < N^{d+}$ the coupling terms are null. Hence N^{d+} is the index of the last term for which there is a remainder of $\Gamma^L_{\alpha,\beta}p$.

We have to distinguish between two situations:

-*if* $M+1 > N^{d+} \geq \widehat{L}$ *then* $\mathcal{R}^L_{\alpha,\beta} = 0$ *(see Table* 2.1*)*, and we do not need Step 3, which implies that

$$H^{L,L,0,l}_{\alpha,\beta,\gamma,\delta} = H^{-k,0,U,U}_{\alpha,\beta,\gamma,\delta} = 0 \qquad \text{for each } k,l; \tag{2.4.25}$$

Table 2.2: L jump rate, $N^{d+} \neq M+1$, $\mathcal{R}^L_{\alpha,\beta} > 0$, $\bar{l} \geq \widehat{L}$

l	$J^{L,l}_{\alpha,\beta}$ $\wedge$ $J^{L,l}_{\gamma,\delta}$	$=$ $H^{L,L,l,l}_{\alpha,\beta,\gamma,\delta}$
$\bar{l}$	$\times$	
$\vdots$	$\vdots$	$J^{L,l}_{\gamma,\delta} = \Gamma^l_{\gamma,\delta} p$
$\widehat{L}$	$\times$	

$$N^{d+} = \widehat{L} - 1$$

-if we do not reach N^{d+} at $\widehat{L}$ yet, that is the minimum given by $H^{L,L,,l,l}_{\alpha,\beta,\gamma,\delta}$ is always the second term, since we have decided to stop at $\widehat{L}^{th}$-step of the coupling construction, we put $N^{d+} = \widehat{L}-1$. We need Step 3 in order to solve the attractiveness problem (Table 2.2 *when $\mathcal{R}^L_{\alpha,\beta} > 0$).*
In both cases, we have

$$H^{L,L,l,l}_{\alpha,\beta,\gamma,\delta} = 0 \qquad \text{for } l < \widehat{L}. \tag{2.4.26}$$

• If $\widehat{L} \leq 0$ we have no lower attractiveness problem and we put $N^{d+} = M+1$.
Suppose γ is U-bad.
• In a symmetric way we start from formula (2.4.20) if $U \geq \widehat{L}$ and $\widehat{U} \leq L$. In this case $\bar{k} = L$.
• If $\widehat{U} > L$, that is $\gamma - U < \alpha - L$, then we will come back to $\Gamma^U_{\gamma,\delta}p$ at Step 3. Notice that in this case $\mathcal{R}^U_{\gamma,\delta} > 0$ and $\bar{k} < \widehat{U}$, thus $N^{d-} = M+1$ by Definition (2.4.13).
• If $U < \widehat{L}$, that is $\beta + L > \delta + U$, then $\bar{k} = U + \delta - \beta$ by (2.2.44), $U < L$, β is L-bad also, so that we start from

$$H^{\bar{k},\bar{k},U,U}_{\alpha,\beta,\gamma,\delta} = J^{\bar{k},U}_{\alpha,\beta} \wedge J^{\bar{k},U}_{\gamma,\delta} = (\Gamma^{\bar{k}}_{\alpha,\beta} \wedge \Gamma^U_{\gamma,\delta})p. \tag{2.4.27}$$

In both cases (2.4.20) and (2.4.27), that is if $\bar{k} \geq \widehat{U}$, we go downwards in k from $\bar{k}$,

$$H^{k,k,U,U}_{\alpha,\beta,\gamma,\delta} = J^{k,U}_{\alpha,\beta} \wedge J^{k,U}_{\gamma,\delta} \qquad \text{if } k \geq \widehat{U}$$

$$J^{k,U}_{\alpha,\beta} = \Gamma^k_{\alpha,\beta}p - \sum_{l'>U} H^{k,k,l',l'}_{\alpha,\beta,\gamma,\delta} - \sum_{l'>U} H^{k,k,-l',0}_{\alpha,\beta,\gamma,\delta} = \Gamma^k_{\alpha,\beta}p \tag{2.4.28}$$

$$\begin{aligned} J^{k,U}_{\gamma,\delta} &= \Gamma^U_{\gamma,\delta}p - \sum_{\bar{k} \geq k' > k} H^{k',k',U,U}_{\alpha,\beta,\gamma,\delta} - \sum_{k'>k} H^{0,k',U,U}_{\alpha,\beta,\gamma,\delta} = \Gamma^U_{\gamma,\delta}p - \sum_{\bar{k} \geq k' > k} H^{k',k',U,U}_{\alpha,\beta,\gamma,\delta} \\ &= \Gamma^U_{\gamma,\delta}p - \sum_{\bar{k} \geq k' > k \vee N^{d-}} \Gamma^{k'}_{\alpha,\beta}p \end{aligned} \tag{2.4.29}$$

by (2.4.28) and Definition (2.4.13).
• If $\widehat{U} \leq 0$ we have no higher attractiveness problem, and we put $N^{d-} = M+1$.

The construction starting from (2.4.20) or (2.4.27) is detailed in Tables 2.3 and 2.4.
We explained the meaning of formulas (2.4.12) and (2.4.13) in Definition 2.38. Notice

Table 2.3: U jump rate, $N^{d-} \neq M+1$, $\mathcal{R}^U_{\gamma,\delta} = 0$, $\bar{k} \geq \widehat{U}$

k	$J^{k,U}_{\alpha,\beta}$	$\wedge$	$J^{k,U}_{\gamma,\delta}$	$=$ $H^{k,k,U,U}_{\alpha,\beta,\gamma,\delta}$
$\bar{k}$	$\times$			
$\vdots$	$\vdots$			$J^{k,U}_{\alpha,\beta} = \Gamma^k_{\alpha,\beta} p$
$N^{d-}+1$	$\times$			
N^{d-}			$\times$	$J^{k,U}_{\gamma,\delta} = \Gamma^U_{\gamma,\delta} p - \sum_{\bar{k} \geq k' > N^{d-}} \Gamma^{k'}_{\alpha,\beta} p \geq 0$
$N^{d-}-1$			$\times$	
$\vdots$			$\vdots$	$J^{k,U}_{\gamma,\delta} = 0$
$\widehat{U}$			$\times$	

Table 2.4: U jump rate, $N^{d-} \neq M+1$, $\mathcal{R}^U_{\gamma,\delta} > 0$, $\bar{k} \geq \widehat{U}$

k	$J^{k,U}_{\alpha,\beta}$	$\wedge$	$J^{k,U}_{\gamma,\delta}$	$=$ $H^{k,k,U,U}_{\alpha,\beta,\gamma,\delta}$
$\bar{k}$	$\times$			
$\vdots$	$\vdots$			$J^{k,U}_{\alpha,\beta} = \Gamma^k_{\alpha,\beta} p$
$\widehat{U}$	$\times$			

$$N^{d-} = \widehat{U} - 1$$

that the construction works if either (β is L-bad and γ is U-bad), by using Tables from 2.1 to 2.4, and if only one of them is bad, by referring to the corresponding tables.

Proof of Proposition 2.39 *i)–ii)*.
Claims i) (resp. ii)) follow from Definition 2.38 of N^{d+} (resp. N^{d-}), (2.4.8) and (2.4.24) (resp. (2.4.9) and (2.4.29)). $\square$

Step 2) We consider lower birth coupling rates when β is L-bad (death coupling rates if γ is U-bad work in a symmetric way). We refer to Tables 2.5 and 2.6.

Step 2a) Suppose $l \geq (\gamma - \alpha) \wedge U$ (notice that if γ is U-bad then $(\gamma - \alpha) \wedge U = U$). We refer to Table 2.5: $\Pi^{0,L}_{\alpha,\beta} p$ can be coupled only with $\Pi^{0,l}_{\gamma,\delta}$ if $l > (\gamma - \alpha) \wedge U$. If $l = (\gamma - \alpha) \wedge U$ we can choose between a jump or a birth rate, but we give priority to births. Remember that by Definition (2.2.21), $H^{0,L,0,l}_{\alpha,\beta,\gamma,\delta} = B^{L,l}_{\alpha,\beta} \wedge B^{L,l}_{\gamma,\delta}$ with

$$B^{L,l}_{\gamma,\delta} = \Pi^{0,l}_{\gamma,\delta} p - \sum_{k'>L} H^{0,k',0,l}_{\alpha,\beta,\gamma,\delta} - \sum_{k'>L} H^{k',k',0,l}_{\alpha,\beta,\gamma,\delta} = \Pi^{0,l}_{\gamma,\delta} p \tag{2.4.30}$$

Table 2.5: L birth rate, $N^B \geq (\gamma-\alpha)\wedge U$

l	$B^{L,l}_{\alpha,\beta}$	$\wedge$ $B^{L,l}_{\gamma,\delta}$	$=$ $H^{0,L,0,l}_{\alpha,\beta,\gamma,\delta}$
M		$\times$	
$\vdots$		$\vdots$	$B^{L,l}_{\gamma,\delta}=\Pi^{0,l}_{\gamma,\delta}p$
N^B+1		$\times$	
N^B	$\times$		$B^{L,l}_{\alpha,\beta}=\Pi^{0,L}_{\alpha,\beta}p-\sum_{l'>N^B}\Pi^{0,l'}_{\gamma,\delta}p\geq 0$
N^B-1	$\times$		
$\vdots$	$\vdots$		$B^{L,l}_{\alpha,\beta}=0$
$(\gamma-\alpha)\wedge U$	$\times$		

and by Definition (2.4.14) and (2.4.30)

$$B^{L,l}_{\alpha,\beta} =\Pi^{0,L}_{\alpha,\beta}p-\sum_{l'>l}H^{0,L,0,l'}_{\alpha,\beta,\gamma,\delta}=\Pi^{0,L}_{\alpha,\beta}p-\sum_{l'>l\vee N^B}\Pi^{0,l'}_{\gamma,\delta}p \tag{2.4.31}$$

when $l\geq(\gamma-\alpha)\wedge U$ by (2.2.19) because of Definition (2.2.6). While the minimum in (2.2.21) is the second term we go on downwards in l. As soon as the minimum is the first term, we have no remainder of $\Pi^{0,L}_{\alpha,\beta}p$, so the lower problem is solved and N^B corresponds to this first such l. Then for l smaller $B^{L,l}_{\alpha,\beta}=H^{0,L,0,l}_{\alpha,\beta,\gamma,\delta}=0$ and by Definitions (2.2.22) and (2.2.21)

$$H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta}\leq B^{L,l}_{\alpha,\beta}-H^{0,L,0,l}_{\alpha,\beta,\gamma,\delta}=0. \tag{2.4.32}$$

We can meet different situations:
-*if we reach N^B before or at $(\gamma-\alpha)\wedge U$, we solve the attractiveness problem without coupling with any upper configuration jump rate. We do not have to pass through Step 2b), then*

$$H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta}=0 \qquad \textit{for each } l \tag{2.4.33}$$

and we set $N^{Bd}=M+1$, see Table 2.5.
-*if we do not reach N^B at $(\gamma-\alpha)\wedge U$ yet, we pass to Step 2b).*

Step 2b) We refer to Tables 2.6 (a), (b) and Tables 2.7 (a), (b). By Definitions (2.2.21), (2.2.22) and (2.4.30), if $l\leq(\gamma-\alpha)\wedge U$

$$H^{0,L,0,l}_{\alpha,\beta,\gamma,\delta} =B^{L,l}_{\alpha,\beta}\wedge(\Pi^{0,l}_{\gamma,\delta}p) \tag{2.4.34}$$
$$H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta} =[B^{L,l}_{\alpha,\beta}-B^{L,l}_{\alpha,\beta}\wedge(\Pi^{0,l}_{\gamma,\delta}p)]\wedge[J^{L,l}_{\gamma,\delta}-J^{L,l}_{\alpha,\beta}\wedge J^{L,l}_{\gamma,\delta}] \tag{2.4.35}$$

We couple the remainder of $\Pi^{0,L}_{\alpha,\beta}p$ after Step 2a), that is $\Pi^{0,L}_{\alpha,\beta}p-\sum_{l'\geq(\gamma-\alpha)\wedge U}\Pi^{0,l'}_{\gamma,\delta}p$, alternatively with upper configuration birth and jump rates.
When $l=(\gamma-\alpha)\wedge U$, and the minimum in (2.4.34) is the second term, we pass to (2.4.35),

Table 2.6: L birth rate, $N^B < (\gamma-\alpha)\wedge U$ and $N^B = N^{Bd}$

a)

l	$B^{L,l}_{\alpha,\beta}$	$\wedge\ B^{L,l}_{\gamma,\delta}$	$= H^{0,L,0,l}_{\alpha,\beta,\gamma,\delta}$
$(\gamma-\alpha)\wedge U$		$\times$	
$\vdots$		$\vdots$	$B^{L,l}_{\gamma,\delta} = \Pi^{0,l}_{\gamma,\delta} p$
N^B+1		$\times$	
N^B	$\times$		$B^{L,l}_{\alpha,\beta} \geq 0$
N^B-1	$\times$		
$\vdots$	$\vdots$		$B^{L,l}_{\alpha,\beta} = 0$
0	$\times$		

b)

l	$[B^{L,l}_{\alpha,\beta} - B^{L,l}_{\alpha,\beta}\wedge B^{L,l}_{\gamma,\delta}]\wedge$	$[J^{L,l}_{\gamma,\delta} - J^{L,l}_{\alpha,\beta}\wedge J^{L,l}_{\gamma,\delta}]$	$= H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta}$
$(\gamma-\alpha)\wedge U$		$\times$	
$\vdots$		$\vdots$	$J^{L,l}_{\gamma,\delta} - J^{L,l}_{\alpha,\beta}\wedge J^{L,l}_{\gamma,\delta}$
N^B+1		$\times$	
$N^B = N^{Bd}$	$\times$		$B^{L,l}_{\alpha,\beta} - B^{L,l}_{\alpha,\beta}\wedge B^{L,l}_{\gamma,\delta} = 0$
N^B-1	$\times$		
$\vdots$	$\vdots$		$B^{L,l}_{\alpha,\beta} - B^{L,l}_{\alpha,\beta}\wedge B^{L,l}_{\gamma,\delta} = 0$
0	$\times$		

and we couple the remainder of $\Pi^{0,L}_{\alpha,\beta}p$ with the remainder of a jump rate from Step 1 (if $\bar{l} < \widehat{L}$ we did not pass Step 1, then we use the original rates). If the minimum is still the second term, we come back to (2.4.34) with $l = (\gamma-\alpha)\wedge U - 1$ and pass to (2.4.35) if necessary. We go on downwards in l until either in (2.4.34) (see Tables 2.6 (a) and (b)) or in (2.4.35) (see Tables 2.7 (a) and (b)) the minimum is the first term, that is until we solve the attractiveness problem. There N^B (resp. N^{Bd}) corresponds to the l^{th} step when it happens for (2.4.34) (resp. for (2.4.35)).

Lemma 2.41 *If* $N^B < (\gamma-\alpha)\wedge U$

$$N^{Bd} \geq N^B \geq N^{Bd} - 1.$$

If $N^D < (\delta-\beta)\wedge L$

$$N^{Dd} \geq N^D \geq N^{Dd} - 1.$$

Proof .
Suppose $N^B < (\gamma-\alpha)\wedge U$ (otherwise $N^{Bd} = M+1$). By Definition (2.4.14)

$$B^{L,N^B}_{\alpha,\beta} - B^{L,N^B}_{\alpha,\beta}\wedge B^{L,N^B}_{\gamma,\delta} = B^{L,N^B}_{\alpha,\beta} - B^{L,N^B}_{\alpha,\beta} = 0.$$

Table 2.7: L birth rate, $N^B < (\gamma-\alpha)\wedge U$ and $N^B = N^{Bd}-1$

a)

l	$B^{L,l}_{\alpha,\beta}$ $\wedge$	$B^{L,l}_{\gamma,\delta}$	$= H^{0,L,0,l}_{\alpha,\beta,\gamma,\delta}$
$(\gamma-\alpha)\wedge U$		$\times$	$B^{L,l}_{\gamma,\delta} = \Pi^{0,l}_{\gamma,\delta}p$
$\vdots$		$\vdots$	
$N^B+1 = N^{Bd}$		$\times$	
N^B	$\times$		$B^{L,l}_{\alpha,\beta} = 0$
N^B-1	$\times$		$B^{L,l}_{\alpha,\beta} = 0$
$\vdots$	$\vdots$		
0	$\times$		

b)

l	$[B^{L,l}_{\alpha,\beta} - B^{L,l}_{\alpha,\beta}\wedge B^{L,l}_{\gamma,\delta}]\wedge$	$[J^{L,l}_{\gamma,\delta}p - J^{L,l}_{\alpha,\beta}\wedge J^{L,l}_{\gamma,\delta}]$	$= H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta}$
$(\gamma-\alpha)\wedge U$		$\times$	$J^{L,l}_{\gamma,\delta}p - J^{L,l}_{\alpha,\beta}\wedge J^{L,l}_{\gamma,\delta}$
$\vdots$		$\vdots$	
$N^{Bd}+1$		$\times$	
N^{Bd}	$\times$		$B^{L,l}_{\alpha,\beta} - B^{L,l}_{\gamma,\delta} \geq 0$
$N^{Bd}-1 = N^B$	$\times$		$B^{L,l}_{\alpha,\beta} - B^{L,l}_{\alpha,\beta}\wedge B^{L,l}_{\gamma,\delta} = 0$
$\vdots$	$\vdots$		
0	$\times$		

and $H^{0,L,N^B,N^B}_{\alpha,\beta,\gamma,\delta} = 0$ by Definition (2.2.22). Therefore $N^{Bd} \geq N^B$.
Suppose $N^{Bd} > N^B$. By Definitions (2.2.19), (2.2.20) and (2.2.22)

$$\begin{aligned} B^{L,N^{Bd}-1}_{\alpha,\beta} =& B^{L,N^{Bd}}_{\alpha,\beta} - H^{0,L,0,N^{Bd}}_{\alpha,\beta,\gamma,\delta} - H^{0,L,N^{Bd},N^{Bd}}_{\alpha,\beta,\gamma,\delta} \\ =& B^{L,N^B}_{\alpha,\beta} - B^{L,N^{Bd}}_{\alpha,\beta}\wedge B^{L,N^{Bd}}_{\gamma,\delta} - (B^{L,N^B}_{\alpha,\beta} - B^{L,N^{Bd}}_{\alpha,\beta}\wedge B^{L,N^{Bd}}_{\gamma,\delta}) = 0. \end{aligned}$$

Therefore $N^B \geq N^{Bd}-1$.
The relations between N^D and N^{Dd} are proved in the same way. □

Remark 2.42 *If the first choice of the first term is given by* (2.4.34) *(resp.* (2.4.35)*), then at the next step* $H^{0,L,N^B,N^B}_{\alpha,\beta,\gamma,\delta} = 0$ *and* $N^B = N^{Bd}$ *(resp.* $H^{0,L,0,N^{Bd}-1}_{\alpha,\beta,\gamma,\delta} = 0$ *and* $N^B = N^{Bd}-1$*). Hence by construction at least one term between* $H^{0,L,0,N^B}_{\alpha,\beta,\gamma,\delta}$ *and* $H^{0,L,N^{Bd},N^{Bd}}_{\alpha,\beta,\gamma,\delta}$ *must be zero.*

Notice that once we solve the attractiveness problem, the remaining rates are null.
In other words if $l > N^B$ then $H^{0,L,0,l}_{\alpha,\beta,\gamma,\delta} = \Pi^{0,l}_{\gamma,\delta}p$; if $l = N^B$ then $H^{0,L,0,l}_{\alpha,\beta,\gamma,\delta} = B^{L,l}_{\alpha,\beta}$ and if $l < N^B$ then $H^{0,L,0,l}_{\alpha,\beta,\gamma,\delta} = 0$. Moreover if $l > N^{Bd}$ then $H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta} = J^{L,l}_{\gamma,\delta} - J^{L,l}_{\alpha,\beta}\wedge J^{L,l}_{\gamma,\delta}$; if $l = N^{Bd}$ then $H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta} = B^{L,l}_{\alpha,\beta} - B^{L,l}_{\alpha,\beta}\wedge B^{L,l}_{\gamma,\delta}$ and if $l < N^{Bd}$ then $H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta} = 0$. The

symmetric case for death rates is explained in Tables 2.8, 2.9 and 2.10.

We explained the meaning of formulas (2.4.14)–(2.4.15) for birth rates and (2.4.16)–

Table 2.8: U death rate, $N^D \geq (\delta-\beta)\wedge L$

k	$D^{k,U}_{\alpha,\beta}$ $\wedge$ $D^{k,U}_{\gamma,\delta}$	$=$ $H^{-k,0,-U,0}_{\alpha,\beta,\gamma,\delta}$
M	$\times$	
$\vdots$	$\vdots$	$D^{k,U}_{\alpha,\beta}=\Pi^{-k,0}_{\alpha,\beta}p$
N^D+1	$\times$	
N^D	$\times$	$D^{k,U}_{\gamma,\delta}=\Pi^{-U,0}_{\gamma,\delta}p-\sum_{k'>N^D}\Pi^{-k',0}_{\alpha,\beta}p\geq 0$
N^D-1	$\times$	
$\vdots$	$\vdots$	$D^{k,U}_{\gamma,\delta}=0$
$(\delta-\beta)\wedge L$	$\times$	

(2.4.17) for death rates in Definition 2.38. Remember that if β is not L-bad (resp. if γ is not U-bad) we set $N^B=N^{Bd}=M+1$ (resp. $N^D=N^{Dd}=M+1$).

Proof of Proposition 2.39 *iii)–iv)–v)–vi).*
iii) We prove this result (and the following ones) by contradiction. Suppose $l>N^B\geq(\gamma-\alpha)\wedge U$. By (2.4.30), Definition (2.4.14) of N^B and (2.4.31)

$$B^{L,l}_{\gamma,\delta}=\Pi^{0,l}_{\gamma,\delta}p=H^{0,L,0,l}_{\alpha,\beta,\gamma,\delta}<B^{L,l}_{\alpha,\beta}=\Pi^{0,L}_{\alpha,\beta}p-\sum_{l'>l}\Pi^{0,l'}_{\gamma,\delta}p. \tag{2.4.36}$$

Suppose $\widehat{L}>N^B$. Then $B^{L,\widehat{L}}_{\alpha,\beta}>B^{L,\widehat{L}}_{\gamma,\delta}$ if and only if

$$\Pi^{0,L}_{\alpha,\beta}-\sum_{l'\geq\widehat{L}}\Pi^{0,l'}_{\gamma,\delta}>0 \tag{2.4.37}$$

but this contradicts (2.2.13). Indeed by taking $K=1$, $m_1=0$, $l_1=\widehat{L}-1$, since $\widehat{L}>N^B\geq(\gamma-\alpha)\wedge U$, $I_a=\{0\geq k'>L-1\}=\emptyset$, $I_b=\{(\gamma-\alpha)\wedge U\geq l'>\widehat{L}-1\}=\emptyset$ and Condition (2.2.13) reduces to

$$\Pi^{0,L}_{\alpha,\beta}\leq\sum_{l'\geq\widehat{L}}\Pi^{0,l'}_{\gamma,\delta}.$$

iv) Suppose $N^B<(\gamma-\alpha)\wedge U$ and $N^B\vee N^{Bd}<\widehat{L}$. If $(\gamma-\alpha)\wedge U\leq\widehat{L}$ what we did for *iii*) is still valid, hence a contradiction. Suppose $(\gamma-\alpha)\wedge U>\widehat{L}$; by Definition (2.4.15) of N^{Bd}, (2.4.23) and (2.4.30), if $l>N^{Bd}$

$$\begin{aligned}H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta}&=J^{L,l}_{\gamma,\delta}-J^{L,l}_{\alpha,\beta}\wedge J^{L,l}_{\gamma,\delta}=\Gamma^l_{\gamma,\delta}p-J^{L,l}_{\alpha,\beta}\wedge(\Gamma^l_{\gamma,\delta}p)\\&<B^{L,l}_{\alpha,\beta}-B^{L,l}_{\alpha,\beta}\wedge B^{L,l}_{\gamma,\delta}=B^{L,l}_{\alpha,\beta}-B^{L,l}_{\gamma,\delta}\\&=B^{L,l}_{\alpha,\beta}-\Pi^{0,l}_{\gamma,\delta}p.\end{aligned} \tag{2.4.38}$$

Table 2.9: U death rate, $N^D < (\delta-\beta)\wedge L$ and $N^D = N^{Dd}$

a)

k	$D^{k,U}_{\alpha,\beta}$ $\wedge$	$D^{k,U}_{\gamma,\delta}$	$= \quad H^{-k,0,-U,0}_{\alpha,\beta,\gamma,\delta}$
$(\delta-\beta)\wedge L$	$\times$		
$\vdots$	$\vdots$		$D^{k,U}_{\alpha,\beta} = \Pi^{-k,0}_{\alpha,\beta}p$
N^D+1	$\times$		
N^D		$\times$	$D^{k,U}_{\gamma,\delta} \geq 0$
N^D-1		$\times$	
$\vdots$		$\vdots$	$D^{k,U}_{\gamma,\delta} = 0$
0		$\times$	

b)

k	$[J^{k,U}_{\alpha,\beta} - J^{k,U}_{\gamma,\delta}\wedge J^{k,U}_{\alpha,\beta}]\wedge$	$[D^{k,U}_{\gamma,\delta} - D^{k,U}_{\gamma,\delta}\wedge D^{k,U}_{\alpha,\beta}]$	$= H^{k,k,-U,0}_{\alpha,\beta,\gamma,\delta}$
$(\delta-\beta)\wedge L$	$\times$		
$\vdots$	$\vdots$		$J^{k,U}_{\alpha,\beta} - J^{k,U}_{\gamma,\delta}\wedge J^{k,U}_{\alpha,\beta}$
$N^{Dd}+1$	$\times$		
$N^{Dd} = N^D$		$\times$	$D^{k,U}_{\gamma,\delta} - D^{k,U}_{\gamma,\delta}\wedge D^{k,U}_{\alpha,\beta} = 0$
$N^{Dd}-1$		$\times$	
$\vdots$		$\vdots$	$D^{k,U}_{\gamma,\delta} - D^{k,U}_{\gamma,\delta}\wedge D^{k,U}_{\alpha,\beta} = 0$
0		$\times$	

We compute $B^{L,l}_{\alpha,\beta}$. By (2.2.19), since $l > N^{Bd} \geq N^B$ by Lemma 2.41

$$\begin{aligned} B^{L,l}_{\alpha,\beta} =& \Pi^{0,L}_{\alpha,\beta}p - \sum_{l'>l} H^{0,L,0,l'}_{\alpha,\beta,\gamma,\delta} - \sum_{U\wedge(\gamma-\alpha)\geq l'>l} H^{0,L,l',l'}_{\alpha,\beta,\gamma,\delta} \\ =& \Pi^{0,L}_{\alpha,\beta}p - \sum_{l'>l} \Pi^{0,l'}_{\gamma,\delta}p - \sum_{U\wedge(\gamma-\alpha)\geq l'>l} (J^{L,l'}_{\gamma,\delta} - J^{L,l'}_{\alpha,\beta}\wedge J^{L,l'}_{\gamma,\delta}). \end{aligned}$$

Remark 2.43 *When $l > N^{Bd}$ the term $H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta}$ depends on Step 1 (see Tables 2.1 and 2.2). By (2.4.23),*
-if $l > N^{d+}$ then $J^{L,l}_{\alpha,\beta}\wedge J^{L,l}_{\gamma,\delta} = J^{L,l}_{\gamma,\delta}$ and

$$H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta} = 0; \tag{2.4.39}$$

-if $l = N^{d+} \geq \widehat{L}$ and $\mathcal{R}^L_{\alpha,\beta} = 0$ (see Table 2.1), then $J^{L,l}_{\alpha,\beta}\wedge J^{L,l}_{\gamma,\delta} = J^{L,l}_{\alpha,\beta}$

$$H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta} = J^{L,l}_{\gamma,\delta} - J^{L,l}_{\alpha,\beta} = \sum_{\bar{l}\geq l'\geq N^{d+}} \Gamma^{l'}_{\gamma,\delta}p - \Gamma^{L}_{\alpha,\beta}p; \tag{2.4.40}$$

-if $\widehat{L} \leq l < N^{d+}$ then $J^{L,l}_{\alpha,\beta}\wedge J^{L,l}_{\gamma,\delta} = J^{L,l}_{\alpha,\beta} = 0$; hence

$$H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta} = J^{L,l}_{\gamma,\delta} = \Gamma^{l}_{\gamma,\delta}p \qquad \text{if } \widehat{L} \leq l < N^{d+}$$

Table 2.10: U death rate, $N^D < (\delta-\beta)\wedge L$ and $N^D = N^{Dd}-1$

a)

k	$D_{\alpha,\beta}^{k,U}$	$\wedge\ D_{\gamma,\delta}^{k,U}$	$= H_{\alpha,\beta,\gamma,\delta}^{-k,0,-U,0}$
$(\delta-\beta)\wedge L$	$\times$		$D_{\alpha,\beta}^{k,U} = \Pi_{\alpha,\beta}^{-k,0}p$
$\vdots$	$\vdots$		
$N^D+1 = N^{Dd}$	$\times$		
N^D		$\times$	$D_{\gamma,\delta}^{k,U} = 0$
N^D-1		$\times$	$D_{\gamma,\delta}^{k,U} = 0$
$\vdots$		$\vdots$	
0		$\times$	

b)

k	$[J_{\alpha,\beta}^{k,U} - J_{\gamma,\delta}^{k,U}\wedge J_{\alpha,\beta}^{k,U}]\wedge$	$[D_{\gamma,\delta}^{k,U} - D_{\gamma,\delta}^{k,U}\wedge D_{\alpha,\beta}^{k,U}]$	$=H_{\alpha,\beta,\gamma,\delta}^{k,k,-U,0}$
$(\delta-\beta)\wedge L$	$\times$		$J_{\alpha,\beta}^{k,U} - J_{\gamma,\delta}^{k,U}\wedge J_{\alpha,\beta}^{k,U}$
$\vdots$	$\vdots$		
$N^{Dd}+1$	$\times$		
N^{Dd}		$\times$	$D_{\gamma,\delta}^{k,U} - D_{\alpha,\beta}^{k,U} \geq 0$
$N^{Dd}-1 = N^D$		$\times$	$D_{\gamma,\delta}^{k,U} - D_{\gamma,\delta}^{k,U}\wedge D_{\alpha,\beta}^{k,U} = 0$
$\vdots$		$\vdots$	
0		$\times$	

-If $N^{d+} = M+1$, that is we did not pass Step 1, and if $l \geq \widehat{L} > U$, then $H_{\alpha,\beta,\gamma,\delta}^{0,L,l,l} = 0$. In other words even if we choose the second term of $H_{\alpha,\beta,\gamma,\delta}^{0,L,l,l}$, we could have a null coupling rate, when solving the lower attractiveness problem left no remainder of $\Gamma_{\gamma,\delta}^{l}p$. It means that positive coupling terms begin below N^{d+}.

Therefore by previous remark, if $l > N^{Bd}$ and $l \geq \widehat{L}$

$$
\begin{aligned}
B_{\alpha,\beta}^{L,l} =& \Pi_{\alpha,\beta}^{0,L}p - \sum_{l'>l}\Pi_{\gamma,\delta}^{0,l'}p - \sum_{(\gamma-\alpha)\wedge U\geq l'>l}\Big\{ 1\!\!1_{\{l'=N^{d+}\geq \widehat{L}\}}\big(\sum_{\bar{l}\geq l'\geq N^{d+}}\Gamma_{\gamma,\delta}^{l'}p - \Gamma_{\alpha,\beta}^{L}p\big) \\
&+ 1\!\!1_{\{N^{d+}>l'\}}\Gamma_{\gamma,\delta}^{l'}p\Big\} \\
=& \Pi_{\alpha,\beta}^{0,L}p - \sum_{l'>l}\Pi_{\gamma,\delta}^{0,l'}p - 1\!\!1_{\{(\gamma-\alpha)\wedge U\geq N^{d+}>(l\vee(\widehat{L}-1))\}}\big(\sum_{\bar{l}\geq l'\geq N^{d+}}\Gamma_{\gamma,\delta}^{l'}p - \Gamma_{\alpha,\beta}^{L}p\big) \\
&- \sum_{(\gamma-\alpha)\wedge U\wedge(N^{d+}-1)\geq l'>l}\Gamma_{\gamma,\delta}^{l'}p
\end{aligned}
$$

$$
J_{\gamma,\delta}^{L,l} - J_{\alpha,\beta}^{L,l}\wedge J_{\gamma,\delta}^{L,l} = 1\!\!1_{\{N^{d+}\geq l\}}\Big\{\Gamma_{\gamma,\delta}^{l}p - 1\!\!1_{\{l=N^{d+}\}}\big(\Gamma_{\alpha,\beta}^{L}p - \sum_{\bar{l}\geq l'>N^{d+}}\Gamma_{\gamma,\delta}^{l'}p\big)\Big\}
$$

Hence if $l = \widehat{L} > N^{Bd}$, (2.4.38) becomes

$$\mathbb{1}_{\{N^{d+}\geq\widehat{L}\}}\Big\{\Gamma^{\widehat{L}}_{\gamma,\delta}p - \mathbb{1}_{\{N^{d+}=\widehat{L}\}}(\Gamma^{L}_{\alpha,\beta}p - \sum_{\bar{l}\geq l'>N^{d+}}\Gamma^{l'}_{\gamma,\delta}p)\Big\} < \Pi^{0,L}_{\alpha,\beta}p - \sum_{l'\geq\widehat{L}}\Pi^{0,l'}_{\gamma,\delta}p$$
$$- \mathbb{1}_{\{(\gamma-\alpha)\wedge U\geq N^{d+}>\widehat{L}\}}(\sum_{\bar{l}\geq l'\geq N^{d+}}\Gamma^{l'}_{\gamma,\delta}p - \Gamma^{L}_{\alpha,\beta}p) - \sum_{(\gamma-\alpha)\wedge U\wedge(N^{d+}-1)\geq l'>\widehat{L}}\Gamma^{l'}_{\gamma,\delta}p \qquad (2.4.41)$$

• Suppose $N^{d+} \geq \widehat{L}$.
If $(\gamma-\alpha)\wedge U \geq N^{d+} > \widehat{L}$ or $N^{d+} = \widehat{L}$ then (2.4.41) becomes

$$\Pi^{0,L}_{\alpha,\beta}p + \Gamma^{L}_{\alpha,\beta}p - \sum_{l'\geq\widehat{L}}\Pi^{0,l'}_{\gamma,\delta}p - \sum_{\bar{l}\geq l'\geq\widehat{L}}\Gamma^{l'}_{\gamma,\delta}p > 0 \qquad (2.4.42)$$

which contradicts (2.2.13) with $K = 1$, $l_1 = \widehat{L} - 1$, $m_1 = \bar{l}$.
If $N^{d+} > (\gamma-\alpha)\wedge U \geq \widehat{L}$ then (2.4.41) reduces to

$$\Pi^{0,L}_{\alpha,\beta}p - \sum_{l'\geq\widehat{L}}\Pi^{0,l'}_{\gamma,\delta}p - \sum_{(\gamma-\alpha)\wedge U\geq l'\geq\widehat{L}}\Gamma^{l'}_{\gamma,\delta}p > 0$$

By adding $\mathcal{R}^{L}_{\alpha,\beta} = \Gamma^{L}_{\alpha,\beta}p - \sum_{\bar{l}\geq l'>N^{d+}}\Gamma^{l'}_{\gamma,\delta}p \geq 0$ (since $N^{d+} > \widehat{L}$),

$$\Pi^{0,L}_{\alpha,\beta}p + \Gamma^{L}_{\alpha,\beta}p - \sum_{l'\geq\widehat{L}}\Pi^{0,l'}_{\gamma,\delta}p - \sum_{\{\bar{l}\geq l'>N^{d+}\}\cup\{(\gamma-\alpha)\wedge U\geq l'\geq\widehat{L}\}}\Gamma^{l'}_{\gamma,\delta}p > 0$$

which contradicts (2.2.13) with $K = 2$, $l_1 = \widehat{L} - 1$, $m_1 = 0$, $l_2 = N^{d+}$, $m_2 = \bar{l}$.

• Suppose $N^{d+} = \widehat{L} - 1$.
In this case (2.4.41) becomes (2.4.37). Notice that $N^{d+} = \widehat{L} - 1$ means $\mathcal{R}^{L}_{\alpha,\beta} > 0$, that is $\Gamma^{L}_{\alpha,\beta}p - \sum_{\bar{l}\geq l'\geq\widehat{L}}\Gamma^{l'}_{\gamma,\delta}p > 0$. By adding this inequality to (2.4.37) we get a contradiction as in (2.4.42).
Claims *v*) and *vi*) are symmetric with respect to *iii*) and *iv*) and we prove them in a similar way by using symmetric Tables and Remarks. □

Remark 2.44 *As a consequence, Tables* 2.5 *to* 2.10 *do not contain any coupling term breaking the partial order between configurations.*

Remark 2.45 *By* (2.4.39), *if* $N^{d+} = \widehat{L} - 1$ *then* $H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta} = 0$ *for each* $l \geq \widehat{L}$ *(and for each* $l \geq 0$ *by Definition* (2.2.22)*) but since* $N^{B} \geq \widehat{L}$, *then* $N^{B} = N^{Bd}$ *and by Definition* (2.4.15)

$$B^{L,N^B}_{\alpha,\beta} = \Pi^{0,L}_{\alpha,\beta} - \sum_{l'>N^B}\Pi^{0,l'}_{\gamma,\delta} - \sum_{U\wedge(\gamma-\alpha)\geq l'>N^B}H^{0,L,l',l'}_{\alpha,\beta,\gamma,\delta} = \Pi^{0,L}_{\alpha,\beta} - \sum_{l'>N^B}\Pi^{0,l'}_{\gamma,\delta}. \qquad (2.4.43)$$

Step 3) Finally, we need to pass through Step 3 if $\mathcal{R}^{L}_{\alpha,\beta} > 0$ ($\mathcal{R}^{U}_{\gamma,\delta} > 0$ in the symmetric case). We refer to Table 2.11 (Table 2.12). We couple the remainder of the lower

Table 2.11: Third step, $\mathcal{R}^L_{\alpha,\beta} > 0$

l	$T^{L,l}_{\alpha,\beta}$ $\wedge$	$[B^{L,l}_{\gamma,\delta} - B^{L,l}_{\alpha,\beta} \wedge B^{L,l}_{\gamma,\delta}]$ $=$	$H^{L,L,0,l}_{\alpha,\beta,\gamma,\delta}$
M		$\times$	
$\vdots$		$\vdots$	$B^{L,l}_{\gamma,\delta} - B^{L,l}_{\alpha,\beta} \wedge B^{L,l}_{\gamma,\delta}$
$N^{dB}+1$		$\times$	
N^{dB}	$\times$		$T^{L,l}_{\alpha,\beta} \geq 0$
$N^{dB}-1$	$\times$		
$\vdots$	$\vdots$		$T^{L,l}_{\alpha,\beta} = 0$
0	$\times$		

Table 2.12: Third step, $\mathcal{R}^U_{\gamma,\delta} > 0$

k	$[D^{k,U}_{\alpha,\beta} - D^{k,U}_{\alpha,\beta} \wedge D^{k,U}_{\gamma,\delta}]$ $\wedge$	$T^{k,U}_{\gamma,\delta}$ $=$	$H^{-k,0,U,U}_{\alpha,\beta,\gamma,\delta}$
M	$\times$		
$\vdots$	$\vdots$		$D^{k,U}_{\alpha,\beta} - D^{k,U}_{\alpha,\beta} \wedge D^{k,U}_{\gamma,\delta}$
$N^{dD}+1$	$\times$		
N^{dD}		$\times$	$T^{k,U}_{\gamma,\delta} \geq 0$
$N^{dD}-1$		$\times$	
$\vdots$		$\vdots$	$T^{k,U}_{\gamma,\delta} = 0$
0		$\times$	

configuration L-jump rate with the remaining part of the upper configuration birth rate. In other words, we begin from $H^{L,L,0,M}_{\alpha,\beta,\gamma,\delta} = T^{L,M}_{\alpha,\beta} \wedge [B^{L,M}_{\gamma,\delta} - B^{L,M}_{\gamma,\delta} \wedge B^{L,M}_{\alpha,\beta}]$. Notice that $T^{L,M}_{\alpha,\beta} = \mathcal{R}^L_{\alpha,\beta}$ by Definitions (2.2.29) and (2.4.8). If the minimum is the second term we proceed downwards in l with terms $H^{L,L,0,l}_{\alpha,\beta,\gamma,\delta} = T^{L,l}_{\alpha,\beta} \wedge [B^{L,l}_{\gamma,\delta} - B^{L,l}_{\gamma,\delta} \wedge B^{L,l}_{\alpha,\beta}]$ until the minimum is the first term, in which case there is no remainder of $\Gamma^L_{\alpha,\beta}p$, so the problem is solved. N^{dB} corresponds to the l^{th} step when it happens. In other words when $l > N^{dB}$, the coupling term is the remainder of the upper configuration l-birth rate, when $l = N^{dB}$ it is the remainder of the lower configuration L jump rate, and when $l < N^{dB}$ the terms are null.
If $\mathcal{R}^L_{\alpha,\beta} = 0$, that is if $M+1 > N^{d+} \geq \widehat{L}$, we put $N^{dB} = M+1$. We give it the same value if β is not L-bad. The corresponding term for deaths rates is N^{dD} and we can do the same remarks by using Table 2.12 involving symmetric rates.
So we explained Formulas (2.4.18) and (2.4.19) in Definition 2.38.

Proof of Proposition 2.39 *vii)–viii)*.
vii) Since $\mathcal{R}^L_{\alpha,\beta} > 0$, $N^{d+} = \widehat{L} - 1$ and Remark 2.45 holds.

Remark 2.46 *If $l > N^{dB}$ then $H^{L,L,0,l}_{\alpha,\beta,\gamma,\delta} = B^{L,l}_{\gamma,\delta} - B^{L,l}_{\gamma,\delta} \wedge B^{L,l}_{\alpha,\beta}$ depends on Step 2. By using Tables 2.5, 2.6, 2.7 and (2.4.31):*
-if $l > N^B$ then $B^{L,l}_{\alpha,\beta} \wedge B^{L,l}_{\gamma,\delta} = B^{L,l}_{\gamma,\delta}$ and

$$H^{L,L,0,l}_{\alpha,\beta,\gamma,\delta} = 0;$$

-if $l = N^B$ and $N^B \geq (\gamma - \alpha) \wedge U$ (see Table 2.5) or $N^{Bd} = N^B < (\gamma - \alpha) \wedge U$ (see Table 2.6) then $B^{L,l}_{\alpha,\beta} \wedge B^{L,l}_{\gamma,\delta} = B^{L,l}_{\alpha,\beta}$ and by (2.4.43)

$$\begin{aligned} H^{L,L,0,l}_{\alpha,\beta,\gamma,\delta} =& B^{L,l}_{\gamma,\delta} - B^{L,l}_{\alpha,\beta} = \Pi^{0,l}_{\gamma,\delta} p - B^{L,l}_{\alpha,\beta} \\ =& \sum_{l' \geq l} \Pi^{0,l'}_{\gamma,\delta} p - \Pi^{0,L}_{\alpha,\beta} p; \end{aligned}$$

-if $l = N^B = N^{Bd} - 1$ (see Table 2.7) then $B^{L,l}_{\alpha,\beta} \wedge B^{L,l}_{\gamma,\delta} = B^{L,l}_{\alpha,\beta} = 0$ and

$$H^{L,L,0,l}_{\alpha,\beta,\gamma,\delta} = B^{L,l}_{\gamma,\delta} = \Pi^{0,l}_{\gamma,\delta} p;$$

-if $l < N^B$ then $B^{L,l}_{\alpha,\beta} \wedge B^{L,l}_{\gamma,\delta} = B^{L,l}_{\alpha,\beta} = 0$ and

$$H^{L,L,0,l}_{\alpha,\beta,\gamma,\delta} = B^{L,l}_{\gamma,\delta} = \Pi^{0,l}_{\gamma,\delta} p;$$

In other words even if we choose the second term of $H^{L,L,0,l}_{\alpha,\beta,\gamma,\delta}$, we could have a null coupling rate, when solving the lower attractiveness problem left no remainder of $\Pi^{0,l}_{\gamma,\delta} p$. It means that positive coupling terms begin below $N^B \vee N^{Bd}$, with $N^B \vee N^{Bd} \geq N^{dB}$.

If $l > N^{dB}$, By Definition (2.4.18) of N^{dB}, (2.4.30), (2.4.31) and Remark 2.46,

$$\begin{aligned} H^{L,L,0,l}_{\alpha,\beta,\gamma,\delta} =& B^{L,l}_{\gamma,\delta} - B^{L,l}_{\alpha,\beta} \wedge B^{L,l}_{\gamma,\delta} = \Pi^{0,l}_{\gamma,\delta} p - B^{L,l}_{\alpha,\beta} \wedge (\Pi^{0,l}_{\gamma,\delta} p) \\ =& \Pi^{0,l}_{\gamma,\delta} p - \mathbb{1}_{\{l = N^B\}} (\Pi^{0,L}_{\alpha,\beta} p - \sum_{l' > N^B} \Pi^{0,l'}_{\gamma,\delta} p) < T^{L,l}_{\alpha,\beta}. \end{aligned} \tag{2.4.44}$$

We compute $T^{L,l}_{\alpha,\beta}$ if $l > N^{dB}$. By Definition (2.2.27) and Remark 2.46

$$\begin{aligned} T^{L,l}_{\alpha,\beta} =& \Gamma^L_{\alpha,\beta} p - \sum_{\bar{l} \geq l' \geq \widehat{L}} \Gamma^{l'}_{\gamma,\delta} p - \sum_{l' > l} H^{L,L,0,l'}_{\alpha,\beta,\gamma,\delta} \\ =& \Gamma^L_{\alpha,\beta} p - \sum_{\bar{l} \geq l' \geq \widehat{L}} \Gamma^{l'}_{\gamma,\delta} p - \sum_{l' > l} \big\{ \mathbb{1}_{\{l' < N^B\}} \Pi^{0,l'}_{\gamma,\delta} p + \mathbb{1}_{\{l' = N^B\}} H^{L,L,0,N^B}_{\alpha,\beta,\gamma,\delta} \big\} \\ =& \Gamma^L_{\alpha,\beta} p - \sum_{\bar{l} \geq l' \geq \widehat{L}} \Gamma^{l'}_{\gamma,\delta} p - \sum_{N^B > l' > l} \Pi^{0,l'}_{\gamma,\delta} p \\ & - \mathbb{1}_{\{N^B > l\}} \big(\Pi^{0,N^B}_{\gamma,\delta} p - (\Pi^{0,L}_{\alpha,\beta} p - \sum_{l' > N^B} \Pi^{0,l'}_{\gamma,\delta} p) \big). \end{aligned} \tag{2.4.45}$$

Suppose $N^{dB} < \widehat{L}$. By (2.4.44)

$$\begin{aligned} \Pi^{0,\widehat{L}}_{\gamma,\delta} p - \mathbb{1}_{\{\widehat{L} = N^B\}} (\Pi^{0,L}_{\alpha,\beta} p - \sum_{l' > N^B} \Pi^{0,l'}_{\gamma,\delta} p) <& \Gamma^L_{\alpha,\beta} p - \sum_{\bar{l} \geq l' \geq \widehat{L}} \Gamma^{l'}_{\gamma,\delta} p - \sum_{N^B > l' > \widehat{L}} \Pi^{0,l'}_{\gamma,\delta} p \\ & + \mathbb{1}_{\{N^B > \widehat{L}\}} (\Pi^{0,L}_{\alpha,\beta} p - \sum_{l' \geq N^B} \Pi^{0,l'}_{\gamma,\delta} p). \end{aligned} \tag{2.4.46}$$

Since $N^B \geq \widehat{L}$, if either $N^B = \widehat{L}$ or $N^B > \widehat{L}$ then (2.4.46) is identical to

$$\Gamma^L_{\alpha,\beta}p + \Pi^{0,L}_{\alpha,\beta}p > \sum_{\bar{l} \geq l' \geq \widehat{L}} \Gamma^{l'}_{\gamma,\delta}p - \sum_{l' \geq \widehat{L}} \Pi^{0,l'}_{\gamma,\delta}p$$

that is a contradiction as in (2.4.42).
viii) is proved exactly as (*vii*) by using symmetric rates. □

Remark 2.47 *As a consequence, Tables* 2.11 *and* 2.12 *do not contain coupling terms breaking the partial order between configurations.*

2.4.4 Proof of Proposition 2.37

i) Assume β is L-bad and $l < \widehat{L}$. If $l < \widehat{L}$ then $H^{L,L,l,l}_{\alpha,\beta,\gamma,\delta} = 0$ by Definition (2.2.18). We refer to Tables 2.5, 2.6 and 2.7.
• Suppose $N^B \geq U \wedge (\gamma - \alpha)$. If $l < N^B$ then $H^{0,L,0,l}_{\alpha,\beta,\gamma,\delta} = 0$; since $N^B \geq \widehat{L}$ by Proposition 2.39 *iii*), then $H^{0,L,0,l}_{\alpha,\beta,\gamma,\delta} = 0$ for each $l < \widehat{L}$. In this case $H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta} = 0$ for each l by (2.4.33).
• Suppose $N^B < (\gamma - \alpha) \wedge U$. If $l < N^B = N^{Bd}$ then $H^{0,L,0,l}_{\alpha,\beta,\gamma,\delta} = H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta} = 0$. Since $N^B \geq \widehat{L}$ by Proposition 2.39 *iv*) then $H^{0,L,0,l}_{\alpha,\beta,\gamma,\delta} = H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta} = 0$ for each $l < \widehat{L}$.
If $l \leq N^B = N^{Bd} - 1$, that is $l < N^{Bd}$, then $H^{0,L,0,l}_{\alpha,\beta,\gamma,\delta} = H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta} = 0$. Since $N^{Bd} \geq \widehat{L}$ by Proposition 2.39 *iv*) then $H^{0,L,0,l}_{\alpha,\beta,\gamma,\delta} = H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta} = 0$ for each $l < \widehat{L}$.
If $\mathcal{R}^L_{\alpha,\beta} > 0$, Proposition 2.39 *vii*) ensures that $H^{L,L,0,l}_{\alpha,\beta,\gamma,\delta} = 0$ if $l < \widehat{L}$, since $N^{dB} \geq \widehat{L}$ (see Table 2.11).
ii) Assume γ is U-bad and $k < \widehat{U}$. Relations involving deaths and jumps from α are proved in the same way as in *i*) by using Proposition 2.39 *ii*), *v*), *vi*) and *viii*) and Tables 2.8, 2.9, 2.10 and 2.12. □

2.4.5 Proof of Proposition 2.25

i) If β is L-bad, we have to prove that $\overline{\Gamma}^L_{\alpha,\beta} = 0 = \overline{\Pi}^{0,L}_{\alpha,\beta}$. By Definitions 2.24, (2.2.31) *i*), (2.2.19), (2.2.32) *i*), (2.2.27) and by (2.4.10), (2.4.26) we get

$$\begin{aligned}
\overline{\Gamma}^L_{\alpha,\beta}p =& \Gamma^L_{\alpha,\beta}p - \sum_{l'>0} H^{L,L,l',l'}_{\alpha,\beta,\gamma,\delta} - \sum_{l'>0} H^{L,L,0,l'}_{\alpha,\beta,\gamma,\delta} \\
=& \Gamma^L_{\alpha,\beta}p - \sum_{l' \geq \widehat{L}} H^{L,L,l',l'}_{\alpha,\beta,\gamma,\delta} - \sum_{l'>0} H^{L,L,0,l'}_{\alpha,\beta,\gamma,\delta} = H^{L,L,0,0}_{\alpha,\beta,\gamma,\delta}; \\
\overline{\Pi}^{0,L}_{\alpha,\beta}p =& \Pi^{0,L}_{\alpha,\beta}p - \sum_{l'>0} H^{0,L,0,l'}_{\alpha,\beta,\gamma,\delta} - \sum_{l'>0} H^{0,L,l',l'}_{\alpha,\beta,\gamma,\delta} \\
=& \Pi^{0,L}_{\alpha,\beta}p - \sum_{l'>0} H^{0,L,0,l'}_{\alpha,\beta,\gamma,\delta} - \sum_{(\gamma-\alpha)\wedge U \geq l' > 0} H^{0,L,l',l'}_{\alpha,\beta,\gamma,\delta} = H^{0,L,0,0}_{\alpha,\beta,\gamma,\delta}.
\end{aligned}$$

Since $\beta + L > \delta + 0$, by Proposition 2.37 *i*), $0 = H^{L,L,0,0}_{\alpha,\beta,\gamma,\delta} = \overline{\Gamma}^L_{\alpha,\beta}p$ and $0 = H^{0,L,0,0}_{\alpha,\beta,\gamma,\delta} = \overline{\Pi}^{0,L}_{\alpha,\beta}p$.
ii) In a symmetric way, if γ is U-bad we have to prove that $\overline{\Gamma}^U_{\gamma,\delta} = \overline{\Pi}^{-U,0}_{\gamma,\delta} = 0$. By Definitions 2.24, (2.2.31) *ii*) and (2.2.33) *ii*) we get $\overline{\Gamma}^U_{\gamma,\delta}p = H^{0,0,U,U}_{\alpha,\beta,\gamma,\delta}$ and $\overline{\Pi}^{-U,0}_{\gamma,\delta}p = H^{0,0,-U,0}_{\alpha,\beta,\gamma,\delta}$. Since $\gamma - U < \alpha + 0$, by Proposition 2.37 *ii*), $0 = H^{0,0,U,U}_{\alpha,\beta,\gamma,\delta} = \overline{\Gamma}^U_{\gamma,\delta}p$, and $0 = H^{0,0,-U,0}_{\alpha,\beta,\gamma,\delta} = \overline{\Pi}^{-U,0}_{\gamma,\delta}p$. □

2.4.6 Proof of Proposition 2.26

We prove, by checking all possible cases, that the system $\overline{\mathcal{S}}$ satisfies Conditions (2.2.13)–(2.2.14), that is: for all I_a, I_b, I_c, I_d in Definition 2.14,

$$\sum_{k>\delta-\beta+l_1} \overline{\Pi}^{0,k}_{\alpha,\beta} + \sum_{k\in I_a} \overline{\Gamma}^{k}_{\alpha,\beta} \le \sum_{l>l_1} \overline{\Pi}^{0,l}_{\gamma,\delta} + \sum_{l\in I_b} \overline{\Gamma}^{l}_{\gamma,\delta} \tag{2.4.47}$$

$$\sum_{k>k} \overline{\Pi}^{-k,0}_{\alpha,\beta} + \sum_{k\in I_d} \overline{\Gamma}^{k}_{\alpha,\beta} \ge \sum_{l>\gamma-\alpha+k} \overline{\Pi}^{-l,0}_{\gamma,\delta} + \sum_{l\in I_c} \overline{\Gamma}^{l}_{\gamma,\delta}. \tag{2.4.48}$$

We prove (2.4.47). Since, by using the symmetry of the system, the proof of (2.4.48) is similar, we skip it.

Remark 2.48 *Let $A = \{i \le K : l_i \ge \widehat{L}\}$, then for all $i \le j \le K$ we have $\widehat{L} \le l_j$ and $1\!\!1_{\{m_j \ge k > \delta-\beta+l_j\}} \overline{\Gamma}^{k}_{\alpha,\beta} = 0$ by Table 2.1 and Definition 2.24. Let*

$$K_A = \begin{cases} \min A & \text{if } A \neq \emptyset \\ K+1 & \text{otherwise} \end{cases}$$

given $K \ge 0$, we define

$$I_a^{K_A} = \bigcup_{i=1}^{K_A-1} \{m_i \ge k > \delta-\beta+l_i\}, \qquad I_b^{K_A} = \bigcup_{i=1}^{K_A-1} \{\gamma-\alpha+m_i \ge l > l_i\} \tag{2.4.49}$$

then if

$$\sum_{k>\delta-\beta+l_1} \overline{\Pi}^{0,k}_{\alpha,\beta} + \sum_{k\in I_a^{K_A}} \overline{\Gamma}^{k}_{\alpha,\beta} \le \sum_{l>l_1} \overline{\Pi}^{0,l}_{\gamma,\delta} + \sum_{l\in I_b^{K_A}} \overline{\Gamma}^{l}_{\gamma,\delta}$$

we have

$$\begin{aligned} \sum_{k>\delta-\beta+l_1} \overline{\Pi}^{0,k}_{\alpha,\beta} + \sum_{k\in I_a} \overline{\Gamma}^{k}_{\alpha,\beta} &= \sum_{k>\delta-\beta+l_1} \overline{\Pi}^{0,k}_{\alpha,\beta} + \sum_{k\in I_a^{K_A}} \overline{\Gamma}^{k}_{\alpha,\beta} \le \sum_{l>l_1} \overline{\Pi}^{0,l}_{\gamma,\delta} + \sum_{l\in I_b^{K_B}} \overline{\Gamma}^{l}_{\gamma,\delta} \\ &\le \sum_{l>l_1} \overline{\Pi}^{0,l}_{\gamma,\delta} + \sum_{l\in I_b} \overline{\Gamma}^{l}_{\gamma,\delta}. \end{aligned}$$

Hence we can suppose without loss of generality

$$\widehat{L} > l_K. \tag{2.4.50}$$

As a consequence, by Proposition 2.39, if β is L-bad

$$\left(N^{d+} 1\!\!1_{\{\mathcal{R}^L_{\alpha,\beta}=0\}} + N^{dB} 1\!\!1_{\{\mathcal{R}^L_{\alpha,\beta}>0\}}\right) \wedge \left(N^B 1\!\!1_{\{N^B>\gamma-\alpha\}} + (N^B \vee N^{Bd}) 1\!\!1_{\{N^B<\gamma-\alpha\}}\right) \ge \widehat{L} > l_K. \tag{2.4.51}$$

If γ is U-bad the same remark involving $\widehat{U}$ and variables N^{d-}, N^D, N^{Dd} and N^{dD} holds symmetrically.

Let $B = \{L+\gamma-\alpha \ge l > U\}$, then $1\!\!1_{\{l\in B\}} \overline{\Gamma}^{l}_{\gamma,\delta} = 0$. We define

$$I_a^B := I_a \cup \{L \ge k > U+\delta-\beta\} \tag{2.4.52}$$

$$I_b^B := I_b \cup B \tag{2.4.53}$$

then if

$$\sum_{k>\delta-\beta+l_1}\overline{\Pi}^{0,k}_{\alpha,\beta}+\sum_{k\in I_a^B}\overline{\Gamma}^k_{\alpha,\beta}\leq\sum_{l>l_1}\overline{\Pi}^{0,l}_{\gamma,\delta}+\sum_{l\in I_b^B}\overline{\Gamma}^l_{\gamma,\delta}$$

we have

$$\begin{aligned}\sum_{k>\delta-\beta+l_1}\overline{\Pi}^{0,k}_{\alpha,\beta}+\sum_{k\in I_a}\overline{\Gamma}^k_{\alpha,\beta}\leq\sum_{k>\delta-\beta+l_1}\overline{\Pi}^{0,k}_{\alpha,\beta}+\sum_{k\in I_a^B}\overline{\Gamma}^k_{\alpha,\beta}\leq\sum_{l>l_1}\overline{\Pi}^{0,l}_{\gamma,\delta}+\sum_{l\in I_b^B}\overline{\Gamma}^l_{\gamma,\delta}\\ =\sum_{l>l_1}\overline{\Pi}^{0,l}_{\gamma,\delta}+\sum_{l\in I_b}\overline{\Gamma}^l_{\gamma,\delta}.\end{aligned}$$

Hence we can suppose without loss of generality

$$I_a=\{L\geq k>U+\delta-\beta\}\cup I_a^K \tag{2.4.54}$$

where I_a^K and I_b^K are given by Definition 2.14 *and $K\leq U+\delta-\beta$.*
Notice that if $\widehat{L}\leq U$ (that is $U+\delta-\beta\geq L$), that is when we pass Step 1*, nothing changed, and this remark will be useful when $\widehat{L}>U$.*

Remark 2.49 *By choosing $\delta-\beta+l_i=m_i$ for all $i=1,\dots K$, by Definition* 2.14 *Condition* (2.2.13) *becomes*

$$\sum_{k>\delta-\beta+l_1}\Pi^{0,k}_{\alpha,\beta}\leq\sum_{l>l_1}\Pi^{0,l}_{\gamma,\delta}+\sum_{l\in I_b}\Gamma^l_{\gamma,\delta} \tag{2.4.55}$$

A symmetric remark holds for Condition (2.2.14)*.*

If $\widehat{L}\leq 0$ ($\beta+L\leq\delta$) then the left hand side of (2.4.47) is null and the inequality is trivially satisfied.
Suppose $\widehat{L}>0$ that is we have a lower attractiveness problem. Remember that if $\alpha-L\leq\gamma-U$ we begin the coupling construction from (2.4.20) with $\bar{l}=U$, otherwise we start from (2.4.22) with $\bar{l}=L+\gamma-\alpha$. In this case if there is a higher attractiveness problem, then $\Gamma^U_{\gamma,\delta}$ can be coupled only with death rates in Step 3 of the construction. Notice moreover that we pass Step 1 only if $\widehat{L}=\bar{l}$ (see Step 1, Section 2.4.3).
By Proposition 2.25, $\overline{\Pi}^{0,L}_{\alpha,\beta}=0$ and by Definition 2.24, $\overline{\Pi}^{0,l}_{\alpha,\beta}=\Pi^{0,l}_{\alpha,\beta}$ for each $l<L$. Therefore

$$\sum_{k>\delta-\beta+l_1}\overline{\Pi}^{0,k}_{\alpha,\beta}p=\sum_{k>\delta-\beta+l_1}\Pi^{0,k}_{\alpha,\beta}p-\Pi^{0,L}_{\alpha,\beta}p \tag{2.4.56}$$

By Proposition 2.25, $\overline{\Gamma}^L_{\alpha,\beta}=0$. Moreover $H^{k,k,-U,0}_{\alpha,\beta,\gamma,\delta}=0$ for each $k>\delta-\beta+l_i\geq\delta-\beta$ by Definition (2.2.26). Then by Definition 2.24 and (2.4.50) we have

$$\sum_{k\in I_a}\overline{\Gamma}^k_{\alpha,\beta}p=\sum_{k\in I_a\setminus\{L\}}\left(\Gamma^k_{\alpha,\beta}p-H^{k,k,U,U}_{\alpha,\beta,\gamma,\delta}\right). \tag{2.4.57}$$

where $I_a\setminus\{L\}$ is the shorthand for $I_a\mathbb{1}_{\{L\notin I_a\}}+(I_a\setminus\{L\})\mathbb{1}_{\{L\in I_a\}}$.

Case A Suppose β is L-bad and γ is U-bad, that is $U>\gamma-\alpha$.
The right hand side of (2.4.47) is given by Definition 2.24,

$$\begin{aligned}\sum_{l>l_1}\overline{\Pi}^{0,l}_{\gamma,\delta}p+\sum_{l\in I_b}\overline{\Gamma}^l_{\gamma,\delta}p=&\sum_{l>l_1}\left(\Pi^{0,l}_{\gamma,\delta}p-H^{0,L,0,l}_{\alpha,\beta,\gamma,\delta}-H^{L,L,0,l}_{\alpha,\beta,\gamma,\delta}\right)\\&+\sum_{l\in I_b\setminus\{U\}}\left(\Gamma^l_{\gamma,\delta}p-H^{L,L,l,l}_{\alpha,\beta,\gamma,\delta}-H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta}\right).\end{aligned} \tag{2.4.58}$$

Notice that $\overline{\Gamma}^U_{\gamma,\delta} = 0$ by Proposition 2.25 since γ is U-bad; moreover $H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta} = 0$ if $l > \gamma - \alpha$ by Definition (2.2.22) and for all l if $N^B \geq \gamma - \alpha$ by (2.4.33). Since $\widehat{U} > 0$, $\{\gamma - \alpha \geq l > l_1\} \subseteq I_b$ then $\sum_{\{l\in I_b\}\cap\{l\neq U\}} H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta} = 1\!\!1_{\{N^B<\gamma-\alpha\}} \sum_{\gamma-\alpha\geq l>l_1} H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta}$. Moreover using Tables 2.1, 2.5 and 2.6

$$\sum_{l>l_1} \big(\Pi^{0,l}_{\gamma,\delta} p - H^{0,L,0,l}_{\alpha,\beta,\gamma,\delta}\big) = \sum_{N^B\geq l>l_1} \big(\Pi^{0,l}_{\gamma,\delta} p - H^{0,L,0,l}_{\alpha,\beta,\gamma,\delta}\big); \tag{2.4.59}$$

$$\begin{aligned}
\sum_{l\in I_b\setminus\{U\}} \big(\Gamma^l_{\gamma,\delta} p - H^{L,L,l,l}_{\alpha,\beta,\gamma,\delta}\big) &= \sum_{l\in I_b\setminus\{U\}} \Gamma^l_{\gamma,\delta} p - \sum_{l\in I_b\setminus\{U\}} \Gamma^l_{\gamma,\delta} p 1\!\!1_{\{\bar{l}\geq l>N^{d+}\}} \\
&\quad - \sum_{l\in I_b\setminus\{U\}} \big(\Gamma^L_{\alpha,\beta} p - \sum_{\bar{l}\geq l>N^{d+}} \Gamma^l_{\gamma,\delta} p\big) 1\!\!1_{\{M+1>l=N^{d+}\geq\widehat{L}\}} \\
&= \sum_{l\in I_b\setminus\{U\}} \Gamma^l_{\gamma,\delta} p - \sum_{l\in I_b\setminus\{U\}} \Gamma^l_{\gamma,\delta} p 1\!\!1_{\{\bar{l}\geq l>N^{d+}\}} \\
&\quad - \big(\Gamma^L_{\alpha,\beta} p - \sum_{\bar{l}\geq l>N^{d+}} \Gamma^l_{\gamma,\delta} p\big) 1\!\!1_{\{M+1>N^{d+}\geq\widehat{L},N^{d+}\in I_b\setminus\{U\}\}}.
\end{aligned} \tag{2.4.60}$$

Notice that (2.4.60) still holds if $N^{d+} = \widehat{L} - 1$, namely (see Table 2.2)

$$\sum_{l\in I_b\setminus\{U\}} \big(\Gamma^l_{\gamma,\delta} p - H^{L,L,l,l}_{\alpha,\beta,\gamma,\delta}\big) = \sum_{l\in I_b\setminus\{U\}} \Gamma^l_{\gamma,\delta} p - \sum_{l\in I_b\setminus\{U\}} \Gamma^l_{\gamma,\delta} p 1\!\!1_{\{\bar{l}\geq l\geq\widehat{L}\}} \tag{2.4.61}$$

and if $N^{d+} = M + 1$, then $H^{L,L,l,l}_{\alpha,\beta,\gamma,\delta} = 0$ for all l, thus

$$\sum_{l\in I_b\setminus\{U\}} \big(\Gamma^l_{\gamma,\delta} p - H^{L,L,l,l}_{\alpha,\beta,\gamma,\delta}\big) = \sum_{l\in I_b\setminus\{U\}} \Gamma^l_{\gamma,\delta} p. \tag{2.4.62}$$

Therefore using (2.4.25), (2.4.59) and (2.4.60) ((2.4.62) if $N^{d+} = M+1$), (2.4.58) becomes

$$\begin{aligned}
\sum_{l>l_1} \overline{\Pi}^{0,l}_{\gamma,\delta} p + \sum_{l\in I_b} \overline{\Gamma}^l_{\gamma,\delta} p &= \sum_{N^B\geq l>l_1} \big(\Pi^{0,l}_{\gamma,\delta} p - H^{0,L,0,l}_{\alpha,\beta,\gamma,\delta}\big) - 1\!\!1_{\{N^B<\gamma-\alpha\}} \sum_{\gamma-\alpha\geq l>l_1} H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta} \\
&\quad + \sum_{l\in I_b\setminus\{U\}} \Gamma^l_{\gamma,\delta} p - \sum_{l\in I_b\setminus\{U\}} \Gamma^l_{\gamma,\delta} p 1\!\!1_{\{\bar{l}\geq l>N^{d+}\}} \\
&\quad - \big(\Gamma^L_{\alpha,\beta} p - \sum_{\bar{l}\geq l>N^{d+}} \Gamma^l_{\gamma,\delta} p\big) 1\!\!1_{\{N^{d+}\geq\widehat{L},N^{d+}\in I_b\setminus\{U\}\}} \\
&\quad - 1\!\!1_{\{\{N^{d+}=\widehat{L}-1\}\cup\{N^{d+}=M+1\}\}} \sum_{l>l_1} H^{L,L,0,l}_{\alpha,\beta,\gamma,\delta}.
\end{aligned} \tag{2.4.63}$$

Since by Proposition 2.39, *iii*) (resp. *iv*)), $N^B \geq \widehat{L}$ (resp. $N^B \wedge N^{Bd} \geq \widehat{L}$), with $|N^B - N^{Bd}| \leq 1$ and $\widehat{L} > l_K \geq l_1$ by (2.4.50), we can suppose $N^B > l_1$.

• If $N^B = N^{Bd}$, by using Tables 2.5, 2.6 and Definition (2.2.19)

$$\begin{aligned}
&\sum_{N^B\geq l>l_1} \big(\Pi^{0,l}_{\gamma,\delta} p - H^{0,L,0,l}_{\alpha,\beta,\gamma,\delta}\big) - 1\!\!1_{\{N^B<\gamma-\alpha\}} \sum_{\gamma-\alpha\geq l>l_1} H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta} \\
&= \sum_{N^B\geq l>l_1} \Pi^{0,l}_{\gamma,\delta} p - B^{L,N^B}_{\alpha,\beta} - 1\!\!1_{\{N^B<\gamma-\alpha\}} \sum_{\gamma-\alpha\geq l>N^B} H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta} \\
&= \sum_{l>l_1} \Pi^{0,l}_{\gamma,\delta} p - \Pi^{0,L}_{\alpha,\beta} p.
\end{aligned} \tag{2.4.64}$$

• If $N^B = N^{Bd} - 1$, by using Table 2.7 and Definitions (2.2.19)–(2.2.20), then

$$\begin{aligned}
&\sum_{N^B \geq l > l_1} (\Pi^{0,l}_{\gamma,\delta} p - H^{0,L,0,l}_{\alpha,\beta,\gamma,\delta}) - \mathbb{1}_{\{N^B < \gamma-\alpha\}} \sum_{\gamma-\alpha \geq l > l_1} H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta} \\
&= \sum_{N^B \geq l > l_1} \Pi^{0,l}_{\gamma,\delta} p - \mathbb{1}_{\{N^B < \gamma-\alpha\}} \sum_{\gamma-\alpha \geq l > N^{Bd}} H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta} - (B^{L,N^{Bd}}_{\alpha,\beta} - B^{L,N^{Bd}}_{\gamma,\delta}) \\
&= \sum_{l > l_1} \Pi^{0,l}_{\gamma,\delta} p - \Pi^{0,L}_{\alpha,\beta} p. \qquad (2.4.65)
\end{aligned}$$

A.1 Suppose $\bar{l} = L + \gamma - \alpha$ or ($\bar{l} = U \geq \widehat{L}$ and $N^{d-} = L$). In these cases we pass Step 1. If $\bar{l} = L + \gamma - \alpha$ then $H^{k,k,U,U}_{\alpha,\beta,\gamma,\delta} = 0$ for each $k \leq L$ since $\gamma - U < \alpha - L \leq \alpha - k$ (see Definition 2.2.18). If $\bar{l} = U$, $\Gamma^L_{\alpha,\beta} > \Gamma^U_{\gamma,\delta}$ (by (2.4.13) and (2.4.20) since $N^{d-} = L$), then $H^{k,k,U,U}_{\alpha,\beta,\gamma,\delta} = 0$ for each $k < L$ (see Table 2.3). Therefore by (2.4.57)

$$\sum_{k \in I_a} \overline{\Gamma}^k_{\alpha,\beta} = \sum_{k \in I_a \setminus \{L\}} \Gamma^k_{\alpha,\beta}. \qquad (2.4.66)$$

A.1.1 Suppose $\mathcal{R}^L_{\alpha,\beta} = 0$. Then $M + 1 > N^{d+} \geq \widehat{L} > l_K$ by (2.4.51). Notice that if $\bar{l} = U$, then $\{U\} \cup \{\bar{l} \geq l > N^{d+}\} = \{U \geq l > N^{d+}\}$.
We use (2.4.56), (2.4.63), (2.4.64), (2.4.65) and (2.4.66) in different cases to rewrite Condition (2.4.47), then we will check it is satisfied under Condition (2.2.13).

A.1.1.1 Suppose $N^{d+} \notin I_b$, that is N^{d+} does not belong to any of the corresponding sets of I_b. Since $N^{d+} > l_K$, we have also $N^{d+} > \gamma - \alpha + m_K$, hence since $U \geq \bar{l}$, $I_b \setminus \{\{U\} \cup \{\bar{l} \geq l > N^{d+}\}\} = I_b$ which implies that $\sum_{l \in I_b \setminus \{U\}} \Gamma^l_{\gamma,\delta} \mathbb{1}_{\{\bar{l} \geq l > N^{d+}\}} = 0$ and Condition (2.4.47) becomes

$$\sum_{k > \delta-\beta+l_1} \Pi^{0,k}_{\alpha,\beta} + \sum_{k \in I_a \setminus \{L\}} \Gamma^k_{\alpha,\beta} \leq \sum_{l > l_1} \Pi^{0,l}_{\gamma,\delta} + \sum_{l \in I_b} \Gamma^l_{\gamma,\delta}$$

which holds by Condition (2.2.13).

A.1.1.2 If $N^{d+} \in I_b$, then

$$\gamma - \alpha + m_K \geq N^{d+} \geq \widehat{L} > l_K, \qquad (2.4.67)$$

hence (see Table 2.1) Condition (2.4.47) is

$$\begin{aligned}
\sum_{k > \delta-\beta+l_1} \Pi^{0,k}_{\alpha,\beta} + \sum_{k \in I_a \setminus \{L\}} \Gamma^k_{\alpha,\beta} &\leq \sum_{l > l_1} \Pi^{0,l}_{\gamma,\delta} + \sum_{l \in I_b \setminus \{U\}} \Gamma^l_{\gamma,\delta} - \sum_{l \in I_b \setminus \{U\}} \Gamma^l_{\gamma,\delta} \mathbb{1}_{\{\bar{l} \geq l > N^{d+}\}} \\
&\quad - \Gamma^L_{\alpha,\beta} + \sum_{\bar{l} \geq l > N^{d+}} \Gamma^l_{\gamma,\delta} \\
&= \sum_{l > l_1} \Pi^{0,l}_{\gamma,\delta} + \sum_{l \in \{I_b \setminus \{U\}\} \cup \{\bar{l} \geq l > N^{d+}\}} \Gamma^l_{\gamma,\delta} - \Gamma^L_{\alpha,\beta}. \qquad (2.4.68)
\end{aligned}$$

A.1.1.2.1 Suppose $\bar{l} = U$ or ($\bar{l} = L + \gamma - \alpha$ and $U \notin I_b$), hence

$$\{I_b \setminus \{U\}\} \cup \{\bar{l} \geq l > N^{d+}\} = I_b \cup \{\bar{l} \geq l > N^{d+}\}.$$

• If $L \in I_a$, (2.4.68) becomes

$$\sum_{k>\delta-\beta+l_1} \Pi^{0,k}_{\alpha,\beta} + \sum_{k\in I_a} \Gamma^k_{\alpha,\beta} \le \sum_{l>l_1} \Pi^{0,l}_{\gamma,\delta} + \sum_{l\in I_b\cup\{\bar{l}\ge l>N^{d+}\}} \Gamma^l_{\gamma,\delta}$$

which holds by (2.2.13).
• If $L \notin I_a$, then (2.4.68) is

$$\sum_{k>\delta-\beta+l_1} \Pi^{0,k}_{\alpha,\beta} + \sum_{k\in I_a\cup\{L\}} \Gamma^k_{\alpha,\beta} \le \sum_{l>l_1} \Pi^{0,l}_{\gamma,\delta} + \sum_{l\in I_b\cup\{\bar{l}\ge l>N^{d+}\}} \Gamma^l_{\gamma,\delta} \tag{2.4.69}$$

Denote by

$$\widehat{I}_a = I_a \cup \{L \ge l \ge \delta-\beta+(L-\delta+\beta)\} = I_a \cup \{L \ge l \ge L\} \tag{2.4.70}$$
$$\widehat{I}_b = I_b \cup \{L+\gamma-\alpha \ge l \ge L-\delta+\beta\} \tag{2.4.71}$$

then by Condition (2.2.13) applied to $\widehat{I}_a$, $\widehat{I}_b$

$$\sum_{k>\delta-\beta+l_1} \Pi^{0,k}_{\alpha,\beta} + \sum_{k\in \widehat{I}_a} \Gamma^k_{\alpha,\beta} \le \sum_{l>l_1} \Pi^{0,l}_{\gamma,\delta} + \sum_{l\in \widehat{I}_b} \Gamma^l_{\gamma,\delta}. \tag{2.4.72}$$

By (2.4.67), $\widehat{I}_b = I_b \cup \{L+\gamma-\alpha \ge l > N^{d+}\}$ and (2.4.72) is identical to (2.4.69) if $\bar{l} = L+\gamma-\alpha$. If $\bar{l} = U$ then (2.4.72) implies (2.4.69).

A.1.1.2.2 Suppose $\bar{l} = L+\gamma-\alpha < U \in I_b$, then $\bar{l} = \gamma-\alpha+L < U \le \gamma-\alpha+m_K$, that is $L < m_K$ (and we also have (2.4.67)). By Condition (2.2.13) applied to

$$I_a^{K-1} \cup \{L \ge k > \delta-\beta+l_K\} = \widetilde{I}_a$$
$$I_b^{K-1} \cup \{\gamma-\alpha+L \ge l > l_K\} = \widetilde{I}_b$$

and (2.4.68)

$$\sum_{k>\delta-\beta+l_1} \Pi^{0,k}_{\alpha,\beta} + \sum_{k\in I_a\cup\{L\}} \Gamma^k_{\alpha,\beta} = \sum_{k>\delta-\beta+l_1} \Pi^{0,k}_{\alpha,\beta} + \sum_{k\in \widetilde{I}_a} \Gamma^k_{\alpha,\beta}$$
$$\le \sum_{l>l_1} \Pi^{0,l}_{\gamma,\delta} + \sum_{l\in \widetilde{I}_b} \Gamma^l_{\gamma,\delta} \le \sum_{l>l_1} \Pi^{0,l}_{\gamma,\delta} + \sum_{l\in\{I_b\setminus\{U\}\}\cup\{\gamma-\alpha+L\ge l>N^{d+}\}} \Gamma^l_{\gamma,\delta}$$

since, by (2.4.67), $I_b^{K-1}\cup\{\gamma-\alpha+L \ge l > l_K\} \subset I_b^{K-1}\cup\{\{\gamma-\alpha+m_K \ge l > l_K\}\setminus\{U\}\}\cup\{\gamma-\alpha+L \ge l > N^{d+}\} = \{I_b\setminus\{U\}\}\cup\{\gamma-\alpha+L \ge l > N^{d+}\}$. Therefore (2.4.47) holds also in this case.

Remark 2.50 *This is the key passage where a Definition of $I_a, \dots I_d$ as a single set instead of a union of several sets does not work.*

A.1.2 Suppose $\mathcal{R}^L_{\alpha,\beta} > 0$, that is $N^{d+} = \widehat{L}-1$ since we pass Step 1.
By using (2.4.61), (2.4.63), Tables 2.2, 2.11 and (2.4.45), since $N^{dB} \ge \widehat{L} > l_K \ge l_1$ by

(2.4.51) and Proposition 2.39 *vii*) we get

$$\begin{aligned}
&\sum_{l>l_1}\overline{\Pi}^{0,l}_{\gamma,\delta}p+\sum_{l\in I_b}\overline{\Gamma}^{l}_{\gamma,\delta}p\\
&=\sum_{l>l_1}\Pi^{0,l}_{\gamma,\delta}p-\Pi^{0,L}_{\alpha,\beta}p+\sum_{l\in I_b\setminus\{\{U\}\cup\{\bar{l}\geq l\geq\widehat{L}\}\}}\Gamma^{l}_{\gamma,\delta}p-\sum_{l>l_1}H^{L,L,0,l}_{\alpha,\beta,\gamma,\delta}\\
&=\sum_{l>l_1}\Pi^{0,l}_{\gamma,\delta}p-\Pi^{0,L}_{\alpha,\beta}p+\sum_{l\in I_b\setminus\{\{U\}\cup\{\bar{l}\geq l\geq\widehat{L}\}\}}\Gamma^{l}_{\gamma,\delta}p-\sum_{l>N^{dB}}H^{L,L,0,l}_{\alpha,\beta,\gamma,\delta}-T^{L,N^{dB}}_{\alpha,\beta}\\
&=\sum_{l>l_1}\Pi^{0,l}_{\gamma,\delta}p-\Pi^{0,L}_{\alpha,\beta}p+\sum_{l\in I_b\setminus\{\{U\}\cup\{\bar{l}\geq l\geq\widehat{L}\}\}}\Gamma^{l}_{\gamma,\delta}p+\sum_{\bar{l}\geq l\geq\widehat{L}}\Gamma^{l}_{\gamma,\delta}p-\Gamma^{L}_{\alpha,\beta}p\\
&=\sum_{l>l_1}\Pi^{0,l}_{\gamma,\delta}p-\Pi^{0,L}_{\alpha,\beta}p+\sum_{l\in\{I_b\setminus\{U\}\}\cup\{\bar{l}\geq l\geq\widehat{L}\}}\Gamma^{l}_{\gamma,\delta}p-\Gamma^{L}_{\alpha,\beta}p
\end{aligned}$$

and Condition (2.4.47) becomes (remember (2.4.66)) (2.4.68).

A.1.2.1 If $U=\bar{l}$ or ($\bar{l}=L+\gamma-\alpha<U\notin I_b$), then we proceed as in case A.1.1.2.1.

A.1.2.2 If $U>\bar{l}$ and $U\in I_b$ we work as in case A.1.1.2.2.

A.2 Suppose $\bar{l}=U=N^{d+}\geq\widehat{L}$ (which is excluded to case A.1 if $L\neq U$ by Remark 2.40). Again we pass Step 1. Therefore $H^{L,L,l,l}_{\alpha,\beta,\gamma,\delta}=0$ for each $l<U$ by Definition (2.4.12) and Table 2.1 and $H^{L,L,0,l}_{\alpha,\beta,\gamma,\delta}=0$ for each $l\leq M$ by (2.4.25).
Hence (2.4.60) becomes,

$$\sum_{l\in I_b\setminus\{U\}}\Gamma^{l}_{\gamma,\delta}p-\sum_{l\in I_b\setminus\{U\}}H^{L,L,l,l}_{\alpha,\beta,\gamma,\delta}=\sum_{l\in I_b\setminus\{U\}}\Gamma^{l}_{\gamma,\delta}p. \tag{2.4.73}$$

By using (2.4.56), (2.4.57), (2.4.58), (2.4.64), (2.4.65) (notice that those last two equalities are still valid) and (2.4.73) then Condition (2.4.47) is identical to

$$\sum_{k>\delta-\beta+l_1}\Pi^{0,k}_{\alpha,\beta}p+\sum_{k\in I_a\setminus\{L\}}(\Gamma^{k}_{\alpha,\beta}p-H^{k,k,U,U}_{\alpha,\beta,\gamma,\delta})\leq\sum_{l>l_1}\Pi^{0,l}_{\gamma,\delta}p+\sum_{l\in I_b\setminus\{U\}}\Gamma^{l}_{\gamma,\delta}p \tag{2.4.74}$$

We are left with evaluating the second term on the left hand side, that is (2.4.57). Since $N^{d+}=U$, then $N^{d-}<L$ by Remark 2.40. This means, see Tables 2.3 and 2.4 that if $k>N^{d-}$ then $H^{k,k,U,U}_{\alpha,\beta,\gamma,\delta}=\Gamma^{k}_{\alpha,\beta}p$ and working as for (2.4.60)

$$\begin{aligned}
\sum_{k\in I_a\setminus\{L\}}(\Gamma^{k}_{\alpha,\beta}p-H^{k,k,U,U}_{\alpha,\beta,\gamma,\delta})&=\sum_{k\in I_a\setminus\{L\}}\Gamma^{k}_{\alpha,\beta}p-\sum_{k\in I_a\setminus\{L\}}\Gamma^{k}_{\alpha,\beta}p\mathbb{1}_{\{\bar{k}\geq k>N^{d-}\}}\\
&\quad-\sum_{k\in I_a\setminus\{L\}}\Big(\Gamma^{U}_{\gamma,\delta}p-\sum_{\bar{k}\geq k>N^{d-}}\Gamma^{k}_{\alpha,\beta}p\Big)\mathbb{1}_{\{k=N^{d-}\geq\widehat{U}\}}\\
&=\sum_{k\in I_a\setminus\{L\}}\Gamma^{k}_{\alpha,\beta}p-\sum_{k\in I_a\setminus\{L\}}\Gamma^{k}_{\alpha,\beta}p\mathbb{1}_{\{\bar{k}\geq k>N^{d-}\}}\\
&\quad-\mathbb{1}_{\{N^{d-}\geq\widehat{U},N^{d-}\in I_a\setminus\{L\}\}}\Big(\Gamma^{U}_{\gamma,\delta}p-\sum_{\bar{k}\geq k>N^{d-}}\Gamma^{k}_{\alpha,\beta}p\Big).
\end{aligned}\tag{2.4.75}$$

Indeed, since $\bar{k}=L$, by Remark 2.31, we begin the coupling from the maximal possible jump and

$$\{\bar{k}\geq k>N^{d-}\}=\{L\geq k>N^{d-}\}. \tag{2.4.76}$$

A.2.1 Suppose $N^{d-} \in I_a \setminus \{L\}$ and $M+1 > N^{d-} \geq \widehat{U}$. Then there exists $j \leq K$ such that $m_j \geq N^{d-} > \delta - \beta + l_j$, and by (2.4.76), formula (2.4.75) is identical to

$$\sum_{k \in I_a \setminus \{L \geq k > N^{d-}\}} \Gamma^k_{\alpha,\beta} - \Gamma^U_{\gamma,\delta} + \sum_{L \geq k > N^{d-}} \Gamma^k_{\alpha,\beta} = \sum_{k \in I_a^{j-1} \cup \{L \geq k > \delta-\beta+l_j\}} \Gamma^k_{\alpha,\beta} - \Gamma^U_{\gamma,\delta}. \quad (2.4.77)$$

Notice that since $\bar{l} = U$, then $L+\gamma-\alpha \geq U$. We denote by $\widehat{I}_a = I_a^{j-1} \cup \{L \geq k > \delta-\beta+l_j\}$, then summing rates $\Gamma^l_{\gamma,\delta}$ for $l \in \widehat{I}_b = I_b^{j-1} \cup \{L+\gamma-\alpha \geq l > l_j\}$ or for $l \in I_b^{j-1} \cup \{U \geq l > l_j\}$ is the same. Moreover $m_j \geq N^{d-} \geq U - \gamma + \alpha$, that is $\gamma - \alpha + m_j \geq U$ and $U = N^{d+} \geq \widehat{L} > l_K \geq l_j$ by (2.4.51). Therefore $\gamma-\alpha+m_j \geq U > l_j$ implies that summing rates $\Gamma^l_{\gamma,\delta}$ for $l \in I_b = I_b^{j-1} \cup \bigcup_{i=j}^K \{\gamma-\alpha+m_i \geq l > l_i\}$ and for $l \in I_b^{j-1} \cup \{U \geq l > l_j\} = \widehat{I}_b$ is the same. Therefore (2.4.74) becomes

$$\sum_{k>\delta-\beta+l_1} \Pi^{0,k}_{\alpha,\beta} + \sum_{k \in \widehat{I}_a} \Gamma^k_{\alpha,\beta} - \Gamma^U_{\gamma,\delta} \leq \sum_{l>l_1} \Pi^{0,l}_{\gamma,\delta} + \sum_{l \in \widehat{I}_b} \Gamma^l_{\gamma,\delta} - \Gamma^U_{\gamma,\delta} \mathbb{1}_{\{U \in I_b\}} \quad (2.4.78)$$

which holds both if $U \in I_b$ and $U \notin I_b$ by Condition (2.2.13) applied to sets $\widehat{I}_a$ and $\widehat{I}_b$.

A.2.2 Suppose $N^{d-} \notin I_a$ and $N^{d-} \geq \widehat{U}$, then by (2.4.75) Condition (2.4.74) is

$$\sum_{k>\delta-\beta+l_1} \Pi^{0,k}_{\alpha,\beta} + \sum_{k \in I_a \setminus \{L \geq k > N^{d-}\}} \Gamma^k_{\alpha,\beta} \leq \sum_{l>l_1} \Pi^{0,l}_{\gamma,\delta} + \sum_{l \in I_b} \Gamma^l_{\gamma,\delta} - \Gamma^U_{\gamma,\delta} \mathbb{1}_{\{U \in I_b\}} \quad (2.4.79)$$

Since $N^{d-} \notin I_a$, either $N^{d-} \leq \delta-\beta+l_1$ or there exists j such that $\delta-\beta+l_j \geq N^{d-} > m_{j-1}$ (or possibly $N^{d-} > m_K$).
If $N^{d-} \leq \delta - \beta + l_1$ the second sum on the left hand side of (2.4.79) is null by Definition (2.2.9). By (2.2.13) with $K = 1$, $m_1 = 0$ we get

$$\sum_{k>\delta-\beta+l_1} \Pi^{0,k}_{\alpha,\beta} \leq \sum_{l>l_1} \Pi^{0,l}_{\gamma,\delta} + \sum_{\gamma-\alpha \geq l > l_1} \Gamma^l_{\gamma,\delta}. \quad (2.4.80)$$

Since γ is U-bad, then $U > \gamma - \alpha$ and (2.4.79) holds by Definition (2.2.10).
Suppose there exists j such that $\delta - \beta + l_j \geq N^{d-} > m_{j-1}$ (or possibly $N^{d-} > m_K$). Then summing rates $\Gamma^k_{\alpha,\beta}$ for $k \in I_a = I_a^{j-1} \bigcup_{i=j}^K \{m_i \geq k > \delta - \beta + l_i\}$ is bounded by the sum onf rates $\Gamma^k_{\alpha,\beta}$ for $k \in I_a^{j-1} \cup \{L \geq k > N^{d-}\}$. In case that $N^{d-} > m_K$ then $I_a \subseteq I_a \cup \{L \geq k > N^{d-}\}$ and the proof works by setting $j = K+1$.
We treat separately the cases $\widehat{U} > m_{j-1}$ and $\widehat{U} \leq m_{j-1}$.

- If $\widehat{U} > m_{j-1}$, then $U > \gamma - \alpha + m_{j-1}$ and $U \notin I_b^{j-1}$. By Condition (2.2.13) applied to I_a^{j-1} and I_b^{j-1}

$$\begin{aligned}\sum_{k>\delta-\beta+l_1} \Pi^{0,k}_{\alpha,\beta} + \sum_{k \in I_a \setminus \{L \geq k > N^{d-}\}} \Gamma^k_{\alpha,\beta} &= \sum_{k>\delta-\beta+l_1} \Pi^{0,k}_{\alpha,\beta} + \sum_{k \in I_a^{j-1}} \Gamma^k_{\alpha,\beta} \\ \leq \sum_{l>l_1} \Pi^{0,l}_{\gamma,\delta} + \sum_{l \in I_b^{j-1}} \Gamma^l_{\gamma,\delta} &\leq \sum_{l>l_1} \Pi^{0,l}_{\gamma,\delta} + \sum_{l \in I_b} \Gamma^l_{\gamma,\delta} - \Gamma^U_{\gamma,\delta} \mathbb{1}_{\{U \in I_b\}}\end{aligned}$$

which is (2.4.79).

- Suppose $\widehat{U} \leq m_{j-1}$. Let $\widehat{I}_a = I_a^{j-1} \cup \{L \geq k > N^{d-}\}$, then the second sum in the left hand side of (2.4.79) satisfies

$$\sum_{k \in I_a \setminus \{L \geq k > N^{d-}\}} \Gamma^k_{\alpha,\beta} \leq \sum_{k \in \widehat{I}_a \setminus \{L \geq k > N^{d-}\}} \Gamma^k_{\alpha,\beta} = \sum_{k \in \widehat{I}_a} \Gamma^k_{\alpha,\beta} - \sum_{k \geq N^{d-}} \Gamma^k_{\alpha,\beta}. \quad (2.4.81)$$

Since $\bar{l} = U$, $N^{d-} - \delta + \beta \leq U - \delta + \beta \leq U$. The set corresponding to $\widehat{I}_a$ is $\widehat{I}_b = I_b^{j-1} \cup \{L + \gamma - \alpha \geq l \geq N^{d-} - \delta + \beta\}$ and the sum of $\Gamma^l_{\gamma,\delta}$ for $l \in \widehat{I}_b$ or for $l \in I_b^{j-1} \cup \{U \geq l \geq N^{d-} - \delta + \beta\}$ is the same. Notice that since $N^{d-} > m_{j-1} \geq \delta - \beta + l_{j-1}$, then $N^{d-} - \delta + \beta > l_{j-1}$ (hence $U > \delta - \beta + l_{j-1} > l_{j-1}$); moreover $\gamma - \alpha + m_{j-1} \geq U$ since $\widehat{U} \leq m_{j-1}$. Therefore $U \in I_b$ and the sum of $\Gamma^l_{\gamma,\delta}$ for $l \in I_b = I_b^{j-1} \cup \bigcup_{i=j}^{K} \{\gamma - \alpha + m_i \geq l > l_i\}$ or for $l \in I_b^{j-1} \cup \{U \geq l > l_j\} = \widehat{I}_b$ is the same. Hence the second sum on the right hand side of (2.4.79) is

$$\sum_{l \in I_b} \Gamma^l_{\gamma,\delta} - \Gamma^U_{\gamma,\delta} = \sum_{l \in \widehat{I}_b} \Gamma^l_{\gamma,\delta} - \Gamma^U_{\gamma,\delta}. \tag{2.4.82}$$

Remember that we are in case $M + 1 > N^{d-} \geq \widehat{U}$, that is $\mathcal{R}^U_{\gamma,\delta} = 0$, which by Definition (2.4.13) and (2.4.29) means that $\Gamma^U_{\gamma,\delta} - \sum_{\bar{k} \geq k > N^{d-}} \Gamma^k_{\alpha,\beta} < \Gamma^{N^{d-}}_{\gamma,\delta}$, implying that the right hand side of (2.4.82) satisfies, since $\bar{k} \leq L$

$$\sum_{l \in \widehat{I}_b} \Gamma^l_{\gamma,\delta} - \sum_{L \geq k \geq N^{d-}} \Gamma^k_{\alpha,\beta} \leq \sum_{l \in \widehat{I}_b} \Gamma^l_{\gamma,\delta} - \Gamma^U_{\gamma,\delta}. \tag{2.4.83}$$

Condition (2.4.79) holds by (2.4.81), Condition (2.2.13) applied to sets $\widehat{I}_a$ and $\widehat{I}_b$, (2.4.83) and finally (2.4.82).

A.2.3 Finally, if $N^{d-} = \widehat{U} - 1$, by (2.4.74), (2.4.75) and (2.4.76) Condition (2.4.47) becomes

$$\sum_{k > \delta - \beta + l_1} \Pi^{0,k}_{\alpha,\beta} + \sum_{k \in I_a \setminus \{L \geq k > N^{d-}\}} \Gamma^k_{\alpha,\beta} \leq \sum_{l > l_1} \Pi^{0,l}_{\gamma,\delta} + \sum_{l \in I_b} \Gamma^l_{\gamma,\delta} - \Gamma^U_{\gamma,\delta} \mathbb{1}_{\{U \in I_b\}}. \tag{2.4.84}$$

• If $N^{d-} \in I_a$, there exists j such that $m_j \geq N^{d-} \geq \delta - \beta + l_j$, that is $\gamma - \alpha + m_j \geq U - 1 \geq l_j$. We use arguments as in case A.2.2: by Condition (2.2.13) applied to $\widehat{I}_a = I_a^{j-1} \cup \{N^{d-} \geq k > \delta - \beta + l_j\} = I_a \setminus \{L \geq k > N^{d-}\}$ and $\widehat{I}_b = I_b^{j-1} \cup \{\gamma - \alpha + N^{d-} \geq k > l_j\}$, since $N^{d-} = \widehat{U} - 1$ and $\gamma - \alpha + m_j \geq U - 1 \geq l_j$

$$\begin{aligned}
\sum_{k > \delta - \beta + l_1} \Pi^{0,k}_{\alpha,\beta} + \sum_{k \in I_a \setminus \{L \geq k > N^{d-}\}} \Gamma^k_{\alpha,\beta} &= \sum_{k > \delta - \beta + l_1} \Pi^{0,k}_{\alpha,\beta} + \sum_{k \in \widehat{I}_a} \Gamma^k_{\alpha,\beta} \\
&\leq \sum_{l > l_1} \Pi^{0,l}_{\gamma,\delta} + \sum_{l \in I_b^{j-1} \cup \{\gamma - \alpha + N^{d-} \geq k > l_j\}} \Gamma^l_{\gamma,\delta} \\
&= \sum_{l > l_1} \Pi^{0,l}_{\gamma,\delta} + \sum_{l \in I_b^{j-1} \cup \{U - 1 \geq k > l_j\}} \Gamma^l_{\gamma,\delta} \\
&\leq \sum_{l > l_1} \Pi^{0,l}_{\gamma,\delta} + \sum_{l \in I_b} \Gamma^l_{\gamma,\delta} - \Gamma^U_{\gamma,\delta} \mathbb{1}_{\{U \in I_b\}}
\end{aligned}$$

and (2.4.84) holds.

• Suppose $N^{d-} \notin I_a$ and there exists j such that $\delta - \beta + l_j \geq U - \gamma + \alpha - 1 = N^{d-} > m_{j-1}$ (or possibly $N^{d-} > m_K$, in which case we set $j = K+1$). This means that $U > \gamma - \alpha + m_{j-1} + 1$, that is $U \notin I_b^{j-1}$ and we get by Condition (2.2.13) on sets I_a^{j-1} and I_b^{j-1}

$$\begin{aligned}
\sum_{k > \delta - \beta + l_1} \Pi^{0,k}_{\alpha,\beta} + \sum_{k \in I_a \setminus \{L \geq k > N^{d-}\}} \Gamma^k_{\alpha,\beta} &= \sum_{k > \delta - \beta + l_1} \Pi^{0,k}_{\alpha,\beta} + \sum_{k \in I_a^{j-1}} \Gamma^k_{\alpha,\beta} \\
&\leq \sum_{l > l_1} \Pi^{0,l}_{\gamma,\delta} + \sum_{l \in I_b^{j-1}} \Gamma^l_{\gamma,\delta} \leq \sum_{l \in I_b} \Gamma^l_{\gamma,\delta} - \Gamma^U_{\gamma,\delta} \mathbb{1}_{\{U \in I_b\}}.
\end{aligned}$$

If $N^{d-}\notin I_a$ and $N^{d-}\le \delta-\beta+l_1$, then $I_a\subseteq\{L\ge k>N^{d-}\}$ and the second sum in the left hand side of (2.4.84) is null. As in case A.2.2 we can start from (2.4.80) and treat different cases exactly in the same way to check that (2.4.84) holds.

A.3 Suppose $\bar{l}=U<\widehat{L}$, that is we do not pass Step 1 and $N^{d+}=M+1$. This means that $L>U+\delta-\beta=\bar{k}$. By (2.4.51), $N^{dB}>l_K$ and by (2.4.21), $H^{L,L,l,l}_{\alpha,\beta,\gamma,\delta}=0$ for all l. Hence, by Definition (2.2.27) we get $T^{L,l}_{\alpha,\beta}=\Gamma^L_{\alpha,\beta}p-\sum_{l>l}H^{L,L,0,l}_{\alpha,\beta,\gamma,\delta}$.
Since by Proposition 2.39 *iii*), $N^B\ge\widehat{L}>U$, by (2.4.33) then $H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta}=0$ for all l. By (2.4.64), (2.4.65), Table 2.11 and since $N^{d+}=M+1$ then (2.4.63) becomes

$$\begin{aligned}\sum_{l>l_1}\overline{\Pi}^{0,l}_{\gamma,\delta}p+\sum_{l\in I_b}\overline{\Gamma}^{l}_{\gamma,\delta}p&=\sum_{N^B\ge l>l_1}\big(\Pi^{0,l}_{\gamma,\delta}p-H^{0,L,0,l}_{\alpha,\beta,\gamma,\delta}\big)+\sum_{l\in I_b\setminus\{U\}}\Gamma^l_{\gamma,\delta}p-\sum_{l>l_1}H^{L,L,0,l}_{\alpha,\beta,\gamma,\delta}\\&=\sum_{l>l_1}\Pi^{0,l}_{\gamma,\delta}p-\Pi^{0,L}_{\alpha,\beta}p+\sum_{l\in I_b\setminus\{U\}}\Gamma^l_{\gamma,\delta}p-\sum_{l>N^{dB}}H^{L,L,0,l}_{\alpha,\beta,\gamma,\delta}-T^{L,N^{dB}}_{\alpha,\beta}\\&=\sum_{l>l_1}\Pi^{0,l}_{\gamma,\delta}p-\Pi^{0,L}_{\alpha,\beta}p+\sum_{l\in I_b\setminus\{U\}}\Gamma^l_{\gamma,\delta}p-\Gamma^L_{\alpha,\beta}p.\end{aligned}\tag{2.4.85}$$

By (2.4.56), (2.4.57) and (2.4.85), Condition (2.4.47) reduces to

$$\sum_{k>\delta-\beta+l_1}\Pi^{0,k}_{\alpha,\beta}p+\sum_{k\in I_a\setminus\{L\}}(\Gamma^k_{\alpha,\beta}p-H^{k,k,U,U}_{\alpha,\beta,\gamma,\delta})\le\sum_{l>l_1}\Pi^{0,l}_{\gamma,\delta}p+\sum_{l\in I_b\setminus\{U\}}\Gamma^l_{\gamma,\delta}p.\tag{2.4.86}$$

A.3.1 Suppose $N^{d-}\ge\widehat{U}$ and remember that we are in case $M+1>N^{d-}$.
• If $N^{d-}\in I_a$, as in case A.2.1, there exists j such that $m_j\ge N^{d-}>\delta-\beta+l_j$. By working as in (2.4.75) and (2.4.77), Condition (2.4.47) is equivalent to

$$\sum_{k>\delta-\beta+l_1}\Pi^{0,k}_{\alpha,\beta}+\sum_{k\in I_a\setminus\{L\}\cup\{\bar{k}\ge k>N^{d-}\}}\Gamma^k_{\alpha,\beta}\le\sum_{l>l_1}\Pi^{0,l}_{\gamma,\delta}+\sum_{l\in I_b\setminus\{U\}}\Gamma^l_{\gamma,\delta}+\Gamma^U_{\gamma,\delta}$$

Notice that we can suppose $I_a=I^K_a\cup\{L\ge k>U+\delta-\beta\}$ by (2.4.54) and $\bar{k}=U+\delta-\beta$, therefore summing $\Gamma^k_{\alpha,\beta}$ for $k\in I_a\cup\{\bar{k}\ge k>N^{d-}\}$ or for $k\in I^{j-1}_a\cup\{L\ge k>\delta-\beta+l_j\}$ is the same. Moreover $m_j\ge N^{d-}\ge U-\gamma+\alpha$, hence $m_j+\gamma-\alpha\ge U$ and the sum of $\Gamma^l_{\gamma,\delta}$ for $l\in I_b$ or for $l\in I^{j-1}_b\cup\{L+\gamma-\alpha\ge l>l_j\}$ is the same, since $L+\gamma-\alpha>L>U$ (because $\widehat{L}>U$). Therefore, by Condition (2.2.13) applied to $\widehat{I}_a$ and $\widehat{I}_b$

$$\begin{aligned}\sum_{k>\delta-\beta+l_1}\Pi^{0,k}_{\alpha,\beta}+\sum_{k\in I_a\cup\{\bar{k}\ge k>N^{d-}\}}\Gamma^k_{\alpha,\beta}&=\sum_{k>\delta-\beta+l_1}\Pi^{0,k}_{\alpha,\beta}+\sum_{k\in I^{j-1}_a\cup\{L\ge k>\delta-\beta+l_j\}}\Gamma^k_{\alpha,\beta}\\&\le\sum_{l>l_1}\Pi^{0,l}_{\gamma,\delta}+\sum_{l\in I^{j-1}_b\cup\{U\ge l>l_j\}}\Gamma^l_{\gamma,\delta}\\&=\sum_{l>l_1}\Pi^{0,l}_{\gamma,\delta}+\sum_{l\in I_b\setminus\{U\}}\Gamma^l_{\gamma,\delta}+\Gamma^U_{\gamma,\delta}\end{aligned}$$

and we are done.

• If $N^{d-}\notin I_a$, condition (2.4.86) is equivalent to

$$\sum_{k>\delta-\beta+l_1}\Pi^{0,k}_{\alpha,\beta}+\sum_{k\in I_a\setminus\{\bar{k}\ge k>N^{d-}\}}\Gamma^k_{\alpha,\beta}\le\sum_{l>l_1}\Pi^{0,l}_{\gamma,\delta}+\sum_{l\in I_b}\Gamma^l_{\gamma,\delta}-\Gamma^U_{\gamma,\delta}\mathbb{1}_{\{U\in I_b\}}\tag{2.4.87}$$

by working as in case A.2.2 to get (2.4.79).
In order to check (2.4.87) and Condition (2.4.86) in case $N^{d-} = \widehat{U} - 1$ we work respectively as in case A.2.2 to check (2.4.79) and as in case A.2.3. We can repeat the same passages, where we take $I_a \setminus \{\bar{k} \geq l > N^{d-}\}$ instead of $I_a \setminus \{L \geq l > N^{d-}\}$ and, by (2.4.54), we can use $I_a^j \cup \{L \geq l > U + \delta - \beta\}$ instead of I_a^j without loss of generality.

Case B Suppose β is L-bad and γ is U-good, that is $U > \gamma - \alpha$.

B.1 If $N^{d+} < U = \bar{l}$, as in case A the first sum on the left hand side of (2.4.47) is given by (2.4.56) and the right hand side by (2.4.63), (2.4.64) and (2.4.65). The different term is in the second sum on the left hand side. Since the first coupling rate of Step 1 is $\Gamma^L_{\alpha,\beta} \wedge \Gamma^U_{\gamma,\delta} = \Gamma^U_{\gamma,\delta}$, then the second term of the left hand side of (2.4.47) is given by (2.4.66), and nothing changes with respect to the case in which γ is U-bad, $N^{d+} < U = \bar{l}$ and $N^{d-} = L$, since N^{d-} did not play any role in that case. Therefore (2.4.47) can be proved in the same way.

B.2 If $N^{d+} = \bar{l} = U$, the left hand side is given by (2.4.56) and by (2.4.66) as in B.1.
By Definition 2.24 when γ is U-good, the right hand side is

$$\begin{aligned}\sum_{l>l_1} \overline{\Pi}^{0,l}_{\gamma,\delta} p + \sum_{l \in I_b} \overline{\Gamma}^l_{\gamma,\delta} p = &\sum_{l>l_1} \Pi^{0,l}_{\gamma,\delta} p - \sum_{l>l_1} H^{0,L,0,l}_{\alpha,\beta,\gamma,\delta} - 1\!\!1_{\{N^B < \gamma-\alpha\}} \sum_{(\gamma-\alpha)\wedge U \geq l > l_1} H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta} \\ &+ \sum_{l \in I_b} \Gamma^l_{\gamma,\delta} p - \sum_{l \in I_b} H^{L,L,l,l}_{\alpha,\beta,\gamma,\delta} - \sum_{l>l_1} H^{L,L,0,l}_{\alpha,\beta,\gamma,\delta}. \end{aligned} \tag{2.4.88}$$

Notice that the difference with respect to case A is that $\overline{\Gamma}^U_{\gamma,\delta}$ has the same definition of $\overline{\Gamma}^k_{\gamma,\delta}$ when $k < U$, and it could be null or not. Indeed by Proposition 2.25 only the higher rates that cause an attractiveness problem must be null in the new system. Therefore this is not the case for $\overline{\Gamma}^U_{\gamma,\delta}$.
We work as for (2.4.64) and (2.4.65) to get

$$\sum_{l>l_1} \Pi^{0,l}_{\gamma,\delta} p - \sum_{l>l_1} H^{0,L,0,l}_{\alpha,\beta,\gamma,\delta} - 1\!\!1_{\{N^B < \gamma-\alpha\}} \sum_{\gamma-\alpha \geq l > l_1} H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta} = \sum_{l>l_1} \Pi^{0,l}_{\gamma,\delta} p - \Pi^{0,L}_{\alpha,\beta} p.$$

Since $N^{d+} = U \geq \widehat{L}$, by Table 2.1 and (2.4.25)

$$\sum_{l \in I_b} \Gamma^l_{\gamma,\delta} p - \sum_{l \in I_b} H^{L,L,l,l}_{\alpha,\beta,\gamma,\delta} - \sum_{l>l_1} H^{L,L,0,l}_{\alpha,\beta,\gamma,\delta} = \sum_{l \in I_b} \Gamma^l_{\gamma,\delta} p - \Gamma^L_{\alpha,\beta} p 1\!\!1_{\{U \in I_b\}}$$

and Condition (2.4.47) becomes

$$\sum_{k>\delta-\beta+l_1} \Pi^{0,k}_{\alpha,\beta} + \sum_{k \in I_a} \Gamma^k_{\alpha,\beta} - \Gamma^L_{\alpha,\beta} 1\!\!1_{\{L \in I_a\}} \leq \sum_{l>l_1} \Pi^{0,l}_{\gamma,\delta} + \sum_{l \in I_b} \Gamma^l_{\gamma,\delta} - \Gamma^L_{\alpha,\beta} 1\!\!1_{\{U \in I_b\}}.$$

Since $\widehat{L} > l_K$ and $\beta + L \leq \delta + U$, then $U \geq \widehat{L} > l_K$. If ($L \notin I_a$ and $U \notin I_b$) or ($L \in I_a$ and $U \notin I_b$) it holds by Condition (2.2.13).
Suppose $U \in I_b$ and $L \notin I_a$. Since γ is U-good, $U < \gamma - \alpha < L + \gamma - \alpha$ and $U \geq \widehat{L} > l_K$, then $1\!\!1_{\{L+\gamma-\alpha > l > \widehat{L}\}} \Gamma^k_{\gamma,\delta} = 1\!\!1_{\{U \geq l > \widehat{L}\}} \Gamma^k_{\gamma,\delta} \leq 1\!\!1_{\{U \geq l > l_K\}} \Gamma^k_{\gamma,\delta}$. Therefore

$$\begin{aligned}\sum_{k>\delta-\beta+l_1} \Pi^{0,k}_{\alpha,\beta} + \sum_{k \in I_a} \Gamma^k_{\alpha,\beta} + \Gamma^L_{\alpha,\beta} \leq &\sum_{k>\delta-\beta+l_1} \Pi^{0,k}_{\alpha,\beta} + \sum_{k \in I_a \cup \{L \geq k \geq L\}} \Gamma^k_{\alpha,\beta} \\ \leq \sum_{l>l_1} \Pi^{0,l}_{\gamma,\delta} + \sum_{l \in I_b \cup \{U \geq l \geq l_K\}} \Gamma^l_{\gamma,\delta} = &\sum_{l>l_1} \Pi^{0,l}_{\gamma,\delta} + \sum_{l \in I_b} \Gamma^l_{\gamma,\delta}\end{aligned}$$

since $U \in I_b$.

B.3 Suppose $U < \bar{l}$. Since γ is U-good and we do not pass Step 1, $\overline{\Gamma}^l_{\gamma,\delta} = \Gamma^l_{\gamma,\delta}$ for each $l \leq U$ by Definition 2.24 and $\overline{\Gamma}^k_{\alpha,\beta} = \Gamma^k_{\alpha,\beta}$ for each $k < L$, since $\overline{\Gamma}^L_{\alpha,\beta} = 0$ by Proposition 2.25 i). Notice that (2.4.85) still holds with $\sum_{l\in I_b} \Gamma^l_{\gamma,\delta}$ instead of $\sum_{l\in I_b\setminus\{U\}} \Gamma^l_{\gamma,\delta}$. By (2.4.56) Condition (2.4.47) is identical to

$$\sum_{k>\delta-\beta+l_1} \Pi^{0,k}_{\alpha,\beta} + \sum_{k\in I_a\setminus\{L\}} \Gamma^k_{\alpha,\beta} \leq \sum_{l>l_1} \Pi^{0,l}_{\gamma,\delta} + \sum_{l\in I_b} \Gamma^l_{\gamma,\delta}$$

which holds by Condition (2.2.13) and the proof of (2.4.47) is complete. $\square$

2.4.7 Proof of Proposition 2.27

In order to prove that $\mathcal{H}$ is increasing, we have to show that all coupling rates breaking the partial order between configurations are null. We denote by $\overline{\mathcal{H}}$ the coupling of Definition 2.21 on the system $\overline{\mathcal{S}}$ and we prove that $\mathcal{H}$ and $\overline{\mathcal{H}}$ differ only on rates involving L or U particles. The claim follows by hypothesis on $\overline{\mathcal{H}}$ and by Proposition 2.37.

Case A Suppose β is L-bad and γ is U-bad for $\mathcal{S}$, then by Proposition 2.25, $\overline{N} = \max\{\overline{L}, \overline{U}\} \leq \max\{L-1, U-1\} = N-1$. In other words

$$\overline{H}^{L,L,l,l}_{\alpha,\beta,\gamma,\delta} = \overline{H}^{L,L,-l,0}_{\alpha,\beta,\gamma,\delta} = \overline{H}^{L,L,0,l}_{\alpha,\beta,\gamma,\delta} = \overline{H}^{k,k,U,U}_{\alpha,\beta,\gamma,\delta} = \overline{H}^{0,k,U,U}_{\alpha,\beta,\gamma,\delta} = \overline{H}^{-k,0,U,U}_{\alpha,\beta,\gamma,\delta} = 0$$

for each k and l. Remember that $\bar{l} = U$ if $\widehat{L} \leq U$, otherwise $\bar{l} = L + \gamma - \alpha$, and $\bar{k} = L$ if $L \geq \widehat{U}$, otherwise $\bar{k} = U + \delta - \beta$.

Step 1. We begin from Step 1. Suppose β is $(L-1)$-bad for $\overline{\mathcal{S}}$. Therefore $\overline{H}^{L-1,L-1,-l,0}_{\alpha,\beta,\gamma,\delta} = 0$ for each l by Definition (2.2.26), and by Table 2.1, Definition 2.21 on $\overline{\mathcal{S}}$ and Definition 2.24, if $U > l \geq \widehat{L}$, we get

$$\begin{aligned}
\overline{H}^{L-1,L-1,l,l}_{\alpha,\beta,\gamma,\delta} =&[\overline{\Gamma}^{L-1}_{\alpha,\beta} p - \sum_{\bar{l}-1\geq l'>l} \big(\overline{H}^{L-1,L-1,l',l'}_{\alpha,\beta,\gamma,\delta} + \overline{H}^{L-1,L-1,-l',0}_{\alpha,\beta,\gamma,\delta}\big)] \wedge [\overline{\Gamma}^l_{\gamma,\delta} p] \\
=&[\Gamma^{L-1}_{\alpha,\beta} p - H^{L-1,L-1,U,U}_{\alpha,\beta,\gamma,\delta} - H^{L-1,L-1,-U,0}_{\alpha,\beta,\gamma,\delta} - \sum_{\bar{l}-1\geq l'>l} \overline{H}^{L-1,L-1,l',l'}_{\alpha,\beta,\gamma,\delta}] \\
&\wedge [\Gamma^l_{\gamma,\delta} p - H^{L,L,l,l}_{\alpha,\beta,\gamma,\delta} - H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta}].
\end{aligned} \tag{2.4.89}$$

• If $\bar{l} = U$ then the first coupling term between jumps in the new system is

$$\begin{aligned}
\overline{H}^{L-1,L-1,U-1,U-1}_{\alpha,\beta,\gamma,\delta} =&[\Gamma^{L-1}_{\alpha,\beta} p - H^{L-1,L-1,U,U}_{\alpha,\beta,\gamma,\delta} - H^{L-1,L-1,-U,0}_{\alpha,\beta,\gamma,\delta}] \\
&\wedge [\Gamma^{U-1}_{\gamma,\delta} p - H^{L,L,U-1,U-1}_{\alpha,\beta,\gamma,\delta} - H^{0,L,U-1,U-1}_{\alpha,\beta,\gamma,\delta}] \\
=&H^{L-1,L-1,U-1,U-1}_{\alpha,\beta,\gamma,\delta}
\end{aligned} \tag{2.4.90}$$

We get the same conclusion if we suppose γ to be $(U-1)$-bad for $\overline{\mathcal{S}}$ by starting from Formula (2.2.18).

• If $\bar{l} = L + \gamma - \alpha < U$ then $\alpha - L > \gamma - U$ implies $\alpha - (L-1) > \gamma - (U-1)$, that is $\overline{H}^{L-1,L-1,U-1,U-1}_{\alpha,\beta,\gamma,\delta} = H^{L-1,L-1,U-1,U-1}_{\alpha,\beta,\gamma,\delta} = 0$ by Definition 2.21. By (2.4.89) the first positive term is

$$\begin{aligned}\overline{H}^{L-1,L-1,\bar{l}-1,\bar{l}-1}_{\alpha,\beta,\gamma,\delta} =&[\overline{\Gamma}^{L-1}_{\alpha,\beta}p - \sum_{l'>\bar{l}-1}\big(\overline{H}^{L-1,L-1,l',l'}_{\alpha,\beta,\gamma,\delta} + \overline{H}^{L-1,L-1,-l',0}_{\alpha,\beta,\gamma,\delta}\big)] \wedge [\overline{\Gamma}^{\bar{l}-1}_{\gamma,\delta}p]\\ =&[\Gamma^{L-1}_{\alpha,\beta}p] \wedge [\Gamma^{\bar{l}-1}_{\gamma,\delta}p - H^{L,L,\bar{l}-1,\bar{l}-1}_{\alpha,\beta,\gamma,\delta} - H^{0,L,\bar{l}-1,\bar{l}-1}_{\alpha,\beta,\gamma,\delta}]\\ =&H^{L-1,L-1,\bar{l}-1,\bar{l}-1}_{\alpha,\beta,\gamma,\delta}.\end{aligned}$$

In both cases all terms $\overline{H}^{L-1,L-1,l,l}_{\alpha,\beta,\gamma,\delta}$ for $\bar{l}-1 > l > \widehat{L}-1$ and $\overline{H}^{k,k,U-1,U-1}_{\alpha,\beta,\gamma,\delta}$ for $L-1 > k > \widehat{U}-1$ are defined recursively starting from $\overline{H}^{L-1,L-1,\bar{l}-1,\bar{l}-1}_{\alpha,\beta,\gamma,\delta}$. Since the higher terms are equal to the original ones, all terms defined downwards recursively starting from that are also equal.
Therefore $\overline{H}^{L-1,L-1,l,l}_{\alpha,\beta,\gamma,\delta} = H^{L-1,L-1,l,l}_{\alpha,\beta,\gamma,\delta}$, for each $\bar{l}-1 \geq l \geq \widehat{L}-1$ and $\overline{H}^{k,k,U-1,U-1}_{\alpha,\beta,\gamma,\delta} = H^{k,k,U-1,U-1}_{\alpha,\beta,\gamma,\delta}$, for each $\bar{k}-1 \geq k \geq \widehat{U}-1$.

In a similar way one can prove that if γ is $(U-1)$-bad then $\overline{H}^{k,k,U-1,U-1}_{\alpha,\beta,\gamma,\delta} = H^{k,k,U-1,U-1}_{\alpha,\beta,\gamma,\delta}$, for each $\bar{k}-1 \geq k \geq \widehat{U}-1$.
Notice that if β is $(L-1)$-good (respectively γ is $(U-1)$-good), then $\overline{H}^{L-1,L-1,l,l}_{\alpha,\beta,\gamma,\delta} = 0 = H^{L-1,L-1,l,l}_{\alpha,\beta,\gamma,\delta}$ for any $l > 0$ ($\overline{H}^{k,k,U-1,U-1}_{\alpha,\beta,\gamma,\delta} = 0 = H^{k,k,U-1,U-1}_{\alpha,\beta,\gamma,\delta}$ for any $k > 0$) by Definition 2.21.
Therefore we proved that if either $\widehat{L} > 0$ or $\widehat{U} > 0$, or both, then

$$\overline{H}^{L-1,L-1,l,l}_{\alpha,\beta,\gamma,\delta} = H^{L-1,L-1,l,l}_{\alpha,\beta,\gamma,\delta} \qquad \text{for each } l > 0 \tag{2.4.91}$$

$$\overline{H}^{k,k,U-1,U-1}_{\alpha,\beta,\gamma,\delta} = H^{k,k,U-1,U-1}_{\alpha,\beta,\gamma,\delta} \qquad \text{for each } k > 0. \tag{2.4.92}$$

Step 2. We now consider birth rates. We begin from Step 2a). Remember that $\overline{H}^{0,L,0,l}_{\alpha,\beta,\gamma,\delta} = \overline{H}^{L,L,0,l}_{\alpha,\beta,\gamma,\delta} = 0$. Suppose β is $(L-1)$-bad for $\overline{\mathcal{S}}$. We get

$$\begin{aligned}\overline{H}^{0,L-1,0,l}_{\alpha,\beta,\gamma,\delta} =&[\overline{\Pi}^{0,L-1}_{\alpha,\beta}p - \sum_{l'>l}\overline{H}^{0,L-1,0,l'}_{\alpha,\beta,\gamma,\delta} - \sum_{(U-1)\wedge(\gamma-\alpha)\geq l'>l}\overline{H}^{0,L-1,l',l'}_{\alpha,\beta,\gamma,\delta}] \wedge [\overline{\Pi}^{0,l}_{\gamma,\delta}p]\\ =&[\Pi^{0,L-1}_{\alpha,\beta}p - \sum_{l'>l}\overline{H}^{0,L-1,0,l'}_{\alpha,\beta,\gamma,\delta} - \sum_{(U-1)\wedge(\gamma-\alpha)\geq l'>l}\overline{H}^{0,L-1,l',l'}_{\alpha,\beta,\gamma,\delta}] \wedge [\Pi^{0,l}_{\gamma,\delta}p - H^{0,L,0,l}_{\alpha,\beta,\gamma,\delta} - H^{L,L,0,l}_{\alpha,\beta,\gamma,\delta}].\end{aligned}$$

If $l = M$ then

$$\overline{H}^{0,L-1,0,M}_{\alpha,\beta,\gamma,\delta} = [\Pi^{0,L-1}_{\alpha,\beta}p] \wedge [\Pi^{0,M}_{\gamma,\delta}p - H^{0,L,0,M}_{\alpha,\beta,\gamma,\delta} - H^{L,L,0,M}_{\alpha,\beta,\gamma,\delta}] = H^{0,L-1,0,M}_{\alpha,\beta,\gamma,\delta}.$$

This implies, by recursion, that $\overline{H}^{0,L-1,0,l}_{\alpha,\beta,\gamma,\delta} = H^{0,L-1,0,l}_{\alpha,\beta,\gamma,\delta}$ for each $U \wedge (\gamma - \alpha) \leq l \leq M$.
Now we move to Step 2b). In a similar way we notice that $\overline{H}^{0,L,l,l}_{\alpha,\beta,\gamma,\delta} = \overline{H}^{L,L,l,l}_{\alpha,\beta,\gamma,\delta} = 0$ and

$$\begin{aligned}\overline{H}^{0,L-1,l,l}_{\alpha,\beta,\gamma,\delta} =&[\overline{\Pi}^{0,L-1}_{\alpha,\beta}p - \sum_{l'\geq l}\overline{H}^{0,L-1,0,l'}_{\alpha,\beta,\gamma,\delta} - \sum_{(U-1)\wedge(\gamma-\alpha)\geq l'>l}\overline{H}^{0,L-1,l',l'}_{\alpha,\beta,\gamma,\delta}] \wedge [\overline{\Gamma}^{l}_{\gamma,\delta}p - \overline{H}^{L-1,L-1,l,l}_{\alpha,\beta,\gamma,\delta}]\\ =&[\Pi^{0,L-1}_{\alpha,\beta}p - \sum_{l'\geq l}\overline{H}^{0,L-1,0,l'}_{\alpha,\beta,\gamma,\delta} - \sum_{(U-1)\wedge(\gamma-\alpha)\geq l'>l}\overline{H}^{0,L-1,l',l'}_{\alpha,\beta,\gamma,\delta}]\\ &\wedge [\Gamma^{l}_{\gamma,\delta}p - H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta} - H^{L,L,l,l}_{\alpha,\beta,\gamma,\delta} - H^{L-1,L-1,l,l}_{\alpha,\beta,\gamma,\delta}].\end{aligned}$$

If $l=\gamma-\alpha$, then

$$\begin{aligned}\overline{H}^{0,L-1,\gamma-\alpha,\gamma-\alpha}_{\alpha,\beta,\gamma,\delta} =&[\Pi^{0,L-1}_{\alpha,\beta}p-\sum_{l'\geq N}H^{0,L-1,0,l'}_{\alpha,\beta,\gamma,\delta}]\\ &\wedge[\Gamma^{\gamma-\alpha}_{\gamma,\delta}p-H^{0,L,\gamma-\alpha,\gamma-\alpha}_{\alpha,\beta,\gamma,\delta}-H^{L,L,\gamma-\alpha,\gamma-\alpha}_{\alpha,\beta,\gamma,\delta}-H^{L-1,L-1,\gamma-\alpha,\gamma-\alpha}_{\alpha,\beta,\gamma,\delta}]\\ =&H^{0,L-1,\gamma-\alpha,\gamma-\alpha}_{\alpha,\beta,\gamma,\delta}.\end{aligned}$$

As in Step 1 they are equal on each term defined downwards recursively starting from the first ones. Therefore $\overline{H}^{0,L-1,l,l}_{\alpha,\beta,\gamma,\delta}=H^{0,L-1,l,l}_{\alpha,\beta,\gamma,\delta}$ and $\overline{H}^{0,L-1,0,l}_{\alpha,\beta,\gamma,\delta}=H^{0,L-1,0,l}_{\alpha,\beta,\gamma,\delta}$, for each $l\leq\gamma-\alpha$. If β is $(L-1)$-good for $\overline{\mathcal{S}}$, by Definition 2.21 we know that $\overline{H}^{0,L-1,0,l}_{\alpha,\beta,\gamma,\delta}=0=H^{0,L-1,0,l}_{\alpha,\beta,\gamma,\delta}$ and $\overline{H}^{0,L-1,l,l}_{\alpha,\beta,\gamma,\delta}=0=H^{0,L-1,l,l}_{\alpha,\beta,\gamma,\delta}$.
In a symmetric way we prove that $\overline{H}^{-M,0,-(U-1),0}_{\alpha,\beta,\gamma,\delta}=H^{-M,0,-(U-1),0}_{\alpha,\beta,\gamma,\delta}$, which implies

$$\overline{H}^{-k,0,-(U-1),0}_{\alpha,\beta,\gamma,\delta}=H^{-k,0,-(U-1),0}_{\alpha,\beta,\gamma,\delta}$$

for each $k\geq\delta-\beta$ and that $\overline{H}^{\delta-\beta,\delta-\beta,-(U-1),0}_{\alpha,\beta,\gamma,\delta}=H^{\delta-\beta,\delta-\beta,-(U-1),0}_{\alpha,\beta,\gamma,\delta}$, which implies

$$\overline{H}^{-k,0,-(U-1),0}_{\alpha,\beta,\gamma,\delta}=H^{-k,0,-(U-1),0}_{\alpha,\beta,\gamma,\delta},\qquad \overline{H}^{k,k,-(U-1),0}_{\alpha,\beta,\gamma,\delta}=H^{k,k,-(U-1),0}_{\alpha,\beta,\gamma,\delta}$$

for each $k\leq\delta-\beta$.
If γ is $(U-1)$-good for $\overline{\mathcal{S}}$, by Definition 2.21, $\overline{H}^{-k,0,-(U-1),0}_{\alpha,\beta,\gamma,\delta}=0=H^{-k,0,-(U-1),0}_{\alpha,\beta,\gamma,\delta}$ and $\overline{H}^{k,k,-(U-1),0}_{\alpha,\beta,\gamma,\delta}=0=H^{k,k,-(U-1),0}_{\alpha,\beta,\gamma,\delta}$. In other words we proved that if β is L-bad and γ is U-bad for $\mathcal{S}$

$$\overline{H}^{0,L-1,0,l}_{\alpha,\beta,\gamma,\delta}=H^{0,L-1,0,l}_{\alpha,\beta,\gamma,\delta},\qquad \overline{H}^{0,L-1,l,l}_{\alpha,\beta,\gamma,\delta}=H^{0,L-1,l,l}_{\alpha,\beta,\gamma,\delta}\qquad\text{for each } l>0, \tag{2.4.93}$$
$$\overline{H}^{-k,0,-(U-1),0}_{\alpha,\beta,\gamma,\delta}=H^{-k,0,-(U-1),0}_{\alpha,\beta,\gamma,\delta},\quad \overline{H}^{k,k,-(U-1),0}_{\alpha,\beta,\gamma,\delta}=H^{k,k,-(U-1),0}_{\alpha,\beta,\gamma,\delta}\qquad\text{for each } k>0. \tag{2.4.94}$$

Step 3. Finally we consider Step 3. Remember that $\overline{H}^{L,L,0,l}_{\alpha,\beta,\gamma,\delta}=\overline{H}^{-k,0,U,U}_{\alpha,\beta,\gamma,\delta}=0$. By Definition 2.21 and (2.4.90) in Step 1, if β is $(L-1)$-bad for $\overline{\mathcal{S}}$

$$\begin{aligned}\overline{H}^{L-1,L-1,0,l}_{\alpha,\beta,\gamma,\delta}=&[\overline{\Gamma}^{L-1}_{\alpha,\beta}p-\sum_{l'\geq L-1-\delta+\beta}\overline{H}^{L-1,L-1,l',l'}_{\alpha,\beta,\gamma,\delta}-\sum_{l'>l}\overline{H}^{L-1,L-1,0,l'}_{\alpha,\beta,\gamma,\delta}]\\ &\wedge[\overline{\Pi}^{0,l}_{\gamma,\delta}p-\overline{H}^{0,L-1,0,l}_{\alpha,\beta,\gamma,\delta}]\\ =&[\Gamma^{L-1}_{\alpha,\beta}p-\sum_{l'\geq L-1-\delta+\beta}H^{L-1,L-1,l',l'}_{\alpha,\beta,\gamma,\delta}-\sum_{l'>l}\overline{H}^{L-1,L-1,0,l'}_{\alpha,\beta,\gamma,\delta}]\\ &\wedge[\Pi^{0,l}_{\gamma,\delta}p-H^{L,L,0,l}_{\alpha,\beta,\gamma,\delta}-H^{0,L,0,l}_{\alpha,\beta,\gamma,\delta}-H^{0,L-1,0,l}_{\alpha,\beta,\gamma,\delta}].\end{aligned}$$

If $l=M$,

$$\begin{aligned}\overline{H}^{L-1,L-1,0,M}_{\alpha,\beta,\gamma,\delta}=&[\Gamma^{L-1}_{\alpha,\beta}p-\sum_{l'\geq L-1-\delta+\beta}H^{L-1,L-1,l',l'}_{\alpha,\beta,\gamma,\delta}]\\ &\wedge[\Pi^{0,M}_{\gamma,\delta}p-H^{L,L,0,M}_{\alpha,\beta,\gamma,\delta}-H^{0,L,0,M}_{\alpha,\beta,\gamma,\delta}-H^{0,L-1,0,l}_{\alpha,\beta,\gamma,\delta}]\\ =&H^{L-1,L-1,0,M}_{\alpha,\beta,\gamma,\delta}.\end{aligned}$$

By symmetric arguments we prove that $\overline{H}_{\alpha,\beta,\gamma,\delta}^{-M,0,U-1,U-1} = H_{\alpha,\beta,\gamma,\delta}^{-M,0,U-1,U-1}$. This implies by recursion as in previous steps that

$$\begin{aligned}
\overline{H}_{\alpha,\beta,\gamma,\delta}^{L-1,L-1,0,l} &= H_{\alpha,\beta,\gamma,\delta}^{L-1,L-1,0,l} && \text{for each } l > 0,\\
\overline{H}_{\alpha,\beta,\gamma,\delta}^{-k,0,U-1,U-1} &= H_{\alpha,\beta,\gamma,\delta}^{-k,0,U-1,U-1} && \text{for each } k > 0
\end{aligned}$$

which holds also if β is $(L-1)$-good and γ is $(U-1)$-good for $\overline{\mathcal{S}}$ by Definition 2.21 as in previous steps.
Since $\overline{\mathcal{H}}$ and $\mathcal{H}$ are defined recursively by using the same rules, the equality of the higher terms implies the equality of all terms. Thus

$$\begin{aligned}
\overline{H}_{\alpha,\beta,\gamma,\delta}^{k,k,l,l} &= H_{\alpha,\beta,\gamma,\delta}^{k,k,l,l} \quad k < L, \quad l < U; & \overline{H}_{\alpha,\beta,\gamma,\delta}^{0,k,0,l} &= H_{\alpha,\beta,\gamma,\delta}^{0,k,0,l} \quad k < L, \quad l \leq M;\\
\overline{H}_{\alpha,\beta,\gamma,\delta}^{0,k,l,l} &= H_{\alpha,\beta,\gamma,\delta}^{0,k,l,l} \quad k < L, \quad l < U; & \overline{H}_{\alpha,\beta,\gamma,\delta}^{-k,0,-l,0} &= H_{\alpha,\beta,\gamma,\delta}^{-k,0,-l,0} \quad k \leq M, \quad l < U;\\
\overline{H}_{\alpha,\beta,\gamma,\delta}^{k,k,-l,0} &= H_{\alpha,\beta,\gamma,\delta}^{k,k,-l,0} \quad k < L, \quad l < U; & \overline{H}_{\alpha,\beta,\gamma,\delta}^{k,k,0,l} &= H_{\alpha,\beta,\gamma,\delta}^{k,k,0,l} \quad k < L, \quad l \leq M;\\
\overline{H}_{\alpha,\beta,\gamma,\delta}^{-k,0,l,l} &= H_{\alpha,\beta,\gamma,\delta}^{-k,0,l,l} \quad k \leq M, \quad l < U.
\end{aligned}$$

Since by hypothesis $\overline{\mathcal{H}}$ is increasing, those terms for $\mathcal{H}$ do not break the partial order between configurations. Fot the higher terms involving L or U, we use Proposition 2.37. Hence, for each k, l such that $\beta + k > \delta + l$ or $\gamma - l < \alpha - k$

$$\begin{aligned}
H_{\alpha,\beta,\gamma,\delta}^{k,k,l,l} = H_{\alpha,\beta,\gamma,\delta}^{0,k,0,l} &= H_{\alpha,\beta,\gamma,\delta}^{0,k,l,l} = 0;\\
H_{\alpha,\beta,\gamma,\delta}^{-k,0,-l,0} = H_{\alpha,\beta,\gamma,\delta}^{k,k,-l,0} &= H_{\alpha,\beta,\gamma,\delta}^{k,k,0,l} = H_{\alpha,\beta,\gamma,\delta}^{-k,0,l,l} = 0.
\end{aligned}$$

Case B Suppose β is L-bad, but γ is U-good for $\mathcal{S}$ (which implies $\gamma - l \leq \alpha$ for each l). In this case we do not have to couple death rates and the coupling is easier. The system $\overline{\mathcal{S}}$ is such that $\overline{L} \leq L - 1$ by Proposition 2.25. Therefore

$$\overline{H}_{\alpha,\beta,\gamma,\delta}^{L,L,l,l} = \overline{H}_{\alpha,\beta,\gamma,\delta}^{0,L,0,l} = \overline{H}_{\alpha,\beta,\gamma,\delta}^{0,L,l,l} = \overline{H}_{\alpha,\beta,\gamma,\delta}^{L,L,0,l} = 0 \qquad \text{for each } l.$$

We can repeat the same but easier steps of case A. Since γ is U-good, then $\bar{l} = U$ and (2.4.90) holds. This implies (2.4.91). We consider Step 2 only for birth rates and we get as in previous case

$$\overline{H}_{\alpha,\beta,\gamma,\delta}^{0,L-1,0,l} = H_{\alpha,\beta,\gamma,\delta}^{0,L-1,0,l}; \qquad \overline{H}_{\alpha,\beta,\gamma,\delta}^{0,L-1,l,l} = H_{\alpha,\beta,\gamma,\delta}^{0,L-1,l,l} \qquad \text{for each } l > 0.$$

We move to Lower Step 3 (since γ is good we do not have to do the Upper one), which works in the same way. We get $\overline{H}_{\alpha,\beta,\gamma,\delta}^{L-1,L-1,0,l} = H_{\alpha,\beta,\gamma,\delta}^{L-1,L-1,0,l}$ for each $l > 0$. Since $\overline{\mathcal{H}}$ and $\mathcal{H}$ are defined recursively by using the same rules, the equality of the higher terms implies the equality of all terms. We get the result by using the induction hypothesis and Proposition 2.37 as in case A.

Case C Suppose β is L-good but γ is U-bad for $\mathcal{S}$. This is symmetric to Case B and it is proved in a symmetric way. $\square$

2.4.8 Proof of Lemma 2.28

i) Since $\widehat{L} = 1$, if $l_1 > 0$ then $\mathbb{1}_{\{k > \delta - \beta + l_1\}}(\Gamma_{\alpha,\beta}^k + \Pi_{\alpha,\beta}^{0,k}) = 0$ and (2.2.13) is satisfied. Suppose $l_1 = 0$, then

$$\sum_{k > \delta - \beta} \Pi_{\alpha,\beta}^{0,k} + \sum_{k \in I_a} \Gamma_{\alpha,\beta}^k = \Pi_{\alpha,\beta}^{0,L} + \mathbb{1}_{\{L \in I_a\}} \Gamma_{\alpha,\beta}^L.$$

By taking $K = 1$, (2.2.13) becomes (2.2.37). Since $\sum_{m_1\geq k>\delta-\beta}\Gamma^k_{\alpha,\beta} = \sum_{k\in I_a}\Gamma^k_{\alpha,\beta}$, Condition (2.2.13) holds for any $K > 1$. By symmetric arguments we get Formula (2.2.38).
The explicit formulation of $\mathcal{H}$ is given by Definition 2.21. We prove that it is increasing.
By Proposition 2.37 *i*) we get $H^{L,L,l,l}_{\alpha,\beta,\gamma,\delta} = H^{0,L,0,l}_{\alpha,\beta,\gamma,\delta} = H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta} = H^{L,L,0,l}_{\alpha,\beta,\gamma,\delta} = 0$ for any l such that $l < \widehat{L}$. Since L is the only k such that $\beta + k > \delta$ then all the lower attractiveness problems are solved.
By Proposition 2.37 *ii*) we get $H^{k,k,U,U}_{\alpha,\beta,\gamma,\delta} = H^{-k,0,-U,0}_{\alpha,\beta,\gamma,\delta} = H^{k,k,-U,0}_{\alpha,\beta,\gamma,\delta} = H^{-k,0,U,U}_{\alpha,\beta,\gamma,\delta} = 0$ for any k such that $k < \widehat{U}$. Since $\widehat{U} = 1$ is the only l such that $\gamma - l < \alpha$ then all the higher attractiveness problems are solved and the attractiveness of $\mathcal{H}$ follows.
ii) Formula (2.2.37) holds as in *i*). Here $\widehat{U} \leq 0$ implies that

$$\sum_{\gamma-\alpha+m_1\geq l>0}\Gamma^l_{\gamma,\delta} = \sum_{l>0}\Gamma^l_{\gamma,\delta}$$

for all m_1. Therefore if $m_1 \geq L$ then (2.2.37) becomes (2.2.39), which implies

$$\Pi^{0,L}_{\alpha,\beta} \leq \sum_{l>0}\Pi^{0,l}_{\gamma,\delta} + \sum_{l>0}\Gamma^l_{\gamma,\delta}$$

that is the case $m_1 < L$.
Coupling $\mathcal{H}$ is given by Definition 2.21. We prove it is increasing. We only have to check that $H^{L,L,l,l}_{\alpha,\beta,\gamma,\delta} = H^{0,L,0,l}_{\alpha,\beta,\gamma,\delta} = H^{0,L,l,l}_{\alpha,\beta,\gamma,\delta} = H^{L,L,0,l}_{\alpha,\beta,\gamma,\delta} = 0$ for any l such that $l < \widehat{L}$ since L is the only k such that $\beta + k > \delta$. This follows by Proposition 2.37 Formula (2.4.10). Since $\widehat{U} \leq 0$, there is no higher attractiveness problem.
iii) This is the symmetric of *ii*) and we can proceed exactly in the same way by using Proposition 2.37 and (2.4.11). □

2.4.9 Proof of Lemma 2.29

i) Since $\widehat{U} \leq 0$ there are no higher attractiveness problems and (2.2.13)–(2.2.14) reduce to (2.2.13). We prove that $\mathcal{H}$ is increasing by induction on L. If $\widehat{L} = 1$, this is true by Lemma 2.28 *ii*). Suppose the result holds for all systems $\widetilde{\mathcal{S}}$ with $L(\widetilde{\mathcal{S}}) := \widetilde{L} \leq L-1$. Starting from $\mathcal{S}$ we can define a new system $\overline{\mathcal{S}}$ by Definition 2.24 (in case $\widehat{U} \leq 0$). By Proposition 2.25 then $\overline{L} \leq L - 1$ and $\overline{\mathcal{S}}$ satisfies the Condition (2.2.13) by Proposition 2.37. We define a coupling for $\overline{\mathcal{S}}$ as in Definition 2.21 and by induction hypothesis it is increasing. We call such a coupling $\overline{\mathcal{H}}$. By Proposition 2.27 the attractiveness of $\mathcal{H}$ follows.
ii) This is the symmetric case and we prove it in a similar way by using Lemma 2.28 *iii*) as induction basis. □

2.4.10 Proof of Proposition 2.23

If β is L-good or γ is U-good the attractiveness of $\mathcal{H}$ is given by Lemma 2.29. Suppose $\widehat{L} > 0$ and $\widehat{U} > 0$. We work by induction on $N = \max\{L, U\}$. Suppose the result holds for each system $\widetilde{\mathcal{S}}$ such that $N(\widetilde{\mathcal{S}}) := \widetilde{N} \leq N-1$. We define a new system $\overline{\mathcal{S}}$ by Definition 2.24. By Proposition 2.25 this is a system such that $\overline{N} \leq N - 1$ which satisfies Condition (2.2.13)–(2.2.14) by Proposition 2.37. We define a coupling $\overline{\mathcal{H}}$ for $\overline{\mathcal{S}}$ as in Definition 2.21 and by induction hypothesis it is increasing. By Proposition 2.27 the attractiveness of $\mathcal{H}$ follows if the induction basis is satisfied.
Notice that the induction basis is not always given by the case $N = 1$.
At each step $j = 1, \ldots, n$ we define a new system $\mathcal{S}_j$ such that β is L_j-bad and γ is U_j-bad and $N_j \leq N_{j+1} - 1$ until:

i) $L_n = \delta - \beta + 1$ and $U_n = \gamma - \alpha + 1$. In this case the induction basis is given by Lemma 2.28 *i*).
ii) $L_n > \delta - \beta$ and $U_n = \gamma - \alpha$. In this case the induction basis is given by Lemma 2.29 *i*).
iii) $L_n = \delta - \beta$ and $U_n > \gamma - \alpha$. In this case the induction basis is given by Lemma 2.29 *ii*). $\square$

Chapter 3

Applications

3.1 Individual recovery epidemic model

Generalizations of the contact process have been studied to understand the spread of a disease in different settings. Schinazi [40], in order to investigate the role of social clusters in the transmission of diseases, introduced a model with a population of N individuals on each site in $\mathbb{Z}^d$ with different infection rates within and between groups. The recovery mechanism is *global*: after an exponential time all individuals in the same cluster recover. Belhadji [5] worked with modifications of this model in compact and non compact state spaces.

We focus on one of them, the *individual recovery epidemic model* in a compact state space: each sick individual recovers after an exponential time and each cluster contains at most N individuals.

The model is a continuous-time Markov process in which the state at time t is a function $\eta_t : \mathbb{Z}^d \to \{0, 1, \dots, N\} \subset \mathbb{N}$, where N denotes the common size of the clusters and $\eta_t(x)$ counts the number of particles present in the cluster located at site x at time $t \geq 0$. We denote by $S = \mathbb{Z}^d$ the set of possible sites, $X = \{0, 1, \dots, N\}$ the set of possible states for each site and we call $\Omega = X^S$ the space of configurations.

A particle represents an infected individual. The cluster at site $y \in \mathbb{Z}^d$ is said *healthy* at time $t \geq 0$ if $\eta_t(y) = 0$, and *infected* otherwise. Given λ, γ, β and ϕ real positive numbers, the state of site y flips according to the transitions given in (1.16): a healthy cluster gets infected at rate λ times the numbers of infected individuals present in the neighbouring clusters plus γ which represents a spontaneous birth of the infection. If there are i infected individuals in cluster y, then a new individual gets infected in y at rate β times the number of infected individuals in the neighbourhood plus $i\phi$, that is ϕ times the number of infected individuals in the same cluster. Finally, each individual recovers at rate 1 regardless of the number of infected individuals in its cluster.

Referring to notations in Section 1.1.3, the infinitesimal generator $\mathcal{L}$ of the process $(\eta_t)_{t\geq 0}$ on $\Omega = X^S$ is given, for a local function f and for $\eta \in \Omega$, by

$$\begin{aligned}\mathcal{L}f(\eta) = \sum_{y\in S} \Big\{ &\mathbb{1}_{\{\eta(y)=0\}} (\lambda \sum_{z\sim y} \eta(z) + \gamma) \Big(f(S_y^1 \eta) - f(\eta) \Big) + \eta(y) \Big(f(S_y^{-1}\eta) - f(\eta) \Big) \\ &+ \mathbb{1}_{\{N-1\geq \eta(y)\geq 1\}} (\beta \sum_{z\sim y} \eta(z) + \eta(y)\phi) \Big(f(S_y^1 \eta) - f(\eta) \Big) \Big\}. \end{aligned} \tag{3.1.1}$$

In other words for each $\eta(x) \in X$, $\eta(y) \in X$,

$$\Gamma^k_{\eta(x),\eta(y)} = 0 \qquad \text{for all } k \in \mathbb{N} \tag{3.1.2}$$

$$R^{0,k}_{\eta(x),\eta(y)} = \begin{cases} 2d\lambda\eta(x) & k = 1 \text{ and } \eta(y) = 0 \\ 2d\beta\eta(x) & k = 1 \text{ and } 1 \leq \eta(y) \leq N-1 \\ 0 & k \geq 2 \text{ and } k \leq -1 \end{cases} \tag{3.1.3}$$

$$P^k_{\eta(y)} = \begin{cases} \gamma & k = 1 \text{ and } \eta(y) = 0 \\ \phi\eta(y) & k = 1 \text{ and } 1 \leq \eta(y) \leq N-1 \\ \eta(y) & k = -1 \text{ and } 1 \leq \eta(y) \leq N \\ 0 & |k| \geq 2 \end{cases} \tag{3.1.4}$$

$$p(x,y) = \frac{1}{2d}\mathbb{1}_{\{y \sim x\}} \qquad \text{for each } (x,y) \in S^2. \tag{3.1.5}$$

By setting $\gamma = 0$, we get the epidemic model in [5], where the author analyses the system with $N < \infty$ and $N = \infty$ and shows [5, Theorem 14] that different phase transitions occur with respect to λ and ϕ. Finally, as an application of [12, Theorem 14.3], one can prove Theorem 1.6 (see [5, Theorem 15]).

In Section 3.1.1 we use the u-criterion (Section 1.1.2) to improve the sufficient condition in Theorem 1.6 and we get a condition dependent on the cluster size; in Section 3.1.2 we give some ergodicity results for the model with $\gamma > 0$.
First of all we observe that

Proposition 3.1 *Generator* (3.1.1) *defines an attractive system for all* λ, β, γ, ϕ, N.

Proof. We refer to notation of Section 2.2. Since $M = 1$, Conditions (2.2.13)–(2.2.14) reduce to (2.3.15)–(2.3.16) in Section 2.3.5. Namely, given two configurations $\eta \in \Omega$, $\xi \in \Omega$ with $\xi \leq \eta$, necessary and sufficient conditions for attractiveness are

$$\text{if } \xi(x) < \eta(x) \text{ and } \xi(y) = \eta(y), \qquad \begin{cases} R^{0,1}_{\xi(x),\xi(y)} + P^1_{\xi(y)} \leq R^{0,1}_{\eta(x),\eta(y)} + P^1_{\eta(y)}; \\ P^{-1}_{\xi(y)} \geq P^{-1}_{\eta(y)}. \end{cases}$$

If $\xi(y) = \eta(y) \geq 1$, $2d\beta\xi(x) \leq 2d\beta\eta(x)$; if $\xi(y) = \eta(y) = 0$, $2d\lambda\xi(x) \leq 2d\lambda\eta(x)$. In all cases the condition holds since $\xi \leq \eta$. $\square$

The transitions of at most one particle per time make the attractiveness conditions easier to understand. The key point for attractiveness is that $R^{0,1}_{\eta(x),\eta(y)}$ is increasing in $\eta(x)$.
The main result, that we will prove through u-criterion, is

Theorem 3.2 *Suppose*

$$\lambda \vee \beta < \frac{1-\phi}{2d(1-\phi^N)}, \tag{3.1.6}$$

with $\phi < 1$ *and either i)* $\gamma = 0$*, or ii)* $\gamma > 0$ *and* $\beta - \lambda \leq \gamma/(2d)$.
Then the system is ergodic.

Notice that if $\gamma > 0$ and $\beta \leq \lambda$ hypothesis *ii*) is trivially satisfied.
Given $(\alpha(x))_{x\in\mathbb{Z}^d}$ a sequence which satisfies (1.4), for all η and ξ, $\rho_\alpha(\eta,\xi) = \rho(\eta,\xi)$ defined by (1.5) is a well defined metric on S if $F(\cdot,\cdot)$ is a metric on X.
We need a *good choice* of $F(\cdot,\cdot)$ to get good ergodicity conditions.

Given $\epsilon > 0$ and $\{u_l(\epsilon)\}_{l\in X}$ such that $u_l(\epsilon) > 0$ for all $l \in X$, let $F : X \times X \to \mathbb{R}^+$ be defined by

$$F_\epsilon(x, y) = \mathbb{1}_{\{x\neq y\}} \sum_{j=0}^{|y-x|-1} u_j(\epsilon) \tag{3.1.7}$$

for all $x, y \in X$. Since $u_l(\epsilon) > 0$, this is a metric. When not necessary we omit the dependence on ϵ and we simply write $F(x, y) = \mathbb{1}_{\{x\neq y\}} \sum_{j=0}^{|y-x|-1} u_j$.

As explained in Section 1.1.2, we look for conditions on transition rates such that

$$\widetilde{\mathbb{E}}\Big(\lim_{t\to\infty} F(\eta_t^0(x), \eta_t^N(x))\Big) = 0 \tag{3.1.8}$$

uniformly with respect to $x \in S$, where $\widetilde{\mathbb{E}}$ is the expected value with respect to a coupling probability measure $\widetilde{\mathbb{P}}$, $\eta_0^0 \in E^0$ and $\eta_0^N \in E^N$.

Since $F(\cdot, \cdot)$ depends on a sequence $\{u_l\}_{l\in X}$, the idea is to take a sequence which gives the best sufficient conditions for ergodicity. The following properties will be useful,

Lemma 3.3 *The metric $F(\cdot, \cdot)$ satisfies:*
If x, y belong to X,
i) $F(x, x) = 0$;
ii) $F(x, y) = F(0, y - x) = F(y - x, 0)$.
If $x, y, k, x + k, y + k$ belong to X,
iii) $F(x + k, y + k) = F(x - k, y - k) = F(x, y)$.
If $x, y, x + 1, y - 1$ belong to X,
iv) $F(x + 1, y) - F(x, y) = F(x, y - 1) - F(x, y) = -u_{y-x-1}$;
v)$F(x + 1, y - 1) - F(x, y) = -u_{y-x-1} - u_{y-x-2}$.
If $x, y, x - 1, y + 1$ belong to X,
vi) $F(x - 1, y) - F(x, y) = F(x, y + 1) - F(x, y) = u_{y-x}$;
vii) $F(x - 1, y + 1) - F(x, y) = u_{y-x-1} + u_{y-x}$.

The proof follows by direct computation starting from Definition (3.1.7). By Lemma 3.3 *ii)* with a slight abuse of notation we write $F(y - x)$ instead of $F(x, y) = F(y - x, 0)$.
The sequences $\{u_l(\epsilon)\}_{l\in X}$ in Definition (3.1.7) are given by

Definition 3.4 *Given $\epsilon > 0$ and $U > 0$, we set $u_0(\epsilon) = U$ and we define $(u_l)_{l\in X}$ recursively through*

$$u_l(\epsilon) = \frac{1}{\phi l}\Big(-\epsilon \sum_{j=0}^{l-1} u_j(\epsilon) - U(\lambda \vee \beta)2dl + l u_{l-1}(\epsilon)\Big) \qquad \textit{for each } l \in X, l \neq 0. \tag{3.1.9}$$

Proposition 3.5 *If $\phi < 1$ and*

$$\frac{1-\phi}{2d} < \lambda \vee \beta < \frac{1-\phi}{2d(1-\phi^N)}$$

then there exists $\bar{\epsilon} > 0$ such that $u_l(\epsilon)$ is positive, decreasing in l and in ϵ for each $l \in X$ and $0 < \epsilon \leq \bar{\epsilon}$.

In order to prove Proposition 3.5 we work by induction in l. We need the following lemma to use the induction hypothesis:

Lemma 3.6 *If for $l \in X$,*

$$\lambda \vee \beta < \frac{1-\phi}{2d(1-\phi^l)}$$

and there exists $\bar{\epsilon}$ such that $u_k(\epsilon) > 0$ for each $k \leq l-1$ and $0 < \epsilon \leq \bar{\epsilon}$, then there exists $0 < \epsilon^* \leq \bar{\epsilon}$ such that

$$U < \frac{u_{l-1}(\epsilon^*)}{2d(\lambda \vee \beta)}. \tag{3.1.10}$$

Proof of Lemma 3.6.
We prove by a downwards induction on $0 \leq k \leq l-1$ that:
if there exists $0 < \epsilon^* \leq \bar{\epsilon}$ such that

$$J(k, \epsilon^*) := -(\lambda \vee \beta)2dU(1 + \phi + \ldots + \phi^k) + u_{l-k-1}(\epsilon^*) > 0 \tag{3.1.11}$$

then $J(k-1, \epsilon^*) > 0$.
Indeed, if $k = l-1$ then (3.1.11) is the assumption $(\lambda \vee \beta)2d(1-\phi^l) < 1-\phi$.
Suppose there exists $0 < \epsilon^* \leq \bar{\epsilon}$ such that $J(k, \epsilon^*) > 0$. By Definition 3.4, (3.1.11) is equivalent to

$$-(\lambda \vee \beta)2dU(1+\ldots+\phi^k) + \frac{-\epsilon^* \sum_{j=0}^{l-k-2} u_j(\epsilon^*) + (l-k-1)\big(-U(\lambda \vee \beta)2d + u_{l-k-2}(\epsilon^*)\big)}{\phi(l-k-2)} > 0$$

that is

$$-(\lambda \vee \beta)2dU(l-k-1)\big[(1+\ldots+\phi^k)\phi + 1\big] - \epsilon^* \sum_{j=0}^{l-k-2} u_j(\epsilon^*) + (l-k-1)u_{l-k-2}(\epsilon^*) > 0.$$

Therefore

$$J(k-1, \epsilon^*) > -(\lambda \vee \beta)2dU(1 + \phi + \ldots + \phi^{k+1}) + u_{l-k-2}(\epsilon^*) > \frac{\epsilon^*}{l-k-1} \sum_{j=0}^{l-k-2} u_j(\epsilon^*)$$

Since by hypothesis $u_j(\epsilon) > 0$ for each $0 < \epsilon \leq \bar{\epsilon}$, then $J(k-1, \epsilon^*) > 0$ and by induction (3.1.11) holds for each $0 \leq k \leq l-1$.
By taking $k = 0$ we get

$$-(\lambda \vee \beta)2dU + u_{l-1}(\epsilon^*) > 0$$

which is the claim. □

Proof of Proposition 3.5.
We prove by induction on $l \in X$ that there exists $\bar{\epsilon}_l$ such that for each $0 < \epsilon < \bar{\epsilon}_l$ and for each $0 \leq j \leq l$ then $u_j(\epsilon)$ is positive, decreasing in j, $U < \dfrac{u_j(\epsilon)}{2d(\lambda \vee \beta)}$ and

$$0 > \frac{d}{d\epsilon} u_j(\epsilon) \geq -C_U(j) \tag{3.1.12}$$

with $C_U(j)$ the solution of

$$C_U(1) = \frac{U}{\phi}, \quad C_U(j) = \frac{C_U(j-1) + U}{\phi} \tag{3.1.13}$$

that is

$$C_U(j) = \frac{U}{\phi^j}(1 + \phi + \phi^2 + \ldots + \phi^{j-1}) = U\frac{1-\phi^j}{\phi^j(1-\phi)} \tag{3.1.14}$$

hence $u_j(\epsilon) > 0$ is also decreasing in ϵ.
We prove the induction basis when $l = 1$. By Definition 3.4

$$u_1(\epsilon) = \frac{-\epsilon u_0 - U(\lambda \vee \beta)2d + u_0}{\phi} = u_0\frac{-\epsilon - (\lambda \vee \beta)2d + 1}{\phi}.$$

Since $(\lambda \vee \beta)2d > 1 - \phi$ we can take $\epsilon < \epsilon_1$ small enough to have $1 - (\lambda \vee \beta)2d - \epsilon < \phi$, that is $u_1(\epsilon) < u_0 = U$; since $(\lambda \vee \beta)2d < 1$ then $U < U/(2d(\lambda \vee \beta))$ and by taking $\epsilon < \epsilon_1^*$ small enough $u_1(\epsilon)$ is positive; moreover notice that $u_1(\epsilon)$ is decreasing in ϵ and $\frac{d}{d\epsilon}u_1(\epsilon) = -\frac{U}{\phi} = -C_U(1)$ for each ϵ. Hence the induction basis as well as the hypothesis of Lemma 3.6 are satisfied for $l = 1$: there exists $\bar{\epsilon}_1 = \epsilon_1 \wedge \epsilon_1^*$ such that if $\epsilon \le \bar{\epsilon}_1$ then $U > u_1(\epsilon) > 0$, $u_1(\epsilon)$ is decreasing in ϵ and (3.1.10) holds.
Suppose there exists $\bar{\epsilon}_{l-1} > 0$ such that for each $0 < \epsilon \le \bar{\epsilon}_{l-1}$, then $0 > \frac{d}{d\epsilon}u_j(\epsilon) \ge -C_U(j)$, $u_j(\epsilon)$ is decreasing in j for each $j \le l-1$, $U < \dfrac{u_j(\epsilon)}{2d(\lambda \vee \beta)}$ and $u_j(\epsilon) > 0$ for each $j \le l-1$.
First of all we prove that there exists $\epsilon_l > 0$ such that if $0 < \epsilon < \epsilon_l$ then $u_l(\epsilon) < u_{l-1}(\epsilon)$. By Definition 3.4

$$u_l(\epsilon) = u_{l-1}(\epsilon)\Big(\frac{-\epsilon\sum_{j=0}^{l-1} u_j(\epsilon) - U(\lambda \vee \beta)2dl + lu_{l-1}(\epsilon)}{\phi l u_{l-1}(\epsilon)}\Big). \tag{3.1.15}$$

By induction hypothesis, if $\epsilon \le \bar{\epsilon}_{l-1}$ then $U > u_j(\epsilon) > 0$ for each $j \le l-1$ and we get

$$\begin{aligned}\frac{-\epsilon\sum_{j=0}^{l-1} u_j(\epsilon) - U(\lambda \vee \beta)2dl + lu_{l-1}(\epsilon)}{\phi l u_{l-1}(\epsilon)} &< \frac{-U(\lambda \vee \beta)2dl + lu_{l-1}(\epsilon)}{\phi l u_{l-1}(\epsilon)} \\ &< \frac{1}{\phi} - \frac{(\lambda \vee \beta)2d}{\phi} < 1\end{aligned}$$

that is $u_l(\epsilon) < u_{l-1}(\epsilon)$ by (3.1.15) and we set $\epsilon_l = \bar{\epsilon}_{l-1}$.
By (3.1.14), $C_U(j)$ is always positive and increasing in j. We prove that (3.1.12) holds for $j = l$. By Definition 3.4

$$\frac{d}{d\epsilon}u_l(\epsilon) = \frac{1}{\phi l}\Big(- \epsilon\frac{d}{d\epsilon}\sum_{j=0}^{l-1} u_j(\epsilon) - \sum_{j=0}^{l-1} u_j(\epsilon) + l\frac{d}{d\epsilon}u_{l-1}(\epsilon)\Big). \tag{3.1.16}$$

We begin from the right inequality involving the derivative of $u_l(\epsilon)$ in (3.1.12). By induction hypothesis, if $0 < \epsilon \le \bar{\epsilon}_{l-1}$ by (3.1.13), (3.1.14) and (3.1.16) on the one hand

$$\phi l\frac{d}{d\epsilon}u_l(\epsilon) > -\epsilon \times 0 - lU - lC_U(l-1) = -C_U(l)l\phi.$$

and on the other hand

$$\begin{aligned}\phi l\frac{d}{d\epsilon}u_l(\epsilon) &\le \epsilon\sum_{j=0}^{l-1} C_U(j) - \sum_{j=0}^{l-1} u_j(\epsilon) + l\frac{d}{d\epsilon}u_{l-1}(\epsilon) \\ &< \epsilon\sum_{j=0}^{l-1} U\frac{1-\phi^j}{\phi^j(1-\phi)} - \sum_{j=0}^{l-1} u_j(\epsilon)\end{aligned}$$

since $0 < \epsilon < \bar{\epsilon}_{l-1}$ and $u_j(\epsilon)$ is positive and decreasing in ϵ, as ϵ approaches 0 the first sum on the right hand side goes to 0, while the second sum is positive and increasing: therefore for each $j \leq l-1$, we can take $\hat{\epsilon}_l$ small enough so that $\frac{d}{d\epsilon}u_l(\epsilon) < 0$ for each $0 < \epsilon \leq \hat{\epsilon}_l$.
Now we prove that there exists $\epsilon_l^* > 0$ such that $u_l(\epsilon) > 0$ for each $0 < \epsilon \leq \epsilon_l^*$. By Definition 3.4, $u_l(\epsilon) > 0$ if

$$-U(\lambda \vee \beta)2dl + lu_{l-1}(\epsilon) > \epsilon \sum_{j=0}^{l-1} u_j(\epsilon), \tag{3.1.17}$$

By induction hypothesis, assumptions of Lemma 3.6 are satisfied, then there exists $0 < \epsilon^* \leq \bar{\epsilon}_{l-1}$ such that (3.1.10) is satisfied. Hence $-U(\lambda \vee \beta)2dl + lu_{l-1}(\epsilon^*) > 0$ and we can choose $\epsilon_l^* \leq \epsilon^*$ such that

$$-U(\lambda \vee \beta)2dl + lu_{l-1}(\epsilon^*) > \epsilon_l^* lU \tag{3.1.18}$$

If $\epsilon \leq \epsilon_l^* (\leq \bar{\epsilon}_{l-1})$

$$\begin{aligned}-U(\lambda \vee \beta)2dl + lu_{l-1}(\epsilon) >& - U(\lambda \vee \beta)2dl + lu_{l-1}(\epsilon^*)\\ >& \epsilon_l^* lU > \epsilon_l^* \sum_{j=0}^{l-1} u_j(\epsilon) > \epsilon \sum_{j=0}^{l-1} u_j(\epsilon)\end{aligned}$$

which is (3.1.17).
By taking $0 < \epsilon < \bar{\epsilon}_0 \wedge \ldots \bar{\epsilon}_{l-1} \wedge \hat{\epsilon}_l \wedge \epsilon_l^*$, then $u_l(\epsilon) > 0$ and is decreasing in l and ϵ and the claim follows. □

Now we treat separately the cases $\gamma = 0$ and $\gamma > 0$.

3.1.1 Individual recovery model without spontaneous birth

A spontaneous birth of the infection is not possible, that is $\gamma = 0$ which implies that the Dirac measure $\delta_{\underline{0}}$ concentrated on the configuration $\eta^0 \in E^0$ is an invariant measure.
We denote by $\mathbb{P}$ the independent coupling measure and we choose $\widetilde{\mathbb{P}} = \mathbb{P}$. Moreover we denote $\widetilde{\mathbb{E}}(\cdot) = \mathbb{E}(\cdot)$.
We know that if (1.17) holds then the system is ergodic, see Theorem 1.6, Section 1.1.4. This is true either in non compact cases or in compact cases with cluster size N. Here we improve the sufficient condition for ergodicity in a compact state space: if $\gamma = 0$, Theorem 3.2 becomes

Proposition 3.7 *Assume $\phi < 1$ and (3.1.6), then the system is ergodic.*

Remark 3.8 *As N goes to infinity, Condition (3.1.6) converges to (1.17).*

The key result is:

Proposition 3.9 (u-criterion) *If there exists $\epsilon > 0$ and a sequence $\{u_l(\epsilon)\}_{l \in X}$ such that for any $l \in X$*

$$\begin{cases} \phi l u_l(\epsilon) - l u_{l-1}(\epsilon) \leq -\epsilon \sum_{j=0}^{l-1} u_j(\epsilon) - \bar{u}(\epsilon)(\lambda \vee \beta)2dl \\ u_l(\epsilon) > 0 \end{cases} \tag{3.1.19}$$

where $\bar{u}(\epsilon) := \max_{l \in X} u_l(\epsilon)$, we write by convention $u_{-1} = 0$ and $u_0 = U > 0$, then the system is ergodic.

Proof . We fix $x \in S$ and we compute the generator (3.1.1) on $F(0, \eta_t^N(x)) = F(\eta_t^N(x))$ in Definition (3.1.7), where $\{u_l(\epsilon)\}_{l\in X}$ satisfies Hypothesis (3.1.19). Let $t \geq 0$, we denote by $\eta_t^N(x) = l$. By Lemma 3.3 *iv*) and *v*) and (3.1.19)

$$\begin{aligned}
\mathcal{L}F(0, \eta_t^N(x)) =& \mathcal{L}F(l) = \mathbb{1}_{\{l=0\}}\Big[\lambda \sum_{y\sim x} \eta_t^N(y)(F(1) - F(0))\Big] \\
&+ \mathbb{1}_{\{1\leq l\leq N-1\}}\Big[(\beta \sum_{y\sim x} \eta_t^N(y) + l\phi)(F(l+1) - F(l)) + l(F(l-1) - F(l))\Big] \\
&+ \mathbb{1}_{\{l=N\}}\Big[N(F(N-1) - F(N))\Big] \\
=& \mathbb{1}_{\{l=0\}}\Big[\lambda \sum_{y\sim x} \eta_t^N(y) u_0\Big] + \mathbb{1}_{\{1\leq l\leq N-1\}}\Big[(\beta \sum_{y\sim x} \eta_t^N(y) + l\phi)u_l - lu_{l-1}\Big] \\
&+ \mathbb{1}_{\{l=N\}}\Big[-Nu_{N-1}\Big] \\
\leq& \mathbb{1}_{\{l=0\}}\Big[(\lambda \sum_{y\sim x} \eta_t^N(y) + l\phi)u_l - lu_{l-1}\Big] + \mathbb{1}_{\{1\leq l\leq N\}}\Big[(\beta \sum_{y\sim x} \eta_t^N(y) + l\phi)u_l - lu_{l-1}\Big] \\
\leq& \mathbb{1}_{\{0\leq l\leq N\}}\Big[((\lambda\vee\beta) \sum_{y\sim x} \eta_t^N(y) + l\phi)u_l - lu_{l-1}\Big] \\
\leq& (\lambda\vee\beta)\bar{u} \sum_{y\sim x} \eta_t^N(y) + l(\phi u_l - u_{l-1}) \\
\leq& (\lambda\vee\beta)\bar{u} \sum_{y\sim x} \eta_t^N(y) - \epsilon \sum_{j=0}^{l-1} u_j - (\lambda\vee\beta)2d\bar{u}l.
\end{aligned} \tag{3.1.20}$$

Let $\eta_0^N \in E^N$. By translation invariance and Definition (3.1.7)

$$\begin{aligned}
\mathbb{E}(\mathcal{L}F(\eta_t^N(x))) \leq& \mathbb{E}\Big((\lambda\vee\beta)\bar{u} \sum_{y\sim x} \eta_t^N(y) - \epsilon \sum_{j=0}^{\eta_t^N(x)-1} u_j - (\lambda\vee\beta)2d\bar{u}\eta_t^N(x)\Big) \\
=& \mathbb{E}\Big((\lambda\vee\beta)\bar{u}2d\eta_t^N(x)\Big) - \epsilon\mathbb{E}\Big(F(\eta_t^N(x))\Big) - \mathbb{E}\Big((\lambda\vee\beta)2d\bar{u}\eta_t^N(x)\Big)
\end{aligned}$$

that is

$$\frac{d}{dt}\mathbb{E}(F(\eta_t^N(x))) = \mathbb{E}(\mathcal{L}F(\eta_t^N(x))) \leq -\epsilon\mathbb{E}(F(\eta_t^N(x))).$$

By Gronwall's Lemma (see [13])

$$\mathbb{E}(F(\eta_t^N(x))) \leq \mathbb{E}(F(\eta_0^N(x)))e^{-\epsilon(t-1)} \leq \bar{u}Ne^{-\epsilon(t-1)}$$

since $\eta_0^N \in E^N$. As t goes to infinity

$$\mathbb{E}\Big(\limsup_{t\to\infty} F(\eta_t^N(x))\Big) \leq \limsup_{t\to\infty} \mathbb{E}\Big(F(\eta_t^N(x))\Big) \leq \limsup_{t\to\infty} \bar{u}Ne^{-\epsilon(t-1)} = 0 \tag{3.1.21}$$

uniformly with respect to x. Therefore, by (1.7) and (3.1.21)

$$\mathbb{E}\rho(\underline{0}, \lim_{t\to\infty} \eta_t^N) \leq 0$$

and ergodicity follows since $\rho(\cdot,\cdot)$ is a metric and the dynamics is attractive: this implies in particular that $\lim_{t\to\infty} \eta_t^N$ exists. $\square$

Hence we are left with checking the existence of $\epsilon > 0$ and positive $\{u_l(\epsilon)\}_{l\in X}$ which satisfy (3.1.19). Such a choice is not unique.

Remark 3.10 *Given $U = 1 > 0$, if $u_l(\epsilon) = 1$ for each $l \in X$ then Condition (3.1.19) reduces to the existence of $\epsilon > 0$ such that $\phi l - l \leq -\epsilon l - (\lambda \vee \beta)2dl$ for all $l \in X$, that is $\epsilon \leq 1 - \phi - (\lambda \vee \beta)2d$, thus Condition (1.17). Notice that in this case*

$$F(x,y) = \sum_{j=0}^{|y-x|-1} 1 = |y-x|.$$

Definition (3.1.9) gives a better choice of $\{u_l(\epsilon)\}_{l\in X}$, indeed the u-criterion is satisfied under more general conditions.

Proof of Proposition 3.7.
Let $\{u_l(\epsilon)\}_{l\in X}$ be given by Definition (3.1.9). By Proposition 3.5 we can choose $0 < \epsilon < \bar{\epsilon}$ such that $u_l(\epsilon) > 0$ for each $l \in X$ and Proposition 3.7 follows from Proposition 3.9. □

3.1.2 Individual recovery model with spontaneous birth

A spontaneous birth of the infection is possible, that is $\gamma > 0$ which implies that $\delta_{\underline{0}}$ is not any more an invariant measure. In this case we use the metric F in Definition (3.1.7) to evaluate the distance between the process starting from $\xi_0 \in E^0$ and the one starting from $\eta_0 \in E^N$ coupled in a particular way: by attractiveness we know that they converge respectively to the lower and the upper invariant measure. Theorem 3.2 becomes

Proposition 3.11 *If $\gamma > 0$ and*

$$i)\quad (\lambda \vee \beta) \leq \frac{1-\phi}{2d(1-\phi^N)}; \qquad\qquad ii)\quad \beta - \lambda \leq \frac{\gamma}{2dN} \tag{3.1.22}$$

the process $(\eta_t)_{t\geq 0}$ with generator (3.1.1) is ergodic.

Notice that if $\gamma > 0$ and $\beta \leq \lambda$, then $\beta - \lambda \leq 0 < \gamma/(2dN)$ and Condition (3.1.22) *ii)* is satisfied. From a biological point of view it makes sense to assume $\beta > \lambda$, since λ and β are the infection rates from infected clusters respectively to healthy and to infected ones. In this case if the pure birth rate γ is large enough the process is ergodic, that is there is only one invariant measure. Since $\gamma > 0$ this does not mean the extinction of the disease. First of all we prove

Lemma 3.12 *Let $(\xi_t)_{t\geq 0}$, $(\eta_t)_{t\geq 0}$ be two processes with generator (3.1.1) such that $\xi_0 \leq \eta_0$ and*

$$\beta - \lambda \leq \frac{\gamma}{2dN}.$$

Let F as in (3.1.7) with $\{u_l\}_{l\in X}$ non increasing in l, and let $x \in \mathbb{Z}^d$. Then

$$\begin{aligned}\mathcal{L}F(\xi_t(x), \eta_t(x)) \leq& \lambda \sum_{y\sim x} (\eta_t(y) - \xi_t(y)) u_{\eta_t(x)-\xi_t(x)} \\ &+ (\eta_t(x) - \xi_t(x))\big(\phi u_{\eta_t(x)-\xi_t(x)} - u_{\eta_t(x)-\xi_t(x)-1}\big).\end{aligned} \tag{3.1.23}$$

Proof. We couple the two processes through basic coupling, which gives the detailed coupling rates

$$
\begin{array}{ll}
\xi(x)=0 & \begin{cases}
\eta(x)=0 & \begin{cases}
\begin{array}{lll}
(\xi(x),\eta(x)) & \to (\xi(x)+1,\eta(x)+1) & \lambda\sum_{y\sim x}\xi(y)+\gamma \\
 & \to (\xi(x),\eta(x)+1) & \lambda\sum_{y\sim x}[\eta(y)-\xi(y)]^{+} \\
 & \to (\xi(x)+1,\eta(x)) & 0
\end{array}
\end{cases} \\[2ex]
\begin{array}{l}\eta(x)>0\\ \eta(x)<N\end{array} & \begin{cases}
\begin{array}{lll}
(\xi(x),\eta(x)) & \to (\xi(x)+1,\eta(x)) & \lambda\sum_{y\sim x}\xi(y)+\gamma \\
 & \to (\xi(x),\eta(x)-1) & \eta(x) \\
 & \to (\xi(x),\eta(x)+1) & \beta\sum_{y\sim x}\eta(y)+\eta(x)\phi
\end{array}
\end{cases} \\[2ex]
\eta(x)=N & \begin{cases}
\begin{array}{lll}
(\xi(x),\eta(x)) & \to (\xi(x)+1,\eta(x)) & \lambda\sum_{y\sim x}\xi(y)+\gamma \\
 & \to (\xi(x),\eta(x)-1) & \eta(x)
\end{array}
\end{cases}
\end{cases} \\[6ex]
\begin{array}{l}\xi(x)>0\\ \xi(x)<N\end{array} & \begin{cases}
\eta(x)<N & \begin{cases}
\begin{array}{lll}
(\xi(x),\eta(x)) & \to (\xi(x)+1,\eta(x)+1) & \beta\sum_{y\sim x}\xi(y)+\xi(x)\phi \\
 & \to (\xi(x),\eta(x)+1) & \beta\sum_{y\sim x}[\eta(y)-\xi(y)]^{+}+ \\
 & & +[\eta(x)-\xi(x)]^{+}\phi \\
 & \to (\xi(x)+1,\eta(x)) & 0 \\
 & \to (\xi(x)-1,\eta(x)-1) & \xi(x) \\
 & \to (\xi(x),\eta(x)-1) & [\eta(x)-\xi(x)]^{+} \\
 & \to (\xi(x)-1,\eta(x)) & 0
\end{array}
\end{cases} \\[2ex]
\eta(x)=N & \begin{cases}
\begin{array}{lll}
(\xi(x),\eta(x)) & \to (\xi(x)+1,\eta(x)) & \beta\sum_{y\sim x}\xi(y)+\xi(x)\phi \\
 & \to (\xi(x)-1,\eta(x)-1) & \xi(x) \\
 & \to (\xi(x),\eta(x)-1) & [\eta(x)-\xi(x)]^{+} \\
 & \to (\xi(x)-1,\eta(x)) & 0
\end{array}
\end{cases}
\end{cases} \\[6ex]
\xi(x)=N & \begin{cases}
\eta(x)=N & \begin{cases}
(\xi(x),\eta(x))\to(\xi(x)-1,\eta(x)-1) \quad N.
\end{cases}
\end{cases}
\end{array}
$$

By Definition (3.1.1) and Lemma 3.3, if we denote by $k=\xi_t(x)$, $l=\zeta_t(x):=\eta_t(x)-\xi_t(x)$ and $k+l=\eta_t(x)$,

$$
\begin{aligned}
\mathcal{L}F(\xi_t(x),\eta_t(x)) =& \mathbb{1}_{\{k=l=0\}}\Big[\lambda\sum_{y\sim x}\zeta_t(y)u_l\Big]+\mathbb{1}_{\{k=0,0<l\leq N-1\}}\Big[(\lambda\sum_{y\sim x}\xi_t(y) \\
&+\gamma)(-u_{l-1})+\eta_t(x)(-u_{l-1})+(\beta\sum_{y\sim x}\eta_t(y)+l\phi)u_l\Big] \\
&+\mathbb{1}_{\{k=0,l=N\}}\Big[(\lambda\sum_{y\sim x}\xi_t(y)+\gamma)(-u_{l-1})+N(-u_{N-1})\Big] \\
&+\mathbb{1}_{\{k>0,k+l\leq N-1\}}\Big[(\beta\sum_{y\sim x}\zeta_t(y)+l\phi)u_l+l(-u_{l-1})\Big] \\
&+\mathbb{1}_{\{k\leq N-1,k+l=N\}}\Big[(\beta\sum_{y\sim x}\xi_t(y)+\xi_t(x)\phi)(-u_{l-1})+l(-u_{l-1})\Big].
\end{aligned}
\tag{3.1.24}
$$

• We prove that

$$\mathcal{L}F(k,k+l) \leq 1\!\!1_{\{k=0,0<l\leq N-1\}}\Big[-(\lambda\sum_{y\sim x}\xi_t(y)+\gamma)u_{l-1}-lu_{l-1}+(\beta\sum_{y\sim x}\eta_t(y)+l\phi)u_l\Big]$$
$$+1\!\!1_{\{\{k=0,0<l\leq N-1\}^c\}}\Big[(\lambda\vee\beta)\sum_{y\sim x}\zeta_t(y)u_l+l\phi u_l-lu_{l-1}\Big]. \qquad (3.1.25)$$

Notice that if $k=0$ then $\eta_t(x)=k+l=l$.
If $l=0$ and $k=N$, that is $\xi_t(x)=\eta_t(x)=N$, then (3.1.24) is null and we get

$$0=l\leq\beta\sum_{y\sim x}\zeta_t(y)u_l=\beta\sum_{y\sim x}\zeta_t(y)u_l+l\phi u_l-lu_{l-1}.$$

If $k>0$ and $k+l=N$, again

$$-\beta\sum_{y\sim x}\xi_t(y)u_{l-1}-\xi_t(x)\phi u_{l-1}-lu_{l-1}\leq\beta\sum_{y\sim x}\zeta_t(y)u_l+l\phi u_l-lu_{l-1}.$$

If $k=0$ and $l=0$,

$$\lambda\sum_{y\sim x}\zeta_t(y)u_l=\lambda\sum_{y\sim x}\zeta_t(y)u_l+l\phi u_l-lu_{l-1}.$$

If $k=0$ and $l=N$,

$$-\lambda\sum_{y\sim x}\xi_t(y)u_{l-1}-(\gamma+N)u_{l-1}\leq\beta\sum_{y\sim x}\zeta_t(y)u_l+l\phi.u_l-lu_{l-1}$$

Therefore (3.1.25) holds.
• Since u_l is non increasing in l then

$$-(\lambda\sum_{y\sim x}\xi_t(y)+\gamma)u_{l-1}\leq-(\lambda\sum_{y\sim x}\xi_t(y)+\gamma)u_l$$

Suppose $\beta\leq\lambda$:

$$\begin{aligned}&-(\lambda\sum_{y\sim x}\xi_t(y)+\gamma)u_{l-1}-lu_{l-1}+(\beta\sum_{y\sim x}\eta_t(y)+l\phi)u_l\\ &\leq l\phi u_l-lu_{l-1}+\lambda\sum_{y\sim x}\zeta_t(y)u_l\end{aligned}$$

and by (3.1.25) the claim follows.
Suppose $\beta>\lambda$:

$$\begin{aligned}&l\phi u_l-lu_{l-1}+\beta\sum_{y\sim x}\eta_t(y)u_l-\lambda\sum_{y\sim x}\xi_t(y)u_{l-1}-\gamma u_{l-1}\\ &=l\phi u_l-lu_{l-1}+\lambda\sum_{y\sim x}\zeta_t(y)u_l+\Big((\beta-\lambda)\sum_{y\sim x}\eta_t(y)-\gamma\Big)u_l. \qquad (3.1.26)\end{aligned}$$

Since $\beta-\lambda\leq\gamma/(2dN)$, then $(\beta-\lambda)\sum_{y\sim x}\eta_t(y)-\gamma\leq0$ and the claim follows from (3.1.25) and (3.1.26). $\square$

The key result to prove Proposition 3.11 is

Proposition 3.13 (u-criterion) *If there exists $\epsilon > 0$, $U > 0$ and a sequence $\{u_l(\epsilon)\}_{l\in X}$, non increasing in l such that $u_l(\epsilon) > 0$ and*

$$\begin{cases} u_0(\epsilon) = U \\ u_l(\epsilon) \le \dfrac{1}{l\phi}\Big(-\epsilon \displaystyle\sum_{j=0}^{l-1} u_j(\epsilon) - \lambda U 2dl + l u_{l-1}(\epsilon)\Big) \end{cases} \tag{3.1.27}$$

for each $l \in X$, then the process with $\gamma > 0$ is ergodic.

Proof . Let $(\xi_t, \eta_t)_{t\ge 0}$ be a coupled process through the basic coupling probability measure $\widetilde{\mathbb{P}}$ such that $\xi_0 \in E^0$ and $\eta_0 \in E^N$. We fix $x \in \mathbb{Z}^d$ and we denote by $k = \xi_t(x)$, $l = \zeta_t(x) := \eta_t(x) - \xi_t(x)$ and $k + l = \eta_t(x)$. By Lemma 3.12, (3.1.27) and translation invariance

$$\begin{aligned} \widetilde{\mathbb{E}}\Big(\mathcal{L}F(k, k+l)\Big) \le & \widetilde{\mathbb{E}}\Big(l\phi u_l - l u_{l-1} + \lambda \sum_{y\sim x} \zeta_t(y) u_l\Big) \\ \le & \widetilde{\mathbb{E}}\Big(-\epsilon \sum_{j=0}^{l-1} u_j - \lambda U 2dl + \lambda \sum_{y\sim x} \zeta_t(y) U\Big) \\ = & -\epsilon \widetilde{\mathbb{E}}\Big(F(k, k+l)\Big). \end{aligned}$$

Therefore

$$\frac{d}{dt}\widetilde{\mathbb{E}}\Big(F(\zeta_t(x))\Big) \le -\epsilon \widetilde{\mathbb{E}}\Big(F(\zeta_t(x))\Big)$$

and the claim follows by Gronwall's Lemma, translation invariance and attractiveness as in proof of Proposition 3.9. □

Proof of Proposition 3.11.
Let $\{u_l\}_{l\in X}$ be given by Definition 3.4. Since

$$\lambda \le \frac{1-\phi}{2d(1-\phi^N)},$$

the sequence $\{u_l\}_{l\in X}$ is positive, non increasing in l by Proposition 3.5 and satisfies Hypothesis (3.1.27). Ergodicity follows from Proposition 3.13. □

3.2 Metapopulation dynamics models

We investigate survival and extinction of species in many models of metapopulation dynamics. See Section 1.1.5 for a biological introduction to the problem and the main notations.
We recall the infinitesimal generator $\mathcal{L}$ of the Markov process $(\eta_t)_{t\ge 0}$. Given $\eta \in \Omega$,

$$\begin{aligned} \mathcal{L}f(\eta) = \sum_{x\in X} \Big\{ & B(\eta(x))\big(f(S_x^1\eta) - f(\eta)\big) + D(\eta(x))\big(f(S_x^{-1}\eta(x)) - f(\eta)\big) \\ & + \sum_{y\sim x}\sum_{k\in X} J(\eta(x), \eta(y), k)\big(f(S_{x,y}^{-k,k}\eta) - f(\eta)\big)\Big\} \end{aligned} \tag{3.2.1}$$

where $B(\cdot)$, $D(\cdot)$ and $J(\cdot,\cdot,\cdot)$ were introduced in Section 1.1.5.
We focus our analysis on 4 different models:

Model I: We take the birth rate larger than the death rate, but we fix a capacity N so that more than N individuals are not allowed on a site. A migration of one individual from some site x towards a nearest neighbor site is allowed when the population on x reaches the maximal number of individuals. We prove that in some cases we can have coexistence and in other cases extinction.

Model II: We introduce the Allee effect in Model I. Each site has a capacity N, but the death rate is larger than the birth rate for small densities. Migration works exactly as in Model I. We prove that for all possible capacities, growth and migration rates there exists an Allee effect large enough for the species to get extinct.

Model III: We introduce mass migration as Allee effect solution. In Model II we proved that for all parameters we are able to find an Allee effect large enough for the species to get extinct. This is not true if we allow a migration of flocks of M individuals. A migration of large flocks avoids small densities in a new environment which are bad for survival. Indeed we prove that even if the Allee effect is the strongest one, if the species lives and migrates in flocks large enough survival is possible.

Model IV: Instead of fixing a capacity N, we consider a little more realistic model. In any environment there is not a maximal size, but there is a kind of self-mechanism of birth control such that the death rate is larger when the number of individuals oversize a population size N. In other words if there are less than N individuals birth rate is larger than death rate, otherwise death rate is the larger one. A migration of one or more individuals is allowed when there are more than N individuals in a site towards a site with few individuals. We prove that in some cases we can have coexistence but on each site the population does not explode even if there is no capacity. Namely on each site the expected value of the number of individuals is finite. In other cases the species gets extinct.

3.2.1 Model I

We begin from the easier model. Let $N < \infty$ be a positive integer and $X = \{0, 1, \dots, N\}$, Model I is an interacting particle system $(\eta_t)_{t\geq 0}$ on $\Omega = X^{\mathbb{Z}^d}$ with births, deaths and migrations of at most one particle per time. Birth and death rates at time t in a cluster x depend only on the size $\eta_t(x)$, but when a population reaches the maximal capacity N, a migration of one individual is possible.
Let $\delta \geq 0$ and $\lambda \geq 0$ be real numbers. Referring to Definition (3.2.1), the dynamics is ruled by the transition rates

$$\begin{aligned} B(l) &:= l\mathbb{1}_{\{l\leq N-1\}} \\ D(l) &:= \delta l \\ J(l,m,k) &:= \lambda M(m)\mathbb{1}_{\{l=N\}}. \end{aligned} \tag{3.2.2}$$

We assume $M(l)$ non increasing in l and $M(N) = 0$. See Section 1.1.5 (*Model I: the basic model*) for more about the biological motivations.
Since we are dealing with births, deaths and migrations of at most one particle per time and the migration rate from $\eta_t(x)$ to $\eta_t(y)$ is increasing in $\eta_t(x)$ and decreasing in $\eta_t(y)$, attractiveness conditions are satisfied (see Section 2.3.3).

We prove that in some cases there exists at least one non-trivial invariant measure.

The first result corresponds to Theorem 1.8 for non catastrophic times model and it is proved in a similar way.

Theorem 3.14 *Suppose $d \geq 2$, $\lambda > 0$ and $\delta < 1$. There exists a critical value $N_c(\lambda, \delta)$ such that if $N > N_c(\lambda, \delta)$, then starting from $\eta_0 \in \Omega$ such that $|\eta_0| \geq 1$ the process has a positive probability of survival. Moreover if $\eta_0 \in E^N$ the process converges to a non-trivial invariant measure with positive probability.*

Proof. We skip it, since the result is a corollary of Theorem 3.23 in Section 3.2.3. We can get an easier proof that the process has a positive probability of surviving by slightly modifying [43, Proof of Theorem 2]. The differences are that we consider a migration instead of a birth from x to $y \sim x$ and the migration rate from x to y is non increasing in $\eta_t(y)$. Such changes are not relevant for the proof. □

As we can expect aggregation is good for Model I, as in non catastrophic times model.

Remark 3.15 *If $N = 1$ the process dies out, since each individual can only migrate or die.*
This suggests that there exists a critical value N_c (depending on λ and δ) such that if $N > N_c(\lambda, \delta)$ the process survives and if $N < N_c(\lambda, \delta)$ it dies out. However since a migration is allowed only when the population size reaches the capacity N, we do not have a monotonicity property with respect to N. Namely, given $N_1 \leq N_2$, it is not possible to couple two processes $(\eta_t)_{t\geq 0}$ with capacity N_1 and $(\xi_t)_{t\geq 0}$ with capacity N_2 in such a way that $\eta_t(x) \leq \xi_t(x)$ for each x, $t \geq 0$.

The model admits a phase transition with respect to the death rate δ for each $N \geq 2$: if migrations are not allowed the process dies out almost surely. This means that migrations are good for survival of a species. This is not surprising, since contact interactions are also good. Contact interactions and migrations work in a similar way, but small differences are present. From a mathematical point of view in this model an increase of migration rate does not favor ergodicity.
Notice that the system has the following monotonicity property.

Proposition 3.16 *Let ξ_t and η_t be two processes with respective parameters (δ_1, λ, N) and (δ_2, λ, N) such that $\delta_1 < \delta_2$. Then ξ_t is stochastically larger than η_t.*

Proof. We prove that if $\eta_0 \leq \xi_0$, then $\eta_t \leq \xi_t$ for each $t > 0$.
This is an application of Theorem 2.15; since $M = 1$ we check Conditions (2.3.15) and (2.3.16) in Section 2.3.5. Given $p(x, y) = (2d)^{-1}\mathbb{1}_{\{y\sim x\}}$, the corresponding rates for the process $(\eta_t)_{t\geq 0}$ are

$$\Pi^{0,1}_{\eta_t(x),\eta_t(y)} = P^1_{\eta_t(y)} = \eta_t(y)\mathbb{1}_{\{\eta_t(y)\leq N-1\}}; \quad \Pi^{-1,0}_{\eta_t(x),\eta_t(y)} = P^{-1}_{\eta_t(x)} = \delta_2\eta_t(x)\mathbb{1}_{\{\eta_t(y)\leq N\}};$$
$$\Gamma^1_{\eta_t(x),\eta_t(y)} = 2d\lambda M(\eta_t(y))\mathbb{1}_{\{\eta_t(x)=N\}}.$$

The rates of $(\xi_t)_{t\geq 0}$ are the same with δ_1 instead of δ_2.
If $\eta_t(y) = \xi_t(y)$ and $\eta_t(x) < \xi_t(x)$, Condition (2.3.15) is

$$\eta_t(y)\mathbb{1}_{\{\eta_t(y)\leq N-1\}} + 2d\lambda M(\eta_t(y))\mathbb{1}_{\{\eta_t(x)=N\}} \leq \xi_t(y)\mathbb{1}_{\{\xi_t(y)\leq N-1\}} + 2d\lambda M(\xi_t(y))\mathbb{1}_{\{\xi_t(x)=N\}}$$

that is $\mathbb{1}_{\{\eta_t(x)=N\}} \leq \mathbb{1}_{\{\xi_t(x)=N\}}$, which is true since $\eta_t(x) < \xi_t(x)$. Condition (2.3.16) is

$$\delta_1\eta_t(x)\mathbb{1}_{\{\eta_t(x)\leq N-1\}}+2d\lambda M(\eta_t(y))\mathbb{1}_{\{\eta_t(x)=N\}} \geq \delta_2\xi_t(x)\mathbb{1}_{\{\xi_t(x)\leq N-1\}}+2d\lambda M(\xi_t(y))\mathbb{1}_{\{\xi_t(x)=N\}}$$

which holds since $\delta_1 > \delta_2$ and $2d\lambda M(\eta_t(y))\mathbb{1}_{\{\eta_t(x)=N\}} \geq 2d\lambda M(\xi_t(y))\mathbb{1}_{\{\xi_t(x)=N\}}$ because $M(\cdot)$ is non increasing. □

Corollary 3.1 *Given $(\eta_t^\zeta)_{t\geq 0}$ such that $\eta_0^\zeta = \zeta$, then*

$$\mathbb{P}(|\eta_t^\zeta| \geq 1 \text{ for all } t \geq 0)$$

is decreasing in δ for each $\zeta \in \Omega$.

Remark 3.17 *There is no stochastic order between systems with different values of N or λ. Indeed in these cases conditions of Theorem* 2.15 *are not satisfied.*

The system admits a critical value δ_c, namely

Theorem 3.18 *i) For all $\lambda > 0$, $1 < N < \infty$ there exists $\delta_c^1(\lambda, N) > 0$ such that if $\delta < \delta_c^1(\lambda, N)$ the process has a positive probability of survival.*

ii) There exists $\delta_c^2 > 0$ such that if $\delta > \delta_c^2$ the process dies out for all λ, N.

iii) For all $\lambda > 0$, $1 < N < \infty$ there exists $\delta_c(\lambda, N) > 0$ such that if $\delta < \delta_c(\lambda, N)$ the process starting from η_0 with $|\eta_0| \geq 1$ has a positive probability of survival and if $\delta > \delta_c(\lambda, N)$ the process dies out. Moreover if $\delta < \delta_c(\lambda, N)$ and $\eta_0 \in E^N$ the process converges to a non-trivial invariant measure with positive probability.

Proof. *i)* We use the comparison technique with oriented percolation explained in Section 1.1.2 as in [42]. We think of the process as being generated by the graphical representation, see [22] for such a construction. Suppose $d = 2$. The proof in higher dimension is similar but the notation more complicated. Denote by

$$\begin{aligned} &e_1 = (1, 0), && \mathcal{N} = \{(m, n) \in \mathbb{Z}^2 : m + n \text{ is even}\} \\ &B = (-4L, 4L)^2 \times [0, T], && B_{m,n} = (2mLe_1, nT) + B \\ &I = [-L, L]^2, && I_m = 2mLe_1 + I \end{aligned}$$

where L and T are integers to be chosen later.
In other words $B_{m,n}$ is the cube that we get by applying a translation of $(2mLe_1, nT)$ to B and I_m is the square that we get by applying a translation of $2mLe_1$ to I. Roughly speaking, the idea consists in constructing boxes large enough so that with large probability the species survives inside a box, and then to compare this evolution with an oriented percolation model (see Section 1.1.2).
We say that (m, n) is *wet* if the process restricted to $B_{m,n}$ and starting at time nT with at least one individual in I_m is such that there is at least one individual in I_{m-1} and one individual in I_{m+1} at time $(n + 1)T$. Otherwise the site is *dry*. In other words, if (m, n) is wet with large probability a site in state 1 at time nT is duplicated at time $(n + 1)T$ in I_{m-1} and I_{m+1}. We want to prove that we can choose L and T such that the probability of a site (m, n) being wet can be made arbitrarily close to 1. By translation invariance it is enough to show it for site $(0, 0)$.
We start by setting $\delta = 0$. This means that each individual survives forever.
Let $(\eta_t)_{t\geq 0}$ be the process defined by generator (3.2.1) and rates given by (3.2.2). We

fix $L > 0$. Assume that at time 0 there is at least one site of I in state 1, that is a site $(i, j) \in \mathbb{Z}^2$ with $-L \leq i \leq L$, $-L \leq j \leq L$ in state 1 at time 0. After a sum of independent exponential times with rates $1, 2, \dots, N-1$ we get N individuals and again after an independent exponential time with rate $\lambda M(0)$ a migration from (i, j) to an empty neighboring site occurs. Let $\{B_j\}_{1\leq j\leq N-1}$ be such that $B_j \sim Exp(j)$ are independent random variables and $W = \sum_{j=1}^{N-1} B_j$. The law of W is the hypoexponential distribution (see Appendix A).

We choose a preferential path $(i, j), (i+1, j), \dots, (L, j), (L+1, j)$ and we prove that there exists T such that with large probability there is at least one individual in $(L+1, j)$ at time T.

Each time t the population in $x = (i, j) \in \mathbb{Z}^2$ reaches size N, there is at least one individual on $(i+1, j) = y \in \mathbb{Z}^2$ either if $\eta_t(y) \geq 1$ or if a migration from x to y happens, that is if the clock of an exponential random variable $Y_y \sim Exp(\lambda M(0))$ rings before the clock of all the other migrations, which is distributed as $X_x(\eta_t) \sim Exp\big(\sum_{z\sim x, z\neq y} \lambda M(\eta_t(z))\big)$, since the minimum of independent exponential distributions is still exponential with rate given by the sum of all rates.

Notice that if $X \sim Exp(\alpha)$ and $Y \sim Exp(\beta)$ are independent then

$$\mathbb{P}(X < Y) = \frac{\alpha}{\alpha + \beta} \tag{3.2.3}$$

Denote by $\gamma(\eta_t) = \mathbb{P}(Y_y < X_x(\eta_t) | \eta_t(x) = N)$. Therefore, for each η

$$\gamma(\eta_t) = \frac{\lambda M(0)}{\lambda M(0) + \sum_{z\sim x, z\neq y} \lambda M(\eta_t(z))} \geq \frac{\lambda M(0)}{\lambda M(0) + \lambda(2d-1)M(0)} = \frac{1}{2d} > 0$$

Denote by $Z \sim Geom(1/2d)$ and $Z_{\eta_t} \sim Geom(\gamma(\eta_t(x)))$, then $\mathbb{P}(Z < k) \leq \mathbb{P}(Z_{\eta_t} < k)$. Let $\tau_{i,j}$ be the time of the first migration from $(i, j) = x$ to $(i+1, j) = y$. Then the first time $\tau(y)$ such that $\eta_t(y) > 0$ is smaller or equal than $\tau_{i,j}$. We get $\tau_{i,j}$ by summing W (the time needed to reach N), Z_{η_t} times an exponential random variable $X_x(\eta_t)$ (the time spent to migrate to $z \sim x$, $z \neq y$) followed by B_{N-1} (the time needed to return to N on site x after this migration), and finally the time of the migration on y with law Y_y. Let $\{B_{N-1}(i)\}_{i\in\mathbb{N}}$ be independent identically distributed random variables such that $B_{N-1}(i) \sim B_{N-1}$ and let $\{X_x(i)\}_{i\in\mathbb{N}}$ be independent identically distributed random variables such that $X_x(i) \sim Exp\big((2d-1)\lambda M(0)\big)$. Therefore $\mathbb{P}(X_x(i) < s) \leq \mathbb{P}(X_x(\eta_t) < s)$ for each i and we get

$$\begin{aligned}
\mathbb{P}(\tau_{i,j} < s) =& \mathbb{P}(W + \sum_{i=0}^{Z_{\eta_t}} ((X_x(\eta_t))(i) + B_{N-1}(i)) + Y_y < s) \\
\geq& \mathbb{P}(W + \sum_{i=0}^{Z} (X_x(i) + B_{N-1}(i)) + Y_y < s) \\
\geq& \mathbb{P}(W + \sum_{i=0}^{k} (X_x(i) + B_{N-1}(i)) + Y_y < s)\mathbb{P}(Z < k).
\end{aligned}$$

For each $\bar{\epsilon} > 0$, since for each $k < \infty$ the random variable $W + \displaystyle\sum_{i=0}^{k} (B_{N-1}(i) + X_x(i)) + Y_y$ is a sum of a finite number of independent random variables with finite mean, we can take k and $s = \overline{T}$ large such that $\mathbb{P}(Z < k) > 1 - \bar{\epsilon}/2$ and $\mathbb{P}(W + \sum_i^k (B_{N-1}(i) + X_x(i)) + Y_y <$

$\overline{T}) > 1 - \bar{\epsilon}/2$. Therefore for each $\bar{\epsilon}$ there exists $\overline{T}$ large such that

$$\mathbb{P}(\tau(y) < \overline{T}) \geq \mathbb{P}(\tau_{i,j} < \overline{T}) > 1 - \bar{\epsilon}. \tag{3.2.4}$$

Now we consider one individual at site $(i,j) \in \mathbb{Z}^2$. For each $\delta \geq 0$, we define $\tau_L(\delta)$ (respectively $\tau'_L(\delta)$) to be the random variable which measures the time to reach $(L+1,j)$ (resp. $(-(L+1),j)$); since we are in case $\delta = 0$, we denote by $\tau_L := \tau_L(0)$.
After the first migration from (i,j) to $(i+1,j)$, we wait for another migration from $(i+1,j)$ to $(i+2,j)$ and so on until we reach $(L+1,j)$. We prove that for all $\epsilon > 0$ we can choose T sufficiently large so that the probability that the position of the rightmost particle is after L is larger than $1-\epsilon$. Let $R_N(t)$ (respectively $R'_N(t)$) be the position of the righmost (leftmost) particle at time t.
Notice that $\mathbb{P}(R_N(t) > L) \geq \mathbb{P}(\tau_L \leq t)$. Therefore by (3.2.4) for each $L > 0$ and $\epsilon > 0$ there exists $T = L\overline{T}$ such that

$$\begin{aligned}
\mathbb{P}(R_N(L\overline{T}) > L) \geq & \mathbb{P}(\tau_L < L\overline{T}) \geq \mathbb{P}\Big(\sum_{k=0}^{2L} \tau_{i+k,j} < L\overline{T}\Big) \\
\geq & \prod_{k=0}^{2L} \mathbb{P}\Big(\tau_{i+k,j} < \overline{T}\Big) \geq (1-\bar{\epsilon})^{2L} \\
\geq & 1 - \epsilon/4.
\end{aligned}$$

In a similar way we prove that by taking the same T the probability that the leftmost particle reaches $(-(L+1),j)$ is as large as we want. We conclude that if $\delta = 0$ for each $\epsilon > 0$ there exists $T = L\overline{T}$ such that

$$\mathbb{P}\Big((0,0) \text{ is wet}|\delta = 0\Big) > 1 - \epsilon/2. \tag{3.2.5}$$

Now we show that for each $\lambda > 0$ and $0 < N < \infty$ there exists $\delta(\lambda, N) > 0$ such that if $\delta \leq \delta(\lambda, N)$, then (3.2.5) holds.
We start with one individual at site $(0,0)$. Let A_L be the time of the first death on one of the sites $(i,0)$, $-L \leq i \leq L$. The random variable A_L is the minimum of at most $2L$ independent exponential random variables with rate at most δN: therefore $\mathbb{P}(A_L > t) \geq \exp(-\delta N 2Lt)$. We take $T = L\overline{T}$ large in order to get

$$\begin{aligned}
\mathbb{P}\Big((0,0) \text{ is wet}\Big) \geq & \mathbb{P}\Big(\{\tau_L(\delta) < T\} \cap \{\tau'_L(\delta) < T\} \cap \{A_L > T\}\Big) \\
\geq & \mathbb{P}\Big(\{\tau_L(\delta) < T\} \cap \{\tau'_L(\delta) < T\} | A_L > T\Big)\mathbb{P}(A_L > T) \\
\geq & \mathbb{P}\Big((0,0) \text{ is wet}|\delta = 0\Big) e^{-2\delta NLT} \\
\geq & (1-\epsilon/2)e^{-2\delta NLT}.
\end{aligned} \tag{3.2.6}$$

By taking $\delta > 0$ small so that $e^{-2\delta NLT} \geq 1 - \epsilon/2$ then (3.2.6) is larger than $1-\epsilon$. Hence there exists $\delta_c^1(\lambda, N) > 0$ such that if $\delta \leq \delta_c^1(\lambda, N)$

$$\mathbb{P}\Big((0,0) \text{ is wet}\Big) > 1 - \epsilon. \tag{3.2.7}$$

For each $\epsilon > 0$ we can fix L, T and δ_c^1 such that for each $(m,n) \in \mathcal{N}$ the event $G_{m,n} = \{(m,n) \text{ is wet}\}$ satisfies $\mathbb{P}(G_{m,n}) > 1 - \epsilon$. Moreover by construction $G_{m,n}$ depends only on the graphical construction in $\mathcal{B}_{m,n}$.

Now we are ready to compare the process with an oriented percolation process (see Section 1.1.2) and we can choose ϵ small enough so that percolation occurs, by Theorem 1.2. By (1.2) this implies the survival of the process for any $\delta \leq \delta_c^1(\lambda, N)$.

ii) Assume $\delta > 1$ and $\eta_0(x) = N$ for each $x \in \mathbb{Z}^2$. By using generator (3.2.1) with rates (3.2.2) and translation invariance on $\mathbb{Z}^2$

$$\begin{aligned}\mathbb{E}\big(\mathcal{L}\eta_t(x)\big) =&\mathbb{E}\Big(\eta_t(x)\mathbb{1}_{\{\eta_t(x)\leq N-1\}} - \delta\eta_t(x) + \sum_{y\sim x}\lambda M(\eta_t(x))\mathbb{1}_{\{\eta_t(y)=N\}} \\ &- \sum_{y\sim x}\lambda M(\eta_t(y))\mathbb{1}_{\{\eta_t(x)=N\}}\Big) = \mathbb{E}\Big(\eta_t(x)\mathbb{1}_{\{\eta_t(x)\leq N-1\}} - \delta\eta_t(x)\Big) \\ \leq&(1-\delta)\mathbb{E}(\eta_t(x)).\end{aligned}$$

Since $\delta > 1$, there exists $\epsilon > 0$ such that for each $t > 0$

$$\frac{d}{dt}\mathbb{E}(\eta_t(x)) = \mathbb{E}(\mathcal{L}\eta_t(x)) \leq -\epsilon\mathbb{E}(\eta_t(x))$$

and by Gronwall's Lemma the process converges to 0 uniformly with respect to x, therefore the process dies out (see Section 1.1.2). We conclude by Proposition 3.1 that the process dies out for each $\delta > 1 =: \delta_c^2$.

iii) The claim follows from Proposition 3.1, Theorem 3.18 *i)* and *ii)* and from the attractiveness of the model.
Starting from $\eta_0 \in E^N$, the existence of the upper non-trivial invariant measure follows from step *i)*, the comparison technique and (1.3) in Section 1.1.2. □

Remark 3.19 *The effect of a migration is to move an individual from a site in state N, where there is no possibility to give birth to other individuals, to a site with less than N individuals, where it may reproduce itself. Therefore even if there is no monotonicity with respect to λ (cf Remark 3.17), this suggests that an increase of λ is good for survival.*

3.2.2 Model II: the Allee effect

We introduce the mathematical version of the Allee effect in Model I (see Section 1.1.5). As in Model I, we fix a capacity N for all sites, but we assume the death rate to be larger than the birth rate when the density is small.
Namely, by referring to Definition 3.2.1, for a fixed $N_A(\leq N)$ positive integer and δ_A, δ, λ positive real numbers the transitions are given by

$$\begin{aligned}B(l) :=&l\mathbb{1}_{\{l\leq N-1\}} \\ D(l) :=&l\big(\delta_A\mathbb{1}_{\{l\leq N_A\}} + \delta\mathbb{1}_{\{N_A<l\}}\big) \\ J(l,m,k) :=&\lambda M(m)\mathbb{1}_{\{l=N\}}\end{aligned} \tag{3.2.8}$$

where the function M is defined as in Model I. We assume $\delta_A \geq 1$, $\delta_A \geq \delta$, $0 \leq N_A \leq N$. Notice that if $\delta_A = \delta$ the process reduces to Model I. See Section 1.1.5 *(Model II: the Allee effect)* for more about the biological meaning of the model.
Since only births, deaths and migrations of at most one particle are allowed and the migration rate from $\eta_t(x)$ to $\eta_t(y)$ is increasing in $\eta_t(x)$, attractiveness conditions are satisfied.

One can prove that Proposition 3.1 still holds for Model II either with respect to δ_A or δ.

We prove that the Allee effect changes the behaviour of the system: for any possible growth and migration rates there exists an Allee effect large enough for the species to get extinct.

Theorem 3.20 *Assume $\delta_A > 1$.*

i) For all $\delta < 1$, $\lambda > 0$, $0 < N < \infty$, $0 \leq N_A \leq N$ there exists $\delta_c^A(\delta, \lambda, N, N_A)$ such that if $\delta_A > \delta_c^A(\delta, \lambda, N, N_A)$ the process converges to the trivial invariant measure $\delta_{\underline{0}}$ for any initial configuration $\eta_0 \in \Omega$.

ii) If $\delta > 1$ the process dies out for all δ_A, N, N_A, λ and any initial configuration.

Proof. We follow the proofs of [44, Theorem 4] and [50, Theorem 4.4] and we compare our system with a subcritical percolation process. We think of the process as being constructed through the graphical representation, see [22] for such a construction. We prove $i)$ when $d = 2$ in order to simplify the notation (the same proof works for all $d \geq 1$).
Let $\delta > 1$ and $(\eta_t)_{t\geq 0}$ be a process with generator (3.2.1), rates (3.2.8) and $\eta_0(x) = N$ for all $x \in S$. We define the space-time regions

$$\mathcal{A} = [-2L, 2L]^2 \times [0, 2T]; \qquad \mathcal{B} = [-L, L]^2 \times [T, 2T].$$

We call $\mathcal{C}$ the boundary of $\mathcal{A}$ and

$$\begin{aligned}
\mathcal{C}_b =& \{(m, n, t) \in \mathcal{A} : t = 0\}\\
\mathcal{C}_s =& \{(m, n, t) \in \mathcal{A} : |m| = 2L \text{ or } |n| = 2L\}\\
\mathcal{C} =& \mathcal{C}_b \cup \mathcal{C}_s = \{(m, n, t) \in \mathcal{A} : |m| = 2L \text{ or } |n| = 2L \text{ or } t = 0\}.
\end{aligned}$$

We construct a percolation process on $\mathcal{N} = \mathbb{Z}^2 \times \mathbb{Z}_+$ starting from $(\eta_t)_{t\geq 0}$. We say that

a site $(k, m, n) \in \mathcal{N}$ is wet if the process $(\eta_t)_{t\geq 0}$ restricted to $\mathcal{A} + (k, m, n)$ has no individuals in $\mathcal{B} + (k, m, n)$ regardless of the configuration on the boundary $\mathcal{C} + (k, m, n)$. (3.2.9)

Therefore it must be true even if all sites in $\mathcal{C}+(k, m, n)$ are in state N for all t. Indeed a site in state N in $\mathcal{C} + (k, m, n)$ improves the probability of a migration to a site $x \in [-2L, 2L]^2$ and reduces the probability of emigrations from $[-2L, 2L]^2$, which are null since all sites on the boundary are full. A site is dry if it is not wet.
In order to have one individual in $\mathcal{B} + (k, m, n)$, it must exist a path of individuals starting either at $\mathcal{C}_b$ or at $\mathcal{C}_s$.
• First of all notice that the process $(\eta_t)_{t\geq 0}$ is stochastically smaller than a birth and death process $(\overline{\eta}_t)_{t\geq 0}$ on $\{0, 1, \dots N\}$ with rates:

$$\begin{aligned}
\overline{B}(l) :=& \mathbb{1}_{\{l\leq N-1\}}(l + 2d\lambda M(0))\\
\overline{D}(l) :=& l\big(\delta_A \mathbb{1}_{\{l\leq N_A\}} + \delta \mathbb{1}_{\{N_A<l\}}\big).
\end{aligned} \tag{3.2.10}$$

In other words the birth rate of $\overline{\eta}_t(x)$ is the original birth rate plus the largest immigration rate on $\eta_t(x)$, and the death rate of $\overline{\eta}_t(x)$ is the original death rate plus the smallest emigration rate on $\eta_t(x)$, which is null. Since $M = 1$, we check Conditions (2.3.15) and (2.3.16). They are satisfied because, for each $\eta \in \Omega$,

$$\begin{aligned} B(\eta(y)) + J(\eta(x),\eta(y),1) \leq & \mathbb{1}_{\{\eta(y)\leq N-1\}}(\eta(y) + 2d\lambda M(0)) = \overline{B}(\eta(y)) \\ D(\eta(x)) + J(\eta(x),\eta(y),1) \geq & \mathbb{1}_{\{\eta(x)\leq N_A\}}\delta_A\eta(x) + \mathbb{1}_{\{N_A<\eta(x)\}}\delta\eta(x) = \overline{D}(\eta(x)). \end{aligned} \tag{3.2.11}$$

Therefore if $\eta_0 = \overline{\eta}_0$ then $\eta_t(x) \leq \overline{\eta}_t(x)$ for each $x \in S$ and $t \geq 0$. Let $0 < \delta_A < \infty$. First of all we prove that there exists a time W at which with large probability there is at most 1 individual per site on $[-2L, 2L]^2$. Denote by

$$S_x^W := \Big| \{s : 0 < s \leq W \text{ and } \overline{\eta}_s(x) = 0, \overline{\eta}_{s^-}(x) > 0\} \Big|$$

the random variable which counts the number of visits to 0 of $\overline{\eta}_t(x)$ before W. Then for each $x \in [-2L, 2L]^2$

$$\mathbb{P}\Big(\overline{\eta}_W(x) > 1\Big) = \sum_{k=0}^{\infty} \mathbb{P}\Big(\overline{\eta}_W(x) > 1 | S_x^W = k\Big)\mathbb{P}(S_x^W = k). \tag{3.2.12}$$

Since $\overline{\eta}_t(x)$ is a birth and death process on a finite interval X, we can take W large enough so that

$$\mathbb{P}(S_x^W = 0) \leq \frac{\epsilon}{6(4L)^2}. \tag{3.2.13}$$

We split the remainder of (3.2.12) in two parts:

$$\begin{aligned} \sum_{k=1}^{\infty} \mathbb{P}\Big(\overline{\eta}_W(x) > 1 | S_x^W = k\Big)\mathbb{P}(S_x^W = k) = & \sum_{k=1}^{K} \mathbb{P}\Big(\overline{\eta}_W(x) > 1 | S_x^W = k\Big)\mathbb{P}(S_x^W = k) \\ & + \sum_{k=K+1}^{\infty} \mathbb{P}\Big(\overline{\eta}_W(x) > 1 | S_x^W = k\Big)\mathbb{P}(S_x^W = k) \\ =: & I_1 + I_2 \end{aligned}$$

We begin with I_2. Notice that in order to have more than K hits to 0 we must have at least K births at 0, that is at least K births when $\overline{\eta}_s(x) = 0$ for $s < W$. Denote by $Z_K \sim Er(K, 2d\lambda M(0))$ the sum of K independent exponential variables with rate $2d\lambda M(0)$, which is an Erlang random variable. For each $\epsilon > 0$ and W there exists K such that

$$\begin{aligned} I_2 \leq & \mathbb{P}(Z_K < W) \leq 1 - e^{-2d\lambda M(0)W} \sum_{j=0}^{K-1} \frac{(2d\lambda M(0)W)^j}{j!} \\ \leq & e^{-2d\lambda M(0)W} \sum_{j=K}^{\infty} \frac{(2d\lambda M(0)W)^j}{j!} < \frac{\epsilon}{12(4L)^2} \end{aligned} \tag{3.2.14}$$

since for each W we can take K large enough for the queue of the sum to be as small as we want.

We now consider I_1. In order to have at least two individuals in a site after a visit to 0, it must happen that an exponential clock with rate $1 + 2d\lambda M(0)$ (the birth rate if $\overline{\eta}_t(x) = 1$) rings before an exponential clock with rate δ_A (the death rate if $\overline{\eta}_t(x) = 1$). Let $\{Y_i\}_{i\in\mathbb{N}}$

and $\{X_i\}_{i\in\mathbb{N}}$ be two collections of independent identically distributed random variables such that $Y_i \sim Exp(1+2d\lambda M(0))$ and $X_i \sim Exp(\delta_A)$ for each $i \in \mathbb{N}$.

$$\begin{aligned}
I_1 =& \sum_{k=1}^{K} \mathbb{P}\Big(\overline{\eta}_W(x) > 1 | S_x^W = k\Big)\mathbb{P}(S_x^W = k) \\
\leq& \sum_{k=1}^{K} \mathbb{P}(\exists i \in \{1,2,\dots k\} : Y_i < X_i) \leq K^2 \frac{1+2d\lambda M(0)}{1+2d\lambda M(0)+\delta_A}
\end{aligned} \tag{3.2.15}$$

and for each $\epsilon > 0$ and K we can take δ_A large enough for $I_1 < \frac{\epsilon}{12(4L)^2}$. Denote by $B_L(t) = \{$All paths such that $\eta_t(x) \leq 1$ for each $x \in [-2L, 2L]^2\}$. By (3.2.13), (3.2.14) and (3.2.15) for each $\epsilon > 0$, $L > 0$ there exists W and δ_A large enough so that

$$\mathbb{P}\big((B_L(W))^c\big) \leq (4L)^2 \sup_{x\in[-2L,2L]^2} \mathbb{P}(\overline{\eta}_W(x) > 1) \leq \epsilon/4. \tag{3.2.16}$$

Therefore with large probability at time W the process $(\overline{\eta}_t(x))_{t\geq 0}$ and hence the process $(\eta_t(x))_{t\geq 0}$ is smaller or equal to 1 for each $x \in [-2L, 2L]^2$.

Now we fix a time $\overline{W}$ and we prove that we can take δ_A large enough so that the probability that after W all individuals in $[-(3/2)L, (3/2)L]^2$ die before any immigration from the boundary $\mathcal{C}_s$ and any birth is large. This is the probability that $(3L)^2$ exponential clocks with rate δ_A ring before the minimum between $(4L)^2$ exponential random variables with rate $2d\lambda M(0) + 1$.
Let $\overline{B}_L(t) = \{$All paths such that $\eta_t(x) = 0$ for each $x \in [-(3/2)L, (3/2)L]^2\}$, $Y \sim Exp(\delta_A)$ and $X \sim Exp\Big((2d\lambda M(0)+1)(4L)^2\Big)$, then for each $\epsilon > 0$ there exists δ_A large enough for

$$\mathbb{P}(\overline{B}_L(W+\overline{W})|B_L(W)) \geq \Big(\mathbb{P}(Y<X)\Big)^{(3L)^2} = \Big(\frac{\delta_A}{\delta_A + (2d\lambda M(0)+1)(4L)^2}\Big)^{(3L)^2} \geq 1-\epsilon/4.$$

Therefore for each $\epsilon > 0$ there exists δ_A large enough so that

$$\mathbb{P}(\eta_{W+\overline{W}}(x) > 0 \text{ for some } x \in [-L, L]^2 | B_L(W)) \leq \epsilon/4 \tag{3.2.17}$$

(L and λ are fixed).
Let $T = W + \overline{W}$, by (3.2.16) and (3.2.17) for all $\epsilon > 0$, L there exists δ_A and W large enough for

$$\begin{aligned}
\mathbb{P}((\overline{B}_L(T))^c) =& \mathbb{P}((\overline{B}_L(T))^c | B_L(W))\mathbb{P}(B_L(W)) \\
&+ \mathbb{P}((\overline{B}_L(T))^c | (B_L(W))^c)\mathbb{P}((B_L(W))^c) \\
\leq& \epsilon/4 + \epsilon/4 = \epsilon/2
\end{aligned} \tag{3.2.18}$$

Therefore $\eta_T(x) = 0$ for each $x \in [-(3/2)L, (3/2)L]^2$ with large probability at time T. Since $B(0) = 0$, the only way to get an individual in $[-L, L]^2$ from time T to $2T$ is that a migration from $y \in ([-(3/2)L, (3/2)L]^2)^c$ gives birth to a chain of individuals which reaches $[-L, L]^2$ in a time smaller than T. The number of migrations M_{yx} from a site y to a site $x \sim y$ in a large but finite time T is larger than a finite number $\widetilde{K}$ with probability smaller than $\epsilon/(48L)$. Each time that a migration from y to $x \in [-(3/2)L, (3/2)L]^2$ occurs, with probability smaller than

$$\frac{\delta_A}{\delta_A + 1 + 2d\lambda M(0)}$$

there is a new birth or a new immigration at x before th death of the individual. Therefore, by taking δ_A large enough for $\widetilde{K}(1+2d\lambda M(0))/(\delta_A+1+2d\lambda M(0)) < \epsilon/4$ we get

$$\begin{aligned}\mathbb{P}((0,0,0) \text{ is dry }) \leq & \mathbb{P}((0,0,0) \text{ is dry } |B_L(T)) + \epsilon/2 \\ \leq & 12L\Big(\mathbb{P}(M_{yx} > \widetilde{K}) + \widetilde{K}\frac{1+2d\lambda M(0)}{\delta_A+1+2d\lambda M(0)}\Big) + \epsilon/2 \\ \leq & \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned} \tag{3.2.19}$$

• Now we can construct a locally dependent percolation model such that the probability of a site to be wet is as large as we want. For each (k, m, n) and (x, y, z) in $\mathcal{N}$ such that $n \leq z$ and the intersection between $(kL, mL, nL) + \mathcal{A}$ and $(xL, yL, zL) + \mathcal{A}$ is not empty we draw an oriented edge.

Notice that by (3.2.9) the probability of a site to be wet depends only on the graphical construction inside $\mathcal{A}$ and the events that depend on disjoint regions on the graphical construction are independent. Moreover the number of sites connected with a site (x, y, z) is a fixed finite number for each site. This means that percolation on $\mathcal{N}$ is dependent but it has an interaction with only a finite number of sites. For this model a path of dry sites is a chain of connected oriented edges which moves along dry sites only. We follow [50, Proof of Theorem 4.4] to prove that there exists a random time after which there will never be any individual.

We call graph theoretic distance between two sites the distance on the undirected graph, that is the graph $\mathcal{N}$ with the same but non-oriented edges. Because of the finite range of the interactions, there exists a constant C depending only on the dimension d so that any set of sites with distance larger than C between any pair of sites are independent.

Let $x \in \mathbb{Z}^2$ and let T_x be the supremum of all times such that $\eta_t(x) > 0$. By translation invariance we can take without loss of generality $x = 0$. Suppose $T_0 > TS$, then there exists m such that $(0, 0, m)$ is the endpoint of an oriented dry path of $\mathcal{N}$ with starting point $(x, y, 0)$ for some pair of sites $(x, y) \in \mathbb{Z}^2$.

Notice that there exists positive finite constants δ, ν such that the number of self-avoiding paths on $\mathcal{N}$ having length r at any given endpoint is no larger than δ^r.

Moreover any self-avoiding path of length r contains at least νr sites such that the distance between any pair of which is larger than C. Let $\gamma = \mathbb{P}((0, 0, 0) \text{ is dry})$, then

$$\mathbb{P}(T_0 > TS) \leq \sum_{m \geq S-1} \sum_{r \geq m} \delta^r \gamma^{\nu r}. \tag{3.2.20}$$

By (3.2.19) we can choose δ_c small enough such that if $\delta_A \geq \delta_c$ and $S \geq 2$ the right hand side of (3.2.20) is finite. In this case the right hand side approaches to 0 as $S \to \infty$. This means that T_0 is a.s. finite. Let I be a finite subset of $\mathbb{Z}^d$ and $T_I := \max\{T_x, x \in I\}$. Since T_x is a.s. finite for each $x \in I$ and I is finite, T_I is a.s. finite. By construction, T_I may be chosen uniformly in the initial configuration η_0.

Denote by $B(y, R)$ the ball with radius R centered at $y \in \mathbb{Z}^d$. Therefore for all $\eta_0 \in \Omega$, $y \in \mathbb{Z}^d$ and $R < \infty$,

$$\bar{\nu}(\xi \in \Omega : \xi(x) > 0 \text{ for some } x \in B(y, R)) = 0.$$

Let $D = \{y \in \mathbb{Z}^d : y = \{2Ry_1, 2Ry_2, \dots, 2Ry_d\}$ and let $\eta_0 \in E^N$: by translation invariance,

$$\begin{aligned}\bar{\nu}(\Omega \setminus \{\underline{0}\}) = & \bar{\nu}\Big(\bigcup_{y\in D} \{\xi \in \Omega : \xi(x) > 0 \text{ for some } x \in B(y, R)\}\Big) \\ \leq & \sum_{y \in D} \bar{\nu}(\xi \in \Omega : \xi(x) > 0 \text{ for some } x \in B(y, R)) = 0.\end{aligned}$$

By attractiveness the claim follows for any initial configuration.

ii) If $\delta = 1+\epsilon$ and $\delta_A = \delta$ the process reduces to Model I. By Theorem 3.18 *ii)* this process dies out. By the monotonicity property in δ_A (cf Proposition 3.16) the process dies out for each $\delta_A \geq 1+\epsilon$. Since we can take ϵ arbitrarily small the process dies out for all $\delta > 1$, $\delta_A > 1$ for all N_A,N, λ. $\square$

Now we change the roles of δ_A and δ and we prove

Theorem 3.21 *Assume $\delta_A < 1$ and $\delta > 1$.*

i) For all $1 < N_A < N$, $\lambda > 0$, $\delta > 1$ there exists $\delta_c^A(\lambda, N_A, N, \delta)$ such that if $\delta_A < \delta_c^A(\lambda, N_A, N, \delta)$ the process survives for any initial configuration η_0 such that $|\eta_0| \geq 1$ with positive probability.

ii) For all $1 < N_A < N$, $\lambda > 0$, $\delta_A < 1$ there exists $\delta_c(\lambda, N_A, N, \delta_A)$ such that if $\delta > \delta_c(\lambda, N_A, N, \delta_A)$ the process dies out for any initial distribution.

Proof . We work as in proof of Theorem 3.18 and we compare the process to an oriented percolation model (see Section 1.1.2). We suppose $d = 2$. If $d \neq 2$ the proof works in a similar way. Denote by

$$\begin{array}{ll} e_1 = (1,0), & \mathcal{N} = \{(m,n) \in \mathbb{Z}^2 : m+n \text{ is even}\} \\ B = (-4L, 4L)^2 \times [0,T], & B_{m,n} = (2mLe_1, nT) + B \\ I = [-L, L]^2, & I_m = 2mLe_1 + I. \end{array}$$

where L and T are integers to be chosen later. We say that (m,n) is wet if the process restricted to $B_{m,n}$ and starting at time nT with at least one individual in I_m is such that there is at least one individual in I_{m-1} and one individual in I_{m+1} at time $(n+1)T$. A site is dry if it is not wet. We prove that we can choose L and T such that the probability of a site (m,n) being wet can be made arbitrarily close to 1.
We suppose $\delta_A = 0$ and we start with one individual at $x = (i,j) \in I$. We fix $L > 0$ and we define $\tau_{i,j}$ to be the random variable which measures the time of the first migration from (i,j) to $(i+1,j)$.
After a time Y identical to the sum of N_A independent exponential random variables with mean 1 we get $\eta_Y(x) = N_A + 1$. Since $\delta_A = 0$, $\widetilde{A} = \{N_A, N_A+1, \dots, N\}$ is an absorbing set for $(\eta_t(x))_{t\geq 0}$ for each $x \in \mathbb{Z}^d$.
Moreover the process $(\eta_t(x))_{t\geq 0}$ on $\widetilde{A}$ is stochastically smaller than a birth and death process $(\xi_t(x))_{t\geq 0}$ on $\widetilde{A}$ with birth rate $\xi_t(x)$ and death rate $\delta\xi_t(x)$ when $N_A < \xi_t(x) < N$; when $\xi_t(x) = N_A$ after an exponential time with mean 1 the process comes back to N_A+1 and when $\xi_t(x) = N$ the process comes back to $N-1$ after an exponential time with mean $\delta + 2d\lambda M(0)$. One can check Conditions (2.3.15) and (2.3.16) to prove that $\eta_t \leq \xi_t$, where $(\xi_t)_{t\geq 0}$ is defined by independent copies of $(\xi_t(x))_{t\geq 0}$ for each $x \in \mathbb{Z}^d$.
By Lemma $A.1$, $\xi_t(x)$ starting at N_A+1 reaches N before N_A with probability $\pi_1(\delta) = (1-\delta)/(1-\delta^{(N-N_A)})$. If this is not the case, ξ_t reaches N_A before N, after an exponential time with mean 1 it comes back to N_A+1 and it tries to reach N again. When the process reaches N, by (3.2.3), with probability $\pi_2(\delta) = \lambda/((2d-1)\lambda+\delta)$ a migration to $(i+1,j)$ occurs. Let $T_A(j)$ be the time of the first meeting of N_A or N corresponding to the j^{th} visit to N_A+1, which is a.s. finite by Lemma $A.2$.
Let $K(t)$ be the number of visits to N_A before t. Since the chain is recurrent for each $\epsilon_1 > 0$ and $M > 0$ there exists t large such that $\mathbb{P}(K(t) \geq M) \geq 1-\epsilon_1$. Let G be a

geometric random variable with parameters $\pi(\delta) = \pi_1(\delta)\pi_2(\delta)$, which is the probability that ξ_t reaches N then migrates to $(i+1,j)$ before hitting N_A. Then after the sum of G random variables independent identically distributed with law $T_A(j)$ a migration from (i,j) to $(i+1,j)$ occurs. For each $\tilde{\epsilon}_0 > 0$ there exists W and M large such that

$$\begin{aligned}\mathbb{P}(\tau_{i,j} < W) \geq & \mathbb{P}(\tau_{i,j} < W | K(W) \geq M)\mathbb{P}(K(W) \geq M) \\ \geq & \mathbb{P}(G < M)(1-\epsilon_1) \geq 1 - \tilde{\epsilon}_0\end{aligned}$$

by taking δ large.
We choose a preferential direction $(i,j),(i+1,j),\ldots,(L,j)$ (and the symmetric ones from (i,j) to $(-L,j)$), we repeat the same calculations for each (k,j) where $i \leq k \leq L$ to prove that for each $\epsilon_0 > 0$ there exists $\tilde{\epsilon}_0$, W, M and $T = 4LW$ such that $\eta_T(x) \geq N_A$ for each $x \in [-2L, 2L]$. Therefore

$$\mathbb{P}((0,0) \text{ is wet} | \delta_A = 0) \geq 1 - \epsilon_0.$$

• Now we suppose $\delta_A > 0$. We start from one individual at $x = (i,j) \in I$, by taking δ_A small, with large probability $\eta_t(x)$ reaches $N_A + 1$ before the first death. As in case $\delta_A = 0$, $\eta_t \leq \xi_t$. Each time ξ_t hits N_A, with probability $1/(1+\delta_A)$ there is a b irth before a death and ξ_t comes back to $N_A + 1$. Thus for each $\tilde{\epsilon}$ there exists W, M large and δ_A small such that

$$\begin{aligned}\mathbb{P}(\tau_{i,j} < W) \geq & \mathbb{P}(\tau_{i,j} < W | K(W) \geq M)\mathbb{P}(K(W) \geq M)\Big(\frac{1}{1+\delta_A}\Big)^M \\ \geq & \mathbb{P}(G < M)(1-\epsilon_1)(1-\epsilon_2) \geq 1 - \tilde{\epsilon}.\end{aligned}$$

We can work as in case $\delta_A = 0$, choose a preferential direction from (i,j) to (L,j) (and symmetrically to $(-L,j)$) and prove that for each $\epsilon > 0$ there exists $\tilde{\epsilon}$, M, W, $T = 4LW$ and δ_A such that get

$$\mathbb{P}((0,0) \text{ is wet}) \geq 1 - \epsilon.$$

Therefore for each L and $\epsilon > 0$ we can take T and δ_c^A large such that for each $(m,n) \in \mathcal{N}$ the event $G_{m,n} = \{(m,n) \text{ is wet}\}$ is dependent only on the configuration on $\mathcal{B}_{m,n}$ and $\mathbb{P}(G_{m,n}) > 1 - \epsilon$. By comparison arguments with oriented percolation (see Section 1.1.2) we get the result.

ii) We work as in proof of Theorem 3.20.
The idea is that even if δ_A is small, there exists δ large so that the probability that the population size reaches N and then one individual migrates is small. We suppose $d = 2$ and we define the space-time regions

$$\mathcal{A} = [-2L, 2L]^2 \times [0, 2T]; \qquad \mathcal{B} = [-L, L]^2 \times [T, 2T].$$

We call $\mathcal{C}$ the boundary of $\mathcal{A}$, that is

$$\mathcal{C} = \{(m,n,t) \in \mathcal{A} : |m| = 2L \text{ or } |n| = 2L \text{ or } t = 0\}.$$

We construct a percolation process on $\mathcal{N} = \mathbb{Z}^2 \times \mathbb{Z}_+$. A site $(k,m,n) \in \mathcal{N}$ is wet if the process $(\eta_t)_{t\geq 0}$ restricted to $\mathcal{A} + (k,m,n)$ has no individuals in $\mathcal{B} + (k,m,n)$ regardless of the configuration on the boundary $\mathcal{C} + (k,m,n)$. In order to have one individual in $\mathcal{B} + (k,m,n)$, it must exist a path of individuals starting from $\mathcal{C}$. A site is dry if it is not wet.

The process $(\eta_t)_{t\geq 0}$ is stochastically smaller than a birth and death process $(\overline{\eta}_t)_{t\geq 0}$ on $\{0, 1, \dots N\}$ with rates given by (3.2.10).
One can prove that there exists a time W such that with large probability there are at most $N_A + 1$ individuals per site on $[-2L, 2L]^2$ at time W by repeating the same steps we used to prove (3.2.16): the difference is that we replace 1 with $N_A + 1$.
Therefore with large probability $\overline{\eta}_W(x) \leq N_A + 1$ for each $x \in [-2L, 2L]^2$.
Now we prove that there exists $T = \overline{W} + W$ such that $\eta_T(x) = 0$ for each $x \in [-L, L]^2$.
Let $\bar{\theta}_x$ (resp. θ_x) be the first time such that $\overline{\eta}_{\bar{\theta}_x}(x) = 0$ (resp. $\eta_{\theta_x}(x) = 0$) if $\overline{\eta}_0(x) = N_A$ (resp. $\eta_0(x) = N_A$). Then $\theta_x \leq \bar{\theta}_x$.
Let $x \in [-L, L]^2$ such that $\overline{\eta}_W(x) = N_A+1$; there is a death before a birth with probability smaller than $\gamma = (1+2d\lambda M(0))/(1+2d\lambda M(0)+\delta)$, which is as small as we want by taking δ large. By Lemma $A.1$, $\overline{\eta}_t(x)$ starting at N_A reaches $N_A + 1$ before 0 with probability

$$1 - \pi_1(\delta_A) = 1 - \frac{1 - \big(\delta_A/(1 + 2d\lambda M(0))\big)^{N_A}}{1 - \big(\delta_A/(1 + 2d\lambda M(0))\big)^{N_A+1}}.$$

If this is not the case, $\overline{\eta}_t(x)$ reaches 0 before $N_A + 1$, with probability $\pi_1(\delta_A)$. When the process reaches $N_A + 1$, with probability γ there is a death before a birth. Let $T_A(j)$ be the first hitting time to 0 or $N_A + 1$ corresponding to the j^{th} visit to N_A, which is a.s. finite by Lemma $A.2$.
As in Step i), let $K(t)$ be the number of visits to $N_A + 1$ before t. Since the chain is recurrent for each $\epsilon_1 > 0$ and $M > 0$ there exists t large enough for $\mathbb{P}(K(t) \geq M) \geq 1 - \epsilon_1$.
Let G be a geometric random variable with parameter $\pi_1(\delta_A)$, which is the probability that $\overline{\eta}_t(x)$ reaches 0 before hitting $N_A + 1$. Then if there is a death before a birth for each visit to $N_A + 1$, after the sum of G random variables independently identically distributed with law $T_A(j)$ the process hits 0. For each $\tilde{\epsilon} > 0$ there exists ϵ_1, ϵ_γ), W and M large enough for

$$\begin{aligned}\mathbb{P}(\bar{\theta}_x < W) \geq& \mathbb{P}(\bar{\theta}_x < W | K(W) \geq M)\mathbb{P}(K(W) \geq M)\gamma^M \\ \geq& \mathbb{P}(G < M)(1 - \epsilon_1)(1 - \epsilon_\gamma) \geq 1 - \tilde{\epsilon}\end{aligned}$$

by taking δ large. Therefore for each $\epsilon > 0$ there exists $\tilde{\epsilon}$, K, M, W, $\overline{W}$ and δ large such that

$$\mathbb{P}(\bar{\theta}_x < W \text{ for each } x \in [-(3/2)L, (3/2)L]^2) \geq \Big(1 - \tilde{\epsilon}\Big)^{(3L)^2} \geq 1 - \epsilon \tag{3.2.21}$$

which implies that $\mathbb{P}(\theta_x < W \text{ for each } x \in [-(3/2)L, (3/2)L]^2) \geq 1 - \epsilon$.
Fixed $\overline{W}$, W and $\epsilon > 0$, the number of migrations from a site $y \notin [-(3/2)L, (3/2)L]^2$ to a site $x \in [-(3/2)L, (3/2)L]^2$ after W and before $W + \overline{W}$ is smaller than $\widetilde{M}$ with large probability. Hence $\eta_t(x)$ may reach N a large but finite number of times M with large probability. Therefore we can choose δ large such that M times a death occurs before the migration to a neighbor site with large probability each time the process reaches N. We can repeat the same calculations that we used to prove (3.2.19) to get that with large probability there are no migrations to a site $x \in [-L, L]^2$ after W and before $W + \overline{W}$. By (3.2.21), for each L, $\epsilon > 0$ there exists $T = W + \overline{W}$ and δ large such that

$$\mathbb{P}((0,0) \text{ is wet}) \geq 1 - \epsilon. \tag{3.2.22}$$

The comparison with a locally dependent percolation gives the result as in proof of Theorem 3.20. □

3.2.3 Model III: Mass migration as Allee effect solution

We propose *mass migration* as a biological solution to the Allee effect. We show that, at least in theory, *migrations of large flocks of individuals improve the probability of survival for any possible Allee effect.*

We define positive parameters δ_A, δ, N_A, N such that $0 \leq N_A \leq N$, $0 \leq \delta \leq 1 \leq \delta_A$. The infinitesimal generator of the process is (3.2.1) and the transition rates are given by

$$\begin{aligned} B(l) :=& l \mathbb{1}_{\{l \leq N-1\}} \\ D(l) :=& l\big(\delta_A \mathbb{1}_{\{l \leq N_A\}} + \delta \mathbb{1}_{\{N_A < l\}}\big) \\ J(l,m,k) :=& \begin{cases} \lambda & \text{if } l-k \geq N-M \text{ and } m+k \leq N, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{3.2.23}$$

In other words we take birth and death rates as in Model II and a local population $\eta(x) = l$ does not migrate if $l < N - M$. See Section 1.1.5 *(Model III: mass migration)* for more about the biological motivations.

We prove that if the population size N is large enough, for any death rate $\delta_A > 1$, there exists M such that if we allow a migration of M individuals the process converges to a non-trivial invariant measure.

First of all we prove that the process is attractive.

Proposition 3.22 *The process* $(\eta_t)_{t\geq 0}$ *defined by generator (*3.2.1*) with rates* (3.2.23) *is attractive.*

Proof . We check sufficient conditions of attractiveness given by Theorem 2.15. Referring to notations in Section 2.2, we suppose $p(x,y) = 1/(2d)$, thus birth and death rates are given by

$$\Pi^{0,k}_{\alpha,\beta} = \begin{cases} \beta & \text{if } k=1 \text{ and } \beta < N \\ 0 & \text{otherwise;} \end{cases} \qquad \Pi^{0,-k}_{\alpha,\beta} = \begin{cases} \beta\delta_A & \text{if } k=1 \text{ and } \beta \leq N_A \\ \beta\delta & \text{if } k=1 \text{ and } \beta > N_A \\ 0 & \text{otherwise.} \end{cases}$$

The migration rates are

$$\Gamma^{k}_{\alpha,\beta} = \begin{cases} 2d\lambda & \text{if } \alpha - k \geq N-M \text{ and } \beta + k \leq N \\ 0 & \text{otherwise.} \end{cases}$$

We evaluate the terms in Condition (2.2.13). The birth rates give

$$\begin{aligned} \sum_{k>\delta-\beta+l_1} \Pi^{0,k}_{\alpha,\beta} =& \mathbb{1}_{\{1>\delta-\beta+l_1\}} \Pi^{0,1}_{\alpha,\beta} = \beta \mathbb{1}_{\{\beta=\delta, l_1=0\}} \\ \sum_{l>l_1} \Pi^{0,l}_{\gamma,\delta} =& \mathbb{1}_{\{1>l_1\}} \Pi^{0,1}_{\gamma,\delta} = \beta \mathbb{1}_{\{l_1=0\}} \end{aligned}$$

thus

$$\sum_{k>\delta-\beta+l_1} \Pi^{0,k}_{\alpha,\beta} \leq \sum_{l>l_1} \Pi^{0,l}_{\gamma,\delta}. \tag{3.2.24}$$

The death rates give

$$\begin{aligned} \sum_{l>\gamma-\alpha+k_1} \Pi^{-l,0}_{\gamma,\delta} =& \mathbb{1}_{\{1>\gamma-\alpha+k_1\}} \Pi^{-1,0}_{\gamma,\delta} = \gamma \mathbb{1}_{\{\gamma=\alpha, k_1=0\}} \Big(\delta_A \mathbb{1}_{\{\gamma \leq N_A\}} + \delta \mathbb{1}_{\{N_A<\gamma\}}\Big) \\ \sum_{k>k_1} \Pi^{-k,0}_{\alpha,\beta} =& \mathbb{1}_{\{1>k_1\}} \Pi^{-1,0}_{\alpha,\beta} = \alpha \mathbb{1}_{\{k_1=0\}} \Big(\delta_A \mathbb{1}_{\{\alpha \leq N_A\}} + \delta \mathbb{1}_{\{N_A<\alpha\}}\Big) \end{aligned}$$

therefore

$$\sum_{l>\gamma-\alpha+k_1} \Pi_{\gamma,\delta}^{-l,0} \leq \sum_{k>k_1} \Pi_{\alpha,\beta}^{-k,0}. \tag{3.2.25}$$

Now we consider the migration rates

$$\begin{aligned}\sum_{k\in I_a} \Gamma_{\alpha,\beta}^{k} &= \sum_{k\in I_a} 2d\lambda \mathbb{1}_{\{\alpha-k\geq N-M,\beta+k\leq N\}} = \sum_{k\in I_a} 2d\lambda \mathbb{1}_{\{k\leq(\alpha-N+M)\wedge(N-\beta)\}}\\ \sum_{l\in I_b} \Gamma_{\gamma,\delta}^{l} &= \sum_{l\in I_b} 2d\lambda \mathbb{1}_{\{\gamma-l<N-M,\delta+l\leq N\}} = \sum_{l\in I_b} 2d\lambda \mathbb{1}_{\{l\leq(\gamma-N+M)\wedge(N-\delta)\}}.\end{aligned}$$

By Definition 2.14, setting $l = k - \delta + \beta$

$$\begin{aligned}&\sum_{k\in I_a} 2d\lambda \mathbb{1}_{\{k\leq(\alpha-N+M)\wedge(N-\beta)\}}\\ &\quad=2d\lambda \left|\bigcup_{i=1}^{K}\{m_i \geq k > \delta-\beta+l_i\} \cap \{\delta-\beta \leq k \leq (\alpha-N+M)\wedge(N-\beta)\}\right|\\ &\quad=2d\lambda \left|\bigcup_{i=1}^{K}\{m_i-\delta+\beta \geq l > l_i\} \cap \{0 \leq l \leq (\alpha-N+M-\delta+\beta)\wedge(N-\delta)\}\right|\\ &\quad\leq 2d\lambda \left|\bigcup_{i=1}^{K}\{\gamma-\alpha+m_i \geq l > l_i\} \cap \{0 \leq l \leq (\gamma-N+M)\wedge(N-\delta)\}\right|\\ &\quad=\sum_{l\in I_b} 2d\lambda \mathbb{1}_{\{l\leq(\gamma-N+M)\wedge(N-\delta)\}}\end{aligned}$$

since $\delta \geq \beta$ and $\gamma \geq \alpha$. Therefore

$$\sum_{k\in I_a} \Gamma_{\alpha,\beta}^{k} \leq \sum_{l\in I_b} \Gamma_{\gamma,\delta}^{l}. \tag{3.2.26}$$

In a similar way we notice that

$$\begin{aligned}\sum_{k\in I_d} \Gamma_{\alpha,\beta}^{k} &= \sum_{k\in I_d} 2d\lambda \mathbb{1}_{\{k\leq(\alpha-N+M)\wedge(N-\beta)\}}\\ \sum_{l\in I_c} \Gamma_{\gamma,\delta}^{l} &= \sum_{l\in I_c} 2d\lambda \mathbb{1}_{\{l\leq(\gamma-N+M)\wedge(N-\delta)\}}\end{aligned}$$

then, if $k = l - \gamma + \alpha$,

$$\begin{aligned}&\sum_{l\in I_c} 2d\lambda \mathbb{1}_{\{l\leq(\gamma-N+M)\wedge(N-\delta)\}}\\ &\quad=2d\lambda \left|\bigcup_{i=1}^{K}\{m_i \geq l > \gamma-\alpha+k_i\} \cap \{\gamma-\alpha \leq l \leq (\gamma-N+M)\wedge(N-\delta)\}\right|\\ &\quad=2d\lambda \left|\bigcup_{i=1}^{K}\{m_i-\gamma+\alpha \geq k > k_i\} \cap \{0 \leq k \leq (\alpha-N+M)\wedge(N-\delta-\gamma+\alpha)\}\right|\\ &\quad\leq 2d\lambda \left|\bigcup_{i=1}^{K}\{\delta-\beta+m_i \geq k > k_i\} \cap \{0 \leq k \leq (\alpha-N+M)\wedge(N-\beta)\}\right|\\ &\quad=\sum_{k\in I_d} 2d\lambda \mathbb{1}_{\{k\leq(\alpha-N+M)\wedge(N-\beta)\}}\end{aligned}$$

since $N-\delta-\gamma+\alpha \leq N-\beta$. Therefore

$$\sum_{k\in I_d}\Gamma^k_{\alpha,\beta} \geq \sum_{l\in I_c}\Gamma^l_{\gamma,\delta}. \tag{3.2.27}$$

We get Condition (2.2.13) from (3.2.24) and (3.2.26), Condition (2.2.14) from (3.2.25) and (3.2.27) and the claim follows. $\square$

As in Models I and II there is a monotonicity property with respect to the death parameters, namely Proposition 3.1 still holds with respect to δ_A and δ.

The main result is

Theorem 3.23 *i) For all $\delta < 1$, $\lambda > 0$, $N_A \geq 0$, there exists $N_c(\delta,\lambda,N_A)$ such that for each $N > N_c(\delta,\lambda,N_A)$, there exists $M(N_A)$ so that the process starting from η_0 such that $|\eta_0| \geq 1$ has a positive probability of survival for each $\delta_A < \infty$. Moreover if $\eta_0 \in E^N$ the process converges to a non-trivial invariant measure for each $\delta_A < \infty$.*

ii) If $\delta > 1$, for all N, N_A, λ, δ_A, M the process dies out.

In order to prove Theorem 3.23 we use largely the preliminary results about gambler's ruin problem in Appendix A.2. We follow the idea of the proof of [43, Theorem 2]. On each site if we do not consider migrations the process moves as a birth and death process on a finite interval. Migration rates are different from 0 to $N-M$, where only immigrations are possible and from $N-M$ to N, where one can have either an immigration or an emigration. Hence we treat the two regions in different ways.

Proof of Theorem 3.23.
Let $A=\{y\in X : y\leq N_A\}$. Suppose "$\delta_A=\infty$". Such a notation refers to a process with the maximum possible Allee effect. In other words a birth on a site x with $\eta_t(x)\in A$ is not possible and the effect of a migration of k individuals from a site x to y such that $\eta_t(y)+k\in A$ is the death of k individuals at x.
Namely the set of the initial configurations of the process is given by

$$\Omega_\infty := \{\eta\in\Omega:\ \text{for all } x\in\mathbb{Z}^d, \eta(x)=\xi(x)\mathbb{1}_{\{\xi(x)>N_A\}} \text{ for some } \xi\in\Omega\};$$

the transitions are

$$\begin{aligned}
\eta_t(x) &\to \eta_t(x)+1 && \text{at rate } \eta_t(x)\mathbb{1}_{\{N_A<\eta_t(x)<N\}}\\
\eta_t(x) &\to \eta_t(x)-1 && \text{at rate } \delta\eta_t(x)\mathbb{1}_{\{N_A+1<\eta_t(x)\}}\\
\eta_t(x) &\to 0 && \text{at rate } \delta\eta_t(x)\mathbb{1}_{\{N_A+1=\eta_t(x)\}}
\end{aligned}$$

and if $y\sim x$,

$$\begin{aligned}
(\eta_t(x),\eta_t(y)) &\to (\eta_t(x)-k,\eta_t(y)+k) && \text{at rate } J(\eta_t(x),\eta_t(y),k)\mathbb{1}_{\{\eta_t(x)-k>N_A,\eta_t(y)+k>N_A\}}\\
(\eta_t(x),0) &\to (\eta_t(x)-k,0) && \text{at rate } J(\eta_t(x),\eta_t(y),k)\mathbb{1}_{\{\eta_t(x)-k>N_A,\eta_t(y)+k\leq N_A\}}\\
(\eta_t(x),\eta_t(y)) &\to (0,\eta_t(y)+k) && \text{at rate } J(\eta_t(x),\eta_t(y),k)\mathbb{1}_{\{\eta_t(x)-k\leq N_A,\eta_t(y)+k>N_A\}}\\
(\eta_t(x),0) &\to (0,0) && \text{at rate } J(\eta_t(x),\eta_t(y),k)\mathbb{1}_{\{\eta_t(x)-k\leq N_A,\eta_t(y)+k\leq N_A\}}.
\end{aligned}$$

Roughly speaking when $\eta_t(x)$ reaches the state N_A, $\eta_t(x)$ becomes immediately 0. The only way for the population on x to leave the state 0 is an immigration of N_A+1 individuals from a neighboring site.

We take N, M such that $N-M>N_A$. We fix $x\in\mathbb{Z}^d$ and we start from an initial configuration $\eta_0(x)=N-M$ and $\eta_0(z)=0$ for each $z\neq x$. We prove that starting from η_0, after a finite time there is a migration of the largest flock of M ($N_A<M<N-N_A$) individuals into a site $y\sim x$ which will give birth to $N-M$ individuals in the new site with large probability regardless of the configuration on $z\sim x$, $z\neq y$. By monotonicity the result will be true for each δ_A. The proof is composed by several steps.

For each $x\in\mathbb{Z}^d$ the process $(\eta_t(x))_{t\geq 0}$ is always larger than a birth and death process $(\xi_t)_{t\geq 0}$ with birth rate ξ_t and death rate given by $\delta\xi_t$ plus the maximal emigration rate of $\eta_t(x)$. The transitions are

$$\begin{aligned}
\xi_t \to & \xi_t+1 && \text{at rate } \xi_t 1\!\!1_{\{N_A<\xi_t\leq N-1\}}\\
\xi_t \to & \xi_t-1 && \text{at rate } \xi_t\big(\delta_A 1\!\!1_{\{\xi_t\leq N_A\}}+\delta 1\!\!1_{\{N_A<\xi_t\}}\big)\\
\xi_t \to & N-M && \text{at rate } 2dM\lambda 1\!\!1_{\{\xi_t>N-M\}}.
\end{aligned}\tag{3.2.28}$$

In other words we can couple $\eta_t(x)$ and ξ_t in such a way that each emigration of $0<k\leq M$ particles from $\eta_t(x)$ corresponds to a multiple death, which we call *mass death*, on ξ_t and the birth and death process comes back to the size $N-M$. The rate of the mass death is the maximal emigration rate of $\eta_t(x)$, which is $2dM\lambda$ and corresponds to the migration rate when $\eta_t(x)=N$, that is the minimum of $2dM$ exponential random variables with rate λ.
One can check that the stochastic order conditions are satisfied and $\xi_t\leq\eta_t(x)$. We show that the number of visits to N of ξ_t before reaching N_A is large.

I. First of all we prove that the number of visits to $N-M+1$ before reaching N_A is large. Since we do not allow births of more than one particle, ξ_t must visit $N-M+1$ before visiting N. Once it reaches $N-M+1$ there are three possibilities:
-At $N-M+1$ there is a birth and ξ_t reaches N and a mass death happens in N before visiting $N-M$. In particular this means that it reaches N without any mass death before the last one, since the effect of each mass death on a population of size $N-M+k$ is to come back to state $N-M$.
-At $N-M+1$ there is a birth, but ξ_t reaches $N-M$ before the largest mass death of M individuals in state N.
-At time $N-M+1$ there is a death or a mass death of just one individual.
This means that ξ_t comes back to $N-M$ after visiting $N-M+1$ with probability 1. Let T_M be the random variable which measure the time that ξ_t needs to come back to $N-M$. Since each time there is a mass death, ξ_t comes back to $N-M$, the law of T_M satisfies,

$$\mathbb{P}(T_M>t)\leq e^{-2dM\lambda t}\tag{3.2.29}$$

which is the rate of the mass death of k individuals on $N-M+k$ with the effect to come back to $N-M$.
Since after visiting $N-M+1$ the random walk comes back to $N-M$ in a time smaller than an exponential random variable, we count the number of visits to $N-M+1$ of the process $(\zeta_t)_{t\geq 0}$ on $\{N_A,N_A+1,\dots N-M+1\}$ with the same rates of ξ_t but reflecting state at $N-M+1$. This is a birth and death chain such that neither mutiple births nor mass deaths are possible, since $\xi_t\leq N-M$. Let $R^{\zeta}_{N-M+1}(N-M)$ be the random variable which counts the number of visits of $(\zeta_t)_{t\geq 0}$ to $N-M+1$ before reaching N_A starting at $N-M$. Since after each visit to $N-M+1$ the process ξ_t will come back to $N-M$, eventually by hitting some more times $N-M+1$,

$$\mathbb{P}(R^{\zeta}_{N-M+1}(N-M)\geq r)\leq\mathbb{P}(R^{\xi}_{N-M+1}(N-M)\geq r)\tag{3.2.30}$$

where $R^{\xi}_{N-M+1}(N-M)$ counts the number of visits of $(\xi_t)_{t\geq 0}$ to $N-M+1$ before reaching N_A starting at $N-M$.
We prove that by taking N large enough the number of visits to $N-M+1$ of ζ_t before reaching N_A is large with large probability. By using (3.2.30) and Lemma $A.2$ with $r_1 = N_A$, $r_2 = N-M+1$, $i = N-M$, we get

$$\lim_{N\to\infty} \mathbb{P}(R^{\xi}_{N-M+1}(N-M) \geq N^3) = 1. \tag{3.2.31}$$

II. Step I states that by taking N large ξ_t visits $N-M+1$ at least N^3 times before reaching N_A with large probability. Now we prove that the number of visits to N are at least N^2 with large probability. Each time the walker hits $N-M+1$ it tries to reach N before coming back to $N-M$, which is possible only if there are no mass deaths. We drop the explicit dependence on the initial state and on ξ when not necessary and we simply write $R_N = R^{\xi}_N(N-M+1)$ and $R_{N-M+1} = R^{\xi}_{N-M+1}(N-M)$. Let $o(1)$ be such that $\lim_{N\to\infty} o(1) = 0$. By (3.2.31)

$$\begin{aligned}\mathbb{P}(R_N < N^2) =&\mathbb{P}(R_N < N^2|R_{N-M+1} \geq N^3)\mathbb{P}(R_{N-M+1} \geq N^3)\\&+\mathbb{P}(R_N < N^2|R_{N-M+1} < N^3)\mathbb{P}(R_{N-M+1} < N^3)\\=&\mathbb{P}(R_N < N^2|R_{N-M+1} \geq N^3)\mathbb{P}(R_{N-M+1} \geq N^3) + o(1)\\=&\mathbb{P}(R_N < N^2|R_{N-M+1} \geq N^3) + o(1)\end{aligned} \tag{3.2.32}$$

hence we are left with evaluating the probability of more than N^3 visits to $N-M+1$ but less than N^2 to N. This is the probability that at least $N^3 - N^2$ times the walk starting at $N-M+1$ reaches $N-M$ before N.
Now we show that such an event has small probability. When $N-M+1 \leq \xi_t \leq N$, we can have births, deaths and mass deaths. The probability of reaching N without any mass death depends on the time the walk spends between $N-M$ and N.
We consider the process without mass deaths, that is we only take into account the rings of birth and death clocks.
By (3.2.3), the probability to have a birth before a death is $p = 1/(1+\delta)$. Let τ_N be the number of steps that a random walk on $\{N-M,\dots,N\}$ with $p = 1/(1+\delta)$ and $q = \delta/(1+\delta)$ needs to reach N starting at $N-M+1$ without visiting $N-M$ and let τ_{N-M} be the number of steps that the random walk needs to reach $N-M$ starting at $N-M+1$ without visiting N. Let τ^* be the minimum between τ_N and τ_{N-M}, that is in the gambler's language the duration of the game. Thus

$$\begin{aligned}\mathbb{P}(\tau^* > N^{1/2}) =&\mathbb{P}\Big(\{\tau_N > N^{1/2}\} \cap \{\tau_N \leq \tau_{N-M}\}\Big)\\&+\mathbb{P}\Big(\{\tau_{N-M} > N^{1/2}\} \cap \{\tau_N > \tau_{N-M}\}\Big).\end{aligned}$$

Moreover by Lemma $A.2$ with $q = 1-p$, $n = M$, $r_1 = N-M$, $r_2 = N$ and $\alpha = 1-(p^n+q^n)$, we get

$$\mathbb{P}\Big(\{\tau_N > N^{1/2}\} \cap \{\tau_N \leq \tau_{N-M}\}\Big) \leq \mathbb{P}(\tau^* > N^{1/2}) \leq \alpha^{\frac{N^{1/2}}{M}} \tag{3.2.33}$$

which converges to zero as N goes to infinity.
The probability of a mass death, if $N-M+1 \leq \xi_t \leq N$, is given by the probability that an exponential clock with rate $2d\lambda M$ rings before the minimum between two exponential clocks with rates $\delta\xi_t$ and ξ_t. By (3.2.3) this is equal to

$$\frac{2d\lambda M}{(1+\delta)\xi_t + 2d\lambda M}.$$

Denote by $A_N = \Big\{$A mass death clock rings before that the process ξ_t reaches $N\Big\}$, and remember that τ^* involves the process without mass death clocks. Then

$$\mathbb{P}(A_N|\tau^* < N^{1/2}) \leq N^{1/2}\frac{2d\lambda M}{(1+\delta)(N-M)+2d\lambda M} \leq \frac{C(d,\lambda,M,\delta)}{N^{1/2}} \tag{3.2.34}$$

which tends to zero as N goes to infinity. Therefore $\mathbb{P}(A_N|\tau^* < N^{1/2})$ is as small as we want by taking N large.
Let $\{\tau_i^*\}_{i\geq 1}$ be a family of random variables such that at time τ_i^* there is the first visit to N or $N-M$ corresponding to the i^{th} visit to $N-M+1$ without mass deaths. Then τ_i^* are independent identically distributed random variables with the same law of τ^*. We define independent identically distributed random variables

$$B_i := \begin{cases} 1 & \text{if } (\xi_t)_{t\geq 0} \text{ such that } \xi_0 = N-M+1 \text{ reaches } N \text{ before } N-M, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2.35}$$

This means that $\mathbb{P}(R_N < N^2) = \mathbb{P}\Big(\sum_{i=1}^{R_{N-M+1}} B_i < N^2\Big)$ and

$$\begin{aligned}\mathbb{P}(R_N < N^2|R_{N-M+1} \geq N^3) =& \mathbb{P}\Big(\sum_{i=1}^{R_{N-M+1}} B_i < N^2|R_{N-M+1} \geq N^3\Big) \leq \mathbb{P}\Big(\sum_{i=1}^{N^3} B_i < N^2\Big)\\ =& \mathbb{P}\Big(\Big\{\sum_{i=1}^{N^3} B_i < N^2\Big\}\bigcap\Big\{\exists i : \tau_i^* \geq N^{1/2}\Big\}\Big)\\ &+ \mathbb{P}\Big(\Big\{\sum_{i=1}^{N^3} B_i < N^2\Big\}\bigcap\Big\{\tau_i^* < N^{1/2} \text{ for all } i\Big\}\Big).\end{aligned} \tag{3.2.36}$$

By (3.2.33)

$$\begin{aligned}\mathbb{P}\Big(\Big\{\sum_{i=1}^{N^3} B_i < N^2\Big\}\bigcap\Big\{\exists i : \tau_i^* \geq N^{1/2}\Big\}\Big) &\leq \sum_{i=1}^{N^3}\mathbb{P}(\tau_i^* \geq N^{1/2})\\ &= N^3\mathbb{P}(\tau^* \geq N^{1/2}) \leq N^3\alpha^{N^{1/2}/M}\end{aligned} \tag{3.2.37}$$

which converges to zero as N goes to infinity. Hence

$$\mathbb{P}\Big(\sum_{i=1}^{N^3} B_i < N^2\Big) = \mathbb{P}\Big(\Big\{\sum_{i=1}^{N^3} B_i < N^2\Big\}\bigcap\Big\{\tau_i^* \leq N^{1/2} \text{ for all } i\Big\}\Big) + o(1). \tag{3.2.38}$$

Let $Y \sim Bin\Big(N^3, \mathbb{P}(B_i = 1|\tau_i^* \leq N^{1/2})\Big)$, by (3.2.33)

$$\begin{aligned}\mathbb{P}\Big(\sum_{i=1}^{N^3} B_i < N^2\Big) =& \mathbb{P}\Big(\sum_{i=1}^{N^3} B_i < N^2|\tau_i^* \leq N^{1/2} \text{ for all } i\Big)\mathbb{P}\Big(\tau_i^* \leq N^{1/2} \text{ for all } i\Big) + o(1)\\ =& \mathbb{P}\Big(\sum_{i=1}^{N^3} B_i < N^2|\tau_i^* \leq N^{1/2} \text{ for all } i\Big) + o(1)\\ =& \mathbb{P}(Y < N^2) + o(1)\end{aligned} \tag{3.2.39}$$

since with large probability $\tau_i^* \leq N^{1/2}$ for all i. Notice that

$$\begin{aligned}\mathbb{P}(B_i = 0|\tau_i^* \leq N^{1/2}) =&\mathbb{P}(\{B_i = 0\} \cap A_N|\tau_i^* \leq N^{1/2}) + \mathbb{P}(\{B_i = 0\} \cap A_N^c|\tau_i^* \leq N^{1/2})\\ =&\mathbb{P}(A_N|\tau_i^* \leq N^{1/2}) + \mathbb{P}(\{B_i = 0\} \cap A_N^c|\tau_i^* \leq N^{1/2}).\end{aligned} \tag{3.2.40}$$

The second sum of the right hand side, that is $\mathbb{P}(\{B_i = 0\} \cap A_N^c|\tau_i^* \leq N^{1/2})$ is the probability that the final state of a gambler's game with $r_1 = N - M$, $r_2 = N$ starting at $N - M + 1$ is $N - M$, under the condition that the duration of the game is smaller than $N^{1/2}$. By Lemma $A.1$ and (3.2.33)

$$\begin{aligned}\mathbb{P}\Big(\Big\{B_i = 0\Big\} \cap A_N^c|\tau_i^* \leq N^{1/2}\Big) \leq&\frac{\mathbb{P}(\{B_i = 0\} \cap A_N^c)}{\mathbb{P}(\tau_i^* \leq N^{1/2})} \leq \frac{1 - 1/\delta}{1 - (1/\delta)^M}\frac{1}{1 - o(1)}\\ \leq&\frac{1 - 1/\delta}{1 - (1/\delta)^M} + o(1)\end{aligned} \tag{3.2.41}$$

Therefore, by (3.2.34), (3.2.40) and (3.2.41)

$$\mathbb{P}(B_i = 0|\tau_i^* \leq N^{1/2}) \leq \frac{C(d, \lambda, M, \delta)}{N^{1/2}} + \frac{1 - 1/\delta}{1 - (1/\delta)^M} + o(1)$$

and

$$p(N) := 1 - \mathbb{P}(B_i = 0|\tau_i^* \leq N^{1/2}) \geq \frac{1 - \delta}{1 - \delta^M} - \frac{C(d, \lambda, M, \delta)}{(1 + \delta)N^{1/2}} - o(1).$$

Since the right hand side converges to $\frac{1-\delta}{1-\delta^M}$ as N goes to infinity, for N large enough there exists $\bar{p} < 1$ such that $1 - \mathbb{P}(B_i = 0|\tau_i^* \leq N^{1/2}) \geq \bar{p}$.
Let $(\overline{B}_i)_{i\in\mathbb{N}}$ be a family of independent identically distributed random variables such that $\overline{B}_i \sim Bernoulli(\bar{p})$ with mean $\bar{p}$ and variance σ^2 for each i. Notive that $\bar{p}$ and σ^2 do not depend on N. Denote by $\overline{Y} \sim Bin(N^3, \bar{p})$. Since $\bar{p} \leq p(N)$, for each N large enough,

$$\mathbb{P}(Y < N^2) \leq \mathbb{P}(\overline{Y} < N^2). \tag{3.2.42}$$

By the central limit theorem

$$\frac{\sum_{i=1}^{N^3} \overline{B}_i - N^3\bar{p}}{\sigma\sqrt{2}N^{3/2}} \to Z \sim \mathcal{N}(0, 1).$$

Therefore

$$\begin{aligned}\mathbb{P}\Big(\sum_{i=1}^{N^3} \overline{B}_i < N^2\Big) =&\mathbb{P}\Big(\frac{\sum_{i=1}^{N^3} \overline{B}_i - N^3\mu}{\sigma\sqrt{2}N^{3/2}} < \frac{N^2 - N^3\bar{p}}{\bar{p}(1 - \bar{p})\sqrt{2}N^{3/2}}\Big)\\ \leq&\mathbb{P}\Big(\frac{\sum_{i=1}^{N^3} \overline{B}_i - N^3\mu}{\sigma\sqrt{2}N^{3/2}} < \frac{CN^2(1 - N)}{N^{3/2}}\Big)\\ =&\mathbb{P}\Big(Z \leq CN^{1/2}(1 - N)\Big) + o(1)\end{aligned} \tag{3.2.43}$$

which converges to zero as N goes to infinity since C is a constant which does not depend on N.
Finally, by (3.2.32), (3.2.36), (3.2.38), (3.2.39), (3.2.42) and (3.2.43)

$$\begin{aligned}\mathbb{P}(R_N < N^2) =&\mathbb{P}(R_N < N^2|R_{N-M+1} > N^3) + o(1)\\ =&\mathbb{P}\Big(\sum_{i=1}^{N^3} B_i < N^2\Big) + o(1) = \mathbb{P}(Y < N^2) + o(1)\\ \leq&\mathbb{P}(\overline{Y} < N^2) + o(1) \leq \mathbb{P}(Z \leq CN^{1/2}(1 - N)) + o(1)\\ =&o(1).\end{aligned}$$

This implies that for each $x \in \mathbb{Z}^d$ the process $(\eta_t)_{t\geq 0}$ satisfies

$$\lim_{N\to\infty} \mathbb{P}(R_N^{\eta(x)} \geq N^2) \geq \lim_{N\to\infty} \mathbb{P}(R_N^{\xi} \geq N^2) = 1 \tag{3.2.44}$$

where $R_N^{\eta(x)}$ counts the number of visits to N of the process $(\eta_t(x))_{t\geq 0}$.
In other words for each $\epsilon > 0$ we are able to take N large enough such that with probability larger than $1-\epsilon$ we reach N at least N^2 times. We prove that in this case, with large probability there is a migration of M individuals from x to y at least $N^{1/2}$ times.

III. Each time $(\eta_t(x))_{t\geq 0}$ visits N there are three possibilities: we may have a migration of M individuals from the site x onto the site y (at rate λ), a death at x or a migration onto $z \sim x$, $z \neq y$ at rate smaller than $N\delta + \lambda M(2d-1) + \lambda(M-1)$, that is the death rate plus the migration rate on a site $z \neq y$ or a migration on y with less than M particles. Thus, by (3.2.3) the probability of a migration to y before $\eta_t(x)$ leaves the state N is larger than

$$\frac{\lambda}{(M-1)\lambda + N\delta}.$$

Moreover, at each visit to N the first exponential clock which rings is independent of what has happened at the preceeding visit. Let $E_N = E_N(x,y)$ be the number of emigrations from x onto y of M particles before $\eta_t(x)$ reaches N_A. Conditioning on $\{R_N^{\eta(x)} \geq N^2\}$, E_N is larger than a binomial random variable V_N with parameters N^2 and $\frac{\lambda}{(M-1)\lambda+N\delta}$. For all $a > 0$ real value,

$$\begin{aligned}\lim_{N\to\infty} \frac{Var(V_N)}{N^{1+2a}} &= \lim_{N\to\infty} \frac{N^2}{N^{1+2a}} \frac{\lambda}{(M-1)\lambda + N\delta} \frac{(M-2)\lambda + N\delta}{(M-1)\lambda + N\delta} \\ &\leq \lim_{N\to\infty} C \frac{N^2}{NN^{1+2a}} = 0\end{aligned}$$

where C is a constant which does not depend on N.
Therefore, as N goes to infinity,

$$\mathbb{E}\Big(\frac{V_N - \mathbb{E}(V_N)}{N^{1/2+a}}\Big)^2 = \frac{Var(V_N)}{N^{1+2a}} \to 0$$

and $(V_N - \mathbb{E}(V_N))/(N^{1/2+a})$ converges to 0 in L^2 and hence in probability. In particular,

$$\lim_{N\to\infty} \mathbb{P}\Big(V_N > \mathbb{E}(V_N) - N^{1/2+a}\Big) = 1.$$

We choose $a \in (0, 1/2)$ such that

$$\lim_{N\to\infty} \mathbb{P}(V_N > N^{1/2}) = 1.$$

We conclude that, since

$$\begin{aligned}\mathbb{P}(E_N > N^{1/2}) \geq& \mathbb{P}(E_N > N^{1/2} | R_N^{\eta(x)} \geq N^2)\mathbb{P}(R_N^{\eta(x)} \geq N^2) \\ \geq& \mathbb{P}(V_N > N^{1/2})\mathbb{P}(R_N^{\eta(x)} > N^2)\end{aligned}$$

and since each term on the right hand side converges to 1 as N goes to infinity,

$$\lim_{N\to\infty} \mathbb{P}(E_N > N^{1/2}) = 1.$$

IV. We show that given that there are at least $N^{1/2}$ emigrations at site y, at least one of these flocks of individuals generates, by internal births only, $N-M+1$ individuals so that y eventually reaches the state N. Every time there is a migration of M individuals to y, since $M(N_A) > N_A$, a birth and death chain $(\xi_t)_t \geq 0$ with transition rates:

$$\begin{aligned} \xi_t \to \xi_t + 1 \quad & \text{at rate } \xi_t \mathbb{1}_{\{N_A < \xi_t \leq N-M+1\}} \\ \xi_t \to \xi_t - 1 \quad & \text{at rate } \xi_t \delta \mathbb{1}_{\{N_A < \xi_t \leq N-M+1\}} \end{aligned}$$

starts. Take the same chain on $\{N_A, \dots, \infty\}$. Since $\delta < 1$, the chain is a transient birth and death chain (see [38, Proposition I.4.1]) and therefore there is a positive probability $q(\delta)$ that starting at $n > N_A$ the chain will never be in state N_A and will go on to infinity. Thus, the probability that the process on y reaches N is at least as large as the probability that one of the birth and death chains goes to infinity.
For x and y nearest neighbors, let E_{xy} be the event that, given that x starts in state $N-M+1$, it gives birth to at least one individual on y whose associated birth and death chain is transient. We have

$$\begin{aligned} \mathbb{P}(E_{xy}) \geq & \mathbb{P}(E_{xy} | E_N > N^{1/2}) \mathbb{P}(E_N > N^{1/2}) \\ \geq & (1 - (1 - q(\delta))^{N^{1/2}}) \mathbb{P}(E_N > N^{1/2}). \end{aligned}$$

As N goes to infinity $\mathbb{P}(E_{xy})$ converges to 1. Therefore for each $\epsilon > 0$ there exists N such that

$$\mathbb{P}(E_{xy}) \geq 1 - \epsilon. \tag{3.2.45}$$

V. Now we consider the time $T_{x,y}$ that the process $(\eta_t)_{t\geq 0}$ needs to reach on $y \sim x$ a configuration with $N-M$ individuals starting at x with $N-M$ individuals at time 0, namely

$$T_{x,y} = \inf_{t \geq 0} \{t : \eta_t(y) = N - M | \eta_0(x) = N - M\}$$

By Step I with large probability there are at least N^3 visits to $N-M+1$. After each visit, the process needs a time T_M smaller than an exponential random variable with rate $2d\lambda M$ (see (3.2.29)) to come back to $N-M$.
Let $\{T_M(i)\}_{i\in\mathbb{N}}$ be a family of independent identically distributed random variables such that $T_M(i) \stackrel{d}{=} T_M$ for all i, that is the return time to $N-M$ corresponding to the i^{th} visit to $N-M+1$. Denote by T_{N-M} the time that the process needs to reach $N-M+1$ starting at $N-M$. The law of T_{N-M} satisfies $(A.1)$ for a game with $r_1 = N_A$ and $r_2 = N-M+1$. For all $x \in \mathbb{Z}^d$, denote by $I_N = \Big\{\{R_N^{\eta(x)} \geq N^2\} \bigcap \{R_{N-M+1}^{\eta(x)} \geq N^3\}\Big\}$, then by (3.2.31) and (3.2.44)

$$\begin{aligned} \mathbb{P}(T_{x,y} > t) \leq & \mathbb{P}\Big(T_{x,y} > t | I_N\Big) \mathbb{P}(I_N) + \mathbb{P}\Big(T_{x,y} > t | (I_N)^c\Big) \mathbb{P}((I_N)^c) \\ = & \mathbb{P}\Big(T_{x,y} > t | I_N\Big) + o(1). \end{aligned}$$

We know that if $R_{N-M+1} \geq N^3$ and $R_N > N^2$, with large probability one of the visit to N on x gives births to a walk on y that reaches $N-M$. Suppose this is the last visit. We get an upper estimation of $\mathbb{P}(T_{x,y} > t)$ by summing the time that the process needs to visit the state R_{N-M+1} at least N^3 times (for which we use $(A.1)$), the time to come back to $N-M$ at least N^3 times (for which we use (3.2.29)) and the time $\widehat{T}_y$ to reach $N-M$ in the new chain on y. Such a time is smaller (since we could have some immigration) than the ending time of a game with $r_1 = N_A$, $r_2 = N - M$ starting at $N_A + 1$ with a success of the player 2. By $(A.1)$

$$\mathbb{P}(\widehat{T}_y > t) \leq \alpha^{[t/N]}$$

where $\alpha = 1 - (p^n + q^n)$, $p = \frac{1}{1+\delta}$, $q = 1 - p$.
Therefore,

$$\begin{aligned}\mathbb{P}(T_{x,y} > t) \leq & \mathbb{P}\Big(\sum_{i=1}^{N^3}(T_{N-M}(i) + T_M(i)) + \widehat{T}_y > t\Big)\\ \leq & \mathbb{P}\Big(3\sum_{i=1}^{N^3} T_{N-M}(i) > t\Big) + \mathbb{P}\Big(3\sum_{i=1}^{N^3} T_M(i) > t\Big) + \mathbb{P}\Big(3\sum_{i=1}^{N^3} \widehat{T}_y(i) > t\Big).\end{aligned}$$

Since each term on the right hand side has an hypoexponential distribution, for each N fixed, for each $\epsilon > 0$ we can take t large enough such that

$$\mathbb{P}(T_{x,y} \leq t) \geq 1 - \epsilon. \tag{3.2.46}$$

VI. Now we make a comparison with an oriented percolation model. As in [43, proof of Theorem 2] we follow [32]. Suppose $d \geq 2$. Between any two nearest neighbor sites x and y in $\mathbb{Z}^d$ we draw a directed edge from x to y. We declare the directed edge open if the event E_{xy} happens. Remember that E_{xy} happens when, given that $\eta_t(x)$ is in state $N - M$, there exists a migration from x to y of M particles which gives birth to $N - M$ individuals on y in state 0. By construction such a probability is as large as we want. This defines a locally dependent random graph since the event E_{xy} depends only on the graphical representations on $I_x = \{(x, y) : y \sim x\}$. Moreover the probability of the directed edge xy to be open is the same for all edges xy and E_{xy} and E_{zt} are independent if $x \neq z$.
Given $I \subseteq I_x$, let $p(I)$ be the probability that each edge in I is closed. We call $p(\cdot)$ the *zero function* of the locally dependent random graph, which we denote by (G, p).
We say that percolation occurs if there exists an infinite path of directed open edges in $\mathbb{Z}^d$.
We begin from one individual at $x \in \mathbb{Z}^d$. For each $\delta_A < \infty$, with positive probability $\eta_t(x)$ reaches $N - M$ before 0 and we can start our construction.
Suppose there are $N - M$ individuals at site x and there is an infinite path of open sites for the percolation model starting at x, that we call $\{(x, x_1) = e_1, (x_1, x_2) = e_2, \ldots, e_k, \ldots\}$. If e_1 is open, the species reaches N, migrates to x_1 and gives birth to $N - M$ individuals on x_1 before dieing out. Then also e_2 is open, therefore if x_1 is in state $N - M$ there is an individual which reaches x_2 and gives birth to $N - M$ individuals. But x_1 reached state $N - M$ since e_1 was open. By going on in this way we notice that the existence of an infinite path in the percolation model implies the existence of an infinite path of individuals. Now we prove that the existence of an infinite path in percolation model has positive probability if $\mathbb{P}(E_{xy})$ is large enough.
We follow [32, Theorem 3.2]) and we compare the process to a a site percolation model. Here we need $d \geq 2$, otherwise the construction does not work.
We define a new locally dependent random graph (G', q) on $\mathbb{Z}^d$. Each pair of edges (x, y) and (z, t) are independently open if $x \neq z$ with the same distribution, but a dependence is present if $x = z$. We fix $w_1 \sim 0$ and $w_2 \sim 0$ non collinear vertices and we denote by $e_1 = (0, w_1)$ and $e_2 = (0, w_2)$ the corresponding edges. Let $I \subseteq I_x$, then the zero function $q(\cdot)$ satisfies

$$\begin{aligned} q(I) =& 1 && \text{if } I \cap \{e_1, e_2\} = \emptyset \\ q(I) =& \mathbb{P}((E_{xy})^c) && \text{if } I \cap \{e_1, e_2\} \neq \emptyset. \end{aligned}$$

In other words each edge different from e_1 and e_2 is closed with probability one; if either $e_1 \in I$ or $e_2 \in I$, the probability that each edge in I is closed is $1 - \mathbb{P}(E_{xy})$.
Now we compare the zero functions of (G, p) and $(G'q)$: if either $e_1 \in I$ or $e_2 \in I$ then $p(I) = \mathbb{P}(\bigcap_{(x,y)\in I} (E_{xy})^c) \leq \mathbb{P}((E_{xy})^c) = q(I)$, otherwise $p(I) \leq 1 = q(I)$.
Therefore $p(I) \leq q(I)$ and by [32, Theorem 2.1], if percolation occurs for (G', q) then percolation occurs for (G, p).
Notice that (G', q) is essentially the oriented site percolation model on the square lattice with both edges from a site open with probability $\pi = 1 - \mathbb{P}((E_{xy})^c)$, which is as large as we want. It is known that for such a model percolation occurs if π is large enough. Let $P^{\pi}(|\mathcal{C}_0| = \infty)$ be the probability of an infinite path of open sites containing 0. If $\eta_0(y) = \mathbb{1}_{\{y=0\}}(y)$ we get

$$\liminf_{t\to\infty} \eta_0 T(t)\{\eta : \eta_t(x) > 0 \text{ for some } x \in \mathbb{Z}^d\} \geq P^{\pi}(|\mathcal{C}_0| = \infty) > 0 \tag{3.2.47}$$

and a survival is possible.
Assume $\eta_0 \in E^N$, then there exists an invariant measure $\bar{\nu}$ which is not concentrated on the Dirac measure $\delta_{\underline{0}}$ by (3.2.47) and the claim follows.

ii) The proof is similar to the one of Theorem 3.18 *ii)*, then we skip it. □

Remark 3.24 *The proof states that in order to have survival we can take $M(N_A) = N_A + 1$. If $N_A = 0$, by taking $M(N_A) = N_A + 1 = 1$, if N sufficiently large the survival is possible. Notice that if $N_A = 0$ and $M(N_A) = 1$, only a migration of one individual is possible and the process reduces to a Model I type process with $M(l) = 1$ for each $l \in X$. This proves Theorem* 3.14 *when $M(l) = 1$ for each $l \in X$. One can work in a similar way to prove that Theorem* 3.14 *holds for a general non increasing function $M(\cdot)$.*

3.2.4 Model IV: ecological equilibrium

Finally in Model IV we do not impose a maximal size on each site, but after a population size N the death rate is always larger than the birth rate. Migrations are allowed only when there are more than N individuals on a site towards sites with less than N individuals. Given δ, $\widetilde{\delta}$ and λ positive real numbers such that $\widetilde{\delta} > 1$, the generator of the process is given by (3.2.1) and the transition rates are

$$\begin{aligned} B(l) =& l \\ D(l) =& l(\delta \mathbb{1}_{\{l \leq N\}} + \widetilde{\delta} \mathbb{1}_{\{l > N\}}) \\ J(l, m, k) =& \lambda M(m) \mathbb{1}_{\{l \geq N, k=1\}} \end{aligned}$$

where $M : X \to \mathbb{R}$ is a non increasing function such that $M(l) = 0$ for each $l \geq N$. See Section 1.1.5 for a biological interpretation of the process.
As explained is Section 1.1.5 the process takes place on a set $\Omega_0 \subseteq \Omega$ since the state space is non compact, see Definition (1.1) Section 1.1.1.
One can check that sufficient conditions for the existence of the process on Ω_0 (see [13, Theorem 13.1]) are satisfied.
Since births, deaths and migrations involve only one particle and the migration rate is increasing in $\eta_t(x)$ and decreasing in $\eta_t(y)$ the process is attractive as in Model I.
As in previous models a monotonicity property (see Proposition 3.1) holds in δ and in $\widetilde{\delta}$

for each initial configuration $\zeta \in \Omega_0$.

We prove that in some cases the process survives but it does not explode, that is it does not die out but the expected value on each site is finite.

Theorem 3.25 *Suppose $|\eta_0| \geq 1$ and there exists $n < \infty$ such that $\eta_0(x) \leq n$ for each $x \in \mathbb{Z}^d$.*

i) For all $\lambda > 0$, $1 < N < \infty$ there exists $\delta_c^1(\lambda, N) > 0$ and $K < \infty$ such that if $\delta < \delta_c^1(\lambda, N)$ the process has a positive probability of survival for each $\widetilde{\delta} > 1$ and $\lim_{t\to\infty} \mathbb{E}(\eta_t(x)) \leq K$ for all $x \in \mathbb{Z}^d$.

ii) There exists $\delta_c^2 > 0$ such that if $\delta > \delta_c^2$ the process dies out for all λ, N, $\widetilde{\delta} > 1$ and any initial configuration.

iii) For all $\lambda > 0$, $1 < N < \infty$, $\widetilde{\delta}$ there exists $\delta_c(\lambda, N, \widetilde{\delta}) > 0$ such that, if $\delta < \delta_c(\lambda, N, \widetilde{\delta})$ the process survives and if $\delta > \delta_c(\lambda, N, \widetilde{\delta})$ the process dies out.

Proof. *i*) Let $\xi_t = \xi_t(\delta', \lambda')$ be a process defined by (3.2.1) with rates given by (3.2.2), that is a Model I type process. Let $\eta_t = \eta_t(\delta, \widetilde{\delta}, \lambda)$ be a Model IV type process. If $\delta = \delta'$ and $\lambda = \lambda'$ the process $(\eta_t)_{t\geq 0}$ is stochastically larger than $(\xi_t)_{t\geq 0}$ and the claim follows by Theorem 3.18.
Now we prove that the expected value is finite.
Let $(\eta_t^N)_{t\geq 0}$ be a process with N immortal particles with transition rates

$$\begin{aligned} \eta_t^N(x) \to& \eta_t^N(x) + 1 && \text{at rate } \eta_t^N(x) \\ \eta_t^N(x) \to& \eta^N(x) - 1 && \text{at rate } \widetilde{\delta}\eta_t^N(x)\mathbb{1}_{\{\eta_t^N(x) > N\}}. \end{aligned}$$

Given an initial configuration $\eta_0^N \in \Omega$ such that $N \leq \eta_0^N(x) \leq n$ for each $x \in \mathbb{Z}^d$, each process $(\eta_t^N(x))_{t\geq 0}$ is a birth and death process independent of the configurations on the other sites with state space $X^N = \{x \in \mathbb{N} : x \geq N\}$. Moreover $(\eta_t^N)_{t\geq 0}$ is stochastically larger than $(\eta_t)_{t\geq 0}$.
If we prove that $\lim_{t\to\infty} \mathbb{E}(\eta_t^N(x)) < \infty$ we are done.
Let $(\xi_t)_{t\geq 0}$ be a birth and death process on $\mathbb{N}$ with birth rate $N + \xi_t$, death rate $\widetilde{\delta}(N + \xi_t)\mathbb{1}_{\{\xi_t > 0\}}$ and $\xi_0 = \eta_0^N(x) - N$. Then $\eta_t^N(x) \stackrel{d}{=} \xi_t + N$ for each $x \in \mathbb{Z}^d$ and

$$\frac{d}{dt}\mathbb{E}(\xi_t) \leq \mathbb{E}(\xi_t + N) - \widetilde{\delta}\mathbb{E}((\xi_t + N)\mathbb{1}_{\{\xi_t > 0\}}) \leq N - (\widetilde{\delta} - 1)\mathbb{E}(\xi_t)$$

and by Gronwall's Lemma,

$$\mathbb{E}(\xi_t) \leq \mathbb{E}(\xi_0) + N/(\widetilde{\delta} - 1).$$

Therefore if $\xi_0 \leq n$ there exists $M = M(n, N, \widetilde{\delta})$ such that $\mathbb{E}(\xi_t) \leq M$ for each $t \geq 0$ and

$$\lim_{t\to\infty} \mathbb{E}(\xi_t) \leq M.$$

By taking $K = M + N$ we get $\lim_{t\to\infty} \mathbb{E}(\eta_t^N) \leq M + N = K$ and the claim follows for $(\eta_t)_{t\geq 0}$.
ii) We can take $\delta > 1$ and prove the result as in Model I, Proposition 3.18, *ii*).
iii) It follows from step *i*), *ii*) and the monotonicity with respect to δ. □

Remark 3.26 *In a similar way we can consider a Model III type process without any a priori bounds by adding a death rate* $\widetilde{\delta} > 1$ *when the number of individuals in a local population is larger than* N. *By comparison arguments, even if a strong Allee effect is present a mass migration of large flocks of individuals leads to the survival of the species through non-exploding local populations.*

Part II

Random Walks on Small World Graphs

Chapter 1

Introduction

1.1 Introduction et résultats principaux (français)

Plusieurs structures réelles peuvent être modélisées par des graphes avec des connexions aléatoires entre les sites: la connectivité d'internet, le graphe des collaborations des acteurs, des réseaux neuronaux et sociaux. Dans plusieurs réseaux réels il y a beaucoup de connexions entre les individus dans la même région; de plus, il y a aussi des connexions entre des sites éloignés, qui causent une diminution de la distance moyenne entre les individus.
On peut construire un graphe aléatoire à partir d'un graphe déterministe soit en rajoutant des connexions aléatoires, comme dans notre modèle, soit en enlevant des connexions au hasard, comme dans le modèle de percolation.
Les graphes *small world* sont des graphes aléatoires tels que chaque site peut avoir soit des connexions avec des sites dans son voisinage (aléatoires ou déterministes), appelées *connexions short-range* soit des connexions avec des éléments choisis au hasard dans le graphe, appelées *connexions long-range*.
En suivant cette idée, avec plusieurs constructions différentes, on peut obtenir un graphe aléatoire où la distance moyenne entre les sites est petite.
Bollobas et Chung [9] ont noté qu'en rajoutant une connexion aléatoire dans un cycle, on peut réduire le diamètre, qui est la distance moyenne entre les sites. Watts et Strogatz [51] ont introduit un modèle pour des applications biologiques (WS small world). Ils ont pris un anneau avec n sommets et ils ont connecté chaque paire de sommets de distance mutuelle plus petite que m avec une connexion short range: les connexions long range ont été construites à partir d'une connexion short range en remplaçant avec probabilité p un site de la connexion avec un autre site choisi au hasard uniformément dans le graphe.
Un modèle plus étudié est le graphe introduit par Newmann et Watts [37] (NW small world): ils ont pris les même connexions short range que pour le WS small world, mais ils ont rajouté une densité p de connexions long range aléatoires entre les sites.
La distance moyenne entre les sites et le coefficient de regroupement du graphe small world ont été déjà bien examinés ([1], [4], [51]). Voir [24] pour une introduction historique aux graphes small world et les résultats principaux.

Récemment des auteurs ont travaillé sur des processus sur les graphes small world. Durrett et Jung [25] ont étudié le processus de contact. Ils ont considéré une version du small world donnée à partir du tore d-dimensionnel $\Lambda^d(L) = \mathbb{Z}^d \mod 2L$ avec des connexions short range entre chaque paire de sommets de distance plus petite que m. Ils ont choisi une partition de $2L$ sommets en L paires avec probabilité uniforme: pour cette

raison il faut avoir un nombre pair de sommets. En connectant chaque paire de la partition nous obtenons un graphe small world possible.
Nous allons travailler sur le même graphe aléatoire, que nous appelons BC small world, puisque c'est une généralisation du modèle de Bollobas et Chung (voir la Section 2.1 pour plus de détails sur la construction). Puisque tous les individus ont le même nombre de connexions short et long range, le modèle est moins réaliste que le modèle de Watts et Strogatz, mais cette propriété rend le graphe homogène et donc plus facile à étudier. L'avantage principal d'une telle construction est que nous pouvons associer au graphe aléatoire un graphe non-aléatoire invariant par translation, appelé *big world* (BW) (voir [25] où le big world a été introduit). Voir la Section 2.2.1 pour plus de détails sur ce graphe et sur sa relation avec le small world.

Nous sommes intéressés par les marches aléatoires coalescentes sur les graphes small world. Grosso modo, les marches aléatoires coalescentes sur un graphe G sont des processus de Markov dans lesquelles n particules exécutent des marches aléatoires indépendantes soumises à la règle: si une particule saute sur un site déjà occupé, les deux particules s'unissent et ils deviennent une seule particule. Si G est fini et connecté, après un temps fini τ appelé *temps de coalescence* le processus qui a initialement une particule sur chaque site se réduit à seulement une particule.
La première variable aléatoire que nous devons étudier pour comprendre le comportement des marches aléatoires coalescentes est le temps de rencontre de deux particules.

Le temps de rencontre de marches aléatoires a été étudié sur plusieurs graphes finis et infinis. En ce qui concerne la marche aléatoire simple symétrique en temps continu $(X_t)_{t\geq 0}$ sur le tore d-dimensionnel $\Lambda^d(L)$ qui part de $x \in \Lambda^d(L)$, Cox [16, Théorème 4] si $d \geq 2$ a prouvé le résultat suivant sur le temps dont une seule particule a besoin pour retourner à l'origine

Théorème 1.1 *[16, Théorèm 4]*
Supposons $d \geq 2$, $a_L \to \infty$ et $a_L = o(L)$ pour $L \to \infty$. Si $d = 2$ on suppose en plus que $a_L\sqrt{\log L}/L \to \infty$. Alors uniformément en $t \geq 0$ et $x_L \in \Lambda^d(L)$ avec $|x_L| \geq a_L$,

$$\mathbb{P}^{x_L}(W_L/s_L > t) \to \exp(-t/G) \tag{1.1}$$

où

$$G = \begin{cases} 2/\pi & d = 2, \\ \displaystyle\sum_{n=0}^{\infty} p^{(n)}(0,0) & d \geq 3. \end{cases} \qquad s_L = \begin{cases} L^2 \log L & d = 2, \\ L^d & d \geq 3. \end{cases}$$

On peut obtenir le même résultat pour la marche aléatoire de distribution initiale uniforme (cela a été prouvé dans [26, Théorème 6.1] en temps discret) comme corollaire.
Durrett (voir [17, Théorème 2]) a démontré un résultat dans le cas 2-dimensionnel sous des conditions plus générales sur le point de départ et sur la matrice de transition pour la marche aléatoire en temps discret. À savoir, la matrice de transition est $p(x, y) = (1 - \nu)\mathbb{1}_{\{x=y\}} + \nu q(y - x)$, $(x, y) \in \Lambda^2(L) \times \Lambda^2(L)$, $0 < \nu < 1$ où $q(z)$ est une distribution de probabilité sur $\mathbb{Z}^2$ avec domaine fini, symétrique, irréducible telle que $q((0, 0)) = 0$ et $q(z) = q(-z)$.

Théorème 1.2 *[17, Théorème 2]*
Supposons que $d = 2$ et x_L satisfait $\lim_{L\to\infty}(\log^+ |x_L|)/\log L = \delta \in [0,1]$. Alors, pour tout $t \geq 0$, pour $L \to \infty$, uniformément en $\nu \in (0,1]$,

$$\mathbb{P}^{x_L}\Big(\frac{W_L}{L^2 \log L} > t\Big) \to \delta \exp(-\pi\nu\sigma^2 t) \tag{1.2}$$

où $\sigma^2 = \sum_{(z_1,z_2)\in\mathbb{Z}^2} z_1^2 q(z)$.

Le cas $d = 1$ est légèrement différent; le résultat suivant a été démontré par Flatto, Odlyzko et Wales [26, Théorèm 6.1] pour le temps de retour W_L de la marche aléatoire en temps discret de distribution initiale uniforme π sur $\Lambda^1(L)$,

Théorème 1.3 *[26, Théorèm 6.1]*
Soit $d = 1$. Pour tout $t \geq 0$,

$$\mathbb{P}^{\pi}(W_L/L^2 > t) \to (2/\pi^2)\sum_{n=0}^{\infty}\frac{\exp\Big(-2\pi^2(n+1/2)^2 t\Big)}{(n+1/2)^2} \tag{1.3}$$

Nous pouvons obtenir les Théorèmes 1.2 et 1.3 en temps continu en travaillant comme dans la preuve du Lemma 2.38 (voir la Section 2.4.1).
Notons que, par l'invariance par translation de la marche aléatoire considérée sur le tore, on peut montrer que le temps de rencontre T_L de deux marches aléatoires indépendantes X_t et Y_t sur le tore, telles que $X_0 = x$ et $Y_0 = y$, coïncide avec la loi (par rapport à une marche qui part de $x - y$) de $2W_L$. Donc les Théorèmes 1.1, 1.2 et 1.3 donnent aussi les comportements asymptotiques du temps de rencontre de deux particules.

Notons que le BC small world est un tore d-dimensionnel avec des raccourcis: donc nous nous attendons à ce que l'"effet small world" apparaisse comme un temps de rencontre plus petit que sur le tore.
Pour expliquer l'idée plus clairement, soit S^L une des réalisations possibles du graphe small world $\mathcal{S}^L$ et soit $\mathcal{S}^L(\widetilde{\Omega})$ l'ensemble de toutes les réalisations possibles de $\mathcal{S}^L$. Nous introduisons une matrice de transition P_{S^L} qui dépend de S^L. On note $\mathbb{P}^{\mu,\nu}_{S^L}$ la loi des deux marches aléatoires indépendantes X_t et Y_t gouvernées par P_{S^L} sur S^L de probabilités initiales μ et ν sur $\Lambda^d(L)$ (voir la Section 2.1). Puisque les marches sont définies sur un graphe aléatoire, tout d'abord nous devons fixer une réalisation possible du graphe et ensuite regarder le processus sur cette réalisation. Nous pouvons chercher des résultats sur chaque graphe S dans un ensemble de grande probabilité (point de vue "*quenched*"): notons qu'il n'est pas possible d'obtenir des résultats pour tout S) ou des résultats en moyenne sur toutes les réalisations S du graphe (point de vue "*annealed*"). Nous étudions les marches aléatoires en temps soit discret soit continu. Dans la Section 2.1 nous donnons les définitions formelles.
Durrett [24, Chapitre 6] a étudié les marches aléatoires coalescentes sur une version 1-dimensionnelle du BC small world. Il a prouvé, pour une grande classe de graphes aléatoires avec N sommets, que pour chaque suite $\{S^N\}_N$ dans un ensemble de grande probabilité le temps de rencontre T de deux particules de distribution initiale uniforme converge vers la loi exponentielle, à savoir il existe $C > 0$ (qui dépend de la structure locale du graphe) tel que

$$\mathbb{P}^{\pi,\pi}_{S^N}\Big(\frac{T}{CN} > t\Big) \overset{N\to\infty}{\longrightarrow} e^{-t}. \tag{1.4}$$

En particulier (1.4) est vérifiée pour le BC small world 1-dimensionnel avec $N = 2L$.
Nous utilisons la technique de la transformée de Laplace (comme dans [16]) pour prouver des résultats plus précis sous des conditions initiales plus générales sur le temps de rencontre T_L. De tels résultats seront utiles pour avoir une meilleure comparaison avec le temps de rencontre de deux particules sur le tore et pour obtenir des résultats sur les marches aléatoires coalescentes.
Un instrument fondamental consiste à construire une application aléatoire du graphe big world sur le graphe aléatoire small world. Comme nous l'expliquons dans la Section 2.2.1, à travers cette application nous pouvons associer à chaque site $x \in \Lambda^d(L)$ un site particulier $+(x)$ dans le big world. De plus nous associons àchaque marche aléatoire sur le small world, une marche aléatoire sur le big world et nous notons sa loi $\mathbb{P}_{BW}$.
Le résultat suivant est valable en temps soit discret soit continu et concerne le temps de rencontre de deux particules qui partent de 0 et x_L. Nous donnons la loi limite de $T_L/(2L)^d$ pour L qui tend vers l'infini pour un point de départ $x_L \in \Lambda^d(L)$ tel que $|x_L|$ tend vers l'infini et pour $x_L = x$ qui ne dépend pas de L.

Théorème 1.4 *1. (Limite annealed, $x_L \to x$). Soit $x_L \in \Lambda(L)$ pour tous L tels que $x_L = x$ pour tout L suffisamment grand. Alors si $x \neq 0$, uniformément en $t \geq 0$*

$$\mathbb{P}^{x_L,0}\left(\frac{T_L}{(2L)^d} > t\right) \stackrel{L\to\infty}{\to} \left(1 - \frac{G(+(x),0)}{G(0,0)}\right) \exp\left(-\frac{t}{G(0,0)}\right). \tag{1.5}$$

Si $x = 0$ alors uniformément en $t \geq 0$

$$\mathbb{P}^{0,0}\left(\frac{T_L}{(2L)^d} > t\right) \stackrel{L\to\infty}{\to} \frac{1}{G(0,0)} \exp\left(-\frac{t}{G(0,0)}\right). \tag{1.6}$$

2. (Limite annealed, $x_L \to \infty$). Si $\alpha_L \to \infty$, alors uniformément en $t \geq 0$ et x_L tels que $|x_L| \geq \alpha_L$,

$$\mathbb{P}^{x_L,0}\left(\frac{T_L}{(2L)^d} > t\right) \to \exp\left(-\frac{t}{G(0,0)}\right). \tag{1.7}$$

3. (Limite quenched, $x_L \to x$). Soit $x_L \in \Lambda(L)$ pour tous L tels quet $x_L = x \neq 0$ pour tout L suffisamment grand. Soit $t \geq 0$: pour tout $\varepsilon > 0$

$$\mathbf{P}\left(S \in \mathcal{S}^L(\widetilde{\Omega}) : \left|\mathbb{P}_S^{x_L,0}\left(\frac{T_L}{(2L)^d} > t\right) - \left(1 - \frac{G(+(x),0)}{G(0,0)}\right) \exp\left(-\frac{t}{G(0,0)}\right)\right| > \varepsilon\right) \stackrel{L\to\infty}{\to} 0. \tag{1.8}$$

Si $x = 0$ alors

$$\mathbf{P}\left(S \in \mathcal{S}^L(\widetilde{\Omega}) : \left|\mathbb{P}_S^{0,0}\left(\frac{T_L}{(2L)^d} > t\right) - \frac{1}{G(0,0)} \exp\left(-\frac{t}{G(0,0)}\right)\right| > \varepsilon\right) \stackrel{L\to\infty}{\to} 0. \tag{1.9}$$

4. (Limite quenched, $x_L \to \infty$). On choisit $\alpha_L > (\log\log L)^2$. Soit $t \geq 0$: pour tout $\varepsilon > 0$

$$\mathbf{P}\left(S \in \mathcal{S}^L(\widetilde{\Omega}) : \sup_{\{x_L : d_S(0,x_L) \geq \alpha_L\}} \left|\mathbb{P}_S^{x_L,0}\left(\frac{T_L}{(2L)^d} > t\right) - \exp\left(-\frac{t}{G(0,0)}\right)\right| > \varepsilon\right) \stackrel{L\to\infty}{\to} 0, \tag{1.10}$$

où

$$G(+(x),0) = \begin{cases} \displaystyle\sum_{n=0}^{\infty} \mathbb{P}_{BW}^{+(x)}(X_{2n}=0) & \textit{si la marche aléatoire sur le small world} \\ & \textit{se déplace en temps discret,} \\ \displaystyle\frac{1}{2}\sum_{n=0}^{\infty} \mathbb{P}_{BW}^{+(x)}(X_{n}=0) & \textit{si la marche aléatoire sur le small world} \\ & \textit{se déplace en temps continu,} \end{cases}$$

où avec 0 *on denote* $+(0)$ *et* $d_S(0,x)$ *est la longueur de la trajectoire la plus petite qui joint* x *à* 0 *sur le graphe small world* S.

On peut utiliser des techniques similaires pour obtenir des résultats sur le temps de retour à l'origine d'une seule particule (voir la Section 2.4.1, Théorèmes 2.37 et 2.37). Comme corollaire nous obtenons la loi du temps de rencontre de deux marches aléatoires et la loi du temps de retour à l'origine d'une seule marche de distribution initiale uniforme (voir la Section 2.4.1, Corollaire 2.39). Notons que si nous travaillons avec une chaîne de Markov réversible sur un graphe invariant par translation nous pouvons facilement obtenir des résultats sur le temps de rencontre de deux marches aléatoires à partir du temps de retour d'une seule marche, mais ce n'est pas le cas. La clé dans les preuves est que l'on peut construire une application aléatoire entre le big world et le small world qui est une bijection sur la structure locale du graphe small world.

Le Théorème 1.4 établit que pour avoir la convergence du temps de rencontre renormalisé sur le small world, nous devons renormaliser par un facteur $(2L)^d$. En comparant avec les résultats sur le tore, si $d \leq 2$ l'effet small world est évident (la convergence a un taux plus rapide); si $d \geq 3$ l'effet small world n'est pas si évident, puisque la convergence a le même taux et nous devons avoir plus d'informations sur la constante. L'effet de raccourcis est plus grand si la dimension est petite et moins évident si d est grand.
Soit $G_{\mathbb{Z}^d}(0,0)$ l'espérance du nombre de retours à l'origine d'une marche aléatoire simple symétrique en temps discret sur $\mathbb{Z}^d$ qui part de $0 \in \mathbb{Z}^d$. Le tableau suivant donne une comparaison entre la marche aléatoire simple symétrique en temps continu sur le tore et celle sur le BC small world:

Point de départ	d	Facteur de renormalization R: $R(Tore)$	$R(SW)$	Constante C: $C(Tore)$	$C(SW)$
Distribution uniforme	1	L^2	L	$1/2$	$G_{BW}(0,0)$
$(\log^+ \lvert x_L\rvert)/\log L \to \delta$ ou					
$\lvert x_L\rvert \geq \alpha_L$, $\alpha_L \frac{\sqrt{\log L}}{L} \to \infty$	2	$L^2 \log L$	L^2	$\pi^2/6$	$G_{BW}(0,0)$
$\lvert x_L\rvert \geq \alpha_L$, $\alpha_L \to \infty$	≥ 3	L^d	L^d	$G_{\mathbb{Z}^d}(0,0)$	$G_{BW}(0,0)$

Notons que si $d = 1$ alors $W_L/(C(Tore)R(Tore))$ et $W_L/C(SW)R(SW))$ convergent vers des lois différentes (voir les Théorèmes 1.3 et 1.2): une comparaison est possible puisque les facteurs de renormalization sont différents.
Si $d \geq 3$ la lois limite et les facteurs de renormalisation sont identiques, donc nous devons comparer les constantes $G_{\mathbb{Z}^d}(0,0)$ et $G_{BW}(0,0)$. L'ordre relatif dépend de la probabilité β de la marche aléatoire de prendre une connexion long range. Nous avons le résultat suivant pour β petit ou grand.

Proposition 1.5 *Supposons $d \geq 3$.*
i) Il existe $\beta_1 > 0$ tel que $G_{\mathbb{Z}^d}(0,0) < G_{BW}(0,0)$ pour tout $\beta \in [\beta_1, 1]$.
ii) Il existe $\beta_2 > 0$ tel que $G_{\mathbb{Z}^d}(0,0) > G_{BW}(0,0)$ pour tout $\beta \in (0, \beta_2]$.

En d'autres termes si β est petit nous avons toujours l'effet small world. Si β est grand le temps de rencontre sur le small world est plus grand que sur le tore. De plus, la preuve de la Proposition 1.5 (voir la Section 2.4.2) établit que $G_{BW}(0,0) \overset{\beta \to 0}{\to} G_{\mathbb{Z}^d}(0,0)$ et $G_{BW}(0,0) \overset{\beta \to 1}{\to} \infty$. Donc la fonction $G_{BW}(0,0)$ n'est pas une fonction monotone en β.

Dans [24, Chapitre 6], l'auteur esquisse une preuve que le nombre de particules d'un processus de n marches aléatoires coalescentes (c'est-à-dire avec n particules au temps 0) renormalisées qui partent de la distribution stationnaire sur le BC small world 1-dimensionnel avec connexions entre plus proches voisins converge vers un processus coalescent de Kingman (voir la Section 3). Brièvement, le coalescent de Kingman est un processus de Markov qui a initialement N individus sans structure spatiale: pour chaque couple il y a une horloge exponentielle avec taux 1 et si celle-ci sonne les deux particules s'unissent et elles deviennent une.
Nous utilisons le Théorème 1.4 pour obtenir de nouvelles informations sur le nombre de particules $(|\xi_t(A)|)_{t\geq 0}$ des marches aléatoires coalescentes $(\xi_t(A))_{t\geq 0}$ qui partent de $A = \{x_1, \ldots, x_n\}$, $x_i \in \Lambda^d(L)$ pour $1 \leq i \leq n$ en temps continu, généralisant le résultat précédent aux graphes small world d-dimensionnels avec probabilités de transition et distance initiale entre les particules plus générales. Le résultat est

Théorème 1.6 ***(Théorème 3.6 Section 3)***
Soit $h_L \geq (\log\log L)^2$ tel que $\lim_{L\to\infty} M^{h_L}/(2L)^d = 0$, alors pour tout $A = \{x_1, \ldots, x_n\} \subset \Lambda^d(L)$ tel que $|x_i - x_j| \geq h_L$ pour $i \neq j$ il existe une suite d'ensembles $\{H_L\}_L$ de graphes small world telle que $\mathbf{P}(H_L) \overset{L\to\infty}{\to} 1$ et pour toutes les suites $\{S^L\}_L$, $S^L \in H_L$, uniformément en $t \geq 0$

$$\left|\mathbb{P}^A_{S^L}\left(|\xi_{s_L t}(A)| < k\right) - P_n\Big(D_t < k\Big)\right| \overset{L\to\infty}{\to} 0. \tag{1.11}$$

où $s_L = (2L)^d G_{BW}(0,0)/2$, M est une constante qui dépend du nombre de connexions short range par site et D_t est le nombre de particules du processus coalescent de Kingman à l'instant $t \geq 0$.

Nous avons travaillé sur des graphes avec une seule connexion long range par site. On peut montrer de façon similaire les mêmes résultats pour des graphes aléatoires avec un nombre $K > 1$ (qui ne dépend pas de L) connexions long range. On conjecture que la limite exponentielle aura un paramètre différent qui serait le temps de retour sur un graphe big world avec une structure différente.

Le Théorème 1.6 concerne le temps de coalescence de n (qui ne dépend pas de L) particules. Un développement ultérieur consiste à travailler avec un processus de marches aléatoires coalescentes sur small world qui part avec une particule par site. La loi du temps de coalescence pour un tel processus donne des informations sur le *temps de consensus du modèle du votant*, qui est la variable aléatoire qui mesure le temps dont le modèle du votant sur small world a besoin pour atteindre une configuration de tous 1's ou tous 0's. Voir [33] pour plus de détails sur le modèle du votant et la relation de dualité entre le modèle du votant et les marches aléatoire coalescentes.

La deuxième partie est organisée comme suit: dans la Section 2.1 nous donnons les définitions nécessaires dans la suite et nous construisons le graphe aléatoire. Dans la Section 2.2 nous expliquons les techniques utilisées. Pour obtenir des résultats sur le temps de rencontre de deux particules, nous utilisons la technique de la transformée de Laplace et nous traitons la loi de la marche aléatoire de façon différente pour les temps petits et pour les temps grands. Pour les temps petits, la structure du graphe aléatoire resemble au big world. Dans la Section 2.2.1 nous présentons l'application du big world sur le small world: la Proposition 2.10 établit qu'ils ne diffèrent pas dans une boule de rayon $\log\log L$ si L est sufisamment grand. Pour les temps grands la loi de la marche aléatoire est proche de la distribution stationnaire. Dans la Section 2.2.3 nous rappelons des estimations bien connues qui donnent des informations sur la vitesse de convergence à l'équilibre d'une marche aléatoire, concernant la constante isopérimetrique.
Une estimation utile de la constante isopérimetrique pour un ensemble de graphes small world avec grande probabilité est donné par le Théorème 2.13, Section 2.2.2.
Dans la Section 2.3 nous utilisons la comparaison avec le big world pour les temps petits et la convergence vers l'équilibre pour les temps grands pour démontrer les lemmes principaux concernant la transfomée de Laplace du temps de rencontre de deux particules. Nous donnons les résultats en temps continu et en temps discret dans des paragraphes différents de cette section.
Le résultat principal sur le temps de rencontre de deux particules est prouvé dans la Section 2.4.1. Ici nous donnons aussi un résultat similaire sur le temps de retour d'une seule marche aléatoire et un corollaire concernant des marches aléatoires qui partent de la distribution uniforme. Dans la Section 2.4.2 nous prouvons la Proposition 1.5 qui nous permet de comparer nos résultats avec ceux sur le temps de rencontre sur le tore d-dimensionnel avec $d \geq 3$. Dans la Section 3 nous introduisons le processus des marches aléatoires coalescentes et nous prouvons le théorème de convergence vers le coalescent de Kingman.

1.1 Introduction and main results

Many real world structures can be modelized by graphs with random connections between sites: the connectivity of Internet, the collaboration graph of film actors, some neural and social networks. In many real networks there are many connections between individuals in the same area; moreover, there are also some connections between distant sites, which leads to a small mean distance between individuals.
One can construct a random graph starting from a deterministic graph either by adding random connections, as in our model, or by removing some connections randomly, as in percolation.
Small world graphs are random graphs such that each site can have both connections with another site in the neighbourhood (random or deterministic), called *short range connections*, and connections with some elements chosen at random from the graph, called *long range connection*.
By following this idea, with several different constructions, we can get a random graph with small mean distance between sites.
Bollobas and Chung [9] first noted that by adding a random matching in a cycle, one can reduce the diameter, defined as the average distance between sites. Watts and Strogatz [51] introduce a model with biological applications (WS small world). They start from a ring of n vertices and connect each pair of vertices with distance less than m with a short range connection: the long range connections are constructed by taking the short range connections and by moving with probability p one of the end sites at random to a new one chosen uniformly from the graph.
A more studied model is the graph introduced by Newmann and Watts [37] (NW small world): they take the same deterministic short range connection of the WS small world, but they add a density p of long range connections between random sites.
Average distance between sites and clustering coefficient of small world graphs have been well investigated ([1],[4], [51]). See [24] for a historical introduction of small world graphs and main results.

Recently some authors have been focusing on processes taking place on small world graphs.
Durrett and Jung [25] have studied the contact process. Their version of the small world is given by the d-dimensional torus $\Lambda^d(L) = \mathbb{Z}^d \mod 2L$ with short range connections between each pair of vertices with distance less than m. They take with uniform probability a partition of the $2L$ vertices in L pairs: for this reason the number of vertices is required to be even. Connecting each pair of the partition we get a possible small world. We are going to work on the same random graph, which we call BC small world, since it is a generalization of Bollobas and Chung model (see Section 2.1 for more details about the construction). Since all individuals have the same number of short and long range connections, the model is less realistic than the Watts and Strogatz version, but this property makes the graph homogeneous and then easier to study. The main advantage of such a costruction is that we can associate to the random graph a non-random translation invariant graph, called *big world* (BW)(see [25] where the big world was first introduced). See Section 2.2.1 for more details on this deterministic graph and on its relationship with the small world.

We are interested in the coalescing random walk on the small world. Roughly speaking, the coalescing random walk on a graph G is a Markov process in which n particles

perform independent random walks subject to the rule that when one particle jumps onto an already occupied site, the two particles coalesce to one. If G is finite and connected, after a finite time τ called *coalescing time* the process starting from one particle on each site coalesces to only one particle.
The first random variable we need to investigate in order to understand the behaviour of the coalescing random walk is the meeting time of two particles.

Meeting time of random walks has been studied on several finite and infinite graphs. For the simple symmetric continuous time random walk $(X_t)_{t\geq 0}$ on the d-dimensional torus $\Lambda^d(L)$ starting from $x \in \Lambda^d(L)$, Cox [16, Theorem 4] for $d \geq 2$ proved the following result on the time a single particle needs to hit the origin $W_L = \inf\{s > 0 : X_s = 0\}$.

Theorem 1.1 *[16, Theorem 4]*
If $d \geq 2$, $a_L \to \infty$ and $a_L = o(L)$ as $L \to \infty$. For $d = 2$ suppose in addition that $a_L\sqrt{\log L}/L \to \infty$. Then uniformly in $t \geq 0$ and $x_L \in \Lambda^d(L)$ with $|x_L| \geq a_L$,

$$\mathbb{P}^{x_L}(W_L/s_L > t) \to \exp(-t/G) \tag{1.1}$$

where

$$G = \begin{cases} 2/\pi & d = 2, \\ \displaystyle\sum_{n=0}^{\infty} p^{(n)}(0,0) & d \geq 3. \end{cases} \qquad\qquad s_L = \begin{cases} L^2 \log L & d = 2, \\ L^d & d \geq 3. \end{cases}$$

One can also get the same result for the random walk starting from the stationary distribution (this was proved in [26, Theorem 6.1] in discrete time) as a corollary.
Durrett (see [17, Theorem 2]) proved a result in the 2 dimensional case under more general conditions on the starting point and on the transition matrix for a discrete time random walk. Namely, the transition matrix is $p(x,y) = (1-\nu)1\!\!1_{\{x=y\}} + \nu q(y-x)$, $(x,y) \in \Lambda^2(L) \times \Lambda^2(L)$, where $q(z)$ is a symmetric finite range irreducibile probability distribution on $\mathbb{Z}^2$ such that $q((0,0)) = 0$ and $q(z) = q(-z)$.

Theorem 1.2 *[17, Theorem 2]*
Suppose that $d = 2$ and x_L satisfies $\lim_{L\to\infty}(\log^+ |x_L|)/\log L = \delta \in [0,1]$. Then, for any $t \geq 0$, as $L \to \infty$ uniformly for $\nu \in (0,1]$,

$$\mathbb{P}^{x_L}\Big(\frac{W_L}{L^2 \log L} > t\Big) \to \delta \exp(-\pi\nu\sigma^2 t) \tag{1.2}$$

where $\sigma^2 = \sum_{(z_1,z_2)\in\mathbb{Z}^2} z_1^2 q(z)$.

The case $d = 1$ is slightly different; the following result has been proved by Flatto, Odlyzko and Wales [26, Theorem 6.1] for the hitting time W_L of the discrete time random walk starting from the uniform distribution π on $\Lambda^1(L)$,

Theorem 1.3 *[26, Theorem 6.1]*
Let $d = 1$. For each $t \geq 0$,

$$\mathbb{P}^{\pi}(W_L/L^2 > t) \to (2/\pi^2)\sum_{n=0}^{\infty} \frac{\exp\Big(-2\pi^2(n+1/2)^2 t\Big)}{(n+1/2)^2} \tag{1.3}$$

We can get Theorems 1.2 and 1.3 in continuous time working as in proof of Lemma 2.38 (see Section 2.4.1).
Note that, by the translation invariance of the considered random walks on the torus, it is easy to show that the meeting time T_L of two independent random walks X_t and Y_t on the torus, with respect to $X_0 = x$ and $Y_0 = y$, coincides with the law (with respect to a starting site $x - y$) of $2W_L$. Therefore Theorems 1.1, 1.2 and 1.3 give also the asymptotic behaviors of the meeting time of two particles.

Notice that the BC small world is a d-dimensional torus with shortcuts: therefore we expect that the small world effect appears as a smaller meeting time with respect to the one on the torus.
To explain the idea more clearly, let S^L be one of the possible realizations of the small world graph $\mathcal{S}^L$ and let $\mathcal{S}^L(\widetilde{\Omega})$ be the set of all possible realizations of $\mathcal{S}^L$. We introduce a transition matrix P_{S^L} depending on S^L. We denote by $\mathbb{P}^{\mu,\nu}_{S^L}$ the law of the two independent random walks X_t and Y_t ruled by P_{S^L} on S^L starting from the probability distributions μ and ν on $\Lambda^d(L)$ (see Section 2.1). Since the walkers are on a random graph, first of all we have to fix a possible graph and then move the process on it. We can look for results for each graph S in a set of large probability ("*quenched*" point of view: note that it is not possible to obtain results for all S) or average results on all realizations S of the graph ("*annealed*" point of view). We study random walks both in discrete and continuous time. In Section 2.1 we give the formal definitions.
Durrett [24, Chapter 6] studied the coalescing random walk on a one dimensional version of BC small world. He proved, for a large class of random graphs with N vertices, that for each sequence of $\{S^N\}_N$ in a set of large probability the rescaled meeting time T of two particles starting from the stationary distribution converges to the exponential distribution, namely there exists $C > 0$ (depending on the local structure of the graph) such that

$$\mathbb{P}^{\pi,\pi}_{S^N}\Big(\frac{T}{CN} > t\Big) \overset{N\to\infty}{\longrightarrow} e^{-t}. \tag{1.4}$$

In particular (1.4) holds for the 1 dimensional BC small world with $N = 2L$.
We use the Laplace transform technique (as in [16]) in order to prove more accurate results under more general initial conditions on the meeting time T_L. Such results will be useful both for a better comparison with the meeting time of two particles on torus and for getting results on coalescing random walk.
A fundamental tool consists in constructing a random map from the deterministic big world graph onto the small world random graph. As we explain in Section 2.2.1, through this map we can associate to each site $x \in \Lambda^d(L)$ a particular site $+(x)$ in the big world. Moreover we associate to the random walk on the small world, a random walk on the big world, and we denote its law by $\mathbb{P}_{BW}$.
The following result holds both in discrete and continuous time and involves the meeting time of two particles starting from 0 and x_L. We give the limit law of $T_L/(2L)^d$ as L goes to infinity both for starting points $x_L \in \Lambda^d(L)$ such that $|x_L|$ goes to infinity and for $x_L = x$ which does not depend on L.

Theorem 1.4 *1. (Annealed limit, $x_L \to x$). Let $x_L \in \Lambda(L)$ for all L such that $x_L = x$ for all L sufficiently large. Then if $x \neq 0$, uniformly in $t \geq 0$*

$$\mathbb{P}^{x_L,0}\left(\frac{T_L}{(2L)^d} > t\right) \overset{L\to\infty}{\to} \left(1 - \frac{G(+(x),0)}{G(0,0)}\right) \exp\left(-\frac{t}{G(0,0)}\right). \tag{1.5}$$

If $x = 0$ then uniformly in $t \geq 0$

$$\mathbb{P}^{0,0}\left(\frac{T_L}{(2L)^d} > t\right) \stackrel{L\to\infty}{\to} \frac{1}{G(0,0)} \exp\left(-\frac{t}{G(0,0)}\right). \tag{1.6}$$

2. *(Annealed limit, $x_L \to \infty$). If $\alpha_L \to \infty$, then uniformly in $t \geq 0$ and x_L such that $|x_L| \geq \alpha_L$,*

$$\mathbb{P}^{x_L,0}\left(\frac{T_L}{(2L)^d} > t\right) \to \exp\left(-\frac{t}{G(0,0)}\right). \tag{1.7}$$

3. *(Quenched limit, $x_L \to x$). Let $x_L \in \Lambda(L)$ for all L such that $x_L = x \neq 0$ for all L sufficiently large. Let $t \geq 0$: for all $\varepsilon > 0$*

$$\mathbf{P}\left(S \in \mathcal{S}^L(\widetilde{\Omega}) : \left|\mathbb{P}_S^{x_L,0}\left(\frac{T_L}{(2L)^d} > t\right) - \left(1 - \frac{G(+(x),0)}{G(0,0)}\right) \exp\left(-\frac{t}{G(0,0)}\right)\right| > \varepsilon\right) \stackrel{L\to\infty}{\to} 0. \tag{1.8}$$

If $x = 0$ then

$$\mathbf{P}\left(S \in \mathcal{S}^L(\widetilde{\Omega}) : \left|\mathbb{P}_S^{0,0}\left(\frac{T_L}{(2L)^d} > t\right) - \frac{1}{G(0,0)} \exp\left(-\frac{t}{G(0,0)}\right)\right| > \varepsilon\right) \stackrel{L\to\infty}{\to} 0. \tag{1.9}$$

4. *(Quenched limit, $x_L \to \infty$). Choose $\alpha_L > (\log\log L)^2$. Let $t \geq 0$: for all $\varepsilon > 0$*

$$\mathbf{P}\left(S \in \mathcal{S}^L(\widetilde{\Omega}) : \sup_{\{x_L : d_S(0,x_L) \geq \alpha_L\}} \left|\mathbb{P}_S^{x_L,0}\left(\frac{T_L}{(2L)^d} > t\right) - \exp\left(-\frac{t}{G(0,0)}\right)\right| > \varepsilon\right) \stackrel{L\to\infty}{\to} 0, \tag{1.10}$$

where

$$G(+(x),0) = \begin{cases} \sum_{n=0}^{\infty} \mathbb{P}_{BW}^{+(x)}(X_{2n} = 0) & \text{if the random walk on the small world moves in discrete time,} \\ \frac{1}{2}\sum_{n=0}^{\infty} \mathbb{P}_{BW}^{+(x)}(X_n = 0) & \text{if the random walk on the small world moves in continuous time,} \end{cases}$$

by 0 we denote $+(0)$ and $d_S(0,x)$ is the length of the shortest path connecting x to 0 in the small world S.

Using similar techniques we get results about the return time of a single particle to the origin (see Section 2.4.1, Theorems 2.37 and 2.37). As a corollary we get the law of the meeting time of two random walks and the law of the hitting time to the origin of a single walker starting from the uniform distribution (see Section 2.4.1, Corollary 2.39).
Note that if we work with a reversible Markov chain on a translation invariant graph we can easily get results on meeting time of two random walks from the results on the hitting time of a single random walk, but this is not the case. The key in the proofs is

that in most sites the local structure can be mapped through a bijection into the big world.

Theorem 1.4 states that in order to have convergence of the rescaled meeting time in the small world, we need to rescale with $(2L)^d$. Comparing with the results on the torus, if $d \leq 2$ the small world effect is clear (convergence has a faster rate); if $d \geq 3$ the small world effect is not so evident, since the convergence has the same rate and we need to know more about the rescaling constant. As we can expect, the effect of shortcuts is larger for lower dimension and less evident when d is large.
Let $G_{\mathbb{Z}^d}(0,0)$ be the expected number of returns to the origin of a discrete time simple symmetric random walk on $\mathbb{Z}^d$ starting from $0 \in \mathbb{Z}^d$. The following table gives a comparison between the continuous time simple symmetric random walk on the torus and the one on the BC small world:

Starting points	d	Rescaling Factor R		Constant C	
		$R(Torus)$	$R(SW)$	$C(Torus)$	$C(SW)$
Uniform distribution	1	L^2	L	$1/2$	$G_{BW}(0,0)$
$(\log^+ \lvert x_L \rvert)/\log L \to \delta$ or					
$\lvert x_L \rvert \geq \alpha_L$, $\alpha_L \frac{\sqrt{\log L}}{L} \to \infty$	2	$L^2 \log L$	L^2	$\pi^2/6$	$G_{BW}(0,0)$
$\lvert x_L \rvert \geq \alpha_L$, $\alpha_L \to \infty$	≥ 3	L^d	L^d	$G_{\mathbb{Z}^d}(0,0)$	$G_{BW}(0,0)$

Comparison Table

Note that if $d = 1$ then $W_L/(C(Torus)R(Torus))$ and $W_L/C(SW)R(SW))$ converge to different laws (see Theorems 1.3 and 1.2): a comparison is possible since the rescaling factors are different.
If $d \geq 3$ the limit laws and the rescaling factors are identical in both cases, thus we need to compare the constants $G_{\mathbb{Z}^d}(0,0)$ and $G_{BW}(0,0)$. The relative order depends on the probability β to move across a long range connection. We have the following result for β either small or large.

Proposition 1.5 *Suppose $d \geq 3$.*
i) There exists $\beta_1 > 0$ such that $G_{\mathbb{Z}^d}(0,0) < G_{BW}(0,0)$ for each $\beta \in [\beta_1, 1]$.
ii) There exists $\beta_2 > 0$ such that $G_{\mathbb{Z}^d}(0,0) > G_{BW}(0,0)$ for each $\beta \in (0, \beta_2]$.

In other words if β is small we still have the small world effect. If β is large the meeting time on the small world is larger than on the torus. Moreover the proof of Proposition 1.5 (see Section 2.4.2) states that $G_{BW}(0,0) \overset{\beta \to 0}{\to} G_{\mathbb{Z}^d}(0,0)$ and $G_{BW}(0,0) \overset{\beta \to 1}{\to} \infty$. Therefore the function $G_{BW}(0,0)$ is not monotone in β.

In [24, Chapter 6], the author sketches a proof that the number of particles of a normalized n-coalescing random walk (that is with n particles at time 0) starting from the stationary distribution on one dimensional BC nearest neighbors small world converges to

the Kingman's coalescent (see Section 3). Briefly, the Kingman's coalescent is a Markov process starting from N individuals without spatial structure: each couple has an exponential clock with mean 1 after which the two particles coalesce.
We use Theorem 1.4 to get new information about the number of particles $(|\xi_t(A)|)_{t\geq 0}$ of the coalescing random walk $(\xi_t(A))_{t\geq 0}$ starting from $A = \{x_1, \dots, x_n\}$, $x_i \in \Lambda^d(L)$ for $1 \leq i \leq n$ in continuous time, extending the previous result to d-dimensional BC small world with general transition probabilities and more general initial distance between particles. The result is

Theorem 1.6 ***(Theorem 3.6 Section 3)***
Let $h_L \geq (\log\log L)^2$ such that $\lim_{L\to\infty} M^{h_L}/(2L)^d = 0$, then for each $A = \{x_1, \dots, x_n\} \subset \Lambda^d(L)$ with $|x_i - x_j| \geq h_L$ for $i \neq j$ there exists a sequence of sets $\{H_L\}_L$ of small world graphs such that $\mathbf{P}(H_L) \stackrel{L\to\infty}{\to} 1$ and for each sequence $\{S^L\}_L$, $S^L \in H_L$, uniformly in $t \geq 0$

$$\left|\mathbb{P}^A_{S^L}\left(|\xi_{s_L t}(A)| < k\right) - P_n\Big(D_t < k\Big)\right| \stackrel{L\to\infty}{\to} 0. \tag{1.11}$$

where $s_L = (2L)^d G_{BW}(0,0)/2$, M is a constant depending on the number of short range connections per site and D_t is the number of particles in a Kingman's coalescent at time $t \geq 0$.

We work on graphs with a single long range connection per site. One can show the same results for random graphs with fixed $K > 1$ (not depending on L) random long range connections in the same way. The exponential limit will have a different parameter which we guess would be the return time on a different big world structure.

Theorem 1.6 involves the coalescing time of n (which does not depend on L) particles. A further development consists in working with a coalescing random walk on small world which starts with one particle per site. The law of the coalescing time for such a process gives some information on the *voter model consensus time*, that is the random variable which measures the time that the voter model on small world needs to reach a configuration of all 1's or all 0's. See [33] for more details on the voter model and the duality between the voter model and the coalescing random walk.

We proceed as follows: in Section 2.1 we give the definitions needed in the sequel and we construct the random graph. In Section 2.2 we explain the techniques used. In order to get results on the meeting time of two particles, we largely use the Laplace transform technique and we treat the law of the walkers in different ways for small or large time. For small time, the random graph structure is similar to the big world. In Section 2.2.1 we introduce the map from the big world graph onto the small world: Proposition 2.10 states that they do not differ when L is large in a ball with radius $\log\log L$. When time is large the law of the random walk is close to the stationary distribution. In Section 2.2.3 we remind some well known estimations for the speed of convergence to equilibrium of a random walk, involving the isoperimetric constant. A useful estimation of the isoperimetric constant for a set with large probability of small world graph is given by Theorem 2.13, Section 2.2.2.
In Section 2.3 we use the comparison with the big world for small time and the convergence to equilibrium for large time to prove the main lemmas involving the Laplace transform of the meeting time of two particles. We give results in continuous time and in discrete time in different paragraphs of this section.
The main result on the meeting time of two particles is proved in Section 2.4.1. Here

we also give a similar result for the hitting time of a single random walk and a corollary involving random walks starting from the uniform distribution. In Section 2.4.2 we prove Proposition 1.5 which allows us to compare our results with the ones on the meeting time on the d-dimensional torus with $d \geq 3$. In Section 3 we introduce the coalescing random walk and we prove the convergence theorem to Kingman's coalescent.

Chapter 2

Meeting time of random walks

2.1 BC Small world

The vertices of the random graph are the ones of the d-dimensional torus, which we denote by

$$\Lambda(L) = \Lambda(L,d) = (\mathbb{Z} \mod 2L)^d,$$

when there is no ambiguity, we will omit the dependence on d.

The set of edges $\mathcal{E}^L$ of the graph is partly deterministic (short range connections) and partly random (long range connections). Note that we consider nonoriented edges, that is, if $(x,y) \in \mathcal{E}^L$ then also $(y,x) \in \mathcal{E}^L$ (thus we identify edges with subsets of order two).

We will consider two kinds of short range connections, one between neighbours (i.e. vertices x, y such that $\|x-y\|_1 = 1$), and the other between vertices x, y such that $\|x-y\|_\infty \leq m$: the corresponding neighbourhoods are

$$\mathcal{N}(x) = \{y \in \Lambda(L) : \|x-y\|_1 = 1\}, \qquad x \in \Lambda(L),$$

$$\mathcal{N}_m^\infty(x) = \{y \in \Lambda(L) : \|x-y\|_\infty \leq m\}, \quad x \in \Lambda(L).$$

For all $x, y \in \Lambda(L)$ we denote by $d(x,y)$ the graph distance between x and y. Let $\widetilde{\Omega}$ be the set of all partitions of the set of $\Lambda(L)$ into $(2L)^d/2$ subsets of cardinality two. Let $\mathbf{P}$ be the uniform probability on $\mathcal{P}(\widetilde{\Omega})$: the random choice of $\widetilde{\omega} \in \widetilde{\Omega}$ represents the choice of the set of long range connections (some of which may coincide with short range ones). Note that both $\widetilde{\Omega}$ and $\mathbf{P}$ depend on L.

Definition 2.1 *Let $\mathcal{G}_L$ be the family of all graphs with set of vertices $\Lambda(L)$. The small world $\mathcal{S}^L$ is a random variable $\mathcal{S}^L(\widetilde{\omega}) : \widetilde{\Omega} \to \mathcal{G}_L$ such that $\mathcal{S}^L(\widetilde{\omega}) = \big(\Lambda(L), \mathcal{E}^L(\widetilde{\omega})\big)$, where*

$$\mathcal{E}^L(\widetilde{\omega}) = \widetilde{\omega} \cup \{\{x,y\} : x \in \Lambda(L), y \in \mathcal{N}(x)\}.$$

The set of edges of the small world $\mathcal{S}^L(\widetilde{\Omega})_m$ is defined as

$$\mathcal{E}_m^L(\widetilde{\omega}) = \widetilde{\omega} \cup \{\{x,y\} : x \in \Lambda(L), y \in \mathcal{N}_m^\infty(x)\}.$$

We denote by $\mathcal{S}^L(\widetilde{\Omega}) = \{\mathcal{S}^L(\tilde{\omega}) : \tilde{\omega} \in \widetilde{\Omega}\}$ and by $\mathcal{S}_m^L(\widetilde{\Omega}) = \{\mathcal{S}_m^L(\tilde{\omega}) : \tilde{\omega} \in \widetilde{\Omega}\}$.
For any fixed $\widetilde{\omega}$, given two short range neighbours x and y, we write $x \sim y$; if they are long range neighbours we write $x \frown y$ (it may happen that $x \sim y$ and $x \frown y$ at the same time).

Note that $\mathbf{P}$ clearly defines a probability measure on $\mathcal{G}_L$: with a slight abuse of notation we denote this measure with $\mathbf{P}$ as well. Given $\widetilde{\omega}$, we will also call "small world" the graph $\mathcal{S}^L(\widetilde{\omega})$. For the sake of simplicity we will focus here on the case $\mathcal{S}^L$, but our proofs can be adapted to $\mathcal{S}^L_m$. Moreover, when there is no ambiguity, we will write $\mathcal{S}$ and $\mathcal{S}_m$ instead of $\mathcal{S}^L$ and $\mathcal{S}^L_m$.

Remark 2.2 *We note that the small world could be defined imposing that we consider as probability space $\Theta \subset \widetilde{\Omega}$, the family of partitions where no couple is a short range connection (thus the random graph has fixed degree), instead of $\widetilde{\Omega}$.*

Given a small world, we consider a random walk on it. We assume that the discrete time random walk is assigned through a family of adapted (i.e. transition from x to y may occur only if they are connected by an edge), symmetric transition matrices $\{P_S = (p_S(x,y))_{x,y\in\Lambda(L)}\}_{S\in\mathcal{S}^L(\widetilde{\Omega})}$, with the property that $p_S(0,y) = p_S(x,x+y)$ whenever y and $x+y$ are short range neighbours of 0 and x respectively (which implies that the probability from a site towards its long range neighbour is fixed as well), with the assumption that the transition probabilities towards a short range neighbour which is also a long range neighbour is the sum of the two corresponding probabilities.

The transition matrix P_S we will consider is given by

$$p_S(x,y) = \begin{cases} 1-2dp-\beta & \text{if } x=y, \\ p & \text{if } x\sim y, \text{ and } x \not\frown y \\ \beta & \text{if } x\frown y, \text{ and } x \not\sim y \\ p+\beta & \text{if } x\sim y, \text{ and } x\frown y \\ 0 & \text{otherwise,} \end{cases} \tag{2.1.1}$$

where $p \in (0, 1/2d)$ and $\beta \in (0, 1-2dp]$ (on $\mathcal{S}_m$ substitute $|\mathcal{N}^\infty_m(0)|$ for $2d$). Nevertheless our results hold also for transition matrices with a different distribution among short range neighbours (we only need symmetry and translation invariance).

Definition 2.3 *Given a probability measure μ on $\Lambda(L)$, a small world $S = \mathcal{S}(\widetilde{\omega})$ and a transition matrix P_S, we denote by $\mathbb{P}^\mu_S$ the law of the discrete time random walk on S with initial probability μ and transitions ruled by P_S. If $\mu = \delta_{x_0}$ we write $\mathbb{P}^{x_0}_S$.*

Definition 2.4 *Given a probability measure μ on $\Lambda(L)$, and a family of transition matrices $\{P_S\}_{S\in\mathcal{S}^L(\widetilde{\Omega})}$, we denote by $\mathbb{P}^\mu$ the product of $\mathbf{P}$ and $\mathbb{P}^\mu_S$, that is*

$$\mathbb{P}^\mu(A, \mathcal{C}(x_0,\ldots,x_n)) = \sum_{\mathcal{S}\in A} \mathbf{P}(\mathcal{S})\mu(x_0)p_{\mathcal{S}}(x_0,x_1)\cdots p_{\mathcal{S}}(x_{n-1},x_n),$$

where $A \subset \mathcal{G}_L$ and $\mathcal{C}(x_0,\ldots,x_n)$ is the cylinder with base $(x_0,\ldots,x_n)$.

We construct the continuous time version $\widetilde{X}_t$ of the random walk X_t by continuization. In other words we define $\widetilde{X}_t :\overset{d}{=} X_{N_t}$ where N_t has Poisson distribution with mean t independent of X_t

Definition 2.5 *Given a probability measure μ on $\Lambda(L)$ and a discrete time random walk X_t with law $\mathbb{P}^\mu_S$ on S the law of the continuous time random walk $\widetilde{X}_t$ on S is given by*

$$\mathbb{P}^\mu_S(\widetilde{X}_t = y) = \sum_{k=0}^{\infty} \frac{e^{-t}t^k}{k!}\mathbb{P}^\mu_S(X_k = y) \tag{2.1.2}$$

We define a family of translations on $\widetilde{\Omega}$.

Definition 2.6 *Let $h \in \Lambda(L)$. The map $T_h : \widetilde{\Omega} \to \widetilde{\Omega}$ is such that*

$$\{v_1, v_2\} \in T_h(\widetilde{\omega}) \quad \Longleftrightarrow \exists \{s_1, s_2\} \in \widetilde{\omega}, v_i = s_i + h, i = 1, 2.$$

The sum is in $(\mathbb{Z}^d \mod 2L)^d$. With a slight abuse of notation we denote by $T_h(\mathcal{S})$ the random graph $T_h(\mathcal{S})(\widetilde{\omega}) := \mathcal{S}(T_h(\widetilde{\omega}))$.

From now on the family of transition matrices $\{P_S\}_{S \in \mathcal{S}^L(\widetilde{\Omega})}$ is considered fixed. It is not difficult to prove the following proposition: the first four assertions follow from the symmetry of P_S; the last two from the fact that it depends only on the “type of relation” between the two sites (short range and/or long range neighbours). The continuous time claims follow from the discrete ones and Definition 2.5.

Proposition 2.7 *Let X_t and Y_t be two discrete time random walks on $\mathcal{S}$ and let π be the uniform probability on $\Lambda(L)$. Then the following hold:*

1. $\mathbb{P}^x(X_t = y) = \mathbb{P}^y(X_t = x)$;
2. $\mathbb{P}^{x,y}(X_t = Y_t) = \mathbb{P}^x(X_{2t} = y)$;
3. *π is the stationary probability measure;*
4. $\mathbb{P}^{\pi,\pi}(X_t = Y_t) = \frac{1}{(2L)^d}$;
5. $\mathbb{P}^y(X_t = x) = \mathbb{P}^0(X_t = y - x)$;
6. $\mathbb{P}^{x,y}(X_t = Y_t) = \mathbb{P}^{0,y-x}(X_t = Y_t)$.

The same statements hold for two continuous time walks $\widetilde{X}_t$ and $\widetilde{Y}_t$.

2.2 Techniques

2.2.1 The mapping into the big world

The small worlds $\mathcal{S}^L$ and $\mathcal{S}^L_m$ (which are random graphs) can be mapped into a non-random graph, the *big worlds* $\mathcal{B}^L$ and $\mathcal{B}^L_m$ respectively, as in [25]. We recall here its construction. The sites are all vectors $\pm(z_1, \ldots, z_n)$, with $n \geq 1$ components, $z_j \in \mathbb{Z}^d$ and $z_j \neq 0$ for $j < n$. The edges in $\mathcal{B}^L$ are drawn between $+(z_1, \ldots, z_n)$ and $+(z_1, \ldots, z_n + y)$ if and only if $y \in \mathcal{N}(0)$; for $\mathcal{B}^L_m$ we consider $y \in \mathcal{N}^\infty_m(0)$ (these edges correspond to the short range connections). The same is done for vectors with a minus sign.

Moreover $+(z_1, \ldots, z_n)$ has a long range neighbour, namely

$$\begin{array}{ll} +(z_1, \ldots, z_n, 0) & \text{if } z_n \neq 0, \\ +(z_1, \ldots, z_{n-1}) & \text{if } z_n = 0, \\ -(0) & \text{if } z_n = 0, n = 1. \end{array}$$

Analogously one defines the long range neighbour of $-(z_1, \ldots, z_n)$. Note that the big world is a vertex transitive graph (i.e. the automorphism group acts transitively). We denote by $|x|$ the graph distance on the big world from x to $+(0)$ and we also write 0 instead of $+(0)$. To each small world we associate a map onto it, from the big world.

Definition 2.8 *Given a small world $\mathcal{S}$ and $x \in \Lambda(L)$, let $LR_{\mathcal{S}}(x)$ be the long range neighbour of x. The map $\phi : \widetilde{\Omega} \to \Lambda(L)^{\mathcal{B}^L}$ is recursively defined as follows:*

$$\begin{aligned}
\phi(\widetilde{\omega})(+(z)) &= z \mod (2L),\\
\phi(\widetilde{\omega})(-(z)) &= LR_{\mathcal{S}(\widetilde{\omega})}(0) + z \mod (2L),\\
\phi(\widetilde{\omega})(\pm(z_1,\dots,z_n)) &= LR_{\mathcal{S}(\widetilde{\omega})}\left(\phi(\widetilde{\omega})(\pm(z_1,\dots,z_{n-1}))\right) + z_n \mod (2L).
\end{aligned}$$

Note that the transition matrix $P = (p(x,y))_{x,y\in\mathcal{B}^L}$, defined by

$$p(x,y) = \begin{cases} 1-2dp-\beta & \text{if } x = y,\\ p & \text{if } x \text{ and } y \text{ are short range neighbours}\\ \beta & \text{if } x \text{ and } y \text{ are long range neighbours}\\ p+\beta & \text{if } x \text{ and } y \text{ are short and long range neighbours}\\ 0 & \text{otherwise,}\end{cases}$$

naturally induces the transition matrix in (2.1.1) on the small world S. Analogously one can proceed on $\mathcal{B}^L_m$ if the neighbouhood relation used in $\mathbb{Z}^d$ is the one given by $\mathcal{N}^\infty_m$. The random walk $(\mathcal{B}^L, P)$ is symmetric and translation invariant; moreover the discrete version is aperiodic if $\beta \in (0, 1-2dp)$, with period 2 if $\beta = 1-2dp$ and it is not difficult to prove that for all $x \in \mathcal{B}^L$ and $n \in \mathbb{N}$,

$$\begin{aligned}
\mathbb{P}^x_{BW}(X_{2n} = 0) &\le \mathbb{P}^0_{BW}(X_{2n} = 0);\\
\mathbb{P}^x_{BW}(X_{2n+1} = 0) &\le \mathbb{P}^0_{BW}(X_{2n} = 0).
\end{aligned} \tag{2.2.1}$$

Indeed using Cauchy-Schwarz's inequality, the symmetry and the translational invariance of the walk,

$$\begin{aligned}
\mathbb{P}^x_{BW}(X_{2n} = 0) &= \sum_w \mathbb{P}^x_{BW}(X_n = w)\mathbb{P}^w_{BW}(X_n = 0)\\
&\le \sqrt{\sum_w \left(\mathbb{P}^x_{BW}(X_n = w)\right)^2}\sqrt{\sum_w \left(\mathbb{P}^w_{BW}(X_n = 0)\right)^2}\\
&= \sqrt{\mathbb{P}^x_{BW}(X_{2n} = x)}\sqrt{\mathbb{P}^0_{BW}(X_{2n} = 0)} = \mathbb{P}^0_{BW}(X_{2n} = 0).
\end{aligned}$$

On the other hand

$$\begin{aligned}
\mathbb{P}^x_{BW}(X_{2n+1} = 0) &= \sum_w \mathbb{P}^x_{BW}(X_{2n} = w)\mathbb{P}^w_{BW}(X_1 = 0)\\
&\le \mathbb{P}^0_{BW}(X_{2n} = 0)\sum_w \mathbb{P}^0_{BW}(X_1 = w) = \mathbb{P}^0_{BW}(X_{2n} = 0).
\end{aligned}$$

Using (2.1.2) we get for each $t \ge 0$ on the continuous time version

$$\widetilde{\mathbb{P}}^x_{BW}(\widetilde{X}_{2t} = 0) \le \widetilde{\mathbb{P}}^0_{BW}(\widetilde{X}_{2t} = 0). \tag{2.2.2}$$

We denote by $G_{BW}(x,0) := \sum_{n=0}^\infty \mathbb{P}^x_{BW}(X_n = 0)$ the expected number of visits to 0 of the discrete time random walk on the big world (starting from x and associated to $\{P_S\}_S$); by $G^{even}_{BW}(x,0) := \sum_{n=0}^\infty \mathbb{P}^x_{BW}(X_{2n} = 0)$ the expected number of visits to 0 made in an even number of steps. We can prove, starting from (2.1.2), that

$$\begin{aligned}
G_{BW}(x,0) &= \int_0^\infty \widetilde{\mathbb{P}}^x_{BW}(X_t = 0)dt\\
\frac{1}{2}G_{BW}(x,0) &= \int_0^\infty \widetilde{\mathbb{P}}^x_{BW}(X_{2t} = 0)dt.
\end{aligned}$$

Clearly $G^{even}_{BW}(x,0) \leq G_{BW}(x,0)$ and they coincide if the random walk has period 2 (in which case they are nonzero only if $|x|$ is even). Note that in all the cases considered here, for all x, $G_{BW}(x,0)$ is finite since the random walk on the big world is transient. Indeed if $m=1$ and $d=1$ the big world is the homogeneous tree $\mathbb{T}_3$ and any translation invariant random walk on it is transient (the big world is a free product - $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ - and the random walk is adapted, see for instance [52]).
If $m=1$ and $d \geq 2$ then the big world is the Cayley graph of $\mathbb{Z}^d * \mathbb{Z}_2$ and again the random walk is transient. If $m \geq 2$ the big world is the Cayley graph of $\widetilde{\mathbb{Z}}^d * \mathbb{Z}_2$, where $\widetilde{\mathbb{Z}}^d$ has the m-neighbourhood relation, and one has the same result (this can be proven via the flow criterion, see [52]).
Moreover, by (2.2.1), $G^{even}_{BW}(x,0) \leq G^{even}_{BW}(0,0)$.

We are interested in the event where *locally the small world does not differ from the big world.*

Definition 2.9 *If $x \in \Lambda(L)$ and $t > 0$, we denote by $I(x,t)$ the event in $\widetilde{\Omega}$*

$$I(x,t) := \{\phi_{|B_{BW}(x,t)} \text{ is injective}\},$$

where $B_{BW}(x,t)$ is the ball of radius t centered at x in the big world.

Clearly $\mathbf{P}(I(x,t))$ does not depend on x.

Proposition 2.10

$$\mathbf{P}(I(x, \log\log L)) \geq 1 - \frac{CM^{3\log\log L}}{L^d} \stackrel{L\to\infty}{\to} 1,$$

where C and M are positive constants depending on the neighbourhood structure we consider.

Proof Denote by K_t the number of long range connections in $B_{BW}(0,t)$, and by J_t the total number of sites in $B_{BW}(0,t)$. Clearly $K_t \leq J_t$ and $|\{x \in \Lambda(L) : d(0,x) \leq t\}| \leq J_t$. Since each site has M neighbours ($M = (2m+1)^d$ in $\mathcal{B}^L_m$ and $M = 2d+1$ in $\mathcal{B}^L$), then $J_t \leq CM^t$. Enumerate the long range connections in $B_{BW}(0,t)$ from 1 to K_t and construct the mapping ϕ. Note that $I(0,t)$ contains the set A of $\widetilde{\omega}$ such that the long range connections in the image of $B_{BW}(0,t)$ in the small world $\mathcal{S}$ are all sites at distance at least $2t$ on $\Lambda(L)$. Thus $\mathbf{P}(I(0,t)) \geq \mathbf{P}(A)$ and

$$\begin{aligned}\mathbf{P}(A) &\geq \frac{(2L)^d - J_{2t}}{(2L)^d}\frac{(2L)^d - 2J_{2t}}{(2L)^d}\cdots\frac{(2L)^d-(K_t-1)J_{2t}}{(2L)^d}\\ &= \prod_{i=1}^{K_t-1}\left(1-\frac{iJ_{2t}}{(2L)^d}\right) = \exp\left(\sum_{i=1}^{K_t-1}\log\left(1-\frac{iJ_{2t}}{(2L)^d}\right)\right).\end{aligned}$$

Pick $\varepsilon > 0$ and note that $\log(1-x) \geq -(1+\varepsilon)x$ if $x \in (0,\bar{x}_\varepsilon)$. Choosing $t = \log\log L$ we get

$$\frac{iJ_{2t}}{(2L)^d} \leq \frac{K_t J_{2t}}{(2L)^d} \stackrel{L\to\infty}{\longrightarrow} 0,$$

thus for L sufficiently large we have

$$\mathbf{P}(A) \geq \exp\left(-(1+\varepsilon)\frac{J_{2t}K_t^2}{(2L)^d}\right) \geq \exp\left(-\frac{CM^{3\log\log L}}{L^d}\right) \stackrel{L\to\infty}{\longrightarrow} 1.$$

□

By $d_{\mathcal{S}}(x,y)$ we denote the (random) graph distance between x and y. Depending on $\widetilde{\omega}$, x and y, it may happen that $d_{\mathcal{S}}(x,y) = d(x,y)$ or $d_{\mathcal{S}}(x,y) < d(x,y)$. The following proposition provides probability estimates of these events.

Proposition 2.11 *a. If $d(0,x) \le \log\log L$, then*

$$\mathbf{P}(d_{\mathcal{S}}(0,x) = d(0,x)) \ge 1 - \frac{CM^{3\log\log L}}{L^d}. \tag{2.2.3}$$

b. If $d(0,x) > \log\log L$, then

$$\mathbf{P}(d_{\mathcal{S}}(0,x) > \log\log L) \ge 1 - \frac{CM^{3log\log L}}{L^d}. \tag{2.2.4}$$

Proof

a. It suffices to note that the event $(d_{\mathcal{S}}(0,x) = d(0,x))$ contains the event A of the previous proposition.

b. We note that the event $(d_{\mathcal{S}}(0,x) > \log\log L)$ contains C_x which is the event that all the $2K_{\log\log L}$ long range connections in $B_{BW}(0, \log\log L/2)$ and $B_{BW}(x, \log\log L/2)$ are mapped by ϕ into vertices of $\Lambda(L)$ at distance at least $\log\log L$ from each other and from the balls of radius $\log\log L$ centered at 0 and at x in $\Lambda(L)$. As in the previous proposition, we write $t = \log\log L$ and estimate

$$\mathbf{P}(C_x) \ge \frac{(2L)^d - 2J_t}{(2L)^d}\frac{(2L)^d - 3J_t}{(2L)^d}\cdots\frac{(2L)^d - (2K_{t/2}-1)J_t}{(2L)^d}.$$

Proceeding as in the previous proposition, we get the thesis.

□

We will compare the random walk on the small world with the associated random walk on the big world, whose law we denote by $\mathbb{P}_{BW}$.

2.2.2 Isoperimetric constant

Estimates of the distance between the random walk and the equilibrium measure involve the isoperimetric constant. Thus we will get bounds for the *edge isoperimetric constant*

$$\iota = \min_{|V|\le n/2} \frac{e(V, V^{\complement})}{|V|}, \tag{2.2.5}$$

where n is the total number of vertices in the graph and $e(V, V^{\complement})$ is the total number of edges between V and $V^{\complement}$.

The following result is essentially Theorem 6.3.2 of [24]: there it was stated that there is a lower bound for ι on the complement of a set whose probability is $o(1)$. We slightly modify the proof in order to get a "bad set" of probability which is $o(n^{-l})$ for l positive integer.

Proposition 2.12 *Consider a random regular graph of n vertices and degree r. Then for all $l \in \mathbb{N}$ there exists $\alpha_l > 0$ independent of n and r (one may choose $\alpha_l = 1/10l$) such that $\mathbb{P}(\iota \le \alpha_l) = o(n^{-l})$ ($\mathbb{P}$ being the probability associated to the choice of the graph).*

Proof Let $P(u,s)$ be the probability that there exists a subset of vertices U such that $|U| = u$ and $e(U, U^{\complement}) = s$. Note that

$$\mathbb{P}(\iota \leq \alpha_l) \leq \sum_{\substack{u \leq n/2 \\ s \leq \alpha_l u}} P(u,s) \leq C\alpha_l n^2 \sup_{\substack{1/\alpha_l \leq u \leq n/2 \\ 1 \leq s \leq \alpha_l u}} P(u,s).$$

By equations (6.3.2), (6.3.3), (6.3.4) and (6.3.5) in [24] we have the upper bound

$$P(u,s) \leq C\sqrt{n} e^u \left(\frac{e^2 ru}{s}\right)^s \left(\frac{u}{n}\right)^{\gamma ru} \left(1 - \frac{ru+s}{rn}\right)^{(rn-ru-s)/2},$$

where $\gamma = \frac{1}{2}\left(1 - \frac{s}{ru}\right) - \frac{1}{r}$. Let

$$G_s(u) = e^u \left(\frac{e^2 ru}{s}\right)^s \left(\frac{u}{n}\right)^{\gamma ru} \left(1 - \frac{ru+s}{rn}\right)^{(rn-ru-s)/2},$$

it suffices to prove that there exists $\alpha_l > 0$ such that for all $1 \leq s \leq \alpha_l u$

$$\sup_{1/\alpha_l \leq u \leq n/2} G_s(u) = o(n^{-l-5/2}).$$

First we write $G_s(u)$ as a function of $\alpha = s/ru$ (note that $2/rn \leq \alpha \leq \alpha_l/r$):

$$G_s(u) = e^u \left(\frac{e^2}{\alpha}\right)^{\alpha ur} \left(\frac{u}{n}\right)^{ru\left(\frac{1-\alpha}{2} - \frac{1}{r}\right)} \left(1 - \frac{u}{n}(1+\alpha)\right)^{\frac{rn-ru}{2}(1+\alpha)}.$$

In [24] it is shown that G_s is convex, so it is enough to estimate it in $n/2$ and $1/\alpha_l$. It is easy to show that for some $C \in (0,1)$

$$G_s(n/2) \leq C^n,$$

while

$$G_s(1/\alpha_l) \leq C n^{1 - \frac{r}{\alpha_l}(1/2 - 1/r)}.$$

Choosing $\alpha_l < 1/(7+2l)$ (for instance $\alpha_l = 1/10l$) we get the thesis. $\square$

Now we use this result to prove the analog for the BC small world. The ideas are taken from Theorem 6.3.4 of [24].

Proposition 2.13 *Consider the small world $\mathcal{S}^L$ and its (random) edge isoperimetric constant ι. Then for all $l \in \mathbb{N}$ if $\alpha_l = 1/10l$ then $\mathbf{P}(\iota \leq \alpha_l) = o(L^{-dl})$.*

Proof First, we partition the set $\Lambda(L)$ in $n = \lfloor (2L)^d/3 \rfloor$ subsets of cardinality three (let $\{I_j\}_{j=1}^n$ be their collection) plus eventually one subset of cardinality one or two. We associate to $\mathcal{S}^L$ the random regular graph of degree three and n vertices: join j with k whenever there exist $x \in I_j$ and $y \in I_k$ such that x is the long range neighbour of y.

Given $A \subset \Lambda(L)$ define J_A as the family of indices j such that $I_j \cap A \neq \emptyset$. Note that $A \subset \bigcup_{j \in J_A} I_j$, $|J_A| \geq |A|/3$ and that if there is an edge between J_A and $J_A^{\complement}$ in the random regular graph then there is a long range connection between A and $A^{\complement}$.

Suppose that $|J_A| \leq n/2$, then by Proposition 2.12, outside a set of probability $o(n^{-l}) = o(L^{-dl})$ we have

$$\frac{e(A, A^{\complement})}{|A|} \geq \frac{e(J_A, J_A^{\complement})}{|A|} \geq \frac{3\alpha_l |J_A|}{|J_A|}.$$

In the case that $|J_A| > n/2$ we exchange J_A with $J_A^{\complement}$ and we are done. $\square$

2.2.3 Convergence to equilibrium

Note by symmetry that the reversible distribution of the walk on $\mathcal{S}^L$ is the uniform probability.

We recall that given a discrete time random walk on a finite set, with transition matrix P and reversible measure the uniform measure π, a result of Sinclair and Jerrum [46] gives an estimate of the speed of convergence to equilibrium. Indeed in this case P has all real eigenvalues, namely $1 = \lambda_0 > \lambda_1 \geq \cdots \geq \lambda_{n-1}$. Let $\lambda = \max\{|\lambda_i| : i = 1, \dots, n-1\}$. It is well known that $\lambda < 1$. Let $\mathbb{N}_0 = \{n \in \mathbb{Z} : n \geq 0\}$; then for all $t \in \mathbb{N}_0$

$$\max_{x,y} \left| p^{(t)}(x,y) - \pi(y) \right| \leq \lambda^t \leq \exp(-t(1-\lambda)),$$

where $p^{(t)}(x,y)$ is a t-step probability of the walk.

We are interested in estimates for λ. If $\lambda = \lambda_1$ then the following (which is known as Cheeger's inequality (see [24, Theorem 6.2.1]), is useful

$$\frac{1}{2}\iota^2 \left(\min_{x,y:p(x,y)>0} p(x,y) \right)^2 \leq 1 - \lambda_1 \leq 2\iota. \tag{2.2.6}$$

A sufficient condition for $\lambda = \lambda_1$ is that all the eigenvalues are positive, which for instance holds when we consider a lazy random walk, that is one which stays put with probability 1/2.

It is thus clear that for any small world S such that $\iota(S) > \alpha$, a random walk X_t on S with symmetric transition matrix P_S such that $\lambda = \lambda_1$,

$$\max_{x,y} |\mathbb{P}_S^x(X_t = y) - \pi(y)| \leq \exp(-c\alpha^2 t), \tag{2.2.7}$$

where c depends only on $\min_{x,y:p_S(x,y)>0} p_S(x,y)$. Moreover by Proposition 2.12 with $\alpha_l = c'/l$

$$\begin{aligned}\max_{x,y} |\mathbb{P}^x(X_t = y) - \pi(y)| &\leq \sum_S \mathbf{P}(S) \max_{x,y} |\mathbb{P}_S^x(X_t = y) - \pi(y)| \\ &\leq \exp(-c(c'/l)^2 t)\mathbf{P}(\iota > c'/l) + 2\mathbf{P}(\iota \leq c'/l) \\ &\leq \exp(-c(c'/l)^2 t) + o(L^{-dl}).\end{aligned} \tag{2.2.8}$$

It is easy starting from (2.1.2) to prove that (2.2.7) and (2.2.8) still hold in continuous time with a different constant in the exponential. Namely, one has to replace $c\alpha_l^2$ with $1 - \exp(-c\alpha_l^2)$.

$$\max_{x,y} \left| \mathbb{P}_S^x(\widetilde{X}_t = y) - \pi(y) \right| \leq \exp\big(- (1 - e^{-c\alpha^2})t \big) \tag{2.2.9}$$

$$\max_{x,y} \left| \mathbb{P}^x(\widetilde{X}_t = y) - \pi(y) \right| \leq \exp\big(- (1 - e^{-c\alpha^2})t \big) + o(L^{-dl}). \tag{2.2.10}$$

We are able to prove that (2.2.7) and (2.2.8) hold for any random walk (not just for the lazy one). The following proposition is the analog of the result of Sinclair and Jerrum for a general discrete time random walk.

Proposition 2.14 *Let $X = \{X_t\}_{t \in \mathbb{N}_0}$ be the random walk on the small world $\mathcal{S}^L$, defined by the family of transition matrices $\{P_S\}_{S \in \mathcal{S}^L(\widetilde{\Omega})}$. There exists $c > 0$ such that for all $l \in \mathbb{N}$ and $h_l = c/l^2$*

$$\max_{x,y} \left| \mathbb{P}^x(X_t = y) - \frac{1}{(2L)^d} \right| \leq \exp(-th_l) + o(L^{-dl}). \tag{2.2.11}$$

Moreover, for any fixed small world S such that $\iota(S) > \alpha$, there exists $c > 0$ such that

$$\max_{x,y} \left| \mathbb{P}^x_S(X_t = y) - \frac{1}{(2L)^d} \right| \le \exp(-(1 - e^{-c\alpha^2})t). \tag{2.2.12}$$

Proof We couple X with a random walk $Y = \{Y_t\}_{t\ge 0}$ with transition matrix Q such that $q(x,x) = (1 + p(x,x))/2$, $q(x,y) = p(x,y)/2$: let $\{B_j\}_{j\ge 1}$ be a Bernoulli process of parameter $1/2$, independent of X. We move Y along with X when the corresponding Bernoulli equals 1, otherwise we stay put. Formally, $Y_0 = X_0$, $Y_t = X_{T_t}$ where $T_t = \sum_{j=1}^t B_j$. Note that the uniform measure is the reversible probability for Q as well.

Let Z_t be the total number of 0s in the Bernoulli process, before the t-th success: then

$$(X_t = y) = \bigcup_{k=0}^{\infty} (Y_{t+k} = y, Z_t = k),$$

and the union is disjoint. Thus

$$\mathbb{P}^x(X_t = y) = \sum_{k=0}^{\infty} \mathbb{P}^x(Y_{t+k} = y) \binom{t+k-1}{k} 2^{-t+k}$$
$$\sum_{\mathcal{S}} \mathbf{P}(\mathcal{S}) \sum_{k=0}^{\infty} \mathbb{P}^x_{\mathcal{S}}(Y_{t+k} = y) \binom{t+k-1}{k} 2^{-(t+k)}.$$

Now we use the estimate for the random walk Y.

$$\begin{aligned} |\mathbb{P}^x(X_t = y) - \pi(y)| &\le \sum_{\mathcal{S}} \mathbf{P}(\mathcal{S}) \sum_{k=0}^{\infty} |\mathbb{P}^x_{\mathcal{S}}(Y_{t+k} = y) - \pi(y)| \binom{t+k-1}{k} 2^{-(t+k)} \\ &\le o(L^{-dl}) + \sum_{\mathcal{S}:\iota(S)>\alpha_l} \mathbf{P}(\mathcal{S}) \sum_{k=0}^{\infty} \exp(-c(t+k)\alpha_l^2) \binom{t+k-1}{k} 2^{-(t+k)} \\ &\le o(L^{-dl}) + \exp(-c\alpha_l^2 t). \end{aligned}$$

Clearly the proof may be repeated for any small world S with $\iota(S) > \alpha$, to obtain (2.2.12).
□

2.3 Laplace transform estimates

Let $T_L = \inf\{s > 0 : X_s = Y_s\}$ be the first time, after time 0, that two independent random walks X_t and Y_t on the random graph S meet. Clearly the law of T_L (with respect to either $\mathbb{P}_S$ or $\mathbb{P}$) depends on the starting sites of the walkers. Without loss of generality, we assume that $Y_0 = 0$ and $X_0 = x$ (if we need to stress the dependence on L, we write $X_0 = x_L$).

We consider the following (annealed) Laplace transforms in discrete time, defined for $\lambda > 0$:

$$G_L(x,\lambda) := \sum_{t=0}^{\infty} e^{-\lambda t} \mathbb{P}^{x,0}(X_t = Y_t) = \sum_{t=0}^{\infty} e^{-\lambda t} \mathbb{P}^x(X_{2t} = 0),$$
$$F_L(x,\lambda) := \sum_{t=0}^{\infty} e^{-\lambda t} \mathbb{P}^{x,0}(T_L = t) = \sum_{t=1}^{\infty} e^{-\lambda t} \mathbb{P}^{x,0}(T_L = t),$$

Let $\widetilde{T}_L = \inf\{s > 0 : \widetilde{X}_s = \widetilde{Y}_s\}$ be the continuous time version of T_L, we introduce the following (annealed) Laplace transforms in continuous time,

$$\begin{aligned}\widetilde{G}_L(x,\lambda) &:= \int_0^\infty e^{-\lambda t}\widetilde{\mathbb{P}}^{x,0}(\widetilde{X}_t = \widetilde{Y}_t)dt = \int_0^\infty e^{-\lambda t}\widetilde{\mathbb{P}}^{x}(\widetilde{X}_{2t} = 0)dt,\\ \widetilde{F}_L(x,\lambda) &:= \int_0^\infty e^{-\lambda t}\widetilde{\mathbb{P}}^{x,0}(\widetilde{T}_L \in dt),\end{aligned}$$

where $\mathbb{P}^{x,0}$ denotes the product law of the two walkers.
We define two other (quenched) transforms, both in discrete and continuous time: given $S \in \mathcal{S}^L(\widetilde{\Omega})$

$$\begin{aligned}G_L^S(x,\lambda) &:= \sum_{t=0}^\infty e^{-\lambda t}\mathbb{P}_S^{x,0}(X_t = Y_t), &\qquad \widetilde{G}_L^S(x,\lambda) &:= \int_0^\infty e^{-\lambda t}\widetilde{\mathbb{P}}_S^{x,0}(\widetilde{X}_t = \widetilde{Y}_t)dt,\\ F_L^S(x,\lambda) &:= \sum_{t=0}^\infty e^{-\lambda t}\mathbb{P}_S^{x,0}(T_L = t), &\qquad \widetilde{F}_L^S(x,\lambda) &:= \int_0^\infty e^{-\lambda t}\widetilde{\mathbb{P}}_S^{x,0}(\widetilde{T}_L \in dt).\end{aligned}$$

With a slight abuse of notation we omit the superscript $\sim$ on the continuous time random walk when not necessary and we use X_t, Y_t, T_L, $\mathbb{P}_S^\mu$, $G_L^S(x,\lambda)$ and $F_L^S(x,\lambda)$ both in discrete and continuos time version of the process.
We are interested in the asymptotic behaviour, as $L \to \infty$, of $T_L/(2L)^d$ in discrete and continuous time, thus we study the the Laplace transforms

$$G_L\left(x, \frac{\lambda}{(2L)^d}\right),\ F_L\left(x, \frac{\lambda}{(2L)^d}\right),\ G_L^S\left(x, \frac{\lambda}{(2L)^d}\right),\ F_L^S\left(x, \frac{\lambda}{(2L)^d}\right).$$

2.3.1 Estimates for G: discrete time

We first note that the evaluation of the limit of the annealed transforms can be done considering only small worlds with large isoperimetric constants. Given $\alpha > 0$, we define

$$Q_L^\alpha := (S \in \mathcal{S}^L(\widetilde{\Omega}) : \iota(S) > \alpha) \tag{2.3.1}$$

Let $\mathcal{K} := \{K \subset \mathbb{R} : \inf K > 0\}$.

Lemma 2.15 *There exists $\alpha > 0$ such that $\mathbf{P}(Q_L^\alpha) \stackrel{L\to\infty}{\to} 1$. Moreover if*

$$\begin{aligned}g_L &:= \sum_{S\in Q_L^c}\mathbf{P}(S)\sum_{t=0}^\infty e^{-\frac{\lambda t}{(2L)^d}}\mathbb{P}_S^x(X_{2t} = 0),\\ f_L &:= \sum_{S\in Q_L^c}\mathbf{P}(S)\sum_{t=0}^\infty e^{-\frac{\lambda t}{(2L)^d}}\mathbb{P}_S^{x,0}(T_L = t),\end{aligned}$$

then $g_L \stackrel{L\to\infty}{\to} 0$ and $f_L \stackrel{L\to\infty}{\to} 0$ (for each $K \in \mathcal{K}$, uniformly for $\lambda \in K$).

Proof We choose $\alpha = 1/20$: by Proposition 2.13 we have $\mathbf{P}(\iota \le \alpha_2) = o(L^{-2d})$. Moreover

$$0 \le f_L \le g_L \le \mathbf{P}(Q_L^c)\sum_{t=0}^\infty e^{-\frac{\lambda t}{(2L)^d}} = \mathbf{P}(Q_L^c)\frac{(2L)^d}{\lambda}(1 + o(1)) \stackrel{L\to\infty}{\to} 0.$$

□

The limit of the sum defining G, from $\lfloor \log\log L \rfloor$ to infinity does not depend on the sequence of small worlds, provided that they are chosen with large isoperimetric constant. From now on, if not otherwise stated, we write $t_L = \lfloor \log\log L \rfloor$ and $Q_L = (S \in \mathcal{S}^L : \iota(S) > 1/20)$.

Lemma 2.16 *If for all L we choose $S \in Q_L$ and $x_L \in \Lambda(L)$, then for all $\lambda > 0$*

$$\lim_{L\to\infty} \sum_{t=t_L}^{\infty} e^{-\frac{\lambda t}{(2L)^d}} \mathbb{P}_S^{x_L}(X_{2t} = 0) = \frac{1}{\lambda}.$$

Moreover, the convergence is uniform with respect to the choice of the sequences $S \in Q_L$, $x_L \in \Lambda(L)$ (and of λ).

Proof Note that

$$\begin{aligned}
&\sum_{t=t_L}^{\infty} e^{-\frac{\lambda t}{(2L)^d}} \mathbb{P}_S^{x_L}(X_{2t} = 0) \\
&= \sum_{t=t_L}^{\infty} e^{-\frac{\lambda t}{(2L)^d}} \frac{1}{(2L)^d} + \sum_{t=t_L}^{\infty} e^{-\frac{\lambda t}{(2L)^d}} \left(\mathbb{P}_S^{x_L}(X_{2t} = 0) - \frac{1}{(2L)^d} \right).
\end{aligned} \tag{2.3.2}$$

The limit of the first term is uniform in λ:

$$\lim_{L\to\infty} \frac{1}{\lambda} e^{-\lambda t_L/(2L)^d}(1 + o(1)) = \frac{1}{\lambda}. \tag{2.3.3}$$

We now consider the second term of (2.3.2). Since S is chosen in Q_L, by (2.2.12) there exists a constant $c > 0$ such that

$$\begin{aligned}
\sum_{t=t_L}^{\infty} e^{-\frac{\lambda t}{(2L)^d}} \left| \mathbb{P}_S^{x}(X_{2t} = 0) - \frac{1}{(2L)^d} \right| &\le \sum_{t=t_L}^{\infty} e^{-\frac{\lambda t}{(2L)^d}} e^{-c\alpha^2 t} \\
&\le \frac{e^{-(\lambda/(2L)^d + c\alpha^2)t_L}}{1 - e^{-(\lambda/(2L)^d + c\alpha^2)}},
\end{aligned}$$

which tends to 0 as L goes to infinity (uniformly with respect to all the choices of the statement). $\square$

Recall that, given a vertex $x \in \Lambda(L)$, there is a unique vertex $+(x)$ in the big world. If L is sufficiently large, for a wide choice of S (i.e. S in a set with **P**-probability which tends to 1 as L increases to infinity), we have that $G_L^S(x_L, \lambda/(2L)^d)$ is close to $\frac{1}{\lambda} + G_{BW}^{even}(+(x_L), 0)$.

Theorem 2.17 *Let*

$$h_L^S(\lambda) = \left| G_L^S(x_L, \lambda/(2L)^d) - \frac{1}{\lambda} - G_{BW}^{even}(+(x_L), 0) \right|.$$

For all $\varepsilon > 0$ there exists $\widetilde{L}$ such that for all λ, x_L and $L \ge \widetilde{L}$ we have that $Q_L \cap I(0, 2t_L) \subset (S : h_L^S(\lambda) \le \varepsilon)$.
If $d_S(0, x_L) > 2t_L$ then $Q_L \subset (S : h_L^S(\lambda) \le \varepsilon)$.

Proof If $S \in I(0, 2t_L)$ or $d_S(0, x_L) > 2t_L$ (in this second case both terms are zero) we have

$$\sum_{t=0}^{t_L} e^{-\frac{\lambda t}{(2L)^d}} \mathbb{P}_S^{x_L}(X_{2t} = 0) = \sum_{t=0}^{t_L} e^{-\frac{\lambda t}{(2L)^d}} \mathbb{P}_{BW}^{+(x_L)}(X_{2t} = 0).$$

Then if either $S \in I(0, 2t_L)$ or $d_S(0, x_L) > 2t_L$

$$h_L^S(\lambda) \le \left|\sum_{t=t_L}^{\infty} e^{-\frac{\lambda t}{(2L)^d}} \mathbb{P}_S^{x_L}(X_{2t}=0) - \frac{1}{\lambda}\right| + \sum_{t=t_L}^{\infty} \mathbb{P}_{BW}^{+(x_L)}(X_{2t}=0) + \sum_{t=0}^{t_L}(1 - e^{-\frac{\lambda t}{(2L)^d}})\mathbb{P}_{BW}^{+(x_L)}(X_{2t}=0).$$

By Lemma 2.16, if $S \in Q_L$, the first term is smaller than $\varepsilon/3$ if L is sufficiently large. By the Dominated Convergence Theorem and (2.2.1), the second and the third term are both smaller than $\varepsilon/3$ if L is sufficiently large. □

Theorem 2.18 *For all $K \in \mathcal{K}$, $\varepsilon > 0$ there exists $\widetilde{L}$ such that for all $L \ge \widetilde{L}$, $x_L \in \Lambda(L)$, and $\lambda \in K$,*

$$\left|G_L(x_L, \lambda/(2L)^d) - \frac{1}{\lambda} - G_{BW}^{even}(+(x_L), 0)\right| \le \varepsilon.$$

Proof Recall that

$$G_L(x_L, \lambda/(2L)^d) = \sum_S \mathbf{P}(S) G_L^S(x_L, \lambda/(2L)^d).$$

By Theorem 2.17 there exists $\widetilde{L}$ such that for all $L \ge \widetilde{L}$,

$$\left|\sum_{S \in Q_L \cap I(0,2t_L)} \mathbf{P}(S) G_L^S(x_L, \lambda/(2L)^d) - \frac{1}{\lambda} - G_{BW}^{even}(+(x_L), 0)\right| \le \varepsilon/3.$$

Thus, since $\mathbf{P}(Q_L^c)$ and $\mathbf{P}(I(0, 2t_L)^c)$ are both small if L is large, we may choose $\widetilde{L}$ such that for all $\lambda \in K$

$$\sum_{S \in Q_L^c \cup (I(0,2t_L))^c} \mathbf{P}(S) \left(\frac{1}{\lambda} + G_{BW}^{even}(+(x_L), 0)\right) \le \varepsilon/3.$$

Now we only need to prove that

$$\sum_{S \in Q_L^c \cup I(0,2t_L)^c} \mathbf{P}(S) G_L^S(x_L, \lambda/(2L)^d) \le \varepsilon/3.$$

By Lemma 2.15 we know that $\sum_{S \in Q_L^c} \mathbf{P}(S) G_L^S(x_L, \lambda/(2L)^d) \le \varepsilon/6$ for all $L \ge \widetilde{L}$ and $\lambda > 0$. Finally, by Proposition 2.10 and Lemma 2.16, for some $C > 0$ and L sufficiently large

$$\begin{aligned}
&\sum_{S \in Q_L \cap I(0,2t_L)^c} \mathbf{P}(S) G_L^S(x_L, \lambda/(2L)^d) \\
&= \sum_{S \in Q_L \cap I(0,2t_L)^c} \mathbf{P}(S) \left(\sum_{t=0}^{t_L} e^{-\frac{\lambda t}{(2L)^d}} \mathbb{P}_S^{x_L}(X_{2t}=0) + \sum_{t=t_L}^{\infty} e^{-\frac{\lambda t}{(2L)^d}} \mathbb{P}_S^{x_L}(X_{2t}=0)\right) \\
&\le \left(t_L + \frac{1}{\lambda} + C\right) \mathbf{P}(I(0, 2t_L)^c) \le \varepsilon/6.
\end{aligned}$$

□

2.3.2 Estimates for G: continuous time

We prove the same result in continuous time

Lemma 2.19 *Let*

$$\tilde{g}_L := \sum_{S \in Q_L^c} \mathbf{P}(S) \int_0^\infty e^{-\frac{\lambda t}{(2L)^d}} \mathbb{P}_S^x(X_{2t} = 0)dt,$$

$$\tilde{f}_L := \sum_{S \in Q_L^c} \mathbf{P}(S) \int_0^\infty e^{-\frac{\lambda t}{(2L)^d}} \mathbb{P}_S^{x,0}(T_L \in dt),$$

then $\tilde{g}_L \stackrel{L\to\infty}{\to} 0$ *and* $\tilde{f}_L \stackrel{L\to\infty}{\to} 0$ *(for each* $K \in \mathcal{K}$*, uniformly for* $\lambda \in K$*).*

Proof By Proposition 2.13 we have $\mathbf{P}(\iota \leq \alpha) = o(L^{-2d})$. Moreover

$$0 \leq \tilde{f}_L \leq \tilde{g}_L \leq \mathbf{P}(Q_L^c) \int_0^\infty e^{-\frac{\lambda t}{(2L)^d}} dt = \mathbf{P}(Q_L^c) \frac{(2L)^d}{\lambda} \stackrel{L\to\infty}{\to} 0.$$

□

Lemma 2.20 *If for all* L *we choose* $S \in Q_L$ *and* $x_L \in \Lambda(L)$*, then for all* $\lambda > 0$

$$\lim_{L\to\infty} \int_{t_L}^\infty e^{-\frac{\lambda t}{(2L)^d}} \mathbb{P}_S^{x_L}(X_{2t} = 0)dt = \frac{1}{\lambda}.$$

Moreover, the convergence is uniform with respect to the choice of the sequences $S \in Q_L$*,* $x_L \in \Lambda(L)$ *(and of* λ*).*

Proof We repeat the proof of Lemma 2.16, by taking integrals instead of sums and by using estimation (2.2.7) in continuous time. □

Theorem 2.21 *Let*

$$h_L^S(\lambda) = \left| G_L^S(x_L, \lambda/(2L)^d) - \frac{1}{\lambda} - \frac{1}{2} G_{BW}(+(x_L), 0) \right|.$$

For all $\varepsilon > 0$ *there exists* $\widetilde{L}$ *such that for all* λ*,* x_L *and* $L \geq \widetilde{L}$ *we have that* $Q_L \cap I(0, t_L^2) \subset (S : h_L^S(\lambda) \leq \varepsilon)$*.*
If $d_S(0, x_L) > t_L^2$ *then* $Q_L \subset (S : h_L^S(\lambda) \leq \varepsilon)$*.*

Proof Notice that

$$\frac{1}{2} G_{BW}(+(x_L), 0) = \frac{1}{2} \int_0^\infty \mathbb{P}_{BW}^{+(x_L)}(X_v = 0)dv = \int_0^\infty \mathbb{P}_{BW}^{+(x_L)}(X_{2t} = 0)dt \tag{2.3.4}$$

Let $Z(t)$ be the Poisson random variable which counts the number of rings of exponential clocks before t. By Markov's inequality

$$\mathbb{P}\big(Z(t) \geq k\big) \leq \frac{2t^2}{k^2} \tag{2.3.5}$$

If $d_S(0,x_L) > t_L^2$ or $S \in I(0,t_L^2)$ we have

$$\left|\int_0^{t_L} e^{-\frac{\lambda t}{(2L)^d}} \mathbb{P}_S^{x_L}(X_{2t}=0)dt - \int_0^{t_L} e^{-\frac{\lambda t}{(2L)^d}} \mathbb{P}_{BW}^{+(x_L)}(X_{2t}=0)dt\right| \le 2t_L \mathbb{P}(Z(t) \ge t_L^2) \le \frac{2}{t_L} \tag{2.3.6}$$

since if $d_S(0,x_L) > t_L^2$ (if $S \in I(0,t_L^2)$) the probabilities on S and on the big world differ only if the clock rings at least t_L^2 times in a time t_L.
Then if either $S \in I(0,t_L^2)$ or $d_S(0,x_L) > t_L^2$

$$h_L^S(\lambda) \le \left|\int_{t_L}^{\infty} e^{-\frac{\lambda t}{(2L)^d}} \mathbb{P}_S^{x_L}(X_{2t}=0)dt - \frac{1}{\lambda}\right| + \left|\int_0^{t_L} e^{-\frac{\lambda t}{(2L)^d}} \big(\mathbb{P}_S^{x_L}(X_{2t}=0)dt - \mathbb{P}_{BW}^{+(x_L)}(X_{2t}=0)\big)dt\right|$$
$$+ \int_{t_L}^{\infty} \mathbb{P}_{BW}^{+(x_L)}(X_{2t}=0)dt + \int_0^{t_L} (1-e^{-\frac{\lambda t}{(2L)^d}})\mathbb{P}_{BW}^{+(x_L)}(X_{2t}=0)dt.$$

By Lemma 2.20, if $S \in Q_L$, the first term is smaller than $\varepsilon/4$ if L is sufficiently large. By (2.3.6) the second term is smaller than $\varepsilon/4$. By the Dominated Convergence Theorem and (2.2.1), the second and the third term are both smaller than $\varepsilon/4$ if L is sufficiently large.
□

Theorem 2.22 *For all $K \in \mathcal{K}$, $\varepsilon > 0$ there exists $\widetilde{L}$ such that for all $L \ge \widetilde{L}$, $x_L \in \Lambda(L)$, and $\lambda \in K$,*

$$\left|G_L(x_L, \lambda/(2L)^d) - \frac{1}{\lambda} - \frac{1}{2}G_{BW}(+(x_L),0)\right| \le \varepsilon.$$

Proof We repeat the proof of Theorem 2.18 by using Theorem 2.21. □

2.3.3 From G to F: discrete time

We note that if $x_L \neq 0$ then $G_L^S(x_L, \lambda/(2L)^d)$ may be written as

$$\sum_z \sum_{q=0}^{\infty} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}_S^z(X_{2q}=z) \sum_{s=0}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L,0}(T_L=s, X_s=z), \tag{2.3.7}$$

while $G_L^S\left(0, \frac{\lambda}{(2L)^d}\right)$ is equal to

$$1 + \sum_z \sum_{t=0}^{\infty} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}_S^z(X_{2q}=z) \sum_{t=0}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{0,0}(T_L=s, X_s=z).$$

Define H_1, H_2 and H_3 (which depend on S, x_L and L) by

$$H_1 := \sum_z \sum_{q=0}^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}_S^z(X_{2q}=z) \sum_{s=0}^{t_L} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L,0}(T_L=s, X_s=z)$$
$$H_2 := \sum_z \sum_{q=0}^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}_S^z(X_{2q}=z) \sum_{s=t_L+1}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L,0}(T_L=s, X_s=z)$$
$$H_3 := \sum_z \sum_{q=t_L+1}^{\infty} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}_S^z(X_{2q}=z) \sum_{s=0}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L,0}(T_L=s, X_s=z).$$

By Lemma 2.15, if $x_L \neq 0$ for all L sufficiently large and if the limit exists,

$$\lim_{L\to\infty} G_L\left(x_L, \frac{\lambda}{(2L)^d}\right) = \lim_{L\to\infty} \sum_{S\in Q_L} \mathbf{P}(S)(H_1 + H_2 + H_3). \tag{2.3.8}$$

Clearly if $x_L = 0$ for all L sufficiently large we only need to add 1 to the previous limit. We now study each of the three summands separately, in order to obtain the limit of F_L as a function of the limit of G_L.

Lemma 2.23 *If $S \in I(0, 2t_L)$ and $x_L \in \Lambda(L)$, then*

$$H_1 = \sum_{t=0}^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}^0_{BW}(X_{2q} = 0) \sum_{t=0}^{t_L} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}^{x_L,0}_S(T_L = s). \tag{2.3.9}$$

This equality also holds whenever $d_S(0, x_L) > 2t_L$, in which case $H_1 = 0$. Moreover, uniformly with respect to the choice of the sequence $\{x_L\}_L$ and of λ,

$$\lim_{L\to\infty} \sum_{S\in(I(0,2t_L))^c} \mathbf{P}(S) \sum_z \sum_{t=0}^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}^z_S(X_{2q} = z) \sum_{t=0}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}^{x_L,0}_S(T_L = s, X_s = z) = 0. \tag{2.3.10}$$

Proof Since in H_1, z is the site where the two random walks X_t and Y_t meet at a time $s \in [0, t_L]$, and since $S \in I(0, 2t_L)$ or $d_S(0, x_L) > 2t_L$, then

$$\mathbb{P}^z_S(X_{2q} = z) = \mathbb{P}^0_{BW}(X_{2q} = 0),$$

which proves (2.3.9). Note that for some $C > 0$

$$\begin{aligned}\sum_{S\in I(0,2t_L)^c} \mathbf{P}(S) \sum_z \sum_{t=0}^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}^z_S(X_{2q} = z) \sum_{t=0}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}^{x_L,0}_S(T_L = s, X_s = z) \\ \leq C t_L \, \mathbf{P}(I(0, 2t_L)^c) \, F_L(x_L, \lambda/(2L)^d),\end{aligned} \tag{2.3.11}$$

which, by Proposition 2.10 and since $F_L(x, \lambda) \leq 1$ for all λ and x, goes to 0, uniformly in x_L and λ, as L goes to infinity. This proves (2.3.10). □

Lemma 2.24 *For all $K \in \mathcal{K}$, $\varepsilon > 0$ there exists $\widetilde{L}$ such that for all $L \geq \widetilde{L}$, x_L and $\lambda \in K$,*

$$\left| \sum_{S\in Q_L} \mathbf{P}(S) H_2 - \sum_{t=0}^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}^0_{BW}(X_{2q} = 0) \sum_{S\in Q_L} \mathbf{P}(S) \sum_{s=t_L+1}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}^{x_L,0}_S(T_L = s) \right| \leq \varepsilon. \tag{2.3.12}$$

Proof Note that $\sum_{S\in Q_L} \mathbf{P}(S) H_2$ can be written as

$$\begin{aligned}&\sum_z \sum_{S\in Q_L\cap I(z,2t_L)} \mathbf{P}(S) \sum_{q=0}^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}^z_S(X_{2q} = z) \sum_{s=t_L+1}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}^{x_L,0}_S(T_L = s, X_s = z) \\ &+ \sum_z \sum_{S\in Q_L\cap I(z,2t_L)^c} \mathbf{P}(S) \sum_{q=0}^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}^z_S(X_{2q} = z) \sum_{s=t_L+1}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}^{x_L,0}_S(T_L = s, X_s = z) \\ &= H_{2,1} + H_{2,2}.\end{aligned}$$

We prove that $H_{2,2} \to 0$, indeed since $\exp(-\lambda q/(2L)^d)\mathbb{P}^z_S(X_{2q} = z) \le 1$ then

$$H_{2,2} \le t_L \sum_z \sum_{S \in Q_L \cap I(z,2t_L)^c} \mathbf{P}(S) \left\{ \sum_{s=t_L+1}^{\lfloor \log L \rfloor} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}^{x_L,0}_S(X_s = Y_s = z) \right.$$

$$\left. + \sum_{s=\lfloor \log L \rfloor +1}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}^{x_L,0}_S(X_s = Y_s = z) \right\} =: H_{2,2,1} + H_{2,2,2}.$$

Note that $H_{2,2,1}$ is

$$\begin{aligned} & t_L \sum_z \sum_{S \in Q_L \cap I(z,2t_L)^c} \mathbf{P}(S) \sum_{s=t_L+1}^{\lfloor \log L \rfloor} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}^{x_L,0}_S(X_s = Y_s = z) \\ & \le t_L \sum_{S \in Q_L} \mathbf{P}(S) \sum_{s=t_L+1}^{\lfloor \log L \rfloor} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}^{x_L,0}_S(X_s = Y_s) \\ & = t_L \sum_{S \in Q_L} \mathbf{P}(S) \sum_{s=t_L+1}^{\lfloor \log L \rfloor} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}^{0}_S(X_{2s} = x_L). \end{aligned}$$

We write $\mathbb{P}^0_S(X_{2s} = x_L) \le \left|\mathbb{P}^0_S(X_{2s} = x_L) - 1/(2L)^d\right| + 1/(2L)^d$, which by (2.2.12) is smaller or equal to $e^{-cs} + 1/(2L)^d$ for some $c > 0$. It is thus only a matter of computation to show that $H_{2,2,1}$ goes to zero (uniformly in x_L and λ) as L goes to infinity.

Now we consider $H_{2,2,2}$, that is

$$t_L \sum_z \sum_{S \in Q_L \cap I(z,2t_L)^c} \mathbf{P}(S) \sum_{s=\lfloor \log L \rfloor+1}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}^{x_L,0}_S(X_s = Y_s = z) ds.$$

Note that by symmetry of the random walk

$$\mathbb{P}^{x_L,0}_S(X_s = Y_s = z) = \mathbb{P}^z_S(X_s = 0)\mathbb{P}^z_S(Y_s = x_L).$$

Write

$$\mathbb{P}^z_S(X_s = 0) = \mathbb{P}^z_S(X_s = 0) - \frac{1}{(2L)^d} + \frac{1}{(2L)^d}$$

and do the same for $\mathbb{P}^z_S(Y_s = x_L)$. Using (2.2.12) and Proposition 2.10, we have that $H_{2,2,2}$ is smaller or equal to

$$\begin{aligned} & t_L \sum_z \sum_{S \in Q_L \cap I(z,2t_L)^c} \mathbf{P}(S) \sum_{s=\lfloor \log L \rfloor+1}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \left(e^{-cs} + \frac{1}{(2L)^d}\right)^2 \\ & \le C t_L L^d \frac{M^{2t_L}}{L^d} \sum_{s=\lfloor \log L \rfloor+1}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \left(e^{-2cs} + \frac{1}{(2L)^{2d}} + \frac{2e^{-cs}}{(2L)^d}\right) \end{aligned}$$

which goes to 0 (uniformly in x_L and $\lambda \in K$ for each $K \in \mathcal{K}$) as $L \to \infty$.

We now consider $H_{2,1}$. As already observed, if $S \in I(z, 2t_L)$ and $q \in [0, t_L]$, then $\mathbb{P}^z_S(X_{2q} = z) = \mathbb{P}^0_{BW}(X_{2q} = 0)$, thus

$$H_{2,1} = \sum_{q=0}^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}^0_{BW}(X_{2q} = 0)$$

$$\sum_z \sum_{S \in Q_L \cap I(z,2t_L)} \mathbf{P}(S) \sum_{s=t_L+1}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}^{x_L,0}_S(T_L = s, X_s = z).$$

We split

$$\begin{aligned}
&\sum_z \sum_{S\in Q_L\cap I(z,2t_L)} \mathbf{P}(S) \sum_{s=t_L+1}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L,0}(T_L = s, X_s = z) \\
&= \sum_{S\in Q_L} \sum_z \mathbf{P}(S) \sum_{s=t_L+1}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L,0}(T_L = s, X_s = z) \\
&- \sum_z \sum_{S\in Q_L\cap I(z,2t_L)^c} \mathbf{P}(S) \sum_{s=t_L+1}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L,0}(T_L = s, X_s = z).
\end{aligned}$$

The first summand is

$$\sum_{S\in Q_L} \mathbf{P}(S) \sum_{s=t_L+1}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L,0}(T_L = s),$$

while the second summand converges to 0 by the same arguments we used to prove that $H_{2,2}$ converges to 0. This completes the proof. $\square$

Lemma 2.25 *Define*

$$a_L^\lambda(S) := H_2 - \sum_{q=0}^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}_{BW}^0(X_{2q} = 0) \sum_{s=t_L+1}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L,0}(T_L = s). \tag{2.3.13}$$

Then $a_L^\lambda > 0$ and $a_L^\lambda \to 0$ in probability (for each $K \in \mathcal{K}$, uniformly in x_L and $\lambda \in K$), that is for all $K \in \mathcal{K}$, $\varepsilon > 0$ and $\delta > 0$ there exists $\widetilde{L}$ such that for all $L \geq \widetilde{L}$ and x_L

$$\mathbf{P}(A_L^\varepsilon(K)) := \mathbf{P}(S : a_L^\lambda(S) \leq \varepsilon, \forall\lambda \in K) \geq 1 - \delta.$$

Proof We first note that for all z and S, $\mathbb{P}_S^z(X_{2q} = z) \geq \mathbb{P}_{BW}^0(X_{2q} = 0)$, hence $a_L^\lambda(S) \geq 0$. Suppose by contradiction that there exist K, $\varepsilon > 0$ and $\delta > 0$ such that $\mathbf{P}(A_L^\varepsilon(K)) \leq 1-\delta$ infinitely often. Then infinitely often

$$\sum_S \mathbf{P}(S) a_L^\lambda(S) > \delta\varepsilon.$$

By Lemmas 2.24 and 2.15, there exists $\widetilde{L}$ such that $\sum_S \mathbf{P}(S) a_L^\lambda(S) < \delta\epsilon$ for each $L \geq \widetilde{L}$, x_L, $\lambda \in K$, whence the contradiction. $\square$

Lemma 2.26 *For all $K \in \mathcal{K}$ and $\varepsilon > 0$ there exists $\widetilde{L}$ such that for all $L \geq \widetilde{L}$, $S \in Q_L$, x_L and $\lambda \in K$,*

$$\left| H_3 - \frac{1}{\lambda} F_L^S(x_L, \lambda/(2L)^d) \right| \leq \varepsilon. \tag{2.3.14}$$

Proof By (2.2.12) we have that for some $c > 0$,

$$\left| \mathbb{P}_S^z(X_{2q} = z) - \frac{1}{(2L)^d} \right| \leq e^{-cq},$$

thus

$$H_3 = \sum_z \sum_{q=t_L+1}^{\infty} \left(\mathbb{P}_S^z(X_{2q} = z) - \frac{1}{(2L)^d} \right) e^{-\frac{\lambda q}{(2L)^d}} \sum_{s=0}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L,0}(T_L = s, X_s = z)$$
$$+ \sum_{q=t_L+1}^{\infty} e^{-\frac{\lambda q}{(2L)^d}} \frac{1}{(2L)^d} F_L^S(x_L, \lambda/(2L)^d).$$

Then the modulus of the first member of (2.3.14) does not exceed

$$\sum_{q=t_L+1}^{\infty} e^{-\frac{\lambda q}{(2L)^d} - cq} \sum_{s=0}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L,0}(T_L = s) \leq C \exp(-ct_L).$$

The claim follows since for L sufficiently large

$$\left| \frac{1}{(2L)^d} \sum_{q=t_L+1}^{\infty} e^{-\frac{\lambda q}{(2L)^d}} - \frac{1}{\lambda} \right| < \varepsilon/2.$$

□

Theorem 2.27 *Let*

$$b_L^\lambda(S) := \left| F_L^S \left(x_L, \frac{\lambda}{(2L)^d} \right) - \frac{G_{BW}^{even}(+(x_L), 0) + \frac{1}{\lambda} - \mathbf{1}_{\{0\}}(+(x_L))}{G_{BW}^{even}(0,0) + \frac{1}{\lambda}} \right|.$$

Then $b_L^\lambda \to 0$ in probability, for each $K \in \mathcal{K}$ uniformly in $x_L \in \Lambda(L)$ and $\lambda \in K$, namely for all $\varepsilon > 0$, $(S : b_L^\lambda(S) \leq \varepsilon, \forall \lambda \in K) \supset Q_L \cap I(0, 2t_L) \cap (A_L^{\varepsilon/2}(K))$ for all L sufficiently large.
Moreover for all $\varepsilon > 0$, $(S : b_L^\lambda(S) \leq \varepsilon) \supset Q_L \cap (S : d_S(0, x_L) > 2t_L) \cap A_L^{\varepsilon/2}(K)$ for all L sufficiently large.

Proof Consider

$$G_L^S(x_L, \lambda/(2L)^d) - \mathbf{1}_{\{0\}}(+(x_L)) - \left(G_{BW}^{even}(0,0) + \frac{1}{\lambda} \right) F_L^S(x_L, \lambda/(2L)^d).$$

Writing $G_L^S(x_L, \lambda/(2L)^d) = \mathbf{1}_{\{0\}}(+(x_L)) + H_1 + H_2 + H_3$, using Lemmas 2.23, 2.25 and 2.26 it is not difficult to prove that the previous difference is smaller than ε when L is sufficiently large and $S \in Q_L \cap I(0, 2t_L) \cap A_L^{\varepsilon/2}(K)$ or $S \in Q_L \cap (S : d_S(0, x_L) > 2t_L) \cap A_L^{\varepsilon/2}(K)$. By Theorem 2.17 we have the conclusion. □

Theorem 2.28 *For each $\epsilon > 0$, $K \in \mathcal{K}$ there exists $\widetilde{L}$ such that for each $L > \widetilde{L}$, $\lambda \in K$ and for each sequence $\{x_L\}_L$ such that $x_L \in \Lambda(L)$*

$$\left| F_L \left(x_L, \frac{\lambda}{(2L)^d} \right) - \frac{G_{BW}^{even}(+(x_L), 0) + \frac{1}{\lambda} - \mathbf{1}_{\{0\}}(+(x_L))}{G_{BW}^{even}(0,0) + \frac{1}{\lambda}} \right| \leq \varepsilon.$$

Proof To keep notation simple we deal only with the case $+(x_L) \neq 0$ (the case $+(x_L) = 0$ is completely analogous). Let

$$Q^{\lambda}_{\varepsilon,L} = \left\{ S : b^{\lambda}_L \leq \varepsilon \right\}, \tag{2.3.15}$$

(b^{λ}_L was defined in Theorem 2.27). By Theorem 2.27 there exists $\widetilde{L}$ such that for all $L \geq \widetilde{L}$ we have $\mathbf{P}(Q^{\lambda}_{\varepsilon,L}) > 1 - \varepsilon$. Then since both $F^S_L(x_L, \lambda/(2L)^d)$ and $\frac{G^{even}_{BW}(+(x_L),0)+1/\lambda}{G^{even}_{BW}(0,0)+1/\lambda}$ are in $[0, 1]$, for all $L \geq \widetilde{L}$

$$\sum_S \mathbf{P}(S) \left| F^S_L\left(x_L, \frac{\lambda}{(2L)^d}\right) - \frac{G^{even}_{BW}(+(x_L), 0) + \frac{1}{\lambda}}{G^{even}_{BW}(0, 0) + \frac{1}{\lambda}} \right| \leq 2\mathbf{P}((Q^{\lambda}_{\varepsilon,L})^c) + \varepsilon \leq 3\varepsilon.$$

□

Remark 2.29 *Clearly if* $\frac{G^{even}_{BW}(+(x_L),0)+\frac{1}{\lambda}-\mathbf{1}_{\{0\}}(+(x_L))}{G^{even}_{BW}(0,0)+\frac{1}{\lambda}}$ *has a limit* $f(\lambda)$ *then we have that* $F_L\left(x_L, \frac{\lambda}{(2L)^d}\right)$ *has limit* $f(\lambda)$. *Regarding* F^S_L *we can have existence of the limit provided that the sequence of small worlds* S *is chosen wisely. Indeed let* $K_n = [1/n, +\infty)$. *For all* n *we know that there exists* L_n *such that* $\mathbf{P}(S : b^{\lambda}_L(S) \leq 1/n, \forall \lambda \in K_n)$ *for all* $L \geq L_n$. *Thus if for all* $L \in [L_n, L_n + 1)$ *we choose* $S \in (S : b^{\lambda}_L(S) \leq 1/n, \forall \lambda \in K_n)$ *we get that also* $F^S_L\left(x_L, \frac{\lambda}{(2L)^d}\right)$ *has limit* $f(\lambda)$ *for all* $\lambda > 0$ *(uniformly with respect to* x_L *if* $\frac{G^{even}_{BW}(+(x_L),0)+\frac{1}{\lambda}-\mathbf{1}_{\{0\}}(+(x_L))}{G^{even}_{BW}(0,0)+\frac{1}{\lambda}}$ *converges uniformly with respect to* x_L*).*

2.3.4 From G to F: continuous time

Suppose $x_L \neq 0$. As in discrete time $G^S_L(x_L, \lambda/(2L)^d)$ may be written as

$$\sum_z \int_0^{\infty} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}^z_S(X_{2q} = z) dq \int_0^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}^{x_L,0}_S(T_L \in ds, X_s = z). \tag{2.3.16}$$

If $G^S_L\left(0, \frac{\lambda}{(2L)^d}\right)$ is equal to

$$1 + \sum_z \int_0^{\infty} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}^z_S(X_{2q} = z) dq \int_0^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}^{0,0}_S(T_L \in ds, X_s = z).$$

We define the continuous versions of H_1, H_2 and H_3 (which depend on S, x_L and L) by

$$H_1 := \sum_z \int_0^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}^z_S(X_{2q} = z) dq \int_0^{t_L} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}^{x_L,0}_S(T_L \in ds, X_s = z)$$

$$H_2 := \sum_z \int_0^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}^z_S(X_{2q} = z) dq \int_{t_L}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}^{x_L,0}_S(T_L \in ds, X_s = z)$$

$$H_3 := \sum_z \int_{t_L}^{\infty} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}^z_S(X_{2q} = z) dq \int_0^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}^{x_L,0}_S(T_L \in ds, X_s = z).$$

By Lemma 2.19, for all L sufficiently large and if the limit exists,

$$\lim_{L\to\infty} G_L\left(x_L, \frac{\lambda}{(2L)^d}\right) = \lim_{L\to\infty} \sum_{S \in Q_L} \mathbf{P}(S)(H_1 + H_2 + H_3). \tag{2.3.17}$$

Clearly if $x_L = 0$ for all L sufficiently large we only need to add 1 to the previous limit. We now study each of the three summands separately as in discrete time case.

Lemma 2.30 *If $S \in I(0,t_L^2)$ and $x_L \in \Lambda(L)$, for each $\varepsilon > 0$ there exists $\widetilde{L}$ such that for each $L > \widetilde{L}$ then*

$$\left| H_1 - \int_0^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}^0_{BW}(X_{2q}=0)dq \int_0^{t_L} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}^{x_L,0}_S(T_L \in ds) \right| < \varepsilon. \tag{2.3.18}$$

This inequality also holds whenever $d_S(0,x_L) > t_L^2$. Moreover, uniformly with respect to the choice of the sequence $\{x_L\}_L$ and of λ,

$$\lim_{L\to\infty} \sum_{S\in(I(0,t_L^2))^c} \mathbf{P}(S) \sum_z \int_0^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}^z_S(X_{2q}=z)dq \int_0^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}^{x_L,0}_S(T_L \in ds, X_s = z) = 0. \tag{2.3.19}$$

Proof Since in H_1, z is the site where the two random walks X_t and Y_t meet at a time $s \in [0,t_L]$, and since $S \in I(0,t_L^2)$ or $d_S(0,x_L) > t_L^2$, then

$$\left|\mathbb{P}^z_S(X_{2q}=z) - \mathbb{P}^0_{BW}(X_{2q}=0)\right| \le \mathbb{P}(Z(t) \ge t_L^2) \le \frac{2}{t_L^2},$$

which proves (2.3.18), since $\int_0^{t_L} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}^{x_L,0}_S(T_L = s)ds \le 1$. Formula (2.3.19) is proved as in discrete time case. □

Lemma 2.31 *For all $K \in \mathcal{K}$, $\varepsilon > 0$ there exists $\widetilde{L}$ such that for all $L \ge \widetilde{L}$, x_L and $\lambda \in K$,*

$$\left| \sum_{S\in Q_L} \mathbf{P}(S) H_2 - \int_0^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}^0_{BW}(X_{2q}=0)dq \sum_{S\in Q_L} \mathbf{P}(S) \int_{t_L}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}^{x_L,0}_S(T_L \in ds) \right| \le \varepsilon. \tag{2.3.20}$$

Proof As in discrete case $\sum_{S\in Q_L} \mathbf{P}(S)H_2$ can be written as

$$\sum_z \sum_{S\in Q_L\cap I(z,t_L^2)} \mathbf{P}(S) \int_0^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}^z_S(X_{2q}=z)dq \int_{t_L}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}^{x_L,0}_S(T_L \in ds, X_s = z)$$

$$+\sum_z \sum_{S\in Q_L\cap I(z,t_L^2)^c} \mathbf{P}(S) \int_0^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}^z_S(X_{2q}=z)dq \int_{t_L}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}^{x_L,0}_S(T_L \in ds, X_s = z)$$
$$= H_{2,1} + H_{2,2}.$$

We prove that $H_{2,2} \to 0$, indeed since $\exp(-\lambda q/(2L)^d)\mathbb{P}^z_S(X_{2q}=z) \le 1$ then

$$\begin{aligned} H_{2,2} \le t_L \sum_z \sum_{S\in Q_L\cap I(z,t_L^2)^c} \mathbf{P}(S) \Bigg\{ &\int_{t_L}^{\log L} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}^{x_L,0}_S(X_s = Y_s = z)ds \\ &+ \int_{\log L}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}^{x_L,0}_S(X_s = Y_s = z)ds \Bigg\} =: H_{2,2,1} + H_{2,2,2}. \end{aligned}$$

We can repeat the discrete time proof in order to show that $H_{2,2,1}$ and $H_{2,2,2}$ go to zero (uniformly in x_L and λ) as L goes to infinity: the only differences are that we work on

$I(z,t_L^2)$ instead of $I(z,2t_L)$ and we use estimation (2.2.7) in continuous time.
We now consider $H_{2,1}$.

$$H_{2,1} = \int_0^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}_S^z(X_{2q}=z)dq$$
$$\sum_z \sum_{S\in Q_L\cap I(z,t_L^2)} \mathbf{P}(S) \int_{t_L}^\infty e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L,0}(T_L\in ds, X_s=z).$$

We split

$$\begin{aligned}
&\sum_z \sum_{S\in Q_L\cap I(z,t_L^2)} \mathbf{P}(S) \int_{t_L}^\infty e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L,0}(T_L\in ds, X_s=z)\\
&= \sum_{S\in Q_L}\sum_z \mathbf{P}(S) \int_{t_L}^\infty e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L,0}(T_L\in ds, X_s=z)\\
&-\sum_z \sum_{S\in Q_L\cap I(z,t_L^2)^c} \mathbf{P}(S) \int_{t_L}^\infty e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L,0}(T_L\in ds, X_s=z).
\end{aligned}$$

the second summand converges to 0 by the same arguments we used to prove that $H_{2,2}$ converges to 0; we replace the first one in $H_{2,1}$ and we get

$$\int_0^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}_S^z(X_{2q}=z)dq \sum_{S\in Q_L} \mathbf{P}(S) \int_{t_L}^\infty e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L,0}(T_L\in ds).$$

If $S\in I(z,t_L^2)$ and $q\in[0,t_L]$, then $|\mathbb{P}_S^z(X_{2q}=z)-\mathbb{P}_{BW}^0(X_{2q}=0)|\le \mathbb{P}(Z(t_L)\ge t_L^2)\le 2/t_L^2$, thus

$$\begin{aligned}
&\Big|\int_0^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}_S^z(X_{2q}=z)dq \sum_{S\in Q_L} \mathbf{P}(S) \int_{t_L}^\infty e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L,0}(T_L\in ds)\\
&-\int_0^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}_{BW}^0(X_{2q}=0)dq \sum_{S\in Q_L} \mathbf{P}(S) \int_{t_L}^\infty e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L,0}(T_L\in ds)\Big|\\
&\le \int_0^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \Big|\mathbb{P}_S^z(X_{2q}=z)dq-\mathbb{P}_{BW}^0(X_{2q}=0)\Big|dq \sum_{S\in Q_L} \mathbf{P}(S) \int_{t_L}^\infty e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L,0}(T_L\in ds)\\
&\le \frac{2}{t_L}F(x_L,\lambda/(2L)^d)
\end{aligned}$$

which can be taken as small as we want if L is large. □

Lemma 2.32 *Let*

$$a_L^\lambda(S) := H_2 - \int_0^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}_{BW}^0(X_{2q}=0)dq \int_{t_L}^\infty e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L,0}(T_L\in ds). \qquad (2.3.21)$$

Then $a_L^\lambda>0$ and $a_L^\lambda\to 0$ in probability (for each $K\in\mathcal{K}$, uniformly in x_L and $\lambda\in K$), that is for all $K\in\mathcal{K}$, $\varepsilon>0$ and $\delta>0$ there exists $\widetilde{L}$ such that for all $L\ge\widetilde{L}$ and x_L

$$\mathbf{P}(A_L^\varepsilon(K)) := \mathbf{P}(S: a_L^\lambda(S)\le\varepsilon, \forall\lambda\in K)\ge 1-\delta.$$

Proof We get the result by contradiction with Lemma 2.31 as in proof of Lemma 2.25.
□

Lemma 2.33 *For all $K \in \mathcal{K}$ and $\varepsilon > 0$ there exists $\widetilde{L}$ such that for all $L \geq \widetilde{L}$, $S \in Q_L$, x_L and $\lambda \in K$,*

$$\left| H_3 - \frac{1}{\lambda} F_L^S(x_L, \lambda/(2L)^d) \right| \leq \varepsilon. \tag{2.3.22}$$

Proof Nothing change with respect to the proof of Lemma 2.26. □

Theorem 2.34 *Let*

$$b_L^\lambda(S) := \left| F_L^S \left(x_L, \frac{\lambda}{(2L)^d} \right) - \frac{\frac{1}{2} G_{BW}(+(x_L), 0) + \frac{1}{\lambda} - \mathbf{1}_{\{0\}}(+(x_L))}{\frac{1}{2} G_{BW}(0,0) + \frac{1}{\lambda}} \right|.$$

Then $b_L^\lambda \to 0$ in probability, for each $K \in \mathcal{K}$ uniformly in $x_L \in \Lambda(L)$ and $\lambda \in K$, namely for all $\varepsilon > 0$ $(S : b_L^\lambda(S) \leq \varepsilon, \forall \lambda \in K) \supset Q_L \cap I(0, t_L^2) \cap A_L^{\varepsilon/2}(K)$ for all L sufficiently large. Moreover for all $\varepsilon > 0$, $(S : b_L^\lambda(S) \leq \varepsilon, \forall \lambda \in K) \supset Q_L \cap (S : d_S(0, x_L) > t_L^2) \cap A_L^{\varepsilon/2}(K)$ for all L sufficiently large.

Proof Notice that

$$\begin{aligned}
\int_0^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}_{BW}^0(X_{2q} = 0) dq =& \frac{1}{2} \int_0^{2t_L} e^{-\frac{\lambda v}{2(2L)^d}} \mathbb{P}_{BW}^0(X_v = 0) dv \\
=& \frac{1}{2} G_{BW}(0,0) + o(1)
\end{aligned}$$

by Dominated Convergence Theorem and we get the result on $S \in Q_L \cap I(0, t_L^2) \cap A_L^{\varepsilon/2}(K)$ or $S \in Q_L \cap (S : d_S(0, x_L) > t_L^2) \cap A_L^{\varepsilon/2}(K)$ by using Lemmas 2.30, 2.32, 2.33 and Theorem 2.21 as in discrete time case. □

Theorem 2.35 *For each $\varepsilon > 0$, $K \in \mathcal{K}$ there exists $\widetilde{L}$ such that for each $L > \widetilde{L}$, $\lambda \in K$ and for each sequence $\{x_L\}_L$ such that $x_L \in \Lambda(L)$,*

$$\left| F_L \left(x_L, \frac{\lambda}{(2L)^d} \right) - \frac{\frac{1}{2} G_{BW}(+(x_L), 0) + \frac{1}{\lambda} - \mathbf{1}_{\{0\}}(+(x_L))}{\frac{1}{2} G_{BW}(0,0) + \frac{1}{\lambda}} \right| \leq \varepsilon.$$

Proof The proof is completely analogous to the one of Theorem 2.28 starting from Theorem 2.34. □

Notice that Remark 2.29 still holds in continuous time.

2.4 Hitting time of random walks

2.4.1 The limit in law of $T_L/(2L)^d$

It is clear that, if $G_{BW}^{even}(+(x_L), 0)$ has a limit as L goes to infinity, then Theorems 2.27 and 2.28 provide the results for the limit of $F_L^S(x_L, \lambda/(2L)^d)$ and $F_L(x_L, \lambda/(2L)^d)$. The limit of $G_{BW}^{even}(+(x_L), 0)$ exists for instance in two particular cases: $x_L = x$ for all L sufficiently large, or $|x_L| \to \infty$. In the first case clearly $\lim_{L \to \infty} G_{BW}^{even}(+(x_L), 0) = G_{BW}^{even}(+(x), 0)$. In the second case, since $G_{BW}^{even}(+(x_L), 0) = \sum_{2n=|x_L|} p^{(2n)}(+(x_L), 0)$, by the Dominated Convergence Theorem, $\lim_{L \to \infty} G_{BW}^{even}(+(x_L), 0) = 0$. Similar remarks hold in continuous

time case for $G_{BW}(+(x_L), 0)$.
We are now ready to prove Theorem 1.4.

Proof We prove the claim in discrete time. The proof in continuous time works in a similar way.

1. By Theorem 2.28 we know that for all $\lambda > 0$

$$F_L\left(x_L, \frac{\lambda}{(2L)^d}\right) \overset{L\to\infty}{\to} \frac{\lambda G^{even}_{BW}(+(x), 0) + 1 - \lambda \mathbf{1}_{\{0\}}(+(x))}{\lambda G^{even}_{BW}(0,0) + 1}. \tag{2.4.1}$$

Since for each L, F_L is a monotone function of λ and so is the right hand side of (2.4.1), which is also continuous in λ, it follows that (2.4.1) holds uniformly in $\lambda \geq 0$. Thus, if $x \neq 0$, $T_L/(2L)^d$ converges in law (with respect to $\mathbb{P}^{x,0}$) to

$$\frac{G^{even}_{BW}(+(x), 0)}{G^{even}_{BW}(0,0)}\delta_0 + \left(1 - \frac{G^{even}_{BW}(+(x), 0)}{G^{even}_{BW}(0,0)}\right) \exp\left(\frac{1}{G^{even}_{BW}(0,0)}\right),$$

while if $x = 0$ then it converges to

$$\left(\frac{1}{G^{even}_{BW}(0,0)}\right)\delta_0 + \frac{1}{G^{even}_{BW}(0,0)} \exp\left(\frac{1}{G^{even}_{BW}(0,0)}\right).$$

Then (1.5) and (1.6) hold, and by monotonicity they hold uniformly in $t \geq 0$.

2. It follows as in the previous step using the fact that $G^{even}_{BW}(+(x_L), 0) \to 0$ uniformly in $\{x_L\}_L$ such that $|x_L| \geq \alpha_L$. Indeed

$$G^{even}_{BW}(+(x_L), 0) = \sum_n \mathbb{1}_{\{n \geq |x_L|/2\}} \mathbb{P}^0_{BW}(X_{2n} = +(x_L)),$$

which goes to 0 by the Dominated Convergence Theorem since $\mathbb{P}^0_{BW}(X_{2n} = +(x_L)) \leq \mathbb{P}^0_{BW}(X_{2n} = 0)$ and $\sum_n \mathbb{P}^0_{BW}(X_{2n} = 0) = G^{even}_{BW}(0,0) < \infty$. Thus

$$G^{even}_{BW}(+(x_L), 0) \leq \sum_{n \geq \alpha_L/2} \mathbb{P}^0_{BW}(X_{2n} = 0).$$

3. As said in Remark 2.29, choosing for all $L \in [L_n, L_{n+1})$ the corresponding set of small worlds $S \in H_L = (b^\lambda_L(S) \leq 1/n, \forall \lambda \in [1/n, \infty))$, we have that for all $\lambda > 0$

$$F^S_L\left(x_L, \frac{\lambda}{(2L)^d}\right) \overset{L\to\infty}{\to} \frac{\lambda G^{even}_{BW}(+(x), 0) + 1}{\lambda G^{even}_{BW}(0,0) + 1}. \tag{2.4.2}$$

This, by an argument as in step 1, proves that

$$\mathbb{P}^{x_L,0}_S\left(\frac{T_L}{(2L)^d} > t\right) \overset{L\to\infty}{\to} \left(1 - \frac{G^{even}_{BW}(+(x), 0)}{G^{even}_{BW}(0,0)}\right) \exp\left(-\frac{t}{G^{even}_{BW}(0,0)}\right),$$

uniformly in $t \geq 0$. Thus if $L \in [L_n, L_{n+1})$, the event in (1.8) contains H_L and $\mathbf{P}(H_L) \overset{n\to\infty}{\to} 1$ implies the assertion.

4. Choosing S as in previous step, uniformly with respect to $\{x_L\}$ such that either $|x_L| \geq \alpha_L$ or $d_S(0, x_L) \geq \alpha_L$ we get

$$\mathbb{P}^{x_L,0}_S\left(\frac{T_L}{(2L)^d} > t\right) \overset{L\to\infty}{\to} \exp\left(-\frac{t}{G^{even}_{BW}(0,0)}\right),$$

uniformly in $t \geq 0$. This proves the claim.

$\square$

Remark 2.36 *Theorem* 1.4.4 *holds if we fix* $0 \in \Lambda(L)$ *and we consider the supremum over all possible* $x_L \in \Lambda(L)$ *such that* $d_S(x_L, 0) \geq \alpha_L$. *We can repeat the same proof to show that the result still holds if we take the supremum over all possible pairs* $(x_L, y_L) \in \Lambda(L) \times \Lambda(L)$ *such that* $d_S(x_L, y_L) \geq \alpha_L$. *Namely, choose* $\alpha_L > t_L^2$ *in continuous time and* $\alpha_L > t_L$ *in discrete time. Let* $t \geq 0$, *then for all* $\varepsilon > 0$

$$\mathbf{P}\left(S \in \mathcal{S}^L(\widetilde{\Omega}) : \sup_{(x_L,y_L)\in\Lambda(L)^2 : d_S(x_L,y_L)\geq\alpha_L} \left|\mathbb{P}_S^{x_L,y_L}\left(\frac{T_L}{(2L)^d} > t\right) - \exp\left(-\frac{t}{G(0,0)}\right)\right| > \varepsilon\right) \overset{L\to\infty}{\to} 0, \tag{2.4.3}$$

We observe that the same technique we employed to determine the asymptotic behaviour of the first encounter time of two random walkers, one starting at x_L and the other at 0, may be used to obtain similar results for the first time that a single random walker starting at x_L hits 0.

Theorem 2.37 *Let* W_L *be the first time that a random walk starting at* x_L *hits 0, in discrete or continuous time.*

1. *(Annealed limit,* $x_L \to x$*). Let* $x_L \in \Lambda(L)$ *for all* L *such that* $x_L = x$ *for all* L *sufficiently large. Then if* $x \neq 0$, *uniformly in* $t \geq 0$
$$\mathbb{P}^{x_L}\left(\frac{W_L}{(2L)^d} > t\right) \overset{L\to\infty}{\to} \left(1 - \frac{G_{BW}(+(x),0)}{G_{BW}(0,0)}\right)\exp\left(-\frac{t}{G_{BW}(0,0)}\right). \tag{2.4.4}$$
If $x = 0$ *then uniformly in* $t \geq 0$
$$\mathbb{P}^{0}\left(\frac{W_L}{(2L)^d} > t\right) \overset{L\to\infty}{\to} \frac{1}{G_{BW}(0,0)}\exp\left(-\frac{t}{G_{BW}(0,0)}\right). \tag{2.4.5}$$

2. *(Annealed limit,* $x_L \to \infty$*). If* $\alpha_L \to \infty$, *then uniformly in* $t \geq 0$ *and* x_L *such that* $|x_L| \geq \alpha_L$,
$$\mathbb{P}^{x_L}\left(\frac{W_L}{(2L)^d} > t\right) \to \exp\left(-\frac{t}{G_{BW}(0,0)}\right). \tag{2.4.6}$$

3. *(Quenched limit,* $x_L \to x$*). Let* $x_L \in \Lambda(L)$ *for all* L *such that* $x_L = x \neq 0$ *for all* L *sufficiently large. Let* $t \geq 0$*: for all* $\varepsilon > 0$
$$\mathbf{P}\left(S \in \mathcal{S}^L(\widetilde{\Omega}) : \left|\mathbb{P}_S^{x}\left(\frac{W_L}{(2L)^d} > t\right) - \left(1 - \frac{G_{BW}(+(x),0)}{G_{BW}(0,0)}\right)\exp\left(-\frac{t}{G_{BW}(0,0)}\right)\right| > \varepsilon\right) \overset{L\to\infty}{\to} 0. \tag{2.4.7}$$
If $x = 0$ *then*
$$\mathbf{P}\left(S \in \mathcal{S}^L(\widetilde{\Omega}) : \left|\mathbb{P}_S^{0}\left(\frac{W_L}{(2L)^d} > t\right) - \frac{1}{G_{BW}(0,0)}\exp\left(-\frac{t}{G_{BW}(0,0)}\right)\right| > \varepsilon\right) \overset{L\to\infty}{\to} 0. \tag{2.4.8}$$

4. *(Quenched limit,* $x_L \to \infty$*). Choose* $\alpha_L > t_L^2$ *in continuous time and* $\alpha_L > t_L$ *in discrete time. Let* $t \geq 0$*: for each* $\varepsilon > 0$
$$\mathbf{P}\left(S \in \mathcal{S}^L(\widetilde{\Omega}) \sup_{x_L : d_S(x_L,0)\geq\alpha_L} : \left|\mathbb{P}_S^{x_L}\left(\frac{W_L}{(2L)^d} > t\right) - \exp\left(-\frac{t}{G_{BW}(0,0)}\right)\right| > \varepsilon\right) \overset{L\to\infty}{\to} 0. \tag{2.4.9}$$

Proof (Discrete time) The proof is analogous to the one of Theorem 1.4 but easier, since we consider the return time of one walk instead of the meeting time and we do not have to use chains with return time on even steps. Notice that in this case the constant is the expected number of visits to 0 of the discrete time random walk on the big world starting at 0.
(Continuous time) By coupling we can show that, given $\widetilde{W}_L$ the first time that a continuous time random walk hits 0, namely

$$\widetilde{W}_L = \sum_{j=1}^{W_L} X_j$$

where $\{X_j\}_{j=1,\dots,\infty}$ are independent identically distributed exponential random variables with mean 1, then

Lemma 2.38 *If $W_L/\alpha_L \xrightarrow{d} X$, then $\widetilde{W}_L/\alpha_L \xrightarrow{d} X$.*

Proof By Slutsky First Theorem it is enough to prove that

$$\frac{W_L - \widetilde{W}_L}{\alpha_L} \xrightarrow{d} 0 \qquad L \to \infty$$

Notice that

$$\frac{W_L - \widetilde{W}_L}{\alpha_L} = \frac{1}{\alpha_L}\sum_{j=1}^{W_L}(X_j - 1) = \frac{\sum_{j=1}^{W_L}(X_j - 1)}{W_L^{1/2}}\frac{W_L^{1/2}}{\alpha_L}$$

We observe that $W_L^{1/2}/\alpha_L \xrightarrow{d} 0$. Indeed, for each $x > 0$

$$\mathbb{P}(\frac{W_L^{1/2}}{\alpha_L} > x) = \mathbb{P}(\frac{W_L}{\alpha_L} > \alpha_L x^2)$$

and for each $M > 0$, since $\alpha_L \to \infty$, there exists L such that $\alpha_L x^2 > M$

$$\mathbb{P}(\frac{W_L}{\alpha_L} > \alpha_L x^2) \leq \mathbb{P}(\frac{W_L}{\alpha_L} > M) \stackrel{L\to\infty}{\longrightarrow} \mathbb{P}(X > M)$$

which is as small as we want by taking M large.
By Slutsky Second Theorem we are left with proving that there exists the limit in law of

$$\frac{\sum_{j=1}^{W_L}(X_j - 1)}{W_L^{1/2}}.$$

We note that, as $L \to \infty$,

$$\mathbb{P}(W_L < \lfloor \alpha_L^{1-\epsilon} \rfloor) = \mathbb{P}(\frac{W_L}{\alpha_L} < \frac{\lfloor \alpha_L^{1-\epsilon} \rfloor}{\alpha_L}) \to 0 \quad \mathbb{P}(W_L > \lfloor \alpha_L^{1+\epsilon} \rfloor) = \mathbb{P}(\frac{W_L}{\alpha_L} > \frac{\lfloor \alpha_L^{1+\epsilon} \rfloor}{\alpha_L}) \to 0 \tag{2.4.10}$$

hence

$$
\begin{aligned}
\frac{\sum_{j=1}^{W_L}(X_j-1)}{W_L^{1/2}} &= \sum_{n=1}^{\infty} \mathbb{P}\left(\frac{\sum_{j=1}^{n}(X_j-1)}{n^{1/2}} \le x\right) \mathbb{P}(W_L = n) \\
&= \sum_{n=\lfloor \alpha_L^{1-\epsilon}\rfloor}^{\lfloor \alpha_L^{1+\epsilon}\rfloor} \mathbb{P}\left(\frac{\sum_{j=1}^{n}(X_j-1)}{n^{1/2}} \le x\right) \mathbb{P}(W_L = n) + o(1) \\
&= \sum_{n=1}^{\infty} \mathbb{1}_{(\lfloor \alpha_L^{1-\epsilon}\rfloor, \lfloor \alpha_L^{1+\epsilon}\rfloor)}(n) \mathbb{P}\left(\frac{\sum_{j=1}^{n}(X_j-1)}{n^{1/2}} \le x\right) \mathbb{P}(\frac{W_L}{\alpha_L} = \frac{n}{\alpha_L}) + o(1) \\
&= \int_{\lfloor \alpha_L^{1-\epsilon}\rfloor/\alpha_L}^{\lfloor \alpha_L^{1+\epsilon}\rfloor/\alpha_L} \mathbb{P}\left(\frac{\sum_{j=1}^{\lfloor t\alpha_L\rfloor}(X_j-1)}{\lfloor t\alpha_L\rfloor^{1/2}} \le x\right) d\nu_L(t) + o(1)
\end{aligned}
$$

where $\nu_L(t)$ is such that $\nu_L(\frac{n}{\alpha_L}) := \mathbb{P}(\frac{W_L}{\alpha_L} = \frac{n}{\alpha_L})$ for all n. Notice that $t\alpha_L \ge \alpha_L^{1-\epsilon}$; hence for each $\delta > 0$ there exists $\widetilde{L}$ such that for all $t \in [\lfloor \alpha_L^{1-\epsilon}\rfloor/\alpha_L, \lfloor \alpha_L^{1+\epsilon}\rfloor/\alpha_L]$ and for all $L \ge \widetilde{L}$,

$$
\left| \mathbb{P}\left(\frac{\sum_{j=1}^{\lfloor t\alpha_L\rfloor}(X_j-1)}{\lfloor t\alpha_L\rfloor^{1/2}} \le x\right) - \Phi(x) \right| < \delta
$$

and

$$
\begin{aligned}
&\left| \int_0^{\infty} \mathbb{P}\left(\frac{\sum_{j=1}^{\lfloor t\alpha_L\rfloor}(X_j-1)}{\lfloor t\alpha_L\rfloor^{1/2}} \le x\right) d\nu_L(t) - \Phi(x) \right| \\
&\le \left| \Phi(x) \left(\int_0^{\lfloor \alpha_L^{1-\epsilon}\rfloor/\alpha_L} d\nu_L(t) + \int_{\lfloor \alpha_L^{1+\epsilon}\rfloor/\alpha_L}^{\infty} d\nu_L(t) \right) + \delta \right|
\end{aligned}
$$

where $\Phi(x)$ is the law of the standard normal distribution. Since δ is arbitrarily small and by (2.4.10) we get

$$
\frac{\sum_{j=1}^{W_L}(X_j-1)}{W_L^{1/2}} \xrightarrow{d} \mathcal{N}(0,1)
$$

and we are done. □

For claims 1, 2, 3 and 4 we repeat the proof of Theorem 1.4. □

As a corollary of Theorems 1.4 and 2.37 we get a similar convergence result for random walkers starting from the stationary distribution π, both in discrete and in continuous time.

Corollary 2.39 *1. Uniformly in $t \ge 0$*

$$
\mathbb{P}^{\pi}\left(\frac{W_L}{(2L)^d} > t\right) \stackrel{L\to\infty}{\to} \exp\left(-\frac{t}{G_{BW}(0,0)}\right) \tag{2.4.11}
$$

$$
\mathbb{P}^{\pi}\left(\frac{T_L}{(2L)^d} > t\right) \stackrel{L\to\infty}{\to} \exp\left(-\frac{t}{G(0,0)}\right). \tag{2.4.12}
$$

2. For all $t \geq 0$ *and* $\varepsilon > 0$

$$\mathbf{P}\left(S \in \mathcal{S}^L(\widetilde{\Omega}) : \left|\mathbb{P}_S^{\pi}\left(\frac{W_L}{(2L)^d} > t\right) - \exp\left(-\frac{t}{G_{BW}(0,0)}\right)\right| > \varepsilon\right) \overset{L\to\infty}{\to} 0 \qquad (2.4.13)$$

$$\mathbf{P}\left(S \in \mathcal{S}^L(\widetilde{\Omega}) : \left|\mathbb{P}_S^{\pi}\left(\frac{T_L}{(2L)^d} > t\right) - \exp\left(-\frac{t}{G(0,0)}\right)\right| > \varepsilon\right) \overset{L\to\infty}{\to} 0. \qquad (2.4.14)$$

Proof 1) Let $t_L = \log\log L$, by Theorem 2.37 *(2)* for each $\epsilon > 0$ there exists $\widetilde{L}$ such that for each $L \geq \widetilde{L}$

$$\begin{aligned}
&\left|\mathbb{P}^{\pi}\left(\frac{W_L}{(2L)^d} > t\right) - \exp\left(-\frac{t}{G_{BW}(0,0)}\right)\right| \\
&\qquad = \left|\sum_{x_L \in \Lambda(L)} \mathbb{P}^{x_L}\left(\frac{W_L}{(2L)^d} > t\right)\frac{1}{(2L)^d} - \exp\left(-\frac{t}{G_{BW}(0,0)}\right)\right| \\
&\qquad \leq \sum_{x_L : d(0,x_L) < t_L} \frac{1}{(2L)^d}\left|\mathbb{P}^{x_L}\left(\frac{W_L}{(2L)^d} > t\right) - \exp\left(-\frac{t}{G_{BW}(0,0)}\right)\right| \\
&\qquad\quad + \sum_{x_L : d(0,x_L) \geq t_L} \frac{1}{(2L)^d}\left|\mathbb{P}^{x_L}\left(\frac{W_L}{(2L)^d} > t\right) - \exp\left(-\frac{t}{G_{BW}(0,0)}\right)\right| \\
&\qquad \leq \frac{2(2t_L)^d}{(2L)^d} + \epsilon/2 \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

since $(2t_L)^d/(2L)^d \to 0$ as L goes to infinity. We can prove the result involving the meeting time in a similar way by using Theorem 1.4 and Remark 2.36.

2) The proof is slightly different. Let $t_L = \log\log L$. We fix $\delta > 0$. By Theorem 1.4.4, for each $\varepsilon > 0$, there exists $\widetilde{L}$ large such that for each $L \geq \widetilde{L}$

$$\begin{aligned}
&\mathbf{P}\Big(S \in \mathcal{S}^L : \left|\mathbb{P}_S^{\pi}\left(\frac{W_L}{(2L)^d} > t\right) - \exp\left(-\frac{t}{G_{BW}(0,0)}\right)\right| > \delta\Big) \\
&\leq \mathbf{P}\Big(S \in \mathcal{S}^L : \Big(\sum_{x : d_S(0,x_L) < t_L} + \sum_{x : d_S(0,x_L) \geq t_L}\Big)\left|\mathbb{P}_S^{x}\left(\frac{W_L}{(2L)^d} > t\right) - \exp\left(-\frac{t}{G_{BW}(0,0)}\right)\right|\frac{1}{(2L)^d} > \delta\Big) \\
&\leq \mathbf{P}\Big(S \in \mathcal{S}^L : \frac{M^{t_L}}{(2L)^d} + \sup_{\{x_L : d(x_L,0) \geq t_L\}}\left|\mathbb{P}_S^{x_L}\left(\frac{W_L}{(2L)^d} > t\right) - \exp\left(-\frac{t}{G_{BW}(0,0)}\right)\right| > \delta\Big) \\
&\leq \mathbf{P}\Big(S \in \mathcal{S}^L : \sup_{\{x_L : d(x_L,0) \geq t_L\}}\left|\mathbb{P}_S^{x_L}\left(\frac{W_L}{(2L)^d} > t\right) - \exp\left(-\frac{t}{G_{BW}(0,0)}\right)\right| > \delta/2\Big) \\
&+ \mathbf{P}\Big(S \in \mathcal{S}^L : \frac{M^{t_L}}{(2L)^d} > \delta/2\Big) \leq \varepsilon
\end{aligned}$$

since we can choose L large such that $\frac{M^{t_L}}{(2L)^d} \leq \delta/2$.
We can prove results involving the meeting times in a similar way by using Theorem 1.4 and Remark 2.36. □

2.4.2 Comparison with the d-dimensional torus

As we observed in the introduction, in order to have convergence of the rescaled meeting time on the d-dimensional torus and on the small world with $d \geq 3$ the rescaling factor is $(2L)^d$. In order to understand whether two particles meet faster on the torus or on the small world, we need to compare the constants $G_{\mathbb{Z}^d}(0,0)$ and $G_{BW}(0,0)$. Proposition 1.5 gives some information in this direction.

Proof of Proposition 1.5

i) Since $\mathbb{P}^0_{BW}(X_{2n}=0) \geq \beta^{2n}$ we get

$$G_{BW}(0,0) \geq \sum_{n=0}^{\infty} \beta^{2n} = \frac{1}{1-\beta^2}.$$

Since $d \geq 3$, $G_{\mathbb{Z}^d}(0,0) < \infty$ therefore there exists $\beta_1 > 0$ close to 1 such that $\frac{1}{1-\beta^2} > G_{\mathbb{Z}^d}(0,0)$ for each $\beta \in [\beta_1, 1]$ and the claim follows. In particular

$$\lim_{\beta \to 1^-} G_{BW}(0,0) = \infty.$$

ii) Given an irreducibile Markov chain (Y, Q) we define

$$G(z) = \sum_{n=0}^{\infty} \mathbb{P}^0(Y_n = 0) z^n$$
$$F(z) = \sum_{n=0}^{\infty} \mathbb{P}^0(Y_n = 0, Y_s \neq 0 \text{ for all } s < n) z^n.$$

By [52, Proposition 9.10], there exists $r > 0$ and a function $\Phi(\cdot)$ such that

$$G(z) = \Phi(zG(z)), \qquad z \in [0, r). \tag{2.4.15}$$

Moreover there exists Φ' and Φ'' and $\Phi(\cdot)$ is strictly increasing and strictly convex.
Let P be the transition matrix on the big world defined in Section 2.2.1 in nearest neighbor case. We denote by $\Phi_{\mathbb{Z}^d * \mathbb{Z}_2}$, $\Phi_{\mathbb{Z}^d}$ and $\Phi_{\mathbb{Z}_2}$ the functions which satisfiy (2.4.15) respectively for the Markov chain (X, P) on the big world, for the simple random walk on $\mathbb{Z}^d$ and for the simple random walk on $\mathbb{Z}_2$.
The function $\Phi_{\mathbb{Z}_2}(t)$ can be computed explicitly,

$$\Phi_{\mathbb{Z}_2}(t) = \frac{1}{2}(1 + \sqrt{1+4t^2}).$$

By [52, Theorem 9.19]

$$\Phi_{\mathbb{Z}^d * \mathbb{Z}_2}(t) = \frac{1}{2}(1 + \sqrt{1+4\beta^2 t^2}) + \Phi_{\mathbb{Z}^d}((1-\beta)t) - 1.$$

By choosing $t = G_{\mathbb{Z}^d * \mathbb{Z}_2}(1)$ we get $\Phi_{\mathbb{Z}^d * \mathbb{Z}_2}(t) = t$ by (2.4.15). We denote by $G_\beta = G_{\mathbb{Z}^d * \mathbb{Z}_2}(1) = G_{BW}(0,0)$, then

$$G_\beta = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\beta^2 G_\beta^2} + \Phi_{\mathbb{Z}^d}((1-\beta)G_\beta).$$

We denote by $G = G_{\mathbb{Z}^d}(1) = G_{\mathbb{Z}^d}(0,0)$. By (2.4.15), G is a fixed point of $\Phi_{\mathbb{Z}^d}$, then $\lim_{\beta\to 0} G_\beta = G$. Notice that as $\beta \to 0$

$$\sqrt{1+4\beta^2 G_\beta^2} = 1 + 2\beta^2 G_\beta^2 + o(\beta^3 G_\beta^3).$$

and by Taylor series of $\Phi_{\mathbb{Z}^d}$ centered at G with Lagrange form of the remainder:

$$\Phi_{\mathbb{Z}^d}((1-\beta)G_\beta) = \Phi_{\mathbb{Z}^d}(G) + \Phi'_{\mathbb{Z}^d}(G)\left[(1-\beta)G_\beta - G\right] + \frac{1}{2}\Phi''_{\mathbb{Z}^d}(y)(y-G)^2,$$

where y is between G and $(1-\beta)G_\beta$. Two useful formulas for Φ' and Φ'' can be found in [52, p. 99]:

$$\Phi'(t) = 1/(z + G(z)/G'(z)), \qquad \Phi''(t) = (G(z)/(G(z)+zG'(z)))^3 F''(z),$$

where z is such that $t = zG(z)$. If $t = G$ then

$$\Phi'_{\mathbb{Z}^d}(G) = \frac{G'}{G'+G},$$

where $G' = \frac{d}{dz}G_{\mathbb{Z}^d}(z)|_{z=1^-}$. By convexity $\Phi'' > 0$. Moreover

$$\Phi''_{\mathbb{Z}^d}(G) = \Big(\frac{G}{G+G'}\Big)^3 F''(1).$$

Therefore

$$G_\beta = \beta^2 G_\beta^2 + G + \frac{G'}{G'+G}\Big[G_\beta - G - \beta G_\beta\Big] + \frac{1}{2}\Phi''_{\mathbb{Z}^d}(y)(y-G)^2 + o(\beta^3 G_\beta^3)$$

Since $(y-G)^2 \le (G_\beta - G - \beta G_\beta)^2$ we get

$$(G_\beta - G)\frac{G}{G'+G} \le \beta^2 G_\beta^2 - \beta G_\beta \frac{G'}{G'+G} + C_\beta\Big[(G_\beta - G)^2 + \beta^2 G_\beta^2 - 2\beta G_\beta (G_\beta - G) + o(\beta^3 G_\beta^3)\Big]$$

where $C_\beta = \frac{1}{2}\Phi''_{\mathbb{Z}^d}(y)(y-G)^2$. Thus

$$(G_\beta - G)\Big[\frac{G}{G'+G} + \Phi''_{\mathbb{Z}^d}(y)\beta G_\beta\Big] \le -\beta G_\beta \frac{G'}{G'+G} + C_\beta (G_\beta - G)^2 + \beta^2 G_\beta^2 (1 + C_\beta + o(\beta G_\beta)$$

$$(G_\beta - G)\Big[\frac{G}{G'+G} + \Phi''_{\mathbb{Z}^d}(y)\beta G_\beta - C_\beta (G_\beta - G)\Big] \le -\beta G_\beta \Big[\frac{G'}{G'+G} + \beta G_\beta (1 + C_\beta + o(\beta G_\beta)\Big]$$

By continuity of $\Phi''_{\mathbb{Z}^d}$ we get that $\Phi''_{\mathbb{Z}^d}(y) \overset{\beta\to 0}{\to} \Phi''_{\mathbb{Z}^d}(1)$, then the coefficient of $(G_\beta - G)$ on the left hand side as $\beta \to 0$ is asymptotically $\frac{G}{G+G'}$, which is strictly positive; the coefficient of βG_β on the right hand side is asymptotically $-G'/(G+G')$, which is strictly negative and the claim follows. $\square$

Chapter 3

Coalescing random walk on small world

The goal of this chapter is to prove a convergence result for coalescing random walk of n particles on BC small world. From now on we refer to continuous time process.

Let $\mathcal{I}(n) = \{\{x_1, \dots, x_n\} : x_i \in \Lambda(L), x_i \neq x_j\}$. Given $A \in \mathcal{I}(n)$, let $\{(X_t^S(x_i))_{t\geq 0}\}_{x_i \in A}$ be a family of independent random walks on small world $S \in \mathcal{S}^L$ such that $(X_t^S(x_i))_{t\geq 0}$ starts from x_i. We define for each $(x_i, x_j) \in \Lambda(L) \times \Lambda(L)$ and $S \in \mathcal{S}^L$

$$\tau_S(i,j) := \inf\{s > 0 : X_s^S(x_i) = X_s^S(x_j)\} \tag{3.1}$$

and for each $A \in \mathcal{I}(n)$

$$\tau_S(A) := \inf_{\{x_i, x_j\} \subseteq A} \{\tau_S(i,j)\}. \tag{3.2}$$

We fix $A \in \mathcal{I}(n)$, $S \in \mathcal{S}^L$ and we move n random walks starting from $x_i \in A$, $i = 1, \dots, n$ independently until the time $\tau_S(A)$, that is until a couple reaches the same site. After the meeting, the two particles coalesce and move together. We remain with $n-1$ walks that move independently until two of them reach the same site and proceed this way until we get only one particle. The time that all particles need to coalesce into one is the *coalescing time* and the process is the *coalescing random walk*.
The Kingman's coalescent is a Markov chain D_t on $\{0, 1, \dots, n\}$, where $n \leq \infty$, with transition mechanism

$$n \to n-1 \text{ at rate } \binom{n}{2}.$$

The law $P_n(D_t = k) = q_{n,k}(t)$ is given by

$$q_{n,k}(t) = \sum_{j=k}^{n} \frac{(-1)^{j+k}(2j-1)(j+k-2)!\binom{n}{j}}{k!(k-1)!(j-k)!\binom{n+j-1}{j}} \exp\left(-t\binom{j}{2}\right);$$

$$q_{\infty,k}(t) = \sum_{j=k}^{\infty} \frac{(-1)^{j+k}(2j-1)(j+k-2)!}{k!(k-1)!(j-k)!} \exp\left(-t\binom{j}{2}\right).$$

See [16], [18] and [49] for more about Kingman's coalescent.
We define

$$\mathcal{A}^L(h,n) := \Big\{A \in \mathcal{I}_n : d(x_i, x_j) > h, \text{ for all } i \neq j\Big\} \tag{3.3}$$

$$\mathcal{A}_S^L(h,n) := \Big\{A \in \mathcal{I}_n : d_S(x_i, x_j) > h, \text{ for all } i \neq j\Big\} \tag{3.4}$$

the set of n-uples with distance larger than h respectively on $\Lambda(L)$ and on a fixed small world S. Notice that $\mathcal{A}^L_S(h,n) \subseteq \mathcal{A}^L(h,n)$ for all $S \in \mathcal{S}^L$. Given $A \in \mathcal{A}^L(h,n)$, we introduce

$$\mathcal{D}(A) := \Big\{ S \in \mathcal{E}^L : A \in \mathcal{A}^L(h,n) \setminus \mathcal{A}^L_S(h,n) \Big\}. \tag{3.5}$$

Remember that we focus on the nearest neighbor case, but all the results can be extended to the case with neihbourhood structure given by $\mathcal{N}^\infty_m$.
From now on when not necessary we omit the dependence on S and we simply write $X_t(x_i)$, $\tau(i,j)$ and $\tau(A)$.
Given a probability measure μ on $\Lambda(L)^n$, we denote by $\mathbb{P}^\mu_S$ the law of the coalescing random walk on S with initial probability μ and transition ruled by P_S. If $\mu = \delta_A$ with $|A| = n$, we write $\mathbb{P}^A_S$.
In [24, Theorem 6.9.4] the author sketched a proof that coalescing random walk on one dimensional small world starting from the stationary distribution converges to Kingman's coalescent. We follow the same idea in order to prove a slightly more general result. A similar approach has been used to prove [16, Theorem 5] and in [18].
We begin from n particles in $A \in \mathcal{A}^L(h,n)$. We prove that by taking a particular $h := h_L$ and L large we get that $A \in \mathcal{A}^L_S(h,n)$ with large probability. We assume

$$h_L \geq t_L^2 \tag{3.6}$$

$$\lim_{L\to\infty} \frac{M^{h_L}}{(2L)^d} = 0 \tag{3.7}$$

where $M = (2m+1)^d$ or $M = 2d+1$ depending on the neighbourhood structure we work with.

Remark 3.1 *If $h_L = t_L^2$ hypothesis* (3.6) *and* (3.7) *are satisfied.*

Lemma 3.2 *Let h_L be such that* (3.6) *and* (3.7) *hold. For each $n \leq N < \infty$, $\epsilon > 0$ there exists $\widetilde{L}$ such that for each $L > \widetilde{L}$ and $A \in \mathcal{A}^L(h_L,n)$,*

$$\mathbf{P}\big(\mathcal{D}(A)\big) < \epsilon. \tag{3.8}$$

Proof Let $A \in \mathcal{A}^L(h_L,n)$. Notice that if $S \in \mathcal{D}(A)$, then there exists at least one pair of elements $(x_i,x_j) \in A \times A$, $i \neq j$, such that $d_S(x_i,x_j) < h_L$. By (3.5) and (2.2.4)

$$\begin{aligned}
\mathbf{P}\big(\mathcal{D}(A)\big) =& \mathbf{P}\big(S \in \mathcal{E}^L : \exists (x_i,x_j) \in A \times A : d_S(x_i,x_j) \leq h_L\big) \\
\leq& \sum_{i=1}^n \sum_{j=1, j\neq i}^n \mathbf{P}\big(S \in \mathcal{E}^L : d_S(x_i,x_j) \leq h_L, (x_i,x_j) \in A \times A\big) \\
\leq& n^2 \frac{CM^{h_L}}{L^d}.
\end{aligned}$$

Since n is fixed, by hypothesis (3.7) we can take L large such that the claim follows. $\square$

Therefore if we fix $A \in \mathcal{A}^L(h_L, n)$, we know that with large probability $A \in \mathcal{A}^L_S(h_L, n)$. By Remark 2.36, there exists a sequence $\{H_L\}_L$ with $H_L \subseteq \mathcal{S}^L$ such that $\mathbf{P}(H_L) \stackrel{L\to\infty}{\to} 1$ and for each sequence $\{S^L\}_L$ with $S^L \in H_L$

$$\sup_{(x_L,y_L):d_S(x_L,y_L)\geq \alpha_L} \left| \mathbb{P}_S^{x_L,y_L}\left(\frac{T_L}{(2L)^d} > t\right) - \exp\left(-\frac{t}{G(0,0)}\right)\right| \stackrel{L\to\infty}{\to} 0 \tag{3.9}$$

where $\alpha_L \geq t_L^2$. Notice that (3.9) still holds for the sequence $\{Q_L \cap H_L\}_L$ and $\mathbf{P}(Q_L \cap H_L) \stackrel{L\to\infty}{\to} 1$. In order to avoid cumbersome notations, we write H_L instead of $H_L \cap Q_L$. The following lemma states that, starting from 4 particles in a set of small world with large probability, when two particles meet the others are distant.

Lemma 3.3 *Suppose h_L satisfies (3.7) and $\lim_{L\to\infty} h_L = \infty$. For each $\epsilon > 0$ there exists $\widetilde{L}$, $\delta > 0$ such that for each $L > \widetilde{L}$, $S \in H_L$ and $A \in \mathcal{A}^L_S(h_L, 4)$,*

$$\int_0^\infty \mathbb{P}_S^A\Big(\tau(1,2) \in ds, d_S(X_s(x_1), X_s(x_3)) \leq h_L\Big) < \epsilon, \tag{3.10}$$

$$\int_0^\infty \mathbb{P}_S^A\Big(\tau(1,2) \in ds, d_S(X_s(x_3), X_s(x_4)) \leq h_L\Big) < \epsilon. \tag{3.11}$$

Proof We prove (3.10). We can obtain (3.11) in a similar way. We follow the idea of [16, (3.5)].
Let $\gamma = 1 - e^{-c\alpha^2} > 0$, where c and α are introduced in Section 2.2.3. We split the integral in two parts. By Theorem 1.4.4 and by hypothesis (3.7), there exists $\widetilde{L}$ such that for each $L > \widetilde{L}$

$$\begin{aligned}
&\int_0^{\frac{d}{\gamma}\log(2L)} \mathbb{P}_S^A\big(\tau(1,2) \in ds, d_S(X_s(x_1), X_s(x_3)) \leq h_L\big) \leq \int_0^{\frac{d}{\gamma}\log(2L)} \mathbb{P}_S^A\big(\tau(1,2) \in ds\big) \\
&= \mathbb{P}_S^{x_1,x_2}\Big(T_L \leq \frac{d}{\gamma}\log(2L)\Big) = \mathbb{P}_S^{x_1,x_2}\Big(\frac{T_L}{(2L)^d} \leq \frac{d\log(2L)}{\gamma(2L)^d}\Big) \\
&= 1 - \exp\Big(-\frac{2d\log(2L)}{\gamma G_{BW}(0,0)(2L)^d}\Big) + \epsilon/6 \\
&< \epsilon/6 + \epsilon/6 = \epsilon/3.
\end{aligned} \tag{3.12}$$

The second part is

$$\begin{aligned}
&\int_{\frac{d}{\gamma}\log(2L)}^\infty \mathbb{P}_S^A\big(\tau(1,2) \in ds, d_S(X_s(x_1), X_s(x_3)) \leq h_L\big) \\
&= \int_{\frac{d}{\gamma}\log(2L)}^\infty \sum_{y\in\Lambda(L)} \mathbb{P}_S^A\big(\tau(1,2) \in ds, X_s(x_1) = y\big) \sum_{z:d_S(y,z)\leq h_L} \mathbb{P}_S^A\big(X_s(x_3) = z\big) \\
&\leq \int_{\frac{d}{\gamma}\log(2L)}^\infty \sum_{y\in\Lambda(L)} \mathbb{P}_S^A\big(\tau(1,2) \in ds, X_s(x_1) = y\big) \sum_{z:d_S(y,z)\leq h_L} \Big|\mathbb{P}_S^{x_3}(X_s = z) - \frac{1}{(2L)^d}\Big| \\
&+ \int_{\frac{d}{\gamma}\log(2L)}^\infty \sum_{y\in\Lambda(L)} \mathbb{P}_S^A\big(\tau(1,2) \in ds, X_s(x_1) = y\big) \sum_{z:d_S(y,z)\leq h_L} \frac{1}{(2L)^d} := I(1) + I(2).
\end{aligned}$$

Notice that the number of sites z such that $d_S(y,z) \le h_L$ is at most M^{h_L} for each $y \in \Lambda(L)$. Hence

$$\begin{aligned}
I(2) \le & \int_{\frac{d}{\gamma}\log(2L)}^{\infty} \sum_{y\in\Lambda(L)} \mathbb{P}_S^A\big(\tau(1,2)\in ds, X_s(x_1)=y\big)\frac{M^{h_L}}{(2L)^d} \le \int_{\frac{d}{\gamma}\log(2L)}^{\infty} \mathbb{P}_S^A\big(\tau(1,2)\in ds\big)\frac{M^{h_L}}{(2L)^d} \\
=& \mathbb{P}_S^{x_1,x_2}\big(T_L > \frac{d}{\gamma}\log(2L)\big)\frac{M^{h_L}}{(2L)^d} \le \frac{M^{h_L}}{(2L)^d} < \epsilon/3 \qquad (3.13)
\end{aligned}$$

by (3.7). By (2.2.9) and since if $s \ge \frac{d}{\gamma}\log(2L)$ then $e^{-\gamma s} \le \frac{1}{(2L)^d}$ the last term is

$$\begin{aligned}
I(1) =& \int_{\frac{d}{\gamma}\log(2L)}^{\infty} \sum_{y\in\Lambda(L)} \mathbb{P}_S^A\big(\tau(1,2)\in ds, X_s(x_1)=y\big) \sum_{z:d_S(y,z)\le h_L} \Big|\mathbb{P}_S^{x_3}(X_s=z) - \frac{1}{(2L)^d}\Big| \\
\le & \int_{\frac{d}{\gamma}\log(2L)}^{\infty} \sum_{y\in\Lambda(L)} \mathbb{P}_S^A\big(\tau(1,2)\in ds, X_s(x_1)=y\big) \sum_{z:d_S(y,z)\le h_L} e^{-\gamma s} \\
\le & \int_{\frac{d}{\gamma}\log(2L)}^{\infty} \sum_{y\in\Lambda(L)} \mathbb{P}_S^A\big(\tau(1,2)\in ds, X_s(x_1)=y\big)\frac{M^{h_L}}{(2L)^d} \le \frac{M^{h_L}}{(2L)^d} \le \epsilon/3 \qquad (3.14)
\end{aligned}$$

and the claim follows by (3.12), (3.13) and (3.14). $\square$

Remark 3.4 *Lemma* 3.3 *holds if for all L we choose $S \in H_L$; therefore it is still true if for all $A \in \mathcal{A}^L(h_L,n)$ we choose $S \in H_L \cap \mathcal{D}(A)^c$. Moreover by Lemma* 1.4 *and Lemma* 3.2

$$\mathbf{P}((H_L \cap \mathcal{D}(A)^c)^c) \le \mathbf{P}(H_L^c) + \mathbf{P}(\mathcal{D}(A)) \qquad (3.15)$$

which is as small as we want by taking L large.

Let $\{\xi_t^S(A)\}_{t\ge 0}$ be the coalescing random walk starting from $A \in \mathcal{I}(n)$ on $S \in \mathcal{S}^L$ and let $|\xi_t^S(A)|$ be the number of particles of $\xi_t^S(A)$ at time t. Since S is a fixed small world on H_L, we will omit the dependency on S. We prove that the number of particles in the coalescing random walk at time $s_L t$, where $s_L = (2L)^d G_{BW}(0,0)/2$ converges in law to the number of particles of a Kingman's coalescent $\{D_t\}_{t\ge 0}$. We prove the result by induction on the number of particles n. If $n=2$, the induction basis is given by Theorem 1.4.4. The following lemma shows that the assertion is true before the first collision of two particles.

Lemma 3.5 *Assume Hypothesis* (3.6) *and* (3.7). *For each $n \in \mathbb{N}$, $A \in \mathcal{A}^L(h_L,n)$, and $\epsilon > 0$ there exists $\widetilde{L}$ such that for each $L > \widetilde{L}$, $S \in H_L \cap \mathcal{D}(A)^c$ and $t \ge 0$,*

$$\Big|\mathbb{P}_S(|\xi_{s_L t}(A)| = n) - \exp\Big(-\binom{n}{2}t\Big)\Big| < \epsilon$$

where $s_L = (2L)^d G_{BW}(0,0)/2$.

Proof We follow the proof of [16, Equation (3.1)]. Notice that $\mathbb{P}_S(|\xi_{s_L t}(A)| = n)$ and $\exp\Big(-\binom{n}{2}t\Big)$ are non-increasing monotone t functions. We define, for each couple $\{i,j\} \subseteq \{1,2,\ldots,n\}$,

$$H_t(i,j) = \{\tau(i,j) \le s_L t\} \qquad (3.16)$$

$$q_t = q_t(A) = \mathbb{P}(\tau(A) \le s_L t). \qquad (3.17)$$

For all $S \in H_L \cap \mathcal{D}(A)^c$,

$$\mathbb{P}_S^A\big(H_t(i,j)\big) = \mathbb{P}_S^A\big(\tau = \tau(i,j) \le s_L t\big) + \sum_{\{k,l\}\neq\{i,j\}} \int_0^{s_L t} \mathbb{P}_S^A\big(\tau = \tau(k,l) \in ds, \tau(i,j) \le s_L t\big) \tag{3.18}$$

We begin from the second term on the right hand side,

$$\begin{aligned}
&\int_0^{s_L t} \mathbb{P}_S^A\big(\tau = \tau(k,l) \in ds, \tau(i,j) \le s_L t\big)\\
&= \int_0^{s_L t} \sum_y \sum_z \mathbb{P}_S^A\big(\tau = \tau(k,l) \in ds, X_s(x_i) = y, X_s(x_j) = z, \tau(i,j) \le s_L t\big)\\
&= \int_0^{s_L t} \sum_y \sum_z \mathbb{P}_S^A\big(\tau = \tau(k,l) \in ds, X_s(x_i) = y, X_s(x_j) = z\big)\mathbb{P}_S^{y,z}\big(T_L \le s_L t - s\big).
\end{aligned}$$

By Lemma 3.3 for all L sufficiently large

$$\begin{aligned}
&\int_0^{s_L t} \sum_y \sum_{z: d_S(y,z)\le h_L} \mathbb{P}_S^A\big(\tau = \tau(k,l) \in ds, X_s(x_i) = y, X_s(x_j) = z, \tau(i,j) \le s_L t\big)\\
&\le \int_0^{\infty} \mathbb{P}_S^A\big(\tau = \tau(k,l) \in ds, d_S(X_s(x_i), X_s(x_j)) \le h_L\big) \le \varepsilon/(8n^4)
\end{aligned}$$

for all choices of $S \in H_L \cap \mathcal{D}(A)^c$, $\{i,j\} \subseteq \{1,\dots n\}$ and $t \ge 0$.
By Remark 2.36, $|\mathbb{P}_S^{y,z}(T_L \le s_L t - s) - 1 + \exp(-t + s/s_L)| < \varepsilon/(8n^4)$ for all L sufficiently large and for all choices of $S \in H_L \cap \mathcal{D}(A)^c$, y and z such that $d_S(y,z) \ge h_L$, $0 \le s \le t$. Then evaluate the remaining part of the integral, it does not differ by more than $\varepsilon/(8n^4)$ from

$$\begin{aligned}
&= \int_0^{s_L t} \sum_y \sum_{z: d_S(y,z)\ge h_L} \mathbb{P}_S^A\big(\tau = \tau(k,l) \in ds, X_s(x_i) = y, X_s(x_j) = z\big)\Big(1 - \exp(-t + s/s_L)\Big)\\
&= \int_0^{s_L t} \sum_{y,z} \mathbb{P}_S^A\big(\tau = \tau(k,l) \in ds, X_s(x_i) = y, X_s(x_j) = z\big)\Big(1 - \exp(-t + s/s_L)\Big)\\
&= \int_0^{s_L t} \mathbb{P}_S^A\big(\tau = \tau(k,l) \in ds\big)\Big(1 - \exp(-t + s/s_L)\Big).
\end{aligned} \tag{3.19}$$

Integrating by parts and changing variables, we get

$$\begin{aligned}
&\int_0^{s_L t} \mathbb{P}_S^A\big(\tau = \tau(k,l) \in ds\big)\Big(1 - \exp(-t + s/s_L)\Big)\\
&= \int_0^{s_L t} \mathbb{P}_S^A\big(\tau = \tau(k,l) \le s\big)\frac{1}{s_L}\exp\Big(-t + s/s_L\Big)ds\\
&= \int_0^{t} \mathbb{P}_S^A\big(\tau = \tau(k,l) \le s_L u\big)\exp\Big(-(t-u)\Big)du.
\end{aligned} \tag{3.20}$$

For all L sufficiently large $|\mathbb{P}_S^A\big(H_t(i,j) \le t) - (1 - e^{-t})| \le \varepsilon/(4n^2)$ for all $S \in H_L \cap \mathcal{D}(A)^c$,

$(i,j) \subseteq \{1,\dots n\}$ and $t \geq 0$. Summing over all pairs of i and j on (3.18) and using (3.20)

$$\begin{aligned}
q(t) =& \sum_{\{i,j\}} \mathbb{P}_S^A\big(\tau = \tau(i,j) \leq s_L t\big) \\
=& \sum_{i,j} \mathbb{P}_S^A\big(H_t(i,j)\big) - \sum_{\{i,j\}} \sum_{\{k,l\}\neq\{i,j\}} \int_0^{s_L t} \mathbb{P}_S^A\big(\tau = \tau(k,l) \in ds, \tau(i,j) \leq s_L t\big) \\
=& \sum_{\{i,j\}} (1-e^{-t}) - \sum_{\{i,j\}} \sum_{\{k,l\}\neq\{i,j\}} \int_0^{t} \mathbb{P}_S^A\big(\tau = \tau(k,l) \leq s_L s\big) e^{-t+s} ds + R \\
=& \binom{n}{2}(1-e^{-t}) - \left(\binom{n}{2} - 1\right) e^{-t} \int_0^t q(s) e^s ds + R
\end{aligned}$$

where the modulus of R, for all L sufficiently large for all choices of $S \in H_L \cap \mathcal{D}(A)^c$, y and z such that $d_S(y,z) \geq h_L$ and $t \geq 0$ is smaller than $\varepsilon/2$. We know (see [18, Lemma 2]) that if

$$u^L(t) = \binom{n}{2}(1-e^{-t}) - \left(\binom{n}{2} - 1\right) e^{-t} \int_0^t u^L(s) e^s ds + R$$

the for L sufficiently large $u^L(t)$ does not differ by more than $\varepsilon/2$ from $u(t)$ the solution of

$$u(t) = \binom{n}{2}(1-e^{-t}) - \left(\binom{n}{2} - 1\right) e^{-t} \int_0^t u(s) e^s ds$$

which is

$$u(t) = 1 - \exp\left(-\binom{n}{2} t\right)$$

and the claim follows. $\square$

We can restate Theorem 1.6 as follows.

Theorem 3.6 *Under hypothesis* (3.6) *and* (3.7)*, for each $A \in \mathcal{A}^L(h_L, n)$ and $\varepsilon > 0$ there exists $\widetilde{L}$ such that for each $L > \widetilde{L}$, $S \in H_L \cap \mathcal{D}(A)^c$ and $t \geq 0$*

$$\left|\mathbb{P}_S^A\Big(|\xi_{s_L t}(A)| < k\Big) - P_n\Big(D_t < k\Big)\right| < \varepsilon \tag{3.21}$$

where $s_L = (2L)^d G_{BW}(0,0)/2$.

Proof We fix $A \in \mathcal{A}^L(h_L, n)$ and we show (3.21) by induction. Notice that if $k = n$ the proof is given by Lemma 3.5, because $\mathbb{P}_S^A(|\xi_{s_L t}(A)| < n) = 1 - \mathbb{P}_S^A(|\xi_{s_L t}(A)| = n)$.
We work by induction on n. Theorem 1.4 gives the result when $n = 2$ for all k (in this case the only possible value for k is 2) and Lemma 3.5 gives the result for n and $k = n$. Suppose the result holds for $n-1$ for all k. We have to prove it for n and $k < n$.

$$\begin{aligned}
\mathbb{P}_S^A(|\xi_{s_L t}(A)| < k) =& \int_0^{s_L t} \mathbb{P}_S^A(\tau \in ds, |\xi_{s_L t}(A)| < k) \\
=& \int_0^{s_L t} \sum_{B \in I(n-1)} \mathbb{P}_S^A(\tau \in ds, \xi_s(A) = B) \mathbb{P}_S^B(|\xi_{s_L t - s}(B)| < k).
\end{aligned}$$

Using Lemma 3.3 we can only consider the sets $B \in \mathcal{A}_S^L(h_L, n-1)$. Indeed, if $B \notin \mathcal{A}_S^L(h_L, n-1)$, for all L sufficiently large

$$\begin{aligned}
&\int_0^{s_L t} \sum_{B \notin \mathcal{A}_S^L(h_L,n-1)} \mathbb{P}_S^A(\tau \in ds, \xi_s(A) = B)\mathbb{P}_S^B(|\xi_{s_L t - s}(B)| < k) \\
&\leq \int_0^{s_L t} \sum_{B \notin \mathcal{A}_S^L(h_L,n-1)} \mathbb{P}_S^A(\tau \in ds, \xi_s(A) = B) \\
&= \sum_{\{i,j\}} \int_0^{s_L t} \sum_{B \notin \mathcal{A}_S^L(h_L,n-1)} \mathbb{P}_S^A(\tau = \tau(i,j) \in ds, \xi_s(A) = B) \\
&\leq \sum_{\{i,j\}} \int_0^{s_L t} \sum_{B \notin \mathcal{A}_S^L(h_L,n-1)} \mathbb{P}_S^A(\tau(i,j) \in ds, \xi_s(A) = B) \\
&= \sum_{\{i,j\}} \int_0^{s_L t} \mathbb{P}_S^A(\tau(i,j) \in ds, \exists \{k,l\} \neq \{i,j\} : d_S(X_s(x_k), X_s(x_l)) \leq h_L) \\
&\leq \sum_{\{i,j\}} \sum_{\{k,l\} \neq \{i,j\}} \int_0^{s_L t} \mathbb{P}_S^A(\tau(i,j) \in ds, d_S(X_s(x_k), X_s(x_l) \leq h_L) < \varepsilon/3
\end{aligned}$$

since n is fixed, for each $S \in H_L \cap \mathcal{D}(A)^c$, $t \geq 0$.
Changing variables, setting $s = s_L v$, we have

$$\begin{aligned}
&\int_0^{s_L t} \sum_{B \in I(n-1)} \mathbb{P}_S^A(\tau \in ds, \xi_s(A) = B)\mathbb{P}_S^B(|\xi_{s_L t - s}(B)| < k) \\
&= \int_0^{t} \sum_{B \in I(n-1)} \mathbb{P}_S^A(\tau \in s_L dv, \xi_{s_L v}(A) = B)\mathbb{P}_S^B(|\xi_{s_L (t-v)}(B)| < k).
\end{aligned}$$

By induction hypothesis, for all L sufficiently large

$$|\mathbb{P}_S^B(|\xi_{s_L(t-s)}(B)| < k) - P_{n-1}(D_{t-s} < k)| < \varepsilon/3$$

for $B \in \mathcal{A}_S^L(h_L, n-1)$ and for each $S \in H_L \cap \mathcal{D}(A)^c$ and $0 \leq s \leq t$. Thus the last term of the previous integral differs at most by ε from

$$\int_0^t \mathbb{P}_S^A\left(\frac{\tau}{s_L} \in dv\right) P_{n-1}(D_{t-v} < k).$$

Integrating by parts

$$\begin{aligned}
&\int_0^t \mathbb{P}_S^A\Big(\frac{\tau}{s_L} \in dv\Big) P_{n-1}(D_{t-v} < k) \\
&= \mathbb{P}_S\Big(\frac{\tau}{s_L} \leq t\Big) P_{n-1}(D_0 < k) - \int_0^t \mathbb{P}_S^A\Big(\frac{\tau}{s_L} \leq v\Big) \frac{d}{dv} P_{n-1}(D_{t-v} < k) dv \\
&= -\int_0^t \mathbb{P}_S^A\Big(\frac{\tau}{s_L} \leq v\Big) \frac{d}{dv} P_{n-1}(D_{t-v} < k) dv
\end{aligned}$$

since for $k < n$ we have $P_{n-1}(D_0 < k) = 0$.
Since $v \to P_{n-1}(D_{t-v} = k)$ is a continuous function, by definition of Kingman's coalescent

and because the right hand side $P_n(D_t < k)$ is finite, we get (see [16])

$$\begin{aligned}\mathbb{P}_S^A(|\xi_{s_L t}(A)| < k) =& \sum_{i=1}^{k-1} \int_0^t \binom{n}{2} \exp\Big(- \binom{n}{2} v\Big) P_{n-1}(D_{t-v} = k) dv + R \\ =& \sum_{i=1}^{k-1} P_n(D_t = k) + R \\ =& P_n(D_t < k) + R\end{aligned}$$

where the modulus of R, for all L sufficiently large for all choices of $S \in H_L \cap \mathcal{D}(A)^c$ and $t \geq 0$ is smaller than ε. □

Remark 3.7 *In Theorem* 3.6 *we fix $A \in \mathcal{A}^L(h_L, n)$ and the result holds in a sequence of small world graphs depending on A. One can prove that the same result holds for the sequence $(H_L)_L$ uniformly in $\mathcal{A}^L_S(h_L, n)$ and $S \in H_L$. Namely, under hypothesis* (3.6) *and* (3.7)*, for each $\varepsilon > 0$ there exists $\widetilde{L}$ such that for each $L > \widetilde{L}$, $S \in H_L$, $A \in \mathcal{A}^L_S(h_L, n)$ and $t \geq 0$*

$$\left| \mathbb{P}_S^A\Big(|\xi_{s_L t}(A)| < k\Big) - P_n\Big(D_t < k\Big)\right| < \varepsilon. \tag{3.22}$$

Remark 3.8 *By summing over all small worlds, we can use Theorem* 3.6 *to prove the annealed result: assume hypothesis* (3.6) *and* (3.7)*, for each $A \in \mathcal{A}^L(h_L, n)$ and $\varepsilon > 0$ there exists $\widetilde{L}$ such that for each $L > \widetilde{L}$ and $t \geq 0$*

$$\left| \mathbb{P}(|\xi_{s_L t}(A)| = k) - P(D_n < k)\right| < \varepsilon. \tag{3.23}$$

Appendix A

Hypoexponential law and ruin problem

A.1 Hypoexponential law

Let $I \subset \mathbb{N}$ be a set of indexes with $|I| = N$ and $\{Y_j\}_{j\in I}$ a family of independent random variables such that $Y_j \sim Exp(\mu_j)$ for each $j \in I$. Denote by $\theta = (1, 0, \dots, 0)$ a row vector of size N, $\mathbb{I}$ a column vector of 1's with size N,

$$\Theta = \begin{bmatrix} -\mu_1 & \mu_1 & 0 & 0 & \dots & 0 \\ 0 & -\mu_2 & \mu_2 & 0 & \dots & 0 \\ & & & \ddots & & \\ 0 & \dots & & 0 & -\mu_{N-1} & \mu_{N-1} \\ 0 & \dots & & & 0 & -\mu_N \end{bmatrix}$$

The law of the *hypoexponential random variable* $W := \sum_{j=1}^{N} Y_j$ is

$$\mathbb{P}(W \leq t) = 1 - \theta e^{\Theta t}\mathbb{I}$$

and satisfies

$$\mathbb{E}(W) = \sum_{j=1}^{N} \frac{1}{\mu_j}; \qquad\qquad \sigma_W^2 = \frac{2}{N+1} \sum_{j=1}^{N} \frac{1}{\mu_j} \sum_{i=1}^{j} \frac{1}{\mu_i}$$

where $e^{\Theta t} = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}(\Theta t)^n$. We write $W \sim Hexp(\Theta)$.
If $\mu_j = \mu$ for all $j \in I$ then the random variable has an Erlang distribution. Namely

$$\mathbb{P}(W \leq t) = 1 - e^{-\mu t} \sum_{n=0}^{N-1} \frac{(\mu t)^n}{n!}$$

and we write $W \sim Er(N, \mu)$.

A.2 Ruin problem

We give some classical results involving random walks on a finite interval. Let X_t be a discrete time random walk on $\{0, 1, \ldots, n\}$ with absorbing state 0 such that

$$\begin{array}{ll} i \to i+1 \text{ with probability } p & i \in \{1, \ldots, n-1\} \\ i \to i-1 \text{ with probability } q & i \in \{1, \ldots, n\}. \end{array}$$

then

Lemma A.1 (Ruin Problem Formula) *Let $P_n(j)$ be the probability that the random walk starting at $j \in \{1, \ldots, n-1\}$ reaches the state n before reaching the absorbing state 0 and let $P_0(j)$ be the probability that the random walk visits the state 0 before n. Then*

$$\begin{aligned} P_n(j) =& \frac{1-(q/p)^j}{1-(q/p)^n} \\ P_0(j) =& 1 - P_n(j). \end{aligned}$$

Proof . See proof in [38, (4.4), Section I.4]. □

We call 0 and n respectively r_1 and r_2, that is the ruin of the first and the second players. The game ends when either the random walk reaches n or it reaches 0.

Lemma A.2 (Problem 6.1, Chapter I.6 of [38]) . *Let $\tau_n(i)$ be the number of steps that a random walk needs to reach n starting from i without visiting 0 and $\tau_0(i)$ be the number of steps that a random walk need to reach 0 starting from i without visiting n. Let $\tau = \tau_n(i) \wedge \tau_0(i)$, that is the duration of the game. Then*

$$\mathbb{P}(\tau > t) \leq \alpha^{[\frac{t}{n}]} \tag{A.1}$$

where $\alpha = 1 - (p^n + q^n) < 1$ and $[x]$ is the integer parts of x.

Lemma A.3 *Let $R_n(i)$ be the random variable which counts the number of visits to n before reaching 0 starting at $i \in \{1, \ldots, n\}$. Suppose $p > q$, then for each $i = n - k > 0$, holds*

$$\lim_{n\to\infty} \mathbb{P}(R_n(i) > n^3) = 1. \tag{A.2}$$

Proof . The probability that, starting at $n-k$, the random walk returns to n before visiting 0 is given by Lemma A.1

$$a(p, n-k) := P_n(n-k) = \frac{1-(q/p)^{n-k}}{1-(q/p)^n}.$$

Since n is a reflecting state, the walk comes back to state $n-1$. The probability that, starting at $n-1$, the random walk returns to n before visiting 0 is given by

$$a(p, n-1) := P_n(n-1) = \frac{1-(q/p)^{n-1}}{1-(q/p)^n}.$$

By the Markov property

$$\mathbb{P}(R_n(n-k) \geq r) = a(p, n-k)a(p, n-1)^{r-1}.$$

Notice that $a(p, n-1) \geq 1 - (q/p)^n (q/p)^{-1} \geq 1 - (q/p)^n (q/p)^{-k}$ and $a(p, n-k) \geq 1 - (q/p)^{n-k}$ since $p > q$. Therefore

$$\mathbb{P}(R_n(n-k) \geq r) \geq (1 - (q/p)^{n-k})^r = e^{r \log(1-(q/p)^{n-k})} \geq e^{-Cr(p/q)^n}$$

and the claim follows. $\square$

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