Quelques aspects de la géométrie non commutative en liaison avec la géométrie différentielle

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Mémoire d'habilitation à diriger des recherches

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Introduction générale

Pour rédiger ce mémoire d'habilitation, j'ai opéré deux choix :

❍ sur le fond, j'ai opté pour ne faire apparaître qu'une partie de mes travaux de recherche menés depuis plus de 10 ans dans le domaine de la géométrie non commutative ;

❍ sur la forme, j'ai repris, comme chapitres principaux de ce mémoire, deux articles de revue que j'ai écrit respectivement en 2006 et 2007 ([Masson, 2008a] et [Masson, 2008c]), qui sont des comptes-rendus de conférences données lors de rencontres internationales.

Le but de cette introduction est de replacer dans leurs contextes respectifs ces deux revues, et de les compléter sommairement sur certains points.

La géométrie non commutative

La **géométrie non commutative** a été conçue à la fois pour répondre à des besoins en mathématiques et pour permettre d'aborder certains problèmes de physique théorique.

En mathématique, il s'agit de généraliser les outils de la géométrie ordinaire qui ont été développés depuis plus d'un siècle : structures différentiables, métriques, actions de groupes, fibrations, connexions… Ces constructions mathématiques sont désormais largement utilisées en physique théorique, et l'essentiel des théories modernes (le Modèle Standard des particules élémentaires, la Relativité Générale, la théorie quantique des champs) se fondent sur des propriétés mathématiques fines élaborées dans ce contexte. C'est à A. Connes que l'on doit, en 1985, d'avoir donné les premières voies concrètes de recherches dans le domaine de la géométrie non commutative, en définissant et en étudiant la cohomologie cyclique [Connes, 1985]. Il montrait ainsi que la notion de calcul différentiel sur les variétés admet un équivalent non commutatif, au sens expliqué ci-dessous.

En physique, dès le milieu des années 1950, les travaux sur les théories quantiques des champs ont permis de faire émerger des notions devant faire cohabiter structures géométriques et structures algébriques : d'un côté les algèbres d'opérateurs, de l'autre les théories de jauge, c'est à dire la théorie des connexions sur les fibrés… Cependant, la plus forte motivation reste encore aujourd'hui l'espoir d'écrire une théorique quantique de la gravitation avec cette mathématique, puisque le principe fondateur de la géométrie non commutative est de fusionner dans un même cadre conceptuel l'aspect opératoriel de la mécanique quantique et l'aspect géométrique de la Relativité Générale (et des théories de jauges).

L'idée maîtresse de la géométrie non commutative est d'abord de caractériser une classe d'espaces « géométriques » bien particulière par un type d'algèbres de fonctions adapté, en munissant ces algèbres d'outils algébriques appropriés. Par exemple, il est possible de caractériser un espace topologique compact et séparé par son algèbre de fonctions continues bornées, et un espace mesurable par son algèbre des classes de fonctions mesurables bornées. Dans les cas favorables, ou bien ces outils algébriques n'utilisent pas explicitement la commutativité des algèbres de fonctions, ou bien des outils plus algébriques équivalents existent, ce qui rend alors possible l'étude des algèbres non commutatives du même type à l'aide de ces outils algébriques, sans avoir à mentionner d'espace sous-jacent.

Cette démarche est largement encouragées par de nombreux résultats mathématiques : caractérisations des algèbres de von Neumann commutatives, théorème de Gelfand-Naïmark sur les C^*-algèbres, théorème de Serre-Swan sur les fibrés, cohomologie cyclique de l'algèbre $C^\infty(M)$, relations entre opérateurs de Dirac et métriques riemanniennes, K-théorie des C^*-algèbres...

Le Chapitre 1 est une revue non exhaustive à vocation pédagogique de ces considérations générales sur la géométrie non commutative. En particulier, l'effort pédagogique a été porté sur les objets mathématiques de nature géométrique qui admettent une généralisation non commutative, comme par exemple le caractère de Chern, pierre angulaire de nombreux résultats en géométrie non commutative.

Ce chapitre ne constitue qu'une introduction à certaines structures algébriques et géométriques qui fondent la géométrie non commutative. On trouvera dans mon ouvrage de 635 pages, *Une introduction aux (co)homologies*, publié en 2008 aux éditions Hermann ([Masson, 2008b]), de quoi largement compléter ce qui est exposé ici. [1] En particulier, on trouvera dans les nombreux exercices proposés des exemples concrets d'utilisation de ces outils dans la démarche de la géométrie non commutative. Pour des introductions plus complètes, je renvoie à [Connes, 1994], [Landi, 1997], [Madore, 1999] et [Gracia-Bondía et al., 2001].

De plus, ce chapitre ne contient aucune contribution personnelle aux mathématiques de la géométrie non commutative sur lesquelles j'ai pu travailler par le passé : problèmes cohomologiques sur des algèbres associatives ([Dubois-Violette and Masson, 1996a]), notion de sous variétés non commutatives et de variétés quotient non commutatives dans le cadre de la géométrie non commutative basée sur les dérivations ([Masson, 1996]), étude de la notion d'opérateur différentiel du premier ordre sur un bimodule en géométrie non commutative ([Dubois-Violette and Masson, 1996b]), structure du calcul différentiel basé sur les dérivations de l'algèbre des matrices ([Masson, 1995], [Masson, 2008b])...

Gravitation et géométrie non commutative

Une des motivations fortes de la géométrie non commutative est d'obtenir un cadre mathématique cohérent dans lequel il serait possible d'écrire une gravitation quantique. Bien que ce thème ne soit pas repris plus loin dans ce mémoire, il me semble important de mentionner et discuter quelques uns des travaux menés dans cette direction.

Une des pistes explorées a été de généraliser le formalisme utilisé dans le cadre de la Relativité Générale d'Einstein à des situations opératorielles. Pour ce faire, une première étape incontournable consiste d'abord à considérer les notions ordinaires de la géométrie riemannienne de façon plus algébrique, et à envisager ensuite de se passer de la commutativité de l'algèbre des fonctions C^∞.

J'ai mené par le passé, avec différents collaborateurs, des travaux dans cette direction. Ils consistaient à étudier une notion de *connections linéaires non commutatives*. Les connections linéaires sont, dans le formalisme de la Relativité Générale, l'objet mathématique derrière les symboles de Christoffel. La définition en elle-même des *connections linéaires non commutatives* ne pose pas de problème, car elle est déjà de nature très algébrique en géométrie ordinaire. On trouvera dans [Dubois-Violette et al., 1996] et [Dubois-Violette et al., 1995a] des considérations générales à propos de ces aspects algébriques.

Les géométries non commutatives sur lesquelles ces notions ont été testées sont très diverses : al-

1. Même si cet ouvrage, comme son titre l'indique, n'est pas un livre sur la géométrie non commutative.

gèbre des matrices ([Madore et al., 1995]), plan quantique ([Dubois-Violette et al., 1995b]), groupes quantiques ([Georgelin et al., 1996] et [Georgelin et al., 1997]). Depuis, d'autres auteurs ont exploré d'autres classes d'exemples, comme le plan quantique h-déformé ([Cho et al., 1998]), des géométries de réseaux ([Dimakis and Mueller-Hoissen, 2003]), des géométries avec métriques pseudo-riemaniennes ([Dimakis and Mueller-Hoissen, 2000])...

De tous ces travaux, il ressort que les contraintes non commutatives sont tellement fortes que l'espace des connections linéaires se réduit très souvent à un espace de paramètres de dimension finie. Ceci est souvent relié au fait que le centre de ces algèbres est lui même très petit, un espace vectoriel de dimension finie. Cette situation est peu encourageante dans l'espoir de faire de la Relativité Générale au « sens ordinaire ».

Une autre approche possible consiste à envisager que la quantification de la Relativité Générale induise une notion d'espace non commutatif, ayant pour paramètre de déformation la longueur de Planck. Il ne s'agit donc pas de réécrire d'une façon ou d'une autre la Relativité Générale directement, mais d'en extraire d'éventuelles conséquences quantiques et de les encoder dans un espace-temps non commutatif. Souvent, cet espace-temps non commutatif est étudié d'une point de vue de sa géométrie « riemannienne », ou bien il est le support de théories des champs dont on étudie les propriétés. Je renvoie à [Madore, 1999] pour de plus amples renseignements sur ce domaine de recherche.

Enfin, une autre voie de recherche pour écrire une théorie de la gravitation au moyen de la géométrie non commutative a été initiée et très largement explorée par A. Connes et A. Chamseddine (voir [Chamseddine and Connes, 1997] pour la première proposition, et [Chamseddine et al., 2007] pour la dernière version de ce modèle et les références bibliographiques). L'ambition de cette démarche, qui aboutit à un modèle assez complet dans le dernier article cité, est de reconstruire le Modèle Standard de la physique des particules élémentaires couplé à la gravitation, à partir de principes issus d'idées de la géométrie non commutative.

Cette démarche ne cherche pas, contrairement aux précédentes, à envisager d'utiliser la géométrie non commutative pour écrire une « gravitation quantique », puisque le modèle auquel elle aboutit est classique. Dans un soucis de comparaison avec les autres approches mentionnés dans les chapitres de ce mémoire (sur les théories de Yang-Mills-Higgs non commutatives), il est utile de rappeler ici, en quelques lignes, en quoi consiste cette démarche.

La géométrie non commutative appliquée à la physique qu'A. Connes a développée est très différente de celle qui sera expliquée au chapitre 2. En effet, la construction d'A. Connes repose de façon essentielle sur un *triplet spectral*, constitué d'une algèbre topologique, d'un espace de Hilbert sur lequel cette algèbre se représente, et d'un opérateur de Dirac sur cet espace de Hilbert. Ces trois objets satisfont à des relations de compatibilité, qui ne sont, ni plus, ni moins, que ce qu'il faut pour que l'algèbre des fonctions C^∞ sur une variété compacte à spin, muni de l'espace de Hilbert obtenu par complétion des sections L^2 d'un fibré des spineurs et de l'opérateur de Dirac naturel, soit un tel triplet spectral. C'est le modèle commutatif de cette construction.

Cette notion de triplet spectral se transforme rapidement en un quintuplet spectral lorsqu'on lui ajoute une éventuelle $\mathbb{Z}_2$-graduation et une notion de réalité. Ces deux concepts, formalisés par deux opérateurs γ et J sur l'espace de Hilbert, trouvent leurs origines dans le Modèle Standard des particules : γ est relié à la chiralité à travers l'opérateur usuel γ^5, et J se relie à la conjugaison de charge. Il ne s'agit pas ici d'exposer en détail ces constructions, de nombreux articles et ouvrages en donnent des exposés très précis ([Connes, 1994] ou [Gracia-Bondía et al., 2001] sont de bons points de départ).

Dans cette approche, l'opérateur de Dirac représente la métrique, et ses fluctuations (non commutatives) s'identifient à des champs de jauge. On peut, jusqu'à un certain point, comprendre ces propriétés par l'argumentaire heuristique suivant.

La métrique sur une variété différentiable riemannienne est donnée par l'*invariant de longueur* $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$. Il est bien connu que l'action naturelle que l'on écrit dans le cadre de la théorie de la relativité, pour une particule libre (voir par exemple [Landau and Lifchitz, 1989]), est

$$S[\gamma] = \int_\gamma d\tau$$

où γ est un chemin paramétré sur la variété, et $d\tau = \sqrt{ds^2}$ est ici le temps propre infinitésimal le long de ce chemin (dans une signature Minkowskienne). Donc $S[\gamma]$ est le temps propre de l'objet le long de la trajectoire γ.

En mécanique quantique relativiste, on est amené à remplacer le formalisme des points et des trajectoires par le formalisme des fonctions d'ondes et de leurs équations différentielles, grâce à la « procédure de quantification » $p_\mu \mapsto -i\frac{\partial}{\partial x^\mu}$. Dans l'espace des impulsions, l'invariant correspondant à ds^2 est $m^2 = g^{\mu\nu} p_\mu p_\nu$, qui, quantifié, donne lieu au Laplacien Δ (généralisé, au sens où il peut s'agir du d'Alembertien). Il est possible de reproduire la démarche qui conduit à l'action dans l'espace des positions en considérant un opérateur D tel que $D^2 = \Delta$, c'est à dire un opérateur de Dirac. Une action naturelle serait alors de la forme $\int D$, qui n'a de sens que dans le cadre de la théorie spectrale d'une certaine classe d'opérateurs. L'action spectrale proposée par A. Connes et A. Chamseddine est de la forme

$$S[D] = \int f(D)$$

où f est une fonction définie sur le spectre de D.

Imaginons maintenant que l'on permette à D d'avoir des fluctuations dans un espace suffisamment « grand », par exemple de telle façon que les couplages minimaux, qui consistent à « faire fluctuer » les p_μ sous la forme $p_\mu + A_\mu$ (A_μ étant un champs de Yang-Mills), soient possibles. L'action écrite ci-dessus sera alors une action contenant les champs $g_{\mu\nu}$ et A_μ. La structure de cette action impose que $S[D]$ produise des invariants mathématiques à partir de ces champs. Aussi, le travail de A. Chamseddine, A. Connes et M. Marcolli, dans [Chamseddine et al., 2007], a consisté à permettre les bonnes fluctuations de D (en choisissant, en particulier, l'algèbre et l'espace de Hilbert du triplet spectral) de telle façon que cette action reproduise une version du Modèle Standard des particules élémentaires (il s'agit d'un modèle inspiré des modèles *see-saw*, contenant un mélange des neutrinos).

Il est bien connu que cette démarche soulève de nombreux problèmes. Techniquement, l'opérateur de Dirac qu'il est possible de considérer dans cette approche est à résolvante compacte, ce qui exclu de fait les opérateurs de Dirac de la physique sur espace-temps de Minkowski. Seuls les cas euclidiens peuvent donner lieu à de telles modélisations, ce qui rend l'intérêt physique de cette démarche, dans son formalisme actuel, assez peu encourageante. [2] Certains auteurs ont commencé à explorer la possibilité d'un opérateur non elliptique. Pour cela, il a été suggéré d'amplifier encore le tri(quintu)plet spectral en lui adjoignant un opérateur supplémentaire, un peu à la manière où un espace de Krein est un espace de Hilbert muni d'un opérateur métrique qui relie le produit scalaire indéfini au produit scalaire défini positif…

2. Même si de trop nombreux physiciens pensent qu'une « simple » rotation de Wick suffirait à basculer d'une signature à une autre, ce que les mathématiques ne semblent pas étayer : combien de théorèmes valables dans le cas elliptiques le sont aussi dans le cas hyperbolique ?

Une autre difficulté de la construction d'A. Connes est qu'elle requiert une certaine forme de compacité pour l'« espace » non commutatif. Dans [Gayral et al., 2004], la définition d'un triplet spectral non compact a été proposée, et l'exemple d'une algèbre de Moyal (il y a de nombreuses définitions possibles de l'« algèbre de Moyal ») y est donné. Cette approche requiert encore une fois d'élargir le triplet, en ajoutant une algèbre topologique non unitale…

Je reviendrai plus loin sur les points plus positifs de cette approche. Il ressort cependant de ce (très) bref tour d'horizon que la « gravitation quantique » est pour l'instant encore à un stade d'investigation relativement préliminaire dans le cadre de la géométrie non commutative.

Relations entre la géométrie non commutative et la géométrie ordinaire

La géométrie non commutative se veut une généralisation de la géométrie différentielle ordinaire. Aussi, il n'est pas surprenant de s'attendre à ce qu'elle permette de reconsidérer les objets de la géométrie ordinaire dans son propre langage. Il en est ainsi, par exemple, de la notion de feuilletage : A Connes a montré qu'on pouvait associer et étudier une certaine C^*-algèbre à un feuilletage, même (et surtout !) singulier. C'est l'un des succès majeurs de la géométrie non commutative d'avoir pu prendre en considération des situations singulières de la géométrie ordinaire qu'il était très difficile, voire impossible, à manœuvrer dans le formalisme géométrique usuel.

En collaboration avec M. Dubois-Violette d'abord, puis avec E. Sérié ensuite, j'ai exploré dans différents articles une autre situation où la géométrie non commutative rejoint et jette un regard nouveau sur la géométrie ordinaire. La publication [Masson, 2008c] est une revue de l'ensemble de ce qui a été considéré jusqu'à présent sur ce sujet et constitue le Chapitre 2 de ce mémoire. Je donne un bref aperçu, dans ce qui suit, des motivations derrière ce travail et des résultats essentiels obtenus.

Le point de départ des recherches publiées dans [Dubois-Violette and Masson, 1998] consistait à généraliser des travaux antérieurs sur la géométrie non commutative basées sur les dérivations de l'algèbre des fonctions à valeurs matricielles. Cette algèbre, étudiée par M. Dubois-Violette, R. Kerner et J. Madore, avait montré que la partie purement non commutative (l'algèbre des matrices) renfermait, du point de vue des théories de jauge non commutative, des degrés de liberté assimilables, au moins dans des modèles simples, à des champs de Higgs.

La généralisation que nous avons considérée dans [Dubois-Violette and Masson, 1998] repose sur la constatation que cette algèbre de fonctions à valeurs matricielles, isomorphe à $C^\infty(\mathcal{M}) \otimes M_n(\mathbb{C})$ ($\mathcal{M}$ est une variété différentiable paracompacte), n'est rien d'autre que l'algèbre des endomorphismes du fibré vectoriel $\mathcal{M} \times \mathbb{C}^n$. Dans cette situation, ce fibré est trivial. Nous avons donc amorcé l'étude de la géométrie non commutative basée sur les dérivations de l'algèbre des endomorphismes d'un fibré vectoriel orientable $\mathcal{E}$ de fibre $\mathbb{C}^n$, qu'on notera de façon générale $\mathbf{A}$ par la suite. Le groupe de structure d'un tel fibré pouvant être réduit à $SU(n)$, nous avons souvent appelé cette algèbre l'*algèbre des endomorphismes d'un fibré $SU(n)$*.

Dans l'article [Dubois-Violette and Masson, 1998], nous avons montré que l'aspect non trivial du fibré $\mathcal{E}$ modifiait en substance certains des résultats obtenus pour l'algèbre des fonctions à valeurs dans les matrices. Par exemple, et ceci joue une rôle crucial par la suite, l'algèbre de Lie des dérivations de cette dernière algèbre se scinde, en tant qu'algèbre de Lie et module sur le centre, en une partie purement non commutative (liée aux dérivations intérieures de l'algèbre des matrices) et une partie géométrique (l'algèbre de Lie des champs de vecteurs sur $\mathcal{M}$). Cette décomposition canonique permet de considérablement simplifier l'analyse de cette géométrie. Au contraire, dans le cas de l'algèbre associée à un fibré $\mathcal{E}$ non trivial, cette décomposition n'a plus lieu. Cependant, nous avons montré qu'une connexion ordinaire sur $\mathcal{E}$ permet de scinder l'algèbre de Lie des dérivations en tant que module sur le centre, mais pas en tant qu'algèbre de Lie, en une composante non commutative (le sous espace des dérivations intérieures de l'algèbre) et une composante « commutative » (l'algèbre de Lie des champs de vecteurs sur $\mathcal{M}$). L'obstruction à la possibilité de scinder cette algèbre de Lie en tant qu'algèbre de Lie est exactement mesurée par la courbure de la connexion choisie.

Le point essentiel de cette publication est que l'**espace (affine) des connexions ordinaires sur $\mathcal{E}$ est un sous espace de l'espace (vectoriel) des connexions non commutatives de cette algèbre des endomorphismes** (voir le Théorème 2.5.2). Les degrés de liberté ainsi disponibles en sus de ceux d'une connexion ordinaire sont, comme dans le cas trivial, assimilables à des champs de Higgs.

D'autre part, nous avons relié cette géométrie à l'algébroïde de Lie d'Atiyah sur $\mathcal{M}$.

Dans la publication [Masson, 1999], j'ai poursuivi l'étude de cette algèbre, en montrant en quoi elle généralisait convenablement, sur de nombreux points, la géométrie ordinaire du fibré principal $\mathcal{P}$ sous-jacent au fibré $\mathcal{E}$. La possibilité de relier entre elles ces deux géométries repose sur l'utilisation de mes travaux antérieurs sur les sous variétés non commutatives et les variétés quotients non commutative dans le cadre des calculs différentiels basés sur les dérivations ([Masson, 1996]). En effet, on peut considérer que les deux géométries mentionnées, la géométrie ordinaire de $\mathcal{P}$ et la géométrie non commutative de l'algèbre **A**, sont des géométries de « variétés quotient non commutatives » de la géométrie non commutative de l'algèbre **B** des fonctions sur $\mathcal{P}$ à valeurs dans $M_n(\mathbb{C})$. Cette dernière géométrie est triviale au sens du fibré vectoriel sous-jacent (voir la Section 2.6). J'ai ainsi pu montrer que la réécriture des connexions ordinaires, comme connexions non commutatives dans le cadre de l'algèbre **A**, complétait parfaitement le schéma géométrique habituel des connexions qui consistait jusqu'à présent à les définir comme formes sur $\mathcal{P}$ (équivariantes et horizontales, à valeurs dans une algèbre de Lie) ou comme une famille de formes locales sur $\mathcal{M}$ (à valeurs dans une algèbre de Lie et satisfaisant à des relations de recollement non homogènes), en définissant une telle connexion comme forme non commutative globale sur $\mathcal{M}$ « à valeurs dans **A** ». Au niveau des courbures, cette dernière caractérisation existait auparavant (2-formes sur $\mathcal{M}$ à valeurs dans un fibré associé à $\mathcal{P}$). J'ai donc montré que la forme de connexion pouvait elle aussi se définir à ce niveau intermédiaire (entre une notion locale sur $\mathcal{M}$ et une notion globale sur $\mathcal{P}$) en ayant recourt à des structures non commutatives (voir la Remarque 2.4.9).

Dans cette publication [Masson, 1999], j'ai aussi considéré la structure de l'espace de cohomologie des formes différentielles non commutatives, et j'ai pu démontrer une **généralisation non commutative du théorème de Leray** sur la cohomologie d'un fibré principal (Théorème 2.7.3).

En collaboration avec mon étudiant de thèse, E. Sérié, nous avons défini et étudié dans [Masson and Sérié, 2005] la notion de connexions non commutatives invariantes sur cette géométrie non commutative. Les connexions (ordinaires) invariantes sous l'action d'un groupe de Lie jouent un rôle important en géométrie ordinaire et en physique des théories de jauge non abéliennes. En effet, elles permettent bien souvent de trouver des solutions explicites des équations du mouvement en présence d'un principe de symétrie réduisant considérablement les degrés de liberté.

La définition que nous avons prises pour les connexions non commutatives invariantes sous l'action d'un groupe de Lie est une généralisation de la notion habituelle, au sens où l'espace des connexions non commutatives contient l'espace (affine) des connexions ordinaires, et que les deux notions de connexions invariantes coïncident sur ce sous espace.

Nous avons en outre caractérisé l'espace des connexions non commutatives invariantes et donné plusieurs exemples, en particulier une généralisation de l'ansatz de Witten. La démarche a consisté à caractériser les connexions non commutatives invariantes comme des objets au niveau de la grande algèbre **B**. L'espace des connexions non commutatives invariantes est alors constitué de deux parties : une algèbre et son calcul différentiel, et un module sur cette algèbre. On retrouve ainsi la décomposition usuelle en une partie « connexion réduite » et une autre partie « champs scalaire ». Le langage géométrico-algébrique permet de considérer et manipuler ces objets de façon beaucoup plus aisée que dans le cas de la géométrie ordinaire (voir la Section 2.8).

Le Chapitre 2 est une revue complète de tous ces travaux. En particulier, j'y insiste sur le lien entre la géométrie ordinaire du fibré $\mathcal{P}$ et la géométrie non commutative de l'algèbre **A**. Dans cet exposé, on trouvera aussi des résultats (non publiés par ailleurs) sur la façon de définir les classes caractéristiques de $\mathcal{E}$ en terme de l'algèbre de Lie des dérivations de **A**. En effet, dans cette revue, je montre que l'**obstruction au fait que cette algèbre de Lie se scinde en tant qu'algèbre de Lie contient les informations permettant de calculer les classes caractéristiques de** $\mathcal{E}$. Cette démarche repose sur la notion de classes de cohomologie associées à une suite exacte courte d'algèbres de Lie, définie et étudiée par Lecomte. On ne trouvera pas les démonstrations originales dans cette revue, et je renvoie aux articles originaux pour les détails.

On peut résumer brièvement ce qui a été appris sur cette géométrie non commutative, en disant qu'elle constitue un outil idéal pour le physicien désireux de considérer des modèles de type Yang-Mills-Higgs. En effet, la géométrie différentielle des théories de jauge sur des fibrés $SU(n)$ est une *sous géométrie* d'une géométrie non commutative de l'algèbre des endomorphismes de $\mathcal{E}$. Cette caractérisation des théories de jauge permet de mieux comprendre l'origine et la place des champs de Higgs vis-à-vis de la géométrie ordinaire, puisqu'ils s'interprètent, de façon tout à fait naturelle, comme les degrés de liberté dans les directions purement non commutatives de la géométrie de cette algèbre d'endomorphismes. Ils s'agit donc d'une partie non visible à travers la géométrie usuelle. Il faut noter que dans le Modèle Standard élaboré par A. Connes évoqué plus haut, les champs de Higgs sont aussi les composantes des connexions dans des directions purement non commutatives (la « géométrie finie » de ce modèle). Ceci semble à la fois répondre à la question physique de l'origine des Higgs, et à la question mathématique du statut exact de ces champs scalaires dans les modèles de jauge à brisures spontanées de symétrie, qui jusqu'à présent n'avaient pas été identifiés mathématiquement comme le sont aujourd'hui les champs de jauge (en tant que connexions).

La géométrie non commutative et les théories des champs

Ceci nous amène directement à l'un des thèmes de recherches les plus développés dans le domaine de la géométrie non commutative. Il concerne les théories de jauge non commutatives. Rappelons que pour définir une telle théorie, c'est à dire définir des connexions non commutatives, il faut trois ingrédients :

① une algèbre associative **A**, souvent unitale pour des raisons pratiques ;

② un calcul différentiel qui jouera le rôle de « géométrie différentielle non commutative », sans lequel la notion de formes différentielles n'a pas de sens ;

③ un module à droite sur l'algèbre **A**, qui sert de support à l'action de la connexion non commutative (dans les modèles de physique des particules, il correspond à l'espace des champs de matière).

Bien souvent, on simplifie le problème, tout en conservant une certaine généralité, en prenant pour module sur **A** l'algèbre **A** elle-même, considérée comme module à droite sur elle-même.

C'est par exemple avec cette démarche qu'avec mes collaborateurs j'ai étudié les connexions ordinaires comme connexions non commutatives sur l'algèbre des endomorphismes d'un fibré $SU(n)$ (voir Section 2.5).

Une autre direction possible de recherche repose sur la constatation que les équivalents non commutatifs des théories de jauge abéliennes sont des théories de jauge non commutatives dont les degrés de libertés s'interprètent comme des champs de jauge non abéliens ! Par exemple, en prenant l'algèbre **A** comme module sur elle-même, la structure non commutative de **A** correspond, en un certain sens, au groupe de jauge : dans le cas $\mathbf{A} = C^\infty(\mathcal{M}) \otimes M_n(\mathbb{C})$, on trouve pour groupe de jauge $C^\infty(\mathcal{M}) \otimes SU(n)$, ce qui rapproche la théorie de jauge (non commutative) considérée d'une théorie de jauge de type Yang-Mills avec comme groupe de structure $SU(n)$. Or, prendre l'algèbre elle-même dans le cadre commutatif ($\mathbf{A} = C^\infty(\mathcal{M})$) revient à considérer une théorie abélienne de type Maxwell !

C'est dans cet esprit qu'en collaboration avec R. Kerner et notre étudiant de thèse E. Sérié, nous avons introduit des théories de type Born-Infeld non abéliennes. En effet, le caractère non abélien correspond au caractère non commutatif de l'algèbre $\mathbf{A} = C^\infty(\mathcal{M}) \otimes M_n(\mathbb{C})$, et nous avons alors généralisé, dans une démarche purement non commutative, l'équivalent de l'action (abélienne) de Born-Infeld. Ces travaux ont donné lieu à plusieurs publications. Dans [Kerner et al., 2003], en guise de mise en place de cette démarche, nous avons explicitement calculé le lagrangien dans le cas $SU(2)$. Nous avons étudié (numériquement) les solutions statiques à symétrie sphérique de cette action de Born-Infeld. Nous avons exhibé une famille à un paramètre de solutions d'énergie finie, dont nous avons décrit les propriétés essentielles. Dans [Kerner et al., 2004], nous avons réduit le type de lagrangien considéré précédemment à un ansatz ne faisant intervenir qu'un seul champ scalaire. L'action obtenue est de type Dirac-Born-Infeld. Nous avons étudié des solutions de cette action, dans des cas simples à symétrie sphérique. Il en ressort que ces solutions ne sont pas stables.

Dans [Cagnache et al., 2008], en collaboration avec E. Cagnache et J.-C. Wallet, nous avons appliqué la technologie des calculs différentiels basés sur les dérivations à l'algèbre de Moyal, afin d'étudier les conséquences possibles, au niveau des théories de jauge, du choix de ce calcul. En effet, l'algèbre de Moyal, qu'on peut considérer sur certains points comme une algèbre généralisant l'algèbre des matrices $M_n(\mathbb{C})$ largement évoquées ci-dessus, n'admet que des dérivations intérieures. Or, le calcul différentiel basé sur les dérivations qui a été utilisé jusqu'à présent dans ce cadre, n'utilise pour espace de dérivations qu'un espace de dimension deux (pour simplifier, on ne traite dans ce résumé que l'algèbre de Moyal sur $\mathbb{R}^2$) : les « dérivées » dans les directions usuelles de $\mathbb{R}^2$. Comme l'algèbre de Lie complète des dérivations est de dimension infinie, il semble très restrictif de ne considérer que ces deux directions, tout en gardant à l'esprit qu'en vue de construire des théories de jauge, il est raisonnable et souhaitable de se restreindre à une sous algèbre de Lie de dimension finie. Aussi, nous avons montré qu'un choix tout aussi naturel serait de considérer une algèbre de Lie de dimension 5, où, aux deux dérivations précédentes, on ajoute les directions du groupe des symplectomorphismes. Cette algèbre de Lie est la plus grande sous algèbre de Lie des dérivations de l'espace de Moyal pour laquelle les dérivations sont aussi des champs de vecteurs usuels sur les fonctions ordinaires (dont certaines, comme par exemples les fonctions polynomiales, sont dans l'algèbre de Moyal considérée). Nous avons étudié des théories de jauge dans cette situation, et montré que les degrés de liberté supplémentaires introduits sur les champs de jauge dans ces trois nouvelles directions pouvaient jouer le rôle de champs de Higgs.

La géométrie non commutative face aux « théories unificatrices »

Je voudrais terminer cette introduction en donnant mon point de vue sur la place de la géométrie non commutative dans le grand programme de recherche qui mobilise les esprits, assez vainement malheureusement, depuis plus d'une cinquantaine d'années maintenant, et qui consiste à élaborer une théorie dans laquelle les interactions électrofaibles, fortes et gravitationnelles trouveraient leur place d'une façon unifiée.

Contrairement à d'autres approches du type « unificatrice » actuellement explorées, comme par exemple la théorie des cordes ou la théorie des boucles quantiques [3], la géométrie non commutative ne se place pas au niveau d'un modèle particulier dans une cadre conceptuel déjà pré-défini (les méthodes de quantification usuelles, la théorie quantique des champs et sa théorie des perturbations...).

Dans les faits et les articles publiés, cette assertion peut largement sembler inexacte, puisque bon nombre de chercheurs en géométrie non commutative n'adhèrent pas à ce point de vue. Pourtant, il me semble que cette conception soit la seule défendable lorsqu'on travaille avec cette nouvelle approche : elle cherche en effet, avant tout, à puiser aux sources mêmes des théories afin de dénicher ce qui les rassemble, dans le but de les réconcilier (on connaît déjà depuis bien longtemps ce qui les divise !). Malheureusement, une large part des travaux se contente de reproduire et de répéter des recettes bien trop classiques face aux enjeux que lancent cette unification, et dans la plupart du temps éloignées de la philosophie même qui motive la géométrie non commutative. Cette dernière cherche avant tout à fournir un nouveau cadre dans lequel penser et écrire une nouvelle physique, susceptible d'accueillir à la fois les aspects les plus pertinents de la théorie quantique des champs, et les caractéristiques les plus utiles des théories de nature géométriques. Par conséquent, les « recettes » habituelles doivent aussi y être repensées profondément.

La géométrie non commutative n'a pas encore atteint cet objectif, car les mathématiques qui la sous-tendent sont encore largement en gestation, et les physiciens ne se sont pas encore appropriés cette nouvelle démarche. [4] Il n'est donc pas possible, à l'heure actuelle, de la comparer à d'autres approches unificatrices.

Néanmoins, la possibilité de reformuler le Modèle Standard des particules en utilisant des outils algébriques nouveaux est un indéniable et très encourageant succès de la « reconstruction » effectuée par A. Connes et ses collaborateurs. En effet, l'élaboration des modèles de particules repose, depuis les années 60, sur l'utilisation d'espaces de représentations de groupes, et tout l'art du « model builder », grande spécialité de ces 40 dernières années, transmise de chercheurs aux thésards dans la bonne vieille tradition du compagnonnage, [5] se révèle dans les choix adéquats des groupes de symétrie et des espaces de représentation. Le Modèle Standard est la grande victoire de cet Art. Mais depuis la fin des années 70, cette démarche semble s'essouffler, non pas dans la littérature, mais dans ses succès, aussi bien pour dépasser le Modèle Standard (qui a bien besoin d'une cure de jouvence depuis la récente confirmation que les neutrinos sont massifs par exemple), que dans sa capacité à apporter des perspectives nouvelles grâce auxquelles des théories unificatrices sont concevables.

Aussi, l'approche non commutative pourrait être un nouveau souffle dans ce contexte. Dans tous les modèles évoqués ci-dessus, aussi bien celui très « réaliste » construit par A. Connes et ses collaborateurs que les modèles plus bruts issus des considérations du Chapitre 2, la notion de groupe de structure disparaît : elle s'efface derrière celle d'algèbre associative (la non abélianité des groupes se transformant en non commutativité des algèbres...), et les représentations sont remplacées par des modules, pour lesquels l'irréductibilité est une contrainte plus forte. Enfin, la non abélianité/non commutativité apporte avec elle, de façon structurellement inévitable, des champs supplémentaires aux champs de jauge « ordinaires », interprétables comme des champs de Higgs dans les bons cas.

3. Bien que la gravitation quantique à boucles ne se revendique aucunement « unificatrice », elle présuppose que la gravitation puisse être quantifiée avec des méthodes très semblables à celles utilisées pour manoeuvrer les autres interactions, ce qui, de ce point de vue, est déjà un principe d'unification.

4. Il faudrait d'ailleurs pour cela que nombre d'entre eux acceptent un changement de paradigme, ce qui n'est pas nécessairement le plus facile, compte-tenu des phénomènes de mode ou des attachements psychologiques parfois rigides à certaines démarches...

5. Il est étonnant de voir combien l'apprentissage de la science à travers l'itinéraire des « post-docs » ressemble à la démarche du compagnonnage.

La démarche d'A. Connes devrait être suivie avec beaucoup d'attention par les physiciens des particules. En effet, la construction même de sa géométrie non commutative est intimement liée, comme il l'avoue lui-même, aux structures géométrico-algébriques sous-jacentes au Modèle Standard, qu'elles aient été introduites en toute conscience par les physiciens, ou qu'elles soient le résultat, au contraire, de contraintes fortes et inévitables face aux données expérimentales. Ainsi, les deux opérateurs mentionnés auparavant, la graduation et la réalité, sont directement issus de ce cheminement. Ils permettent de mieux comprendre la structure mathématique globale que semble avoir le Modèle Standard (si le modèle de Chamseddine-Connes-Marcolli est « relativement » correct).

D'autre part, il est admis que la « gravitation quantique » [6] est un grand défi d'un point de vue technique, en particulier parce que nombre de méthodes pertinentes dans d'autres contextes de quantification semblent inefficientes dans le traitement de la gravitation. Cependant, il serait temps d'admettre que le défi à relever est aussi et surtout de nature plus conceptuel. Je renvoie à l'excellente revue de C. Isham [Isham, 1995] pour un exposé très complet et argumenté de ces questions fondamentales, qui ne semblent pas trop embarrasser les tenants d'approches « classiques », mais auxquelles il faudra pourtant bien apporter des réponses satisfaisantes avant même de vouloir s'attaquer aux problèmes techniques « concrets ». Ainsi, le statut de l'espace-temps est l'un des problèmes les plus profonds et pertinents qu'on ait à traiter et à résoudre. Comme le rappelle C. Isham, toutes les tentatives actuellement mises en œuvre pour élaborer une théorie quantique de la gravitation (l'approche euclidienne, les cordes ou les boucles quantiques) présupposent une structure spatio-temporelle très « classique » sous forme d'une variété différentiable on ne peut plus ordinaire… Face à ce type de difficultés, la géométrie non commutative me semble beaucoup mieux armée car elle ouvre des perspectives inconcevables par ailleurs. En effet, comme je l'ai rappelé dans cette introduction, et comme l'illustre le Chapitre 1, c'est le concept même de variété différentiable que la géométrie non commutative se propose, entre autres, de généraliser (avec succès !), sans qu'il soit nécessaire d'avoir recours aux notions usuelles et embarrassantes (dans le contexte quantique) de points et de trajectoires. C'est pourquoi je renouvelle mon espoir que la géométrie non commutative, après avoir enrichi considérablement les mathématiques en jetant des regards nouveaux et originaux sur bien des considérations, inspire enfin, un jour, des idées fondamentalement nouvelles en physique.

Aussi, la piste géométrico-algébrique empruntée actuellement par la géométrie non commutative, qui ne s'intéresse pour l'instant qu'aux théories de jauge, n'a pas encore rejoint la véritable finalité qu'elle se propose d'atteindre en physique : permettre de penser une théorie (avant même de la construire…) éclairant de façon élégante et pertinente les deux aspects antagonistes de la Nature : le quantique et le géométrique…

6. L'usage des guillemets me permet d'avoir à éviter de préciser d'avantage de quoi il s'agit précisément…

1

Ideas and concepts of noncommutative geometry

1.1 Introduction

Once upon a time, in a perfect land, the idea of point was conceived.

This was a beautiful concept, full of potentiality, especially in Natural Science: how easy is it to say where objects are when one has introduced such a precise definition of localization! How easy is it to describe the kinematics of bodies when one assigns to them a point at each time... Well, at least if time is there too! And then laws were found for the interactions of moving bodies, and then predictions were formulated: Pluto, the former planet, was *where* it was calculated to be! Better: generalized geometries where conceived, in which parallels can meet. And you know it: physicists (one of them at least!) where fool enough to show us how useful these geometries can be to describe gravitation.

But nature seems often more subtle than human mind. And the dream ceased when quantum mechanics entered the game. We are no more allowed to say where an electron is exactly located on its "orbit" around the proton in the hydrogen atom. What is the photon trajectory in the Young's double slit experiment? It is forbidden to know! Knowing destroys the diffraction pattern on the target screen.

The main feature of quantum mechanics which exposes us to this annoying situation is the non-commutativity of observables.

How can we accommodate this? Well, one of the reasonable answers can be found in mathematics. Surprisingly enough, mathematicians discovered, not so long ago, that we can speak about spaces without even mentioning them. The trick is to use algebraic objects, and the surprise is that spaces (some of them at least, miracles are not the prerogative of mathematicians!) can be *reconstructed* from them. In the language of mathematics, one has an equivalence of categories...

The algebraic objects we need to deal with are associative algebras, not only with their friendly product but also with other structures, like involutions and norms. There, quantum mechanics is at home: observables are special operators on a Hilbert space, so that they live in such an algebra! Where are the "points" which were mentioned? Take a normal operator (it commutes with its adjoint) in such an operator algebra, consider the smallest subalgebra it generates. This subalgebra is a *commutative* algebra, which can be shown to be the algebra of continuous functions on the spectrum of this normal operator. Associate to this element as many other normal operators as you can find, on the condition that they commute among themselves, and you get another algebra of continuous functions on a topological space. Yes, we get it: a topological space from pure algebraic objects!

This is one of the main results behind *noncommutative geometry*. The idea is the following: if commutative algebras are ordinary topological spaces (in the category of C^*-algebra to be precise), what are the noncommutative ones? How can we study them using the machinery that we are used to manipulate the topological spaces? Is there somewhere another category of algebras in which commutative algebras are differential functions on a differential manifold? If not (for the moment, it is no!), can we manipulate them with some kind of differential structures?

I hope to show you in the following that these questions make sense, and that some answers can be formulated. In section 1.2 we introduce C^*-algebras, and we make the precise statement about commutative C^*-algebras. In section 1.3, we show that one of the machineries developed on topological spaces can be used on their noncommutative counterparts, the C^*-algebras. In section 1.4, cyclic homology is shown to be a good candidate to fool us enough into thinking that we manipulate differential structures on algebras. Section 1.5 is devoted to the Chern character, an object which can convince the more commutative geometer that noncommutative geometry does not only make sense, but also is one of the most beautiful developments in modern mathematics.

1.2 C^*-algebras for topologists

In this section, we will explore some aspects of the theory of C^*-algebras. The main result, we would like to explain, is the theorem by Gelfand and Neumark about commutative C^*-algebras.

1.2.1 General definitions and results

In order to be concise, only algebras over the field $\mathbb{C}$ will be considered.

Definition 1.2.1 (Involutive, Banach and C^* algebras)
An involutive algebra $\mathbf{A}$ is an associative algebra equipped with map $a \mapsto a^*$ such that

$$a^{**} = a \qquad (a+b)^* = a^* + b^* \qquad (\lambda a)^* = \overline{\lambda} a^* \qquad (ab)^* = b^* a^*$$

for any $a, b \in \mathbf{A}$ and $\lambda \in \mathbb{C}$, and where $\overline{\lambda}$ denotes the ordinary conjugation on the complex numbers.

A Banach algebra $\mathbf{A}$ is an associative algebra equipped with a norm $\|\cdot\| : \mathbf{A} \to \mathbb{R}^+$ such that the topological space $\mathbf{A}$ is complete for this norm and such that

$$\|ab\| \leq \|a\|\,\|b\|$$

If the algebra is unital, with unit denoted by $\mathbb{1}$, then it is also required that $\|\mathbb{1}\| = 1$.

A C^*-algebra is an involutive and a Banach algebra $\mathbf{A}$ such that the norm satisfies the C^*-condition

$$\|a^* a\| = \|a\|^2 \tag{1.2.1}$$

♦

A C^*-algebra is then a normed complete algebra equipped with an involution and a compatibility condition between the norm and the involution. One can show that this C^*-condition (1.2.1) implies that $\|a\| \leq \|a^*\| \leq \|a^{**}\| = \|a\|$. The adjoint is then an isometry in any C^*-algebra.

Definition 1.2.2 (self-adjoint, normal, unitary and positive elements)
An element a in a C^*-algebra $\mathbf{A}$ is self-adjoint if $a^* = a$, normal if $a^*a = aa^*$, unitary if $a^*a = aa^* = \mathbb{1}$ when $\mathbf{A}$ is unital, and positive if it is of the form $a = b^*b$ for some $b \in \mathbf{A}$. ♦

Self-adjoint and unitary elements are obviously normal, and positive elements are obviously self-adjoint.

Example 1.2.3 (The algebra of matrices)
Let $M_n(\mathbb{C})$ be the algebra of $n \times n$ matrices over $\mathbb{C}$. This is an involutive algebra for the adjoint. This algebra is complete for the three equivalent norms

$$\|a\|_{\max} = \max\{|a_{ij}| \ / \ i,j = 1,\dots,n\} \qquad \text{max norm}$$
$$\|a\| = \sup\{\|av\| \ / \ v \in \mathbb{C}^n, \ \|v\| \leq 1\} \qquad \text{operator norm}$$
$$\|a\|_{\sum} = \sum_{i,j} |a_{ij}| \qquad \text{sum norm}$$

which are related by the inequalities

$$\|a\|_{\max} \leq \|a\| \leq \|a\|_{\sum} \leq n^2 \|a\|_{\max}$$

The algebra $M_n(\mathbb{C})$ is then a Banach algebra for any of these norms. Only the operator norm defines on $M_n(\mathbb{C})$ a C^*-algebraic structure. ♦

Example 1.2.4 (The algebra of bounded linear operators)
Let $\mathcal{H}$ be a separable Hilbert space, and $\mathcal{B}(\mathcal{H}) = \mathcal{B}$ the algebra of bounded linear operators on $\mathcal{H}$. Equipped with the adjointness operation and the operator norm

$$\|a\| = \sup\{\|au\| \ / \ u \in \mathcal{H}, \ \|u\| \leq 1\}$$

this algebra is a C^*-algebra. In the finite dimensional case, one recovers $M_n(\mathbb{C})$. ♦

Example 1.2.5 (The algebra of compact operators)
Let $\mathcal{H}$ be an Hilbert space. A finite rank operator $a \in \mathcal{B}$ is an operator such that $\dim \operatorname{Ran} a < \infty$. Let $\mathcal{B}_F$ denote the subalgebra of finite rank operators in $\mathcal{B}$. The algebra $\mathcal{K}(\mathcal{H}) = \mathcal{K}$ of compact operators is the closure of $\mathcal{B}_F$ for the topology of the operator norm. The algebra $\mathcal{K}$ is a C^*-algebra, which is not unital when $\mathcal{H}$ is infinite dimensional. In case $\mathcal{H}$ is finite dimensionnal, $\mathcal{K} = M_n(\mathbb{C})$ for $n = \dim \mathcal{H}$.

For any integer $n \geq 1$, $M_n(\mathbb{C})$ is identified as a subalgebra of $\mathcal{B}$, as the operators which act only on the first n vectors of a fixed orthonormal basis of $\mathcal{H}$. Then one gets a direct system of C^*-algebras inside $\mathcal{B}$, $i_n : M_n(\mathbb{C}) \hookrightarrow M_{n+1}(\mathbb{C})$ with $i_n(a) = \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$. One has

$$\mathcal{K} = \varinjlim M_n(\mathbb{C})$$

Using this identification, it is easy to see that $\mathcal{K}$ is an ideal in $\mathcal{B}$. The quotient C^*-algebra $\mathcal{Q} = \mathcal{B}/\mathcal{K}$ is the Calkin algebra. ♦

Example 1.2.6 (The algebra of continuous functions)
Let X be a compact Hausdorff space. Denote by $C(X)$ the (commutative) algebra of continuous functions on X, for pointwise addition and multiplication of functions. Define the involution $f \mapsto \overline{f}$ and the sup norm

$$\|f\|_\infty = \sup_{x \in X} |f(x)| \tag{1.2.2}$$

With these definitions, $C(X)$ is a C^*-algebra.

If the topological space X is only a locally compact Hausdorff space, one defines $C_0(X)$ to be the algebra of continous functions on X vanishing at infinity : for any $\varepsilon > 0$, there exists a compact $K \subset X$ such that $f(x) < \varepsilon$ for any $x \in X \backslash K$. Equipped with the same norm and involution as $C(X)$, this is a C^*-algebra. ♦

Example 1.2.7 (The tensor product)
One can perform a lot of operations on C^*-algebras, some of them being described in the following examples.

Let us mention that there exist some well defined tensor products on C^*-algebras (see Chapter 11 in [Kadison and Ringrose, 1997] or Appendix T in [Wegge-Olsen, 1993]). In the following, we denote by $\widehat{\otimes}$ the spatial tensor product. One of its properties is that $C(X) \widehat{\otimes} C(Y) = C(X \times Y)$ for any compact spaces X, Y. ♦

Example 1.2.8 (The algebra $M_n(\mathbf{A})$)
Let $\mathbf{A}$ be a C^*-algebra. We denote by $M_n(\mathbf{A})$ the set of $n \times n$ matrices with entries in $\mathbf{A}$. This is naturally an algebra. One can define the max norm and the sum norm on this algebra using $\|a_{ij}\|$ instead of $|a_{ij}|$ as in Example 1.2.3. For the operator norm, the situation is more subtle. One has to take an injective representation $\rho : \mathbf{A} \to \mathcal{B}(\mathcal{H})$, which induces an injective representation $\rho_n : M_n(\mathbf{A}) \to \mathcal{B}(\mathcal{H}^n)$. The operator norm of $a \in M_n(\mathbf{A})$ is defined as $\|a\| = \|\rho_n(a)\|$ where the last norm is on $\mathcal{B}(\mathcal{H}^n)$. One can then show that this norm is independent of the choice of the injective representation ρ and gives $M_n(\mathbf{A})$ a structure of C^*-algebra.

This construction corresponds also to define $M_n(\mathbf{A})$ as $M_n(\mathbb{C}) \widehat{\otimes} \mathbf{A}$.

The natural inclusion $M_n(\mathbf{A}) \hookrightarrow M_{n+1}(\mathbf{A})$ defines a direct system of C^*-algebras. One can show that $\mathbf{A} \widehat{\otimes} \mathcal{K} = \varinjlim M_n(\mathbf{A})$. ♦

Example 1.2.9 (The algebra $C_0(X, \mathbf{A})$)
Let X be a locally compact topological space and $\mathbf{A}$ a C^*-algebra. The space $C_0(X, \mathbf{A})$ of continuous functions $a : X \to \mathbf{A}$ vanishing at infinity, equipped with the involution induced by the involution on $\mathbf{A}$ and the sup norm $\|a\|_\infty = \sup_{x \in X} \|a(x)\|$, is a C^*-algebra. Using the spatial tensor product, one has $C_0(X, \mathbf{A}) = C_0(X) \widehat{\otimes} \mathbf{A}$.

If X is compact, we denote it by $C(X, \mathbf{A})$. If $\mathbf{A}$ is unital and X is compact, this algebra is unital. ♦

Example 1.2.10 (The convolution algebra)
The algebra $L^1(\mathbb{R})$ for the convolution product

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y) g(y) dy$$

with the norm

$$\|f\|_1 = \int_{\mathbb{R}} |f(x)| dx$$

and equipped with the involution

$$f^*(x) = \overline{f(-x)}$$

is a Banach algebra with involution but is not a C^*-algebra. ♦

Definition 1.2.11 (Fréchet algebra)
A semi-norm of algebras p on $\mathbf{A}$ is a semi-norm on the vector space $\mathbf{A}$ for which $p(ab) \leq p(a)p(b)$ for any $a, b \in \mathbf{A}$.

A Fréchet algebra is a topological algebra for the topology of a numerable set of algebra semi-norms, which is complete. ♦

Example 1.2.12 (The Fréchet algebra $C^\infty(M)$)
Let M be a C^∞ finite dimension locally compact manifold. Then the algebra $C^\infty(M)$ of differentiable functions on M is an involutive algebra but is not a Banach algebra for the sup norm because it is not complete. Nevertheless, this algebra can be equipped with a family of semi-norms $p_{K_r,N}$ to make it a Fréchet algebra. These semi-norms are defined as follows. For any $\alpha = (\alpha_1, \dots, \alpha_n)$ where $\alpha_r \in \mathbb{N}$, let us use the compact notation $D^\alpha = \left(\frac{\partial}{\partial x^1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x^n}\right)^{\alpha_n}$ with $|\alpha| = \alpha_1 + \dots + \alpha_n$. Then, for any compact subspace $K \subset M$ and any integer $N \geq 0$, define $p_{K,N}(f) = \max\{|(D^\alpha f)(x)| \ / \ x \in K, \ |\alpha| \leq N\}$. With a increasing numerable family of compact spaces $K_r \subset M$ such that $\bigcup_{r\geq 0} K_r = M$, one gets a numerable family of semi-norms $p_{K_r,N}$ for the topology of which $C^\infty(M)$ is complete. ♦

Example 1.2.13 (The irrational rotation algebra)
Let θ be an irrational number. On the Hilbert space $L^2(\mathbb{S}^1)$, consider the two unitary operators

$$(Uf)(t) = e^{2\pi i t} f(t) \qquad\qquad (Vf)(t) = f(t-\theta)$$

where $f : \mathbb{S}^1 \to \mathbb{C}$ is considered as a periodic function in the variable $t \in \mathbb{R}$. Then one has $UV = e^{2\pi i\theta} VU \in \mathcal{B}(L^2(\mathbb{S}^1))$. The C^*-algebra $\mathcal{A}_\theta$ generated by U and V is called the irrational rotation algebra or the noncommutative torus.

Let us consider the Schwartz space $\mathcal{S}(\mathbb{Z}^2)$ of sequences $(a_{m,n})_{m,n\in\mathbb{Z}}$ of rapid decay *i.e.* $(|m| + |n|)^q |a_{m,n}|$ is bounded for any $q \in \mathbb{N}$. We define the algebra $\mathcal{A}_\theta^\infty$ as the set of elements in $\mathcal{A}_\theta$ which can be written as $\sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n$ for a sequence $(a_{m,n})_{m,n\in\mathbb{Z}} \in \mathcal{S}(\mathbb{Z}^2)$. The family of semi-norms $p_q(a) = \sup_{m,n\in\mathbb{Z}} \{(1+|m|+|n|)^q |a_{m,n}|\}$ gives to $\mathcal{A}_\theta^\infty$ a structure of Fréchet algebra.

This algebra admits two continuous non inner derivations δ_i, $i = 1,2$, defined by $\delta_1(U^m) = 2\pi i m U^m$, $\delta_1(V^n) = V^n$, $\delta_2(U^m) = U^m$ and $\delta_2(V^n) = 2\pi i n V^n$.

Using Fourier analysis, $\mathcal{S}(\mathbb{Z}^2)$ is isomorphic to the space $C^\infty(\mathbb{T}^2)$ where $\mathbb{T}^2$ is the two-torus. The algebra $\mathcal{A}_\theta^\infty$ is then the equivalent of smooth functions on the noncommutative torus $\mathcal{A}_\theta$. ♦

Let us mention some general results about C^*-algebras:

Proposition 1.2.14 (Unitarisation)
Any C^*-algebra $\mathbf{A}$ is contained in a unital C^*-algebra $\mathbf{A}_+$ as a maximal ideal of codimension one.

The construction of the unitarization $\mathbf{A}_+$ is as follows: as a vector space, $\mathbf{A}_+ = \mathbf{A} + \mathbb{C}$; as an algebra, $(a+\lambda)(b+\mu) = ab + \lambda b + \mu a + \lambda\mu$; as an involutive algebra, $(a+\lambda)^* = a^* + \bar{\lambda}$; as a normed algebra, $\|(a+\lambda)\| = \sup\{\|ab + \lambda b\| \ / \ b \in \mathbf{A}, \ \|b\| \leq 1\}$.

Theorem 1.2.15
For any C^*-algebra $\mathbf{A}$, there exist a Hilbert space $\mathcal{H}$ and an injective representation $\mathbf{A} \to \mathcal{B}(\mathcal{H})$. Then every C^*-algebra is a subalgebra of the bounded operators on a certain Hilbert space.

The construction of this Hilbert space, which is not necessarily separable, is performed through the GNS construction. This theorem implies that any C^*-algebra can be concretely realized as an algebra of operators on a Hilbert space. Obviously, the converse is not true: there are many algebras of operators on Hilbert spaces which are not C^*-algebras.

Proposition 1.2.16
Any morphism between two C^*-algebras is norm decreasing.

The norm on a C^*-algebra is unique. An isomorphism of C^*-algebras is an isometry.

1.2.2 The Gelfand transform

Let $\mathbf{A}$ be a unital Banach algebra. Let us use the notation $z = z\mathbb{1} \in \mathbf{A}$ for any $z \in \mathbb{C}$.

Definition 1.2.17 (Resolvant, spectrum and spectral radius)
Let a be an element in $\mathbf{A}$.

The resolvant of a, denoted by $\rho(a)$, is the subspace of $\mathbb{C}$:

$$\rho(a) = \{z \in \mathbb{C} \ / \ (a - z)^{-1} \in \mathbf{A}\}$$

The spectrum of a, denoted by $\sigma(a)$, is the complement of $\rho(a)$ in $\mathbb{C}$: $\sigma(a) = \mathbb{C} \backslash \rho(a)$. One can show that $\sigma(a)$ is a compact subspace of $\mathbb{C}$ contained in the disk $\{z \in \mathbb{C} \ / \ |z| \leq \|a\|\}$.

The spectral radius of a is defined as

$$r(a) = \sup\{|z| \ / \ z \in \sigma(a)\}$$ ♦

For any $a \in M_n(\mathbb{C})$, the spectrum of a contains the set of eigenvalues of a, but can contain other values not associated to eigenvectors.

The spectrum of $a \in \mathbf{A}$ depends on the algebra $\mathbf{A}$. Nevertheless, we will see exceptions to that.

The spectral radius can be computed using the relation

$$r(a) = \lim_{n \to \infty} \|a^n\|^{1/n}$$

which can look surprising at first: on the left, the radius is defined using only the algebraic structure of $\mathbf{A}$ (is an element $(a - z)$ invertible?), on the right it is related to the norm…

Here are some interesting results about the spectrum of particular elements.

Proposition 1.2.18
If a is self-adjoint, then $\sigma(a) \subset \mathbb{R}$. If a is unitary, then $\sigma(a) \subset \mathbb{S}^1$. If a is positive, then $\sigma(a) \subset \mathbb{R}_+$.

One can show the following:

Theorem 1.2.19 (Gelfand-Mazur)
Any unital Banach algebra in which every non zero element is invertible is isomorphic to $\mathbb{C}$.

Definition 1.2.20 (The spectrum of an algebra and the Gelfand transform)
Let $\mathbf{A}$ be a Banach algebra. A (continuous) character on $\mathbf{A}$ is a non zero continuous morphism of algebras $\chi : \mathbf{A} \to \mathbb{C}$. If $\mathbf{A}$ is unital, we require $\chi(\mathbb{1}) = 1$.

The spectrum of $\mathbf{A}$, denoted by $\Delta(\mathbf{A})$, is the set of characters of $\mathbf{A}$. The spectrum $\Delta(\mathbf{A})$ is a topological space for the topology induced by the pointwise convergence $\chi_n \xrightarrow{n\to\infty} \chi \Leftrightarrow \forall a \in \mathbf{A}$, $\chi_n(a) \xrightarrow{n\to\infty} \chi(a)$.

There is a natural map $\mathbf{A} \to C(\Delta(\mathbf{A}))$ defined by $a \mapsto \widehat{a}$ where $\widehat{a}(\chi) = \chi(a)$. This is the Gelfand transform of $\mathbf{A}$. ♦

So, now that we have associated to elements in a Banach algebra, and to the algebra itself, some topological spaces, it is time to introduce some functional spaces on them! The next example gives us an insight to what will happen in the general case.

Example 1.2.21 (The spectrum of $C(X)$)
Let $\mathbf{A} = C(X)$ as in Example 1.2.6. For any function $f \in C(X)$, $\sigma(f)$ is the set of values of f: $\sigma(f) = f(X) \subset \mathbb{C}$. Any point $x \in X$ defines a character $\chi_x \in \Delta(\mathbf{A})$ by $\chi_x(f) = f(x)$, so that $X \subset \Delta(\mathbf{A})$. The topologies on X and $\Delta(\mathbf{A})$ makes this inclusion a continuous application. ♦

Example 1.2.22 (Unital commutative Banach algebra)
What happens when the Banach algebra is unital and commutative? In that case, one can show that the maximal ideals in $\mathbf{A}$ are in a one-to-one correspondence with characters on $\mathbf{A}$. Indeed, it is easy to associate to any character $\chi \in \Delta(\mathbf{A})$, the maximal ideal $\mathbf{I}_\chi = \operatorname{Ker}\chi$. In the other direction, for any maximal ideal $\mathbf{I}$, one can show that every non zero element in the algebra $\mathbf{A}/\mathbf{I}$ is invertible, which means, by the Gelfand-Mazur's theorem, that $\mathbf{A}/\mathbf{I} = \mathbb{C}$. Associate to $\mathbf{I}$ the projection $\mathbf{A} \to \mathbf{A}/\mathbf{I}$. This is the desired character.

Then, one can show that $\Delta(\mathbf{A})$ is a compact Hausdorff space. The Gelfand transform connects two commutative unital Banach algebras $\mathbf{A} \to C(\Delta(\mathbf{A}))$ by a continuous morphism of algebras. What is now a pleasant surprise, is that the spectrum of a in $\mathbf{A}$ is exactly the spectrum of $\widehat{a}$ in $C(\Delta(\mathbf{A}))$, which is the set of values of the function $\widehat{a}$ on $\Delta(\mathbf{A})$:

$$\sigma(a) = \sigma(\widehat{a}) = \{\widehat{a}(\chi) = \chi(a) \ / \ \chi \in \Delta(\mathbf{A})\}$$ ♦

When the commutative Banach algebra is not unital, the Gelfand transform realizes a continuous morphism of algebras $\mathbf{A} \to C_0(\Delta(\mathbf{A}))$. One important result is that $\Delta(\mathbf{A})^+ = \Delta(\mathbf{A}_+)$ where on the left $\Delta(\mathbf{A})^+$ is the one-point compactification of the topological space $\Delta(\mathbf{A})$ and on the right $\mathbf{A}_+$ is the unitarization of $\mathbf{A}$.

It is now possible to state the main theorem in this section:

Theorem 1.2.23 (Gelfand-Neumark)
For any commutative C^*-algebra $\mathbf{A}$, the Gelfand transform is an isomorphism of C^*-algebras.

In the unital case, one gets $\mathbf{A} \simeq C(\Delta(\mathbf{A}))$ and in the non unital case, $\mathbf{A} \simeq C_0(\Delta(\mathbf{A}))$ and $\mathbf{A}_+ \simeq C(\Delta(\mathbf{A})^+)$.

In the language of categories, this theorem means that the category of locally compact Hausdorff spaces is equivalent to the category of commutative C^*-algebras.

1.2.3 Functional calculus

The demonstration of the Gelfand-Neumark theorem relies on some constructions largely known as functional calculus. These constructions are very important to understand the relations between commutative C^*-algebras and topological spaces. Their understanding opens the door to the comprehension of noncommutative geometry.

The first example we consider is the polynomial functional calculus. This gives us the general idea. Let $a \in \mathbf{A}$, where $\mathbf{A}$ is any unital associative algebra. To every polynomial function $p \in \mathbb{C}[x]$ in the real variable x, we can associate $p(a) \in \mathbf{A}$ as the element obtained by the replacement $x^n \mapsto a^n$ in p. In particular, for the polynomial $p(x) = x$ (resp. $p(x) = 1$), one gets $p(a) = a$ (resp. $p(a) = \mathbb{1}$).

For algebras with supplementary structures, this can be generalized using other algebras of functions.

First, consider an involutive unital algebra $\mathbf{A}$, and let $a \in \mathbf{A}$ be a normal element. To every polynomial function $p \in \mathbb{C}[z, \overline{z}]$ of the complex variable z and its conjugate $\overline{z}$, we associate $p(a) \in \mathbf{A}$

through the replacements $z^n \mapsto a^n$ and $\overline{z}^n \mapsto (a^*)^n$. Because a is normal, $p(a)$ is a well defined element in $\mathbf{A}$.

Let $\mathbf{A}$ be now a unital Banach algebra. For any $\lambda \notin \sigma(a)$ one introduces the resolvant of a at λ:

$$R(a,\lambda) = \frac{1}{\lambda - a} \in \mathbf{A}$$

Consider any holomorphic function $f : U \to \mathbb{C}$, with U an open subset of $\mathbb{C}$ which strictly contains the compact subspace $\sigma(a)$, and $\Gamma : [0,1] \to \mathbb{C}$ a closed path in U such that $\sigma(a)$ is strictly inside Γ. The usual Cauchy formula $f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\lambda)}{\lambda - z} d\lambda \in \mathbb{C}$ can be generalized in the form

$$f(a) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\lambda)}{\lambda - a} d\lambda \in \mathbf{A}$$

Indeed, $\lambda \mapsto R(a,\lambda)$ is a function which takes its values in a Banach space, and integration of $f(\lambda)R(a,\lambda)$ is meaningful along Γ. What can be shown is that this integration does not depend on the choice of the surrounding closed path Γ.

This relation defines what is called the holomorphic functional calculus on $\mathbf{A}$. In case f is a polynomial function (of the variable z, but not of the variable $\overline{z}$), $f(a)$ coincides with the polynomial functional calculus.

Let us consider now a unital C^*-algebra $\mathbf{A}$. In that case, one would like to mix the two previous functional calculi on involutive and Banach algebras.

In order to do that, consider a normal element $a \in \mathbf{A}$. One can introduce $C^*(a)$, the smallest unital C^*-subalgebra of $\mathbf{A}$ which contains a and a^* (and $\mathbb{1}$ since it is unital). Because $\mathbb{1}, a$ and a^* commute among themselves, the C^*-algebra $C^*(a)$ is a commutative C^*-algebra. Let us summarize some facts about this algebra:

Proposition 1.2.24
The spectrum of a in $C^*(a)$ is the same as the spectrum of a in $\mathbf{A}$. It will be denoted by $\sigma(a)$.

The spectrum of the algebra $C^*(a)$ is the spectrum of the element a, $\Delta(C^*(a)) = \sigma(a)$, so that

$$C^*(a) = C(\sigma(a))$$

The Gelfand transform maps a into the continuous function $\widehat{a} : \sigma(a) \to \mathbb{C}$ which is identity: $\sigma(a) \ni z \mapsto z \in \mathbb{C}$.

The inverse of the Gelfand transform associates to any continuous function $f : \sigma(a) \to \mathbb{C}$ a unique element $f(a) \in C^*(a) \subset \mathbf{A}$ such that

$$\|f(a)\| = \|f\|_\infty \qquad\qquad \sigma(f(a)) = f(\sigma(a)) \subset \mathbb{C}$$

In particular, the norm of $f(a)$ in $C^*(a)$ is the norm of $f(a)$ in $\mathbf{A}$.

The association $f \mapsto f(a)$ in this Proposition is the continuous functional calculus associated to the normal element a. In case f is a polynomial function in the variables z and $\overline{z}$ (resp. f is an holomorphic function), one recovers the polynomial functional calculus (resp. the holomorphic functional calculus).

There are a lot of interesting normal elements in a C^*-algebras (self-adjoint, unitary, positive…) for which the continuous functional calculus is very convenient. The next example illustrates such a situation.

Example 1.2.25 (Absolute value in A)
One can associate to any element $a \in \mathbf{A}$ its absolute value using the functional calculus associated to the normal (and positive) element a^*a. Consider the continuous function $\mathbb{R}_+ \ni x \mapsto f(x) = |x|^{1/2}$ and define $|a| \in \mathbf{A}^+$ by $|a| = f(a^*a)$. ♦

What do we learn from these constructions? The main result here is that it is not necessary to consider a commutative C^*-algebra in order to manipulate some topological spaces. Just consider some normal elements commuting among themselves, build upon them the smallest C^*-algebra they generate, and you have in hand a Hausdorff space!

The idea of noncommutative topology is to study C^*-algebras from the point of view that they are "continuous functions on noncommutative spaces". In order to do that, one needs some tools that are common to the topological situation and to the algebraic one.

Such tools exist! One of them is K-theory.

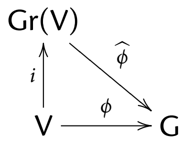
1.3 K-theory for beginners

The K-theory groups are defined through a universal construction, the Grothendieck group associated to an abelian semigroup.

Definition 1.3.1 (Grothendieck group of an abelian semigroup)
An abelian semigroup is a set V equipped with an internal associative and abelian law $\boxplus : \mathsf{V} \times \mathsf{V} \to \mathsf{V}$. A unit element is an element 0 such that $v \boxplus 0 = v$ for any $v \in \mathsf{V}$. Any abelian group is a semigroup.

The Grothendieck group associated to V is the abelian group $(\mathsf{Gr}(\mathsf{V}), +)$ which satisfies the following universal property. There exits a semigroup map $i : \mathsf{V} \to \mathsf{Gr}(\mathsf{V})$ such that for any abelian group $(\mathsf{G}, +)$ and any morphism of abelian semigroups $\phi : \mathsf{V} \to \mathsf{G}$, there exists a unique morphism of abelian groups $\widehat{\phi} : \mathsf{Gr}(\mathsf{V}) \to \mathsf{G}$ such that $\phi = \widehat{\phi} \circ i$. ♦

This means that the following diagram can be completed with $\widehat{\phi}$ to get a commutative diagram:



It is convenient to have in mind one of the possible constructions of the Grothendieck group associated to V. On the set $\mathsf{V} \times \mathsf{V}$ consider the equivalence relation: $(v_1, v_2) \sim (v'_1, v'_2)$ if there exists $v \in \mathsf{V}$ such that $v_1 \boxplus v'_2 \boxplus v = v'_1 \boxplus v_2 \boxplus v \in \mathsf{V}$. Denote by $\langle v_1, v_2 \rangle$ an equivalence class in $\mathsf{V} \times \mathsf{V}$ for this relation and let $\mathsf{Gr}(\mathsf{V}) = (\mathsf{V} \times \mathsf{V})/\sim$. The group structure on $\mathsf{Gr}(\mathsf{V})$ is defined by $\langle v_1, v_2 \rangle + \langle v'_1, v'_2 \rangle = \langle v_1 \boxplus v'_1, v_2 \boxplus v'_2 \rangle$, the unit is $\langle v, v \rangle$ for any $v \in \mathsf{V}$, the inverse of $\langle v_1, v_2 \rangle$ is $\langle v_2, v_1 \rangle$. The morphism of abelian semigroups $i : \mathsf{V} \to \mathsf{Gr}(\mathsf{V})$ is $v \mapsto \langle v \boxplus v', v' \rangle$ (independent of the choice of v'). Notice that $\langle v_1 + v, v_2 + v \rangle = \langle v_1, v_2 \rangle$ in $\mathsf{Gr}(\mathsf{V})$.

Then one can show that $\mathsf{Gr}(\mathsf{V})$ is indeed the Grothendieck group associated to V and that

$$\mathsf{Gr}(\mathsf{V}) = \{ i(v) - i(v') \ / \ v, v' \in \mathsf{V} \}$$

A useful relation is that $i(v + w) - i(v' + w) = i(v) - i(v') \in \mathsf{Gr}(\mathsf{V})$ for any $w \in \mathsf{V}$.

Example 1.3.2 (The semigroup $\mathbb{N}$)
The set of natural numbers defines an abelian semigroup $(\mathbb{N},+)$. Its Grothendieck's group is $(\mathbb{Z},+)$. In this situation, the morphism $i:\mathbb{N}\to\mathbb{Z}$ is injective. This is not always the case. Another particular property is that any relation $n_1+n=n_2+n$ in $\mathbb{N}$ can be simplified into $n_1=n_2$. This is the simplification property, which is not satisfied by all abelian semigroups. ♦

Example 1.3.3 (The semigroup $\mathbb{N}\cup\{\infty\}$)
Consider the set $\mathbb{N}\cup\{\infty\}$ with the ordinary additive law for two elements in $\mathbb{N}$ and the new law $\infty+n=\infty+\infty=\infty$. Then its Grothendieck group is 0. Indeed, all couples (n,m), (n,∞), (∞,m) and (∞,∞) are equivalent. ♦

1.3.1 The topological K-theory

It is useful to recall some facts and constructions about vector bundles over topological spaces. We will restrict ourselves to locally trivial complex vector bundles over Hausdorff spaces.

Definition 1.3.4
Let $\pi:E\to X$ and $\pi':E'\to X$ two vector bundles over X, of rank n and n'. Then one defines the Whitney sum $E\oplus E'\to X$ of rank $n+n'$ by $E\oplus E'=\cup_{x\in X}(E_x\oplus E'_x)=\{(e,e')\in E\times E' \ / \ \pi(e)=\pi'(e')\}\subset E\times E'$ and the tensor product $E\otimes E'=\cup_{x\in X}(E_x\otimes E'_x)$ of rank nn'.

We denote by $\underline{\mathbb{C}}^n=X\times\mathbb{C}^n\to X$ the trivial vector bundles of rank n.

For any continuous map $f:Y\to X$ and any vector bundle $E\to X$, we define the pull-back $f^*E\to Y$ as $f^*E=\{(y,e)\in Y\times E \ / \ f(y)=\pi(e)\}\subset Y\times E$. ♦

When $i:Y\hookrightarrow X$ is an inclusion, the pull-back $i^*E=E_{|Y}$ is just the restriction of E to $Y\subset X$. When $f_0,f_1:Y\to X$ are homotopic, the two vector bundles f_0^*E and f_1^*E are isomorphic.

We will use the following very important result in the theory of vector bundles:

Theorem 1.3.5 (Serre-Swan)
Let X be a compact topological space. For any vector bundle $E\to X$ there exist an integer N and a second vector bundle $E'\to X$ such that $E\oplus E'\simeq\underline{\mathbb{C}}^N$.

We introduce $\mathsf{V}(X)$, the set of isomorphic classes of vector bundles over X. Let use the notation $[E]$ for the isomorphic class of E. The set $\mathsf{V}(X)$ is an abelian semigroup for the law induced by the Whitney sum: $[E]+[E']=[E\oplus E']$.

For any continuous map $f:Y\to X$, the pull-back construction defines a morphism of abelian semigroups $f^*:\mathsf{V}(X)\to\mathsf{V}(Y)$, which depends only on the homotopic class of f.

Definition 1.3.6 ($K^0(X)$ for X compact)
For any compact topological space X, we define $K^0(X)$ as the Grothendieck group of $\mathsf{V}(X)$. ♦

Remark 1.3.7 (Representatives in $K^0(X)$)
From the construction of the Grothendieck group, any element in $K^0(X)$ can be realized as a formal difference $[E]-[F]$ of two isomorphic classes in $\mathsf{V}(X)$. Adding the same vector bundle to E and F does not change this element in the Grothendieck group. So, one can always find a representative of the form $[E]-[\underline{\mathbb{C}}^n]$ for some integer n. ♦

For any continuous map $f:Y\to X$, the morphism $f^*:\mathsf{V}(X)\to\mathsf{V}(Y)$ induces a morphism of abelian groups $f^\sharp:K^0(X)\to K^0(Y)$.

Example 1.3.8 ($K^0(*) = \mathbb{Z}$)
Let $*$ denote the space reduced to a point. In that case any vector bundle $E \to *$ is just a finite dimensional vector space. It is well known that the isomorphic classes of finite dimensional vector spaces are classified by their dimension, so that $\mathsf{V}(*) = \mathbb{N}$, with the abelian semigroup structure of Example 1.3.2. Then one gets $K^0(*) = \mathbb{Z}$. Obviously this result is true for any contractible topological space. ♦

Let $x_0 \in X$ be a fixed point. Denote by $i : * = \{x_0\} \to X$ the inclusion, and $p : X \to *$ the projection. Then $p \circ i$ is Id_*. These maps define two morphisms

$$p^\sharp : \mathbb{Z} = K^0(*) \to K^0(X) \qquad\qquad i^\sharp : K^0(X) \to K^0(*) = \mathbb{Z}$$

Because $i^\sharp p^\sharp = \mathrm{Id}_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$, $p^\sharp$ is injective.

Definition 1.3.9 (Reduced *K*-group for pointed compact spaces)
We define the reduced K-group of the pointed compact topological space X by

$$\widetilde{K}^0(X) = \mathrm{Ker}(i^\sharp : K^0(X) \to \mathbb{Z}) = K^0(X)/p^\sharp\mathbb{Z}$$ ♦

The injective morphism $p^\sharp$ splits the short exact sequence of abelian groups

$$0 \longrightarrow \widetilde{K}^0(X) \longrightarrow K^0(X) \xrightarrow{i^\sharp} K^0(*) = \mathbb{Z} \longrightarrow 0$$

so that $K^0(X) = \widetilde{K}^0(X) \oplus \mathbb{Z}$. The reduced K-theory, like the reduced singular homology, is the natural K-theory of pointed compact spaces.

Remark 1.3.10 (Interpretation of $\widetilde{K}^0(X)$)
An element in $\widetilde{K}^0(X)$ is a formal difference $[E] - [F]$ where now E and F have the same rank because they must coincide over x_0 in order to be in the kernel of $i^\sharp$. It is possible to choose $F = \underline{\mathbb{C}}^N$ with $N = \mathrm{rank} E$. ♦

Definition 1.3.11 ($K^0(X)$ for any X)
Let X be a locally compact topological space X, not necessarily compact. Denote by X^+ its one-point compactification. Then one defines $K^0(X) = \widetilde{K}^0(X^+)$. ♦

Let us make some comments about this construction.

Remark 1.3.12
In this situation, the natural fixed point in X^+ is the point at infinity, so that $i : * \to X^+$ sends $*$ into ∞. The compactification adds a point to X and the reduced K-group construction removes the contribution from this point.

In case X is compact, it is then easy to verify that $\widetilde{K}^0(X^+)$ identifies with the K-group as in Definition 1.3.6. ♦

Remark 1.3.13 (Interpretation of $K^0(X)$)
In $\widetilde{K}^0(X^+)$, an element is a formal difference $[E] - [F]$ with $\mathrm{rank} E = \mathrm{rank} F$. Because this element is in the kernel of $i^\sharp$, one has $E_{|\infty} = F_{|\infty}$. So these two vector bundles are also isomorphic in a neighborhood of $\infty \in X^+$. By definition of the one-point compactification, such a neighborhood is the complement of a compact in X, so that E and F can be considered as vector bundles over X which coincide outside some compact $K \subset X$.

One can go a step further. Because E and F coincide outside some compact K, one can add to them a third vector bundle such that outside K the sums are isomorphic to a trivial vector bundle. Adding such a vector bundle does not change the element $[E]-[F] \in K^0(X)$. Therefore, any element in $K^0(X)$ can be represented by a formal difference $[E]-[F]$ where E and F are not only isomorphic outside some compact, but also trivial.

Owing to this interpretation, this definition of $K^0(X) = \widetilde{K}^0(X^+)$ is also called, in the literature, the K-theory with compact support (the non trivial part of the vector bundles is inside a compact). For instance, it is denoted by K_{cpt} in [Lawson and Michelsohn, 1989]. ♦

The indice 0 in the definition of the K-group suggests that others K-groups can be defined. This is indeed the case, but we will see that there are not so many!

Definition 1.3.14 (Higher orders K-groups)
Let X be a locally compact topological space X. For any $n \geq 1$, we define $K^{-n}(X) = K^0(X \times \mathbb{R}^n)$. ♦

The corresponding reduced K-group is $\widetilde{K}^{-n}(X) = \widetilde{K}^0(X \wedge \mathbb{S}^n)$, where we recall that for two pointed compact spaces (X, x_0) and (Y, y_0), their wedge product is $X \wedge Y = X \times Y / (\{x_0\} \times Y \cup X \times \{y_0\})$. For any n, one can show that $K^{-n}(X) = \widetilde{K}^{-n}(X^+)$.

Proposition 1.3.15 (Long exact sequences)
Let X be a locally compact space and $Y \subset X$ a closed subspace. Then there exist boundary maps $\delta : K^{-n}(Y) \to K^{-n+1}(X \backslash Y)$ and a long exact sequence

$$\cdots \xrightarrow{\delta} K^{-n}(X \backslash Y) \longrightarrow K^{-n}(X) \longrightarrow K^{-n}(Y) \xrightarrow{\delta} K^{-n+1}(X \backslash Y) \longrightarrow \cdots$$
$$\cdots \xrightarrow{\delta} K^0(X \backslash Y) \longrightarrow K^0(X) \longrightarrow K^0(Y)$$

In reduced K-theory, for pointed compact spaces $Y \subset X$, one has the corresponding long exact sequence

$$\cdots \xrightarrow{\delta} \widetilde{K}^{-n}(X/Y) \longrightarrow \widetilde{K}^{-n}(X) \longrightarrow \widetilde{K}^{-n}(Y) \xrightarrow{\delta} \widetilde{K}^{-n+1}(X/Y) \longrightarrow \cdots$$
$$\cdots \xrightarrow{\delta} \widetilde{K}^0(X/Y) \longrightarrow \widetilde{K}^0(X) \longrightarrow \widetilde{K}^0(Y)$$

Proposition 1.3.16 (The ring structure)
The tensor product of vector bundles induces a ring structure on $K^0(X)$ and $\widetilde{K}^0(X)$.

The external tensor product induces graded ring structures on $K^\bullet(X) = \bigoplus_{n \geq 0} K^{-n}(X)$ and on $\widetilde{K}^\bullet(X) = \bigoplus_{n \geq 0} \widetilde{K}^{-n}(X)$ which extend the ring structures on $K^0(X)$ and $\widetilde{K}^0(X)$.

Example 1.3.17 (The 2-sphere)
Any vector bundle on the 2-sphere is characterized by its clutching function on the equator. This is a continuous map $\mathbb{S}^1 \to U(n)$ for a vector bundle of rank n. In order to consider all the possible ranks at the same time, the maps to consider are $\mathbb{S}^1 \to U(\infty) = \varinjlim U(n)$. Studying these functions, in particular their homotopic equivalence classes, gives the following result. Let H denote the tautological vector bundle of rank 1 over $\mathbb{CP}^1 = \mathbb{S}^2$. Then one has $(H \otimes H) \oplus \underline{\mathbb{C}} \simeq H \oplus H$ and as rings $K^0(\mathbb{S}^2) \simeq \mathbb{Z}[H]/\langle (H - \underline{\mathbb{C}})^2 \rangle$ where $\langle (H - \underline{\mathbb{C}})^2 \rangle$ is the ideal in $\mathbb{Z}[H]$ generated by $(H - \underline{\mathbb{C}})^2$, so that $K^0(\mathbb{S}^2) \simeq \mathbb{Z} \cdot \underline{\mathbb{C}} \oplus \mathbb{Z} \cdot (H - \underline{\mathbb{C}}) \simeq \mathbb{Z} \oplus \mathbb{Z}$. Because $H - \underline{\mathbb{C}} \in \operatorname{Ker}(i^\sharp : K^0(\mathbb{S}^2) \to \mathbb{Z})$, one has $\widetilde{K}^0(\mathbb{S}^2) = \mathbb{Z} \cdot (H - \underline{\mathbb{C}}) \simeq \mathbb{Z}$ with a null product. ♦

Example 1.3.18 (The ring $K^\bullet(*)$)
The ring structure of $K^\bullet(*)$ is easy to describe. One can show that $K^{-2}(*) = K^0(\mathbb{R}^2) = \widetilde{K}^0(\mathbb{S}^2) = \mathbb{Z}$. Denote by ξ the generator of $K^{-2}(*)$. Then, one can show that $K^\bullet(*) = \mathbb{Z}[\xi]$. As ξ is of degree -2, one has $K^{-2n}(*) = \mathbb{Z}$ and $K^{-(2n+1)}(*) = 0$. ♦

Here is now the main result in K-theory:

Theorem 1.3.19 (Bott periodicity)
For any locally compact space X, one has a natural isomorphism

$$K^0(X \times \mathbb{R}^2) = K^{-2}(X) \simeq K^0(X)$$

For any pointed compact space X, one has a natural isomorphism

$$\widetilde{K}^0(X \wedge \mathbb{S}^2) = \widetilde{K}^{-2}(X) \simeq \widetilde{K}^0(X)$$

In reduced K-theory, for two pointed compact spaces X, Y, there is a natural (graded) product

$$\widetilde{K}^\bullet(X) \otimes \widetilde{K}^\bullet(Y) \to \widetilde{K}^\bullet(X \wedge Y)$$

Using $Y = \mathbb{S}^2$, this product gives us an isomorphism

$$\widetilde{K}^0(X) \otimes \widetilde{K}^0(\mathbb{S}^2) \xrightarrow{\simeq} \widetilde{K}^0(X \wedge \mathbb{S}^2)$$

which is exactly the Bott periodicity. Indeed, we saw in Example 1.3.17 that $\widetilde{K}^0(\mathbb{S}^2)$ is generated by $H - \underline{\mathbb{C}}$ (with $(H - \underline{\mathbb{C}})^2 = 0$). The Bott periodicity is the isomorphism

$$\beta : \widetilde{K}^0(X) \xrightarrow{\simeq} \widetilde{K}^0(X \wedge \mathbb{S}^2) = \widetilde{K}^{-2}(X)$$
$$a \mapsto (H - \underline{\mathbb{C}}) \cdot a$$

Example 1.3.20 (K-theories of spheres)
One has $\mathbb{S}^n \wedge \mathbb{S}^m \simeq \mathbb{S}^{n+m}$, so that $\widetilde{K}^0(\mathbb{S}^{2n}) = \widetilde{K}^0(\mathbb{S}^{2n-2} \wedge \mathbb{S}^2) = \widetilde{K}^0(\mathbb{S}^2) = \mathbb{Z}$. For odd degrees, one only needs to know $\widetilde{K}^0(\mathbb{S}^1)$. Using standard arguments from topology of fiber bundles (see [Steenrod, 1951] for instance), there are no non trivial (complex) vector bundles over $\mathbb{S}^1$, so that $\mathsf{V}(\mathbb{S}^1) = \mathbb{N}$ and then $K^0(\mathbb{S}^1) = \mathbb{Z}$ and $\widetilde{K}^0(\mathbb{S}^1) = 0$. This shows that $\widetilde{K}^0(\mathbb{S}^{2n+1}) = \widetilde{K}^0(\mathbb{S}^1) = 0$. Notice that because $\mathbb{R}^+ = \mathbb{S}^1$ (one-point compactification), one has $K^0(\mathbb{R}) = 0$. ♦

Proposition 1.3.21 (Six term exact sequences in K-theory)
The Bott periodicity reduces the long exact sequences of Proposition 1.3.15 into two six term exact sequences

$$\begin{array}{ccccc} K^0(X\backslash Y) & \longrightarrow & K^0(X) & \longrightarrow & K^0(Y) \\ \delta\uparrow & & & & \downarrow\delta \\ K^{-1}(Y) & \longleftarrow & K^{-1}(X) & \longleftarrow & K^{-1}(X\backslash Y) \end{array} \tag{1.3.3}$$

for locally compact spaces $Y \subset X$ with Y closed, and

$$\begin{array}{ccccc} \widetilde{K}^0(X/Y) & \longrightarrow & \widetilde{K}^0(X) & \longrightarrow & \widetilde{K}^0(Y) \\ \delta\uparrow & & & & \downarrow\delta \\ \widetilde{K}^{-1}(Y) & \longleftarrow & \widetilde{K}^{-1}(X) & \longleftarrow & \widetilde{K}^{-1}(X/Y) \end{array}$$

for pointed compact spaces $Y \subset X$ with Y closed.

Example 1.3.22 (K-groups for some topological spaces)
Here is a table of some known K-groups for ordinary topological spaces.

Topological space	K^0	K^{-1}
$*$, compact contractible Hausdorff space	$\mathbb{Z}$	0
$]0,1]$	0	0
$\mathbb{R}$, $]0,1[$	0	$\mathbb{Z}$
$\mathbb{R}^{2n}$, $n \geq 1$	$\mathbb{Z}$	0
$\mathbb{R}^{2n+1}$, $n \geq 0$	0	$\mathbb{Z}$
$\mathbb{S}^{2n}$, $n \geq 1$	$\mathbb{Z} \oplus \mathbb{Z}$	0
$\mathbb{S}^{2n+1}$, $n \geq 0$	$\mathbb{Z}$	$\mathbb{Z}$
$\mathbb{T}^n$	$\mathbb{Z}^{2^{n-1}}$	$\mathbb{Z}^{2^{n-1}}$

Remark 1.3.23 (Real topological K-theory)
We have introduced the topological K-theory using the complex vector bundles over topological spaces. It is possible to define a real topological K-theory in exactly the same way using real vector bundles. The theory is different. For instance, there are non trivial real vector bundles over $\mathbb{S}^1$ (think at the Moebius trip), but there are no non trivial complex vector bundles. In real K-theory the Bott periodicity is of period 8, and the six term exact sequence is replaced by a 24 terms exact sequence. ♦

Remark 1.3.24 (The origin of Bott periodicity)
The first paper mentioning Bott periodicity, [Bott, 1959], was concerned with the homotopy of classical groups, in particular $U(n)$ and $O(n)$. What Bott discovered is that if one denotes by $U(\infty) = \varinjlim U(n)$ and $O(\infty) = \varinjlim O(n)$ the inductive limits for the natural inclusions $U(n) \hookrightarrow U(n+1)$ and $O(n) \hookrightarrow O(n+1)$, then

$$\pi_k(U(\infty)) = \pi_{k+2}(U(\infty)) \qquad \pi_k(O(\infty)) = \pi_{k+8}(O(\infty))$$

In fact, for n large enough, $\pi_k(U(n)) = \pi_k(U(n+1))$ for $n > k/2$, so that this periodicity expresses itself before infinity. The period 2 for the complex case $U(\infty)$ (resp. 8 for the real case $O(\infty)$) is related to the period 2 for K-theory (resp. real K-theory). See [Karoubi, 2005] for a review and references. ♦

1.3.2 K-theory for C^*-algebras

Topological K-theory is defined using some explicit geometrical constructions on vector bundles over a compact topological space X. These constructions, except the tensor product of vector bundles, can be described using the C^*-algebra $C(X)$.

Indeed, it is a well known fact that continuous sections of a vector bundle $E \to X$ is a $C(X)$-module. Recall the following definitions about modules. From now on, every modules are left modules.

Definition 1.3.25 (Finite projective modules)
Let $\mathbf{A}$ be a unital associative algebra.

$\boldsymbol{M}$ is a free $\mathbf{A}$-module if it admits a free basis.

$\boldsymbol{M}$ is a projective $\mathbf{A}$-module if there exists a $\mathbf{A}$-module $\boldsymbol{N}$ such that $\boldsymbol{M} \oplus \boldsymbol{N}$ is a free module.

$\boldsymbol{M}$ is a finite projective $\mathbf{A}$-module if there exist a $\mathbf{A}$-module $\boldsymbol{N}$ and an integer N such that $\boldsymbol{M} \oplus \boldsymbol{N} \simeq \mathbf{A}^N$. ♦

Theorem 1.3.5 can then be written in the following algebraic form:

Theorem 1.3.26 (Serre-Swan, algebraic version)
The functor "continuous sections" realizes an equivalence of categories between the category of vector bundles over a compact topological space X and the category of finite projective modules over $C(X)$.

Any finite projective module is characterized by the morphism of $\mathbf{A}$-modules $p : \mathbf{A}^N \to \mathbf{A}^N$ which projects onto $\boldsymbol{M}$. This morphism is representable as a projection $p \in M_N(\mathbf{A})$, $p^2 = p$, $p^* = p$, such that $\boldsymbol{M} = \mathbf{A}^N p$. In particular, any vector bundle over X is given by a projection $p \in M_N(C(X)) = C(X, M_n(\mathbb{C}))$ (see Example 1.2.9).

We have defined the topological K-theory via the isomorphic classes of vector bundles. In the algebraic language, isomorphic classes correspond to some equivalence classes on projections. Let us define some possible equivalence relations on projections in C^*-algebras.

Let us denote by $\mathcal{P}(\mathbf{A}) = \{p \in \mathbf{A} \ / \ p^2 = p^* = p\}$ the set of projections in a unital C^*-algebra $\mathbf{A}$.

Definition 1.3.27 (Equivalences of projections)
A partial isometry is an element $v \in \mathbf{A}$ such that $v^* v \in \mathcal{P}(\mathbf{A})$. In that case, one can show that $vv^* \in \mathcal{P}(\mathbf{A})$. An isometry is an element $v \in \mathbf{A}$ such that $v^* v = \mathbb{1}$. Unitaries are in particular isometries.

Two projections $p, q \in \mathcal{P}(\mathbf{A})$ are orthogonal if $pq = qp = 0 \in \mathbf{A}$. This means that they project on direct summands of $\mathbf{A}$. In this situation $p \oplus q \in \mathcal{P}(\mathbf{A})$ is well defined.

There are three notions of equivalence for two projections $p, q \in \mathcal{P}(\mathbf{A})$:

homotopic equivalence: $p \sim_h q$ if there exists a continuous path of projections in $\mathbf{A}$ connecting p and q.

unitary equivalence: $p \sim_u q$ if there exists a unitary element $u \in \mathbf{A}$ such that $u^* p u = q$.

Murray-von Neumann equivalence: $p \sim_{\text{M. v.N.}} q$ if there exists a partial isometry $v \in \mathbf{A}$ such that $v^* v = p$ and $vv^* = q$. ♦

One can show that

$$p \sim_h q \Longrightarrow p \sim_u q \Longrightarrow p \sim_{\text{M. v.N.}} q$$

Define $\mathcal{P}_n(\mathbf{A}) \subset M_n(\mathbf{A})$, the set of projections in $M_n(\mathbf{A})$. The natural inclusions $i_n : M_n(\mathbf{A}) \hookrightarrow M_{n+1}(\mathbf{A})$ with $i_n(a) = \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$ permits one to define $M_\infty(\mathbf{A}) = \bigcup_{n \geq 1} M_n(\mathbf{A})$ and $\mathcal{P}_\infty(\mathbf{A}) = \bigcup_{n \geq 1} \mathcal{P}_n(\mathbf{A})$. The three equivalence relations defined above are well defined on $\mathcal{P}_\infty(\mathbf{A})$.

Proposition 1.3.28 (Stabilisation of the equivalence relations)
In $\mathcal{P}_\infty(\mathbf{A})$, the three equivalence relations coincide.

We will denote this relation by $\sim$.

Definition 1.3.29 ($K_0(\mathbf{A})$ for unital C^*-algebra)
Let $\mathsf{V}(\mathbf{A})$ denote the set of equivalence classes in $\mathcal{P}_\infty(\mathbf{A})$ for the relation $\sim$. This is an abelian semigroup for the addition $p \oplus q = \left(\begin{smallmatrix} p & 0 \\ 0 & q \end{smallmatrix}\right) \in \mathcal{P}_\infty(\mathbf{A})$.

The group $K_0(\mathbf{A})$ is the Grothendieck group associated to $(\mathsf{V}(\mathbf{A}), \oplus)$. ♦

Example 1.3.30 ($K_0(M_n(\mathbb{C}))$)
Let us look at the algebra $\mathbf{A} = \mathbb{C}$. In that case, a projection $p \in \mathcal{P}_\infty(\mathbb{C})$ is represented by a projection $p \in M_N(\mathbb{C})$ for a sufficiently large N. Such a projection defines the vector space $\operatorname{Ran} p \subset \mathbb{C}^N$ of dimension $\operatorname{rank} p$. It is easy to see that the equivalence relation $\sim$ detects only this dimension, so that $\mathsf{V}(\mathbb{C}) = \mathbb{N}$ and $K_0(\mathbb{C}) = \mathbb{Z}$.

Let us consider now $\mathbf{A} = M_n(\mathbb{C})$. One has $M_N(M_n(\mathbb{C})) = M_{Nn}(\mathbb{C})$, so that $\mathcal{P}_\infty(M_n(\mathbb{C})) = \mathcal{P}_\infty(\mathbb{C})$, with the same equivalence relation. Then one has $K_0(M_n(\mathbb{C})) = \mathbb{Z}$. This result is an example of Morita invariance of the K-theory. ♦

More generally, by the same argument, one can show:

Proposition 1.3.31 (Morita invariance of the K-theory)

$$K_0(M_n(\mathbf{A})) = K_0(\mathbf{A})$$

Example 1.3.32 ($K_0(C(X))$)
Let X be a compact topological space. Then by Theorem 1.3.26 one has

$$K_0(C(X)) = K^0(X)$$ ♦

Any morphism of C^*-algebras $\phi : \mathbf{A} \to \mathbf{B}$ gives rise to a natural map $\phi : \mathcal{P}_\infty(\mathbf{A}) \to \mathcal{P}_\infty(\mathbf{B})$ compatible with the relation $\sim$ on both sides. This induces a morphism of semigroups $\mathsf{V}(\mathbf{A}) \to \mathsf{V}(\mathbf{B})$ and a morphism of abelian groups $\phi_\sharp : K_0(\mathbf{A}) \to K_0(\mathbf{B})$.

When the algebra $\mathbf{A}$ is not unital, consider its unitarization $\mathbf{A}_+$. Then one has the short exact sequence of C^*-algebras

$$0 \longrightarrow \mathbf{A} \xrightarrow{\ i\ } \mathbf{A}_+ \xrightarrow{\ \pi\ } \mathbb{C} \longrightarrow 0$$

Definition 1.3.33 (K_0(A) for non unital C^*-algebra)
For a non unital algebra $\mathbf{A}$, one defines

$$K_0(\mathbf{A}) = \operatorname{Ker}(\pi_\sharp : K_0(\mathbf{A}_+) \to K_0(\mathbb{C}) = \mathbb{Z})$$ ♦

Remark 1.3.34
Exactly as in Remark 1.3.12, this construction adds a point (the unity) and removes its contribution afterwards. In case $\mathbf{A}$ is unital, one can show that the two definitions coincide. More generally, as abelian groups, one has $K_0(\mathbf{A}_+) = K_0(\mathbf{A}) \oplus \mathbb{Z}$. ♦

Remark 1.3.35 (Interpretation of K_0(A))
An element in $K_0(\mathbf{A})$ is a difference $[p] - [q]$ for some projections $p, q \in \mathcal{P}_n(\mathbf{A}_+)$, for large enough n, such that $[\pi(p)] - [\pi(q)] = 0$. In fact, it is possible to choose p and q such that $p - q \in M_n(\mathbf{A}) \subset M_n(\mathbf{A}_+)$ ($p - q$ is not a projection in this relation!). Adding a common projection, one can always represent an element in $K_0(\mathbf{A})$ as $[p] - [\mathbb{1}_n]$, where $\mathbb{1}_n \in M_n(\mathbf{A}_+)$ is the unit matrix. ♦

Example 1.3.36 ($K_0(\mathcal{B})$)
For any integer n and infinite dimensional separable Hilbert space $\mathcal{H}$, one has $M_n(\mathcal{B}) \simeq \mathcal{B}$ (because $\mathcal{H}^n \simeq \mathcal{H}$ here), so we only need to consider projections in $\mathcal{B}$. Two projections in $\mathcal{B}$ are equivalent precisely when their ranges are isomorphic. So that only the dimension (possibly infinite) is an invariant, and one gets $\mathsf{V}(\mathcal{B}) = \mathbb{N} \cup \{\infty\}$, which produces $K_0(\mathcal{B}) = 0$ (see Example 1.3.3). ♦

Definition 1.3.37 (Higher orders K-groups)
Let $\mathbf{A}$ be a C^*-algebra. The suspension of $\mathbf{A}$ is the C^*-algebra $S\mathbf{A} = C_0(\mathbb{R}, \mathbf{A})$ (see Example 1.2.9).
For any $n \geq 1$, we define $K_n(\mathbf{A}) = K_0(S^n\mathbf{A})$ where $S^n\mathbf{A} = S(S^{n-1}\mathbf{A})$ is the n-th suspension of $\mathbf{A}$. ◆

This definition leads to the following useful result:

Proposition 1.3.38 (Long exact sequences)
For any short exact sequence of C^*-algebras

$$0 \longrightarrow \mathbf{I} \longrightarrow \mathbf{A} \longrightarrow \mathbf{A}/\mathbf{I} \longrightarrow 0$$

there exist boundary maps $\delta : K_n(\mathbf{A}/\mathbf{I}) \to K_{n-1}(\mathbf{I})$ and a long exact sequence

$$\cdots \xrightarrow{\delta} K_n(\mathbf{I}) \longrightarrow K_n(\mathbf{A}) \longrightarrow K_n(\mathbf{A}/\mathbf{I}) \xrightarrow{\delta} K_{n-1}(\mathbf{I}) \longrightarrow \cdots$$
$$\cdots \xrightarrow{\delta} K_0(\mathbf{I}) \longrightarrow K_0(\mathbf{A}) \longrightarrow K_0(\mathbf{A}/\mathbf{I})$$

Remark 1.3.39 (Other definition of $K_1(\mathbf{A})$)
Let us introduce the following groups

- $GL_n(\mathbf{A}_+)$, invertible elements in $M_n(\mathbf{A}_+)$
- $GL_n^+(\mathbf{A}) = \{a \in GL_n(\mathbf{A}_+) \ / \ \pi(a) = \mathbb{1}_n\}$ where $\pi : \mathbf{A}_+ \to \mathbb{C}$ is the projection associated to the unitarization
- $\mathcal{U}_n(\mathbf{A}_+)$, unitaries in $M_n(\mathbf{A}_+)$
- $\mathcal{U}_n^+(\mathbf{A}) = \{u \in \mathcal{U}_n(\mathbf{A}_+) \ / \ \pi(u) = \mathbb{1}_n\}$

These groups define some direct systems for the natural inclusion $g \mapsto \begin{pmatrix} g & 0 \\ 0 & \mathbb{1} \end{pmatrix}$. Denote by $GL_\infty(\mathbf{A}_+)$, $GL_\infty^+(\mathbf{A})$, $\mathcal{U}_\infty(\mathbf{A}_+)$ and $\mathcal{U}_\infty^+(\mathbf{A})$ their respective inductive limits.
One can show that for any C^*-algebra $\mathbf{A}$, one has

$$\begin{aligned} K_1(\mathbf{A}) &= GL_\infty(\mathbf{A}_+)/GL_\infty(\mathbf{A}_+)_0 = GL_\infty^+(\mathbf{A})/GL_\infty^+(\mathbf{A})_0 \\ &= \mathcal{U}_\infty(\mathbf{A}_+)/\mathcal{U}_\infty(\mathbf{A}_+)_0 = \mathcal{U}_\infty^+(\mathbf{A})/\mathcal{U}_\infty^+(\mathbf{A})_0 \end{aligned}$$

where the index 0 means the connected component of the unit element. ◆

Proposition 1.3.40 (Continuity for direct systems)
Let $(\mathbf{A}_i, \alpha_i)$ be a direct system of C^*-algebras. Then for any n one has $K_n(\varinjlim A_i) = \varinjlim K_n(A_i)$.

Example 1.3.41 ($K_0(\mathcal{K})$)
The algebra of compact operators is the direct limit $\mathcal{K} = \varinjlim M_n(\mathbb{C})$. As $K_0(M_n(\mathbb{C})) = \mathbb{Z}$ is a stationary system, one has $K_0(\mathcal{K}) = \mathbb{Z}$. Explicitly, the isomorphism is realized as the trace $[p] \mapsto \mathrm{Tr}(p)$. ◆

More generally we have:

Proposition 1.3.42 (Morita invariance of K-theory)
For any C^*-algebra $\mathbf{A}$, and any n, one has $K_n(\mathbf{A} \widehat{\otimes} \mathcal{K}) = K_n(\mathbf{A})$.

Here is the version of Bott periodicity for K-theory of C^*-algebras:

Theorem 1.3.43 (Bott periodicity)
For any C^*-algebra $\mathbf{A}$, one has

$$K_0(S^2\mathbf{A}) = K_2(\mathbf{A}) \simeq K_0(\mathbf{A})$$

Proposition 1.3.44 (Six term exact sequence)
The Bott periodicity theorem reduces the long exact sequence associated to any short exact sequence of C^*-algebras to a six term exact sequence

$$\begin{array}{ccccc} K_0(\mathbf{I}) & \longrightarrow & K_0(\mathbf{A}) & \longrightarrow & K_0(\mathbf{A}/\mathbf{I}) \\ {\scriptstyle\delta}\uparrow & & & & \downarrow{\scriptstyle\delta} \\ K_1(\mathbf{A}/\mathbf{I}) & \longleftarrow & K_1(\mathbf{A}) & \longleftarrow & K_1(\mathbf{I}) \end{array}$$

Remark 1.3.45 (K-groups via homotopy groups)
We have seen in Remark 1.3.39 that the K_1-group can be defined using the 0-th homotopy group as $K_1(\mathbf{A}) = \pi_0(\mathcal{U}_\infty(\mathbf{A}_+))$. It is possible to show that more generally

$$K_n(\mathbf{A}) = \pi_{n-1}(\mathcal{U}_\infty(\mathbf{A}_+))$$

Bott periodicity is then directly equivalent to

$$\pi_{n+2}(\mathcal{U}_\infty(\mathbf{A}_+)) \simeq \pi_n(\mathcal{U}_\infty(\mathbf{A}_+))$$

Because $\mathcal{U}_n(\mathbf{A}_+)$ and $GL_n(\mathbf{A}_+)$ have the same topology (one is the retraction of the other), these relations make sense with $GL_\infty(\mathbf{A}_+)$. ♦

Example 1.3.46 (K-groups for some C^*-algebras)
Here is a table of some known K-groups for ordinary C^*-algebras.

Algebra	K_0	K_1
$\mathbb{C}$, $M_n(\mathbb{C})$, $\mathcal{K}(\mathcal{H})$ (compacts op.)	$\mathbb{Z}$	0
$\mathcal{B}(\mathcal{H})$ (bounded op.)	0	0
$\mathcal{Q}(\mathcal{H})$ (Calkin's alg.)	0	$\mathbb{Z}$
$\mathcal{T}$ (Tœplitz' alg.)	$\mathbb{Z}$	0
$\mathcal{O}_n$, $n \geq 2$ (Cuntz' alg.)	$\mathbb{Z}_{n-1}$	0
$\mathcal{A}_\theta$, θ irrationnal	$\mathbb{Z}^2 \simeq \theta\mathbb{Z} + \mathbb{Z}$	$\mathbb{Z}^2$
$C^*(\mathbb{F}_n)$ ($\mathbb{F}_n$ free group with n generators)	$\mathbb{Z}$	$\mathbb{Z}^n$
$M_n(\mathbf{A})$, $\mathbf{A} \widehat{\otimes} \mathcal{K}$ (stabilisation)	$K_0(\mathbf{A})$	$K_1(\mathbf{A})$
$\mathbf{A}_+$ (unitarization)	$K_0(\mathbf{A}) \oplus \mathbb{Z}$	$K_1(\mathbf{A})$
$S\mathbf{A} = C_0(]0,1[, \mathbf{A})$ (suspension)	$K_1(\mathbf{A})$	$K_0(\mathbf{A})$
$C\mathbf{A} = C_0(]0,1], \mathbf{A})$ (cone)	0	0

Remark 1.3.47 (K-theory computed on dense subalgebras)
For a lot of examples, one can compute the K-groups of a C^*-algebra $\mathbf{A}$ using a dense subalgebra $\mathbf{B}$. For instance, for any compact finite dimensional manifold M, the K-theory of $C(M)$ (continuous functions) is the same as the K-theory of the Fréchet algebra $C^\infty(M)$. The same situation occurs for the irrational rotation algebra: $K_n(\mathcal{A}_\theta) = K_n(\mathcal{A}_\theta^\infty)$.

In the geometric situation, it is possible to understand this result. Smooth structures are sufficiently dense in continuous structures: any continuous vector bundle can be deformed into a smooth one…

Here is a description of some more general situations. Let $\mathbf{A}$ be C^*-algebra (or a Banach algebra) and $\mathbf{A}^\infty \subset \mathbf{A}$ a dense subalgebra (but not necessarily a C^*-subalgebra). The exponent ∞ does not mean that we consider "differentiable" functions, even if in practice this can happen: think about $\mathcal{A}_\theta^\infty \subset \mathcal{A}_\theta$ as a typical example. Let $\mathbf{A}_+$ and $\mathbf{A}_+^\infty$ their unitarizations. Suppose that $\mathbf{A}_+^\infty$ is stable under holomorphic functional calculus, which means that for any $a \in \mathbf{A}_+^\infty$ and any holomorphic function f in a neighborhood of the spectrum of a, $f(a) \in \mathbf{A}_+^\infty$.

Using the topologies induced on $\mathbf{A}_+^\infty$ and $GL_n(\mathbf{A}_+^\infty)$ by the topologies on $\mathbf{A}_+$ and $GL_n(\mathbf{A}_+)$, it is possible to define K-groups by using the relations in Remark 1.3.45. Then one has the density theorem: the inclusion $i : \mathbf{A}^\infty \to \mathbf{A}$ induces isomorphisms

$$i_\sharp : K_n(\mathbf{A}^\infty) \xrightarrow{\simeq} K_n(\mathbf{A})$$

for any $n \geq 0$. ♦

Remark 1.3.48 (*K*-homology)
As for many other ordinary homologies, there exists a dual version of the K-theory of C^*-algebras, named K-homology, which we outline here.

A Fredholm module over the C^*-algebra $\mathbf{A}$ is a triplet $(\mathcal{H}, \rho, F)$ where $\mathcal{H}$ is a Hilbert space, ρ is an involutive representation of $\mathbf{A}$ in $\mathcal{B}(\mathcal{H})$, and F is an operator on $\mathcal{H}$ such that for any $a \in \mathbf{A}$, $(F^2 - 1)\rho(a)$, $(F - F^*)\rho(a)$ and $[F, \rho(a)]$ are in $\mathcal{K}$. Such a Fredholm module is called odd.

A $\mathbb{Z}_2$-graded Fredholm module is a Fredholm module $(\mathcal{H}, \rho, F)$ such that $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ and $\rho(a)$ is of even parity in this decomposition, and F is of odd parity: $\rho(a) = \begin{pmatrix} \rho^+(a) & 0 \\ 0 & \rho^-(a) \end{pmatrix}$ and $F = \begin{pmatrix} 0 & U^+ \\ U^- & 0 \end{pmatrix}$. With these notations, $U^\pm : \mathcal{H}^\mp \to \mathcal{H}^\pm$ are essentially adjoint (adjoint modulo compact operators). Such a Fredholm module is called even.

In the even case, one has a natural grading map $\gamma : \mathcal{H} \to \mathcal{H}$ defined by $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on the decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$. It satisfies $\gamma = \gamma^*$, $\gamma^2 = 1$, $\gamma\rho(a) = \rho(a)\gamma$ and $\gamma F = -F\gamma$.

Two Fredholm modules $(\mathcal{H}, \rho, F)$ and $(\mathcal{H}', \rho', F')$ are unitary equivalent if there exists a unitary map $U : \mathcal{H}' \to \mathcal{H}$ such that $\rho' = U^*\rho U$ and $F' = U^* F U$. This defines an equivalence relation $\sim_U$ of Fredholm modules.

A homotopy of Fredholm modules is a familly $t \mapsto (\mathcal{H}, \rho, F_t)$ with $[0,1] \ni t \mapsto F_t$ continuous for the operator norm in $\mathcal{H}$. Two Fredholm modules are homotopic equivalent if they are connected by a homotopy of Fredholm modules. This defines an equivalence relation $\sim_h$.

The direct sum of two Fredholm modules $(\mathcal{H}, \rho, F)$ and $(\mathcal{H}', \rho', F')$ is defined by

$$(\mathcal{H}, \rho, F) \oplus (\mathcal{H}', \rho', F') = \left(\mathcal{H} \oplus \mathcal{H}', \begin{pmatrix} \rho & 0 \\ 0 & \rho' \end{pmatrix}, \begin{pmatrix} F & 0 \\ 0 & F' \end{pmatrix}\right)$$

The K-homology group $K^0(\mathbf{A})$ of $\mathbf{A}$ is the Grothendieck group of the abelian semigroup of equivalence classes of even Fredholm modules for $\sim_U$ and $\sim_h$. The unit for the addition is the class of the Fredholm module $(0,0,0)$, the inverse of the class of $(\mathcal{H}, \rho, F)$ is the class of $(\mathcal{H}, \rho, -F)$.

The K-homology group $K^1(\mathbf{A})$ is defined in the same manner using odd Fredholm modules.

A degenerated Fredholm module is a Fredholm module for which $(F - F^*)\rho(a) = 0$, $(F^2 - 1)\rho(a) = 0$ and $[F, \rho(a)] = 0$ for any $a \in \mathbf{A}$. The equivalence class of such a Fredholm module is zero.

In each equivalence class, there is a representative for which $F^* = F$ (self-adjoint Fredholm module) and $F^2 = 1$ (involutive Fredholm module).

For $\mathbf{A} = \mathbb{C}$, the representation ρ defines a projection $p = \rho(1)$ on $\mathcal{H}$, and one can show that modulo compact operators, one has $(\mathcal{H}, \rho, F) = (p\mathcal{H}, \rho, pFp) \oplus ((1-p)\mathcal{H}, \rho, (1-p)F(1-p))$. The representation for the second Fredholm module is zero, so that its class is zero. pFp is an ordinary Fredholm operator on $\mathcal{H}$, and its index induces an isomorphism $\mathrm{Ind} : K^0(\mathbb{C}) \xrightarrow{\simeq} \mathbb{Z}$.

For any C^*-algebra $\mathbf{A}$, let p be a projection in $\mathcal{P}_n(\mathbf{A})$, and $(\mathcal{H}, \rho, F)$ a Fredholm module. Then, in the Hilbert space $\rho(p)(\mathcal{H} \otimes \mathbb{C}^n)$, the operator $\rho(p)(F \otimes \mathbb{1}_n)\rho(p)$ is a Fredholm operator, and its index defines a pairing between $K_0(\mathbf{A})$ and $K^0(\mathbf{A})$: $\langle [p], [(\mathcal{H}, \rho, F)] \rangle = \mathrm{Ind}\rho(p)(F \otimes \mathbb{1}_n)\rho(p) \in \mathbb{Z}$.

For more developments in K-homology, see [Blackadar, 1998] and [Higson and Roe, 2004]. ♦

1.3.3 Algebraic K-theory

Until now, K-theory has been defined using topological structures, either at the level of a space or at the level of an algebra (remember that they are the manifestations of the same topological structure in the commutative case).

Nevertheless the group $K_0(\mathbf{A})$ can be defined in the pure algebraic context. Indeed, to any ring $\mathbf{A}$, which we take unital from now on, one can associate its category of finite projective modules.

Definition 1.3.49 ($K_0^{\mathrm{alg}}(\mathbf{A})$ for unital ring A)
The group $K_0^{\mathrm{alg}}(\mathbf{A})$ is the Grothendieck group associated to the semigroup of isomorphic classes of finite projective modules on $\mathbf{A}$, on which the additive law is induced by the direct sum of modules. ♦

As in the topological case, every finite projective $\mathbf{A}$-module $\boldsymbol{M}$ is characterized by a (non unique) projector $p \in M_m(\mathbf{A})$. Be aware of the different terminologies that are used. "Projection" is reserved to the C^*-algebra context, because in that case p satisfies $p^2 = p$ and $p^* = p$. "Projector" is more general, in that case p satisfies only $p^2 = p$.

The equivalence relation we use on these projectors is the following:

Definition 1.3.50 (Equivalence relation on projectors)
Two projectors $p \in M_m(\mathbf{A})$ and $q \in M_n(\mathbf{A})$ are equivalent if there exist an integer $r \geq m, n$ and an invertible $u \in GL_r(\mathbf{A})$ such that $p, q \in M_r(\mathbf{A})$ are conjugated by u: $p = u^{-1}qu$. We donote by $\sim$ this equivalence relation. ♦

If $p \sim q$, then they define isomorphic finite projective modules.

In case $\mathbf{A}$ is a C^*-algebra, one can show that in the equivalence class of any projector p one can find a projection. This means that the two semigroups which define the K-theories are the same :

$$K_0^{\mathrm{alg}}(\mathbf{A}) = K_0(\mathbf{A}) \text{ (as a } C^*\text{-algebra)}$$

For higher order groups, the situation is no more equivalent. The Definition 1.3.37 (or their equivalent ones given in Remark 1.3.45) uses extensively the topological structure of the algebra, either to define continuous functions $\mathbb{R} \to \mathbf{A}$ or to compute the homotopy groups of the spaces $\mathcal{U}_\infty(\mathbf{A}_+)$ in Remark 1.3.45.

Nevertheless, one can define $K_1^{\mathrm{alg}}(\mathbf{A})$ as follows:

Definition 1.3.51 ($K_1^{\mathrm{alg}}(\mathbf{A})$ for unital ring A)
One defines

$$K_1^{\mathrm{alg}}(\mathbf{A}) = GL_\infty(\mathbf{A}) / [GL_\infty(\mathbf{A}), GL_\infty(\mathbf{A})] = GL_\infty(\mathbf{A})_{\mathrm{ab}}$$ ♦

Let $\mathbf{A}$ be a C^*-algebra. A well known fact about invertibles is that if $u, v \in GL_\infty(\mathbf{A})$ then uv and vu are homotopic. So that there is a natural morphism of groups

$$K_1^{\mathrm{alg}}(\mathbf{A}) \to K_1(\mathbf{A}) \text{ (as a } C^*\text{-algebra)}$$

which factors out by the homotopic relation.

For every $n \geq 2$, there is a definition which surprisingly uses some topological objects: $K_n^{\mathrm{alg}}(\mathbf{A})$ is the π_n group of a topological space associated to the classifying space $BGL_\infty(\mathbf{A})$ where $GL_\infty(\mathbf{A})$ is considered as a discrete group.

In algebraic K-theory, there is no Bott periodicity, but there are some other beautiful and powerful results which are beyond the scope of this introduction: in the following, we will only make use of $K_0^{\mathrm{alg}}(\mathbf{A})$ and $K_1^{\mathrm{alg}}(\mathbf{A})$. For a review, see [Karoubi, 2003] or [Rosenberg, 1994].

1.4 Cyclic homology for (differential) geometers

Now we have in hand some tools to characterize noncommutative topological spaces. But topology is not everything in life. Differential geometry has to be considered also! In this section, we will explore other concepts that look very much like differential forms.

1.4.1 Differential calculi

Differential forms on a differentiable manifold define a differential graded commutative algebra. This concept can be generalised:

Definition 1.4.1 (Differential calculus on an algebra)
Let $\mathbf{A}$ be an associative algebra. A differential calculus on $\mathbf{A}$ is a graded differential algebra $(\Omega^\bullet, d)$ such that $\Omega^0 = \mathbf{A}$. ♦

Remember the definition of a graded differential algebra: $\Omega^\bullet$ is a graded algebra on which the differential satisfies $d(\omega\eta) = (d\omega)\eta + (-1)^{|\omega|}\omega(d\eta)$ for any $\omega, \eta \in \Omega^\bullet$ and where $|\omega|$ is the degree of ω.

In this definition, one does not suppose this graded algebra to be a graded commutative algebra.

Example 1.4.2 (de Rham differential calculus)
Let M be a finite dimensional differential manifold. The graded differential algebra $(\Omega^\bullet(M), d)$ of differential forms is a differential calculus on $C^\infty(M)$. ♦

There are many possibilities to define a differential calculus on an algebra. One can summarize the questing after noncommutative differential geometry to the search of some reasonable definition of such a differential calculus on any algebra. Many propositions have been made, depending on the context: associative algebra without any additional structure, topological or involutive algebras, quantum groups...

Some examples will be given after we introduce three main examples which are the universal differential calculi.

Example 1.4.3 (Universal differential calculus for associative algebra)
This differential calculus is defined to be the free graded differential algebra generated by $\mathbf{A}$ as elements in degree 0. It is denoted by $(\Omega^\bullet(\mathbf{A}), d)$.

Because it is freely generated, it has the following universal property: for any differential calculus $(\Omega^\bullet, d)$ on $\mathbf{A}$, there exists a unique morphism of differential calculi $\phi : \Omega^\bullet(\mathbf{A}) \to \Omega^\bullet$ (of degree 0) such that $\phi(a) = a$ for any $a \in \mathbf{A} = \Omega^0(\mathbf{A}) = \Omega^0$.

This implies that if $(\Omega^\bullet, d)$ is generated (possibly with relations) by $\mathbf{A} = \Omega^0$ then it is a quotient of $(\Omega^\bullet(\mathbf{A}), d)$ by a differential two-side ideal.

Concretely, any element in $\Omega^n(\mathbf{A})$ is a sum of terms either of the form $adb_1 \dots db_n$ or of the form $db_1 \dots db_n$. This property gives us an identification of left $\mathbf{A}$-modules

$$\Omega^n(\mathbf{A}) = \mathbf{A}_+ \otimes \mathbf{A}^{\otimes n}$$

by the morphism $adb_1 \dots db_n \mapsto (0 + a) \otimes b_1 \otimes \cdots \otimes b_n$ and $db_1 \dots db_n \mapsto (1 + 0) \otimes b_1 \otimes \cdots \otimes b_n$, where $(0 + a)$ and $(1 + 0)$ are elements in $\mathbf{A}_+ = \mathbb{C} \oplus \mathbf{A}$. Be aware of the fact that this identification is not an identification of graded differential algebras, neither of bimodules. ♦

In the differential calculus $(\Omega^\bullet(\mathbf{A}), d)$, if $\mathbf{A}$ is unital, $d\mathbb{1}$ is not zero, because it is identified with $1 \otimes \mathbb{1} \in \mathbf{A}_+ \otimes \mathbf{A}$. It is the aim of the following example to show that in the more restrictive situation where $\mathbf{A}$ is unital, one can promote the unit of $\mathbf{A}$ to a unit of the differential calculus.

Example 1.4.4 (Universal differential calculus for associative unital algebra)
Let $\mathbf{A}$ be an associative unital algebra. The differential calculus $(\Omega^\bullet_U(\mathbf{A}), d_U)$ is defined to be the free unital graded differential algebra generated by $\mathbf{A}$ in degree 0. The indice U stands for unital.

Because this algebra is required to have an unit, this unit is necessarily the unit in $\mathbf{A} = \Omega^0_U(\mathbf{A})$. Then the derivative law for d_U gives $d_U\mathbb{1} = 0$. This differential calculus admits a universal property as the previous one does: for any unital differential calculus $(\Omega^\bullet, d)$ on $\mathbf{A}$, there exists a unique morphism of unital differential calculi $\phi : \Omega^\bullet_U(\mathbf{A}) \to \Omega^\bullet$ (of degree 0) such that $\phi(a) = a$ for any $a \in \mathbf{A} = \Omega^0_U(\mathbf{A}) = \Omega^0$.

Because $(\Omega^\bullet_U(\mathbf{A}), d_U)$ is a differential calculus generated by $\mathbf{A}$, it is a quotient of $(\Omega^\bullet(\mathbf{A}), d)$. This quotient reveals itself in the concrete identification of $\Omega^\bullet(\mathbf{A})$: any element in $\Omega^n_U(\mathbf{A})$ is a sum of terms of the form $ad_Ub_1 \cdots d_Ub_n$. Here a can be $\mathbb{1}$, and in this case $\mathbb{1}d_Ub_1 \cdots d_Ub_n = d_Ub_1 \cdots d_Ub_n$. If one of the b_k is proportional to $\mathbb{1}$, one has $ad_Ub_1 \cdots d_Ub_n = 0$. This leads to the identification of left modules

$$\Omega^n_U(\mathbf{A}) = \mathbf{A} \otimes \overline{\mathbf{A}}^{\otimes n}$$

by the map $ad_Ub_1 \dots d_Ub_n \mapsto a \otimes \overline{b_1} \otimes \cdots \otimes \overline{b_n}$, where $\overline{b}$ is the projection of $b \in \mathbf{A}$ onto the vector space $\overline{\mathbf{A}} = \mathbf{A}/\mathbb{C}\mathbb{1}$. ♦

In these two examples, even if the algebra $\mathbf{A}$ is commutative, the graded algebras are not graded commutative. For commutative algebras, it is possible to construct a differential calculus with a graded commutative algebra.

Example 1.4.5 (Kähler differential calculus for commutative unital algebra)
Let $\mathbf{A}$ be an associative commutative unital algebra over a field $\mathbb{K}$. The Kähler differential calculus $(\Omega^\bullet_{\mathbf{A}|\mathbb{K}}, d_K)$ is defined to be the free unital graded commutative differential algebra generated by $\mathbf{A}$ in degree 0.

One can show that the algebra $\Omega^\bullet_{\mathbf{A}|\mathbb{K}}$ is an exterior algebra over $\mathbf{A}$: $\Omega^\bullet_{\mathbf{A}|\mathbb{K}} = \bigwedge^\bullet_{\mathbf{A}} \Omega^1_{\mathbf{A}|\mathbb{K}}$. Moreover, let $\mathbf{I} \subset \mathbf{A} \otimes \mathbf{A}$ be the kernel of the product map $\mu : \mathbf{A} \otimes \mathbf{A} \to \mathbf{A}$. Consider $\mathbf{A} \otimes \mathbf{A}$ as an algebra (commutative) and introduce $\mathbf{I}^2$, generated by the products of elements in $\mathbf{I}$. Then one has the explicit construction $\Omega^1_{\mathbf{A}|\mathbb{K}} = \mathbf{I}/\mathbf{I}^2$.

We denote by $\pi_\bullet : \Omega^\bullet(\mathbf{A}) \to \Omega^\bullet_{\mathbf{A}|\mathbb{C}}$ the universal projection, given explicitly by

$$\pi_n(a_0 da_0 \cdots da_n) = a_0 d_K a_1 \wedge \cdots \wedge d_K a_n \qquad \pi_n(da_0 \cdots da_n) = d_K a_1 \wedge \cdots \wedge d_K a_n$$

The cohomology of this differential algebra is denoted by $H^\bullet_{\mathrm{dR}}(\mathbf{A})$, and is called the de Rham cohomologie of the commutative unital algebra $\mathbf{A}$. This terminology comes from the fact that this differential calculus looks very much like the de Rham differential calculus (see Example 1.4.6). ♦

Example 1.4.6 (Polynomial algebra)
Let V be a finite dimensional vector space over $\mathbb{C}$. Consider the commutative algebra $\mathsf{S}V$ of polynomials on V. Then one has the identification $\mathsf{S}V \otimes \bigwedge^n V = \Omega^n_{\mathsf{S}V|\mathbb{C}}$ by the map $a \otimes v \mapsto a d_K v$ in degree 1. This differential calculus is the "restriction" of the de Rham differential calculus of C^∞ functions to the subalgebra of polynomial functions. ♦

Example 1.4.7 (Spectral triplet)
Let $\mathbf{A}$ be an involutive unital associative algebra. A spectral triplet on $\mathbf{A}$ is a triplet $(\mathbf{A}, \mathcal{H}, D)$ where $\mathcal{H}$ is a Hilbert space on which an involutive representation ρ of $\mathbf{A}$ is given, and D is a self-adjoint operator on $\mathcal{H}$ (not necessarily bounded), whose resolvant is compact, and such that $[D, \rho(a)]$ is bounded for any $a \in \mathbf{A}$. The operator D is called a Dirac operator.

The map $\pi : \Omega^\bullet_U(\mathbf{A}) \to \mathcal{B}(\mathcal{H})$ defined by $\pi(a_0 d_U a_1 \cdots d_U a_n) = a_0 [D, a_1] \cdots [D, a_n]$ is an involutive representation of $\Omega^\bullet_U(\mathbf{A})$ on $\mathcal{H}$.

Define $J_0 = \bigoplus_{n \geq 0} (\operatorname{Ker} \pi \cap \Omega^n_U(\mathbf{A}))$. One can show that $J = J_0 + d_U J_0$ is a differential two-sided ideal in $\Omega^\bullet_U(\mathbf{A})$. The differential calculus defined by the spectral triplet $(\mathbf{A}, \mathcal{H}, D)$ is the graded differential algebra $\Omega^\bullet_D(\mathbf{A}) = \Omega^\bullet_U(\mathbf{A})/J$.

This construction is inspired by the definition of Fredholm modules, which are the building blocks of K-homology (see Remark 1.3.48).

See [Connes, 1994] and [Gracia-Bondía et al., 2001] for more details and examples. ♦

Example 1.4.8 (Derivations based differential calculus for associative algebra)
Let $\mathbf{A}$ be an associative algebra. The space of derivations on $\mathbf{A}$,

$$\operatorname{Der}(\mathbf{A}) = \{X : \mathbf{A} \to \mathbf{A} \ / \ X \text{ linear map and } X(ab) = (Xa)b + a(Xb)\}$$

is a Lie algebra and a module over the center $\mathcal{Z}(\mathbf{A})$ of $\mathbf{A}$.

Let $\underline{\Omega}^n_{\mathrm{Der}}(\mathbf{A})$ be the set of $\mathcal{Z}(\mathbf{A})$-multilinear antisymmetric maps $\operatorname{Der}(\mathbf{A})^n \to \mathbf{A}$. Define on $\underline{\Omega}^\bullet_{\mathrm{Der}}(\mathbf{A}) = \bigoplus_{n \geq 0} \underline{\Omega}^n_{\mathrm{Der}}(\mathbf{A})$ the product

$$(\omega\eta)(X_1, \dots, X_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in \mathfrak{S}_{p+q}} (-1)^{\operatorname{sign}(\sigma)} \omega(X_{\sigma(1)}, \dots, X_{\sigma(p)}) \eta(X_{\sigma(p+1)}, \dots, X_{\sigma(p+q)})$$

for any $X_i \in \operatorname{Der}(\mathbf{A})$, any $\omega \in \underline{\Omega}^p_{\mathrm{Der}}(\mathbf{A})$ and any $\eta \in \underline{\Omega}^q_{\mathrm{Der}}(\mathbf{A})$. Introduce on this graded algebra the differential $\hat{d} : \underline{\Omega}^n_{\mathrm{Der}}(\mathbf{A}) \to \underline{\Omega}^{n+1}_{\mathrm{Der}}(\mathbf{A})$:

$$\hat{d}\omega(X_1, \dots, X_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} X_i \omega(X_1, \dots \overset{i}{\vee} \dots, X_{n+1}) + \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} \omega([X_i, X_j], \dots \overset{i}{\vee} \dots \overset{j}{\vee} \dots, X_{n+1})$$

Then $(\underline{\Omega}^\bullet_{\mathrm{Der}}(\mathbf{A}), \hat{d})$ is a differential calculus on $\mathbf{A}$.

This differential calculus is not *a priori* generated by $\mathbf{A}$ in degree 0. The differential calculus generated by $\mathbf{A}$ in $(\underline{\Omega}^\bullet_{\mathrm{Der}}(\mathbf{A}), \hat{d})$ is denoted by $(\Omega^\bullet_{\mathrm{Der}}(\mathbf{A}), \hat{d})$.

For $\mathbf{A} = C^\infty(M)$, the Lie algebra $\mathrm{Der}(\mathbf{A})$ is the Lie algebra of vector fields on the manifold M, and this differential calculus (the two coincide here) is the de Rham differential calculus.

For $\mathbf{A} = M_n(\mathbb{C})$, the Lie algebra $\mathrm{Der}(\mathbf{A})$ identifies with $\mathfrak{sl}_n(\mathbb{C})$, and the differential calculus identifies with the Lie complex $M_n(\mathbb{C}) \otimes \bigwedge^\bullet \mathfrak{sl}_n(\mathbb{C})^*$ for the adjoint representation of $\mathfrak{sl}_n(\mathbb{C})$ on $M_n(\mathbb{C})$.

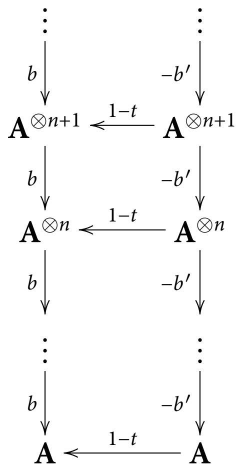
See [Dubois-Violette, 1988], [Dubois-Violette et al., 1990b], [Dubois-Violette et al., 1990a], [Dubois-Violette and Masson, 1998], [Masson, 1999], [Masson and Sérié, 2005] for more details, examples and applications. ♦

1.4.2 Hochschild homology

The Hochschild homology will not be presented here in its full generality. We refer to [Pierce, 1982], [Loday, 1998] and [Gerstenhaber and Schack, 1988] (for instance) to get further developments. What will be presented here is the relation between Hochschild homology with values in the algebra itself and the differential calculi introduced above. These constructions are necessary to introduce and understand cyclic homology.

Let $\mathbf{A}$ be an associative algebra, not necessarily unital. As usual we denote by $\mathbf{A}_+ = \mathbb{C} \oplus \mathbf{A}$ its unitarization.

The Hochschild homology we are interested in is defined using the following bicomplex, denoted by $CC^{(2)}_{\bullet,\bullet}(\mathbf{A})$, with only two non zero columns:



$t : \mathbf{A}^{\otimes n} \to \mathbf{A}^{\otimes n}$ is the cyclic operator:

$$t(a_1 \otimes \cdots \otimes a_n) = (-1)^{n+1} a_n \otimes a_1 \otimes \cdots \otimes a_{n-1}$$

$b : \mathbf{A}^{\otimes n+1} \to \mathbf{A}^{\otimes n}$ is the Hochschild boundary for the Hochschild complex with values in $\mathbf{A}$:

$$b(a_0 \otimes \cdots \otimes a_n) = \sum_{p=0}^{n-1} (-1)^p a_0 \otimes \cdots \otimes a_p a_{p+1} \otimes \cdots \otimes a_n + (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}$$

$b' : \mathbf{A}^{\otimes n+1} \to \mathbf{A}^{\otimes n}$ is the first part of the Hochschild boundary b:

$$b'(a_0 \otimes \cdots \otimes a_n) = \sum_{p=0}^{n-1} (-1)^p a_0 \otimes \cdots \otimes a_p a_{p+1} \otimes \cdots \otimes a_n$$

One can show the relations

$$b^2 = b'^2 = 0 \qquad\qquad b(1-t) = (1-t)b'$$

The total complex of this bicomplex is given in degree n by $CC^{(2)}_n(\mathbf{A}) = \mathbf{A}^{\otimes n+1} \oplus \mathbf{A}^{\otimes n}$, with the total differential

$$b_H = \begin{pmatrix} b & 1-t \\ 0 & -b' \end{pmatrix}$$

in matrix form.

Now, notice that $CC^{(2)}_n(\mathbf{A}) = \mathbf{A}^{\otimes n+1} \oplus \mathbf{A}^{\otimes n} = \mathbf{A}_+ \otimes \mathbf{A}^{\otimes n} = \Omega^n(\mathbf{A})$ in degree $n \geq 1$ and $CC^{(2)}_0(\mathbf{A}) = \mathbf{A}$. In this identification, the differential b_H takes the very simple expression

$$b_H(\omega da) = (-1)^n [\omega, a]$$

Definition 1.4.9 (Hochschild homology with values in the algebra)
Let $\mathbf{A}$ be an associative algebra. The Hochschild homology $HH_\bullet(\mathbf{A})$ is the homology of the total complex of the bicomplex $CC^{(2)}_{\bullet,\bullet}(\mathbf{A})$ defined above, *i.e.* the homology of the complex $(\Omega^\bullet(\mathbf{A}), b_H)$. ♦

This second complex takes the form

$$\Omega^0(\mathbf{A}) \xleftarrow{b_H} \cdots \xleftarrow{b_H} \Omega^n(\mathbf{A}) \xleftarrow{b_H} \Omega^{n+1}(\mathbf{A}) \xleftarrow{b_H} \cdots$$

Notice that b_H is of degree -1, but the differential, which has not appeared in this construction, is of degree 1.

Remark 1.4.10 (The unital case)
When the algebra is unital, the second column (the one with b') is exact: it admits the homotopy

$$s(a_1 \otimes \cdots \otimes a_n) = \mathbb{1} \otimes a_1 \otimes \cdots \otimes a_n \tag{1.4.4}$$

Using standard spectral sequence arguments on this bicomplex, the homology of the total complex is then the homology of the first column. In this case, one recovers the definition of the Hochschild complex which is usually given in textbooks:

$$\mathbf{A} \xleftarrow{b} \cdots \xleftarrow{b} \mathbf{A}^{\otimes n} \xleftarrow{b} \mathbf{A}^{\otimes n+1} \xleftarrow{b} \cdots \tag{1.4.5}$$

One can even go a step further. It is possible to consider a quotient of this complex, called the normalized complex, and to show, by standard arguments coming from the theory of simplicial modules, that its homology is the same as the homology of the previous complex.

This normalized complex is defined by removing any contributions coming from elements proportional to the unit in the last n factors in $\mathbf{A}^{\otimes n+1}$. The first factor is not affected because it is in fact the $\mathbf{A}$-bimodule in which the Hochschild homology takes its values. The normalized complex is then defined on the spaces $\mathbf{A} \otimes \overline{\mathbf{A}}^{\otimes n}$ (where as before $\overline{\mathbf{A}} = \mathbf{A}/\mathbb{C}\mathbb{1}$) on which it is easy to check that the differential b is well-defined. But now, one has the identification $\Omega^n_U(\mathbf{A}) = \mathbf{A} \otimes \overline{\mathbf{A}}^{\otimes n}$, so that in the unital case, the Hochschild homology can be computed from the complex

$$\Omega^0_U(\mathbf{A}) \xleftarrow{b} \cdots \xleftarrow{b} \Omega^n_U(\mathbf{A}) \xleftarrow{b} \Omega^{n+1}_U(\mathbf{A}) \xleftarrow{b} \cdots \tag{1.4.6}$$

♦

Definition 1.4.11 (The trace map)
The trace map $\mathrm{Tr} : CC^{(2)}_n(M_n(\mathbf{A})) \to CC^{(2)}_n(\mathbf{A})$ is the morphism of complexes defined by

$$\mathrm{Tr}(\alpha_0 \otimes \cdots \otimes \alpha_n) = \sum_{(i_0,\dots,i_n)} a_{0,i_0 i_1} \otimes \cdots \otimes a_{n,i_n i_0}$$

where $\alpha_r = (a_{r,ij})_{i,j} \in M_n(\mathbf{A})$. ♦

Proposition 1.4.12 (Morita invariance of Hochschild homology)
For any unital algebra $\mathbf{A}$ *and any integer* n*, the trace map induces an isomorphism*

$$HH_\bullet(M_n(\mathbf{A})) \simeq HH_\bullet(\mathbf{A})$$

In fact, Morita invariance of the Hochschild homology of unital algebras is stronger than the one presented here. It is invariant for the Morita equivalence which we now define:

Definition 1.4.13 (Morita equivalence of algebras)
One says that two algebras $\mathbf{A}$ and $\mathbf{B}$ are Morita equivalent if there exist an $\mathbf{A}$-$\mathbf{B}$-module $\boldsymbol{M}$ and a $\mathbf{B}$-$\mathbf{A}$-module $\boldsymbol{N}$ such that $\mathbf{A} \simeq \boldsymbol{M} \otimes_{\mathbf{B}} \boldsymbol{N}$ and $\mathbf{B} \simeq \boldsymbol{N} \otimes_{\mathbf{A}} \boldsymbol{M}$ as bimodules over $\mathbf{A}$ and $\mathbf{B}$ respectively. ♦

For instance, $\mathbf{A}$ is Morita equivalent to $\mathbf{B} = M_n(\mathbf{A})$ using $\boldsymbol{M} = \mathbf{A}^n$ written as a row and $\boldsymbol{N} = \mathbf{A}^n$ written as a column.

Morita invariance of the Hochschild homology can be extended to the class of H-unital algebras, which contains the unital algebras.

Definition 1.4.14 (H-unital algebras)
A H-unital algebra is an algebra $\mathbf{A}$ for which the complex $(\mathbf{A}^{\otimes\bullet}, b')$ has trivial homology. ♦

Example 1.4.15 (The algebra $\mathbb{C}$)
In the case $\mathbf{A} = \mathbb{C}$, one has

$$HH_0(\mathbb{C}) = \mathbb{C} \qquad\qquad HH_n(\mathbb{C}) = 0 \text{ for } n \geq 1$$ ♦

Example 1.4.16 (Tensor algebra)
Let V be a finite dimensional vector space and $\mathbf{A} = \mathcal{T}V$ the tensor algebra over V. Denote by t the cyclic permutation acting on $\mathbf{A}$ in each degree (the t defining the bicomplex). Then

$$\begin{aligned}
&HH_0(\mathbf{A}) = \oplus_{m\geq 0}\left(V^{\otimes m}/\operatorname{Ran}(1-t)\right) && \text{co-invariants under the action of } t\\
&HH_1(\mathbf{A}) = \oplus_{m\geq 1}\left(V^{\otimes m}\right)^t && \text{invariants under the action of } t\\
&HH_n(\mathbf{A}) = 0 && \text{for } n \geq 2
\end{aligned}$$ ♦

Example 1.4.17 (Relation with Lie algebra homology)
Any associative algebra $\mathbf{A}$ gives rise to a Lie algebra $\mathbf{A}_{\text{Lie}}$ where the vector space is $\mathbf{A}$ and the Lie bracket is the commutator: $[a,b] = ab - ba$. In the following, $\mathbf{A}$ is supposed to be unital.

The permutation group $\mathfrak{S}_n$ acts on $\mathbf{A}^{\otimes n}$ by $\sigma(a_1 \otimes \cdots \otimes a_n) = a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(n)}$. Let us define $\varepsilon_n = \sum_{\sigma\in\mathfrak{S}_n}(-1)^{\operatorname{sign}(\sigma)}\sigma : \mathbf{A}^{\otimes n} \to \mathbf{A}^{\otimes n}$ the total antisymmetrisation. It induces a natural morphism

$$\begin{aligned}
\varepsilon_n : \textstyle\bigwedge^n \mathbf{A} &\to \mathbf{A}^{\otimes n}\\
a_1 \wedge \cdots \wedge a_n &\mapsto \varepsilon_n(a_1 \otimes \cdots \otimes a_n)
\end{aligned}$$

which can be shown to commute with the boundary ∂ of the Lie algebra complex $\bigwedge^\bullet \mathbf{A}_{\text{Lie}}$ and the boundary b of the Hochschild complex, so that the morphism of differential complexes $\varepsilon_\bullet$: $(\bigwedge^\bullet \mathbf{A}_{\text{Lie}}, \partial) \to (\mathbf{A}^{\otimes\bullet}, b)$ induces a morphism in homologies

$$\varepsilon_\sharp : H_\bullet(\textstyle\bigwedge^\bullet \mathbf{A}_{\text{Lie}}, \partial) \to HH_\bullet(\mathbf{A})$$

If one consider the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$, one can show that $HH_\bullet(\mathcal{U}(\mathfrak{g})) \simeq H_\bullet(\mathfrak{g}; \mathcal{U}(\mathfrak{g}))$ where on the right it is the ordinary Lie algebra homology defined with the complex $(\bigwedge^\bullet \mathfrak{g}, \partial)$. ♦

Example 1.4.18 (The commutative case)
Let us suppose that $\mathbf{A}$ is a commutative unital algebra.

Consider the constructions of Example 1.4.17. Here the Lie structure on $\mathbf{A}_{\mathrm{Lie}}$ is trivial, so that $\partial = 0$, and the morphism $\varepsilon_\sharp$ is in fact a morphism $\varepsilon_\sharp : \bigwedge^\bullet \mathbf{A} \to HH_\bullet(\mathbf{A})$.

One can show that there is a natural map

$$\begin{aligned} \textstyle\bigwedge^n \mathbf{A} &\to \Omega^n_{\mathbf{A}|\mathbb{C}} \\ a_1 \wedge \cdots \wedge a_n &\mapsto da_1 \wedge \cdots \wedge da_n \end{aligned}$$

through which $\varepsilon_\sharp$ factors. One then get a natural map (also denoted by $\varepsilon_\sharp$):

$$\begin{aligned} \varepsilon_\sharp : \Omega^\bullet_{\mathbf{A}|\mathbb{C}} &\to HH_\bullet(\mathbf{A}) \\ da_1 \wedge \cdots \wedge da_n &\mapsto [\varepsilon_n(a_1 \otimes \cdots \otimes a_n)] \end{aligned}$$

where on the right hand side the brackets mean the homology class. ♦

Example 1.4.19 (Polynomial algebra)
For a finite dimensional vector space V and the commutative unital algebra $\mathsf{S}V$ of polynomials on V, one has

$$HH_\bullet(\mathsf{S}V) = \mathsf{S}V \otimes \textstyle\bigwedge^\bullet V = \Omega^\bullet_{\mathsf{S}V|\mathbb{C}}$$ ♦

This is a particular situation of a more general theorem for which we need the following definition:

Definition 1.4.20 (Smooth algebras)
A commutative algebra $\mathbf{A}$ is a smooth algebra if for any algebra $\mathbf{B}$ and any ideal $\mathbf{I}$ in $\mathbf{B}$ such that $\mathbf{I}^2 = 0$, the map $\mathsf{Hom}_{\mathbb{C}}(\mathbf{A}, \mathbf{B}) \to \mathsf{Hom}_{\mathbb{C}}(\mathbf{A}, \mathbf{B}/\mathbf{I})$ is surjective. This means that every morphism of algebras $\mathbf{A} \to \mathbf{B}/\mathbf{I}$ can be lifted to a morphism of algebras $\mathbf{A} \to \mathbf{B}$. ♦

Then one has the following result:

Theorem 1.4.21 (Hochschild-Kostant-Rosenberg)
For any unital smooth commutative algebra $\mathbf{A}$*, the map* $\varepsilon_\sharp : \Omega^\bullet_{\mathbf{A}|\mathbb{C}} \to HH_\bullet(\mathbf{A})$ *of Example 1.4.18 is an isomorphism of graded commutative algebras:*

$$HH_\bullet(\mathbf{A}) \simeq \Omega^\bullet_{\mathbf{A}|\mathbb{C}}$$

The natural map which identifies a differential forms in $\Omega^n(\mathbf{A})$ to a differential form in $\Omega^n_{\mathbf{A}|\mathbb{C}}$ is explicitly given by $a_0 da_0 \cdots da_n \mapsto \frac{1}{n!} a_0 d_K a_0 \wedge \cdots \wedge d_K a_n$. Notice the extra factor $\frac{1}{n!}$ compared to the universal projection π_n of Example 1.4.5. This factor is required to get a further identification of the differential on the Kähler differential calculus with the B operator in cyclic homology (see [Loday, 1998]) and to get a morphism of graded commutative algebras.

Remark 1.4.22 (Extension to topological algebras)
One can generalize the definition of the Hochschild homology given above to take into account some topological structure on the algebra $\mathbf{A}$. In order to do that, one defines the spaces $\mathbf{A}^{\otimes n}$ using a tensor product adapted to the topological structure on the algebra. The Hochschild homology one obtains in this way is called the continuous Hochschild homology.

For Fréchet algebras, such a continuous homology is well defined and leads to the next two very interesting examples. ♦

Example 1.4.23 (The Fréchet algebra $C^\infty(M)$)
Let M be a C^∞ finite dimensional locally compact manifold. Then Connes computed its continuous Hochschild homology in [Connes, 1985] and found the following result which generalizes the Hochschild-Kostant-Rosenberg theorem:

$$HH_\bullet^{\mathrm{Cont}}(C^\infty(M)) = \Omega_{\mathbb{C}}^\bullet(M) \text{ (complexified de Rham forms)}$$

For reasons that will be explained later, this isomorphism between vector spaces, which we denote by ϕ, is explicitly given in terms of universal forms by

$$\begin{aligned}\Omega^{2k}(C^\infty(M)) &\to \Omega_{\mathbb{C}}^{2k}(M) \\ \omega &\mapsto \left(\frac{i}{2\pi}\right)^k \frac{1}{(2k)!}\pi_{2k}(\omega)\end{aligned} \qquad \begin{aligned}\Omega^{2k+1}(C^\infty(M)) &\to \Omega_{\mathbb{C}}^{2k+1}(M) \\ \omega &\mapsto \left(\frac{i}{2\pi}\right)^{k+1} \frac{1}{(2k+1)!}\pi_{2k+1}(\omega)\end{aligned}$$

where $\pi_\bullet : \Omega^\bullet(C^\infty(M)) \to \Omega_{\mathbb{C}}^\bullet(M)$ is the universal map defined in Example 1.4.5. ♦

Example 1.4.24 (The irrational rotation algebra)
The continuous Hochschild homology of the Fréchet algebra $\mathcal{A}_\theta^\infty$ (θ irrational) has been computed by Connes in [Connes, 1985]. Let $\lambda = \exp(2i\pi\theta)$.

If λ satisfies some diophantine condition (there exists an integer k such that $|1-\lambda^n|^{-1}$ is $O(n^k)$), then

$$HH_0^{\mathrm{Cont}}(\mathcal{A}_\theta^\infty) = \mathbb{C} \qquad HH_1^{\mathrm{Cont}}(\mathcal{A}_\theta^\infty) = \mathbb{C}^2$$

For any λ:

$$HH_2^{\mathrm{Cont}}(\mathcal{A}_\theta^\infty) = \mathbb{C} \qquad HH_n^{\mathrm{Cont}}(\mathcal{A}_\theta^\infty) = 0 \text{ for } n \geq 3$$

If λ does not satisfy some diophantine condition, $HH_0^{\mathrm{Cont}}(\mathcal{A}_\theta^\infty)$ and $HH_1^{\mathrm{Cont}}(\mathcal{A}_\theta^\infty)$ are infinite dimensional. ♦

Definition 1.4.25 (Hochschild cohomology)
Recall that the dual $\mathbf{A}^*$ of an algebra $\mathbf{A}$ is a bimodule on $\mathbf{A}$ for the definition $(a\phi b)(c) = \phi(bca)$ for any $\phi \in \mathbf{A}^*$ and $a, b, c \in \mathbf{A}$. The Hochschild complex $(C^\bullet(\mathbf{A}), \delta)$ for the Hochschild cohomology with values in the bimodule $\mathbf{A}^*$ is defined by

$$C^n(\mathbf{A}) = \mathrm{Hom}(\mathbf{A}^{\otimes n}, \mathbf{A}^*) = \mathrm{Hom}(\mathbf{A}^{\otimes n+1}, \mathbb{C})$$

and by

$$\begin{aligned}\delta\phi(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{p=0}^{n} (-1)^p \phi(a_0 \otimes a_1 \otimes \cdots \otimes a_p a_{p+1} \otimes \cdots \otimes a_{n+1}) \\ + (-1)^{n+1}\phi(a_{n+1}a_0 \otimes a_1 \otimes \cdots \otimes a_n)\end{aligned}$$

By construction, δ is the adjoint to b in homology.

The cohomology of this complex is denoted by $HH^\bullet(\mathbf{A})$. ♦

1.4.3 Cyclic homology

Cyclic homology is defined using a bicomplex $CC_{\bullet,\bullet}(\mathbf{A})$ constructed using the bicomplex $CC^{(2)}_{\bullet,\bullet}(\mathbf{A})$ of the Hochschild homology. In order to do that, we need a new operator.

$N : \mathbf{A}^{\otimes n} \to \mathbf{A}^{\otimes n}$ is the norm operator defined by

$$N = 1 + t + \cdots + t^n$$

Then one has the relations

$$(1-t)N = N(1-t) = 0 \qquad\qquad b'N = Nb$$

The bicomplex $CC_{\bullet,\bullet}(\mathbf{A})$ is a repetition of the bicomplex $CC^{(2)}_{\bullet,\bullet}(\mathbf{A})$ infinitely on the right, using N to connect them. In terms of the algebra $\mathbf{A}$, one has

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \\
\mathbf{A}^{\otimes n+1} & \xleftarrow{1-t} & \mathbf{A}^{\otimes n+1} & \xleftarrow{N} & \mathbf{A}^{\otimes n+1} & \xleftarrow{1-t} & \mathbf{A}^{\otimes n+1} & \xleftarrow{N} & \cdots \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \\
\mathbf{A}^{\otimes n} & \xleftarrow{1-t} & \mathbf{A}^{\otimes n} & \xleftarrow{N} & \mathbf{A}^{\otimes n} & \xleftarrow{1-t} & \mathbf{A}^{\otimes n} & \xleftarrow{N} & \cdots \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \\
\vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \\
\mathbf{A} & \xleftarrow{1-t} & \mathbf{A} & \xleftarrow{N} & \mathbf{A} & \xleftarrow{1-t} & \mathbf{A} & \xleftarrow{N} & \cdots
\end{array}$$

Definition 1.4.26 (Cyclic homology)
Let $\mathbf{A}$ be an associative algebra. The cyclic homology $HC_\bullet(\mathbf{A})$ of $\mathbf{A}$ is the homology of the total complex of the bicomplex $CC_{\bullet,\bullet}(\mathbf{A})$ defined above. ♦

Any morphism of algebras $\varphi : \mathbf{A} \to \mathbf{B}$ induces a natural map $CC_{\bullet,\bullet}(\mathbf{A}) \to CC_{\bullet,\bullet}(\mathbf{B})$ of bicomplexes, so that one gets an induced map in cyclic homology $\varphi_\sharp : HC_\bullet(\mathbf{A}) \to HC_\bullet(\mathbf{B})$.

Remark 1.4.27 (The Connes complex)
In [Connes, 1985], Connes introduced cyclic cohomology, a dual version of cyclic homology. The way he introduced it did not rely on a bicomplex, but on a subcomplex of the Hochschild complex for cohomology. Some details of this construction are given in Example 1.4.53. In a dual version, one can define the Connes complex to compute cyclic homology as a quotient of the Hochschild complex for homology for a unital algebra.

To the bicomplex defined above, add a column on the left whose spaces are the cokernels of the morphisms $(1-t) : \mathbf{A}^{\otimes n+1} \to \mathbf{A}^{\otimes n+1}$, which we denote by $C^\lambda_n(\mathbf{A}) = \mathbf{A}^{\otimes n+1} / \mathrm{Ran}(1-t)$. One can then check that the operator b is a well-defined operator on $C^\lambda_\bullet(\mathbf{A}) = \oplus_{n\geq 0} C^\lambda_n(\mathbf{A})$ which satisfies $b^2 = 0$. Denote by $H^\lambda_\bullet(\mathbf{A})$ the homology of this complex. The total complex $TCC_\bullet(\mathbf{A})$ of $CC_{\bullet,\bullet}(\mathbf{A})$ projects onto the complex $C^\lambda_\bullet(\mathbf{A})$, sending the column $p = 0$ onto $C^\lambda_\bullet(\mathbf{A})$ and the other columns onto $\mathbf{0}$. One then gets a morphism in homology

$$HC_\bullet(\mathbf{A}) \to H^\lambda_\bullet(\mathbf{A})$$

When the field over which the algebra is defined contains $\mathbb{Q}$, this is an isomorphism. To show that, one introduces an explicit homotopy for the horizontal operators which shows that the horizontal homology of $CC_{\bullet,\bullet}(\mathbf{A})$ is trivial. By standard arguments on bicomplexes, this proves the assertion.⧫

Remark 1.4.28 (The horizontal homology of $CC_{\bullet,\bullet}(\mathbf{A})$)
One can show that for any algebra $\mathbf{A}$, the homology of any row of $CC_{\bullet,\bullet}(\mathbf{A})$ is the group homology $H_\bullet(\mathfrak{C}_{n+1};\mathbf{A}^{\otimes n+1})$ of the cyclic group $\mathfrak{C}_{n+1}$ with values in the $\mathfrak{C}_{n+1}$-module $\mathbf{A}^{\otimes n+1}$ (for the action induced by t). ⧫

We have seen that the total complex of $CC^{(2)}_{\bullet,\bullet}(\mathbf{A})$ can be written in terms of the universal differential calculus $\Omega^\bullet(\mathbf{A})$ with the boundary operator b_H. We can do something similar here. Every grouping of two columns isomorphic to $CC^{(2)}_{\bullet,\bullet}(\mathbf{A})$ can be "compressed" as we did for the Hochschild bicomplex. The operators b, b' and $(1-t)$ are then replaced by the unique operator $b_H : \Omega^n(\mathbf{A}) \to \Omega^{n-1}(\mathbf{A})$. The operator N is replaced by a new operator $B : \Omega^n(\mathbf{A}) \to \Omega^{n+1}(\mathbf{A})$, which takes the matrix form

$$B = \begin{pmatrix} 0 & 0 \\ N & 0 \end{pmatrix}$$

in the decomposition $\Omega^n(\mathbf{A}) = \mathbf{A}^{\otimes n+1} \oplus \mathbf{A}^{\otimes n}$. In order to have a pleasant diagram representing the new bicomplex, lift vertically each column on the right in proportion to its degree in the horizontal direction. We then get the following (triangular) bicomplex

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
{\scriptstyle b_H}\downarrow & & {\scriptstyle b_H}\downarrow & & {\scriptstyle b_H}\downarrow & & {\scriptstyle b_H}\downarrow \\
\Omega^{n+1}(\mathbf{A}) & \xleftarrow{B} & \Omega^{n}(\mathbf{A}) & \xleftarrow{B} \cdots \xleftarrow{B} & \Omega^{1}(\mathbf{A}) & \xleftarrow{B} & \Omega^{0}(\mathbf{A}) \\
{\scriptstyle b_H}\downarrow & & {\scriptstyle b_H}\downarrow & & {\scriptstyle b_H}\downarrow & & \\
\Omega^{n}(\mathbf{A}) & \xleftarrow{B} & \Omega^{n-1}(\mathbf{A}) & \xleftarrow{B} \cdots \xleftarrow{B} & \Omega^{0}(\mathbf{A}) & & \\
{\scriptstyle b_H}\downarrow & & {\scriptstyle b_H}\downarrow & & & & \\
 & & & & & & \\
\vdots & & \vdots & & \vdots & & \\
{\scriptstyle b_H}\downarrow & & {\scriptstyle b_H}\downarrow & & {\scriptstyle b_H}\downarrow & & \\
\Omega^{2}(\mathbf{A}) & \xleftarrow{B} & \Omega^{1}(\mathbf{A}) & \xleftarrow{B} & \Omega^{0}(\mathbf{A}) & & \\
{\scriptstyle b_H}\downarrow & & {\scriptstyle b_H}\downarrow & & & & \\
\Omega^{1}(\mathbf{A}) & \xleftarrow{B} & \Omega^{0}(\mathbf{A}) & & & & \\
{\scriptstyle b_H}\downarrow & & & & & & \\
\Omega^{0}(\mathbf{A}) & & & & & &
\end{array}$$

The total homology of this bicomplex is again the cyclic homology of $\mathbf{A}$.

Definition 1.4.29 (Mixed bicomplex)
A mixed bicomplex is a $\mathbb{N}$-graded complex $M_\bullet = \bigoplus_{n\geq 0} M_n$ equipped with a differential b_M of degree -1 and a differential B_M of degree $+1$ such that

$$b_M B_M + B_M b_M = 0$$

The homology of the complex (M, b_M) is called the Hochschild homology of the mixed bicomplex, and it is denoted by $HH_\bullet M = H_\bullet(M, b_M)$.

We associate to such a mixed bicomplex the $\mathbb{N}$-graded complex $\widetilde{M}_\bullet$ defined by

$$\widetilde{M}_n = \bigoplus_{p \geq 0} M_{n-2p}$$

on which we introduce the differential operator $B'_M - b_M$ where $B'_M : \widetilde{M}_n \to \widetilde{M}_{n-1}$ is B_M on M_{n-2p} such that $0 < 2p \leq n$ and 0 on M_n. The cyclic homology of the mixed bicomplex is the homology of this differential complex: $HC_\bullet M = H_\bullet(\widetilde{M}, B'_M - b_M)$

Two mixed bicomplexes $(M_\bullet, b_M, B_M)$ and $(N_\bullet, b_N, B_N)$ are said to be b-quasi-isomorphic if there exists a morphism of mixed bicomplexes $\varphi : (M_\bullet, b_M, B_M) \to (N_\bullet, b_N, B_N)$ (φ is of degree 0 and commutes with the b's and B's) which induces an isomorphism in Hochschild homology. ♦

Proposition 1.4.30
Two b-quasi-isomorphic mixed bicomplexes have the same cyclic homology.

Example 1.4.31 (The mixed bicomplex $(\Omega^\bullet(\mathbf{A}), b_H, B)$)
The motivation for the above definition is the example of the $\mathbb{N}$-graded module $\Omega^\bullet(\mathbf{A})$ with two differentials b_H and B. The definitions of Hochschild and cyclic homology reproduce the ones we introduced before. ♦

Example 1.4.32 (The mixed bicomplex $(HH_\bullet(\mathbf{A}), 0, B_\sharp)$)
Because the differential B commutes with the differential b_H, it defines a morphism $B_\sharp$ on the Hochschild homology $HH_\bullet(\mathbf{A})$. With this induced morphism, the triplet $(HH_\bullet(\mathbf{A}), 0, B_\sharp)$ is a mixed bicomplex, whose Hochschild homology is the Hochschild homology of $\mathbf{A}$. Indeed, when taking Hochschild homology, b_H is mapped to the zero operator.

Now, using standard argument on the spectral sequence constructed on the filtration by vertical degree, one can see that this mixed complex computes the cyclic homology of $\mathbf{A}$. ♦

Remark 1.4.33 (Some ideas to compute cyclic homology)
Example 1.4.32 tells us that in order to compute cyclic homology, one can first compute Hochschild homology, then look at the operator $B_\sharp$ induced by B, and compute the $B_\sharp$-homology. Many examples of concrete computations of cyclic homology are performed this way. Obviously, this supposes that Hochschild homology is computable!

Another approach is to consider the simplest possible differential complex which computes the Hochschild homology of the algebra, and then guess a B operator on it in order to build a mixed bicomplex b-quasi-isomorphic to one of the standard mixed bicomplexes given here. Because Hochschild homology can be defined through projective resolutions, such simple differential complexes are usually possible to find. ♦

Example 1.4.34 (Mixed bicomplexes for the unital case)
Let us suppose now that the algebra $\mathbf{A}$ is unital. Then we know that the Hochschild homology can be computed with the complex (1.4.5). Using the ideas of Remark 1.4.33, one can find a B operator on this complex in order to make it into a mixed bicomplex.

This operator is defined by $B = (1-t)sN : \mathbf{A}^{\otimes n} \to \mathbf{A}^{\otimes(n+1)}$ where t and N have been introduced before, and $s : \mathbf{A}^{\otimes n} \to \mathbf{A}^{\otimes(n+1)}$ is the homotopy (1.4.4). Explicitly, one has

$$B(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{p=0}^{n-1} \big[(-1)^{np} \mathbb{1} \otimes a_p \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{p-1} \\ - (-1)^{n(p-1)} a_{p-1} \otimes \mathbb{1} \otimes a_p \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{p-2}\big]$$

In low degrees, these expressions take the following forms

$$\begin{aligned} B(a_0) &= \mathbb{1} \otimes a_0 + a_0 \otimes \mathbb{1} \\ B(a_0 \otimes a_1) &= (\mathbb{1} \otimes a_0 \otimes a_1 - \mathbb{1} \otimes a_1 \otimes a_0) + (a_0 \otimes \mathbb{1} \otimes a_1 - a_1 \otimes \mathbb{1} \otimes a_0) \end{aligned}$$

This mixed bicomplex $(\mathbf{A}^{\otimes(\bullet+1)}, b, B)$ is represented by the diagram

$$\begin{array}{ccccc} \vdots & & \vdots & & \vdots \\ \downarrow b & & \downarrow b & & \downarrow b \\ \mathbf{A}^{\otimes 3} & \xleftarrow{B} & \mathbf{A}^{\otimes 2} & \xleftarrow{B} & \mathbf{A} \\ \downarrow b & & \downarrow b & & \\ \mathbf{A}^{\otimes 2} & \xleftarrow{B} & \mathbf{A} & & \\ \downarrow b & & & & \\ \mathbf{A} & & & & \end{array}$$

Now, one can perform this procedure with the complex (1.4.6). Then one obtains the same operator B and the mixed bicomplex $(\Omega^\bullet_U(\mathbf{A}), b, B)$ which is b-quasi-isomorphic to the mixed bicomplex of Example 1.4.31 through the natural projection $\Omega^\bullet(\mathbf{A}) \to \Omega^\bullet_U(\mathbf{A})$. ♦

Example 1.4.35 (The commutative case)
Let us consider the notations and results of Example 1.4.18, where the algebra is over the field $\mathbb{C}$. One can introduce the mixed bicomplex $(\Omega^\bullet_{\mathbf{A}|\mathbb{C}}, 0, d_K)$ based on the Kähler differential calculus, which takes the diagrammatic form

$$\begin{array}{ccccc} \vdots & & \vdots & & \vdots \\ \downarrow 0 & & \downarrow 0 & & \downarrow 0 \\ \Omega^2_{\mathbf{A}|\mathbb{C}} & \xleftarrow{d_K} & \Omega^1_{\mathbf{A}|\mathbb{C}} & \xleftarrow{d_K} & \Omega^0_{\mathbf{A}|\mathbb{C}} \\ \downarrow 0 & & \downarrow 0 & & \\ \Omega^1_{\mathbf{A}|\mathbb{C}} & \xleftarrow{d_K} & \Omega^0_{\mathbf{A}|\mathbb{C}} & & \\ \downarrow 0 & & & & \\ \Omega^0_{\mathbf{A}|\mathbb{C}} & & & & \end{array}$$

One can show that there is a natural morphism of mixed bicomplexes $(\Omega^\bullet_U(\mathbf{A}), b, B) \to (\Omega^\bullet_{\mathbf{A}|\mathbb{C}}, 0, d_K)$, so that there is a natural map

$$HC_n(\mathbf{A}) \to \Omega^n_{\mathbf{A}|\mathbb{C}} / d_K \Omega^{n-1}_{\mathbf{A}|\mathbb{C}} \oplus H^{n-2}_{\mathrm{dR}}(\mathbf{A}) \oplus H^{n-4}_{\mathrm{dR}}(\mathbf{A}) \oplus \cdots$$

the last term being $H^0_{\mathrm{dR}}(\mathbf{A})$ or $H^1_{\mathrm{dR}}(\mathbf{A})$ depending on the parity of n.

Using Theorem 1.4.21, for any smooth unital commutative algebra $\mathbf{A}$, one has the isomorphism

$$HC_\bullet(\mathbf{A}) \simeq \Omega^\bullet_{\mathbf{A}|\mathbb{C}} / d_K \Omega^{\bullet-1}_{\mathbf{A}|\mathbb{C}} \oplus H^{\bullet-2}_{\mathrm{dR}}(\mathbf{A}) \oplus H^{\bullet-4}_{\mathrm{dR}}(\mathbf{A}) \oplus \cdots$$

In particular, this is the case for the polynomial algebra $\mathbf{A} = \mathrm{S}V$ over a finite dimensional vector space V. ♦

There are a lot of structural properties on the cyclic homology groups which help a lot to compute them. We refer to [Loday, 1998] to explore them. Let us just mention the following result:

Proposition 1.4.36 (Connes long exact sequence)
There are morphisms I and S which induce the following long exact sequence

$$\cdots \longrightarrow HH_n(\mathbf{A}) \xrightarrow{I} HC_n(\mathbf{A}) \xrightarrow{S} HC_{n-2}(\mathbf{A}) \xrightarrow{B} HH_{n-1}(\mathbf{A}) \xrightarrow{I} \cdots$$

In low degrees, one gets

$$\cdots \longrightarrow HH_2(\mathbf{A}) \xrightarrow{I} HC_2(\mathbf{A}) \xrightarrow{S} HC_0(\mathbf{A}) \xrightarrow{B} HH_1(\mathbf{A}) \xrightarrow{I} HC_1(\mathbf{A}) \xrightarrow{S} 0$$

and the isomorphism

$$0 \xrightarrow{B} HH_0(\mathbf{A}) \xrightarrow{I} HC_0(\mathbf{A}) \xrightarrow{S} 0$$

This long exact sequence is a direct consequence of the fact that the bicomplex $CC^{(2)}_{\bullet,\bullet}(\mathbf{A})$ is included as pairs of columns in the bicomplex $CC_{\bullet,\bullet}(\mathbf{A})$. This inclusion gives rise to the short exact sequence of bicomplexes

$$0 \longrightarrow CC^{(2)}_{\bullet,\bullet}(\mathbf{A}) \xrightarrow{I} CC_{\bullet,\bullet}(\mathbf{A}) \xrightarrow{S} CC_{\bullet-2,\bullet}(\mathbf{A}) \longrightarrow 0$$

which defines I and S. In homology this short exact sequence produces Connes long exact sequence. The morphism S is called the periodic morphism.

Example 1.4.37 ($HC_\bullet(\mathbb{C})$)
Using the results of Example 1.4.15 and the mixed bicomplex of Example 1.4.32, one easily gets

$$HC_{2n}(\mathbb{C}) = \mathbb{C} \qquad\qquad HC_{2n+1}(\mathbb{C}) = 0$$

Using Connes long exact sequence, one has an isomorphism $S : HC_n(\mathbb{C}) \to HC_{n-2}(\mathbb{C})$. Denote by $u_n \in HC_{2n}(\mathbb{C}) = \mathbb{C}$ the canonical generator. Then, one can show that there is an isomorphism of coalgebras $HC_\bullet(\mathbb{C}) \xrightarrow{\simeq} \mathbb{C}[u]$ explicitly given by $u_n \mapsto u^n$, where the coproduct on $\mathbb{C}[u]$ is $\Delta(u^n) = \sum_{p+q=n} u^p \otimes u^q$.

For any algebra $\mathbf{A}$, $HC_\bullet(\mathbf{A})$ is a comodule over the coalgebra $HC_\bullet(\mathbb{C})$:

$$\begin{aligned} HC_\bullet(\mathbf{A}) &\to HC_\bullet(\mathbf{A}) \otimes \mathbb{C}[u] \\ x &\mapsto \sum_{p \geq 0} S^p(x) \otimes u^p \end{aligned}$$

where S^p is the p-th iteration of S. This is a concrete interpretation of the periodic morphism S. ♦

Let us now define the periodic and negative cyclic homologies. In order to do that, let us introduce the bicomplex $CC^{\mathrm{per}}_{\bullet,\bullet}(\mathbf{A})$, infinite in the two horizontal directions:

$$
\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
\cdots & \xleftarrow{N} & \mathbf{A}^{\otimes n+1} & \xleftarrow{1-t} & \mathbf{A}^{\otimes n+1} & \xleftarrow{N} & \mathbf{A}^{\otimes n+1} & \xleftarrow{1-t} & \mathbf{A}^{\otimes n+1} & \xleftarrow{N} & \cdots \\
 & & \downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
\cdots & \xleftarrow{N} & \mathbf{A}^{\otimes n} & \xleftarrow{1-t} & \mathbf{A}^{\otimes n} & \xleftarrow{N} & \mathbf{A}^{\otimes n} & \xleftarrow{1-t} & \mathbf{A}^{\otimes n} & \xleftarrow{N} & \cdots \\
 & & \downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
\cdots & \xleftarrow{N} & \mathbf{A} & \xleftarrow{1-t} & \mathbf{A} & \xleftarrow{N} & \mathbf{A} & \xleftarrow{1-t} & \mathbf{A} & \xleftarrow{N} & \cdots \\
 & & & & & & & & & & \\
p= & & -2 & & -1 & & 0 & & 1 & & \cdots
\end{array}
$$

The bicomplex $CC_{\bullet,\bullet}(\mathbf{A})$ is naturally included in $CC^{\mathrm{per}}_{\bullet,\bullet}(\mathbf{A})$ as the sub-bicomplex for which $p \geq 0$. Denote by $CC^{-}_{\bullet,\bullet}(\mathbf{A})$ the sub-bicomplex defined by $p \leq 1$.

Definition 1.4.38 (Periodic and negative cyclic homology)
The periodic cyclic homology $HP_\bullet(\mathbf{A})$ of $\mathbf{A}$ is the homology of the total complex (for the product) defined from $CC^{\mathrm{per}}_{\bullet,\bullet}(\mathbf{A})$ by

$$TCC^{\mathrm{per}}_n(\mathbf{A}) = \prod_{p+q=n} CC^{\mathrm{per}}_{p,q}(\mathbf{A})$$

for any $n \in \mathbb{Z}$.

The negative cyclic homology $HC^{-}_\bullet(\mathbf{A})$ of $\mathbf{A}$ is the homology of the total complex (for the product) defined from $CC^{-}_{\bullet,\bullet}(\mathbf{A})$ by

$$TCC^{-}_n(\mathbf{A}) = \prod_{\substack{p+q=n \\ (p\leq 1)}} CC^{-}_{p,q}(\mathbf{A}) \qquad \blacklozenge$$

Let us recall that in this situation, as the two bicomplexes we consider are infinite in the left direction, direct sum and product do not coincide. An element in the direct sum contains only a finite number of non zero elements in the spaces $CC^{\mathrm{per}}_{p,q}(\mathbf{A})$ for $p+q=n$, but an element in the product can be non zero in all of these spaces. If we were using direct sums to define their total complexes, then it would be possible to show that the associated homologies were trivial if the base field contains $\mathbb{Q}$.

Using an adaptation of the procedure described for the cyclic homology, one can define the cyclic periodic homology of a mixed bicomplex, as well as its cyclic negative homology. Then one has:

Proposition 1.4.39
Two b-quasi-isomorphic mixed bicomplexes have the same cyclic periodic homology and the same cyclic negative homology.

The natural inclusion and the natural projection

$$I : CC^-_{\bullet,\bullet}(\mathbf{A}) \to CC^{\text{per}}_{\bullet,\bullet}(\mathbf{A}) \qquad\qquad p : CC^{\text{per}}_{\bullet,\bullet}(\mathbf{A}) \to CC_{\bullet,\bullet}(\mathbf{A})$$

induce morphisms in homology

$$I : HC^-_n(\mathbf{A}) \to HP_n(\mathbf{A}) \qquad\qquad p : HP_n(\mathbf{A}) \to HC_n(\mathbf{A})$$

Proposition 1.4.40 (2-periodicity of $HP_\bullet(\mathrm{A})$)
The periodic map S defined on the periodic bicomplex by translating on the left through two columns is an isomorphism. It induces the natural 2-periodicity:

$$HP_n(\mathbf{A}) \simeq HP_{n-2}(\mathbf{A})$$

which means that $HP_\bullet(\mathbf{A})$ is $\mathbb{Z}_2$-graded.

From now on, we will use the notation $HP_\nu(\mathbf{A})$, with $\nu = 0, 1$.

We saw a $\mathbb{Z}_2$-graded situation earlier for complex K-theory. Here is another similitude proved in [Cuntz and Quillen, 1997]:

Proposition 1.4.41 (Six term exact sequence)
For any short exact sequence of associative algebras $0 \longrightarrow \mathbf{I} \longrightarrow \mathbf{A} \longrightarrow \mathbf{A}/\mathbf{I} \longrightarrow 0$, one has the six term exact sequence

$$\begin{array}{ccccc} HP_0(\mathbf{I}) & \longrightarrow & HP_0(\mathbf{A}) & \longrightarrow & HP_0(\mathbf{A}/\mathbf{I}) \\ \delta\uparrow & & & & \downarrow\delta \\ HP_1(\mathbf{A}/\mathbf{I}) & \longleftarrow & HP_1(\mathbf{A}) & \longleftarrow & HP_1(\mathbf{I}) \end{array}$$

Proposition 1.4.42 (Morita invariance)
For any integer $n \geq 1$, the trace map defined in Definition 1.4.11 induces an isomorphism $\mathrm{Tr}_\sharp : HP_\nu(M_n(\mathbf{A})) \xrightarrow{\simeq} HP_\nu(\mathbf{A})$.

Notice that we did not mention such a result for cyclic homology, because it is not true! There is a Morita invariance for cyclic homology on H-unital algebras (see Definition 1.4.14), but not on all algebras.

Example 1.4.43 ($HP_\nu(\mathbb{C})$ and $HC^-_\bullet(\mathbb{C})$)
One has

$$HP_0(\mathbb{C}) = \mathbb{C} \qquad\qquad HP_1(\mathbb{C}) = 0$$

There is an isomorphism of algebras $HC^-_\bullet(\mathbb{C}) \simeq \mathbb{C}[\nu]$ for a generator $\nu \in HC^-_{-2}(\mathbb{C})$. The product by ν corresponds to the operation $S : HC^-_n(\mathbb{C}) \to HC^-_{n-2}(\mathbb{C})$. ♦

Example 1.4.44 (Tensor algebra)
Let us use the notations of Example 1.4.16. The inclusion $\mathbb{C} \to \mathcal{T}V$ induces an isomorphism in periodic cyclic homology:

$$HP_0(\mathcal{T}V) = \mathbb{C} \qquad\qquad HP_1(\mathcal{T}V) = 0$$ ♦

Example 1.4.45 (The Laurent polynomials)
Let $\mathbf{A} = \mathbb{C}[z, z^{-1}]$ be the Laurent polynomials for the variable z. Then one has

$$HP_0(\mathbb{C}[z,z^{-1}]) = \mathbb{C} \qquad HP_1(\mathbb{C}[z,z^{-1}]) = \mathbb{C}$$ ♦

Example 1.4.46 (Unital smooth commutative algebras)
For any unital smooth commutative algebra $\mathbf{A}$, one has

$$HP_0(\mathbf{A}) = H^{\text{even}}_{\text{dR}}(\mathbf{A}) = \prod_{p\geq 0} H^{2p}_{\text{dR}}(\mathbf{A}) \qquad HP_1(\mathbf{A}) = H^{\text{odd}}_{\text{dR}}(\mathbf{A}) = \prod_{p\geq 0} H^{2p+1}_{\text{dR}}(\mathbf{A})$$

If we compare this result with the cyclic homology groups given at the end of Example 1.4.35, one sees that periodic cyclic homology is not ill with the edge effects on the left of the cyclic bicomplex which produce the contributions $\Omega^{\bullet}_{\mathbf{A}|\mathbb{C}}/d_K\Omega^{\bullet-1}_{\mathbf{A}|\mathbb{C}}$. ♦

Remark 1.4.47 (Extension to topological algebras)
As for Hochschild homology, one can generalize the definitions given above to take into account some topological structure on the algebra $\mathbf{A}$, by replacing the tensor products with topological tensor products. We then obtain "continuous" versions of these cyclic homologies.

In [Cuntz, 1997], Cuntz proved a six term exact sequence as in Proposition 1.4.41 for a restricted class of topological algebras, called m-algebras (see also [Cuntz et al., 2004]). ♦

Example 1.4.48 (Continuous cyclic homology of Banach algebras)
On Banach algebras, the continuous cyclic homologies are not interesting. For instance, for commutative C^*-algebras, one gets

$$HP^{\text{cont}}_0(C(X)) = \{\text{bounded measures on } X\} \qquad HP^{\text{cont}}_1(C(X)) = 0$$ ♦

This example shows that cyclic homology is not a very powerful theory for noncommutative *topological* spaces. The following result confirms this fact. We need a

Definition 1.4.49 (Diffeotopic morphisms)
Let $\mathbf{A}$ and $\mathbf{B}$ be two associative algebras. Two morphisms of algebras $\varphi_0, \varphi_1 : \mathbf{A} \to \mathbf{B}$ are said to be diffeotopic if there exists a morphism of algebras $\varphi : \mathbf{A} \to \mathbf{B} \otimes C^\infty([0,1])$ such that φ_t coincides with φ_0 (resp. φ_1) when evaluated at $t = 0$ (resp. at $t = 1$) in the target algebra. Notice that the tensor product $\mathbf{B} \otimes C^\infty([0,1])$ is purely algebraic. ♦

Proposition 1.4.50 (Diffeotopic invariance)
If φ_0 and φ_1 are diffeotopic then they induce the same morphism $HP_v(\mathbf{A}) \to HP_v(\mathbf{B})$.

There is no general homotopic invariance result on periodic cyclic homology.

If you need a result more to convince you that cyclic homology is well adapted to differential structures, here is the main result, obtained by Connes in [Connes, 1985], which is a generalisation of Example 1.4.46:

Example 1.4.51 (Continuous periodic cyclic homology of $C^\infty(M)$)
Let M be a C^∞ finite dimensional locally compact manifold. Then one has

$$HP^{\text{cont}}_0(C^\infty(M)) = H^{\text{even}}_{\text{dR}}(M) \qquad HP^{\text{cont}}_1(C^\infty(M)) = H^{\text{odd}}_{\text{dR}}(M)$$ ♦

Remark 1.4.52 (Comparing cyclic homology with K-theory)
In the next section, we will establish a very strong relation between K-theory and periodic cyclic homology. $\mathbb{Z}_2$-graduation, Morita invariance and the six term exact sequence give us obvious similarities between these two theories.

But, we would like to make it clear that the two theories are very different on an essential point. We notice in Remark 1.3.47 that K-theory can be computed using some dense subalgebra (stable by holomorphic functional calculus). The situation is clearly not the same for periodic cyclic homology: compare Example 1.4.48 with Example 1.4.51.

K-theory is an homology theory for noncommutative topological spaces. Periodic cyclic homology is an homology theory for algebras concealing some differentiable properties. ♦

Example 1.4.53 (First version of cyclic cohomology)
We define here the cyclic cohomology using the differential complex $(C^\bullet_\lambda(\mathbf{A}), \delta)$, which is the first version of cyclic cohomology exposed in [Connes, 1985].

By definition,

$$\begin{aligned} C^n_\lambda(\mathbf{A}) &= \{\phi \in \mathsf{Hom}(\mathbf{A}^{\otimes n+1}, \mathbb{C}) \ / \ \phi(a_1 \otimes \cdots \otimes a_n \otimes a_0) = (-1)^n \phi(a_0 \otimes a_1 \otimes \cdots \otimes a_n)\} \\ &\subset C^n(\mathbf{A}) \end{aligned}$$

and δ is the Hochschild boundary operator for the complex $C^\bullet(\mathbf{A})$ restricted to this subspace (see Definition 1.4.25). Indeed, the main remark made by Connes to define its cyclic complex as a subcomplex of a Hochschild complex was that the cyclic condition defining the ϕ's in $C^n(\mathbf{A})$ which are elements of $C^n_\lambda(\mathbf{A})$ is compatible with the boundary δ.

The cohomology of the complex $(C^\bullet_\lambda(\mathbf{A}), \delta)$ is the cyclic cohomology $HC^\bullet(\mathbf{A})$ of $\mathbf{A}$.

For $n = 0$, a cycle ϕ is a trace on $\mathbf{A}$, because $(\delta\phi)(a_0 \otimes a_1) = \phi(a_0 a_1) - \phi(a_1 a_0) = 0$. The cyclic complex of Connes is explicitly defined to generalize this property to higher degrees. So, cyclic cohomology is a theory of generalized traces.

On this cohomology, the inclusion $C^\bullet_\lambda(\mathbf{A}) \hookrightarrow C^\bullet(\mathbf{A})$ at the level of complexes induces a map $I : HC^\bullet(\mathbf{A}) \to HH^\bullet(\mathbf{A})$ and Connes long exact sequence

$$\cdots \longrightarrow HH^n(\mathbf{A}) \xrightarrow{B} HC^{n-1}(\mathbf{A}) \xrightarrow{S} HC^{n+1}(\mathbf{A}) \xrightarrow{I} HH^{n+1}(\mathbf{A}) \xrightarrow{B} \cdots$$

In this long exact sequence, the two maps B and S are not so easy to define as in the previous construction for cyclic homology.

Nevertheless, one can show that the periodic map $S : HC^n(\mathbf{A}) \to HC^{n+2}(\mathbf{A})$ can be used to define periodic cyclic cohomology as

$$HP^0(\mathbf{A}) = \varinjlim (HC^{2n+1}(\mathbf{A}), S) \qquad HP^1(\mathbf{A}) = \varinjlim (HC^{2n}(\mathbf{A}), S)$$

which explains the name "periodic" for the cohomology group and the map S. ♦

One defines the dual bicomplex $CC^{\bullet,\bullet}(\mathbf{A})$ of $CC_{\bullet,\bullet}(\mathbf{A})$ replacing in each bidegree (p,n) the space $\mathbf{A}^{\otimes n+1}$ by the space $\mathsf{Hom}(\mathbf{A}^{\otimes n}, \mathbf{A}^*) = \mathsf{Hom}(\mathbf{A}^{\otimes n+1}, \mathbb{C})$, and adjoining the four maps b, b', t and N. Recall that the adjoint of b is the δ map introduced in Definition 1.4.25.

In the same manner, one defines the bicomplex $CC^{\bullet,\bullet}_{\text{per}}(\mathbf{A})$ from the bicomplex $CC^{\text{per}}_{\bullet,\bullet}(\mathbf{A})$. With these bicomplexes one can define cyclic cohomology in its generality.

Definition 1.4.54 (Cyclic cohomology)
The cyclic cohomology $HC^\bullet(\mathbf{A})$ of $\mathbf{A}$ is the cohomology of the total complex of the bicomplex $CC^{\bullet,\bullet}(\mathbf{A})$.

The cyclic periodic cohomology $HP^\bullet(\mathbf{A})$ of $\mathbf{A}$ is the cohomology of the total complex of the bicomplex $CC^{\bullet,\bullet}_{\text{per}}(\mathbf{A})$. Here, the total complex is constructed using direct sums. ♦

As for periodic cyclic homology, $HP^\bullet(\mathbf{A})$ is $\mathbb{Z}_2$-graded.

Remark 1.4.55 (Entire cyclic cohomology)
In the definition of $HP^\bullet(\mathbf{A})$, one uses the direct sum to construct the total complex. This is the dual version of the direct product used for periodic cyclic homology. Indeed, one can show that the direct product would produce a trivial cohomology. Using direct sum in periodic cyclic cohomology permits one to define a natural pairing with periodic cyclic homology: cochains in periodic cyclic cohomology have finite support, so that only a finite number of terms are non zero when evaluated on a (infinite) chain in cyclic periodic homology.

Let $\mathbf{A}$ be a Banach algebra. Then one defines a norm on $\mathsf{Hom}(\mathbf{A}^{\otimes n+1},\mathbb{C})$ by $\|\phi_n\| = \sup\{|\phi(a_0 \otimes \cdots \otimes a_n)| \;/\; \|a_i\| \leq 1\}$.

Denote by $TCC^\bullet_{\prod}(\mathbf{A})$ the total complex of $CC^{\bullet,\bullet}_{\text{per}}(\mathbf{A})$ obtained using direct product. Each element in $TCC^p_{\prod}(\mathbf{A})$ is an infinite sequence (ϕ_{2n}) or (ϕ_{2n+1}) according to parity of p. One defines a subcomplex $ECC^\bullet(\mathbf{A})$ of $TCC^\bullet_{\prod}(\mathbf{A})$ imposing a growing condition on such an infinite sequence: the radius of convergence of the series $\sum_{n\geq 0} \|\phi_{2n}\| z^n/n!$ (resp. $\sum_{n\geq 0} \|\phi_{2n+1}\| z^n/n!$) is infinity.

The entire cyclic cohomology $HE^\bullet(\mathbf{A})$ is defined as the cohomology of the complex $ECC^\bullet(\mathbf{A})$. One can show that $HE^\bullet(\mathbf{A})$ is $\mathbb{Z}_2$-graded, as is the periodic cyclic cohomology.

Any cochain defining an element in $HP^\bullet(\mathbf{A})$ has finite support, so that there is an natural map $HP^\bullet(\mathbf{A}) \to HE^\bullet(\mathbf{A})$. This map is an isomorphism in some cases, for instance $\mathbf{A} = \mathbb{C}$, but not in general. See [Khalkhali, 1994] for examples of such isomorphisms. ♦

Example 1.4.56 (The irrational rotation algebra)
For irrational θ, one has

$$HP_0^{\text{cont}}(\mathcal{A}_\theta^\infty) = \mathbb{C}^2 \qquad\qquad HP_1^{\text{cont}}(\mathcal{A}_\theta^\infty) = HH_1(\mathcal{A}_\theta^\infty)/\operatorname{Ran} B_\sharp = \mathbb{C}^2$$

There is no need to mention any diophantine condition here (see Example 1.4.24).

In periodic cyclic cohomology, one of the two classes in $HP^\nu_{\text{cont}}(\mathcal{A}_\theta^\infty) = \mathbb{C}^2$ is $S\tau$ where τ is the unique normalised trace on $\mathcal{A}_\theta^\infty$, $\tau(\sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n) = a_{0,0}$, and the second one is expressed in terms of the continuous derivations δ_1 and δ_2:

$$\varphi(a_0 \otimes a_1 \otimes a_2) = \frac{1}{2i\pi} \tau[a_0(\delta_1(a_1)\delta_2(a_2) - \delta_2(a_1)\delta_1(a_2))]$$ ♦

Remark 1.4.57 (Pairing with K-theories)
A Fredholm module $(\mathcal{H},\rho,F)$ over the C^*-algebra $\mathbf{A}$ is called p-summable if $[F,a] \in \mathcal{L}^p(\mathcal{H})$ for any $a \in \mathbf{A}$ (we write a for $\rho(a)$ from now on). The space $\mathcal{L}^p(\mathcal{H}) = \{T \in \mathcal{K} \;/\; \sum_{n=0}^\infty \mu_n(T)^p < \infty\}$ with $\mu_n(T)$ the n-th eigenvalue of $|T| = (T^*T)^{1/2}$, is the Schatten class. It is a two-sided ideal in $\mathcal{B}$, and a Banach space for the norm $\|T\|_p = (\sum_{n=0}^\infty \mu_n(T)^p)^{1/p} = \operatorname{Tr}(|T|^p)^{1/p}$. For any $S \in \mathcal{L}^p(\mathcal{H})$ and $T \in \mathcal{L}^q(\mathcal{H})$, one has $ST \in \mathcal{L}^r(\mathcal{H})$ for $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and $\|ST\|_r \leq \|S\|_p \|T\|_q$.

Let $(\mathcal{H},\rho,F)$ be a p-summable Fredholm module (odd or even according to the parity of $p-1$) and denote by γ its grading map if it is even. For any operator T on $\mathcal{H}$ such that $FT + TF \in \mathcal{L}^1(\mathcal{H})$, let us define $\mathrm{Tr}'(T) = \frac{1}{2}\mathrm{Tr}(F(FT+TF))$. For any $n \geq 0$ such that $2n+1 \geq p-1$ in the odd case and $2n \geq p-1$ in the even case, and for any $a_i \in \mathbf{A}$, the expressions

$$\varphi_{2n+1}(a_0 \otimes \cdots \otimes a_{2n+1}) = \mathrm{Tr}'(a_0[F,a_1]\cdots[F,a_{2n+1}]) \text{ in the odd case}$$
$$\varphi_{2n}(a_0 \otimes \cdots \otimes a_{2n}) = \mathrm{Tr}'(\gamma a_0[F,a_1]\cdots[F,a_{2n}]) \text{ in the even case}$$

make sense and define an odd or an even cocyle in $HC^{\bullet}(\mathbf{A})$, which depends only on the K-homology class of the Fredholm module. In fact, using correct normalizations, this defines a natural pairing $HP_{\nu}(\mathbf{A}) \times K^{\nu}(\mathbf{A}) \to \mathbb{C}$ for $\nu = 0,1$. See [Connes, 1994] and [Gracia-Bondía et al., 2001] for details.

In the next section, the Chern character will realize a pairing $HP^{\nu}(\mathbf{A}) \times K_{\nu}(\mathbf{A}) \to \mathbb{C}$. ♦

1.5 The not-missing link: the Chern character

The Chern character is a special characteristic class defined first in the topological context. It was used to related the K-theory of a topological space to its cohomology. When Connes introduced cyclic homology, he saw immediately that a purely algebraic generalisation was possible, which connects the K-theory for algebras and the periodic cyclic homology. Now, the Chern character is extensively studied, because it helps interpret a lot of previous results in different areas of mathematics, which where not so well understood.

1.5.1 The Chern character in ordinary differential geometry

Let us recall some basic facts about characteristic classes for vector bundles.

Let G be a topological group. Then one has:

Proposition 1.5.1 (Classifying space BG)
There exists a G-principal fibre bundle $EG \to BG$ such that for any G-principal fibre bundle P over a topological space X, there exists a continuous map $f_P : P \to BG$ such that $P = f_P^* EG$ (the pull-back fibre bundle). BG is called the classifying space of the topological group G and f_P the classifying map of P.

Recall that the pull-back $P = f^*Q$ of a fibre bundle $Q \to Y$ through a continuous map $f : X \to Y$ is defined by $P_x = Q_{f(x)}$ for any $x \in X$. If $g : X \to Y$ is homotopic to f then f^*Q and g^*Q are isomorphic.

One can show that EG is a contractible space, so that its homology is not very interesting. The important object in this proposition is BG:

Proposition 1.5.2 (Classification of G-principal fibre bundles)
The space of isomorphic classes of G-principal fibre bundles over X is $[X;BG]$, the space of homotopic classes of continuous maps $X \to BG$.

This space is not easy to compute, so that this classification remains just an identification without any practical utility in general. This leads us to consider other objects to try to classify G-principal fibre bundles, in terms of cohomology classes:

Definition 1.5.3 (Characteristic classes)
A characteristic class of P in a cohomology group $H^{\bullet}(X;\mathsf{A})$ with coefficient in the abelian group A, is the pull-back by f_P of any cohomology class $c \in H^{\bullet}(BG;\mathsf{A})$. ♦

Characteristic classes depend upon the coefficient group A, which is often essential to make some concrete interpretations of certain characteristic classes.

If one is interested in vector bundles instead of principal bundles, the previous construction can be performed with the associated principal bundle. Any vector bundle is the pull-back through the classifying map f_P of a canonical vector bundle over BG. So that for any vector bundle $E \xrightarrow{\pi} X$ with structure group G, one can introduce its characteristic classes as pull-back of classes in $H^\bullet(BG;\mathsf{A})$. We will use the notation $c(E)$ for the pull-back of $c \in H^\bullet(BG;\mathsf{A})$ in $H^\bullet(X;\mathsf{A})$.

Proposition 1.5.4 (Functoriality of characteristic classes)
Let $\varphi : X \to Y$ be a continuous map, and $E \to Y$ a vector bundle on Y. Then for any characteristic class c one has $c(\varphi^* E) = \varphi^\sharp c(E)$ where $\varphi^\sharp : H^\bullet(Y) \to H^\bullet(X)$ is the ring morphism induced in cohomology.

Example 1.5.5 (Discrete groups)
In the case of a discrete group G, one can show that $BG = K(G,1)$ is the Eilenberg-MacLane space of type $(G,1)$, so that $H^\bullet(BG;\mathbb{Z})$ is the ordinary cohomology of groups $H^\bullet(G)$. ♦

It is possible to construct explicitly the classifying spaces BG for a large class of groups. Here are some examples.

Example 1.5.6 (Some usual classifying spaces)

G	$\mathbb{Z}$	$\mathbb{Z}^n$	$\mathbb{Z}_2$	$U(1)=\mathbb{S}^1$	$U(n)$	$O(n)$
EG	$\mathbb{R}$	$\mathbb{R}^n$	$\mathbb{S}^\infty$			
BG	$\mathbb{S}^1$	$\mathbb{T}^n$	$\mathbb{RP}^\infty$	$\mathbb{CP}^\infty$	$\mathbb{G}(n,\mathbb{C}^\infty)$	$\mathbb{G}(n,\mathbb{R}^\infty)$

♦

$\mathbb{S}^\infty$ is the sphere in $\mathbb{R}^\infty$, $\mathbb{RP}^\infty = \varinjlim \mathbb{RP}^n$, $\mathbb{CP}^\infty = \varinjlim \mathbb{CP}^n$, $\mathbb{G}(n,\mathbb{C}^\infty) = \varinjlim \mathbb{G}(n,\mathbb{C}^p)$ where $\mathbb{G}(n,\mathbb{C}^p)$ is the complex Grassmanian manifold...

Example 1.5.7 (Cohomology groups of some classifying spaces)
Here are some examples of cohomology groups of some classifying spaces.

We denote by $\mathsf{A}[a_1,\dots,a_p]$ the graded commutative ring generated over the abelian groups A by the p elements a_i (whose degrees will be given):

$$H^\bullet(BU(n);\mathbb{Z}) = \mathbb{Z}[c_1,c_2,\dots,c_n] \qquad H^\bullet(BSU(n);\mathbb{Z}) = \mathbb{Z}[c_2,\dots,c_n]$$

where $\deg c_k = 2k$. The class c_k is the k-th Chern class. The class $c = 1 + c_1 + c_2 + \cdots + c_n$ is the total Chern class. It satisfies $c(E \oplus E') = c(E)c(E')$ for any vector bundles E and E'.

$$H^\bullet(BO(n);\mathbb{Z}) = \mathbb{Z}[p_1,p_2,\dots,p_{[n/2]}]$$

where $\deg p_k = 4k$ and $[n/2]$ is the integer part of $n/2$. The class p_k is the k-th Pontrjagin class. The class $p = 1 + p_1 + p_2 + \cdots + p_{[n/2]}$ is the total Pontrjagin class which satisfies $p(E \oplus E') = p(E)p(E')$.

$$H^\bullet(BSO(2m+1);\mathbb{Z}) = \mathbb{Z}[p_1,p_2,\dots,p_m] \qquad H^\bullet(BSO(2m);\mathbb{Z}) = \mathbb{Z}[p_1,p_2,\dots,p_{m-1},e]$$

where $\deg p_k = 4k$ and $\deg e = 2m$. The class e is called the Euler class, it satisfies $e(E \oplus E') = e(E)e(E')$.

$$H^\bullet(BO(n);\mathbb{Z}_2) = \mathbb{Z}_2[w_1,\dots,w_n] \qquad H^\bullet(BSO(n);\mathbb{Z}_2) = \mathbb{Z}_2[w_2,\dots,w_n]$$

where $\deg w_k = k$ is the k-th Stiefel-Whitney class. The class $w = 1 + w_1 + w_2 + \cdots + w_n$ is the total Stiefel-Whitney class which satisfies $w(E \oplus E') = w(E)w(E')$. ♦

Example 1.5.8 (Interpretation of w_1 and w_2)
Let M be a locally compact finite dimensional manifold. M is orientable if and only if the first Stiefel-Whitney class $w_1(TM)$ of its tangent space TM is zero. If M is orientable, it admits a spin structure if and only if the second Stiefel-Whitney class $w_2(TM)$ is zero. ♦

Example 1.5.9 (Classification of complex line vector bundles)
The first Chern class of $c_1(L) \in H^2(X;\mathbb{Z})$ of a complex line vector bundle $L \to X$ is a total invariant in the space of isomorphic classes of line vector bundles over X. ♦

Example 1.5.10 (Compact connected Lie groups)
For any compact connected Lie group G, one has

$$H^{2n}(BG;\mathbb{R}) = \mathcal{P}_I^n(\mathfrak{g}) \qquad\qquad H^{2n+1}(BG;\mathbb{R}) = 0$$

where $\mathfrak{g}$ is the Lie algebra of G and $\mathcal{P}_I^\bullet(\mathfrak{g})$ is the graded algebra of invariant polynomials on $\mathfrak{g}$.

For the compact Lie groups in Example 1.5.7, these invariant polynomials are generated by the formulas:

$$\det(\lambda\mathbb{1} + \frac{i}{2\pi}X) = \lambda^n + c_1(X)\lambda^{n-1} + c_2(X)\lambda^{n-2} + \cdots + c_n(X)$$

for any $X \in \mathfrak{u}(n)$;

$$\det(\lambda\mathbb{1} - \frac{1}{2\pi}X) = \lambda^n + p_1(X)\lambda^{n-2} + p_2(X)\lambda^{n-4} + \cdots + p_m(X)\lambda^{n-2m}$$

for any $X \in \mathfrak{o}(n)$;

$$e(X) = \frac{(-1)^m}{2^{2m}\pi^m m!}\sum_{i_1,\ldots,i_{2m}} \varepsilon_{i_1 i_2 \ldots i_{2m-1} i_{2m}} X_{i_1 i_2} \ldots X_{i_{2m-1} i_{2m}}$$

for any $X \in \mathfrak{so}(2m)$, where $\varepsilon_{i_1 i_2 \ldots i_{2m-1} i_{2m}}$ is completely antisymmetric with $\varepsilon_{12\ldots 2m} = 1$. The quantity $\mathrm{Pf}(X) = (2\pi)^m e(X)$ is called the Pfaffian of X. It is a square root of the determinant. The Euler class is then associated to a very particular invariant polynomial. ♦

Example 1.5.11 (Characteristic classes through connections)
It is possible to construct characteristic classes directly using invariant polynomials in $\mathcal{P}_I^\bullet(\mathfrak{g})$. In order to do that, consider a differentiable principal fibre bundle $P \to M$ over a differential manifold with structure group G. Let us denote by $\omega \in \Omega^1(P) \otimes \mathfrak{g}$ a connection on P and Ω its curvature. Recall that ω is a covariant object for the action $\widetilde{R}_g$ of G on P by right multiplication and the adjoint action Ad on $\mathfrak{g}$: $\widetilde{R}_g^*\omega = Ad_{g^{-1}}\omega$ for any $g \in G$. Its curvature is also covariant, $\widetilde{R}_g^*\Omega = Ad_{g^{-1}}\Omega$, and satisfies the Bianchi identity $d\Omega + [\omega, \Omega] = 0$.

Let (U,ϕ) be a local trivialisation of P, where U is an open subset of M and $\phi : U \times G \to P_{|U}$ is a diffeomorphism which intertwines the actions of G on P and G. Define by $s_U(x) = \phi(x,e)$ the section which trivializes $P_{|U}$ and by $A^U = s_U^*\omega$ and $F^U = s_U^*\Omega$ the local connection 1-form and the local curvature 2-form. If (V,ψ) is a second trivialization of P, with $U \cap V \neq \varnothing$, then one has the relations

$$A^V = g_{UV}^{-1}A^U g_{UV} + g_{UV}^{-1}dg_{UV} \qquad\qquad F^V = g_{UV}^{-1}F^U g_{UV}$$

where $g_{UV} : U \cap V \to G$ is the transition function between the two trivializations. The local forms F^U satisfy to a Bianchi identity.

Let us consider p an invariant polynomial on $\mathfrak{g}$, of degree k. Then one can define $p(F^U,\dots,F^U)$ as a local $2k$-form on U by evaluating p on the values of F^U in $\mathfrak{g}$. Because p is invariant, one has $p(F^U,\dots,F^U) = p(F^V,\dots,F^V)$ so that it defines a global $2k$-form on M. Using the Bianchi identity, one can then show that its differential is zero. We then have associated to p a cohomology class in $H^{2k}(M;\mathbb{R})$, which can be shown to be independent of the choice of the connection ω. This map $\mathcal{P}_I^k(\mathfrak{g}) \to H^{2k}(M;\mathbb{R})$ is the Chern-Weil map.

This class is exactly the characteristic class given by the invariant polynomial p in the identification $H^{2n}(BG;\mathbb{R}) = \mathcal{P}_I^n(\mathfrak{g})$ in Example 1.5.10.

One does not really need to express the connection 1-form and its curvature 2-form locally on an open set of the base space M. Indeed, $p(\Omega,\dots,\Omega)$ makes sense as a $2k$-form on P. Using the properties of the curvature 2-form Ω and the invariance of the polynomial p, one can show that it is a basic form for the action of G on P, and as such, it identifies with a $2k$-form on the base space M. ♦

Proposition 1.5.12 (Decomposition principle)
Let $E_1,\dots,E_p \to X$ be p complex vector bundles. Then there exist a manifold $\mathbb{F}$ and a continuous map $\sigma : \mathbb{F} \to X$ such that the pull-backs $\sigma^* E_i \to \mathbb{F}$ are all decomposed as direct sum of complex line vector bundles, and such that the map induced in cohomology $\sigma^\sharp : H^\bullet(X) \to H^\bullet(\mathbb{F})$ is injective.

Why decompose a vector bundle in a direct sum of line vector bundles? The answer is in the following construction.

Let $R(c(E_1),\dots,c(E_p))$ be a polynomial relation in $H^\bullet(X)$ between the Chern classes of the vector bundles E_i. We would like to establish the relation $R(c(E_1),\dots,c(E_p)) = 0$ for any vector bundles over X, and for any X. Using the decomposition map $\sigma : \mathbb{F} \to X$ and the functoriality of the Chern classes (and the fact that the relation is a polynomial relation) we have

$$\sigma^\sharp(R(c(E_1),\dots,c(E_p))) = R(c(\sigma^* E_1),\dots,c(\sigma^* E_p))$$

Now, let us assume that for any base space Y and any vector bundles F_i over Y which are direct sum of line vector bundles, the relation $R(c(F_1),\dots,c(F_p)) = 0$ can be established. Then, for any E_i over X, the $F_i = \sigma^* E_i$ over $Y = \mathbb{F}$ are direct sums of line vector bundles, so that the relation is true for them. The right hand side of the relation is then zero, which implies by injectivity of $\sigma^\sharp : H^\bullet(X) \to H^\bullet(\mathbb{F})$ that the relation is also zero for the E_i's.

So, in order to establish an abstract relation between the Chern classes, it is sufficient to show it for *any* vector bundle decomposed as a direct sum of line vector bundles over *any* space.

Example 1.5.13 (Chern classes and elementary symmetric polynomials)
Let us apply the relation $c(E \oplus E') = c(E)c(E')$, where c is the total Chern class, to a direct sum of line vector bundles $E = \ell_1 \oplus \cdots \oplus \ell_n$. Then $c(E) = c(\ell_1)\cdots c(\ell_n)$. For a line vector bundle, one has $c(\ell) = 1 + c_1(\ell)$. Denote by $x_i = c_1(\ell_i)$ the first Chern classes of these line vector bundles. Then one has

$$c(E) = \prod_{i=1}^{n}(1 + x_i) = \sum_{j=0}^{n} \sigma_j(x_1,\dots,x_n)$$

where the functions σ_j are the elementary symmetric polynomials of total degree j. They are explicitly given in terms of the n (commuting) variables X_i by

$$\sigma_0(X_1,\ldots,X_n)=1 \qquad \sigma_1(X_1,\ldots,X_n)=\sum_{1\le i\le n} X_i \qquad \sigma_2(X_1,\ldots,X_n)=\sum_{1\le i<j\le n} X_iX_j$$
$$\cdots \qquad \sigma_n(X_1,\ldots,X_n)=\prod_{1\le i\le n} X_i$$

Any symmetric polynomial (resp. any formal symmetric series) in the n variables X_i can be expressed as a polynomial (resp. a formal series) in these elementary symmetric polynomials:

$$\begin{aligned}\mathbb{C}[X_1,\ldots,X_n]^{\mathfrak{S}_n} &= \{p\in\mathbb{C}[X_1,\ldots,X_n] \;/\; p(X_1,\ldots,X_n)=p(X_{\sigma^{-1}(1)},\ldots,X_{\sigma^{-1}(n)})\}\\ &=\mathbb{C}[\sigma_1,\ldots,\sigma_n]\end{aligned}$$

The previous computation shows us that the Chern classes can be written as $c_j(E)=\sigma_j(x_1,\ldots,x_n)$ when E is decomposed. If E is not decomposed, then use σ^*E over $\mathbb{F}$. ♦

Example 1.5.14 (Characteristic class associated to a symmetric polynomial)
The previous Example gives us another application of the decomposition principle, which is to construct a new characteristic class in terms of the Chern classes, but writing it down explicitly only in terms of the first Chern classes and a symmetric polynomial. Indeed, let $p(X_1,\ldots,X_n)$ be a symmetric polynomials. Then it is a polynomial of the form $R(\sigma_1,\ldots,\sigma_n)$. For any vector bundle $E\to X$ decomposed as $E=\ell_1\oplus\cdots\oplus\ell_n$, define the characteristic class $c_p(E)=p(x_1,\ldots,x_n)$ where $x_i=c_1(\ell_i)$. Then $c_p(E)=R(\sigma_1(x_1,\ldots,x_n),\ldots,\sigma_n(x_1,\ldots,x_n))=R(c_1(E),\ldots,c_n(E))$. Now, if E is not decomposed as a direct sum of line vector bundles, the last relation can be used to define, without ambiguities, the class $c_p(E)$, thanks to the decomposition principle and the functoriality of the Chern classes.

This construction can be generalised to any invariant formal series in n variables. ♦

Definition 1.5.15 (The Chern character)
Let E be a vector bundle over X. The Chern character $\mathrm{ch}(E)$ of E is defined to be the characteristic class associated to the formal series

$$p(x_1,\ldots,x_n)=e^{x_1}+\cdots+e^{x_n}=n+\sum_{i=1}^{n}x_i+\frac{1}{2}\sum_{i=1}^{n}(x_i)^2+\cdots$$

Notice that the coefficient group for the cohomology of this class is necessarily $\mathbb{Q}$, because the defining expression for $\mathrm{ch}(E)$ makes use of rational numbers. ♦

Example 1.5.16 (The invariant polynomial of the Chern character)
We saw in Example 1.5.11 that characteristic classes can be defined using a connection on the vector bundle and an invariant polynomial. The Chern character is a particular characteristic class, and its invariant polynomial (in fact an invariant formal series) is $P(X)=\mathrm{Tr}\exp(\frac{i}{2\pi}X)$, so that $\mathrm{ch}(E)=\mathrm{Tr}\circ\exp\left(\frac{iF}{2\pi}\right)$ for any local curvature 2-form of a connection on E.

As a form on the principal fibre bundle, this expression is

$$\mathrm{ch}(\omega)=\mathrm{Tr}\circ\exp\left(\frac{i\Omega}{2\pi}\right)=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{i}{2\pi}\right)^k\mathrm{Tr}(\Omega^k)$$ ♦

Proposition 1.5.17 (Product and additive properties of ch)
Using the decomposition principle, one can show that for any vector bundles E and E':

$$\mathrm{ch}(E \oplus E') = \mathrm{ch}(E) + \mathrm{ch}(E') \qquad\qquad \mathrm{ch}(E \otimes E') = \mathrm{ch}(E)\,\mathrm{ch}(E')$$

Theorem 1.5.18 (The Chern character as an isomorphism)
The Chern Character defines a natural morphism of rings $\mathrm{ch} : K^0(X) \to H^{even}(X;\mathbb{Q})$ *which induces an isomorphism*

$$\mathrm{ch} : K^0(X) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\simeq} H^{even}(X;\mathbb{Q})$$

for locally compact finite dimensional manifolds X. In that case, the Chern character can be extended to a isomorphism $\mathrm{ch} : K^1(X) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\simeq} H^{odd}(X;\mathbb{Q})$.

Example 1.5.19 (The Chern character $K^{-1}(M) \to H^{\mathrm{odd}}(M;\mathbb{Q})$)
It is possible to give an expression of the Chern character in odd degrees using connections. Let ω_0 and ω_1 be connections on the principal fibre bundle P. Then $\omega_t = \omega_0 + t(\omega_1 - \omega_0)$ is also a connection for any $t \in [0,1]$. We denote by Ω_t its curvature. One can show that the Chern-Simons form

$$\mathrm{cs}(\omega_0,\omega_1) = \int_0^1 dt\, \mathrm{Tr}\left((\omega_1 - \omega_0) \exp\left(\frac{i\Omega_t}{2\pi} \right) \right)$$

satisfies $d\,\mathrm{cs}(\omega_0,\omega_1) = \mathrm{ch}(\omega_1) - \mathrm{ch}(\omega_0)$ where $\mathrm{ch}(\omega)$ is given as in Example 1.5.16.

Let $g : M \to U(n)$ be a smooth map. Consider the trivial fibre bundle $P = M \times U(n)$, with the two connections $\omega_0 = 0$ and $\omega_1 = g^{-1}dg$. Then one defines

$$\mathrm{ch}(g) = \mathrm{cs}(0, g^{-1}dg) = \sum_{k=0}^{\infty} (-1)^k \frac{k!}{(2k+1)!} \left(\frac{i}{2\pi} \right)^{k+1} \mathrm{Tr}((g^{-1}dg)^{2k+1})$$

This defines a map from the class of g in $K^{-1}(M)$ into $H^{\mathrm{odd}}(M;\mathbb{Q})$. ♦

1.5.2 Characteristic classes and Chern character in noncommutative geometry

It is possible to construct some characteristic classes, and in particular the Chern character, using the algebraic setting of modules and differential calculi. The construction of these classes are based upon some generalisation of the construction of the Chern classes in terms of the curvature of some connection. In order to do that, one need to define the so-called noncommutative connections.

Let $(\Omega^\bullet, d)$ be a differential calculus on an associative unital algebra $\mathbf{A}$, and let $\boldsymbol{M}$ be a finite projective left module over $\mathbf{A} = \Omega^0$.

Definition 1.5.20 (Noncommutative connection)
A noncommutative connection on $\boldsymbol{M}$ for the differential calculus $(\Omega^\bullet, d)$ is linear map $\nabla : \boldsymbol{M} \to \Omega^1 \otimes_{\mathbf{A}} \boldsymbol{M}$ such that $\nabla(am) = da \otimes m + a(\nabla m)$ for any $m \in \boldsymbol{M}$ and $a \in \mathbf{A}$. ♦

Let us introduce $\widetilde{\boldsymbol{M}} = \Omega^\bullet \otimes_{\mathbf{A}} \boldsymbol{M}$ as a graded left module over $\Omega^\bullet$, and $\mathrm{End}^\bullet_\Omega(\widetilde{\boldsymbol{M}})$ the graded algebra of $\Omega^\bullet$-linear endomorphisms on $\widetilde{\boldsymbol{M}}$ (any $T \in \mathrm{End}^k_\Omega(\widetilde{\boldsymbol{M}})$ satisfies $T(\eta) \in \Omega^{m+k} \otimes_{\mathbf{A}} \boldsymbol{M}$ and $T(\omega\eta) = (-1)^{nk}\omega T(\eta)$ for any $\eta \in \Omega^m \otimes_{\mathbf{A}} \boldsymbol{M}$ and any $\omega \in \Omega^n$).

Let $\boldsymbol{M}'$ be a left module such that $\boldsymbol{M} \oplus \boldsymbol{M}' = \mathbf{A}^N$ and denote by $p : \mathbf{A}^N \to \boldsymbol{M}$ the projection and $\phi : \boldsymbol{M} \to \mathbf{A}^N$ the inclusion. Then $p\phi = \mathrm{Id}_{\boldsymbol{M}}$.

Using right multiplication on $\widetilde{\mathbf{A}^N}$, one can make the identification $\mathrm{End}^\bullet_\Omega(\widetilde{\mathbf{A}^N}) = M_N(\Omega^\bullet)$.

Proposition 1.5.21 (General properties of noncommutative connections)
Any noncommutative connection ∇ can be extended into a map of degree +1 on $\widetilde{\boldsymbol{M}}$ such that for any $m \in \boldsymbol{M}$ and $\omega \in \Omega^n$,

$$\nabla(\omega \otimes_{\mathbf{A}} m) = (d\omega) \otimes_{\mathbf{A}} m + (-1)^n \omega(\nabla m)$$

The space of connections is an affine space over $\mathrm{End}^1_\Omega(\widetilde{\boldsymbol{M}})$.

Definition 1.5.22 (Curvature of a noncommutative connection)
The curvature of ∇ is the map $\Theta = \nabla^2 = \nabla \circ \nabla$. ♦

We define the linear map $\delta : \mathrm{End}^\bullet_\Omega(\widetilde{\boldsymbol{M}}) \to \mathrm{End}^\bullet_\Omega(\widetilde{\boldsymbol{M}})$ by the relation $\delta(T) = \nabla T - (-1)^k T\nabla$, where $T \in \mathrm{End}^k_\Omega(\widetilde{\boldsymbol{M}})$. One can easily show that δ is a graded derivation of degree +1 on the graded algebra $\mathrm{End}^\bullet_\Omega(\widetilde{\boldsymbol{M}})$.

Proposition 1.5.23
One has $\Theta \in \mathrm{End}^2_\Omega(\widetilde{\boldsymbol{M}})$, $\delta(T) = \Theta T - T\Theta = [\Theta, T]_{\mathrm{gr}}$ and the Bianchi identity $\delta(\Theta) = 0$.

Example 1.5.24 (Existence of connections)
Let $e \in \mathrm{End}_{\mathbf{A}}(\mathbf{A}^N)$ be a projector, and define the left module $\boldsymbol{M} = e(\mathbf{A}^N)$. e is also a projector in $\mathrm{End}^0_\Omega(\Omega^\bullet \otimes_{\mathbf{A}} \mathbf{A}^N)$ which is naturally extended by $\Omega^\bullet$-linearity. Then, if ∇ is a connection on $\mathbf{A}^N$, the map $m \mapsto e(\nabla m)$ is a connection on $\boldsymbol{M}$.

On $\mathbf{A}^N$ there is a natural connection given by the differential map of the differential calculus: $\mathbf{A}^N \ni (a_1, \dots, a_N) \mapsto (da_1, \dots, da_n) \in \Omega^1 \otimes_{\mathbf{A}} \mathbf{A}^N = (\Omega^1)^N$. Then any finite projective module on $\mathbf{A}$ admits at least one connection. ♦

Example 1.5.25 (Direct sum of connections)
Let $(\boldsymbol{M}, \nabla^{\boldsymbol{M}})$ and $(\boldsymbol{N}, \nabla^{\boldsymbol{N}})$ be two finite projective modules over $\mathbf{A}$ for the same differential calculus. Then $\nabla : \boldsymbol{M} \oplus \boldsymbol{N} \to \Omega^1 \otimes_{\mathbf{A}} (\boldsymbol{M} \oplus \boldsymbol{N})$ defined by $\nabla(m \oplus n) = (\nabla^{\boldsymbol{M}} m) \oplus (\nabla^{\boldsymbol{N}} n)$ is a connection on $\boldsymbol{M} \oplus \boldsymbol{N}$ which we denote by $\nabla^{\boldsymbol{M}} \oplus \nabla^{\boldsymbol{N}}$. ♦

Definition 1.5.26 (Graded trace)
Let $V^\bullet$ be a graded vector space. A graded trace on $\Omega^\bullet$ with values in $V^\bullet$ is a linear morphism of degree 0, $\tau : \Omega^\bullet \to V^\bullet$, such that $\tau(\omega\eta) = (-1)^{mn}\tau(\eta\omega)$ for any $\omega \in \Omega^m$ and $\eta \in \Omega^n$. ♦

Notice that the restriction $\tau : \mathbf{A} = \Omega^0 \to V^0$ is an ordinary trace on $\mathbf{A}$.

Proposition 1.5.27 (The universal trace)
If we denote by $[\Omega^\bullet, \Omega^\bullet]_{\mathrm{gr}}$ the subspace of $\Omega^\bullet$ lineary generated by the graded commutators, then the graded vector space $\widehat{\Omega}^\bullet = \Omega^\bullet / [\Omega^\bullet, \Omega^\bullet]_{\mathrm{gr}}$ inherits the differential of $\Omega^\bullet$, which we denote by $\widehat{d}$, and the projection $\tau_\Omega : \Omega^\bullet \to \widehat{\Omega}^\bullet$ is a graded trace which commutes which the differentials.

For any graded trace $\tau : \Omega^\bullet \to V^\bullet$ there is a factorisation $\tau = \overline{\tau}\tau_\Omega$ for a morphism $\overline{\tau} : \widehat{\Omega}^\bullet \to V^\bullet$. This is why τ_Ω is called the universal trace on $\Omega^\bullet$.

Example 1.5.28 (The trace on $\mathrm{End}^\bullet_\Omega(\widetilde{\boldsymbol{M}})$)
Because of the identification $\mathrm{End}^\bullet_\Omega(\widetilde{\mathbf{A}^N}) = M_N(\Omega^\bullet)$, there is a natural trace on $\mathrm{End}^\bullet_\Omega(\widetilde{\mathbf{A}^N})$ with values in $\Omega^\bullet$ induced by the trace on the matrix algebra $M_N(\mathbb{C})$, which we denote by Tr. For any $T \in \mathrm{End}^\bullet_\Omega(\widetilde{\boldsymbol{M}})$, one has $\widehat{T} = \phi T p \in \mathrm{End}^\bullet_\Omega(\widetilde{\mathbf{A}^N})$, so that we can define $\tau_\Omega(\mathrm{Tr}(\widehat{T})) \in \widehat{\Omega}^\bullet$. This map is a graded trace which does not depend upon p, ϕ and N. The trace $\mathrm{End}^\bullet_\Omega(\widetilde{\boldsymbol{M}}) \to \widehat{\Omega}^\bullet$ will be denoted by Tr_Ω. It satisfies $\mathrm{Tr}_\Omega\, \delta = \widehat{d}\, \mathrm{Tr}_\Omega$. ♦

Definition 1.5.29 (Characteristic classes of M)
For any integer k, the cohomology class of $\mathrm{Tr}_\Omega(\Theta^k)$ in $H^{2k}(\widehat{\Omega}^\bullet, \widehat{d})$ is independent of the connection ∇. This is the k-th characteristic class of $\boldsymbol{M}$ for the differential calculus $(\Omega^\bullet, d)$. ♦

Definition 1.5.30 (The Chern character of M)
We define the Chern character $\mathrm{ch}(\boldsymbol{M}) \in H^{\mathrm{even}}(\widehat{\Omega}^\bullet, \widehat{d})$ associated to $\boldsymbol{M}$ by

$$\mathrm{ch}_k(\boldsymbol{M}) = \left[\frac{(-1)^k}{(2i\pi)^k k!}\mathrm{Tr}_\Omega(\Theta^k)\right] \in H^{2k}(\widehat{\Omega}^\bullet, \widehat{d})$$

$$\mathrm{ch}(\boldsymbol{M}) = \sum_{k\geq 0}\mathrm{ch}_k(\boldsymbol{M}) = \mathrm{Tr}_\Omega \circ \exp\left(\frac{i\Theta}{2\pi}\right) \in H^{\mathrm{even}}(\widehat{\Omega}^\bullet, \widehat{d})$$ ♦

Obviously, this definition is just an algebraic rephrasing of the expression that was given in Example 1.5.16.

Example 1.5.31 (The connection induced by a projector)
We saw in Example 1.5.24 that there is a natural connection on $\boldsymbol{M} = e(\mathbf{A}^N)$ expressed in terms of the projector $e \in \mathrm{End}_{\mathbf{A}}(\mathbf{A}^N)$. From now on, e will be identified with an element in the matrix algebra $M_N(\Omega^\bullet)$, which acts on $\mathbf{A}^N$ by multiplication on the right. One can compute explicitly the curvature of this connection using this matrix algebra, and then one finds, for any $a \in \boldsymbol{M} \subset \mathbf{A}^N$,

$$\Theta(a) = -a(de)(de)e$$

This expression can be used to express the Chern character of $\boldsymbol{M}$ in terms of the matrix e :

$$\mathrm{ch}(\boldsymbol{M}) = \sum_{k\geq 0}\left[\frac{1}{(2i\pi)^k k!}\tau_\Omega \mathrm{Tr}(e(de)^{2k})\right]$$ ♦

Proposition 1.5.32 (Additive properties of ch)
For any two finite projective left modules $\boldsymbol{M}$ and $\boldsymbol{N}$ on $\mathbf{A}$, one has

$$\mathrm{ch}(\boldsymbol{M} \oplus \boldsymbol{N}) = \mathrm{ch}(\boldsymbol{M}) + \mathrm{ch}(\boldsymbol{N})$$

Remark 1.5.33 (No product property!)
There is no product property which could satisfy this Chern character, because there is no possibility to define a tensor product of two finite projective left modules $\boldsymbol{M}$ and $\boldsymbol{N}$...

Example 1.5.34 (The geometric Chern Character)
In the case $\mathbf{A} = C^\infty(M)$ and $\Omega^\bullet = \Omega^\bullet(M)$, the de Rham differential calculus, one has $\widehat{\Omega}^\bullet = \Omega^\bullet$ because $\Omega^\bullet(M)$ is graded commutative. Then the Chern character takes its values in the even de Rham cohomology of M. By the Serre-Swan theorem in its algebraic version, Theorem 1.3.26, any finite projective module on $C^\infty(M)$ is the space of smooth sections of a vector bundle E over M. It is easy to verify that a noncommutative connection is then an ordinary connection on E, seen as a covariant derivative maps on sections. This identification uses the natural isomorphism $\Omega^\bullet \otimes_{\mathbf{A}} \boldsymbol{M} = \Omega^\bullet(M, E)$, where $\Omega^\bullet(M, E)$ is the space of differential forms on M with values in E. The curvature is then an element in $\Omega^2(M, \mathrm{End}(E)) = \Omega^2(M) \otimes_{C^\infty(M)} \mathrm{End}_{C^\infty(M)}(\Gamma(E))$, the space of 2-forms with values in the associated vector bundle $\mathrm{End}(E) = E \otimes E^*$. In this context, one has $\mathrm{End}^\bullet_\Omega(\Omega^\bullet \otimes_{\mathbf{A}} \boldsymbol{M}) = \Omega^\bullet(M, \mathrm{End}(E))$ and the trace is the ordinary trace on the fibres of $\mathrm{End}(E)$.

Using these considerations and the explicit formulas defining them, the two definitions of the Chern characters coincide.

As an exercise, one can show that the relation $\delta(\Theta) = 0$ is indeed the Bianchi identity! ♦

1.5.3 The Chern character from algebraic K-theory to periodic cyclic homology

The definition of the (algebraic) Chern character we will use rests upon the two results concerning the algebras $\mathbb{C}$ and $\mathbb{C}[z,z^{-1}]$ given in Examples 1.4.43 and 1.4.45: $HP_0(\mathbb{C})=\mathbb{C}$ and $HP_1(\mathbb{C}[z,z^{-1}])=\mathbb{C}$. Recall that the trace map Tr defined in Definition 1.4.11 induces the Morita isomorphism in periodic cyclic homology.

Let $\mathbf{A}$ be an associative unital algebra. Let $p\in M_N(\mathbf{A})$ be a projector. Then it defines a morphism of algebras $i_p:\mathbb{C}\to M_N(\mathbf{A})$ by $\lambda\mapsto\lambda p$. Indeed, $1\in\mathbb{C}$ is mapped to p, and the relation $p^2=p$ is the required compatibility with $1^2=1$. This morphism is not a morphism of unital algebras.

Let $u\in M_N(\mathbf{A})$ be an invertible element. Then it defines a morphism of algebras $i_u:\mathbb{C}[z,z^{-1}]\to M_N(\mathbf{A})$ completely given by $z\mapsto u$ and $1\mapsto\mathbb{1}_N$.

Definition 1.5.35 (The algebraic Chern character)
With the previous notations, the Chern character $[\mathrm{ch}_0(p)]\in HP_0(\mathbf{A})$ of the projector p is the image of $1\in HP_0(\mathbb{C})$ in the composite map

$$HP_0(\mathbb{C})\xrightarrow{i_{p\sharp}}HP_0(M_N(\mathbf{A}))\xrightarrow{\mathrm{Tr}_\sharp}HP_0(\mathbf{A})$$

The Chern character $[\mathrm{ch}_1(u)]\in HP_1(\mathbf{A})$ of the invertible u is the image of $1\in HP_1(\mathbb{C}[z,z^{-1}])$ in the composite map

$$HP_1(\mathbb{C}[z,z^{-1}])\xrightarrow{i_{u\sharp}}HP_1(M_N(\mathbf{A}))\xrightarrow{\mathrm{Tr}_\sharp}HP_1(\mathbf{A})$$ ♦

Proposition 1.5.36 (The Chern character on algebraic K-theory)
The class $[\mathrm{ch}_0(p)]\in HP_0(\mathbf{A})$ (resp. $[\mathrm{ch}_1(u)]\in HP_1(\mathbf{A})$) depends only on the class of p in $K_0^{alg}(\mathbf{A})$ (resp. on the class of u in $K_1^{alg}(\mathbf{A})$).

The Chern character is a map $\mathrm{ch}:K_v^{alg}(\mathbf{A})\to HP_v(\mathbf{A})$ for $v=0,1$.

Example 1.5.37 (Explicit formula for $\mathrm{ch}_0(p)$ in $\Omega^\bullet(\mathbf{A})$)
In order to give an explicit formula for the representative $\mathrm{ch}_0(p)$ of the class of the Chern character in the mixed complex $(\Omega^\bullet(\mathbf{A}),b_H,B)$, one has to explicitly write down the generator $1\in HP_0(\mathbb{C})=\mathbb{C}$. It is convenient to do that in the same mixed bicomplex $(\Omega^\bullet(\mathbb{C}),b_H,B)$. In order to make notations clear, let us denote by $e\in\mathbb{C}$ the unit element. Then one can show, using explicit formulas on b_H and B, that

$$e+\sum_{n\geq 1}(-1)^n\frac{(2n)!}{n!}\left(e-\frac{1}{2}\right)(de)^{2n}$$

is the generator of the class 1, in the total complex of the mixed complex $(\Omega^\bullet(\mathbb{C}),b_H,B)$.

Using the composite map at the level of mixed bicomplexes (i_p and Tr), one gets

$$\mathrm{ch}_0(p)=\mathrm{Tr}(p)+\sum_{n\geq 1}(-1)^n\frac{(2n)!}{n!}\mathrm{Tr}\left(\left(p-\frac{1}{2}\right)(dp)^{2n}\right)$$ ♦

Example 1.5.38 (Explicit formula for $\mathrm{ch}_1(u)$ in $\Omega_U^\bullet(\mathbf{A})$)
In the mixed bicomplex $(\Omega_U^\bullet(\mathbf{A}),b_H,B)$, we can give an explicit formula for the representative $\mathrm{ch}_1(u)$ using the following expression of the representative of $1\in HP_1(\mathbb{C}[z,z^{-1}])=\mathbb{C}$:

$$\sum_{n\geq 0}n!z^{-1}dz(dz^{-1}dz)^n$$

Then the element $\mathrm{ch}_1(u)$ takes the form

$$\mathrm{ch}_1(u) = \sum_{n\geq 0} n!\,\mathrm{Tr}\left(u^{-1}du(du^{-1}du)^n\right)$$ ♦

Remark 1.5.39 (What is really a representative of the Chern character?)
The Chern character is well defined only in the periodic cyclic homology of the algebra. But it is convenient to manipulate it as a cycle in the complex computing this homology.

But which complex to consider? Indeed, as we saw before, there are many possibilities, at least as many mixed bicomplexes that are b-quasi-isomorphic (Proposition 1.4.39). So that one can expect some representative cycles in the complexes $CC^{\mathrm{per}}_{\bullet,\bullet}(\mathbf{A})$, $(\Omega^\bullet(\mathbf{A}), b, B)$, $(\Omega^\bullet_U(\mathbf{A}), b, B)$, and even $(\Omega^\bullet_{\mathbf{A}|\mathbb{C}}, 0, d_K)$ if the algebra is a smooth commutative algebra…

The representatives given in Examples 1.5.37 and 1.5.38 are then only particular expressions. For instance, for the algebra $\mathbf{A} = \mathrm{S}V$, one can use the Kähler differential calculus, in which any element of degree $\geq \dim V$ is 0. In that case, the Chern character is represented by a finite sum of differential forms of odd or even degrees.

The expressions we gave above have the advantage that they are written in the universal differential calculi, in which all the degrees can be represented. Let us give another expression for the generator $1 \in HP_1(\mathbb{C}[z,z^{-1}])$ in the bicomplex $CC^{\mathrm{per}}_{\bullet,\bullet}(\mathbb{C}[z,z^{-1}])$. In order to do that, define the family of elements

$$\begin{aligned}
\alpha_n &= (n+1)!(z^{-1}-1)\otimes(z-1)\otimes[(z^{-1}-1)\otimes(z-1)]^{\otimes 2n} && \in \mathbb{C}[z,z^{-1}]^{\otimes 2n+2}\\
\beta_n &= (n+1)!(z-1)\otimes[(z^{-1}-1)\otimes(z-1)]^{\otimes 2n} && \in \mathbb{C}[z,z^{-1}]^{\otimes 2n+1}
\end{aligned}$$

then the representative of the generator is

$$c = \sum_{n\geq 0} \alpha_n \oplus \beta_n \in TCC^{\mathrm{per}}_1(\mathbb{C}[z,z^{-1}])$$

Using the identification $\Omega^{2n+1}(\mathbb{C}[z,z^{-1}]) = \mathbb{C}[z,z^{-1}]^{\otimes 2n+2} \oplus \mathbb{C}[z,z^{-1}]^{\otimes 2n+1}$, this generator is also directly written as a generator in the mixed bicomplex $(\Omega^\bullet(\mathbb{C}[z,z^{-1}]), b, B)$.

Finally, notice that the explicit development of the Chern character in one of the complexes mentioned above is completely determined by the lowest degrees, in which a normalisation is imposed, and the condition to be a cycle in the periodic complex. Hence this object is a very canonical one. ♦

Proposition 1.5.40 (Naturality of the Chern character)
For any short exact sequence of associative algebras $0 \longrightarrow \mathbf{I} \longrightarrow \mathbf{A} \longrightarrow \mathbf{A}/\mathbf{I} \longrightarrow 0$, *one has the commutative diagram*

$$\begin{array}{ccccccccccc}
K_1^{alg}(\mathbf{I}) & \longrightarrow & K_1^{alg}(\mathbf{A}) & \longrightarrow & K_1^{alg}(\mathbf{A}/\mathbf{I}) & \xrightarrow{\delta} & K_0^{alg}(\mathbf{I}) & \longrightarrow & K_0^{alg}(\mathbf{A}) & \longrightarrow & K_0^{alg}(\mathbf{A}/\mathbf{I}) \\
\downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} \\
HP_1(\mathbf{I}) & \longrightarrow & HP_1(\mathbf{A}) & \longrightarrow & HP_1(\mathbf{A}/\mathbf{I}) & \xrightarrow{\delta} & HP_0(\mathbf{I}) & \longrightarrow & HP_0(\mathbf{A}) & \longrightarrow & HP_0(\mathbf{A}/\mathbf{I})
\end{array} \tag{1.5.7}$$

Remark 1.5.41 (The topological case)
When the algebra is a topological algebra, one can show that the Chern character is in fact a map from the K-groups defined on topological algebras and the continuous cyclic periodic homology. Indeed, one can show that it is homotopic invariant.

Nevertheless, remember that in Remark 1.4.52 we mentioned that K-theory is well adapted to C^*-algebras and continuous functional calculus in general, but that cyclic periodic homology is only useful for topological algebras underlying some smooth structures...

If one wants to connect K-theory and cyclic periodic homology directly at the level of representative cycles, one has to consider some intermediate algebras between "algebraic" and "C^*", for instance Fréchet or locally convex algebras. In these cases, unfortunately, the K-groups are not defined using projections and unitaries, so that the interpretation of the Chern character is not at all transparent whereas it looks so clear in the algebraic version...

When the Bott periodicity takes place in K-theory, the commutative diagram (1.5.7) connects in reality the two six term exact sequences of Propositions 1.3.44 and 1.4.41. But there is a defect in this closed relation, a factor $2\pi i$ is necessary in the morphism $\delta : K_0(\mathbf{A}/\mathbf{I}) \to K_1(\mathbf{I})$ to get a commutative diagram (see [Cuntz et al., 2004]).

Remark 1.5.42 (The Chern character as an isomorphism)
In Theorem 1.5.18, we saw that the Chern character realizes an isomorphism between K-theory of topological spaces (in fact its torsion-free part) and the ordinary cohomology of the underlying topological space.

In [Solleveld, 2005], it is shown that the Chern character for topological algebras realizes an equivalent isomorphism for a large class of Fréchet algebras in the following form

$$\mathrm{ch} \otimes \mathrm{Id} : K_\bullet(\mathbf{A}) \otimes \mathbb{C} \xrightarrow{\simeq} HP_\bullet(\mathbf{A})$$

In particular, the Fréchet algebras $C^\infty(M)$ for a locally compact manifold M is in this class. ♦

Remark 1.5.43 (Chern character and cyclic cohomology)
For a Banach algebra $\mathbf{A}$, the Chern character can be realized as a pairing $K_\nu(\mathbf{A}) \times HP^\nu(\mathbf{A}) \to \mathbb{C}$, using the natural pairing between periodic cyclic cohomology and periodic cyclic homology.

Let us consider the case $\nu = 0$. In Example 1.4.53, we defined cyclic cohomology using the Connes complex. Let $\phi \in C_\lambda^{2n}(\mathbf{A})$ be a cyclic cocycle and $p \in M_N(\mathbf{A})$ a projector. Define

$$\langle [p], [\phi] \rangle = \frac{1}{(2i\pi)^n n!} \sum_{i_1,\dots,i_N} \phi(p_{i_1 i_2}, p_{i_2 i_3}, \dots, p_{i_N i_1})$$

One can show that this pairing is well defined at the level of the K_0 group and at the level of $HC^{2n}(\mathbf{A})$, and that it satisfies $\langle [p], S[\phi] \rangle = \langle [p], [\phi] \rangle$. Because the periodic cyclic cohomology group $HP^0(\mathbf{A})$ can be defined as an inductive limit through the periodic operator S on the $HC^{2n}(\mathbf{A})$ spaces, the previous pairing is indeed a pairing between $K_0(\mathbf{A})$ and $HP^0(\mathbf{A})$.

Using this construction, no extra structure is required. One then recovers that the Chern character is indeed a canonical object in the context of K-theory and periodic cyclic (co)homology. There is a similar expression for $\nu = 1$. ♦

Remark 1.5.44 (Comparing Chern characters)
The expressions in Examples 1.5.16, 1.5.19, 1.5.37 and 1.5.38 look very similar. But there are differences which are important to be noted. In order to make them clear, we will call "geometric Chern character" the expressions given in Examples 1.5.16, 1.5.19 (and also 1.5.31), and "algebraic Chern character" the expressions given in Examples 1.5.37 and 1.5.38.

First, the spaces on which these Chern characters are expressed as "differential forms" are not the same in the two situations. In the geometric one, it is the de Rham differential calculus. In the algebraic one, it is one of the universal differential calculi.

In order to compare them, one has to take into account a situation in which they both make sense, the case of the algebra $\mathbf{A} = C^\infty(M)$ for instance. In that case, one knows that the identification of the Hochschild homology with de Rham forms can be expressed as in Example 1.4.23. Using these expressions, one easily show that the following squares are commutative, where the vertical isomorphisms concerning K-theories express the Serre-Swan Theorem 1.3.26,

$$\begin{array}{ccc} K_0(C^\infty(M)) & \xrightarrow{\mathrm{ch}_{\mathrm{alg}}} & HP_0^{\mathrm{cont}}(C^\infty(M)) \\ \downarrow \simeq & & \simeq \downarrow \phi \\ K^0(M) & \xrightarrow{\mathrm{ch}_{\mathrm{geom}}} & H^{\mathrm{even}}(M;\mathbb{Q}) \end{array} \qquad \begin{array}{ccc} K_1(C^\infty(M)) & \xrightarrow{\mathrm{ch}_{\mathrm{alg}}} & HP_1^{\mathrm{cont}}(C^\infty(M)) \\ \downarrow \simeq & & \simeq \downarrow \phi \\ K^{-1}(M) & \xrightarrow{\mathrm{ch}_{\mathrm{geom}}} & H^{\mathrm{odd}}(M;\mathbb{Q}) \end{array}$$

This explains the extra factors used in the isomorphism ϕ in Example 1.4.23. Notice that the two definitions of the Chern characters are constrained: the geometric case is normalised in such a way that it is a ring morphism, the algebraic one is expressed as an infinite cycle in cyclic periodic homology, so that all the terms are normalised by the first one. The only degree of freedom in this square is the isomorphism of vector spaces ϕ (and fortunately not an isomorphism of algebras since $HP_v(\mathbf{A})$ has no natural structure of algebra). ♦

1.6 Conclusion

There cannot be any conclusion to a subject that is still full of vivacity! Thousands of mathematicians try everyday to conquest some new landmarks in this extraordinary vast and rich world. In this lecture, only some selected aspects of this theory have been presented. For instance, no mention has been made about "noncommutative measure theory", in which von Neumann algebras play the role of C^*-algebras for measurable spaces.

We have seen that one can reasonably manipulate "noncommutative topological spaces" using the K-theory of C^*-algebras. One can convince oneself that differentiable structures are available in the heart of cyclic homology.

Nevertheless this research project is facing a challenge which have not yet been solved: what is the noncommutative counterpart of smooth functions? Does it exist? We have made it clear that cyclic homology sees some smooth structures, but the right category of "noncommutative smooth algebras" has not yet been identified. Some paths have been investigated. For instance, Cuntz has considered m-algebras (see [Fragoulopoulou, 2005]), some kind of locally convex algebras, on which he succeed to enrich K-theory and cyclic homology (see [Cuntz et al., 2004]). But what is still missing is a Gelfand-Neumark theorem for smooth functions.

2

$SU(n)$-principal fiber bundles and noncommutative geometry

2.1 Introduction

The geometry of fiber bundles is now widely used in the physical literature, mainly through the concept of connections, which are interpreted as gauge fields in particle physics. It is worth to recall why the structure of these gauge theories leads to this mathematical identification. The main points which connect these two concepts are the common expression for gauge transformations and the field strength of the gauge fields recognized as the curvature of the connection.

Since the introduction of the Higgs mechanics, some attempts have been made to understand its geometrical origin in a same satisfactory and elegant way as the gauge fields. The reduction of some higher dimensional gauge field theories to some more "conventional" dimensions has been proposed to reproduce the Higgs part of some models.

Nevertheless, one of the more convincing constructions from which Higgs fields emerged naturally and without the need to perform some dimensional reduction of some extra *ad-hoc* afterward arbitrary distortion of the model, was firstly exposed in [Dubois-Violette et al., 1990a], and highly popularized in subsequent work by A. Connes in its noncommutative standard model (see [Chamseddine and Connes, 2008] for a review of the recent developments in this direction). What the pioneer work by Dubois-Violette, Kerner and Madore revealed is that the Higgs fields can be identified with the purely noncommutative part of a noncommutative connection on an noncommutative algebra "containing" an ordinary smooth algebra of functions over a manifold and a purely noncommutative algebra.

The algebra used there is the tensor product $C^\infty(\mathcal{M}) \otimes M_n$ of smooth functions on some manifold $\mathcal{M}$ and the matrix algebra of size n. This "trivial" product does not reveal the richness of this approach when some more intricate algebra is involved. In this review, we consider the algebra of endomorphisms of a $SU(n)$-vector bundle. This algebra reduces to the previous situation for a trivial vector bundle. This non triviality gives rise to some elegant and powerful constructions we exposed in a series of previous papers, and to some results nowhere published before.

The first part deals with some reviews of the ordinary geometry of fiber bundles and connections. We think this is useful to fix notations, but also to highlight what the noncommutative differential geometry defined in the following extends from these constructions.

We then define the general settings of our noncommutative geometry, which is based on derivations. The notion of noncommutative connections is exposed, and some important examples are then given to better understand the general situation.

The algebra we are interested in is then introduced as the algebra of endomorphisms of a $SU(n)$-vector bundle. We show how it is related to ordinary geometry, and how ordinary connections plays an essential role to study its noncommutative geometry.

The noncommutative connections on this algebra are then studied, and here we recall why the purely noncommutative part can be identified with Higgs fields.

Then it is shown that this algebra is indeed related, through the algebraic notion of Cartan operations on a bigger algebra, to the geometry of the $SU(n)$-principal fiber bundle underlying the geometry of the $SU(n)$-vector bundle.

Some considerations about the cohomology behind the endomorphism algebra are then exposed, in particular a new construction of the Chern classes of the $SU(n)$-vector bundle which are obtained from a short exact sequence of Lie algebras of derivations.

The last section is concerned with the symmetric reduction of noncommutative connections, which generalizes a lot of previous works about symmetric reduction of ordinary connections.

2.2 A brief review of ordinary fiber bundle theory

The noncommutative geometry we will consider in the following contains, and relies in an essential way to the ordinary differential geometry of the $SU(n)$-fiber bundles theory. This section is devoted to some aspects of this differential geometry. Its aim is to fix notations but also to present some constructions which will be generalized or completed by the noncommutative geometry introduced later on.

2.2.1 Principal and associated fiber bundles

Let $\mathcal{M}$ be a smooth manifold and G a Lie group. Denote by $G \longrightarrow \mathcal{P} \xrightarrow{\pi} \mathcal{M}$ a (locally trivial) principal fiber bundle for the right action of G on $\mathcal{P}$, denoted by $p \mapsto p{\cdot}g = \widetilde{R}_g p$.

For any $p \in \mathcal{P}$, one defines $V_p = \operatorname{Ker}(T_p\pi : T_p\mathcal{P} \to T_{\pi(p)}\mathcal{M})$, the vertical subspace of $T_p\mathcal{P}$. For any $X \in \mathfrak{g}$, let

$$X^v(p) = \left(\frac{d}{dt} p \cdot \exp(tX) \right)_{t=0}$$

Then $V_p = \{X^v(p) / X \in \mathfrak{g}\}$ and one has $\widetilde{R}_{g*} V_p = V_{p\cdot g}$.

This defines vertical vector fields over $\mathcal{P}$ and horizontal differential forms, which are differential forms on $\mathcal{P}$ which vanish when one of its arguments is vertical.

Let (U, ϕ) be a local trivialisation of $\mathcal{P}$ over an open subset $U \subset \mathcal{M}$, which means that there exists a isomorphism $\phi : U \times G \xrightarrow{\simeq} \pi^{-1}(U)$ such that $\pi(\phi(x,h)) = x$ and $\phi(x, hg) = \phi(x,h){\cdot}g$ for any $x \in U$ and $g, h \in G$.

If (U_i, ϕ_i) and (U_j, ϕ_j) are two local trivialisations such that $U_i \cap U_j \neq \varnothing$, then there exists a differentiable map $g_{ij} : U_i \cap U_j \to G$ such that, if $\phi_i(x, h_i) = \phi_j(x, h_j)$ for $h_i, h_j \in G$, then $h_i = g_{ij}(x) h_j$ for any $x \in U_i \cap U_j$. The g_{ij} are called the transition functions for the system $\{(U_i, \phi_i)\}_i$ of local trivialisations. They satisfy $g_{ij}(x) = g_{ji}^{-1}(x)$ for any $x \in U_i \cap U_j$ and the cocycle condition $g_{ij}(x) g_{jk}(x) g_{ki}(x) = e$ for any $x \in U_i \cap U_j \cap U_k \neq \varnothing$.

Now, let $\mathcal{F}$ be a manifold on which G acts on the left: $\varphi \mapsto \ell_g \varphi$. On the manifold $\mathcal{P} \times \mathcal{F}$ we define the right action $(p, \varphi) \mapsto (p{\cdot}g, \ell_{g^{-1}} \varphi)$, and we denote by $\mathcal{E} = (\mathcal{P} \times \mathcal{F})/G$ the orbit space for this action. This is the associated fiber bundle to $\mathcal{P}$ for the couple $(\mathcal{F}, \ell)$. It is denoted by $\mathcal{E} = \mathcal{P} \times_\ell \mathcal{F}$, and $[p,\varphi] \in \mathcal{E}$ is the projection of (p,φ) in the quotient $\mathcal{P}\times\mathcal{F} \to (\mathcal{P}\times\mathcal{F})/G$. By construction, one has $[p\cdot g,\varphi] = [p,\ell_g\varphi]$.

A (smooth) section of $\mathcal{E}$ is a (differentiable) map $s : \mathcal{M} \to \mathcal{E}$ such that $\pi\circ s(x) = x$ for any $x \in \mathcal{M}$. We denote by $\Gamma(\mathcal{E})$ the space of differentiable sections of $\mathcal{E}$.

The main point of this construction is the fact that one can show that $\Gamma(\mathcal{E})$ identifies with the space $\mathcal{F}_G(\mathcal{P},\mathcal{F}) = \{\Phi : \mathcal{P} \to \mathcal{F} \ / \ \Phi(p\cdot g) = \ell_{g^{-1}}\Phi(p)\}$ of G-equivariant maps $\mathcal{P} \to \mathcal{F}$. This result will be useful later on.

Let (U,ϕ) be a local trivialisation of $\mathcal{P}$ over U. Then the smooth map $s_U : U \to \pi^{-1}(U)$ given by $s_U(x) = \phi(x,e)$ is a local section of $\mathcal{P}$.

Any section s of $\mathcal{P}$ is locally given by a local map $h : U \to G$ such that $s(x) = s_U(x)\cdot h(x) = \phi(x,h(x))$. We call s_U a local gauge over U for $\mathcal{P}$, because it is used as a local reference in $\mathcal{P}$ to decompose sections on $\mathcal{P}$.

In the same way, any section s of $\mathcal{E}$ is locally given by a local map $\varphi : U \to \mathcal{F}$ such that $s(x) = [s_U(x),\varphi(x)]$. This means that the local gauge s_U can also be used to decompose sections of any associated fiber bundle.

Let (U_i,ϕ_i) and (U_j,ϕ_j) be two local trivialisations such that $U_i \cap U_j \neq \varnothing$. Then one has

$$s_j(x) = \phi_j(x,e) = \phi_i(x,g_{ij}(x)) = \phi_i(x,e)\cdot g_{ij}(x) = s_i(x)\cdot g_{ij}(x)$$

so that on $\mathcal{P}$, if $s(x) = s_i(x)\cdot h_i(x) = s_j(x)\cdot h_j(x)$, then

$$h_i(x) = g_{ij}(x)h_j(x)$$

On $\mathcal{E}$, if $s(x) = [s_i(x),\varphi_i(x)] = [s_j(x),\varphi_j(x)]$ for $x \in U_i \cap U_j \neq \varnothing$, then

$$\varphi_i(x) = \ell_{g_{ij}(x)}\varphi_j(x)$$

These are the transformation laws for the local decompositions of sections in $\mathcal{P}$ and $\mathcal{E}$.

Let F be a vector space and ℓ a representation (linear action) of G. In that case, the associated fiber bundle $\mathcal{E}$ for the couple (F,ℓ) is called a vector bundle. The space of smooth sections $\Gamma(\mathcal{E})$ is then a $C^\infty(\mathcal{M})$-module for the pointwise multiplication: $f(x)s(x)$ for any $f \in C^\infty(\mathcal{M})$, $s \in \Gamma(\mathcal{E})$ and $x \in \mathcal{M}$.

Moreover, if $\mathcal{E}$ and $\mathcal{E}'$ are vector bundles, then $\mathcal{E}^*$ (dual), $\mathcal{E}\oplus\mathcal{E}'$ (Whitney sum), $\mathcal{E}\otimes\mathcal{E}'$ (tensor product) and $\bigwedge^\bullet\mathcal{E}$ (exterior product) are defined. They are associated respectively to (F^*,ℓ^*), $(F\oplus F',\ell\oplus\ell')$, $(F\otimes F',\ell\otimes\ell')$ and $(\bigwedge^\bullet F,\bigwedge\ell)$.

Here are now the main examples which will be at the root of the noncommutative geometry introduced in the following, and will permits one to make connections between this noncommutative geometry and the pure geometrical context.

Example 2.2.1 (Tangent and cotangent spaces)
The tangent space $T\mathcal{M} \to \mathcal{M}$, and the cotangent space $T^*\mathcal{M} \to \mathcal{M}$ are canonical vector bundles over $\mathcal{M}$.

The space $\Gamma(T\mathcal{M})$ will be denoted by $\Gamma(\mathcal{M})$. It is the $C^\infty(\mathcal{M})$-module of vector fields on $\mathcal{M}$. It is also a Lie algebra for the bracket $[X,Y]\cdot f = X\cdot Y\cdot f - Y\cdot X\cdot f$ for any $f \in C^\infty(\mathcal{M})$.

One the other hand, by duality, $\Gamma(T^*\mathcal{M}) = \Omega^1(\mathcal{M})$ is the space of 1-forms on $\mathcal{M}$. Extending this construction, $\Gamma(\bigwedge^\bullet T^*\mathcal{M}) = \Omega^\bullet(\mathcal{M})$ is the algebra of (de Rham) differential forms on $\mathcal{M}$.

If $\mathcal{E}$ is a vector bundle over $\mathcal{M}$, then $\Gamma(T^*\mathcal{M}\otimes\mathcal{E}) = \Omega^\bullet(\mathcal{M},\mathcal{E})$ is the space of differential forms with values in the vector bundle $\mathcal{E}$, which means that for any $\omega \in \Omega^p(\mathcal{M},\mathcal{E})$, $X_i \in \Gamma(\mathcal{M})$ and $x \in \mathcal{M}$, $\omega(X_1,\dots,X_p)(x) \in \mathcal{E}_x$. ♦

Example 2.2.2 (The endomorphism bundle)
Consider the case where F is a finite dimensional vector space. Then $\mathcal{E}^* \otimes \mathcal{E}$ is associated to $\mathcal{P}$ for the couple $(F^* \otimes F, \ell^* \otimes \ell)$.

One has the identification $F^* \otimes F \simeq \text{End}(F)$ by $(\alpha \otimes \varphi)(\varphi') = \alpha(\varphi')\varphi$, where $\text{End}(F)$ is the space of endomorphisms of F.

The vector bundle $\text{End}(\mathcal{E}) = \mathcal{E}^* \otimes \mathcal{E}$ is called the endomorphism fiber bundle of $\mathcal{E}$.

There is a natural pairing $\Gamma(\mathcal{E}^*) \otimes \Gamma(\mathcal{E}) \to C^\infty(\mathcal{M})$ denoted by $x \mapsto \langle \alpha(x), s(x) \rangle$. One can show that $\Gamma(\mathcal{E}^* \otimes \mathcal{E}) = \Gamma(\text{End}(\mathcal{E}))$ is an algebra, which identifies with $\Gamma(\mathcal{E}^*) \otimes_{C^\infty(\mathcal{M})} \Gamma(\mathcal{E})$ and with the space of $C^\infty(\mathcal{M})$-module maps $\Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ by $(\alpha \otimes s)(s')(x) = \langle \alpha(x), s'(x) \rangle s(x)$. ♦

Example 2.2.3 (The gauge group and its Lie algebra)
The group G acts on itself by conjugaison: $\alpha_g(h) = ghg^{-1}$. The associated fiber bundle $\mathcal{P} \times_\alpha G$ has G as fiber but is not a principal fiber bundle. In particular, this fiber bundle has a global section, defined in any trivialisation by $x \mapsto e$, where $e \in G$ is the unit element. But one knows that the existence of a global section on $\mathcal{P}$ is equivalent to $\mathcal{P}$ being trivial.

Denote by $\mathcal{G} = \Gamma(\mathcal{P} \times_\alpha G)$ the space of smooth sections. It is a group, called the gauge group of $\mathcal{P}$: it is the sub-group of vertical automorphisms in $\text{Aut}(\mathcal{P})$, the group of all automorphisms of $\mathcal{P}$. Indeed, any element in $\mathcal{G}$ is also a G-equivariant map $\Phi : \mathcal{P} \to G$, which defines the vertical automorphism $p \mapsto p \cdot \Phi(p)$. The compatibility condition is ensured by the G-equivariance: $p \cdot g \mapsto (p \cdot g) \cdot \Phi(p \cdot g) = (p \cdot g) \cdot (g^{-1} \Phi(p) g) = (p \cdot \Phi(p)) \cdot g$.

By construction, one has the short exact sequence of groups:

$$1 \longrightarrow \mathcal{G} \longrightarrow \text{Aut}(\mathcal{P}) \longrightarrow \text{Aut}(\mathcal{M}) \longrightarrow 1$$

G acts on the vector space $\mathfrak{g}$ by the adjoint action Ad. Denote by $\text{Ad}\mathcal{P} = \mathcal{P} \times_{\text{Ad}} \mathfrak{g}$ the associated vector bundle. The vector space $\Gamma(\text{Ad}\mathcal{P})$ is the Lie algebra of the gauge group $\mathcal{G}$, denoted hereafter by $\text{Lie}\mathcal{G}$. ♦

2.2.2 Connections

Let $G \longrightarrow \mathcal{P} \xrightarrow{\pi} \mathcal{M}$ be a principal fiber bundle, and let $\mathcal{E} \to \mathcal{M}$ be an associated vector bundle. There is at least three ways to define a connection in this context:

Geometrical definition: A connection on $\mathcal{P}$ is a smooth distribution H in $T\mathcal{P}$ such that for any $p \in \mathcal{P}$ and $g \in G$:

$$T_p\mathcal{P} = V_p \oplus H_p \quad \text{and} \quad \widetilde{R}_{g*} H_p = H_{p \cdot g}$$

This defines horizontal vector fields and vertical differential forms (forms which vanish when one of its arguments is horizontal).

One gets the geometrical notion of horizontal lifting of vector fields on $\mathcal{M}$, which we denote by $\Gamma(\mathcal{M}) \ni X \mapsto X^h \in \Gamma(\mathcal{P})$.

Algebraic definition: A connection on $\mathcal{P}$ is a 1-form on $\mathcal{P}$ taking values in the Lie algebra $\mathfrak{g}$, $\omega \in \Omega^1(\mathcal{P}) \otimes \mathfrak{g}$, such that for any $g \in G$ and $X \in \mathfrak{g}$:

$$\widetilde{R}_g^* \omega = \text{Ad}_{g^{-1}} \omega \text{ (equivariance)} \quad \text{and} \quad \omega(X^v) = X \text{ (vertical condition)}$$

The associated horizontal distribution is $H_p = \text{Ker}\, \omega_{|p}$.

Analytic definition: A connection on $\mathcal{E}$ is a linear map $\nabla^{\mathcal{E}}_X : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ defined for any $X \in \Gamma(\mathcal{M})$, such that for any $f \in C^\infty(\mathcal{M})$, $s \in \Gamma(\mathcal{E})$, $X, Y \in \Gamma(\mathcal{M})$:

$$\nabla^{\mathcal{E}}_X(fs) = (X{\cdot}f)s + f\nabla^{\mathcal{E}}_X s \qquad \nabla^{\mathcal{E}}_{fX} s = f\nabla^{\mathcal{E}}_X s \qquad \nabla^{\mathcal{E}}_{X+Y} s = \nabla^{\mathcal{E}}_X s + \nabla^{\mathcal{E}}_Y s$$

If $s \in \Gamma(\mathcal{E})$ corresponds to $\Phi \in \mathcal{F}_G(\mathcal{P}, \mathcal{F})$, then $\nabla^{\mathcal{E}}_X s$ corresponds to $X^h{\cdot}\Phi$.

The equivariance of the connection 1-form ω implies the relation

$$L_{X^v}\omega + [X, \omega] = 0$$

for any $X \in \mathfrak{g}$.

For each of these three definitions, the curvature of a connection can be introduced:

Geometrical definition: There exists a geometrical interpretation of the curvature as the obstruction to the closure of horizontal lifts of "infinitesimal" closed paths on $\mathcal{M}$.

Let $\gamma : [0,1] \mapsto \mathcal{M}$ be a closed path and let $p \in \mathcal{P}$. There exists a unique path $\gamma^h : [0,1] \mapsto \mathcal{P}$ such that $\gamma^h(0) = p$ and $\dot{\gamma}^h(t) \in H_{\gamma^h(t)}$ for any $t \in [0,1]$. γ^h is a horizontal lifting of γ. One has $\gamma^h(1) \neq \gamma^h(0) = p$ *a priori*, but they are in the same fiber, so that the deficiency is in G.

When the path γ is shrunk to an infinitesimal path, the deficiency is an element in $\mathfrak{g}$ which depends only on $\dot{\gamma}(0)$ and $\dot{\gamma}(1)$. This is the curvature.

Algebraic definition: The curvature is the equivariant horizontal 2-form $\Omega \in \Omega^2(\mathcal{P}) \otimes \mathfrak{g}$ defined for any $\mathcal{X}, \mathcal{Y} \in \Gamma(\mathcal{P})$ by

$$\Omega(\mathcal{X}, \mathcal{Y}) = \mathrm{d}\omega(\mathcal{X}, \mathcal{Y}) + [\omega(\mathcal{X}), \omega(\mathcal{Y})]$$

It satisfies the Bianchi identity

$$\mathrm{d}\Omega + [\omega, \Omega] = 0$$

Analytic definition: Given $\nabla^{\mathcal{E}}_X : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$, the curvature $R^{\mathcal{E}}(X, Y)$ is the map defined for any $X, Y \in \Gamma(\mathcal{M})$ by

$$R^{\mathcal{E}}(X, Y) = \nabla^{\mathcal{E}}_X \nabla^{\mathcal{E}}_Y - \nabla^{\mathcal{E}}_Y \nabla^{\mathcal{E}}_X - \nabla^{\mathcal{E}}_{[X,Y]} : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$$

The remarkable fact is that this particular combinaison is a $C^\infty(\mathcal{M})$-module map.

One can connect these definitions by the following relations. Let η be the representation of $\mathfrak{g}$ on F induced by the representation ℓ of G. If $s \in \Gamma(\mathcal{E})$ corresponds to $\Phi \in \mathcal{F}_G(\mathcal{P}, \mathcal{F})$, then $R^{\mathcal{E}}(X, Y)s$ corresponds to $\eta(\Omega(\mathcal{X}, \mathcal{Y})){\cdot}\Phi$ for any $\mathcal{X}, \mathcal{Y}$ such that $\pi_*\mathcal{X} = X$ and $\pi_*\mathcal{Y} = Y$.

Let $\omega \in \Omega^1(\mathcal{P}) \otimes \mathfrak{g}$ be a connection 1-form on $\mathcal{P}$, and Ω its curvature. Let (U, ϕ) be local trivialisation of $\mathcal{P}$, and s its associated local section.

One can define the local expression of the connection and the curvature in this trivialisation as the pull-back of ω and Ω by $s : U \to \mathcal{P}$:

$$A = s^*\omega \in \Omega^1(U) \otimes \mathfrak{g} \qquad F = s^*\Omega \in \Omega^2(U) \otimes \mathfrak{g}$$

If (U_i, ϕ_i) and (U_j, ϕ_j) are two local trivialisations, on $U_i \cap U_j \neq \varnothing$ one has the well-known relations

$$A_j = g_{ij}^{-1} A_i g_{ij} + g_{ij}^{-1} \mathrm{d}g_{ij} \qquad F_j = g_{ij}^{-1} F_i g_{ij} \tag{2.2.1}$$

Globally on $\mathcal{P}$	**Locally on $\mathcal{M}$**
$\omega\in\Omega^1(\mathcal{P})\otimes\mathfrak{g}$, equivariant, vertical condition.	Family of local 1-forms $\{A_i\}_i$, $A_i\in\Omega^1(U_i)\otimes\mathfrak{g}$, satisfying gluing non homogeneous relations.
$\Omega\in\Omega^2(\mathcal{P})\otimes\mathfrak{g}$, equivariant and horizontal.	Family of local 2-forms $\{F_i\}_i$, $F_i\in\Omega^2(U_i)\otimes\mathfrak{g}$, satisfying gluing homogeneous relations.

Table 2.1: The two ordinary constructions of the connections and curvature, the global one on $\mathcal{P}$ and the local one on $\mathcal{M}$.

with obvious notations. A family of 1-forms $\{A_i\}_i$ satisfying these gluing relations defines a connection 1-form on $\mathcal{P}$. This is (too) often used in the physical literature as a possible definition of a connection and its curvature.

Remark 2.2.4 (Intermediate construction)
We summarize in Table 2.1 the two common ways to introduce a connection as differential objects, either as a global 1-form on $\mathcal{P}$ or as a family of local 1-forms on $\mathcal{M}$.

It is well known that, using the homogeneous gluing relations for the F_i's, or using the equivariant and horizontal property of Ω, one can show that the curvature is also a section of the associated vector bundle $\bigwedge^2 T^*\mathcal{M}\otimes \mathrm{Ad}\mathcal{P}$, *i.e.* a global 2-forms on $\mathcal{M}$ with values in the vector bundle $\mathrm{Ad}\mathcal{P}=\mathcal{P}\times_{\mathrm{Ad}}\mathfrak{g}$. We denote by $\mathbb{F}\in\Omega^2(\mathcal{M},\mathrm{Ad}\mathcal{P})$ this 2-form.

Because of the inhomogeneous gluing relations for the A_i's, the connection cannot be the section of such an "intermediate" construction between forms on $\mathcal{P}$ and local forms on the U_i's.

Let us mention here that in the noncommutative geometry introduced in the following, this intermediate construction is possible also for the connection 1-form. See Remark 2.4.9. ♦

2.2.3 Gauge transformations

We saw that the gauge group $\mathcal{G}=\Gamma(\mathcal{P}\times_\alpha G)$ acts on $\mathcal{P}$. To any $a\in\mathcal{G}$ one can associate a G-equivariant map $\Phi:\mathcal{P}\to G$. The corresponding vertical diffeomorphism $\mathcal{P}\to\mathcal{P}$ defined by a is also denoted by a

Let $\omega\in\Omega^1(\mathcal{P})\otimes\mathfrak{g}$ be a connection on $\mathcal{P}$. Then one can show that the pull-back $a^*\omega$ is also a connection and $a^*\Omega$ is its curvature. Explicitly, one can establish the formulae $a^*\omega=\Phi^{-1}\omega\Phi+\Phi^{-1}\mathrm{d}\Phi$ and $a^*\Omega=\Phi^{-1}\Omega\Phi$, which look very similar to (2.2.1), but are not the same: here we perform some active transformation on the space of connections while in (2.2.1) we look at the same connection in different trivialisations. This is the difference between active and passive transformation laws.

In order to get the action of the Lie algebra of the gauge group, consider $\Phi=\exp(\xi)$ with $\xi:\mathcal{P}\to\mathfrak{g}$, G-equivariant, so that ξ defines an element in $\mathrm{Lie}\mathcal{G}=\Gamma(\mathrm{Ad}\mathcal{P})$. Then the infinitesimal action on connections and curvatures take the form:

$$\omega\mapsto \mathrm{d}\xi+[\omega,\xi] \qquad\qquad \Omega\mapsto[\Omega,\xi]$$

2.3 Derivation-based noncommutative geometry

In this section, we introduce the algebraic context in which the noncommutative geometry we are interested in is constructed. The differential calculus we consider here has been introduced in [Dubois-Violette, 1988] and has been exposed and studied for various algebras, for instance in [Dubois-Violette et al., 1990b], [Dubois-Violette et al., 1990a], [Masson, 1996], [Masson, 1995], [Dubois-Violette and Masson, 1998],[Masson, 1999], [Dubois-Violette and Michor, 1994], [Dubois-Violette and Michor, 1996], [Dubois-Violette and Michor, 1997].

2.3.1 Derivation-based differential calculus

Let $\mathbf{A}$ be an associative algebra with unit $\mathbb{1}$. Denote by $\mathcal{Z}(\mathbf{A})$ the center of $\mathbf{A}$.

Definition 2.3.1 (Vector space of derivations of A)
The vector space of derivations of $\mathbf{A}$ is the space $\mathrm{Der}(\mathbf{A}) = \{\mathfrak{X} : \mathbf{A} \to \mathbf{A} \ / \ \mathfrak{X} \text{ linear}, \mathfrak{X}(ab) = \mathfrak{X}(a)b + a\mathfrak{X}(b), \forall a, b \in \mathbf{A}\}$ ♦

The essential properties of this space are contained in the following:

Proposition 2.3.2 (Structure of Der(A))
$\mathrm{Der}(\mathbf{A})$ *is a Lie algebra for the bracket* $[\mathfrak{X},\mathfrak{Y}]a = \mathfrak{X}\mathfrak{Y}a - \mathfrak{Y}\mathfrak{X}a$ $(\forall \mathfrak{X},\mathfrak{Y} \in \mathrm{Der}(\mathbf{A}))$ *and a* $\mathcal{Z}(\mathbf{A})$*-module for the product* $(f\mathfrak{X})a = f(\mathfrak{X}a)$ $(\forall f \in \mathcal{Z}(\mathbf{A}), \forall \mathfrak{X} \in \mathrm{Der}(\mathbf{A}))$.

The subspace $\mathrm{Int}(\mathbf{A}) = \{\mathrm{ad}_a : b \mapsto [a,b] \ / \ a \in \mathbf{A}\} \subset \mathrm{Der}(\mathbf{A})$, *called the vector space of inner derivations, is a Lie ideal and a* $\mathcal{Z}(\mathbf{A})$*-submodule.*

With $\mathrm{Out}(\mathbf{A}) = \mathrm{Der}(\mathbf{A})/\mathrm{Int}(\mathbf{A})$, *there is a short exact sequence of Lie algebras and* $\mathcal{Z}(\mathbf{A})$*-modules*

$$0 \longrightarrow \mathrm{Int}(\mathbf{A}) \longrightarrow \mathrm{Der}(\mathbf{A}) \longrightarrow \mathrm{Out}(\mathbf{A}) \longrightarrow 0 \tag{2.3.2}$$

In case $\mathbf{A}$ has an involution $a \mapsto a^*$, one can define real derivations:

Definition 2.3.3 (Real derivations for involutive algebras)
If $\mathbf{A}$ is an involutive algebra, the derivation $\mathfrak{X} \in \mathrm{Der}(\mathbf{A})$ is real if $(\mathfrak{X}a)^* = \mathfrak{X}a^*$ for any $a \in \mathbf{A}$. We denote by $\mathrm{Der}_{\mathbb{R}}(\mathbf{A})$ the space of real derivations. ♦

Definition 2.3.4 (The graded differential algebra $\underline{\Omega}^\bullet_{\mathrm{Der}}(\mathbf{A})$)
Let $\underline{\Omega}^n_{\mathrm{Der}}(\mathbf{A})$ be the set of $\mathcal{Z}(\mathbf{A})$-multilinear antisymmetric maps from $\mathrm{Der}(\mathbf{A})^n$ to $\mathbf{A}$, with $\underline{\Omega}^0_{\mathrm{Der}}(\mathbf{A}) = \mathbf{A}$, and let

$$\underline{\Omega}^\bullet_{\mathrm{Der}}(\mathbf{A}) = \bigoplus_{n\geq 0} \underline{\Omega}^n_{\mathrm{Der}}(\mathbf{A})$$

We introduce on $\underline{\Omega}^\bullet_{\mathrm{Der}}(\mathbf{A})$ a structure of $\mathbb{N}$-graded differential algebra using the product

$$(\omega\eta)(\mathfrak{X}_1, \dots, \mathfrak{X}_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in \mathfrak{S}_{p+q}} (-1)^{\mathrm{sign}(\sigma)} \omega(\mathfrak{X}_{\sigma(1)}, \dots, \mathfrak{X}_{\sigma(p)}) \eta(\mathfrak{X}_{\sigma(p+1)}, \dots, \mathfrak{X}_{\sigma(p+q)})$$

and using the differential d (of degree 1) defined by the Koszul formula

$$\begin{aligned} \mathrm{d}\omega(\mathfrak{X}_1, \dots, \mathfrak{X}_{n+1}) = & \sum_{i=1}^{n+1} (-1)^{i+1} \mathfrak{X}_i \omega(\mathfrak{X}_1, \dots \overset{i}{\vee} \dots, \mathfrak{X}_{n+1}) \\ & + \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} \omega([\mathfrak{X}_i, \mathfrak{X}_j], \dots \overset{i}{\vee} \dots \overset{j}{\vee} \dots, \mathfrak{X}_{n+1}) \end{aligned}$$ ♦

Definition 2.3.5 (The graded differential algebra $\Omega^\bullet_{\mathrm{Der}}(\mathbf{A})$)
Denote by $\Omega^\bullet_{\mathrm{Der}}(\mathbf{A}) \subset \underline{\Omega}^\bullet_{\mathrm{Der}}(\mathbf{A})$ the sub differential graded algebra generated in degree 0 by $\mathbf{A}$. ♦

Notice that by definition, every element in $\Omega^n_{\text{Der}}(\mathbf{A})$ is a sum of terms of the form $a_0 \text{d}a_1 \cdots \text{d}a_n$ for $a_0, \dots, a_n \in \mathbf{A}$.

The previous definitions are motivated by the following important example which shows that these definitions are correct generalisations of the space of ordinary differential forms on a manifold:

Example 2.3.6 (The algebra $\mathbf{A} = C^\infty(\mathcal{M})$)
Let $\mathcal{M}$ be a smooth manifold and let $\mathbf{A} = C^\infty(\mathcal{M})$. The center of this algebra is $\mathbf{A}$ itself: $\mathcal{Z}(\mathbf{A}) = C^\infty(\mathcal{M})$. The Lie algebra of derivations is exactly the Lie algebra of smooth vector fields on $\mathcal{M}$: $\text{Der}(\mathbf{A}) = \Gamma(\mathcal{M})$. In that case, there is no inner derivations, $\text{Int}(\mathbf{A}) = 0$, so that $\text{Out}(\mathbf{A}) = \Gamma(\mathcal{M})$.

The two graded differential algebras coincide with the graded differential algebra of de Rham forms on $\mathcal{M}$: $\Omega^\bullet_{\text{Der}}(\mathbf{A}) = \underline{\Omega}^\bullet_{\text{Der}}(\mathbf{A}) = \Omega^\bullet(\mathcal{M})$ ♦

In the previous definitions of the graded differential calculi, one is not bounded to consider the entire Lie algebra of derivations:

Definition 2.3.7 (Restricted derivation-based differential calculus)
Let $\mathfrak{g} \subset \text{Der}(\mathbf{A})$ be a sub Lie algebra and a sub $\mathcal{Z}(\mathbf{A})$-module. The restricted derivation-based differential calculus $\underline{\Omega}^\bullet_{\mathfrak{g}}(\mathbf{A})$ associated to $\mathfrak{g}$ is defined as the set of $\mathcal{Z}(\mathbf{A})$-multilinear antisymmetric maps from $\mathfrak{g}^n$ to $\mathbf{A}$ for $n \geq 0$, using the previous formulae for the product and the differential. ♦

Now, let $\mathfrak{g}$ be any Lie subalgebra of $\text{Der}(\mathbf{A})$. Then $\mathfrak{g}$ defines a natural operation in the sense of H. Cartan on the graded differential algebra $(\underline{\Omega}^\bullet_{\text{Der}}(\mathbf{A}), \text{d})$. The interior product is the graded derivation of degree -1 on $\underline{\Omega}^\bullet_{\text{Der}}(\mathbf{A})$ defined by

$$i_{\mathfrak{X}} : \underline{\Omega}^n_{\text{Der}}(\mathbf{A}) \to \underline{\Omega}^{n-1}_{\text{Der}}(\mathbf{A}) \qquad (i_{\mathfrak{X}}\omega)(\mathfrak{X}_1, \dots, \mathfrak{X}_{n-1}) = \omega(\mathfrak{X}, \mathfrak{X}_1, \dots, \mathfrak{X}_{n-1})$$

$\forall \mathfrak{X} \in \mathfrak{g}$, $\forall \omega \in \underline{\Omega}^n_{\text{Der}}(\mathbf{A})$ and $\forall \mathfrak{X}_i \in \text{Der}(\mathbf{A})$. By definition, $i_{\mathfrak{X}}$ is 0 on $\underline{\Omega}^0_{\text{Der}}(\mathbf{A}) = \mathbf{A}$.

The associated Lie derivative is the graded derivation of degree 0 on $\underline{\Omega}^\bullet_{\text{Der}}(\mathbf{A})$ given by

$$L_{\mathfrak{X}} = i_{\mathfrak{X}}\text{d} + \text{d}i_{\mathfrak{X}} : \underline{\Omega}^n_{\text{Der}}(\mathbf{A}) \to \underline{\Omega}^n_{\text{Der}}(\mathbf{A})$$

One can easily verify the relations defining a Cartan operation:

$$i_{\mathfrak{X}} i_{\mathfrak{Y}} + i_{\mathfrak{Y}} i_{\mathfrak{X}} = 0 \qquad L_{\mathfrak{X}} i_{\mathfrak{Y}} - i_{\mathfrak{Y}} L_{\mathfrak{X}} = i_{[\mathfrak{X},\mathfrak{Y}]}$$
$$L_{\mathfrak{X}} L_{\mathfrak{Y}} - L_{\mathfrak{Y}} L_{\mathfrak{X}} = L_{[\mathfrak{X},\mathfrak{Y}]} \qquad L_{\mathfrak{X}}\text{d} - \text{d}L_{\mathfrak{X}} = 0$$

One can then associate to this operation the following subspaces of $\underline{\Omega}^\bullet_{\text{Der}}(\mathbf{A})$:

- The horizontal subspace is the kernel of all the $i_{\mathfrak{X}}$ for $\mathfrak{X} \in \mathfrak{g}$. This is a graded algebra.
- The invariant subspace is the kernel of all the $L_{\mathfrak{X}}$ for $\mathfrak{X} \in \mathfrak{g}$. This is a graded differential algebra.
- The basic subspace is the kernel of all the $i_{\mathfrak{X}}$ and $L_{\mathfrak{X}}$ for $\mathfrak{X} \in \mathfrak{g}$. This is a graded differential algebra.

For instance, $\mathfrak{g} = \text{Int}(\mathbf{A})$ defines such an operation.

2.3.2 Noncommutative connections and their properties

Noncommutative connections play a central role in noncommutative differential geometry. They are all based on some generalisation of what we called the analytic definition of ordinary connections, where one replaces the C^∞-module of sections of a vector bundle by a more general (finitely projective) module over the algebra. Various definitions has been proposed, for instance to take into account some bimodule structures. Here we only consider right $\mathbf{A}$-modules.

Definitions and general properties

Let $\boldsymbol{M}$ be a right $\mathbf{A}$-module.

Definition 2.3.8 (Noncommutative connections, curvature)
A noncommutative connection on $\boldsymbol{M}$ for the differential calculus based on derivations is a linear map $\widehat{\nabla}_{\mathfrak{X}} : \boldsymbol{M} \to \boldsymbol{M}$, defined for any $\mathfrak{X} \in \mathrm{Der}(\mathbf{A})$, such that $\forall \mathfrak{X}, \mathfrak{Y} \in \mathrm{Der}(\mathbf{A})$, $\forall a \in \mathbf{A}$, $\forall m \in \boldsymbol{M}$, $\forall f \in \mathcal{Z}(\mathbf{A})$:

$$\widehat{\nabla}_{\mathfrak{X}}(ma) = m(\mathfrak{X}a) + (\widehat{\nabla}_{\mathfrak{X}}m)a, \qquad \widehat{\nabla}_{f\mathfrak{X}}m = f\widehat{\nabla}_{\mathfrak{X}}m, \qquad \widehat{\nabla}_{\mathfrak{X}+\mathfrak{Y}}m = \widehat{\nabla}_{\mathfrak{X}}m + \widehat{\nabla}_{\mathfrak{Y}}m$$

The curvature of $\widehat{\nabla}$ is the linear map $\hat{R}(\mathfrak{X}, \mathfrak{Y}) : \boldsymbol{M} \to \boldsymbol{M}$ defined for any $\mathfrak{X}, \mathfrak{Y} \in \mathrm{Der}(\mathbf{A})$ by

$$\hat{R}(\mathfrak{X}, \mathfrak{Y})m = [\widehat{\nabla}_{\mathfrak{X}}, \widehat{\nabla}_{\mathfrak{Y}}]m - \widehat{\nabla}_{[\mathfrak{X},\mathfrak{Y}]}m$$ ♦

This definition is an adaptation to the derivation-based noncommutative calculus of the ordinary (analytic) definition of connections. Notice that we have to make use of the center $\mathcal{Z}(\mathbf{A})$ of the algebra $\mathbf{A}$ for one of the above relations, which means that we have to differentiate the respective roles of the algebra and of its center.

Proposition 2.3.9 (General properties)
The space of connections is an affine space modeled over the vector space $\mathsf{Hom}^{\mathbf{A}}(\boldsymbol{M}, \boldsymbol{M} \otimes_{\mathbf{A}} \underline{\Omega}^1_{\mathrm{Der}}(\mathbf{A}))$ *(right* $\mathbf{A}$*-module morphisms from* $\boldsymbol{M}$ *to* $\boldsymbol{M} \otimes_{\mathbf{A}} \underline{\Omega}^1_{\mathrm{Der}}(\mathbf{A})$*).*
$\hat{R}(\mathfrak{X}, \mathfrak{Y}) : \boldsymbol{M} \to \boldsymbol{M}$ *is a right* $\mathbf{A}$*-module morphism.*

Definition 2.3.10 (The gauge group)
The gauge group of $\boldsymbol{M}$ is the group of automorphisms of $\boldsymbol{M}$ as a right $\mathbf{A}$-module. ♦

Proposition 2.3.11 (Gauge transformations)
For any Φ *in the gauge group of* $\boldsymbol{M}$ *and any noncommutative connection* $\widehat{\nabla}$*, the map* $\widehat{\nabla}^{\Phi}_{\mathfrak{X}} = \Phi^{-1} \circ \widehat{\nabla}_{\mathfrak{X}} \circ \Phi : \boldsymbol{M} \to \boldsymbol{M}$ *is a noncommutative connection.*
This defines the action of the gauge group on the space of noncommutative connections.

Suppose now that $\mathbf{A}$ is an involutive algebra and let as before $\boldsymbol{M}$ be a right $\mathbf{A}$-module.

Definition 2.3.12 (Hermitean structure, compatible noncommutative connections)
A Hermitean structure on $\boldsymbol{M}$ is a sesquilinear form $\langle -, - \rangle : \boldsymbol{M} \times \boldsymbol{M} \to \mathbf{A}$ such that, $\forall m_1, m_2 \in \boldsymbol{M}$, $\forall a_1, a_2 \in \mathbf{A}$,

$$\langle m_1, m_2 \rangle^* = \langle m_2, m_1 \rangle \qquad \langle m_1 a_1, m_2 a_2 \rangle = a_1^* \langle m_1, m_2 \rangle a_2$$

A noncommutative connection $\widehat{\nabla}$ is compatible with $\langle -, - \rangle$ if, $\forall m_1, m_2 \in \boldsymbol{M}$, $\forall \mathfrak{X} \in \mathrm{Der}_{\mathbb{R}}(\mathbf{A})$,

$$\mathfrak{X}\langle m_1, m_2 \rangle = \langle \widehat{\nabla}_{\mathfrak{X}} m_1, m_2 \rangle + \langle m_1, \widehat{\nabla}_{\mathfrak{X}} m_2 \rangle$$ ♦

Definition 2.3.13 ("Unitary" gauge transformations)
An element Φ in the gauge group is compatible with the Hermitean structure if, for any $m_1, m_2 \in \boldsymbol{M}$, one has $\langle \Phi(m_1), \Phi(m_2) \rangle = \langle m_1, m_2 \rangle$. In that case, we refer to such a gauge transformation as a "unitary" gauge transformation. ♦

Lemma 2.3.14
The space of compatible noncommutative connections with $\langle -,- \rangle$ is stable under "unitary" gauge transformations.

The right A-module $M = \mathbf{A}$

As a special case of the previous general situation, we consider the right $\mathbf{A}$-module $\boldsymbol{M} = \mathbf{A}$. Let $\widehat{\nabla}_{\mathfrak{X}} : \mathbf{A} \to \mathbf{A}$ be a noncommutative connection.

Proposition 2.3.15 (Noncommutative connections on $M = \mathbf{A}$)
The noncommutative connection $\widehat{\nabla}$ is completely determined by $\widehat{\nabla}_{\mathfrak{X}} \mathbb{1} = \omega(\mathfrak{X})$, with $\omega \in \underline{\Omega}^1_{\text{Der}}(\mathbf{A})$, by the relation

$$\widehat{\nabla}_{\mathfrak{X}} a = \mathfrak{X} a + \omega(\mathfrak{X}) a$$

The curvature of $\widehat{\nabla}$ is the multiplication on the left on $\mathbf{A}$ by the noncommutative 2-form

$$\Omega(\mathfrak{X}, \mathfrak{Y}) = \mathrm{d}\omega(\mathfrak{X}, \mathfrak{Y}) + [\omega(\mathfrak{X}), \omega(\mathfrak{Y})]$$

The gauge group is identified with the invertible elements $g \in \mathbf{A}$ by $\Phi_g(a) = ga$ and the gauge transformations on $\widehat{\nabla}$ take the following form on ω and Ω:

$$\omega \mapsto \omega^g = g^{-1} \omega g + g^{-1} \mathrm{d}g \qquad\qquad \Omega \mapsto \Omega^g = g^{-1} \Omega g$$

$\widehat{\nabla}^0_{\mathfrak{X}}$, defined by $a \mapsto \mathfrak{X} a$, is a noncommutative connection on $\mathbf{A}$.

The gauge transformations on the noncommutative forms ω and Ω are clearly of the same nature as the one encountered in ordinary differential geometry. Nevertheless, the relations are different: the differential operator is the noncommutative differential here.

In the particular case when $\mathbf{A}$ is involutive, one can define a canonical Hermitean structure on $\boldsymbol{M}$ by $\langle a, b \rangle = a^* b$. Then, $U(\mathbf{A}) = \{ u \in \mathbf{A} \ / \ u^* u = u u^* = \mathbb{1} \}$, the group of unitary elements of $\mathbf{A}$, identifies with the unitary gauge group.

Let us stress the following important point.

Remark 2.3.16 (Vector space *versus* gauge transformations)
We saw that the space of noncommutative connections is an affine space, but here it looks like the vector space $\underline{\Omega}^1_{\text{Der}}(\mathbf{A})$. In fact, one can show that gauge transformations are not compatible with this linear structure:

$$\begin{aligned}(\lambda_1 \omega_1 + \lambda_2 \omega_2)^u &= u^{-1}(\lambda_1 \omega_1 + \lambda_2 \omega_2) u + u^{-1} \mathrm{d}u \\ \lambda_1 \omega_1^u + \lambda_2 \omega_2^u &= \lambda_1 (u^{-1} \omega_1 u + u^{-1} \mathrm{d}u) + \lambda_2 (u^{-1} \omega_2 u + u^{-1} \mathrm{d}u)\end{aligned}$$

are not equal except for $\lambda_1 + \lambda_2 = 1$. ♦

The following proposition applies in some important examples:

Proposition 2.3.17 (Canonical gauge invariant noncommutative connection)
If there exists a noncommutative 1*-form* $\xi \in \underline{\Omega}^1_{\text{Der}}(\mathbf{A})$ *such that* $\text{d}a = [\xi, a]$ *for any* $a \in \mathbf{A}$*, then the canonical noncommutative connection defined by* $\widehat{\nabla}^{-\xi}_{\mathfrak{X}} a = \mathfrak{X}a - \xi(\mathfrak{X})a$ *can be written as* $\widehat{\nabla}^{-\xi}_{\mathfrak{X}} a = -a\xi(\mathfrak{X})$*.*

Moreover, this canonical noncommutative connection is gauge invariant.

PROOF One has $\mathfrak{X}a = [\xi(\mathfrak{X}), a]$, so that $\widehat{\nabla}^{-\xi}_{\mathfrak{X}} a = [\xi(\mathfrak{X}), a] - \xi(\mathfrak{X})a = -a\xi(\mathfrak{X})$.

Let $u \in U(\mathbf{A})$ be a unitary gauge transformation. Its action on the noncommutative 1-form $-\xi$ is $(-\xi)^u = -u^{-1}\xi u + u^{-1}\text{d}u = u^{-1}(-\xi u + [\xi, u]) = u^{-1}(-u\xi) = -\xi$, which shows that this noncommutative connection is indeed gauge invariant. ■

As can be immediately seen, this situation can't occur in the commutative case (ordinary differential geometry) because for any 1-form ξ, one has $[\xi, a] = 0$. Below, we will encounter a situation where such a noncommutative 1-form exists, in the context of the algebra $M_n(\mathbb{C})$ of complex matrices. An other important example where such an invariant noncommutative connection makes its appearance is the Moyal algebra. These two examples share in common that they only have inner derivations. They are highly noncommutative situation in this respect, even if the Moyal algebra can be considered as a deformation of some commutative algebra of ordinary smooth functions.

The right A-module $\boldsymbol{M} = \mathbf{A}^N$

As an other special case of right $\mathbf{A}$-modules, we consider now the case where the right $\mathbf{A}$-module is $\boldsymbol{M} = \mathbf{A}^N$. Denote by $e_i = (0, \dots, \mathbb{1}, \dots, 0)$, for $i = 1, \dots, N$, a canonical basis of the right module $\mathbf{A}^N$. We look at $m = e_i a^i \in \boldsymbol{M}$ as a column vector for the a^i's, so that we use some matrix product notations. We also use the notation $\mathfrak{X}m = e_i(\mathfrak{X}a^i)$ for any derivation $\mathfrak{X}$ of $\mathbf{A}$.

Let $\widehat{\nabla}_{\mathfrak{X}} : \mathbf{A}^N \to \mathbf{A}^N$ be a noncommutative connection.

Proposition 2.3.18 (Noncommutative connections on $\boldsymbol{M} = \mathbf{A}^N$)
The noncommutative connection $\widehat{\nabla}$ *is completely determined by* N^2 *noncommutative* 1*-forms* $\omega^j_i \in \underline{\Omega}^1_{\text{Der}}(\mathbf{A})$ *defined by* $\widehat{\nabla}_{\mathfrak{X}} e_i = e_j \omega^j_i(\mathfrak{X})$*, through the relation* $\widehat{\nabla}_{\mathfrak{X}} m = \mathfrak{X}m + \omega(\mathfrak{X})m$*, with* $\omega = (\omega^j_i) \in M_N(\underline{\Omega}^1_{\text{Der}}(\mathbf{A}))$*.*

The curvature of $\widehat{\nabla}$ *is the multiplication on the left on* $\mathbf{A}^N$ *by the matrix of noncommutative* 2*-forms* $\Omega = \text{d}\omega + [\omega, \omega] \in M_N(\underline{\Omega}^2_{\text{Der}}(\mathbf{A}))$*.*

The gauge group of $\mathbf{A}^N$ *is* $GL_N(\mathbf{A})$ *(invertibles in* $M_N(\mathbf{A})$*), which acts by left (matrix) multiplication. The gauge transformations take the forms* $\omega^g = g^{-1}\omega g + g^{-1}\text{d}g$ *and* $\Omega^g = g^{-1}\Omega g$ *in matrix notations.*

$\widehat{\nabla}^0_{\mathfrak{X}}$*, defined by* $m \mapsto \mathfrak{X}m$*, is a noncommutative connection on* $\mathbf{A}^N$*.*

In the particular case when $\mathbf{A}$ is involutive, the natural Hermitean structure on $\boldsymbol{M}$ is defined by $\langle (a^i), (b^j) \rangle = \sum_{i=1}^N (a^i)^* b^i$. Then, $U_N(\mathbf{A}) = \{u \in M_N(\mathbf{A}) \ / \ u^*u = uu^* = \mathbb{1}_N\}$, the group of unitary elements of $M_N(\mathbf{A})$, is the unitary gauge group.

The projective finitely generated right A-modules

From the Serre-Swan theorem, one knows that any vector bundle on a smooth manifold $\mathcal{M}$ is characterised by its space of smooth sections as a projective finitely generated right module (p.f.g.m.) over $C^\infty(\mathcal{M})$. The natural generalisation of vector bundles in noncommutative geometry is then taken to be the projective finitely generated right $\mathbf{A}$-modules.

Let $\boldsymbol{M}$ be such a projective finitely generated right $\mathbf{A}$-modules. $\boldsymbol{M}$ is a direct summand in $\mathbf{A}^N$, so that there exists a projection $p \in M_N(\mathbf{A})$ such that $\boldsymbol{M} = p\mathbf{A}^N$.

Proposition 2.3.19 (Noncommutative connections on p.f.g.m.)
If $\widehat{\nabla}$ is a noncommutative connection on the right $\mathbf{A}$-module $\mathbf{A}^N$, then $m \mapsto p\widehat{\nabla}_{\mathfrak{X}} m$ defines a noncommutative connection on $\boldsymbol{M}$, where $m \in \boldsymbol{M} \subset \mathbf{A}^N$.

The curvature of the noncommutative connection obtained this way from the canonical noncommutative connection $\widehat{\nabla}^0_{\mathfrak{X}}$ of Proposition 2.3.18, is the multiplication on the left on $\boldsymbol{M} \subset \mathbf{A}^N$ by the matrix of noncommutative 2-forms $p\mathrm{d}p\mathrm{d}p$.

Example 2.3.20 (The algebra $\mathbf{A} = C^\infty(\mathcal{M})$)
We saw that the noncommutative derivation-based differential calculus is the ordinary de Rham calculus. Using the equivalence given in the Serre-Swan theorem, the definitions of (ordinary) connections and of noncommutative connections coincide. ♦

2.3.3 Two important examples

The algebra $\mathbf{A} = M_n(\mathbb{C}) = M_n$

Let us consider the case $\mathbf{A} = M_n(\mathbb{C}) = M_n$, the finite dimensional algebra of $n \times n$ complex matrices. This is an involutive algebra for the adjointness of matrices.

First, we summarize the general properties of its derivation-based differential calculus, which is described in [Dubois-Violette, 1988], [Dubois-Violette et al., 1990b] and [Masson, 1995].

Proposition 2.3.21 (General properties of the differential calculus)
One has the following results:

- $\mathcal{Z}(M_n) = \mathbb{C}$.
- $\mathrm{Der}(M_n) = \mathrm{Int}(M_n) \simeq \mathfrak{sl}_n = \mathfrak{sl}(n, \mathbb{C})$ *(traceless matrices). The explicit isomorphism associates to any $\gamma \in \mathfrak{sl}_n(\mathbb{C})$ the derivation $\mathrm{ad}_\gamma : a \mapsto [\gamma, a]$.*

  $\mathrm{Der}_{\mathbb{R}}(M_n) = \mathfrak{su}(n)$ *and* $\mathrm{Out}(M_n) = 0$.
- $\underline{\Omega}^\bullet_{\mathrm{Der}}(M_n) = \Omega^\bullet_{\mathrm{Der}}(M_n) \simeq M_n \otimes \bigwedge^\bullet \mathfrak{sl}_n^*$, *with the differential d' coming from the differential of the differential complex of the Lie algebra $\mathfrak{sl}_n$ represented on M_n by the adjoint representation (commutator).*
- *There exits a canonical noncommutative 1-form $i\theta \in \Omega^1_{\mathrm{Der}}(M_n)$ such that for any $\gamma \in M_n(\mathbb{C})$*

  $$i\theta(\mathrm{ad}_\gamma) = \gamma - \tfrac{1}{n}\mathrm{Tr}(\gamma)\mathbb{1}$$

  This noncommutative 1-form $i\theta$ makes the explicit isomorphism $\mathrm{Int}(M_n(\mathbb{C})) \xrightarrow{\simeq} \mathfrak{sl}_n$.
- *$i\theta$ satisfies the relation $\mathrm{d}'(i\theta) - (i\theta)^2 = 0$. This makes $i\theta$ look very much like the Maurer-Cartan form in the geometry of Lie groups (here $SL_n(\mathbb{C})$).*
- *For any $a \in M_n$, one has $\mathrm{d}'a = [i\theta, a] \in \Omega^1_{\mathrm{Der}}(M_n)$. This relation is no more true in higher degrees.*

Let us now introduce a particular basis of this algebra, which permits one to perform explicit computations. Denote by $\{E_k\}_{k=1,\dots,n^2-1}$ a basis for $\mathfrak{sl}_n$ of hermitean matrices. Then, it defines a basis for the Lie algebra $\mathrm{Der}(M_n) \simeq \mathfrak{sl}_n$ through the n^2-1 derivations $\partial_k = ad_{iE_k}$, which are real derivations. Adjoining the unit $\mathbb{1}$ to the E_k's, one gets a basis for M_n.

Let us define the θ^ℓ's in $\mathfrak{sl}_n^*$ by duality: $\theta^\ell(\partial_k) = \delta_k^\ell$. Then $\{\theta^\ell\}$ is a basis of 1-forms in $\bigwedge^\bullet \mathfrak{sl}_n^*$. By definition, they anticommute: $\theta^\ell \theta^k = -\theta^k \theta^\ell$ in this exterior algebra.

Define the structure constants by $[E_k, E_\ell] = -iC_{k\ell}^m E_m$. Then one can show that the differential d' takes the explicit form:

$$\mathrm{d}'\mathbb{1} = 0 \qquad \mathrm{d}'E_k = -C_{k\ell}^m E_m \theta^\ell \qquad \mathrm{d}'\theta^k = -\frac{1}{2} C_{\ell m}^k \theta^\ell \theta^m$$

The noncommutative 1-form $i\theta$ can be written as $i\theta = iE_k\theta^k \in M_n \otimes \bigwedge^1 \mathfrak{sl}_n^*$. It is obviously independent of the chosen basis.

Proposition 2.3.22 (The cohomology of the differential calculus)
The cohomology of the differential algebra $(\Omega_{\mathrm{Der}}^\bullet(M_n), \mathrm{d}')$ *is*

$$H^\bullet(\Omega_{\mathrm{Der}}^\bullet(M_n), d') = \mathcal{I}(\textstyle\bigwedge^\bullet \mathfrak{sl}_n^*)$$

the algebra of invariant elements for the natural Lie derivative.

Recall that the algebra $\mathcal{I}(\bigwedge^\bullet \mathfrak{sl}_n^*)$ *is the graded commutative algebra generated by elements* c_{2r-1}^n *in degree* $2r-1$ *for* $r \in \{2, 3, \dots, n\}$.

Let us introduce the symmetric matrix $g_{k\ell} = \frac{1}{n}\mathrm{Tr}(E_k E_\ell)$. Then the $g_{k\ell}$'s define a natural metric (scalar product) on $\mathrm{Der}(M_n)$ with the relation $g(\partial_k, \partial_\ell) = g_{k\ell}$.

Now, one can show that every differential form of maximal degree $\omega \in \Omega_{\mathrm{Der}}^{n^2-1}(M_n)$ can be written uniquely in the form

$$\omega = a\sqrt{|g|}\theta^1 \cdots \theta^{n^2-1}$$

where $a \in M_n$ and where $|g|$ is the determinant of the matrix $(g_{k\ell})$.

Definition 2.3.23 (Noncommutative integration)
One defines a noncommutative integration

$$\int_{\mathrm{n.c.}} : \Omega_{\mathrm{Der}}^\bullet(M_n) \to \mathbb{C}$$

by $\int_{\mathrm{n.c.}} \omega = \frac{1}{n}\mathrm{Tr}(a)$ if $\omega \in \Omega_{\mathrm{Der}}^{n^2-1}(M_n)$ written as above, and 0 otherwise.

This integration satisfies the closure relation

$$\int_{\mathrm{n.c.}} \mathrm{d}'\omega = 0$$ ♦

Let us now consider the right **A**-module $\boldsymbol{M} = \mathbf{A}$.

The noncommutative 1-form $-i\theta$ defines a canonical noncommutative connection by the relation $\widehat{\nabla}_{\mathfrak{X}}^{-i\theta} a = \mathfrak{X}a - i\theta(\mathfrak{X})a$ for any $a \in \mathbf{A}$.

Proposition 2.3.24 (Properties of $\widehat{\nabla}^{-i\theta}$)
For any $a \in M_n$ *and* $\mathfrak{X} = \mathrm{ad}_\gamma \in \mathrm{Der}(M_n)$ *(with* $\mathrm{Tr}\,\gamma = 0$*), one has*

$$\widehat{\nabla}_{\mathfrak{X}}^{-i\theta} a = -ai\theta(\mathfrak{X}) = -a\gamma$$

$\widehat{\nabla}^{-i\theta}$ *is gauge invariant.*

The curvature of the noncommutative connection $\widehat{\nabla}^{-i\theta}$ *is zero.*

PROOF This is a consequence of the existence of the canonical gauge invariant noncommutative connection implied by the relation $d'a = [i\theta, a]$ (Proposition 2.3.17).

The curvature is the noncommutative 2-form $\Omega(\mathfrak{X},\mathfrak{Y}) = d'(-i\theta)(\mathfrak{X},\mathfrak{Y}) + [(-i\theta)(\mathfrak{X}),(-i\theta)(\mathfrak{Y})] = -(d'i\theta(\mathfrak{X},\mathfrak{Y}) - (i\theta)^2(\mathfrak{X},\mathfrak{Y})) = 0$. ∎

Let us now consider the right $\mathbf{A}$-module $M = M_{r,n}$, the vector space of $r \times n$ complex matrices with the obvious right module structure and the Hermitean structure $\langle m_1, m_2\rangle = m_1^* m_2 \in M_n$.

Proposition 2.3.25 ($\widehat{\nabla}^{-i\theta}$, flat noncommutative connections)
The noncommutative connection $\widehat{\nabla}^{-i\theta}_{\mathfrak{X}} m = -mi\theta(\mathfrak{X})$ is well defined, it is compatible with the Hermitean structure and its curvature is zero.

Any noncommutative connection can be written $\widehat{\nabla}_{\mathfrak{X}} a = \widehat{\nabla}^{-i\theta}_{\mathfrak{X}} a + A(\mathfrak{X})a$ for $A = A_k\theta^k$ with $A_k \in M_r$. The curvature of $\widehat{\nabla}$ is the multiplication on the left by the M_r-valued noncommutative 2-form

$$F = \frac{1}{2}([A_k, A_\ell] - C^m_{k\ell}A_m)\theta^k\theta^\ell$$

This curvature vanishes if and only if $A : \mathfrak{sl}_n \to M_r$ is a representation of the Lie algebra $\mathfrak{sl}_n$.

Two flat connections are in the same gauge orbit if and only if the corresponding Lie algebra representations are equivalent.

For the proof, we refer to [Dubois-Violette et al., 1990b].

The algebra $\mathbf{A} = C^\infty(\mathcal{M}) \otimes M_n$

As a second important example, we consider now the mixed of the two algebras $C^\infty(\mathcal{M})$ and $M_n(\mathbb{C})$ studied before, in the form of matrix valued functions on a smooth manifold $\mathcal{M}$ ($\dim \mathcal{M} = m$).

The derivation-based differential calculus for this tensor product algebra was first considered in [Dubois-Violette et al., 1990a]:

Proposition 2.3.26 (General properties of the differential calculus)
One has the following results:

- $\mathcal{Z}(\mathbf{A}) = C^\infty(\mathcal{M})$.
- $\mathrm{Der}(\mathbf{A}) = [\mathrm{Der}(C^\infty(\mathcal{M})) \otimes \mathbb{1}] \oplus [C^\infty(\mathcal{M}) \otimes \mathrm{Der}(M_n)] = \Gamma(\mathcal{M}) \oplus [C^\infty(\mathcal{M}) \otimes \mathfrak{sl}_n]$ *as Lie algebras and $C^\infty(\mathcal{M})$-modules. In the following, we will use the notations: $\mathfrak{X} = X + \mathrm{ad}_\gamma$ with $X \in \Gamma(\mathcal{M})$ and $\gamma \in C^\infty(\mathcal{M}) \otimes \mathfrak{sl}_n = \mathbf{A}_0$ (traceless elements in $\mathbf{A}$).*

  One can identify $\mathrm{Int}(\mathbf{A}) = \mathbf{A}_0$ *and* $\mathrm{Out}(\mathbf{A}) = \Gamma(\mathcal{M})$.
- $\underline{\Omega}^\bullet_{\mathrm{Der}}(\mathbf{A}) = \Omega^\bullet_{\mathrm{Der}}(\mathbf{A}) = \Omega^\bullet(\mathcal{M}) \otimes \Omega^\bullet_{\mathrm{Der}}(M_n)$ *with the differential* $\widehat{d} = d + d'$, *where* d *is the de Rham differential and* d' *is the differential introduced in the previous example.*
- *The noncommutative 1-form $i\theta$ is defined as $i\theta(X + \mathrm{ad}_\gamma) = \gamma$. It splits the short exact sequence of Lie algebras and $C^\infty(\mathcal{M})$-modules*

$$0 \longrightarrow \mathbf{A}_0 \underset{\curvearrowleft i\theta}{\longrightarrow} \mathrm{Der}(\mathbf{A}) \longrightarrow \Gamma(\mathcal{M}) \longrightarrow 0 \qquad (2.3.3)$$

- *Noncommutative integration is a well-defined map of differential complexes*

$$\int_{n.c.} : \Omega^\bullet_{\mathrm{Der}}(\mathbf{A}) \to \Omega^{\bullet-(n^2-1)}(\mathcal{M}) \qquad \int_{n.c.} \widehat{d}\omega = d\int_{n.c.} \omega$$

Using a metric h on $\mathcal{M}$ and the metric $g_{k\ell} = \frac{1}{n}\mathrm{Tr}(E_k E_\ell)$ on the matrix part, one can define a metric on $\mathrm{Der}(\mathbf{A})$ as follows,

$$\widehat{g}(X+\mathrm{ad}_\gamma, Y+\mathrm{ad}_\eta) = h(X,Y) + \tfrac{1}{m^2} g(\gamma\eta)$$

where m is a positive constant which measures the relative "weight" of the two "spaces". In physical natural units, it has the dimension of a mass.

Consider now the right $\mathbf{A}$-module $\boldsymbol{M} = \mathbf{A}$. As for the algebra M_n, the noncommutative 1-form $-i\theta$ defines a canonical noncommutative connection by the relation $\widehat{\nabla}^{-i\theta}_{\mathfrak{X}} a = \mathfrak{X}a - i\theta(\mathfrak{X})a$ for any $a \in \mathbf{A}$.

Proposition 2.3.27 (Properties of $\widehat{\nabla}^{-i\theta}$)
For any $a \in \mathbf{A}$ and $\mathfrak{X} = X + \mathrm{ad}_\gamma \in \mathrm{Der}(\mathbf{A})$, one has $\widehat{\nabla}^{-i\theta}_{\mathfrak{X}} a = X \cdot a - a\gamma$.

The curvature of the noncommutative connection $\widehat{\nabla}^{-i\theta}$ is zero.

The gauge transformed connection $\widehat{\nabla}^{-i\theta g}$ by $g \in C^\infty(\mathcal{M}) \otimes GL_n(\mathbb{C})$ is associated to the noncommutative 1-form $\mathfrak{X} \mapsto -i\theta(\mathfrak{X}) + g^{-1}(X\cdot g) = -\gamma + g^{-1}(X\cdot g)$.

2.4 The endomorphism algebra of a vector bundle

The second example of the previous section mixes together two geometries: the de Rham ordinary differential geometry, and the noncommutative derivation-based differential geometry of the matrix algebra. This last geometry is very similar to the ordinary geometry of the Lie group $SL_n(\mathbb{C})$.

It is common in physics to consider the geometry of a based manifold with the geometry of a Lie group (especially Lie groups of the type $SU(n)$): indeed, this is the geometry underlying gauge theories as they are used in the Standard Model of particle physics. This kind of geometry is well understood in the context of principal fiber bundles (see Section 2.2).

This section is devoted to the definition of a noncommutative geometry which generalizes and contains in a precise meaning (see Section 2.6) some essential aspects of the ordinary geometry of $SU(n)$-principal fiber bundles.

2.4.1 The algebra and its derivations

Let $\mathcal{E}$ be a $SU(n)$-vector bundle over $\mathcal{M}$ with fiber $\mathbb{C}^n$. Consider $\mathrm{End}(\mathcal{E})$, the fiber bundle of endomorphisms of $\mathcal{E}$ (see Example 2.2.2). We denote by $\mathbf{A}$ the algebra of sections of $\mathrm{End}(\mathcal{E})$. This is the algebra we will study using noncommutative differential geometry.

For later references, the trivial case is the situation where $\mathcal{E} = \mathcal{M} \times \mathbb{C}^n$ is the trivial fiber bundle. In that case, one has $\mathbf{A} = C^\infty(\mathcal{M}) \otimes M_n$. Its noncommutative geometry is the one exposed as the second example of the previous section. In general, $\mathbf{A}$ is (globally) more complicated.

Let us motivate the importance of this algebra by the following remarks:

Remark 2.4.1 (Relation to ordinary geometry)
The endomorphism fiber bundle $\mathrm{End}(\mathcal{E})$ is associated to a $SU(n)$-principal fiber bundle $\mathcal{P}$ for the couple (M_n, Ad).

Because $G = SU(n) \subset M_n(\mathbb{C})$ and $\mathfrak{g} = \mathfrak{su}(n) \subset M_n(\mathbb{C})$, one has $\mathcal{P} \times_\alpha G \subset \mathrm{End}(\mathcal{E})$ and $\mathrm{Ad}\mathcal{P} = \mathcal{P} \times_{\mathrm{Ad}} \mathfrak{g} \subset \mathrm{End}(\mathcal{E})$ where $\alpha_g(h) = g^{-1}hg$ for any $g, h \in G$.

This implies that the gauge group $\mathcal{G} = \Gamma(\mathcal{P} \times_\alpha G)$ and its Lie algebra $\text{Lie}\mathcal{G} = \Gamma(\text{Ad}\mathcal{P})$ (see Example 2.2.3) are subspaces of $\mathbf{A}$.

We will see in the following that (ordinary) connections are also related to this noncommutative geometry. ♦

Locally, using trivialisations of $\mathcal{E}$, the algebra $\mathbf{A}$ looks like $C^\infty(U) \otimes M_n$. This is very useful to study some objects defined on $\mathbf{A}$.

Proposition 2.4.2 (Basic properties)
One has $\mathcal{Z}(\mathbf{A}) = C^\infty(\mathcal{M})$.

Involution, trace map and determinant ($\text{Tr}, \det : \mathbf{A} \to C^\infty(\mathcal{M})$*), are well defined fiberwise.*

Let us define $SU(\mathbf{A})$ *as the unitaries in* $\mathbf{A}$ *of determinant* 1, *and* $\mathfrak{su}(\mathbf{A})$ *as the traceless antihermitean elements. Then* $\mathcal{G} = SU(\mathbf{A})$ *and* $\text{Lie}\mathcal{G} = \mathfrak{su}(\mathbf{A})$.

This identifies exactly the gauge group and its Lie algebra as natural and canonical subspaces of $\mathbf{A}$.

Let $\rho : \text{Der}(\mathbf{A}) \to \text{Der}(\mathbf{A})/\text{Int}(\mathbf{A}) = \text{Out}(\mathbf{A})$ be the projection of the short exact sequence (2.3.2). This projection has an natural interpretation in this context:

Proposition 2.4.3 (The derivations of A)
One has $\text{Out}(\mathbf{A}) \simeq \text{Der}(C^\infty(\mathcal{M})) = \Gamma(\mathcal{M})$ *and* ρ *is the restriction of derivations* $\mathfrak{X} \in \text{Der}(\mathbf{A})$ *to* $\mathcal{Z}(\mathbf{A}) = C^\infty(\mathcal{M})$. $\text{Int}(\mathbf{A})$ *is isomorphic to* $\mathbf{A}_0$, *the traceless elements in* $\mathbf{A}$.

The short exact sequence of Lie algebras and $C^\infty(\mathcal{M})$*-modules of derivations looks like*

$$0 \longrightarrow \text{Int}(\mathbf{A}) \longrightarrow \text{Der}(\mathbf{A}) \xrightarrow{\rho} \Gamma(\mathcal{M}) \longrightarrow 0$$
$$\mathfrak{X} \longmapsto X$$

Real inner derivations are given by the ad_ξ *with* $\xi \in \text{Lie}\mathcal{G} = \mathfrak{su}(\mathbf{A})$.

The short exact sequence in this proposition describes the general situation which generalises the splitting for the trivial situation encountered in (2.3.3). There is no a *priori* canonical splitting in the non trivial case. Moreover, the noncommutative 1-form $i\theta$ is no more defined here. But one can define a map of $C^\infty(\mathcal{M})$-modules:

$$i\theta : \text{Int}(\mathbf{A}) \to \mathbf{A}_0 \qquad \text{ad}_\gamma \mapsto \gamma - \tfrac{1}{n}\text{Tr}(\gamma)\mathbb{1}$$

Here is an important result which can be proved using local trivialisations:

Proposition 2.4.4

$$\underline{\Omega}^\bullet_{\text{Der}}(\mathbf{A}) = \Omega^\bullet_{\text{Der}}(\mathbf{A})$$

The next proposition will be used in the study of ordinary connections on $\mathcal{E}$ and their relations to the noncommutative geometry of $\mathbf{A}$:

Proposition 2.4.5 (Horizontal forms for the operation of Int(A))
The space of sections $\Gamma(\bigwedge^\bullet T^*\mathcal{M} \otimes \text{End}(\mathcal{E}))$ *is the graded algebra of noncommutative horizontal forms in* $\Omega^\bullet_{\text{Der}}(\mathbf{A})$ *for the operation of* $\text{Int}(\mathbf{A})$ *on* $\Omega^\bullet_{\text{Der}}(\mathbf{A})$.

2.4.2 Ordinary connections

Let us now show how this noncommutative geometry is well adapted to not only study ordinary connections on the vector bundle $\mathcal{E}$, but also,as will be seen in the next section, to allow to some natural generalisations of these connections.

Let $\nabla^{\mathcal{E}}$ be any (usual) connection on the vector bundle $\mathcal{E}$. One can define the two associated connections $\nabla^{\mathcal{E}^*}$ on $\mathcal{E}^*$ and ∇ on $\mathrm{End}(\mathcal{E})$ by the relations

$$X\cdot\langle\varphi,s\rangle = \langle\nabla^{\mathcal{E}^*}_X\varphi,s\rangle + \langle\varphi,\nabla^{\mathcal{E}}_X s\rangle \qquad\qquad \nabla_X(\varphi\otimes s) = (\nabla^{\mathcal{E}^*}_X\varphi)\otimes s + \varphi\otimes(\nabla^{\mathcal{E}}_X s)$$

with $X\in\Gamma(\mathcal{M})$, $\varphi\in\Gamma(\mathcal{E}^*)$ and $s\in\Gamma(\mathcal{E})$

In the following, we will use the notation $X=\rho(\mathfrak{X})\in\Gamma(\mathcal{M})$ for any $\mathfrak{X}\in\mathrm{Der}(\mathbf{A})$.

Proposition 2.4.6 (The noncommutative 1-form α)
For any $X\in\Gamma(\mathcal{M})$, ∇_X is a derivation of $\mathbf{A}$.

For any $\mathfrak{X}\in\mathrm{Der}(\mathbf{A})$, the difference $\mathfrak{X}-\nabla_X$ is an inner derivation. This permits one to introduce $\mathfrak{X}\mapsto\alpha(\mathfrak{X}) = -i\theta(\mathfrak{X}-\nabla_X)$. By construction, α is a noncommutative 1-form $\alpha\in\Omega^1_{\mathrm{Der}}(\mathbf{A})$ which gives the decomposition

$$\mathfrak{X} = \nabla_X - \mathrm{ad}_{\alpha(\mathfrak{X})}$$

For any $\gamma\in\mathbf{A}_0$, one has $\alpha(\mathrm{ad}_\gamma) = -\gamma$, for any $\mathfrak{X}\in\mathrm{Der}(\mathbf{A})$, one has $\mathrm{Tr}\,\alpha(\mathfrak{X}) = 0$, and for any $\mathfrak{X}\in\mathrm{Der}_{\mathbb{R}}(\mathbf{A})$ one has $\alpha(\mathfrak{X})^* + \alpha(\mathfrak{X}) = 0$.

Notice that by the decomposition given in this proposition, $X\mapsto\nabla_X$ is a splitting as $C^\infty(\mathcal{M})$-modules of the short exact sequence

$$0\longrightarrow\mathbf{A}_0\longrightarrow\mathrm{Der}(\mathbf{A})\underset{\nabla}{\overset{\longleftarrow}{\longrightarrow}}\Gamma(\mathcal{M})\longrightarrow 0 \qquad (2.4.4)$$

The obstruction to be a splitting of Lie algebras is nothing but the curvature of ∇ which we denote by $R(X,Y) = [\nabla_X,\nabla_Y]-\nabla_{[X,Y]}$.

Remark 2.4.7 (α extends $-i\theta$)
The relation $\alpha(\mathrm{ad}_\gamma) = -\gamma$ shows that α extends $-i\theta:\mathrm{Int}(\mathbf{A})\to\mathbf{A}_0$. As will be seen in Proposition 2.4.8, any such extension is indeed related to a choice of an ordinary connection of $\mathcal{E}$. ♦

One can then introduce the main result which connects the ordinary geometry of $\mathcal{E}$ and the noncommutative differential geometry of $\mathbf{A}$:

Proposition 2.4.8 (Ordinary connections and noncommutative forms)
The map $\nabla^{\mathcal{E}}\mapsto\alpha$ is an isomorphism between the affine spaces of $SU(n)$-connections on $\mathcal{E}$ and the traceless antihermitean noncommutative 1-forms on $\mathbf{A}$ such that $\alpha(\mathrm{ad}_\gamma) = -\gamma$.

The noncommutative 2-form $(\mathfrak{X},\mathfrak{Y})\mapsto\Omega(\mathfrak{X},\mathfrak{Y}) = \hat{d}\alpha(\mathfrak{X},\mathfrak{Y}) + [\alpha(\mathfrak{X}),\alpha(\mathfrak{Y})]$ depends only on the projections X and Y of $\mathfrak{X}$ and $\mathfrak{Y}$. This means that it is a horizontal noncommutative 2-form for the operation of $\mathrm{Int}(\mathbf{A})$ on $\Omega^\bullet_{\mathrm{Der}}(\mathbf{A})$.

The curvature $R^{\mathcal{E}}$ of $\nabla^{\mathcal{E}}$, considered as a section of $\bigwedge^2 T^*\mathcal{M}\otimes\mathrm{Ad}\mathcal{P}\subset\bigwedge^2 T^*\mathcal{M}\otimes\mathrm{End}(\mathcal{E})$ (see Proposition 2.4.5), is exactly the horizontal noncommutative 2-form Ω.

Remark 2.4.9 (The intermediate construction in ordinary geometry)
We saw in Remark 2.2.4 that in the ordinary geometry of a principal fiber bundle, one is used to introduce connections as 1-forms $\omega \in \Omega^1(\mathcal{P}) \otimes \mathfrak{g}$, with two conditions: vertical normalisation and equivariance. Its curvature is then a 2-form in $\Omega^2(\mathcal{P}) \otimes \mathfrak{g}$, equivariant and horizontal. The other possibility is to introduce a family of local 1-forms $A \in \Omega^1(U) \otimes \mathfrak{g}$ on open subsets U of trivialisations of $\mathcal{P}$, with some non homogeneous gluing relations. The curvature is represented by a family of 2-forms $F \in \Omega^2(U) \otimes \mathfrak{g}$ satisfying some homogeneous gluing relations.

Using the "top" construction (equivariant and horizontal properties) or the "bottom" one (homogeneous gluing relations), one can show that the curvature is indeed a section of the vector bundle $\bigwedge^2 T^*\mathcal{M} \otimes \mathrm{Ad}\mathcal{P} \subset \bigwedge^2 T^*\mathcal{M} \otimes \mathrm{End}(\mathcal{E})$.

This proposition shows that this "intermediate" construction (the curvature as a section of a vector bundle) can be completed at the level of the connection 1-form, at the price of using noncommutative geometry (the noncommutative 1-form α) in order to take into account the non homogeneous gluing relations of the local connection 1-forms (see Remark 2.4.11). The vertical normalisation and the equivariant conditions at the level of $\mathcal{P}$ are replaced by a unique condition on inner derivations at the level of $\mathbf{A}$. ♦

Let us now look at gauge transformations. Let $u \in \mathcal{G} = SU(\mathbf{A})$ and $\xi \in \mathrm{Lie}\mathcal{G} = \mathfrak{su}(\mathbf{A})$.

Proposition 2.4.10 (Gauge transformations)
The noncommutative 1-form α^u corresponding to the gauge transformed connection $\nabla^{\mathcal{E}u}$ is given by the suggestive expression

$$\alpha^u = u^*\alpha u + u^*\hat{\mathrm{d}}u$$

The infinitesimal gauge transformation induced by ξ is

$$\alpha \mapsto -\hat{\mathrm{d}}\xi - [\alpha, \xi] = L_{\mathrm{ad}_\xi}\alpha$$

This means that we can interpret infinitesimal gauge transformations on connections on $\mathcal{E}$ as Lie derivative of real inner derivations on $\mathbf{A}$.

Remark 2.4.11 (Local expressions)
It is instructive to look at the noncommutative 1-form α in some local trivialisation of $\mathcal{E}$. Let $U_i \subset \mathcal{M}$ be a local trivialisation system of $\mathcal{E}$, and so of $\mathrm{End}(\mathcal{E})$. We denote by $a_i^{\mathrm{loc}} : U_i \to M_n$ the restriction of the global section $a \in \mathbf{A}$ looked at in a local trivialisation.

Over $U_i \cap U_j \neq \varnothing$, one has the homogeneous gluing relations $a_j^{\mathrm{loc}} = \mathrm{Ad}_{g_{ij}^{-1}} a_i^{\mathrm{loc}} = g_{ij}^{-1} a_i^{\mathrm{loc}} g_{ij}$, with $g_{ij} : U_i \cap U_j \to SU(n)$ the transition functions.

Locally a derivation $\mathfrak{X} \in \mathrm{Der}(\mathbf{A})$ can be written as $\mathfrak{X}_i^{\mathrm{loc}} = X_i + \mathrm{ad}_{\gamma_i}$, with $\gamma_i : U_i \to M_n$ (traceless) and X_i a vector fields on U_i. Using the map ρ, one gets that X_i is the restriction of $X = \rho(\mathfrak{X})$ to U_i, so that we can write $X = X_i$.

Using compatibility with the homogeneous gluing relations for sections, one finds that the γ_i's satisfy some non homogeneous gluing relations

$$\gamma_j = g_{ij}^{-1}\gamma_i g_{ij} + g_{ij}^{-1} X \cdot g_{ij}$$

The noncommutative 1-form α is then locally given by the expressions

$$\alpha_i^{\text{loc}}(X + \text{ad}_{\gamma_i}) = A_i(X) - \gamma_i$$

where the A_i's form the family of local trivialisation of the connection 1-form. It is then easy to check that

$$\begin{aligned}\alpha_j^{\text{loc}}(X + \text{ad}_{\gamma_j}) &= A_j(X) - \gamma_j = (g_{ij}^{-1} A_i(X) g_{ij} + g_{ij}^{-1} X \cdot g_{ij}) - (g_{ij}^{-1} \gamma_i g_{ij} + g_{ij}^{-1} X \cdot g_{ij}) \\ &= g_{ij}^{-1}(A_i(X) - \gamma_i) g_{ij} = g_{ij}^{-1} \alpha_i^{\text{loc}}(X + \text{ad}_{\gamma_i}) g_{ij}\end{aligned}$$

so that these expressions indeed define a global section in $\mathbf{A}$.

As can be noticed here, the global existence of the noncommutative 1-form α relies on the fact that the A_i's and the γ_i's share the same non homogeneous gluing relations. ♦

2.5 Noncommutative connections on A

In this section, we study noncommutative connections on the right $\mathbf{A}$-module $\boldsymbol{M} = \mathbf{A}$ equipped with the canonical Hermitean structure $(a, b) \mapsto a^* b$.

2.5.1 Main properties

As we saw in Proposition 2.3.15, a noncommutative connection $\widehat{\nabla}$ on the right $\mathbf{A}$-module $\boldsymbol{M} = \mathbf{A}$ is given by a noncommutative 1-form $\omega \in \Omega^1_{\text{Der}}(\mathbf{A})$ by the relation $\widehat{\nabla}_{\mathfrak{X}} a = \mathfrak{X} a + \omega(\mathfrak{X}) a$. This implies that studying $\widehat{\nabla}$ is equivalent to studying ω.

Let us first look at some particular noncommutative connections:

Proposition 2.5.1 (The noncommutative connection associated to $\boldsymbol{\alpha}$)
Let $\nabla^{\mathcal{E}}$ be a $SU(n)$-connection on $\mathcal{E}$, and denote by α its associated noncommutative 1-form. Then, the noncommutative connection $\widehat{\nabla}^\alpha$ defined by the noncommutative 1-form α is given by

$$\widehat{\nabla}^\alpha_{\mathfrak{X}} a = \nabla_X a + a\alpha(\mathfrak{X}) \tag{2.5.5}$$

In particular, for any $X \in \Gamma(\mathcal{M})$, one has $\widehat{\nabla}^\alpha_{\nabla_X} a = \nabla_X a$.

This noncommutative connection $\widehat{\nabla}^\alpha$ is compatible with the canonical Hermitean structure.

The curvature of $\widehat{\nabla}^\alpha$ is $\widehat{R}^\alpha(\mathfrak{X}, \mathfrak{Y}) = R^{\mathcal{E}}(X, Y)$.

A gauge transformation induced by $u \in \mathcal{G} = SU(\mathbf{A})$ on the connection $\nabla^{\mathcal{E}}$ induces a (noncommutative) gauge transformation on $\widehat{\nabla}^\alpha$.

Proof Recall that by definition, one has $\mathfrak{X} = \nabla_X - \text{ad}_{\alpha(\mathfrak{X})}$ and $\widehat{\nabla}^\alpha_{\mathfrak{X}} a = \mathfrak{X} a + \alpha(\mathfrak{X}) a$. This proves (2.5.5).

One the other hand, the curvature of $\widehat{\nabla}^\alpha$ is the noncommutative 2-form $\widehat{d}\alpha(\mathfrak{X}, \mathfrak{Y}) + [\alpha(\mathfrak{X}), \alpha(\mathfrak{Y})]$ which has been identified with the curvature of ∇ in Proposition 2.4.8.

In a gauge transformation, one has $\alpha^u = u^* \alpha u + u^* \widehat{d} u$, which is also the noncommutative gauge transformation applied to $\widehat{\nabla}^\alpha$. ■

We now arrive at the main result of this report:

Theorem 2.5.2 (Ordinary connections as noncommutative connections)
The space of noncommutative connections on the right $\mathbf{A}$*-module* $\mathbf{A}$ *compatible with the Hermitean structure* $(a,b)\mapsto a^*b$ *contains the space of ordinary* $SU(n)$*-connections on* $\mathcal{E}$*.*

This inclusion is compatible with the corresponding definitions of curvature and gauge transformations.

From now on, one can consider that an ordinary connection is a noncommutative connection on the right $\mathbf{A}$-module $\mathbf{A}$. In this respect, this point of view generalizes the notion of connection through the intermediate construction.

A natural question is: what are noncommutative connections from a physical point of view?

2.5.2 Decomposition of noncommutative connections on the module $\mathbf{A}$

In order to answer the above question, one can look at some natural decompositions of noncommutative connections, and compare these decompositions to "ordinary" connections.

Let us fix a connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$, and denote by α its associated noncommutative 1-form. Then any noncommutative connection $\widehat{\nabla}$ can be decomposed as

$$\widehat{\nabla}_{\mathfrak{X}}a=\widehat{\nabla}^{\alpha}_{\mathfrak{X}}a+\mathcal{A}(\mathfrak{X})a$$

with $\mathcal{A}\in\Omega^1_{\text{Der}}(\mathbf{A})$, so that $\omega=\alpha+\mathcal{A}$ is the noncommutative 1-form for $\widehat{\nabla}$.

Using the relation $\mathfrak{X}=\nabla_X-\text{ad}_{\alpha(\mathfrak{X})}$, one splits $\mathcal{A}$ as $\mathcal{A}(\mathfrak{X})=\mathfrak{a}(X)-\mathfrak{b}(\alpha(\mathfrak{X}))$, where $\mathfrak{b}:\mathbf{A}_0\to\mathbf{A}$ is defined by $\mathfrak{b}(\gamma)=\mathcal{A}(\text{ad}_\gamma)$.

A straightforward computation shows that the curvature of $\widehat{\nabla}$ can then be written as

$$\begin{aligned}\widehat{R}(\mathfrak{X},\mathfrak{Y})&=R^{\mathcal{E}}(X,Y)+\nabla_X\mathcal{A}(\mathfrak{Y})-\nabla_Y\mathcal{A}(\mathfrak{X})-\mathcal{A}([\mathfrak{X},\mathfrak{Y}])+[\mathcal{A}(\mathfrak{X}),\mathcal{A}(\mathfrak{Y})]\\&=R^{\mathcal{E},\mathfrak{a}}(X,Y)-\nabla^{\mathfrak{a}}_X\mathfrak{b}(\alpha(\mathfrak{Y}))+\nabla^{\mathfrak{a}}_Y\mathfrak{b}(\alpha(\mathfrak{X}))\\&\qquad+[\mathfrak{b}(\alpha(\mathfrak{X})),\mathfrak{b}(\alpha(\mathfrak{Y}))]+\mathfrak{b}(\alpha([\mathfrak{X},\mathfrak{Y}]))\end{aligned}$$

where $R^{\mathcal{E},\mathfrak{a}}$ is the curvature of the connection $\nabla^{\mathcal{E},\mathfrak{a}}_X s=\nabla^{\mathcal{E}}_X s+\mathfrak{a}(X)s$ on $\mathcal{E}$ and $\nabla^{\mathfrak{a}}$ is its associated connection on $\text{End}(\mathcal{E})$.

Performing a gauge transformation with $u\in\mathcal{G}=SU(\mathbf{A})$, one has

$$\mathcal{A}^u=u^*\mathcal{A}u+u^*(\nabla u)\qquad\qquad \mathfrak{a}^u=u^*\mathfrak{a}u+u^*(\nabla u)\qquad\qquad \mathfrak{b}^u=u^*\mathfrak{b}u$$

Notice the replacement of the differential by ∇ in these expressions.

Remark 2.5.3 (Local expressions)
In Remark 2.4.11, we looked at local expressions of the noncommutative 1-form α. Let us now look at the previous decomposition in a local trivialisation of $\mathcal{E}$. The noncommutative connection $\widehat{\nabla}$ takes the local expression:

$$\widehat{\nabla}^{\text{loc}}_{\mathfrak{X}^{\text{loc}}}a^{\text{loc}}=X\cdot a^{\text{loc}}+\left[A(X)+\mathfrak{a}^{\text{loc}}(X)-\mathfrak{b}^{\text{loc}}(A(X))\right]a^{\text{loc}}+\mathfrak{b}^{\text{loc}}(\gamma)a^{\text{loc}}-a^{\text{loc}}\gamma$$

where A is the local connection 1-form of $\nabla^{\mathcal{E}}$ and $\mathfrak{X}^{\text{loc}}=X+\text{ad}_\gamma$ as before.

In a change of local trivialisation, the two local maps $\gamma\mapsto\mathfrak{b}^{\text{loc}}(\gamma)$ and $X\mapsto\mathfrak{a}^{\text{loc}}(X)$ transform as

$$\gamma\mapsto\mathfrak{b}'^{\text{loc}}(\gamma)=g^{-1}\mathfrak{b}^{\text{loc}}(g\gamma g^{-1})g\qquad\qquad X\mapsto\mathfrak{a}'^{\text{loc}}(X)=g^{-1}\mathfrak{a}^{\text{loc}}(X)g$$

which are both homogeneous gluing relations. ♦

In order to be more explicit, consider now the trivial situation $\mathcal{E} = \mathcal{M}\times\mathbb{C}^n$ and $\mathbf{A} = C^\infty(\mathcal{M})\otimes M_n$.

As a reference (ordinary) connection, one can take $\nabla^{\mathcal{E}}_X s = X\cdot s$, so that, using the local expression of α, one has

$$\alpha(\mathfrak{X}) = \alpha(X + \mathrm{ad}_\gamma) = -\gamma = -i\theta(\mathfrak{X})$$

with $\mathrm{Tr}\,\gamma = 0$. Then $\widehat{\nabla}^\alpha = \widehat{\nabla}^{-i\theta}$. Moreover, $\mathfrak{b}(\alpha(\mathfrak{X})) = \mathfrak{b}(-\gamma) = -\mathfrak{b}(\gamma)$, so that

$$\widehat{\nabla}_{\mathfrak{X}} a = X\cdot a + \mathfrak{a}(X)a + \mathfrak{b}(\gamma)a - a\gamma = \overline{\nabla}^{\mathfrak{a}}_X a + \mathfrak{b}(\gamma)a - a\gamma$$

where $\overline{\nabla}^{\mathfrak{a}}$, defined in some local trivialization by $\overline{\nabla}^{\mathfrak{a}}_X a = X\cdot a + \mathfrak{a}(X)a$, is an ordinary connection on $\mathrm{End}(\mathcal{E})$, but is not $\nabla^{\mathfrak{a}}$, which takes the explicit local form $\nabla^{\mathfrak{a}}_X a = X\cdot a + [\mathfrak{a}(X), a]$.

$X \mapsto \mathfrak{a}(X)$ behaves like a gauge potential with respect to gauge transformations (here $\nabla = \mathrm{d}$). The difference between ordinary connections and noncommutative connections is the presence of $\mathfrak{b}$, which represents some additional fields in physics. These fields have homogeneous gauge transformations.

The curvature can be written, for $\mathfrak{X} = X + \mathrm{ad}_\gamma$ and $\mathfrak{Y} = Y + \mathrm{ad}_\eta$,

$$\widehat{R}(\mathfrak{X},\mathfrak{Y}) = R^{\mathcal{E},\mathfrak{a}}(X,Y) + (\widetilde{\nabla}^{\mathfrak{a}}_X \mathfrak{b})(\eta) - (\widetilde{\nabla}^{\mathfrak{a}}_Y \mathfrak{b})(\gamma) + [\mathfrak{b}(\gamma), \mathfrak{b}(\eta)] - \mathfrak{b}([\gamma,\eta])$$

where $\widetilde{\nabla}^{\mathfrak{a}}$ is the connection $(\widetilde{\nabla}^{\mathfrak{a}}_X \mathfrak{b})(\eta) = X\cdot\mathfrak{b}(\eta) - \mathfrak{b}(X\cdot\eta) + [\mathfrak{a}(X), \mathfrak{b}(\eta)]$ on the space of $C^\infty(\mathcal{M})$-linear maps $\mathbf{A}_0 \to \mathbf{A}$.

2.5.3 Yang-Mills-Higgs Lagrangian on the module A

Consider, as before, the trivial case $\mathbf{A} = C^\infty(\mathcal{M})\otimes M_n$ and the right $\mathbf{A}$-module $\mathbf{A}$. Let $\mathfrak{a} = \mathfrak{a}_\mu \mathrm{d}x^\mu$ and $\mathfrak{b} = \mathfrak{b}_k\theta^k$, with $\mathfrak{a}_\mu, \mathfrak{b}_k \in C^\infty(\mathcal{M})\otimes M_n$.

The curvature is then the noncommutative 2-form

$$\widehat{R} = \tfrac{1}{2}(\partial_\mu\mathfrak{a}_\nu - \partial_\nu\mathfrak{a}_\mu + [\mathfrak{a}_\mu, \mathfrak{a}_\nu])\mathrm{d}x^\mu\mathrm{d}x^\nu + (\partial_\mu\mathfrak{b}_k + [\mathfrak{a}_\mu, \mathfrak{b}_k])\mathrm{d}x^\mu\theta^k + \tfrac{1}{2}([\mathfrak{b}_k, \mathfrak{b}_\ell] - C^m_{k\ell}\mathfrak{b}_m)\theta^k\theta^\ell$$

Using a metric (here euclidean) on $\mathrm{Der}(\mathbf{A})$ and an associated Hodge star operation, one can define a Lagrangian. Using ordinary and noncommutative integration, one then defines the action:

$$S(\widehat{R}) = \int \mathrm{d}x\,\mathrm{Tr}\Big\{\sum_{\mu,\nu}\tfrac{1}{4}(\partial_\mu\mathfrak{a}_\nu - \partial_\nu\mathfrak{a}_\mu + [\mathfrak{a}_\mu,\mathfrak{a}_\nu])^2 + m^2\sum_{\mu,k}(\partial_\mu\mathfrak{b}_k + [\mathfrak{a}_\mu, \mathfrak{b}_k])^2 + m^4\sum_{k,\ell}\tfrac{1}{4}([\mathfrak{b}_k,\mathfrak{b}_\ell] - C^m_{k\ell}\mathfrak{b}_m)^2\Big\}$$

The integrand is zero when

$$\mathfrak{a} \text{ gauge equivalent to } 0 \qquad \mathrm{d}\mathfrak{b} = 0 \qquad [\mathfrak{b}_k, \mathfrak{b}_\ell] = C^m_{k\ell}\mathfrak{b}_m$$

so that the $\mathfrak{b}_k$'s are constant and induce a representation of $\mathfrak{sl}_n$ in M_n.

For the right $\mathbf{A}$-module $\boldsymbol{M} = C^\infty(\mathcal{M})\otimes M_{r,n}$, one would get similar results: flat connections are classified by inequivalent representations of $\mathfrak{sl}_n$ in M_r.

Remark 2.5.4 (Physical interpretation)
From a fields theory point of view, one can notice that the $\mathfrak{a}_\mu$ fields behave like ordinary Yang-Mills fields, for a $SU(n)$ gauge theory. On the other hand, the interesting point is that the $\mathfrak{b}_k$ fields behave

as Higgs fields: in the above action, the vacuum states can be non trivial and the Higgs mechanism of mass generation is possible. Finally, the coupling between these fields is a covariant derivative in the adjoint representation. ♦

For a more general situation where **A** is not the trivial case, one can proceed in the same line:

- One has to use a reference connection on $\mathcal{E}$ to help to decompose noncommutative connections.
- The curvature looks similar except for the presence of the reference connection.
- The Hodge star operator is defined.
- The action splits into three terms, and the vacuum states are related to the global structure of the vector fiber bundle $\mathcal{E}$.

2.6 Relations with the principal fiber bundle

It is possible to look at the noncommutative geometry of **A** using the ordinary geometry of the underlying $SU(n)$-principal fiber bundle $\mathcal{P}$ and the noncommutative geometry of a bigger algebra, hereafter denoted by **B**.

2.6.1 The algebra B

As before, let $\mathcal{P}$ be the $SU(n)$-principal fiber bundle to which $\mathcal{E}$ is associated, and consider the associative algebra $\mathbf{B} = C^\infty(\mathcal{P}) \otimes M_n$. This algebra is an example of the trivial situation mentioned in 2.3.3, so that one has immediately the following facts: the center of **B** is $\mathcal{Z}(\mathbf{B}) = C^\infty(\mathcal{P})$, its Lie algebra and $\mathcal{Z}(\mathbf{B})$-module of derivations splits, $\text{Der}(\mathbf{B}) = \Gamma(\mathcal{P}) \oplus [C^\infty(\mathcal{P}) \otimes \mathfrak{sl}_n]$, and its noncommutative differential calculus is the tensor product of the two differential calculi associated to $\mathcal{P}$ and M_n: $\Omega^\bullet_{\text{Der}}(\mathbf{B}) = \Omega^\bullet(\mathcal{P}) \otimes \Omega^\bullet_{\text{Der}}(M_n)$ with the differential $\widehat{\text{d}} = \text{d} + \text{d}'$.

One can embed the real Lie algebra $\mathfrak{su}(n)$ as a subalgebra of $\text{Der}(\mathbf{B})$ in two ways:

$$\xi \mapsto \xi^v \text{ vertical vector field on } \mathcal{P} \qquad\qquad \xi \mapsto \text{ad}_\xi \text{ inner derivation}$$

This permits one to introduce the following two Lie subalgebras of $\text{Der}(\mathbf{B})$:

$$\mathfrak{g}_{\text{ad}} = \{\text{ad}_\xi \ / \ \xi \in \mathfrak{su}(n)\} \qquad\qquad \mathfrak{g}_{\text{equ}} = \{\xi^v + \text{ad}_\xi \ / \ \xi \in \mathfrak{su}(n)\}$$

Proposition 2.6.1
*The algebra $C^\infty(\mathcal{P})$ (resp. **A**) is the set of invariant elements for the action of $\mathfrak{g}_{\text{ad}}$ (resp. $\mathfrak{g}_{\text{equ}}$) on **B**.*

Proof $C^\infty(\mathcal{P})$ is the invariants of $\mathfrak{g}_{\text{ad}}$ because $\text{ad}_\xi b = 0$ for any $\xi \in \mathfrak{su}(n)$ implies $b \in \mathcal{Z}(\mathbf{B})$. **A** is the invariants of $\mathfrak{g}_{\text{equ}}$ because **A** is the set of sections of $\text{End}(\mathcal{E})$, which is $\mathcal{F}_{SU(n)}(\mathcal{P}, M_n)$, the space of $SU(n)$-equivariant maps from $\mathcal{P}$ to M_n. The relation $\xi^v \cdot b + \text{ad}_\xi b = 0$ for any $\xi \in \mathfrak{su}(n)$ is the infinitesimal version of this equivariance. ■

The two Lie subalgebras $\mathfrak{g}_{\text{ad}}$ and $\mathfrak{g}_{\text{equ}}$ define Cartan operations on $(\Omega^\bullet_{\text{Der}}(\mathbf{B}), \widehat{\text{d}})$. The previous proposition tells us that the algebras **B**, $C^\infty(\mathcal{P})$ and **A** are related by these two operations.

Moreover, $C^\infty(\mathcal{M})$ is itself the set of invariant elements for $\xi \mapsto \xi^v$ in $C^\infty(\mathcal{P})$ and the invariants in **A** for the operation of $\text{Int}(\mathbf{A})$.

Proposition 2.6.2 (Relations between the differential calculi)
At the level of differential calculi, all these relations generalize in the following structure:

$$\begin{array}{ccc}
\Omega^\bullet(\mathcal{P}) \otimes \Omega^\bullet_{\mathrm{Der}}(M_n) & \xleftarrow[\mathfrak{su}(n) \ni \xi \mapsto \mathrm{ad}_\xi]{\text{basic elements}} & \Omega^\bullet(\mathcal{P}) \\
\uparrow {\scriptstyle \text{basic elements}\ \mathfrak{su}(n) \ni \xi \mapsto \xi^v + \mathrm{ad}_\xi} & & \uparrow {\scriptstyle \text{basic elements}\ \mathfrak{su}(n) \ni \xi \mapsto \xi^v} \\
\Omega^\bullet_{\mathrm{Der}}(\mathbf{A}) & \xleftarrow[\mathrm{Int}(\mathbf{A})]{\text{basic elements}} & \Omega^\bullet(\mathcal{M})
\end{array}$$

In order to show these relations, one need the concept of noncommutative quotient manifold introduced in [Masson, 1996]. We refer to [Masson, 1999] for the complete proof.

Notice that this proposition contains a well known result in ordinary differential geometry, which says that the space of tensorial forms in $\Omega^\bullet(\mathcal{P}) \otimes \mathfrak{g}$ (horizontal and equivariant for the action induced by right multiplication on $\mathcal{P}$ and the adjoint action on the Lie algebra $\mathfrak{g}$) is the space $\Omega^\bullet(\mathcal{M}, \mathrm{Ad}\mathcal{P})$ of forms on the base manifold $\mathcal{M}$ with values in the vector bundle $\mathrm{Ad}\mathcal{P}$. This result permits one to indentify the curvature of a connection on $\mathcal{P}$ to a form in $\Omega^2(\mathcal{M}, \mathrm{Ad}\mathcal{P})$ (see Proposition 2.4.8 and Remark 2.4.9).

In Proposition 2.3.26, we saw that the noncommutative integration is well defined on algebras like $\mathbf{B}$. This induces a map

$$\int_{\text{n.c.}} : \Omega^r_{\mathrm{Der}}(\mathbf{B}) \to \Omega^{r-(n^2-1)}(\mathcal{P})$$

which has the following properties:

Proposition 2.6.3 (Noncommutative integration)
If $\omega \in \Omega^r_{\mathrm{Der}}(\mathbf{B})$ is a horizontal (resp. basic) noncommutative form for one of the operations of $\mathfrak{g}_{\mathrm{ad}}$ or $\mathfrak{g}_{\mathrm{equ}}$, then $\int_{n.c.} \omega \in \Omega^{r-(n^2-1)}(\mathcal{P})$ is horizontal (resp. basic) for the corresponding operation restricted to $\Omega^\bullet(\mathcal{P}) \subset \Omega^\bullet_{\mathrm{Der}}(\mathbf{B})$.

This noncommutative integration then restricts to a "noncommutative integration along the noncommutative fiber" $\Omega^\bullet_{\mathrm{Der}}(\mathbf{A}) \to \Omega^{\bullet-(n^2-1)}(\mathcal{M})$.

This noncommutative integration is compatible with the differentials, and it induces maps in cohomologies

$$\int_{n.c.} : H^\bullet(\Omega^\bullet_{\mathrm{Der}}(\mathbf{B}), \widehat{\mathrm{d}}) \to H^{\bullet-(n^2-1)}_{dR}(\mathcal{P})$$
$$\int_{n.c.} : H^\bullet(\Omega^\bullet_{\mathrm{Der}}(\mathbf{A}), \widehat{\mathrm{d}}) \to H^{\bullet-(n^2-1)}_{dR}(\mathcal{M})$$

This situation looks very similar to the integration along the fibers of compactly supported (along the fibers) differential forms in the theory of vector bundles.

2.6.2 Ordinary *vs.* noncommutative connections

It is instructive to identify ordinary connections in this setting. Let $\widehat{\nabla}$ be a noncommutative connection on the right $\mathbf{A}$-module $\mathbf{A}$, and denote by $\alpha \in \Omega^1_{\mathrm{Der}}(\mathbf{A})$ it associated noncommutative 1-form.

As a basic noncommutative 1-form in $\Omega^1_{\mathrm{Der}}(\mathbf{B})$ for the operation of $\mathfrak{g}_{\mathrm{equ}}$, one can write

$$\alpha = \omega - \phi \in [\Omega^1(\mathcal{P}) \otimes M_n] \oplus [C^\infty(\mathcal{P}) \otimes M_n \otimes \mathfrak{sl}_n^*]$$

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \mathcal{Z}_{\mathrm{Der}}(\mathbf{A}) & \longrightarrow & \Gamma(V\mathcal{P}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Int}(\mathbf{A}) & \longrightarrow & \mathcal{N}_{\mathrm{Der}}(\mathbf{A}) & \xrightarrow{\rho_{\mathcal{P}}} & \Gamma_{\mathcal{M}}(\mathcal{P}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\pi_*} & & \\
0 & \longrightarrow & \mathrm{Int}(\mathbf{A}) & \longrightarrow & \mathrm{Der}(\mathbf{A}) & \xrightarrow{\rho} & \Gamma(\mathcal{M}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Figure 2.1: Some relations between the derivations of $\mathbf{B}$ and $\mathbf{A}$ and some vector fields on $\mathcal{P}$ and $\mathcal{M}$.

The basic condition implies the relations

$$(L_{\xi^v} + L_{\mathrm{ad}_\xi})\omega = 0 \qquad (L_{\xi^v} + L_{\mathrm{ad}_\xi})\phi = 0 \qquad i_{\xi^v}\omega - i_{\mathrm{ad}_\xi}\phi = 0$$

for any $\xi \in \mathfrak{su}(n)$.

Proposition 2.6.4 (Ordinary connection)
Let $\nabla^{\mathcal{E}}$ be an ordinary connection on $\mathcal{E}$ and $\alpha \in \Omega^1_{\mathrm{Der}}(\mathbf{A})$ its associated noncommutative 1-form. Then, as a basic element in $\Omega^1_{\mathrm{Der}}(\mathbf{B})$, one has

$$\alpha = \omega - i\theta$$

where $\omega \in \Omega^1(\mathcal{P}) \otimes \mathfrak{su}(n) \subset \Omega^1(\mathcal{P}) \otimes M_n$ is the connection 1-form on $\mathcal{P}$ associated to $\nabla^{\mathcal{E}}$ and $i\theta$ is the canonical noncommutative 1-form defined in $\Omega^1_{\mathrm{Der}}(\mathbf{B})$ (Proposition 2.3.26).

In order to prove this formula, one has to use the equivariance and the vertical condition for ω, and some of the properties listed before on $i\theta$.

Notice that this inclusion of ordinary connection into the space of basic 1-forms on $\mathbf{B}$ is canonical, since the noncommutative 1-form $i\theta$ is itself canonical.

2.6.3 Splittings coming from connections

The previous considerations show how the differential calculi connect together through some Cartan operations. There also exist some strong relations between the derivations of $\mathbf{A}$, some derivations of $\mathbf{B}$, and some vector fields on $\mathcal{P}$ and $\mathcal{M}$. They are summarised in the diagram of Fig. 2.1.

In this diagram, one has the following short exact sequences of Lie algebras and $C^\infty(\mathcal{M})$-modules:

- $0 \longrightarrow \mathrm{Int}(\mathbf{A}) \longrightarrow \mathrm{Der}(\mathbf{A}) \xrightarrow{\rho} \Gamma(\mathcal{M}) \longrightarrow 0$

  This is the short exact sequence which relates vector fields on $\mathcal{M}$, derivations on $\mathbf{A}$ and inner derivations on $\mathbf{A}$ given in Proposition 2.4.3.

- $0 \longrightarrow \mathcal{Z}_{\mathrm{Der}}(\mathbf{A}) \longrightarrow \mathcal{N}_{\mathrm{Der}}(\mathbf{A}) \xrightarrow{\tau} \mathrm{Der}(\mathbf{A}) \longrightarrow 0$

  $\mathcal{N}_{\mathrm{Der}}(\mathbf{A}) \subset \mathrm{Der}(\mathbf{B})$ is the subset of derivations on $\mathbf{B}$ which preserve $\mathbf{A} \subset \mathbf{B}$.
  $\mathcal{Z}_{\mathrm{Der}}(\mathbf{A}) \subset \mathrm{Der}(\mathbf{B})$ is the subset of derivations on $\mathbf{B}$ which vanish on $\mathbf{A}$. This is a Lie ideal in $\mathcal{N}_{\mathrm{Der}}(\mathbf{A})$, and τ is the quotient map.
  The Lie algebra $\mathcal{Z}_{\mathrm{Der}}(\mathbf{A})$ is generated as a $C^\infty(\mathcal{P})$-module by the elements $\xi^v + \mathrm{ad}_\xi$ for any $\xi \in \mathfrak{su}(n)$.

- $0 \longrightarrow \Gamma(V\mathcal{P}) \longrightarrow \Gamma_{\mathcal{M}}(\mathcal{P}) \overset{\pi_*}{\longrightarrow} \Gamma(\mathcal{M}) \longrightarrow 0$

  These are pure geometrical objects:
  $\Gamma(V\mathcal{P})$ is the Lie algebra of vertical vector fields on $\mathcal{P}$.
  $\Gamma_{\mathcal{M}}(\mathcal{P}) = \{\mathcal{X} \in \Gamma(\mathcal{P}) / \pi_*\mathcal{X}(p) = \pi_*\mathcal{X}(p')\ \forall p, p' \in \mathcal{P} \text{ s.t. } \pi(p) = \pi(p')\}$ is the Lie algebra of vector fields on $\mathcal{P}$ which can be mapped to vector fields on $\mathcal{M}$ using the tangent maps $\pi_* : T_p\mathcal{P} \to T_{\pi(p)}\mathcal{M}$.

- $0 \longrightarrow \mathrm{Int}(\mathbf{A}) \longrightarrow \mathcal{N}_{\mathrm{Der}}(\mathbf{A}) \overset{\rho_{\mathcal{P}}}{\longrightarrow} \Gamma_{\mathcal{M}}(\mathcal{P}) \longrightarrow 0$

  Here, the elements in $\mathrm{Int}(\mathbf{A})$ are identified to the ad_γ for $\gamma \in \mathbf{A}_0 \subset \mathbf{B}$. $\mathrm{Int}(\mathbf{A})$ is then a Lie subalgebra of $\mathcal{N}_{\mathrm{Der}}(\mathbf{A})$.
  $\rho_{\mathcal{P}}$ is the restriction to $\mathcal{N}_{\mathrm{Der}}(\mathbf{A})$ of the projection on the first term in the splitting $\mathrm{Der}(\mathbf{B}) = \Gamma(\mathcal{P}) \oplus [C^\infty(\mathcal{P}) \otimes \mathrm{Der}(M_n)]$.

An ordinary connection $\omega \in \Omega^1(\mathcal{P}) \otimes \mathfrak{su}(n)$ splits these short exact sequences. Let us look more closely at the central square of the diagram of Fig. 2.1. One can define splittings as follows:

$$
\begin{array}{ccccc}
\mathcal{N}_{\mathrm{Der}}(\mathbf{A}) & & \xrightarrow{\quad\rho_{\mathcal{P}}\quad} & & \Gamma_{\mathcal{M}}(\mathcal{P}) \\
 & (\pi_*\mathcal{X})^h + \omega(\mathcal{X})^v + \mathrm{ad}_{\omega(\mathcal{X})} & \longleftarrow\!\!\dashv & \mathcal{X} & \\
 & \rho(\mathfrak{X})^h - \mathrm{ad}_{\alpha(\mathfrak{X})^{\mathbf{B}}} & & X^h & \\
\tau \downarrow & \uparrow & & \uparrow & \downarrow \pi_* \\
 & \mathfrak{X} & & X & \\
 & \nabla_X & \longleftarrow\!\!\dashv & X & \\
\mathrm{Der}(\mathbf{A}) & & \xrightarrow{\quad\rho\quad} & & \Gamma(\mathcal{M})
\end{array}
$$

where:

- $\Gamma(\mathcal{M}) \to \mathrm{Der}(\mathbf{A}),\ X \mapsto \nabla_X$:

  This is the splitting mentioned in Proposition 2.4.6, which lifts vector fields on $\mathcal{M}$ into derivations on $\mathbf{A}$.

- $\Gamma(\mathcal{M}) \to \Gamma_{\mathcal{M}}(\mathcal{P}),\ X \mapsto X^h$:

  This splitting lifts vector fields on $\mathcal{M}$ into horizontal vector fields X^h on $\mathcal{P}$ through the ordinary geometrical procedure. Using its equivariance, one can easily verify that the vector field X^h is indeed a π_*-projectable vector field. In fact, for any $\mathcal{X} \in \Gamma_{\mathcal{M}}(\mathcal{P})$, one has $\mathcal{X} = (\pi_*\mathcal{X})^h + \mathcal{X}^v$, where $\mathcal{X}^v$ is the vertical projection of $\mathcal{X}$, explicitly given by the formula $\mathcal{X}^v = \omega(\mathcal{X})^v$.

- $\mathrm{Der}(\mathbf{A}) \to \mathcal{N}_{\mathrm{Der}}(\mathbf{A}),\ \mathfrak{X} \mapsto \rho(\mathfrak{X})^h - \mathrm{ad}_{\alpha(\mathfrak{X})^{\mathbf{B}}}$:

  This lifts derivations on $\mathbf{A}$ into derivations on $\mathbf{B}$. Here, $\alpha(\mathfrak{X})^{\mathbf{B}}$ is the basic element in $\mathbf{B}$ associated to $\alpha(\mathfrak{X}) \in \mathbf{A}$ and $\rho(\mathfrak{X})^h \in \Gamma(\mathcal{P})$ is the horizontal lift of the vector field $\rho(\mathfrak{X})$. By construction, one has $\mathrm{ad}_{\alpha(\mathfrak{X})^{\mathbf{B}}} \in \mathcal{N}_{\mathrm{Der}}(\mathbf{A})$. On the other hand, one verifies that for any $X \in \Gamma(\mathcal{M})$ and any $\xi \in \mathfrak{su}(n)$, $[\xi^v + \mathrm{ad}_\xi, X^h] = 0$ (as an element in $\mathrm{Der}(\mathbf{B})$), which shows that $X^h \in \mathcal{N}_{\mathrm{Der}}(\mathbf{A})$. The relation $\tau(\rho(\mathfrak{X})^h - \mathrm{ad}_{\alpha(\mathfrak{X})^{\mathbf{B}}}) = \mathfrak{X}$ relies on the two following facts: in the identification of $a \in \mathbf{A}$ as an equivariant map $a^{\mathbf{B}} : \mathcal{P} \to M_n$, one has the identification of $\nabla_X a$ with $X^h \cdot a^{\mathbf{B}}$, and one has the decomposition $\mathfrak{X} = \nabla_{\rho(\mathfrak{X})} - \mathrm{ad}_{\alpha(\mathfrak{X})}$.

- $\Gamma_{\mathcal{M}}(\mathcal{P}) \to \mathcal{N}_{\text{Der}}(\mathbf{A})$, $\mathcal{X} \mapsto (\pi_*\mathcal{X})^h + \omega(\mathcal{X})^v + \text{ad}_{\omega(\mathcal{X})}$:

  Here, we lift π_*-projectable vector fields $\mathcal{X}$ on $\mathcal{P}$ into derivations on $\mathbf{B}$. Notice that for any $\mathcal{X} \in \Gamma_{\mathcal{M}}(\mathcal{P})$, one has the decomposition $\mathcal{X} = (\pi_*\mathcal{X})^h + \omega(\mathcal{X})^v$. The inner derivation $\text{ad}_{\omega(\mathcal{X})}$ is there in order that $\omega(\mathcal{X})^v + \text{ad}_{\omega(\mathcal{X})} \in \mathcal{N}_{\text{Der}}(\mathbf{A})$ (we know from the previous result that $(\pi_*\mathcal{X})^h \in \mathcal{N}_{\text{Der}}(\mathbf{A})$). In fact, one has the more interesting result that

$$\omega(\mathcal{X})^v + \text{ad}_{\omega(\mathcal{X})} \in \mathcal{Z}_{\text{Der}}(\mathbf{A})$$

In order to better understand the two liftings ending in $\mathcal{N}_{\text{Der}}(\mathbf{A})$, it is useful to characterize derivations in $\mathcal{N}_{\text{Der}}(\mathbf{A})$. Such a derivation can be decomposed, as an element in $\text{Der}(\mathbf{B})$, as $\widehat{\mathfrak{X}} = \mathcal{X} + \text{ad}_b$, with $\mathcal{X} \in \Gamma(\mathcal{P})$ and $b \in \mathbf{B}_0 = C^\infty(\mathcal{P}) \otimes \mathfrak{sl}_n$. Using the fact that $\rho_{\mathcal{P}}$ is just the restriction of $\widehat{\mathfrak{X}}$ to $C^\infty(\mathcal{P})$, one has $\rho_{\mathcal{P}}(\widehat{\mathfrak{X}}) = \mathcal{X} \in \Gamma_{\mathcal{M}}(\mathcal{P})$. The condition $\widehat{\mathfrak{X}} \in \mathcal{N}_{\text{Der}}(\mathbf{A})$ implies that $[\xi^v + \text{ad}_\xi, \widehat{\mathfrak{X}}] \in \mathcal{Z}_{\text{Der}}(\mathbf{A})$ for any $\xi \in \mathfrak{su}(n)$. Using the structure of $\mathcal{Z}_{\text{Der}}(\mathbf{A})$, one can write $[\xi^v + \text{ad}_\xi, \widehat{\mathfrak{X}}] = f^i(\eta_i^v + \text{ad}_{\eta_i})$ for some $f^i \in C^\infty(\mathcal{P})$ and $\eta_i \in \mathfrak{su}(n)$, which can be decomposed into two parts: $[\xi^v, \mathcal{X}] = f^i \eta_i^v$ and $\xi^v \cdot b + [\xi, b] = f^i \eta_i$.

Denote by $L_\xi^{\text{equ}} = L_{\xi^v} + \text{ad}_\xi$ the Lie derivative associated to the Cartan operation of the Lie algebra $\mathfrak{g}_{\text{equ}}$ on $(\Omega^\bullet_{\text{Der}}(\mathbf{B}), \widehat{\text{d}})$. The second relation is then $L_\xi^{\text{equ}} b = f^i \eta_i$. Applying now the connection 1-form ω on the first relation, one gets $\omega([\xi^v, \mathcal{X}]) = f^i \eta_i$, which can be written, using the equivariance of ω: $L_\xi^{\text{equ}} \omega(\mathcal{X}) = f^i \eta_i$. The difference $a(\widehat{\mathfrak{X}}) = \omega(\mathcal{X}) - b$ is then L^{equ}-invariant, which means that $a(\widehat{\mathfrak{X}}) \in \mathbf{A}$. With $\mathfrak{X} = \tau(\widehat{\mathfrak{X}})$, this is exactly the element $\alpha(\mathfrak{X}) \in \mathbf{A}$ identified as an element in $\mathbf{B}$, where α is the noncommutative 1-form associated to the connection ω.

Using these constructions, one has the following decomposition of any $\widehat{\mathfrak{X}} \in \mathcal{N}_{\text{Der}}(\mathbf{A})$:

$$\widehat{\mathfrak{X}} = \mathcal{X} + \text{ad}_b = (\pi_*\mathcal{X})^h + \underbrace{\omega(\mathcal{X})^v + \text{ad}_{\omega(\mathcal{X})}}_{\in \mathcal{Z}_{\text{Der}}(\mathbf{A})} - \underbrace{\text{ad}_{a(\widehat{\mathfrak{X}})}}_{\in \text{Int}(\mathbf{A})}$$

2.7 Cohomology and characteristic classes

In ordinary differential geometry, it is possible to relate the cohomology of a fiber bundle to the cohomology of its base manifold using a spectral sequence based on a Čech-de Rham bicomplex constructed using differential forms. We will show that such a construction can be performed with the space of noncommutative differential forms.

Using the noncommutative geometry structures described above, it is also possible to recover the Chern characteristic classes of the vector bundle $\mathcal{E}$. The construction we present in the following is purely algebraic, and relies on an adaptation of some work by Lecomte about characteristic classes associated to splitting of short exact sequence of Lie algebras.

2.7.1 The cohomology of $\Omega^\bullet_{\text{Der}}(\mathbf{A})$

Let us recall the Leray theorem in ordinary differential geometry.

Theorem 2.7.1 (Leray)
For any fiber bundle $\mathcal{F} \to \mathcal{E} \xrightarrow{\pi} \mathcal{M}$, there exists a spectral sequence $\{\boldsymbol{E}_r\}$ converging to the cohomology of the total space $H^\bullet_{dR}(\mathcal{E})$ with

$$\boldsymbol{E}_2^{p,q} = H^p(\mathfrak{U}; \mathcal{H}^q)$$

$$
\begin{array}{ccccccccccc}
 & \vdots & & \vdots & & \vdots & & \cdots & \vdots & & \cdots \\
 & \mathrm{d}\uparrow & & \mathrm{d}\uparrow & & \mathrm{d}\uparrow & & & \mathrm{d}\uparrow & & \\
0 \to & \Omega^q(\mathcal{M}) & \xrightarrow{\delta} & \prod \Omega^q(U_{\alpha_0}) & \xrightarrow{\delta} & \prod \Omega^q(U_{\alpha_0\alpha_1}) & \xrightarrow{\delta} & \cdots \xrightarrow{\delta} & \prod \Omega^q(U_{\alpha_0\dots\alpha_p}) & \xrightarrow{\delta} & \cdots \\
 & \mathrm{d}\uparrow & & \mathrm{d}\uparrow & & \mathrm{d}\uparrow & & & \mathrm{d}\uparrow & & \\
\vdots & \vdots & & \vdots & & \vdots & & \cdots & \vdots & & \cdots \\
 & \mathrm{d}\uparrow & & \mathrm{d}\uparrow & & \mathrm{d}\uparrow & & & \mathrm{d}\uparrow & & \\
0 \to & \Omega^2(\mathcal{M}) & \xrightarrow{\delta} & \prod \Omega^2(U_{\alpha_0}) & \xrightarrow{\delta} & \prod \Omega^2(U_{\alpha_0\alpha_1}) & \xrightarrow{\delta} & \cdots \xrightarrow{\delta} & \prod \Omega^2(U_{\alpha_0\dots\alpha_p}) & \xrightarrow{\delta} & \cdots \\
 & \mathrm{d}\uparrow & & \mathrm{d}\uparrow & & \mathrm{d}\uparrow & & & \mathrm{d}\uparrow & & \\
0 \to & \Omega^1(\mathcal{M}) & \xrightarrow{\delta} & \prod \Omega^1(U_{\alpha_0}) & \xrightarrow{\delta} & \prod \Omega^1(U_{\alpha_0\alpha_1}) & \xrightarrow{\delta} & \cdots \xrightarrow{\delta} & \prod \Omega^1(U_{\alpha_0\dots\alpha_p}) & \xrightarrow{\delta} & \cdots \\
 & \mathrm{d}\uparrow & & \mathrm{d}\uparrow & & \mathrm{d}\uparrow & & & \mathrm{d}\uparrow & & \\
0 \to & \Omega^0(\mathcal{M}) & \xrightarrow{\delta} & \prod \Omega^0(U_{\alpha_0}) & \xrightarrow{\delta} & \prod \Omega^0(U_{\alpha_0\alpha_1}) & \xrightarrow{\delta} & \cdots \xrightarrow{\delta} & \prod \Omega^0(U_{\alpha_0\dots\alpha_p}) & \xrightarrow{\delta} & \cdots \\
 & & & i\uparrow & & i\uparrow & & & i\uparrow & & \\
 & & & C^0(\mathfrak{U};\mathbb{R}) & \xrightarrow{\delta} & C^1(\mathfrak{U};\mathbb{R}) & \xrightarrow{\delta} & \cdots \xrightarrow{\delta} & C^p(\mathfrak{U};\mathbb{R}) & \xrightarrow{\delta} & \cdots \\
 & & & \uparrow & & \uparrow & & & \uparrow & & \\
 & & & 0 & & 0 & & \cdots & 0 & &
\end{array}
$$

Figure 2.2: The ordinary Čech-de Rham bicomplex associated to a fiber bundle $\mathcal{F} \longrightarrow \mathcal{E} \xrightarrow{\pi} \mathcal{M}$

where $\mathcal{H}^q(U) = H^q_{dR}(\pi^{-1}U)$ is a locally constant presheaf on the good covering $\mathfrak{U}$ of $\mathcal{M}$.

If $\mathcal{M}$ is simply connected and $H^q_{dR}(\mathcal{F})$ is finite dimensional, then

$$\boldsymbol{E}_2^{p,q} = H^p_{dR}(\mathcal{M}) \otimes H^q_{dR}(\mathcal{F})$$

One of the proofs of this theorem relies on the construction of a Čech-de Rham bicomplex as illustrated in the diagram of Fig. 2.2 (see [Bott and Tu, 1995] for instance):

$$\boldsymbol{K}^{p,q} = \prod_{\alpha_0 < \cdots < \alpha_p} \Omega^q(\mathcal{E}_{U_{\alpha_0 \dots \alpha_p}}) = \prod_{\alpha_0 < \cdots < \alpha_p} \Omega^q(\pi^{-1} U_{\alpha_0 \dots \alpha_p})$$

with $U_{\alpha_0 \dots \alpha_p} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ for $U_{\alpha_i} \in \mathfrak{U}$, where $\mathfrak{U}$ is a good cover of $\mathcal{M}$, $\mathrm{d} : \boldsymbol{K}^{p,q} \to \boldsymbol{K}^{p,q+1}$ is the ordinary de Rham differential on the spaces $\Omega^\bullet(\mathcal{E}_{U_{\alpha_0 \dots \alpha_p}})$, and $\delta : \boldsymbol{K}^{p,q} \to \boldsymbol{K}^{p+1,q}$ is the Čech differential

$$(\delta\omega_p)_{\alpha_0 \dots \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \omega_{\alpha_0 \dots \widehat{\alpha_i} \dots \alpha_{p+1} | U_{\alpha_0 \dots \alpha_{p+1}}}$$

One can introduce a noncommutative Čech-de Rham bicomplex for $\mathbf{A}$. In order to do that, denote by $\mathbf{A}(U) \simeq C^\infty(U) \otimes M_n$ the sections of $\mathrm{End}(\mathcal{E})$ restricted over a local trivialisation $U \subset \mathcal{M}$ with $U \in \mathfrak{U}$, where as before $\mathfrak{U}$ is a good cover of $\mathcal{M}$. Denote by $g_{UV} : U \cap V \to SU(n)$ the transition functions for $\mathcal{E}$.

For any noncommutative p-form $\omega = a_0 \widehat{\mathrm{d}} a_1 \cdots \widehat{\mathrm{d}} a_p \in \Omega^p_{\mathrm{Der}}(\mathbf{A}(U))$ and any differential function $g : U \to SU(n)$, define the action of g on ω by $\omega^g = (g^{-1} a_0 g) \widehat{\mathrm{d}}(g^{-1} a_1 g) \cdots \widehat{\mathrm{d}}(g^{-1} a_p g)$.

Lemma 2.7.2 (The presheaf $\Omega^\bullet_{\mathrm{Der}}(\mathbf{A}(U))$)
For any $V \subset U$, the maps $i^V_U : \Omega^\bullet_{\mathrm{Der}}(\mathbf{A}(U)) \to \Omega^\bullet_{\mathrm{Der}}(\mathbf{A}(V))$ given by $\omega \mapsto (\omega_{|V})^{g_{UV}}$ (restriction to V and action of g_{UV}) give to $U \mapsto \Omega^\bullet_{\mathrm{Der}}(\mathbf{A}(U))$ a structure of presheaf on $\mathcal{M}$, which we denote by $\mathcal{F}$.

Using this presheaf, one can introduce the bicomplex

$$\boldsymbol{C}^{p,q}(\mathfrak{U};\mathcal{F}) = \prod_{\alpha_0<\cdots<\alpha_p} \Omega^q_{\mathrm{Der}}(\mathbf{A}(U_{\alpha_0\ldots\alpha_p}))$$

where by convention the trivialisation over $U_{\alpha_0\ldots\alpha_p}$ is the one over U_{α_p}. Let $\widehat{\mathrm{d}} : \boldsymbol{C}^{p,q} \to \boldsymbol{C}^{p,q+1}$ be the noncommutative differential, and define $\delta : \boldsymbol{C}^{p,q}(\mathfrak{U};\mathcal{F}) \to \boldsymbol{C}^{p+1,q}(\mathfrak{U};\mathcal{F})$ by (here $g_{\alpha\beta} = g_{U_\alpha U_\beta}$)

$$(\delta\omega)_{\alpha_0\ldots\alpha_{p+1}} = \sum_{i=0}^{p} (-1)^i (\omega_{\alpha_0\ldots\widehat{\alpha_i}\ldots\alpha_{p+1}})_{|U_{\alpha_0\ldots\alpha_{p+1}}} + (-1)^{p+1} (\omega_{\alpha_0\ldots\alpha_p})^{g_{\alpha_p\alpha_{p+1}}}_{|U_{\alpha_0\ldots\alpha_{p+1}}}$$

Notice that in the last term, the action of $g_{\alpha_p\alpha_{p+1}}$ performs the change of trivialisation from the one above U_{α_p} to the one above $U_{\alpha_{p+1}}$.

Denote by $\boldsymbol{C}^{-1,q}(\mathfrak{U};\mathcal{F}) = \Omega^q_{\mathrm{Der}}(\mathbf{A})$ and define $\delta : \boldsymbol{C}^{-1,q}(\mathfrak{U};\mathcal{F}) \to \boldsymbol{C}^{0,q}(\mathfrak{U};\mathcal{F})$ as the restrictions to the trivialisations of the good cover.

One has the following results about the cohomology of $\Omega^\bullet_{\mathrm{Der}}(\mathbf{A})$:

Theorem 2.7.3 (Noncommutative Leray theorem)
The cohomology of the total complex of the bicomplex $(\boldsymbol{C}^{\bullet,\bullet}(\mathfrak{U};\mathcal{F}), \widehat{\mathrm{d}}, \delta)$ *is the cohomology of* $\Omega^\bullet_{\mathrm{Der}}(\mathbf{A})$.

The spectral sequence $\{\boldsymbol{E}_r\}$ *associated to the filtration*

$$F^p\boldsymbol{C}(\mathfrak{U};\mathcal{F}) = \oplus_{s\geq p}\oplus_{q\geq 0}\boldsymbol{C}^{s,q}(\mathfrak{U};\mathcal{F})$$

converges to the cohomology of $\Omega^\bullet_{\mathrm{Der}}(\mathbf{A})$ *and satisfies*

$$\boldsymbol{E}_2 = H^\bullet_{dR}(\mathcal{M}) \otimes \mathcal{I}(\textstyle\bigwedge^\bullet \mathfrak{sl}_n^*)$$

Recall that the structure of $\mathcal{I}(\bigwedge^\bullet \mathfrak{sl}_n^*)$ is known. One can find the proof of this result in [Masson, 1999].

2.7.2 Characteristic classes and short exact sequences of Lie algebras

Let us now show that the splitting (2.4.4) of the short exact sequence of derivations contains all the informations needed to recover the Chern characteristic classes of the fiber bundle $\mathcal{E}$. In order to do that, one has first to introduce a construction by Lecomte (see [Lecomte, 1985]).

Let $0 \longrightarrow \mathfrak{i} \longrightarrow \mathfrak{g} \overset{\pi}{\longrightarrow} \mathfrak{h} \longrightarrow 0$ be a short exact sequence of Lie algebras, and let $\varphi : \mathfrak{h} \to \mathfrak{g}$ be a morphism which splits it as vector spaces. Define $R_\varphi = \mathrm{d}_\mathfrak{h}\varphi + \frac{1}{2}[\varphi,\varphi] : \bigwedge^2\mathfrak{h}^* \otimes \mathfrak{g}$ with $\mathrm{d}_\mathfrak{h}$ the differential on $\bigwedge^\bullet\mathfrak{h}^* \otimes \mathfrak{g}$ for the trivial representation of $\mathfrak{h}$ on $\mathfrak{g}$. For any $x,y \in \mathfrak{h}$, the quantity $R_\varphi(x,y) = -\varphi([x,y]) + [\varphi(x),\varphi(y)]$ is exactly the obstruction on φ to be a Lie algebra morphism, *i.e.* a splitting of Lie algebras.

It is worth to note that R_φ looks like a curvature, and indeed, the following construction treats it as if it were a curvature. One can show that R_φ belongs to $\bigwedge^2\mathfrak{h}^* \otimes \mathfrak{i}$ and that it satisfies a Bianchi identity $d_\mathfrak{h}R_\varphi + [\varphi, R_\varphi] = 0$.

Now, let V be a vector space and ρ a representation of $\mathfrak{h}$ on V. Denote by $\mathrm{S}^q_\rho(\mathfrak{i}, V)$ the space of linear symmetric maps $\otimes^q\mathfrak{i} \to V$ which intertwine the adjoint representation $\mathrm{ad}^{\otimes^q}$ of $\mathfrak{g}$ on $\otimes^q\mathfrak{i}$ and the representation $\rho\circ\pi$ of $\mathfrak{g}$ on V. Let ε be the antisymmetrisation map $\otimes^\bullet\mathfrak{h}^* \to \bigwedge^\bullet\mathfrak{h}^*$.

One has the following result, shown in [Lecomte, 1985]:

Proposition 2.7.4 (Characteristic classes of a short exact sequence of Lie algebras)
For any $\alpha \in \mathsf{S}^q_\rho(\mathfrak{i}, V)$, let $\alpha_\varphi = \varepsilon \circ \alpha(R_\varphi \otimes \cdots \otimes R_\varphi) \in \bigwedge^{2q}\mathfrak{h}^* \otimes V$. Then one has $\mathrm{d}\alpha_\varphi = 0$ where d is the differential of the complex $\bigwedge^\bullet \mathfrak{h}^* \otimes V$.

The cohomology class of α_φ in $H^{2q}(\mathfrak{h}; V)$ does not depends on the choice of φ.

If the short exact sequence is split exact as a Lie algebra short exact sequence then this cohomology class is zero.

Let us adapt this construction to the short exact sequence

$$0 \longrightarrow \mathrm{Int}(\mathbf{A}) \longrightarrow \mathrm{Der}(\mathbf{A}) \xrightarrow{\rho} \Gamma(\mathcal{M}) \longrightarrow 0$$

It is possible to generalise the previous construction in order to take into account the extra structures of $\mathcal{Z}(\mathbf{A})$-modules.

We identify $\mathrm{Int}(\mathbf{A})$ with $\mathbf{A}_0$. The adjoint representation of $\mathrm{Der}(\mathbf{A})$ on $\mathrm{Int}(\mathbf{A})$ is explicitly given by $\mathrm{ad}_{\mathfrak{X}}(\mathrm{ad}_a) = [\mathfrak{X}, \mathrm{ad}_a] = \mathrm{ad}_{\mathfrak{X}(a)}$ so that it is $(\mathfrak{X}, a) \mapsto \mathfrak{X}(a)$ on $\mathbf{A}_0$.

The vector space (and $\mathcal{Z}(\mathbf{A})$-module) we consider is $\mathcal{Z}(\mathbf{A})$ itself, on which the representation ρ is $(\mathfrak{X}, f) \mapsto \rho(\mathfrak{X})\cdot f$.

Let $\mathsf{S}^q_{\mathcal{Z}(\mathbf{A})}(\mathbf{A}_0, \mathcal{Z}(\mathbf{A}))$ be the space of $\mathcal{Z}(\mathbf{A})$-linear symmetric maps $\otimes^q_{\mathcal{Z}(\mathbf{A})}\mathbf{A}_0 \to \mathcal{Z}(\mathbf{A})$ which intertwine the adjoint representation $\mathrm{ad}^{\otimes^q}$ of $\mathrm{Der}(\mathbf{A})$ on $\otimes^q_{\mathcal{Z}(\mathbf{A})}\mathrm{Int}(\mathbf{A}) = \otimes^q_{\mathcal{Z}(\mathbf{A})}\mathbf{A}_0$ and the representation ρ of $\mathrm{Der}(\mathbf{A})$ on $\mathcal{Z}(\mathbf{A})$.

Notice that, thanks to the $\mathcal{Z}(\mathbf{A})$-linearity, maps in $\mathsf{S}^q_{\mathcal{Z}(\mathbf{A})}(\mathbf{A}_0, \mathcal{Z}(\mathbf{A}))$ are local on $\mathcal{M}$, so that one can look at them in local trivialisations of $\mathcal{E}$. In such a trivialisation over an open set U, the intertwining relations can be written, with the usual notation $\mathfrak{X}^{\mathrm{loc}} = X + \mathrm{ad}_\gamma$:

$$\sum_{i=1}^{q} \phi(a_1 \otimes \cdots \otimes X\cdot a_i \otimes \cdots \otimes a_q) = X\cdot\phi(a_1 \otimes \cdots \otimes a_q)$$
$$\sum_{i=1}^{q} \phi(a_1 \otimes \cdots \otimes [\gamma, a_i] \otimes \cdots \otimes a_q) = 0$$

for any $a_i : U \to \mathfrak{sl}_n$.

Proposition 2.7.5 (Characteristic classes of $\mathcal{E}$)
The space $\mathsf{S}^q_{\mathcal{Z}(\mathbf{A})}(\mathbf{A}_0, \mathcal{Z}(\mathbf{A}))$ is well defined, which means that the $\mathcal{Z}(\mathbf{A})$-linearity and the intertwining condition are compatible, and one has

$$\mathsf{S}^q_{\mathcal{Z}(\mathbf{A})}(\mathbf{A}_0, \mathcal{Z}(\mathbf{A})) = \mathcal{P}^q_I(\mathfrak{sl}_n)$$

the space of invariant polynomials on the Lie algebra $\mathfrak{sl}_n$.

The differential complex in which the characteristic classes for the splitting are defined is

$$\mathsf{Hom}_{\mathcal{Z}(\mathbf{A})}(\textstyle\bigwedge^\bullet_{\mathcal{Z}(\mathbf{A})}\Gamma(\mathcal{M}), \mathcal{Z}(\mathbf{A}))$$

which is the de Rham complex of differential forms on $\mathcal{M}$.

The characteristic classes one computes in this way are precisely the ordinary Chern characteristic classes of the vector bundle $\mathcal{E}$ (or of the principal fiber bundle $\mathcal{P}$).

The last statement relies on the fact that any ordinary connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$ gives rise to a splitting of the short exact sequence whose curvature is exactly the obstruction to be a morphism of Lie algebras. The construction based on $\mathsf{S}^q_{\mathcal{Z}(\mathbf{A})}(\mathbf{A}_0, \mathcal{Z}(\mathbf{A})) = \mathcal{P}^q_I(\mathfrak{sl}_n)$ is then the ordinary Chern-Weil morphism.

2.8 Invariant noncommutative connections

Many works have been done in the theory of ordinary connections which are symmetric with respect to the action of a Lie group. This leads to understand some ansatz used to get exact solutions of Yang-Mills theories, and recover or introduce some Yang-Mills models coupled with scalar fields through these symmetric reductions.

In this section, we generalize these considerations to noncommutative connections on the algebra **A**, and show that the geometrico-algebraic structures introduced so far are very natural in the theory of symmetric reductions.

This exposé is based on [Masson and Sérié, 2005], and we refer to this paper for more details and references.

2.8.1 Action of a Lie group on a principal fiber bundle

Let us recall some general constructions that were introduced in the theory of symmetric reduction of connections.

Let $G \longrightarrow \mathcal{P} \xrightarrow{\pi} \mathcal{M}$ be a G-principal fiber bundle, and let H be a Lie group acting on the left on $\mathcal{P}$, such that the action commutes with the right action of G.

Then, the action of H on $\mathcal{P}$ induces a left action of H on $\mathcal{M}$. In the following, we assume that this action is simple, which means that $\mathcal{M}$ admits the fiber bundle structure $H/H_0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}/H$ where H_0 is an isotropy subgroup: $H_0 = H_{x_0} = \{h \in H \ / \ h{\cdot}x_0 = x_0\}$. In particular, all the isotropy subgroups are isomorphics to one of them. We fix H_0 as such an isotropy subgroup.

Then we introduce the following spaces:

- $\mathcal{N} = \{x \in \mathcal{M} \ / \ H_x = H_0\}$ is the space of points in $\mathcal{M}$ whose isotropy subgroup is exactly H_0.
- $\mathcal{N}_H(H_0) = \{h \in H \ / \ hH_0 = H_0 h\}$ is the normalizer of H_0 in H.
- H_0 is a normal subgroup of $\mathcal{N}_H(H_0)$, and one has the principal fiber bundle

  $$\mathcal{N}_H(H_0)/H_0 \longrightarrow \mathcal{N} \longrightarrow \mathcal{M}/H$$

  The fiber bundle $H/H_0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}/H$ is associated to this bundle for the natural action of $\mathcal{N}_H(H_0)/H_0$ on H/H_0 by (right) multiplication.

Define $S = H \times G$. This group acts on the right on $\mathcal{P}$ by the following relation: $(h,g){\cdot}p = h^{-1}{\cdot}p{\cdot}g$. For any $p \in \mathcal{P}$, let $\lambda_p : H_{\pi(p)} \to G$ be defined such that $h{\cdot}p = p{\cdot}\lambda_p(h)$. Then one can show the followings:

- $S_p = \{(h, \lambda_p(h)) \ / \ h \in H_{\pi(p)}\}$ is the isotropy subgroup of $p \in \mathcal{P}$ for the action of S. This implies that the action of S on $\mathcal{P}$ is simple.
- Fix an isotropy subgroup S_0 and let $\mathcal{Q} = \{p \in \mathcal{P} \ / \ S_p = S_0\}$. Then

  $$S/S_0 \longrightarrow \mathcal{P} \longrightarrow \mathcal{M}/H \text{ is associated to } \mathcal{N}_S(S_0)/S_0 \longrightarrow \mathcal{Q} \longrightarrow \mathcal{M}/H$$

  as for the (simple) action of H on $\mathcal{M}$.

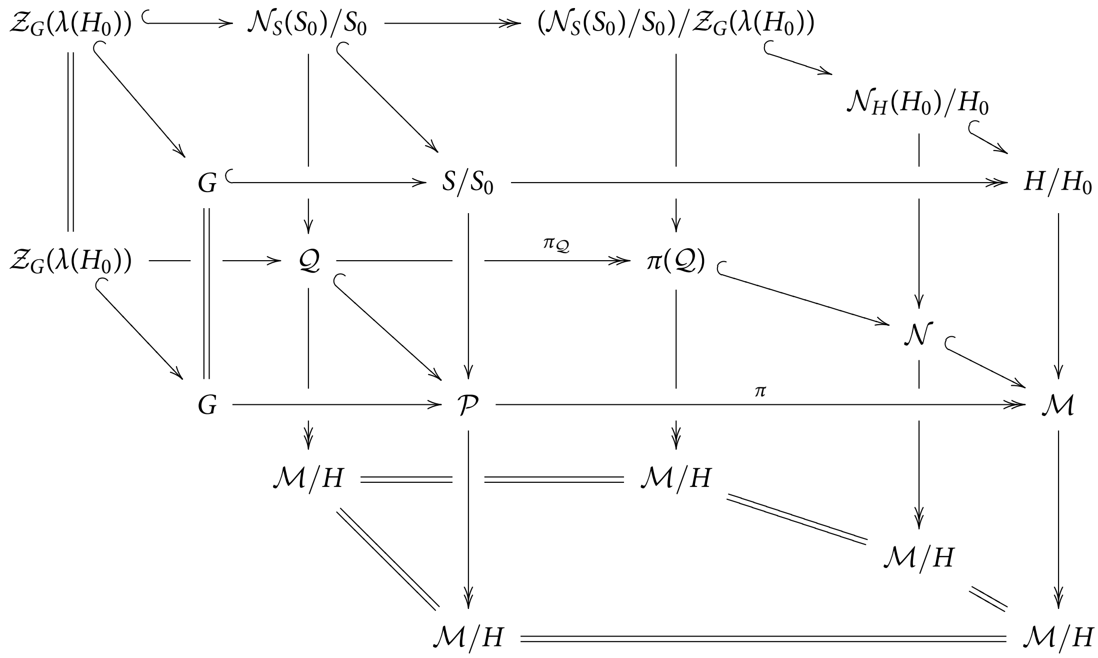


Figure 2.3: In this diagram, some arrows represent true applications and other arrows are part of diagrams of fibrations, most of them explicitly given before. Some horizontal arrows correspond to the action of G (or subgroups of G) and some vertical arrows correspond to actions of groups related to H and S.

Proposition 2.8.1 (Some properties of $\mathcal{Q}$ and λ)
The map $\lambda_p : H_{\pi(p)} \to G$ such that $h{\cdot}p = p{\cdot}\lambda_p(h)$ satisfies

$$\lambda_{p\cdot g}(h) = g^{-1}\lambda_p(h)g$$

For any $q \in \mathcal{Q}$, λ_q depends only on $\pi(q) \in \mathcal{M}$: $\lambda_q(h) = \lambda_{q\cdot g}(h)$. For a fixed x_0 in $\mathcal{M}$ whose isotropy group is H_0, we denote this map restricted to $\mathcal{Q}$ by $\lambda : H_0 \to G$.

The projection $\pi : \mathcal{P} \to \mathcal{M}$ induces the fiber bundle structure

$$\mathcal{Z}_G(\lambda(H_0)) \longrightarrow \mathcal{Q} \xrightarrow{\pi_\mathcal{Q}} \pi(\mathcal{Q})$$

with $\mathcal{Z}_G(\lambda(H_0)) = \{g \in G \,/\, g\lambda(h_0) = \lambda(h_0)g, \forall h_0 \in H_0\}$, the centralizer of $\lambda(H_0)$ in G, and $\pi(\mathcal{Q}) \subset \mathcal{N}$.

We summarize in Fig. 2.3 all the fibrations one can obtain relating the spaces introduced so far. In the following, we will concentrate more precisely on the diagram of fibrations:

$$\begin{array}{ccccc}
\mathcal{N}_S(S_0)/S_0 & \longrightarrow & \mathcal{Q} & \longrightarrow & \mathcal{M}/H \\
\downarrow & & \downarrow & & \| \\
S/S_0 & \longrightarrow & \mathcal{P} & \longrightarrow & \mathcal{M}/H
\end{array}$$

In order to connect this construction to some differential structures, we are now interested in the Lie algebras of the different groups introduced before.

Let us look at the structure of the Lie algebra $\mathfrak{h}$ of the group H. One can introduce the following spaces:

- $\mathfrak{h}_0$ is the Lie algebra of H_0, the once for all fixed isotropy group.
- $\mathfrak{k}$ is the Lie algebra of the quotient group $\mathcal{N}_H(H_0)/H_0$.
- $\mathfrak{n}_0 = \mathfrak{h}_0 \oplus \mathfrak{k}$ is the Lie algebra of $\mathcal{N}_H(H_0)$, the normalizer of H_0 in H.
- $\mathfrak{l}$ is the vector space in the orthogonal decomposition $\mathfrak{h} = \mathfrak{n}_0 \oplus \mathfrak{l}$ such that $[\mathfrak{n}_0, \mathfrak{l}] \subset \mathfrak{l}$ (this is called a reductive decomposition of $\mathfrak{h}$ along $\mathfrak{n}_0$).

Denote by $\mathfrak{g}$ the Lie algebra of the group G. As before, we can introduce the following spaces:

- $\mathfrak{z}_0$ the Lie algebra of $\mathcal{Z}_G(\lambda(H_0))$, the centralizer of $\lambda(H_0)$ in G.
- $\mathfrak{m}$ the vector space in the orthogonal and reductive decomposition $\mathfrak{g} = \mathfrak{z}_0 \oplus \mathfrak{m}$ ($[\mathfrak{z}_0, \mathfrak{m}] \subset \mathfrak{m}$).

The Lie algebra of the group $S = H \times G$ is $\mathfrak{s} = \mathfrak{h} \oplus \mathfrak{g}$, and one has

- $\mathfrak{s}_0 = \{(x_0, \lambda_* x_0) \ / \ x_0 \in \mathfrak{h}_0\}$ is the Lie algebra of S_0, the fixed isotropy group.
- $\mathfrak{s}_0 \oplus \mathfrak{k} \oplus \mathfrak{z}_0$ is the Lie algebra of $\mathcal{N}_S(S_0)$, the normalizer of S_0 in S.
- $\mathfrak{k} \oplus \mathfrak{z}_0$ is the Lie algebra of the quotient group $\mathcal{N}_S(S_0)/S_0$.

With these spaces, one has the following result:

Proposition 2.8.2 (Decomposition of $T\mathcal{P}$)
For any $q \in \mathcal{Q}$, one has $\mathfrak{k}_q^{\mathcal{Q}} \oplus \mathfrak{z}_{0q}^{\mathcal{Q}} \subset T_q\mathcal{Q}$ and $T_q\mathcal{P} = T_q\mathcal{Q} \oplus \mathfrak{l}_q^{\mathcal{P}} \oplus \mathfrak{m}_q^{\mathcal{P}}$.

In this proposition, we use the following compact notation: $\mathfrak{a}_q^{\mathcal{R}}$ is the space of tangent vectors over q associated to elements $x \in \mathfrak{a} \subset \mathfrak{h}$ or $\mathfrak{g}$ through the fundamental vector fields on $\mathcal{R} = \mathcal{Q}$ or $\mathcal{P}$ for the action of the corresponding group H or G.

2.8.2 Invariant noncommutative connections

It is now possible to mix together the geometrical constructions of the previous subsection and the noncommutative algebraic considerations on the endomorphism algebra associated to a G-principal fiber bundle $\mathcal{P}$ with $G = SL(n)$ or $G = SU(n)$. Let then as before H be a compact connected Lie group acting on $\mathcal{P}$.

Proposition 2.8.3 (Operations of $\mathfrak{h}$ on $\Omega^\bullet_{\mathrm{Der}}(\mathbf{B})$ and $\Omega^\bullet_{\mathrm{Der}}(\mathbf{A})$)
The operation of $\mathfrak{h}$ on $\Omega^\bullet(\mathcal{P})$ induced by the action of H on $\mathcal{P}$ extends to an operation of $\mathfrak{h}$ on $\Omega^\bullet_{\mathrm{Der}}(\mathbf{B}) = \Omega^\bullet(\mathcal{P}) \otimes \Omega^\bullet_{\mathrm{Der}}(M_n)$ (using a trivial action on the second factor). This operation commutes with the operations of $\mathfrak{g}_{\mathrm{ad}}$ and $\mathfrak{g}_{\mathrm{equ}}$, and so reduces to the operation of $\mathfrak{h}$ on $\Omega^\bullet(\mathcal{P})$ and to an operation of $\mathfrak{h}$ on $\Omega^\bullet_{\mathrm{Der}}(\mathbf{A})$.

Definition 2.8.4 (Invariant noncommutative connection)
The operation of $\mathfrak{h}$ on $\Omega^\bullet_{\mathrm{Der}}(\mathbf{A})$ obtained in Proposition 2.8.3 is our definition of the (noncommutative) action of H on the algebra $\mathbf{A}$.

A noncommutative connection $\widehat{\nabla}$ on the right $\mathbf{A}$-module $\boldsymbol{M} = \mathbf{A}$ is said to be $\mathfrak{h}$-invariant if, $\forall y \in \mathfrak{h}$, $\forall \mathfrak{X} \in \mathrm{Der}(\mathbf{A})$ and $\forall a \in \mathbf{A}$, one has $L_y(\widehat{\nabla}_{\mathfrak{X}} a) = \widehat{\nabla}_{[y,\mathfrak{X}]} a + \widehat{\nabla}_{\mathfrak{X}}(L_y a)$, where L_y is the Lie derivative of the operation of $\mathfrak{h}$ on $\Omega^\bullet_{\mathrm{Der}}(\mathbf{A})$. ♦

Using this definition, one obtains the equivalent characterization:

Proposition 2.8.5 (Invariance of the noncommutative 1-form α)
The noncommutative connection $\widehat{\nabla}$ is $\mathfrak{h}$-invariant if and only if its noncommutative 1-form α is invariant: $L_y\alpha = 0$ for all $y \in \mathfrak{h}$.

This last proposition, combined with the relations between the noncommutative geometries of the algebras **A** and **B**, permits one to reduce the problem of finding the $\mathfrak{h}$-invariant noncommutative connections $\widehat{\nabla}$ on the right **A**-module $\boldsymbol{M} = \mathbf{A}$ to the following problem: find all the noncommutative 1-forms written as $\alpha = \omega - \phi \in [\Omega^1(\mathcal{P}) \otimes M_n] \oplus [C^\infty(\mathcal{P}) \otimes M_n \otimes \mathfrak{sl}_n^*]$ satisfying the four relations

$$(L_{\xi^v} + L_{\mathrm{ad}_\xi})\omega = 0 \qquad (L_{\xi^v} + L_{\mathrm{ad}_\xi})\phi = 0 \qquad i_{\xi^v}\omega - i_{\mathrm{ad}_\xi}\phi = 0 \qquad L_y(\omega - \phi) = 0$$

for all $\xi \in \mathfrak{g} = \mathfrak{sl}_n$ and $y \in \mathfrak{h}$. The three first relations express the basicity of α for the operation of $\mathfrak{g}_{\mathrm{equ}}$ on $\Omega^\bullet_{\mathrm{Der}}(\mathbf{B})$, and the last one is the $\mathfrak{h}$-invariance.

This last relation decomposes into two independent equations $L_y\omega = 0$ and $L_y\phi = 0$ for all $y \in \mathfrak{h}$. This implies in particular that one can restrict the study of ω and ϕ defined over $\mathcal{P}$ to the submanifold $\mathcal{Q} \subset \mathcal{P}$. For all $q \in \mathcal{Q}$, one has then to characterize the maps

$$\omega_q : T_q\mathcal{P} = T_q\mathcal{Q} \oplus \mathfrak{l}_q^{\mathcal{P}} \oplus \mathfrak{m}_q^{\mathcal{P}} \to M_n \qquad\qquad \phi_q : \mathfrak{g} \to M_n$$

Now, the relation $i_{\xi^v}\omega - i_{\mathrm{ad}_\xi}\phi = 0$ for all $\xi \in \mathfrak{g}$, says that $\phi_q(\xi)$ is completely determined by $\omega_q(\xi_q^v)$. It is then sufficient to study ω_q.

Let us first consider the $T_q\mathcal{Q}$ part of $T_q\mathcal{P}$. Denote by $\mu_q : T_q\mathcal{Q} \to M_n$ the restriction of ω_q to $T_q\mathcal{Q}$. One has $\mathfrak{z}_{0q}^{\mathcal{Q}} \subset T_q\mathcal{Q}$, so that μ and ϕ are both defined on $\mathfrak{z}_0 \subset \mathfrak{g}$, where they coincide: $\mu(z^{\mathcal{Q}}) = \phi(z)$ for any $z \in \mathfrak{z}_0$. Denote by η_q the restriction of ϕ_q to $\mathfrak{z}_0$, which is then also the restriction of μ_q to $\mathfrak{z}_{0q}^{\mathcal{Q}}$. It is possible to write down these relations in a compact way through the following result:

Proposition 2.8.6 (The algebra W)
Let $\mathbf{W} = \mathcal{Z}_{M_n}(\lambda_*\mathfrak{g}_0)$ be the centralizer of $\lambda_*\mathfrak{g}_0$ in M_n. It is an associative algebra and $\mathfrak{z}_0 \subset \mathrm{Der}(\mathbf{W})$. Let $\Omega^\bullet_{\mathfrak{z}_0}(\mathbf{W}) = \mathbf{W} \otimes \bigwedge^\bullet \mathfrak{z}_0^*$ be the restricted derivation-based differential calculus associated to it.

There is a natural operation of $\mathfrak{z}_0$ on $\Omega^\bullet(\mathcal{Q}) \otimes \Omega^\bullet_{\mathfrak{z}_0}(\mathbf{W})$, and $\mu - \eta \in \left(\Omega^\bullet(\mathcal{Q}) \otimes \Omega^\bullet_{\mathfrak{z}_0}(\mathbf{W})\right)^1_{\mathfrak{z}_0\text{-basic}}$.

In order to take into account the remaining part of μ_q in $T_q\mathcal{Q}$, we introduce the following bigger differential calculus:

Proposition 2.8.7 (The differential calculus $\Omega^\bullet_{\mathfrak{k}\oplus\mathfrak{z}_0}(\mathcal{M}/H; \mathbf{W})$)
There are natural operations of $\mathfrak{k} \subset \mathfrak{h}$ and $\mathfrak{z}_0 \subset \mathfrak{g}$ on the differential algebra $\Omega^\bullet(\mathcal{Q}) \otimes \Omega^\bullet_{\mathfrak{z}_0}(\mathbf{W}) \otimes \bigwedge^\bullet \mathfrak{k}^*$.

Define

$$\Omega^\bullet_{\mathfrak{k}\oplus\mathfrak{z}_0}(\mathcal{M}/H; \mathbf{W}) = \left(\Omega^\bullet(\mathcal{Q}) \otimes \Omega^\bullet_{\mathfrak{z}_0}(\mathbf{W}) \otimes \textstyle\bigwedge^\bullet \mathfrak{k}^*\right)_{\mathfrak{k}\oplus\mathfrak{z}_0\text{-basic}}$$

The algebra $\mathbf{C} = \Omega^0_{\mathfrak{k}\oplus\mathfrak{z}_0}(\mathcal{M}/H; \mathbf{W}) = (C^\infty(\mathcal{Q}) \otimes \mathbf{W})_{\mathfrak{k}\oplus\mathfrak{z}_0\text{-invariants}}$ is the algebra of sections of a $\mathbf{W}$-fiber bundle associated to the principal fiber bundle

$$\mathcal{N}_S(S_0)/S_0 \longrightarrow \mathcal{Q} \longrightarrow \mathcal{M}/H$$

Recall that $\mathfrak{k}_q^{\mathcal{Q}} \subset T_q\mathcal{Q}$. For any $k \in \mathfrak{k}$, let us define $\nu_q(k) = \mu_q(k_q^{\mathcal{Q}})$, so that $\nu \in C^\infty(\mathcal{Q}) \otimes \mathbf{W} \otimes \bigwedge^1 \mathfrak{k}^*$. This element contains the dependence of μ_q over $\mathfrak{k}_q^{\mathcal{Q}} \subset T_q\mathcal{Q}$, but ν and $\mu_{|\mathfrak{k}^{\mathcal{Q}}}$ are not equal as elements in $\left(\Omega^\bullet(\mathcal{Q}) \otimes \Omega^\bullet_{\mathfrak{z}_0}(\mathbf{W}) \otimes \bigwedge^\bullet \mathfrak{k}^*\right)^1$ (they do not have the same tri-graduation).

Proposition 2.8.8 (The $T_q\mathcal{Q}$ part of $T_q\mathcal{P}$)
One has $\mu - \eta - \nu \in \Omega^1_{\mathfrak{k}\oplus\mathfrak{z}_0}(\mathcal{M}/H;\mathbf{W})$, and this expression contains all the information about the restriction of ω to $T\mathcal{Q}$.

Let us now look at the $\mathfrak{l}^{\mathcal{P}}_q \oplus \mathfrak{m}^{\mathcal{P}}_q$ part of $T_q\mathcal{P}$.

Recall that $[\mathfrak{h}_0\oplus\mathfrak{k},\mathfrak{l}]\subset\mathfrak{l}$ and $[\mathfrak{z}_0,\mathfrak{m}]\subset\mathfrak{m}$, so that there are natural actions $[\mathfrak{h}_0,\mathfrak{l}\oplus\mathfrak{m}]\subset\mathfrak{l}\oplus\mathfrak{m}$ and $[\mathfrak{k}\oplus\mathfrak{z}_0,\mathfrak{l}\oplus\mathfrak{m}]\subset\mathfrak{l}\oplus\mathfrak{m}$. On the other hand, recall that $\mathfrak{s}_0\oplus\mathfrak{k}\oplus\mathfrak{z}_0$ is the Lie algebra of $\mathcal{N}_S(S_0)$ and $\mathfrak{k}\oplus\mathfrak{z}_0$ is the Lie algebra of $\mathcal{N}_S(S_0)/S_0$, with $\mathfrak{s}_0=\{(x_0,\lambda_* x_0)\ /\ x_0\in\mathfrak{h}_0\}$.

On the restriction of ω to $\mathcal{Q}$, the H-invariance and the G-invariance combine together into a $\mathcal{N}_S(S_0)$-invariance. One can treat this invariance in two steps: one for S_0 and the other one for $\mathcal{N}_S(S_0)/S_0$.

In order to encode the S_0-invariance, let us define the vector space of S_0-invariant linear maps $\mathfrak{l}\oplus\mathfrak{m}\to M_n$:

$$F=\{f:\mathfrak{l}\oplus\mathfrak{m}\to M_n\ /\ f([x_0,v])-[\lambda_* x_0,f(v)]=0,\ \forall x_0\in\mathfrak{h}_0,\ \forall v\in\mathfrak{l}\oplus\mathfrak{m}\}$$

on which $\mathfrak{k}\oplus\mathfrak{z}_0$ acts naturally using the Lie derivative $(L_{k+z}f)(v)=-f([k,v])+[z,f(v)]$ for any $k\in\mathfrak{k}$ and $z\in\mathfrak{z}_0$.

The $\mathcal{N}_S(S_0)/S_0$-invariance is then encoded into the space $\boldsymbol{M}=(C^\infty(\mathcal{Q})\otimes F)_{\mathfrak{k}\oplus\mathfrak{z}_0\text{-invariants}}$.

Proposition 2.8.9 (The $\mathfrak{l}^{\mathcal{P}}_q \oplus \mathfrak{m}^{\mathcal{P}}_q$ part of $T_q\mathcal{P}$)
$\boldsymbol{M}$ is the space of sections of the F-fiber bundle associated to the principal fiber bundle

$$\mathcal{N}_S(S_0)/S_0 \longrightarrow \mathcal{Q} \longrightarrow \mathcal{M}/H$$

It is a $\mathbf{C}$-bimodule.

The restriction of ω to the subspaces $\mathfrak{l}^{\mathcal{P}}_q \oplus \mathfrak{m}^{\mathcal{P}}_q$ is in $\boldsymbol{M}$.

Using the two previous decomposition of ω, one get the final identification:

Theorem 2.8.10 (The space of H-invariant noncommutative connections)
The space of H-invariant noncommutative connections on the endomorphism algebra $\mathbf{A}$ over the right $\mathbf{A}$-module $\mathbf{A}$ is the space $\Omega^1_{\mathfrak{k}\oplus\mathfrak{z}_0}(\mathcal{M}/H;\mathbf{W})\oplus\boldsymbol{M}$.

It is important to notice the following facts:

Remark 2.8.11 (Naturality of the spaces)
In this result, all the spaces are constructed from the principal fiber bundle

$$\mathcal{N}_S(S_0)/S_0 \longrightarrow \mathcal{Q} \longrightarrow \mathcal{M}/H$$

with the help of geometrical or algebraic methods which are natural in this noncommutative framework:

- $\mathbf{C}=(C^\infty(\mathcal{Q})\otimes\mathbf{W})_{\mathfrak{k}\oplus\mathfrak{z}_0\text{-invariants}}$ is modeled on the finite dimensional algebra $\mathbf{W}\subset M_n$. It looks like a "reduced algebra" constructed from $\mathbf{A}$, as $\mathbf{A}$ itself is a reduced algebra for $\mathbf{B}$.

- $\Omega^\bullet_{\mathfrak{k}\oplus\mathfrak{z}_0}(\mathcal{M}/H;\mathbf{W})$ is a natural differential calculus over $\mathbf{C}$.

- $\boldsymbol{M}=(C^\infty(\mathcal{Q})\otimes F)_{\mathfrak{k}\oplus\mathfrak{z}_0\text{-invariants}}$ is a natural $\mathbf{C}$-bimodule.

- The space $SU(\mathbf{C})$ acts naturally on the space $\Omega^1_{\mathfrak{k}\oplus\mathfrak{z}_0}(\mathcal{M}/H;\mathbf{W}) \oplus \boldsymbol{M}$ as restriction of noncommutative gauge transformations.
- All these spaces are sections of fiber bundles over the base space $\mathcal{M}/H$. ♦

We refer to [Masson and Sérié, 2005] for examples of such symmetric noncommutative restrictions, in particular the noncommutative generalisation of the well studied situation of spherical $SU(2)$ gauge fields over a flat four dimensional space-time. For purely noncommutative situations, the problem reduces to the study of the decomposition of some representation of G in M_n into irreducible representations.

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