Quantitative properties of real algebraic varieties

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Chapter 1

Introduction

1.1 Français

L'arrivée récente des ordinateurs dans tous les domaines des sciences a motivé beaucoup de recherches sur les aspects quantitatifs des objets mathématiques. Cette voie fut à l'origine suivie par les constructivistes au début du $20^{\text{ième}}$ siècle, mais ce n'est que récemment que les théories quantitatives trouvent un large domaine d'application. La transformation de théorèmes d'existence en algorithmes concrets a même créé une nouvelle activité de recherche connue sous le nom de mathématiques expérimentales. En opposition à l'analyse et à la géométrie différentielle, le monde informatique ne consiste que de données discrètes, ce qui fait de l'algèbre (discrète) et de la géométrie algébrique des outils naturels pour les technologies de l'information.
Les mathématiques quantitatives ne se préoccupent pas seulement de rendre des théorèmes effectifs, dans le sens de l'existence d'un algorithme, elles sont en fait principalement orientées vers la mise au point d'algorithmes rapides. Le temps et l'espace (mémoire) sont importants car avoir un algorithme qui donnera un résultat dans une centaine d'années ou bien qui requiert des millions de teraoctets de mémoire équivaut en pratique à ne pas avoir d'algorithme du tout. Dans ce contexte, il devient important de comprendre la complexité intrinsèque des objets que l'on manipule afin d'éviter de s'obstiner à modifier un algorithme en espérant résoudre rapidement un problème qui est en fait trop compliqué pour être traité en un temps raisonnable.
Cette thèse s'intéresse aux variétés semi-algébriques réelles ainsi qu'aux singularités qui apparaissent inévitablement lorsqu'on les manipule. Bien évidemment, même en s'étant restraint à ce domaine, le champ de recherche reste gigantesque. Les sujets abordés dans ce travail ont donc été choisis selon les orientations de mes encadrants. Je pense qu'il est raisonnable de dire que cette thèse explore les relations entre six principaux objets: la topologie et les triangulations, les techniques de subdivision, les stratifications, la complexité géométrique, la transversalité, et la densité. Topologie et triangulation sont groupées car dans un con-

texte informatique, l'encodage de la topologie par une triangulation est sans doute la seule représentation universelle qui existe. La topologie/triangulation est le sujet des quatres premiers chapitres 2, 3, 4 et 5. Le chapitre 2 présente un algorithme pour trianguler les courbes planes, et les outils utilisés sont donc relativement élémentaires. La technique qui soutend l'approche du chapitre 2 est la subdivision du domaine. Le chapitre 3 présente aussi une méthode de subdivision pour calculer la topologie d'une hypersurface (discutée dans le cas des surfaces). Cependant, la complexité géométrique des singularités pour des variétés de dimension supérieure ou égale à 2 ne permet pas d'aisément les trianguler. Par conséquent l'algorithme ne fournit une triangulation que lorsque la variété est lisse. D'autre part, la complexité de l'algorithme est estimée dans le cas lisse en fonction de la complexité géométrique de la surface. Lorsque la variété est singulière l'algorithme permet d'isoler la partie singulière aussi précisément que souhaité, mais ne garanti pas que le complexe simplicial obtenu est en effet une triangulation homéomorphe à l'hypersurface. Afin de traiter ces problèmes de singularité, les notions de stratification et de la transversalité sont introduites dans le chapitre 4 pour analyser la topologie des ensembles semi-algébriques dans $\mathbb{R}^n$ en général. Stratification et transversalité jouent un rôle important dans ce chapitre et ceux qui suivent (4, 5, 6). Dans le chapitre 4 elles sont utilisées pour donner une version étendue du théorème de Thom-Mather afin de donner une procédure générale de triangulation des objets semi-algébriques par une technique de subdivision. Le chapitre 5 explore une autre approche que la subdivision pour trianguler la variété. Il s'appuie aussi sur des stratifications mais utilise une méthode de balayage à la place d'une méthode de subdivision. Son domaine d'application est réduit aux variétés algébriques de $\mathbb{R}^3$ afin de pouvoir écrire un algorithme complet de triangulation (sans aucune restriction sur la variété). Le chapitre final 6, donne une information topologique plus abstraite: une borne uniforme sur un nombre de composantes connexes pour un germe de variété analytique. Cette borne peut à son tour être utilisée pour obtenir une borne sur la densité du germe et de manière plus générale sur ses invariants de Lipschitz-Killing. Ces quantitées sont reliées à la complexité géometrique du germe.

L'obtention d'une triangulation (possiblement approchée) d'un ensemble défini par des équations polynomiales est un problème qui a reçu beaucoup d'attention depuis l'avènement de l'ère informatique. Le chapitre 3 présente un algorithme qui fournit toujours une triangulation proche de l'hypersurface algébrique. La présentation des techniques existantes pour obtenir des triangulations approchées des variétés algébriques est donc faite en son début.

Le rapport entre stratification, transversalité et topologie est bien connu. La notion de stratification fut introduite par H. Whitney dans [141] et fut rapidement utilisée par R. Thom dans [132] pour analyser des propriétées topologiques. Le théorème de trivialité topologique qui est central dans la théorie est connu sous le nom de théorème de Thom-Mather et il est prouvé dans [132] et dans [95] de manière plus détaillée. L'histoire ne s'arrête pas là, et une présentation plus circonstanciée de l'historique est donnée dans la section 4.1 où elle s'intègre mieux à la discussion.

Le rapport entre la densité et l'analyse de la complexité intrinsèque des objets géométriques est plus récente. Ce rapport est explicité dans le chapitre 5 du livre [144]. Le livre discute en détail du rapport entre la quantification de la transversalité et la complexité des objets géométriques. La densité est aussi liée aux stratifications car elle peut-être utilisé comme critère de stratification en relation avec l'équisingularité (voir [38, 39]). De nouveau, une présentation plus détaillée de l'historique est faite dans l'introduction du chapitre 6.
La présentation précédente était du point de vue des concepts manipulé dans cette thèse. Pour conclure cette introduction, voici une présentation plus linéaire du contenu, chapitre par chapitre :

- Le chapitre 2 traite d'un problème computationnel sur lequel j'ai travaillé à la fin de ma thèse. C'est essentiellement une version amélioré de [4]. Il y est expliqué une procédure de subdivision permettant de trianguler une courbe algébrique réelle dans le plan. Les outils mathématiques clefs sont le degré topologique, aussi connu sous le nom de degré de l'application de Gauss, ainsi que la représentation des polynômes dans la base de Bernstein. Ces deux outils sont alliés dans une méthode de subdivision récursive. L'algorithme décrit a été implémenté et présente une efficacité nettement supérieure aux autres méthodes existantes tout en fournissant un résultat certifié.

- Le chapitre 3 est basé sur l'article [2] qui fut publié avant le début de ma thèse. Il introduit naturellement la problématique du traitement des parties singulières, ce qui motive l'usage des stratifications pour les traiter. Il présente aussi les techniques usuelles utilisées dans une méthode de subdivision telles que les octrees et la base de Bernstein. Il montre aussi comment une analyse de complexité peut-être formulée en terme d'invariants géométriques.

- Le chapitre 4 discute du problème avec lequel j'ai commencé mon travail de thèse. Il se situe donc à la suite de mon stage de DEA. À cette époque, j'avais programmé une version améliorée de l'algorithme du chapitre 3 qui utilisait les stratifications de Whitney afin de trianguler les surfaces algébriques réelles en 3 dimensions par une méthode de subdivision. Le programme souffrait cependant de difficultés techniques et théoriques. Durant ma thèse j'ai poursuivi ce travail en changeant de perspective. L'approche originelle était centrée sur une méthode de subdivision, mais le coeur du travail dans ce chapitre est finalement une théorie quantitative de la transversalité à une application semi-algébrique non nécessairement lisse. La description d'un algorithme devient une application de ces résultats. Le développement de cette théorie quantitative donne lieu à une version quantitative du théorème de trivialité topologique de Thom-Mather (théoréme 4.1.10) qui s'obtient par le contrôle précis des champs de vecteurs intégrés. Elle est développée dans le cadre des ensembles stratifiés de Whitney, même si elle peut probablement en partie être étendue à un cadre plus général comme celui des stratifications C-régulières de Bekka

[15]. Cette théorie donne lieu à une version plus raffinée et "métriquement stable" du théorème de structure conique local et plus généralement de l'existence d'un "tube de Milnor" autour des strates d'une stratification de Whitney. Elle est finalement appliquée à la mise en oeuvre d'un algorithme de triangulation utilisant des partitions de Voronoĭ. La mise en place d'un algorithme effectif n'est pas complète car le problème de l'estimation effective de la transversalité n'est pas traité. Il semble cependant raisonnable de s'attendre à pouvoir le faire en utilisant des calculs algébriques dans l'éclatement de la variété par exemple.

- Le chapitre 5 présente une méthode pour trianguler une variété algébrique quelconque dans $\mathbb{R}^3$. Elle repose sur une approche par la théorie de Morse stratifiée [67] et le théorème de Thom-Mather. Afin de rendre les théorèmes de cette théorie effectifs, des techniques de calcul de résultants sont mises en œuvre. Ce chapitre est une version adaptée de [5] qui est lui même une version finalisée de [106].

- Le chapitre 6 présente une borne sur les nombres de Betti dans une section d'un germe analytique réel par un espace affine générique. La borne est polynômiale en la multiplicité du germe et exponentielle en la dimension de l'espace. Contrairement au cas complexe, ces deux paramètres ne suffisent pas toujours à borner ce nombre de composantes connexes. Le résultat est donc prouvé sous certaines conditions, et des contre-exemples où la multiplicité est constante alors que le nombre de composantes connexes tend vers l'infini sont donnés afin de montrer que les conditions proposées sont optimales dans un sens précis.

1.2 English

The recent advent of computers in every area of science has prompted a lot of research on quantitative aspects of mathematical objects. The trend was started with constructivism at the beginning of the 20^{th} century, but it is only now that quantitative theories find a vast field of application. The turning of existential theorems into concrete algorithms has even created a new research activity known as experimental mathematics. As opposed to calculus and differential geometry, information technology deals with discrete data only, which makes (discrete) algebra and algebraic geometry natural tools for this latter field.
Quantitative mathematics not only deals with making theorems effective, in the sense of the existence of an algorithm, it is in fact mainly oriented towards the designing of efficient algorithms. Time and space (memory space) are important issues since having an algorithm that will yield a result in a hundred years or one that requires millions of terabytes of memory concretely amounts to having no algorithm at all. In this context, it becomes important to understand the intrinsic complexity of the objects one manipulates so as to avoid stubbornly trying to solve quickly a problem that is in fact too complicated to ever be

solved efficiently.
This thesis is concerned with real semi-algebraic varieties and the singularities that inevitably arise when manipulating them. Of course, even having restricted oneself to this field, the area of research is still immense. The aspects this work focuses on thus derive from the direction in which I was oriented by my advisers. I believe it is fair to say that this thesis explores the interrelations of six main objects: topology and triangulations, stratifications, subdivision techniques, geometric complexity, transversality, and density. Topology and triangulations are grouped together since, in an information technology context, triangulations are the only universal representations for topology available. Topology is the concern of the four first chapters 2, 3, 4, and 5. Chapter 2 gives an algorithm to triangulate a planar curve, and the tools it uses are rather elementary. The strategy that underlies this approach is the subdivision of the domain. Chapter 3 also presents a subdivision method to compute the topology of a hypersurface. The discussion is made for three-space, but it can be generalized to any dimension without obstructions. However, this method only produces approximate triangulations for hypersurfaces that are singular. This is due to the complexity of singularities as soon as the variety is of dimension 2 or higher. Consequently, the algorithm is guaranteed to yield a triangulation homeomorphic to the hypersurface only when it is smooth. In addition, the complexity of the algorithm is assessed for the smooth case in terms of the geometric complexity of the variety. When the variety is singular, the algorithm isolates the singular locus with any given level of precision, but does not guarantee that the triangulation has same topology as the original variety. So as to treat those singularity problems, the concepts of stratification and transversality are introduced in chapter 4, and serve to analyze the topology of semi-algebraic sets in $\mathbb{R}^n$ in general. These two concepts will then be used until the end of the thesis, chapters 4, 5 and 6. In chapter 4 they are utilized to give an extended version of Thom-Mather's theorem in order to describe a general subdivision-based triangulation procedure for semi-algebraic objects. Chapter 5 explores another approach than subdivision to triangulate the variety; it uses a sweeping method instead. It also relies on stratifications but its scope is limited to algebraic varieties in $\mathbb{R}^3$ so as to write a complete algorithm to triangulate them (without any restriction on the variety). The final chapter 6 gives a more abstract topological information: a bound on a number of connected components for a real analytic germ. This bound can in turn be used to yield a bound on the density of a germ (and more generally on its Lipschitz-Killing invariants). These quantities are related to the geometric complexity of the germ.
Obtaining a triangulation (possibly an approximate one) of a set defined by polynomial equations is a problem which received a lot of attention since the beginning of the computer era. Chapter 3 presents an algorithm that always produces a simplicial complex that is close to the algebraic hypersurface. We thus present at its beginning the techniques that exist to compute approximate triangulations of algebraic varieties.
The connection between stratification, transversality and topology is well-known. The notion of stratification was introduced by H. Whitney in [141] and was soon

used by R. Thom in [132] to analyze topological properties. The central topological triviality theorem of the theory is known as Thom-Mather's theorem as was proved in [132] and in greater details in [95]. The history does not stop at this point, and a more circumstanced discussion of this background is made when it is most relevant in section 4.1.
The connection between density and the analysis of the intrinsic complexity of geometric objects is more recent. This connection is made in chapter 5 of the book [144]. The book makes a general discussion of the connection between quantitative transversality and the complexity of geometric objects. Density is also related to stratifications as it can be used as a stratification criterion in relation to equisingularity (see [38, 39]). Again a more detailed review of the background is made in the introduction of chapter 6.
We have made a review of the content of this thesis from the viewpoint of the concepts it involves. To conclude this introduction, let us make a more linear presentation of its content, chapter by chapter:

- Chapter 2 is about a computer related problem I worked on toward the end of my Ph.D. It is essentially an improved version of [4]. It explains a subdivision method to triangulate a real algebraic planar curve. The key mathematical tools involved are the topological degree, otherwise known as the degree of the Gauss map, and the representation of polynomials in the Bernstein basis. These two tools come together under the framework of a recursive subdivision method. The algorithm that it describes has been implemented and features a much higher efficiency than other currently existing methods, while providing a certified output.

- Chapter 3 is based on the article [2] which was published before I entered my Ph. D program. This article naturally introduces the problems posed by the singular locus, and motivates the use of stratifications to handle it. It also presents techniques involved in subdivision methods such as octrees and Bernstein basis. It also shows how a complexity analysis in terms of geometric invariants can be carried out.

- Chapter 4 presents the problem on which I started working at the beginning of my Ph.D preparation. It presents the continuation of my DEA (the equivalent of a Master's degree) internship. I had then programmed an algorithm based on Whitney stratifications to triangulate real 3-dimensional algebraic surfaces. However, the algorithm had several technical and theoretical flaws. During my Ph.D preparation I followed on that work and switched viewpoint. The original perspective was that of a subdivision method, but the core of chapter 4 is now a quantitative theory of the transversality to a semi-algebraic map which is not necessarily smooth. The description of a triangulation algorithm thus became an application of those results.
  The development of this quantitative theory gives rise to a quantitative version of Thom-Mather's topological triviality theorem 4.1.10 which is obtained through the fine control of the integrated vector fields. It is

developed in the theoretical framework of Whitney stratified sets, even though some of the results can be extended to a more general setting such as C-regular Bekka stratifications [15]. This theory allows us to give refined and "metrically stable" versions of the local conic structure theorem and of the existence of a "Milnor tube" around the strata of a Whitney stratification. Finally the theory is applied to devising a triangulation algorithm using Voronoĭ partitions. However, a complete algorithm is not fully described as this would require an effective means to estimate the transversality. At the time of writing, this had not yet been done. Nonetheless it is expected that such a thing can be achieved through the algebraic manipulation of the blow-up along the strata and the use of classical real algebraic quantitative techniques.

- Chapter 5 presents a triangulation for arbitrary algebraic varieties in three-space. It relies on stratified Morse theory and the Thom-Mather theorem. In order to make the theorems of this theory effective, computations of the resultant are carried out. This chapter is a revised version of [5] which is in turn a finalized version of [106].

- Chapter 6 presents a bound on the Betti numbers of the section of a germ by a generic affine space. The bound is polynomial in the multiplicity of the germ and exponential in the dimension of the ambient space. Contrary to the complex case, those two parameters are not always enough to bound this number of connected components. The result is thus proved under some conditions. We show that those conditions are optimal in a precise sense by giving counter-examples where the multiplicity is constant while the number of connected components diverges to infinity.

Chapter 2

Fast and certified topology computations for planar curves

The goal of this chapter is to present a method to determine the topology of an algebraic planar curve inside a rectangle $[a, b] \times [c, d]$ of $\mathbb{R}^2$. The way it represents the topology is by outputting a topological complex (vertices and straight line segments between them) that can be continuously deformed into the algebraic curve inside the rectangle. This presentation is essentially a revised version of the article [6] that is to be published in the CAGD journal, this article was in turn a revised version of the Pacific Graphics 2007 paper [4]. Another more general paper on topology computation [3] had been published earlier that year. The part that deals with the 2d topology computation in [3] is along the same lines as the following exposition. This article was produced during the preparation of my Ph. D, but due to the considerable overlap of its content with the rest of this thesis, it is not reproduced here.
As of today there are two main types of algorithms that enable one to carry out topology computations for algebraic curves: subdivision type algorithms and Cylindrical Algebraic Decomposition (CAD) type algorithms. We will discuss their general behaviors in chapters 3 (for subdivision) and 4 (for CAD). For now we make a specialized discussion of them for planar curves. The need for a specific discussion in this case arises from the fact that, because the dimension is very low, the prominent features of the algorithms' behavior differ greatly from the general case in higher dimensions.

The CAD algorithm first appears in [36]. The common feature that CAD-like algorithms share is that they proceed by projection. We can think of the way they function as a conceptual sweeping line perpendicular to some axis that detects the critical topological events: tangents to the sweeping line and singularities. They involve the exact computation of critical points, genericity condition tests and adjacency tests. They assume exact input equations and

rely on the analysis of the curve at the critical values of its projection. From an algebraic point of view, they involve the computation of (sub)-resultant polynomials and of their roots which are algebraic numbers. This can be a bottleneck in many examples with large degree and large coefficients, for which the resultant is difficult to compute, and its real roots even harder to manipulate. There is unfortunately no easy fix for this problem as CAD-like methods are very delicate to apply using approximate computation because of the numerical instability introduced by multiple points (e.g. tangent point to the sweeping line, cusps, ...).
Even if the problems that arise from the cost of exact algebraic operations are overcome, CAD-like methods suffer a more intrinsic drawback: as these algorithms work by projection, they have to compute every point in the fibers above the points in the projection. In other words, most points that they compute are actually useless for the computation of the final topological description.
The complexity of the algorithm can also vary wildly, depending on the direction of projection we choose. Also, non-degeneracy conditions have to be checked (which can be difficult by itself) to ensure the correctness of the algorithm. The problem is that the choice of the projection direction is not at all related to the geometry of the curve. This is why the Cylindrical Algebraic Decomposition methods are hardly efficient in practice.

The other type of methods relies on subdivision techniques of the original domain. This process is most commonly used to get approximations of the curve in terms of Hausdorff distance. The most famous family of algorithms using this approach is the marching cube algorithms family [93]. It does not give any guarantee on the topological correctness of its output, but it has inspired some algorithms that do certify that their output has the same topology as the curve under some conditions (usually in the smooth case). They have already been used for solving several complicated equations. See [125, 52] and the recent improvements proposed in [104], exploiting preconditioning techniques. Extensions of this approach to higher dimensional objects have also been considered [123, 80, 73, 78, 86, 117], and we will return to this in chapter 3. In their original form, these subdivision methods are not completely reliable when singular points exist in the domain. If a floor for the size of the cells of the subdivision is not set, these algorithms do not always terminate. Indeed at singularities, no matter the scale of approximation, the shape and topology of an algebraic object remains similar and if the algorithm is unable to terminate past a certain resolution, it never will.

Our algorithm, like the one in [118], is hybrid. It is a subdivision method and thus adapts to the geometry of the surface, meanwhile it enjoys the guarantee on the correctess of the output afforded by CAD-like methods. It subdivides the domain $\mathcal{D}_0$ into regular regions in which the curve is smooth and regions that contain singular points. In the regular regions, we can approximate the curve as precisely as we want and the "singular" regions can be made as small as required. The algorithm computes the topology inside the regions by using what happens on their frontiers and we use enveloping techniques to efficiently treat large input equations. The fact that one can recover the topology inside

the regions from what happens on their frontiers renders useless the exact representation of the singular points. This makes possible the use of numerical approximations and hereby fast floating point computation with machine numbers.
Nonetheless, although the method described hereafter does not rely on exact representations of algebraic numbers it is important to notice that it requires a subroutine which computes arbitrarily small isolation boxes for the singular points. Isolation of roots (here singular points) is a classical problem. Several techniques are known to find such isolating boxes quickly in most cases [125, 52, 104]. However, the most degenerate cases will require to go back to a purely algebraic approach such as Rational Univariate Representation of roots. It would thus be a misconception to consider that the method presented here gets rid of every algebraic tool, but it reduces greatly, if not completely, their usage, and restricts it to the well-known and much worked-on problem of root isolation. A short presentation of a simple way to implement such a root isolation method is given at the end of the section. It is intended for completeness of the presentation, but is by far not the most efficient way to carry out the task. As explained before the root isolation routine is easily separable from the rest of the algorithm and can be replaced by any other more efficient isolation method. In practice, the algorithm displays outstanding performances in comparison to any heavily algebraic method, even with the most naive implementation of root isolation.

2.1 Notations and definitions

In this first section we set the basic notations and definitions we use throughout this chapter. Then the following sections present the specifics of the method for topology computation, and for manipulating arrangements.

The implicit curves we manipulate are defined by squarefree polynomials in $\mathbb{Q}[x, y]$. For $f \in \mathbb{Q}[x, y]$, $\mathcal{Z}(f) = \{(x, y) \in \mathbb{R}^2 | f(x, y) = 0\}$ will denote its zero set. But when we deal with only one curve (i.e. the zero set of a single function), we will simply refer to the curve as $\mathcal{C}$ and to its equation as f. The rectangular domain in which we carry out all of our computations is denoted by $\mathcal{D}_0 := [a, b] \times [c, d] \subset \mathbb{R}^2$.

The set of *singular points* of $\mathcal{C}$ is denoted $\mathrm{Sing}(\mathcal{C}) := \{(x, y) \in \mathbb{R}^2 | f(x, y) = \partial_x f(x, y) = \partial_y f(x, y) = 0\}$. *Smooth points* of $\mathcal{C}$ are all the points of $\mathcal{C}$ that are not singular.

The set of *critical* or *extremal points* of f is denoted $\mathcal{Z}_e(f) := \{(x, y) \in \mathbb{R}^2 | \partial_x f(x, y) = \partial_y f(x, y) = 0\}$.

The set of *x-critical points* (resp. *y-critical points*) is the set of points such that $\delta_y f = 0$ (resp. $\delta_x f = 0$). In other words, x-critical points are points where $\mathcal{C}$ has a vertical tangent, and y-critical points are points where $\mathcal{C}$ has a horizontal tangent.

We recall that a tangent to the curve $\mathcal{C}$ is a line, which intersects $\mathcal{C}$ with multiplicity ≥ 2. In particular, any line through a singular point of $\mathcal{C}$ is tangent

to $\mathcal{C}$.

For a subset $S \subset \mathbb{R}^2$, we denote by S° its *interior*, by $\overline{S}$ its *closure*, and by ∂S its *topological frontier*. We call *domain* any compact set $\mathcal{D}$ such that $\mathcal{D}^\circ \neq \emptyset$ and $\mathcal{D}$ is simply connected. And we call *region* any open set R which is a connected component of the complement of an algebraic curve.

We call *branch* (relative to a domain $\mathcal{D}$), any smooth closed segment (i.e. C^∞ diffeomorphic to $[0,1]$) whose endpoints are on $\partial\mathcal{D}$.

We call *half branch* at a point $p \in \mathcal{D}^\circ$ or half branch originating from $p \in \mathcal{D}^\circ$, any smooth closed segment which has one endpoint on $\partial\mathcal{D}$ and whose other endpoint is p.

2.2 Overview

Our objective is this section is to determine the topology of an algebraic curve $\mathcal{C}$ inside a rectangular domain $\mathcal{D}_0$. To do this, we find a partition of $\mathcal{D}_0$ into what we call *simple domains* $\mathcal{D}_i$ for which we can compute the topology. Then we can piece together the topologies of the simple domain by gluing them on their boundaries. For each kind of simple domain, we have a *connection algorithm* that computes a piecewise linear approximation of the curve inside simple domains of that type. Finally, to be able to reconstruct the global topology in $\mathcal{D}_0$ we have to ensure that the approximations on the $\mathcal{D}_i$ agree on the boundaries. Our connection algorithms have this property at no extra cost.

Our approach is iterative, which means we do not construct a partition in simple domains in one pass. Instead we guess such a partition, test it, and if it doesn't work, we refine it by splitting the subdomains that are not yet simple domains. Each type of simple domain is defined by a set of type conditions and we have test algorithms to effectively check them.

We distinguish three different types of simple domains: x-regular domains, y-regular domains and simply singular domains.

Definition 2.2.1. *A domain $\mathcal{D}$ is* x-regular *(resp.* y-regular*) for $\mathcal{C}$ if $\mathcal{C}$ is smooth in $\mathcal{D}$ and it has no vertical (resp. horizontal) tangent in $\mathcal{D}$. This is algebraically formulated as the following condition: $\mathcal{Z}(f, \partial_y f) \cap \mathcal{D} = \emptyset$ (resp. $\mathcal{Z}(f, \partial_x f) \cap \mathcal{D} = \emptyset$).*

We might equivalently say that the curve $\mathcal{C}$ is x-regular (resp. y-regular) in $\mathcal{D}$ instead of saying that $\mathcal{D}$ is x-regular (resp. y-regular) for $\mathcal{C}$ in $\mathcal{D}$.

Remark 2.2.2. *Pay attention to the fact that x-regularity is a condition on the partial derivative with respect to y. It ensures that the orthogonal projection to the x-axis is a submersion. A similar remark applies to y-regularity.*

Finally we say for short that a curve is *regular* in $\mathcal{D}$, or equivalently that $\mathcal{D}$

is *regular* for $\mathcal{C}$ if $\mathcal{C}$ is x-regular *or* y-regular in $\mathcal{D}$.

Definition 2.2.3. *A domain $\mathcal{D}$ is* simply singular *for $\mathcal{C}$ if* $\mathrm{Sing}(\mathcal{C}) \cap \mathcal{D} = \{p\}$ *and if the number n of half branches of $\mathcal{C}$ at the singular point p is equal to $\sharp(\partial\mathcal{D} \cap \mathcal{C})$, the number of points of $\mathcal{C}$ on the frontier of $\mathcal{D}$.*

We now give a description of the content of the subsequent subsections: Section 2.3 contains the description of a connection algorithm for x-regular and y-regular domains and a test for the x and y-regularity conditions based on Bernstein basis representation of polynomials. In section 2.4, we introduce the topological degree in order to compute the number n of half branches originating from a singular point, from this we deduce a test of regularity. Section 2.5 puts together the elements introduced in the previous sections and describes two different strategies to find a partition of $\mathcal{D}$ in simple domains. We isolate the roots of a bivariate polynomial system, using either a Bernstein subdivision solver to approximate efficiently $\mathcal{C}$ or algebraic techniques to certify the result. Section 2.6 shows some experimental results.

2.3 Regular domains

In this section, we consider a curve $\mathcal{C}$ in $\mathbb{R}^2$, defined by the equation $f(x, y) = 0$ with $f \in \mathbb{Q}[x, y]$ and a domain $\mathcal{D} = [a, b] \times [c, d] \subset \mathbb{R}^2$.

We are going to show that if $\mathcal{C}$ is x-regular in $\mathcal{D}$, then its topology can be deduced from its intersection with the frontier $\partial\mathcal{D}$. By symmetry the same applies when $\mathcal{C}$ is y-regular. We only require that $\partial\mathcal{D} \cap \mathcal{C}$ be 0-dimensional. This is a very mild requirement that can be easily taken care of when choosing a partition of the initial domain.

Remark 2.3.1. *This is well defined because we required that $\partial_y f$ does not vanish at any point of $\mathcal{C}$ in $\mathcal{D}$.*

Definition 2.3.2. *For a point $p \in \mathcal{C} \cap \partial\mathcal{D}$, and a sufficiently small neighborhood U of p, by the implicit function theorem, $\mathcal{C}$ is a function graph over the x-axis because $\partial_y f(p) \neq 0$. We define the local right branch at p relative to U as the portion of $\mathcal{C}$ in the half plane $x > x_p$. We define the local left branch at p relative to U as the portion of $\mathcal{C}$ in the half plane $x < x_p$.*

Definition 2.3.3. *For a point $p \in \mathcal{C} \cap \partial\mathcal{D}$, we define its x-index.*

\+ *if $\mathcal{C}$ enters $\mathcal{D}$ locally: there exists a local left (resp. right) tangent lying outside (resp. inside) $\mathcal{D}$.*

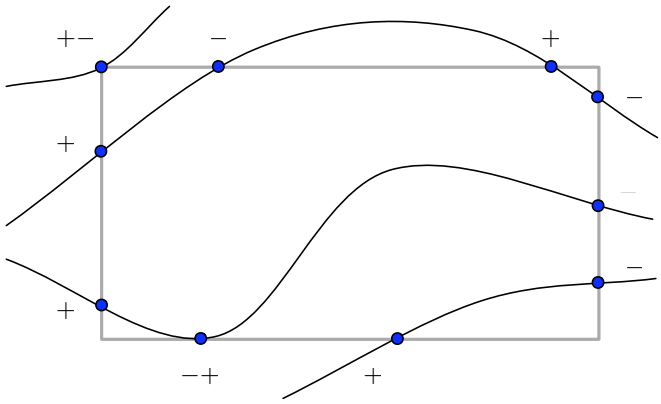


Figure 2.1: x-indices of an x-regular domain

- *if $\mathcal{C}$ exits $\mathcal{D}$ locally: there exists a local left (resp. right) tangent lying inside (resp. outside) $\mathcal{D}$.*

+- *if $\mathcal{C}$ is tangent to $\mathcal{D}$ and does not enter it locally: $\mathcal{C} - \{p\}$ locally lies outside $\mathcal{D}$.*

-+ *if $\mathcal{C}$ is tangent to $\mathcal{D}$ and does not exit it locally: $\mathcal{C} \subset \mathcal{D}$.*

Remark 2.3.4. *This is well defined because if there exists a local left (resp. right) tangent lying outside (resp. inside) $\mathcal{D}$, then there cannot exist a local left (resp. right) tangent lying inside (resp. outside) $\mathcal{D}$.*
And we necessarily fall into one of these cases because $\partial\mathcal{D} \cap \mathcal{C}$ is 0-dimensional.

These conditions can be effectively tested using the sign s_y of $\partial_y f$, the order k of the first x derivative of f that does not vanish, and the sign s_x of $\partial_x^k f$. The integer k is well defined because if all these partial derivatives were 0, the whole horizontal line would be included in $\mathcal{C}$ which would mean $\mathcal{C} \cap \partial\mathcal{D}$ is not 0-dimensional.

This table summarizes how to obtain the x-index I_p of a point $p \in \mathcal{D}$ from these 3 numbers for a box $\mathcal{D} = [a, b] \times [c, d]$. The individual tables are laid out so that their positions on the page corresponds to the position of p on the boundary of the box.

$p = (a,d)$

$s_y s_x (-1)^k > 0$	$I_p = +$
$s_y s_x (-1)^k < 0$	$I_p = +-$

$p \in](a,d),(b,d)[$

k odd	$s_y s_x > 0$	$I_p = -$
k odd	$s_y s_x < 0$	$I_p = +$
k even	$s_y s_x > 0$	$I_p = -+$
k even	$s_y s_x < 0$	$I_p = +-$

$p = (b,d)$

$s_y s_x (-1)^k > 0$	$I_p = -$
$s_y s_x (-1)^k < 0$	$I_p = +-$

$p \in](a,c),(a,d)[$	$I_p = +$

$p \in](b,c),(b,d)[$	$I_p = -$

$p = (a,c)$

$s_y s_x (-1)^k < 0$	$I_p = +$
$s_y s_x (-1)^k > 0$	$I_p = +-$

$p \in](a,c),(b,c)[$

k odd	$s_y s_x > 0$	$I_p = +$
k odd	$s_y s_x < 0$	$I_p = -$
k even	$s_y s_x > 0$	$I_p = +-$
k even	$s_y s_x < 0$	$I_p = -+$

$p = (b,c)$

$s_y s_x (-1)^k < 0$	$I_p = -$
$s_y s_x (-1)^k > 0$	$I_p = +-$

In the following the points with double index (+- or -+) are considered as double points, one with "smaller x component" than the other (although they correspond to a single point that has only one x component). The one with smaller x component gets the left part of the double index, and the one to its right (bigger x component) gets the right part.

Lemma 2.3.5. *If $\mathcal{C}$ is x-regular in $\mathcal{D}$, then a branch of $\mathcal{C} \cap \mathcal{D}$ connects a point p of x-index $+$ to a point q of x-index $-$, such that $x_p < x_q$.*

Proof. As the curve is x-regular, it has no vertical tangent and thus no closed loop in $\mathcal{D}$. Consequently, each of the interior connected components of $\mathcal{C} \cap \mathcal{D}$ intersects $\partial\mathcal{D}$ in two distinct points $p, q \in \mathcal{C} \cap \partial\mathcal{D}$ (with $x_p \leq x_q$).

Assume that the x-indices of p and q are the same. Suppose that this index is $+$. Then for an analytic parameterization $s \in [0,1] \mapsto (x(s), y(s))$ of the branch $[p,q]$ with $(x(0), y(0)) = p$, $(x(1), y(1)) = q$, we have $\partial_s x(0) > 0$, $\partial_s x(1) < 0$. This implies that for a value $0 < s_0 < 1$, $x(s_0) > x(1) = x_q \geq x(0) = x_p$ and that there exists $s_0' \in]0,1[$ such that $x(s_0') = x(1)$. We deduce that $\partial_s x(s)$ vanishes in $[0,1]$ and that the branch $[p,q]$ of $\mathcal{C}$ has a vertical tangent, which is excluded by hypothesis. If the index of p and q is $-$, we exchange the role of p and q and obtain the same contradiction. As $\partial_s x(s) > 0$ for $s \in [0,1]$, we have $x_p < x_q$, which proves the lemma. □

Lemma 2.3.6. *Suppose that $\mathcal{C}$ is x-regular in $\mathcal{D}$ and let p, q be two consecutive points of $\mathcal{C} \cap \partial\mathcal{D}$ with: q such that x_q is minimal among the points with x-index$= -$, and $x_p < x_q$, then p, q belong to the same branch of $\mathcal{C} \cap \mathcal{D}$.*

Proof. Suppose that p and q are not on the same branch. Let p' be the other endpoint of the branch going to q. Let q' be the other endpoint of the branch

starting from p. By lemma 2.3.5, x-index$(p') = +$ and $x_{p'} < x_q$. By that same lemma, x-index$(q') = -$ and $x_p < x_{q'}$.

The branch (p', q) separates $\mathcal{D}$ in two connected components. We call C_r the one whose frontier $B_r = \partial C_r$ contains the point p.

Because (p', q) and (p, q') do not intersect, p and q' are in the same connected component of $\mathcal{D} - (p', q)$ and on B_r.

Consider the topological sub-frontier $\{x \geq x_q\} \cap B_r$. It must be connected. Otherwise the branch (p', q) would intersect $x = x_q$ in two distinct points and the curve would have an x-critical point in between. We denote by $q, \tilde{q}$, the endpoints of $\{x \geq x_q\} \cap B_r$ (with possibly $q = \tilde{q}$). We decompose B_r as the union of arcs $B_r = (p', q) \cup (q, \tilde{q}) \cup (\tilde{q}, p')$ with $(q, \tilde{q}) \subset \partial\mathcal{D}$, $(\tilde{q}, p') \subset \partial\mathcal{D}$.

By minimality of x_q, we have $x_{q'} \geq x_q$ so that $q' \in \{x \geq x_q\} \cap B_r = (q, \tilde{q})$. Because $x_p < x_q$ and $p \in B_r$ and $p \notin (p', q)$, we have $p \in (\tilde{q}, p') \subset \partial\mathcal{D}$.

This proves that p is in between p' and q' and q' is in between p and q on $\partial\mathcal{D}$. Therefore, p and q cannot be consecutive points of $\mathcal{C}$ on $\partial\mathcal{D}$. By way of contradiction, we conclude that p and q must be on the same branch of $\mathcal{C}$. □

Proposition 2.3.7. *Let $\mathcal{C} = \mathcal{Z}(f)$. If $\mathcal{D}$ is an x-regular domain, the topology of $\mathcal{C}$ in $\mathcal{D}$ is uniquely determined by its intersection $\mathcal{C} \cap \partial\mathcal{D}$ with the frontier of $\mathcal{D}$.*

Proof. We prove the proposition by induction on the number $N(\mathcal{C})$ of points on $\mathcal{C} \cap \partial\mathcal{D}$. We denote this set of points by $\mathcal{L}$.
Since the curve has no vertical tangent in $\mathcal{D}$ and has no closed loop, each of the connected components of $\mathcal{C} \cap \mathcal{D}^\circ$ have exactly two distinct endpoints on $\partial\mathcal{D}$. Thus if $N(\mathcal{C}) = 0$, then there is no branch of $\mathcal{C}$ in $\mathcal{D}$.
Assume now that $N(\mathcal{C}) > 0$, and let us find two consecutive points p, q of $\mathcal{L}$ with x-index$(p) = +$, x-index$(q) = -$, $x_p < x_q$ and x_q minimal. By lemma 2.3.6, the points p, q are the endpoints of the branch of $\mathcal{C}$.
Removing this branch from $\mathcal{C}$, we obtain a new curve $\mathcal{C}'$ which is still x-regular and such that $N(\mathcal{C}') < N(\mathcal{C})$. We conclude by the induction hypothesis, that the topology of $\mathcal{C}'$ and thus of $\mathcal{C}$ is uniquely determined. □

Proposition 2.3.8. *If $\mathcal{C}$ has at most one x-critical or y-critical point in $\mathcal{D}$, which is also smooth, then its topology in $\mathcal{D}$ is uniquely determined by its intersection with the frontier of $\mathcal{D}$.*

Proof. Suppose $\mathcal{C}$ has at most one x-critical point in $\mathcal{D}$, which is smooth, then the curve is smooth in $\mathcal{D}$ and has no closed loop inside $\mathcal{D}$ (otherwise the number of x-critical points would be at least 2). Therefore, the branches are intersecting $\partial\mathcal{D}$ in two points. If there is no branch has an x-critical point, by lemma 2.3.5 their x-index $\in \{-, +\}$ are distinct. If the branch has an x-critical point of even multiplicity (i.e. $\min\{k \in \mathbb{N} \mid \delta_y^k f = 0\}$ is even), then the x-indices of the end-points of the branch in $\mathcal{C}$ are the same. If there are only two points

of $\mathcal{C}$ on ∂D, then this branch is connecting the two points. As the curve is smooth, the branches do not intersect. If there are more points, and thus at least 2 branches, the branch with the even x-critical point is separating the set of branches into two disjoint subsets of branches with no x-critical points. Changing the orientation of the x-axis if necessary, we can find consecutive points p, q on ∂D which satisfy the hypothesis of lemma 2.3.6. By this lemma, they are necessarily on the same branch of one of these two subsets. Removing this branch from $\mathcal{C}$ and processing recursively in this way, we end up either with no point on $\partial\mathcal{D}$ or two points on $\partial\mathcal{D}$ with the same x-index. These points are necessarily connected by the branch containing the x-critical point of $\mathcal{C}$ in $\mathcal{D}$. □

This leads to the following algorithm:

Algorithm 2.3.1: Connection for an x-regular domain

Input: an algebraic curve $\mathcal{C}$ and a domain $\mathcal{D} = [a, b] \times [c, d] \subset \mathbb{R}^2$ such that $\mathcal{C}$ has no vertical tangent in $\mathcal{D}$
Output: the set $\mathcal{B}$ of branches of $\mathcal{C}$ in $\mathcal{D}$
Isolate the points $\mathcal{C} \cap \partial\mathcal{D}$ and compute their x-index ;
Order the points of $\mathcal{C} \cap \partial\mathcal{D}$ with nonzero x-indices clockwise and store them in the circular list $\mathcal{L}$;
while $\mathcal{L} \neq \emptyset$ **do**
    Take a point q such that x_q is minimal among the points in $\mathcal{L}$ with x-index$= -$;
    Take the point p that follows or precedes q in $\mathcal{L}$ such that $x_p < x_q$ (thus x-index$(p) = +$) ;
    Add the arc $[p, q]$ to the set $\mathcal{B}$ of branches and remove p, q from $\mathcal{L}$;
end

Notice that a sufficient condition for the x (resp. y) regularity of f in a domain $\mathcal{D}$ is that the coefficients of ∂_y (resp. $\partial_x f$) in the Bernstein basis on $\mathcal{D}$ are all > 0 or < 0. In this case the connection algorithm can be simplified even further. This condition is discussed in more details in section 3.3.1 where it is referred to as x or y-regularity.

2.4 Simply singular domains

In this section we deal with simply singular domains (definition 2.2.3). We will assume here that $\mathcal{D}$ contains a unique critical point p of f and that the curve passes through it (i.e. it is a singular point of $\mathcal{C}$). We will see in section 2.5, how to compute such a domain.

In the following subsection we explain how using topological degree, [87] one can count the number of half branches of $\mathcal{C}$ at p and check if it is the same as the number of points in $\partial\mathcal{D} \cap \mathcal{C}$.

Finally, in the second subsection, we show that the topology in simply singular domains (i.e. satisfying the above conditions) is conic and we derive a straightforward connection algorithm from that fact.

2.4.1 Topological Degree

In this section, we recall the definition of the topological degree in two dimensions and how it can be computed. See [87, 127] for more details.

Let $\mathcal{D}$ be a bounded open domain of $\mathbb{R}^2$ and $F = (f_1, f_2) : U \to \mathbb{R}^2$ a bivariate analytic mapping which is two times continuously differentiable on an open a neighborhood U of $\overline{Dc}$.
A point $p \in \mathbb{R}^2$ is said to be a *regular value* of F on $\mathcal{D}$ if the roots of the equation $F(x, y) = p$ in $\mathcal{D}$ are simple roots, i.e. the determinant of the Jacobian J_F of F at these roots is non-zero).

Definition 2.4.1. *Let $p \in \mathbb{R}^2$ and suppose further that the roots of the equation $F(x, y) = p$, are not located on $\mathcal{D}$, the topological frontier of $\mathcal{D}$.*

Then the topological degree of F at p relative to $\mathcal{D}$, *denoted by* $\deg[F, \mathcal{D}, p]$, *is defined by*

$$\deg[F, \mathcal{D}, p] = \sum_{\mathbf{x} \in \mathcal{D} : F(\mathbf{x}) = q} \text{sign det } J_F(\mathbf{x}),$$

for q a regular value of F on $\mathcal{D}$ in the connected component of $\mathbb{R}^2 - F(\partial\mathcal{D})$ containing p.

The sum in the definition is finite since F is analytic and $\mathcal{D}$ is relatively compact.
It can be proved that this construction does not depend on the regular value q in the same connected component of $\mathbb{R}^2 - F(\partial\mathcal{D})$ as p [87]. The point p needs not be a regular value of F on $\mathcal{D}$, but if it is we can take $q = p$ in the above formula.

Remark 2.4.2. *The topological degree has a geometric interpretation known as the degree of the "Gauss map". Since $\mathcal{D}$ is a domain and J_F does not identically vanish on $\mathcal{D}$, the image $F(\mathcal{D})$ of $\mathcal{D}$ by F is also a domain (i.e. $F(\mathcal{D})^\circ \neq \emptyset$, and $F(\mathcal{D})$ is compact and simply connected). Let $x(t) : [0, 1] \to \partial\mathcal{D}$ be a parametrization of $\partial\mathcal{D}$. The topological degree is the number of times $F(x(t))$ goes around $F(\mathcal{D})$ (i.e. the index of $F(x(t))$ for any point in $F(\mathcal{D})$), when the point $x(t)$ goes around $\mathcal{D}$ one time. The sign of the topological degree is positive is F preserves the orientation of $\mathcal{D}$, it is negative if it reverses it.*
The red arrows in fig. 2.2 illustrate the values $F(x(t))$ as $x(t)$ runs over the frontier of $\mathcal{D}$. This viewpoint allows to use the strong geometric intuition behind the gradient field when F is the gradient map of f.

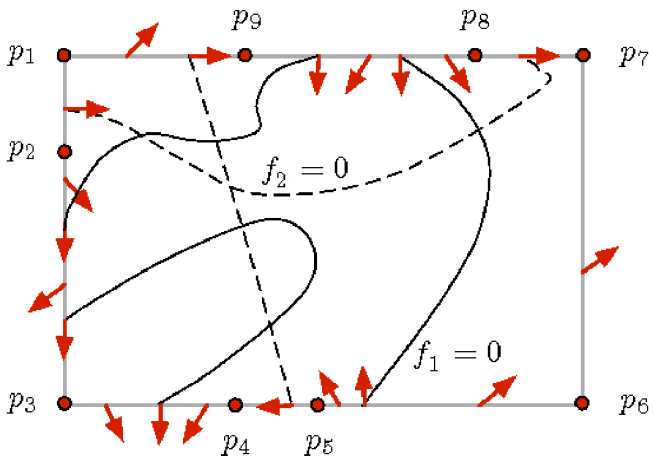


Figure 2.2: Computing the topological degree

Let us now give a more explicit formula for computing this topological degree, which involves only information on the frontier of $\mathcal{D}$.

Proposition 2.4.3. *[127] Assume here that the frontier $\mathcal{D}$ is a polygon and that it is decomposed in reverse clockwise order into the union of segments*

$$\partial\mathcal{D} = \cup_{i=1}^{g}[p_i, p_{i+1}], \quad p_{g+1} = p_1,$$

in such a way that one of the components f_{σ_i} ($\sigma_i \in \{1,2\}$) of $F = (f_1, f_2)$ has a constant sign ($\neq 0$) on $[p_i, p_{i+1}]$. Then

$$\deg[F, \mathcal{D}, (0,0)] = \frac{1}{8}\sum_{i=1}^{g}(-1)^{\sigma_i - 1}\begin{vmatrix} \mathrm{sg}(f_{\sigma_i}(p_i)) & \mathrm{sg}(f_{\sigma_i}(p_{i+1})) \\ \mathrm{sg}(f_{\sigma_i+1}(p_i)) & \mathrm{sg}(f_{\sigma_i+1}(p_{i+1})) \end{vmatrix} \tag{2.1}$$

where $f_1 = f_3$ and $\mathrm{sg}(x)$ denotes the sign of x.

Thus in order to compute the topological degree of F on a domain $\mathcal{D}$ bounded by a polygon, we need to separate the roots of f_1 from the roots of f_2 on $\partial\mathcal{D}$ by points $p_1, \ldots, p_{g+1}$ at which we compute the sign of f_1 and f_2. This will be performed on each segment of the frontier of $\mathcal{D}$, by a univariate root isolation method working simultaneously on f_1 and f_2, that we will described in the next section.

Figure 2.2 shows a sequence of points $p_1, \ldots, p_9$, which decomposes ∂D into segments on which one of the two functions ($f_1 = 0$ and $f_2 = 0$ are represented by the plain and dash curves) has a constant sign. Computing the sign of these functions and applying formula (2.1) yields the topological degree of $F = (f_1, f_2)$ relative to $\mathcal{D}$ at $(0,0)$.

2.4.2 Counting the number of branches

Let us consider a curve $\mathcal{C}$ in a domain $\mathcal{D} \subset \mathbb{R}^2$, defined by the equation $f(x, y) = 0$ with $f(x, y) \in \mathbb{R}[x, y]$. Let $\nabla f = (\partial_x f, \partial_y f)$ be the gradient of f. A point $p \in \mathcal{C}$ is singular if $\nabla f(p) = 0$. We defined a real half branch of $\mathcal{C}$ at p, as a connected component of $\mathcal{C} - \{p\} \cap \mathcal{D}(p, \epsilon)$ for $\epsilon > 0$ small enough.

The topological degree of ∇f can be used to count the number of half branches at a singular point, based on the following theorem:

Theorem 2.4.4. (Khimshiashvili [81, 10, 128]) *Suppose that p is the only root of $\nabla f = 0$ in $\mathcal{D}$. Then the number N of real half branches at p of the curve defined by $f(x, y) = f(p)$ is*

$$N = 2\,(1 - \deg[\nabla f, \mathcal{D}, (0, 0)]). \tag{2.2}$$

We will denote by $N(f, \mathcal{D})$ the number given by Formula (2.2).

In order to count the number of branches of $\mathcal{C}$ at a singular point $p \in \mathcal{C}$, first we isolate the singular point p in a domain $\mathcal{D}$, so that ∇f does not vanish elsewhere in $\mathcal{D}$. Then we compute the topological degree $\deg[\nabla f, \mathcal{D}, (0, 0)]$, as described previously, by isolating the roots of $\partial_x f$ and $\partial_y f$ on $\partial\mathcal{D}$.

Let us describe now the algorithm used to compute the topological degree of ∇f in a domain $\mathcal{D} = [a, b] \times [c, d]$. According to formula (2.1), this reduces to separating the roots of the product $\partial_x f \partial_y f$ on the frontier of $\mathcal{D}$, which consists in 2 horizontal and 2 vertical segments. The problem can thus be transformed into isolating the roots of univariate polynomials on a given interval. Hereafter, these polynomials will be called $g_1(t), g_2(t)$ and the interval $[u, v] \subset \mathbb{R}$. For instance, one the 4 cases to consider will be $g_1(t) = \partial_x f(t, c), g_2(t) = \partial_y f(t, c)$, $u = a, v = b$. We recall briefly the subdivision method described in [109, 105, 54], which can be used for this purpose. First we express our polynomials $g_1(t)$, $g_2(t)$ of degree d_1, d_2 in the Bernstein bases $(B^i_{d_k}(t; u, v))_{i=0,\ldots,d_k}$ $(k = 1, 2)$, on the interval $[u, v]$:

$$g_k = \sum_{i=0}^{d_k} \lambda_{k,i}\, B^i_{d_k}(t; u, v), k = 1, 2,$$

where $B^i_d(t; u, v) = \binom{d}{i}(t - u)^i (v - t)^d (v - u)^{-d}$. The number of sign variations of the sequence $\lambda_k = [\lambda_{k,0}, \ldots, \lambda_{k,d_k}]$ $(k = 1, 2)$ is denoted $V(g_k; [u, v])$. By a variant of Descartes rule [13], it bounds the number of roots of g_k on the interval $[u, v]$ and is equal modulo 2 to it. Thus if $V(g_k; [u, v]) = 0$, g_k has no root in the interval $[u, v]$, if $V(g_k; [u, v]) = 1$, g_k has exactly one root in the interval $[u, v]$. This is the main ingredient of the subdivision algorithm [54], which splits the interval using the de Casteljau algorithm [55] if $V(g_k; [u, v]) > 1$; store the interval if $V(g_k; [u, v]) = 1$ and remove it otherwise. It iterates the process on each subintervals until the number of sign variations is 0 or 1. The complexity analysis of the algorithm is described in [54]. See also [49].

In our case, we need to compute intervals on which one of the polynomials g_1 or g_2 has a constant sign. Thus we replace the subdivision test by the following:

- if $V(g_1;[u,v])=0$ or $V(g_2;[u,v])=0$, we store the interval $[u,v]$;
- otherwise we split it and compute the Bernstein representation of g_k ($k=1,2$) on the two subintervals using the de Casteljau algorithm and repeat the process.

This yields the following algorithm for computing the topological degree of $\nabla f = (f_1(x,y), f_2(x,y))$ on $\mathcal{D}$:

Algorithm 2.4.1: Topological degree of (f_1, f_2)

Input: a polynomial $f(x,y) \in \mathbb{Q}[x,y]$ and a domain $\mathcal{D} = [a,b] \times [c,d]$
Output: N the topological degree of ∇f on $\mathcal{D}$ at $(0,0)$
$\mathcal{B} := \{\}$ (a circular list representing the frontier $\partial\mathcal{D}$) ;
foreach *side segment I of the box $\mathcal{D}$* **do**
  Compute the restriction $g_1(t)$ (resp. $g_2(t)$) of f_1 (resp. f_2) on this side segment I and its representation in the Bernstein basis ;
  $\mathcal{L} := \{I\}$;
  while $\mathcal{L} \neq \emptyset$ **do**
    pop up an interval $[p,q]$ from $\mathcal{L}$;
    if $V(g_1;p,q)=0$ *or* $V(g_2;p,q)=0$ **then**
      insert p, q clock-wise in the circular list $\mathcal{B}$;
    else
      split $[p,q]$ in half and insert the two subintervals in $\mathcal{L}$;
    end
  end
end
Compute N given by formula (2.1) for the points in the circular list $\mathcal{B}$;

If we assume that $\partial_x f$ and $\partial_y f$ have no common root on the frontier of $\mathcal{D}$, it can be proved (by the same arguments as those used in [13, 105, 54]) that this algorithm terminates and outputs a sequence of intervals on which one of the functions g_1, g_2 has no sign variation. The complexity analysis of this method is described in [109]. This analysis can be improved by exploiting the recent results in [54].

2.4.3 Conic structure and connection algorithm

Finally we prove that the topology in a simply singular domain $\mathcal{D}$ is conic and write a connection algorithm for these domains.

Let $A \subset \mathbb{R}^n$ and $p \in \mathbb{R}^n$. We call the cone over A with center p the set $p \star A := \bigcup_{q \in A} [p,q]$.

Proposition 2.4.5. *Let $\mathcal{D}$ be a convex simply singular domain, i.e. $\mathcal{D}$ is convex such that there is a unique singular point s and no other critical point of f in $\mathcal{D}$, and such that the number of half branches of $\mathcal{C}$ at s is $\sharp(\partial\mathcal{D} \cap \mathcal{C})$. Then the*

topology of $\mathcal{D}$ is conic, i.e. for any point p in the inside of $\mathcal{D}$, $Z(f) \cap \mathcal{D}$ can be deformed into $p \star (\partial\mathcal{D} \cap \mathcal{C})$.

Proof. The point s is the unique critical point of f in $\mathcal{D}$. If the endpoint of a half branch at s is not on $\partial\mathcal{D}$, the half branch has to be a closed loop inside $\mathcal{D}$. In that case, f would be extremal at some point p ($\neq s$) inside the loop, and p would be another critical point of f inside $\mathcal{D}$. Thus, by way of contradiction, the endpoints of half branches at s have to be on $\partial\mathcal{D}$.

Let us show that the number of half branches at s is exactly $\sharp(\partial\mathcal{D} \cap \mathcal{C})$. Since two half branches cannot have the same endpoint on $\partial\mathcal{D}$ (that would be another singular point in $\mathcal{D}$), all points on $\partial\mathcal{D}$ are endpoints of half branches at s. Thus, at this point, we know that the connected component of s inside $\mathcal{D}$ is conic.

But in fact, there is no other connected component. Suppose we have another connected component α of $\mathcal{C}$ intersecting $\mathcal{D}$. Since all the points of $\partial\mathcal{D} \cap \mathcal{C}$ are connected to s, we have $\alpha \subset \mathcal{D}$. Because s is the only singular point in $\mathcal{D}$, α is a smooth 1-dimensional manifold. Therefore α is a closed loop inside $\mathcal{D}$ and f has an extremum in it. The function f vanishes on α and it is not the constant nil function, this shows that there is an extremum inside α where f is non-zero. As f vanishes at s, this extremum cannot be s. This contradicts the unicity of s inside $\mathcal{D}$ as extremal point of f. By way of contradiction we have proved that there is not such component as α and thus $\mathcal{C} \cap \mathcal{D}$ is connected.

This concludes our argument as we have proved that $\mathcal{C} \cap \mathcal{D}$ is exactly the connected component of s inside $\mathcal{D}$ and that it has the topology of a cone over $\partial\mathcal{D} \cap \mathcal{C}$, which is what we claimed. □

Remark 2.4.6. *We do not have to suppose that $\mathcal{D}$ is convex, simply connected would suffice. But we only work with convex sets (boxes) and the denomination "conic topology" originates from the convex case.*

In the end the connection algorithm is extremely simple. We just proved that the topology inside these domains is conic, that is $\mathcal{C} \cap \mathcal{D}$ can be deformed into a cone over $\mathcal{C} \cap \partial\mathcal{D}$. Therefore the connection algorithm for (convex) simply singular domains is to first compute the points q_i of $\mathcal{C} \cap \partial\mathcal{D}$, then choose an arbitrary point p inside $\mathcal{D}$ and finally for every q_i, connect q_i and p by a half branch segment $\mathfrak{b}_i = [p, q_i]$.

2.5 Isolating the interesting points

Let $\mathcal{D}_0 = [a, b] \times [c, d]$ be a domain of $\mathbb{R}^2$. The goal of this section is to describe effective methods to partition $\mathcal{D}_0$ into simple domains. The difficult step of this approach is to isolate the roots of

$$\mathcal{Z}_e(f) = \{(x, y) \in \mathcal{D}, \partial_x f(x, y) = 0, \partial_y f(x, y) = 0\}.$$

which are on $\mathcal{C}$, with the following property:

- There is only one point p of $\mathcal{Z}_e(f)$ in each isolating domain $\mathcal{D}$ (and it is on $\mathcal{C}$, that is singular)
- The number of points in $\mathcal{C} \cap \partial\mathcal{D}_0$ is the number of half-branches at the singular point p (that is $N(f, \mathcal{D}) = 2\,(1 - \deg[\nabla f, \mathcal{D}, 0])$).

We present two approaches. The first one exploits the Bernstein representation of f and subdivision techniques to isolate the roots of $\mathcal{Z}_e(f)$, while identifying domains where the curve is regular. It outputs an approximation of $\mathcal{C}$ to a precision that is given as input to the algorithm. We prove that, for a sufficiently high precision, the algorithm output has the same topology as $\mathcal{C}$. The second algorithm is based on algebraic techniques (namely Rational Univariate Representation) and is guaranteed to output the correct topology.

The two following methods do the isolation work in a different way but they share the test described in section 2.4 to count the number of half branches at a singular point.

2.5.1 Subdivision method

We describe here the subdivision method used to obtain such isolating domains, which is a specialization of the approach used in [104]. See also [125, 52]. It is based on the Bernstein basis representation of polynomials. This method which we recall here for polynomials in $\mathbb{Q}[x, y]$ applies for general multivariate polynomials. In chapter 3 we will make a multi-variate use of this representation. We are going to consider the system $f(x, y) = 0$, $\partial_x f(x, y) = 0$, $\partial_y f(x, y) = 0$ in the domain $\mathcal{D}_0 = [a, b] \times [c, d]$.

Each of these polynomials is expressed in the Bernstein basis on $\mathcal{D}_0$:

$$h(x, y) = \sum_{i=0}^{d_x} \sum_{j=0}^{d_y} \gamma_{i,j}\, B^i_{d_x}(x; a, b)\, B^j_{d_y}(y; c, d),$$

where $h \in \{f, \partial_x f, \partial_y f\}$ and d_x is the degree of h in x, d_y the degree of h in y. By using a method described in [104] we can quickly generate a set of boxes where the curve is x or y-regular and a small set of boxes of size smaller than a given precision $\epsilon > 0$ that isolates the part of the curve where we don't yet know what is happening.

The principle of this method is to either reduce a box by using convexity inequalities on Bernstein bases or to split the boxes if the inequalities do not apply. This is the main loop of the subdivision algorithm, which is combined with preconditioning techniques to improve the performance of the solver. The computation is iterated until the size of the box is smaller than ϵ.

When the domain is reduced in one direction, one of the functions f, $\partial_x f$, $\partial_y f$ does not vanish in the regions which are removed. Thus the curve $\mathcal{C}$ in these regions is regular and according to section 2.3, its topology can be deduced from the intersection of the curve with the frontier of the region.

This method can be adapted to our implicit curve problem, and yields the following algorithm:

Algorithm 2.5.1: Subdivision algorithm for the topology of $\mathcal{C}$

Input: a curve $\mathcal{C}$ defined by $f(x,y)=0$, $\mathcal{D}_0=[a,b]\times[c,d]$, a rendering precision $\epsilon>0$ and a computation precision ν with $\epsilon\geq\nu>0$

Output: A graph of points $\in\mathcal{D}$ connected by segments

$\mathcal{L}=\{\mathcal{D}_0\}$; $\mathcal{S}=\{\}$;

while $\mathcal{L}\neq\emptyset$ **do**

  Pop up a domain $\mathcal{D}$ from $\mathcal{L}$;

  if $\mathcal{D}>\nu$ **then**

    reduce or split the domain $\mathcal{D}$ according to the Bernstein coefficients of $f,\partial_x f,\partial_y f$ and insert the resulting domains in $\mathcal{L}$; apply the connection algorithm of regular domain 2.3.1 on the removed regions ;

  else

    add $\mathcal{D}$ to the set of singular domains $\mathcal{S}$ and update its connected components;

  end

end

foreach *minimal box* $\mathcal{D}$ *containing a connected component of* $\mathcal{S}$ **do**

  if $|\mathcal{D}|<\epsilon$ *and* $\mathcal{D}$ *does not intersect such another minimal box and if* $\sharp(\mathcal{C}\cap\partial\mathcal{D})=2\,(1-\deg[\nabla f,\mathcal{D},(0,0)])$ **then**

    apply the algorithm of connection 2.4.3 in $\mathcal{D}$;

  else

    replace ν by $\frac{\nu}{2}$ and apply the same algorithm on $\mathcal{D}$.

  end

end

This algorithm decomposes the initial domain into regions where the topology is known and a set of non-intersecting boxes of size $\leq \epsilon$ where $\sharp(\partial\mathcal{D} \cap \mathcal{C}) = 2\,(1 - \deg[\nabla f, \mathcal{D}, 0])$ (this is (2.2)). If ϵ corresponds to the size of a pixel, the visualization of the curve will be correct, except in these pixel boxes, which we call singular regions. Inside them equation (2.2) holds, and if in addition there is a unique critical point of f, which is also on $\mathcal{C}$, then the computed topology is correct.

During the subdivision process we have to zoom on domains or equivalently to scale the variables ($x := \lambda\, x,\ y := \lambda\, y$). In order to handle the numerical instability problems, which may happen in this scaling step or when we have to deal polynomials with large coefficients and degrees, we use the following enveloping techniques, which allows us to compute with fixed precision numbers: To analyze the curve $\mathcal{C}$ defined by the polynomial $f \in \mathbb{Q}[x, y]$ on a domain $\mathcal{D} = I \times J$,

- we convert f to the Bernstein basis on the domain $\mathcal{D}$ using exact arithmetic:
  $f(x, y) = \sum_{i,j}\ \gamma_{i,j}\, B^i_{d_x}(x; I)\, B^j_{d_y}(y; J)$
- we round up and down to the nearest machine precision number $\underline{\gamma_{i,j}} \leq \gamma_{i,j} \leq \overline{\gamma_{i,j}}$, so that we have $\underline{f}(x, y) \leq f(x, y) \leq \overline{f}(x, y)$ on $\mathcal{D}$.
- We use the interval coefficients $[\underline{\gamma_{i,j}}, \overline{\gamma_{i,j}}]$ to test the sign conditions and to remove the regular regions.

It can be proved that if ϵ is small enough, then this algorithm compute the topology of $\mathcal{C}$ (but for space limitation reasons, we do not include the proof here).

Remark that if $Z(f)$ is smooth in a domain $\mathcal{D}$, this algorithm can be run with $\epsilon = 0$ and will terminate (and output the correct topology) as every subdomain will ultimately be x-regular or y-regular.

2.5.2 Rational univariate representation

Choosing the precision parameter ϵ smaller than some bound was enough to certify the output of the previous algorithm. The drawback is that the bounds are difficult to compute and are bad because uniform. The algebraic technique we present hereafter, namely RUR (rational univariate representation), is guaranteed to yield the correct topology. It allows the algorithm to use coarser approximations of roots (when the critical points of f are far away from each others).

We explain in short what RUR is in the bivariate case (see [13] for more details). When given a system of equations $E = \{f_1 = 0, f_2 = 0\}$ in $\mathbb{R}^2$ with 0-dimensional solution space, it is possible to find polynomials $P, P_1, P_2 \in \mathbb{R}[u]$ so that we have $\mathcal{Z}(E) = \left\{ \left(\frac{P_1}{P'}(\alpha), \frac{P_2}{P'}(\alpha) \right) \mid \alpha \in \mathbb{R}, \quad P(\alpha) = 0 \right\}$ where P is squarefree and P' is its derivative. In other words, the roots of E are the image of the roots of P by a rational map. An RUR of the roots of E can be computed

by finding a separating linear function and using resultant or Groebner basis techniques.

In our case the following problem arises: $\mathcal{Z}_e(f)$ can have 1-dimensional components. Because we are dealing with curves in $\mathbb{R}^2$, we can easily separate the 1-dimensional part from the 0-dimensional part by computing $g := \gcd(\partial_x f, \partial_y f)$. We define

$$\mathcal{Z}_e^1(f) = \mathcal{Z}(g), \quad \mathcal{Z}_e^0(f) = \mathcal{Z}\left(\frac{\partial_x f}{g}, \frac{\partial_y f}{g}\right).$$

Among the points in $\mathcal{Z}_e^0(f)$ we want to be able to tell those that are in $\mathcal{C}$, that is those which are singular points of $\mathcal{Z}(f)$. This way we can isolate the singular points of $\mathcal{C}$ from the rest of $\mathcal{Z}_e(f)$. Since f is square-free, the singular locus $\mathrm{Sing}(f)$ of f is 0-dimensional and $\mathcal{Z}_e^0(f) \cap \mathcal{C} = \mathcal{Z}_e^1(f) \cap \mathcal{C}$.

Therefore we compute (P, F_1, F_2) an RUR for $\mathcal{Z}_e^0(f)$ instead of $\mathcal{Z}_e(f)$ to isolate the critical points of f. And to tell which points are on $\mathcal{C}$ we compute $Q = \gcd(P, \mathrm{num}\, f(F_1, F_2))$ where num takes the numerator of an irreducible rational fraction. It can be checked easily that (Q, F_1, F_2) is an RUR for $\mathcal{Z}_e^0(f) \cap \mathcal{C}$ by using the fact that P' and P have no common roots.

Now, we use this RUR to isolate the roots of a square-free polynomial P using a univariate solver (see e.g. [54]). By using interval arithmetic one can find isolating intervals for the roots of $\mathcal{Z}_e^0(f)$ by computing the images of the isolating intervals of the roots of P by $F_1 := \frac{P_1}{P'}$ and $F_2 := \frac{P_2}{P'}$. This generates boxes containing these roots. If the boxes intersect we refine the isolating intervals of the roots of P until the boxes do not intersect anymore. Finally, using again interval arithmetic, we check that g does not vanish in these isolating boxes. Otherwise we refine them until it doesn't.

Keeping the boxes which correspond to roots of Q, we obtain isolating boxes which contain a single singular point. For each isolating boxes $\mathcal{D}$, we compute the topological degree. If $N(f, \mathcal{D})$ is not the number of points of $\mathcal{C} \cap \partial\mathcal{D}$, we refine the isolating box.

This yields isolating boxes for the singular points of $\mathcal{C}$, which are simply singular. The complement of the isolating boxes is divided into boxes on which we apply the previous subdivision algorithm for smooth curves.

2.6 Examples

Algorithms presented in this chapter have been completely implemented with the algebraic geometric modeling software Axel[1]. The efficiency of the topology algorithm presented here allows a real time manipulation of algebraic objects within the software, whereas current solutions usually only propose ray tracing algorithm for visualization. Here are some significant illustrations. More examples [2] and videos [3] can be found on the software's website.

[1] `http://axel.inria.fr`
[2] `http://axel.inria.fr/user/screenshots`
[3] `http://axel.inria.fr/user/screencasts`

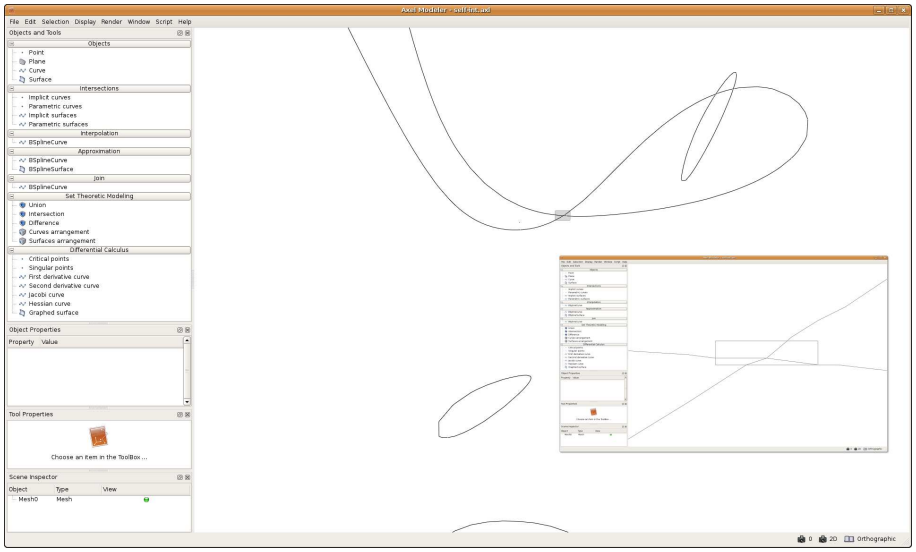


Figure 2.3: Topology of a high degree curve in Axel.

The curve in figure 2.3 is the preimage in the parameter space of a self-intersection point of a bicubic surface. Its equation has been obtained by resultant computation. It is of total degree 76 and of degree 44 in each parameter. Its coefficients are of maximal bit size 590. It takes 7s to visualize this curve.

Figure 2.4 shows the discriminant curve of a bivariate system with few monomials that gives a counter-example to Kushnirenko's conjecture [45]. It is of degree 47 in x and y, and the maximal bit size of its coefficient is of order 300. It takes less that 10 seconds to visualize it within the modeler.

The area that is emphasized looks like a cusp point, but when we blow it up in the thumbnail shown in bottom right corner of figure 2.4, we see that it is actually made of 3 cusp points and 3 crossings. The counter-example comes from this area.

Figure 2.5 illustrates how the algorithm behaves with a large amount of implicit curves of degree up to 9 in degenerate cases, that is to say with tangential intersection points and coincident singular points. In this example, the subdivision process takes no longer than 7.224 seconds, the corresponding quadtree depth is 10.

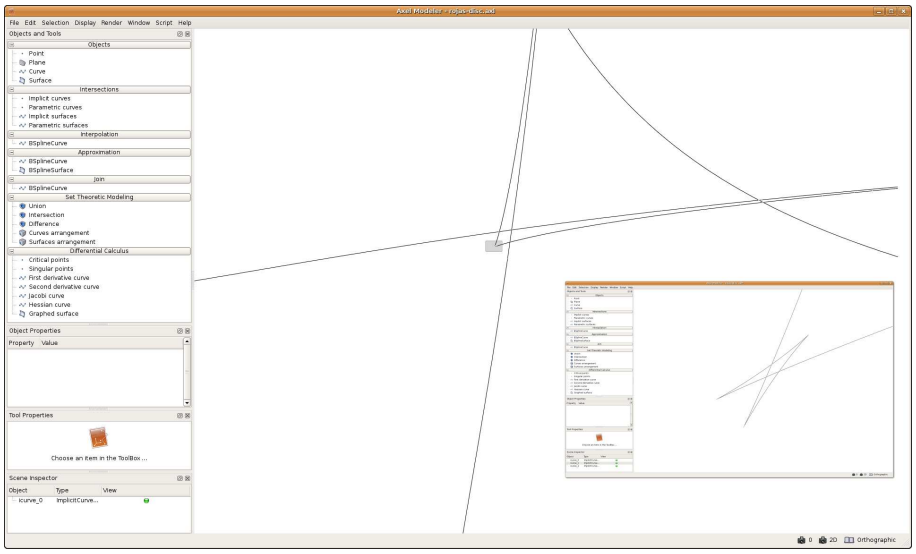


Figure 2.4: Topology of a curve with hidden cusp points.

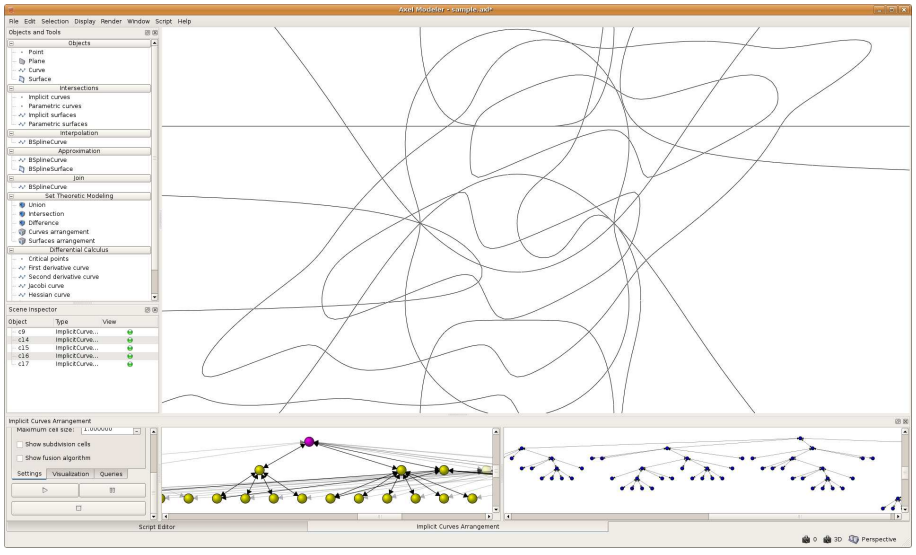


Figure 2.5: Computing an arrangement in Axel.

Chapter 3

Triangulating smooth algebraic varieties

This chapter is based for its most part on the article [2] which served as starting point for this Ph. D thesis. The chapter presents an algorithm to triangulate surfaces defined by an implicit equation. It is able to isolate the singular points of the surface to guarantee the topology in the smooth part, while producing a number of triangles which is related to geometric invariants of the surface. We prove its termination and correctness and give complexity bounds, based on metric entropy analysis. The method applies to surfaces defined by a polynomial equation or a spline equation. Although the presentation is carried out in three-space, there is in fact no obstruction to applying the same approach to smooth hypersurfaces in any dimension, and with little additional work, to smooth complete intersection varieties. We use Bernstein bases to represent the defining function of the surface in a box and we subdivide this representation according to a generalization of Descartes's rule, until the problem in each box boils down to determining whether the implicit object is isotopic to its linear approximation in the cell, or whether the size of the cell is smaller than a parameter ε. This ensures that the topology of the implicit surface is caught within a precision ε. When the surface is smooth, ε can be chosen small enough for the topology to be entirely determined (i.e. we have a triangulation of the surface). Experimentations on classical examples from the classification of singularities show the efficiency of the approach.

3.1 Background on meshing implicit surfaces

Several methods have been developed over the last decades to visualize or to mesh an implicit surface. The terms mesh and triangulation have related meanings in that they refer to a discretization of a geometric into simplexes (triangles, segments, and points in three-space). The use of the word triangulation emphasizes that the discretization has the same topology as the original object,

whereas the term mesh denotes that one aims at obtaining a discretization that is metrically close to the original object, for visualization or integration purposes for instance. The method we present in this chapter presents both aspects and we use both words interchangeably. On smooth surfaces our algorithm will give a triangulation when the precision parameter ε is small enough. On singular surfaces, it will yield a mesh of the surface. In this section we discuss meshing methods. As chapter 4 discusses the triangulation of general semi-algebraic sets, we defer to this chapter the discussion of triangulation methods in particular. The meshing methods we will now present are:

- ray tracing[1],
- marching cube,
- marching polygonizer,
- sample methods,
- deformation methods,
- subdivision methods.

Used to detect the visibility of objects, "ray tracing" methods [79] compute the intersection between the ray from the eye of the observer and the first object of the scene, for each pixel of the image. The rendering is very good, but the computation time is significant. It depends on the resolution of the scene to be viewed, and can produce only images in 2 dimensions. Moreover, computation cannot (a priori) be re-used for other views. Other techniques, such as particle sampling, also use clouds of points lying on the surface, to visualize it. See for instance [9]. However, it only yields an approximation of the surface [143] without the topological structure, nor any guarantee on the visual quality of the result.

The "marching cube" algorithm [93] developed in order to reconstruct images in 3 dimensions starting from medical data, is very much used for visualization of level sets of functions. The principle of this algorithm is simple: the domain of interest is divided in several cells, generally boxes of the same size. At the corners of each cell, the values of the function f are calculated and a triangular mesh is then deduced according to the sign of the function at these corners. This triangulation may not capture the topology of the surface, if it is not supported by additional calculation. Several triangulations are possible for the same combination of signs. Some partial solutions exist to avoid some of these ambiguities [85]. The covering of all the space of study increases the computing time considerably. Indeed the boxes not cut by the surface are not useful. Despite its defects, the "marching cube" method remains a reference for its simplicity and its ease of adaptation.

The marching polygonizer method brings a significant improvement to the marching cube method. The principal idea of this method is to calculate only

[1] see `http://www.algebraicsurface.net/`

the "useful" cells, that is, those which cut the surface. The algorithm starts from a valid cube (or tetrahedron), and propagate toward the connected cells, which cut the surface [21], [22], [72], [1]. Thus it is necessary to start from a cell intersecting the surface. Again, for self intersecting surfaces or surfaces with singularities, the result might be erroneous. The algorithm is rather effective, but it does not make it possible to mesh any surface correctly, if this one has, for example, several connected components, without using external tools such as "topological skeleton " [20].

Another approach are the so-called "sample" methods. One type uses moving particles on the surface, with repulsion forces which make them spread over the surface [143]. Another class of methods start from an initial set of sample points on the surface and refine it by inserting new points of the surface in order to improve the approximation level. Techniques based on Delaunay triangulations of these points have been used for this purpose [28, 43]. Although they prove effective on examples to give a good vision of very regular varieties, they completely fail to detect lone lower dimensional components and miss small details of the variety.

Deformation methods exploit results from Morse theory, in order to correctly follow the transformation of the level-sets $f(x) = t$ [1]. See also [25] for a connected approach, which applies for smooth surfaces. These methods assume implicitly that the function is a Morse function, i.e. the critical points that are traversed during the deformation are not degenerate. This is not always the case nor is it straightforward to check.

Here we describe a subdivision algorithm to mesh an implicit surface, which is able to isolate the singular points of the surface, to guarantee the topology in the smooth part (that is triangulate these parts), while producing a number of linear pieces, related to the Vitushkin variations of the surface. It applies to surfaces defined by a polynomial equation or a B-spline equation and can be generalized to hypersurfaces in any dimension. Our method has similarities with the one presented in [99], but we go further by describing a new and guaranteed subdivision criterion, hence enabling us to triangulate smooth objects. In addition, we analyze in detail the complexity of the subdivision algorithm in terms of the entropy of the surface, which yields a bound on the number of cells produced by the method in terms of geometric invariants of the surface. This gives a proof of termination, correctness and complexity of the algorithm when used for a smooth surface. Its extension for the treatment of singularities, using a local conic structure theorem is the subject of chapter 4.

Regarding the technical aspects, we use Bernstein bases to represent the function in a box and subdivide this representation according to a generalization of Descartes's rule, until the problem in each box boils down to the case where either the implicit object is proved to be homeomorphic to the computed linear approximation in the cell or the size of the cell is smaller than ϵ. This ensures that the topology of the implicit surface is caught within a precision ϵ, where ϵ is a tunable parameter. Experimentations on classical examples from the classification of singularities show the efficiency of the approach.

3.2 Algebraic ingredients

For any point $p \in \mathbb{R}^3$, and any set $A \subset \mathbb{R}^3$, $\mathrm{dist}(p, A)$ denotes the minimal Euclidean distance between p and points $q \in A$. We define $\mathrm{dist}_x(p, A)$ as the minimal Euclidean distance between p and a point $q \in A$ with the same (y, z)-coordinates, if it exists and $+\infty$ otherwise. The distances $\mathrm{dist}_y(p, \mathrm{A})$, $\mathrm{dist}_z(p, A)$ are defined similarly.

3.2.1 Representation of polynomials

Let us recall that a univariate polynomial $f(x)$ of degree d can be represented in the Bernstein basis by:

$$f(x) = \sum_{i=0}^{d} b_i \; B_d^i(x),$$

where $B_d^i(x) = \binom{d}{i} x^i (1-x)^{d-i}$. The sequence $\mathbf{b} = [b_i]_{i=0,\ldots,d}$ is called the set of control coefficients on $[0, 1]$. The polynomials B_d^i form the Bernstein basis on $[0, 1]$. Similarly, we will say that a sequence $\mathbf{b}$ represents the polynomial f on the interval $[a, b]$ if:

$$f(x) = \sum_{i=0}^{d} b_i \; \binom{d}{i} \frac{1}{(b-a)^n} \; (x-a)^i (b-x)^{d-i}.$$

The polynomials

$$B_d^i(x; a, b) := \binom{d}{i} \frac{1}{(b-a)^n} \; (x-a)^i (b-x)^{d-i}$$

form the Bernstein basis on $[a, b]$. Hereafter, we are going to consider the sequence of values $\mathbf{b}$ together with the corresponding interval $[a, b]$. A first property of this representation is that the derivative f' of f, is represented by the control coefficients:

$$d\Delta \boldsymbol{b} := d(b_{i+1} - b_i)_{0 \leqslant i \leqslant d-1}.$$

Another fundamental algorithm that we will use on such a representation is the De Casteljau algorithm [55]:

$$\begin{aligned} b_i^0 &= b_i & i &= 0, \ldots, d \\ b_i^r &= (1-t)\, b_i^{r-1} + t\, b_{i+1}^{r-1}(t) & i &= 0, \ldots, d-r \end{aligned}$$

It allows us to subdivide the representation of f into the two subrepresentations on the intervals $[a, (1-t)a + tb]$ and $[(1-t)a + tb, b]$. For a complete list of methods on this representation, we refer for instance to [55].

By a direct extension to the multivariate case, any polynomial $f(x, y, z)$ of degree d_1 in x ,d_2 in y, d_3 in z, can be decomposed as:

$$f(x, y, z) = \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} \sum_{k=0}^{d_3} b_{i,j,k} B_{d_1}^i(x; a_1, b_1)\, B_{d_2}^j(y; a_2, b_2)\, B_{d_3}^k(z; a_3, b_3),$$

where $(B^i_{d_1}(x;a_1,b_1)\ B^j_{d_2}(y;a_2,b_2)\ B^k_{d_3}(z;a_3,b_3))_{0\le i\le d_1,0\le j\le d_2,0\le k\le d_3}$ is the tensor product Bernstein basis on the domain $\mathcal{D} := [a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]$. The polynomial f is represented in this basis by the third order tensor of control coefficients $\mathbf{b} = (b_{i,j,k})_{0\le i\le d_1,0\le j\le d_2,0\le k\le d_3}$.

Hereafter, we will denote by a *cell*, the pair of the box $[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]$ together with the control coefficients $\boldsymbol{b}$, representing f. The size of a cell will be $\max\{|b_1-a_1|,|b_2-a_2|,|b_3-a_3|\}$.

The De Casteljau algorithm also applies in the x, y or z-direction. Because of this tensor product representation, the control coefficients of the derivative $\partial_x f(x,y,z)$ are given by:

$$d\,(b_{i+1,j,k}-b_{i,j,k})_{0\le i\le d_1-1,0\le j\le d_2,0\le k\le d_3}$$

and similarly for the derivatives $\partial_y f$, $\partial_z f$.

Notice that the univariate Bernstein representation also extends to a so-called triangular Bernstein basis. This representation can also be used in our approach, but we will concentrate on the tensor product one.

3.2.2 Univariate solver

The subdivision criterion that we are going to use, is based on Descartes's rule for a univariate polynomial with control coefficients $\mathbf{b}$ in the Bernstein basis. The number of sign changes of a sequence $\mathbf{b}$, also called the sign variation of $\mathbf{b}$, is denoted hereafter by $V(\mathbf{b})$.

Proposition 3.2.1. *[55], [120] The number of sign changes $V(\mathbf{b})$ of the control coefficients $\mathbf{b} = [b_i]_{i=0,\dots,d}$ of a univariate polynomial on $[0,1]$ bounds its number of real roots in $[0,1]$ and is equal to it modulo 2.*

Thus, by this proposition,

- if $V(\mathbf{b}) = 0$, the number of real roots in $[0,1]$ is 0;
- if $V(\mathbf{b}) = 1$, the number of real roots in $[0,1]$ is 1.

This yields the following simple but efficient algorithm:

Algorithm 3.2.2.
INPUT*: A precision ϵ and a polynomial f represented in the Bernstein basis of an interval $[b,a]$: $f = (\mathbf{b},[a,b])$.*

- *Compute the number of sign changes $V(\mathbf{b})$.*
- *If $V(\mathbf{b}) > 1$ and $|b-a| > \epsilon$, subdivide the representation into two sub-representations $\mathbf{b}^-,\mathbf{b}^+$, corresponding to the two halves of the input interval and apply recursively the algorithm to them.*
- *If $V(\mathbf{b}) > 1$ and $|b-a| < \epsilon$, output the $\epsilon/2$-root $(a+b)/2$ with multiplicity $V(\mathbf{b})$.*

- *If $V(\mathbf{b}) = 0$, remove the interval $[a, b]$.*
- *If $V(\mathbf{b}) = 1$, the interval contains one root, that can be isolated within the precision ϵ.*

OUTPUT*: list of subintervals of $[a, b]$ containing exactly one real root of f or of ϵ-roots with their multiplicities.*

In the presence of a multiple root, the number of sign changes of a representation containing a multiple root is bigger than 2, and the algorithm splits the box until its size is smaller than ϵ.

In order to analyze the behavior of the algorithm, we used a partial inverse of Descartes's rule [109] (see also [97]), to show that if $f(x) = 0$ has only simple roots on $[a, b]$, an upper bound of the number of recursion steps of the algorithm 3.2.2 is

$$l = \lceil \log_2 \left(\frac{1 + \sqrt{3}}{2s} \right) \rceil,$$

where s is the minimal distance between the complex roots of f and $\lceil x \rceil$ denotes x rounded up to the next integer.

Notice that this localization algorithm extends naturally to B-splines, which are piecewise polynomial functions [55].

3.3 Toward a guaranteed method

The aim of this section is to describe the method, which allows us to build a mesh (i.e. triangulation, see 3.1) of the surface $f(x, y, z) = 0$ in a domain $\mathcal{D} = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \subset \mathbb{R}^3$, having the same topology as the surface. The set of points (x, y, z) in $\mathcal{D}$ such that $f(x, y, z) = 0$ will be denoted by $S := Z(f) \cap \mathcal{D}$. The set of singular points of S (where $f = \partial_x f = \partial_y f = \partial_z f = 0$) will be denoted by S_{sing}, the set of smooth (non-singular) points of S by S_{smooth}.

3.3.1 Description of the algorithm

The general scheme of the meshing algorithm is as follows:

- Represent the polynomial $f(x, y, z)$ in the Bernstein basis adapted to the domain $\mathcal{D} = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ as follows:

$$f(x, y, z) = \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} \sum_{k=0}^{d_3} b_{i,j,k} \, B_{d_1}^i(x) \, B_{d_2}^j(y) \, B_{d_3}^k(z),$$

- Subdivide the box into smaller boxes (using the De Casteljau algorithm) until the topology in these boxes can be certified or the size of the box is smaller than ϵ.

It leads to the following scheme:

Algorithm 3.3.1.

```
L := [Cell(f,D)];
while (L is not empty)
{
 C := first_cell_of(L);
 if(topology_guarantee(C) &&  size(C)<epsilon_smooth)
   insert C at the head in the list of solutions;
 else if(not(topology_guarantee(C)) && size(C)<epsilon_sing)
   insert C at the head in the list of (unmeshed) solutions;
 else {
   subdivide the cell C;
   insert  the new generated cells at the tail of L;
   remove  C from  L;
 }
}
```

The two parameters involved here are:

- $\varepsilon_{\text{smooth}}$ which is the maximal size of the cells where the topology is guaranteed,
- $\varepsilon_{\text{sing}}$, which is the minimal size after which we consider that the cell contains a singular point.

Hereafter, to simplify the analysis of the algorithm, we will take $\varepsilon = \varepsilon_{\text{smooth}} = \varepsilon_{\text{sing}}$. In practice, it could be interesting to have $\varepsilon_{\text{smooth}} > \varepsilon_{\text{sing}}$, in order to compute large boxes in the smooth part and small boxes around the singularities. This explains why we consider these two parameters.

This subdivision scheme produces a sequence of boxes F of decreasing size. It corresponds to the construction of an octree, level by level. An octree is a tree structure where every node has 8 children nodes (or leaves). It is relevant to subdivision methods in $\mathbb{R}^3$ since it corresponds to cutting a box by 3 planes, one in every direction, which divides the box into 8 sub-boxes. An advantage of the octree data structure is the fast localization of points and of faces or edges shared by several cells [122].

The subdivision criterion we will use is based on an extension of Descartes's rule for a polynomial with control coefficients $\mathbf{b} = (b_{i,j,k})_{0 \leqslant i \leqslant d_1, 0 \leq j \leq d_2, 0 \leq k \leq d_3}$ in the Bernstein basis.

In order to test whether we have to split a cell, we check if the number of sign changes in one of the directions x, y, z is 0 or 1 and that the sign variation of the control coefficients of the derivative in this direction is 0. More precisely, the sign variation of f in the x direction is the maximum for all j, k with $0 \leq j \leq d_2$, $0 \leq k \leq d_3$, of the sign variations of the sequences $\mathbf{b}_{j,k} = (b_{i,j,k})_{0 \leq i \leq d_1}$.

The termination criterion that we use is the following:

Definition 3.3.2 (x-regular cell)**.** *The cell C is x-regular (resp. y, z-regular) for f, if the sign variation of $\mathbf{b}$ in the x (resp. y, z) direction is* 0 *or* 1 *and if the coefficients of the derivative in this direction have a constant sign.*

A similar definition applies to control coefficients of polynomials in two variables on two-dimensional boxes.

Lemma 3.3.3. *Let (u, v, w) be any permutation of (x, y, z). Assume that a (u, v)-face F of a cell is u-regular for f. Then the topology of the surface $f = 0$ on the face F is uniquely determined by its intersection points (counted with multiplicity) with the edges of the face.*

Proof. Since $f = 0$ has no singular point on F, the trace of $f = 0$ on F is a set of arc segments (possibly of length 0) intersecting the edges of the face. They project along the u-direction on the other axis (say the v-axis) as a set of non-overlapping intervals. Consequently, the topology of $f = 0$ on F, is the same as those of the set of segments connecting the points of S on the edges, sorted according to their v coordinates and taken by pairs. This proves that the topology of the surface $f = 0$ on the faces of the cell is determined by its points on the edges. □

Proposition 3.3.4. *Let (u, v, w) be any permutation of (x, y, z). Assume that C is u-regular and that the topology of $f = 0$ on the two (v, w)-faces of C is known. Then the topology of the surface $f = 0$ in the box C is uniquely determined by its intersection points (counted with multiplicity) with the edges of the box.*

Proof. We can assume without loss of generality, that f is u-regular on F, for $u = x$ or $u = y$ or $u = z$. According to the previous lemma, the topology of $f = 0$ on all the faces of the cell C is determined. As inside the cell C the surface S is the graph of a function in the u-direction, and as there are no singular points of $f = 0$ in C, $S \cap C$ is topologically homeomorphic to a set of discs which are determined by the projection of the segments of the faces, on a (v, w)-plane along the u -direction. This concludes the proof. □

The previous lemma and proposition imply that checking the regularity of f in the box B and on faces, and computing the points of the surface on the edges of the box allows us to deduce the topology of the surface in the box. To compute the mesh in a regular cell, we need to compute the points of S (counted with their multiplicity) on the edges of the boxes. This is performed by the univariate solver (see algorithm 3.2.2).

This criterion implies that in the valid cells, the derivative of f in one direction is of constant sign and on the two faces transversal to this direction, another derivative is of constant sign. This may be difficult to obtain, when a point of the surface where two derivatives vanish is on (or near) the border of the cell. In order to avoid this situation, we weaken the criterion and improve the subdivision in the following way:

- We check that a derivative of f in one of the directions x, y, z has a constant sign in the cell C. If not, the cell is subdivided.

- For the two faces transversal to this direction, we apply the same algorithm on the faces (in 2 dimensions), in order to get polygons representing the trace of $f = 0$ on these faces.

In such a case, the topology of the set S in the cell C is guaranteed: It is the graph of a function, say in the direction u for which the derivative has a constant sign. The polygons of $f = 0$ on all the faces define closed curves on the border of C. Applying proposition 3.3.4, we are able to compute the topology of $f = 0$ in C.

Notice that if precautions are not taken, the trace of $f = 0$ on the border of C might be a singular curve. To avoid this situation, we simply precompute the critical points of S for the projection in the directions $(x, y), (x, z), (y, z)$. These points are defined by the equations $f = \partial_x f = \partial_y f = 0$, $f = \partial_x f = \partial_z f = 0$, $f = \partial_y f = \partial_z f = 0$. In the case of a smooth surface, after a generic change of coordinates (or simply a generic translation) the number of such points is finite (it is bounded by $3\,d(d-1)^2$ where $d = \deg(f)$, by Bezout's theorem). We avoid these points when the cells are subdivided, by choosing adequately the position of subdivision (applying the De Casteljau algorithm for a value of t in between critical values). In order to apply recursively the algorithm in dimension 2, we take the parameters $\varepsilon_{smooth} = \varepsilon_{sing} = \varepsilon$.

These adaptations allow us to prove that for a smooth surface and ε small enough, the algorithm stops with the correct topology. By the structure of the algorithm, we are able to detect if ε is not small enough.

To prove termination and correctness, we need the following definition and result on the approximation of a function by the control polygon. Let $K_2(f) = \max_{p\in\mathcal{D}} \|H_2(f)(p)\|$ where $H_2(f)(p)$ is the Hessian of f at p. Let C be a cell of size ε.

Let $d_1 = \deg_x(f), d_2 = \deg_y(f), d_3 = \deg_z(f)$. The polynomial f can be represented in the Bernstein basis by coefficients $(c_{i,j,k})_{0\leq i\leq d_1,\ 0\leq j\leq d_2,\ 0\leq k\leq d_3}$. Let $(s_{i,j,k})_{0\leq i\leq d_1,\ 0\leq j\leq d_2,\ 0\leq k\leq d_3}$ be the control points of f in the Bernstein basis. Those points are the vertices of a regular lattice in C: scaling C back to the standard cube $[0, 1]^3$ by an affine mapping ϕ, we simply have

$$\phi(s_{i,j,k}) = \left(\frac{i}{d_1}, \frac{j}{d_2}, \frac{k}{d_3}\right).$$

Then there exists $\gamma_2(\boldsymbol{d}) = \gamma_2(d_1, d_2, d_3)$ depending of such that

$$|f(s_{i,j,k}) - c_{i,j,k}| < \gamma_2(\boldsymbol{d})K_2(f)\varepsilon^2. \tag{3.1}$$

See eg. [112], [119], [94] for a proof and more details on this result. We denote $\kappa_2(f) = \gamma_2(\boldsymbol{d})K_2(f)$.

First, we analyze the cells which are rejected by the algorithm. We denote $\Gamma_f(r) = \{p \in \mathcal{D}, |f(p)| \leqslant r\}$.

Proposition 3.3.5. *Let C be a cell of size ε, outside $\Gamma_f(\kappa_2(f)\varepsilon^2)$. Then the control coefficients of f on C are of constant sign.*

Proof. As C is outside $\Gamma_f(\kappa_2(f)\varepsilon^2)$, f does not vanish in C, so that it has a constant sign. Assume, without loss of generality, that $f > 0$ so that $f > \kappa_2(f)\varepsilon^2 > 0$ in C. Then by (3.1), we have

$$c_{i,j,k} = f(s_{i,j,k}) - (f(s_{i,j,k}) - c_{i,j,k}) > \kappa_2(f)\varepsilon^2 - \kappa_2(f)\varepsilon^2 = 0.$$

□

In consequence, such a cell will not be kept by the algorithm.

Theorem 3.3.6. *If the surface S defined by $f(x, y, z) = 0$ is smooth in $\mathcal{D}$, then the algorithm 3.3.1 stops for $\varepsilon_{smooth} > \varepsilon_{sing}$ small enough, and output a mesh homeomorphic to S.*

Proof. By equation (3.1), for $\varepsilon := \varepsilon_{smooth}$ small enough, the cells C which are kept by the algorithm intersect $\Gamma_f(\kappa_2(f)\varepsilon^2)$.

Let us denote by x_0 a point of $\Gamma_f(\kappa_2(f)\varepsilon^2) \cap C$. For any $x \in C$, we have

$$|f(x) - f(x_0)| \leq \kappa_1(f)\,||x - x_0||_\infty \leq \kappa_1(f)\varepsilon$$

where $\kappa_1(f) = \max_{p\in\mathcal{D}} ||(\partial_x f(p), \partial_y f(p), \partial_z f(p))||_1$. As $x_0 \in \Gamma_f(\kappa_2(f)\varepsilon^2)$, we have

$$|f(x))| \leq \kappa_1(f)\varepsilon + \kappa_2(f)\varepsilon^2,$$

which implies that $C \subset \Gamma_f(\kappa_1(f)\varepsilon + \kappa_2(f)\varepsilon^2)$. As S is smooth, for ε small enough, we have

$$\begin{aligned}&\Gamma_f(\kappa_1(f)\varepsilon + \kappa_2(f)\varepsilon^2)\\ &\quad\cap\quad \Gamma_{\partial_x f}(\kappa_2(\partial_x f)\varepsilon^2) \cap \Gamma_{\partial_y f}(\kappa_2(\partial_y f)\varepsilon^2) \cap \Gamma_{\partial_z f}(\kappa_2(\partial_z f)\varepsilon^2) = \emptyset.\end{aligned}$$

This implies, for ε small enough, for any cell C of size ε kept by the algorithm and for all $x \in C$, either $|\partial_x f(x)| > \kappa_2(\partial_x f)\varepsilon^2$ or $|\partial_y f(x)| > \kappa_2(\partial_y f)\varepsilon^2$ or $|\partial_z f(x)| > \kappa_2(\partial_z f)\varepsilon^2$. By equation (3.1), either $\partial_x(f)$ or $\partial_y(f)$ or $\partial_z(f)$ has its Bernstein coefficients of the same sign in C. A similar proof applies for the trace of f on the transversal faces, since we have avoided the critical sections, for which the trace of f on the face is singular. Consequently, for ε_{smooth} and ε_{sing} small enough the algorithm stops on cells, in which the topology of f is guaranteed. □

3.3.2 Complexity analysis

In this section, we analyze the behavior of the algorithm as the size of the cells goes to 0. Let A be a subset of the surface S in the domain $\mathcal{D}$.

Definition 3.3.7 (Octree covering complexity)**.** *We denote by $C(\varepsilon, A)$ the minimal union of cells of size $\leqslant \varepsilon$ in the octree, covering A. Let $N(\varepsilon, A)$ be the number of cells involved in $C(\varepsilon, A)$.*

In order to analyze the number of boxes $N(\varepsilon, A)$, we connect it to the following notion [58], [144]:

Definition 3.3.8 (ε-entropy)**.** *For any set A in $\mathbb{R}^3$, let $E(\varepsilon, A)$ be the minimal number of closed balls of radius ε, covering A.*

We will first show that $N(\varepsilon, A)$ is of the same order as the *entropy* $E(\varepsilon, A)$ of $A \subset \mathbb{R}^3$:

Proposition 3.3.9. *$E(\varepsilon, A) \leqslant N(\varepsilon, A) \leqslant \gamma_0 E(\varepsilon, A)$ where $\gamma_0 = \mu(4\sqrt{3})$ and $\mu(r)$ is the minimal number of balls of radius 1, covering a ball of radius r.*

Proof. Since a cell of size ε is covered by a ball of radius ε, and $C(\varepsilon, A)$ covers A, we have $E(\varepsilon, A) \leqslant N(\varepsilon, A)$.

Since the Hausdorff distance between A and $C(\varepsilon, A)$ is at most the length $\sqrt{3}\,\varepsilon$ of the diagonal of the cube, we have (see [144])

$$E(2\sqrt{3}\varepsilon, C(\varepsilon, A)) \leqslant E(\sqrt{3}\,\varepsilon, A).$$

On the other hand, $N(\nu, A) \leqslant E(\frac{\nu}{2}, C(\nu, A))$ since a cell of size ν, cannot be covered by a single ball of radius $\frac{\nu}{2}$, so that we have:

$$N(\varepsilon, A) \quad \leqslant \quad E(\frac{\varepsilon}{2}, C(\varepsilon, A)) \leqslant \mu(4\sqrt{3})E(2\sqrt{3}\,\varepsilon, C(\varepsilon, A))$$

since $E(\frac{\nu}{\lambda}, A) \leq \mu(\lambda)E(\nu, A)$ for $\lambda > 0, \nu > 0$. We deduce that

$$N(\varepsilon, A) \quad \leqslant \quad \mu(4\sqrt{3})E(\sqrt{3}\,\varepsilon, A) \leqslant \mu(4\sqrt{3})E(\varepsilon, A),$$

since $E(\sqrt{3}\,\varepsilon, A) \leq E(\varepsilon, A)$ □

Next we will use the relations between the ε-entropy and the Vitushkin variations, defined as follows:

Definition 3.3.10 (Vitushkin variations)**.** *For any set $S \subset \mathbb{R}^3$, let $V_0(S)$ be the number of connected components of S, and*

$$V_i(S) = c(i) \int_{L \in \mathcal{G}_{3-i}} V_0(S \cap L)\, \mathrm{dL},$$

where $\mathcal{G}_k$ is the Grassmannian of affine spaces of dimension k in $\mathbb{R}^3$, dL is the canonical measure on $\mathcal{G}_{3-i}$, and $c(i) = \dfrac{1}{\displaystyle\int_{L \in \mathcal{G}_{3-i}} V_0([0,1]^i \cap L)\, \mathrm{dL}}$, (so that $V_i([0,1]^i) = 1$ and $c(3) = 1$).

Our aim is now to relate the number of boxes produced by the algorithm to geometric invariants of the surface, such as the variations $V_i(S)$:

Theorem 3.3.11. *Suppose that the surface $S \subset \mathcal{D}$ defined by $f(x, y, z) = 0$ is smooth in $\mathcal{D}$ (i.e. f and df do not vanish simultaneously at any point in $\mathcal{D}$). Then the number N of cells produced by the algorithm for $\varepsilon = \varepsilon_{smooth}$ is bounded by*

$$N \leqslant \gamma_0 \left(V_0(S) + \frac{1}{\varepsilon} V_1(S) + \frac{1}{\varepsilon^2} V_2(S) \right). \tag{3.2}$$

where $\gamma_0 \in \mathbb{R}_{>0}$ is a universal constant.

Proof. We use the following property [77], [144]:

$$E(\varepsilon, S) \leqslant \left(V_0(A) + \frac{1}{\varepsilon} V_1(S) + \frac{1}{\varepsilon^2} V_2(S) + \frac{1}{\varepsilon^3} V_3(S) \right),$$

and the property that $V_3(S) = 0$ since S is of dimension 2.

Now by proposition 3.3.9, we have

$$N(\varepsilon, S) \leqslant \gamma_0' \left(V_0(S) + \frac{1}{\varepsilon} V_1(S) + \frac{1}{\varepsilon^2} V_2(S) \right).$$

By proposition 3.3.5, in every cell outside $\Gamma_f(\kappa_2(f)\varepsilon^2)$, the Bernstein coefficients of f have the same sign. Thus such a cell is not kept by the algorithm. Consequently, we have

$$N \leq N(\varepsilon, \Gamma_f(\kappa_2(f)\varepsilon^2)).$$

As S is smooth in $\mathcal{D}$, the function $x \in \mathcal{D} \mapsto \frac{\operatorname{dist}(x,S)}{|f(x)|}$ is well defined and bounded by a constant $\kappa_1(f)$.

Thus, for ϵ small enough

$$\Gamma_f(\kappa_2(f)\varepsilon^2)) \subset S_\epsilon = \{x \in \mathcal{D}; \operatorname{dist}(x, S) < \varepsilon\}$$

We deduce that

$$N \leq N(\varepsilon, S_\epsilon) \leq 27 N(\varepsilon, S),$$

since by surrounding each of the cells covering S, by its 26 neighbors cells we cover the points of S_ϵ at distance ϵ from S. This proves inequality (3.2), with $\gamma_0 = 27\gamma_0'$. □

Notice that we can link $V_1(S)$ to the curvature of S, since there exists a universal constant c_1 such that

$$V_1(S) \leqslant c_1 \int_{S_{smooth}} |k_1(p)| + |k_2(p)| dp,$$

where $k_1(p), k_2(p)$ are the principal curvatures of S at p.

Similarly, we have

$$V_2(S) = \text{Area}(S).$$

See [144], [84] for more details.

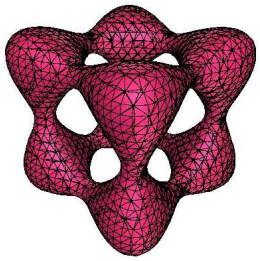
3.3.3 Singularities

The approach we have described so far is unable to handle correctly the singular locus of the variety. However there are wide classes of sets (e.g. algebraic, semi-algebraic, sub-analytic or more generally a set definable in some o-minimal structure) that have a local conic structure at any of their points (see theorem 4.3.3 for a precise statement). This structure gives a natural way to triangulate the variety by triangulating the boundary of the box first and then taking the cone over the boundary. This conic structure property is a consequence of the existence of Whitney stratifications for these sets and of general properties of topological uniform finiteness. All the previous remarks lead to the work presented in chapter 4.

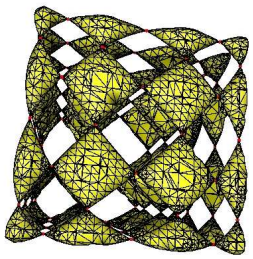
3.4 Experimentation

We present here some experimentations on surfaces related to the classification of singularities [11]. The implementation is available in the library AXEL (Algebraic Software-Components for gEometric modeLing)[2]. The table reports on the number of triangles, of cells including the singular one denoted by **nt** (represented by boxes in the pictures) and the timing. The tests have been run on a Pentium IV 2.4 Ghz workstation. We consider a smooth case, a case with finitely many singular points, with a self intersection curve and with the singular points containing an isolated curve arc[3]. The parameters used for the subdivision are $\varepsilon_{smooth} = 2^{-5}\,|\mathcal{D}|$ and $\varepsilon_{sing} = 2^{-8}\,|\mathcal{D}|$. The pictures show the corresponding mesh. In order to get a better rendering we could compute the normal at points on the surface. This is direct from the implicit equation, but is not done in the following visualization. Notice also that once the topology is certified, the triangulation can be improved in the smooth boxes, according to geometric criteria [35], [27].

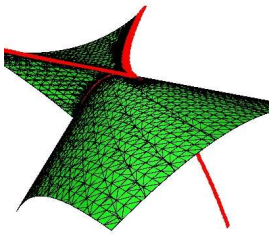
Equation:
$x^4 - 5\,x^2 + y^4 - 5\,y^2 + z^4 - 5\,z^2 + 11.8 = 0$
Nb of triangles: 8881
Nb of cells: 4165
Time (s): 1.53 s



Equation:
$32\,x^8 - 64\,x^6 + 40\,x^4 - 8\,x^2 + 1 + 32\,y^8$
$-64\,y^6 + 40\,y^4 - 8\,y^2 + 32\,z^8 - 64\,z^6$
$+40\,z^4 - 8\,z^2 = 0$
Nb of triangles: 45680
Nb of cells: 14555 + 594 nt
Time (s): 8.13 s



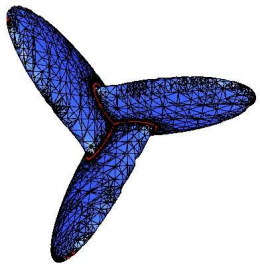
Equation:
$-4\,z^3\,y^2 - 27\,y^4 + 16\,x\,z^4 - 128\,x^2\,z^2$
$+144\,x\,y^2\,z + 256\,x^3 = 0$
Nb of triangles: 21354
Nb of cells: 7752 + 4684 nt
Time (s): 53.39 s



[2] `http://www-sop.inria.fr/galaad/software/axel`
[3] More examples can be found at `http://www-sop.inria.fr/galaad/data/surface/`

Equation:

$$-2749.231165\,x^3\,z\,y^2 - 1832.820776\,y\,z^2\,x^2$$
$$+648\,z\,x^2\,y^2 - 1620\,z^2\,x^2\,y^2$$
$$+1832.820776\,y\,z^3\,x^2 - 4123.846747\,y^4\,x\,z$$
$$+916.4103882\,z^3\,x\,y^2 + 64\,z^3 + 432\,z^5 - 216\,z^6$$
$$-729\,x^6 - 729\,y^6 - 144\,z^2\,x^2 - 288\,z^4$$
$$-144\,z^2\,y^2 + 324\,z^4\,x^2 + 324\,z^4\,y^2$$
$$+324\,z\,x^4 + 324\,z\,y^4 + 610.9402588\,y^3\,z^2$$
$$-810\,z^2\,y^4 - 610.9402588\,y^3\,z^3 - 2187\,x^4\,y^2$$
$$-2187\,x^2\,y^4 + 1374.615582\,x^5\,z$$
$$-305.4701294\,z^3\,x^3 = 0$$



Nb of triangles: 26184
Nb of cells: 7924 + 1616 nt
Time (s): 9.66 s

Chapter 4

Whitney stratifications, Transversality, and Triangulations

The work in chapter 3 originated from reflections on how to effectively compute a triangulation of a real algebraic hypersurface in a hypercube. Chapter 3 was thus focused on obtaining an efficient triangulation algorithm even at the cost of losing generality and not being able to handle singularities properly. This chapter sheds the limitations of chapter 3 and tackles the problem of triangulating semi-algebraic sets in its full generality. It thus takes a more abstract perspective on the matter, which turns out in the end to be at the expense of not being able to give a fully effective and efficient triangulation algorithm. Its contribution is to introduce new tools that are sufficiently abstract and general to be easily related to the geometry of the semi-algebraic set we want to triangulate. The concepts that it relates are: transversality, pointwise approximation of an object, stratifications, Voronoi partitions. The key to bridging together these concepts is a new version of the Thom-Mather topological triviality theorem.
The question we want to answer can be formulated as follows:
The central question.
Let $C \subseteq \mathbb{R}^n$ be a convex compact set, and $(f_{i,j})_{(i,j)\in\{1,...,k\}\times\{1,...,l\}} \in \mathbb{R}[X_1, ..., X_n]$.
Let $V(f_{i,j}) \subset \mathbb{R}^n$ be the zero set of $f_{i,j}$ $\big($where $(i,j) \in \{1,\dots,k\} \times \{1,\dots,l\}\big)$.
Can we compute effectively and efficiently a simplicial complex T such that T is isotopic to $\cup_{i=1}^{k} \left(\cap_{j=1}^{l} V(f_{i,j})\right) \bigcap C$?
By isotopic we mean the following:

Definition 4.0.1 (Isotopy)**.** *$\forall E, F \subset \mathbb{R}^n$, E is said to be isotopic to F iff $\exists h : [0,1] \times E \to \mathbb{R}^n$ such that h is continuous, $h(0,E) = E$, $h(1,E) = F$ and $\forall t \in [0,1]$, $h(t,.)$ is a homeomorphism on its image.*

We are thus aiming to triangulate the set $V(f) \cap C$. Notice that, as opposed to the notion of isotopic triangulation we aim at, the usual notion of

triangulation only requires that the simplicial complex be homeomorphic to the variety. In the following introductory passage we will be concerned with existing background on the subject, therefore the notion of triangulation should thus be understood as being only homeomorphic to the variety. However, the material presented afterward is concerned with isotopic triangulations.
Triangulations have a long history. The first proofs of the existence of homeomorphic triangulations of semi-analytic sets (hence algebraic sets also) were given by S. Lojasiewicz and B. Giesecke in the 60's ([89], sect.3, thm.1 and [64] (in German)). Ten years later H. Hironaka in [76] gave a simplified proof for the semi-algebraic case that relies on the fact that the projections of semi-algebraic sets are also semi-algebraic. This simplified approach can be adapted to the more general setting of sub-analytic sets (see [92] for instance). This stability by projection translates, in model theoretic terms, into the possibility to eliminate quantifiers in semi-algebraic formulas. This property is commonly referred to as the Tarski-Seidenberg quantifier elimination principle. The principle first appears in [129, 130, 124]. A recent proof of several formulations of this theorem can be found in [41], sect.1.3.1. The effective computation of the projection of semi-algebraic sets is in itself a difficult problem. In 1975, G. Collins in [36, 33] presented an algorithm to carry out such a computation. This algorithm is known as the Cylindrical Algebraic Decomposition (CAD for short). However its complexity is doubly exponential in the dimension of the ambient space. This algorithm recursively split the space into cylinders so that the semi-algebraic set above the base of the cylinders is a function graph. It loses track of the adjacency information between the cylinders, but with a little more work this approach can retain the adjacency information and produce a triangulation of the semi-algebraic set (for a complete exposition see [41], thm.3.12). More recent work in [96], explains an efficient strategy to obtain the adjacency information, this could possibly be applied to the computation of triangulation and yield a lighter procedure than the one presented in [41]. In a less computer-oriented direction, recursive projections to describe the structure of sets can be used to create a triangulation procedure for the wider class of o-minimal structures (defined for instance in [40], sect.1.3). They are essentially structures which allow boolean operations, are locally finite, and are stable by projection. The existence of triangulations in the setting of o-minimal structure is proved in [40], thm.4.4. Global and compact sub-analytic sets are one such example as A.M. Gabrièlov proved in [62] in 1968. Therefore the presentation through o-minimal structures is really inclusive of all the previously mentioned theoretical results.

We now go back to our original problem. In the following the objects we consider are compact semi-algebraic sets. When we talk about algebraic varieties, we will refer to the variety intersected with a fixed semi-algebraic convex set (rather than to a compact algebraic variety). As explained above, a refined version of the CAD algorithm can compute a triangulation. However, its complexity is very high and it is hopeless to use it as such on actual examples that come from application fields, such as motion planning, computer aided ge-

ometric design (see [23] for instance), or arrangement of algebraic varieties (see [70, 108] for the treatment of arrangements of hypersurfaces). Several methods have been developed to avoid this overly costly CAD algorithm in various contexts. For instance, the simple problem of deciding whether a semi-algebraic set is empty is addressed in [12] using a critical point method instead of recursive projections as in the CAD.

We focus here on triangulation, as opposed to meshes which do not necessarily have the same topology as the original variety. We have dealt with techniques for meshing the variety in chapter 3 already. Nevertheless, let us quickly go through the methods it mentions to highlight why they cannot be used to triangulate the variety. Raytracing methods simply do not generate meshes, let alone triangulations, they generate pixel images. The marching cube algorithm [93] chops up space, and create triangles in each cube. When the variety is smooth, there exists a cube size such that the generated triangulation is indeed homeomorphic to the variety. However, there is never any guarantee that a given cube size is small enough. The same problem occurs in other related algorithms such as those described in [21], [22], [72], [20]. Improvements made to the marching cube algorithm have primarily focused on speed. The marching polygonizer methods thus suffer the same drawbacks as the marching cube method. Consequently, these algorithms, do not enable us to ensure that we have produced a triangulation homeomorphic to the variety, not even when it is smooth (as opposed to the algorithm in chapter 3). Deformation methods do produce triangulations, but their scope is very limited to date. Finally "sample" methods which work by spreading points over the surface are inaccurate and do not produce triangulations. However, as mentioned earlier, the way the points are spread is related to Delaunay triangulations which are closely linked to Voronoĭ partitions. We will use those partitions later (from definition 4.5.5 and on) and the present work thus opens the perspective of using sample methods in a certified context.

To date, the only methods that have been successfully applied to produce triangulations in the general case are all improvements of the Cylindrical Algebraic Decomposition (CAD) algorithm. The improvements all aim at speeding up the very slow original CAD algorithm. Such improvements are often made by reducing its scope to a particular type of sets. In [33] for instance, the topology of algebraic curves or surfaces is analyzed, even in singular cases. The approach has been applied successfully to curves in 2D, 3D, 4D [68, 80, 65, 63, 8] and to surfaces [60, 34, 107]. They use projection techniques based on a conceptual sweeping line/plane perpendicular to some axis, and detect the critical topological events, such as tangents to the sweeping planes and singularities. This is a geometric rephrasing of the condition imposed by the CAD algorithm in low dimension, and is of course reminiscent of Morse theory. Because of the way they function, those methods are called sweeping methods. They involve the exact computation of critical points, genericity tests and adjacency tests. The final output of these methods is a topological complex of points, segments, triangles isotopic to the curve or the surface. We have mentioned already (chapter 2) that one factor that slows down CAD-based methods is that fact that they require

exact computations. But when the dimension of the ambient space increases another serious drawback appears. Those methods proceed by iterated projections, each projection being defined by sign conditions on polynomials. This leads to sequences of sets of polynomials, one for each dimension of the ambient space. The polynomial sequences are obtained by (sub)-resultant computations, corresponding to successive projections from $\mathbb{R}^{k+1}$ to $\mathbb{R}^k$. This was not playing an important role for planar curves as only one projection was needed, but in higher dimension this becomes a major problem. The degree of the polynomials in these sequences is bounded by $\mathcal{O}(d^{2^{n-1}})$ and their number by $\mathcal{O}((md)^{3^{n-1}})$, where m is the number of polynomials defining the semi-algebraic set S, d is a bound on the degree of these polynomials and n the number of different variables appearing in these polynomials [13]. Even for the case of implicit surfaces in $\mathbb{R}^3$ ($m = 1, n = 3$), this yields a bound of $\mathcal{O}(d^4 \times d^9) = \mathcal{O}(d^{13})$ points to compute in order to get the topology of the surface. What makes the picture even grimmer is that, as mentioned before in this introduction, the CAD algorithm does not directly yield a triangulation, nor any global topological information on the set S because the representation lacks information about the adjacency of the cells. Additional work is required to obtain a triangulation of S (see [41], [13], [96]). Recently, the CAD approach has been further investigated in [7]. It is shown how to analyze the topology of critical sections of an implicit surface, by exploiting the properties of delineability.

The work that is presented in this thesis revolved for a large part around the objective of providing such triangulation algorithms for real algebraic varieties in $\mathbb{R}^n$. This is in fact an important research topic of the team GALAAD at the INRIA of Sophia-Antipolis. The algorithm in chapter 2 treats the triangulation of varieties in $\mathbb{R}^2$. In $\mathbb{R}^3$, the approach described in chapter 5 allows to compute the topology of surfaces $Z = V(f)$ (without any assumptions on Z such as smoothness). It is classified as a sweeping method such as the one mentioned above: a sweeping plane records the topology changes as the plane moves, and then the algorithm makes the connections between different cuts of the variety according to the topology change information. This is thus in some way a much improved version of Morse theory. Chapter 5 shows that the connections are correct by using a famous result of topological triviality (the Thom-Mather theorem 4.1.10) for Whitney stratifications (definition 4.1.7). We will introduce this theorem and an improvement of it in the current chapter. Unfortunately there is no way to compute Whitney stratifications efficiently in general. The only method readily available is the application of their definition which is a first order formula (definition 4.1.7), on which we would use an effective, but overly costly, quantifier elimination algorithm such as the CAD.

The current chapter elaborates on the idea of Whitney stratifications. The purely algebraic approaches such as the CAD have failed to prove efficient (this much so that they are ineffective on practical examples). The algorithms in chapter 3 and 5 show that subdivision and Whitney stratification, that is the introduction of the geometric properties of the variety, are relevant to the problem of triangulating efficiently real algebraic varieties. The current piece of work

has developed in that direction. In the course of the work it appeared that the relevant setting for it was semi-algebraic geometry of compact sets without any further restrictions. Because we wanted to triangulate the sets by means of subdivision, we needed to obtain a minute partition of the initial domain in which the compact semi-algebraic set lies. It is obviously not possible to partition $\mathbb{R}^n$ into arbitrarily small pieces without allowing them to be singular. But as soon as we allow the parts to be singular, very simple partitions of $\mathbb{R}^n$ exist: partitions in hypercubes are one such example. In that context, the usual Thom-Mather theorem (thm. 4.1.10) did not apply satisfactorily as the boundaries of the pieces of the partition could be singular, thus the maps associated to them were not smooth which is a requisite to apply this theorem (the maps associated to convex sets are convex gauges as by definitions 4.3.6). We now discuss in greater details how and why the obstruction to using the classical Thom-Mather theorem arises.

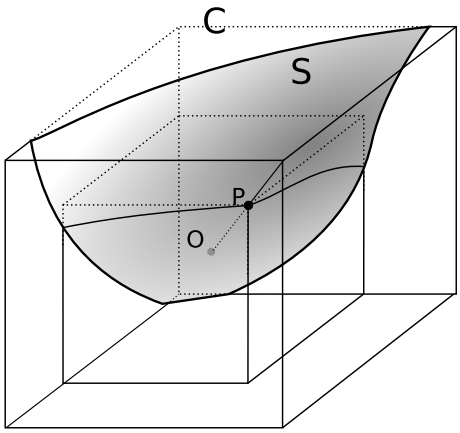


Figure 4.1: Problem with the usual Thom-Mather theorem

We know that a given semi-algebraic set has conic structure in a sufficiently small convex set (see 4.3 for a proof). This gives rise to a simple way to triangulate a semi-algebraic set by means of a partition: if we can find a partition such that the topology in every piece of the partition is conic, then we can obtain a triangulation by recursively triangulating the boundary of the pieces and then taking the cone over the boundary. To use the Thom-Mather theorem to prove that the variety is conic in a convex set, one has to find a family of convex sets which contract to a point while staying transverse to the variety. In figure 4.1, one sees two points, O and P, which lie on the surface S. Because of the curvature of S, the segment OP lies behind S and its extension after P sticks out in front of S. When contracting the larger cube C to O (by dilations centered at O on the picture), the corners of the smaller cubes will eventually touch S at the point P. In fact a corner of the contracted cubes will have to go through S for any contraction of C to O such that the corners of the contracted cubes stay away from $T_O S$, the tangent space to S at O. This is because, since O is

on S and S is smooth at O, for a cube small enough half of the corners of the cube will be on one side of S and the other half will be on the other side of S (this can be seen by considering the two extremities of a diagonal of C, one will lie on one side of $T_O S$, and the other will be on its other side). But C has six corners on one side of S and two on the other side. Therefore in a family of cubes contracting from C to O, a cube corner will have to cross S (two of them in fact). The classical Thom-Mather theorem will find an obstruction at any corner P that crosses S, because a point cannot be transverse to anything else than the whole space (not S in particular). Thus the Thom-Mather theorem will not enable us to conclude that S can be triangulated as a cone over the boundary. Nevertheless it is clear from the figure that the surface S *does* have conic structure in C. In fact, if the cube C were smoothed at the edges, the obstruction created by the corner would disappear and we could apply the usual Thom-Mather theorem. The previous idea of smoothing is intuitive, but is not in fact applicable in practice. This is due to the fact that to recover the singular case from the smooth case, one has to make the smoothed cubes degenerate to the singular original cube. But the Thom-Mather theorem does not provide any global bound on the Lipschitz coefficients of the isotopies it gives. It is thus impossible to control the behavior of the isotopies as the smoothed cubes degenerate to the original cube, and no result can be obtained by a process of limits.

This situation prompted the need to develop a new tool that would generalize the usual Thom-Mather theorem to handle the type of cases such as the one found in figure 4.1 and allow to control the Lipschitz coefficient of the isotopies. Thus, in a way, this extension of the Thom-Mather theorem solves twice the problem presented in figure 4.1, because it allows the corner of the cube to go through S and because it allows us to smooth the cube while uniformly controlling the Lipschitz coefficients of the isotopies and thus to make them degenerate to an isotopy for the original cube. This extension of the Thom-Mather topological trivialization theorem is presented in section 4.1. The basic definitions and theorems about stratification theory are recalled in the first part of the subsection (4.1.1), then the extension of the theory itself is presented and the new results are proved (subsection 4.1.2). The extension of the Thom-Mather theorem enables us to control the isotopies but it is not easy to manipulate as such. Subsection 4.2 provides some tools in that direction. We define some measures of the transversality of maps and stratified sets and prove a few lemmas that enable us to conclude that a map and a stratified set are transverse with quantitative arguments. The next subsection (4.3) uses the previous results and definitions to create a tool to triangulate a variety in a small enough convex set. This tool is a "metrically" stable version of the classical conic structure theorem (theorem 4.3.3) that can be derived from Hardt's theorem. The background is recalled in subsection 4.3.1, and the stable version of the conic structure theorem is proved next (subsection 4.3.2). So far, the theorems work for any type of convex deformations (definition 4.3.4), that is families that contract a convex set to a point. Subsection 4.4 takes care of relating all the convex deformations to the Euclidean distance. This allows us to carry out an analysis in more nat-

ural terms as in most settings results are formulated for the Euclidean distance. Finally the last subsection (4.5) presents a tentative procedure to triangulate a Whitney stratified set. It is tentative in that it requires a certain type of approximation (definition 4.5.2) of the variety with points, and a procedure to obtain such approximations is not thoroughly worked out although we point out some possible solutions. Also, to be effective, it requires knowledge of a transversality measure (definition 4.4.4) and this thesis does not give a means to estimate that invariant (still, we show that if the point approximation is fine enough the transversality measure has the necessary properties to allow us to obtain a triangulation). Therefore the procedure is not yet ready to be turned into an implementation. Nevertheless, the interest of the procedure we describe lies in the fact that it is abstract from any specialized way of obtaining the point approximation of the variety and relies on a metrically intrinsic object: Voronoĭ partitions (definition 4.5.5). Another interesting point is that although Whitney stratifications of the variety are instrumental in the proof, they do not intervene explicitly in the algorithm since only a point approximation of them is required. In view of our experience with other metric techniques, it seems possible that one can compute a point approximation for some Whitney stratification and estimate the transversality invariant without having an actual algebraic knowledge of such a stratification (and this will probably turn out to be much faster than computing such an algebraic representation). In case a Whitney stratification is available, its algebraic description will, in all likeliness, allow us to estimate effectively the transversality invariant and to generate suitable pointwise approximation. In some special situations there are effective algorithms to compute Whitney stratification: as will be described in section 5.5.1, there is an algorithm to compute Whitney stratifications of surfaces in $\mathbb{R}^3$ and, in [138], an algorithm for semi-algebraic sets which are a union of intersection of transverse hypersurfaces is described.

Now let us start with the presentation of the new results in stratification and topological triviality theory.

4.1 Stratification and topological stability

This subsection presents an extension of the Thom-Mather theorem about topological trivialization. The first part of the section (subsection 4.1.1) recalls basic definitions and theorems about stratifications of varieties and strives to position the current work with respect to this background. The second part of the section (subsection 4.1.2) deals with the actual presentation and proof of the new result. The result is obtained by integration of vector fields as in the literature, but the integrated vector fields are better controlled here which enables us to obtain a quantitative version of the Thom-Mather theorem (theorems 4.1.30 and 4.1.44) that can be used to obtain uniform bounds on the Lipschitz coefficients of the isotopies and that applies to a wider class of mappings than the class of smooth mappings. It is proved primarily for semi-algebraic mappings but the

result seems extensible to sub-analytic maps without obstruction.

4.1.1 Notations and background on stratifications

These first notations are very usual and will be used throughout the text.

Notations 4.1.1. *For a set E in a topological space*

- *The interior of E is denoted by E°.*
- *The closure of E is denoted by $\overline{E}$.*
- *The boundary of E is denoted by ∂E.*
- *Let S be a topological space, and $S' \subset S$ a subspace of S. The notation $p_i \xrightarrow{S'} p$ will mean that the sequence p_i of points in S' converges to a point $p \in S$.*

A stratification of a set $Z \subset \mathbb{R}^n$ is a partition of Z into smooth submanifolds called strata. This notion first clearly emerged in [139]. There are many ways to choose stratifications. Consider the example of two lines crossing at the origin. One could choose the following stratification: the first stratum is the first line, and the second stratum is what is left of the second line. However, stratifications have better properties when their strata connect smoothly to each other. Even if the previous sentence is not very precise it is fairly intuitive. For instance on the previous example of the two lines, we see that the proposed stratification is quite unnatural since on the first stratum, nothing distinguishes the origin from any other point on the line. To date, several formal conditions have been considered to make precise this intuitive idea of smooth connection.
We now introduce classical properties that one requires of stratifications to make them useful mathematical objects. To characterize the relative positions of strata we use the degeneration ordering:

Definition 4.1.2 (Degeneration ordering)**.** *Let $X, Y \subset \mathbb{R}^n$, Y is said to degenerate to X iff $X \subset \partial Y$. In such case we write $X < Y$.*
We will also say that Y is adjacent to X.

Obviously the previous relation is a partial ordering on sets. The following notion of S-decomposition enforces some structure on the partitions of the variety that we want to call stratifications:

Definition 4.1.3 (S-decomposition)**.** *Let $Z \subset \mathbb{R}^n$, and let $\mathcal{S}$ be a partition of Z, then $\mathcal{S}$ is called an S-decomposition of Z iff*

- *every $X \in \mathcal{S}$ is a smooth submanifold of $\mathbb{R}^n$.*
- $\forall X \neq Y \in \mathcal{S}$,
$$(X \bigcap \overline{Y} \neq \emptyset) \;\Rightarrow\; X < Y.$$

- *$\mathcal{S}$ is locally finite. This means that for all $K \subset \mathbb{R}^n$ compact set, the subset of $\mathcal{S}$, $\{X \in \mathcal{S} \mid X \bigcap K \neq \emptyset\}$ is finite.*
- *And of course*

$$\bigcup_{X \in \mathcal{S}} X = Z.$$

The elements of $\mathcal{S}$ are called strata and Z is said to be stratified (by $\mathcal{S}$). Notice that the definition does not require the strata to be connected components has some authors impose. In fact, the only time we will need an explicit stratification in section 4.5, it will prove far more convenient to consider S-decompositions of Z of the form $\mathcal{S} = (\sigma_0, \ldots, \sigma_{\dim Z})$ where the σ_i are pure i-dimensional strata (considered in definition 4.5.2 and on).
Finally, the most important point for a stratification to be an interesting object is to define what conditions of smooth connection between strata we want to enforce. Several notions of smooth connection have been considered up to this day, but all of them rely on the enforcement of conditions on tangent spaces:

Notation 4.1.4 (Tangent space)**.** *Let M be a C^∞ manifold, and $p \in M$ one of its points, then the tangent space to M at p will be denoted by T_pM.*
Let $Z \subset \mathbb{R}^n$ be a stratified set by an S-decomposition $\mathcal{S}$, and $p \in Z$. Then the tangent space T_pZ to Z at p for $\mathcal{S}$ is defined as T_pX where $X \in \mathcal{S}$ is the stratum containing X.

Definition 4.1.5 (Tangent bundle)**.** *Let M be a C^∞ manifold, the tangent bundle to M will be denoted TM.*
Let $Z \subset \mathbb{R}^n$ be a stratified set by an S-decomposition $\mathcal{S}$. For any $x \in Z$, the tangent space to Z at X is defined as

$$T_x^{\mathcal{S}} Z := T_x \sigma,$$

where σ_p is the stratum of $\mathcal{S}$ containing p. The tangent bundle $T^{\mathcal{S}}Z$ to Z is defined as

$$T^{\mathcal{S}} Z = \coprod_{p \in Z} T_p^{\mathcal{S}} \sigma_p.$$

When the stratification of Z that we consider is clear from the context we will drop the superscript for T and simply write T_xZ and TZ.
The topology on TZ is defined as follows. For any $p \in Z$, T_pZ is identified with $\mathbb{R}^i \times \{0\}^{n-i} \subset \mathbb{R}^n$ where $i = \dim T_pZ$. At any point p, since $\mathcal{S}$ is locally finite, there is an open set U small enough such that for any stratum $X \in \mathcal{S}$, the tangent bundle $TX|_U$ is trivial. Therefore $TZ|_U$ can be identified with a subset of $U \times \mathbb{R}^n$. The topology on $TZ|_U$ is given by the trace of the product topology of $U \times \mathbb{R}^n$ on $TZ|_U$.

Definition 4.1.6 (A-regular stratifications)**.** *Let $Z \subset \mathbb{R}^n$ be a stratified set by an S-decomposition $\mathcal{S}$. Let $A : \mathcal{S}^2 \to \{\text{True}, \text{False}\}$ be a property on pairs of strata of $\mathcal{S}$. We say that two strata $X, Y \in \mathcal{S}$ meet the A condition, or*

alternately that X, Y are A-regular iff $A(X,Y) = \text{True}$. We say that $\mathcal{S}$ is an A-regular stratification or alternatively that $\mathcal{S}$ is an A-stratification iff

$$\forall X, Y \in \mathcal{S},\ A(X,Y) = \text{True}.$$

Also, conditions on pairs of strata (i.e. in the form of A) will be also be referred to as regularity conditions.

The regularity condition between strata we use in this thesis has been defined by H. Whitney in [141]. It is known as Whitney's condition (b).

Definition 4.1.7 (Whitney's condition (b), [141], sect.19)**.** *Let $X, Y \subset \mathbb{R}^n$ be two submanifolds, let $p \in X$, the pair (X,Y) satisfies Whitney's condition (b) at p iff*

$$\forall p_n \xrightarrow{X} p,\ \forall q_n \xrightarrow{Y} p,$$
$$\left(\lim_{n\to\infty} \overline{p_n q_n} = l \text{ and } \lim_{n\to\infty} T_{q_n} Y = \tau\right) \Rightarrow l \subseteq \tau$$

where $\overline{p_n q_n}$ denotes the straight line that goes through p_n and q_n.

The pair (X,Y) is said to satisfy Whitney's condition (b) or alternatively, to be Whitney regular iff it satisfies condition (b) at every point $p \in X \cap \overline{Y}$.

According to definition 4.1.6, a Whitney stratification is an S-decomposition whose pairs of strata all meet condition (b), that is, a (b)-regular stratification. The notion of Whitney stratification is very useful since, for any semi-algebraic (resp. semi-analytic, sub-analytic) sets, there exist semi-algebraic (resp. semi-analytic, sub-analytic) (b)-regular stratifications and under a very natural additional condition (a smooth mapping must be submersive) we have a topological triviality theorem known as the Thom-Mather theorem (see 4.1.10).

We will digress a bit before returning in more length to the Thom-Mather theorem and its implications in order to present several regularity conditions that have been considered in the literature. This presentation aims to link the content of this chapter to its background and explain why, with regard to this background, the context of semi-algebraic varieties endowed with a Whitney stratification is the most suitable setting for the rest of this chapter. Here is a point list of the main types of stratifications that have been considered up to now:

- Together with condition (b) in [141], sect.19, H. Whitney defined condition (a) which seems to be the most basic requirement one can expect of a regularity condition: it asks that any limit of tangent spaces at points in a stratum includes the tangent space at the limit of the sequence of points. Condition (a) is implied by condition (b). Unfortunately, condition (a) is too weak and there are no results for (a)-regular stratifications that entail topological triviality.

- J-L. Verdier introduced condition (w) in [137] (def.1.4) and proved (thm.2.2) that there exist (w)-regular semi-algebraic (resp. semi-analytic, sub-analytic)

stratifications for any semi-algebraic (resp. semi-analytic, sub-analytic) varieties and that such stratifications enjoy a topological trivialization theorem identical to theorem 4.1.10 (see [137], thm.4.14) except that the set is stratified with condition (w) and the trivialization homeomorphism comes from the integration of a "rugose" vector field (see [137], def.4.1). This construction gives rise to a weaker trivialization homeomorphism than the one obtained with Whitney's condition (b).
Condition (w) was inspired from condition (r), formerly introduced by T.-C. Kuo in [82] (def.1.1) as a sufficient condition for Whitney's condition (b) (theorem [82], thm.1) in the semi-analytic case. Conditions (r) and (w) have to do with the distance between tangent spaces and are somewhat related to the measures of transversality we will introduce. Definition 4.2.4 makes that link.

- Yet another condition is Mostowski's condition (L). It was introduced in [101, 100] for analytic sets where it is proved that one can stratify analytic varieties with condition (L) and obtain a topological trivialization theorem identical to theorem 4.1.10 with the additional requirement that the stratification should be (L)-regular and the additional consequence that the trivializing homeomorphism is Lipschitz. Later, in [115], it is proved that the same results (existence of stratification and trivialization theorem) hold for semi-analytic sets, and finally in [116] the results are furthermore shown to hold for the sub-analytic case (see [116], thm.1.4 for the existence of stratifications and [116], thm.1.6 for the topological trivialization theorem). Condition (L) ensures that one can extend Lipschitz vector fields on a stratum to a Lipschitz vector field on an adjacent stratum provided they meet condition (L) (see [116], prop.1.3 for the sub-analytic case).

- A last type of stratifying condition is Bekka's condition (C) which is defined in [15] (def.1.1) (originally in [14]). This condition is the minimal condition that allows to extend a continuous controlled vector field on a stratum to a continuous controlled vector field on an adjacent stratum ([15], thm.2.5 and its corollary). For sets stratified by condition (C), theorem 4.1.10 still holds. In fact, condition (C) is implied by Whitney's condition (b) (see [15], rem.5, and [133] for more detail).

In the following we will focus our interest on condition (b) for it is the most adequate setting to present our results. The following summarizes why it is so:

- As mentioned above, condition (a) is too weak to yield any of the results that we will prove and use in this chapter.

- Condition (L) is unnecessarily strong for our purpose as we do not need that the vector fields be Lipschitz. Also, this condition has a rather unwieldy formulation. In theorem 4.1.30 we will in fact be able to enforce that the isotopies we obtain be Lipschitz, but not because the integrated vector fields they come from are Lipschitz.

- In the same way as condition (L), condition (w) is also unnecessarily strong as we will control the distance between tangent spaces by a transversality function (see section 4.2), and using a finer stratification (such as a (w)-regular one) than a (b)-regular one, would only worsen the properties of the transversality function.
This fact can be surprising at first. The reason is that the transversality function is not continuous when passing from one stratum to another. Therefore, adding strata to refine a stratification will introduce unnecessary discontinuities in the transversality function.

- Finally condition (C) is too minimalistic to be used for the triangulation procedure (condition (b) is required for the proof of lemma 4.4.5), although it is sufficient to obtain the trivializing isomorphism of this section.

After this historical summary, we take back the focus to Whitney's condition (b) and the notions we will need in the rest of this chapter, especially the Thom-Mather topological triviality theorem 4.1.10 and related results.
As mentioned earlier, Whitney's condition naturally stratifies semi-algebraic varieties (see [90, 91]). That is, one can obtain a Whitney stratification of a semi-algebraic set Z through the following procedure:

Algorithm 4.1.8.
- *Let $X_0 = Z$. Set $i := 0$.*
- *Define*

  $$X_{i+1} = \mathrm{Sing}(X_i) \cup \{p \in X_i \mid \exists\ j < i,\ (X_i, Y_j) \text{ not Whitney regular at } p\}.$$

  Set $Y_i = X_i \setminus X_{i+1}$.
- *If $X_{i+1} = \emptyset$, then end the procedure. Else, increment i by one, and go back to the previous step.*

When the procedure ends, $\mathcal{S} = \{Y_j \mid j \in \{0, \ldots, i\}\}$ is a Whitney stratification of the semi-algebraic set Z. As said earlier, other classes of sets are known to be Whitney stratifiable, such as semi-analytic and sub-analytic sets.

The classical Thom-Mather theorem tells us that Whitney stratified sets behave nicely with respect to topological triviality. A proof of this theorem can be found in J. Mather's lecture notes [95]. We recall this theorem here.

Definition 4.1.9 (Proper stratified submersion)**.** *Let $Z \subset \mathbb{R}^n$ be a set stratified by $\mathcal{S}$ and M a manifold. Let $f : Z \to M$. If $\forall X \in \mathcal{S}$, $f(X)$ is a smooth submanifold and $f|_X$ is a submersion on its image, then f is called a stratified submersion of Z for $\mathcal{S}$.*
If in addition, f is proper, then f is a proper stratified submersion.

Theorem 4.1.10 (Thom-Mather)**.** *$\forall Z \subset \mathbb{R}^n$, $\forall \mathcal{S}$ Whitney stratification of Z, let $g : Z \to \mathbb{R}^m$ be a proper stratified submersion for $\mathcal{S}$. Let x be any point in $\mathbb{R}^m$. Let $\mathcal{S}'$ be the stratification of $g^{-1}(x) \times \mathbb{R}^m$ formed by the set of strata*

$\left(g^{-1}(x) \cap \sigma\right) \times \mathbb{R}^m$ *for* σ *running over the strata of* $\mathcal{S}$.
Then $\exists h : Z \to (g^{-1}(x) \times \mathbb{R}^m)$ *a homeomorphism, such that* h *is smooth on each stratum of* Z, $\Pi_2 \circ h = g$ *where* Π_2 *is the projection to the second component, and* h *is stratum preserving (that is* $h(\sigma) \subset \left(g^{-1}(x) \cap \sigma\right) \times \mathbb{R}^m$*).*

A straightforward consequence of the previous theorem, that is maybe geometrically more intuitive, is the "moving the wall" theorem as formulated by M. Goresky and R. MacPherson in [67], thm.1. It is formulated with the use of the notion of transversality.

Definition 4.1.11 (Transversality)**.** *Let* $Z, Z' \subset \mathbb{R}^n$ *be two sets which are respectively stratified by* $\mathcal{S}$ *and* $\mathcal{S}'$. *Let* $p \in Z \cap Z'$, $X \in \mathcal{S}$ *and* $X' \in \mathcal{S}'$ *be such that* $p \in X$ *and* $p \in X'$. *Then* Z *and* Z' *are said to be transverse at* p *iff* $T_pX + T_pX' = \mathbb{R}^n$.
Globally, Z *and* Z' *are said to be transverse iff they are transverse at any point* $p \in Z \cap Z'$.

Lemma 4.1.12 (Moving the Wall, [67], thm.1)**.** *Let* $Z \subset \mathbb{R}^n$ *be a set which is Whitney stratified by an S-decomposition* $\mathcal{S}$, *and let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a smooth map. Let* $Z_t := f^{-1}(t) \cap \sigma$, *and for any stratum* $\sigma \in \mathcal{S}$ *and* $t \in \mathbb{R}$, *let* $\sigma_t := f^{-1}(t) \cap \sigma$. *We endow* $Z_0 \times \mathbb{R}$ *with the stratification*

$$\mathcal{S}_0 = \{\sigma_0 \times \mathbb{R} \mid \sigma \in \mathcal{S}\}.$$

If $f|_Z$ *is proper, and* $\forall t \in \mathbb{R}$, Z *and* $f^{-1}(t)$ *are transverse, then* $\mathcal{S}_0$ *is a Whitney stratification for* $Z_0 \times \mathbb{R}$ *and there exists a homeomorphism*

$$h : Z \ \to \ Z_0 \times \mathbb{R},$$

which is smooth on the strata of Z *and such that* $\Pi_2 \circ h = f$ *with* Π_2 *the projection to the second component. In addition* h *is stratum preserving, that is, for any* $\sigma \in \mathcal{S}$, *we have* $h(\sigma) \subset (\sigma_0 \times \mathbb{R})$ *(notice* $(\sigma_0 \times \mathbb{R})$ *belongs to* $\mathcal{S}_0$*).*

Remark 4.1.13. *If the reader is familiar with [67], the previous version of the moving the wall theorem can seem simplified in comparison to the version in [67]. It is in fact easily equivalent to it by considering the graph of the function* g *in [67] intersected with* $Z \times \mathbb{R}^{m+1}$ *and* $\mathbb{R}^n \times Y$, *and taking* f *in our version to be the projection to the image space of the graph.*

The idea behind the name of the theorem is that the sets $f^{-1}(t)$ act like a moving wall, in a much similar fashion to Morse theory.
An important consequence of the moving the wall theorem is that for any $t \in \mathbb{R}$, Z_t and Z_0 are homeomorphic. This is straightforward because $\Pi_2 \circ h = f$ and therefore by h we have:

$$Z_t \overset{h}{\cong} Z_0 \times \{t\} \cong Z_0.$$

4.1.2 An extension of the moving the wall theorem

This section follows the same approach as J. Mather in his lecture notes ([95]) to prove an extended version of the moving the wall theorem in the previous section (theorem 4.1.12). As in J. Mather's notes, we obtain topological triviality along a one parameter group by means of integration of a controlled vector field. However, we use a result by Du Plessis to construct a continuous controlled vector field (see [48] for the original result, and [111] for a more recent and comprehensive treatment of this subject). This vector field is constructed so that it is "transverse" to a continuous map (transverse here is used by analogy with the vector fields constructed for the usual moving the wall theorem 4.1.12, see remark 4.1.28 for what we exactly mean by this). The continuity of the vector field allows us to better control the trivializing homeomorphism we construct and simplifies the proof of the existence of this homeomorphism.

The generalized notion of transversality we introduce for non-smooth maps is formalized by means of cones on the tangent space of the manifold.

Definition 4.1.14 (Cones)**.** *An open cone of a $\mathbb{R}$ vector space is an open set which is stable by multiplication by elements of $\mathbb{R}_+^*$ and by addition. The cones are also required to be proper in the sense that they do not contain 0 (this is equivalent to saying that they are not the whole space).*
The space of all open cones of a vector space E is denoted $\mathrm{C}(E)$.

Definition 4.1.15 (Cone bundle)**.** *For a manifold M, a cone bundle on M is a map :*

$$\mathcal{C} : x \in M \mapsto \mathrm{C}(T_x M).$$

For a stratified set $Z \subset \mathbb{R}^n$ stratified by a set of smooth strata Σ, a cone bundle on Z is a map

$$\mathcal{C} : x \in \sigma \mapsto \mathrm{C}(T_x \sigma),$$

where $\sigma \in \Sigma$ is the stratum of Z that contains x.

The following regularity conditions on the cone bundles will be needed when lifting continuous vector fields to adjacent strata.

Definition 4.1.16 (Upper semi-continuous cone bundle)**.** *A cone bundle $\mathcal{C}$ on a stratified set Z is said to be upper semi-continuous iff for all $x \in Z$, there exists a neighborhood $U \subset TZ$ of $\mathcal{C}(x)$ such that for any base point y of an element of U, we have:*

$$U \cap T_y Z \subset \mathcal{C}(y).$$

(the tangent bundle TZ and its topology are defined in definition 4.1.5).

Remark 4.1.17. *This condition is stronger than asserting that $\mathcal{C}(x)$ is included in the adherence of the neighboring cones $\mathcal{C}(x')_{x' \neq x}$. It means that for any sequence of points $y_i \in T_{x_i} M$ converging to $y \in \mathcal{C}(x)$, y_i ultimately lies in $\mathcal{C}(\overline{x_i})$.*

Definition 4.1.18 (Non-vanishing cone bundle)**.** *A cone bundle $\mathcal{C}$ on a manifold M is said to be non-vanishing if every fiber of $\mathcal{C}$ is not empty. In other words $\mathcal{C}(x) \neq \emptyset$ $(\forall x \in M)$.*

The first result is that, once a manifold M has been endowed with a non-vanishing upper semi-continuous cone bundle, it is possible to define a global non-vanishing smooth vector field on M with value in that cone bundle. This means in particular that not all varieties can be endowed with such a structure: by Poincaré's theorem, a $2n$-sphere and in general any compact manifold with non-zero Euler characteristic cannot carry such a cone bundle.

Definition 4.1.19 (Vector field with value in a cone bundle)**.** *A vector field $v : M \to TM$ on a manifold M is said to have value in a cone bundle $\mathcal{C}$ on M iff $\forall\ x \in M$, $v(x) \in \mathcal{C}(x)$.*

Theorem 4.1.20 (Existence of vector fields with value in a cone bundle)**.** *It is possible to associate to any non-vanishing upper semi-continuous cone bundle $\mathcal{C}$ on a manifold M, a smooth vector field v with value in $\mathcal{C}$.*

Proof. Assume that for any $x \in M$ we are able to construct a vector field with value in $\mathcal{C}$ in a neighborhood of x. Then as M is a locally compact space countable at infinity, by means of a partition of unity we can create a global vector field on M. This new vector field will still have values in $\mathcal{C}$ because every cone in the cone bundle is stable by addition and multiplication by a positive constant. The resulting vector field will be non-vanishing as the cones in $\mathcal{C}$ do not contain 0 (by definition of cone). Therefore we only have to prove the result locally at any $x \in M$.

To construct the vector field locally, first pick a (non-zero) vector $v(x) \in \mathcal{C}(x)$ which is possible as $\mathcal{C}$ is non-vanishing. Locally in a neighborhood of x we can identify to $\mathbb{R}^n$ all the tangent spaces to M. In that context the fact that U in the definition of upper semi-continuity is open, means that there is an open neighborhood N of $v(x)$ in T_xM, and an open neighborhood V of $x \in M = \mathbb{R}^n$ such that $N \times V \subset U$. For $y \in V$, the condition of upper semi-continuity is that we have $U \cap T_yM \subset \mathcal{C}(y)$. Therefore it implies that $N \subset \mathcal{C}(y)$, and we can take $v(y) = v(x)$ everywhere in V. □

By applying the previous reasoning to varieties endowed with a Whitney stratification, it is thus proved that one can construct a smooth nonzero vector field on any stratum of a Whitney stratified set Z while constraining it to have values in a given upper semi-continuous cone bundle on Z. Now one has to take care of what happens where the strata connect together. To do so the vector fields are constrained when approaching lower dimensional strata. Those constraints are encoded in a control tube system and the same compatibility conditions for tube systems as in J. Mather's lecture notes ([95]) are used here. Although the general line of reasoning is similar to the one followed by J. Mather in his notes, substantial changes are introduced because we allow more general maps and because we use Du Plessis's result to create *continuous* trivializing vector fields.

As a reminder J. Mather constructs a controlled tube system so that it is compatible with a smooth map, then he constructs a controlled vector field that

is also compatible with that map, and then uses the fact that both objects are controlled to prove that the generated one parameter group is continuous.
The approach here is different as no compatibility condition with a map is enforced on the control tube system, we only need the commutation relations between the projection maps and the distance maps. But as the controlled vector fields will all be continuous thanks to Du Plessis's work ([48], and in more depth [111]), the one parameter groups obtained by their integration will also be continuous (J. Mather recovered continuity of the one parameter groups by a different argument since the vector fields he considered were not necessarily continuous). We will furthermore impose that the vector fields have values in a fixed cone bundle.
Of course in both cases the vector field is smooth on each stratum and controlled, therefore the existence of the one parameter group is not a problem no matter the approach. The essential difficulty is to prove the continuity of the global one parameter group where the strata come together.

The first lemma that follows is a quote of a weaker version of J. Mather's lemma which states that, given a Whitney stratified set, one can find a system of control tubes. J. Mather's result includes the compatibility of the tube system with a smooth map, here only commutation relations are needed.

Lemma 4.1.21 (Existence of control tube system)**.** *Let Z be a set in $\mathbb{R}^n$ which is Whitney stratified by Σ. There exists a system of tubes*

$$(T_\sigma)_{\sigma\in\Sigma}, \quad (\pi_\sigma : T_\sigma \to \sigma)_{\sigma\in\Sigma}, \quad (\rho_\sigma)_{\sigma\in\Sigma} : T_\sigma \to \mathbb{R}^+,$$

where the T_σ are open neighborhoods of the σ's in $\mathbb{R}^n$, the π_σ are orthogonal projections to the σ's, and the ρ_σ are the distance functions to the σ's, such that for any $X \in \Sigma$, $Y \in \Sigma$ with $Y > X$ the following commutation relations hold:

- $\pi_X = \pi_X \circ \pi_Y$ *on* $T_X \cap T_Y$
- $\rho_X = \rho_X \circ \pi_Y$ *on* $T_X \cap T_Y$

Such systems of tubes are called control tube systems for Σ.

The Whitney condition comes into play again for the next theorem that states that a smooth vector field with value in a cone bundle can be extended continuously to a neighboring stratum (what J. Mather does not need to do).

Definition 4.1.22 (Controlled vector field)**.** *Let Z be a set which is Whitney stratified by Σ. Let*

$$(T_\sigma)_{\sigma\in\Sigma}, \quad (\pi_\sigma : T_\sigma \to \sigma)_{\sigma\in\Sigma}, \quad (\rho_\sigma)_{\sigma\in\Sigma} : T_\sigma \to \mathbb{R}^+$$

be a control tube system for Σ. Let $\zeta : Z \to TZ$ be a tangent vector field to Z. We say that ζ is a controlled vector field for $(T_\sigma, \pi_\sigma, \rho_\sigma)_{\sigma\in\Sigma}$ iff ζ meets the two following conditions for any stratum $X \in \Sigma$ and $z \in Z \cap T_X$:

$$\begin{cases} \pi_X^*(\zeta(z)) = \zeta(\pi_X^*(z)), \\ (d\rho_X)_z(\zeta(z)) = 0, \end{cases}$$

where $\pi_X^* : T(Z \cap T_X) \to TX$ is the transpose of π_X, and $(d\rho_X)_z$ is the differential of ρ_X at z.

Theorem 4.1.23. *Let M be a manifold, and $Y > X$ two disjoint smooth manifolds that satisfy Whitney's condition (b). Let $\mathcal{C}$ be a non-vanishing upper semi-continuous cone bundle on $Y \cup X$ (considered as a stratified set). If v_X is a continuous vector field on X with values in $\mathcal{C}$, then it extends to a continuous controlled vector field with values in $\mathcal{C}$ which is defined on a neighborhood of X in $X \cup Y$.*

This theorem is in fact a straightforward consequence of A. Du Plessis's theorem ([48] and [111], thm.3) which states that a continuous controlled vector fields on a stratum can be extended to all the adjacent strata.

Theorem 4.1.24 (Du Plessis [48], and [111], thm.3)**.** *Let M be a manifold and $X, Y \subset M$ be submanifolds such that $Y > X$ and the pair (X, Y) is Whitney (b)-regular. If v_X is a continuous controlled vector field on X, there exists N a neighborhood of X in $X \cup Y$, and v a continuous controlled vector field on N, such that $v|_X = v_X$.*

Proof of theorem 4.1.23. First, we extend v_X to v in a neighborhood N of X in $X \cup Y$ by Du Plessis's theorem 4.1.24. At any point $x \in X$, $\mathcal{C}$ is upper semi-continuous. Therefore, there exists an open neighborhood U_x of $v_X(x)$ that is included in the cone bundle $\mathcal{C}|_N$. Because the vector field v is continuous, the preimage N_x of U_x by $v : x \in N \mapsto v(x) \in \mathcal{C}(x)$ is an open neighborhood of x included in N. By taking the union N' of the N_x for x running over X, we obtain an open neighborhood $N' \subset N$ of X where v has values in $\mathcal{C}$. In addition, since v was obtained from theorem 4.1.24, it is continuous and controlled on N', which is a neighborhood of X in $X \cup Y$. Q.E.D. □

This result for a pair extends naturally to a global one. This is what is proven in the following.

Theorem 4.1.25 (Existence of continuous controlled vector fields with values in a cone bundle)**.** *For Z a Whitney stratified set with stratification Σ and $\mathcal{C}$ an upper semi-continuous cone bundle, there exists a continuous controlled vector field on Z with values in $\mathcal{C}$.*

Proof. We build it by induction on the degeneration partial ordering $<$ on strata (definition 4.1.2). A minimal element for this order is a locally compact manifold, therefore by theorem 4.1.20 one can build a smooth vector field with values in $\mathcal{C}$ on it. The vector field can be chosen smooth because $\mathcal{C}$ is upper semi-continuous. As the stratum is minimal for $<$ there is no stratum that lies in its adherence and the control conditions are trivially satisfied (i.e. we have a controlled vector field).

Now take Y such that we have already built a continuous controlled vector field v with values in $\mathcal{C}$ on $\overline{Y} \setminus Y$. For every $X < Y$, theorem 4.1.23 enables us to obtain an extension v_X of v to an open neighborhood N_X of X. Let $N'_X \subset N_X$

be a locally compact neighborhood of X. Using a partition of unity for N_X and N'_X we can assume that v_X connects smoothly to the 0 vector field on Y/N_X. Therefore v_X takes values in $\mathcal{C}$ over N'_X, and outside N'_X it is either with values in $\mathcal{C}$ or 0 (by definition of cones, they do not contain the 0 vector). Consider $\mathcal{X}$ the set of strata X that are maximal for the ordering $<$ among the strata X such that $X < Y$. Because they are maximal, for any two such X and X', one can reduce further N_X and $N_{X'}$ so that $N_X \cap N_{X'} = \emptyset$ (because $\overline{X} \cap X' = \emptyset$ by maximality so we can subtract $\overline{X}$ from $N_{X'}$, and symmetrically for N_X). We thus choose the N_X's for $X \in \mathcal{X}$ so that they are disjoint pairwise. For any $y \in Y$ there is at most one $X \in \mathcal{X}$ such that $y \in N_X$. If there is none, then by theorem 4.1.20 one can construct a vector field with values in $\mathcal{C}$ in a neighborhood of y. This vector field is automatically controlled because there is no lower control tube of a lower dimensional stratum in the neighborhood of y.

If there is one such X with $y \in N'_X$, by theorem 4.1.23 we have already created v_X which a continuous controlled vector field in a neighborhood of y with values in $\mathcal{C}$ (v_X does not vanish on N'_X). The theorem insures that the vector field is continuous and controlled for the pair of strata Y, X, but as a consequence of the axioms of controlled tube system it is controlled for any Y, X' (with $Y > X > X'$). Indeed, if y happens to be in a $N'_{X'}$ (therefore $X > X'$), the control conditions are automatically satisfied for the pair Y, X' because by the induction hypothesis v is already controlled for the pair of strata X, X':

$$d\rho_{X'}(v(y)) = d\rho_{X'} d\pi_X(v(y)) = d\rho_{X'}(v(\pi_X(y))) = 0,$$

$$d\pi_{X'}(v(y)) = d\pi_{X'} d\pi_X(v(y)) = d\pi_{X'} v(\pi_X(y)) = v(\pi_{X'}\pi_X(y)) = v(\pi_{X'}(y)).$$

Finally we have proved that it is possible to construct a continuous controlled vector field with values in $\mathcal{C}$ in a neighborhood of any $y \in Y$, and by means of a partition of unity, we can glue all these local vector fields into a global one. The resulting global vector field still has values in $\mathcal{C} \cup \{0\}$ because cones are stable by addition: the positive linear combinations due to partitions of unity leave $\mathcal{C}$ stable. The resulting vector field is also non-vanishing since at any y the summands are always in $\mathcal{C}(y)$ or null (when they come from the v_X for instance), and there is always at least one summand that is nonzero. □

We now explained how to recover topological triviality from a continuous controlled vector field. Notice again that unlike in J. Mather's lecture notes, the vector fields here are not compatible with a smooth map (in the sense that there is *a priori* no $f : M \to N$ such that, given w a vector field on N, it is possible to lift w into a controlled vector field v on M so that $df(v(x)) = w(f(x))$). Here the constraint is put on the map f which is required to be compatible with the cone bundle.

The key point here, and the reason for all this construction is that the map can be taken continuous (with some restrictions of course) instead of smooth. The result will be applied afterward in a semi-algebraic context.

Definition 4.1.26 (Compatibility of a map with a cone bundle)**.** *Let S be a set stratified by an S-decomposition Σ. Let $f : S \to \mathbb{R}$ be a continuous function. For any stratum $\sigma \in \Sigma$, any point $x \in \sigma$ and any $\alpha \in \mathbb{R}$, a vector $v \in T_x\sigma$ is α-compatible with f^Σ iff for any function $\delta \in \mathcal{C}(\mathbb{R}, \sigma)$, such that $\delta(0) = x$ and $\delta'(0) = v$, $f \circ \delta$ meets the following local transversality condition:*

$$\exists \epsilon > 0,\ \forall t \in\ ]-\epsilon, \epsilon[\backslash\{0\},\ \frac{f \circ \delta(t) - f(x)}{t} > \alpha\ \|v\|_2\ . \quad (*)$$

For a cone bundle $\mathcal{C}$ on S, $f|_Z$ is said to be α-compatible with $\mathcal{C}$ iff for any $x \in S$, all the vectors $v \in \mathcal{C}(x)$ are α-compatible with $f|_Z$.
We will simply say that $\mathcal{C}$ is compatible with f^Σ when $\alpha = 0$ in $(*)$, that is:

$$\exists \epsilon > 0,\ \forall t \in\ ]-\epsilon, \epsilon[\backslash\{0\},\ f \circ \delta(t) - f(x) > 0.$$

Remark 4.1.27. *Notice that for given $x \in \sigma$, and $v \in T_x\sigma$, since σ is a manifold, there always exists $\delta : \mathbb{R} \to \sigma$ such that $\delta(0) = x$, $\delta'(0) = v$. Therefore, if a function f is α-compatible with $\mathcal{C}$, there automatically exist paths δ for which f meets condition $(*)$ in the previous definition.*

Remark 4.1.28. *By analogy with the original moving the wall theorem 4.1.12, vector fields with value in a cone bundle compatible with a map f can be said to be transverse to f. This is because the vector fields that the proof of the Thom-Mather theorem creates in the context of the moving the wall theorem are exactly the vector fields with value in a cone bundle compatible with the submersion that is used in the Thom-Mather theorem.*

The reader may have noticed that in the previous definition 4.1.26 we used the letter S instead of Z to refer to the stratified set. From its definition, one sees that the notion of α-compatible vector is tied to the stratification of the underlying set we have chosen. However there is a conceptual benefit in separating the notion of α-compatible and the stratification of the set we want to analyze. The way we achieve this is by using the stratification of the ambient space S to characterize α-compatible vectors on the one hand, and imposing that the stratified set Z we want to analyze refines the stratification of S. In the following, S will represent the ambient space $\mathbb{R}^n$, stratified by $\{\mathbb{R}^n\}$, and Z be a set, Whitney stratified by a stratification Σ, to which we will apply our new version of the Thom-Mather theorem, and it will induce the stratification of $\mathbb{R}^n$ given by $\Sigma \cup (\mathbb{R}^n \setminus Z)$.
However, for this separation to be valid, we have to make sure that α-compatible vectors for the stratification of S will be α-compatible for the stratification associated to Z. This is what the following proposition guarantees:

Proposition 4.1.29 (α-compatibility is preserved by restriction)**.** *Let S be a set stratified by two stratifications Σ and Σ'. Assume that Σ refines Σ' (i.e. every stratum of Σ' is a union of strata of Σ). Let $f : S \to \mathbb{R}$ be a continuous function. For any point $x \in S$, and any vector $v \in T_x^\Sigma S$, we can naturally consider v as a vector of $T_x^{\Sigma'} S'$ since Σ refines Σ'. Based on these identification then we have for any $\alpha \in \mathbb{R}$,*

- *if v is α-compatible with $f^{\Sigma'}$ <u>then</u> v is α-compatible with f^{Σ}.*

- *if the cone bundle $\mathcal{C}$ is α-compatible with $f^{\Sigma'}$ <u>then</u> $\mathcal{C}$ is α-compatible with f^{Σ}.*

Proof. This is straighforward. Since Σ refines Σ', for any $x \in S$, and $v \in T_x^{\Sigma} S$, there exists $\sigma \in \Sigma$, and $\sigma' \in \Sigma'$ such that $x \in \sigma \subset \sigma'$. Therefore, for any path $\delta \in \mathcal{C}(\mathbb{R}, \sigma)$, we have $\delta \in \mathcal{C}(\mathbb{R}, \sigma')$.
If v is α-compatible with $f^{\Sigma'}$, inequality $(*)$ in definition 4.1.26 is satisfied for any path $\delta \in \mathcal{C}(\mathbb{R}, \sigma')$ and thus *a fortiori* for any path $\delta \in \mathcal{C}(\mathbb{R}, \sigma)$. Consequently, v is also α-compatible with f^{Σ}. This proves the first item of the proposition.
The second item follows immediately since all the vectors of $\mathcal{C}$ are α-compatible with f^{Σ} by the first point. □

Thanks to the previous proposition 4.1.29, for any set S stratified by Σ, and any continous function $f : S \to \mathbb{R}$ we will now use the expression α-compatible with f to mean α-compatible with f^{Σ}, and consider sub-stratified sets Z of S without worrying that we lose the α-compatibility by restriction.

We now have the necessary definition to state the fundamental version of our new version of the moving the wall theorem that applies to non-smooth maps and provides Lipschitz control of the homeomorphism:

Theorem 4.1.30 (Quantitative Moving the Wall)**.** *Let $f : Z \to \mathbb{R}$ be a continous map, Z be a Whitney stratified set, and let $\alpha > 0$. Let $\mathcal{C}$ be an α-compatible cone bundle for f. If we have:*

- *$\mathcal{C}$ is upper semi-continuous.*

- *$\mathcal{C}$ is non-vanishing.*

Then there exists a homeomorphism $h : f^{-1}(0) \times \mathbb{R} \to Z$ such that $\pi_2 = f \circ h$, where π_2 is the projection to the second parameter.
In addition, we have for any $x, y \in f^{-1}(0)$ and $t, t' \in \mathbb{R}$:

$$\| h(x,t) - h(y,t') \|_2 \leq \frac{2 \max(t,t')}{\alpha} + \| x - y \|_2,$$

where $\| v \|_2$ denotes the Euclidean norm of v.

The next proof follows the same lines as the one in J. Mather's lecture notes ([95]). The two main differences are that the vector field is continuous therefore the induced one parameter group is automatically continuous, and the other difference is that the vector field has value in a cone bundle instead of being compatible with a map. The former point only simplifies the proof, but the latter makes a proof necessary, although it turns out that the same arguments as in J. Mather's notes remain valid.

Proof. By theorem 4.1.25 we have a continuous controlled vector field v on Z with values in $\mathcal{C}$ (as $\mathcal{C}(x) \neq \emptyset$ at every x). Therefore v gives rise to a one parameter group $\delta : Z \times \mathbb{R} \to Z$ such that $\delta(\sigma \times \mathbb{R}) \subset \sigma$ for any stratum σ of the stratification of Z. The proof that such a one parameter group is well-defined is done by J. Mather in his lecture notes (proposition 10.1 in [95]). Notice that by renormalization of v we can assume that $\| v(x) \|_2 = 1$ at any point x, this is important to obtain a Lipschitz homeomorphism.

We now show that for any $t > 0$ (the case $t < 0$ is symmetrical), and any $x \in f^{-1}(0)$, there exist $t_x \in \mathbb{R}$ and $y_x \in f^{-1}(t)$ such that $\delta(y_x, -t_x) = x$ (or equivalently $\delta(x, t_x) = y_x$). Let $g : s \in \mathbb{R}^+ \to f(\delta(x, s))$. Since $\mathcal{C}(x)$ is α-compatible with f, and $\| v \|_2 = 1$, g meets the following condition:

$$\forall s > 0,\ \exists \epsilon > 0,\ \forall x, y \in]-\epsilon, \epsilon[, \| g(x) - g(y) \|_2 \geq \alpha |y - x|. \quad (*)$$

It is then easy to prove that for all $s > 0$, $g(s) > \alpha s$. Consider the maximal interval $[0, s_0[$ on which $g(s) > \alpha s$, it is non empty by applying $(*)$ at $s = 0$, and s_0 cannot be anything else than $+\infty$ because otherwise we could apply $(*)$ at $s = s_0$ and obtain a contradiction.

What precedes shows that for any t, and any $x \in f^{-1}(0)$, there exist $t_x \in \mathbb{R}$ and $y_x \in f^{-1}(t)$ such that $\delta(y_x, -t_x) = x$ (or equivalently $\delta(x, t_x) = y_x$). The continuity of δ insures that t_x is continuous with y, and so is y_x (y_x can be seen as a function of y). The unicity of t_x and y_x stems from the fact that $f \circ \delta$ is increasing. Hence we can define the map $\phi_t : x \in f^{-1}(0) \mapsto y_x = \delta(x, t_x) \in f^{-1}(t)$ which is a continuous injection of $f^{-1}(0)$ and $f^{-1}(t)$. Because $\delta(\delta(x, t), -t) = x$, it is also surjective, hence a bijection. Finally, $h : (x, t) \in f^{-1}(0) \times \mathbb{R} \to \phi_t(x)$ is the trivializing homeomorphism announced in the theorem.

Now we prove the Lipschitz-like inequality on the homeomorphism h. As the vector field δ comes from is normalized (i.e. $\| v(x) \|_2 = 1$ at any $x \in Z$), the fact that the vectors of the cone bundle are α-compatible implies that

$$\alpha \| \delta'(x, s) \|_2 = \alpha < \frac{f(\delta(x, s'')) - f(\delta(x, s'))}{s'' - s'}$$

for $s'' > s'$ sufficiently close to s, where δ' is the differential of δ with respect to its second parameter. This shows that $d^+(s) \geq \alpha$ (similarly for $d^-(s)$) and thus $\exists t_x \in [0, t/\alpha]$ such that $f(\delta(x, t_y)) = t$. In other words $\| \phi_t(x) - \phi_0(x) \|_2 \leq t/\alpha$. By applying this to two points x, y we can conclude:

$$\begin{aligned} \| h(x,t) - h(y,t') \|_2 &\leq \| h(x,t) - h(x,0) \|_2 + \| h(x,0) - h(y,0) \|_2 + \| h(y,0) - h(y,t') \|_2 \\ &\leq \| \phi_t(x) - \phi_t(0) \|_2 + \| x - y \|_2 + \| \phi_{t'}(y) - \phi_0(y) \|_2 \\ &\leq \frac{2 \max(t, t')}{\alpha} + \| x - y \|_2 . \end{aligned}$$

□

Let us contrast the previous theorem 4.1.30 with its classical form.

First let us consider the condition that the cone bundle must be non-vanishing. On the one hand, it is easy to see that if a smooth map is not a submersion at some point x, for any $\alpha > 0$, there cannot be any α-compatible vector with f at x. On the other hand it is fairly easy to prove that if f is globally a submersion, then there exists a non-vanishing α-compatible cone bundle with f (proposition 4.1.33). Therefore the notion of non-vanishing cone bundle contains and generalizes the notion of submersion in the context of the Thom-Mather theorem.
One may also notice that there is no need for f to be proper in our quantitative version. This is because the positive constant $\alpha > 0$ controls the vector field in a Lipschitz way, and thus prevents the integral lines from diverging in finite time.

Finally, one sees a new condition that did not appear in the original moving the wall theorem: the upper semi-continuity of the cone bundle. In fact, this condition is not really about the relation between f and Z as is the other condition on the cone bundle being non-vanishing. This condition is actually about f itself and the way its level lines degenerate. This is why this condition is trivial in the original setting of the theorem: the level lines of a submersion are all smooth.
This condition is difficult to manipulate because the geometry of the cone bundle that are α-compatible with general continuous maps can be with very intricate. The next subsection will address this issue by defining more tools to construct upper semi-continuous α-compatible cone bundles. We will then come back to the case of smooth maps in remark 4.1.45 to tie the current discussion to the new tools introduced in the next subsection.

4.1.3 A more practical form of the extended theorem

The previous theorem 4.1.30 has a very wide scope. However it fails to be easily applicable on any specific example because, for a given continous map f, we lack any practical means to construct an α-compatible upper semi-continous cone bundle and to check that it is non-vanishing. To address those two points we restrict the class of maps we consider to what we call wall maps (definition 4.1.34). From their definition one will realize that the restriction is really moderate. For this class of maps, we show how to construct a canonical α-compatible upper semi-continuous cone bundle that we call induced cone bundle (definition 4.1.36) ; this is the first objective of this subsection. The second objective is to make it easy to prove that those cone bundles are non-vanishing. This aspect is only treated partially in this subsection: as we will discuss (remark 4.1.32) we want the notion of transversality to only depend on the geometry of the level lines of the function, and this implies that the α in α-compatibility should have no role in it. Abstracting this dependency on α is what this subsection deals with, and it is achieved by showing that, for wall maps, the existence of a 0-compatible cone bundle is a sufficient condition to apply the quantitative version of moving the wall theorem (proposition 4.1.43). Dealing with the relation between the geometry of the level lines of f and the stratified set Z (or, in other words, 0-compatible cone bundles) is the goal of the next section 4.2.

More precisely, the construction of an α-compatible upper semi-continous cone bundle proceeds in two steps. The first consists in characterizing the maximal bundle of α-compatible vectors through the notion of growth of a map (definition 4.1.31). This bundle is upper semi-continuous on an open dense set of Z, but fails to be so globally, and, in addition, it is not a cone bundle (the fibers are not convex). The second step consists in defining the notion of (f, α)-induced cone bundle (definition 4.1.36). This object is an upper semi-continuous α-compatible cone bundle (propositions 4.1.37 and 4.1.42), achieving our first objective. Finally, we prove that if f is proper and the $(0, f)$-induced cone bundle is non-vanishing, there exists an $\alpha > 0$ such that the (f, α)-induced cone bundle is also non-vanishing (proposition 4.1.43). This allows us to end the section with a much more practical version (theorem 4.1.44) of theorem 4.1.30 that only requires an analysis of transversality between Z and the level lines of f (which is the subject of the next section 4.2).

We now transform α into a function of $v \in TZ$ by considering the supremum of all the α such that v is α-compatible with f. This quantity we call the growth of f along v:

Definition 4.1.31 (Growth of a map)**.** *Let $f : Z \to \mathbb{R}$ be a continuous function on a set Z stratified by Σ. For any $x \in Z$ and $v \in T_x Z \setminus \{0\}$, the growth of f along v (at x) is defined as*

$$\mathfrak{g}_f(v) = \sup \left\{ \alpha \in \mathbb{R} \cup \{\pm\infty\} \mid v \text{ is } \alpha\text{-compatible with } f \right\}.$$

This quantity gives a lower bound on how fast f grows locally at x along any differentiable path tangent to v at x that runs in the same stratum as x. It allows us to naturally define the maximal bundle of α-compatible vectors as $\mathfrak{g}_f^{-1}(]\alpha, +\infty))$.

Remark 4.1.32. *The previous notion could seem a good way to measure and generalize the notion of transversality of Z to the level lines of f since, when f is a submersion at x, $\mathfrak{g}_f(v) > 0$ if and only if the level line $f = f(x)$ is transverse to Z at x.*
This is not the case because this quantity depends on more than the geometry of the level lines of f only: if we compose f with an increasing function $g : \mathbb{R} \to \mathbb{R}$, $g \circ f$ will have the same level lines as f but its growth along v can be completely different. The next section 4.2 about the measure of transversality will address this concern and define measures that are only dependent on the shape of the level lines of f. But for now, it should be clear that the notion of growth we have just defined is not intrinsic to the geometry of Z, and includes some foreign independent component introduced by f.

It is not true that the growth of a map is an upper semi-continuous function on TZ, but if we look at an open set of TZ whose base points are in an open set over which f is smooth, then $\mathfrak{g}_f$ is smooth. In fact, the growth has a simple explicit form over those sets:

Proposition 4.1.33 (Growth when f is smooth)**.** *Let Z be a stratified set and U an open set of Z. Assume $f \in \mathcal{C}^\infty(U, \mathbb{R})$. Then for any $x \in U$ and $v \in T_x Z \setminus \{0\}$ we have*

$$\mathfrak{g}_f(v) = \frac{df_x(v)}{\| v \|_2}.$$

Note that this proves that the bundle of the $\mathfrak{g}_f^{-1}(]\alpha, +\infty))$ is a continuous cone bundle since $\mathfrak{g}_f - \alpha$ is a continous concave function.

Proof. This is immediate from the definition of differentiation. Let $\alpha = \mathfrak{g}_f(x)$. We have on the one hand, for any path γ such that $\gamma(0) = x$ and $\gamma'(0) = v$,

$$\begin{aligned} d(f \circ \gamma)_0(1) &= df_x(\gamma'(0)) \\ &= df_x(v), \end{aligned}$$

and

$$\begin{aligned} d(f \circ \gamma)_0(1) &= \lim_{h \to 0} \frac{f(\gamma(h)) - f(x)}{h} \\ &\geq \alpha \| v \|_2 \qquad \text{by definition of } \alpha - \text{compatibility.} \end{aligned}$$

On the other hand, for any $\epsilon > 0$, there exists a path γ such that

$$\begin{aligned} d(f \circ \gamma)_0(1) &= \lim_{h \to 0} \frac{f(\gamma(h)) - f(x)}{h} \\ &< (\alpha + \epsilon) \| v \|_2 \qquad \forall \epsilon > 0 \quad \text{because the growth is a supremum} \\ &\leq \alpha \| v \|_2 \end{aligned}$$

Q.E.D. □

We now give the definition of what we call wall map.

Definition 4.1.34 (Wall map)**.** *Let Z be a set stratified by Σ, and $f : Z \to \mathbb{R}$. We say that f is a wall map iff*

- *There exists a dense open set U in Z such that $f|_U \in \mathcal{C}^\infty(U, \mathbb{R})$.*
- *f is non degenerate, that is* $\inf_{x \in U} \| df_x \|_2 > 0$.

Remark 4.1.35. *Wall maps are called in this way because their level lines play the role of the moving wall in the moving the wall theorem, and not because the conditions they meet enforce that the level lines are geometrically "like a wall" (if $Z = \mathbb{R}^n$ for instance, this would be homeomorphic to a hyperplane). In fact, this latter condition will be reflected in the vanishing of the induced cone bundle (see next definition 4.1.36).*

We now define a notion of induced cone bundle which characterizes the maximum cone bundle that can be compatible with a given map.

Definition 4.1.36 (Induced cone bundle)**.** *Let $f : Z \to \mathbb{R}$ be a wall map on Z, a set Whitney stratified by Σ. Let U be the dense open set such that $f \in \mathcal{C}^\infty(U, \mathbb{R})$ as by definition 4.1.34. Let $\sigma \in \Sigma$ the stratum of Z containing x, and define $\mathcal{C}$, the (f, α)-induced cone bundle on σ at x by*

$$\mathcal{C}(x) = \mathfrak{g}_f^{-1}\big(]\alpha, +\infty)\big).$$

Since by definition of wall map all the points in Z lie in the adherence of U, where we have already defined $\mathcal{C}$, we can extend it by the following formula:

$$\mathcal{C}(x) = \bigcap_{x_i \in U \to x} \lim_{i \to \infty} \mathcal{C}(x_i),$$

where $x_i \in U \to x$ denotes all the sequences converging to x with $x_i \in U$, and where lim *represent the usual set theoretical limit of open fibers $V_i \subset T_{x_i}Z$, that is:*

$$\lim V_i = \left((\overline{\cup_i V_i}) \bigcap T_x Z \right)^\circ$$

When $\alpha = 0$ we will call f-induced cone bundle the $(f, 0)$-induced cone bundle.

The first important property of the (f, α)-induced cone bundle is that it is upper semi-continuous.

Proposition 4.1.37. *For any Z stratified set, any $f : Z \to \mathbb{R}$ wall map on Z, and any $\alpha \in \mathbb{R}$, the (f, α)-induced cone bundle is upper semi-continuous.*

Proof. This is actually immediate since proposition 4.1.33 proves the (f, α)-induced cone bundle is an upper semi-continous cone bundle on U, the open set where f is smooth, and the definition enforces that for all $x \notin U$, $\mathcal{C}$ lies in the intersection of all the limits of nearby fibers (which is the very definition of upper semi-continuity). □

Another conceptually important point about the induced cone bundle is that, although its definition relies on an arbitrary dense open set U associated to f, the resulting induced cone bundle does <u>not</u> depend on the choice of U. As a side note, notice this fact is irrelevant when it comes to using induced cone bundles in applications since even without the next proposition we could still fix some open set U to work with beforehand. In any case, the following proposition shows that the induced cone bundle does only depend on f and Z:

Proposition 4.1.38. *Let Z be a Whitney stratified set. Let $f : Z \to \mathbb{R}$ be a wall map for two different dense open sets U and U'. Then for any $\alpha \in \mathbb{R}$, the two (f, α)-induced cone bundles based on U and U' are the same.*

Proof. Let $\mathcal{C}$ the (f, α)-induced cone bundle associated to U and $\mathcal{C}'$ the one associated to U', and let x be any point in U. We prove that $\mathcal{C}'(x) = \mathcal{C}(x)$. This is sufficient to prove the proposition since this will mean that the induced

cone bundle based on U' and the one based on $U \cup U'$ coincide. By symmetry of role between U and U' this will also mean the induced cone bundles based on U and $U \cup U'$ coincide, hence proving the proposition.
On the one hand, since f is smooth over U, we can use the explicit form of the induced cone bundle in U (proposition 4.1.33). This form makes it clear that for any given sequence x_i converging to x, since U is a neighborhood of x, we have

$$\mathcal{C}(x) = \lim_{i \to \infty} \mathcal{C}(x_i). \qquad (*)$$

Since U' and U are dense in Z, so is their intersection V. We can thus choose a sequence $x_i \in V$ converging to x. Since f is smooth on V, by definition of the induced cone bundle, $\mathcal{C}$ and $\mathcal{C}'$ coincide on V. If $x \in U'$, then $x \in V$ and we are done: $\mathcal{C}'(x) = \mathcal{C}(x)$. If $x \notin U'$, $\mathcal{C}'(x)$ is defined as the intersection of all the limits of fibers with base point in U', therefore we have

$$\begin{aligned}
\mathcal{C}'(x) &= \bigcap_{y_i \in U' \to x} \lim_{i \to \infty} \mathcal{C}'(y_i) && \text{by definition of } \mathcal{C}'(x), \\
\mathcal{C}'(x) &\subset \lim_{i \to \infty} \mathcal{C}'(x_i) && \text{since } x_i \text{ is one of the sequences } y_i \in U' \to x, \\
\mathcal{C}'(x) &\subset \lim_{i \to \infty} \mathcal{C}(x_i) && \text{because } \mathcal{C} \text{ and } \mathcal{C}' \text{ coincide on } V, \\
\mathcal{C}'(x) &\subset \mathcal{C}(x) && \text{by equality } (*).
\end{aligned}$$

On the other hand, since equality $(*)$ holds for any sequence x_i converging to x, and that any sequence converging to x in $\overline{U'}$ ultimately lies in U (U is a neighborhood of x), we have the reverse inclusion:

$$\begin{aligned}
\mathcal{C}(x) &= \bigcap_{x_i \in U \to x} \lim_{i \to \infty} \mathcal{C}(x_i) && \text{by } (*), \\
&\subset \bigcap_{y_i \in U' \to x} \lim_{i \to \infty} \mathcal{C}(y_i) && \text{by intersecting over less sequences}, \\
&\subset \mathcal{C}'(x) && \text{by definition of } \mathcal{C}'(x).
\end{aligned}$$

□

Another desirable property the (f, α)-induced cone bundle has is that it is decreasing with α:

Proposition 4.1.39 (Monotonicity and continuity in α of induced cone bundles)**.** *Let Z be a Whitney stratified set. Let $f : Z \to \mathbb{R}$ be a wall map on Z. Let $\mathcal{C}_\alpha$ be the (f, α)-induced cone bundle on Z. Then for any $x \in Z$, and any two $\alpha < \alpha' \in \mathbb{R}$, $\mathcal{C}_\alpha(x) \supset \mathcal{C}_{\alpha'}(x)$.*
In addition for any $x \in Z$, any $\alpha \in \mathbb{R}$ and any decreasing sequence α_i converging to α, we have the following continuity equality

$$\mathcal{C}_\alpha = \cup_{i \in \mathbb{N}} \mathcal{C}_{\alpha_i}$$

Proof. Thanks to the explicit form of $\mathcal{C}$ (proposition 4.1.33), all the above properties are clear on U, the open set where f is smooth. At any other point $x \notin U$, let us resort to the definition of $\mathcal{C}(x)$:

$$\lim \mathcal{C}(x_i) = \bigcap_{x_i \in U \to x} \left(\left(\overline{\bigcup_i \mathcal{C}(x_i)} \right) \bigcap T_x Z \right)^\circ$$

From this definition the fact that $\mathcal{C}_\alpha$ is decreasing with α is clear, since for any two families of sets S_i, S_i', $(\forall i,\ S_i \subset S_i') \to \cup_i S_i \subset \cup_i S_i'$.
Let us now prove that the non-degeneracy of f entails the continuity equality. For any point $x \in Z$, and vector v in $\mathcal{C}_\alpha(x)$, there exists a compact neighborhood K of v in $\mathcal{C}_\alpha$ (K is thus a neighborhood of v in the tangent bundle, not just in the tangent space $T_x Z$). As a wall map, f is non-degenerate. Therefore, by considering the explicit form of the induced cone bundle over smooth points (proposition 4.1.33), $K \subset \mathcal{C}_\alpha$ implies that for α_i close enough to α, we still have $K \subset \mathcal{C}_{\alpha_i}$. This shows that v is every limit of cone bundles for any $\alpha' \leq \alpha_i$ since cone bundles are decreasing with α. Therefore v is in $\mathcal{C}_{\alpha_i}$ and thus $v \in \cup_{i \in \mathbb{N}} \mathcal{C}_{\alpha_i}$.
Q.E.D. □

Let us now consider the α-compatibility of the (f, α)-induced cone bundle. From the definition it is clear that the vectors of the (f, α)-induced cone bundle are α-compatible when their base points are in the dense open set where f is smooth, since the fibers of $\mathcal{C}$ are defined as $\mathfrak{g}_f^{-1}\big(]\alpha, +\infty)\big)$ over those base points. We now have to prove that the vectors of the cone bundle are also α-compatible over the remaining base points.
That the whole induced cone bundle is α-compatible with f is the most delicate property to prove, essentially because the class of wall maps is large. Imposing stronger conditions such as analyticity would simplify the proof, however for the sake of generality we prove it for the wider class of wall maps. We begin with two elementary lemmas, then proceed to state and prove the property itself.

Lemma 4.1.40. *Let $f : \mathbb{R} \to \mathbb{R}$, assume there exists an open set U dense in some $]-\eta, \eta[$, such that $\forall x \in U,\ \mathfrak{g}_f(x) \geq \alpha$. Then for all $x, y \in U$ distinct, we have*

$$\frac{f(y) - f(x)}{y - x} \geq \alpha.$$

Proof. This is a fairly intuitive result. First we prove that the result holds for any connected component V of U. For any $z \in V$, $\mathfrak{g}_f(z) \geq \alpha$ implies that for any $\alpha' < \alpha$, there exists a neighborhood V_z of z such that

$$\forall y \in V_z,\ \frac{f(y) - f(z)}{y - z} > \alpha'. \qquad (*)$$

It is sufficient to prove the property for $y > x$ since x and y can be swapped in the property that we want to prove, that is

$$\frac{f(y) - f(x)}{y - x} = \frac{f(x) - f(y)}{x - y}.$$

As the segment $[x, y]$ is compact, there is finitely many V_z that cover it. We call them $z_1 < \ldots < z_k$, with $x \in V_{z_1}$, $y \in V_{z_k}$, and $z'_1 < \ldots z'_{k-1}$ such that $z'_i \in V_{z_i} \cap V_{z_{i+1}}$. We also set $z'_0 = x$ and $z'_k = y$. This yields

$$\begin{aligned} f(y) - f(x) &= \sum_{i=0}^{k-1} f(z'_{i+1}) - f(z'_i) \\ &> \sum_{i=0}^{k-1} \alpha'(z'_{i+1} - z'_i) \qquad \text{by } (*) \text{ for every pair of points } z'_{i+1},\ z'_i \\ &> \alpha'(z'_k - z'_0) = \alpha'(y - x). \end{aligned}$$

Since this is true for any $\alpha' < \alpha$, we obtain $f(y) - f(x) \geq \alpha(y - x)$ as required.

Finally, since U is dense in $]-\eta, \eta[$, for any point $x \notin U$, there is a connected component V of U adherent to x. By symmetry we can suppose V is "after" x (i.e. $\forall v \in V,\ v > x$). For any $v \in V$, by what we just proved we have for any $y \in V$ such that $y < v$, $f(v) - f(y) \geq \alpha(v - y)$. We let y tend to x and by continuity of f we obtain $f(v) - f(x) \geq \alpha(v - x)$. Q.E.D. □

Lemma 4.1.41. *Let S be a relatively compact subset of $\mathbb{R}^n$, and let V be an open set dense in S. Let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the orthogonal projection to the first $n-1$ components. We define*

$$D = \{x \in \mathbb{R}^{n-1} \mid (\pi^{-1}(x) \cap V)^\circ \text{ is dense in } (\pi^{-1}(x) \cap S) \text{ or is empty}\}.$$

Then D contains a dense set of $\mathbb{R}^{n-1}$.

Proof. This is not easy to prove unless we resort to integration theory. For any subset X of $\mathbb{R}^n$, let $\mathbf{1}_X : \mathbb{R}^n \to \mathbb{R}$ be the function such that $\mathbf{1}_X(x) = 1$ if $x \in X$ and $\mathbf{1}_X(x) = 0$ if $x \notin X$.

Since V is dense in S, $\int_{\mathbb{R}^n} \mathbf{1}_S = \int_{\mathbb{R}^n} \mathbf{1}_V$, and this quantity is finite since S is relatively compact. Since $\mathbf{1}_V$ is a non negative function we can apply Fubini's theorem and we conclude that

$$\int_{x \in \mathbb{R}^{n-1}} \int_{t \in \mathbb{R}} \mathbf{1}_V(x, t) = \int_{x \in \mathbb{R}^{n-1}} \int_{t \in \mathbb{R}} \mathbf{1}_S(x, t).$$

This equality implies that

$$\int_{x \in \mathbb{R}^{n-1}} \left(\left(\int_{t \in \mathbb{R}} \mathbf{1}_S(x, t) \right) - \left(\int_{t \in \mathbb{R}} \mathbf{1}_V(x, t) \right) \right) = 0.$$

Since $S \supset V$, $\mathbf{1}_S \geq \mathbf{1}_V$, and thus this means that $\int_{t \in \mathbb{R}} \mathbf{1}_V(x, t) = \int_{t \in \mathbb{R}} \mathbf{1}_S(x, t)$ almost everywhere (as functions of x), that is in particular on a dense subset D' of $\mathbb{R}^{n-1}$. In addition, when $\int_{t \in \mathbb{R}} \mathbf{1}_V(x, t) = \int_{t \in \mathbb{R}} \mathbf{1}_S(x, t)$ is satisfied for a fixed x, the equality means that $\mathbf{1}_V(x, t) = \mathbf{1}_S(x, t)$ almost everywhere (as functions of t), in particular this means that $\pi^{-1}(x) \cap V$ contains a dense set of $\pi^{-1}(x) \cap S$. Since V is open, we have the stronger property that $\pi^{-1}(x) \cap V$ contains a dense open set of $\pi^{-1}(x) \cap S$.

We have thus found a set D' dense in $\mathbb{R}^{n-1}$ which is included in D, hence proving D itself is dense. □

Proposition 4.1.42 (Induced cone bundles are α-compatible)**.** *Let $f : Z \to \mathbb{R}$ be a wall map on Z, a set Whitney stratified by Σ. Let $\mathcal{C}$ be the (f, α)-induced cone bundle on Z. Then any vector $v \in \mathcal{C}$ is α-compatible with f.*

Proof. As we just mentioned, the only thing to check is when v lies in some $T_x Z$ where $\mathcal{C}$ has been defined as

$$\mathcal{C}(x) = \bigcap_{x_i \in U \to x} \lim_{i \to \infty} \mathcal{C}(x_i),$$

where U is a dense open set such that $f \in \mathcal{C}^\infty(U, \mathbb{R})$. We proceed by induction on the degeneration order $<$. The proof thus proceeds in two parts, first we prove the result holds for the maximal strata, then we use the induction hypothesis to prove the result on the rest of the strata. Let σ be the stratum of Z to which x belongs.

If σ is maximal for $>$, since U is dense in Z, U has to be dense in σ. Since we want to prove a result locally at x, we can identify a sufficiently small neighborhood N of x in σ to $\mathbb{R}^k$ for $k = \dim \sigma$. Let γ be any path in σ such that $\gamma'(0) = v$ and $\gamma(0) = x$. For any $w \in T_x\sigma$ orthogonal to v, since we have assimilated N to $\mathbb{R}^k$, we can define $\Gamma(w, t)$ as $\gamma(t) + w$. After shrinking N if necessary, since $\gamma'(0) \neq 0$ and w is orthogonal to $\gamma'(0)$, the map Γ is a local diffeomorphism of N onto its image. Since U is dense in σ, $\Gamma^{-1}(U)$ is also dense in N. We can assume without loss of generality that N is relatively compact of the form $N' \times]-\eta, \eta[$, and we can thus apply lemma 4.1.41 to conclude that there is a dense open subset W of $\mathbb{R}^{k-1}$ such that $\forall w \in W$, $\Gamma(w, .)^{-1}(U)$ contains a dense open set of $\{w\} \times \mathbb{R}$. As $\mathcal{C}$ is upper semi-continuous, and $\partial_t \Gamma(x, t)$ is continous, we can shrink N further so that $\partial_t \Gamma(w, t) \in \mathcal{C}$ for any $(w, t) \in N$. For those w we can use lemma 4.1.40 to conclude that for any $t, t' \in]-\eta, \eta[$ distinct, $(N = N' \times]-\eta, \eta[)$, we have

$$\frac{f \circ \Gamma(w, t) - f \circ \Gamma(w, t')}{t - t'} \geq \alpha \inf_{t \in]-\eta, \eta[} \| \Gamma'(w, t) \|_2 .$$

Since $0 \in \mathbb{R}^{k-1}$ is adherent to W, we can let w converge to 0 while t is left fixed and t' is set to 0, and we obtain

$$\begin{aligned} \frac{f \circ \Gamma(0, t) - f \circ \Gamma(0, 0)}{t} &= \frac{f \circ \gamma(t) - f \circ \gamma(0)}{t} \\ &\geq \alpha \inf_{t \in]-\eta, \eta[} \| \gamma'(t) \|_2 . \end{aligned}$$

Therefore for any $\alpha' < \alpha$, since $\gamma'(t)$ is continuous, we can choose η small enough such that

$$\frac{f \circ \gamma(t) - f \circ \gamma(0)}{t} \geq \alpha' \inf_{t \in]-\eta, \eta[} \| \Gamma'(t) \|_2 .$$

This shows that v is α'-compatible, and thus the growth of f along v is greater or equal to α. This concludes the case when σ is maximal for $>$.

If σ is not maximal for $>$, the induction hypothesis is that for all $\sigma' > \sigma$, and any $y \in \sigma'$, $\mathcal{C}(y)$ is α-compatible with f. Let $\delta : \mathbb{R} \to Z$ be any path on σ such that $\delta(0) = x$ and $\delta'(0) = v$. Let σ' be any stratum such that $\sigma' > \sigma$. As the strata are part of a Whitney stratification we can create a wing at x along σ in σ'. A wing is, a smooth function $\delta : (s,t) \in \mathbb{R}^+ \times \mathbb{R} \mapsto \delta_s(t) \in \sigma \cup \sigma'$ such that $\delta_0 = \delta$ and for any $s > 0$, δ_s is a path in σ'. To create it we use the fact that the projection π to σ and the distance function ρ of the controlled tube system give rise to a submersion $x \in \sigma' \mapsto (\rho(x), \pi(x)) \in \mathbb{R}^+ \times \sigma$ (this is lemma 7.3 in [95]). Then we apply the Thom-Mather theorem to an arbitrary lifting of δ in $\sigma' \cap \rho^1(\epsilon)$ for some ϵ small enough (by lifting we mean that π is a diffeomorphism between the lifting and δ). Since the pair (σ, σ') meets Whitney's condition (b), the resulting function is $\mathcal{C}^1$ in s at 0, $\mathcal{C}^\infty$ in t everywhere and $\mathcal{C}^\infty$ in s and t where $s > 0$.
By upper semi-continuity of the cone bundle there exists a neighborhood V of v such that $V \subset \mathcal{C}$. Since Z is Whitney stratified, for any fixed t, $\delta'_s(t)$ $(t > 0)$ converges to $\delta'_0(t)$. Therefore there is a neighborhood N of $(0,0)$ in $[0,\infty) \times \mathbb{R}$ such that $\forall (s,t) \in N$, $\delta'_s(t) \in V$. We can suppose without loss of generality that N is of the form $[0,\eta[\times]-\eta,\eta[$. For any $s \in]0,\eta[$ and $t \in]-\eta,\eta[$, the induction hypothesis tells us that $\mathfrak{g}_f(\delta'_s(t)) \geq \alpha$, which also means that $\mathfrak{g}_{g\circ\delta_s}(1_t) \geq \alpha$ where 1_t denotes the unit positive vector in $T_t\mathbb{R}$. By the previous lemma 4.1.40 we can thus conclude that for all $s \in]0,\eta[$, and $t, t' \in]-\eta,\eta[$ distinct, we have

$$\frac{f(\delta_s(t)) - f(\delta_s(t'))}{t - t'} \geq \alpha \inf_{t \in]-\eta,\eta[} \| \delta'_s(t) \|_2 \ .$$

We set $t' = 0$, fix the value of t, and let s converge to 0. Since δ' is at least continuous (δ is $\mathcal{C}^1$), $\delta'_s(0)$ converges to v, therefore for any $\alpha' < \alpha$, we can find some η small enough such that for all $t \in]-\eta,\eta[\backslash\{0\}$,

$$\alpha' \| v \|_2 \leq \frac{f(\delta_s(t)) - f(\delta_s(0))}{t}.$$

Since f is continuous and so is δ, as s converges to 0 we finally obtain

$$\alpha' \| v \|_2 \leq \frac{f(\delta(t)) - f(\delta(0))}{t} \qquad \text{because } \delta_0 = \delta$$

$$\alpha \| v \|_2 \leq \frac{f(\delta(t)) - f(\delta(0))}{t} \qquad \text{since the previous inequality held for all } \alpha' < \alpha$$

Q.E.D. □

The notion of induced cone bundle is convenient because it allows us to construct an α-compatible cone bundle in an automatic manner. This cone bundle can then be used with the extended moving the wall theorem, after making sure it is non-vanishing.

A strong point of theorem 4.1.30 is that it gives an explicit quantitative control on the Lipschitz coefficient of the isotopy. However, even if it is necessary to study the (f,α)-induced cone bundle to control this Lipschitz coefficient, one is

often only interested in proving the existence of an isotopy. Even in this context, our extended version of the Thom-Mather theorem is still an improvement over its classical counterpart because it can be applied with a wider class of functions. But in this context where we are only aiming at proving the existence of an isotopy (cf. remark 4.1.32), having to prove that some (f, α)-induced cone bundle ($\alpha > 0$) is non-vanishing becomes a hassle. In addition, as we discussed earlier, it is not satisfactory that the proofs must rely on something more than the level sets of f. The following proposition addresses the two previous issues nicely:

Proposition 4.1.43. *Let Z be a set Whitney stratified by Σ. Let $f : Z \to \mathbb{R}$ be a* **proper** *wall map. Assume the f-induced cone bundle (definition 4.1.36) is non-vanishing. Then there exists $\alpha > 0$ such that $\mathcal{C}$ the (f, α)-induced cone bundle is also non-vanishing.*

Proof. Assume that there is a sequence of points $x_i \in Z$ converging to x and decreasing $\alpha_i > 0$ converging to 0 such that, $\mathcal{C}_{\alpha_i}(x_i) = \emptyset$. Let K a neighborhood of $f(x)$ in $\mathbb{R}$. Since f is proper $f^{-1}(K)$ is a compact neighborhood of x. Thus for i large enough, $x_i \in f^{-1}(K)$ and we can assume that the x_i converges to some $x \in Z$ by extraction of a subsequence. By monotonicity and continuity of the induced cone bundle in α (proposition 4.1.39), we conclude that for any $v \in \mathcal{C}_0(x)$ we have $v \in \mathcal{C}_{\alpha_i}(x)$ for i large enough. By proposition 4.1.37, induced cone bundles are upper semi-continuous, therefore we have

$$v \in \lim_{j \to \infty} \mathcal{C}_{\alpha_i}(x_j).$$

Since cone bundles are decreasing with α (proposition 4.1.39) we have $\mathcal{C}_{\alpha_i} \subset \mathcal{C}_{\alpha_j}$ for $j > i$, hence (i is fixed)

$$\begin{aligned} \lim_{j \to \infty} \mathcal{C}_{\alpha_i}(x_j) &\subset \lim_{j \to \infty} \mathcal{C}_{\alpha_j}(x_j). \\ &\subset \emptyset \qquad \text{by the initial assumption of this proof.} \end{aligned}$$

This leads to the conclusion that $v \in \emptyset$, and proves the proposition by way of contradiction. □

Finally, the concepts of wall map and induced cone bundle allow us to state a simplified version of the extended moving the wall theorem to be used when one wants to prove the existence of an isotopy between two sets but has no interest in controlling the Lipschitz coefficient of that isotopy.

Theorem 4.1.44 (Non-Lipschitz extended moving the wall)**.** *Let Z be a Whitney stratified set, and let $f : Z \to \mathbb{R}$ be a wall map on Z. Let $\mathcal{C}$ be the f-induced cone bundle on Z. If f is proper and $\mathcal{C}$ is non-vanishing, there exists a homeomorphism $h : f^{-1}(0) \times \mathbb{R} \to Z$ such that $\pi_2 = f \circ h$, where π_2 is the projection to the second parameter.*
In addition, there exists $\alpha > 0$ such that for any $x, y \in f^{-1}(0)$ and $t, t' \in \mathbb{R}$,

$$\| h(x,t) - h(y,t') \|_2 \leq \frac{2 \max(t,t')}{\alpha} + \| x - y \|_2,$$

where $\|v\|_2$ denotes the Euclidean norm of v.

Proof. Since $\mathcal{C}$ is non-vanishing, by proposition 4.1.43 there exists $\alpha > 0$ such that $\mathcal{C}_\alpha$, the (f, α)-induced cone bundle, is non-vanishing. Then by propositions 4.1.42 and 4.1.37 we know that $\mathcal{C}_\alpha$ is α-compatible with f and upper semi-continuous. We can thus apply the quantitative version of the moving the wall theorem (theorem 4.1.30) which yields the desired result. □

Remark 4.1.45. *Let us come back to the important special case when $f : Z \to \mathbb{R}$ is globally smooth in order to further our comparison with the usual Thom-Mather theorem. As we mentioned at the end of subsection 4.1.2, proposition 4.1.33 shows that the condition that f is submersive is equivalent to the condition that the f-induced cone bundle is non-vanishing.*
In the original version of the Thom-Mather theorem the fact that f is a submersion is the main condition to check, and at the end of the last section we pointed out that having to construct "by hand" an upper semi-continuous α - compatible cone bundle with f was an additional burden in comparison to the original Thom-Mather theorem. This problem is now solved: theorem 4.1.44 only requires us to check that the f-induced cone bundle is non-vanishing and f is proper. Therefore theorem 4.1.44 is a pure generalization of the original Thom-Mather theorem, weakening the hypotheses (f is only required to be a wall map), and strengthening the conclusion (there is some Lipschitz control on the isotopy).

More practical tools to manipulate cone bundles (the f-induced cone bundle in particular), and to use the extended moving the wall theorem are introduced in the following sections. Then the previous theorem 4.1.44 will be used to prove a new stronger version of the local conic structure theorem for semi-algebraic sets:
The classical version of the moving the wall theorem states that in a small enough Euclidean ball the topology of a semi-algebraic set is conic. The new version gives the same result for any type of convex (not only Euclidean balls) and shows that the topology is left invariant by small enough deformations of the boundary of the convex set. This stability under deformation stems from the possibility that we have to control quantitatively the cone bundles by means of the extended moving the wall theorem 4.1.44 of this section.

Remark 4.1.46. *Another interesting application of the previous theorem is to create a vector field which trivializes several maps at once. This is feasible provided the intersection of their induced cone bundles is non-vanishing. This entails that the level lines of the two maps are isotopic.*

4.2 Measuring transversality

The aim of this section is to present different measures of transversality between maps and stratified sets. They enable, by handling simple inequalities, to conclude that a given map and a given stratified set are transverse so as to

apply theorem 4.1.44. These measures of transversality are obtained by defining measures on cones.

Beware that in the context of this section any set stable by multiplication by elements of $\mathbb{R}_*^+$ will be considered to be a cone. Cones that are also stable by addition will be called convex cones (vector spaces are thus an example of convex cones). And strong cones will be those cones that meet all the conditions in the previous section, that is convex cones that are open and do not contain 0. These precisions are of no real importance to the discussion but they are given in order to avoid confusion.

As a reminder from the first subsection on Whitney stratifications (definition 4.1.14) a cone of $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ which is stable by multiplication by positive constants *and stable by addition.*

Definition 4.2.1 (Measures on cones)**.** *Let $C, C' \subset \mathbb{R}^n$ be cones, we set the following definitions and notations:*

- *$\forall v, w \in \mathbb{R}^n - \{0\}$, let the angle distance between v and w be*

$$d_C(v,w) = 1 - \frac{(v|w)}{\| v \|_2 \| w \|_2},$$

  where $(v|w)$ denotes the scalar product of v and w and $\| x \|_2 = \sqrt{(x,x)}$.

- *Let the gap between C and C' be*

$$\Gamma(C, C') = \inf_{v \in C \setminus \{0\},\ w \in C' \setminus \{0\}} d_C(v,w).$$

- *Let the pseudo-distance between C and C' be*

$$\mathcal{H}(C \to C') = \sup_{v \in C' \setminus \{0\}} \inf_{w \in C \setminus \{0\}} d_C(v,w).$$

- *And finally let the distance between C and C' be*

$$d\mathcal{H}(C, C') := \mathcal{H}(C \to C') + \mathcal{H}(C' \to C).$$

All these definitions are usual definitions on sets. Cones can be identified to their intersection with the unit sphere, and one can look at the correspondence with usual definitions: $d_C|_{S(0,1)}$ induces the usual Euclidean topology on the sphere. Indeed if $v, w \in S(0,1)$, then we have $2d_C(v,w) = \| v - w \|_2^2$ by expanding the scalar product $(v-w|v-w)$.
We call $\mathcal{H}(. \to .)$ pseudo-distance because it satisfies the triangle inequality although it is not symmetrical. $\mathcal{H}(. \to .)$ is no more than the usual Hausdorff pseudo-distance induced by d_C for subsets of the unit sphere. One way to interpret $\mathcal{H}(C \to C')$ is to consider it as the minimal thickening radius such that C' is contained in the thickening of C.
This shows that $d\mathcal{H}(.,.)$ restricted to the unit sphere is the square of the usual Hausdorff distance between two sets for the Euclidean distance.

Remark 4.2.2. *Apart from the mathematical relevance of the measures, they are of much interest computer-wise since it is well-known that it is far more efficient to carry out computations with approximate numbers while controlling the rounding errors than to drag along exact representations of the numbers. The estimations and bounds on these measures enable to certify computations that are made using such controlled rounding arithmetic systems.*

There are many relations between the quantities that have just been defined. The one that we will mainly use is the "triangular" inequality for $\mathcal{H}(. \to .)$ and $\Gamma(.,.)$:

Proposition 4.2.3. *For three cones C, C', C'' we have:*

$$\Gamma(C, C') + \mathcal{H}(C'' \to C') \geq \Gamma(C, C'').$$

As a side remark, one can express Kuo-Verdier's condition (w) that we mentioned earlier, in terms of the pseudo-distance function $\mathcal{H}(. \to .)$:

Definition 4.2.4 (Condition (w))**.** *Let $X, Y \subset \mathbb{R}^n$ be two strata such that $X < Y$. We say that (X, Y) satisfy condition (w) at $p \in X$ iff there exists a neighborhood U of p and $C > 0$ such that for any $x \in X \cap U$ and $y \in Y \cap U$, we have:*

$$\mathcal{H}(T_y Y \to T_x X) < C \parallel x - y \parallel_2 .$$

Let us now go back to the definition of our measures of transversality. They involve the key concept of orthogonal cone:

Definition 4.2.5 (Orthogonal cone)**.** *Let E be a $\mathbb{R}$ vector space and C be a cone of E. The orthogonal cone to C is defined as:*

$$C^{\perp} := \{x \in \mathbb{R}^n \mid (x|C) \subset \mathbb{R}^+\},$$

where $\mathbb{R}^+ = \{x \in \mathbb{R} | x \geq 0\}$ and $(x|C)$ is the set of all scalar products $(x|y)$ for $y \in C$.

Remark 4.2.6. *Notice that when C is a vector space, $C^{\perp}$ as defined above and $C^{\perp}$, the usual orthogonal space, coincide.*

The first measure of transversality for sets is defined as follow:

Definition 4.2.7 (Measure of transversality)**.** *$\forall M, N \subset \mathbb{R}^n$ smooth submanifolds, $\forall x \in M \cap N$, we define the transversality of M and N at x as*

$$\delta_x(M, N) = \Gamma(T_x M^{\perp}, T_x N^{\perp}).$$

If one of the tangent spaces is the whole of $\mathbb{R}^n$, then $\delta_x(M, N) = +\infty$.
We define the transversality of M and N as

$$\delta(M, N) = \inf_{x \in M \cap N} \delta_x(M, N).$$

We have the following property that justifies that we call it a measure of transversality:

Proposition 4.2.8. *$\forall M, N \subset \mathbb{R}^n$ smooth manifolds, $\forall x \in M \cap N$, M and N are transverse at x iff $\delta_x(M, N) > 0$*

Proof. Let $C = T_xM^\perp$ and $C' = T_xN^\perp$.
Then if M and N are transverse $T_xM + T_xN$ spans the whole space, hence $\{0\} = (T_xM + T_xN)^\perp = C \cap C'$ (for two vector spaces A, B we have $(A+B)^\perp = A^\perp \cap B^\perp$). Because $C \cap S(0,1)$ and $C' \cap S(0,1)$ are compact the infimum of d_C over $C \times C'$ is reached for some $v, w \in (C \cap S(0,1)) \times (C' \cap S(0,1))$ (δ is invariant by dilation of its parameters). We have $C \cap C' = \{0\}$, and $C \cap C' \cap S(0,1) = \emptyset$, thus $v \neq w$ and $d_C(v, w) = \delta_x(M, N) > 0$.

Conversely assume $\delta_x(M, N) > 0$. Once again the infimum of d_C is reached for some $v, w \in (C \cap S(0,1)) \times (C' \cap S(0,1))$. Therefore by definition of the infimum, for any $v', w' \in (C \setminus \{0\}) \times (C \setminus \{0\})$ we have: $d_C(v', w') \geq d_C(v, w) > 0$ and thus $v' \neq w'$. This proves that $\{0\} = C \cap C'$. Finally, as C and C' are vector spaces, we have $\{0\} = C \cap C' = (T_xM + T_xN)^\perp$. This proves that $T_xM + T_xN = \mathbb{R}^n$ by taking the orthogonal again (which is an involution on finite dimensional spaces). □

Remark 4.2.9. *Coming back to Whitney stratified set, if Z, Z' are two sets which are Whitney regularly stratified by $\mathcal{S}$ and $\mathcal{S}'$, and if Z and Z' are transverse, then $Z \cap Z'$ is Whitney stratified by $\{X \bigcap X' : (X, X') \in \mathcal{S} \times \mathcal{S}'\}$*

After defining transversality for sets, we can define the transversality of a set to a semi-algebraic map. This notion is simply a way to quantify geometrical properties of the induced cone bundle. This is why we introduce the following notion of transversal acceptance of a map to a set, which is purely geometrical as it only depends on the level lines of f. It is defined by means of measures on orthogonal cones. This is how we define the orthogonal cone bundle and transverse acceptance:

Definition 4.2.10 (Orthogonal cone bundle)**.** *Let Z be a stratified set and $\mathcal{C}$ be a cone bundle on Z. The orthogonal cone bundle $\mathcal{C}^\perp(x)$ is defined at any $x \in Z$ by*

$$\mathcal{C}^\perp(x) = \big(\mathcal{C}(x)\big)^\perp.$$

Definition 4.2.11 (Transverse Acceptance)**.** *Let $Z \subset \mathbb{R}^n$ be a stratified set, $f : \mathbb{R}^n \to \mathbb{R}$ a continuous semi-algebraic map and $x \in Z$. Let $\mathcal{C}$ be the f-induced cone bundle on $\mathbb{R}^n$, the transverse acceptance of f to Z at x is defined as*

$$\tau(f, x) = \Gamma(\mathcal{C}^\perp(x), T_xZ^\perp).$$

The transverse acceptance of f to Z is

$$\tau(f) = \inf_{x \in Z} \Gamma(\mathcal{C}(x)^\perp, T_xZ^\perp).$$

The transverse acceptance is in fact a measure of how far the map f is from inducing a vanishing cone bundle on Z. This observation stems from the next proposition.

Proposition 4.2.12. *Let Z be a stratified set, and let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous semi-algebraic map. For any $x \in Z$ let C be the f-induced cone bundle on $\mathbb{R}^n$ at x. Let $\mathcal{C}(x) = T_xZ \cap C$ be the f-induced cone bundle on Z at x. We have*

$$\tau(f, x) > 0 \quad \underline{\text{iff}} \quad \mathcal{C}(x) \neq \emptyset.$$

Proof. If $\Gamma(C^\perp, T_xZ^\perp) > 0$ then the two cones $C^\perp$ and $T_xZ^\perp$ do not intersect. Therefore, as $T_xZ^\perp$ is a vector space, it is a proper vector space and by Banach's separation theorem it is possible to find a hyperplane H containing $T_xZ^\perp$ that does not intersect $C^\perp$. Consequently there is a non-zero orthogonal vector v to H such that $(v \mid C^\perp) \subset \mathbb{R}^+$ and $(v \mid T_xZ^\perp) = \{0\}$. Because the $^\perp$ operator is an involution for vector spaces, we have that $v \in T_xZ$. As $C^\perp$ is closed it means that $d_C(v, .)$ reaches its minimum on $C^\perp$. This minimum is not 0 as $\{v = 0\}$ avoids $C^\perp$. Therefore v is in the interior of $\left(C^\perp\right)^\perp$, that is in the interior of C. This shows that $v \in \mathcal{C}(x)$ and thus $\mathcal{C}(x)$ is not empty.

Conversely, assume $\tau(f, x) = \Gamma(C^\perp, T_xZ^\perp) = 0$. This means that there is a sequence of $v_i \in C^\perp$ that converges to a non-zero vector $v \in T_xZ^\perp$. If there were a non-zero vector $v' \in \mathcal{C}(x)$, considering that $v \in T_xZ^\perp$, we would have:

$$\lim_{i \to \infty} (v_i \mid v') = (v \mid v') = 0.$$

But all the v_i lie in $C^\perp$, which implies that all the $(v_i \mid v')$ are positive, and if they converge to 0, this implies that v' lies in the boundary of C. This is impossible as this set is open. Hence, by way of contradiction, $\mathcal{C}(x) \subset \{0\}$. As C is open, $C \cap T_xZ = \mathcal{C}(x) = \emptyset$. □

We now introduce the notion of jump from one map to another. The idea behind it is that if a map is far from inducing a vanishing cone bundle then it can be replaced with a map close to it and the new map will still induce a non-vanishing cone bundle. The jump from one map to another measures the cost of replacing one map with another. This idea is formalized by a simple lemma 4.2.14 which shows that when the transverse acceptance is greater than the jump, the transverse acceptance of the new map will remain positive.

Definition 4.2.13 (Jump from a map to another)**.** *Let $Z \subset \mathbb{R}^n$ be a stratified set, let $f : Z \to \mathbb{R}$ and $f' : Z \to \mathbb{R}$ and let $\mathcal{C}$ and $\mathcal{C}'$ be their induced cone bundle on Z, the jump from f to f' is defined as*

$$\mathcal{H}\left(f \to f'\right) = sup_{x \in Z} \mathcal{H}\left(\mathcal{C}'^\perp(x) \to \mathcal{C}^\perp(x)\right).$$

This finally enables us to formulate the following essential lemma that is a mere rephrasing of the basic properties of the measures on cones introduced earlier.

Lemma 4.2.14. *Assume that Z is a Whitney stratified set, and that f and f' are two maps from Z to $\mathbb{R}$, then*

$$\tau(f) + \mathcal{H}(f \to f') \geq \tau(f').$$

Proof. The inequality (4.2.3) about the measures on cones is

$$\Gamma(C, C') + \mathcal{H}(C'' \to C') \geq \Gamma(C, C'').$$

To get the result, it suffices to apply this inequality to $C = T_x Z^\perp$, $C' = \mathcal{C}^\perp(x)$, and $C'' = \mathcal{C}'^\perp(x)$ where x is a point of Z and $\mathcal{C}$ (resp. $\mathcal{C}'$) is the f-induced cone bundle (resp. f') on Z. Taking the infimum of both sides of the equality yields the following inequality

$$\inf_{x\in Z} \Gamma\left(T_x Z^\perp, \mathcal{C}^\perp(x)\right) + \mathcal{H}\left(\mathcal{C}'^\perp(x) \to \mathcal{C}^\perp(x)\right) \quad \geq \quad \inf_{x\in Z} \Gamma\left(T_x Z^\perp, \mathcal{C}'^\perp(x)\right).$$

Because for any set S and functions $a, b : S \to \mathbb{R}$, we have

$$\inf_{x\in S} a(x) + \sup_{x\in S} b(x) \geq \inf_{x\in S}(a(x) + b(x)),$$

the following inequality holds

$$\inf_{x\in Z} \Gamma\left(T_x Z^\perp, \mathcal{C}^\perp(x)\right) + \sup_{x\in Z} \mathcal{H}\left(\mathcal{C}'^\perp(x) \to \mathcal{C}^\perp(x)\right) \geq \inf_{x\in Z} \Gamma\left(T_x Z^\perp, \mathcal{C}'^\perp(x)\right).$$

This is exactly the claim of the lemma after applying the definitions of transverse acceptance and of the jump from f to f'. □

This lemma concludes the discussion of the measures of transversality. We can now apply those tools to prove the conic structure of a set and then to obtain a triangulation procedure.

4.3 Conic structure

This subsection introduces a set of definitions that will give a quantitative sufficient condition for a stratified set to have conic structure (definition 4.3.2) in a given convex set. As mentioned in the introduction, this problem is linked to triangulations by the fact that if the convex set has a piecewise linear boundary, and if we are able to recursively triangulate the trace of Z on the boundary, we obtain a triangulation of Z in the convex set by simply taking the cone over the trace of Z on the boundary.
In the next subsection (4.3.1) we present some background about the well-known local conic structure theorem that serves as an introduction and motivation to the second subsection (4.3.2) that will introduce a stable version of the conic structure theorem. It is shown that the topology inside a convex set does not change when the convex set is slightly deformed. The measure of the deformation is made using the measures of the previous subsection. This is why the new version of the conic structure theorem can be said to be "metrically" stable.

4.3.1 The classical setup of the conic structure theorem

The most well-known theorem about topological triviality of semi-algebraic sets is Hardt's theorem.

Theorem 4.3.1 (Hardt's Theorem)**.** *Let $Z \subset \mathbb{R}^n$ be a semi-algebraic set and let $f : Z \to \mathbb{R}^m$ be a semi-algebraic map. Then there is a finite partition of $\mathbb{R}^m$ into semi-algebraic sets $(\Pi_i)_{i\in I}$ (I finite) such that f is topologically trivial over each Π_i. In addition the trivializing homeomorphism is semi-algebraic.*

One direct consequence of Hardt's theorem is the local "conic structure" theorem. Here is what is meant by "conic structure":

Definitions 4.3.2 (Conic structure)**.** *For $p \in \mathbb{R}^n$ and $B \subset \mathbb{R}^n$, $[p \star B)$ is called the real positive cone with vertex p and base B and it is defined by*

$$p \star B \;=\; \bigcup_{x\in B} [p, x),$$

where $[p, x)$ is the real half-line starting at p and going through x.

For $Z \subset \mathbb{R}^n$ and $C \subset \mathbb{R}^n$ a convex set with non-empty interior C°, $Z \cap C$ is said to have conic structure iff $Z \cap C^\circ$ is homeomorphic to $p\star(Z \cap \partial C)$ for some $p \in C^\circ$.

It is straightforward that the above definition does not depend on $p \in C^\circ$. By this we mean that the two following statements are equivalent (provided $C^\circ \neq \emptyset$):

- The set $Z \cap C^\circ$ is homeomorphic to $p \star (Z \cap \partial C)$ for *some* $p \in C^\circ$.
- The set $Z \cap C^\circ$ is homeomorphic to $p \star (Z \cap \partial C)$ for *any* $p \in C^\circ$.

Theorem 4.3.3 (Local conic structure)**.** *Let $Z \subset \mathbb{R}^n$ be a semi-algebraic set containing 0. There exists $\epsilon > 0$ such that for all $\eta \in (0, \epsilon)$, $Z \cap B(0, \eta)$ has conic structure, where $B(0, \eta)$ is the closed ball centered at 0 with radius η.*

We have formulated the theorem at 0 for simplicity, but of course, it holds at any p by applying it to a translated image of Z. We will give two proofs of this theorem so as to compare them. The first one uses Hardt's theorem:

First proof of theorem 4.3.3. Consider the family $A_\eta = S(0, \eta) \cap Z$ for $\eta \geq 0$ and $A_\eta = \emptyset$ for $\eta < 0$. We regard the family $(A_\eta)_{\eta\in\mathbb{R}}$ as the subset $\cup_{\eta\in\mathbb{R}} (A_\eta \times \{\eta\})$ of $\mathbb{R}^n \times \mathbb{R}$. This set is clearly semi-algebraic since it is the intersection of two semi-algebraic sets: Z and the cone $0 \star (S(0, 1) \times \{1\})$. Consider the projection to the parameter line, that is $f : \; x \in A_\eta \mapsto \eta$. This is a linear function, hence semi-algebraic. By Hardt's theorem it is possible to find a semi-algebraic partition of the real line into finitely many semi-algebraic sets, so that for every element of the partition there is a trivializing homeomorphism. The only semi-algebraic sets of the real line are points and intervals. Therefore there is

an interval adherent to 0 in the partition that contains some $(0,\epsilon)$ for $\epsilon > 0$ small enough.
Let $\alpha \in (0,\epsilon)$ and $h : f^{-1}(0,\epsilon) \to f^{-1}(\alpha) \times (0,\epsilon) \subset \mathbb{R}^n \times \mathbb{R}$ be the trivializing homeomorphism given by Hardt's theorem. The map $\phi : (x,\eta) \in \text{Im } h \mapsto \eta x$ is a homeomorphism on its image. We only prove the key point that $\phi|_{\text{Im } h}$ is an injection. All the points $(x,\eta) \in \text{Im } h$ are such that $x \in S(0,\alpha) \cap Z$, thus $\| x \|_2 = \alpha$ and if $\eta x = \eta' x'$ then $\| \eta x \|_2 = \alpha\eta = \alpha\eta' = \| \eta' x' \|_2$, hence $\eta = \eta'$ and $x = x'$.
We also have that the projection Π of $(A_\eta)_{\eta\in(0,\epsilon)}$ to $\mathbb{R}^n$ is $\big(Z \cap B(0,\epsilon)\big) \setminus \{0\}$, and Π is a homeomorphism on this set. This means that $\big(Z \cap B(0,\epsilon)\big) \setminus \{0\}$ is homeomorphic to $(A_\eta)_{\eta\in(0,\epsilon)}$ which is homeomorphic to $f^{-1}(\alpha) \times (0,\epsilon)$ by h, and finally this last set is homeomorphic to $\big(0 \star (Z \cap S(0,\eta))\big) \setminus \{0\}$ by ϕ. We can clearly add 0 on both ends without altering the resulting homeomorphism $\phi \circ h \circ \Pi|_{A_\eta}{}^{-1}$, and we have the desired result. □

One can see the previous theorem as a statement about the local structure of semi-algebraic sets. But it can also be interpreted from the point of view of transversality and of the Moving the Wall theorem. In this case, the moving wall would be a family of spheres centered at 0 whose radius grows to infinity. This shift in viewpoint gives rise to the following alternate proof of the local conic structure theorem. The last argument of the proof refers to a theorem proved later. But since the comparison we make between the proofs does not involve that argument, this is not a problem for our current discussion.

Second proof of theorem 4.3.3. Consider the cone $C := 0 \star (S(0,1),1)$, where $S(0,1)$ is the unit sphere centered at the origin and $(S(0,1),1)$ denotes the set $S(0,1) \times \{1\} \subset \mathbb{R}^n \times \mathbb{R}$. Consider the following condition on points of $\mathbb{R}^n \times \mathbb{R}$: $Z \times \mathbb{R}$ is not transverse to C at p (as by definition 4.1.11). This condition is satisfied if and only if it is satisfied for any pair of strata. Being transverse for semi-algebraic submanifolds is itself a semi-algebraic condition as it involves the existence of n linearly independent tangent vectors, and being a tangent vector is a semi-algebraic condition. The condition is therefore semi-algebraic and by the Tarski-Seidenberg theorem the image of the points where $Z \times \mathbb{R}$ is not transverse to p by the projection $\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ to the last parameter, is a semi-algebraic set of $\mathbb{R}$. Therefore by the same argument as in the first proof of this theorem, there is an interval $(0,\epsilon)$ where $Z \times \mathbb{R}$ is transverse to C (at every point p in their intersection). By the moving the wall theorem one can conclude all the $Z \cap S(0,\eta)$ ($\eta \in (0,\epsilon)$) have the same topology. It is thus geometrically intuitive that Z has conic structure in any $B(0,\eta)$ ($\eta \in (0,\epsilon)$). Theorem 4.3.13 which is proved later, allows us to conclude that there is an isotopy from $Z \cap B(0,\eta)$ to $0 \star (Z \cap S(0,\eta))$. □

Now let us compare the proofs. The semi-algebraicity of the homeomorphism between $Z \cap B(0,\eta)$ and $0 \star (Z \cap S(0,\eta))$ is lost in the second proof, but the homeomorphism is locally bi-Lipschitz everywhere outside 0. The first proof using Hardt's theorem proved the homeomorphism was semi-algebraic, but said

nothing about whether it is Lipschitz or anywhere in $B(0,\eta)$. We could imagine that to keep the semi-algebraicity of the homeomorphism it is necessary to introduce non-Lipschitz deformations of $Z \cap B(0,\eta)$ into $0 \star (Z \cap S(0,\eta))$. In fact, G. Valette has proved that both properties can be retained at once in [136]: if we use his result in the first proof instead of the usual Hardt theorem, we obtain a locally bi-Lipschitz homeomorphism between $Z \cap B(0,\eta)$ and $0 \star (Z \cap S(0,\eta))$ anywhere in $B(0,\eta) \setminus \{0\}$.
One can also notice that in the two previous proofs the Euclidean balls and spheres could have been replaced by any kind of semi-algebraic convex set (convexity is needed for the notion of conic structure to make sense). From this remark a question arises: if we take two different convex sets C, C' containing 0 in their interior, and we take them small enough, we know that Z has conic structure in them, but do $Z \cap C$ and $Z \cap C'$ have the same topology ? One can arrive at a similar question through a different way: it seems rather intuitive that if the ball is very small one can translate it a little bit in any direction without altering the topology of Z in the ball. Is this true ? This second question is actually a specialization of the previous one as the two convex sets C, C' can be taken a translation of one another.
The answer to these two questions is yes. The local conic structure of semi-algebraic sets is a very stable property that is not altered by deforming the boundary of the ball as long as it stays convex . This stronger version of the local conic structure theorem is a consequence of the results of the next subsection.

4.3.2 A stable version of conic structure

We now introduce the notion of convex deformations. Their purpose is twofold: first it allows us to quantify the size of the convex sets that is sufficient to have conic structure inside them, secondly it allows us to prove that the topology inside a convex set is stable by deformation of the shape of the convex set (and thus by small translation as explained at the end of the previous subsection).

Definition 4.3.4 (Convex deformation)**.** *A convex deformation is a function $D : \mathbb{R}^n \to \mathbb{R}^+$ such that*

- *$\exists p \in \mathbb{R}^n$ such that $D^{-1}(0) = \{p\}$.*
- *The function D is convex. That is*

$$\forall x, y \in \mathbb{R}^n,\ \forall \alpha \in [0,1],\ (1-\alpha)D(x) + \alpha D(y) \geq D\big((1-\alpha)x + \alpha y\big).$$

- *The function D is $\mathcal{C}^\infty$ on an open dense set of $\mathbb{R}^n$.*
- *There exists a constant $\alpha > 0$, such that the radial derivative of D is greater than α, that is*

$$dD_x(x-p) > \alpha \,\|\, x - p \,\|_2 \,.$$

The point $p = D^{-1}(0)$ is referred to as the center of the deformation.

From this definition ensue the first important property of convex deformations: they are proper wall maps.

Proposition 4.3.5 (Convex Deformations are Wall Maps)**.** *Let $D : \mathbb{R}^n \to \mathbb{R}^+$ be a convex deformation, then D is a wall map.*

Proof. By definition D is $\mathcal{C}^\infty$ on an open dense set of $\mathbb{R}^n$, and since its radial derivative is greater than α, so is the norm of its differential. Q.E.D. □

The simplest type of convex deformations are those related to dilations of a convex set. The maps associated to those dilations correspond to the usual notion of convex gauge:

Definition 4.3.6 (Convex set gauge)**.** *$\forall C \subset \mathbb{R}^n$ convex set, $\forall p \in C^\circ$, define the convex set gauge $N_{p,C}$ of (p, C), so that:*

$$\forall x \in \mathbb{R}^n,\ N_{p,C}(x) = \inf\{\alpha > 0 : \frac{x-p}{\alpha} + p \in C\}.$$

Notice that $\text{Graph}(N_{p,C}) = (0,0) \star (\partial C, 1) \subset \mathbb{R}^n \times \mathbb{R}^+$. In particular, $N_{p,C}^{-1}([0,1]) = C$ and $N_{p,C}^{-1}(\{0\}) = \{p\}$.

Proposition 4.3.7. *Let $N_{p,C}$ be the convex gauge of C at p. Assume the frontier ∂C of C is $\mathcal{C}^\infty$ on an open dense set. Then $N_{p,C}$ is a convex deformation.*

Proof. We have $N_{p,C}^{-1}(0) = \{p\}$, and $N_{p,C}$ is clearly convex. In addition, $N_{p,C}$ is $\mathcal{C}^\infty$ at any point x such that $[p, x) \cap \partial C$ is $\mathcal{C}^\infty$. Since by hypothesis this ∂C is $\mathcal{C}^\infty$ on an open dense set of ∂C, $N_{p,C}$ is $\mathcal{C}^\infty$ on an open dense set of $\mathbb{R}^n$.
Finally by definition of the convex gauge, the radial derivative at a point $x \in \mathbb{R}^n \setminus \{0\}$, is the distance to 0 of $x' = [p, x) \cap \partial C$. Since p is in the interior of C, the infimum of the radial derivative over $\mathbb{R}^n \setminus \{0\}$ is positive. □

We now prove that the gauge of a semi-algebraic set remains semi-algebraic. Thus one could readily apply theorem 4.3.3 to recover local conic structure in sufficiently small balls. But again, this does not give any quantification on the size of the ball that is sufficient and does not say anything about what happens when changing the shape and position of the convex set.
The semi-algebraicity of the convex set gauge is a straightforward consequence of the Tarski-Seidenberg quantifier elimination theorem (see [17]):

Proposition 4.3.8. (Semi-algebraic convex set gauge is semi-algebraic) *$\forall p \in \mathbb{R}^n$, $\forall C \subseteq \mathbb{R}^n$ semi-algebraic convex set such that $p \in C^\circ$, $N_{p,C}$ is a semi-algebraic mapping.*

Proof. $\forall p \in C \subset \mathbb{R}^n$, the graph of $N_{p,C}$ is defined by the following condition: $\forall (a, b) \in \mathbb{R}^n \times \mathbb{R}$, $(a, b) \in \text{Graph}(N)$ $\underline{\text{iff}}$ $\forall \lambda > b$, $\frac{a-p}{\lambda} + p \in C$ and $\forall \lambda \in]0, b[$, $\frac{a-p}{\lambda} + p \notin C$.

As C is semi-algebraic, being contained in C is a semi-algebraic condition. Hence the condition is a semi-algebraic first order formula and the graph of N is semi-algebraic as the quantifiers can be eliminated. □

Remark 4.3.9. *Let C be any hypercube with center c. That is C is the isometric image of $\Pi_{i=1}^{n}[-a,a]$ for some $a>0$, and c is the image of 0 by that isometry. Let C' be the ball for the Euclidean distance with same center as C. Let D and D' their respective associated convex gauges. Then a simple computation (the worst case being at the 2^n corners of the hypercube) shows that $\mathcal{H}(D' \to D) = \frac{1}{\sqrt{n}}$.*

To relate two convex sets to one another it is possible to define another type of convex deformation map that we call transition maps:

Definition 4.3.10 (Transition maps)**.** *For two compact convex sets $C, C' \subset \mathbb{R}^n$ and a point $p \in \mathbb{R}^n$ such that $p \in C^\circ$ and $C \subset C'^\circ$, the transition map $D_p(C,C') : \mathbb{R}^n \to \mathbb{R}^+$ from C to C' centered at p is defined by*

- $\forall x \in C$, $D_p(C,C')(x) := N_{p,C}(x)$
- $\forall x \in C' - C$, *let* $q(x) = \big([p,x) \cap \partial C'\big)$ *($\big([p,x) \cap \partial C'\big)$ is a singleton as C' is convex) and let* $\alpha(x) = N_{p,C}(q(x))$. *We set*

$$D_p(C,C')(x) := 1 + \frac{N_{p,C}(x) - 1}{\alpha(x) - 1}.$$

- $\forall x \in C'$, $D_p(C,C')(x) := 2N_{p,C'}(x)$

Transition maps are in fact a straightforward linear interpolation between two convex sets when one is inside the other. It is thus straightforward to check that they are wall maps since convex gauges are wall maps by proposition 4.3.7.

Theorem 4.3.11 (Topological triviality for convex deformation maps)**.** *Given a convex deformation map D with center p and a compact variety Z endowed with a Whitney stratification, assume that $\tau(D, Z \setminus \{p\}) > 0$, then there is a homeomorphism*

$$h : (D_1 \cap Z) \times \mathbb{R}_*^+ \to ((Z \times \mathbb{R}_*^+) \cap \mathrm{Graph}(D)).$$

In addition, h commutes with the projection to the second component (that is, it leaves the $D(\alpha)$ stable).

Proof. By proposition 4.2.12, the induced cone bundle by D on $Z \setminus \{p\}$ is non-vanishing. Therefore it is possible to apply theorem 4.1.44 and the resulting homeomorphism is exactly what is claimed. □

Remark 4.3.12. *The weaker condition that $\tau(D, Z \setminus \{p\}, x) > 0$ at every $x \in Z \setminus \{p\}$ also works, but this is of no interest for semi-algebraic or sub-analytic sets. This weaker form could be used to prove that a logarithmic spiral has conic structure inside a disk.*

This does not yet prove that Z has conic structure inside the $D^{-1}([0,\alpha])$ ($\alpha > 0$), because one only has a homeomorphism so far. But the convex deformation maps have a structure that is naturally compatible with dilations. This makes it possible to construct an isotopy from the previous homeomorphism.

Corollary 4.3.13 (Conic structure for convex deformations)**.** *Let $D : \mathbb{R}^n \to \mathbb{R}$ be a convex deformation map with center p, and Z a stratified set such that $\tau(D, Z \setminus \{p\}) > 0$, then Z has conic structure in any $D^{-1}([0, \alpha])$ $(\alpha > 0)$.*

Proof. First, apply the previous theorem (4.3.11). Let $C := D^{-1}([0, 1])$. Although all the fibers $h^{-1}(\mathbb{R}^n, \alpha)$ are isotopic, there is still to construct an isotopy between $D^{-1}([0, 1])$ and $p \star (\partial C \cap Z)$. It is enough to prove the result for $D^{-1}([0, 1])$ to recover the result for all the $D^{-1}([0, \alpha])$ $(\alpha > 0)$ as it suffices to reparametrize the deformation (i.e. consider $(t \mapsto t/\alpha) \circ D$ instead of D).

Let h be the homeomorphism that we get from the previous theorem. The first remark is that D is an increasing function on any half-line $[p, x)$ for any $x \in \mathbb{R}^n$. Indeed if $y, y' \in [p, x)$ and $\| y - p \|_2 < \| y' - p \|_2$, by convexity of $D^{-1}([0, D(y')[)$, $y \in D^{-1}([0, D(y')[)$ hence $D(y) < D(y')$.

Let $h_1 = \Pi_{\mathbb{R}^n} \circ h$, hence $h = (h_1, D)$. Consider the following mapping

$$\begin{array}{rccc} \Phi : & [0,1] \times [0,1] & \to & [0,1] \\ & u, t & \mapsto & u + (1-u)t \end{array}$$

The map Φ is clearly continuous. For any fixed $u \in]0, 1]$, $\Phi_u := \Phi(u, .)$ is a bijection. Thus $h^{-1}(h_1(x), \Phi(u, D(x))) :]0, 1] \times (Z \cap C) \to (Z \cap C)$ is continuous and for any fixed $u \in]0, 1]$ it is injective. Consider the map

$$\begin{array}{rccc} \rho : & [0,1] \times (C \setminus \{p\}) & \to & C \setminus \{p\} \\ & u, x & \mapsto & [p, x) \cap D^{-1}(u) \end{array}$$

As D is minimal at p and convex, it is increasing on the half-lines $[p, x)$. Therefore $[p, x) \cap D^{-1}(u)$ is a singleton and ρ is well-defined. Let us now prove that ρ is continuous. To do so it suffices to prove that the graph of ρ is closed. Consider the following subsets of $([0, 1] \times (C \setminus \{p\}) \times \mathbb{R}^n)$:

$$\begin{aligned} A &:= \big\{ (u, x, [p, x)) \mid u \in [0, 1], p \in (C \setminus \{p\}) \big\}, \\ B &:= \big\{ (u, x, D^{-1}(u)) \mid u \in [0, 1], p \in (C \setminus \{p\}) \big\}. \end{aligned}$$

The graph of ρ is the intersection of A and B. It is clear that A is closed, and this is also the case for B as D is a convex map. Therefore $A \cap B = \mathrm{Graph}(\rho)$ is closed which is what we wanted.

The map ρ has also the property that $\rho(u, x) = \rho(u', x')$ implies $u = u'$ and if in addition $x \neq p$ and $x' \neq p$, we also have $[p, x) = [p, x')$. This is because if $\rho(u, x) = \rho(u', x')$ then $D^{-1}(u) = D^{-1}(u')$ since by way of contradiction if they where not equal they would be disjoint and it would be impossible for $\rho(u, x)$ and $\rho(u', x')$ to be equal. In the same way if $[p, x)$ and $[p, x')$ were not equal they would be disjoint (when x and x' are different from p), which is impossible as $\rho(u, x) = \rho(u', x')$.

Finally the following mapping is an isotopy:

$$\begin{array}{rccc} \Psi : & [0,1] \times Z \bigcap C & \to & C \\ & u, x & \mapsto & \rho(D(x), h^{-1}(h_1(x), \Phi(u, D(x)))) \text{if} x \neq p \\ & u, p & \mapsto & p \end{array}$$

It is continuous everywhere outside p as the composition of continuous functions. If u_n, x_n converges to u, p, then $D(x)$ converges to 0 and $h^{-1}(h_1(x), \Phi(u, D(x)))$ stays in C which is compact. Therefore $\Psi(u_n, x_n)$ converges to p.
For any fixed $u \in [0,1]$, $\Psi_u = \Psi(u, .)$ is injective. If $\Psi_u(x) = \Psi_u(y)$ then there are two possibilities: $x = y = p$ and there is nothing to prove, or both x and y differ from p. There is no other possibility because $x \neq p$ implies $\Psi_u(x) \neq p$. If both differ from p then as explained previously, $D(x) = D(y) > 0$. Thus we have $\Phi(u, D(x)) = \Phi(u, D(y)) > 0$ and also:

$$\Big(h^{-1}\Big(h_1(x),\ \Phi(u, D(x))\Big), h^{-1}\Big(h_1(y),\ \Phi(u, D(y))\Big)\Big) \in D^{-1}\Big(\Phi(u, D(x))\Big).$$

As was explained before the two following half-lines are the same:

$$\Big[p, h^{-1}\Big(h_1(x),\ \Phi(u, D(x))\Big)\ \Big) = \Big[p, h^{-1}\Big(h_1(y),\ \Phi(u, D(y))\Big)\ \Big).$$

But the $[p, z) \cap D^{-1}(\alpha)$ for any z and α are singletons, therefore

$$h^{-1}\Big(h_1(x),\ \Phi(u, D(x))\Big) = h^{-1}\Big(h_1(y),\ \Phi(u, D(y))\Big),$$

which in turns means that $h_1(x) = h_1(y)$. Hence $h(x) = h_1(x),\ D(x) = h_1(y)$ and $D(y) = h(y)$, which proves that $x = y$.
Finally it is clear that Ψ_0 is the identity map on $Z \bigcap C$. On the other end $\Psi_1(Z \bigcap C) = p \star \partial C$ because:

- For $u = 1$, $\Phi_u(.) = 1$ and thus $h^{-1}\big(h_1(x), 1\big) \in \partial C$ for any $x \in Z \cap C$
- For any $x \in p \star \partial C$, $y = h^{-1}\big(h_1([p, x) \cap \partial C),\ D(x)\big)$ is the preimage of C: First $h_1(y) = h_1\big([p, x) \cap \partial C\big)$, hence

  $$h^{-1}\Big(h_1([p, x) \cap \partial C), 1\Big) = [p, x) \cap \partial C,$$

  and $\rho\big(D(x),\ [p, x) \cap \partial C\big)$ is the only point on $[p, x)$ whose image by D is $D(x)$. Since x itself has this property, we finally conclude that $\rho\big(D(x),\ [p, x) \cap \partial C\big) = x$.

□

4.4 Relating Euclidean distance and transversality

From the work in the previous section, we know that to prove that a semi-algebraic set A has conic structure inside a given convex region $\mathcal{R}$, it suffices to prove that the transverse acceptance of the convex gauge $N_{p,\mathcal{R}}$ to A is positive. The simplest type of convex set one can imagine is the Euclidean ball. The following definitions and lemmas help relate the transverse acceptance of a the convex gauge of an arbitrary convex to set A, and the transverse acceptance of the Euclidean distance to the same set A.

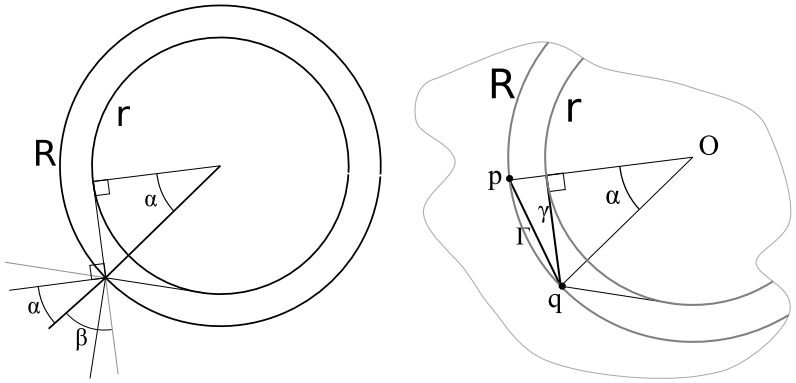


Figure 4.2: Jump from the Euclidean distance to a convex gauge

Lemma 4.4.1. *Let $\mathfrak{R}$ be a convex set such that $B_{0,r} \subset \mathfrak{R}$ and $\mathfrak{R} \subset B_{0,R}$. Let $d(x) = ||x||^2$ and let $N_{0,\mathfrak{R}}$ be the convex gauge associated to $\mathfrak{R}$ at 0. Then*

$$\mathcal{H}\left(d \to N_{0,\mathfrak{R}}\right) \leq \frac{R-r}{R}.$$

Proof. The worst case that can happen is shown on the figure 4.2. The two circles R and r represent a cut of the balls $B_{0,r}$ and $B_{0,R}$. Only a portion of the boundary of the region is drawn in the space between r and R. The tip of this portion of $\mathfrak{R}$ is q. This is the worst possible case as this is the most pointed that the region can be provided it lies in $B_{0,R}$ and contains $B_{0,r}$. On the left-hand figure, the boundary of the region is continued in gray outside $B_{0,R}$ to show how it compares to its orthogonal cone which is drawn in solid black. The line originating from the center of the circles represents the orthogonal cone for the Euclidean distance at q. The angles α and β are what we want to bound: the difference between the orthogonal cones to $\mathfrak{R}$ and $B(0,R)$ at q. Notice that the line determining the upper limit of the orthogonal cone is parallel to the base of the right triangle that has a corner at the origin. Therefore the angle α can be drawn at the origin too. Consequently, to determine the distance between the orthogonal cones of the two maps, we want to know $1-\cos(\alpha)$ (this is according to definition 4.2.1). By taking O as the origin of the reference frame in the right-hand figure, we have:

$$1 - \cos(\alpha) = 1 - (p/R|q/R) = \frac{\|p-q\|_2^2}{2R^2} = \frac{\Gamma^2}{2R^2}.$$

This is easily obtained by considering the segment γ. We have the two following Pythagorean equalities $\gamma^2 = R^2 - r^2$ and $\Gamma^2 = \gamma^2 + (R-r)^2$. Therefore we obtain

$$\frac{\Gamma^2}{2R^2} = \frac{R^2 - r^2 + R^2 - 2Rr + r^2}{2R^2} = \frac{R-r}{R}.$$

□

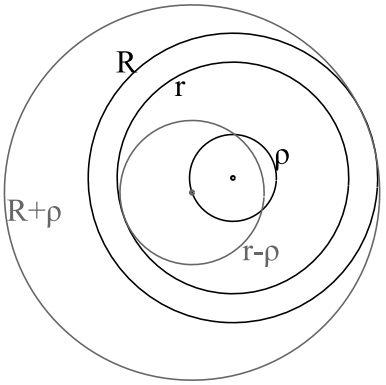


Figure 4.3: Upper bound on the jump to the Euclidean distance to a point in a ball with radius ρ

A crucial dilemma for all computer based methods is to choose how to handle points on the variety. If those points are approximated, it will be challenging to deal with numerical instability. If the points are algebraically encoded, their representation can inflate extremely fast and become impossible to use, even for moderately sized examples. Here we completely avoid that difficulty by allowing points to be approximate and only controlling the quality of the approximation. To use the previous result, we need to adapt it to a setting where we only know the variety is close to our approximate point. This is easily done by considering "the worst transversality" possible for the convex deformation associated to the region and a point running over the interior of the box. This gives us the following lemma:

Lemma 4.4.2. *Let $\mathfrak{R}$ be a convex set such that $B_{0,r} \subset \mathfrak{R} \subset B_{0,R}$. For any $p \in B_{0,r}$, let $d_p(x) = ||x-p||^2$ and $N_{p,\mathfrak{R}}$ be the convex gauge associated to $\mathfrak{R}$ at p. Assume that p always lies in a ball $B_{0,b}$ (with $b < r$), then*

$$\forall p \in B_{0,b},\ \mathcal{H}\left(d \to N_{p,\mathfrak{R}}\right) \leq \frac{R-r+2b}{R+b}.$$

Proof. For a point $p \in B_{0,b}$, let $\rho = ||x-p||$. As $p \in B(0,b)$ we have $\rho \leq b$. Figure 4.3 shows that we have $B_{p,r-\rho} \subset B_{0,r}$ and $B_{0,R} \subset B_{p,R+\rho}$ (this is a simple triangle inequality !). Since $\rho \leq b$, we also have $B_{p,r-b} \subset B_{p,r-\rho}$ and $B_{p,R+\rho} \subset B_{p,R+b}$. As $\mathfrak{R}$ lies between $B_{0,r}$ and $B_{0,R}$ we finally obtain $B_{p,r-b} \subset \mathfrak{R} \subset B_{p,R+b}$. The previous lemma 4.4.1 allows us to conclude that

$$\mathcal{H}\left(d \to N_{p,\mathfrak{R}}\right) \leq \frac{R-r+2b}{R+b}.$$

□

As we are going to relate every convex deformation to the Euclidean distance, it is convenient to specialize the definition of transverse acceptance (definition 4.2.11) to the Euclidean distance:

Definition 4.4.3 (Euclidean acceptance)**.** *Let $Z \subset \mathbb{R}^n$ be a set of $\mathbb{R}^n$ stratified by a set Σ of strata. Let $p \in Z$ and $\rho \in \mathbb{R}^+$. Then we define the Euclidean transverse acceptance of Z at p to the fiber ρ as*

$$\tau(Z, p, \rho) = \inf_{d(x,p)=\rho} \tau(d(p, .), x),$$

where d is the usual Euclidean distance.
In other word $\tau(Z, p, \rho)$ is the infimum of the distances between the orthogonal cones associated to Z and d at a point x running on the sphere centered at p with radius ρ.

We now define the Euclidean acceptance radius. This quantity is interesting as it gives the maximal radius possible for a ball to be transverse to the variety with a given level of transversality.

Definition 4.4.4 (Euclidean Acceptance radius)**.** *Let $Z \subset \mathbb{R}^n$ be an algebraic variety which is Whitney stratified by a set of strata Σ. For any $\alpha \in [0, 1]$ and $p \in Z$ we define the Euclidean acceptance $\mathcal{T}(Z, p, \alpha)$ of Z at p with level α by*

$$\mathcal{T}(Z, p, \alpha) = \inf\{\rho \in \mathbb{R}^+ \mid \tau(Z, \sigma, \rho) \leq \alpha\}.$$

If $K \subset \sigma \in \Sigma$ is a compact subset of the stratum σ, we define the uniform Euclidean acceptance radius $\mathcal{T}(Z, K, \alpha)$ of Z along σ with level α by

$$\mathcal{T}(Z, K, \alpha) = \inf_{p \in K} \mathcal{T}(Z, p, \alpha).$$

The following lemma primarily shows that the Euclidean acceptance tends to 1 as the radius ρ goes to 0. It also gives a characterization of its continuity. An immediate corollary follows which shows that the Euclidean acceptance radius is a lower semi-continuous function along every given stratum.

Lemma 4.4.5. *Let $Z \subset \mathbb{R}^n$ be a set of $\mathbb{R}^n$ Whitney stratified by a set Σ of strata. Let $p \in Z$. Then*

- *the function $(p, \rho) \mapsto \tau(Z, p, \rho)$ is lower semi-continuous. If $\tau(Z, ., .)$ is non-vanishing on a set, it is actually continuous on it.*
- *for any $\sigma \in \Sigma$, and any sequence $p_i \in \sigma$ converging to $p \in \sigma$,*

$$\lim_{i \to \infty,\ \rho \to 0} \tau(Z, p_i, \rho) = 1.$$

Proof. Let us begin with the first point. Let $\mathcal{C}_p$ be the cone bundle induced by the convex deformation $d(p, .)$ on $\mathbb{R}^n$. As we are dealing with the Euclidean distance to a point p, $\mathcal{C}_p^{\perp}(x)$ is nothing else than $\mathbb{R}(x - p)$. Because Z has a Whitney stratification, it meets the Whitney's condition (a), that is, the limit of tangent spaces includes the tangent space at the limit point. This means that the orthogonal tangent spaces $T_x Z^{\perp}$ include the limit of the orthogonal tangent spaces $T_{x_i} Z^{\perp}$ for a sequence of points x_i converging to x. Therefore if

$\Gamma(\mathcal{C}_p^\perp(x), T_x Z^\perp)$ is non-zero, it will not vanish in a neighborhood of x. When τ is non-zero, it means that the spheres centered at p and Z are transverse, therefore the tangent space at x and the orthogonal space to the sphere at x vary continuously (with p and $x \in Z$). Therefore $\Gamma(\mathcal{C}_p^\perp(x), T_x Z^\perp)$ itself is continuous. This makes $(p, x) \mapsto \tau(d(p, .), x)$ a lower semi-continuous function that is continuous on sets where it does not vanish. Taking the infimum of this function on spheres $\{x \mid d(p, x) = \rho\}$ does not alter these properties.
Let us address the second point. Let $\mathcal{C}_i$ be the cone bundle induced on $\mathbb{R}^n$ by the Euclidean distance to the point p_i. If the second statement was false, there would be a sequence of $x_i \in Z$ such that $\Gamma(\mathcal{C}_i^\perp(x_i), T_{x_i} Z^\perp) \leq \alpha$ for some $\alpha < 1$ and for any i. This simply means that the limit of the $T_{x_i} Z$ does not contain the limit of the $\mathcal{C}_i^\perp(x_i) = \mathbb{R}(x_i - p_i)$ (we can assume that the sequences of $T_{x_i} Z$ and $\mathbb{R}(x_i - p_i)$ converge as the set x_i can be taken in the same stratum and the Grassmannians are compact sets). This is the exact contradiction of Whitney's condition (b) at p. Therefore the second point is proven. □

Corollary 4.4.6. *Let $Z \subset \mathbb{R}^n$ be Whitney stratified by Σ. For any $\sigma \in \Sigma$ and $\alpha \in [0, 1)$, the Euclidean acceptance radius with level α is a positive lower semi-continuous function along σ. In other words, the following function is positive and lower semi-continuous:*

$$\phi : s \in \sigma \mapsto \mathcal{T}(Z, s, \alpha) \in \mathbb{R}^+.$$

Proof. Let $\psi : (p, \rho) \in \mathbb{R}^n \times \mathbb{R}_*^+ \mapsto \tau(Z, p, \rho)$. For a given $s \in \sigma$, we have $\phi(s) = \inf\{\rho | \psi(s, \rho) \leq \alpha\}$. The second property of lemma 4.4.5 shows that in a neighborhood of $(\sigma \times \{0\})$ the function ψ is greater than α. This proves that ψ is a positive function in a neighborhood of $(\sigma \times \{0\})$.
By the first point of lemma 4.4.5, ψ is lower semi-continuous, this implies that $U = \psi^{-1}(]\alpha, +\infty))$ is an open set. Thus $(s, \rho) \notin U \Leftrightarrow \psi(s, \rho) \leq \alpha$ and we thus have $\phi(s) = \inf\{\rho \mid (s, \rho) \notin U\}$. To prove that ϕ is lower semi-continuous we want to prove that for any $t \geq 0$, $\phi^{-1}(]t, +\infty)$) is open. Let $s \in \phi^{-1}(]t, +\infty)$). We want to prove that there is a neighborhood of s in $\phi^{-1}(]t, +\infty)$). We have $\phi(s) > t$, thus we can choose $t' \in]t, \phi(s)[$ and consequently $(s,]0, t'[) \subset U$. As U is endowed with the product topology there is a neighborhood $V \subset U$ of $(s,]0, t'[)$ in the form of $(S,]0, t'[)$ where $S \subset \sigma$ is neighborhood of s. Thus for any $s' \in S$, $\phi(s') \geq t' > t$ which proves that $S \subset \phi^{-1}(]t, +\infty)$). □

Here we digress a bit from the subject to prove a version of the existence of a Milnor tube. The classical understanding of Milnor tube is that for a fixed stratum there is a tube (i.e. open set) around it, such that for any sphere centered on the stratum and included in the tube, the variety has conic structure in the sphere. In our version the spheres can be any type of convex set. We derive our version of the existence of Milnor tubes from the lemmas that precede:

Proposition 4.4.7. *Let $Z \subset \mathbb{R}^n$ be a set of $\mathbb{R}^n$ Whitney stratified by a set Σ of strata. Let $\sigma \in \Sigma$. Let $\mathfrak{K}$ be a convex set containing 0. Let $\gamma = N_{0,\mathfrak{K}}$ be the convex gauge of $\mathfrak{K}$ at 0 and let $\gamma_p = N_{p,\mathfrak{K}}$ be its translated gauge to a point*

$p \in \sigma$. Let $\mathfrak{R}(p,t) = \gamma_p^{-1}([0,t])$. Then there exist U an open neighborhood of σ such that for any $p \in \sigma$, if $\mathfrak{R}(p,t) \subset U$ then Z has conic structure in $\mathfrak{R}(p,t)$.

Proof. For any point x, let $d(x,.)$ be the Euclidean distance to x. Let $\alpha < 1$ be the jump from $d(0,.)$ to γ. Obviously the jump from $d(p,.)$ to γ_p is the same. For $p \in \sigma$ and $\rho = 0$, the second property of the previous lemma 4.4.5 shows that in a neighborhood of (p,ρ) the function $\tau(Z,p,\rho)$ is non-zero, therefore by the first point of the lemma it is continuous on that neighborhood. This shows that the set of points $p \in \sigma, \rho \in \mathbb{R}^+$ such that $\tau(Z,p,\rho) > \alpha$ is a neighborhood V of $\sigma \times \{0\}$ in $\sigma \times \mathbb{R}^+$. By the jump inequality (lemma 4.2.14) we can conclude that Z is transverse to $\mathfrak{R}(p,\rho)$ as soon as $(p,\rho) \in V$ and thus by the semi-algebraic moving the wall (theorem 4.1.2) we can conclude that Z has conic structure in $\mathfrak{R}(p,\rho)$.
Now we show that there is a tube U around σ such that if $\mathfrak{R}(p,t)$ is in the tube, then $(p,t) \in V$. Because $\mathfrak{R}$ is a convex set containing 0, it contains some ball $B_{0,r}$. Let $\rho(p) = \sup\{t \in \mathbb{R}^+ \mid \{p\} \times [0,t[\in V\}$. We have $\rho(p) > 0$ for all $p \in \sigma$ as V is a neighborhood of $\sigma \times \{0\}$. Let $\pi : Z \to \sigma$ be the orthogonal projection to σ that is defined in a neighborhood U' of σ. Define U as the set of points x of U' such that $d(\pi(x),x) < r\rho(\pi(x))$. Then U satisfies the condition of the proposition:
Indeed, if $\mathfrak{R}(p,t) \subset U$, as $B_{p,rt} \subset \mathfrak{R}(p,t)$ we can conclude that for all points $x \in B_{p,rt}$ we have $d(\pi(x),x) < r\rho(\pi(x))$. In particular for those points x in the orthogonal space to σ at p, that is $x \in T_p\sigma^\perp$, we have $\pi(x) = p$ and thus $d(p,x) < r\rho(p)$. As the orthogonal space to σ at p cuts the ball through its center p, it is possible to find points x_i in $B_{p,rt} \cap T_p\sigma^\perp$ such that $\lim_{i\to\infty} d(p,x_i) = rt$ the radius of the ball. Finally we have $rt < r\rho(p)$, hence $t < \rho(p)$, which proves that Z has conic structure in $\mathfrak{R}(p,t)$ by what was proven in the first paragraph of this proof. □

After this digression let us go back to our objective to outline a rather intrinsic triangulation procedure.

4.5 A tentative triangulation procedure

All the work that has been done so far enables us to relate topological structure to transversality and in turn to relate transversality to Euclidean distance. From the results on conic structure we know that to triangulate the variety it would be sufficient to partition the space into regions in which Z is "sufficiently transverse", thus has conic structure. We are going to prove that for a certain class of good pointwise "metric" approximations of the variety, the Voronoĭ partition associated to the approximation determines such a partition (proposition 4.5.7). The results of this section thus give a new proof of the triangulability of semi-algebraic sets. Here one has to be aware of the important difference that exists between proving the existence of an object and constructing it. This section not only proves the existence of triangulations for semi-algebraic sets

but aims at constructing them efficiently. As explained earlier in the introduction of this chapter, our approach promises to be more efficient than already existing methods (CAD based) thanks to the use of geometric and metric concepts instead of purely algebraic ones. However, even if the procedure that we describe is clearly intent on effectiveness and efficiency, we are still two steps away from a complete algorithm to construct triangulations. Firstly, we do not completely work out a procedure to produce the good pointwise approximations we need (although we hint at some possible directions). Secondly, and this is probably a more challenging point, we do not give a way to effectively measure the transversality of the regions of the partition, we only prove that for approximations good enough, the transversality condition will be satisfied.

We are going to approximate the variety by a set of points that will be a discrete approximation of the variety. The idea is that if the points are distributed "well enough" the topology will be recoverable from this set of points. To make this precise we need the following definitions; one of them happens to be very similar to a previously introduced definition for cones (definition 4.2.1), hence the identical notation:

Definition 4.5.1. *Let $\Gamma \subset \mathbb{R}^n$ be a set of points in $\mathbb{R}^n$.*

- *For a set S, the necessary thickening $\mathcal{H}(\Gamma \to S)$ of Γ to include S is defined as:*

$$\mathcal{H}(\Gamma \to S) := \inf\{R \in \mathbb{R}^+ \mid S \subset \cup_{p\in\Gamma} B_{p,R}\}.$$

- *For a set S, the necessary thickening $\mathcal{H}(S \to \Gamma)$ of S to include Γ is defined as: $\mathcal{H}(S \to \Gamma) := \sup_{p\in\Gamma} d(S,p)$.*
- *The sparsity $\mathcal{S}(\Gamma)$ of Γ is defined as: $\mathcal{S}(\Gamma) := \min\{d(p,q) \mid p,q \in \Gamma\}$.*

Of course the first and second items define the same function $\mathcal{H}(. \to .)$ and this is why we use the same notation. Nevertheless the definitions are given differently as we will give non symmetrical roles to Γ and S: the set of points Γ will be used to approximate the variety S. On the one hand we will choose the points in Γ so that they are close to S, and on the other hand we will aim at covering S with balls centered at points of Γ. The different definitions account for these two different perspectives.

Definition 4.5.2 (Stratified approximation)**.** *Let $\Sigma = \{\sigma_0, \ldots, \sigma_n\}$ be a Whitney stratification of a relatively compact open set $U \subset \mathbb{R}^n$ where σ_i is a pure i-dimensional submanifold or is the empty set. Let $\Gamma = (\Gamma_0, \ldots, \Gamma_n)$ be a sequence of finite sets of points in $\mathbb{R}^n$ (i.e. $\forall i \in \{0,\ldots,n\}$, $\Gamma_i \subset \mathbb{R}^n$ is finite). Let $b \in \mathbb{R}^+$, and $r_0 \geq \ldots \geq r_n \in \mathbb{R}^+$. Let $\sigma_i' = \{p \in \sigma_i \mid \forall j < i,\ d(p,\sigma_j) > r_j + b\}$. The t-uple Γ is said to be a $(b,(r_i)_i)$-stratified approximation of Σ if it satisfies to the following conditions for all $i \in \{0,\ldots,n\}$:*

1. $\mathcal{H}(\sigma_i' \to \Gamma_i) \leq b$
2. $\mathcal{H}(\Gamma_i \to \sigma_i') \leq r_i + b$

3. $\forall j < i,\ d(\sigma_j, \Gamma_i) \geq r_j + b$

4. $\mathcal{S}(\cup_{j \leq i} \Gamma_j) \geq r_i$

We will also consider Γ *as the subset* $\cup_i \Gamma_i \subset \mathbb{R}^n$ *when this proves convenient.*

Let us now comment on the previous definition. The first condition is not essential for the mathematical analysis of the situation. However its presence makes it possible for an algorithm to actually produce a stratified approximation, as if one were to set $b = 0$ the algorithm would have to deal with points that are actually on the variety with all the difficulties that this entails as was discussed earlier.
Nevertheless, we do require from the points in a "stratum" of a stratified approximation that its points be close to the corresponding stratum of the stratification of the stratified set we want to approximate. This is what the first condition stands for, and the quantity controlling this distance is b. Secondly, the points have to be close to the *whole* variety: no part of the variety has to be forgotten. This is the meaning of the second condition which is controlled by the $(r_i)_i$. Thirdly, the stratified approximation should follow the stratification of $\mathbb{R}^n$ in the sense that the higher "strata" of the approximation stay away from the lower strata of the stratification of $\mathbb{R}^n$. The final fourth condition imposes that the points have to be fairly distributed on the variety and not accumulate at specific spots. This insures that when splitting $\mathbb{R}^n$ into regions, these regions will be fairly round shaped. In the words of the previous subsections, this means that the jump from the convex gauge of the regions to the Euclidean distance is controlled by the $(r_i)_i$.
The reader may have noticed that conditions (1) and (2) make use of the σ_i''s instead of the σ_i's, so that it may seem as though we are not actually enforcing that the points are close to the whole variety. The next short lemma shows that, in fact, the stratified approximation has points close to the whole of the σ_i.

Lemma 4.5.3. *Let* Γ *be a* $(b, (r_i)_i)$*-stratified approximation of a stratification* $\Sigma = (\sigma_0, \ldots, \sigma_n)$ *of* $\mathbb{R}^n$*. Then for any point* $p \in \mathbb{R}^n$ *such that* $d(p, \sigma_k) \leq r_k + b$ *(for some* k*), there exists* $q \in \Gamma_j$ *(for some* $j \leq k$*) that satisfies the inequality*

$$d(p, q) \leq b + r_j + \sum_{i=j}^{k} (b + r_i).$$

We also have a fortiori $d(p, q) \leq b + r_0 + \sum_{i=0}^{n} (b + r_i)$.

Proof. If $k = 0$, then σ_0 is a finite set of points. Therefore there exists a point $s \in \sigma_0$ such that $d(p, s) = d(p, \sigma_0)$ and by hypothesis $d(p, \sigma_0) \leq b + r_0$. For $k = 0$, $\sigma_0 = \sigma_0'$, thus, by definition of Γ_0 (condition 2) there exists $q \in \Gamma_0$ such that $d(q, s) \leq r_0 + b$. Finally the claim for $k = 0$ is proved by the triangle inequality:

$$d(p, q) \leq d(p, s) + d(s, q) = 2r_0 + 2b = b + r_0 + \sum_{i=0}^{0} (b + r_i).$$

Suppose inductively that the result is true for strata σ_j for every $j < k$. We prove the result for k. Let $s \in \overline{\sigma_k}$ be (one of) the closest neighbor(s) of p in $\overline{\sigma_k}$, the closure of σ_k. By hypothesis $d(p,s) \leq r_k + b$. We distinguish the cases $s \in \sigma'_k$ and $s \notin \sigma'_k$. If $s \in \sigma'_k$, by definition of Γ_i (condition 2) there exists a point $q \in \Gamma_k$ such that $d(q,s) \leq r_k + b$. By the triangle inequality we have:

$$d(p,q) \leq d(p,s) + d(s,q) \leq 2r_k + 2b = b + r_k + \sum_{i=k}^{k}(b + r_i).$$

If $s \notin \sigma'_k$, then by definition of σ'_k there exists $j < k$ such that $d(p, \sigma_j) \leq r_j + b$. Hence by the induction hypothesis, there exists $q \in \Gamma_i$ (for some $i \leq j$) such that $d(q,s) \leq b + r_i + \sum_{l=i}^{j}(b + r_l)$. By the triangle inequality

$$d(p,q) \leq d(p,s) + d(s,q) \leq b + r_k + b + r_i + \sum_{l=i}^{j}(b + r_l).$$

Finally as $j < k$ we have $b + r_k + \sum_{l=i}^{j}(b + r_l) \leq \sum_{l=i}^{k}(b + r_l)$ and thus

$$d(p,q) \leq b + r_i + \sum_{l=i}^{k}(b + r_l).$$

□

The next proposition states the existence of stratified approximations of arbitrary precision.

Proposition 4.5.4. *Let $\Sigma = \{\sigma_0, \ldots, \sigma_n\}$ be the set of strata of a Whitney stratification of a relatively compact open set $U \subset \mathbb{R}^n$ where σ_i is pure dimensional of dimension i or is empty. For any $b \in \mathbb{R}^+$ and any sequence $r_0 \geq \ldots \geq r_n \in \mathbb{R}^+$ there exists a $(b, (r_i)_i)$-stratified approximation Γ of Σ.*

Proof. Consider $\Sigma_k = (\sigma_0, \ldots, \sigma_k, \emptyset, \ldots, \emptyset)$ the truncation at σ_k of Σ. We set $\Sigma_{-1} = (\emptyset, \ldots, \emptyset)$. We proceed by induction by assuming that we have managed to produce a $(b, (r_i)_i)$-stratified approximation of Σ_{k-1}. The initialization of the induction is trivial as $\Gamma = (\emptyset, \ldots, \emptyset)$ is a suitable approximation of Σ_{-1}.
We set the Γ_j for $j < k$ to be the same as the Γ_j in the previous step of the induction. We also set the $\Gamma_j := \emptyset$ for $j > k$. We are left with the task to define Γ_k. As U is relatively compact, $s := \overline{\sigma'_k}$, the closure of σ'_k, is compact. We can construct the sequence $(s_j)_j$ of subsets of s by the following induction: $s_0 := s$, choose a point x_j in s_j and let $s_{j+1} = s_j - B(x_j, r_k)$, this until $s_l = \emptyset$ ultimately. The fact that the sequence reaches $s_l = \emptyset$ in a finite number of steps results from the fact that s is compact: because the distance between one x_j and another is always greater than r_k it is not possible to have infinitely many of them in a compact set. We set Γ_k to be the set of the x_i chosen in the previous induction to create $(s_j)_j$.
By induction the conditions in the definition of stratified-approximation are all

satisfied for $i < k$. The conditions for Γ_j $(j > k)$ are also trivially satisfied as $\Gamma_j = \Sigma_j = \emptyset$ from rank $k+1$ and on. Therefore we only have to verify that the four properties in the definition of stratified approximation hold for the rank $i = k$. The first condition is trivially satisfied as the points here are simply inside $\overline{\sigma'_k}$ and thus $\mathcal{H}(\sigma'_k \to \Gamma_k) = 0$. The second condition is clearly satisfied as well because $\emptyset = s_l = \overline{\sigma'_k} - \cup_{j=0}^{l-1} B(x_j, r_k)$, therefore σ'_k is *a fortiori* contained in $\cup_{j=0}^{l-1} B(x_j, r_k + b)$. The third condition is a straightforward consequence of the definition of σ'_k. This set is defined to stay at a distance $r_j + b$ of the σ_j $(j < k)$, and as the $x_j \in \Gamma_k$ are in its adherence, their distance to the σ_j's is also greater than $r_j + b$. As for the fourth condition, the distance between points in $\cup_{j<k}\Gamma_j$ is greater than r_{k-1} by induction, and as $r_k \leq r_{k-1}$ it is greater than r_k (if $k = 0$ this part of the argument should be omitted). Consequently we only have to control the distance between the points of Γ_k and those of Γ_i, as well as the distance of the points of Γ_k between themselves. The third condition for stratified approximations ensures that the distance between a point of Γ_k and σ_j $(j < k)$ is greater than $r_k + b$. As the first condition ensures that the points of Γ_j are within a distance b of σ_j, this proves that the points of Γ_j are at a distance at least r_k from Γ_k. Finally the distance between two points of Γ_k is more than r_k as every x_j is chosen in s_j which consists of points of $\overline{\sigma'_k}$ which are at least at a distance r_k of the previous x_i $(i < j)$. By induction we have thus proved that for $k = n$ there exists a $(b, (r_i)_i)$-stratified approximation of $\Sigma = \Sigma_n$. □

It is our interest to be able to effectively generate such stratified approximations. The above proof is interesting inasmuch as it proves that the approximations exist, but it is of no use when it comes to computationally constructing an approximation. As a matter of fact, b plays no role in it. We have not yet worked out a certified and efficient algorithm to generate such a stratified approximation. We briefly suggest two different techniques that could be utilized in such an algorithm. Subdivision methods have been discussed independently under several variations in this thesis. They give a certified output while retaining a good efficiency. They would allow us to generate stratified approximations provided that we have a representation of the Whitney stratification by radical ideals. The other method we would like to suggest is the so-called "sampling" methods mentioned in the introduction (see [143]). They consist in spraying random points in $\mathbb{R}^n$ and making them evolve by following the gradients of the defining functions of the ideals of the variety. Certifying the accuracy of their results seems *a priori* a challenge but the method is in nature bound to be faster than the subdivision methods. Their speed is thus what makes them appealing. We refer to [121] for an example of the usage of this method.

We shall now explain how the existence of such stratified approximations gives a means to triangulate the variety. We first introduce the definition of Voronoĭ partitions which are the type of partitions we will use to partition the space into regions where the variety has conic structure:

Definition 4.5.5 (Weighted Voronoĭ Partition)**.** *Let $\Gamma \subset \mathbb{R}^n$ be a discrete set of*

points in $\mathbb{R}^n$ and $W : \Gamma \to \mathbb{R}$ any function. We call W the weight function. The weighted Voronoï partition associated to Γ and W is the family $V^W_{\gamma\in\Gamma}$ defined by

$$V^W_\gamma = \{p \in \mathbb{R}^n \mid \forall q \in \Gamma,\ W(q)d(p,q) > W(\gamma)d(p,\gamma)\}.$$

Definition 4.5.6 (Accurate stratified approximation)**.** *For any b, $(r_i)_i$ and stratified approximation Γ of a stratified set Z, we define the weight function $W : p \in \Gamma_j \mapsto r_j^{-1}$. We call the standard partition associated to Γ, the weighted Voronoï partition V^W_Γ associated to Γ and W. We say that Γ is an accurate stratified approximation for Z iff Z has conic structure in every region of the standard partition associated to Γ.*

We can now prove the final proposition:

Proposition 4.5.7. *Let $Z \subset \mathbb{R}^n$ be a semi-algebraic* **compact** *variety which is Whitney stratified by a set of strata $\Sigma = (\sigma_0, \dots, \sigma_d, \emptyset, \dots, \emptyset)$. Then there exists $b_0 > 0$ and $(r_0, \dots, r_n)$ such that for all $b \in [0, b_0]$, and for any $(b, (r_i))$-stratified approximation Γ, Γ is an accurate stratified approximation for Z.*

Proof. This proof goes in three parts. Part I gives a sufficient condition for the regions $\mathfrak{R}$ to have conic structure. Part II gives a condition on stratified approximations to satisfy the requirement given in part I. Finally part III shows that the conditions of part II will be satisfied for some $b, (r_i)_i$ as specified in the theorem.

Part I: Let $d(p,.)$ the Euclidean distance to p. For a given convex region $\mathfrak{R}$, let $J = \mathcal{H}\left(d(p,.) \to N_{0,\mathfrak{R}}\right)$ be the jump from $d(p,.)$ to the convex gauge of $\mathfrak{R}$ at p. We show that to prove the proposition it suffices to prove that $\mathbb{R}^n$ is subdivided in regions $\mathfrak{R}$ such that $\mathfrak{R} \subset B_{p,\mathcal{T}(Z,p,J)}$ for some $p \in \mathfrak{R}$.
Indeed, this condition means that the transversality to the Euclidean distance in the ball $B_{p,\mathcal{T}(Z,p,J)}$ is greater than the jump J. By the jump inequality (lemma 4.2.14), the convex gauge of $\mathfrak{R}$ will be transverse to Z in that ball and by the semi-algebraic moving the wall theorem (4.1.44) Z will have conic structure in $\mathfrak{R}$.

Part II: We define

$$C(R,r,b) = \frac{R - r + 2b}{R + b} \text{ and } K_k = 2(n+2)r_k.$$

Let Γ be a $(b, (r_i)_i)$-stratified approximation for $\Sigma = (\sigma_0, \dots, \sigma_d, \emptyset, \dots, \emptyset, \mathbb{R}^n - Z)$. Recall that $W : \Gamma \to \mathbb{R}$ is the weight function such that for $p \in \Gamma_i$, $W(p) = 1/r_i$ ($i \in \{0, \dots, n\}$). Let $\mathcal{R}$ be the standard Voronoï partition associated to Γ (considered as a set of points in $\mathbb{R}^n$). We prove that for the regions of $\mathcal{R}$ to meet the condition of part I, it suffices that for any $k \in \{0, \dots, n\}$ and $p \in \Gamma_k$, there exists $q \in \sigma'_k$ such that $d(p,q) \leq b$ and

$$K_k + b \leq \mathcal{T}\left(Z, q, C(K_k, \frac{r_k}{2}, b)\right) \tag{4.1}$$

Firstly, let us prove that the region $\mathfrak{R}$ associated to p contains $B(p, \frac{r_k}{2})$. It suffices to prove that if a point x is in this ball, its closest neighbor is p (by

definition of weighted Voronoĭ partitions this will mean that $x \in \mathfrak{R}$). We reason by way of contradiction. If there were a point p' closer to x than p, then we would have $d(x,p') < r_k$ and by the triangle inequality $d(p,p') < r_k$. It is not possible that $p' \in \Gamma_i$ $(i \leq k)$ as it would contradict the fourth condition that defines a $(b,(r_i)_i)$-stratified approximation. It is not possible that $p' \in \Gamma_i$ $(i > k)$ either, because condition (3) guarantees that $d(\sigma_k, p') \geq r_k + b$, but $d(\sigma_k, p) \leq b$, thus $d(p,p') \geq r_k$ (which contradicts $d(p,p') < r_k$ that we have established before). Secondly, let us show that the region $\mathfrak{R}$ is contained in $B(p,K_k)$. Let s be a point in this region. The definition of weighted Voronoĭ partition tells us that for any $j \in \{0,\ldots,n\}$ and $p' \in \Gamma_j$ we have:

$$\frac{d(s,p)}{r_k} \leq \frac{d(s,p')}{r_j} \tag{4.2}$$

Since $(\sigma_0,\ldots,\sigma_d,\emptyset,\ldots,\emptyset,\mathbb{R}^n - Z)$ is a stratification of the whole space, by definition of the σ_i''s we know that s lies within a distance $r_i + b$ of some stratum σ_i'. By lemma 4.5.3 we have for any $j \in \{0,\ldots,n\}$ and $p' \in \Gamma_j$:

$$\begin{aligned}
d(s,p') &\leq b + r_j + \sum_{l=j}^{i}(b + r_l) \\
&\leq b + r_j + \sum_{l=j}^{n}(b + r_l) && \text{by adding more positive terms} \\
&\leq (n-j+2)(b+r_j) && \text{because the } r_i \text{ decrease} \\
&\leq 2(n+2)r_j && \text{because } b \leq r_j \text{ and } (n-j+2) \leq (n+2).
\end{aligned}$$

By combining the inequalities we have obtained we can conclude that:

$$\begin{aligned}
d(s,p) &\leq \frac{r_k}{r_j} d(s,p') && \text{by equation (4.2)} \\
&\leq 2(n+2)r_k = K_k && \text{by the previous inequality.}
\end{aligned}$$

This exactly tells us that $s \in B(p,K_k)$. Hence, we can finally conclude that

$$B\left(p, \frac{r_k}{2}\right) \subset \mathfrak{R} \subset B(p,K_k).$$

We have that $d(p,q) \leq b$, and by lemma 4.4.2 in the previous section we know that the jump from the convex gauge of $\mathfrak{R}$ at q to the Euclidean distance to q is less than $C(K_k, \frac{r_k}{2}, b)$. Because $d(p,q) \leq b$, we have the inclusion $\mathfrak{R} \subset B(q, b + K_k)$. Therefore if equation (4.1) is satisfied we will have

$$\mathfrak{R} \subset B(q, K_k + b) \subset B\left(q, \mathcal{T}\left(Z, q, C(K_k, \frac{r_k}{2}, b)\right)\right).$$

And the previous inclusion corresponds to the condition required by part I.

Part III: We show the existence of b_0 and $(r_i)_i$ such that for all $b \in [0, b_0]$, any $(b, (r_i)_i)$-stratified approximation Γ meets the conditions given in Part II. This will finalize the proof of the proposition.
For convenience let us define

$$L(b, r_k) = C(K_k, \frac{r_k}{2}, b) = \frac{(4n+7)r_k + 4b}{(4n+8)r_k + 2b}.$$

As Z is compact, for given $b, (r_i)_i$, the σ_i''s are relatively compact sets. The function $\mathcal{T}(Z, ., L(b, r_k))$ is known to be lower semi-continuous on strata by corollary 4.4.6, therefore it reaches its minimum $\mathfrak{m}_k$ on every $\overline{\sigma_k'}$. This minimum is non-zero as $\overline{\sigma_i'} \subset \sigma_i'$, and by lemma 4.4.5, at any point $q \in \sigma_i'$, $\tau(Z, q, \rho) \to 1$ as $\rho \to 0$. Notice that $L(b, r_k)$ is a decreasing function of r_k and an increasing function of b. Therefore for a fixed q, $\mathcal{T}(Z, q, L(b, r_k))$ is an increasing function of r_k and a decreasing function of b. This means that if the r_j $(j < k)$ are fixed, $\mathfrak{m}_k$ will increase with r_k and decrease with b.
We are going to build suitable r_i's by induction. Notice that for a given stratum σ_k, the condition of part II equation (4.1) only involves b and r_k. So we can choose r_k stratum by stratum starting from the 0-dimensional one and moving up while reducing b_0 as necessary at each step. The initial step of the induction is trivial, we fix b_0 arbitrarily and we fix no r_i. The induction hypothesis is that we have already fixed a b_0 and $r_0, \ldots, r_{k-1}$, and that in addition, for any $b \leq b_0$, equation 4.1 is satisfied for any stratum σ_i' $(i \leq k-1)$. Let $\alpha \in]L(0,1), 1[$. It is well-defined as for any $r_k > 0$ we have:

$$L(0,1) = L(0, r_k) = \frac{(4n+7)}{(4n+8)} < 1.$$

The curve $\mathcal{C}$ defined by $L(b, r_0) = \alpha$ is a line through 0 that crosses the positive quadrant (that is, $r_k > 0$, $b > 0$). We have already fixed $r_0, \ldots, r_{k-1}$, therefore $\mathfrak{m}_k$ only depends on $\mathcal{T}(Z, ., \alpha)$ as the domain σ_k' on which the infimum is taken is fixed. Because on $\mathcal{C}$, $L(b, r_k)$ is constantly equal to α, $\mathfrak{m}_k$ is constant along this curve. Therefore we can choose (b, r_k) on $\mathcal{C}$ so that b is the maximal value for which we have $2(n+2)r_k + b \leq \mathfrak{m}_k$ (this is well-defined because $\mathcal{C}$ intersects the positive quadrant). If this b_0 is already less than b we leave it unchanged. If b is less than b_0, we set $b_0 := b$. Since $\mathfrak{m}_k$ is the minimum of $\mathcal{T}$ over σ_k', this shows that for this b and r_k, equation (4.1) holds. As mentioned earlier, since we have fixed the $r_0, \ldots, r_{k-1}$, $\mathfrak{m}_k$ is decreasing with b, therefore equation (4.1) will still hold for σ_k' for any $b \in [0, b_0]$. Notice that in addition to the fact that equation (4.1) is satisfied as a point $q \in \sigma_k'$, part II requires that $d(q, p) \leq b$. This holds thanks to the first condition of $(b, (r_i)_i)$-stratified approximations. Finally, the induction hypothesis tells us that equation (4.1) will also hold for any stratum σ_i' $(i < k)$ and for any $b \in [0, b_0]$ (b_0 has only been reduced). This finishes the proof of the induction step and completes the proof of the proposition. □

Remark 4.5.8. *One can wonder whether the constant*

$$L(b, r_k) = C(K_k, \frac{r_k}{2}, b) = \frac{(4n+7)r_k + 4b}{(4n+8)r_k + 2b}$$

that appears in the previous theorem is intrinsic in any way to the variety Z. It is not. It proceeds from a drastic simplification of the bound from lemma 4.5.3 in part II. This simplification was made in order to simplify the argument. However if one were to keep the bound from lemma 4.5.3 unsimplified to define K_k, the argument of part II would work in exactly the same way. In addition, we can consider the $\mathfrak{m}_k$'s as functions of b, α (the acceptance level) and the r_j's ($j < k$) and look at the combinations of those parameters which satisfy equation (4.1). The set of such parameters characterizes intrinsically when the stratified approximations are sufficiently accurate to describe Z (that is, Z has conic structure in the regions of the associated weighted Voronoĭ partition). This set thus characterizes intrinsically the precision required to describe Z by means of a discrete approximation.

Let us finally sum up what the triangulation procedure would be if the previous proposition was made effective. That is, if we had an algorithm that, given a variety Z, generates an accurate stratified approximation for Z (possible solutions are sample methods or subdivision methods as explained earlier in this section). With the stratified approximation given by the algorithm we can construct its associated standard partition (there are very efficient algorithms for constructing weighted Voronoĭ partitions). As weighted Voronoĭ partitions define piecewise linear regions, we can recursively reapply the algorithm on all the faces of the regions until we arrive at 0-dimensional sets. We have applied the lemma recursively to go down from $\dim Z$ to 0. Now we build up a triangulation by recursively creating cones over the boundaries starting from dimension 0 and moving up to $\dim Z$. As taking the cone over a piecewise linear set gives a piecewise linear cone, and 0-dimensional sets are piecewise linear, we have finally created a triangulation for Z as a whole.
To conclude, the previous procedure proves again that semi-algebraic sets are triangulable as we have proved that accurate stratified approximations always exist (by combining propositions 4.5.4 and 4.5.7). This different approach is more straightforward than other existing proofs, which could lead to much more efficient algorithms. However the lack of control on $\mathcal{T}(Z,.,.)$ so far does not allow us to write a complete algorithm. Numerical techniques based on evaluations of the defining functions of Z combined with an algebraic description of the stratification of Z and the blow-ups along the strata may be prove useful to control $\mathcal{T}(Z,.,.)$.

Chapter 5

A sweeping method to triangulate surfaces in three-space

This chapter presents an algorithm for triangulating a real algebraic variety S in a ball $B \subset \mathbb{R}^3$, even in singular cases. As we are working over the reals, such varieties can always be represented by a single polynomial, which is the input of the algorithm. We use algorithms for 2D and 3D algebraic curves and show how one can compute a simplicial complex equivalent to S, and even a simplicial complex isotopic to S by exploiting properties of the contour curve of S (definition 5.1.3). The correctness proof of the algorithm is based on results from stratification theory. The strategy consists in constructing an explicit Whitney stratification of S by resultant computation, and then using Thom's isotopy lemma to deduce the topology of S from a finite number of characteristic points on the surface. An analysis of the complexity of the algorithm and effectiveness issues conclude the chapter.

The classification of singularities [11] provides simple algebraic formulas for complicated shapes, which may be difficult to handle geometrically.

The triangulation can subsequently be utilized for a variety of purposes such as geometric modeling, approximation and simulation purposes, computing the number of connected components, and computing the Betti numbers. For instance, the problem of determining the connected components of a semi-algebraic set and a path between two points of the same connected component has been investigated in [32, 134]. In those articles, polar varieties of the semi-algebraic set and non-explicit Whitney stratifications were used to define so-called roadmaps which provide a path between two given points. Interestingly, we make use of those same tools.

The special case of varieties in $\mathbb{R}^3$ has already received a lot of attention: we refer in particular to [61], [59], [26], [2] (this is chapter 3), but these pieces of work deal only with smooth surfaces. In this context it is also interesting

to mention again the Cylindrical Algebraic Decomposition algorithm [36, 33]. It consists in decomposing a semi-algebraic set S into cells, defined by sign conditions on polynomial sequences. Such polynomial sequences are obtained by (sub)-resultant computations, corresponding to successive projections from $\mathbb{R}^{k+1}$ to $\mathbb{R}^k$. Once again, there is a parallel with our algorithm: we will make use of resultants to compute a Whitney stratification of the variety.

Our aim here is to describe an effective (and efficient) method for the triangulation of the part of a real algebraic surface S of $\mathbb{R}^3$ that lies inside a sphere. It can be generalized to other bounding shapes than spheres such as boxes, but for the sake of clarity we will stick to a spherical bounding shape throughout the rest of the chapter. The method is based on the computation of characteristic points on this surface. As we will see, it requires the computation of $\mathcal{O}(d^7)$ points. We follow a sweeping plane approach and exploit the following idea: after choosing a generic sweeping plane direction, the topology of the sections of the surface with this plane only changes for a discrete set of positions C. Computing this set of critical values (or more precisely a sup-set $C' \supset C$) and the topology of the sections at these critical values, will allow us to recover the topology of the surface. For this purpose, we will use the contour curve of S, which is a 3D curve on S (definition 5.1.3). Our approach exploits results from stratified Morse theory. We give an explicit Whitney stratification of S, involving resultant computation and prove its correctness using equi-singularity arguments. This ensures the cylindrical structure of the surface between the critical sections that we have computed and yields a way to connect them, by "following" the contour curve.

This chapter is organized as follows. In the next section, we recall basic definitions and describe the set of interesting points on the surface that we use to deduce its topology. In section 5.2, we describe how we treat the critical section of surface. In section 5.3, we describe how we compute the topology of the polar curve. In part 5.4, we describe the algorithm for surfaces and in particular how to connect two consecutive sections and obtain a triangulation of this connection while keeping safe the topology. We will see, in particular, how a discrete description of the polar variety allows us to recover the two dimensional faces of a triangulation of the surface. In section 5.5, we prove the correctness of the algorithm, showing as a new result, how a resultant computation yields a Whitney stratification of the surface. The proof of correctness of the connection algorithm is given and the isotopy between the surface S and its triangulation is made explicit. An example is given in section 5.6. Finally, we detail effectiveness and complexity issues in section 5.7.

5.1 Notations

We consider an algebraic surface S defined by the equation $f(x, y, z) = 0$ (with $f \in \mathbb{R}[x, y, z]$) in a given ball B for the Euclidean distance (instead of a ball B, we could also consider a box, but the description of the method is less simple). Hereafter, to simplify the presentation, we will assume that the boundary of B

is not included in S. We denote by $S_B = S \cap B$ the intersection of S with the closed ball B. Our objective is to compute a simplicial complex, isotopic to the surface S_B.

We denote by π_y (resp. π_z, $\pi_{y,z}$), the projection of $\mathbb{R}^3$ along the y direction (resp. the z direction, the y, z plane) on the x, z plane (resp. the x, y plane, the x axis). An (x, y) plane section of a variety V of $\mathbb{R}^3$ will be the intersection of V, with a plane parallel to the (x, y) plane (and similarly for the other variables).

A point $p \in V$ at which ∇f vanishes is called a *singular* point of V, where $\nabla(f) = [\partial_{x_1}(f), \ldots, \partial_{x_n}(f)]$.

The notion of critical point for a projection is a key notion of our approach, this is how we define it.

Definition 5.1.1 (Singular and Critical Points)**.** *Let V be an algebraic variety in $\mathbb{R}^n$ defined by equations $f_1 = 0, \ldots, f_s = 0$ and let $p \in V$.*
The point p is said to be singular *iff*

$$\dim_{\mathbb{R}} \mathrm{Vect}(df_1, \ldots, df_s) < n - \dim(V),$$

where $\dim_{\mathbb{R}}(E)$ denotes the dimension of E as an $\mathbb{R}$-vector space, and $\dim(V)$ denotes the Krull dimension of the variety V.
Let $\pi : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. The point p is said to be a critical *point of π iff p is singular or*

$$\ker(df_1, \ldots, df_s) \subset \ker \pi.$$

A point $p \in \mathbb{R}^3$ of an algebraic variety $V \subset \mathbb{R}^3$ is x-critical (resp. (x, y)-critical) if it is critical for the projection $\pi_{y,z}$ (resp. π_z) on the x axis (resp. (x, y) plane). If $V \subset \mathbb{R}^3$ is defined by the polynomial equations $f_1 = 0, \ldots, f_s = 0$, an x-critical point of V is either a singular point or a point where the tangent space of V at this point is in a plane parallel to the (y, z)-plane i.e the multiplicity of the intersection of the plane with the ideal $(f_1, \ldots, f_s)$ at p is greater or equal to 2. The corresponding x-coordinate of p is called an *x-critical value*. If a value is not x-critical, it is called x-regular. We use similar notations for the other variables.

5.1.1 The contour curve

Hereafter, we will use the properties of the contour curve of $S_B = S \cap B$. The contour curve is in fact a polar curve of S augmented with information to take into account the interference of S with the ball B.

Definition 5.1.2 (Polar curve)**.** *The polar curve of S for the projection π_z in the z-direction is the locus of the critical points of S for the projection along the direction z.*

If S is defined by $f(x, y, z) = 0$, this polar curve is defined by the equations $f(x, y, z) = \partial_z f(x, y, z) = 0$.

In order to take into account the restriction of S to B, we use the following definition for the contour curve:

Definition 5.1.3 (Contour curve)**.** *We denote by $\mathfrak{C}_z(S_B)$ the union of*

- *the set of points $p \in B$ on the polar curve of S in the z-direction,*
- *the intersection of S with the boundary ∂B of the ball B.*

In other words $\mathfrak{C}_z(S_B) = (V(f, \partial_z f) \cap B) \cup (V(f) \cap \partial B)$. We will call it the contour curve of S.

The equations of the intersection of S with the boundary of B are $f(x, y, z) = 0$ and $q(x, y, z) = 0$, where q is the quadratic polynomial of the sphere associated to B. How to compute the topology of the contour curve is described in section 5.3. If we had used a box instead of a ball for the domain B, it would have been necessary to use the restrictions of $f(x, y, z)$ to the faces of the box B and the 2D algorithm (see section 5.2) to compute the topology of the corresponding planar curves.

5.1.2 Characteristic points on the surface

Let $f \in \mathbb{R}[x, y, z]$ be a square-free polynomial and let $S = V(f)$ be the surface it defines. Let $q(x, y, z)$ be the quadratic polynomial associated to the ball B. We denote by $R(x, y) = \mathrm{Res}_z(f(x, y, z), q(x, y, z)\, \partial_z f(x, y, z))$ and by $\Delta(x, y)$ its square-free part. Let $\mathcal{C}_{x,y} \subset \mathbb{R}^2$ be the planar curve defined by $\Delta(x, y) = 0$. For any $x_0 \in \mathbb{R}$, the points of $\mathcal{C}_{x,y} \cap V(x - x_0)$ are the projections on the (x, y)-plane of the points of $S \cap V(x - x_0)$ that are either singular or smooth with a vertical tangent or in ∂B.

The algorithm for computing the topology of S will isolate the real solutions of the following system in $\mathbb{R}^5$:

$$\begin{cases} \Delta(x, y') = 0 \\ \partial_y \Delta(x, y') = 0 \\ f(x, y, z') = 0 \\ q(x, y, z')\, \partial_z f(x, y, z') = 0 \\ f(x, y, z) = 0 \end{cases} \tag{5.1}$$

We denote by $\Xi(f)$ the set of real solutions of this system. This system can be assumed to be 0-dimensional over the complex field (since $V(q)$ is not in S) as we can perform a change of coordinates to put it in general position. An alternative way to say this, is that one can use a coordinate system different from x, y and z. The invariance of the sphere under rotations makes this step easy, but there is no substantial obstruction to developing the same algorithm with another bounding shape provided one is able to take into account the effect of the coordinate change on the bounding shape.

Notice that if $(\alpha, \beta, \gamma, \beta', \gamma') \in \Xi(f)$, then

- (α, β, γ) is a point on S,
- (α, β, γ') is a point on the contour curve of S,

- (α, β') is a critical point of $\mathcal{C}_{x,y} \subset \mathbb{R}^2$ for the projection to the x-axis.

We associate to a solution $(\alpha, \beta, \gamma, \alpha', \beta')$ of the system (5.1) the following index:

- x if $\gamma = \gamma'$ and $\beta = \beta'$,
- c if $\gamma = \gamma'$ and $\beta \neq \beta'$,
- r otherwise.

A point $(\alpha, \beta, \gamma, \alpha', \beta')$ has index x if and only if (α, β, γ) is a point of the contour curve of S, which projects onto an x-critical point of $\mathcal{C}_{x,y}$.

A point $(\alpha, \beta, \gamma, \alpha', \beta')$ has index c if and only if (α, β, γ) is a point of the contour curve $\mathcal{C}$ of S, which projects onto a regular point of $\mathcal{C}_{x,y}$. Thus it is also smooth on $\mathcal{C}$.

For a point $(\alpha, \beta, \gamma, \alpha', \beta')$ with index r, (α, β, γ) is a smooth point of S, on the same vertical line as a point of $\mathcal{C}_{x,y}$ but not on the contour curve.

The intersection of the surface S with a plane $x = \alpha$ where α is the first coordinate of a solution of the system (5.1), will be called an x-critical section of S (at $x = \alpha$) and denoted by S_α.

5.2 Topology of the x-critical sections

In this section, we describe how we compute the topology of the x-critical section $S_\alpha = S \cap V(x - \alpha)$ at an x-critical value α. We use the subdivision approach presented in [4] to determine the topology of S_α. In the following we outline briefly the strong points of the method and how it proceeds.

The algorithm works on a square-free polynomial and $f(\alpha, y, z)$ will always be square-free in our algorithm. Otherwise the contour curve $\mathcal{C}$ would contain the x-critical section S_α, but this can't happen in the generic positions we allow (see definition 5.4.1 later to see how we enforce that condition). To make explanations clearer in the rest of this section we drop the first component of f, and consider it as a function in the y and z variables (i.e. "$f(y, z) = f(\alpha, y, z)$"). The algorithm works by covering the disk $(V(x - \alpha) \cap B)$ in which we want to triangulate $f = 0$ with rectangular boxes in which we know how to compute the topology. An important feature of this subdivision approach is that, unlike sweeping methods, it does not require any genericity assumptions, and will work in the given coordinate system, taking advantage of the potential sparsity of its input. This feature is important for our usage because when cutting over a singular point of $\Delta(x, y)$ there is no need for an additional change of variables (over such points there are often two singular points with the same y-coordinate).

The two categories of boxes for which we can determine the topology are the following:

- Regular boxes where either $\partial_y f(\alpha, y, z)$ or $\partial_z f(\alpha, y, z)$ does not vanish.

- Star-singular boxes where the topology in the box is star like. This means that to triangulate the portion of curve inside the box it suffices to pick any point in the interior of the box and to connect by a straight line to all the points of the curve that lie on the boundary of the box.

For the algorithm to be complete, we need to explain how we can effectively cover the disk with such boxes and explain how we manage to recover the topology for a box when it falls into one of the two above categories.

In the first step, we consider the points of $\Xi_\alpha(f)$ of index c, x and refine their isolating boxes until all the extremal points of f which are not on S lie outside the box. To determine that a box is star-singular we use the following criterion:

Lemma 5.2.1 (Star-singular box)**.** *Let* $\deg(F_1, F_2, D)$ *denote the topological degree of a continuous map* $F : \mathbb{R}^2 \to \mathbb{R}^2$ *in a connected domain* $D \subset \mathbb{R}^2$ *[87]. If* D *is a box containing a singular point* p *such that* p *is the only extremal point of* f *in the box (i.e.* $\forall q \in D,\ \partial_y f(q) = \partial_z f(q) = 0 \Rightarrow p = q$ *and* $f(p) = \partial_y f(p) = \partial_z f(p) = 0$*) and if in addition the number of zeros of* f *on the boundary of* D *is* $2(1 - \deg(\partial_y f, \partial_y f, D))$*, then the topology in* D *is star like.*

This stems from the fact that $2(1 - \deg(\partial_y f, \partial_y f, D))$ is the number of half-branches at p [81]. Computing the topological degree is made possible by a formula that expresses it as a function of the values of f on the boundary of D, therefore it is possible to compute it using univariate solving on the segments of the boundary of D.
The second step is then to refine the isolation of the singular points until the topological degree in the box matches the number of zeros of f on the boundary.
The third step is quite straightforward. We refine the star-singular boxes so that they do not intersect the boundary of the disk ($V(x - \alpha) \cap B$) (the region in which we want to triangulate $V(f)$). Then we create a subdivision that contains all the isolation boxes we have computed so far, and we end up with a set of regular boxes and star-singular boxes. If a singular point unluckily happens to be on the boundary of the disk, it is not a problem as it is possible to handle this case by counting the number of half-branches that lie inside the disk.
In the final step, we compute the topology in star-singular boxes by connecting a point inside the box to the point of the curve on the boundary. For regular boxes, we explain how the connection algorithm works if $\partial_z f$ does not vanish, the treatment of the case where $\partial_y f$ does not vanish is symmetrical. If $\partial_z f$ does not vanish, then there is no vertical tangent, therefore we can orient the curve segments that lie in the box from left to right (i.e. according to their y component). This gives a formal meaning to "entering" the box (leftmost endpoint) and "exiting" the box (rightmost endpoint). Finally, because the curve segments cannot intersect each other, if we take the leftmost exit point, the corresponding entry point has to be the first point encountered to its left on the boundary of the box (because there are only entry points to its left). So we just connect these two points together, remove them from the list of points to be connected, and repeating this process recursively eventually gives the topology of the curve in the box. For more details, see [4].

5.3 Topology of the contour curve

The algorithm to compute the topology of the contour curve, exploits the 2D algorithm [4] described in the previous section, combined with the algorithm in [63]. We use two projections of the 3D curve to recover the connection of the points above these projected planar curves and the points in $\Sigma(f)$ to analyze the critical points of the projected curves. The restriction in [63] that $(f, \partial_z f)$ has to be a radical ideal can be removed, since we deduce the critical points of the 3D curve from the points of $\Sigma(f)$.

Other approaches can be used here to compute the topology of the 3D contour curve. One can use for instance the algorithm in [8] (if $(f, \partial_z f)$ is radical), the main difference being the genericity conditions which are required and the technique used to lift points from the (x, y) or (x, z) plane to 3 dimensional space. In [63], the genericity conditions are related to the projection of the curves on the x-axis, whereas in [8] they are related to the projection on the (x, y)-plane and to the projection of this projection on the x-axis (which is more restrictive). The effective techniques described in [8] to check this genericity condition involve delicate computation such as approximate gcd or absolute factorization, in particular in the presence of singularities. In another recent approach [51], the 3D curve is described by its projection on the plane (x, y) and by a reduced "monoid" equation $a(x)\, z - b(x, y) = 0$. This allows to lift the planar curve and to deduce the connection above the critical points, under some genericity conditions. The polynomial $a(x)$ is obtained from an iterated resultant and may be huge. In another recent work [44], non-reduced curves are treated using rational lifting maps deduced directly from the decomposition of subresultants with respect to the variable z.

As opposed to the aforementioned methods, the approach that we are going to describe here does not require genericity conditions on the projected curves but only pseudo-generic conditions (two branches of the contour curve do not project on the same branch in the (x, y) or (x, z)-plane).

The general idea is to project the contour curve onto the (x, y)-plane and (x, z) plane, and to compute the topology of the projected curves in order to recover the topology of the 3D contour curve.

We will use the subset $\Xi_{\mathcal{C}}(f)$ of points of $\Xi(f)$ that have index c or x. Points of $\Xi(f) \in \mathbb{R}^5$ naturally project to points of S by taking their first three components. The role of the fourth and fifth components is to allow us to label them as x, c, and r points. Once we have this information we can discard the last two components, and to simplify the discussion we will in the following, consider the points in $\Xi_{\mathcal{C}}(f)$ as the points on $S \subset \mathbb{R}^3$ to which they project. In this way, points with index c are smooth points on $\mathcal{C}$ (since their projection to $\mathcal{C}_{x,y}$ is smooth). We will also use the points of $\mathcal{C}$ at intermediate sections $x = \mu$, chosen adequately as we describe now.

Let $\Delta(x, y)$ be the square-free part of $\mathrm{Res}_z(f, \partial_z f\, q)$ and $\mathcal{C}_{x,y}$ the curve it defines in the plane (x, y). We also denote by $\Psi(x, z)$ be the square-free part of $\mathrm{Res}_y(f, q\, \partial_z f)$ and by $\mathcal{C}_{x,z}$ the curve it defines in the plane (x, z).

In a first step, we compute the topology of the curve $\mathcal{C}_{x,y}$ (see section 5.2) in

the projection of the bounding ball B where we want to determine the topology.

Let Σ be the x-critical values of $\mathcal{C}_{x,y}$ and Σ' the x-critical values of $\mathcal{C}_{x,z}$: $\Sigma = \{\sigma_1, \ldots, \sigma_s\}$ with $\sigma_1 < \cdots < \sigma_s$. For each $\sigma_i \in \Sigma$, we compute two (rational) values μ_i, μ_i' such that $\sigma_{i-1} < \mu_i < \sigma_i < \mu_i' < \sigma_{i+1}$ and $\Sigma' \cap [\mu_i, \sigma_i[= \Sigma' \cap]\sigma_i, \mu_i'] = \emptyset$. Note that Σ and Σ' can have points in common, that's why the intervals are open in σ_i (in fact if there is a y,z-critical point they will have a σ_i in common).

In the following, we denote by $\mathcal{C}_{\mu_i}$ the section $\mathcal{C} \cap V(x - \mu_i)$. By construction, above the interval $[\mu_i, \sigma_i[$ the curves $\mathcal{C}_{x,y}$ and $\mathcal{C}_{x,z}$ have no x-critical points. If two points of $\mathcal{C}_{\mu_i}$ have the same y-coordinate, and if the projection $\mathcal{C}_{x,y}$ has no critical point at $x = \mu_i$, then two branches of $\mathcal{C}$ project onto the same branch of $\mathcal{C}_{x,y}$. By a generic change of coordinates, we can avoid this situation. We proceed similarly, if two points of $\mathcal{C}_{\mu_i}$ have the same z-coordinates. Thus we can assume that $\mathcal{C}_{\mu_i}$ projects injectively on the (x,y) and (x,z) planes.

In order to connect the points of $\mathcal{C}_{\mu_i}$ to those of $\mathcal{C}_{\sigma_i}$, we also compute the topology of $\mathcal{C}_{x,z}$ above the interval $[\mu_i, \mu_i']$ using $\Sigma(f)$ (see section 5.2). Notice that by construction of μ_i, μ_i', the projection of $\Sigma(f)$ on the (x,z)-plane contains the z-critical of the projected curve, above the interval $[\mu_i, \mu_i']$. Using the computed topological graph of $\mathcal{C}_{x,y}$ and $\mathcal{C}_{x,z}$, we proceed as follows.
Given a point $p = (\mu_i, v, w) \in \mathcal{C}_{\mu_i}$, its projection (μ_i, v) is connected to a point (σ_i, β) by the topological graph that we have computed for $\mathcal{C}_{x,y}$. Its projection (μ_i, w) is connected to a point (σ_i, γ) by the topological graph of $\mathcal{C}_{x,z}$. As the projections of $\mathcal{C}_{\mu_i}$ onto the planes (x,y) and (x,z) are injective, there is a (unique) branch of $\mathcal{C}$, which connects p to the point $(\sigma_i, \beta, \gamma) \in \mathcal{C}_{\sigma_i}$.

The connection above the interval $[\mu_{i-1}', \mu_i]$ proceeds similarly by using only the topological graph of the curve $\mathcal{C}_{x,y}$, which has no x-critical values in $[\mu_{i-1}', \mu_i]$, since $\mathcal{C}_{\mu_{i-1}'}$ and $\mathcal{C}_{\mu_i}$ project injectively to the (x,y)-plane.

Let us summarize the main steps of this algorithm:

Algorithm 5.3.1 (**Topology of $\mathcal{C}$ defined by** $f_1(x,y,z) = 0, f_2(x,y,z) = 0$**).**

INPUT: *Polynomials* $f_1, f_2 \in \mathbb{Q}[x,y,z]$ *and a box* $D_0 \subset \mathbb{R}^3$

- *Compute the square-free part* $\Delta(x,y)$ *of* $\mathrm{Res}_z(f_1, f_2)$*, defining the projected curve* $\mathcal{C}_{x,y} \subset \mathbb{R}^2$*.*
- *Compute the square-free part* $\Psi(x,z)$ *of* $\mathrm{Res}_y(f_1, f_2)$*, defining the projected curve* $\mathcal{C}_{x,z} \subset \mathbb{R}^2$*.*
- *Compute the topology of* $\mathcal{C}_{x,y}$ *in the projection of* D_0 *on the plane* (x,y)*.*
- *Compute the* x*-critical values* $\Sigma := \{\sigma_1, \ldots, \sigma_k\}$ *with* $\sigma_1 < \cdots < \sigma_k$*. of* $\mathcal{C}_{x,y}$ *and the critical values* Σ' *of* $\mathcal{C}_{x,z}$*.*
- *Choose a (rational)* $\mu_i \in]\sigma_{i-1}, \sigma_i[$ *(resp.* $\mu_i' \in]\sigma_i, \sigma_{i+1}[$*) such that* $[\mu_i, \sigma_i[\cap \Sigma' = \emptyset$ *(resp.* $]\sigma_i, \mu_i'] \cap \Sigma' = \emptyset$*).*
- *Compute the topology of* $\mathcal{C}_{x,z}$ *above* $[\mu_i, \mu_i]$ *in the projection of* D_0 *on the plane* (x,z)*.*

- *Compute the set $\mathcal{C}_{\mu_i}$ of real points of $\mathcal{C}$ at $x = \mu_i$ and check that it is finite and that two points do not have the same y-coordinates (resp. z-coordinates). If this is the case, raise the exception "non-generic position".*
- *Use the topology of $\mathcal{C}_{x,y}$ and $\mathcal{C}_{x,z}$ above $[\mu_i, \sigma_i]$ (resp. $[\sigma_i, \mu'_i]$ to connect the points of $\mathcal{C}_{\sigma_i}$ to $\mathcal{C}_{\mu_i}$ (resp. $\mathcal{C}_{\mu'_i}$).*
- *Use the topology of $\mathcal{C}_{x,y}$ above $[\mu'_{i-1}, \mu_i]$ to connect the points $\mathcal{C}_{\mu'_{i-1}}$ to $\mathcal{C}_{\mu_i}$.*

OUTPUT: *The graph of 3D points of the curve connected by segments, isotopic to the curve $\mathcal{C}$ or the exception "non-generic position".*

This algorithm is applied for $f_1 = f$, $f_2 = q\,\partial_z f$ where $q(x, y, z) = 0$ is the equation of the boundary of B, to get the topology $\mathcal{C} = \mathfrak{C}_z(S_B)$. We need the topology of the contour curve because there are topology changes that come from the interference of the bounding sphere B with S, and this is taken into account by adding $S \cap B$ into the contour curve. The way this comes into play is explained in the next section.

Since we are interested in the topology of $S \cap B$, we only need to compute the topology of the curves $\mathcal{C}_{x,y}$ or $\mathcal{C}_{x,z}$ in boxes of $\mathbb{R}^2$ which are the projection of a box in $\mathbb{R}^3$ containing B.
In order to compute the x-critical values of $\mathcal{C}_{x,y}$ or $\mathcal{C}_{x,z}$, we apply iterated resultant computations. Though the degree of these resultant polynomials can grow quickly, they can be decomposed into explicit factors in order to simplify the computation (see [31]).

5.4 The algorithm for singular algebraic surfaces

5.4.1 The algorithm

We will assume hereafter that the surface is in a generic position:

Definition 5.4.1. *We say that the surface is in generic position if*

- *the system* (5.1) *has a finite number of (complex) solutions.*
- *two distinct arcs of the contour curve do not have the same projection in the (x, y) (resp. (x, z)) plane.*

The first point is checked while solving system (5.1). If it is a zero dimensional system, we assume that the polynomial solver over the complex field yields isolating boxes containing one and only one real root of $\Xi(f)$. The second point is checked while applying the algorithm 5.3.1 to $f(x, y, z)$ and $\partial_z f(x, y, z)\, q(x, y, z)$. If these conditions are not fulfilled, we perform a random change of coordinates and restart the algorithm. There is a high probability to be in generic position after a change of coordinate, and therefore this process eventually stops and yields a surface in generic position.

Let us first outline briefly the algorithm, before going into the details.

The first step consists in computing the contour curve for the projection in the z-direction. We apply algorithm 5.3.1 for 3D-curves with $f_1 = f$, $f_2 = q\,\partial_z f$, which computes a polygonal approximation of the contour curve which is isotopic to it. Doing this, the algorithm computes x-critical values corresponding to x-critical points of the 3D curve and singular points of the projection of the contour curve on the (x, y)-plane and the (x, z)-plane. Let us call Σ this set of x-critical values.

For each σ of Σ, we compute the topology of the corresponding sections of the surface, by applying algorithm for the topology of 2D curves (see section 5.2).

Next, we compute regular values between two critical values and the topology of the corresponding sections. Here again, we use the 2D algorithm for implicit curves (see section 5.2).

The following step consists in connecting two consecutive sections, using the topology of the contour curve (see details in section 5.4.2).

Finally, we mesh the resulting patches of the surface, by computing a set of points, open segments and open triangles, which are not self-intersecting, and which defines a simplicial complex isotopic to the surface (see details in section 5.4.3).

We summarize the algorithm as follows:

Algorithm 5.4.2 (Topology of an algebraic surface S in a ball B)**.**
INPUT*: A polynomial $f(x, y, z)$ defining S and a ball B.*

- *Compute the topology of the contour curve for the projection in the z-direction, using algorithm 5.3.1.*
- *Compute the topology of the sections, using algorithm 5.2.*
- *Connect two consecutive sections, by exploiting the topology of the contour curve.*
- *Triangulate the resulting surface patches, avoiding self-intersection of segments and triangles.*

OUTPUT: *A simplicial complex isotopic to S_B.*

Let us now detail the last two steps of this algorithm.

5.4.2 Connection algorithm

We denote by $\mathcal{V}$ the topological description of $\mathcal{C} = \mathfrak{C}_z(S_B)$ and $\mathcal{K} := \mathcal{V}$ the initial value of the topological complex describing S. The initial value for $\mathcal{K}$ is the result of the curve topology computation done for $\mathfrak{C}_z(S_B)$ by algorithm 5.3.1. We are going to update this complex by explaining how we define the connections between the points of two successive sections of S, a regular one which is regular S_r and a critical one S_c which contains an x-critical point of $\mathfrak{C}_z(S_B)$. By section of S we mean a set $S \cap V(x - \alpha)$ where α is such that

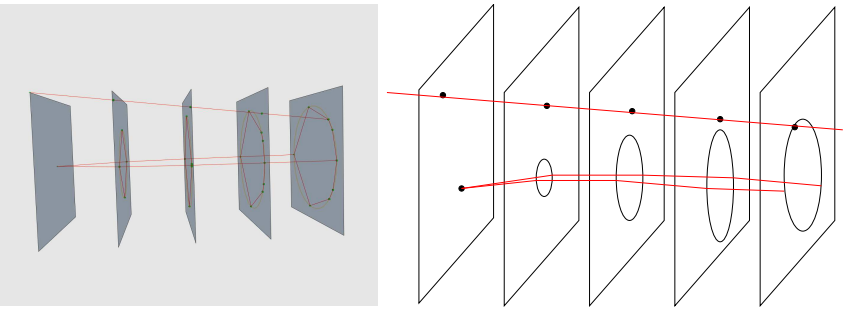


Figure 5.1: Polar variety and first connections for the union of a sphere and a line defined by one equation.

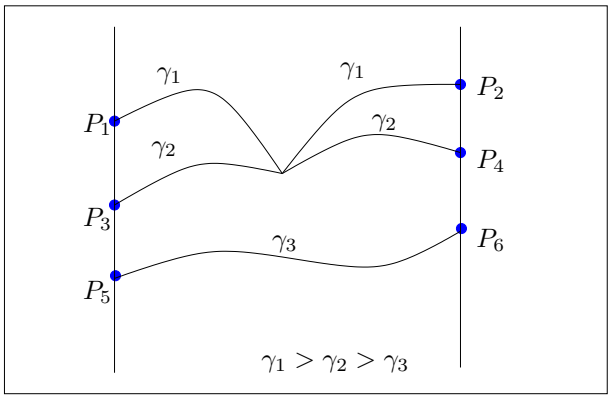


Figure 5.2: Order on the arcs.

$V(x-\alpha)$ contains no x-critical point of $\mathfrak{C}_z(S_B)$ for regular sections S_r and does contain such a point for critical sections S_c.

Let us denote by $p_1, \ldots, p_r$ (resp. $q_1, \ldots, q_s$) the points of $\pi_z(\mathcal{V} \cap S_r)$ (resp. $\pi_z(\mathcal{V} \cap S_c)$) ordered by increasing y-coordinates, which are also on the projection of an arc of $\mathcal{V}$ connecting S_r and S_c.

Hereafter, we use the convention that $p_0, p_{l+1}, q_0, q_{m+1}$ are points on the boundary of the ball B. We denote by $\mathcal{A}_i$ $(i = 0, \ldots, l)$ the set of arcs of S_r which projects onto $[p_i, p_{i+1}]$. We denote by $\mathcal{B}_j$ $(j = 0, \ldots, m)$ the set of arcs of S_c, which connect a point projecting at q_j to a point projecting at q_{j+1}. If, moreover there is a critical point U in between, we require that if this arc is to the k^{th} branch arriving at U on the left, then it is also the k^{th} branch starting from U on the right, if this branch exists.

The arcs in $\mathcal{A}_i$ (resp. $\mathcal{B}_j$) are naturally ordered according to their z-position: an arc is bigger than another if it is above the other (see figure 5.2). We treat

incrementally the points p_i, starting from p_0. Let us denote by $q_{\nu(i)}$ the point connected to p_i by an arc of $\pi_z(\mathcal{K}) \supset \pi_z(\mathcal{V})$.

- If p_{i+1} is connected to $q_{\nu(i)}$ by an arc of $\pi_z(\mathcal{K})$, for any arc $\gamma = (P, P')$ of $\mathcal{A}_i$, such that P is connected by $\mathcal{K}$ to Q, we add the arc (P', Q) and the face (P, P', Q) to $\mathcal{K}$.

- If p_{i+1} is not connected to $q_{\nu(i)}$, it is connected to $q_{\nu(i)+1}$. We consider the smallest arc $\gamma = (P, P')$ of $\mathcal{A}_i$, and the smallest arc $\eta = (Q, Q')$ of $\mathcal{B}_{\nu(i)}$. The arc (P, Q) is in $\mathcal{K}$. We add the arc (P', Q') and the face (P, P', Q', Q) to $\mathcal{K}$. Then we remove these smallest arcs γ and δ, respectively from $\mathcal{A}_i$ and $\mathcal{B}_{\nu(i)}$ and go on until $\mathcal{A}_i$ is empty.

This procedure is applied iteratively, until we reach the point p_r, so that we move to the next section S'_r, S'_c.

5.4.3 Triangulation algorithm

The final step is the triangulation of the different faces computed previously.

Assume that in the connection algorithm (section 5.4.2), we have connected an arc $\gamma = (P_1, P_2)$ of S_r to an arc (or point) $\eta = (Q_1, Q_2)$ in S_c, by a face of $\mathcal{K}$.

The triangulation algorithm works as follows (see figure 5.3):

If $Q_1 = Q_2$ then we connect successively all the points between P_1 and P_2 to Q_1, creating the triangles of our triangulation.

If $Q_1 \neq Q_2$, there can exist at most one critical point U on η. The situation to avoid is described in the figure 5.4. If we do not pay attention to the connections that are created during the triangulation, we can create intersection curves between two faces that do not exist. We will quickly explain by an example what we have to do to avoid that before going back to the general algorithm.
We see on figure 5.4 a situation where the arc γ_1 (resp. γ_2) is connected to an arc η_1 (resp. η_2). The two patches defined respectively by γ_1 and η_1 and by γ_2 and η_2 connect to the arcs P_1Q_1 and P_2Q_2 but do not intersect. To create an intersection, we would need to connect a point shared by γ_1 and γ_2 to a point shared by δ_1 and δ_2. This case corresponds to the drawing in figure 5.4. So what has to be done is simply to connect the point U (the only point different from Q_1 and Q_2 belonging to the two arcs η_1 and η_2) to a point different from P_1 and P_2. This can always be done because there exist intermediate points between P_1 and P_2.
If there exists a y-critical point U on the arc $\eta = (Q_1, Q_2)$ then, we connect the point U to an intermediate point T_1 of γ between P_1 and P_2 (see figure 5.4). Let us now consider the two sub-arcs (P_1, T_1) and (Q_1, U). We start simultaneously from P_1 and Q_1. The two points are connected by an arc of $\mathcal{K}$. We consider the next point on the arc (P_1, U) and the next point on the arc (Q_1, U). We connect them. This process goes on until there are no more points on one of the two arcs. If there are less points on an arc than on the other, we connect the remaining points on one arc by adjacent triangles sharing the same vertex

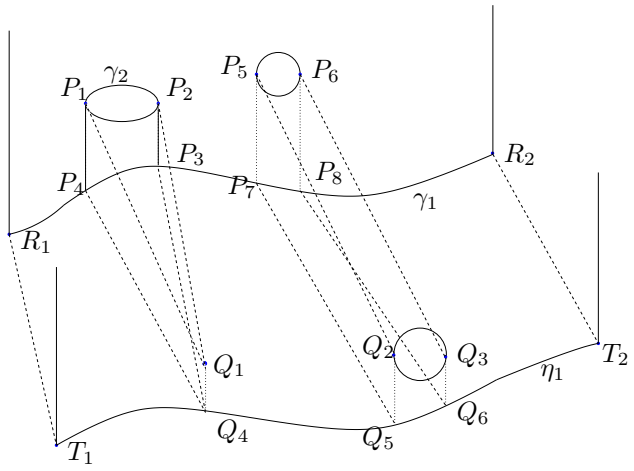


Figure 5.3: Division of the space with vertical walls.

(see figure 5.5). After this step, we obtain triangles or quadrangles, which we subdivide in order to obtain the final triangulation.

5.5 Why we obtain a triangulation

As mentioned previously, the general idea of this sweeping algorithm is to detect where *some topological changes* in the intersection of S with the sweeping plane happen so that in between we have a simple product topology. We are going to prove that in between the events that we have computed in the previous section, the topology is locally trivial and we use this result to describe explicitly the isotopy between the mesh and the surface.

To prove the correctness of the algorithm we will use results from stratified Morse theory ([67], [46] for more details). The concepts that we will use have already been recalled in section 4.1.1, along with the Thom-Mather theorem 4.1.10. In our case, we will apply the theorem for the variety $Z = S_B$ and the projection π to the x-axis. This projection is automatically proper as we work in a ball B which is compact. The only supplemental proposition we will need is the following:

Proposition 5.5.1. *Any semi-algebraic stratum S is Whitney regular along a zero-dimensional stratum.*

This property is very natural but its proof is not utterly straightforward. We refer to [46][Lemma 1.10, p.5] for a proof, using the Curve Selection Lemma.

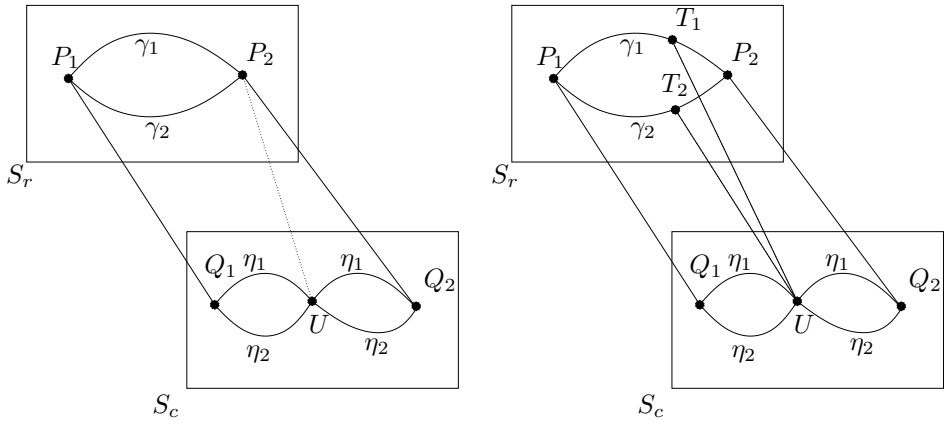


Figure 5.4: Incorrect connection and its correction

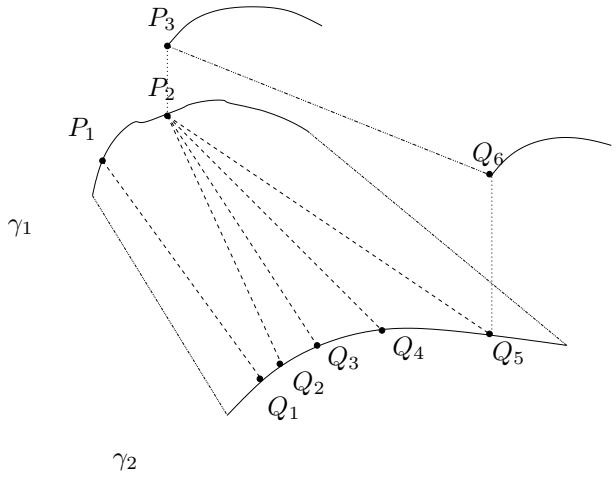


Figure 5.5: Meshing.

5.5.1 Computation of a Whitney stratification

For a projection π_z in the direction z on the (x, y) plane, we recall that $\Delta(x, y)$ is the square-free part of the resultant $\mathrm{Res}_z(f, q\,\partial_z f)$ and that its associated zero set $V(\Delta(x, y))$ is $\mathcal{C}_{x,y}$.

Theorem 5.5.2. *For a generic projection π_z, let*

- *S^0 be the set of points of $\mathfrak{C}_z(S_B)$ that project by π_z onto singular points of $\mathcal{C}_{x,y}$, each point is considered as a stratum,*
- *S^1 the set of the connected components of $\mathfrak{C}_z(S_B) - S^0$, (each connected component is a stratum),*
- *S^2 the set of the connected components of $S - \mathfrak{C}_z(S_B)$ (each connected component is a stratum).*
- *S^3 the set of connected components of $\mathbb{R}^3 - S$ (each connected component is a stratum).*

Then (S^0, S^1, S^2, S^3) is a Whitney stratification of $\mathbb{R}^3$ compatible with S.

From proposition 5.5.1 and as the Whitney regularity of any stratum of S^1 or S^2 along a stratum of S^3 is always fulfilled, we deduce that showing that (S^0, S^1, S^2, S^3) is a Whitney stratification of $\mathbb{R}^3$ compatible with S boils down to showing that (S^1, S^2) is Whitney-regular.

Depending on whether we consider the polynomial f defining S over $\mathbb{R}$ or $\mathbb{C}$, we obtain a real variety $S = S_{\mathbb{R}}$ or a complex variety $S_{\mathbb{C}}$, as the set of zeros of f. We will use the results of equi-singularity along $\mathbb{C}$ and the notion of "permissible" projection to prove the proposition. Speder gave in [126] a definition of permissible projection, stronger than Zariski's original one [145]. We will consider only the case of codimension 1, for which both definitions coincide, so hereafter we will just consider the definition of permissible projection of Zariski:

Definition 5.5.3. *A permissible direction of projection for the pair (X, Y) with $Y \subset \overline{X}$ at $Q \in Y$ is an element of $\mathbb{PC}^3$ so that the line passing through Q defined by this direction is neither included in a neighborhood of Q nor in the tangent space to Q at Y.*

Proposition 5.5.4. *For a given algebraic surface S, a generic direction of projection is permissible for (S^1, S^2) at every point of S^1.*

Proof. For an algebraic variety, the local inclusion of a line into the surface is equivalent to a global inclusion. We deduce that the directions of projection to avoid are included into the union of:

- directions of lines included into the surface
- directions of the tangents to the smooth part of the singular locus of the variety.

We consider the first set of directions of lines included in the surface S, defined by one equation $f(x, y, z) = 0$. We consider the surface embedded in projective space. The directions of lines included in S considered as points of projective space are included in the intersection of S with the hyperplane at infinity which is a projective curve. Thus the directions corresponding to the first set are included in a set of dimension 1 and are generically avoided.

Now let us consider the second set. We consider an arc of the smooth part of the contour curve (there exists a finite number of such arcs for an algebraic surface). We consider a semi-algebraic parameterization of this arc $(x(s), y(s), z(s))$. Thus we obtain a semi-algebraic parameterization $(x'(s), y'(s), z'(s))$ of a set of unit vectors corresponding to the directions of the tangents to the curve. We deduce that the set to avoid (for the tangency condition) corresponds to a semi-algebraic curve on the unit sphere of $\mathbb{R}^3$ and is generically avoided. □

Proposition 5.5.5. *If the surface S is in generic position (see definition 5.4.1) then the projection π_z along the z-axis is a permissible projection.*

Proof. First, there is no line parallel to the z-axis in S because if this were the case, all the vertical line would be included in the contour curve and we would not be in generic position. The second point to check is that the z-direction is not a direction of a tangent of S^1. This is the case as by construction the points of the contour curve with vertical tangents project onto singular points of $\mathcal{C}_{x,y}$ and are thus in S^0. □

We also recall the notion of equi-singularity (which is defined inductively):

Definition 5.5.6. *[126][p. 577], [145][Def. 4.1 p.981] Let $X \subset \mathbb{C}^n$ a hypersurface, Y a smooth submanifold of X of codimension c, P be a point of Y. We say that X is equi-singular at P along Y if either $c = 0$ and X is smooth at P or $c > 0$, $Y \subset X_{\text{sing}}$ and there exists a so-called permissible projection π such that the critical locus of $\pi|_X$ is equi-singular at $\pi_z(P)$ along $\pi_z(Y)$.*

The main result that we use is the following:

Proposition 5.5.7. *[126][Th. III, p.585] If the hypersurface $X \subset \mathbb{C}^n$ is equi-singular along Y and if the codimension of Y in X is 1, then the pair (X_{smooth}, Y) fulfills the Whitney conditions along Y.*

This allows us to check the Whitney condition over the complex field. We need to check it on $\mathbb{R}$:

Proposition 5.5.8. *If X and Y are two strata of a Whitney stratification of $S_{\mathbb{C}}$ with $\dim X = 2$ and $\dim Y = 1$, then $(X_{\mathbb{R}}, Y_{\mathbb{R}})$ is Whitney regular, where $X_{\mathbb{R}} = X \cap \mathbb{R}^3$ and $Y_{\mathbb{R}} = Y \cap \mathbb{R}^3$.*

Proof. Let P be a point in $Y_{\mathbb{R}} \cap \overline{X_{\mathbb{R}}}$. Consider a sequence x_n of points of $X_{\mathbb{R}}$ and y_n of points of $Y_{\mathbb{R}}$, both sequences converging to P. Note these sequences exist because P is in $Y_{\mathbb{R}} \cap \overline{X_{\mathbb{R}}}$ which means there are points of $X_{\mathbb{R}}$ in a neighborhood

of P (and $P \in Y_{\mathbb{R}}$). Of course $Y_{\mathbb{R}}$ and $X_{\mathbb{R}}$ are disjoint sets because Y and X were already disjoint. We assume that the sequence of secants $\overline{x_n y_n}$ converges to a limit $l \in \mathbb{R}^3$ and the sequence of tangent planes $T_{x_n} X_{\mathbb{R}}$ converges to a limit T. If we consider x_n and y_n in $\mathbb{C}^3$, the sequence of secants converges also to a complex line $l_{\mathbb{C}}$ because $x_n \wedge y_n$ converges to a limit L in $\mathbb{P}(\Lambda^2\mathbb{R}^4)$ which is embedded in $\mathbb{P}(\Lambda^2\mathbb{C}^4)$. The convergence of the sequence $T_{x_n} X_{\mathbb{R}}$ is equivalent to the convergence of $T_{x_n} X$: the sequence of normals defined by the orthogonal vectors ∇f converges equivalently in $\mathbb{R}$ or $\mathbb{C}$. Thus $\lim_{n\to\infty} \overline{x_n y_n}_{\mathbb{R}} \subset \lim_{n\to\infty} \overline{x_n y_n}_{\mathbb{C}} \subset \lim_{n\to\infty} T_{x_n} X$ (since (X, Y) is Whitney regular). We deduce that $\lim_{n\to\infty} \overline{x_n y_n}_{\mathbb{R}} \subset \lim_{n\to\infty} T_{x_n} X \cap \mathbb{R}^3$. We know that $\lim_{n\to\infty} T_{x_n} X_{\mathbb{R}} \subset \lim_{n\to\infty} T_{x_n} X \cap \mathbb{R}^3$. As x_n is a sequence of real points, $\lim_{n\to\infty} T_{x_n} X$ is defined as the orthogonal in $\mathbb{C}$ of a real vector. We deduce that $\lim_{n\to\infty} T_{x_n} X \cap \mathbb{R}^3$ is a real space of dimension less or equal to 2 containing $\lim_{n\to\infty} T_{x_n} X_{\mathbb{R}}$ which is of dimension 2, thus the two linear spaces are equal. So we deduce that $\lim_{n\to\infty} \overline{x_n y_n}_{\mathbb{R}} \subset \lim_{n\to\infty} T_{x_n} X_{\mathbb{R}}$ and that $(X_{\mathbb{R}}, Y_{\mathbb{R}})$ is Whitney regular. □

Proof of Theorem 5.5.2. The stratification defined in (5.5.2) over the complex field, yields a stratification of $S_{\mathbb{C}}$. We consider its restriction to $\mathbb{R}^3$. By proposition 5.5.1, we only need to check the Whitney condition for the 1-dimensional strata $S^1_{\mathbb{R}}$ and the 2-dimensional strata $S^2_{\mathbb{R}}$ of $S_{\mathbb{R}}$. Let $p \in S^1_{\mathbb{R}} \cap \overline{S^2_{\mathbb{R}}}$. If p is a smooth point of S, the Whitney condition is trivially satisfied. If p is singular, by proposition 5.5.7, we have the Whitney condition for $(S^2_{\mathbb{C}}, S^1_{\mathbb{C}})$ at p. And by proposition 5.5.8, we deduce the Whitney condition for $(S^2_{\mathbb{R}}, S^1_{\mathbb{R}})$ at p. This proves that $(S^0_{\mathbb{R}}, S^1_{\mathbb{R}}, S^2_{\mathbb{R}}, S^3_{\mathbb{R}})$ is a Whitney stratification of $\mathbb{R}^3$ compatible with $S_{\mathbb{R}}$. □

5.5.2 Connection of the sections

We have described in section 5.4 an algorithm to connect two successive sections. Now we are going to justify what this algorithm does.

By proposition 5.5.2 and using Thom's lemma (Theorem 4.1.10), we deduce that in between two consecutive critical sections, the topology of the sections is constant. We have computed the topology of regular sections, in between two successive critical ones. So now, in order to prove the isotopy of the surface and the mesh, we have three things to verify:

a) From a topological point of view, we define the good connections between the sections.

b) The triangulation that we construct is valid (i.e. the simplices of the complex do not intersect). Or in other words, the embedding in $\mathbb{R}^3$ of the simplicial complex we have constructed is injective.

c) The mesh is isotopic to the surface.

The point c) will be made explicit in subsection 5.5.3. We now prove the first two points:

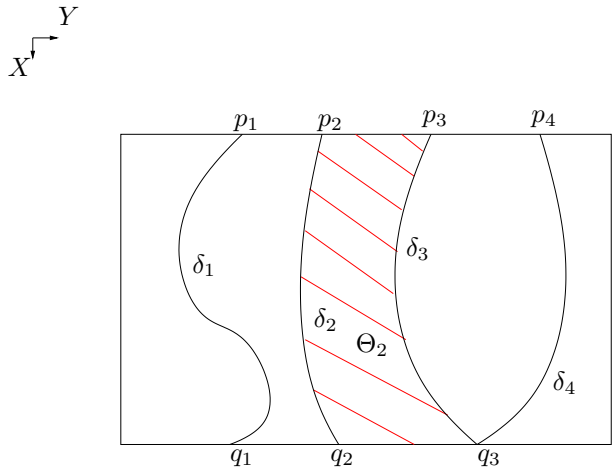


Figure 5.6: Projection of the contour curve

a) We are going to justify the connection algorithm described in section 5.4.2. Let us recall the notations of section 5.4.2.

We denote by $p_1, \ldots, p_l$ (resp. $q_1, \ldots, q_m$) the points of $\pi_z(\mathcal{V} \cap S_r)$ (resp. $\pi_z(\mathcal{V} \cap S_c)$) ordered by increasing y-coordinates, which are on the projection of an arc of $\mathcal{V}$ connecting S_r and S_c. Notice that we have $s \leq r$.

We denote by $\mathcal{A}_i$ $(i = 0, \ldots, l)$ the set of arcs of S_r which project onto $[p_i, p_{i+1}]$ and by $\mathcal{B}_j$ $(j = 0, \ldots, m)$ the set of arcs of S_c which project onto $[q_j, q_{j+1}]$, with the convention that $p_0, p_{l+1}, q_0, q_{m+1}$ are on the boundary of the ball B.

The point p_i is connected to $q_{\nu(i)}$ by an arc δ_i of the projection of $\mathcal{K}$. We note Θ_i the open planar domain between δ_i and δ_{i+1} (dashed part in figure 5.6).

Proposition 5.5.9. *If the topology of $\pi_z^{-1}(\delta_i) \cap S$, S_r, S_c is determined, then algorithm 5.4.2 computes the topology of $\pi_z^{-1}(\delta_{i+1}) \cap S$ and of $\pi_z^{-1}(\Theta_i) \cap S$.*

Proof. Let us consider an arc γ in $\mathcal{A}_i$ connecting a point P to a point P'. If we apply Thom's lemma to $S \cap \pi_z^{-1}(\Theta)$, we deduce that $S \cap \pi_z^{-1}(\Theta)$ is topologically trivial (i.e. made of a family of patches lying one above the other) and that the boundary of each patch contains an arc θ_i in $\pi_z^{-1}(\delta_i)$ and an arc θ_{i+1} in $\pi_z^{-1}(\delta_{i+1})$. We denote hereafter by F the patch associated to γ.

There are two cases to consider:

1. δ_i and δ_{i+1} intersect in $q_{\nu(i)}$.

2. δ_i and δ_{i+1} do not intersect.

In the first case, we denote by $Q = \theta_i \cap \theta_{i+1}$ the point of $\overline{F}$ which projects onto $q_{\nu(i)}$. By induction hypothesis, as the topology of $\overline{F_i} \cap \pi_z^{-1}(\delta_i)$ is determined

by algorithm 5.4.2, the arc θ_i is represented in $\mathcal{K}$ as the connection of P to Q. The arc θ_{i+1} corresponds to the connection (P', Q), produced by the algorithm, as well as the face (P, P', Q) corresponding to F.

We have

$$\pi_z^{-1}(\delta_{i+1}) \cap S = \left(\mathfrak{C}_z(S_B) \cap \pi_z^{-1}(\delta_{i+1})\right) \cup \left(\overline{\pi_z^{-1}(\Theta_i) \cap S} \cap \pi_z^{-1}(\delta_{i+1})\right)$$

According to the previous paragraph, the arcs of $\overline{\pi_z^{-1}(\Theta_i) \cap S} \cap \pi_z^{-1}(\delta_{i+1})$ are thus obtained by algorithm 5.4.2. The arcs of $\mathfrak{C}_z(S_B) \cap \pi_z^{-1}(\delta_{i+1})$ are obtained by algorithm 5.3.1. Thus the algorithm 5.4.2 compute the topology of $\pi_z^{-1}(\delta_i) \cap S$.

In the second case, we denote again by $Q = \theta_i \cap S_c$ the point of $\overline{F}$ which projects onto $q_{\nu(i)}$ and by $Q' = \theta_{i+1} \cap S_c$ the point of $\overline{F}$ which projects onto $q_{\nu(i)+1}$. The intersection $\overline{F} \cap S_c$ is an arc connecting Q to Q', which exists in $\mathcal{K}$, by hypothesis.

Conversely, as the surface is in generic position (see definition 5.4.1), an arc of $S_c \cap \pi_z^{-1}([q_{\nu(i)}, q_{\nu(i)+1}]) = \mathcal{B}_{\nu(i)}$ is in the closure of only one patch defined by an arc in $S_r \cap \pi_z^{-1}([p_i, p_{i+1}]) = \mathcal{A}_i$. So there is a one to one correspondence between the arcs in $\mathcal{A}_i$ and the arcs in $\mathcal{B}_{\nu(i)}$. Moreover, this correspondence respects the z-order on the arcs, since there is no point of $\mathfrak{C}_z(S_B)$ above Θ_i.

In particular, the smallest arc $\gamma = (P, P')$ in $\mathcal{A}_i$ is connected to the smallest arc $\eta = (Q, Q')$ in $\mathcal{B}_{\nu(i)}$ by a face (P, P', Q', Q) corresponding to F, as computed by algorithm 5.4.2.

The arc θ_{i+1} connects the point P' to Q', as computed by the algorithm 5.4.2, so that the topology of $\pi_z^{-1}(\delta_{i+1}) \cap S$ is determined by the algorithm.

This proves that if the topology of $\pi_z^{-1}(\delta_i) \cap S$, S_r, S_c are determined, then algorithm 5.4.2 compute the topology of S above $\overline{\Theta_i}$. □

b) We have to ensure that our triangulation is valid. It is clear that the triangulation we compute does not create holes, because the triangulation refines the topological complex $\mathcal{K}$. Let us check now that we do not create intersection of the open segments and open triangles.

As the algorithm proceeds iteratively on the cylinders $\overline{\pi_z^{-1}(\Theta_i)}$, we have only to check this property above $\overline{\Theta_i}$. By construction, the projection by π_z of open segments and open triangles are either disjoint or included one in the other.

If these projections are disjoint, they cannot self-intersect.

Otherwise, since these are linear objects, their intersection would imply an inversion of the z-position of the corresponding arcs (resp. points) in the section S_r and S_c, which is not possible by Thom's isotopy lemma.

This shows that the triangulation of S is valid.

5.5.3 The isotopy

We are going to detail an explicit isotopy (definition 4.0.1) between the original surface and its polygonal approximation.

We have seen that the projection of the contour curve on the x, y-plane (parallel to the z-direction), partitions the part of the x, y-plane between S_r and S_c (see figure 5.6).

The region $\overline{\Theta_i}$ is defined by the projection of two arcs of the contour curve. We will call δ_{i-1} and δ_i the two projected arc. They correspond to graphs of semi-algebraic functions of x on $[a, b]$, as the restriction of $\pi_{y,z}$ to $\mathfrak{C}_z(S_B)$ is submersive over $[a, b[$ and the arcs are of dimension 1.

The vertical cylinder with base $\overline{\Theta_i}$, cuts the variety along a family of patches and possibly curves which are not included in the closure of a dimension 2 patch. By construction, to each patch corresponds a sub-part of the triangulation with the particular property that the set of all those parts of the triangulation has also a cylindrical structure. More exactly, the patches of the original surface are projected onto $\overline{\Theta_i}$ and the corresponding triangulations project on the same quadrangle or triangle that will be denoted by Δ_i. This is a consequence of the division of the space with vertical walls that we have made (see figure 5.3).

Let us now consider two families $\phi_k, k = 1, \ldots, n$ and $\psi_k, k = 1, \ldots, n$ of graphs of continuous functions defined on the interval $[0, 1]$. Those graphs verify:

1. $\forall x \in]0, 1[\ \phi_1(x) < \cdots < \phi_n(x)\ ,\ \psi_1 < \cdots < \psi_n(x)$.
2. For $x \in \{0, 1\}$, if $\phi_k(x) = \phi_{k+1}(x)$ then $\psi_k(x) = \psi_{k+1}(x)$.

Then there exists an isotopy from $[0, 1] \times \mathbb{R}$ that send each ϕ_k, $(k \in \{1, \ldots, n\}$ onto ψ_k. One can easily verify that the following map is suitable: $(x, y, t) \mapsto (x, g(x, y, t))$ with:

$$
\begin{aligned}
&g(x,y,t) = \mathbf{1}_{]-\infty,\phi_1(x)]}(y + t(\psi_1(x) - \phi_1(x)))\\
&+\mathbf{1}_{]-\Phi_1(x),\phi_2(x)]}((1-t)y + t(\tfrac{y-\phi_1(x)}{\phi_2(x)-\phi_1(x)}\psi_2(x) + \tfrac{\phi_2(x)-y}{\phi_2(x)-\phi_1(x)}\psi_1(x))) + \cdots\\
&+\mathbf{1}_{]-\Phi_{n-1}(x),\phi_n(x)]}((1-t)y + t(\tfrac{y-\phi_{n-1}(x)}{\phi_n(x)-\phi_{n-1}(x)}\psi_n(x) + \tfrac{\phi_n(x)-y}{\phi_n(x)-\phi_{n-1}(x)}\psi_{n-1}(x))))\\
&+\mathbf{1}_{]\phi_n(x),+\infty[}(y + t(\psi_n(x) - \phi_n(x))).
\end{aligned}
$$

For a fixed x, the map sends intervals on intervals.

Let us consider for ϕ_k, the family of arcs defining the Θ_i and for ψ_k, those defining the Δ_i. We deduce from the previous result that applying an isotopy of the form : $(x, y, z, t) \overset{H_1}{\mapsto} (x, g(x, y, t), z)$, we make the projections of the patches and their triangulations on the plane (x, y) coincide. As illustrated in figure 5.8, we have transformed the Θ_i into the Δ_i. Moreover, applying this result on each interval between a regular and a critical section, the isotopies glue together into a global one.

More precisely :

1. In the first step, we have considered a transformation of the form $(x, y, z, t) \overset{H_1}{\mapsto} (x, g(x, y, t), z)$ which does not modify the coordinates x and z. This transformation makes the projections on the plane (x, y) of ϕ_k (patches of surfaces) and of ψ_k (patches of triangulation) coincide. This transformation does not modify the relative order of the graphs.

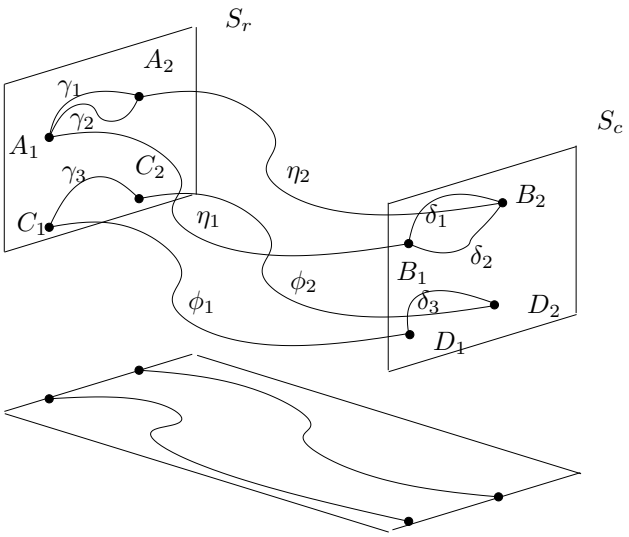


Figure 5.7: Family of patches

2. The second step of the isotopy is a vertical transformation of the form $(x, y, z, t) \overset{H_2}{\mapsto} (x, y, h(x, y, z, t))$ which sends $H_1(\phi_k)$ on ψ_k. It is similar to the previous one, we will not go into any further details here. Above $\overline{\Theta_i}$, there are patches and possibly isolated arcs of the contour curve. If such an isolated arc γ is in between two patches $H_1(\phi_k)$ and $H_1(\phi_{k+1})$, we add a term in the isotopy transformation, corresponding to a virtual pacth F with γ on its boundary and which lies between the two patches $H_1(\phi_k)$ and $H_1(\phi_{k+1})$.

5.6 Example

In this section, we illustrate what the algorithm does on two examples. The first example we chose is known as Whitney's umbrella and the classical normal form for it is $zx^2 - y^2$. We ask the algorithm to compute the topology of this surface in the unit ball.

In the first step the algorithm determines that $z\,x^2 - y^2$ is not in generic position because the line $x = 0$, $y = 0$ is included in the surface. It performs a random change of variable and the surface is now in generic position. The algorithm then computes the projection of the intersection of the unit ball and the umbrella and of the polar variety in the new coordinate system and identifies the x-critical points. Then it performs the connection between the points on the surface. These points have been plotted in green on the pictures below, the blue lines show the polar variety (the vertical z-axis is in it, but it is hidden by the red line), and the red lines that connect the green points illustrate the arcs that

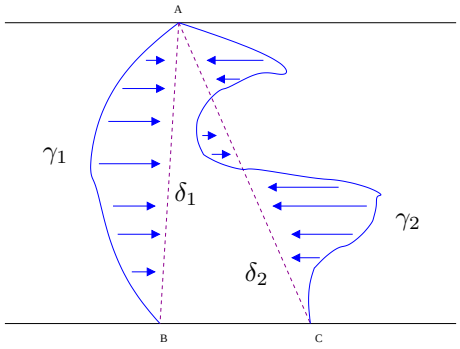
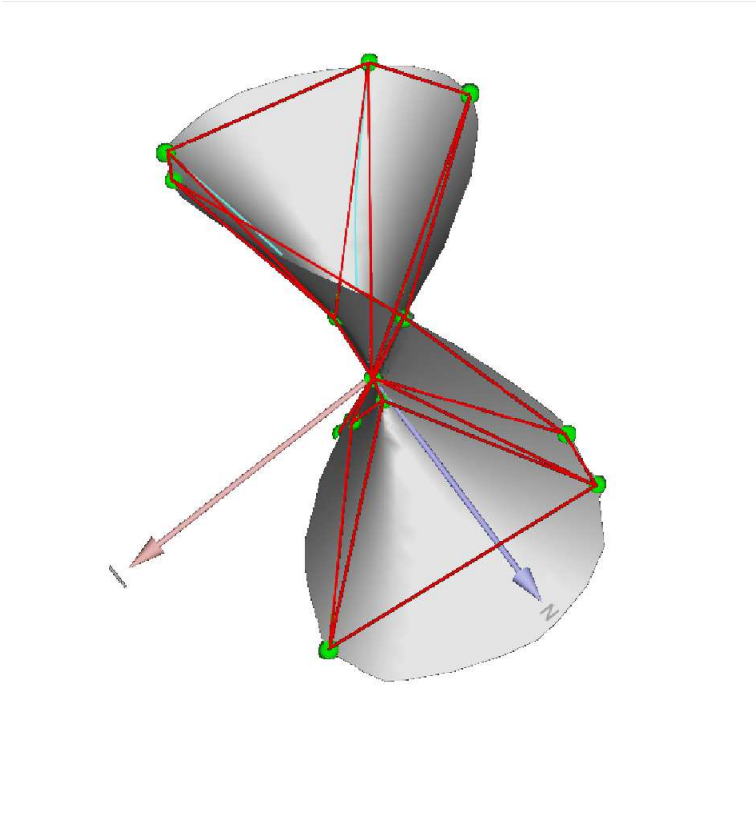


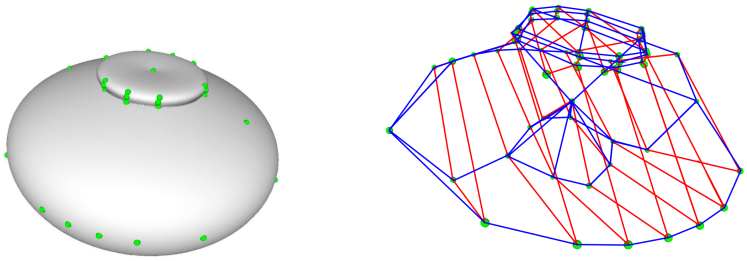
Figure 5.8: First step of the isotopy

connect the green points in the output of the algorithm. In other words, two critical points are connected by a red line if there is a direct path that connect them in the output of the algorithm. We have only represented the critical points in the picture to keep it clear. As a matter of fact on this example the algorithm has to compute more points but the underlying connection structure is the one represented here. The structure has two symmetrical "chip"-like parts, and the stick of the umbrella separates them. Notice that the isolated part of the stick is correctly handled by the algorithm, but the lower "chip"-like part partially hides its endpoint, so one has to look carefully to see the whole stick.



The second example comes from the following equation:

$$\begin{aligned}&4\,z^4x^4+8\,z^4x^2y^2+4\,z^4y^4+8\,z^3x^4+16\,z^3x^2y^2+8\,z^3y^4+19\,z^4x^2+19\,z^4y^2\\&+8\,z^2x^4+8\,z^2y^4+16\,y^2\,z^2x^2-72\,z^3x^2-72\,z^3y^2+4\,zx^4+8\,zx^2y^2+4\,zy^4\\&+121\,z^4-28\,z^2x^2-28\,z^2y^2+x^4+2\,x^2y^2+y^4-308\,z^3+20\,zx^2+20\,zy^2\\&+262\,z^2-3\,x^2-3\,y^2-84\,z+9=0.\end{aligned}$$



The first picture shows what the surface looks like. The second picture is an illustration of the connections. The green points are the same as in the previous

picture: they are the characteristic points that the algorithm uses to recover the topology. They have been computed by the algorithm then displayed. The red lines were added to show the topology of the slices. The blue line represent *some* of the connections between the slices, drawing them all would have made the picture too messy.
From the output of the algorithm one sees that the surface is self-intersecting with a cone, which was not obvious on the first picture.

5.7 Complexity and effectiveness

The algorithm of Cylindrical Algebraic Decomposition (CAD) computes in the case of a polynomial $f(x, y, z) = 0$, at most $\mathcal{O}(d^{3^2})$ polynomials of degree at most $\mathcal{O}(d^{2^2})$ [13][Chap. 11], which yields at most $\mathcal{O}(d^{13})$ points to compute.

With our algorithm, we have the following result.

Proposition 5.7.1. *At most $\mathcal{O}(d^7)$ points on an algebraic surface S of degree d are enough to determine a simplicial complex isotopic to it.*

Proof. As described in the previous sections, we are able to deduce the topology of the surface from the solution of system (5.1) and from the intersection points of the polar curve with planes $V(x - \alpha)$ where the α's lie in between the x-critical values of the planar curves defined by the polynomials $\Delta(x, y) = \mathrm{Res}_z(f(x, y, z), q(x, y, z)\,\partial_z f(x, y, z))$, $\Psi(x, z) = \mathrm{Res}_y(f(x, y, z), q(x, y, z)\,\partial_z f(x, y, z))$.

As $\deg(f) = d$ and $\deg(\partial_z f\, q) = d + 1$, the degree of $\Delta(x, y)$ is bounded by $d\,(d + 1)$. By Bezout theorem, the number of (real) solutions of the system 5.1 is bounded by

$$d\,(d + 1)\,(d\,(d + 1) - 1)\,d\,(d + 1)\,d = \mathcal{O}(d^7).$$

As there are at most $\mathcal{O}(d^4)$ critical values for $\Delta(x, y)$ and $\Psi(x, z)$ (which are of degree d^2), and as the polar curve is of degree $d\,(d-1)$, there are at most $\mathcal{O}(d^6)$ additional points to insert to get the topology of the polar curve and to deduce an isotopic triangulation of the surface. □

Notice that this bound is bigger than the size of a minimal cell decomposition, since several non-isotopic curves or surfaces yield the same size for the minimal decomposition (eg. just take distinct configurations of ovals in the plane) and does not compare with the bounds on connected components (see eg. [24]) or the complexity of the semi-algebraic set [144] or with output size bounds in [18].

From an effectiveness point of view, we have to compute an approximate or exact representation of the real roots of system (5.1) and then to compare their coordinates in order to deduce the connections. This can be performed effectively by using a rational univariate representation of the roots and Sturm (Habicht) sequences [66], [13], [53].

In [113], [114] an analysis of the number of isotopy types of a smooth plane algebraic curve of degree d is given. It is shown that this number is exponentially weakly equivalent[1] to e^{d^2} when $d \to \infty$.

Using the sweeping algorithm in 2D [65], we can prove that the number of isotopy classes for general planar curves of degree d is exponentially weakly bounded by e^{d^3}. The proof is similar to the one that we detail now for surfaces:

Proposition 5.7.2. *The number of isotopy types of an algebraic surface of degree d is exponentially weakly bounded by e^{d^7}.*

Proof. Assume the surface is in generic position (see definition 5.4.1) and that moreover the projected curve $\mathcal{C}_{x,y}$ has at most one x-critical point for each x and that the number of points on $\mathcal{C}$ above an x-critical point of $\mathcal{C}_{x,y}$ is ≤ 2. These conditions can be satisfied by a generic change of coordinates.

As the degree of $\mathcal{C}_{x,y}$ is $\leq d^2$, it has at most d^4 x-critical points. We consider d^4 x-critical sections which intersect $\mathcal{C}_{x,y}$ in at most d^2 points, above which we have at most d points on S. This yields a total of d^7 points. To each of these points, we associate the value

- 0 if it is not in the section of S,
- r if it is a regular point of the section of S,
- c if it is on the contour curve $\mathcal{C}$ and projects onto a regular point of $\mathcal{C}_{x,y}$,
- x if it is on the contour curve and projects onto an x-critical point of $\mathcal{C}_{x,y}$.

By the genericity assumption, there are at most two points with index x on an x-critical section. Similarly, we insert regular sections between these x-critical sections and regular vertical lines between the points of $\mathcal{C}_{x,y}$ at x-critical section. This gives $\mathcal{O}(d^7)$ additional points to which we associate the index 0 if the point is not on S and r otherwise.

From this information, the algorithm determines in a unique way the connections between the points of x-critical section and a consecutive regular section, if there is only one point with index x in the x-critical section. Otherwise, there are $\mathcal{O}(d)$ choices to connect the two points of index x with the other in the x-critical section and $\mathcal{O}(d^2)$ choices to connect them in the next regular sections of $\mathcal{C}$. Once these connections are chosen, they determine a unique topological complex equivalent to the surface. This shows that the number of isotopy classes of algebraic surfaces of degree d is bounded by $d^3\, 4^{\mathcal{O}(d^7)}$, which proves the proposition. □

Acknowledgments: We would like to thank Georges Comte and Michel Merle for very helpful discussions on stratification theory. We thank V. Kharlamov for suggesting this problem on the asymptotic of the number of isotopy classes of curves and surfaces. We also thank the reviewers for their precise and constructive comments.

[1] A function f is said to be exponentially bounded by (resp. weakly equivalent to) g if $\log(f) = \mathcal{O}(\log(g))$ (resp. $\log(f) = \mathcal{O}(\log(g))$ and $\log(f)^{-1} = \mathcal{O}(\log(g)^{-1})$).

Chapter 6

Bound on the local sum of Betti numbers of a germ

This chapter tackles the problem of bounding the local sum of Betti numbers of a generic affine cut of a real or complex analytic germ (see definition 6.1.9) by the multiplicity of the germ (definition 6.1.12). This result can be seen as a localization of the well-known Oleinik-Petrovsky-Thom-Milnor bound on the sum of Betti numbers of an algebraic set (see [98], proof of theorem 2, or [41], thm 4.7, and also [131, 24, 16, 17]). In his paper [88], F. Loeser also obtained results in the direction of localizing the Oleinik-Petrovsky-Thom-Milnor bound for a germ (this bound is also know as Oleinik-Petrovski's bound, hence the name of Loeser's article). His bounds are stated in terms of monodromy and homological invariants rather than in terms of the multiplicity as is our result. There is thus no straighforward relation between his result and ours.
For complex analytic varieties, the starting point is when the dimension of the affine space is the codimension of the germ. In this case the generic intersection contains only points and the sum of the Betti numbers of the cut is simply the number of points in the intersection. This generic number of points is in fact one possible definition of the local multiplicity (this definition is straightforwardly equivalent to H. Whitney's definition in [142],chapt.7,sect.7,def.7J). Therefore in that situation the sum of Betti numbers is, by definition, the multiplicity. This approach to multiplicity is especially common for hypersurfaces, as in that case the affine cutting space is a line and the multiplicity is the degree of the defining polynomial of the hypersurface (see for instance [142] for such an usage). R. N. Draper in [47] carries out a very thorough investigation of multiplicity from another angle, and characterizes it in several different manners: as an intersection number, as the geometric degree of the tangent cone to the germ (see thm.6.4 for the equivalence of these last two), as the multiplicity of the local ring of the germ (see thm.6.4 for the equivalence with this one), and as Lelong number (see thm.7.3 for the equivalence with this last one).
One can further wonder how the Betti numbers evolve when cutting by an affine

space of greater dimension than the codimension of the germ. For instance, it is natural to think that the multiplicity remains a bound for the number of connected components of the germ: if we further restrict the affine cut of the germ to an affine subspace with dimension the codimension of the germ, the affine subspace will generically intersect all the connected components at least once since the complex field is algebraically closed, and the number of intersection point will generically be the multiplicity, hence bounding the number of connected components in the affine cut by the multiplicity.

Experience from the complex case hints at the key role of multiplicity in controlling the local Betti numbers. This work is concerned with the real case, as in this context interesting integral quantities can be derived from the local Betti numbers in a generic affine cut: the local density of the germ (see [37], thm.2.1;[39], thm.2.8), and in greater generality, Lipschitz-Killing curvature invariants ([19, 39, 144], and definition 6.5.2 here) and Vitushkin variations ([77, 144], and definition 3.3.10 here). These quantities play an important role in defining notions of equisingularity for real varieties ([38, 39]). The local number of connected components in a generic affine cut is also relevant to localizing the results on entropy by means of Vitushkin's variations (see [144], thm.3.5). The relation that links the local Betti numbers in a generic affine section and these quantities is an integral known as the local multidimensional Cauchy-Crofton formula ([39], thm.3.1 for the multidimensional case, and [37], thm.2.1; [38], thm.1.10 for the original result about the density only). It inherits its name from the classical Cauchy-Crofton formula (see [57], thm.5.11, or also [58], thm.2.10.15) which is a global quantity. The sum of Betti numbers bounds both the number of connected components which serves to define Vitushkin variations and the Euler characteristic which serves to define Lipschitz-Killing curvature invariants (the density being both a Vitushkin variation and a Lipschitz-Killing invariant). By the local multidimensional Cauchy-Crofton formula it is thus possible to derive local bounds in terms of the multiplicity for the density and the Lipschitz-Killing curvature.

When considering real germs, it turns out that it is not always possible to bound the local sum of Betti numbers in terms of the multiplicity. However we show that, under some conditions (stated in thm. 6.1.21), there still is a polynomial bound in terms of the multiplicity. These conditions distinguish between the situations where a bound exists, and the situations where it cannot exist. We exhibit a bound in the former case, we provide counter-examples for latter case.

In the first place (section 6.1) we introduce the general definitions we use in this chapter. Then we state our main theorem 6.1.21 which bounds the local sum of Betti numbers in a generic affine cut of a real analytic germ by a polynomial in the multiplicity. The degree of this polynomial is the dimension of the cutting affine space. In the following section 6.2 we justify the hypotheses of the main theorem by exhibiting families of counter-examples that show that it is impossible to bound the number of connected components (the first Betti number) in terms of the multiplicity as soon as the conditions of the main theorem are not met. This shows that the conditions we give for our bound to exist are

optimal among conditions only involving the dimension of the variety and of the singular locus of its tangent cone. In section 6.3, we study the case where the intersection of the germ with an affine space is generically zero dimensional. We give a fine description of how the number of points in the intersection behaves. This analysis is required in the proof of the main theorem. This section also proves the part of the main theorem relative to the zero dimensional case. Section 6.4 finalizes the proof of the main theorem. It relies on the work in the previous section 6.3, the classical Oleinik-Petrovsky-Thom-Milnor bound for the sum of Betti numbers (see [98],proof of theorem 2, or [41], thm 4.7, and also [131, 24, 16, 17]), and a result of J. Heintz (see [74], prop.3 for a detailed proof) and D. Mumford (see [110], in proof of theorem 1, for a less detailed presentation). Finally section 6.5 looks at applications of the main theorem to bounding the local density of a real analytic germ and to controlling the Lipschitz-Killing curvature invariants ([37, 38, 39, 144]). It also discusses some algorithms to effectively compute the multiplicity of a germ from the generators of its defining ideal.

6.1 Presentation of the main result

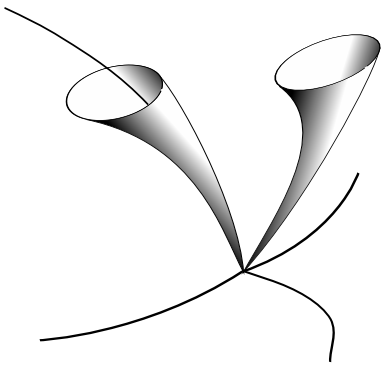


Figure 6.1: An illustration of a germ in three-space.

In this section we will state our main result about the local sum of Betti numbers in a generic affine section of a real analytic germ (theorem 6.1.21). In order to do so we introduce the basic definitions and properties we will use in the rest of this chapter.

Notations 6.1.1.

- *The notation $B_{p,\epsilon}$ will either refer to the n-dimensional open ball in $\mathbb{R}^n$ centered at a point $p \in \mathbb{R}^n$ of radius ϵ or to the complex polydisk of radius ϵ centered at $p \in \mathbb{C}^n$, that is $B_{p,\epsilon} = \{z \in \mathbb{C}^n \mid \sum_{i=1}^n |z_i - p_i|^2 < \epsilon^2\}$. Because $B_{p,\epsilon}$ in $\mathbb{R}^{2n}$ is the same object as $B_{p,\epsilon}$ in $\mathbb{C}^n$ when considering $\mathbb{C}^n$ as a $\mathbb{R}$-vector space, there is no risk of confusion and we can use the*

same notation for complex and real vector spaces.
The notation $S_{p,\epsilon}$ will denote the boundary of $B_{p,\epsilon}$ for the usual metric topology of $\mathbb{R}^n$ or $\mathbb{C}^n$ dependent on the context.

- *To avoid repetitions, in the following the notation $\mathbb{K}$ will be used as a placeholder for $\mathbb{C}$ or $\mathbb{R}$.*

- *When given a function sheaf $\mathcal{J}$ we will denote the associated scheme $V(\mathcal{J})$. Conversely, when given a variety Y we will denote its function sheaf by $\mathcal{O}(Y)$.*

- *We will call cone of $\mathbb{K}^n$ any set that is stable by multiplication by any $\lambda \in \mathbb{K}$ and not empty ($\{0\}$ is a cone).*
  Beware *that throughout this chapter we use this new definition of cone and not anymore the one in the previous section 4.1.14 which is completely different.*

- *We will often consider cones as subsets of projective space and vice-versa. Vector spaces are one such instance. To avoid confusions when it is not clear whether an object Y should be regarded as a cone or as a subset of projective space, we will adopt the convention that we add the subscript C when we want to see it as a vector space in $\mathbb{K}^n$ and we will denote it by Y_C, and we add the subscript $\mathbb{P}$ when we want to see it as a subset of $\mathbb{P}^{n-1}(\mathbb{K})$ and it will be denoted by $Y_{\mathbb{P}}$.*

- *Because of the frequent alternation between complex varieties and real varieties we will use the convention that X will always refer to a real variety, and that Z will always refer to a complex variety. The usage of other letters will remain dependent on the context.*

- *The notation $\mathbb{G}_{k,n}(\mathbb{K})$ will denote the Grassmannian of k-dimensional vector spaces in $\mathbb{K}^n$. The notation $\overline{\mathbb{G}}_{k,n}(\mathbb{K})$ will denote the Grassmannian of k-dimensional affine spaces in $\mathbb{K}^n$. For any element $\overline{P} \in \overline{\mathbb{G}}_{k,n}(\mathbb{K})$ we call canonical decomposition of $\overline{P}$, and write $\mathcal{D}(\overline{P})$ to denote the unique pair $(D,t) \in \mathbb{G}_{k,n}(\mathbb{K}) \times \mathbb{K}^n$ such that $t \in D^\perp$ and $\overline{P} = D + t$.*
  Finally we define $\tilde{\mathbb{G}}_{k,n}(\mathbb{K})$ as the blow-up of $\overline{\mathbb{G}}_{k,n}(\mathbb{K})$ along $\mathbb{G}_{k,n}(\mathbb{K})$. We write $\tilde{\mathbb{G}}^0_{k,n}(\mathbb{K})$ to denote the exceptional divisor of the blow-up. Any $\tilde{P} \in \tilde{\mathbb{G}}^0_{k,n}$ can thus be uniquely represented as a pair $(D, t_{\mathbb{P}}) \in \mathbb{G}_{k,n}(\mathbb{K}) \times \mathbb{P}^{n-1}(\mathbb{K})$ such that $t_{\mathbb{P}} \in D^\perp_{\mathbb{P}}$. We call this pair the canonical decomposition of $\tilde{P}$ and it will be denoted by $\mathcal{D}(\tilde{P})$. To avoid burdening notations, we will also freely identify $\tilde{\mathbb{G}}_{k,n}(\mathbb{K})$ with $\overline{\mathbb{G}}_{k,n}(\mathbb{K})$ outside $\tilde{\mathbb{G}}^0_{k,n}(\mathbb{K})$.
  When it is clear from the context whether $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ we will simply use $\mathbb{G}_{k,n}$, $\overline{\mathbb{G}}_{k,n}$ and $\tilde{\mathbb{G}}_{k,n}$.

- *For a bounded analytic set $Y \subset \mathbb{K}^n$, we will denote by $\mathfrak{b}_i(Y)$ the i^{th} Betti number. We set $\Sigma_{\mathfrak{b}}(Y) = \sum_{i=0}^n \mathfrak{b}_i(Y)$. And we set $\mathcal{C}(Y) = \mathfrak{b}_0(Y)$ the number of connected components of Y.*

We will use two different notions of dimension in the following:

Definition 6.1.2 (Geometric dimension)**.** *For an analytic variety $Y \in \mathbb{K}^n$, the notation* $\dim(Y)$ *will denote its geometric dimension . That is, the maximum of the dimensions of the submanifolds that can be embedded in Y. When Y is an algebraic variety the geometric dimension of Y is the Krull dimension of $\mathcal{O}(Y) \subset \mathbb{K}[X_1, \ldots, X_n]$ (see [41], thm.3.20 or less explicitely [141], thm.19.2).*

Definition 6.1.3 (Vector space dimension)**.** *For an analytic variety $Y \in \mathbb{K}^n$, the notation* $\dim_{\mathbb{K}}(Y)$ *will denote the dimension of $\mathcal{O}(Y)$ as a $\mathbb{K}$ vector space (it is therefore infinite as soon as* $\dim(Y) > 0$*).*

In this exposition, there is a constant interplay between real and complex varieties. To make our discussion clear we need to define how we relate them.

Definition 6.1.4 (Complexification)**.** *Let $X \subset \mathbb{R}^n$ be a real analytic variety. We can consider $\mathbb{R}^n$ as the subset of $\mathbb{C}^n$ of complex number with no imaginary component. By this canonical inclusion X is a subset of $\mathbb{C}^n$ (with no structure at this point). We call complexification of X, the closure of X in $\mathbb{C}^n$ for the complex analytic topology on $\mathbb{C}^n$, and we denote it by $X_{\mathbb{C}}$.*

Proposition 6.1.5. *Let $X \subset \mathbb{R}^n$ be a real analytic variety. Let $\mathcal{I} \subset \mathcal{O}(\mathbb{R}^n)$ be the defining ideal of X, that is, $\mathcal{I} = \mathcal{I}(X)$. By the canonical inclusion of $\mathbb{R}$ into $\mathbb{C}$, $\mathcal{O}(\mathbb{C}^n)$ is a $\mathcal{O}(\mathbb{R}^n)$-module and we can consider $\mathcal{J} = \mathcal{I}\mathcal{O}(\mathbb{C}^n)$. Then we have $X_{\mathbb{C}} = V(\mathcal{J})$.*

Proof. By definition the zero set Z of $\mathcal{J}$ is an analytic variety which clearly contains X, hence Z contains $X_{\mathbb{C}}$ by minimality of $X_{\mathbb{C}}$ among the complex analytic varieties containing X. To prove the reverse inclusion we prove that any function f in $\mathcal{I}(X_{\mathbb{C}}) \subset \mathcal{O}(\mathbb{C}^n)$ lies in $\mathcal{J}$. We can decompose f as its real and complex parts: $f = f_R + if_I$ where f_R and f_I lie in $\mathcal{O}(\mathbb{R}^n)$. As f vanishes on $X_{\mathbb{C}}$, so do f_R and f_I, and as $X \subset X_{\mathbb{C}}$, we can conclude that f_R and f_I vanish on X. Therefore f_R and f_I belong to $\mathcal{I}$. This shows that $f \in \mathcal{I}\mathcal{O}(\mathbb{C}^n) = \mathcal{J}$ as we have decomposed it as the sum of two function of $\mathcal{I}$. □

Definition 6.1.6 (Realisation)**.** *Let $Z \subset \mathbb{C}^n$ be a complex analytic variety. We can consider $\mathbb{C}^n$ as a $\mathbb{R}$ vector space and there is a canonical isometry between $\mathbb{C}^n$ and $\mathbb{R}^{2n}$. We call realization of Z, the image of Z in $\mathbb{R}^{2n}$ by this canonical isometry intersected back with $(\mathbb{R}^n \times \{0\})$. In other words it is the real part of Z. We denote it by $Z_{\mathbb{R}}$.*

Let us now make precise what we mean by the expression generic property of an analytic space:

Definition 6.1.7 (Generic Property)**.** *Let $\mathbf{P} : Y \to \{\text{True}, \text{False}\}$ a property on an analytic variety $Y \subset \mathbb{K}^n$. The property $\mathbf{P}$ is said to be generically true on Y iff $\mathbf{P}^{-1}(\text{False})$ is contained in a proper analytic subset of Y.*

We emphasize that contrary to some other contexts in real geometry, our notion of genericity does not mean true on a set of positive measure, but implies that the property is true almost everywhere.

Let us now define the object that we study: the local sum of Betti numbers in a generic affine section of a germ. The following lemma will show that this notion is well-defined.

Lemma 6.1.8. *Let $Y \subset \mathbb{K}^n$ be an analytic germ at 0, and choose a Betti number $\mathfrak{b}_i$. Let $\tilde{P} \in \tilde{\mathbb{G}}^0_{k,n}(\mathbb{K})$ and let $D \in \mathbb{G}_{k,n}$ and $t \in D^\perp$ such that $(D, (\mathbb{K}t)_\mathbb{P}) = \mathcal{D}(\tilde{P})$. The following function of ϵ and λ is bounded and sub-analytic:*

$$\Phi(\lambda, \epsilon) = \mathfrak{b}_i(Y \cap (\lambda t + D) \cap B_{0,\epsilon}),$$

where ϵ is a positive real number and λ a non-zero element of $\mathbb{K}$.
Therefore the following quantity is sub-analytic and well-defined:

$$\mathfrak{b}_i(Y, \tilde{P}) = \lim_{\epsilon \to 0} \lim_{|\lambda| \to 0} \mathcal{C}(Y \cap (\lambda t + D) \cap B_{0,\epsilon}),$$

where ϵ is a positive real number and λ a non-zero element of $\mathbb{K}$.

Proof. The family of sets $(B_{0,\epsilon})_{\lambda,\epsilon}$ is sub-analytic (not analytic as if $\mathbb{K} = \mathbb{C}$ even $S_{0,\epsilon}$ is not an analytic set). The family $(\lambda t + D)_{\lambda,\epsilon}$ is analytic hence sub-analytic. And the constant family $(Y)_{\lambda,\epsilon}$ is also analytic because Y is analytic, hence it is sub-analytic. Therefore the whole family $(Y \cap (\lambda t + D) \cap B_{0,\epsilon})_{\lambda,\epsilon}$ is sub-analytic. By Hardt's theorem for bounded sub-analytic sets (see [71], thm.4) there is a finite sub-analytic partition of the parameter space $(\mathbb{K} \times \mathbb{R}^+)$ such that the topological type is constant over the element of the partition (the proof in [71] is not very explicit for sub-analytic sets, alternately one can use the fact that sub-analytic sets are o-minimal structures as their complement is sub-analytic (see [62]), and use Hardt's theorem for o-minimal structures (see [40], thm.5.22)).
Thus, in particular for the i^{th} Betti number we have that $\mathfrak{b}_i(Y \cap (\lambda t + D) \cap B_{0,\epsilon})$ is constant over the elements of the partition and is thus a sub-analytic function. The homology groups of a bounded sub-analytic set are finite dimensional as those sets are triangulable ([40], thm.4.4 and also [92]), thus for any fixed pair (λ, ϵ), $\mathfrak{b}_i(Y \cap (\lambda t + D) \cap B_{0,\epsilon})$ is finite. Since there are only finitely many elements in the partition of the parameter space, $\Phi(\lambda, \epsilon)$ is bounded.
The definition of limits can be written as a first order formula, since the category of sub-analytic sets allows quantifier elimination, limits of sub-analytic functions always exist (we allow infinite limits) and are sub-analytic functions. Because $\Phi(\lambda, \epsilon)$ is bounded, we can conclude that the limits are bounded.
Notice that $\mathfrak{b}_i(Y, \tilde{P})$ is defined in terms of $\tilde{P}$ but D and t appear in its definition. Although there is only one D associated to $\tilde{P}$, t is only determined up to an element of $\mathbb{K}^*$. This is not a problem as even when $\mathbb{K} = \mathbb{R}$ we do not fix the sign of λ, so that it does not matter which representative $t \in D^\perp$ we choose for the canonical decomposition $(D, (\mathbb{K}t)_\mathbb{P})$ of $\tilde{P}$. Thus, $\mathfrak{b}_i(Y, \tilde{P})$ is well-defined as a function of $\tilde{P}$ and is finite by the previous paragraph. □

Definition 6.1.9 (Local Betti numbers)**.** *Let $Y \subset \mathbb{K}^n$ be an analytic germ at 0, let $\tilde{P} \in \tilde{\mathbb{G}}^0_{k,n}(\mathbb{K})$ and let $D \in \mathbb{G}_{k,n}$ and $t \in D^\perp$ be such that $(D, (\mathbb{K}t)_{\mathbb{P}}) = \mathcal{D}(\tilde{P})$. By lemma 6.1.8, we can define the local Betti number $\mathfrak{b}_i(Y, \tilde{P})$ for $\tilde{P}$ and Y as*

$$\mathfrak{b}_i(Y, \tilde{P}) = \lim_{\epsilon \to 0} \lim_{|\lambda| \to 0} \mathfrak{b}_i(Y \cap (\lambda t + D) \cap B_{0,\epsilon}),$$

where ϵ is a positive real number and λ a non-zero element of $\mathbb{K}$.
We say that the local Betti number in the intersection of a germ Y with an affine space is generically bounded by a constant c iff $\mathfrak{b}_i(Y, \tilde{P}) \leq c$ generically for $\tilde{P} \in \tilde{\mathbb{G}}^0_{k,n}$.
The infimum of such bounds will be denoted by $\mathfrak{b}_i(Y, k)$ in the following.
We set $\mathcal{C}(Y, \tilde{P}) = \mathfrak{b}_0(Y, \tilde{P})$, and $\mathcal{C}(Y, k) = \mathfrak{b}_0(Y, k)$. We define the local sum of Betti numbers for $\tilde{P}$ and Y as $\Sigma_{\mathfrak{b}}(Y, \tilde{P}) = \sum_{i=0}^n \mathfrak{b}_i(Y, \tilde{P})$. Finally, we define the local sum of Betti numbers $\Sigma_{\mathfrak{b}}(Y, k)$ as the infinimum of the constants $c \in \mathbb{R}$ such that $c \geq \Sigma_{\mathfrak{b}}(Y, \tilde{P})$ for $\tilde{P}$ generic in $\tilde{\mathbb{G}}^0_{k,n}$.

Remark 6.1.10. *Let $\Sigma'_{\mathfrak{b}}(Y, k) = \sum_{i=0}^n \mathfrak{b}_i(Y, k)$. Notice that we have $\Sigma_{\mathfrak{b}}(Y, k) \geq \Sigma'_{\mathfrak{b}}(Y, k)$ since $\Sigma'_{\mathfrak{b}}(Y, k)$ is a sum of infinima and is thus less than $\Sigma_{\mathfrak{b}}(Y, k)$ the infimum of the sum. The interesting point is that equality holds in the complex case because each $\mathfrak{b}_i(Y, \tilde{P})$ is maximal on a generic set of $\tilde{\mathbb{G}}^0_{k,n}$. Hence they are all maximal at once on the intersection of these generic sets (which is generic), and we have $\Sigma_{\mathfrak{b}}(Y, k) \leq \Sigma'_{\mathfrak{b}}(Y, k)$.*

Let us now give a definition for the multiplicity of a germ Y.

Definition 6.1.11 (Point Multiplicity)**.** *For a point $p \in \mathbb{K}^n$ defined by a primary ideal $\mathcal{I}$ (i.e. the radical of $\mathcal{I}$ is the maximal ideal vanishing at p), the multiplicity $\mu(p)$ of p is $\dim_{\mathbb{K}} (\mathcal{O}(\mathbb{K}^n)/\mathcal{I})$.*

Definition 6.1.12 (Germ Multiplicity)**.** *If $Y \subset \mathbb{K}^n$ is an analytic germ of dimension d, the $\mathbb{K}$-multiplicity $\mu_{\mathbb{K}}(Y)$ of Y is $\mathcal{C}(Y, n-d)$ $(= \Sigma_{\mathfrak{b}}(Y, n-d))$.*
When $\mathbb{K} = \mathbb{R}$ we will refer to the $\mathbb{R}$-multiplicity as the real multiplicity. When $\mathbb{K} = \mathbb{C}$ we will refer to the $\mathbb{C}$-multiplicity as the complex multiplicity.
When given a real analytic germ X, the term multiplicity of X will refer to $\mu_{\mathbb{C}}(X_{\mathbb{C}})$, the complex multiplicity of its complexification. We will denote it by $\mu(X)$.

Remark 6.1.13. *The previous definition is two-fold as it applies for $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$ and the meaning of $\mathcal{C}(Y, n-d)$ is dependent on the nature of $\mathbb{K}$. The commonly accepted definition of multiplicity is the complex multiplicity ([142]). In our context we should pay special attention to distinguishing between the real multiplicity $\mu_{\mathbb{R}}(X)$ of a real analytic germ X and its multiplicity $\mu(X) = \mu_{\mathbb{C}}(X_{\mathbb{C}})$.*
We will indeed bound $\mu_{\mathbb{R}}(X)$ by $\mu(X) = \mu_{\mathbb{C}}(X_{\mathbb{C}})$ in the main theorem.

As we mentioned in the introduction, for a real analytic germ X, it is not always possible to bound $\mathcal{C}(X, k)$ (hence $\Sigma_{\mathfrak{b}}$) in terms of $\mu_{\mathbb{C}}(X_{\mathbb{C}})$. The object that will enable us to formulate the condition under which such a bound exists is the algebraic tangent cone:

Definition 6.1.14 (Algebraic tangent cone)**.** *Let $Y \subset \mathbb{K}^n$ be an analytic germ at 0. Let $b : x \in \mathbb{K}^n/\{0\} \mapsto (x, (\mathbb{K}x)_{\mathbb{P}}) \in \mathbb{K}^n \times \mathbb{P}^{n-1}(\mathbb{K})$ be the blow-up map at the origin. Let $\tilde{Y} = \overline{b(Y)} \subset \mathbb{K}^n \times \mathbb{P}^{n-1}(\mathbb{K})$ be the adherence of $b(Y)$ in $\mathbb{K}^n \times \mathbb{P}^{n-1}(\mathbb{K})$, and $\mathcal{I}$ its defining function sheaf ($\tilde{Y}$ is thus the blow-up of Y at the origin). Let $E = \{0\} \times \mathbb{P}^{n-1}(\mathbb{K})$ the exceptional divisor associated to b, and $\mathcal{J}$ the restriction of $\mathcal{I}$ to E. $\mathcal{J}$ is thus a function sheaf on $\mathbb{P}^{n-1}(\mathbb{K})$.*

As we have an analytic variety in $\mathbb{P}^{n-1}(\mathbb{K})$ it is actually algebraic (see [69], thm.7 for a proof). As mentioned earlier, any function sheaf on $\mathbb{P}^{n-1}(\mathbb{K})$ can be regarded as a cone centered at the origin in $\mathbb{K}^n$. The algebraic tangent cone $T(Y)$ to Y is the scheme in $\mathbb{K}^n$ associated to $\mathcal{J}$. In other words we have $T(Y) = V(\mathcal{J})_C$ and $\mathcal{O}(T(Y)_{\mathbb{P}}) = \mathcal{J}$.

Definition 6.1.15. *For a polynomial $f \in \mathbb{K}[X_1, \ldots, X_n]$ the initial part of f is sum of the monomials of f which have the lowest total degree in $X_1, \ldots, X_n$.*

Remark 6.1.16. *This tangent cone is the same as the cone C_3 defined by H.Whitney in [140], def.3 (and also in [142], def.1G). In [140], thm.5.8 it is proven that if $\mathbb{K} = \mathbb{C}$ the ideal $\mathcal{J} \subset \mathcal{O}(\mathbb{C}^n)$ defining $T(Y)$ is exactly the ideal of the initial parts of the functions of the defining ideal of the germ Y. The result in [140] is in fact stronger as it asserts that all the points in $T(Y)$ (as a geometric variety this time) come from a limit of secants at the origin of the germ. The result is reproduced below as proposition 6.1.17.*
Notice that for a real analytic germ X, we have $(X_{\mathbb{C}})_{\mathbb{R}} = X$. Therefore the blow-up of X is the realization of the blow-up of $X_{\mathbb{C}}$. This shows that the same result holds for real analytic germs too, that is, their algebraic tangent cone is defined by the initial parts of their function sheaf. However, beware that in the real case the tangent cone can have points that do not come from limits of secants of the germ, as opposed to the complex case where the cone is exactly the set of limits of secants.

Proposition 6.1.17. *Let $Y \in \mathbb{K}^n$ be an analytic germ at 0. Let $\mathcal{J} = I(X) \subset \mathbb{K}[X_1, \ldots, X_n]$. For any $\epsilon \in \mathbb{K}$, let $J_\epsilon = \{f(\epsilon X_1, \ldots, \epsilon X^n)/\epsilon^{\mu(f)} \mid f \in \mathcal{J}\}$ where $\mu(f)$ is the total degree of the initial part of f (which happens to be its multiplicity at 0). Let $\tilde{Y}_\epsilon = V(\mathcal{J}_\epsilon) \subset \mathbb{K}^n \times \mathbb{K}$. The family $\tilde{Y}_\epsilon$ is clearly analytic and its fiber at 0, $\tilde{Y}_0$, is equal to the algebraic tangent cone of Y, $T(X)_C$ as by definition 6.1.14.*

Proof. This is a mere rephrasing of theorem 5.8, in [140]. □

Definition 6.1.18 (Conic Blow-up)**.** *We call the family defined in the previous lemma, conic blow-up of Y at the origin, and we denote it by $(\tilde{Y}_\epsilon)_{\epsilon \in \mathbb{K}}$.*

Remark 6.1.19. *We emphasize the fact that because the algebraic tangent cone is a scheme and not a geometric variety, it can be singular although the associated variety is smooth. For instance the algebraic tangent cone to $X = V(x^2 - y^3)$ in $\mathbb{R}^2$ is defined by x^2. The algebraic tangent cone to X is therefore entirely singular. Keeping track of the multiplicities in the algebraic tangent cone will allow us to control the way the germ deforms into its tangent*

cone and thus to count the number of connected components in the intersection with the cone instead of the germ itself.

To avoid any possible confusion as to what we mean by singular for the algebraic tangent cone, we give the classical definition of singularity in that case:

Definition 6.1.20 (Singularity of tangent cone)**.** *If $T \subset \mathbb{K}^n$ is an algebraic tangent cone defined by an ideal $\mathcal{J}$, and $p \in \mathbb{K}^n$. Then T is singular at p iff $\dim(T_p) < \dim_{\mathbb{K}}\left(\mathcal{J}/M^2\right)$ where T_p is the localization at p of T and $M \subset \mathcal{O}(\mathbb{K}^n)$ is the maximal ideal at p.*
The singular locus of T will be denoted by $\mathrm{Sing}(T)$.

We can now state the main theorem:

Theorem 6.1.21 (Main Theorem)**.** *Let $X \subset \mathbb{R}^n$ be a real analytic germ at 0. Let T be the algebraic tangent cone to X. Then*

- *If $k = n - \dim(X)$, we have the inequality $\mu_{\mathbb{R}}(X) \leq \mu(X)$. Since $k = n - \dim(X)$, $\mathfrak{b}_0(X,k)$ is the only non-zero Betti number and the previous inequality can be rewritten $\Sigma_{\mathfrak{b}}(X,k) \leq \mu(X)$.*
- *For any $k \in \mathbb{N}$, if $k + \dim(\mathrm{Sing}(T)) < n$ and $X_{\mathbb{C}}$ is pure dimensional, then*

$$\Sigma_{\mathfrak{b}}(X,k) \leq \mu(X)(2\mu(X)-1)^{k-1}.$$

Remark 6.1.22. *The first item in the main theorem 6.1.21 can be seen as a generalization of lemma 1.4 in [135]. In the vocabulary of this chapter, this lemma implies that the real multiplicity is bounded by the complex multiplicity when X is a hypersurface. The first item of the main theorem states the same fact but generalizes it to the case of an arbitrary germ X.*

The second item of the main theorem 6.1.21 requires that $X_{\mathbb{C}}$ be pure dimensional. It is interesting to notice that this is a weaker requirement than assuming that X itself is pure dimensional. This is what the following proposition proves:

Proposition 6.1.23. *Let $X \subset \mathbb{R}^n$ be a real analytic germ at 0. If X is pure dimensional, then $X_{\mathbb{C}}$ is also pure dimensional.*

Proof. Suppose that $X_{\mathbb{C}}$ is not pure dimensional. Let $\mathfrak{I}$ be the set of irreducible components of $X_{\mathbb{C}}$. To prove the proposition it is necessary and sufficient to prove that for any $i \in \mathfrak{I}$, we have $\dim(i) = \dim(X)$.
Firstly, the realization of any irreducible components $i \in \mathfrak{I}$ is non-empty. If it were, by definition of irreducible components, it could be removed from $X_{\mathbb{C}}$, and as i avoids $\mathbb{R}^n$, the new variety would still contain X. This would contradict the definition of $X_{\mathbb{C}}$ as the minimal variety containing X.
For any $i \in \mathfrak{I}$, the dimension of i is the dimension of the tangent space to i at any of its smooth points. We have just proved that i intersects $\mathbb{R}^n$, hence we can find a point $x \in i_{\mathbb{R}}$. Let C be the connected component of $i_{\mathbb{R}}$ which contains x.

In the same way as for any real variety, the points $y \in C$ at which $i_{\mathbb{R}}$ is smooth are generic on $i_{\mathbb{R}}$. We can thus further require that $y \in C$ be smooth on X. As y and x are in the same connected component of $i_{\mathbb{R}}$, they are *a fortiori* in i. By Taylor expansion at y, we can conclude that y is smooth on i, and that the real tangent space to X at y has the same dimension as the complex tangent space to i at y. As X is pure dimensional, we can conclude that

$$\dim(X) = \dim_{\mathbb{R}}(T(X,y)) = \dim_{\mathbb{C}}(T(i,y)) = \dim(i).$$

□

Remark 6.1.24. *The implication in the previous proposition is not an equivalence since some real varieties which are not pure dimensional have pure dimensional complexification. One such example is the variety defined by* $f(x,y) = x^4+4x^3+4x^2-y^2$*: the graph has two humps, one emerges over the plane* $f=0$ *and the other one is tangent to it, which creates an oval and a point. However, the complex variety is pure dimensional as* f *is irreducible (this can be seen by trying to factor* f *by hand as its degree in* y *is only* 2*).*

The following theorem precisely states how theorem 6.1.21 is optimal:

Theorem 6.1.25 (Optimality of Main Theorem)**.** *For any* $n, k \in \mathbb{N}$ *such that* $2 \leq k \leq n-1$*, we have the following:*

- *there exist real analytic germs* $(X_l)_{l\in\mathbb{N}}$ *such that* $\lim_{l\to\infty} \mathcal{C}(X_l,k) = +\infty$ *and such that* $k + \dim(\mathrm{Sing}(T)) < n$*, but the* X_l *are not pure dimensional.*
- *for any* $m \in \mathbb{N}$ *such that* $n \leq m \leq k+n-1$*, there exist pure dimensional real analytic germs* X_l *such that* $\lim_{l\to\infty} \mathcal{C}(X_l,k) = +\infty$ *and such that* $k + \dim(\mathrm{Sing}(T)) = m$ *where* T *is the algebraic tangent cone to* X*.*

Remark 6.1.26. *Notice that the conditions in the optimality theorem are exactly the complement of the conditions of the main theorem. That is, for any possible combination of* $k, n, \dim(\mathrm{Sing}(T)) \in \mathbb{N}$*, there either exist a counter-example family with those characteristics, or the bounds of the main theorem apply.*

We will now move on to the counter-examples section which gives an understanding of why it is not possible to find a bound in terms of the multiplicity for any type of germ. The counter-examples section shows that the conditions that appear in the main theorem are optimal by proving the optimality theorem 6.1.25.

6.2 Counter-examples

In this section we show that the main theorem is optimal. The optimality theorem 6.1.25 is proved by means of counter-examples whose first Betti number is not controllable by the multiplicity of the germ (hence *a fortiori* neither is

$\Sigma_{\mathfrak{b}}$). The first example (subsection 6.2.1) shows why it is necessary to assume that the real germ is pure dimensional. This corresponds to the first item of the optimality theorem 6.1.25. Then (subsection 6.2.2) we show two examples in three-space that show why it is necessary to have $\dim(\mathrm{Sing}(T)) + k < n$ (where T it is the algebraic tangent cone of the germ). These two examples prove the optimality theorem in the case $n = 3$. Finally subsection 6.2.3 generalizes the first example of subsection 6.2.2 and gives two general transformations for families of germs that enable us to prove the second claim of the optimality theorem 6.1.25 for any n.

6.2.1 Necessity of pure dimensionality

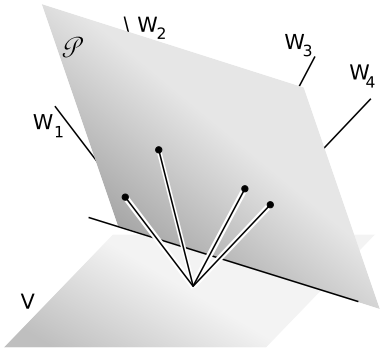


Figure 6.2: The tangent cone is the germ itself, but a generic planar cut of the germ has as many connected components has desired because the W_i do not count toward the multiplicity.

The first example shows a phenomenon that has nothing specific to the real case and that happens similarly in the complex case. It is due to the fact that only the top dimensional components of a germ are taken into account by the multiplicity, but of course, lower dimensional components of the germ can provide connected components when cutting with an affine space whose dimension is greater than the codimension of the germ. Consequently, these lower dimensional components do not change the multiplicity but can make the number of connected components in a generic cut grow arbitrarily large.
The construction of the counter-example is identical for complex and real germs. However we construct it as a real germ since this is what we are interested in to prove the optimality of the main theorem.

Here is one of many ways to construct such germs. For any given $l \in \mathbb{N}$, let V be a d-dimensional vector space. Let $W_1, \ldots, W_l$, and let $k \in \mathbb{N}$ be such that $n - k < d$. We can clearly find $(n-k)$-dimensional vector spaces such that $\forall i \neq j,\ W_i \cap W_j = \{0\}$ and $W_i \cap V = \{0\}$. Let X be the union of V and all the $W_1, \ldots, W_l$. The situation for $n = 3,\ d = 2,\ k = 2,\ l = 4$ is depicted in figure 6.2.

The multiplicity of X is clearly $\mu = 1$ as a $(n-d)$-dimensional affine space will generically avoid all the $W_1, \ldots, W_l$ (as $(n-d)+(n-k) < (n-d)+d = n$), and the number of points in a generic intersection with V is 1. On the other hand, when intersecting X with a generic k-dimensional affine space that does not go through 0, the affine space cuts V and all the $W_1, \ldots, W_l$. As they only intersect at 0 each one of them gives rise to a different connected component of the intersection. Hence there are $l+1$ connected components in the intersection. Notice by the way, that X and its tangent cone (which are here the same thing) have an isolated singularity at 0 and that the intersection with a generic k-dimensional affine space is smooth, therefore the only condition of the main theorem that is not satisfied is that the germ should be pure dimensional. This proves the first claim of the optimality theorem 6.1.25.

6.2.2 Necessity of controlling the tangent cone: 3d case

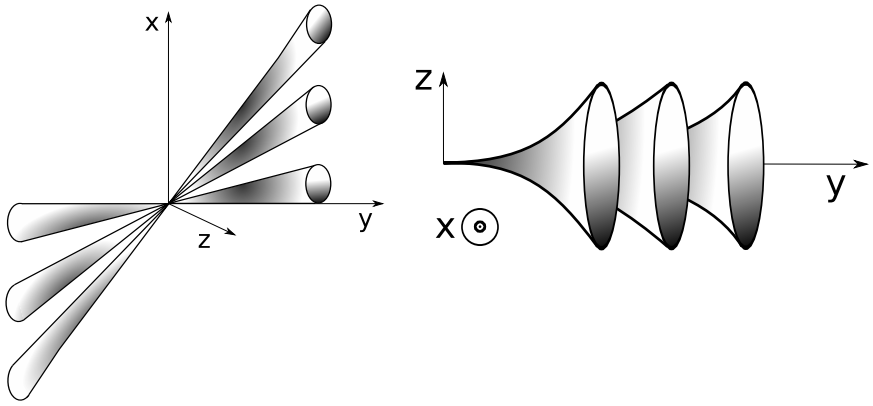


Figure 6.3: The tangent cone is a singular plane, but a generic planar section of the germ has as many connected components as desired because they degenerate.

The first family of counter-examples in $\mathbb{R}^3$ is given by the equations

$$f_l = z^2 + x(x-y)(x-2y)\ldots(x-(2l-1)y),$$

for any positive integer l. For every l, let $X_l = V(f_l)$ be the germ at the origin defined by f_l. As the f_l define hypersurfaces it is an easy computation to check that the f_l have multiplicity 2 by injecting the parameterization of a generic complex line into the equation. Thus we have $\mu(X_l) = 2$ for any l.
Figure 6.3 shows what X_3 looks like. If we look at cuts of X_l by $x = \alpha$ planes as α converges to 0, we can see that $X_l \cap \{y = \alpha\}$ contains l ovals that flatten on the $z = 0$ plane in the z direction, but their width in the y direction decreases linearly with α: their width in the y direction is always α, and the distance between one oval and the next is also always α. This shows how the variety degenerates to its singular tangent cone whose defining ideal is (z^2). It is clear

from this description and the figures that for any plane P that does not go through 0 and cuts the y-axis transversely the intersection of P and X_l contains exactly l ovals. This shows that $\mathcal{C}(X_l, 2) = l$.

Nevertheless the previous example is not entirely satisfactory even in three-space as the second claim of the optimality theorem 6.1.25 states that we can find counter-examples for any $k \in \{2, \ldots, n-1\}$ and $k + \dim(\mathrm{Sing}(T)) = m \in \{n, \ldots, n+k-1\}$. Here $n = 3$, therefore only $k = 2$ is possible, but m can be 3 or 4. The counter-example family we just gave is of type $m = 4 (= 2 + \dim(z^2))$. So we lack a counter-example with $m = 3$, that is, a counter-example where the tangent cone is 1-dimensional. This is slightly more difficult to create as all the connected components have to contract to a single line. The next example shows one way to do it.

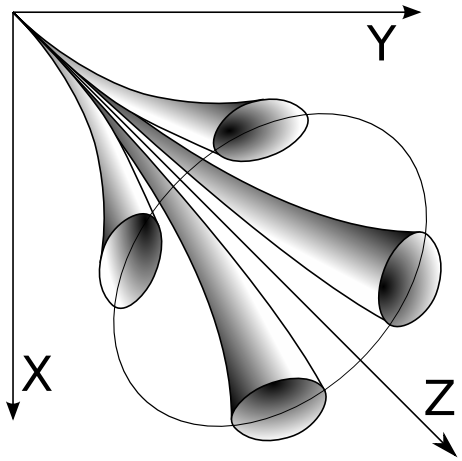


Figure 6.4: The tangent cone is a singular line, but a generic planar section of the germ has as many connected components as desired because they degenerate.

This example is constructed in a similar way as the previous one, but instead of simply degenerating flatly to a line, the ovals will degenerate to a point. Consider the family

$$g_l = (x^2 + y^2 - z^4)^2 + \prod_{i=0}^{2l-1} \left(y - \frac{i - l + 0.5}{l + 0.5} z^2 \right).$$

Let X_l be the associated varieties. By plugging in the parameterization of a complex line it is clear that the multiplicity at 0 of the X_l is 4 (as soon as $l \geq 2$). The tangent cone is $(x^2 + y^2)^2$ which defines a real singular line.
We are going to show that for this family $\mathcal{C}(X_l, 2) = 2l$. Figure 6.4 represents X_2. One can see that the connected components are grouped on a cusped cone that degenerates to the tangent cone defined by $x = y = 0$ (with multiplicity 4). On $z = \alpha$ cuts, the $X_l \cap \{z = \alpha\}$ contain exactly $2l$ ovals that flatten on the cusped

cone. The larger l the faster the ovals will flatten on the cusped cone. But, no matter l, the size of the projections of the ovals to the y line always decreases quadratically with α: their width is always $2\alpha^2/(2l+1)$ and the distance from the projection of one oval to the next is also always $2\alpha^2/(2l+1)$. As l grows, more ovals group on the cusped cone. Here is a more detailed analysis of what happens, and why it happens:
The first summand of g_l defines a cusped cone at the origin. For a fixed z the cone defines a circle centered at $x = y = 0$ of radius z^2. The second summand of g_l is a polynomial of degree $2l$, therefore for a fixed z, it will be positive for

$$y < \left(-1 + \frac{1}{l+0.5}\right) z^2 \text{ and } y > \left(1 - \frac{1}{l+0.5}\right) z^2.$$

When summing the two polynomials for a fixed z, the resulting variety is empty for

$$y \in \left]-\infty, \left(-1 + \tfrac{1}{l+0.5}\right) z^2\right[,$$

$$\left(-1 + \tfrac{2i}{l+0.5}\right) z^2 < y < \left(-1 + \tfrac{2i+1}{l+0.5}\right) z^2 \text{ for } i \in \{1, \ldots, l-1\},$$

$$y \in \left]\left(1 - \tfrac{1}{l+0.5}\right) z^2, \infty\right[.$$

The resulting variety is made of two ovals on each slice of the form

$$y \in \left[\left(-1 + \frac{2i+1}{l+0.5}\right) z^2, \left(-1 + \frac{2i+2}{l+0.5}\right) z^2\right] \text{ for } i \in \{0, \ldots, l-1\}.$$

From this description and figure 6.4 it should be clear that when cutting X_l with a plane P that does not go through the origin and cuts the z-axis transversely, there are exactly $2l$ ovals in $P \cap X_l$.

This gives an example of a family such that $k = 2$, $n = 3$ and $m = 3$. For $n = 3$, we have found an example for each of the two possible combinations of (k, n, m), hence proving the optimality theorem 6.1.25 for $\mathbb{R}^3$. The next subsection shows how to obtain counter-examples for every possible combination of k, n and m.

6.2.3 Necessity of controlling the tangent cone: general case

We extend the first counter-example of the previous subsection to create counter-examples for any (k, n, m) such that $2 \leq k \leq n-1$ and $n \leq m \leq n+k-1$ (as mentioned in the hypotheses of the optimality theorem 6.1.25). The first example was of type $k = 2,\ n = 3,\ m = 4$. From now on we will say that it is of type $(2, 3, 4)$ for simplicity. The second example was such that $k = 2$, $n = 3$, $\dim(\mathrm{Sing}(T)) = 1$, that is of type $(2, 3, 3)$. These are the only possible combination for $n = 3$, and 3 is the smallest possible value for n (as $2 \leq k \leq n-1$). To generate any possible combination of (k, n, m) we define a generalization of the first example and two transformations that we can apply to any family of germs.

We will prove that we can generate counter-examples of any type (k, n, m) with them.

Definition 6.2.1 (Generalized families)**.** *The generalization in $\mathbb{R}^{h+3}$ of the first example ($f_l = z^2 + x(x-y)(x-2y)\dots(x-(2l-1)y)$) is given by the following equation in the variables $x, y, z, t_1, \dots, t_h$:*

$$f'_l = z^2 + x(x-y)(x-2y)\dots(x-(2l-1)y) + \frac{1}{2^{2l}}\prod_{i=1}^{h}(t_i^2 - y^2)^l.$$

The associated varieties of $\mathbb{R}^{h+3}$ are denoted by X'_l.

Proposition 6.2.2. *The generalized family X'_l in $\mathbb{R}^{k+1}$ is of type $(k, k+1, 2k)$ for any $k \geq 2$. That is, $\dim(\mathrm{Sing}(T(X'_l))) = k$ and $\lim_{l\to\infty} \mathcal{C}(X'_l, k) = +\infty$. Also, $\mu(X'_l) = 2$ for all l and the $(X'_l)_{\mathbb{C}}$ are pure dimensional.*

Proof. The affine spaces by which we cut X'_l are hyperplanes (as $k+1$ is the dimension of the ambient space). Therefore a generic hyperplane does not go through 0 and can be linearly parametrized by the variables $x, y, t_1, \dots, t_{k-2}$. In other words, z can be expressed in terms of the other variables. We plug the expression of z into f'_l and we obtain equations of hypersurfaces in $\mathbb{R}^k$ whose topology is the same as the topology of $P \cap X'_l$. In the following we keep the notation z for simplicity, but it has to be regarded as an expression in the other variables.
The equation f'_l can be split into two parts:

$$z^2 + x(x-y)(x-2y)\dots(x-(2l-1)y)$$

and

$$\tfrac{1}{2^{2l}}\textstyle\prod_{i=1}^{k-2}(t_i^2 - y^2)^l.$$

The first part corresponds to what has been studied in the previous example in $\mathbb{R}^3$, it thus defines l ovals in $\mathbb{R}^2$ that are prolonged in the $k-2$ remaining dimensions corresponding to the t_i.
Notice that the function $x(x-y)(x-2y)\dots(x-(2l-1)y)$ is strictly greater than $y^{2l}/2^{2l}$ for any $x \in \{-y/2, 3y/2, 7y/2, \dots, (4l+1)y/2\}$, hence *a fortiori* $z^2 + x(x-y)(x-2y)\dots(x-(2l-1)y)$ has greater values than $y^{2l}/2^{2l}$. But the second part is constructed so that it does not take values greater than $y^{2l}/2^{2l}$ for $t_i \in [-y, y]$. Thus when summing the two parts, the l ovals will always stay apart. On the other hand, the second part diverges to $+\infty$ when the t_i tend to ∞. Consequently, when summing the two parts, the $(k-2)$-dimensional cylinder of l ovals are limited to a compact set of $\mathbb{R}^k$ by the second part. This shows that $P \cap X'_l$ is generically made of l $(k-1)$-spheres. Hence $\lim_{l\to\infty} \mathcal{C}(X'_l, k) = \infty$.

By injecting the parameterization of a generic line it can be seen that the multiplicity of the X'_l is 2. The complexification is pure dimensional as the defining equation f'_l is irreducible (it can be checked by hand as the degree in z is only 2).
By applying the definition of the algebraic tangent cone, or straightforwardly

using proposition 6.1.17, one sees that the tangent cone of X_l' is defined by z^2. As we are in $\mathbb{R}^{k+1}$ this means that $\dim(\mathrm{Sing}(T(X_l'))) = k$ thus $m = 2k$ as required. □

The two transformations we can apply to families of germs are the product transformation and the embedding transformation:

Definition 6.2.3 (Germ Transformations)**.** *If X_l is a family of real germs in $\mathbb{R}^n$, the h-product transformation of X_l is*

$$\mathfrak{P}_h(X_l) = X_l \times \mathbb{R}^h \ \subset \mathbb{R}^{n+h}.$$

The h-embedding transformation of X_l is

$$\mathfrak{E}_h(X_l) = X_l \times \{0\} \ \subset \mathbb{R}^{n+h}.$$

Proposition 6.2.4. *If a family of real germs X_l in $\mathbb{R}^n$ is of type (k, n, m) then $\mathfrak{P}_h(X_l)$ is of type $(k, n+h, m+h)$ and $\mathfrak{E}_h(X_l)$ is of type $(k+h, n+h, m+h)$. Also the multiplicity of the germs is left unchanged after transformation and if the $(X_l)_{\mathbb{C}}$ were pure dimensional, the $\mathfrak{P}_h(X_l)_{\mathbb{C}}$ and $\mathfrak{E}_h(X_l)_{\mathbb{C}}$ will also be pure dimensional.*

Proof. **Treatment of h-product transformations**:
Obviously $\mathfrak{P}_h(X_l)$ lies in $\mathbb{R}^{n+h}$ and $\mathrm{Sing}(T(\mathfrak{P}_h(X_l))) = \mathrm{Sing}(T(X_l)) \times \mathbb{R}^h$. Therefore it suffices to check that $\mathcal{C}(\mathfrak{P}_h(X_l), k) = \mathcal{C}(X_l, k)$ as we know that $\mathcal{C}(X_l, k)$ tends to infinity. Consider $\Pi : \mathbb{R}^{n+h} \to \mathbb{R}^n$ the orthogonal projection to $\mathbb{R}^n$. Consider $\tilde{P} \in \tilde{\mathbb{G}}^0_{k,n+h}$ and $(D, t) \in \mathbb{G}_{k,n+h} \times D^\perp$ such that $\mathcal{D}(\tilde{P}) = (D, (\mathbb{R}t)_{\mathbb{P}})$. As X_l is of type (k, n, m) we know that $k < n$. Therefore for $\tilde{P}$ generic, $\Pi(t) \neq 0$, $D \in \mathbb{G}_{k,n+h}$ is isomorphic to its image in $\mathbb{R}^n$ by Π and so are all the $\lambda t + D$. For such $\tilde{P}$, we define $\Pi(\tilde{P}) \in \tilde{\mathbb{G}}^0_{k,n}$ as the only element in $\tilde{\mathbb{G}}^0_{k,n}$ such that $\mathcal{D}(\Pi(\tilde{P})) = (\Pi(D), \Pi(\mathbb{R}t))$. We have just proved that $\Pi(\tilde{P})$ is defined on a generic set of $\tilde{\mathbb{G}}^0_{k,n+h}$, and the image of a generic subset of $\tilde{\mathbb{G}}^0_{k,n+h}$ by Π is a generic subset of $\tilde{\mathbb{G}}^0_{k,n}$. Consequently, for a generic $\tilde{P} \in \tilde{\mathbb{G}}^0_{k,n+h}$, we have the equalities

$$\mathcal{C}(\mathfrak{P}_h(X_l), k) = \mathcal{C}(\mathfrak{P}_h(X_l), \tilde{P}) = \lim_{\epsilon \to 0} \lim_{|\lambda| \to 0} \mathcal{C}(\mathfrak{P}_h(X_l) \cap (\lambda t + D) \cap B_{0,\epsilon}),$$

$$\mathcal{C}(X_l, k) = \mathcal{C}(X_l, \Pi(\tilde{P})) = \lim_{\epsilon \to 0} \lim_{|\lambda| \to 0} \mathcal{C}(X_l \cap \Pi(\lambda t + D) \cap B_{0,\epsilon}).$$

Since $\mathfrak{P}_h(X_l)$ has product topology and $(\lambda t + D)$ is a function graph over its image by Π, we can conclude that

$$(\lambda t + D) \cap \mathfrak{P}_h(X_l) \overset{\Pi}{\approx} \Pi(\lambda t + D) \cap \Pi(\mathfrak{P}_h(X_l)) = \Pi(\lambda t + D) \cap X_l,$$

where $\overset{\Pi}{\approx}$ means that Π is an isomorphism between the two sets. Therefore we can conclude that the two above limits are equals, hence $\mathcal{C}(X_l, k) = \mathcal{C}(\mathfrak{P}_h(X_l), k)$

which is what we wanted. The same reasoning with complex lines shows that the multiplicity of $\mathfrak{P}_h(X_l)$ is the same as the multiplicity of X_l. If X_l was pure dimensional, as the components of $\mathfrak{P}(X_l)$ are the product of the components of X_l by $\mathbb{R}^h$, $\mathfrak{P}(X_l)$ is pure dimensional too.

Treatment of h-embedding transformations:
Clearly $\mathfrak{E}_h(X_l)$ lies in $\mathbb{R}^{n+h}$ and $\text{Sing}(T(\mathfrak{E}_h(X_l))) = \text{Sing}(T(X_l)) \times \{0\}$. Therefore it suffices to check that $\mathcal{C}(\mathfrak{E}_h(X_l), k+h) = \mathcal{C}(X_l, k)$ as by hypothesis $\mathcal{C}(X_l, k)$ tends to infinity. For any $\tilde{P} \in \tilde{\mathbb{G}}^0_{k+h,n+h}$, let $(D,t) \in \mathbb{G}_{k+h,n+h} \times D^\perp$ such that $\mathcal{D}(\tilde{P}) = (D, (\mathbb{R}t)_\mathbb{P})$. For a vector space $D \in \mathbb{G}_{k+h,n+h}$ transverse to $\mathbb{R}^n \times \{0\}$, we define $C(D) = D \cap (\mathbb{R}^n \times \{0\}) \in \mathbb{G}_{k,n}$ and $C(t)$ such that $(t+D) \cap (\mathbb{R}^n \times \{0\}) = C(t) + C(D)$ (hence $(\lambda t + D) \cap (\mathbb{R}^n \times \{0\}) = \lambda C(t) + C(D)$). For $\tilde{P}$ generic, $C(t) \neq 0$ and $C(D)$ is a k-dimensional vector space as D is $(k+h)$-dimensional in $\mathbb{R}^{n+h}$ and $\mathbb{R}^n \times \{0\}$ is a n-dimensional vector space. For such $\tilde{P} \in \tilde{\mathbb{G}}^0_{k+h,n+h}$ we define $C(\tilde{P})$ as the only element of $\tilde{\mathbb{G}}^0_{k,n}$ such that $\mathcal{D}(C(\tilde{P})) = (C(D), (\mathbb{R}C(t))_\mathbb{P})$. We have just shown that C is defined on a generic set of $\tilde{\mathbb{G}}^0_{k+h,n+h}$ and the image by C of a generic subset of $\tilde{\mathbb{G}}^0_{k+h,n+h}$ is a generic subset of $\tilde{\mathbb{G}}^0_{k,n}$. Consequently for a generic $\tilde{P} \in \tilde{\mathbb{G}}^0_{k+h,n+h}$ we have the equalities

$$\mathcal{C}(\mathfrak{E}_h(X_l), k) = \mathcal{C}(\mathfrak{E}_h(X_l), \tilde{P}) = \lim_{\epsilon \to 0} \lim_{|\lambda| \to 0} \mathcal{C}(\mathfrak{E}_h(X_l) \cap (\lambda t + D) \cap B_{0,\epsilon}).$$

$$\mathcal{C}(X_l, k) = \mathcal{C}(X_l, C(\tilde{P})) = \lim_{\epsilon \to 0} \lim_{|\lambda| \to 0} \mathcal{C}(X_l \cap (\lambda C(t) + C(D)) \cap B_{0,\epsilon}).$$

Since $\mathfrak{E}_h(X_l)$ is already contained in $\mathbb{R}^n \times \{0\}$, the map C is an isomorphism between $\mathfrak{E}_h(X_l) \cap (\lambda t + D) \cap B_{0,\epsilon}$ and $X_l \cap (\lambda C(t) + C(D)) \cap B_{0,\epsilon}$. We can thus conclude that the two above limits are equals, hence $\mathcal{C}(X_l, k) = \mathcal{C}(\mathfrak{E}_h(X_l), k+h)$ which is what we wanted.
The same reasoning with $(n + h - \dim(X_l))$-dimensional complex affine spaces shows that the multiplicity of $\mathfrak{E}_h(X_l)$ is the same as the multiplicity of X_l. As $\mathfrak{E}_h(X_l)$ is a linear embedding of X_l, the dimensionality of the components of X_l is unaffected and if X_l was pure dimensional then so is $\mathfrak{E}_h(X_l)$. □

Let us now prove that the generalized family and the two transformations allow us to generate any type of counter example:

Proposition 6.2.5. *For any combination of $(k,n,m) \in \mathbb{N}^3$ that satisfies to the inequalities $2 \leq k \leq n-1$ and $n \leq m \leq n+k-1$. There exists a family $(X_l)_{l \in \mathbb{N}}$ of real analytic germs in $\mathbb{R}^n$ such that $(X_l)_\mathbb{C}$ is pure dimensional, $\lim_{l \to \infty} \mathcal{C}(X_l, k) = +\infty$ and $\dim(\text{Sing}(T(X_l))) + k = m$ $(\forall l \in \mathbb{N})$.*

Proof. All the families have constant multiplicity and are pure dimensional. The transformations preserve the multiplicity and the pure dimensionality. Therefore all we have to check is that we can create any given combination (k,n,m) that satisfies to the inequalities of the proposition. We will use the capital letters K, N, M to label the three components of the triplet, and the lower case

letters k, n, m to refer to their values. We know by hypothesis that $n \leq m$ and we make the proof by distinguishing between the case $n = m$ and $n < m$:
If $n < m$, we have $1 + m - n \geq 2$. Thus there is a generalized family which is of type $(1+m-n, 2+m-n, 2+2m-2n)$. We have $k-1-m+n \geq 0$ as by hypothesis $m \leq n+k-1$ and we can apply a $(k-1-m+n)$-embedding transformation that will raise the components K, N and M by $(k-1-m+n)$ and we obtain the combination $(k, k+1, k+1+m-n)$. As by hypothesis $k \leq n-1$, we obtain $n-k-1 \geq 0$ and we can apply an $(n-k-1)$-product transformation to raise the component N and $\dim(\mathrm{Sing}(T))$ (thus the component M) by $(n-k-1)$, which gives the new combination (k, n, m).
If $m = n$, we start with the second example we gave in $\mathbb{R}^3$ which is of type $(2, 3, 3)$. We apply a $(k-2)$-embedding transformation ($k \geq 2$ always) which raises the components K, N and M by an amount of $k-2$. We obtain the combination $(k, 1+k, 1+k)$. As by hypothesis $k \leq n-1$, we have that $n-k-1 \geq 0$. We can thus apply a $(n-k-1)$-product transformation that will raise the components N and M by an amount of $(n-k-1)$, and we obtain the combination $(k, n, n) = (k, n, m)$ which is what we wanted. This proves the proposition. □

This concludes the proof of the optimality theorem 6.1.25. The next two sections will prove the main theorem.

6.3 The zero-dimensional case

The aim of this section is two-fold. The first goal is to prove the part of the main theorem that states that if the generic affine section is 0-dimensional, the multiplicity bounds the number of points in the section (proved in corollary 6.3.9). This is not obvious as the section is a *real* section of the germ, and the multiplicity is obtained for a generic *complex* section. The real sections of the germ correspond to a proper algebraic subset of the complex Grassmannian (which is a real algebraic manifold and cannot bear a complex analytic structure). Thus the multiplicity gives *a priori* no information on the number of points in a real section. To address this, the bulk of this section aims at characterizing the set of affine spaces for which the number of points in the intersection with the germ is equal to the multiplicity. We use the tangent cone to the germ at the origin to describe this set. This allows us to show that the complexification of a real affine space is generically in this set, hence proving the inequality for the cut by the complexified affine space, and thus *a fortiori* for the original real affine space by inclusion. A related result appears in [135] as lemma 1.4. It proves a stronger inequality than our result, but it is limited to germs of hypersurfaces.
The second goal of the section is to prove a result that equates the number of points in a generic section of the germ, and the number of points in the same section of the tangent cone (corollary 6.3.3). The latter result will be used to prove the second part of the main theorem 6.1.21 in the next section in theorem 6.1.21. Along the way, we also prove a fine characterization of the multiplicity in terms of the geometric degree of the tangent cone (proposition 6.3.5). Such a

strong characterization is not actually necessary to prove the main theorem but the result is nice in itself, it can be seen as a refinement of prop.6.3 of [47]. We will use this characterization to give an algorithm to compute the multiplicity in the last section 6.5.

To be able to relate the behaviors of the tangent cone and the germ, we need to prove the following flatness property. It proves that under certain conditions on a one parameter family at the 0 fiber, there is no degeneration nor exceptional components. The techniques of proof of the lemma are classical.

Lemma 6.3.1. *Let $Z_\lambda \subset \mathbb{C}^n \times \mathbb{C}$ be a one parameter algebraic family of varieties (not necessarily defined by a radical ideal in $\mathcal{O}(\mathbb{C}^n \times \mathbb{C})$). Assume that $Z_\lambda = \overline{Z_\lambda - Z_0}$, where $\overline{Z_\lambda - Z_0}$ denotes the algebraic closure of the family deprived of its fiber at 0. Assume also that there exists $\alpha > 0$ and $\lambda_0 > 0$ such that $\forall\ |\lambda| < \lambda_0$, $\dim(Z_\lambda) = 0$ and $Z_\lambda \subset B_{0,\alpha}$, then $\dim_{\mathbb{C}}(Z_0) = \lim_{\lambda\to 0} \dim_{\mathbb{C}}(Z_\lambda)$.*

Proof. Let $l : \mathbb{C}^n \to \mathbb{C}$ a linear form. Let $\mathcal{I}$ the ideal defining Z_λ in $\mathbb{C}^n \times \mathbb{C}$. Consider the image Y_λ of Z_λ by $(x, \lambda) \in \mathbb{C}^n \times \mathbb{C} \to (l(x), \lambda)$, its defining ideal is $\mathcal{J} = \mathbb{C}[L, \lambda] \cap (\mathcal{I} + (L - l))$ where L is understood to be the variable associated to the image space of l, and $L - l$ is the defining equation of the graph of l. If we choose l generic, l is an injection of 0-dimensional varieties and we know that the equality $\dim_{\mathbb{C}}(\mathcal{O}(Y_\lambda)) = \dim_{\mathbb{C}}(\mathcal{O}(Z_\lambda))$ holds for a generic λ. We also know that for a generic l, the equality $\dim_{\mathbb{C}}(\mathcal{O}(Y_0)) = \dim_{\mathbb{C}}(\mathcal{O}(Z_0))$ holds. Therefore it is possible to choose l so that both of the previous equalities hold at once. We call such a linear form separating.
The primary decomposition of Y_λ contains 0-dimensional and 1-dimensional components. As we are interested in what happens for $\lambda \to 0$ we can discard the 0-dimensional components of Y_λ that are outside Y_0. The condition $Z_\lambda = \overline{Z_\lambda - Z_0}$ insures that there are no 0-dimensional components of Z_λ that lie in Z_0 (because they would not be in the adherence of $Z_\lambda - Z_0$). As Z_0 is the adherence of a one parameter family of 0-dimensional sets, it is itself 0-dimensional. The linear form l might send some positive dimensional components of Z_λ to 0-dimensional components of Y_λ. In this case they are entirely contained in some Z_λ for a fixed λ, and since $\dim(Z_0) = 0$ such components are away from Z_0, they can also be discarded without altering $\dim_{\mathbb{C}}(Z_0)$ nor $\lim_{\lambda\to 0} Z_\lambda$. Therefore we can consider that Y_λ is a pure 1-dimensional variety of $\mathbb{C}^2$. Its defining ideal $\mathcal{J}$ is thus monogeneous and we have $\mathcal{J} = (f)$ for some $f \in \mathcal{O}(\mathbb{C}^2)$. Furthermore we can choose f to be monic. By definition $\mathcal{O}(Y_\Lambda) = \mathbb{C}[L, \lambda]/(f, \lambda - \Lambda)$. For a fixed λ, let the y_λ^i be the points of Y_λ. As Z_λ is bounded for λ, and that l is continuous, the symmetric functions (in i) of the $l(y_\lambda^i)$, are bounded in a neighborhood of $\lambda = 0$. As f defines the image of Z_λ by l, the roots of f for a given λ are the $l(y_\lambda^i)$. Therefore the coefficients of f are the symmetric functions in the $l(y_\lambda^i)$ and are all bounded in a neighborhood of $\lambda = 0$. This proves that the coefficients of f actually lie in $\mathbb{C}[\lambda]_0$, the localization of $\mathbb{C}[\lambda]$ at 0. As the ground field is $\mathbb{C}$ which is algebraic closed, the number of roots of f is always the same for a $\mathbb{C}[\lambda]_0$-generic λ, that is, λ generic on $\mathbb{C}$ or $\lambda = 0$.

As l is separating, we have the equality $\deg f = \dim_{\mathbb{C}}(\mathcal{O}(Z_\lambda))$ for λ generic on $\mathbb{C}$ or $\lambda = 0$. Consequently $\deg f = \dim_{\mathbb{C}}(Z_0) = \lim_{\lambda\to 0}\dim_{\mathbb{C}}(Z_\lambda)$.

□

The next lemma shows that when the germ is pure dimensional, a generic affine cut of the complex germ degenerates flatly to the tangent cone when approaching the origin. This allows us to relate the multiplicity of the tangent cone to the multiplicity of the complex germ.

Lemma 6.3.2. *Let $Z \in \mathbb{C}^n$ be a pure dimensional complex germ at 0. Let $\tilde{P} \in \tilde{\mathbb{G}}^0_{k,n}$ where $k = n - \dim(Z)$. Let $D \in \tilde{\mathbb{G}}_{k,n}$ and $t \in D^\perp$ such that $\mathcal{D}(\tilde{P}) = (D, (\mathbb{C}t)_{\mathbb{P}})$. Let $T_C = \tilde{Z}_0$ be the tangent cone to Z. Assume $(t+D)\cap T_C$ is 0-dimensional. For any $\eta > 0$, let Y_γ be the family defined as*

$$Y_{u,\epsilon} = (u + D) \cap \tilde{Z}_\epsilon \cap B_{0,\eta},$$

where $u \in D^\perp$ and $\epsilon \in \mathbb{C}$. Let

$$\Pi : (Y_{u,\epsilon})_{u\in D^\perp, \epsilon\in\mathbb{C}} \mapsto (u, \epsilon) \in (D^\perp \times \mathbb{C}).$$

Then we have

$$\forall \eta > 0,\ \exists U \subset (D^\perp \times \mathbb{C}) \text{ neighborhood of } (t, 0) \text{ s.t. } \Pi|^U \text{ is proper and finite.}$$

In the vocabulary of [142], this means that Π is good for $U \times B_{0,\eta}$.
Moreover for any $(u, \epsilon) \in U$ $\dim_{\mathbb{C}}\left(\Pi^{-1}(u, \epsilon)\right) = \dim_{\mathbb{C}}\left(\Pi^{-1}(t, 0)\right)$.

Proof. By hypothesis $\tilde{Z}_0 \cap (t + D)$ is 0-dimensional. Choose $\eta' < \eta$ such that $Y_{t,0} = \tilde{Z}_0 \cap (t + D) \cap B_{0,\eta'}$ (that is we choose a ball smaller than $B_{0,\eta}$ which still contains all the points of $Y_{t,0}$). As there is no points of $Y_{t,0}$ in $B_{0,\eta} - B_{0,\eta'}$, we have that $C := \Pi(D^\perp \times \mathbb{C} \times S_{0,\eta'})$ does not contain $(t, 0)$. As $S_{0,\eta'}$ is compact, Π is proper on $(D^\perp \times \mathbb{C} \times S_{0,\eta'})$ and C is closed. Therefore Π is proper on the complement of C. It is finite on a neighborhood of $(t, 0)$ because the dimension is upper semi-continuous. We call U the neighborhood of $(t, 0)$ over which it is both proper and finite.

Now we prove that the dimension of the fibers is in fact constant. By hypothesis Z is pure dimensional, therefore so is $(\tilde{Z}_\epsilon)_{\epsilon\in\mathbb{C}}$. Choose l a linear separating form on $\mathbb{C}^n$ for $\Pi^{-1}(t, 0)$. By separating we mean that l is an injection of $\Pi^{-1}(t, 0)$ (as a scheme) into $\mathbb{C}$. Such l of course exists because $\Pi^{-1}(t, 0)$ is finite. Because the points in the fibers vary continuously l is also separating for points in a neighborhood V of $(t, 0)$. As l is an injection we have $\dim_{\mathbb{C}}\left(l(\Pi^{-1}(t, 0))\right) = \dim_{\mathbb{C}}\left(\Pi^{-1}(t, 0)\right)$. Notice the equality takes into account the multiplicity of the points. Since $\tilde{Z}_\epsilon$ is pure dimensional and l is locally an injection, $l(Y_{u,\epsilon})$ is a hypersurface and is thus defined by a single function f. As Π is proper and finite over a neighborhood of $(t, 0)$, the map $\pi : l(Y_{u,\epsilon}) \mapsto (u, \epsilon)$ is also proper and finite over $U\cap V$. This means that f is good for $(U\cap V)\times B_{0,\eta}$ (with η as defined for Π earlier). Thus by the Weierstrass preparation theorem

for f (see [142] for instance) there is a monic polynomial P in the variables u and ϵ defining $l(Y_{u,\epsilon})$ for $(u,t) \in (U \cap V)$. Therefore we have for any $(u,t) \in (U \cap V)$

$$\deg P = \dim_C \pi^{-1}(u,t) = \dim_C l(\Pi^{-1}(u,t)) = \dim_C \Pi^{-1}(u,t).$$

As V is a neighborhood of $(t,0)$ we have proved the claim. □

Corollary 6.3.3. *Let $Z \in \mathbb{C}^n$ be a complex germ at 0 of dimension d. Let $T_C = \tilde{Z}_0$ be the tangent cone to Z, and let $T_{<d}$ be the components of the tangent cone with dimension less than d. Let $\tilde{P} \in \tilde{\mathbb{G}}^0_{k,n}$ where $k = n - \dim(Z)$. Let $D \in \tilde{\mathbb{G}}_{k,n}$ and $t \in D^\perp$ such that $\mathcal{D}(\tilde{P}) = (D, (\mathbb{C}t)_\mathbb{P})$. Assume $D_\mathbb{P} \cap T_\mathbb{P} = \emptyset$, $(t+D) \cap T_C$ is 0-dimensional and $(t+D) \cap T_{<d} = \emptyset$. Then*

$$\exists \epsilon_0,\ \textit{such that } \forall \epsilon < \epsilon_0,\ \exists \lambda_0,\ \textit{such that } \forall \lambda < \lambda_0,$$
$$\dim_\mathbb{C}\left((\lambda t + D) \cap Z \cap B_{0,\epsilon}\right) = \dim_\mathbb{C}\left((t+D) \cap T_C\right) = \dim_\mathbb{C}\left((\mathbb{C}t + D)_\mathbb{P} \cap T_\mathbb{P}\right)$$

Proof. As $(t+D)$ avoids $T_{<d}$, there is a neighborhood of 0 for ϵ such that $(t+D)$ avoids the lower dimensional components of $\tilde{Z}_\epsilon$. As we are interested in what happens for ϵ small we can assume that Z is pure dimensional. As $(t+D) \cap T_C$ is 0-dimensional by hypothesis, we can apply lemma 6.3.2. If we fix $u = t$ in that lemma, we obtain that for any given η, there is a neighborhood of 0 for ϵ such that $\dim_\mathbb{C}\left(\Pi^{-1}(t,\epsilon)\right) = \dim_\mathbb{C}\left(\Pi^{-1}(t,0)\right)$. By expanding the definition of Π we obtain

$$\dim_\mathbb{C}\left((t+D) \cap \tilde{Z}_\epsilon \cap B_{0,\eta}\right) = \dim_\mathbb{C}\left((t+D) \cap \tilde{Z}_0 \cap B_{0,\eta}\right).$$

If we take η bigger than some sufficiently large η_0, $B_{0,\eta}$ contains all the points in $(t+D) \cap \tilde{Z}_0$ thus we have the following statement

$$\exists \eta_0,\ \forall \eta > \eta_0,\ \exists \epsilon_1,\ \forall \epsilon < \epsilon_1,\ \dim_\mathbb{C}\left((t+D) \cap \tilde{Z}_\epsilon \cap B_{0,\eta}\right) = \dim_\mathbb{C}\left((t+D) \cap \tilde{Z}_0\right).$$

Set $\epsilon_0 = 1/\eta_0$ and $\lambda_1 = \epsilon_1$, the previous relation becomes

$$\exists \epsilon_0,\ \forall \epsilon < \epsilon_0,\ \exists \lambda_1,\ \forall \lambda < \lambda_1,\ \dim_\mathbb{C}\left((t+D) \cap \tilde{Z}_\lambda \cap B_{0,1/\epsilon}\right) = \dim_\mathbb{C}\left((t+D) \cap \tilde{Z}_0\right).$$

By a dilation h of ratio λ we may rewrite the left-hand side of the equality in the claim of the corollary as:

$$(\lambda t + D) \cap Z \cap B_{0,\epsilon} \overset{h}{\approx} (t+D) \cap \tilde{Z}_\lambda \cap B_{0,\epsilon/\lambda}.$$

Thus we see that for $\lambda_0 < 1/\epsilon^2$ the ball $B_{0,\epsilon/\lambda}$ contains $B_{0,1/\epsilon}$. To prove the claim, it thus suffices to set $\lambda_0 < \min(\lambda_1, 1/\epsilon^2)$ and to prove that for any α, if $B_{0,1/\epsilon} \subset B_{0,\alpha}$ then

$$\dim_\mathbb{C}\left((t+D) \cap \tilde{Z}_\lambda \cap B_{0,1/\epsilon}\right) = \dim_\mathbb{C}\left((t+D) \cap \tilde{Z}_\lambda \cap B_{0,\alpha}\right).$$

This is not true in general, but if we first simplify Z (without altering the quantities we look at) we will see that the result holds for this simplified Z and thus for the original Z as the simplification did not affect the quantities we are interested in. Some of the branches of the one dimensional family $((\lambda t+D)\cap Z)_\lambda$ to not go to the origin, therefore for ϵ_0 sufficiently small, they do not intersect $B_{0,\epsilon}$ and do not count in $\dim_{\mathbb{C}}((\lambda t + D)\cap Z \cap B_{0,\epsilon})$. The way we simplify Z is that we discard those branches and assume all the branches go to the origin.
If we did not have the equality

$$\dim_{\mathbb{C}}\left((t+D)\cap \tilde{Z}_\lambda \cap B_{0,1/\epsilon}\right) = \dim_{\mathbb{C}}\left((t+D)\cap \tilde{Z}_\lambda \cap B_{0,\alpha}\right)$$

for the simplified Z, this would mean that there are points in $(t+D)\cap\tilde{Z}_\lambda\cap B_{0,\alpha}$ that are not in $(t+D)\cap\tilde{Z}_\lambda\cap B_{0,1/\epsilon}$. As ϵ has been chosen small enough (that is, η big enough before the change of notation) so that $(t+D)\cap\tilde{Z}_\lambda\cap B_{0,1/\epsilon}$ contains all the points that belong to branches that go to $(t+D)\cap\tilde{Z}_0$, these extra points belong to branches that converge to a point of $T_{\mathbb{P}}$ outside $(t+D)_{\mathbb{P}}$, that is, inside $D_{\mathbb{P}}$. But by hypothesis $D_{\mathbb{P}}\cap T_{\mathbb{P}}=\emptyset$, thus proving that there is no such extra point.
Finally we have proved that for $\lambda < \min(\lambda_1, 1/\epsilon^2)$ the following equality of dimensions holds:

$$\begin{aligned}\dim_{\mathbb{C}}\left((\lambda t+D)\cap Z\cap B_{0,\epsilon}\right) &= \dim_{\mathbb{C}}\left((t+D)\cap\tilde{Z}_\lambda\cap B_{0,\epsilon/\lambda}\right)\\ &= \dim_{\mathbb{C}}\left((t+D)\cap\tilde{Z}_\lambda\cap B_{0,1/\epsilon}\right)\\ &= \dim_{\mathbb{C}}\left((t+D)\cap\tilde{Z}_0\right).\end{aligned}$$

In the claim of the corollary, the last equality between $\dim_{\mathbb{C}}((t+D)\cap T_C)$ and $\dim_{\mathbb{C}}((\mathbb{C}t+D)_{\mathbb{P}}\cap T_{\mathbb{P}})$ is simple: We have assumed that $(t+D)\cap T_C$ is 0-dimensional and $D_{\mathbb{P}}\cap T_{\mathbb{P}}=\emptyset$, this implies that $(\mathbb{C}t+D)_{\mathbb{P}}\cap T_{\mathbb{P}} = ((t+D)\cap T_C)_{\mathbb{P}}$ and thus $\dim_{\mathbb{C}}((t+D)\cap T_C) = \dim_{\mathbb{C}}((\mathbb{C}t+D)_{\mathbb{P}}\cap T_{\mathbb{P}})$. □

In order to arrive at a strong characterization of multiplicity (proposition 6.3.5) we need the following general lemma about projective pure dimensional varieties. This is again a flatness property, and it ensues from lemma 6.3.1.

Lemma 6.3.4. *Let $Z\subset\mathbb{P}^n(\mathbb{C})$ a complex algebraic projective variety that is purely d-dimensional. Then there is a constant $\delta\in\mathbb{N}$, such that for any linear space L of dimension $n-d$ such that $\dim(Z\cap L)=0$, we have $\dim_{\mathbb{C}}(Z\cap L)=\delta$.*

Proof. Consider an affine chart such that all the points of $Z\cap L$ lie in the chart (there are only a finite number of points as $\dim(Z\cap L)=0$). Consider $Y = Z - L \subset\mathbb{C}^n$, and L_λ a one parameter algebraic family of linear spaces converging to L as $\lambda\to 0$, such that $L_\lambda\cap L=\emptyset$ and $\dim(L_\lambda\cap Z)=0(\forall\lambda\in\mathbb{R}_*^+)$. It is possible to find such linear spaces L_λ that avoid L as we are in $\mathbb{C}^n$, and it is possible to enforce that $\dim(L_\lambda\cap Z)=0$ as this is generically the case since Z is not the whole projective space. By the previous lemma 6.3.1, we have

$\dim_{\mathbb{C}}\left(L\cap\overline{Y}\right)=\lim_{\lambda\to 0}\dim_{\mathbb{C}}\left(L_\lambda\cap Y\right)$ where $\overline{Y}$ denotes the algebraic closure of Y. As L avoids L_λ, we have $L_\lambda\cap Y=L_\lambda\cap Z$. To complete the argument we need to prove that $Z=\overline{Y}$: if it was not, there would be a component that is entirely contained in L and as $\dim\left(Z\cap L\right)=0$, that would mean there is a 0-dimensional component in Z. As Z is pure dimensional that would mean that $\dim\left(Z\right)=0$ and $L=\mathbb{P}^n(\mathbb{C})$ in which case the lemma is trivially true. So we can assume that $Z=\overline{Y}$ and we thus have the following string of equalities:

$$\dim_{\mathbb{C}}\left(L\cap Z\right)=\dim_{\mathbb{C}}\left(L\cap\overline{Y}\right)=\lim_{\lambda\to 0}\dim_{\mathbb{C}}\left(L_\lambda\cap Y\right)=\lim_{\lambda\to 0}\dim_{\mathbb{C}}\left(L_\lambda\cap Z\right).$$

We just proved that outside those linear spaces such that $\dim\left(L\cap Z\right)>0$, the dimension is continuous. As the dimension is a discrete-valued function, this proves that the dimension is constant over the connected components of the set of linear spaces such that $\dim\left(Z\cap L\right)=0$. As the base field is $\mathbb{C}$, algebraic open sets of $\overline{\mathbb{G}}_{n-d,n}(\mathbb{C})$ are connected, therefore there is a unique constant $\delta\in\mathbb{N}$ such that $\dim_{\mathbb{C}}\left(L\cap Z\right)=\delta$ as soon as $\dim\left(L\cap Z\right)=0$. □

This enables us to have the following fine characterization of the multiplicity:

Proposition 6.3.5. *Let $Z\subset\mathbb{C}^n$ a complex analytic germ of dimension d at 0 of multiplicity μ. Let $T:=\tilde{Z}_0$, let T_d be the components of T of dimension d, and let $T_{<d}$ be the components of T of dimension strictly less than d.*
For all linear spaces L in $\mathbb{P}^{n-1}(\mathbb{C})$ such that $T\cap L$ is 0-dimensional we have the equality $\mu=\dim_{\mathbb{C}}\left(T_d\cap L\right)$. Thus, if in addition $L\cap T_{<d}=\emptyset$, then the equality $\mu=\dim_{\mathbb{C}}\left(T\cap L\right)$ holds.

Proof. The definition of multiplicity is that on a generic set $U\subset\tilde{\mathbb{G}}^0_{n-d,n}(\mathbb{C})$, we have $\mathcal{C}(Z,\tilde{P})=\mu\ (\forall\tilde{P}\in U)$. For an element $\tilde{P}\in\tilde{\mathbb{G}}^0_{n-d,n}$, let D and t such that $(D,(\mathbb{C}t)_{\mathbb{P}})=\mathcal{D}(\tilde{P})$. The set of V of the $\tilde{P}\in\tilde{\mathbb{G}}^0_{n-d,n}(\mathbb{C})$ such that $(t+D)_{\mathbb{P}}\cap T_{\mathbb{P}}$ is 0-dimensional, $D_{\mathbb{P}}\cap T_{\mathbb{P}}=\emptyset$, and $(t+D)_{\mathbb{P}}$ avoids $T_{<d}$, is a generic set of $\tilde{\mathbb{G}}^0_{n-d,n}(\mathbb{C})$. As U and V are generic sets, their intersection is generic and thus in particular it is non empty. Pick any $\tilde{P}\in U\cap V$. Let $L=(\mathbb{C}t+D)_{\mathbb{P}}\subset\mathbb{P}^{n-1}(\mathbb{C})$. As V corresponds to the hypotheses of the previous corollary 6.3.3, for λ and ϵ small enough, we have:

$$\dim_{\mathbb{C}}\left((\lambda t+D)\cap Z\cap B_{0,\epsilon}\right)=\dim_{\mathbb{C}}\left((\lambda t+D)\cap T\right)=\dim_{\mathbb{C}}\left(L\cap T_{\mathbb{P}}\right).$$

Since $\tilde{P}\in U$, $\dim_{\mathbb{C}}\left((\lambda t+D)\cap Z\cap B_{0,\epsilon}\right)=\mu$, and by the previous equality, we conclude that $\mu=\dim_{\mathbb{C}}\left(L\cap T_{\mathbb{P}}\right)$. As $\tilde{P}\in V$, L avoids $T_{<d}$, hence $\mu=\dim_{\mathbb{C}}\left(L\cap T_d\right)$.
As T_d is pure dimensional, by the previous lemma 6.3.4, for all $(n-d)$-dimensional projective linear spaces L such that $\dim\left(T_d\cap L\right)=0$, we have $\mu=\dim_{\mathbb{C}}\left(T_d\cap L\right)$. Thus if $L\cap T_{<d}=\emptyset$, $\dim_{\mathbb{C}}\left(L\cap T\right)=\dim_{\mathbb{C}}\left(L\cap T_d\right)=\mu$. □

The previous proposition can be seen as a refinement of prop.6.3 in [47] with the limitation that the previous proposition is only concerned with cutting

linear spaces L, whereas the result in [47] is formulated for any cutting smooth germ at 0 (of the right codimension).

Notice that the above characterization yields immediately the following property for hypersurfaces:

Proposition 6.3.6 (Hypersurface Multiplicity)**.** *If $Z \subset \mathbb{C}^n$ is a complex hypersurface defined by the polynomial $f \in \mathbb{C}[\underline{Z}]$, the multiplicity of Z at 0 is the degree of the non-zero homogeneous component of f expressed in the monomial basis that has the lowest degree.*

Proof. Let $f = \sum_{i=m}^{\infty} h_i$ be the decomposition of f into homogeneous components for the total degree grading. Then the blow-up $\tilde{Z}$ of Z at 0 is defined by h_m the lowest degree homogeneous component of f. Then $Y = V(h_m) \subset \mathbb{P}^{n-1}(\mathbb{C})$ is a hypersurface of degree m, and for any line in $\mathbb{P}^{n-1}(\mathbb{C})$, the number of points (with multiplicity) when $\dim(Y \cap L) = 0$ is $\dim_{\mathbb{C}}(Y \cap L) = m$. By the previous proposition $m = \mu$ the multiplicity of Z. □

Finally the last corollary of this section shows that the multiplicity bounds the number of points in the intersection with the *real* germ. This is a consequence of the following lemma:

Lemma 6.3.7. *Let $X \subset \mathbb{R}^n$ be a real analytic germ at 0 of dimension d and multiplicity μ. Let T be the algebraic tangent cone to X. Let $\tilde{P} \in \tilde{\mathbb{G}}^0_{k,n}(\mathbb{R})$ where $k = n - \dim(X)$. Let $D \in \mathbb{G}_{k,n}(\mathbb{R})$ and $t \in D^{\perp}$ such that $\mathcal{D}(\tilde{P}) = (D, (\mathbb{C}t)_{\mathbb{P}})$. Let T', $\tilde{P}'$, D', and t' their complexification. Assume $D'_{\mathbb{P}} \cap T'_{\mathbb{P}} = \emptyset$, $(\mathbb{C}t' + D')_{\mathbb{P}} \cap T'_{\mathbb{P}}$ is 0-dimensional and $(\mathbb{C}t' + D')_{\mathbb{P}}$ avoids $T'_{<d}$ the components of T' of dimension less than d. Then we have*

$$\begin{aligned} &\exists \epsilon_0, \text{ such that } \forall \epsilon < \epsilon_0,\ \exists \lambda_0, \text{ such that } \forall \lambda \in]0, \lambda_0[, \\ &\quad \dim_{\mathbb{R}}((\lambda t + D) \cap X \cap B_{0,\epsilon}) \leq \dim_{\mathbb{C}}((t' + D')_{\mathbb{P}} \cap T'_{\mathbb{P}}) = \mu \end{aligned}$$

Proof. Under the above conditions, the previous corollary 6.3.3 applies directly and therefore

$$\begin{aligned} &\exists \epsilon_0, \text{ such that } \forall \epsilon < \epsilon_0,\ \exists \lambda_0, \text{ such that } \forall \lambda < \lambda_0, \\ &\quad \dim_{\mathbb{C}}((\lambda t' + D') \cap X_{\mathbb{C}} \cap B_{0,\epsilon}) = \dim_{\mathbb{C}}((t' + D')_{\mathbb{P}} \cap T'_{\mathbb{P}}) \end{aligned}$$

By proposition 6.3.5, $\mu = \dim_{\mathbb{C}}((t' + D')_{\mathbb{P}} \cap T'_{\mathbb{P}})$ as by hypothesis $((t'+D')_{\mathbb{P}} \cap T'_{\mathbb{P}})$ is 0-dimensional. Then by considering the realizations of these varieties, we obtain the inclusion

$$(\lambda t + D) \cap X \cap B_{0,\epsilon} \subset ((\lambda t' + D') \cap X_{\mathbb{C}} \cap B_{0,\epsilon})_{\mathbb{R}}.$$

Thus we can conclude:

$$\begin{aligned} \dim_{\mathbb{R}}((\lambda t + D) \cap X \cap B_{0,\epsilon}) &\leq \dim_{\mathbb{R}}((\lambda t' + D') \cap X_{\mathbb{C}} \cap B_{0,\epsilon})_{\mathbb{R}} \\ &\leq \dim_{\mathbb{C}}((\lambda t' + D') \cap X_{\mathbb{C}} \cap B_{0,\epsilon}) \\ &\leq \dim_{\mathbb{C}}((t' + D')_{\mathbb{P}} \cap T'_{\mathbb{P}}) \ = \ \mu. \end{aligned}$$

□

Remark 6.3.8. *The previous lemma 6.3.7 and lemma 1.4 in [133] are related. The previous lemma shows in particular that*

$$\exists \epsilon_0, \text{ such that } \forall \epsilon < \epsilon_0, \ \exists \lambda_0, \text{ such that } \forall \lambda \in]0, \lambda_0[, \\ \dim_{\mathbb{R}} \left((\lambda t + D) \cap X \cap B_{0,\epsilon} \right) \leq \mu$$

When X is a hypersurface, lemma 1.4 in [133] gives the same inequality. Let us explain how. When we consider D as in the previous lemma, D', the complexification of D, is an excellent 2-plane in the terminology of [133]. Therefore lemma 1.4 gives the following inequality with the multiplicity

$$\exists \epsilon_0, \text{ such that } \forall \epsilon < \epsilon_0, \ \exists \lambda_0, \text{ such that } \forall \lambda \in]0, \lambda_0[, \\ \dim_{\mathbb{C}} \left((t' + D')_{\mathbb{P}} \cap T'_{\mathbb{P}} \right) \leq \mu$$

At this step our lemma is stronger as we have shown that for the D we consider, we actually have equality. But then we lose this aspect when use the inclusion property to obtain the inequality

$$\dim_{\mathbb{R}} \left((\lambda t + D) \cap X \cap B_{0,\epsilon} \right) \leq \dim_{\mathbb{C}} \left((t' + D')_{\mathbb{P}} \cap T'_{\mathbb{P}} \right).$$

In the end, when X is a hypersurface, lemma 1.4 in [133] gives the same inequality between the dimension of the real intersection and the multiplicity as our lemma.
Notice however, that the previous lemma 6.3.7 does not imply lemma 1.4 in [133]. Lemma 1.4 in [133] applies for any real 2-plane cutting the complex hypersurface, our lemma only applies to those 2-planes that are the complexification of a real line.

Corollary 6.3.9. *Let $X \subset \mathbb{R}^n$ a real analytic germ at 0 of dimension d. Then $\Sigma_{\mathfrak{b}}(X, n-d) = \mathcal{C}(X, n-d) \leq \mu$.*

Proof. As the intersection of X and a generic affine space of dimension $(n-d)$ is generically empty, $\Sigma_{\mathfrak{b}}(X, n-d) = \mathcal{C}(X, n-d)$. It thus suffices to prove that with the same definitions as in the previous lemma 6.3.7, the set U of $\tilde{P} \in \tilde{\mathbb{G}}^0_{n-d,n}(\mathbb{R})$ such that $D'_{\mathbb{P}} \cap T'_{\mathbb{P}} = \emptyset$, $(\mathbb{C}t' + D')_{\mathbb{P}} \cap T'_{\mathbb{P}}$ is 0-dimensional and $(\mathbb{C}t' + D')_{\mathbb{P}}$ avoids $T'_{<d}$, is a generic set of $\tilde{\mathbb{G}}^0_{n-d,n}(\mathbb{R})$.
We prove that each of these conditions is generically satisfied for $\tilde{P} \in \tilde{\mathbb{G}}^0_{n-d,n}(\mathbb{R})$. We can represent elements of $\mathbb{G}_{n-d,n}(\mathbb{R})$ as projection matrices of rank d. That is, the matrices $M \in M_n(\mathbb{R})$ such that $M = M^t$ (^{t} denotes the transposition), $M^t M = M^2 = \mathrm{Id}$ and $\mathrm{rk} M = d$. As X is a real germ, by proposition 6.1.17, the equations defining T (and thus T', they are the same) have real coefficients. Therefore even if the conditions involve the complexifications of the objects, they are algebraically defined by equations with real coefficients. This shows that the conditions are degenerate on the reals if and only if they are degenerate on the complex field.
The conditions are not degenerate on $\tilde{\mathbb{G}}^0_{n-d,n}(\mathbb{C})$ as

- for the first condition:

  $$\dim\left(D'_{\mathbb{P}}\right) + \dim\left(T'_{\mathbb{P}}\right) = (n-d-1) + (d-1) < n-1 = \dim\left(\mathbb{P}^{n-1}(\mathbb{C})\right),$$

  which shows that $D'_{\mathbb{P}}$ and $T'_{\mathbb{P}}$ avoid each other generically.

- for the second condition:

  $$\dim\left((\mathbb{C}t' + D')_{\mathbb{P}}\right) + \dim\left(T'_{\mathbb{P}}\right) = (n-d) + d - 1 = n-1 = \dim\left(\mathbb{P}^{n-1}(\mathbb{C})\right),$$

  which shows that $(\mathbb{C}t' + D') \cap T'_{\mathbb{P}}$ is generically 0-dimensional.

- for the third condition:

  $$\dim\left((\mathbb{C}t' + D')_{\mathbb{P}}\right) + \dim\left((T'_{<d})_{\mathbb{P}}\right) < (n-d) + d - 1 = \dim\left(\mathbb{P}^{n-1}(\mathbb{C})\right),$$

  which shows that $(\mathbb{C}t' + D') \cap T'_{\mathbb{P}}$ generically avoid each other.

□

6.4 When the cut of the tangent cone is smooth and pure dimensional

In this section we address the case when the intersection of the affine spaces with the tangent cone to the germ is generically smooth and pure dimensional. In an analogous way as in the previous section, we show that for a generic affine space, the topology of the intersection with the tangent cone is the same as the topology of the intersection with the germ. This is proved using the Thom-Mather topological trivialization theorem (see thm.4.1.10). Then, we use proposition 6.3.7 to prove that the multiplicity of the germ is the geometric degree of the sections of the tangent cone. This allows to use the Heintz-Mumford result (theorem 6.4.15) to bound the degree of the generators of the sections of the tangent cone and thereby to bound the sum of its Betti numbers using the Oleinik-Petrovsky-Thom-Milnor bound (theorem 6.4.13).

We begin (subsection 6.4.1) with the proof that the topology of a generic section of the tangent cone is the same as the topology of the corresponding section of the germ (thm. 6.4.2), hence their Betti numbers are the same. The proof of this theorem relies on lemma 6.4.4 that relates the definition of the local Betti numbers (def. 6.1.9) which is based on affine cuts, to another equivalent characterization that is based on spherical cuts. This technical lemma is proved in the separate subsection 6.4.2. Finally the last subsection 6.4.3 deals with the second part of the main theorem that remains to be proven. As a reminder, the second part of the main theorem corresponds to when the affine cut of the germ is generically smooth and pure dimensional.

6.4.1 Sections of tangent cone and germ have the same topology

We know that analytic sets admit Whitney stratifications ([141], thm.19.2). In the following we will repeatedly consider the transversality of singular analytic varieties (as by definition 4.1.11) without explicitly defining a stratification. It is understood that we stratify the sets we consider with any Whitney stratification the first time we encounter them and all the subsequent proofs are carried out with respect to these arbitrary stratifications.

Definition 6.4.1. *For any $D \in \mathbb{G}_{k,n}(\mathbb{R})$, $t \in D^\perp$ and $\lambda \in \mathbb{R}$, we define:*

$$\begin{aligned}
C_\lambda(t,D) &= (\lambda t + D) \cap S_{0,1},\\
D_\lambda(t,D) &= (\lambda t + D) \cap B_{0,1},\\
\overline{D}_\lambda(t,D) &= (\lambda t + D) \cap \overline{B}_{0,1},\\
H_\lambda(t,D) &= (\mathbb{R}t + D) \cap S_{0,1} \cap \{p \in \mathbb{R}^n \mid (p|t) > \lambda\},\\
\overline{H}_\lambda(t,D) &= (\mathbb{R}t + D) \cap S_{0,1} \cap \{p \in \mathbb{R}^n \mid (p|t) \geq \lambda\}.
\end{aligned}$$

The main theorem of this section runs as follows:

Theorem 6.4.2. *Let $X \in \mathbb{R}^n$ be a real analytic germ at 0 and $\tilde{X}_\epsilon$ its conic blow-up at the origin (def. 6.1.18). For any element $\tilde{P} \in \tilde{\mathbb{G}}^0_{k,n}$, let $D \in \mathbb{G}_{k,n}$ and $t \in D^\perp$ such that $\mathcal{D}(\tilde{P}) = (D, (\mathbb{R}t)_\mathbb{P})$. Assume that*

1. *The projective varieties $D_\mathbb{P}$ and $T_\mathbb{P}$ are transverse (as stratified sets).*
2. *The projective linear space $(\mathbb{R}t + D)_\mathbb{P}$ and $T_\mathbb{P}$ are transverse (as stratified sets).*
3. *The projective linear space $(\mathbb{R}t + D)_\mathbb{P}$ does not contain any singular point of the algebraic tangent cone $T_\mathbb{P}$.*

Then we have

$$\exists \alpha > 0,\ \exists \epsilon_0 > 0,\ \forall \epsilon \in]0, \epsilon_0],\ \forall \lambda \in]0, \alpha],\ \exists h \text{ homeomorphism},$$
$$D_\lambda(t,D) \cap \tilde{X}_\epsilon \overset{h}{\approx} (t + D) \cap \tilde{X}_0$$

Corollary 6.4.3. *Let $X \in \mathbb{R}^n$ be a real analytic germ at 0 and T its tangent cone. Let $\mathfrak{b}_i$ be any local Betti number. For any element $\tilde{P} \in \tilde{\mathbb{G}}^0_{k,n}$ let $D \in \mathbb{G}_{k,n}$ and $t \in D^\perp$ such that $\mathcal{D}(\tilde{P}) = (D, (\mathbb{R}t)_\mathbb{P})$. If the conditions (1),(2) and (3) of theorem 6.4.2 are satisfied, then*

$$\mathfrak{b}_i(X, \tilde{P}) = \mathfrak{b}_i(T, \tilde{P}).$$

Proof. By theorem 6.4.2 applied to the conic blow-up of X at the origin, we have:

$$\begin{aligned}
\mathfrak{b}_i(X, \tilde{P}) &= \lim_{\epsilon \to 0} \lim_{\lambda \to 0} \mathfrak{b}_i((\lambda t + D) \cap X \cap B_{0,\epsilon})\\
&= \lim_{\epsilon \to 0} \lim_{\lambda \to 0} \mathfrak{b}_i(D_\lambda(t,D) \cap \tilde{X}_\epsilon)\\
&= \mathfrak{b}_i((t + D) \cap \tilde{X}_0).
\end{aligned}$$

As T is already a cone, its conic blow-up is the constant family $\tilde{T}_\epsilon = \tilde{X}_0 = T$. In particular $\tilde{T}_0 = \tilde{X}_0 = T$ therefore the three conditions of theorem 6.4.2 are also satisfied for T since they are satisfied for X. By theorem 6.4.2 applied to the conic blow-up of T at the origin we have:

$$\begin{aligned}\mathfrak{b}_i(T, \tilde{P}) &= \lim_{\epsilon \to 0} \lim_{\lambda \to 0} \mathfrak{b}_i((\lambda t + D) \cap T \cap B_{0,\epsilon}) \\ &= \lim_{\epsilon \to 0} \lim_{\lambda \to 0} \mathfrak{b}_i(D_\lambda(t, D) \cap \tilde{T}_\epsilon) \\ &= \mathfrak{b}_i((t + D) \cap \tilde{T}_0) = \mathfrak{b}((t + D) \cap \tilde{X}_0).\end{aligned}$$

We have thus proved that

$$\mathfrak{b}_i(X, \tilde{P}) = \mathfrak{b}_i((t + D) \cap \tilde{X}_0) = \mathfrak{b}_i(T, \tilde{P}).$$

□

We now introduce three lemmas to prove theorem 6.4.2.

Lemma 6.4.4. *Let $X \subset \mathbb{R}^n$ be a real analytic germ at 0 and $\tilde{X}_\epsilon$ its conic blow-up at the origin. For any element $\tilde{P} \in \tilde{\mathbb{G}}^0_{k,n}$, let $D \in \mathbb{G}_{k,n}$ and $t \in D^\perp$ such that $\mathcal{D}(\tilde{P}) = (D, (\mathbb{R}t)_\mathbb{P})$. Assume that condition (1) of theorem 6.4.2 is satisfied. Then we have*

$$\exists \alpha > 0,\ \exists \epsilon_1 > 0,\ \textit{such that } \forall \epsilon \in]0, \epsilon_1],\ \forall \lambda \in]0, \alpha],$$
$$\exists h_1 \textit{ homeomorphism, such that } D_\lambda(t, D) \cap \tilde{X}_\epsilon \overset{h_1}{\approx} H_0(t, D) \cap \tilde{X}_\epsilon$$

We defer the proof of lemma 6.4.4 to the next subsection (6.4.2) as it is technical and rather long.

Lemma 6.4.5. *Let $X \subset \mathbb{R}^n$ be a real analytic germ at 0 and $\tilde{X}_\epsilon$ its conic blow-up at the origin. For any element $\tilde{P} \in \tilde{\mathbb{G}}^0_{k,n}$ let $D \in \mathbb{G}_{k,n}$ and $t \in D^\perp$ such that $\mathcal{D}(\tilde{P}) = (D, (\mathbb{R}t)_\mathbb{P})$. Assume that conditions (1), (2) and (3) of theorem 6.4.2 are satisfied. Then we have:*

$$\exists \epsilon_2 > 0, \textit{ such that } \forall \epsilon \in]0, \epsilon_2],\ \exists h_2 \textit{ homeomorphism, such that}$$
$$H_0(t, D) \cap \tilde{X}_\epsilon \overset{h_2}{\approx} H_0(t, D) \cap \tilde{X}_0.$$

Proof. We prove the lemma using the moving the wall theorem (thm. 4.1.12). To do so we are going to prove that the constant family in ϵ, $\overline{H}_0(t, D)$ is transverse to $\tilde{X}_\epsilon$ on some $]-\epsilon_2, \epsilon_2[$ (we view $\tilde{X}_\epsilon$ as a subset of $\mathbb{R}^n \times \mathbb{R}$). Let $T := \tilde{X}_0$. We denote by $\{\epsilon = 0\}$ the set $\{(x_1, \ldots, x_n, \epsilon) \mid \epsilon = 0\}$.
By condition (3) of theorem 6.4.2 (which is assumed for lemma 6.4.5), there is no singular point of T in $H_0(t, D)$. By construction of T this means that $\tilde{X}_\epsilon$ is smooth in a neighborhood of $T \cap H_0(t, D)$ in $\mathbb{R}^n \times \mathbb{R}$. At every point $p \in T \cap H_0(t, D)$, $\tilde{X}_\epsilon$ is transverse to $\{\epsilon = 0\}$ because if it were not the case, there would be a defining function of $\tilde{X}_\epsilon$ which is a multiple of ϵ. This is impossible by the definition of $\tilde{X}_\epsilon$ as the set of initial parts from which ϵ has been

factored out. By hypothesis (2), $(\mathbb{R}t + D)_C$ is transverse to T_C in $\{\epsilon = 0\}$, therefore since $\tilde{X}_\epsilon$ transverse to $\{\epsilon = 0\}$ and $\tilde{X}_\epsilon \cap \{\epsilon = 0\} = T_C$ we have that $(\mathbb{R}t + D)_C \cap \{\epsilon = 0\}$ is transverse to $\tilde{X}_\epsilon$ in $\mathbb{R}^n \times \mathbb{R}$.
As T is a cone, $T \cap (\mathbb{R}t + D)$ is also a cone and thus $H_0(t, D)$ is transverse to $T \cap (\mathbb{R}t + D)$ in $(\mathbb{R}t + D)$. Since $\tilde{X}_\epsilon$ transverse to $(\mathbb{R}t + D) \cap \{\epsilon = 0\}$ and $\tilde{X}_\epsilon \cap (\mathbb{R}t + D) \cap \{\epsilon = 0\} = T_C \cap (\mathbb{R}t + D)$, we obtain that $H_0(t, D) \cap \{\epsilon = 0\}$ is transverse to $\tilde{X}_\epsilon$ in $\mathbb{R}^n \times \mathbb{R}$. Condition (1) asserts that $T_{\mathbb{P}}$ and $D_{\mathbb{P}}$ are transverse. This means that T_C and $C_0(t, D)$ are transverse in $(\mathbb{R}t+D)$. Consequently, since $\tilde{X}_\epsilon$ transverse to $(\mathbb{R}t+D)\cap\{\epsilon = 0\}$ and $\tilde{X}_\epsilon \cap (\mathbb{R}t+D)\cap\{\epsilon = 0\} = T_C \cap (\mathbb{R}t+D)$, we obtain that $C_0(t, D) \cap \{\epsilon = 0\}$ is transverse to $\tilde{X}_\epsilon$ in $\mathbb{R}^n \times \mathbb{R}$. Finally, since transversality is an open condition and that $\overline{H}_0(t, D)$ is compact, there exists a neighborhood $]-\epsilon_2, \epsilon_2[$ of 0, such that the constant families in ϵ, $H_0(t, D)$ and $C_0(t, D)$ are transverse to $\tilde{X}_\epsilon$ in $\mathbb{R}^n \times]-\epsilon_2, \epsilon_2[$.
We have fulfilled the condition to apply the moving the wall theorem and we can conclude that there exists an isomorphism h_2 between $\tilde{X}_\epsilon \cap H_0(t, D)$ and $T_C \cap H_0(t, D)$ for any $\epsilon \in]-\epsilon_2, \epsilon_2[$. □

Lemma 6.4.6. *Let $C \in \mathbb{R}^n$ be a cone. For any $D \in \mathbb{G}_{k,n}$ and $t \in D^\perp \setminus \{0\}$, we have:*

$$\exists h_3 \text{ homeomorphism, such that } H_0(t, D) \cap C \overset{h_3}{\approx} (t + D) \cap C.$$

Proof. Consider the map $\phi : p \in (t + D) \cap C \ \mapsto \ p/\parallel p \parallel_2$. This map is well defined and continuous as $0 \notin (t + D)$ and $\parallel p \parallel_2$ never vanishes. As C is a cone, $C \cap (\mathbb{R}_*^+ t + D)$ is also a cone, and the image of $C \cap (t + D)$ is included in $C \cap (\mathbb{R}_*^+ t + D)$. Clearly the norm of $\phi(p)$ for any point is always 1, thus $\mathrm{Im}(\phi) \subset S_{0,1}$. This means that $\mathrm{Im}(\phi) \subset C \cap S_{0,1} \cap (\mathbb{R}_*^+ t + D) = C \cap H_0(t, D)$.
Let $\psi : p \in H_0(t, D) \cap C \ \mapsto \parallel t \parallel_2^2 p/(p|t)$. The function ψ is clearly continuous and well-defined as $H_0(t, D)$ avoids $\mathbb{R}t^\perp = D$. We have for any $p \in (t + D)$:

$$\begin{aligned}
\psi(\phi(p)) =& \parallel t \parallel_2^2 \frac{p}{\parallel p \parallel_2} \frac{\parallel p \parallel_2}{(p|t)} && \text{by applying the definitions of } \phi \text{ and } \psi \\
=& \parallel t \parallel_2^2 \frac{p}{(p|t)} && \text{by simplifying the expression} \\
=& \parallel t \parallel_2^2 \frac{p}{\parallel t \parallel_2^2} = p && \text{because } p \in (t + D) \text{ and } t \in D^\perp.
\end{aligned}$$

Hence ψ is a continuous inverse of the continuous function ϕ. This proves that ϕ is an homeomorphism between $H_0(t, D) \cap C$ and $(t + D) \cap C$ as requested. □

We now prove that these lemmas imply theorem 6.4.2:

Proof of theorem 6.4.2. All the hypotheses of the three lemmas are satisfied under the hypotheses of theorem 6.4.2. We keep the α of lemma 6.4.4 and we set $\epsilon_0 = \min(\epsilon_1, \epsilon_2)$ with ϵ_1 as in lemma 6.4.4 and ϵ_2 as in lemma 6.4.5. In lemma 6.4.6 we choose $\tilde{X}_0$ as the cone C we consider. Finally we set $h = h_3 \circ h_2 \circ h_1$.

Consequently we have that

$$\begin{aligned}
\forall \epsilon \in \ &]0,\epsilon_0],\ \forall \lambda \in]0,\alpha\epsilon], \\
& D_\lambda(t,D)\cap \tilde{X}_\epsilon \overset{h_1}{\approx} H_0(t,D)\cap \tilde{X}_\epsilon && \text{by lemma 6.4.4 as } \epsilon \leq \epsilon_1 \\
& \qquad\qquad \overset{h_2\circ h_1}{\approx} H_0(t,D)\cap \tilde{X}_0 && \text{by lemma 6.4.5 as } \epsilon \leq \epsilon_2 \\
& \qquad\qquad \overset{h}{\approx} (t+D)\cap \tilde{X}_0 && \text{by lemma 6.4.6.}
\end{aligned}$$

□

6.4.2 Proof of lemma 6.4.4

Let us now prove lemma 6.4.4. We will use two additional lemmas to prove it:

Lemma 6.4.7. *Let $X \subset \mathbb{R}^n$ be a real analytic germ at 0 and $\tilde{X}_\epsilon$ its conic blow-up at the origin. For any element $\tilde{P} \in \tilde{\mathbb{G}}^0_{k,n}$, let $D \in \mathbb{G}_{k,n}$ and $t \in D^\perp$ such that $\mathcal{D}(\tilde{P}) = (D, (\mathbb{R}t)_\mathbb{P})$. Assume that condition (1) of theorem 6.4.2 is satisfied. Then we have*

$$\begin{aligned}
&\forall \alpha > 0,\ \forall \eta \in]0,\alpha[,\ \exists \epsilon_1'' > 0,\ \textit{such that} \\
&\qquad \forall \epsilon \in]0,\epsilon_1''[,\ \forall p \in (\tilde{X}_\epsilon \cap B_{0,1}) \text{ s.t. } (p|t) \geq \alpha, \\
&\qquad\qquad \exists v \in (T(\tilde{X}_\epsilon, p)\cap S_{0,1}),\ \textit{such that } (p|v) > \eta/\left\|\, t\,\right\|_2 \text{ and } (t|v) > \eta
\end{aligned}$$

Proof. If the statement of the lemma was not true, there would exist $\alpha > 0$, $\eta \in]0,\alpha[$, a sequence of ϵ_i converging to 0, and a sequence of $p_i \in (\tilde{X}_{\epsilon_i} \cap B_{0,1})$ such that $(p_i|t) \geq \alpha$ ($\forall i \in \mathbb{N}$), and such that for all $v \in (S_{0,1} \cap T(\tilde{X}_{\epsilon_i}, p_i))$, we have $(p_i|v) \leq \eta/\left\|\, t\,\right\|_2$ or $(t|v) \leq \eta$.
As $B_{0,1}$ is relatively compact, we can assume that p_i converges to $p \in \overline{B}_{0,1}$. As ϵ_i converges to 0, the tangent spaces $T(\tilde{X}_{\epsilon_i}, p_i)$ converge to $T(\tilde{X}_0, p)$. Since $\tilde{X}_0$ is a cone, $p \in T(\tilde{X}_0, p)$ and thus we can find vectors $v_i \in T(\tilde{X}_{\epsilon_i}, p_i)$ such that the v_i converge to p.
We can renormalize the v_i and we set $w_i = v_i/\left\|\, v_i\,\right\|_2$. We have assumed that the lemma was false, therefore we conclude that

$$\text{or} \begin{cases} \left\|\, p\,\right\|_2 = (p|p/\left\|\, p\,\right\|_2) = \lim\limits_{i\to\infty} (p_i|w_i) \leq \eta/\left\|\, t\,\right\|_2 \\ \qquad\qquad (t|p/\left\|\, p\,\right\|_2) = \lim\limits_{i\to\infty} (t|w_i) \leq \eta \end{cases}$$

As $(p_i|t) \geq \alpha$, we have $(p|t) \geq \alpha$ too and by Cauchy's inequality we obtain:

$$\left\|\, p\,\right\|_2 \left\|\, t\,\right\|_2 \geq \alpha > \eta.$$

Therefore we cannot have $\left\|\, p\,\right\|_2 \leq \eta/\left\|\, t\,\right\|_2$. As $p_i \in B_{0,1}$, we have $\left\|\, p\,\right\|_2 \leq 1$, and we have $(p|t) \geq \alpha > \eta \geq \left\|\, p\,\right\|_2 \eta$. Thus $(t|p)/\left\|\, p\,\right\|_2 \leq \eta$ cannot happen either. In conclusion, there cannot be any such $\alpha > 0$, $\eta < \alpha$ and sequences ϵ_i and p_i. By way of contradiction the lemma is proved. □

Lemma 6.4.8. *Let $X \subset \mathbb{R}^n$ be a real analytic germ at 0 and $\tilde{X}_\epsilon$ its conic blow-up at the origin. For any element $\tilde{P} \in \tilde{\mathbb{G}}^0_{k,n}$ let $D \in \mathbb{G}_{k,n}$ and $t \in D^\perp$ such that $\mathcal{D}(\tilde{P}) = (D, (\mathbb{R}t)_\mathbb{P})$. Assume that condition (1) of theorem 6.4.2 is satisfied. Then we have:*

$$\begin{array}{l}\exists \beta > 0,\ \exists \epsilon'_1 > 0,\ \textit{such that } \forall \epsilon \in\ ]0, \epsilon'_1[,\ \forall \lambda \in\ ]0, \beta], \\ \exists h'_1\ \textit{homeomorphism, such that} \\ \qquad (\overline{D}_\lambda(t, D) \cap \tilde{X}_\epsilon)_{\lambda \in\ ]0,\beta]} \overset{h'_1}{\approx} (\overline{D}_\beta(t, D) \cap \tilde{X}_\epsilon) \times]0, \beta].\end{array}$$

In addition, $(\overline{D}_\lambda(t, D) \cap \tilde{X}_\epsilon)_{\lambda \in\ ]0,\beta]}$ is Whitney stratified by

$$\textit{and} \qquad \begin{array}{r} I_X = (D_\lambda(t, D) \cap \tilde{X}_\epsilon)_{\lambda \in]0,\beta]}, \\ B_X = (C_\lambda(t, D) \cap \tilde{X}_\epsilon)_{\lambda \in]0,\beta]},\end{array}$$

and $(\overline{D}_\beta(t, D) \cap \tilde{X}_\epsilon) \times]0, \beta]$ is Whitney stratified by

$$\textit{and} \qquad \begin{array}{r} I_\beta = (D_\beta(t, D) \cap \tilde{X}_\epsilon) \times]0, \beta], \\ B_\beta = (C_\beta(t, D) \cap \tilde{X}_\epsilon) \times]0, \beta].\end{array}$$

The homeomorphism h'_1 is a stratum preserving homeomorphism for these stratifications and it commutes with the projection to the parameter space.

Proof. We prove this lemma using the moving the wall theorem (thm. 4.1.12). The family we consider is the $\overline{D}_\lambda(t, D)$. We prove that the $C_\lambda(t, D)$ and $D_\lambda(t, D)$ are transverse to $\tilde{X}_\epsilon$ in $(\mathbb{R}t + D)$ for $\lambda < \beta'$ for some β' that we are going to determine. Once we have proved this, we can choose β in the statement of the lemma to be any number in $]0, \beta'[$.
Condition (1) of theorem 6.4.2 implies that T_C and $C_0(t, D)$ are transverse. As transversality is an open condition and $C_0(t, D)$ is compact, there exists $\beta' > 0$ such that all the $C_\lambda(t, D)$ are transverse to T_C for $\lambda < \beta'$. As the $\tilde{X}_\epsilon$ converge to $\tilde{X}_0 = T_C$, we can find some $\eta > 0$ such that if $\epsilon < \eta$, then the $C_\lambda(t, D)$ are transverse to $\tilde{X}_\epsilon$. In other words, for the β' we already have determined, we have:

$$\exists \eta > 0, \text{ such that } \forall \epsilon < \eta,\ \forall \lambda < \beta',\ C_\lambda(t, D) \text{ and } \tilde{X}_\epsilon \text{ are transverse.}$$

We have thus proved that the sets are transverse in $\mathbb{R}^n$ thus *a fortiori* they are transverse in $(\mathbb{R}t + D)$.
The projection of $((\mathbb{R}t + D) \cap \tilde{X}_\eta \cap B_{0,1})$ along $D^\perp$ to $\mathbb{R}t$ has finitely many critical points, therefore there exists η' such that for $\lambda < \eta'$ the sets $\tilde{X}_{\eta'}$ and $D_\lambda(t, D)$ are transverse in $(\mathbb{R}t + D)$. For any $\epsilon \leq \eta'$ the dilation of ratio ϵ/η' and center 0 is an isomorphism that sends $\tilde{X}_\epsilon$ to $\tilde{X}_{\eta'}$ and that sends any $D_\lambda(t, D)$ to $B_{0,\epsilon/\eta'} \cap (\lambda' t + D)$ where $\lambda' = \lambda\epsilon/\eta'$. The set $D_{\lambda'}(t, D)$ is transverse to $\tilde{X}_{\eta'}$ because $\lambda\epsilon/\eta' < \eta'$ (as $\lambda < \eta'$ and $\epsilon/\eta' \leq 1$). Since $B_{0,\epsilon/\eta'} \cap (\lambda' t + D)$ is an open subset of $D_{\lambda'}(t, D)$, it is also transverse to $\tilde{X}_{\eta'}$. Hence by the dilation $D_\lambda(t, D)$ and $\tilde{X}_\epsilon$ also are transverse.

Set $\epsilon_1' = \min(\eta, \eta')$. We have just proved that for $\lambda < \epsilon_1'$, for any $\epsilon < \epsilon_1'$, both $C_\lambda(t, D)$ and $D_\lambda(t, D)$ are transverse to $\tilde{X}_\epsilon$ in $(\mathbb{R}t + D)$. Choose an arbitrary $\beta \in]0, \beta'[$. We can apply the moving the wall theorem and we obtain a homeomorphism h_1' that sends $(\overline{D}_\lambda(t, D) \cap \tilde{X}_\epsilon)_{\lambda \in\ ]0,\beta'[}$ to $(\overline{D}_\beta(t, D) \cap \tilde{X}_\epsilon) \times]0, \beta'[$. We have that h_1' sends I_X to I_β and B_X to B_β as h_1' commutes to the projection to the parameter space. Hence h_1' is stratum preserving as required. □

Now we can prove lemma 6.4.4:

Definition 6.4.9. *For any given $\alpha \geq 0$ and $t \in \mathbb{R}^n \setminus \{0\}$, we define the family $(S_\gamma)_{\gamma \in \mathbb{R}}$ as*

$$S_\gamma(\alpha) = \{p \in \overline{B}_{0,1} \mid \gamma((p|t) - \alpha) + (1 - \gamma)(\| p \|_2^2 - 1) = 0\}.$$

Remark 6.4.10. *When restricted to some $(\mathbb{R}t + D)$, this family interpolates between $D_\alpha(t, D)$ ($\gamma = 1$) and $H_\alpha(t, D)$ ($\gamma = 0$) while leaving $C_\alpha(t, D)$ stable (i.e. $C_\alpha(t, D)$ is included in every S_γ). In fact the S_γ for $\gamma \in \mathbb{R} \setminus \{1\}$ are the portions of spheres in $\overline{B}_{0,1}$ whose center lies in $\mathbb{R}t$ (for $\gamma = 1$ one could also see the plane $(.|t) = \alpha$ as a degenerate sphere).*
This means that the family $(S_\gamma)_{\gamma \in \mathbb{R}}$ is Whitney stratified by S_γ° (its interior) and $C_\alpha(t, D)$ as a constant family with respect to γ.

Proof of lemma 6.4.4. **Paragraph I:** First we use lemma 6.4.8 that gives us a β and ϵ_1'. We set $\alpha < \beta$ and thus the lemma shows that for any $\epsilon < \epsilon_1'$ we have that $D_\alpha(t, D) \cap \tilde{X}_\epsilon$ and $D_\lambda(t, D) \cap \tilde{X}_\epsilon$ are homeomorphic for any $\lambda \leq \alpha$. We thus only need prove that $D_\alpha(t, D) \cap \tilde{X}_\epsilon$ and $H_0(t, D) \cap \tilde{X}_\epsilon$ are homeomorphic.
Paragraph II: We use the moving the wall theorem (thm. 4.1.12) to prove that $D_\alpha(t, D) \cap \tilde{X}_\epsilon$ and $H_\alpha(t, D) \cap \tilde{X}_\epsilon$ have the same topology. For any $p \in S_\gamma^\circ$ the tangent space at p is the kernel of the differential of

$$\gamma((p|t) - \alpha) + (1 - \gamma)(\| p \|_2^2 - 1).$$

We thus see that the points $v \in T(S_\gamma^\circ, p)$ are exactly the points satisfying $\gamma(v|t) + 2(1 - \gamma)(v|p)$. By lemma 4.1.11 for $\alpha/2$ (α is already fixed), there exists ϵ_1'' and $\eta > 0$ such that for any $\epsilon < \epsilon_1''$ and $p \in B_{0,1}$ such that $(p|t) > \alpha/2$, there exists $v \in T(\tilde{X}_\epsilon, p)$ such that $(v|t) > \eta$ and $(v|p) > \eta$. Therefore there exists a small $\Gamma > 0$, such that $\gamma(v|t) + 2(1-\gamma)(v|p) > 0$ for any $\gamma \in]-\Gamma, 1+\Gamma[$, which means that v lies out of the tangent space to S_γ°. This shows that the deformation is transverse to $\tilde{X}_\epsilon$ as soon as $\epsilon < \epsilon_1''$. The boundary of S_γ is always C_α hence there is no transversality condition to verify for this stratum of the family. Finally by the moving the wall theorem we can conclude that all the $S_\gamma \cap \tilde{X}_\epsilon$ are homeomorphic for $\gamma \in]-\Gamma, 1+\Gamma[$ and thus in particular for $\gamma = 0$ and $\gamma = 1$ we see that $H_\alpha(t, D) \cap \tilde{X}_\epsilon$ and $D_\alpha(t, D) \cap \tilde{X}_\epsilon$ are homeomorphic.
Paragraph III: Finally lemma 6.4.8 tells us that for $\epsilon < \epsilon_1'$, there is a homeomorphism h_1' commuting to the projection to the parameter space between $(C_\lambda(t, D) \cap \tilde{X}_\epsilon)_{\lambda \in\ ]0,\beta]}$ and $(C_\beta(t, D) \cap \tilde{X}_\epsilon) \times]0, \beta]$. Let us use the notation $\mathfrak{H}_\lambda$

for $(H_0(t,D)\cap\tilde{X}_\epsilon)$ for the next sequence of homeomorphisms:

$$\begin{aligned}
\mathfrak{H}_0 &\approx \mathfrak{H}_\beta \cup (C_\lambda(t,D)\cap\tilde{X}_\epsilon)_{\lambda\in\,]0,\beta[} && \text{by definition of } H_\lambda \text{ and } C_\lambda\\
&\approx \mathfrak{H}_\beta \cup (C_\lambda(t,D)\cap\tilde{X}_\epsilon)_{\lambda\in\,]\alpha,\beta[} && \text{by } h_1'\\
&\approx \mathfrak{H}_\alpha && \text{by definition of } H_\lambda \text{ and } C_\lambda.
\end{aligned}$$

Conclusion: We set $\epsilon_1=\min(\epsilon_1',\epsilon_1'')$. For this ϵ_1 and α as chosen above we have shown that $\forall\epsilon<\epsilon_1$ we have for any $\lambda\in\,]0,\alpha]$

$$\begin{aligned}
H_0(t,D)\cap\tilde{X}_\epsilon &\approx H_\alpha(t,D)\cap\tilde{X}_\epsilon && \text{by paragraph III}\\
&\approx D_\alpha(t,D)\cap\tilde{X}_\epsilon && \text{by paragraph II}\\
&\approx D_\lambda(t,D)\cap\tilde{X}_\epsilon && \text{by paragraph I.}
\end{aligned}$$

This is exactly the claim of the lemma. □

6.4.3 Proof of main theorem

Here is the direct corollary that ensues from theorem 6.4.2, which asserts that the local Betti numbers of a generic section of the tangent cone are the same as those of a generic section of the germ.

Corollary 6.4.11. *For $X\subset\mathbb{R}^n$ be a pure dimensional real analytic germ, $\mathfrak{b}_i$ any Betti number and T the tangent cone to X. Assume that $k+\dim(\mathrm{Sing}(T))<n$ for some $k\in\mathbb{N}$, then $\mathfrak{b}_i(X,k)=\mathfrak{b}_i(T,k)$.*

Proof. It suffices to show that the three conditions of theorem 6.4.2 are satisfied for a generic $\tilde{P}\in\tilde{\mathbb{G}}^0_{k,n}$. Recall that $(D,(\mathbb{R}t)_\mathbb{P})=\mathcal{D}(\tilde{P})$.
Consider the following general fact: for a generic linear space $L_\mathbb{P}$ in projective space and any fixed Whitney stratified set $W_\mathbb{P}$, we have that $L_\mathbb{P}$ and $W_\mathbb{P}$ are generically transverse. For $L_\mathbb{P}=D$ and $W_\mathbb{P}=T_\mathbb{P}$ this means that D is generically transverse to $T_\mathbb{P}$, and for $L_\mathbb{P}=(\mathbb{R}t+D)$ and $W_\mathbb{P}=T_\mathbb{P}$ this means that $(\mathbb{R}t+D)$ and $T_\mathbb{P}$ are generically transverse. In the first case this proves that $\tilde{P}$ generically meets condition (1) (as there is no condition imposed on t by the condition (1)). In the second case this proves that $\tilde{P}$ generically satisfies to condition (2) (as there is a one to one correspondence between the $\tilde{P}$ and the $(\mathbb{R}t+D)$).
Let $s=\dim(\mathrm{Sing}(T))$. We have $\dim((\mathbb{R}t+D)_\mathbb{P})=k$ and $\dim(\mathrm{Sing}(T_\mathbb{P}))=s-1$. Since $k+s<n$, $\dim((\mathbb{R}t+D)_\mathbb{P})+\dim(\mathrm{Sing}(T)_\mathbb{P})=k+s-1<n-1$, which proves that $(\mathbb{R}t+D)_\mathbb{P}$ generically avoids $\mathrm{Sing}(T)_\mathbb{P}$. That is, condition (3) is met for a generic $\tilde{P}\in\tilde{\mathbb{G}}^0_{k,n}$. □

Now we can tackle the main theorem of this section: there is a polynomial bound on the local sum of Betti numbers of a generic section of the germ when this section is generically smooth and pure dimensional.

Theorem 6.4.12. *For X an analytic germ of multiplicity μ and of pure dimension d, let s be the dimension of the singular locus of its tangent cone. If $s<n-k$, then $\Sigma_\mathfrak{b}(X,k)\leq\mu(2\mu-1)^{k-1}$.*

In order to prove it we need two auxiliary theorems. The first one is the well-known Oleinik-Petrovsky-Thom-Milnor bound on the number of connected components of a real affine algebraic variety $X \subset \mathbb{R}^n$ or a real projective algebraic variety $X \subset \mathbb{P}^n(\mathbb{R})$ (see [98],proof of theorem 2, or [16, 24]).

Theorem 6.4.13 (Oleinik-Petrovski-Thom-Milnor bound)**.** *If X is an algebraic variety defined by polynomials of degree at most d in $\mathbb{R}^n$ or $\mathbb{P}^n(\mathbb{R})$ then*

$$\Sigma_{\mathfrak{b}}(X) \leq d(2d-1)^{n-1}.$$

The second auxiliary theorem states that if μ is the geometric degree of a pure dimensional affine variety, then the variety can be defined by generators in degree μ.

Definition 6.4.14 (Geometric Degree)**.** *Let $Z \in \mathbb{C}^n$ be a variety of dimension d such that the intersection with a generic $(n-d)$ affine space contains $\delta(Z)$ points (counted with multiplicity). The quantity $\delta(Z)$ is called the geometric degree of Z.*

The theorem is a straighforward consequence of the work of J. Heintz ([74] prop.3). D. Mumford also obtained a similar result that appears in the proof of theorem 1, in [110] (the proof is less detailed than in J. Heintz's presentation).

Theorem 6.4.15 (Generators in bounded degree)**.** *Let $Z \in \mathbb{C}^n$ be a pure dimensional variety with geometric degree δ. Then there exist finitely many generators $(g_i)_{i\in I} \in \mathcal{O}(\mathbb{C}^n)$ such that $\deg g_i \leq \delta$ $(\forall i \in I)$, and $Z = V((g_i)_{i\in I})$.*

Proof. J. Heintz's result in [74] (prop.3) states that for an irreducible variety $Y \subset \mathbb{C}^n$, there exist an ideal $\mathcal{I} = (g_1, \ldots, g_{n+1}) \subset \mathcal{O}(\mathbb{C}^n)$ such that $Y = V(\mathcal{I})$ and $\deg g_i \leq \delta(Y)$ where $\delta(Y)$ is the geometric degree of Y. In this case we say that $\mathcal{I}$ is generated in degree $\delta(Y)$.
As Z is pure dimensional, it can be broken down as the union of irreducible varieties $Y_1, \ldots, Y_l$, of same dimension as Z. Let the $\mathcal{I}_1, \ldots, \mathcal{I}_l$ be the defining radical ideals of the $Y_1, \ldots, Y_l$. As Z was a geometric variety there is no immersed component in it, and thus the ideal $\mathcal{J} = \Pi_{i=1}^{l} \mathcal{I}_i$ is the defining radical ideal of Z. By J. Heintz's theorem we can conclude that each Y_i was generated in degree $\delta(Y_i)$ and thus Z is generated in degree $\sum_{i=1}^{l} \delta(Y_i)$. To prove the theorem it thus suffices to prove that $\delta(Z) = \sum_{i=1}^{l} \delta(Y_i)$.
Let $c := n - \dim(Z)$ be the codimension of Z. This is where we use that Z is pure dimensional: this implies that all the Y_i have the same codimension c as Z. Therefore, for any Y_i, a generic affine space $A \in \overline{\mathbb{G}}_{c,n}$, we have $\dim(A \cap Y_i) = 0$ (the intersection is zero-dimensional) and $\dim_{\mathbb{C}}(A \cap Y_i) = \delta(Y_i)$ (the number of points in the intersection is the geometric degree of Y_i). A generic A will also avoid the intersections $Y_i \cap Y_j$ $(i \neq j)$ as they have strictly greater codimension than c. Therefore for a generic A we have the equality:

$$\sharp(A \cap Z) \leq \sum_{i=1}^{l} \sharp(A \cap Y_i),$$

where $\sharp(S)$ denotes the cardinality of the set S. By definition of the topological degree the previous inequality for a generic $A \in \overline{\mathbb{G}}_{c,n}$ means:

$$\delta(Z) \leq \sum_{i=1}^{l} \delta(Y_i).$$

□

Equipped with these last two theorems we can finally prove the main theorem of the section (which corresponds to the second item of the main theorem of the chapter 6.1.21).

Proof of theorem 6.4.12. The conditions that $s + k < n$ and X is pure dimensional are exactly the necessary conditions to apply the previous corollary (6.4.11), therefore we have $\mathfrak{b}_i(X,k) = \mathfrak{b}_i(T,k)$ for any Betti number $\mathfrak{b}_i$. By summation we obtain $\Sigma_{\mathfrak{b}}(X,k) = \Sigma_{\mathfrak{b}}(T,k)$.
Now we are going to bound the degree of the generators in a generic cut by μ. For any $\tilde{P} \in \tilde{\mathbb{G}}^0_{k,n}(\mathbb{R})$, let $D \in \mathbb{G}_{k,n}(\mathbb{R})$ and $t \in D^\perp$ such that $(D, (\mathbb{C}t)_{\mathbb{P}}) = \mathcal{D}(\tilde{P})$. The algebraic tangent cone is an algebraic variety. As X is pure d-dimensional, all components of T that are not d-dimensional are singular, and by hypothesis they have dimension strictly less than $n-k$. Let T', t' and D' the complexifications of T, t and D. For a generic $\tilde{P}$, $(\mathbb{C}t' + D')$ avoids $T'_{<d}$ (the components of T' of dimension less than d) as it is singular hence of dimension less than $n-k$. Furthermore for a generic $\tilde{P}$, $(t'+D')$ is transverse to T'_C and thus $(t'+D') \cap T'_C$ is pure dimensional of dimension $k+d-n$. Under those two conditions we have: For a generic pair $(\Delta, \tau) \in \tilde{\mathbb{G}}^0_{k,n}(\mathbb{C}) \times \Delta^\perp$ such that $(\tau + \Delta) \subset (t' + D')$, we have that $\Delta_{\mathbb{P}} \cap T'_{\mathbb{P}} = \emptyset$, $(\mathbb{C}\tau + \Delta) \cap T'_{\mathbb{P}}$ is 0-dimensional and also $(\mathbb{C}\tau + \Delta) \cap (T'_{<d})_{\mathbb{P}} = \emptyset$ because $(\mathbb{C}t' + D')$ itself already avoids $T'_{<d}$. Therefore we can apply corollary 6.3.3, and by lemma 6.3.4 we have that for a generic affine space P $(= \tau + \Delta)$ inside $(t' + D')$, $\mathcal{C}(T', P) \leq \mu$ (in fact equality holds as $T' \cap (t' + D')$ is pure dimensional). This means that the geometric degree of $(t' + D') \cap T'$ is less than μ. As we have made sure that $(t' + D') \cap T'$ is pure dimensional, we can apply Heintz-Mumford's result (theorem 6.4.15), and conclude that there exist generators of the defining ideal of $T' \cap (t' + D')$ in degree μ.
Obviously the generators of $T' \cap (t' + D')$ give rise to generators of $T \cap (t + D)$ with the same degree by considering the realization of $T' \cap (t' + D')$. Consequently by the Oleinik-Petrovsky-Thom-Milnor bound (theorem 6.4.13), the number of connected components in $T \cap (t + D)$ is bounded by $\mu(2\mu - 1)^{k-1}$ for a generic $\tilde{P} \in \tilde{\mathbb{G}}^0_{k,n}(\mathbb{R})$. This means that $\Sigma_{\mathfrak{b}}(X,k) = \Sigma_{\mathfrak{b}}(T,k) \leq \mu(2\mu - 1)^{k-1}$. □

6.5 Applications

In this section we present an effective procedure to compute the multiplicity of a germ defined by a set of generators, as well as a bound on the density and the local Lipschitz-Killing invariants of a real analytic germ (definition 6.5.2).

Calculating the multiplicity of a germ is *per se* interesting. We give an algorithm based on Gröbner bases, a classical tool in computer algebra, to compute it. We also give an example on which we carry out the algorithm. The second application to bounding the density and the local Lipschitz-Killing invariants gives at least one reason why knowing the multiplicity is useful. The bound we give on the density and the Lipschitz-Killing invariants of a real analytic germ (proposition 6.5.10) is relevant to the work in [37, 38, 39] which all involve these quantities. The result is obtained from a localized version of the multidimensional Cauchy-Crofton formula found in [39], thm.3.1.

6.5.1 Computing the multiplicity

We present an effective algorithm to compute the multiplicity of a germ given its generators. It is based on a classical tool, namely Gröbner bases, and proposition 6.3.5. Gröbner bases are an effective tool thanks to Buchberger's algorithm that computes them. There are many implementations of this algorithm available, among them are MacCaulay2[1] and Singular[2]. A classical reference on the effective theory of Gröbner bases and Buchberger's algorithm is [42]. The first complete exposition of Buchberger's algorithm appears in [30] and the theory of Gröbner bases has been the object of much subsequent work. An analogous notion for local rings (standard bases) has also been introduced by H. Hironaka in [75]. The most efficient way to compute a Gröbner basis to date is Faugère's F_5 algorithm [56].
To compute the multiplicity, we first give a straightforward and unoptimized algorithm which has the advantage of simplicity. Then we work out a full example using Singular to show how the algorithm behaves in practice. Finally we discuss possible improvements to make the algorithm more efficient.

The initial algorithm

Assume that $Z \subset \mathbb{C}^n$ is defined by a set of generators $f_1, \ldots, f_k \in \mathbb{K}[X_1, \ldots, X_n]$. Let $\mathcal{I}$ the ideal generated by the f_i's. Our starting point is H. Whitney's theorem ([140], thm.5.8) that characterizes the generators of the tangent cone as the initial parts of $\mathcal{I}$. This theorem has been restated earlier in this chapter as theorem 6.1.17. Recall that the initial parts are the homogeneous components with lowest total degree. We denote by $\mathrm{Init}(\mathcal{I})$ the ideal generated by the initial parts of the elements of $\mathcal{I}$. We denote by T the tangent cone, that is, the scheme associated to $\mathrm{Init}(\mathcal{I})$.

For the sake of simplicity we will call the base field $\mathbb{C}$, but of course, what the computer will in fact use is a countable subfield of $\mathbb{C}$ (probably $\mathbb{Q}$), and we will assume computations in this subfield are fast. Now that we have made precise this technical detail, we can look into the mathematical difficulties that computing the multiplicity raises. The first obstacle is that H. Whitney's theorem considers the initial parts of *all* the elements in $\mathcal{I}$, not just of a set of generators.

[1] see http://www.math.uiuc.edu/Macaulay2/
[2] http://www.singular.uni-kl.de/

The example of the ideal generated by $(x^2 - y^3, x^2 - 2y^3)$ shows that it is important to consider not only the generators. If one were to take the initial parts of the generators only, one would conclude that the tangent cone is defined by $(x^2, x^2) = (x^2)$, and this is obviously wrong as $(x^2 - y^3, x^2 - 2y^3) = (x^2, y^3)$ and thus the tangent cone is defined by (x^2, y^3) itself. The ideal situation would be that we find generators of $\mathcal{I}$ whose initial parts generate $\mathrm{Init}(\mathcal{I})$.
Fortunately there is an algorithm that does exactly this. This is Buchberger's algorithm to compute Gröbner bases. Buchberger's algorithm takes generators $f_1, \ldots, f_k$ of an ideal $\mathcal{I}$ as input and outputs a new set of generators $g_1, \ldots, g_l$ that is called Gröbner basis. As a Gröbner basis the g_i's have the key property that

$$(\mathrm{Init}(g_1), \ldots, \mathrm{Init}(g_l)) = \mathrm{Init}(\mathcal{I}) \tag{6.1}$$

This is just what we wanted. To run the algorithm we only need to find a valuation on $\mathbb{C}[X_1, \ldots, X_n]$ such that the initial part of a polynomial has higher priority than the rest of the polynomial. This is done by choosing ν the valuation that gives the order at 0 of a polynomial, and setting that the lower the valuation, the higher the priority. This ordering is called a local ordering. It is not important for our purpose how we order monomials with same valuation, but Buchberger's algorithm needs a total ordering on monomials. In the example of next subsection we will use the reverse lexicographic order on monomials with same valuation (a common choice because it is heuristically good).
The careful reader may think that because we compute a Gröbner basis for a finer valuation than the local ordering, equation 6.5.1 may not hold. As a matter of fact, it does because of the following proposition

Proposition 6.5.1. *Let ν and ν' be two local monomial valuations on $\mathbb{C}[X_1, \ldots, X_n]$ such that ν' is finer than ν (that is $\nu'(p) = \nu'(q)$ implies $\nu(p) = \nu(q)$). For any element p, let $\mathrm{Init}_\nu(p)$ (resp. $\mathrm{Init}_{\nu'}(p)$) be the ν-initial (resp. ν'-initial) part of p (that is, the sum of the monomials of p with highest priority). If $\mathcal{I}$ is an ideal generated by $g_1, \ldots, g_l$ such that*

$$(\mathrm{Init}_{\nu'}(g_1), \ldots, \mathrm{Init}_{\nu'}(g_l)) = \mathrm{Init}_{\nu'}(\mathcal{I}),$$

then $g_1, \ldots, g_l$ also satisfy to

$$(\mathrm{Init}_{\nu}(g_1), \ldots, \mathrm{Init}_{\nu}(g_l)) = \mathrm{Init}_{\nu}(\mathcal{I}).$$

Proof. Let $p \in \mathcal{I}$. Because ν' is finer than ν, the polynomial p can be decomposed as $p = a + b + c$ with all the monomials in a with highest priority for ν', the monomials in $a + b$ with highest priority for ν, and c the rest of p. Since by hypothesis we have

$$(\mathrm{Init}_{\nu'}(g_1), \ldots, \mathrm{Init}_{\nu'}(g_l)) = \mathrm{Init}_{\nu'}(\mathcal{I}).$$

We can write

$$a = \mathrm{Init}_{\nu'}(p) = \sum_{i=1}^{l} \lambda_i \ \mathrm{Init}_{\nu'}(g_i),$$

with the $\lambda_i \in \mathbb{C}[X_1, \ldots, X_n]$. Set $\lambda_{1,i} = \lambda_i$ and let

$$p_1 = p - \sum_{i=1}^{l} \lambda_{1,i}\ g_i.$$

By definition of monomial valuations, p_1 has lower priority than p, and thus ν'-valuation strictly greater than p (they increase because we are looking at a local ordering, therefore the lower the priority, the higher the valuation). The way p_1 is defined implies it lies in $\mathcal{I}$, and thus we can apply again the same procedure to p_1, as we did for p. In this way, we create sequence of $p_1, p_2, p_3, \ldots, p_j, \ldots$ and their associated $\lambda_{j,i}$, such that the p_i's have growing valuations for ν' and the $(p - p_i)$'s are all in $\sum_{i=1}^{l} \mathbb{C}[X_1, \ldots, X_n]g_i$. Eventually for some $\alpha \in \mathbb{N}$, we will have $\nu'(p_\alpha) \geq \nu'(c) > \nu'(p_{\alpha-1})$. As ν' is finer than ν, this shows that the valuation for ν of every monomial in p_α is strictly higher than $\nu(p)$, and thus that $\mathrm{Init}_\nu(p - p_\alpha) = \mathrm{Init}_\nu(p)$. By construction of the p_i for $i \leq \alpha$ we have $\nu(p) = \nu(\lambda_{j,i}\ g_i)$ whenever $\lambda_{j,i} \neq 0$.
It is true in general that for polynomials q, r_n if we have

$$q = \sum_{n=1}^{k} r_n \text{ and } \nu(q) = \nu(r_n)\ (\forall n \in \{1, \ldots, k\}),$$

then

$$\mathrm{Init}_\nu(q) = \sum_{n=0}^{k} \mathrm{Init}_\nu r_n.$$

This is because the initial part of q is equal to the sum of monomials in the r_n with same valuation as q, and because the $\nu(r_n)$ have already the same valuation as q, these monomials are in fact the initial parts of the r_n. We apply this general fact to our case by setting $q = p - p_\alpha$ and making the r_n run over the $\lambda_{j,i}\ g_i$ $(i \in \{1, \ldots, l\}$ and $j \in \{1, \ldots, \alpha\})$. The definition of valuations straighforwardly entails that

$$\mathrm{Init}_\nu(\lambda_{j,i}\ g_i) = \mathrm{Init}_\nu(\lambda_{j,i})\mathrm{Init}_\nu(g_i).$$

This shows that $\mathrm{Init}_\nu(p - p_\alpha)$ lies in the ideal generated by the ν-initial parts of the g_i. Since we have already proved that $\mathrm{Init}_\nu(p - p_\alpha) = \mathrm{Init}_\nu(p)$ we have proved the proposition. □

The initial parts of the elements of the Gröbner basis of Z are thus generators of the defining ideal of the tangent cone T. Proposition 6.3.5 states that for a generic $\overline{P} \in \overline{\mathbb{G}}_{n-d,n}$, we have $\dim_{\mathbb{C}}(\overline{P} \cap T) = \mu(Z)$. In other words the multiplicity of Z is equal to the geometric degree of T. This quantity happens to be the lead coefficient (coefficient of the monomial with highest degree) of the Hilbert polynomial associated to T. In addition it can be computed from a Gröbner basis for the defining ideal of T (see [50], thm.15.1.1 and 15.26, and also [29], chapter 4). The approach described in [29] has been implemented in

the software package "normaliz" by W. Bruns.
Here, we present a more straightforward strategy which is still based on proposition 6.3.5. This direct approach is easier to understand than the approach relying on the Hilbert polynomial and it will be of similar efficiency after the improvements proposed at the end of this subsection. Let d be the (Krull) dimension of the tangent cone. This dimension can directly be read from the Gröbner basis: it is the number of variables X_i for which there is no polynomial in the Gröbner basis whose lead monomial is of the form X_i^α for some $\alpha \in \mathbb{N}$. The fact that for a generic $\overline{P} \in \overline{\mathbb{G}}_{n-d,n}$, we have $\dim_{\mathbb{C}}(\overline{P} \cap T) = \mu(Z)$ can easily be formulated algebraically by introducing the linear forms $l_1, \ldots, l_d$ defined by $l_i = \sum_{j=1}^{n} \alpha_{i,j} X_j$, and the fraction field $F = \mathbb{C}(\alpha_{1,1}, \ldots, \alpha_{1,n}, \ldots, \alpha_{d,1}, \ldots, \alpha_{d,n})$. In that context proposition 6.3.5 reads

$$\dim_F \mathcal{O}(T \cap \overline{P}) = \dim_F \frac{F[X_1, \ldots, X_n]}{(\mathrm{Init}(\mathcal{I}), l_1 - 1, \ldots, l_d - 1)} = \mu(Z).$$

The (-1) we remove from the l_i is there to offset the vector space and make $\{x \in \mathbb{C}^n \mid l_i(x) - 1 = 0\}$ a generic affine space. Once again, we can compute the dimension of this function sheaf by means of Buchberger's algorithm but this time we use the valuation associated to the lexicographic order on variables which gives higher priority to variables with higher degree. We call lead monomial of a polynomial f , the monomial that has higher priority according this ordering. We denote it by $\mathrm{Lead}(f)$. Lead monomials are thus the analogue of initial parts, they only differ in the ordering that is associated to them. The ordering for the lead monomial system is called the usual polynomial ordering. We run Buchberger's algorithm on $(\mathrm{Init}(g_1), \ldots, \mathrm{Init}(g_l), l_1 - 1, \ldots, l_d - 1)$ for the usual polynomial ordering and we obtain a Gröbner basis $g'_1, \ldots, g'_{l'}$ in $F[X_1, \ldots, X_n]$. With the usual polynomial ordering the key property of Gröbner bases reads:

$$(\mathrm{Lead}(g'_1), \ldots, \mathrm{Lead}(g'_{l'})) = \mathrm{Lead}(\mathrm{Init}(\mathcal{I}), l_1 - 1, \ldots, l_d - 1).$$

As the system is 0-dimensional, the above equality means that only finitely many monomials $x_1^{a_1} \ldots x_n^{a_n}$ $(a_i \in \mathbb{N})$ will not be divisible by the lead coefficient of a polynomial in the Gröbner basis. These monomials thus give a vector basis of $\mathcal{O}(T \cap \overline{P})$ over F, and their number is the dimension of $T \cap \overline{P}$ over F, that is the multiplicity.

Unfortunately, the previous procedure does not scale very well. In practice, even if the first Gröbner basis computation was reasonable to some extent, the second one involving computation on a polynomial field clearly confines the procedure to simple examples. This is not to say that this procedure is useless, it works on simple examples such as the one in the next subsection, but it won't run on an example with degree 100 and multiplicity 10 in $\mathbb{R}^4$. Some improvements are suggested after the example.

A full example

Let us now give a non-trivial example run with Singular. The explanations and comments follow.

```
ring r0=0,(x,y,z),ds;
ideal m=(x*(x-z^3)*(x-2*z^2),  y*(y-z^3)*(y-2*z^2),
  (x+y)*(x+y-z^3));

m=
x3-2x2z2-x2z3+2xz5,
y3-2y2z2-y2z3+2yz5,
x2+2xy+y2-xz3-yz3

ideal hm=(x3,  y3,  x2+2xy+y2);

ideal sm= std(m);

sm=
x2+2xy+y2-xz3-yz3,
3xy2+2y3-2x2z2-xyz3-2y2z3+2xz5,
y3-2y2z2-y2z3+2yz5,
8xyz4-24y2z4-3xy2z4+16xyz5+24yz7,
64y2z4-x2yz4+7xy2z4-32xyz5-64yz7

ideal hsm= (x2+2xy+y2,  3xy2+2y3,  y3,  8xyz4-24y2z4,  64y2z4)

ring r2=0, (a,b,c,t1,t2,t3),lp;
map f=r1,a,b,c;
ideal n= f(hm);
n=
a^3,
b^3,
a^2+2*a*b+b^2

ideal o= f(hsm);

o=
a^2+2*a*b+b^2,
3*a*b^2+2*b^3,
b^3,
8*a*b*c^4-24*b^2*c^4,
64*b^2*c^4

ideal sn= std(n);
sn=
c^4*t3^4-4*c^3*t3^3+6*c^2*t3^2-4*c*t3+1,
```

```
3*b*c^2*t1*t2*t3^2-3*b*c^2*t2^2*t3^2-6*b*c*t1*t2*t3+6*b*c*t2^2*t3+
  3*b*t1*t2-3*b*t2^2+c^3*t1*t3^3-3*c^3*t2*t3^3-3*c^2*t1*t3^2+
  9*c^2*t2*t3^2+3*c*t1*t3-9*c*t2*t3-t1+3*t2,
b*c^3*t3^3-3*b*c^2*t3^2+3*b*c*t3-b,
b^2*t1^2-2*b^2*t1*t2+b^2*t2^2-2*b*c*t1*t3+2*b*c*t2*t3+
  2*b*t1-2*b*t2+c^2*t3^2-2*c*t3+1,
2*b^2*c*t1*t2*t3-2*b^2*c*t2^2*t3+b^2*t1^2-4*b^2*t1*t2+3*b^2*t2^2+
  (-1)*b*c^2*t2*t3^2-2*b*c*t1*t3+4*b*c*t2*t3+2*b*t1-3*b*t2+
  c^3*t3^3-2*c^2*t3^2+c*t3,
b^2*c^2*t3^2-2*b^2*c*t3+b^2,
b^3*t2^2-2*b^2*c*t1*t3+5*b^2*c*t2*t3+2*b^2*t1-5*b^2*t2+
  4*b*c^2*t3^2-8*b*c*t3+4*b,
2*b^3*t1-3*b^3*t2-3*b^2*c*t3+3*b^2,
b^3*c*t3-b^3,
b^4,
a*t1+b*t2+c*t3-1,
3*a*c^2*t3^2-6*a*c*t3+3*a+3*b^3*t1*t2-4*b^3*t2^2-3*b^2*c*t1*t3+
  3*b^2*t1+6*b*c^2*t3^2-12*b*c*t3+6*b,
a*b*t2+a*c*t3-a-b^2*t1+2*b^2*t2+2*b*c*t3-2*b,
3*a*b*c*t3-3*a*b-3*b^3*t1+4*b^3*t2+6*b^2*c*t3-6*b^2,
3*a*b^2+2*b^3,
a^2+2*a*b+b^2

ideal so= std(o);
so=
c^2*t3^2-2*c*t3+1,
b*c*t3-b,
b^2,
a*t1+b*t2+c*t3-1,
3*a*c*t3-3*a-3*b^3*c^3*t1*t2*t3^3+4*b^3*c^3*t2^2*t3^3+
  (-1)*3*b^3*c^2*t1*t2*t3^2+4*b^3*c^2*t2^2*t3^2-3*b^3*c*t1*t2*t3+
  4*b^3*c*t2^2*t3-3*b^3*t1*t2+4*b^3*t2^2+15*b^2*c^4*t2*t3^4+
  (-1)*3*b^2*t1+6*b*c*t3-6*b,
3*a*b+3*b^3*c^3*t1*t3^3-4*b^3*c^3*t2*t3^3+3*b^3*c^2*t1*t3^2+
  (-1)*4*b^3*c^2*t2*t3^2+3*b^3*c*t1*t3-4*b^3*c*t2*t3+3*b^3*t1+
  (-1)*4*b^3*t2-15*b^2*c^4*t3^4+6*b^2,
a^2+2*a*b+b^2
```

We start the procedure with the germ given by

$$m = \big(x(x-z^3)(x-2z^2), y(y-z^3)(y-2z^2), (x+y)(x+y-z^3)\big).$$

As one can see, the starting ideal is not trivial, but not all that complicated. In spite of this, the computation already gives rise to a fairly lengthy output. The ideal is given in factorized form so that we can understand it, but of course the computer does not use at all this information (as the output of Singular shows, they are all in developed form). It is probably useful to understand the geometry

of the variety associated to m to better understand the rest of this discussion. The example is built on a simple case of non-complete intersection. In the plane we use the equations $(x(x-1)(x-2), y(y-1)(y-2))$ to define 9 points, out of which we pick $(0,0),(1,0),(0,1)$ by adding the polynomial $(x+y)(x+y-1)$. The construction in three-space is analogous. We associate a branch to those 9 points by changing the 1's to some powers of z. A straight line is associated to the points $(0,0),(1,1),(2,2)$, the branches $(0,z^3,z)$ and $(z^3,0,z)$ are associated to $(0,1)$ and $(1,0)$, the branches $(0,z^2,z)$ and $(z^2,0,z)$ are associated to $(0,2)$ and $(2,0)$ and finally the branches (z^3,z^2,z) and (z^2,z^3,z) are associated to $(1,2)$ and $(2,1)$. By changing the third polynomial to $(x+y)(x+y-z^3)$ we only keep the branches associated to $(0,0),(0,1),(1,0)$. Thus 3 should be the multiplicity.

We now go back to the code in Singular. The line `ring r0 = 0, (x,y,z),ds`, creates the ring $\mathbb{Q}[x,y,z]$ with a local ordering (the first `0` stands for characteristic 0 and `ds` specifies we want the local reverse lexicographic ordering on the monomials). The first pitfall has been mentioned before with the example (x^2-y^3, x^2-2y^3). If we take the initial part of the generators, we do not obtain the tangent cone T, we obtain some supset T' of it. In this case we obtain T' as the zero set of $hm = (x^3, y^3, (x+y)^2)$. We obtain T by using the command `std` that computes a Gröbner basis for m (a synonym of Gröbner basis is standard basis, hence the name of the command) and taking the initial part of its generators for the local ordering. The result of the algorithm is:

$$\begin{aligned} hsm &= (x^2+2xy+y^2, 3xy^2+2y^3, y^3, 8xyz^4-24y^2z^4, 64y^2z^4) \\ &= ((x+y)^2, xy^2, y^3, xyz^4, y^2z^4). \end{aligned}$$

Even if the theory ensures that $hm \subset hsm$ it is always pleasing to check it on a particular example. The generators y^3 and $(x+y)^2$ are shared in hm and hsm. As for $x^3 \in hm$, we use the equality $x^3 = (x-2y)(x+y)^2 + 3xy^2 + 2y^3$. Although the example (x^2-y^3, x^2-2y^3) looked fabricated and it was easy to see what the true tangent cone was, this more realistic example shows that it is not generally so. The elements xyz^4, y^2z^4 in hsm could easily be overlooked. They are however essential as the remainder of the computation will show.

Once we have generators of the tangent cone, the second pitfall is to be too accustomed to the hypersurface case and start looking at the degrees of the generators. While it is true that the product of their degrees gives a bound on the multiplicity, it is not equal to it. In fact, if the generators were in complete intersection, then the tangent cone would be pure dimensional and the product of the degrees of the generators would be the multiplicity. But we have picked an example that was not complete intersection from the start and the tangent cone is not complete intersection either: the number of generators of hsm that far exceeds its codimension clearly demonstrate hsm is not complete intersection. This is where the computation with a generic cutting affine plane comes into play. The line `ring r1=0,(a,b,c,t1,t2,t3),lp` creates $\mathbb{Q}[t_1,t_2,t_3][a,b,c]$ with the usual polynomial ordering. The instruction `map f=r0,a,b,c` simply creates the "identity" map between the rings. Singular cannot change the ordering

on the monomials for a given ring, therefore we have to create a different ring with a different ordering bound to it. Also, we use $\mathbb{Q}[t_1, t_2, t_3]$ as coefficient ring instead of its fraction field. This is because Singular does not handle those fraction fields. This does not really matter however, we just have to keep in mind that the final result of the algorithm is to be considered as a polynomial over $\mathbb{Q}(t_1, t_2, t_3)$. We see that the output of the second Gröbner basis computation is not simple, fortunately Singular orders the monomials of the output with decreasing precedence for the ordering, therefore we only have to look at the first monomial of each polynomial in the output to know the dimension of $T \cap \{(a, b, c) \mid (t_1\ a + t_2\ b + t_3\ c - 1 = 0\}$. The polynomials in the output are also ordered with increasing priority of the leading monomials. For *so* we obtain the leading monomials:

$$c^2t_3^2,\ bct_3,\ b^2,\ at_1,\ 3act_3,\ 3ab,\ a^2.$$

As t_1, t_2, t_3 are invertible, this defines the same ideal c^2, bc, b^2, a. This allows us to conclude that $(1, b, c)$ is a basis of the function sheaf on $T \cap \overline{P}$ for a generic affine space $\overline{P}$. This proves that the dimension of this function sheaf, that is the multiplicity, is 3.
To illustrate the importance of the two additional generators xyz^4, y^2z^4 in *hsm* compared to *hm*, let us look at the leading monomials we obtain for *sn* (which is associated to *hm*):

$$c^4t_3^4, 3bc^2t_1t_2t_3^2, bc^3t_3^3, b^2t_1^2, 2b^2ct_1t_2t_3, b^2c^2t_3^2, b^3t_2^2,$$
$$b^3ct_3, b^4, at_1, 3ac^2t_3^2, abt_2, 3abct_3, 3ab^2, a^2.$$

As t_1, t_2, t_3 are invertible this defines the same ideal as c^4, bc^2, b^2, a, which gives a basis of $(1, b, c, bc, c^2, c^3)$ for the function sheaf on $T' \cap \overline{P}$ where T' is what we could think to be the tangent cone, and $\overline{P}$ a generic affine space. Therefore we would have concluded that the multiplicity of the germ was 6.

Improvements to the algorithm

By looking at the run of the algorithm in the previous example it clearly appears that, as expected, the bottleneck is the introduction of a generic linear form in the second Gröbner basis calculation. The most crucial improvement is thus to avoid working in $F = \mathbb{C}(\alpha_{1,1}, \ldots, \alpha_{1,n}, \ldots, \alpha_{d,1}, \ldots, \alpha_{d,n})$. In proposition 6.3.5, we see that in fact the equality $\dim_{\mathbb{C}}(\overline{P} \cap T_d) = \mu(Z)$ holds for any $\overline{P}$ provided $D_{\mathbb{P}} \cap T_{\mathbb{P}} = \emptyset$ (where D is the direction of $\overline{P}$). Consider ∇ the jacobian matrix of the generators of Z. We assume that Z is defined by a radical ideal. To filter out T_d we can add the $(n - d + 1)$-minors $m_1, \ldots, m_k$, of ∇. We can thus adopt the following strategy: We choose a random $\overline{P} \in \overline{\mathbb{G}}_{n-d,n}$ and compute a Gröbner basis for $\overline{P} \cap T$. This computation is done over $\mathbb{C}$. First we check if the resulting system S is 0-dimensional by checking that for any $i \in \{1, \ldots, n\}$, there are polynomials with leading monomial of the form $x_i^{\alpha_i}$ for some $\alpha_i \in \mathbb{N}$. If the system S is not 0-dimensional we have to take a new random $\overline{P} \in \overline{\mathbb{G}}_{n-d,n}$, until S is 0-dimensional. Let D be the vector space parallel to $\overline{P}$. We also

have to check that $D \cap T = \{0\}$ because the injection of $\overline{P}$ in $\mathbb{P}^{n-1}(\mathbb{C})$ by the canonical quotient is not surjective, so we have to check that we do not miss any branch in T. As $T \cap D$ is a cone, $D \cap T = \{0\}$ iff $T \cap D$ is 0-dimensional, therefore we can test this second condition in the same way as the first one. Again if the condition is not satisfied we pick new random $\overline{P}$ until it is. This trial and error process is guaranteed to end rapidly as $T \cap \overline{P}$ is 0-dimensional for a generic $\overline{P}$. Once we have found a suitable $\overline{P}$ we can proceed. The dimension of S is an upper bound on the multiplicity, but we still have to get rid of the points in $T_{<d}$. This can be done by choosing a basis $B = (b_1, \ldots, b_r)$ for S (generally the monomial basis), and constructing the matrix M of the map

$$(\lambda_1, \ldots, \lambda_k) \rightarrow \sum_{i=1}^{k} \lambda_i m_i \in (\mathbb{C}b_1 + \ldots + \mathbb{C}b_r),$$

which expresses the linear combinations of the m_i's on the basis B. The dimension of the kernel of the transpose of M is $\dim_{\mathbb{C}}(\overline{P} \cap T_d) = \mu$. Notice it would have been a bad idea to directly compute the Gröbner basis with the m_i's as they are big polynomials. If we had done so, the new method would in fact have been worse than the original one. Computing a Gröbner basis for S first allows us to compute the minors in the quotient ring (computing the modulo with a Gröbner basis is a straightforward process), keeping the complexity low. Another factor that gives a good complexity is the fact that the linear algebra involved in determining the dimension of the kernel of M is done over $\mathbb{C}$, since M has coefficients in $\mathbb{C}$.
There are other solutions than the linear algebra option. For instance, we can compute a Rational Univariate Representation of the roots of S. Those representations are simply a rational curve $\gamma : \mathbb{C} \rightarrow \mathbb{C}^n$ that goes through all the roots of S and a univariate polynomial R whose roots are sent to the roots of S by γ. If plug such a parameterizations into ∇ we obtain k univariate polynomials (one for each $(n-d+1)$-minor), and to determine the multiplicity we have to determine the degree of their greatest common divisor. This computation is not too heavy because we do it modulo R whose degree is the number of roots (with multiplicity) in S.

Finally, we see that in the second Gröbner basis computation, we do not really need to have a Gröbner basis, but merely to be able to carry out computations in the 0-dimensional quotient $\mathbb{C}[X_1, \ldots, X_n]/\mathrm{Init}(\mathcal{I})$ (remember $\mathrm{Init}(\mathcal{I})$ is the defining ideal of the tangent cone). Normal forms are a generalization of Gröbner bases that enable to do such calculations at a lesser cost (see [102, 103]). This gives yet another means to improve the efficiency of the algorithm. However, the first Gröbner basis computation is not dispensable as we really need to find polynomials whose initial parts generate the whole of $\mathrm{Init}(\mathcal{I})$, which is the defining property of Gröbner bases.

6.5.2 Bound on the density and the Lipschitz-Killing invariants

In this subsection we bound the k-density and the local Lipschitz-Killing invariants of a real analytic germ by a polynomial in the multiplicity. The result proceeds from the main theorem 6.1.21 of this chapter. In the first place we define the Lipschitz-Killing invariants by a multidimensional Cauchy-Crofton formula (the density is one of the Lipschitz-Killing invariants). We define the *local* Lipschitz-Killing invariants by a process of limit from the original Lipschitz-Killing invariants. Then we quote theorem 3.1 in [37] which gives a direct formula for the local Lipschitz-Killing invariant. This latter formula can be seen as a local version of the multidimensional Cauchy-Crofton formula. Finally we apply the main theorem 6.1.21 to this expression of the local Lipschitz-Killing invariants to bound them. In order to illustrate the usefulness of these new bounds we conclude by using it to bound the 1-density of the example we worked out in the previous subsection.

Definition 6.5.2 (k^{th} Lipschitz-Killing invariant)**.** *Let $\gamma_{k,n}$ be the unit measure $\mathcal{O}_n(\mathbb{R})$-invariant on $\mathbb{G}_{k,n}$, where $\mathcal{O}_n(\mathbb{R})$ is the orthogonal group of $\mathbb{R}^n$. Let $\mathcal{H}^k$ be the usual Lebesgue measure on $\mathbb{R}^k$. For any real sub-analytic set X in $\mathbb{R}^n$, the k^{th} Lipschitz-Killing invariant $\Lambda^k(X)$ is defined by:*

$$\Lambda_k(X) = \beta(k,n)^{-1} \int_{V \in \mathbb{G}_{k,n}} \int_{y \in V} \chi\left(X \cap \pi_V^{-1}(\{y\})\right) d\mathcal{H}^k(y) d\gamma_{k,n}(V),$$

where χ is the Euler characteristic, $\beta(k,n)$ only depends on k and n and π_V is the orthogonal projection to V.

Remark 6.5.3. *One may have noticed the similarity between the definition of Lipschitz-Killing invariants and the definition of Vitushkin variations defined in the previous section (definition 3.3.10). Although the latter have a more straightforward geometric interpretation since their definition is based on the number of connected components, the former have more interesting properties and applications. This is due to the fact that Lipschitz-Killing invariants enjoy the same additivity property as the Euler characteristic from which they are derived (i.e. $\Lambda_k(X \cup Y) = \Lambda_k(X) + \Lambda_k(Y) - \Lambda_k(X \cap Y)$).*

Definition 6.5.4. *For a real sub-analytic germ $X \subset \mathbb{R}^n$ at the origin, the k^{th} local Lipschitz-Killing invariant of X is defined by:*

$$\Lambda_k^{loc}(X) = \lim_{r \to 0} \frac{\Lambda_k\big(X \cap B^n(0,r)\big)}{\mathcal{H}^k\big(B^k(0,r)\big)},$$

where $B^k(0,r)$ stands for the k-dimensional ball centered at the origin of radius r. It is proved in [83] (theorem 2.2) that this limit exists.
When $k = \dim(X)$, the fibers $\pi_V^{-1}(\{y\})$ are generically 0-dimensional, and the Euler characteristic simply counts the number of points in those fibers. In this

case the k^{th} local Lipschitz-Killing invariant of X is called the k-density of X and it is defined as:

$$\Theta_k(X) = \Lambda_k^{loc}(X) = \lim_{r\to 0} \frac{\Lambda_k\big(X \cap B^n(0,r)\big)}{\mathcal{H}^k\big(B^k(0,r)\big)}.$$

At this point, it would already be possible to use the main theorem 6.1.21 to derive a bound on the k-density and the local Lipschitz-Killing invariants. If we did so we would obtain a bound that is not sharp. Conceptually this would amount to approximating the balls by their bounding cubes. To avoid losing this sharpness we use an alternate characterization of the k-density and the local Lipschitz-Killing invariants. It comes from a local version of the multidimensional Cauchy-Crofton formula.

Definition 6.5.5 (Local polar profiles and multiplicities)**.** *Let X a real sub-analytic germ at the origin. Let $\mathrm{Crit}(\pi_V|_X)$ the critical locus of π_V on X_{smooth}, where $X_{\text{smooth}} = X \setminus \mathrm{Sing}(X)$. Let $\mathcal{O}_X^k$ be the vector spaces $V \in \mathbb{G}_{k,n}$ such that $T(X) \cap V^\perp = \{0\}$, where $T(X)$ is the tangent cone to X. The local polar profiles of X for V are the connected components K_j^V ($j \in \{0,\dots,n_V\}$) of the open germ $\pi_V(X) \setminus \pi_V\left(\mathrm{Crit}(\pi_V|_X)\right)$. The definition of $\mathrm{Crit}(\pi_V|_X)$ entails that the topology of the fibers $\big(X \cap \pi^{-1}(y)\big)$ ($y \in V$) is constant when y runs over a given K_j^V. Therefore, the Euler characteristic of $\big(X \cap \pi^{-1}(y)\big)$ is constant over the K_j^V. We call this constant the local polar Euler characteristic χ_j^V associated to K_j^V. Notice that when $\dim(V) = \dim(X)$, the fiber contains finitely many points, and χ_j^V is thus the number of points in each fiber. In this case, we denote χ_j^V by e_j^V.*

Definition 6.5.6 (Local polar invariants)**.** *Let X be a real sub-analytic germ of $\mathbb{R}^n$. The k^{th} local polar invariant associated to X is:*

$$\sigma_k(X) = \int_{V\in\mathcal{O}_X^k} \left(\sum_{j=0}^{n_V} \chi_j^V \Theta_k(K_j^V)\right) d\gamma_{k,n}(V).$$

Theorem 6.5.7 (Local multidimensional Cauchy-Crofton formula)**.** *Let X be a real sub-analytic germ of $\mathbb{R}^n$. Theorem 3.1 in [37] states that there exists an upper-triangular matrix $M \in \mathcal{M}_n(\mathbb{R})$ such that*

$$\begin{pmatrix} \Lambda_1^{loc} \\ \vdots \\ \Lambda_n^{loc} \end{pmatrix} = \begin{pmatrix} M_{1,1} & M_{1,2} & \dots & M_{1,n-1} & M_{1,n} \\ 0 & M_{2,2} & \dots & M_{2,n-1} & M_{2,n} \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 0 & M_{n,n} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_n \end{pmatrix}$$

where $M_{i,i} = 1$, and for $i < j \le n$:

$$M_{i,j} = \frac{\alpha_j}{\alpha_{j-i}\ \alpha_i} C_j^i - \frac{\alpha_{j-1}}{\alpha_{j-1-i}\ \alpha_i} C_{j-1}^i,$$

with α_k the k-dimensional volume of the unit ball in $\mathbb{R}^k$ and C_j^i the usual binomial coefficients.

Remark 6.5.8. *The previous result extends the local Cauchy-Crofton formula for the density in [37] which asserts that for a k-dimensional real analytic germ X at 0 in $\mathbb{R}^n$:*

$$\Theta_k(X) = \int_{V \in \mathcal{O}_X^k} \left(\sum_{j=0}^{n_V} e_j^V \Theta_k(K_j^V) \right) d\gamma_{k,n}(V).$$

This is simply because for $i > k$, the $(n-i)$-dimensional spaces avoid X generically and thus, $\sigma_i = 0$. As M is upper-triangular the only non-zero coefficient involved in the expression of $\Theta_k(X)$ $(= \Lambda_k^{loc}(X))$ is the diagonal coefficient $M_{k,k} = 1$, hence giving the above equality for the density.

The relation given by theorem 6.5.7 between the σ_i and the local Lipschitz-Killing invariants enables us to derive the following bounds on the Λ_i^{loc}:

Lemma 6.5.9 (Bound on the σ_i)**.** *Let $X \subset \mathbb{R}^n$ be a real analytic germ at the origin. Let T be the algebraic tangent cone to X. Then*

1. *If $l = \dim(X)$, we have the inequality*
$$\sigma_l(X) = \Theta_l(X) \leq \mu(X).$$
2. *For any $l \in \mathbb{N}$, if $\dim(\mathrm{Sing}(T)) < l$ and $X_{\mathbb{C}}$ is pure dimensional, then*
$$\sigma_l(X) \leq \mu(X)(2\mu(X) - 1)^{n-l-1}.$$

Proof. We start from the expression of $\sigma_l(X)$. Let $k = n - l$. The bound stems from the simple fact that the sum of Betti numbers for a given set is a bound on the Euler characteristic of that set since this latter one is an alternated sum of Betti numbers. Under the hypothesis of the proposition, we can apply the main theorem 6.1.21, and we can conclude that for a generic $V \in \mathbb{G}_{l,n}$ and $y \in V$, we have

$$\begin{aligned} &\lim_{\lambda \to 0} \chi\left(X \cap \pi_V^{-1}(\lambda y)\right) \leq \mu(X) && \text{in case 1,} \\ &\lim_{\lambda \to 0} \chi\left(X \cap \pi_V^{-1}(\lambda y)\right) \leq \mu(X)(2\mu(X) - 1)^{n-l-1} && \text{in case 2.} \end{aligned}$$

If $\Theta_l(K_j^V)$ is non-zero, its tangent cone has non empty interior and thus the previous inequalities show that

$$\begin{aligned} &\chi_j^V \leq \mu(X) && \text{in case 1,} \\ &\chi_j^V \leq \mu(X)(2\mu(X) - 1)^{l-1} && \text{in case 2.} \end{aligned}$$

The definition of the $\sigma_l(X)$ thus gives the two following inequalities:

$$\sigma_l(X) \leq \mu(X) \int_{V \in \mathcal{O}_X^l} \left(\sum_{j=0}^{n_V} \Theta_l(K_j^V) \right) d\gamma_{l,n}(V) \qquad \text{in case 1,}$$

$$\sigma_l(X) \leq \mu(X)(2\mu(X) - 1)^{l-1} \int_{V \in \mathcal{O}_X^l} \left(\sum_{j=0}^{n_V} \Theta_l(K_j^V) \right) d\gamma_{l,n}(V) \qquad \text{in case 2.}$$

Notice that if $\Theta_l(K_j^V)$ was zero, the inequalities on χ_j^V might not hold, but this does not matter as they do not count in the Cauchy-Crofton expression of $\sigma_l(X)$. Finally by definition of the K_j^V we have:

$$\cup_{j=0}^{n_V} K_j^V = V_0 \setminus \pi_V\left(\mathrm{Crit}(\pi_V|_X)\right),$$

where V_0 denotes the germ at 0 associated to V. Furthermore the previous union is disjoint and $\pi_V\left(\mathrm{Crit}(\pi_V|_X)\right)$ is dense in V_0. This shows that

$$\begin{aligned}\sum_{j=0}^{n_V} \Theta_l(K_j^V) &= \Theta_l\Big(V_0 \setminus \pi_V\big(\mathrm{Crit}(\pi_V|_X)\big)\Big) \\ &= \Theta_l(V_0) \quad = \quad 1.\end{aligned}$$

In the end we obtain the desired result, since $\mathcal{O}_X^l$ is dense in $\mathbb{G}_{l,n}$:

$$\begin{aligned}\sigma_l(X) &\leq \mu(X) \int_{V\in\mathcal{O}_X^l} 1 \; d\gamma_{l,n}(V) \\ &\leq \mu(X) && \text{in case 1,} \\ \sigma_l(X) &\leq \mu(X)(2\mu(X)-1)^{l-1} \int_{V\in\mathcal{O}_X^l} 1 \; d\gamma_{l,n}(V) \\ &\leq \mu(X)(2\mu(X)-1)^{l-1} && \text{in case 2.}\end{aligned}$$

□

Proposition 6.5.10 (Bound on local Lipschitz-Killing invariants)**.** *Let X be an analytic germ at 0 of dimension d such that $\dim(\mathrm{Sing}(T)) < k$. Then we have the following bound on the k^{th} local Lipschitz-Killing invariant of X:*

$$\Lambda_k^{loc}(X) \leq M_{k,d}\ \mu(X) + \sum_{l=k}^{d-1} M_{k,l}\ \mu(X)(2\mu(X)-1)^{l-1},$$

where M is defined as in theorem 6.5.7.

Proof. By the previous lemma 6.5.9, we have a bound in term of the multiplicity for every $\sigma_l(X)$. We can store those bounds in a vector b. According to theorem 6.5.7, the vector $(\Lambda_l(X))$ which is made up of the local Lipschitz-Killing invariants, is the image of the vector $(\sigma_l(X))$ through M. The coefficients of M are all non negative, therefore after applying M to the vector b, we obtain a vector whose components bound the $\Lambda_l(X)$. By expanding the product Mb we obtain the bounds announced in the proposition. The sum ends at $M_{k,d}$ because the $\sigma_l(X)$ for $l > d$ all vanish (as $(n-l)$-dimensional spaces generically avoid X). □

Remark 6.5.11. *Notice that for $k = \dim(X)$, the previous proposition gives:*

$$\Lambda_k^{loc}(X) \leq M_{k,k}\mu(X) = \mu(X).$$

As $\Lambda_k^{loc}(X) = \Theta_k(X)$ by definition, the previous proposition yields the same result as the first item of lemma 6.5.9 in this case.

The bounds we have obtained are much sharper than what we could obtain using the usual Oleinik-Petrovsky-Thom-Milnor bound for the Betti numbers. For the density the Oleinik-Petrovsky-Thom-Milnor bound gives:

Theorem 6.5.12 (The Oleinik-Petrovsky-Thom-Milnor Bound on Density)**.** *For an analytic germ $X \subset \mathbb{R}^n$ of dimension l defined by function $f_1, \ldots, f_k$ of degree $d_1, \ldots, d_k$, theorem 5.5, in [144], gives the following bound*

$$\Theta_l(X) \leq (d+1)(2d+1)^{n-l-1} \quad \text{where} \quad d = \sum_{i=1}^{k} d_i.$$

Remark 6.5.13. *If one compares theorem 5.5 in [144] to the previous theorem, one should pay attention to the fact that in [144], the sets are defined by inequalities, therefore to define an algebraic set in this setting one needs to repeat each function twice. This explains how the seemingly different formula found in [144] gives the bound in the previous theorem.*

Our bound is sharper because it is a localization of the Oleinik-Petrovsky-Thom-Milnor bound. The defining functions may store information on what happens away from the origin, but the multiplicity only accounts for what happens at the origin.
Let us now look at the difference between our bound and the Oleinik-Petrovsky-Thom-Milnor bound on the example introduced in the previous subsection 6.5.1. The defining functions for the germ X were

$$\big(x(x-z^3)(x-2z^2), y(y-z^3)(y-2z^2), (x+y)(x+y-z^3)\big).$$

Those generators have degrees $6, 6, 4$. Therefore by the Oleinik-Petrovsky-Thom-Milnor bound for the density 6.5.12 we have

$$\begin{aligned} \Theta_1(X) &\leq (16+1)(2\times 16+1)^{3-1-1} \\ &\leq 17 \times 33 \quad = \quad 561. \end{aligned}$$

Since we know the multiplicity is 3 by the computation of subsection 6.5.1, our bound given in the first item of lemma 6.5.9 shows that

$$\Theta_1(X) \leq 3.$$

In fact, the intersection of X with a generic plane has exactly 3 points in it, therefore the density of X is equal to 3, and our bound was as sharp as it could be.

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Nomenclature

$<$	Degeneration ordering, page 52
$\overline{\mathbb{G}}_{k,n}$	Affine Grassmannian, page 121
$\mathfrak{b}_i$	Local Betti number, page 124
$\tilde{\mathbb{G}}_{k,n}$	Blow-up of Grassmannian, page 121
$\tilde{X}_\epsilon$	Conic blow-up, page 125
$\tilde{Z}_\epsilon$	Conic blow-up, page 125
∂E	Boundary, page 52
$\overline{E}$	Closure, page 52
$X_{\mathbb{C}}$	Complexification, page 122
$\xrightarrow{S}$	Convergence in a set, page 52
$\mathcal{D}$	Domain, page 14
$\delta(.,.)$	Manifold transversality, page 69
$d\mathcal{H}(,\ )$	Cone measures, page 67
$\mathcal{O}(X)$	Function sheaf, page 121
$\Gamma(.,.)$	Cone measures, page 67
$\mathbb{G}_{k,n}$	Grassmannian, page 121
$\mathcal{H}(. \to .)$	Cone measures, page 67
$\mathcal{H}(. \to .)$	Jump between maps, page 71
$\mathcal{H}(. \to .)$	Necessary thickening, page 85
E°	Interior, page 52
$\mathbb{K}$	$\mathbb{C}$ or $\mathbb{R}$, page 121

$N_{p,C}$ Convex gauge, page 76

S_γ Sphere family, page 149

$T(Y)$ Algebraic tangent cone, page 125

V_γ^W Weighed Voronoĭ partition, page 88

X A real variety, page 121

Z A complex variety, page 121

Λ_k^{loc} k-Local Lipchitz-Killing invariant, page 162

Λ_k k-Lipchitz-Killing invariant, page 162

$\mathcal{T}(.,.,.)$ Euclidean acceptance radius, page 82

$\mathfrak{b}_i$ Betti number, page 121

$\mathfrak{E}_h$ Embedding transformation, page 133

$\mathfrak{P}_h$ Product transformation, page 133

$\deg[F, \mathcal{D}, p]$ Topological degree, page 20

Index

Résumé

L'introduction (section 1) introduit la problématique générale de la thèse: la mesure quantitative des propriétés géométriques des variétés algébriques et particulièrement leur triangulation.

La section 2 explique une procédure de subdivision rapide et certifiée triangulant une courbe algébrique réelle plane. Les outils mathématiques sont le degré topologique, la base des polynômes de Bernstein.

La section 3 est une copie d'un article expliquant la méthode de subdivision pour les surfaces lisses dans $\mathbb{R}^n$. Elle comporte une analyse de complexité.

La section 4 présente une version quantitative du théorème de trivialité topologique de Thom-Mather pour des applications semi-algébriques non lisses. Il en découle: une version "métriquement stable" du théorème de structure conique local et de l'existence d'un "tube de Milnor" autour des strates. Un algorithme de triangulation utilisant des partitions de Voronoĭ (sa mise en place n'est pas complète car l'estimation effective de la transversalité n'est pas complètement traité).

La section 5 est une copie d'un article paru en 2008 sur une méthode de balayage pour calculer la topologie d'une surface singulière de $\mathbb{R}^3$. Elle repose sur l'utilisation du théorème de Thom-Mather.

La section 6 présente une borne sur le nombre générique de composantes connexes dans une section d'un germe analytique réel par un espace affine en fonction de la multiplicité et de la dimension de l'espace. La borne ne s'applique pas toujours et des contre-examples sont donnés.

Summary

The introduction (section 1) presents the general subject-matter of the thesis: the quantitative measurement of the geometric properties of real algebraic varieties, and especially their triangulation.

Section 2 explains a fast and certified subdivision procedure triangulating an algebraic plane curve. The mathematical tools are the topological degree, and the Bernstein's polynomial basis.

Section 3 is a copy of an article explaining the subdivision method for smooth surfaces in $\mathbb{R}^n$. It includes a complexity analysis.

Section 4 presents a quantitative version of Thom-Mather's topological triviality for singular semi-algebraic maps. Stem from it: A "metrically stable" version of the local conic structure theorem and of the existence of a "Milnor tube" around strata. A triangulation algorithm based on Voronoĭ partitions (not completely implementable because the effective estimation of transversality is not completely detailed).

Section 5 is a copy of an article published in 2008 on a sweeping method to compute the topology of singular surfaces in $\mathbb{R}^3$. It is based on Thom-Mathers theorem.

Section 6 presents a bound on the generic number of connected components in an affine section of a real analytic germ in terms of the multiplicity and of the dimension of the ambient space. The bound does not always apply and counter-examples are given in that case.