Elementary subgroups of torsion-free hyperbolic groups

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Chloé Perin Elementary subgroups of torsion-free hyperbolic groups

Plan of the talk

- The first-order theory of free and hyperbolic groups
- 2 Hyperbolic towers
- 3 Elementary embeddings in hyperbolic groups
- Obtaining a non-injective preretraction
- 5 From preretraction to hyperbolic floor

What is a first-order formula?

A first-order formula in the language of groups is a finite formula using the following set of symbols

- Usual mathematical symbols: $=, \neq, \neg, \lor, \land, \forall, \exists, \Rightarrow$.
- Some variables: x, y,...
- $\bullet\,$ Symbols specific to the language of groups: $1,*,^{-1}$.

N.B: A variable always represents an element of the group we're talking about.

Example

$$(x^4 = 1) \land (x^2 \neq 1) \land (x \neq 1)$$

 $(\forall y xy = yx) \Rightarrow (x = 1)$

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 A variable x which appears in a formula φ is free if neither ∀x nor ∃x appears before it in φ.

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- A first-order formula φ is a sentence, or a closed formula, if none of the variables which appear in φ are free.
- A group G satisfies a sentence φ of the language of groups if the interpretation of the formula holds in G. We denote this by G ⊨ φ.



What is NOT a first-order formula?

Most important points to remember: a first-order formula is finite, and we only quantify on one type of element (here, the elements of the group).

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Some properties of a group may be expressed by a first-order formula, others not.

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• If G is abelian, so is G'.

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The **elementary theory** of a group G is the set of first-order sentences satisfied by G. If two groups G and G' have the same elementary theory, we say they are **elementary equivalent** and denote this by $G \equiv G'$. Suppose $G \equiv G'$.

• If G is abelian, so is G'.

$$\forall x \forall y \ xy = yx$$

• If G is finite of order q, so is G'.

 $\exists x_1 \ldots \exists x_q \ (\bigwedge_{i,j} _{i \neq j} x_i \neq x_j) \land (\forall x \ \bigvee_{i=1}^q x = x_i).$ In fact, *G* and *G'* must be isomorphic.

Example

The groups $\mathbb Z$ and $\mathbb Z^2$ are not elementary equivalent. An element in $\mathbb Z$ is either even or odd.

$$\exists x \forall y \exists z \ (y = z^2) \lor (y = z^2 x).$$

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Tarski's problem, 1945

Let \mathbb{F}_k be the free group on k generators. If $k \neq n$, are \mathbb{F}_k and \mathbb{F}_n elementary equivalent (for $k, n \geq 2$)?

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Answer (Sela, 2006)

Yes.

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The first-order theory of free and hyperbolic groups

Hyperbolic towers Elementary embeddings in hyperbolic groups Obtaining a non-injective preretraction From preretraction to hyperbolic floor

Question

If G is a finitely generated group, and $G \equiv \mathbb{F}_2$, what does G look like?

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Theorem (Sela, 2006)

The fundamental group of a closed surface whose Euler characteristic is at most -2 is elementary equivalent to \mathbb{F}_2 .

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Suppose $H \leq G$. We extend the set of symbols we can use in first-order formulas by adding a name $\lceil h \rceil$ for every element h of H.

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Example

Let
$$h \in H$$
 and let $\phi_h : \forall x \lceil h \rceil x = x \lceil h \rceil$.

Such a formula can be interpreted both in G and H:

- $G \models \phi_h \iff h$ is in the centre of G;
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Definition

We say that the embedding $H \subseteq G$ is **elementary**, if for any first-order sentence ϕ in this extended language,

$$H \models \phi \iff G \models \phi.$$

We denote this by $H \preceq G$.

Example

The group $\mathbb Z$ does not contain any proper elementary subgroups.

If
$$\mathbb{Z}=\langle z
angle$$
 and $H=\langle h
angle$, with $h=z^k$, consider the formula

$$\exists x \; x^k = \lceil h \rceil.$$

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Elementary embeddings in hyperbolic towers Obtaining a non-injective preretraction From preretraction to hyperbolic floor

Theorem (Sela, 2006)

The standard embedding $\mathbb{F}_k \leq \mathbb{F}_n$ for $k \leq n$ is elementary.

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Suppose $H \leq \mathbb{F}_n$. What does H look like? In particular, does H have to be a free factor of \mathbb{F}_n ?

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Suppose $H \preceq \Gamma$, where Γ is a torsion-free hyperbolic group. What does the embedding $H \hookrightarrow \Gamma$ look like?

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A hyperbolic tower.

We say
$$(G, G', r)$$
 is a **hyperbolic floor** if
 $r: G \to G'$ is a retraction, $G = \pi_1(X)$ and
 $G' = \pi_1(X')$, where X and X' are complexes
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- X is obtained by gluing hyperbolic surfaces with boundary to X' (gluing = identifying boundary components to non null-homotopic loops in X'),
- the image of the fundamental groups of these surfaces by r is not abelian.


A group G admits a structure of hyperbolic tower over H if there exists a finite sequence $H \le G_0 \le G_1 \le \ldots \le G_k = G$ such that

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- for each *i*, there is a retraction $r_i : G_i \rightarrow G_{i-1}$ such that (G_i, G_{i-1}, r_i) is the floor of a hyperbolic tower.



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Theorem (Sela, 2006)

A finitely generated group G is elementary equivalent to a free group if and only if it is a hyperbolic tower over $\{1\}$.

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Theorem

Let G be a torsion-free hyperbolic group, and suppose we have an elementary embedding $H \hookrightarrow G$. Then G has a structure of hyperbolic tower over H.

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Possible generalisations: relatively hyperbolic groups, elementary closed subgroups (weaker notion) in free/hyperbolic groups.

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For this we use the following criterion

Theorem

Let G be a torsion-free hyperbolic group, freely indecomposable with respect to H, and let Λ be the JSJ decomposition of G with respect to H. If there exists a map $f : G \rightarrow G$ such that

then there is a retraction $r: G \to G'$, with $H \leq G'$, such that (G, G', r) is a hyperbolic floor.

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If there exists a map $f: G \rightarrow G$ such that

• if R is a non surface type vertex group, $f|_R = \operatorname{Conj}(\gamma_R)$;

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We call a map $f: G \to G$ which satisfies 1 and 2 a **preretraction** with respect to Λ .

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• Step 1: Build a non-injective preretraction $f : G \rightarrow G$.

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- Step 1: Build a non-injective preretraction $f: G \rightarrow G$.
- Step 2: By the theorem, get a retraction $r: G \to G'$ such that (G, G', r) is a hyperbolic floor.



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- Step 2: By the theorem, get a retraction $r: G \to G'$ such that (G, G', r) is a hyperbolic floor.
- Step 3: Iterate the process.

To simplify things, we suppose that:

- *H* is finitely generated;
- G is freely indecomposable with respect to H;
- the cyclic JSJ decomposition of G with respect to H has one non surface type vertex group R, and one surface type vertex group S.



Step 1:

Show that there exists a non-injective preretraction $G \rightarrow G$ with respect to the JSJ decomposition Λ of G relative to H.

How do we find a preretraction?

- **(**) state the existence of a **factor set** for maps $G \rightarrow H$;
- weaken this statement to get rid of things that can't be expressed by first-order theory;
- express this new statement as a first-order formula satisfied by H;
- interpret the formula on G, and apply it to the identity map to obtain a non-injective preretraction.

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Theorem (Sela, 2002)

G a t.f. hyperbolic group, freely indecomposable with respect to *H*. There exists a **finite** number of proper quotients $\eta_i : G \to G_i$, s.t. if a morphism $f : G \to G$ is non-injective and fixes *H*, then $f \circ \sigma$ factors through one of the maps η_i , for some **modular automorphism** σ of *G* relative to *H*.



We call the set $\{\eta_1, \ldots, \eta_k\}$ a factor set for $\operatorname{Hom}_{H}(G, G)$.

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In our case, if $\sigma \in \operatorname{Mod}_H(G)$,

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- $\sigma|_S$ sends S isomorphically to a conjugate of itself.

A statement

For any morphism $f: G \rightarrow G$, if f fixes H, f is non-injective,

then $\exists \sigma \in Mod_H(G)$ such that

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• Replace G by H.

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For any morphism $f : G \to H$, if f fixes H, then $\exists f' : G \to H$ a morphism such that • f' kills one of the elements v_i . • $f' = f \circ \sigma$ for an element σ of $Mod_H(G)$;

We want to try and express this as a first-order sentence.

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We want to try and express this as a first-order sentence. **Question:** How do we say 'For any morphism $f : G \to H$ ' or ' $\exists f' : G \to H$ a morphism' in first-order theory?

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Morphisms in first-order

Example

Giving a morphism $f : \mathbb{Z}^2 \to H$ for $\mathbb{Z}^2 = \langle a \rangle \oplus \langle b \rangle$ is the same as giving x, y in H which commute: take $f : a \mapsto x, b \mapsto y$.

Then the image of an element $a^k b^j$ is $x^k y^j$.

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Remark

Giving a morphism $f : G \to H$ for a f.p. group $G = \langle \mathbf{g} | \Sigma_G(\mathbf{g}) = 1 \rangle$ is the same as giving a tuple \mathbf{x} in H which satisfies $\Sigma_G(\mathbf{x}) = 1$: take $f_{\mathbf{x}} : G \to H$ defined by $\mathbf{g} \mapsto \mathbf{x}$.

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Any element w of G is represented by a word $\bar{w}(\mathbf{g})$, then $f_{\mathbf{x}}(w)$ is represented by $\bar{w}(\mathbf{x})$.

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Problem: we cannot express precomposition by an automorphism in first-order theory.

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Solution: express something weaker.

If $f' = f \circ \sigma$ for some σ in $Mod_H(G)$, then recall that

• $\sigma|_R$ is just conjugation by some element β_R . Thus $f'|_R = \operatorname{Conj}(f(\beta_R)) \circ f|_R$.

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Solution: express something weaker.

If $f' = f \circ \sigma$ for some σ in $Mod_H(G)$, then recall that

- $\sigma|_R$ is just conjugation by some element β_R . Thus $f'|_R = \operatorname{Conj}(f(\beta_R)) \circ f|_R$.
- $\sigma|_S$ sends S isomorphically to a conjugate of itself. Thus if f(S) is not abelian, f'(S) is not abelian.

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For any morphism $f : G \to H$, if f fixes H, then $\exists f' : G \to H$ a morphism such that • f' kills one of the elements v_i . • $\exists \gamma_R$ such that $f'|_R = \operatorname{Conj}(\gamma_R) \circ f$; • if f(S) is not abelian, neither is f'(S);

The first-order formula

$$\forall \mathbf{x} \ \Sigma_G(\mathbf{x}) = 1, \text{ if } \bigwedge_{h \in H_0} \lceil h \rceil = \overline{h}(\mathbf{x}), \text{ then } \exists \mathbf{y} \ \Sigma_G(\mathbf{y}) = 1 \text{ s.t.}$$

• $\exists z \ \overline{\mathbf{r}}(\mathbf{y}) = z \overline{\mathbf{r}}(\mathbf{x}) z^{-1};$
• $\neg \{\bigwedge_{i,i} [\overline{s}_i(\mathbf{x}), \overline{s}_j(\mathbf{x})] = 1\} \Rightarrow \neg \{\bigwedge_{i,i} [\overline{s}_i(\mathbf{y}), \overline{s}_j(\mathbf{y})] = 1\};$

•
$$\bigvee_{i=1}^{\prime} \bar{v}_i(\mathbf{y}) = 1$$

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Interpretation over H

For any morphism $f : G \to H$, if f fixes H,

then \exists a morphism $f': G \rightarrow H$ such that

•
$$\exists \gamma_R$$
 such that $f'|_R = \operatorname{Conj}(\gamma_R) \circ f$;

• if
$$f(S)$$
 is not abelian, so is $f'(S)$;

• f' kills one of the elements v_i.

Interpretation over G

For any morphism $f : G \to G$, if f fixes H, then \exists a morphism $f' : G \to G$ such that $\bullet \exists \gamma_R$ such that $f'|_R = \operatorname{Conj}(\gamma_R) \circ f$; \bullet if f(S) is not abelian, so is f'(S);

• f' kills one of the elements v_i.

For $f = \text{Id}_G$, this gives us a morphism $f' : G \to G$ which is precisely a non-injective preretraction.

How did we get a preretraction?

- **①** state the existence of a **factor set** for maps $G \rightarrow H$;
- weaken this statement to get rid of things that can't be expressed by first-order theory;
- express this new statement as a first-order formula satisfied by H;
- interpret the formula on G, and apply it to the identity map to get a non-injective preretraction.

Strategy of the proof:

• Step 1: Build a non-injective preretraction $f: G \rightarrow G$.

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Strategy of the proof:

- Step 1: Build a non-injective preretraction $f: G \rightarrow G$.
- Step 2: Modify f to get a retraction $r: G \to G'$ such that (G, G', r) is a hyperbolic floor.

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Strategy of the proof:

- Step 1: Build a non-injective preretraction $f: G \rightarrow G$.
- Step 2: Modify f to get a retraction $r: G \to G'$ such that (G, G', r) is a hyperbolic floor.
- Step 3: Iterate the process.

Step 2:

Modify the non-injective preretraction f to get a retraction $r: G \rightarrow G'$ such that (G, G', r) is a hyperbolic floor.

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Step 2:

Modify the non-injective preretraction f to get a retraction $r: G \rightarrow G'$ such that (G, G', r) is a hyperbolic floor.

Recall that we assumed that the JSJ of G relative to H has only two vertices with groups R and S, and one edge with group $C = \langle c \rangle$.



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Hypothesis: We have a morphism $f: G \rightarrow G$ such that

- $f|_R$ is just a conjugation;
- f(S) is not abelian;
- f is non-injective.



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We want to show: There exists a retraction $G \rightarrow G'$ which makes (G, G', r) a hyperbolic floor.

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We want to show: There exists a retraction $G \rightarrow G'$ which makes (G, G', r) a hyperbolic floor.

We will show: $f(S) \le R$. Thus the map f itself is a retraction $G \to R$, and (G, R, f) is a hyperbolic floor.

Let T_{Λ} be the *G*-tree corresponding to the decomposition Λ of *G* as $R *_C S$.

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Let T_{Λ} be the *G*-tree corresponding to the decomposition Λ of *G* as $R *_C S$.



Remark: The tree T_{Λ} is 1-acylindrical next to surface type vertices, i.e. if g stabilises $\bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc$ then g = 1.

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Let T_{Λ}^{f} be the same tree, but endowed with the action of G twisted by f. Note that f(R) = R so R stabilises the vertex v_{R} in this action.

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Let T_{Λ}^{f} be the same tree, but endowed with the action of G twisted by f. Note that f(R) = R so R stabilises the vertex v_{R} in this action.

Idea of the proof: look at the action of S on T_{Λ}^{f} .

Case 1: S stabilises v_R in T^f_{Λ} , i.e. $f(S) \leq R$. Then $f(G) \leq R$. Thus f is a retraction $G \to R$.

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Case 1': S stabilises $g \cdot v_R$ in T_{Λ}^f , i.e. $f(S) \leq gRg^{-1}$. Then c stabilises both v_R and $g \cdot v_R$ in T_{Λ}^f .

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Case 2: S stabilises v_S in T_{Λ}^f , i.e. $f(S) \leq S$.

Claim

If $f(S) \leq S$, then f(S) has finite index in S.

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Claim If $f(S) \leq S$, then f(S) has finite index in S.

Lemma

If Σ is a surface with boundary, H f.g. and $[\pi_1(\Sigma) : H] = \infty$ then $H = B_1 * \ldots * B_I * F$ where

- each B_i is a boundary subgroup of S,
- a boundary element of π₁(Σ) contained in H can be conjugated in H into one of the groups B_i
- F is a (possibly trivial) free group.

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Proof of the claim:

Suppose f(S) has infinite index in S. Then
 f(S) = C * C₁ * ... * C_l * F and the decomposition is not contained in C since f(S) is not abelian.

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Proof of the claim:

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 f(S) = C * C₁ * ... * C_l * F and the decomposition is not contained in C since f(S) is not abelian.
- Let T be the f(S)-tree with trivial edge stabilisers corresponding to this decomposition. Then $S = \pi_1(\Sigma)$ acts on T via f.

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Lemma

If Σ is a surface with boundary, if $\pi_1(\Sigma)$ acts minimally on a tree T so that boundary elements are elliptic, there exists a set of disjoint simple closed curves C on Σ such that:

- elements corresponding to curves of C stabilise edges of T;
- the π_1 of connected components of ΣC are elliptic.



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If $f(S) \leq S$, then f(S) has finite index in S.

Proof of the claim:

- Suppose f(S) has infinite index in S. Then
 f(S) = C * C₁ * ... * C_l * F and the decomposition has at least two factors since f(S) is not abelian.
- Let T be the f(S)-tree with trivial edge stabilisers corresponding to this decomposition. Then $S = \pi_1(\Sigma)$ acts on T via f.

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- The lemma gives us a set of curves C whose corresponding elements stabilise edges of T, so these elements have trivial image by f. But we assumed (additional hypothesis) that f does not kill curves.

The claim is proved.

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Case 3: General case, $S = \pi_1(\Sigma)$ acts on T_{Λ} via f, and f(C) = C so C is elliptic in T_{Λ}^f : we get a set C of curves on Σ



such that $S_i = \pi_1(\Sigma_i)$ is elliptic in T^f_{Λ} .

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Thus we get equality, so $S = S_i$ and $f|_S$ is an isomorphism $S \to S$.

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All the images $f(S_i)$ lie in conjugates of R. In fact they must all lie in R by 1-acylindricity. So $f(G) \leq R$, and f is a retraction $G \rightarrow R$. This finishes the proof of Step 2.