Analyse Mathématique De Problèmes En Océanographie Côtière

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ANALYSE MATHÉMATIQUE DE PROBLÈMES EN OCÉANOGRAPHIE CÔTIÈRE

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Introduction générale

La modélisation des flux côtièrs représente actuellement un défi scientifique très important en ingénierie côtière. Les processus hydrodynamiques liés à la transformation des ondes dans des environnements peu profonds tels que les effets non linéaire, dispersifs et les effets bathymétriques sont complexes à étudier. Leur compréhension est essentielle, si l'on veut être en mesure de prévoir l'impact des phénoménes de grande échelle tels que la propagation des vagues de type tsunami. Le problème d'Euler surface libre pour un fluide idéal consiste à décrire le mouvement de la surface libre et l'évolution de la vitesse de fluide parfait, incompressible, irrotationnel sous l'influence de la gravité. La propagation des ondes de surface pour un fluide parfait homogène incompressible est décrit par les équations d'Euler $3D$ combiné avec des conditions limites non linéaires sur la surface libre et le fond. Ce problème est extrémement difficile à résoudre, notamment parce que son domaine fait partie de la solution. La complexité de ce problème a conduit les physiciens, océanographes et mathématiciens de dériver ensembles d'équations simples dans certains régimes physiques spécifiques. Les équations ainsi obtenues peuvent être divisées en deux groupes, à savoir les modèles en eau peu profonde et les modèles en eau profonde. Beaucoup des modèles approchés ont ainsi été dérivés afin de comprendre les processus hydrodynamiques pour des paramètres physiques simplifiés. La construction des modèles classiques de type Boussinesq remonte à la fin du XIXème siècle. Ces équations ont été d'abord obtenues par Boussinesq [13, 14] pour décrire la propagation des ondes de faibles amplitudes et de longues longueurs d'ondes à la surface de l'eau dans un canal. Ces équations se posent également lors de la modélisation de la propagation des ondes longues sur les grands lacs ou les océans et dans d'autres contextes, et elles ont été largement utilisées et améliorées malgré que le problème de la justification mathématique n'a pas été abordé jusqu'à la fin du siècle dernier, [22, 45, 11, 12, 10]. En plus, ils sont souvent limités dans le cadre d'un fond plat, et ne tiennent pas compte de nombreux phénoménes. Le cas des fonds non plat a également été étudié; quelques références significatives sont [61, 19, 18]. Un autre modèle plus important est le modèle de Green–Naghdi (GN) - qui est un modèle largement utilisé en océanographie côtière ([29, 9, 30] (et, par exemple,[40, 48]) - parce qu'il tient compte les effets dispersifs négligés par les équations de Saint-Venant et il est plus non linéaire que les équations de Boussinesq.

L'objectif de cette thèse est de proposer de nouveaux modèles des flux côtièrs en tenant compte l'influence de la topographie du fond sur les ondes de surface, et de justifier mathématiquement certains modèles approchés pour le problème d'Euler surface libre comme le modèle de Green-Naghdi. Pour atteindre cet objectif, nous avons divisé cette thèse en trois parties d'importance égale: la première partie présente la modélisation et l'analyse mathématique de certains modèles et les méthodes de construction en détail. Les modèles obtenus sont aussi systématiquement et rigoureusement justifiés.
La deuxième partie présente la dérivation et l'analyse mathématique de modèle de Green-Naghdi comme un modèle asymptotique du probèlme d'Euler surface libre. Enfin, quelques simulations numériques des modèles de la première partie sont faites et discutées dans la troisième partie.

Tous les travaux de la modélisation nécéssitent d'abord un cadre physique spécifique. Nous nous intéressons ici au cas d'un fluide (l'eau) IDEAL, incompressible, irrotationnel et sous l'influence de la gravité. Ces hypothèses sont classiquement utilisées en océanographie et peuvent être facilement justifiées (voir, par exemple [47, 27]). En effet,
1. L'eau est (presque) incompressible. L'eau qui remplit un certain volume pourra toujours combler ce même volume, quelle que soit la pression appliquée.
2. L'eau n'a (presque) aucune viscosité. Il y a très peu de frottement interne dans l'eau et nous ne nous intéressons pas ici aux zones de surf et de swash, dans lesquelles les effets de rotationel ne sont pas négligeables. Une conséquence importante de ces propriétés est que l'eau ne peut pas prendre de vorticité lorsque les vagues passent; il est impossible à des remous ou tourbillons d'être créés. On appelle souvent un tel fluide irrotationnel. Le domaine d'étude est présenté comme suit: (voir la figure ci-dessous)

Nous nous restreignons ici au cas où la surface est un graphe paramétré par une fonction $\zeta(t,X)$, où t désigne la variable de temps et $X=(X_1,\dots,X_d)\in\mathbb{R}^d$ les variables horizontales. Le fond, non nécessairement plat, est lui aussi paramètré par une fonction $b(X)$, indépendante du temps. Nous notons Ω_t le domaine occupé par le fluide à l'instant t, où

$$\Omega_t=\{(X,z),X\in\mathbb{R}^d,-h_0+b(x)\leq z\leq\zeta(t,X)\}.$$

La propriété d'incompressibilité du fluide est traduite par l'équation

$$divV=0 \quad dans \quad \Omega_t, \qquad t\geq 0, \tag{0.0.1}$$

où $V=(V_1,\dots,V_d,V_{d+1})$ désigne le champ de vitesse ($V_1,\dots,V_d$ étant les composantes horizantales, et V_{d+1} la composante verticale). L'irrotationnalité s'exprime quant à elle par la relation:

$$\nabla_{X,z}\wedge V=0 \quad dans \quad \Omega_t, \qquad t\geq 0. \tag{0.0.2}$$

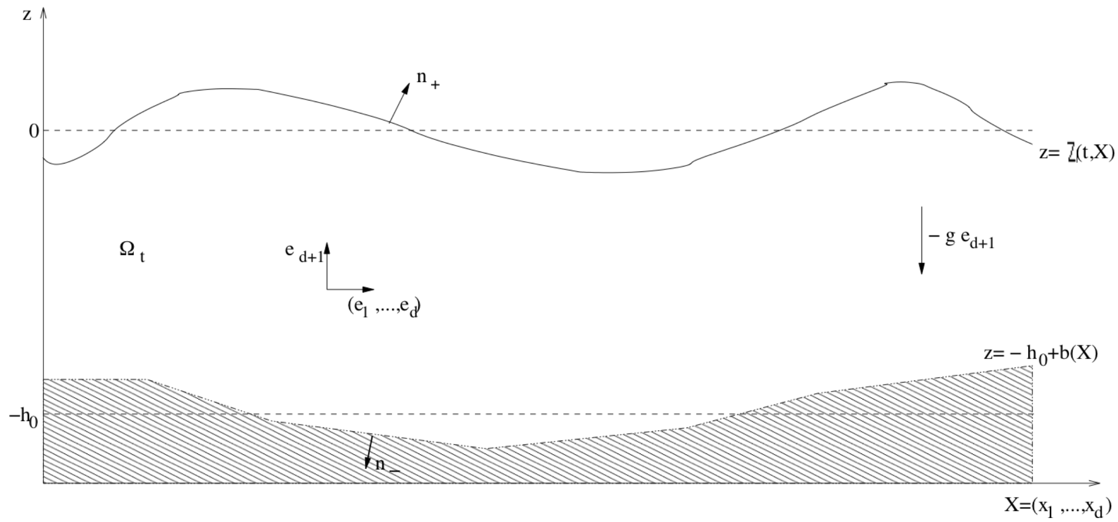

Figure 0.0.1: Domaine du fluide

On complète le système d'équations à l'intérieur du fluide par l'équation d'Euler,

$$\partial_t V + V \cdot \nabla_{X,z} V = -g e_{d+1} - \nabla_{X,z} P \quad dans \quad \Omega_t \qquad t \geq 0. \tag{0.0.3}$$

Où ge_{d+1} est l'accélération de la gravité, et P désigne la pression, qui est également une des inconnues du problème.
Pour clore ce systéme d'èquations, nous devons donner les conditions aux bords satisfaites par la vitesse au fond et à la surface ; elles sont obtenues en traduisant l'hypothèse physique qu'aucune particule de fluide n'est transportée au travers de ces surfaces. Plus précisement, cela s'écrit :

$$V_n \mid_{z=-h_0+b(X)} = n_- \cdot V \mid_{z=-h_0+b(X)} = 0 \quad pour \quad t \geq 0, \quad X \in \mathbb{R}^d, \tag{0.0.4}$$

où $n_- := \dfrac{1}{\sqrt{1+|\nabla b|^2}}(\nabla b, -1)^T$ est le vecteur normal sortant au bord inférieur de Ω_t. A la surface libre, on a

$$\partial_t \zeta - \sqrt{1+|\nabla \zeta|^2} n_+ \cdot V \mid_{z=\zeta(t,X)} = 0, \quad pour \quad t \geq 0, \quad X \in \mathbb{R}^d, \tag{0.0.5}$$

Où $n_+ := \dfrac{1}{\sqrt{1+|\nabla \eta|^2}}(-\nabla \zeta, 1)^T$ désigne le vecteur normal sortant à la surface .
En négligant les effets de tension de surface, P est constante à la surface. Quitte à renormaliser, on peut supposer

$$P \mid_{z=\zeta(t,X)} = 0 \quad pour \quad t \geq 0, \quad X \in \mathbb{R}^d. \tag{0.0.6}$$

Enfin, l'ensemble d'équations est fermé par l'équation d'Euler,

$$\partial_t V + V \cdot \nabla_{X,z} V = -g e_{d+1} - \nabla_{X,z} P \quad \text{in } \Omega_t, \quad t \geq 0, \tag{0.0.7}$$

où $g e_{d+1}$ est l'accélération de la gravité. Le système d'Euler surface libre peut être écrire comme suit:

$$\begin{cases} \partial_t V + V \cdot \nabla_{X,z} V = -g e_{d+1} - \nabla_{X,z} P & \text{dans } \Omega_t, \quad t \geq 0 \ , \\ \nabla_{X,z} \cdot V = 0 & \text{dans } \Omega_t, \quad t \geq 0 \ , \\ \nabla_{X,z} \wedge V = 0 & \text{dans } \Omega_t, \quad t \geq 0 \ , \\ \partial_t \zeta - \sqrt{1 + |\nabla_X \zeta|^2}\, \mathbf{n}_+ \cdot V|_{z=\zeta(t,X)} = 0 & \text{pour } t \geq 0, \ X \in \mathbb{R}^d \ , \\ P_{|_{z=\zeta(t,X)}} = 0 & \text{pour } t \geq 0, \ X \in \mathbb{R}^d \ , \\ \mathbf{n}_- \cdot V|_{z=-h_0+b(X)} = 0 & \text{pour } t \geq 0, \ X \in \mathbb{R}^d \ . \end{cases} \tag{0.0.8}$$

Dans cette thèse, nous avons délibérément choisi de travailler dans le paramètre d'Eulérien (plutôt que lagrangien), car il est le plus facile à manipuler, en particulier lorsque les propriétés asymptotiques des solutions sont concernés. Nous utilisons une autre formulation du problème (0.0.8). Des hypothèses d'incompressibilité et d'irrotationalité (0.0.1) et (0.0.2), il existe un flux potentiel φ tel que $V = \nabla_{X,z}\varphi$ et

$$\Delta_{X,z}\varphi = 0 \quad \text{dans} \quad \Omega_t, \quad t \geq 0; \tag{0.0.9}$$

les conditions aux limites (0.0.4) et (0.0.5) peuvent aussi être exprimées en termes de φ:

$$\partial_{n_-}\varphi \,|_{z=-h_0+b(X)} = 0, \quad \text{pour} \quad t \geq 0, \quad X \in \mathbb{R}^d, \tag{0.0.10}$$

et

$$\partial_t \zeta - \sqrt{1 + |\nabla_X \zeta|^2}\partial_{n_+}\varphi \ = 0, \quad \text{pour} \quad t \geq 0, \quad X \in \mathbb{R}^d, \tag{0.0.11}$$

dont on a utilisé les notations $\partial_{n_-} := n_- \cdot \nabla_{X,z}$ et $\partial_{n_+} := n_+ \cdot \nabla_{X,z}$. Finalement, l' équation d'Euler (0.0.38) s'écrit sous forme de Bernoulli

$$\partial_t \varphi + \frac{1}{2}\left[\, |\nabla\varphi|^2 + |\partial_z \varphi|^2 \,\right] + gz = -P \quad \text{dans } \Omega_t, \quad t \geq 0. \tag{0.0.12}$$

L'ensemble des équations (0.0.8) peut donc être écrit sous la formulation de Bernoulli, en termes de potentiel de la vitesse φ, qui est défini sur

$$\Omega_t = \{(X, z), -h_0 + b(X) < z < \zeta(t, X)\}$$

$$\begin{cases} \partial_t\varphi + \dfrac{1}{2}\left[\,|\nabla\varphi|^2 + |\partial_z\varphi|^2\,\right] + gz = -P & \text{dans } \Omega_t,\ \ t \geq 0\ \ , \\ \Delta\varphi + \partial_z^2\varphi = 0 & \text{dans } \Omega_t,\ \ t \geq 0\ \ , \\ \partial_t\zeta - \sqrt{1+|\nabla\zeta|^2}\,\partial_{\mathbf{n}_+}\varphi|_{z=\zeta(t,X)} = 0 & \text{pour } t \geq 0,\ X \in \mathbb{R}^d\ \ , \\ \partial_{\mathbf{n}_-}\varphi|_{z=-h_0+b(X)} = 0 & \text{pour } t \geq 0,\ X \in \mathbb{R}^d\ \ . \end{cases} \tag{0.0.13}$$

Remark 0.0.1. *Quand aucune confusion ne peut être faite, nous noterons ∇_X par ∇.*

En séparant les dérivées partielles en X et z, et en prenant la trace de la première équation à la surface libre, on obtient le système suivant:

$$\begin{cases} \Delta\varphi + \partial_z^2\varphi = 0 & \text{dans } \Omega_t,\ \ t \geq 0\ \ , \\ \partial_t\varphi + \dfrac{1}{2}\left[\,|\nabla\varphi|^2 + |\partial_z\varphi|^2\,\right] + g\zeta = 0 & \text{pour } z = \zeta(t,X),\ X \in \mathbb{R}^d,\ t \geq 0\ \ , \\ \partial_t\zeta + \nabla\zeta\cdot\nabla\varphi - \partial_z\varphi = 0 & \text{pour } z = \zeta(t,X),\ X \in \mathbb{R}^d,\ t \geq 0\ \ , \\ \nabla b\cdot\nabla\varphi - \partial_z\varphi = 0 & \text{pour } z = -h_0 + b(X),\ X \in \mathbb{R}^d,\ t \geq 0\ \ . \end{cases} \tag{0.0.14}$$

La plupart de l'étude du probème d'Euler surface libre est basée sur l'utilisation de modèles asymptotiques, qui exigent des paramètres de petites tailles. Notre première chose est d'adimensionner les équations du modèle d'Euler. Ceci nécessite d'introduire divers ordres de grandeur liés à un régime physique donné. Plus précisément, nous allons introduire les quantités suivantes: (voir la figure ci-dessous)
a est l'amplitude typique des vagues;
λ est la longueur d'onde des vagues;
b_0 est l'ordre de l'amplitude des variations de la topographie du fond;
h_0 est la profondeur moyenne.

Nous introduisons également les paramètres sans dimensions suivants:

$$\varepsilon = \frac{a}{h_0}, \quad \mu = \frac{h_0^2}{\lambda^2}, \quad \beta = \frac{b_0}{h_0};$$

Le paramètre ε est souvent appelé le paramètre de non-linéarité; tandis que μ est le paramètre de dispersion (ou de faible profondeur). Ce dernier paramètre μ représente le carré du rapport de la profondeur moyenne à la longueur d'onde des vagues, et il

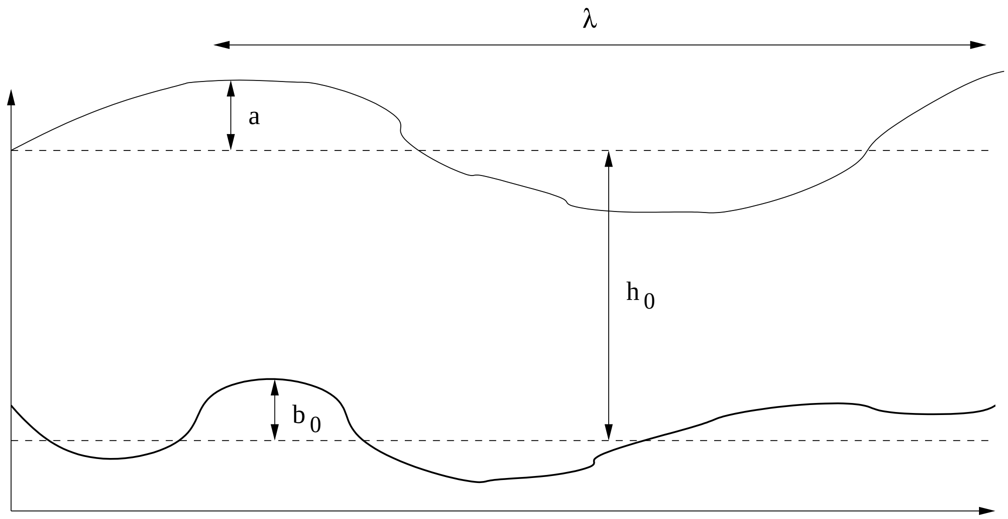

Figure 0.0.2: Régime physique

prend l'ensemble des valeurs de 0 à ∞. La principale distinction à faire est d'imposer $\mu \ll 1$ (en eau peu profonde).
L'intérêt d'avoir introduit les paramètres sans dimensions ε, β et μ est qu'il est souvent possible de déduire de leurs valeurs une idée du comportement de l'écoulement. Plus précisément, il est possible de déduire certains modèles asymptotiques qui sont plus simples pour les simulations numériques et dont les propriétés soient plus transparentes. On peut maintenant à l'aide des relations suivantes:

$$\begin{aligned} &X = \lambda X', && z = h_0 z', \quad \zeta = a\zeta', \\ &\varphi = \frac{a}{h_0}\lambda\sqrt{gh_0}\varphi', && b = b_0 b', \quad t = \frac{\lambda}{\sqrt{gh_0}}t'; \end{aligned} \tag{0.0.15}$$

écrire le problème d'Euler classique (0.0.14) sans dimensions

$$\begin{cases} \mu\partial_x^2\varphi + \mu\partial_y^2\varphi + \partial_z^2\varphi = 0, & \text{pour} \quad -1 + \beta b < z < \varepsilon\zeta, \\ \partial_z\varphi - \mu\beta\nabla b \cdot \nabla\varphi = 0, & \text{pour} \quad z = -1 + \beta b, \\ \partial_t\zeta - \dfrac{1}{\mu}(-\mu\varepsilon\nabla\zeta \cdot \nabla\varphi + \partial_z\varphi) = 0, & \text{pour} \quad z = \varepsilon\zeta, \\ \partial_t\varphi + \dfrac{1}{2}(\varepsilon|\nabla\varphi|^2 + \dfrac{\varepsilon}{\mu}(\partial_z\varphi)^2) + \zeta = 0, & \text{pour} \quad z = \varepsilon\zeta, \end{cases} \tag{0.0.16}$$

Les solutions de ces équations sont très difficiles à décrire. À ce stade, une méthode classique consiste à choisir un régime spécifique, dans lequel nous recherchons des modèles approximatifs, et donc des solutions approchées. Les équations les plus fameuses de la physique mathématique qui ont été historiquement obtenues comme modèle asymptotique des équations d'Euler surface libre sont les équations de Saint-Venant, de Korteweg-de Vries (KdV) et de Kadomtsev-Petviashvili (KP) [47], et les systèmes de Boussinesq [13, 14], etc. Chacun de ces modèles asymptotiques correspond à un régime physique très spécifique dont la game de validité est déterminée en fonction des caractéristiques de l'écoulement du fluide (amplitude, longueur d'onde, anisotropie, la topographie, la profondeur, ...). La dérivation de ces modèles remonte au XIXème siècle, mais l'analyse rigoureuse de ces modèles en tant que modèles approximatifs pour les équations d'Euler surface libre est seulement commencé il y a trois décennies avec les travaux de Ovsjannikov [62, 63], Craig [22], et Kano-Nishida [46, 44, 45] qui sont les premiers à aborder le problème de justification des modèles asymptotique formels. Pour tous les différents modèles asymptotiques, le problème peut être formulé comme suit:
1) le modèle d'Euler surface libre est-il bien posé dans un intervalle de temps pertinent?
2) est ce que ces modèles asymptotiques fournissent une bonne approximation de la solution du problème d'Euler avec surface libre?
Répondre à la première question nécessite des théorèmes d'existence pour les équations d'Euler surface libre dans un intervalle de temps large, tandis que la seconde a besoin d'une dérivation rigoureuse des modèles asymptotiques et un contrôle précis de l'erreur d'approximation. Les travaux pour $1D$ de Ovsjannikov [63] et Nalimov [60] (voir encore Yosihara [79, 80]), Craig [22], et Kano-Nishida [44] donnent une justification pour l'équation de KdV, et pour les systèmes de Boussinesq et de Saint-Venant. Cependant, la compréhension de la théorie de bien posé pour les équations d'Euler surface libre entravait la justification des autres modèles asymptotiques jusqu' à les résultats de S. Wu ([74] et [75] pour les cas 1DH et 2DH respectivement, en profonde infinie et sans hypothèses restrictives). La littérature sur les équations d'Euler surface libre a été très actif: le cas de la profondeur finie a été prouvé par D.Lannes [50], et Lindblad [57, 58] a étudié la surface libre d'un fluide dans le vide avec une gravité zéro. Plus récemment Coutand et Shkoller [21] et Shatah et Zeng [68] ont réussi à supprimer la condition d'irrotationalité et/ou pris en compte les effets de tension superficielle (voir aussi [5] pour 1DH avec la tension de surface). Dans le cadre d'Eulérien qui est plus convenable pour notre objectif, le caractère bien posé de ces équations est finalement démontré par Lannes [50] et, par Alvarez-Samaniego et Lannes [3] sur le temps long. Voir aussi les travaux récents de T. Alazard, N. Burq et C. Zuily [1].

Afin d'examiner les résultats existants de la justification rigoureuse des modèles asymptotiques pour le probleme d'Euler surface libre, il convient de classer les différents régimes physiques à l'aide de deux nombres sans dimensions: ε et μ.

Faible profondeur, amplitude large ($\mu \ll 1, \varepsilon \sim 1$).

Formellement, ce régime conduit au premier ordre aux équations de Saint-Venant et au deuxième ordre aux équations de "Green-Naghdi". La première justification rigoureuse du modèle de Saint-Venant remonte à Ovsjannikov [62, 63] et Kano et Nishida [44] qui ont prouvé la convergence des solutions des ces équations, aux solutions du modèle d'Euler lorsque $\mu \to 0$ en 1DH et, sous certaines conditions. Plus récemment, Y.A. Li [55] a supprimé ces hypothèses et a rigoureusement justifié les équations de Saint-Venant et Green-Naghdi, dans 1DH pour un fond plat. Ensuite, T. Iguchi [36] a justifié dans le cas de 2DH les équations de Saint-Venant, et pour un fond non plat. Enfin, une justification rigoureuse récente pour Green-Naghdi au fond non plat est donnée par [3, 38, 39].

Dans le régime ($\mu \ll 1$), et sans aucune donnée sur ε on peut dériver le modèle du Green-Naghdi (voir [29, 53] pour la dérivation et [3] pour la justification), ces équations sont plus non linéaires que les équations de Boussinesq [54]. Pour les variables sans dimensions, en désignant par $u(t, X)$ la composante horizontale de la vitesse intégrée sur la verticale à l'instant t, les équations peuvent s'écrire

$$\begin{cases} \partial_t \zeta + \nabla \cdot (hu) = 0, \\ (h + \mu h \mathcal{T}[h, \varepsilon b])\partial_t u + h\nabla\zeta + \varepsilon h(u \cdot \nabla)u \\ \quad + \mu\varepsilon\big\{ -\frac{1}{3}\nabla[(h^3((u\cdot\nabla)(\nabla\cdot u) - (\nabla\cdot u)^2)] + h\Re[h, \varepsilon b]u\big\} = 0, \end{cases} \tag{0.0.17}$$

où $h = 1 + \varepsilon(\zeta - b)$ et

$$\mathcal{T}[h, \varepsilon b]W = -\frac{1}{3h}\nabla(h^3\nabla\cdot W) + \frac{\varepsilon}{2h}[\nabla(h^2\nabla b\cdot W) - h^2\nabla b\nabla\cdot W] + \varepsilon^2\nabla b\nabla b\cdot W,$$

tandis que le terme topographique $\Re[h, \varepsilon b]u$ est défini par

$$\begin{aligned} \Re[h, \varepsilon b]u &= \frac{\varepsilon}{2h}[\nabla(h^2(u\cdot\nabla)^2 b) - h^2((u\cdot\nabla)(\nabla\cdot u) - (\nabla\cdot u)^2)\nabla b] \\ &\quad + \varepsilon^2((u\cdot\nabla)^2 b)\nabla b. \end{aligned}$$

Ce modèle est souvent utilisé en océanographie côtière parce qu'il prend en compte les effets dispersifs négligés par les équations de Saint-Venant et il est plus non linéaire que les équations de Boussinesq. Une justification rigoureuse récente du modèle GN a été donnée par Li [55] en $1D$ fond plat, B. Alvarez-Samaniego et D. Lannes et S. Israwi [3, 38, 39] dans le cas général.

Faible profondeur, amplitude petite ($\mu \ll 1, \varepsilon \sim \mu$).

Ce régime (appelé aussi régime d'ondes-longues) provoque un gros intéressant mathématiquement en raison de l'équilibre entre les effets non-linéaires et les effets dispersifs:

Les systèmes de Boussinesq: depuis la première dérivation de Boussinesq, de nombreux modèles formellement équivalents (également nommés d'après Boussinesq) ont été dérivés. W. Craig [22] et Kano-Nishida [45] sont les premiers à donner une justification complète de ces modéles, en 1DH (et pour un fond plat avec données intitiales de taille petite). Notez, toutefois que le résultat de convergence est donné dans [45] sur une échelle de temps trop court pour prendre en compte les effets non-linéaires et dispersifs spécifiques aux systèmes de Boussinesq; les résultats d'existence des solutions sur un intervalle de temps large (et la convergence) pour le problème d'Euler sont donnés dans [22]. La preuve, d'un tel résultat pour les équations d'Euler surface libre est le point le plus délicat dans le processus de justification. En outre, il est la dernière étape nécessaire pour justifier pleinement les systèmes de Boussinesq dans 2DH, cependant, dans [10] (fond plat) et [17] (fond non plat), la propriété de convergence est établie en supposant que les équations de problème d'Euler sont bien posées dans un intervalle de temps large.

Les modèles découplés: au premier ordre, les systèmes de Boussinesq peuvent se réduire en une équation d'onde simple et, dans 1DH, le mouvement de la surface libre peut être décrit comme la somme des deux solitons modulés par l'équation de Korteweg-de Vries (KdV). En 2DH et pour ondes faiblement transervers, un phénomène similaire se produit, mais avec l'équation de Kadomtsev-Petviashvili (KP) qui remplace l'équation de KdV. De nombreux articles ont soulevé le problème de la justification et de la validation du modèle de KdV (e.g. [22, 45, 10, 71, 67, 36]).

Le travail présenté dans cette thèse comporte trois parties.

Nous étudions dans la première partie le problème d'Euler avec surface libre sur un fond non plat et dans un régime fortement non linéaire où l'hypothèse de faible amplitude de l'équation de KdV n'est pas vérifiée. On sait que, pour un tel régime, une généralisation de l'équation de KdV peut être dérivée et justifiée lorsque le fond est plat. Nous généralisons ici ces résultats en proposant une nouvelle classe d'équations prenant en compte des topographies variables. Nous démontrons également que ces nouveaux modèles sont bien posés.

Ensuite, dans la deuxième partie nous améliorons quelques résultats sur l'existence des équations de Green-Naghdi (GN) dans le cas $1D$. Dans le cas de $2D$, nous dérivons et étudions un nouveau système de la même précision que les équations de GN usuelles, mais avec un meilleur comportement mathématique.

Dans la troisième partie, nous étudions numériquement les modèles dérivés en première partie en détaillant les schémas numériques utilisés.

Partie I : Équation de KdV au fond Variable et leurs généralisations à des régimes plus non linéaire

Le chapitre 1 traite le problème d'Euler surface libre et quelques modèles approchés. Récemment, Alvarez-Samaniego et Lannes [3] ont rigoureusement justifié les principaux modéles asymptotiques utilisés en océanographie côtière, y compris: les équations de Saint-Venant, les systèmes de Boussinesq, Kadomtsev-Petviashvili (KP), Green-Naghdi (GN), Serre. Certains de ces modèles traitent l'existence des ondes solitaires et manifestent leurs phénomènes associés [41]. L'exemple le plus famous est l'équation de Korteweg-de Vries (KdV) [49], qui est intégrable et présente un tel phénomène de soliton. Le modèle de KdV a été dérivé sur un fond plat et rigoureusement justifié dans [22, 67, 10, 36]. Lorsque le fond n'est pas plat, différentes généralisations de l'équation de KdV à coefficients non constants, ont été proposées [47, 70, 27, 59, 32, 78, 64, 31, 43, 66]. L'un des objectifs de ce chapitre est de justifier la dérivation de cette équation de Korteweg-de Vries pour un fond variable (KdV-top). Un autre développement asymptotique des équations de problème d'Euler a été développé afin de mieux comprendre les comportement des vagues, tel que la rupture des vagues l'un des aspects les plus fondamentaux de ce problème [28]. En 2008, Constantin et Lannes [20] ont rigoureusement justifié la généralisation en cas fortement non linéaire de l'équation de KdV (ces modèles sont liés à l'équation de Camassa-Holm [15] et à l'équation de Degasperis-Procesi [26]) en tant que modèles approchés pour le problème d'Euler surface libre en faible profondeur. Ils ont prouvé que ces équations peuvent être utilisées pour fournir des approximations pour les équations d'Euler surface libre et dans leur enquête, ils ont mis des procédures asymptotiques (formelle) due à Johnson [42] sur une base robuste et rigoureuse mathématiquement. Cependant, tous ces résultats se vérifient pour fond plat uniquement. L'objectif principal de ce chapitre est que nous généralisons ici ces résultats en proposant une nouvelle classe d'équations prenant en compte des topographies variables. Nous démontrons également que ces nouveaux modèles sont bien posés.

Nous considérons dans cette partie un régime physique particulier. Plus précisément, nous allons introduire les quantités suivantes: a, λ, h_0 et b_0 sont définis comme précédemment et λ/α est la longueur d'onde de la variation du fond (voir la figure ci-dessous). L'intérêt de ce régime est que beaucoup des équations de type KdV au fond variable existantes dans la littérature (par exemple [64, 59]) peuvent être récupérées ici comme des cas particuliers.

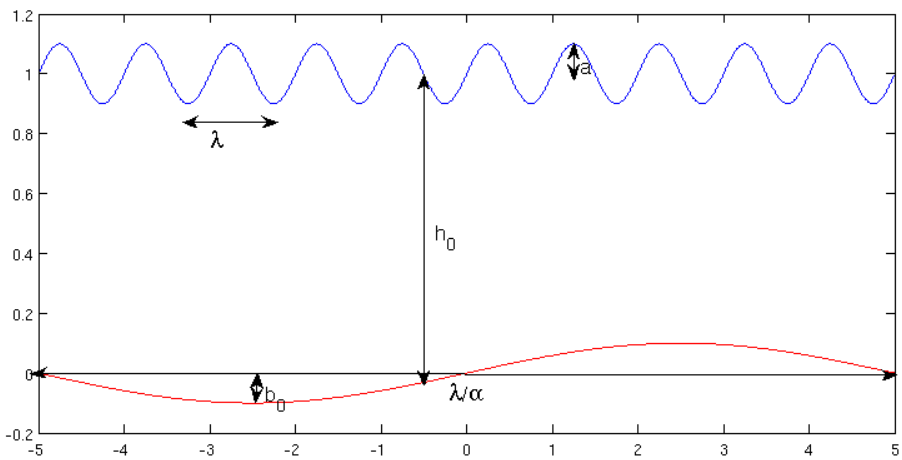

Figure 0.0.3: Régime physique

Alors, les équations (0.0.14) peuvent s'écrire (pour $d = 1$)

$$\begin{cases} \mu\partial_x^2\varphi + \partial_z^2\varphi = 0, & \text{pour} \quad -1+\beta b^{(\alpha)} < z < \varepsilon\zeta, \\ \partial_z\varphi - \mu\beta\alpha\partial_x b^{(\alpha)}\partial_x\varphi = 0 & \text{pour} \quad z = -1+\beta b^{(\alpha)}, \\ \partial_t\zeta - \dfrac{1}{\mu}(-\mu\varepsilon\partial_x\zeta\partial_x\varphi + \partial_z\varphi) = 0, & \text{pour} \quad z = \varepsilon\zeta, \\ \partial_t\varphi + \dfrac{1}{2}(\varepsilon(\partial_x\varphi)^2 + \dfrac{\varepsilon}{\mu}(\partial_z\varphi)^2) + \zeta = 0 & \text{pour} \quad z = \varepsilon\zeta, \end{cases} \tag{0.0.18}$$

où, $b^{(\alpha)}(x) = b(\alpha x)$.
Faisant des hypothèses sur la taille de ε, β, α et μ on peut dériver de (0.0.18) le modèle asymptotique GN.
Pour $1D$ surface et sur un fond non plat ces équations relient l'élévation de la surface libre ζ à la composante horizontale de la vitesse intégrée sur la verticale,

$$u(t,x) = \frac{1}{1+\varepsilon\zeta-\beta b^{(\alpha)}}\int_{-1+\beta b^{(\alpha)}}^{\varepsilon\zeta}\partial_x\varphi(t,x,z)dz \tag{0.0.19}$$

et peuvent s'écrire:

$$\begin{cases} \partial_t\zeta + \partial_x(hu) = 0, \\ (1+\dfrac{\mu}{h}\mathcal{T}[h,\beta b^{(\alpha)}])\partial_t u + \partial_x\zeta + \varepsilon u\partial_x u \\ \quad +\mu\varepsilon\big\{-\dfrac{1}{3h}\partial_x(h^3(u\partial_x^2 u) - (\partial_x u)^2) + \Im[h,\beta b^{(\alpha)}]u\big\} = 0 \end{cases} \tag{0.0.20}$$

où, $h = 1+\varepsilon\zeta-\beta b^{(\alpha)}$ et

$$\mathcal{T}[h,\beta b^{(\alpha)}]W = -\frac{1}{3}\partial_x(h^3\partial_x W) + \frac{\beta}{2}\partial_x(h^2\partial_x b^{(\alpha)})W + \beta^2 h(\partial_x b^{(\alpha)})^2 W,$$

tandis que le terme topographique $\Im[h, \beta b^{(\alpha)}]u$ est défini par:

$$\begin{aligned}\Im[h, \beta b^{(\alpha)}]u &= \frac{\beta}{2h}[\partial_x(h^2(u\partial_x)^2 b^{(\alpha)}) - h^2((u\partial_x^2 u) - (\partial_x u)^2)\partial_x b^{(\alpha)}] \\ &\quad +\beta^2((u\partial_x)^2 b^{(\alpha)})\partial_x b^{(\alpha)}.\end{aligned}$$

Dans le régime

$$\mu \ll 1, \quad \varepsilon = O(\sqrt{\mu}), \tag{0.0.21}$$

les effets non linéaires sont plus forts, l'équation de KdV [49] et BBM [8] sont remplacées par la famille des équations suivante (voir [20, 42]):

$$u_t + u_x + \frac{3}{2}\varepsilon u u_x + \mu(A u_{xxx} + B u_{xxt}) = \varepsilon\mu(E u u_{xxx} + F u_x u_{xx}), \tag{0.0.22}$$

on prouve qu'une bonne généralisation de l'équation (0.0.22) sous le régime (0.0.21) et avec les conditions suivantes sur les variations du fond

$$\beta\alpha = O(\mu), \quad \beta\alpha^{3/2} = O(\mu^2) \quad \beta\alpha\varepsilon = O(\mu^2), \tag{0.0.23}$$

est donnée par:

$$\begin{aligned}&u_t + c u_x + \frac{3}{2}c_x u + \frac{3}{2}\varepsilon u u_x + \mu(\tilde{A} u_{xxx} + B u_{xxt}) \\ &= \varepsilon\mu\tilde{E} u u_{xxx} + \varepsilon\mu\Big(\partial_x(\frac{\tilde{F}}{2}u)u_{xx} + u_x\partial_x^2(\frac{\tilde{F}}{2}u)\Big),\end{aligned} \tag{0.0.24}$$

où $c = \sqrt{1 - \beta b^{(\alpha)}}$ et à cause des effets topographiques les coefficients $\tilde{A}$, $\tilde{E}$, $\tilde{F}$ sont differents des coefficients A, E, F dans (0.0.22)

$$\begin{aligned}\tilde{A} &= Ac^5 - Bc^5 + Bc, \\ \tilde{E} &= Ec^4 - \frac{3}{2}Bc^4 + \frac{3}{2}B, \\ \tilde{F} &= Fc^4 - \frac{9}{2}Bc^4 + \frac{9}{2}B.\end{aligned}$$

Afin de récupérer les nombreuses équations de KdV avec fond variable qui ont été dérivées par les océanographes. On prouve même genre des résultats pour le régime de KdV. Dans cette section de ce chapitre, l'attention est accordée au régime de la variation lente de la topographie du fond sous la condition d'ondes-langues $\varepsilon = O(\mu)$. Nous donnons ici une justification rigoureuse dans le sens de la consistence de l'équation de KdV au fond variable (KdV-top) qui a été dérivée dans [47, 70, 27]. Nous étudions aussi l'existence des solutions pour tous les modèles dérivés dans ce chapitre. Deux approches différentes sont utilisées, en fonction du coefficient de B dans (0.0.24): l'un traite le cas $B < 0$ (dans ce cas, on peut étudier les possibiltés du déferlemnt de la vague), et l'autre traite le cas $B = 0$.

Partie II : Les équations de Green-Naghdi

Dans le chapitre 2, nous considérons le modèle $1D$ de Green-Naghdi qui est couramment utilisé en océanographie côtière pour décrire la propagation des ondes de surface de grande amplitude. Une justification rigoureuse récente du modèle GN au fond plat a été donnée par Li [55] en $1D$, et par B. Alvarez-Samaniego et D. Lannes [3] dans le cas général. Cette dernière référence donne des résultats d'existence pour les équations de GN en basant sur la référence [4] qui montre l'existence d'une solution pour un problÃ¨me d'évolution plus général que celui de GN en utilisant un schéma d'existence de Nash-Moser. Le résultat de [4] couvre à la fois le cas de $1D$ et $2D$ surfaces de GN et pour un fond variable. La raison pour laquelle un schéma de Nash-Moser est utilisé, est dû au fait que les estimations sur les équations linéarisées pertent des dérivés. Toutefois, dans le cas $1D$ avec un fond plat, ces pertes ne se produisent pas et il est possible de construire une solution avec un schéma itératif simple de Picard comme dans [55]. Notre but ici est de montrer qu'il est aussi possible d'utiliser un tel schéma simple dans le cas $1D$ mais avec un fond non plat.
Les équations de Green-Naghdi (0.0.17) peuvent simplifier pour $1D$, sous cette forme: ($\beta = \varepsilon$)

$$\begin{cases} \partial_t\zeta + \partial_x(hu) = 0, \\ (h + \mu h\mathcal{T}[h, \varepsilon b])[\partial_t u + \varepsilon u\partial_x u] + h\partial_x\zeta + \varepsilon\mu h Q[h, \varepsilon b](u) = 0 \end{cases} \tag{0.0.25}$$

où, $h = 1 + \varepsilon(\zeta - b)$ et

$$\mathcal{T}[h, \varepsilon b]w = -\frac{1}{3h}\partial_x(h^3 w_x) + \frac{\varepsilon}{2h}[\partial_x(h^2 b_x w) - h^2 b_x w_x] + \varepsilon^2 b_x^2 w,$$

$$Q[h, \varepsilon b](w) = \frac{2}{3h}\partial_x(h^3 w_x^2) + \varepsilon h w_x^2 b_x + \varepsilon\frac{1}{2h}\partial_x(h^2 w^2 b_{xx}) + \varepsilon^2 w^2 b_{xx} b_x.$$

Nous définissons maintenant les espaces X^s, qui sont les espaces d'énergies pour ce problème.

Definition 0.0.1. *Pour tout $s \geq 0$ et $T > 0$, on note par X^s l'espace $H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R})$ munie de la norme*

$$\forall\, U = (\zeta, u) \in X^s, \quad |U|^2_{X^s} := |\zeta|^2_{H^s} + |u|^2_{H^s} + \mu|\partial_x u|^2_{H^s},$$

tandis que X^s_T représent l'espace $C([0, \dfrac{T}{\varepsilon}]; X^s)$ munie de la norme canonique.

Theorem 0.0.1. *Soit $b \in C_b^\infty(\mathbb{R})$, $t_0 > 1/2$, $s \geq t_0 + 1$. Soit, la condition initiale $U_0 = (\zeta_0, u_0)^T \in X^s$, et vérifie*

$$\exists\, h_{min} > 0, \quad \inf_{x\in\mathbb{R}} h \geq h_{min}, \quad h = 1 + \varepsilon(\zeta - b). \tag{0.0.26}$$

Alors il existe $T_{max} > 0$ maximale uniformément minoré par rapport à $\varepsilon, \mu \in (0,1)$ sachant que le modèle de Green-Naghdi (0.0.25) admet une solution unique $U = (\zeta, u)^T \in X^s_{T_{max}}$ avec la condition initiale $(\zeta_0, u_0)^T$ et vérifie (0.0.26) pour tout $t \in [0, \frac{T_{max}}{\varepsilon})$. En particulier, si $T_{max} < \infty$ on aura

$$|U(t,\cdot)|_{X^s} \longrightarrow \infty \quad \textit{quand} \quad t \longrightarrow \frac{T_{max}}{\varepsilon},$$

ou

$$\inf_{\mathbb{R}} h(t,\cdot) = \inf_{\mathbb{R}} 1 + \varepsilon(\zeta(t,\cdot) - b(\cdot)) \longrightarrow 0 \quad \textit{quand} \quad t \longrightarrow \frac{T_{max}}{\varepsilon}.$$

De plus, on a la conservation d'énergie suivante:

$$\partial_t\Big(|\zeta|_2^2 + (hu, u) + \mu(h\mathcal{T}u, u)\Big) = 0,$$

où, $\mathcal{T} = \mathcal{T}[h, \varepsilon b]$.

Le chapitre 3 est consacré à la dérivation d'un nouveau modèle de $2D$ Green-Naghdi qui modélise la propagation des ondes dispersives non linéaires à la surface de l'eau peu profonde dans deux-directionnes. Ce nouveau modèle a la même précision que le modèle usuel de $2D$ Green-Naghdi. Son intérêt mathématique est qu'il permet un contrôle de la partie rotationelle de la vitesse horizontale intégrée sur la verticale, qui n'est pas le cas pour les équations de Green-Naghdi usuelles. En utilisant cette propriété, nous montrons que la solution de ces nouvelles équations peut être construite par un schéma simple de Picard afin qu'il n'y avait pas eu de pertes de la régularité de la solution par rapport à la condition initiale. Enfin, nous prouvons que le nouveau modèle de Green-Naghdi conserve l'irrotationalité de la vitesse horizontale intégrée sur la verticale.
Pour $2D$ de surface et sur un fond non plat ces équations relient l'élévation de la surface libre ζ à la composante horizontale de la vitesse intégrée sur la verticale,

$$v(t, X) = \frac{1}{1 + \varepsilon\zeta - \beta b} \int_{-1+\beta b}^{\varepsilon\zeta} \nabla\varphi(t, X, z)dz, \tag{0.0.27}$$

et peuvent s'écrire

$$\begin{cases} \partial_t\zeta + \nabla\cdot(hv) = 0, \\ \big(h + \mu\mathcal{T}[h, \beta b]\big)\partial_t v + h\nabla\zeta + \varepsilon\big(h + \mu\mathcal{T}[h, \beta b]\big)(v\cdot\nabla)v \\ \quad + \mu\varepsilon\big\{\frac{2}{3}\nabla[(h^3(\partial_1 v\cdot\partial_2 v^\perp + (\nabla\cdot v)^2)] + \Re[h, \beta b](v)\big\} = 0, \end{cases} \tag{0.0.28}$$

où, $h = 1 + \varepsilon\zeta - \beta b$, $v = (V_1, V_2)^T$ et

$$\begin{aligned} \mathcal{T}[h, \varepsilon b]W &= -\frac{1}{3}\nabla(h^3\nabla\cdot W) + \frac{\beta}{2}[\nabla(h^2\nabla b\cdot W) - h^2\nabla b\nabla\cdot W] \\ &\quad +\beta^2 h\nabla b\nabla b\cdot W, \end{aligned} \tag{0.0.29}$$

tandis que le terme topographique $\Re[h, \beta b](v)$ est défini par:

$$\begin{aligned} \Re[h, \beta b](v) &= \frac{\beta}{2}\nabla(h^2(V_1^2\partial_1^2 b + 2V_1V_2\partial_1\partial_2 b + V_2^2\partial_2^2 b)) \\ &\quad +\beta h^2(\partial_1 v\cdot\partial_2 v^\perp + (\nabla\cdot v)^2)\nabla b \\ &\quad +\beta^2 h(V_1^2\partial_1^2 b + 2V_1V_2\partial_1\partial_2 b + V_2^2\partial_2^2 b)\nabla b, \end{aligned} \tag{0.0.30}$$

où $v^\perp = (-V_2, V_1)^T$.
On remarque que le term $\partial_1 v\cdot\partial_2 v^\perp$ n'est pas côntrolé par la norme $|\cdot|_{Y^s}$ associée à (0.0.28) (voir [4]),

$$|(\zeta, v)|^2_{Y^s} = |\zeta|^2_{H^s} + |v|^2_{H^s} + \mu|\nabla\cdot v|^2_{H^s}.$$

C'est la motivation pour dériver un nouveau modèle de $2D$ Grren-Naghdi a la même précision que le modèle de $2D$ Green-Naghdi usuel (0.0.28) et peut s'écrire de la forme

$$\begin{cases} \partial_t\zeta + \nabla\cdot(hv) = 0, \\ \big(h + \mu\big(\mathcal{T}[h, \beta b] - \nabla^\perp(\mathrm{curl}\ )\big)\big)\partial_t v + h\nabla\zeta \\ \ + \varepsilon\big(h + \mu\big(\mathcal{T}[h, \beta b] - \nabla^\perp(\mathrm{curl}\ )\big)\big)(v\cdot\nabla)v \\ \ + \mu\varepsilon\big\{\frac{2}{3}\nabla[h^3(\partial_1 v\cdot\partial_2 v^\perp + (\nabla\cdot v)^2)] + \Re[h, \beta b](v)\big\} = 0, \end{cases} \tag{0.0.31}$$

dont, $\nabla^\perp = (-\partial_2, \partial_1)^T$, $\mathrm{curl}\, v = \partial_1 V_2 - \partial_2 V_1$, et les opérateures linéaires $\mathcal{T}[h, \varepsilon b]$ et $\Re[h, \beta b]$ sond définies dans (0.0.29) et (0.0.30) respectivement.

La raison pour laquelle les nouveaux termes $(\nabla^\perp(\mathrm{curl}\ ))$ n'affectent pas la précision du modèle est parce que les solutions de (0.0.28) sont presque irrotationnelles (dans le sens que $\mathrm{curl}\, v$ est petit). Cette propriété est bien sûr satisfaite également par notre nouveau modèle (0.0.31). La présence de ces nouveaux termes permettent la définition d'une nouvelle norme d'énergie qui contrôle aussi la partie rotationelle de v. En conséquence, nous montrons qu'il est possible d'utiliser un schéma simple de Picard pour prouver l'existence d'une solution de (0.0.31) afin qu'il n'y avait pas eu de pertes de la régularité de la solution par rapport à la condition initiale.

Nous définissons maintenant les espaces X^s, qui sont les espaces d'énergies pour ce *nouveau* problème.

Definition 0.0.2. *Pour tout $s \geq 0$ et $T > 0$, on note par X^s l'espace $H^s(\mathbb{R}^2) \times (H^{s+1}(\mathbb{R}^2))^2$ munie de la norme*

$$\forall\, U = (\zeta, v) \in X^s, \quad |U|^2_{X^s} := |\zeta|^2_{H^s} + |v|^2_{(H^s)^2} + \mu|\nabla \cdot v|^2_{H^s} + \mu|curl\ v|^2_{H^s},$$

tandis que X^s_T représent l'espace $C([0, \frac{T}{\varepsilon}]; X^s)$ munie de la norme canonique.

Theorem 0.0.2. *Soit $b \in C^\infty_b(\mathbb{R}^2)$, $t_0 > 1$, $s \geq t_0 + 1$. Soit, la condition initiale $U_0 = (\zeta_0, v_0^T)^T \in X^s$, et vérifie*

$$\exists\, h_{min} > 0, \quad \inf_{X \in \mathbb{R}^2} h \geq h_{min}, \quad h = 1 + \varepsilon(\zeta - b). \tag{0.0.32}$$

Alors il existe $T_{max} > 0$ maximale uniformément minoré par rapport à $\varepsilon, \mu \in (0,1)$ sachant que le nouveau *modèle de Green-Naghdi (0.0.31) admet une solution unique $U = (\zeta, v^T)^T \in X^s_{T_{max}}$ avec la condition initiale $(\zeta_0, v_0^T)^T$ et vérifie (**??**) pour tout $t \in [0, \frac{T_{max}}{\varepsilon})$. En particulier, si $T_{max} < \infty$ on aura*

$$|U(t,\cdot)|_{X^s} \longrightarrow \infty \quad quand \quad t \longrightarrow \frac{T_{max}}{\varepsilon},$$

ou

$$\inf_{\mathbb{R}^2} h(t,\cdot) = \inf_{\mathbb{R}^2} 1 + \varepsilon(\zeta(t,\cdot) - b(\cdot)) \longrightarrow 0 \quad quand \quad t \longrightarrow \frac{T_{max}}{\varepsilon}.$$

Partie III : Simulations numériques

Le chapitre 4 (ce travail est effectué en collaboration avec Marc Duruflé) est consacré à étudier numériquement les équations de KdV et “Camassa-Holm-like” dérivées dans [37]/Chapitre 1. L'équation de Korteweg-de Vries (KdV) a été dérivée sur un fond plat [49] est une approximation des équations de Boussinesq, et cette relation a été rigoureusement justifiée dans [22, 67, 10, 36]. Lorsque le fond est non plat, diverses généralisations de l'équation de KdV (KdV-top) à coefficients non constants, ont été proposées dans [47, 70, 27, 59, 32, 78, 64, 31, 43, 66], et rigoureusement justifiées dans [37]. L'un des objectifs de ce chapitre est d'étudier numériquement ces équations (KdV-top), et de les comparer avec les équations de Boussinesq sur un fond non plat. L'équation KdV sur fond plat peut être résolue numériquement par l'utilisation de schémas de différences finies [65, 81], ou par l'utilisation des schémas de type Galerkin discontinue [76]. Elle est traitée avec des schémas des différences finies dans [18] en utilisant un schéma de relaxation de Crank-Nicolson sur le temps introduit par Besse-Bruneau dans [6] et justifié par Besse dans [7]. Notre schéma en différences finies est inspiré de ces travaux. Nous proposons une modification de sorte que le schéma numérique conserve l'énergie discrète, pour le schéma totalement discrets (dans l'espace et le temps). Nous comparons aussi cette approche avec la méthode de Galerkin discontinue [76]. La généralisation de l'équation de KdV en régimes plus fort contient des termes fortement non linéaires ne se trouvent pas dans les équations de KdV et de BBM . En 2008, Constantin et Lannes [20] ont été, rigoureusement justifié ces généralisations de l'équation de KdV dans le cas de fond plat. Ils ont prouvé que ces équations peuvent être utilisées pour fournir des approximations pour les équations d'Euler surface libre. Ces équations (CH) sur le fond plat peut être numériquement étudiées en utilisant des schémas des différences finies [34, 35, 20, 25], ou en utilisant des schémas de Galerkin discontinue [77]. En 2009, et dans [37], nous avons étudié le cas du fond variable dans le même régime que dans [20]. Nous avons dérivé une nouvelle classe d'équations qui prend en compte les effets topographiques et généralise les équations de (CH) dérivées par Constantin-Lannes [20]. Dans le présent chapitre, nous trouvons des schémas des différences finies pour ces nouveaux modèles, de sorte que ces schémas numériques conservent l'énergie discrète. Nous comparons aussi cette approche numérique avec la méthode de Galerkin discontinue [77].

Publications

- *Variable depth KdV equations and generalizations to more nonlinear regimes.* Livre des communications Smai Canum 2009 P: 61.

- *Variable depth KdV equations and generalizations to more nonlinear regimes.* à apparaître dans Esaim: M2AN. (Voir chapitre 1)

- *Large Time existence For 1D Green-Naghdi equations.* Soumis à Nonlinear Analysis. (Voir chapitre 2)

- *Derivation and analysis of A NEW 2D Green-Naghdi system.* Soumis à Nonlinearity. (Voir chapitre 3)

- *A numerical study of variable depth KdV equations and generalizations of Camassa-Holm-like equations.* (Avec M. DURUFlÉ). À soumettre. (Voir chapitre 4).

General introduction

The modeling of coastal flows currently represents a major scientific size in coastal engineering. Hydrodynamic processes related to the transformation of waves in shallow environments involve nonlinear, dispersive phenomena and bathymetric effects are complex to study. Their understanding is essential, if one wants be able to predict the impact of large scale phenomena such as the spread type of tsunami waves. The water waves problem for an ideal liquid consists in describing the motion of the free surface and the evolution of the velocity field of a layer of perfect, incompressible, irrotational fluid under the influence of gravity. The propagation of surface waves through an incompressible homogenous inviscid fluid is described by the 3D Euler equations combined with nonlinear boundary conditions at the free surface and at the bottom. This problem is extremely difficult to solve, in particular because the moving surface boundary is part of the solution. The complexity of this problem led physicists, oceanographers and mathematicians to derive simpler sets of equations in some specific physical regimes. Equations thus obtained may be divided into two groups, namely shallow-water models and deep-water models. Many approximate models have thus been derived in order to understand the hydrodynamic processes in simplified physical settings. The construction of the classical Boussinesq models of this kind goes back to the late XIXth century. Such equations were first derived by Boussinesq [13, 14] to describe the two-way propagation of small-amplitude, long wavelength, gravity waves on the surface of water in a canal. These equations arise also when modeling the propagation of long-crested waves on large lakes or the ocean and in other contexts, and they have been since then widely used and improved although the problem of mathematical justification was not addressed until the end of the last century [22, 45, 11, 12, 10]. Moreover, they are often restricted to the framework of a flat bottom, and do not account for many phenomena. The case of uneven bottoms has also been investigated; some of the significant references are [61, 19, 18]. The other prominent model is the Green-Naghdi equations (GN) – which is a widely used model in coastal oceanography ([29, 9, 30] and, for instance, [40, 48])–This model is often used in coastal oceanography because it takes into account the dispersive effects neglected by the shallow-water (or Saint-Venant) equations and it is more nonlinear than the Boussinesq equations.

The objective of this thesis is to propose new models of coastal flows for reporting the influence of bottom topography on surface waves, and justify mathematically some approximate models for the water waves problem as the Green-Naghdi model. To achieve this goal, we devided this thesis into three parts of equal importance: in the first part modeling and mathematical analysis of some models are addressed, and the construction methods are presented in detail. The obtained models are also systematically and rigorously justified.
In the second part derivation and mathematical analysis of the Green-Naghdi asymptotic model is proposed. Finally some numerical simulation of the models of the first part are implemented and discussed in the third part.

All modeling work first requires the establishment of a specific physical setting. We focus here on a case of a fluid (water) IDEAL, incompressible, irrotational and under the influence of gravity. Such simplifying assumptions are conventionally used in oceanography and can be easily justified (see for intance [47, 27]). Indeed,
1. Water is (almost) incompressible. In other words, it would be impossible to squeeze a gallon of water into your pint-sized water bottle, regardless of how hard you tried. Water that fills a certain volume will always fill that same volume regardless of the pressure applied to it.
2. Water has (almost) no viscosity. There is very little internal friction in water, and we are not interested here in the surf and swash zones, in which rotational effects are not negligible. An important consequence of these properties is that the water can't gain angular momentum as waves pass; it is impossible for eddies or whirlpools to be created. One often calls such a fluid irrotational. The field study is presented as follows:

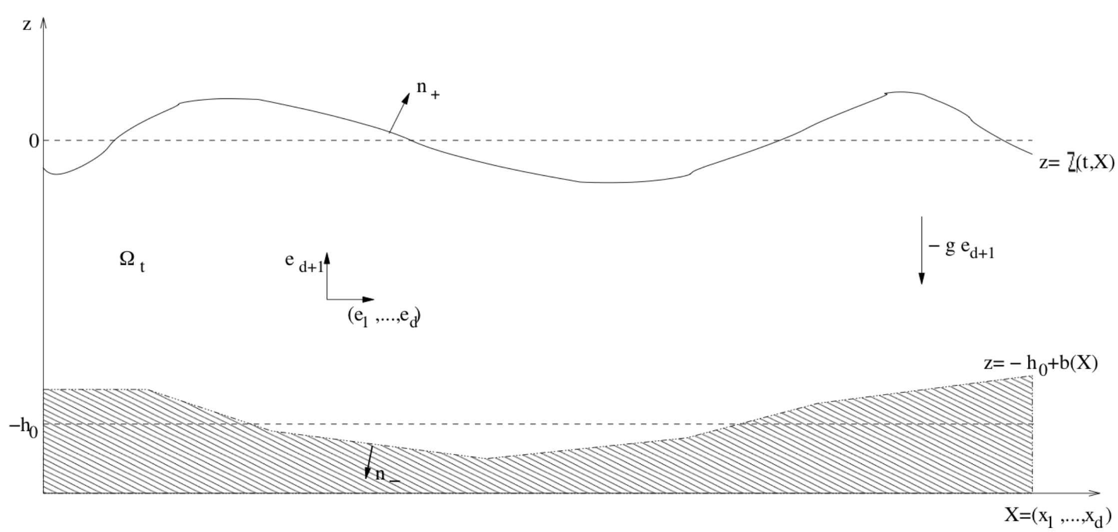

Figure 0.0.4: Fluid domain

We restrict our attention to the case when the surface is a graph parameterized by a function $\zeta(t, X)$, where t denotes the time variable and $X = (X_1, ..., X_d) \in \mathbb{R}^d$ the horizontal spacial variables. But the only physically relevant cases are of course $d = 1$ and $d = 2$. The layer of fluid is also delimited from below by a not necessarily flat bottom parameterized by a time-independent function $-h_0 + b(X)$. We denote by Ω_t the fluid domain at time t. The incompressibility of the fluids is expressed by

$$\nabla_{X,z} \cdot V = 0 \quad \text{in} \quad \Omega_t, \quad t \geq 0, \tag{0.0.33}$$

where $V = (V_1, ..., V_d, V_{d+1})$ denotes the velocity filed ($V_1, ..., V_d$ being the horizontal, and V_{d+1} the vertical components of the velocity). Irrotationality means that

$$\nabla_{X,z} \wedge V = 0 \quad \text{in} \quad \Omega_t, \quad t \geq 0. \tag{0.0.34}$$

The boundary conditions on the velocity at the surface and at the bottom are given by the usual assumption that they are both bounding surfaces, i.e. surfaces across which no fluid particles are transported. At the bottom, this is given by

$$V_n \mid_{z=-h_0+b(X)} = n_- \cdot V \mid_{z=-h_0+b(X)} = 0, \quad \text{for} \quad t \geq 0, \quad X \in \mathbb{R}^d, \tag{0.0.35}$$

where $n_- := \dfrac{1}{\sqrt{1 + |\nabla_X b|^2}} (\nabla_X b, -1)^T$ denotes the outward normal vector to the lower boundary of Ω_t . At the free surface, the boundary condition is kinematic and is given by

$$\partial_t \zeta - \sqrt{1 + |\nabla_X \zeta|^2} V_n \mid_{z=\zeta(t,X)} = 0, \quad \text{for} \quad t \geq 0, \quad X \in \mathbb{R}^d, \tag{0.0.36}$$

where $V_n \mid_{z=\zeta(t,X)} = n_+ \cdot V \mid_{z=\zeta(t,X)}$, with $n_+ := \dfrac{1}{\sqrt{1 + |\nabla_X \zeta|^2}} (-\nabla_X \zeta, 1)^T$ denoting the outward normal vector to the free surface.
Neglecting the effects of surface tension yields that the pressure P is constant at the interface. Up to a renormalization, we can assume that

$$P \mid_{z=\zeta(t,X)} = 0, \quad \text{for} \quad t \geq 0, \quad X \in \mathbb{R}^d. \tag{0.0.37}$$

Finally, the set of equations is closed with Euler's equation within the fluid,

$$\partial_t V + V \cdot \nabla_{X,z} V = -g e_{d+1} - \nabla_{X,z} P \quad \text{in} \ \Omega_t, \ \ t \geq 0, \tag{0.0.38}$$

where $-g e_{d+1}$ is the acceleration of gravity. The water-waves system can thus be written as:

$$\left\{\begin{array}{ll} \partial_t V + V\ \cdot \nabla_{X,z} V = -g e_{d+1} - \nabla_{X,z} P & \text{in } \Omega_t,\ \ t \geq 0\ \ , \\ \nabla_{X,z} \cdot V = 0 & \text{in } \Omega_t,\ \ t \geq 0\ \ , \\ \nabla_{X,z} \wedge V = 0 & \text{in } \Omega_t,\ \ t \geq 0\ \ , \\ \partial_t \zeta - \sqrt{1+|\nabla_X \zeta|^2}\, \mathbf{n}_+ \cdot V|_{z=\zeta(t,X)} = 0 & \text{for } t \geq 0,\ X \in \mathbb{R}^d\ \ , \\ P_{|_{z=\zeta(t,X)}} = 0 & \text{for } t \geq 0,\ X \in \mathbb{R}^d\ \ , \\ \mathbf{n}_- \cdot V|_{z=-h_0+b(X)} = 0 & \text{for } t \geq 0,\ X \in \mathbb{R}^d\ \ . \end{array}\right. \tag{0.0.39}$$

In this thesis, we deliberately chose to work in the Eulerian (rather than Lagrangian) setting, since it is the easiest to handle, especially when asymptotic properties of the solutions are concerned. We use an alternate formulation of the water-waves equations (0.0.39). From the incompressibility and irrotationality assumptions (0.0.33) and (0.0.34), there exists a potential flow φ such that $V = \nabla_{X,z}\varphi$ and

$$\Delta_{X,z}\varphi = 0 \quad \text{in} \quad \Omega_t, \quad t \geq 0, \tag{0.0.40}$$

the boundary conditions (0.0.35) and (0.0.36) can also be expressed in terms of φ:

$$\partial_{n_-}\varphi\ |_{z=-h_0+b(X)} = 0, \quad \text{for} \quad t \geq 0, \quad X \in \mathbb{R}^d, \tag{0.0.41}$$

and

$$\partial_t \zeta - \sqrt{1+|\nabla_X \zeta|^2}\partial_{n_+}\varphi\ = 0, \quad \text{for} \quad t \geq 0, \quad X \in \mathbb{R}^d, \tag{0.0.42}$$

where we used the notation $\partial_{n_-} := n_- \cdot \nabla_{X,z}$ and $\partial_{n_+} := n_+ \cdot \nabla_{X,z}$. Finally, Euler's equation (0.0.38) can be put into bernoulli's form

$$\partial_t \varphi + \frac{1}{2}\left[\, |\nabla\varphi|^2 + |\partial_z\varphi|^2 \,\right] + gz = -P \quad \text{in } \Omega_t,\ \ t \geq 0. \tag{0.0.43}$$

The set of equations (0.0.39) can thus be written under Bernoulli's formulation, in terms of the velocity potential φ defined on $\Omega_t = \{(X,z), -h_0 + b(X) < z < \zeta(t,X)\}$:

$$\left\{\begin{array}{ll} \partial_t \varphi + \dfrac{1}{2}\ \left[\, |\nabla\varphi|^2 + |\partial_z\varphi|^2 \,\right] + gz = -P & \text{in } \Omega_t,\ \ t \geq 0\ \ , \\ \Delta\varphi + \partial_z^2\varphi = 0 & \text{in } \Omega_t,\ \ t \geq 0\ \ , \\ \partial_t \zeta - \sqrt{1+|\nabla\zeta|^2}\, \partial_{\mathbf{n}_+}\varphi|_{z=\zeta(t,X)} = 0 & \text{for } t \geq 0,\ X \in \mathbb{R}^d\ \ , \\ \partial \mathbf{n}_- \varphi|_{z=-h_0+b(X)} = 0 & \text{for } t \geq 0,\ X \in \mathbb{R}^d\ \ . \end{array}\right. \tag{0.0.44}$$

Remark 0.0.2. *When no confusion can be made, we denote ∇_X by ∇.*

By separating the partial derivatives in X and z, and taking the trace of the first equation on the free surface, we obtain the following system:

$$\begin{cases} \Delta\varphi + \partial_z^2\varphi = 0 & \text{in } \Omega_t, \quad t \geq 0 \ , \\ \partial_t\varphi + \dfrac{1}{2}\left[|\nabla\varphi|^2 + |\partial_z\varphi|^2\right] + g\zeta = 0 & \text{at } z = \zeta(t,X), \ X \in \mathbb{R}^d, \ t \geq 0 \ , \\ \partial_t\zeta + \nabla\zeta\cdot\nabla\varphi - \partial_z\varphi = 0 & \text{at } z = \zeta(t,X), \ X \in \mathbb{R}^d, \ t \geq 0 \ , \\ \nabla b\cdot\nabla\varphi - \partial_z\varphi = 0 & \text{at } z = -h_0 + b(X), \ X \in \mathbb{R}^d, \ t \geq 0 \ . \end{cases} \tag{0.0.45}$$

Much of the study of nonlinear water waves is based on the use of perturbation expansions, which require the identification of small parameters. Our first task is to identify the relevant small parameters, which we do by way of non-dimensionalizing the governing equations. This requires the introduction of various orders of magnitude linked to the physical regime under consideration. More precisely, let us introduce the following quantities: (see Figure below)

a is a typical amplitude of the waves;

λ is the wave-length of the waves;

b_0 is the order of amplitude of the variations of the bottom topography;

h_0 is the reference depth.

We also introduce the following dimensionless parameters:

$$\varepsilon = \frac{a}{h_0}, \quad \mu = \frac{h_0^2}{\lambda^2}, \quad \beta = \frac{b_0}{h_0};$$

the parameter ε is often called nonlinearity parameter; while μ is the shallowness parameter. This last parameter μ represents the square of the ratio of reference depth to wavelength, and runs the entire range of values from 0 to ∞. The major distinction to be made is $\mu \ll 1$ (shallow-water). The interest of having introduced the dimensionless parameters ε, β and μ is that it is often possible to deduce from their values some insight on the behavior of the flow. More precisely, it is possible to derive some (much simpler) asymptotic models more amenable to numerical simulations and whose properties are more transparent. We now perform the classical

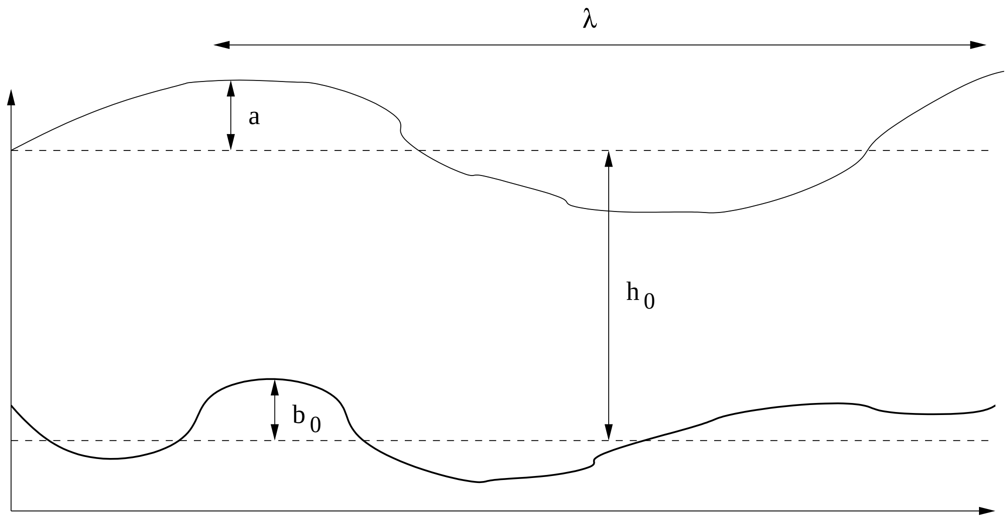

Figure 0.0.5: Physical regime

shallow water non-dimensionalization using the following relations:

$$\begin{aligned} &X = \lambda X', && z = h_0 z', \quad \zeta = a\zeta', \\ &\varphi = \frac{a}{h_0}\lambda\sqrt{gh_0}\varphi', && b = b_0 b', \quad t = \frac{\lambda}{\sqrt{gh_0}}t'; \end{aligned} \tag{0.0.46}$$

so, the equations of motion (0.0.45) then become (after dropping the primes for the sake of clarity):

$$\begin{cases} \mu\partial_x^2\varphi + \mu\partial_y^2\varphi + \partial_z^2\varphi = 0, & \text{at} \quad -1+\beta b < z < \varepsilon\zeta, \\ \partial_z\varphi - \mu\beta\nabla b\cdot\nabla\varphi = 0, & \text{at} \quad z = -1+\beta b, \\ \partial_t\zeta - \dfrac{1}{\mu}(-\mu\varepsilon\nabla\zeta\cdot\nabla\varphi + \partial_z\varphi) = 0, & \text{at} \quad z = \varepsilon\zeta, \\ \partial_t\varphi + \dfrac{1}{2}(\varepsilon|\nabla\varphi|^2 + \dfrac{\varepsilon}{\mu}(\partial_z\varphi)^2) + \zeta = 0, & \text{at} \quad z = \varepsilon\zeta, \end{cases} \tag{0.0.47}$$

The solutions of these equations are very difficult to describe. At this point, a classical method is to choose an asymptotic regime, in which we look for approximate models and hence for approximate solutions. In fact, many of the most famous equations of mathematical physics were historically obtained as formal asymptotic limits of the water-waves equations: the shallow-water equations, the Korteweg-de Vries (KdV)

and Kadomtsev-Petviashvili (KP) equations [47], the Boussinesq systems [13, 14], etc. Each of these asymptotic limits corresponds to a very specific physical regime whose range of validity is determined in terms of the characteristics of the flow (amplitude, wavelength, anisotropy, bottom topography, depth, ...). The derivation of these models goes back to the XIXth century, but the rigorous analysis of their relevance as approximate models for the water-waves equations only began three decades ago with the works of Ovsjannikov [62, 63], Craig [22], and Kano and Nishida [46, 44, 45] who first addressed the problem of justifying the formal asymptotics. For all the different asymptotic models, the problem can be formulated as follows:
1) do the water-waves equations have a solution on the time scale relevant for the asymptotic model?
2) does this model furnish a good approximation of the solution?
Answering the first question requires a large-time existence theorem for the water-waves equations, while the second one requires a rigorous derivation of the asymptotic models and a precise control of the approximation error.

Following the pioneer works for one-dimensional surfaces (1DH) of Ovsjannikov [63] and Nalimov [60] (see also Yosihara [79, 80]), Craig [22], and Kano and Nishida [44] provided a justifcation of the KdV and 1DH Boussinesq and shallow water approximations. However, the comprehension of the well-posedness theory for the water-waves equations hindered the perspective of justifying the other asymptotic regimes until the breakthroughs of S. Wu ([74] and [75] respectively for the 1DH and 2DH case, in infinite depth, and without restrictive assumptions). Since then, the literature on free surface Euler equations has been very active: the case of finite depth was proved in [50], and in the related case of the study of the free surface of a liquid in vacuum with zero gravity, Lindblad [57, 58]. More recently Coutand and Shkoller [21] and Shatah and Zeng [68] managed to remove the irrotationality condition and/or took into account surface tension effects (see also [5] for 1DH water-waves with surface tension). In the Eulerian framework which is more relevant for our purposes, the well-posedness of these equations is finally demonstrated by Lannes [50] and, by Alvarez-Samaniego and Lannes [3] on long-time. See also the recent work by T. Alazard, N. Burq and C. Zuily [1].

In order to review the existing results of the rigorous justification of asymptotic models for water-waves, it is suitable to classify the different physical regimes using two dimensionless numbers: the amplitude parameter ε and the shallowness parameter μ.

Shallow-water, large amplitude ($\mu \ll 1, \varepsilon \sim 1$).

Formally, this regime leads at first order to the well-known "shallow-water equations" (or Saint-Venant) and at second order to the so-called "Green-Naghdi" model, The first rigorous justification of the shallow-water model goes back to Ovsjannikov [62, 63] and Kano and Nishida [44] who proved the convergence of the solutions of the shallow-water equations to solutions of the water-waves equations as $\mu \to 0$ in 1DH, and under some restrictive assumptions (small and analytic data). More recently, Y.A. Li [55] removed these assumptions and rigorously justified the shallow-water and Green-Naghdi equations, in 1DH for flat bottoms, a rigorous work on a 2DH asymptotic model is due to the work by T. Iguchi [36] in which he justified the 2DH shallow-water equations, also allowing non flat bottoms. Finally a very recent rigorous justification of the Green-Naghdi equations over uneven bottoms is given by [3, 38, 39]

In the shallow-water scaling ($\mu \ll 1$), and without smallness assumption on ε one can derive the so-called Green-Naghdi equations (see [29, 53] for a derivation and [3] for a rigorous justification) also called Serre or fully nonlinear Boussinesq equations [54]. In nondimensionalized variables, denoting by $u(t,X)$ the vertically averaged horizontal component of the velocity at time t, the equations read

$$\begin{cases} \partial_t\zeta + \nabla\cdot(hu) = 0, \\ (h+\mu h\mathcal{T}[h,\varepsilon b])\partial_t u + h\nabla\zeta + \varepsilon h(u\cdot\nabla)u \\ \quad +\mu\varepsilon\big\{-\frac{1}{3}\nabla[(h^3((u\cdot\nabla)(\nabla\cdot u)-(\nabla\cdot u)^2)] + h\Re[h,\varepsilon b]u\big\} = 0, \end{cases} \tag{0.0.48}$$

where $h = 1+\varepsilon(\zeta - b)$ and

$$\mathcal{T}[h,\varepsilon b]W = -\frac{1}{3h}\nabla(h^3\nabla\cdot W) + \frac{\varepsilon}{2h}[\nabla(h^2\nabla b\cdot W) - h^2\nabla b\nabla\cdot W] + \varepsilon^2\nabla b\nabla b\cdot W,$$

while the purely topographical term $\Re[h,\varepsilon b]u$ is defined as:

$$\begin{aligned} \Re[h,\varepsilon b]u \;&=\; \frac{\varepsilon}{2h}[\nabla(h^2(u\cdot\nabla)^2 b) - h^2((u\cdot\nabla)(\nabla\cdot u)-(\nabla\cdot u)^2)\nabla b] \\ &\quad +\varepsilon^2((u\cdot\nabla)^2 b)\nabla b. \end{aligned}$$

This model is often used in coastal oceanography because it takes into account the dispersive effects neglected by the shallow-water and it is more nonlinear than the Boussinesq equations. A recent rigorous justification of the GN model was given by Li [55] in $1D$ and for flat bottoms, and by B. Alvarez-Samaniego and D. Lannes [3, 38, 39] in the general case.

Shallow water, small amplitude ($\mu \ll 1, \varepsilon \sim \mu$).

This regime (also called long-waves regime) leads to many mathematically interesting models due to the balance of nonlinear and dispersive effects: Boussinesq systems: since the first derivation by Boussinesq, many formally equivalent models (also named

after Boussinesq) have been derived. W. Craig [22] and Kano and Nishida [45] were the first to give a full justification of these models, in 1DH (and for fat bottoms and small data). Note, however, that the convergence result given in [45] is given on a time scale too short to capture the nonlinear and dispersive effects specific to the Boussinesq systems; in [22], the correct large time existence (and convergence) results for the water-waves equations are given. The proof, of such a large time well-posedness result for the water-waves equations, is the most delicate point in the justification process. Furthermore, it is the last step needed to fully justify the Boussinesq systems in 2DH, owing to [10] (flat bottoms) and [17] (general bottom topography), where the convergence property is proved assuming that the large-time well-posedness theorem holds.
The uncoupled models: at first order, the Boussinesq systems reduce to a simple wave equation and, in 1DH, the motion of the free surface can be described as the sum of two uncoupled counter-propagating waves, slightly modulated by a Korteweg-de Vries (KdV) equation. In 2DH and for weakly transverse waves, a similar phenomenon occurs, but with the Kadomtsev-Petviashvili (KP) equation replacing the KdV equation. Many papers addressed the problem of validating the KdV model (e.g. [22, 45, 10, 71, 67, 36]) and its justification.

The work presented in this thesis has three parts. We study in the first part the water-waves problem for uneven bottoms in a highly nonlinear regime where the small amplitude assumption of the Korteweg-de Vries (KdV) equation is enforced.

In the second part, we consider the $1D$ Green-Naghdi equations. We show that the solution of the Green-Naghdi equations can be constructed by a standard Picard iterative scheme so that there is no loss of regularity of the solution with respect to the initial condition. Our approach does not admit a straightforward generalization to the $2D$ case. Therefore, we derive a new system of the $2D$ Green-Naghdi equations (which is shown to be equialent to the usual Green-Naghdi equations). Then we show that the solution of this new Green-Naghdi equations can be constructed by a standard Picard iterative scheme.

Finally, in the third part we study numerically the KdV-top equation derived in [37]/chapter 1, and compare it with the Boussinesq equations over uneven bottom. We study also numerically the equations derived in the same reference [37]/chapter 1, in the case of stronger nonlinearities and related to the Camassa-Holm-like equation.

Part I : Variable depth KdV equations and generalizations to more nonlinear regimes

The chapter 1 deals with the water-waves problem and their approximate models.

More recently Alvarez-Samaniego and Lannes [3] rigorously justified the relevance of the main asymptotical models used in coastal oceanography, including: shallow-water equations, Boussinesq systems, Kadomtsev-Petviashvili (KP) approximation, Green-Naghdi equations (GN), Serre approximation, full-dispersion model and deep-water equations. Some of these models capture the existence of solitary water waves and the associated phenomenon of soliton manifestation [41]. The most prominent example is the Korteweg-de Vries (KdV) equation [49], that is integrable and relevant for the phenomenon of soliton manifestation. The KdV approximation originally derived over flat bottoms has been rigorously justified in [22, 67, 10, 36]. When the bottom is not flat, various generalizations of the KdV equation with non constant coefficients have been proposed [47, 70, 27, 59, 32, 78, 64, 31, 43, 66]. One of the aims of this chapter is to justify the derivation of this Korteweg-de Vries equation with topography (called KdV-top). Another development of models for water waves was initiated in order to gain insight into wave breaking, one of the most fundamental aspects of water-waves [28]. In 2008 Constantin and Lannes [20] rigorously justified the relevance of more nonlinear generalization of the KdV equations (linked to the Camassa-Holm equation [15] and the Degasperis-Procesi equations [26]) as models for the propogation of shallow water waves. They proved that these equations can be used to furnish approximations to the governing equations for water waves, and in their investigation they put earlier (formal) asymptotic procedures due to Johnson [42] on a firm and mathematically rigorous basis. However, all these results hold for flat bottoms only. The main goal of this chapter is to investigate the same scaling as in [20] and to include topographical effects. To this end, we derive a new variable coefficients class of equations which takes into account these effects and generalizes the CH like equations of Constantin-Lannes [20].

We consider in this part a particular physical regime. More precisely, let us introduce the following quantities: a, λ, h_0 and b_0 are defined as above and λ/α is the wave-length of the bottom variations (see Figure below). The interset of this physical regime is that many variable depth KdV-top equations existing in the literature (e.g. [64, 59]) can be recovered here as particular cases.

So, the equations of motion (0.0.45) then become (for $d = 1$)

$$\begin{cases} \mu\partial_x^2\varphi + \partial_z^2\varphi = 0, & \text{at} \quad -1+\beta b^{(\alpha)} < z < \varepsilon\zeta, \\ \partial_z\varphi - \mu\beta\alpha\partial_x b^{(\alpha)}\partial_x\varphi = 0 & \text{at} \quad z = -1+\beta b^{(\alpha)}, \\ \partial_t\zeta - \dfrac{1}{\mu}(-\mu\varepsilon\partial_x\zeta\partial_x\varphi + \partial_z\varphi) = 0, & \text{at} \quad z = \varepsilon\zeta, \\ \partial_t\varphi + \dfrac{1}{2}(\varepsilon(\partial_x\varphi)^2 + \dfrac{\varepsilon}{\mu}(\partial_z\varphi)^2) + \zeta = 0 & \text{at} \quad z = \varepsilon\zeta, \end{cases} \tag{0.0.49}$$

where $b^{(\alpha)}(x) = b(\alpha x)$. Making assumptions on the size of ε, β, α, and μ one is

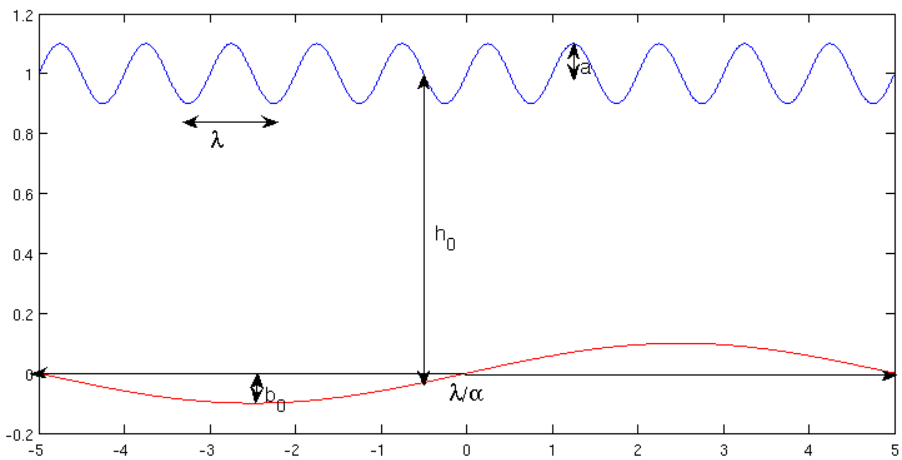

Figure 0.0.6: Physical regime

led to derive the asymptotic GN model from (0.0.49). For one-dimensional surfaces and over uneven bottoms these equations couple the free surface elevation ζ to the vertically averaged horizontal component of the velocity,

$$u(t,x) = \frac{1}{1+\varepsilon\zeta-\beta b^{(\alpha)}}\int_{-1+\beta b^{(\alpha)}}^{\varepsilon\zeta}\partial_x\varphi(t,x,z)dz \tag{0.0.50}$$

and can be written as:

$$\begin{cases} \partial_t\zeta + \partial_x(hu) = 0, \\ (1+\frac{\mu}{h}\mathcal{T}[h,\beta b^{(\alpha)}])\partial_t u + \partial_x\zeta + \varepsilon u\partial_x u \\ \;+\mu\varepsilon\big\{-\frac{1}{3h}\partial_x(h^3(u\partial_x^2 u) - (\partial_x u)^2) + \Im[h,\beta b^{(\alpha)}]u\big\} = 0 \end{cases} \tag{0.0.51}$$

where $h = 1+\varepsilon\zeta-\beta b^{(\alpha)}$ and

$$\mathcal{T}[h,\beta b^{(\alpha)}]W = -\frac{1}{3}\partial_x(h^3\partial_x W) + \frac{\beta}{2}\partial_x(h^2\partial_x b^{(\alpha)})W + \beta^2 h(\partial_x b^{(\alpha)})^2 W,$$

while the purely topographical term $\Im[h,\beta b^{(\alpha)}]u$ is defined as:

$$\begin{aligned} \Im[h,\beta b^{(\alpha)}]u &= \frac{\beta}{2h}[\partial_x(h^2(u\partial_x)^2 b^{(\alpha)}) - h^2((u\partial_x^2 u) - (\partial_x u)^2)\partial_x b^{(\alpha)}] \\ &\quad +\beta^2((u\partial_x)^2 b^{(\alpha)})\partial_x b^{(\alpha)}. \end{aligned}$$

For higher values of ε, the nonlinear effects are stronger; in the regime

$$\mu \ll 1, \quad \varepsilon = O(\sqrt{\mu}), \tag{0.0.52}$$

the KdV [49] and BBM equations [8] should be replaced by the following family (see [20, 42]):

$$u_t + u_x + \frac{3}{2}\varepsilon u u_x + \mu(A u_{xxx} + B u_{xxt}) = \varepsilon\mu(E u u_{xxx} + F u_x u_{xx}), \tag{0.0.53}$$

we show that the correct generalization of the equation (0.0.53) under the scaling (0.0.52) and with the following conditions on the topographical variations:

$$\beta\alpha = O(\mu), \quad \beta\alpha^{3/2} = O(\mu^2) \quad \beta\alpha\varepsilon = O(\mu^2), \tag{0.0.54}$$

is given by:

$$\begin{aligned} u_t + c u_x + \frac{3}{2}c_x u + \frac{3}{2}\varepsilon u u_x + \mu(\tilde{A} u_{xxx} + B u_{xxt}) \\ = \varepsilon\mu \tilde{E} u u_{xxx} + \varepsilon\mu\Big(\partial_x(\frac{\tilde{F}}{2}u)u_{xx} + u_x\partial_x^2(\frac{\tilde{F}}{2}u)\Big), \end{aligned} \tag{0.0.55}$$

where $c = \sqrt{1-\beta b^{(\alpha)}}$ and $\tilde{A}$, $\tilde{E}$, $\tilde{F}$ differ from the coefficients A, E, F in (0.0.53) because of topographic effects:

$$\begin{aligned} \tilde{A} &= Ac^5 - Bc^5 + Bc, \\ \tilde{E} &= Ec^4 - \frac{3}{2}Bc^4 + \frac{3}{2}B, \\ \tilde{F} &= Fc^4 - \frac{9}{2}Bc^4 + \frac{9}{2}B. \end{aligned}$$

The same kind of result is given in the (more restrictive) KdV scaling in order to recover the many variable depth KdV equations formally derived by oceanographers. In this section of this chapter, attention is given to the regime of slow variations of the bottom topography under the long-wave scaling $\varepsilon = O(\mu)$. We give here a rigorous justification in the meaning of consistency of the variable-depth extensions of the KdV equation (called KdV-top) originally derived in [47, 70, 27]. We study also the well posedness of the all models derived in this chapter. Two different approaches are used, depending on the coefficient B in (0.0.55): one deals with the case $B < 0$ (in that case, further investigation on the breaking of waves can be performed), and other treats the case $B = 0$.

Part II : The Green-Naghdi equations

We consider in chapter 2 the $1D$ Green-Naghdi equations that are commonly used in coastal oceanography to describe the propagation of large amplitude surface waves. A recent rigorous justification of the GN model was given by Li [55] in $1D$ and for flat bottoms, and by B. Alvarez-Samaniego and D. Lannes [3] in the general case. This

latter reference relies on well-posedness results for these equations given in [4] and based on general well-posedness results for evolution equations using a Nash-Moser scheme. The result of [4] covers both the case of $1D$ and $2D$ surfaces, and allows for non flat bottoms. The reason why a Nash-Moser scheme is used there is because the estimates on the linearized equations exhibit losses of derivatives. However, in the $1D$ case with flat bottoms, such losses do not occur and it is possible to construct a solution with a standard Picard iterative scheme as in [55]. Our goal here is to show that it is also possible to use such a simple scheme in the $1D$ case with non flat bottoms, thanks to a careful analysis of the linearized equations.
For one dimensional surfaces, the Green-Naghdi equations (0.0.48) can be simplified, after some computations, into: ($\beta = \varepsilon$)

$$\begin{cases} \partial_t \zeta + \partial_x(hu) = 0, \\ (h + \mu h \mathcal{T}[h, \varepsilon b])[\partial_t u + \varepsilon u \partial_x u] + h\partial_x \zeta + \varepsilon \mu h Q[h, \varepsilon b](u) = 0 \end{cases} \tag{0.0.56}$$

where $h = 1 + \varepsilon(\zeta - b)$ and

$$\mathcal{T}[h, \varepsilon b]w = -\frac{1}{3h}\partial_x(h^3 w_x) + \frac{\varepsilon}{2h}[\partial_x(h^2 b_x w) - h^2 b_x w_x] + \varepsilon^2 b_x^2 w,$$

$$Q[h, \varepsilon b](w) = \frac{2}{3h}\partial_x(h^3 w_x^2) + \varepsilon h w_x^2 b_x + \varepsilon \frac{1}{2h}\partial_x(h^2 w^2 b_{xx}) + \varepsilon^2 w^2 b_{xx} b_x.$$

We define now the X^s spaces, which are the energy spaces for this problem.

Definition 0.0.3. *For all $s \geq 0$ and $T > 0$, we denote by X^s the vector space $H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R})$ endowed with the norm*

$$\forall\, U = (\zeta, u) \in X^s, \quad |U|^2_{X^s} := |\zeta|^2_{H^s} + |u|^2_{H^s} + \mu|\partial_x u|^2_{H^s},$$

while X^s_T stands for $C([0, \frac{T}{\varepsilon}]; X^s)$ endowed with its canonical norm.

Theorem 0.0.3. *Let $b \in C_b^\infty(\mathbb{R})$, $t_0 > 1/2$, $s \geq t_0 + 1$. Let also the initial condition $U_0 = (\zeta_0, u_0)^T \in X^s$, and satisfy*

$$\exists\, h_{min} > 0, \quad \inf_{x \in \mathbb{R}} h \geq h_{min}, \quad h = 1 + \varepsilon(\zeta - b). \tag{0.0.57}$$

Then there exists a maximal $T_{max} > 0$, uniformly bounded from below with respect to $\varepsilon, \mu \in (0,1)$, such that the Green-Naghdi equations (0.0.56) admit a unique solution $U = (\zeta, u)^T \in X^s_{T_{max}}$ with the initial condition $(\zeta_0, u_0)^T$ and preserving the nonvanishing depth condition (0.0.57) for any $t \in [0, \frac{T_{max}}{\varepsilon})$. In particular if $T_{max} < \infty$ one has

$$|U(t, \cdot)|_{X^s} \longrightarrow \infty \quad as \quad t \longrightarrow \frac{T_{max}}{\varepsilon},$$

or

$$\inf_{\mathbb{R}} h(t,\cdot) = \inf_{\mathbb{R}} 1 + \varepsilon(\zeta(t,\cdot) - b(\cdot)) \longrightarrow 0 \quad as \quad t \longrightarrow \frac{T_{max}}{\varepsilon}.$$

Morever, the following conservation of energy property holds

$$\partial_t \Big(|\zeta|_2^2 + (hu, u) + \mu(h\mathcal{T}u, u) \Big) = 0,$$

where $\mathcal{T} = \mathcal{T}[h, \varepsilon b]$.

The chapter 3 is devoted to the derivation of a variant of the $2D$ Green-Naghdi equations that model the propagation of two-directional, nonlinear dispersive waves in shallow water. This new model has the same accuracy as the standard $2D$ Green-Naghdi equations. Its mathematical interest is that it allows a control of the rotational part of the (vertically averaged) horizontal velocity, which is not the case for the usual Green-Naghdi equations. Using this property, we show that the solution of these new equations can be constructed by a standard Picard iterative scheme so that there is no loss of regularity of the solution with respect to the initial condition. Finally, we prove that the new Green-Naghdi equations conserves the almost irrotationality of the vertically averaged horizontal component of the velocity.
For two-dimensional surfaces and over uneven bottoms these equations couple the free surface elevation ζ to the vertically averaged horizontal component of the velocity,

$$v(t,X) = \frac{1}{1+\varepsilon\zeta - \beta b} \int_{-1+\beta b}^{\varepsilon\zeta} \nabla\varphi(t,X,z)dz, \tag{0.0.58}$$

and can be written as:

$$\begin{cases} \partial_t\zeta + \nabla\cdot(hv) = 0, \\ \big(h + \mu\mathcal{T}[h,\beta b]\big)\partial_t v + h\nabla\zeta + \varepsilon\big(h + \mu\mathcal{T}[h,\beta b]\big)(v\cdot\nabla)v \\ \quad + \mu\varepsilon\big\{\frac{2}{3}\nabla[(h^3(\partial_1 v\cdot\partial_2 v^{\perp} + (\nabla\cdot v)^2)] + \Re[h,\beta b](v)\big\} = 0, \end{cases} \tag{0.0.59}$$

where $h = 1 + \varepsilon\zeta - \beta b$, $v = (V_1, V_2)^T$ and

$$\begin{aligned} \mathcal{T}[h,\varepsilon b]W &= -\frac{1}{3}\nabla(h^3\nabla\cdot W) + \frac{\beta}{2}[\nabla(h^2\nabla b\cdot W) - h^2\nabla b\nabla\cdot W] \\ &\quad + \beta^2 h\nabla b\nabla b\cdot W, \end{aligned} \tag{0.0.60}$$

while the purely topographic l term $\Re[h,\beta b](v)$ is defined as:

$$\begin{aligned} \Re[h,\beta b](v) &= \frac{\beta}{2}\nabla(h^2(V_1^2\partial_1^2 b + 2V_1V_2\partial_1\partial_2 b + V_2^2\partial_2^2 b)) \\ &\quad + \beta h^2(\partial_1 v\cdot\partial_2 v^{\perp} + (\nabla\cdot v)^2)\nabla b \\ &\quad + \beta^2 h(V_1^2\partial_1^2 b + 2V_1V_2\partial_1\partial_2 b + V_2^2\partial_2^2 b)\nabla b, \end{aligned} \tag{0.0.61}$$

where $v^\perp = (-V_2, V_1)^T$, .
Remarking that for instance, the term $\partial_1 v \cdot \partial_2 v^\perp$ is not controled by the energy norm $|\cdot|_{Y^s}$ naturally associated to (0.0.59) (see [4]),

$$|(\zeta, v)|_{Y^s}^2 = |\zeta|_{H^s}^2 + |v|_{H^s}^2 + \mu |\nabla \cdot v|_{H^s}^2.$$

This is the motivation for the present derivation of a new variant of the $2D$ Green-Naghdi equations (0.0.59). This variant has the same accuracy as the standard $2D$ Green-Naghdi equations (0.0.59) and can be written under the form:

$$\begin{cases} \partial_t \zeta + \nabla \cdot (hv) = 0, \\ \big(h + \mu\big(\mathcal{T}[h, \beta b] - \nabla^\perp(\mathrm{curl}\ )\big)\big)\partial_t v + h\nabla\zeta \\ +\, \varepsilon\big(h + \mu\big(\mathcal{T}[h, \beta b] - \nabla^\perp(\mathrm{curl}\ )\big)\big)(v \cdot \nabla)v \\ +\, \mu\varepsilon\big\{\frac{2}{3}\nabla[h^3(\partial_1 v \cdot \partial_2 v^\perp + (\nabla \cdot v)^2)] + \Re[h, \beta b](v)\big\} = 0, \end{cases} \tag{0.0.62}$$

where $\nabla^\perp = (-\partial_2, \partial_1)^T$, $\mathrm{curl}\, v = \partial_1 V_2 - \partial_2 V_1$, and the linear operators $\mathcal{T}[h, \varepsilon b]$ and $\Re[h, \beta b]$ being defined in (0.0.60) and (0.0.61).
The reason why the new terms (involving $\nabla^\perp(\mathrm{curl}\,)$) do not affect the precision of the model is because the solutions to (0.0.59) are nearly irrotational (in the sense that $\mathrm{curl}\, v$ is small). This property is of course satisfied also by our new model (0.0.62). The presence of these new term allow the definition of a new energy norm that controls also the rotational part of v. Consequently, we show that it is possible to use a standard Picard iterative scheme to prove the well-posedness of (0.0.62), so that there is no loss of regularity of the solution with respect to the initial condition.

We define now the X^s spaces, which are the energy spaces for this *new* GN problem .

Definition 0.0.4. *For all $s \geq 0$ and $T > 0$, we denote by X^s the vector space $H^s(\mathbb{R}^2) \times (H^{s+1}(\mathbb{R}^2))^2$ endowed with the norm*

$$\forall\, U = (\zeta, v) \in X^s, \quad |U|_{X^s}^2 := |\zeta|_{H^s}^2 + |v|_{(H^s)^2}^2 + \mu|\nabla \cdot v|_{H^s}^2 + \mu|curl\ v|_{H^s}^2,$$

while X_T^s stands for $C([0, \frac{T}{\varepsilon}]; X^s)$ endowed with its canonical norm.

Theorem 0.0.4. *Let $b \in C_b^\infty(\mathbb{R}^2)$, $t_0 > 1$, $s \geq t_0 + 1$. Let also the initial condition $U_0 = (\zeta_0, v_0^T)^T \in X^s$ and satisfy*

$$\exists\, h_{min} > 0, \quad \inf_{X \in \mathbb{R}^2} h \geq h_{min}, \quad h = 1 + \varepsilon(\zeta - b). \tag{0.0.63}$$

Then there exists a maximal $T_{max} > 0$, uniformly bounded from below with respect to $\varepsilon, \mu \in (0,1)$, such that the new Green-Naghdi equations (0.0.62) admit a unique solution $U = (\zeta, v^T)^T \in X^s_{T_{max}}$ with the initial condition $(\zeta_0, v_0^T)^T$ and preserving the nonvanishing depth condition (0.0.63) for any $t \in [0, \frac{T_{max}}{\varepsilon})$. In particular if $T_{max} < \infty$ one has

$$|U(t,\cdot)|_{X^s} \longrightarrow \infty \quad as \quad t \longrightarrow \frac{T_{max}}{\varepsilon},$$

or

$$\inf_{\mathbb{R}^2} h(t,\cdot) = \inf_{\mathbb{R}^2} 1 + \varepsilon(\zeta(t,\cdot) - b(\cdot)) \longrightarrow 0 \quad as \quad t \longrightarrow \frac{T_{max}}{\varepsilon}.$$

Part III : Numerical simulation

The chapter 4 (this work is done in collaboration with Marc Duruflé) is devoted to study numerically the variable depth KdV and Camassa-Holm-like equations derived in [37]/chapter 1. The Korteweg-de Vries (KdV) equation originally derived over flat bottoms [49] is an approximation of the Boussinesq equations, and this relation has been rigorously justified in [22, 67, 10, 36]. When the bottom is uneven, various generalizations of the KdV equation with non constant coefficients (called KdV-top) have been proposed [47, 70, 27, 59, 32, 78, 64, 31, 43, 66], and rigorously justified in [37]. One of the aims of this chapter is to study numerically these KdV-top equations, and to compare them with the Boussinesq equations over uneven bottom. The KdV equation on flat bottom can be numerically solved by using finite difference schemes [65, 81], or discontinuous Galerkin schemes [76]. It is treated with finite differences in [18] by using a Crank-Nicolson relaxation method in time introduced by Besse-Bruneau in [6] and justified by Besse in [7]. Our finite-difference scheme is inspired from these earlier works. We propose a modification so that the numerical scheme conserves a discrete energy for the fully discrete scheme (in space and time). We also compare this approach with the discontinuous Galerkin method of [76]. The generalization of the KdV-top equation to more nonlinear regimes (related to Camassa-Holm [15] and Degasperis-Procesi [26] equations) contains higher order nonlinear dispersive/nonlocal balances not present in the KdV and BBM equations. In 2008 Constantin and Lannes [20], rigorously justified these generalizations of the KdV equation in the case of flat bottoms . They proved that these equations can be used to furnish approximations to the governing equations for water waves. These Camassa-Holm (CH) equations on flat bottom can be numerically studied by using finite difference schemes [34, 35, 20, 25], or discontinous Galerkin schemes [77]. In 2009 [37], we investigated the case of variable bottoms in the same scaling as in [20]. We derived a new variable coefficients class of equations which takes into account topographic effects and generalizes the CH-like equations of Constantin-Lannes [20]. In the present chapter, we find many finite difference schemes for these new models, so that the numerical scheme conserves a discrete energy for the fully discrete scheme (in space and time). We also compare this approach with the discontinuous Galerkin method [77].

Publications

- *Variable depth KdV equations and generalizations to more nonlinear regimes.* Livre des communications Smai Canum 2009 P: 61.
- *Variable depth KdV equations and generalizations to more nonlinear regimes.* To appear in Esaim: M2AN. (See chapter 1)
- *Large Time existence For 1D Green-Naghdi equations.* Submitted to Nonlinear Analysis. (See chapter 2)
- *Derivation and analysis of A NEW 2D Green-Naghdi system.* Submitted to Nonlinearity. (See chapter 3)
- *A numerical study of variable depth KdV equations and generalizations of Camassa-Holm-like equations.* (With M. DURUFlÉ). To be submitted. (See chapter 4).

PARTIE I

Variable depth KdV equations and generalizations to more nonlinear regimes

Chapitre 1

Derivation and analysis of the variable depth KdV and Camassa-Holm-like equations

We study in this chapter the water waves problem for uneven bottoms in a highly nonlinear regime where the small amplitude assumption of the Korteweg-de Vries (KdV) equation is enforced. It is known that, for such regimes, a generalization of the KdV equation (somehow linked to the Camassa-Holm equation) can be derived and justified [20] when the bottom is flat. We generalize here this result with a new class of equations taking into account variable bottom topographies. Of course, many variable depth KdV equations existing in the literature are recovered as particular cases. Various regimes for the topography regimes are investigated and we prove consistency of these models, as well as a full justification for some of them. We also study the problem of wave breaking for our new variable depth and highly nonlinear generalizations of the KdV equations.

1.1 Introduction

1.1.1 General Setting

This chapter deals with the water waves problem for uneven bottoms, which consists in studying the motion of the free surface and the evolution of the velocity field of a layer of fluid under the following assumptions: the fluid is ideal, incompressible, irrotationnal, and under the only influence of gravity. Earlier works have set a good theoretical background for this problem. Its well-posedness has been discussed among others by Nalimov [60], Yasihara [79], Craig [22], Wu [74], [75] and Lannes [50]. Nevertheless, the solutions of these equations are very difficult to describe, because of the complexity of these equations. At this point, a classical method is to choose an asymptotic regime, in which we look for approximate models and hence for approximate solutions. More recently Alvarez-Samaniego and Lannes [3] rigorously justified the relevance of the main asymptotical models used in coastal oceanography, including: shallow-water equations, Boussinesq systems, Kadomtsev-Petviashvili (KP) approximation, Green-Naghdi equations (GN), Serre approximation, full-dispersion model and deep-water equations. Some of these models capture the existence of solitary water waves and the associated phenomenon of soliton manifestation [41]. The most prominent example is the Korteweg-de Vries (KdV) equation [49], that is integrable and relevant for the phenomenon of soliton manifestation. The KdV approximation originally derived over flat bottoms has been rigorously justified in [22, 67, 10, 36]. When the bottom is not flat, various generalizations of the KdV equation with non constant coefficients have been proposed [47, 70, 27, 59, 32, 78, 64, 31, 43, 66]. One of the aims of this chapter is to justify the derivation of this Korteweg-de Vries equation with topography (called KdV-top). Another development of models for water waves was initiated in order to gain insight into wave breaking, one of the most fundamental aspects of water waves [28]. In 2008 Constantin and Lannes [20] rigorously justified the relevance of more nonlinear generalization of the KdV equations (linked to the Camassa-Holm equation [15] and the Degasperis-Procesi equations [26]) as models for the propogation of shallow water waves. They proved that these equations can be used to furnish approximations to the governing equations for water waves, and in their investigation they put earlier (formal) asymptotic procedures due to Johnson [42] on a firm and mathematically rigorous basis. However, all these results hold for flat bottoms only. The main goal of this article is to investigate the same scaling as in [20] and to include topographical effects. To this end, we derive a new variable coefficients class of equations which takes into account these effects and generalizes the CH like equations of Constantin-Lannes [20]. The presence of the topography terms induce secular growth effects which do not always allow a full justification of the model. We however give some consistency results for all the models derived here, and then show that under some additional assumptions on the topography variations, the secular terms can be controled and a full justification given.

1.1.2 Presentation of the results

Parameterizing the free surface by $z = \zeta(t,x)$ (with $x \in \mathbb{R}$) and the bottom by $z = -h_0 + b(x)$ (with $h_0 > 0$ constant), one can use the incompressibility and irrotationality conditions to write the water waves equations under Bernoulli's formulation, in terms of a velocity potential φ associated with the flow, and where $\varphi(t,.)$ is defined on $\Omega_t = \{(x,z), -h_0 + b(x) < z < \zeta(t,x)\}$ (i.e. the velocity field is given by

$v = \nabla_{x,z}\varphi)$:

$$\begin{cases} \partial_x^2\varphi + \partial_z^2\varphi = 0, & \text{in} \quad \Omega_t, \\ \partial_n\varphi = 0, & \text{at} \quad z = -h_0 + b, \\ \partial_t\zeta + \partial_x\zeta\partial_x\varphi = \partial_z\varphi, & \text{at} \quad z = \zeta, \\ \partial_t\varphi + \frac{1}{2}((\partial_x\varphi)^2 + (\partial_z\varphi)^2) + g\zeta = 0 & \text{at} \quad z = \zeta, \end{cases} \tag{1.1.1}$$

where g is the gravitational acceleration, $\partial_n\varphi$ is the outward normal derivative at the boundary of the fluid domain. The qualitative study of the water waves equations is made easier by the introduction of dimensionless variables and unknowns. This requires the introduction of various orders of magnitude linked to the physical regime under consideration. More precisely, let us introduce the following quantities: a is a typical amplitude of the waves; λ is the wave-length of the waves; b_0 is the order of amplitude of the variations of the bottom topography; λ/α is the wave-length of the bottom variations; h_0 is the reference depth. We also introduce the following dimensionless parameters:

$$\varepsilon = \frac{a}{h_0}, \quad \mu = \frac{h_0^2}{\lambda^2}, \quad \beta = \frac{b_0}{h_0};$$

the parameter ε is often called nonlinearity parameter; while μ is the shallowness parameter. We now perform the classical shallow water non-dimensionalization using the following relations:

$$\begin{aligned} x &= \lambda x', & z &= h_0 z', \quad \zeta = a\zeta', \\ \varphi &= \frac{a}{h_0}\lambda\sqrt{gh_0}\varphi', & b &= b_0 b', \quad t = \frac{\lambda}{\sqrt{gh_0}}t'; \end{aligned} \tag{1.1.2}$$

so, the equations of motion (1.1.1) then become (after dropping the primes for the sake of clarity):

$$\begin{cases} \mu\partial_x^2\varphi + \partial_z^2\varphi = 0, & \text{at} \quad -1 + \beta b^{(\alpha)} < z < \varepsilon\zeta, \\ \partial_z\varphi - \mu\beta\alpha\partial_x b^{(\alpha)}\partial_x\varphi = 0 & \text{at} \quad z = -1 + \beta b^{(\alpha)}, \\ \partial_t\zeta - \frac{1}{\mu}(-\mu\varepsilon\partial_x\zeta\partial_x\varphi + \partial_z\varphi) = 0, & \text{at} \quad z = \varepsilon\zeta, \\ \partial_t\varphi + \frac{1}{2}(\varepsilon(\partial_x\varphi)^2 + \frac{\varepsilon}{\mu}(\partial_z\varphi)^2) + \zeta = 0 & \text{at} \quad z = \varepsilon\zeta, \end{cases} \tag{1.1.3}$$

where $b^{(\alpha)}(x) = b(\alpha x)$.

Making assumptions on the size of ε, β, α, and μ one is led to derive (simpler) asymptotic models from (1.1.3). In the shallow-water scaling ($\mu \ll 1$), one can derive (when no smallness assumption is made on ε, β and α) the so-called Green-Naghdi equations (see [29, 53] for a derivation and [3] for a rigorous justification). For one-dimensional surfaces and over uneven bottoms these equations couple the free surface elevation ζ to the vertically averaged horizontal component of the velocity,

$$u(t,x) = \frac{1}{1+\varepsilon\zeta-\beta b^{(\alpha)}}\int_{-1+\beta b^{(\alpha)}}^{\varepsilon\zeta} \partial_x\varphi(t,x,z)dz \tag{1.1.4}$$

and can be written as:

$$\begin{cases} \partial_t\zeta + \partial_x(hu) = 0, \\ (1+\frac{\mu}{h}\mathcal{T}[h,\beta b^{(\alpha)}])\partial_t u + \partial_x\zeta + \varepsilon u\partial_x u \\ +\mu\varepsilon\{-\frac{1}{3h}\partial_x(h^3(u\partial_x^2 u)-(\partial_x u)^2) + \Im[h,\beta b^{(\alpha)}]u\} = 0 \end{cases} \tag{1.1.5}$$

where $h = 1+\varepsilon\zeta-\beta b^{(\alpha)}$ and

$$\mathcal{T}[h,\beta b^{(\alpha)}]W = -\frac{1}{3}\partial_x(h^3\partial_x W) + \frac{\beta}{2}\partial_x(h^2\partial_x b^{(\alpha)})W + \beta^2 h(\partial_x b^{(\alpha)})^2 W,$$

while the purely topographical term $\Im[h,\beta b^{(\alpha)}]u$ is defined as:

$$\begin{aligned} \Im[h,\beta b^{(\alpha)}]u &= \frac{\beta}{2h}[\partial_x(h^2(u\partial_x)^2 b^{(\alpha)}) - h^2((u\partial_x^2 u)-(\partial_x u)^2)\partial_x b^{(\alpha)}] \\ &\quad +\beta^2((u\partial_x)^2 b^{(\alpha)})\partial_x b^{(\alpha)}. \end{aligned}$$

If we make the additional assumption that $\varepsilon \ll 1$, $\beta \ll 1$ then the above system reduces at first order to a wave equation of speed ± 1 and any perturbation of the surface splits up into two components moving in opposite directions. A natural issue is therefore to describe more accurately the motion of these two "unidirectional" waves. In the so called long-wave regime

$$\mu \ll 1, \quad \varepsilon = O(\mu), \tag{1.1.6}$$

and for flat bottoms, Korteweg and de Vries [49] found that say, the right-going wave should satisfy the KdV equation:

$$u_t + u_x + \frac{3}{2}\varepsilon u u_x + \frac{\mu}{6}u_{xxx} = 0, \tag{1.1.7}$$

and ($\zeta = u + O(\varepsilon,\mu)$).
At leading order, this equation reduces to the expected transport equation at speed 1. It has been noticed by Benjamin, Bona, Mahoney [8] that the KdV equation belongs to a wider class of equations. For instance, the BBM equation first used by Peregrine [61], and sometimes also called the regularized long-wave equation,

provides an approximation of the exact water waves equations of the same accuracy as the KdV equation and can be written under the form[1]:

$$u_t + u_x + \frac{3}{2}\varepsilon u u_x + \mu(A u_{xxx} + B u_{xxt}) = 0 \quad \text{with} \quad A - B = \frac{1}{6}. \tag{1.1.8}$$

For higher values of ε, the nonlinear effects are stronger; in the regime

$$\mu \ll 1, \quad \varepsilon = O(\sqrt{\mu}), \tag{1.1.9}$$

the BBM equations (1.1.8) should be replaced by the following family (see [20, 42]):

$$u_t + u_x + \frac{3}{2}\varepsilon u u_x + \mu(A u_{xxx} + B u_{xxt}) = \varepsilon\mu(E u u_{xxx} + F u_x u_{xx}), \tag{1.1.10}$$

(with some conditions on A, B, E, and F) in order to keep the same $O(\mu^2)$ accuracy of the approximation. However, all these results only hold for flat bottoms; for the situation of an uneven bottom, various generalizations of the KdV equations with non constant coefficients have been proposed [47, 70, 27, 59, 32, 78, 64, 31, 43, 66]. We justify in this paper the derivation of the generalized KdV equation and also we show that the correct generalization of the equation (1.1.10) under the scaling (1.1.9) and with the following conditions on the topographical variations:

$$\beta\alpha = O(\mu), \quad \beta\alpha^{3/2} = O(\mu^2) \quad \beta\alpha\varepsilon = O(\mu^2), \tag{1.1.11}$$

is given by:

$$\begin{aligned} u_t + c u_x + \frac{3}{2}c_x u + \frac{3}{2}\varepsilon u u_x + \mu(\tilde{A} u_{xxx} + B u_{xxt}) \\ = \varepsilon\mu\tilde{E} u u_{xxx} + \varepsilon\mu\Big(\partial_x(\frac{\tilde{F}}{2}u)u_{xx} + u_x\partial_x^2(\frac{\tilde{F}}{2}u)\Big), \end{aligned} \tag{1.1.12}$$

where $c = \sqrt{1 - \beta b^{(\alpha)}}$ and $\tilde{A}$, $\tilde{E}$, $\tilde{F}$ differ from the coefficients A, E, F in (1.1.10) because of topographic effects:

$$\begin{aligned} \tilde{A} &= Ac^5 - Bc^5 + Bc, \\ \tilde{E} &= Ec^4 - \frac{3}{2}Bc^4 + \frac{3}{2}B, \\ \tilde{F} &= Fc^4 - \frac{9}{2}Bc^4 + \frac{9}{2}B. \end{aligned}$$

Notice that for an equation of the family (1.1.12) to be linearly well-posed it is necessary that $B \leq 0$. In Sec. 1.2, we derive asymptotical approximations of the

[1] The BBM equation corresponds to $A = 0$ in (1.1.8), and KdV to $B = 0$. The intermediate cases can be found in [12].

Green-Naghdi equations over non flat bottoms: equations on the velocity are given in Sect. 1.2.1 and equations on the surface elevation are obtained in Sect. 1.2.2; for these equations, L^∞-consistency results are given (see Definition 1.2.1). In Sect. 1.2.3, the same kind of result is given in the (more restrictive) KdV scaling in order to recover the many variable depth KdV equations formally derived by oceanographers. Section 1.3 is devoted to the study of the well posedness of the equations derived in Section 1.2. Two different approaches are used, depending on the coefficient B in (1.1.12): §1.3.1 deals with the case $B<0$ (in that case, further investigation on the breaking of waves can be performed, see §1.3.2) and §1.3.3 treats the case $B=0$. While secular growth effects prevent is from proving H^s-consistency (see Definition 1.4.1) for the models derived in Section 1.2, we show in Section 1.4 that such results hold if one makes stronger assumptions on the parameters. A full justification of the models can then be given (see Th. 1.4.2).

1.2 Unidirectional limit of the Green-Naghdi equations over uneven bottom in the CH and KdV scalings

We derive here asymptotical approximations of the Green-Naghdi equations over non flat bottoms in the scalings (1.1.11) and (1.1.9). We remark that the Green-Naghdi equations can then be simplified into
(denoting $h = 1 + \varepsilon\zeta - \beta b^{(\alpha)}$):

$$\left\{\begin{array}{l} \zeta_t + [hu]_x = 0 \\ u_t + \zeta_x + \varepsilon u u_x = \dfrac{\mu}{3h}[h^3(u_{xt} + \varepsilon u u_{xx} - \varepsilon u_x^2)]_x, \end{array}\right. \tag{1.2.1}$$

where $O(\mu^2)$ terms have been discarded.

We consider here parameters ε, β, α and μ linked by the relations

$$\varepsilon = O(\sqrt{\mu}),\ \beta\alpha = O(\varepsilon),\ \beta\alpha = O(\mu),\ \beta\alpha^{3/2} = O(\mu^2),\ \beta\alpha\varepsilon = O(\mu^2), \tag{1.2.2}$$

(note that in the case of flat bottoms, one can take $\beta = 0$, so that this set of relations reduce to $\varepsilon = O(\sqrt{\mu})$).
Equations for the velocity u are first derived in §1.2.1 and equations for the surface elevation ζ are obtained in §1.2.2. The considerations we make on the derivation of these equations are related to the approach initiated by Constantin and Lannes [20]. In addition, in §1.2.3 we recover and justify the KdV equation over a slowly varying depth (formally derived in [47, 70, 27]).

1.2.1 Equations on the velocity.

If we want to find an approximation at order $O(\mu^2)$ of the GN equations under the scalings (1.2.2), it is natural to look for u as a solution of (1.1.12) with variable coefficients $\widetilde{A}$, $\widetilde{B}$, $\widetilde{E}$, $\widetilde{F}$ to be determined. We prove in this section that one can associate to the solution of (1.1.12) a family of approximate solutions consistent with the Green-Naghdi equations (1.2.1) in the following sense:

Definition 1.2.1. *Let $\wp$ be a family of parameters $\theta = (\varepsilon, \beta, \alpha, \mu)$ satisfying (1.2.2). A family $(\zeta^\theta, u^\theta)_{\theta\in\wp}$ is L^∞-consistent on $[0, \frac{T}{\varepsilon}]$ with the GN equations (1.2.1), if for all $\theta \in \wp$ (and denoting $h^\theta = 1 + \varepsilon\zeta^\theta - \beta b^{(\alpha)}$),*

$$\begin{cases} \zeta_t^\theta + [h^\theta u^\theta]_x = \mu^2 r_1^\theta \\ u_t^\theta + \zeta_x^\theta + \varepsilon u^\theta u_x^\theta = \dfrac{\mu}{3h^\theta}[(h^\theta)^3(u_{xt}^\theta + \varepsilon u^\theta u_{xx}^\theta - \varepsilon(u_x^\theta)^2)]_x + \mu^2 r_2^\theta, \end{cases}$$

with $(r_1^\theta, r_2^\theta)_{\theta\in\wp}$ bounded in $L^\infty([0, \frac{T}{\varepsilon}] \times \mathbb{R})$.

Remark 1.2.1. *The notion of L^∞-consistency is weaker then the notion of H^s-consistency given in §1.4 (Definition 1.4.1) and does not allow a full justification of the asymptotic models. Since secular growth effects do not allow in general an H^s-consistency, we state here an L^∞-consistency result under very general assumptions on the topography parameters α and β. H^s-consistency and full justification of the models will then be achieved under additional assumptions in §1.4.*

The following proposition shows that there is a one parameter family of equations (1.1.12) L^∞-consistent with the GN equations (1.2.1). (For the sake of simplicity, here and throughout the rest of this chapter, we take an infinitely smooth bottom parameterized by the function b).

Proposition 1.2.1. *Let $b \in H^\infty(\mathbb{R})$ and $p \in \mathbb{R}$. Assume that*

$$A = p, \quad B = p - \frac{1}{6} \quad E = -\frac{3}{2}p - \frac{1}{6}, \quad F = -\frac{9}{2}p - \frac{23}{24}.$$

Then:

- *For any family $\wp$ of parameters satisfying (1.2.2),*
- *For all $s \geq 0$ large enough and $T > 0$,*
- *For any bounded family $(u^\theta)_{\theta\in\wp} \in C([0, \frac{T}{\varepsilon}], H^s(\mathbb{R}))$ solving (1.1.12),*

the familly $(\zeta^\theta, u^\theta)_{\theta\in\wp}$ with (omitting the index θ)

$$\zeta := cu + \frac{1}{2}\int_{-\infty}^{x} c_x u + \frac{\varepsilon}{4}u^2 + \frac{\mu}{6}c^4 u_{xt} - \varepsilon\mu c^4[\frac{1}{6}uu_{xx} + \frac{5}{48}u_x^2], \tag{1.2.3}$$

is L^∞-consistent on $[0, \frac{T}{\varepsilon}]$ with the GN equations (1.2.1).

Remark 1.2.2. *If we take $b = 0$ -i.e if we consider a flat bottom-, then one can recover the equation (7) of [20] and the equations (26a) and (26b) of [42] with $p = -\frac{1}{12}$ and $p = \frac{1}{6}$ respectively.*

Proof. For the sake of simplicity, we denote by $O(\mu)$ any family of functions $(f^\theta)_{\theta\in\wp}$ such that $\frac{1}{\mu}f^\theta$ remains bounded in $L^\infty([0, \frac{T}{\varepsilon}], H^r(\mathbb{R}))$ for all $\theta \in \wp$, (and for possibly different values of r). The same notation is also used for real numbers, e.g $\varepsilon = O(\mu)$, but this should not yield any confusion. We use the notation $O_{L^\infty}(\mu)$ if $\frac{1}{\mu}f^\theta$ remains bounded in $L^\infty([0, \frac{T}{\varepsilon}] \times \mathbb{R})$. Of course, similar notations are used for $O(\mu^2)$ etc. To simplify the text, we also omit the index θ and write u instead of u^θ.
Step 1. We begin the proof by the following Lemma where a new class of equations is deduced from (1.1.12). The coefficients A, B, E, F in this new class of equations are constants (as opposed to $\widetilde{A}$, $\widetilde{E}$ and $\widetilde{F}$ in (1.1.12) that are functions of x).

Lemma 1.2.1. *Under the assumptions of Proposition 1.2.1, there is a family $(R^\theta)_{\theta\in\wp}$ bounded in $L^\infty([0, \frac{T}{\varepsilon}], H^r(\mathbb{R}))$ (for some $r < s$) such that (omitting the index θ)*

$$\begin{aligned} &u_t + cu_x + \frac{3}{2}c_x u + \frac{3}{2}\varepsilon uu_x + \mu c^5 A u_{xxx} + \mu B\partial_x(c^4 u_{xt}) \\ &= \varepsilon\mu c^4 E uu_{xxx} + \varepsilon\mu\frac{1}{2}F(c^4u)_x u_{xx} + \varepsilon\mu\frac{1}{2}F u_x(c^4u)_{xx} + \mu^2 R. \end{aligned} \tag{1.2.4}$$

Proof. Remark that the relation $\alpha\beta = O(\mu)$ and the definitions of $\tilde{A}$, $\tilde{B}$, and $\tilde{E}$ in terms of A, B, E and F imply that

$$\begin{aligned} &\mu(-Bc^5 + Bc)u_{xxx} + \varepsilon\mu(-\frac{3}{2}Bc^4 + \frac{3}{2}B)uu_{xxx} \\ &\quad + \varepsilon\mu\partial_x\Big(\Big(\frac{-\frac{9}{2}Bc^4 + \frac{9}{2}B}{2}\Big)u\Big)u_{xx} + \varepsilon\mu u_x\partial_x^2\Big(\Big(\frac{-\frac{9}{2}Bc^4 + \frac{9}{2}B}{2}\Big)u\Big) \\ &\quad = -\mu B\partial_x(c(c^4-1)\partial_x u_x) - \frac{3}{2}\mu\varepsilon B\partial_x((c^4-1)\partial_x(uu_x)) + O(\mu^2) \\ &\quad = -\mu B\partial_x\Big[(c^4-1)\partial_x(cu_x + \frac{3}{2}\varepsilon uu_x)\Big] + O(\mu^2) \\ &\quad = \mu B\partial_x((c^4-1)u_{xt}) + O(\mu^2), \end{aligned}$$

the last line being a consequence of the identity $u_t = -(cu_x + \frac{3}{2}\varepsilon uu_x) + O(\mu)$ provided by (1.1.12) since we have

$$|c_x u|_{H^r} = \big| -\frac{1}{2c}\beta\alpha b_x^{(\alpha)} u\big|_{H^r} \leq \text{Cst }\alpha\beta\big|\partial_x b^{(\alpha)}\big|_{W^{[r]+1,\infty}}|u|_{H^r} = O(\beta\alpha) = O(\mu)$$

where $[r]$ is the largest integer smaller or equal to r. The equation (1.1.12) can thus be written under the form:

$$\begin{aligned} &u_t + cu_x + \frac{3}{2}c_x u + \frac{3}{2}\varepsilon uu_x + \mu c^5 A u_{xxx} + \mu B\partial_x(c^4 u_{xt}) \\ &= \varepsilon\mu c^4 E uu_{xxx} + \varepsilon\mu\frac{1}{2}F(c^4 u)_x u_{xx} + \varepsilon\mu\frac{1}{2}F u_x (c^4 u)_{xx} + O(\mu^2), \end{aligned}$$

which is exactly the result stated in the Lemma. □

If u satisfies (1.1.12) one also has

$$\begin{aligned} u_t + cu_x + \frac{3}{2}\varepsilon uu_x &= -\frac{3}{2}c_x u + O(\mu) \\ &= O(\mu), \end{aligned} \tag{1.2.5}$$

Differentiating (1.2.5) twice with respect to x, and using again the fact that $c_x f = O(\beta\alpha) = O(\mu)$ for all f smooth enough, one gets

$$cu_{xxx} = -u_{xxt} - \frac{3}{2}\varepsilon\partial_x^2(uu_x) + O(\mu).$$

It is then easy to deduce that

$$\begin{aligned} c^5 u_{xxx} &= -c^4 u_{xxt} - \frac{3}{2}c^4\varepsilon\partial_x^2(uu_x) + O(\mu) \\ &= -\partial_x(c^4 u_{xt}) - \frac{3}{2}\varepsilon c^4(uu_{xxx} + 3u_x u_{xx}) + O(\mu) \end{aligned}$$

so that we can replace the $c^5 u_{xxx}$ term of (1.2.4) by this expression. By using Lemma 1.2.1, one gets therefore the following equation where the linear term in u_{xxx} has been removed:

$$u_t + cu_x + \frac{3}{2}c_x u + \frac{3}{2}\varepsilon uu_x + \mu c^4 a u_{xxt} = \varepsilon\mu c^4[euu_{xx} + du_x^2]_x + O(\mu^2) \tag{1.2.6}$$

with $a = B - A$, $e = E + \frac{3}{2}A$, $d = \frac{1}{2}(F + 3A - E)$.

Step 2. We seek v such that if $\zeta := cu + \varepsilon v$ and u solves (1.1.12) then the second equation of (1.2.1) is satisfied up to a $O(\mu^2)$ term. This is equivalent to checking that

$$\begin{aligned} u_t+[cu+\varepsilon v]_x+\varepsilon uu_x &= \mu(1-\beta b^{(\alpha)})\varepsilon\partial_x(cu)u_{xt}+\frac{\mu}{3}((1-\beta b^{(\alpha)})+\varepsilon cu)^2u_{xxt}\\ &\quad+\frac{\varepsilon\mu}{3}(1-\beta b^{(\alpha)})^2(uu_{xx}-u_x^2)_x+O(\mu^2)\\ &= \frac{\mu}{3}c^4u_{xxt}+\varepsilon\mu\Big(c^3u_xu_{xt}+\frac{2}{3}c^3uu_{xxt}+\frac{c^4}{3}(uu_{xx}-u_x^2)_x\Big)\\ &\quad+O(\mu^2), \end{aligned}$$

where we used the relations $O(\varepsilon^2)=O(\mu)$, $O(\beta\alpha)=O(\mu)$ and the fact that $c^2=1-\beta b^{(\alpha)}$. This condition can be recast under the form

$$\begin{aligned} &\varepsilon v_x+[u_t+cu_x+\frac{3}{2}c_xu+\frac{3}{2}\varepsilon uu_x+\mu c^4au_{xxt}-\varepsilon\mu c^4[euu_{xx}+du_x^2]_x]\\ =\ &\frac{1}{2}c_xu+\frac{\varepsilon}{2}uu_x+\mu c^4(a+\frac{1}{3})u_{xxt}\\ &+\varepsilon\mu c^3\Big(u_xu_{xt}+\frac{2}{3}uu_{xxt}+c[(\frac{1}{3}-e)uu_{xx}-(\frac{1}{3}+d)u_x^2]_x\Big)+O(\mu^2). \end{aligned}$$

Since morever one gets from (1.2.5) that $u_{xt}=-cu_{xx}+O(\mu,\varepsilon)$ and $u_{xxt}=-cu_{xxx}+O(\mu,\varepsilon)$, one gets readily

$$\begin{aligned} &\varepsilon v_x+[u_t+cu_x+\frac{3}{2}c_xu+\frac{3}{2}\varepsilon uu_x+\mu c^4au_{xxt}-\varepsilon\mu[euu_{xx}+du_x^2]_x]\\ &=\frac{1}{2}c_xu+\frac{\varepsilon}{2}uu_x+\mu c^4(a+\frac{1}{3})u_{xxt}-\varepsilon\mu c^4[(e+\frac{1}{3})uu_{xx}+(d+\frac{1}{2})u_x^2]_x+O(\mu^2). \end{aligned}$$

From Step 1, we know that the term between brackets in the lhs of this equation is of order $O(\mu^2)$ so that the second equation of (1.2.1) is satisfied up to $O(\mu^2)$ terms if

$$\varepsilon v_x=\frac{1}{2}c_xu+\frac{\varepsilon}{2}uu_x+\mu c^4(a+\frac{1}{3})u_{xxt}-\varepsilon\mu c^4[(e+\frac{1}{3})uu_{xx}+(d+\frac{1}{2})u_x^2]_x+O(\mu^2). \quad (1.2.7)$$

At this point we need also the following lemma

Lemma 1.2.2. *With u and b as in the statement of Proposition 1.2.1, the mapping $(t,x)\longrightarrow\int_{-\infty}^x c_xu\,dx$ is well defined on $[0,\frac{T}{\varepsilon}]\times\mathbb{R}$.*
Morever one has that

$$\Big|\int_{-\infty}^x c_xu\,dx\Big|_{L^\infty([0,\frac{T}{\varepsilon}]\times\mathbb{R})}\leq \mathrm{Cst}\sqrt{\alpha}\beta|b_x|_2|u|_2.$$

Proof. We used here the Cauchy-Schwarz inequality and the definition $c^2=1-\beta b^{(\alpha)}$ to get

$$\int_{-\infty}^x c_xu\,dx\leq \mathrm{Cst}\ \alpha\beta|(b_x)^{(\alpha)}|_2|u|_2\leq \mathrm{Cst}\ \sqrt{\alpha}\beta|b_x|_2|u|_2<\infty.$$

It is then easy to conclude the proof of the lemma. $\square$

Thanks to this lemma there is a solution $v \in C([0, \frac{T}{\varepsilon}] \times \mathbb{R})$ to (1.2.7), namely

$$\varepsilon v = \frac{1}{2}\int_{-\infty}^{x} c_x u + \frac{\varepsilon}{4}u^2 + \mu c^4(a+\frac{1}{3})u_{xt} - \varepsilon\mu c^4[(e+\frac{1}{3})uu_{xx} + (d+\frac{1}{2})u_x^2]. \quad (1.2.8)$$

Step 3. We show here that it is possible to choose the coefficients A, B, E, F such that the first equation of (1.2.1) is also satisfied up to $O_{L^\infty}(\mu^2)$ terms. This is equivalent to checking that

$$[cu + \varepsilon v]_t + [(1 + \varepsilon(cu + \varepsilon v) - \beta b^{(\alpha)})u]_x = 0 \quad (1.2.9)$$

Remarking that the relations $O(\beta\alpha) = O(\mu)$, $O(\beta\alpha^{3/2}) = O(\mu^2)$, and $O(\beta\alpha\varepsilon) = O(\mu^2)$ imply that

$$\frac{1}{2}\int_{-\infty}^{x} c_x u_t = -\frac{1}{2}cc_x u + O_{L^\infty}(\mu^2),$$

one infers from (1.2.8) that

$$\begin{aligned}
\varepsilon v_t &= \frac{1}{2}\int_{-\infty}^{x} c_x u_t + \frac{\varepsilon}{2}uu_t + \mu c^4(a+\frac{1}{3})u_{xtt} - \varepsilon\mu c^4[(e+\frac{1}{3})uu_{xx} + (d+\frac{1}{2})u_x^2]_t \\
&= -\frac{1}{2}cc_x u - \frac{\varepsilon}{2}u(cu_x + \frac{3\varepsilon}{2}uu_x + \mu c^4 a u_{xxt}) - \mu c^4(a+\frac{1}{3})\partial_{xt}^2(cu_x + \varepsilon\frac{3}{2}uu_x) \\
&\quad + \varepsilon\mu c^5[(e+\frac{1}{3})uu_{xx} + (d+\frac{1}{2})u_x^2]_x + O(\mu^2) + O_{L^\infty}(\mu^2) \\
&= -\frac{1}{2}cc_x u - \varepsilon\frac{1}{2}cuu_x - \varepsilon^2\frac{3}{4}u^2u_x - \mu(a+\frac{1}{3})c^5u_{xxt} \\
&\quad + \varepsilon\mu c^5[(2a+e+\frac{5}{6})uu_{xx} + (\frac{5}{4}a+d+1)u_x^2]_x + O(\mu^2) + O_{L^\infty}(\mu^2).
\end{aligned}$$

Similarly, one gets

$$\varepsilon^2[vu]_x = \varepsilon^2\frac{3}{4}u^2u_x - \varepsilon\mu c^5(a+\frac{1}{3})[uu_{xx}]_x + O(\mu^2) + O_{L^\infty}(\mu^2),$$

so (1.2.9) is equivalent to

$$cu_t + \varepsilon v_t + c^2u_x + 2cc_x u + 2\varepsilon cuu_x + \varepsilon^2[vu]_x = O_{L^\infty}(\mu^2).$$

Multiplying by $\frac{1}{c}$, we get

$$\begin{aligned}
& u_t + cu_x + \frac{3}{2}c_x u + \varepsilon\frac{3}{2}uu_x - \mu c^4(a+\frac{1}{3})u_{xxt} \\
= \;& \varepsilon\mu c^4[-(e+a+\frac{1}{2})uu_{xx} - (\frac{5}{4}a+d+1)u_x^2]_x + O_{L^\infty}(\mu^2).
\end{aligned}$$

Equating the coefficients of this equation with those of (1.2.6) shows that the first equation of (1.2.1) is also satisfied at order $O_{L^\infty}(\mu^2)$ if the following relations hold:

$$a = -\frac{1}{6}, \quad e = -\frac{1}{6}, \quad d = -\frac{19}{48},$$

and the conditions given in the statement of the proposition on A, B, E, and F follows from the expressions of a, e and d given after (1.2.6). $\square$

1.2.2 Equations on the surface elevation.

Proceeding exactly as in the proof of Proposition 1.2.1, one can prove that the family of equations of the surface elevation

$$\begin{aligned}\zeta_t + c\zeta_x + \frac{1}{2}c_x\zeta + \frac{3}{2c}\varepsilon\zeta\zeta_x - \frac{3}{8c^3}\varepsilon^2\zeta^2\zeta_x + \frac{3}{16c^5}\varepsilon^3\zeta^3\zeta_x \\ +\mu(\tilde{A}\zeta_{xxx} + B\zeta_{xxt}) = \varepsilon\mu\tilde{E}\zeta\zeta_{xxx} + \varepsilon\mu\Big(\partial_x(\frac{\tilde{F}}{2}\zeta)\zeta_{xx} + \zeta_x\partial_x^2(\frac{\tilde{F}}{2}\zeta)\Big),\end{aligned} \tag{1.2.10}$$

where

$$\begin{aligned}\tilde{A} &= Ac^5 - Bc^5 + Bc \\ \tilde{E} &= Ec^3 - \frac{3}{2}Bc^3 + \frac{3}{2c}B \\ \tilde{F} &= Fc^3 - \frac{9}{2}Bc^3 + \frac{9}{2c}B,\end{aligned}$$

can be used to construct an approximate solution consistent with the Green-Naghdi equations:

Proposition 1.2.2. *Let $b \in H^\infty(\mathbb{R})$ and $q \in \mathbb{R}$. Assume that*

$$A = q, \quad B = q - \frac{1}{6} \quad E = -\frac{3}{2}q - \frac{1}{6}, \quad F = -\frac{9}{2}q - \frac{5}{24}.$$

Then:

- *For any family $\wp$ of parameters satisfying (1.2.2),*
- *For all $s \geq 0$ large enough and $T > 0$,*
- *For any bounded family $(\zeta^\theta)_{\theta\in\wp} \in C([0, \frac{T}{\varepsilon}], H^s(\mathbb{R}))$ solving (1.2.10),*

the familly $(\zeta^\theta, u^\theta)_{\theta\in\wp}$ with (omitting the index θ)

$$\begin{aligned}u := \frac{1}{c}\Big(\zeta + \frac{c^2}{c^2+\varepsilon\zeta}\Big(-\frac{1}{2}\int_{-\infty}^x \frac{c_x}{c}\zeta - \frac{\varepsilon}{4c^2}\zeta^2 - \frac{\varepsilon^2}{8c^4}\zeta^3 \\ +\frac{3\varepsilon^3}{64c^6}\zeta^4 - \mu\frac{1}{6}c^3\zeta_{xt} + \varepsilon\mu c^2\big[\frac{1}{6}\zeta\zeta_{xx} + \frac{1}{48}\zeta_x^2\big]\Big)\Big)\end{aligned} \tag{1.2.11}$$

is L^∞-consistent on $[0, \frac{T}{\varepsilon}]$ with the GN equations (1.2.1).

Remark 1.2.3. *If we take $b = 0$ -i.e if we consider a flat bottom-, then one can recover the equation (18) of [20].*

Remark 1.2.4. *Choosing $q = \frac{1}{12}$, $\alpha = \varepsilon$ and $\beta = \mu^{3/2}$ the equation (1.2.10) reads after neglecting the $O(\mu^2)$ terms:*

$$\begin{aligned} &\zeta_t + c\zeta_x + \frac{1}{2}c_x\zeta + \frac{3}{2}\varepsilon\zeta\zeta_x - \frac{3}{8}\varepsilon^2\zeta^2\zeta_x + \frac{3}{16}\varepsilon^3\zeta^3\zeta_x \\ &\quad + \frac{\mu}{12}(\zeta_{xxx} - \zeta_{xxt}) = -\frac{7}{24}\varepsilon\mu(\zeta\zeta_{xxx} + 2\zeta_x\zeta_{xx}), \end{aligned} \tag{1.2.12}$$

it is more advantageous to use this equation (1.2.12) to study the pattern of wave-breaking for the variable bottom CH equation (see §3.2 below).

Proof. As in Proposition 1.2.1, we make the proof in 3 steps (we just sketch the proof here since it is similar to the proof of Proposition 1.2.1).
Step 1. We prove that if ζ solves (1.2.10) one gets

$$\begin{aligned} &\zeta_t + c\zeta_x + \frac{1}{2}c_x u + \frac{3}{2c}\varepsilon\zeta\zeta_x - \frac{3}{8c^3}\varepsilon^2\zeta^2\zeta_x + \frac{3}{16c^5}\varepsilon^3\zeta^3\zeta_x + \mu c^4 a\zeta_{xxt} \\ &\quad = \varepsilon\mu c^3[e\zeta\zeta_{xx} + d\zeta_x^2]_x + O(\mu^2) \end{aligned} \tag{1.2.13}$$

with $a = B - A$, $e = E + \frac{3}{2}A$, $d = \frac{1}{2}(F + 3A - E)$.
Step 2. We seek v such that if $u := \frac{1}{c}(\zeta + \varepsilon v)$ and ζ solves (1.2.10) then the first equation of (1.2.1) is satisfied up to a $O_{L^\infty}(\mu^2)$ term. Proceeding as in the proof of Proposition 1.2.1, one can check that a good choice for v is

$$\begin{aligned} \Big(\frac{c^2 + \varepsilon\zeta}{c^2}\Big)\varepsilon v = &-\frac{1}{2}\int_{-\infty}^{x} \frac{c_x}{c}\zeta - \frac{\varepsilon}{4c^2}\zeta^2 - \frac{\varepsilon^2}{8c^4}\zeta^3 + \frac{3\varepsilon^3}{64c^6}\zeta^4 \\ &+\mu c^3 a\zeta_{xt} - \varepsilon\mu c^2[e\zeta\zeta_{xx} + d\zeta_x^2]. \end{aligned} \tag{1.2.14}$$

Step 3. We show here that it is possible to choose the coefficients A, B, E, F such that the second equation of (1.2.1) is also satisfied up to $O_{L^\infty}(\mu^2)$ terms. Replacing u by $\frac{1}{c}(\zeta + \varepsilon v)$ with v given by (1.2.14), one can check that that this condition is equivalent to

$$\begin{aligned} &\zeta_t + c\zeta_x + \frac{1}{2}c_x\zeta + \frac{3}{2c}\varepsilon\zeta\zeta_x - \frac{3}{8c^3}\varepsilon^2\zeta^2\zeta_x + \frac{3}{16c^5}\varepsilon^3\zeta^3\zeta_x - \mu c^4(a + \frac{1}{3})\zeta_{xxt} \\ = \;& \varepsilon\mu c^3[-(e + a - \frac{1}{6})\zeta\zeta_{xx} - (\frac{7}{4}a + d + \frac{1}{3})\zeta_x^2]_x + O_{L^\infty}(\mu^2). \end{aligned}$$

Equating the coefficients of this equation with those of (1.2.13) shows that the second equation of (1.2.1) is also satisfied at order $O_{L^\infty}(\mu^2)$ if the following relations hold:

$$a = -\frac{1}{6}, \quad e = -\frac{1}{6}, \quad d = -\frac{1}{48},$$

and the conditions given in the statement of the proposition on A, B, E, and F follows from the expressions of a, e and d given after (1.2.13). $\square$

1.2.3 Derivation of the KdV equation in the long-wave scaling.

In this subsection, attention is given to the regime of slow variations of the bottom topography under the long-wave scaling $\varepsilon = O(\mu)$. We give here a rigorous justification in the meaning of consistency of the variable-depth extensions of the KdV equation (called KdV-top) originally derived in [47, 70, 27]. We consider the values of ε, β, α and μ satisfying:

$$\varepsilon = O(\mu), \quad \alpha\beta = O(\varepsilon), \quad \alpha^{3/2}\beta = O(\varepsilon^2). \tag{1.2.15}$$

Remark 1.2.5. *Any family of parameters $\theta = (\varepsilon, \beta, \alpha, \mu)$ satisfying (1.2.15), also satisfies (1.2.2) .*

Neglecting the $O(\mu^2)$ terms, one obtains from (1.2.1) the following Boussinesq system:

$$\begin{cases} \zeta_t + [hu]_x = 0 \\ u_t + \zeta_x + \varepsilon u u_x = \dfrac{\mu}{3} c^4 u_{xxt}, \end{cases} \tag{1.2.16}$$

where we recall that $h = 1 + \varepsilon\zeta - \beta b^{(\alpha)}$ and $c^2 = 1 - \beta b^{(\alpha)}$. The next proposition proves that the KdV-top equation

$$\zeta_t + c\zeta_x + \frac{3}{2c}\varepsilon\zeta\zeta_x + \frac{1}{6}\mu c^5 \zeta_{xxx} + \frac{1}{2}c_x\zeta = 0, \tag{1.2.17}$$

is L^∞-consistent with the equations (1.2.16).

Proposition 1.2.3. *Let $b \in H^\infty(\mathbb{R})$. Then:*

- *For any family $\wp'$ of parameters satisfying (1.2.15),*
- *For all $s \geq 0$ large enough and $T > 0$,*
- *For any bounded family $(\zeta^\theta)_{\theta\in\wp'} \in C([0, \frac{T}{\varepsilon}], H^s(\mathbb{R}))$ solving (1.2.17)*

the familly $(\zeta^\theta, u^\theta)_{\theta\in\wp'}$ with (omitting the index θ)

$$u := \frac{1}{c}\Big(\zeta - \frac{1}{2}\int_{-\infty}^{x}\frac{c_x}{c}\zeta - \frac{\varepsilon}{4c^2}\zeta^2 + \mu\frac{1}{6}c^4\zeta_{xx}\Big)$$

is L^∞-consistent on $[0, \frac{T}{\varepsilon}]$ with the equations (4.1.7).

Remark 1.2.6. *Similarly, one can prove that a family $(\zeta^\theta, u^\theta)_{\theta\in\wp'}$ with u^θ solution of the KdV-top equation*

$$u_t + cu_x + \frac{3}{2}\varepsilon uu_x + \frac{1}{6}\mu c^5 u_{xxx} + \frac{3}{2}c_x u = 0, \tag{1.2.18}$$

and ζ^θ given by

$$\zeta := cu + \frac{1}{2}\int_{-\infty}^{x} c_x u + \frac{\varepsilon}{4}u^2 - \frac{\mu}{6}c^5 u_{xx}, \tag{1.2.19}$$

is L^∞-consistent with the equations (4.1.7).

Proof. We saw in the previous subsection that if $(\zeta^\theta)_{\theta\in\wp}$ is a family of solutions of (1.2.10), then the family $(\zeta^\theta, u^\theta)_{\theta\in\wp}$ with u^θ is given by (4.3.2) is L^∞-consistent with the equations (1.2.1). Since $\theta = (\varepsilon, \beta, \alpha, \mu) \in \wp' \subset \wp$ then by taking $q = \dfrac{1}{6}$, we remark that the equations (1.2.10) and (1.2.17) are equivalent in the meaning of L^∞-consistency and the systems (1.2.1) and (1.2.16) are also, so it is clear that $(\zeta^\theta, u^\theta)_{\theta\in\wp'}$ is L^∞-consistent with the equations (1.2.16). $\square$

1.3 Mathematical analysis of the variable bottom models

1.3.1 Well-posedness for the variable bottom CH equation

We prove here the well posedness of the general class of equations

$$\begin{aligned}(1 - \mu m\partial_x^2)u_t + cu_x + kc_x u + \sum_{j\in J}\varepsilon^j f_j u^j u_x + \mu g u_{xxx} \\ = \varepsilon\mu\Big[h_1 uu_{xxx} + \partial_x(h_2 u)u_{xx} + u_x\partial_x^2(h_2 u)\Big],\end{aligned} \tag{1.3.1}$$

where $m > 0$, $k \in \mathbb{R}$, J is a finite subset of $\mathbb{N}^*$ and $f_j = f_j(c)$, $g = g(c)$, $h_1 = h_1(c)$ and $h_2 = h_2(c)$ are smooth functions of c. We also recall that $c = \sqrt{1 - \beta b^{(\alpha)}}$.

Example 1.3.1. *Taking*

$$m = -B, \qquad k = \frac{3}{2}, \qquad J = \{1\}, \qquad f_1(c) = \frac{3}{2},$$
$$g(c) = Ac^5 - Bc^5 + Bc, \qquad h_1(c) = c^4 E - \frac{3}{2}Bc^4 + \frac{3}{2}B,$$
$$h_2(c) = \frac{Fc^4 - \frac{9}{2}Bc^4 + \frac{9}{2}B}{2},$$

the equation (1.3.1) coincides with (1.1.12).

Example 1.3.2. *Taking*

$$m = -B, \qquad k = \frac{1}{2}, \qquad J = \{1, 2, 3\},$$
$$f_1(c) = \frac{3}{2c}, \qquad f_2(c) = -\frac{3}{8c^3}, \qquad f_3(c) = \frac{3}{16c^5},$$
$$g(c) = Ac^5 - Bc^5 + Bc, \qquad h_1(c) = c^3 E - \frac{3}{2}Bc^3 + \frac{3}{2c}B,$$
$$h_2(c) = \frac{Fc^3 - \frac{9}{2}Bc^3 + \frac{9}{2c}B}{2},$$

the equation (1.3.1) coincides with (1.2.10).

More precisely, Theorem 2.2.1 below shows that one can solve the initial-value problem

$$\left| \begin{aligned} &(1 - \mu m \partial_x^2)u_t + cu_x + kc_x u + \sum_{j \in J} \varepsilon^j f_j u^j u_x + \mu g u_{xxx} \\ &\qquad\qquad = \varepsilon\mu \Big[h_1 u u_{xxx} + \partial_x(h_2 u) u_{xx} + u_x \partial_x^2 (h_2 u) \Big], \\ &u_{|t=0} = u^0 \end{aligned} \right. \tag{1.3.2}$$

on a time scale $O(1/\varepsilon)$, and under the condition $m > 0$. In order to state the result, we need to define the energy space X^s $(s \in \mathbb{R})$ as

$$X^{s+1}(\mathbb{R}) = H^{s+1}(\mathbb{R}) \text{ endowed with the norm } |f|^2_{X^{s+1}} = |f|^2_{H^s} + \mu m |\partial_x f|^2_{H^s}.$$

Theorem 1.3.1. *Let $m > 0$, $s > \frac{3}{2}$ and $b \in H^\infty(\mathbb{R})$. Let also $\wp$ be a family of parameters $\theta = (\varepsilon, \beta, \alpha, \mu)$ satisfying (1.2.2). Then for all $u^0 \in H^{s+1}(\mathbb{R})$, there exists $T > 0$ and a unique family of solutions $(u^\theta)_{\theta \in \wp}$ to (1.3.2) bounded in $C([0, \frac{T}{\varepsilon}]; X^{s+1}(\mathbb{R})) \cap C^1([0, \frac{T}{\varepsilon}]; X^s(\mathbb{R}))$.*

Proof. In this proof, we use the generic notation

$$C = C(\varepsilon, \mu, \alpha, \beta, s, |b|_{H^\sigma})$$

for some $\sigma > s + 1/2$ large enough. When the constant also depends on $|v|_{X^{s+1}}$, we write $C(|v|_{X^{s+1}})$. Note that the dependence on the parameters is assumed to be nondecreasing.

For all v smooth enough, let us define the "linearized" operator $\mathcal{L}(v, \partial)$ as

$$\begin{aligned}\mathcal{L}(v,\partial) &= (1-\mu m\partial_x^2)\partial_t + c\partial_x + kc_x + \varepsilon^j f_j v^j \partial_x + \mu g \partial_x^3 \\ &\quad -\varepsilon\mu\big[h_1 v\partial_x^3 + (h_2 v)_x \partial_x^2 + (h_2 v)_{xx}\partial_x\big];\end{aligned}$$

for the sake of simplicity we use the convention of summation over repated indexes, $\sum_{j\in J} \varepsilon^j f_j v^j = \varepsilon^j f_j v^j$. To construct a solution of (1.3.1) using an iterative scheme, we have to study the initial-value problem

$$\begin{cases} \mathcal{L}(v,\partial)u = \varepsilon f, \\ u_{|t=0} = u^0. \end{cases} \tag{1.3.3}$$

If v is smooth enough, it is completely standard to check that for all $s \geq 0$, $f \in L^1_{loc}(\mathbb{R}_t^+; H^s(\mathbb{R}_x))$ and $u^0 \in H^s(\mathbb{R})$, there exists a unique solution $u \in C(\mathbb{R}^+; H^{s+1}(\mathbb{R}))$ to (1.3.3) (recall that $m > 0$). We thus take for granted the existence of a solution to (1.3.3) and establish some precise energy estimates on the solution. These energy estimates are given in terms of the $|\cdot|_{X^{s+1}}$ norm introduced above:

$$|u|^2_{X^{s+1}} = |u|^2_{H^s} + \mu m |\partial_x u|^2_{H^s}.$$

Differentiating $\frac{1}{2}e^{-\varepsilon\lambda t}|u|^2_{X^{s+1}}$ with respect to time, one gets using the equation (1.3.3) and integrating by parts,

$$\begin{aligned}\frac{1}{2}e^{\varepsilon\lambda t}\partial_t(e^{-\varepsilon\lambda t}|u|^2_{X^{s+1}}) &= -\frac{\varepsilon\lambda}{2}|u|^2_{X^{s+1}} - (\Lambda^s(c\partial_x u), \Lambda^s u) - k(\Lambda^s(u\partial_x c), \Lambda^s u) \\ &\quad +\varepsilon(\Lambda^s f, \Lambda^s u) - \varepsilon^j(\Lambda^s(f_j v^j \partial_x u), \Lambda^s u) - \mu(\Lambda^s(g\partial_x^3 u), \Lambda^s u) \\ &\quad -\varepsilon\mu(\Lambda^s(h_1 v\partial_x^3 u), \Lambda^s u) - \varepsilon\mu(\Lambda^s((h_2 v)_x \partial_x u), \Lambda^s \partial_x u),\end{aligned}$$

where $\Lambda = (1-\partial_x^2)^{1/2}$. Since for all constant coefficient skewsymmetric differential polynomial P (that is, $P^* = -P$), and all h smooth enough, one has

$$(\Lambda^s(hPu), \Lambda^s u) = ([\Lambda^s, h]Pu, \Lambda^s u) - \frac{1}{2}([P,h]\Lambda^s u, \Lambda^s u),$$

we deduce (applying this identity with $P=\partial_x$ and $P=\partial_x^3$),

$$\begin{aligned}
\frac{1}{2}e^{\varepsilon\lambda t}\partial_t(e^{-\varepsilon\lambda t}|u|^2_{X^{s+1}}) &= -\frac{\varepsilon\lambda}{2}|u|^2_{X^{s+1}}+\varepsilon(\Lambda^s f,\Lambda^s u)\\
&-\big([\Lambda^s,c]\partial_x u,\Lambda^s u\big)+\frac{1}{2}((\partial_x c)\Lambda^s u,\Lambda^s u\big)-k(\Lambda^s(u\partial_x c),\Lambda^s u)\\
&-\varepsilon^j\big([\Lambda^s,f_j v^j]\partial_x u,\Lambda^s u\big)+\frac{\varepsilon^j}{2}(\partial_x(f_j v^j)\Lambda^s u,\Lambda^s u\big)\\
&-\mu\big([\Lambda^s,g]\partial_x^2 u-\frac{3}{2}g_x\Lambda^s\partial_x u-g_{xx}\Lambda^s u,\Lambda^s\partial_x u\big)-\mu\big([\Lambda^s,g_x]\partial_x^2 u,\Lambda^s u\big)\\
&-\varepsilon\mu\big([\Lambda^s,h_1 v]\partial_x^2 u-\frac{3}{2}(h_1 v)_x\Lambda^s\partial_x u-(h_1 v)_{xx}\Lambda^s u,\Lambda^s\partial_x u\big)\\
&-\varepsilon\mu\big([\Lambda^s,(h_1 v)_x]\partial_x^2 u,\Lambda^s u\big)-\varepsilon\mu\big(\Lambda^s((h_2 v)_x\partial_x u),\Lambda^s\partial_x u\big). \qquad (1.3.4)
\end{aligned}$$

Note that we also used the identities

$$[\Lambda^s,h]\partial_x^3 u=\partial_x\big([\Lambda^s,h]\partial_x^2 u\big)-[\Lambda^s,h_x]\partial_x^2 u$$

and

$$\frac{1}{2}(h_{xxx}\Lambda^s u,\Lambda^s u)=-(h_{xx}\Lambda^s u,\Lambda^s u_x).$$

The terms involving the velocity c (second line in the r.h.s of (1.3.4)) are controlled using the following lemma:

Lemma 1.3.1. *Let $s>3/2$ and $(\varepsilon,\beta,\alpha,\mu)$ satisfy (1.2.2). Then there exists $C>0$ such that*

$$\begin{aligned}
|\big([\Lambda^s,c]\partial_x u,\Lambda^s u\big)| &\leq \varepsilon C|u|^2_{X^{s+1}} && (1.3.5)\\
|\big((\partial_x c)\Lambda^s u,\Lambda^s u\big)| &\leq \varepsilon C|u|^2_{X^{s+1}} && (1.3.6)\\
|\big(\Lambda^s(u\partial_x c),\Lambda^s u\big)| &\leq \varepsilon C|u|^2_{X^{s+1}}. && (1.3.7)
\end{aligned}$$

Proof. • Estimate of $|\big([\Lambda^s,c]\partial_x u,\Lambda^s u\big)|$. One could control this term by a standard commutator estimates in terms of $|c_x|_{H^{s-1}}$; however, one has $|c_x|_{H^{s-1}}=O(\sqrt{\alpha}\beta)$ and not $O(\alpha\beta)$ as needed. We thus write

$$\big([\Lambda^s,c]\partial_x u,\Lambda^s u\big)=\big(([\Lambda^s,c]-\{\Lambda^s,c\})\partial_x u,\Lambda^s u\big)+\big(\{\Lambda^s,c\}\partial_x u,\Lambda^s u\big),$$

where for all function F, $\{\Lambda^s,F\}$ stands for the Poisson bracket,

$$\{\Lambda^s,F\}=-s\partial_x F\Lambda^{s-2}\partial_x.$$

We can then use the following commutator estimate ([51], Theorem 5): for all F and U smooth enough, one has

$$\forall s>3/2,\qquad |([\Lambda^s,F]-\{\Lambda^s,F\})U|_2\leq \mathrm{Cst}\ |\partial_x^2 F|_{H^s}|U|_{H^{s-2}}.$$

Since $|\partial_x^2 c|_{H^s} = O(\alpha\beta) = O(\varepsilon)$, we deduce

$$|\big(([\Lambda^s, c] - \{\Lambda^s, c\})\partial_x u, \Lambda^s u\big)| \leq C\beta\alpha |u|^2_{X^{s+1}} \leq C\varepsilon |u|^2_{X^{s+1}}.$$

Morever

$$|\big(\{\Lambda^s, c\}\partial_x u, \Lambda^s u\big)| \leq |-s\partial(c)\Lambda^{s-2}\partial_x^2 u|_2 |u|_{X^{s+1}} \leq C|\partial(c)|_\infty |u|^2_{X^{s+1}} \leq \varepsilon C |u|^2_{X^{s+1}},$$

and thus (1.3.5) follows easily.
• Estimate of $|\big((\partial_x c)\Lambda^s u, \Lambda^s u\big)|$. Since $|u|_{H^s} \leq |u|_{X^{s+1}}$, one has by the Cauchy-Schwarz inequality

$$\big((\partial_x c)\Lambda^s u, \Lambda^s u\big) \leq |\partial_x c|_\infty |u|^2_{X^{s+1}};$$

therefore, by using the fact that $|\partial_x c|_\infty \leq \mathrm{Cst}\ \alpha\beta |\partial_x b|_\infty$ and $\alpha\beta = O(\varepsilon)$, one gets easily

$$\big((\partial_x c)\Lambda^s u, \Lambda^s u\big) \leq \varepsilon C |u|^2_{X^{s+1}}.$$

• Estimate of $|\big(\Lambda^s(u\partial_x c), \Lambda^s u\big)|$. By Cauchy-Schwarz we get

$$|(\Lambda^s(u\partial_x c), \Lambda^s u)| \leq |u\partial_x c|_{H^s} |u|_{H^s}.$$

We have also that

$$|u\partial_x c|_{H^s} \leq |\partial_x c|_{W^{[s]+1,\infty}} |u|_{X^{s+1}}$$

where $[s]$ is the largest integer smaller or equal to s (this estimate is obvious for s integer and is obtained by interpolation for non integer values of s.) Using this estimate, and the fact that $\beta\alpha = O(\varepsilon)$, it is easy to deduce (1.3.7). $\square$

For the terms involving the f_j, (third line in the r.h.s of (1.3.4)) we use the controls given by:

Lemma 1.3.2. *Under the assumptions of Theorem 2.2.1, one has that*

$$|\varepsilon^j\big([\Lambda^s, f_j v^j]\partial_x u, \Lambda^s u\big)| \leq \varepsilon C(|v|_{X^{s+1}})|u|^2_{X^{s+1}} \qquad (1.3.8)$$

$$|\varepsilon^j\big(\partial_x(f_j v^j)\Lambda^s u, \Lambda^s u\big)| \leq \varepsilon C(|v|_{X^{s+1}})|u|^2_{X^{s+1}}. \qquad (1.3.9)$$

Proof. • Estimate of $\varepsilon^j\big([\Lambda^s, f_j v^j]\partial_x u, \Lambda^s u\big)$. By Cauchy-Schwarz we get

$$|\varepsilon^j\big([\Lambda^s, f_j v^j]\partial_x u, \Lambda^s u\big)| \leq \varepsilon^j |[\Lambda^s, f_j v^j]\partial_x u|_2 |u|_{X^{s+1}}.$$

we use here the well-known Calderon-Coifman-Meyer commutator estimate: for all F and U smooth enough, one has

$$\forall s > 3/2, \qquad |[\Lambda^s, F]U|_2 \leq \mathrm{Cst}|F|_{H^s}|U|_{H^{s-1}};$$

using this estimate, it is easy to check that one gets (1.3.8).
• Estimate of $\varepsilon^j\big(\partial_x(f_j v^j)\Lambda^s u, \Lambda^s u\big)$. It is clear that

$$|\big(\partial_x(f_j v^j)\Lambda^s u, \Lambda^s u\big)| \leq |\partial_x(f_j v^j)|_\infty |u|^2_{X^{s+1}}.$$

Therefore (1.3.9) follows from the continuous embedding $H^s \subset W^{1,\infty}$ $(s > 3/2)$. $\square$

Similarly, the terms involving g (fourth line in the r.h.s of (1.3.4)) are controled using the following lemma:

Lemma 1.3.3. *Under the assumptions of Theorem 2.2.1, one has that*

$$\Big| -\mu\big([\Lambda^s, g]\partial_x^2 u - \frac{3}{2}g_x\Lambda^s\partial_x u - g_{xx}\Lambda^s u, \Lambda^s\partial_x u\big) \qquad (1.3.10)$$
$$-\mu\big([\Lambda^s, g_x]\partial_x^2 u, \Lambda^s u\big)\Big| \leq \varepsilon C|u|^2_{X^{s+1}}.$$

Proof. Since $|u|_{H^s} \leq |u|_{X^{s+1}}$ and $\sqrt{\mu}|\partial_x u|_{H^s} \leq \dfrac{1}{\sqrt{m}}|u|_{X^{s+1}}$, by using the Cauchy-Schwarz inequality and proceeding as for the proofs of Lemmas 1.3.1, 1.3.2 one gets directly (1.3.10). ☐

Finally to control the terms involving h_i, (fifth and sixth lines in the r.h.s of (1.3.4)) let us state the following lemma:

Lemma 1.3.4. *Under the assumptions of Theorem 2.2.1, one has that*

$$| -\varepsilon\mu\big([\Lambda^s, h_1 v]\partial_x^2 u - \frac{3}{2}(h_1 v)_x\Lambda^s\partial_x u - (h_1 v)_{xx}\Lambda^s u, \Lambda^s\partial_x u\big) \qquad (1.3.11)$$
$$-\varepsilon\mu\big([\Lambda^s, (h_1 v)_x]\partial_x^2 u, \Lambda^s u\big)| \leq \varepsilon C(|v|_{X^{s+1}})|u|^2_{X^{s+1}},$$

$$| -\varepsilon\mu\big(\Lambda^s((h_2 v)_x\partial_x u), \Lambda^s\partial_x u\big)| \leq \varepsilon C(|v|_{X^{s+1}})|u|^2_{X^{s+1}}. \qquad (1.3.12)$$

Proof. We remark first that

$$|\mu\big((h_1 v)_{xx}\Lambda^s u, \Lambda^s\partial_x u\big)| \leq |\partial_x(\sqrt{\mu}\partial_x(h_1 v))|_\infty |u|^2_{X^{s+1}}$$

since $s-1 > \dfrac{1}{2}$, so by using the imbedding $H^{s-1}(\mathbb{R}) \subset L^\infty(\mathbb{R})$ we get

$$|\partial_x(\sqrt{\mu}\partial_x(h_1 v))|_\infty \leq C(|v|_{X^{s+1}}).$$

Proceeding now as in the proof of Lemma 1.3.3 one gets directly (1.3.11) and (1.3.12). ☐

Gathering the information provided by the above lemmas, we get

$$e^{\varepsilon\lambda t}\partial_t(e^{-\varepsilon\lambda t}|u|^2_{X^{s+1}}) \leq \varepsilon\big(C(|v|_{X^{s+1}}) - \lambda\big)|u|^2_{X^{s+1}} + 2\varepsilon|f|_{X^{s+1}}|u|_{X^{s+1}}.$$

Taking $\lambda = \lambda_T$ large enough (how large depending on $\sup_{t\in[0,\frac{T}{\varepsilon}]} C(|v(t)|_{X^{s+1}})$ to have the first term of the right hand side negative for all $t \in [0, \dfrac{T}{\varepsilon}]$, one deduces

$$\forall t \in [0, \frac{T}{\varepsilon}], \qquad \partial_t(e^{-\varepsilon\lambda_T t}|u|^2_{X^{s+1}}) \leq 2\varepsilon e^{-\varepsilon\lambda_T t}|f|_{X^{s+1}}|u|_{X^{s+1}}.$$

Integrating this differential inequality yields therefore

$$\forall t \in [0, \frac{T}{\varepsilon}], \qquad |u(t)|_{X^{s+1}} \leq e^{\varepsilon \lambda_T t} |u^0|_{X^{s+1}} + 2\varepsilon \int_0^t e^{\varepsilon \lambda_T (t-t')} |u(t')|_{X^{s+1}} dt'.$$

Thanks to this energy estimate, we conclude classically (see e.g. [2]) the existence of a

$$T = T(|u^0|_{X^{s+1}}) > 0,$$

and of a unique solution $u \in C([0, \frac{T}{\varepsilon}]; X^{s+1}(\mathbb{R}))$ to (1.3.2) as a limit of the iterative scheme

$$u_0 = u^0, \quad \text{and} \quad \forall n \in \mathbb{N}, \quad \begin{cases} \mathcal{L}(u^n, \partial) u^{n+1} = 0, \\ u^{n+1}_{|t=0} = u^0. \end{cases}$$

Since u solves (1.3.1), we have $\mathcal{L}(u, \partial) u = 0$ and therefore

$$(\Lambda^{s-1}(1 - \mu m \partial_x^2)\partial_t u, \Lambda^{s-1}\partial_t u) = -(\Lambda^{s-1}\mathcal{M}(u, \partial)u, \Lambda^{s-1}\partial_t u),$$

with $\mathcal{M}(u, \partial) = \mathcal{L}(u, \partial) - (1 - \mu m \partial_x^2)\partial_t$. Proceeding as above, one gets

$$|\partial_t u|_{X^s} \leq C(|u^0|_{X^{s+1}}, |u|_{X^{s+1}}),$$

and it follows that the family of solution is also bounded in $C^1([0, \frac{T}{\varepsilon}]; X^s)$. $\square$

1.3.2 Explosion condition for the variable bottom CH equation

As in the case of flat bottoms, it is possible to give some information on the blow-up pattern for the equation (1.2.12) for the free surface.

Proposition 1.3.1. *Let $b \in H^\infty(\mathbb{R})$, $\zeta_0 \in H^3(\mathbb{R})$. If the maximal existence time $T_m > 0$ of the solution of (1.2.12) with initial profile $\zeta(0, \cdot) = \zeta_0$ is finite, the solution $\zeta \in C([0, T_m); H^3(\mathbb{R})) \cap C^1([0, T_m); H^2(\mathbb{R})$ is such that*

$$\sup_{t \in [0, T_m), \, x \in \mathbb{R}} \{|\zeta(t, x)|\} < \infty \tag{1.3.13}$$

and

$$\sup_{x \in \mathbb{R}} \{\zeta_x(t, x)\} \uparrow \infty \quad \text{as} \quad t \uparrow T_m. \tag{1.3.14}$$

Remark 1.3.1. *It is worth remarking that even though topography effects are introduced in our equation (1.2.12), wave breaking remains of 'surging' type (i.e the slope grows to $+\infty$) as for flat bottoms. This shows that plunging breakers (i.e the slope decays to -∞) occur for stronger topography variations then those considerd in this chapter.*

Proof. By using the Theorem 1.3.1 given $\zeta_0 \in H^3(\mathbb{R})$, the maximal existence time of the solution $\zeta(t)$ to (1.2.12) with initial data $\zeta(0) = \zeta_0$ is finite if and only if $|\zeta(t)|_{H^3(\mathbb{R})}$ blows-up in finite time. To complete the proof it is enough to show that
(i) the solution $\zeta(t)$ given by Theorem 1.3.1 remains uniformly bounded as long as it is defined;
and
(ii) if we can find some $M = M(\zeta_0) > 0$ with

$$\zeta_x(t,x) \leq M, \qquad x \in \mathbb{R}, \tag{1.3.15}$$

as long as the solution is defined, then $|\zeta(t)|_{H^3(\mathbb{R})}$ remains bounded on $[0, T_m)$.
Remarking that

$$\int_{\mathbb{R}} [(c\zeta_x + \frac{1}{2}c_x\zeta)\zeta]\,dx = -\int_{\mathbb{R}} [(c\zeta)_x\zeta - \frac{1}{2}c_x\zeta^2]\,dx = -\int_{\mathbb{R}} [(c\zeta_x + \frac{1}{2}c_x\zeta)\zeta]\,dx,$$

we can deduce

$$\int_{\mathbb{R}} [(c\zeta_x + \frac{1}{2}c_x\zeta)\zeta]\,dx = 0. \tag{1.3.16}$$

We have also that

$$\int_{\mathbb{R}} [(\zeta^i\zeta_x)\zeta]\,dx = 0, \qquad \forall i \in \mathbb{N}^*. \tag{1.3.17}$$

Item (i) follows at once from (1.3.16), (1.3.17) and the imbedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$ since multiplying (1.2.12) by ζ and integrating on $\mathbb{R}$ yields

$$\partial_t\Big(\int_{\mathbb{R}} [\zeta^2 + \frac{1}{12}\,\mu \int_{\mathbb{R}} \zeta_x^2]\,dx\Big) = 0. \tag{1.3.18}$$

To prove item (ii), as in [20] we multiply the equation (1.2.12) by ζ_{xxxx} and after performing several integrations by parts we obtains the following identity:

$$\begin{aligned}
\partial_t\Big(\int_{\mathbb{R}} [\zeta_{xx}^2 + \frac{1}{12}\,\mu \int_{\mathbb{R}} \zeta_{xxx}^2]\,dx\Big) &= 15\,\varepsilon \int_{\mathbb{R}} \zeta\zeta_{xx}\zeta_{xxx}\,dx \\
-\frac{15}{4}\,\varepsilon^2 \int_{\mathbb{R}} \zeta^2\zeta_{xx}\zeta_{xxx}\,dx &+ \frac{9}{16}\varepsilon^3 \int_{\mathbb{R}} \zeta_x^5\,dx + \frac{15}{8}\,\varepsilon^3 \int_{\mathbb{R}} \zeta^3\zeta_{xx}\zeta_{xxx}\,dx \\
+\frac{7}{4}\,\mu\varepsilon \int_{\mathbb{R}} \zeta_x\zeta_{xxx}^2\,dx &+ I + J.
\end{aligned} \tag{1.3.19}$$

Where,

$$I = \int_{\mathbb{R}} c\zeta_x\zeta_{xxxx}\,dx \qquad J = \frac{1}{2}\int_{\mathbb{R}} c_x\zeta\zeta_{xxxx}\,dx.$$

One can use the Caushy-schwarz inequality to obtain

$$\begin{aligned}
I = -\int_{\mathbb{R}} \zeta_x(c\zeta_x)_{xxx} &= -\frac{1}{2}\int_{\mathbb{R}} c_{xxx}\zeta_x^2 - \frac{3}{2}\int_{\mathbb{R}} c_{xx}\zeta_{xx}\zeta_x \\
-\frac{3}{2}\int_{\mathbb{R}} c_x\zeta_{xxx}\zeta_x &\leq M_1\big(\frac{1}{2}|\zeta_x|_2|\zeta_x|_2 + \frac{3}{2}|\zeta_{xx}|_2|\zeta_x|_2 + \frac{3}{2}|\zeta_{xxx}|_2|\zeta_x|_2\big),
\end{aligned} \tag{1.3.20}$$

and

$$\begin{aligned} J &= -\frac{1}{2}\int_{\mathbb{R}} c_{xx}\zeta\zeta_{xxx} - \frac{1}{2}\int_{\mathbb{R}} c_x\zeta_x\zeta_{xxx} \\ &\leq M_1\big(\frac{1}{2}|\zeta|_2|\zeta_{xxx}|_2 + \frac{1}{2}|\zeta_x|_2|\zeta_{xxx}|_2\big), \end{aligned} \tag{1.3.21}$$

for some $M_1 = M_1(|c|_{W^{3,\infty}}) > 0$. If (1.3.15) holds, let in accordance with (1.3.18) $M_0 > 0$ be such that

$$|\zeta(t,x)| \leq M_0, \qquad x \in \mathbb{R}, \tag{1.3.22}$$

for as long as the solution exists. Using the Cauchy-Schwarz inequality as well as the fact that $\mu \leq 1$, we infer from (1.3.18), (1.3.19), (1.3.20) , (1.3.21) and (1.3.22) that there exists $C(M_0, M_1, M, \varepsilon, \mu)$ such that

$$\partial_t E(t) \leq C(M_0, M_1, M, \varepsilon, \mu)\, E(t),$$

where

$$E(t) = \int_{\mathbb{R}} [\zeta^2 + \frac{1}{12}\,\mu\zeta_x^2 + \zeta_{xx}^2 + \frac{1}{12}\,\mu\zeta_{xxx}^2]\, dx.$$

(Note that we do not give any details for the components of (1.3.19) other than I and J because these components do not involve any topography term and can therefore be handled exactly as in [20]). An application of Gronwall's inequality enables us to obtain the result. □

Our next aim is to show as in the case of flat bottoms that there are solutions to (1.2.12) that blow-up in finite time as surging breakers, that is, following the pattern given in Proposition 1.3.1.

Proposition 1.3.2. *Let $b \in H^\infty(\mathbb{R})$. If the initial wave profile $\zeta_0 \in H^3(\mathbb{R})$ satisfies*

$$\begin{aligned} \Big|\sup_{x\in\mathbb{R}}\{\zeta_0(x)\}\Big|^2 \;\geq\;& \frac{28}{3}\,C_0\,\mu^{-3/4} + \frac{1}{2}\,\varepsilon\,C_0^{3/2}\,\mu^{-3/4} + \frac{1}{4}\,\varepsilon^2\,C_0^2\,\mu^{-3/4} \\ &+\frac{7}{3}\,C_0\,\mu^{-1/2} + \frac{8}{3}\,C_0^{1/2}\,C_1\,\mu^{-3/4}\,\varepsilon^{-1} + \frac{4}{3}\,C_0^{1/2}\,C_1\,\mu^{-3/4}\,\varepsilon^{-1}, \end{aligned}$$

where

$$C_0 = \int_{\mathbb{R}}[\zeta_0^2 + \frac{1}{12}\mu(\zeta_0')^2]\,dx > 0, \quad C_1 = |c|_{W^{1,\infty}} > 0$$

then wave breaking occurs for the solution of (1.2.12) in finite time $T = O(\frac{1}{\varepsilon})$.

Proof. One can adapt the proof of this Proposition in the same way of the proof of the Proposition 6 in [20], and we omit the proof here. □

1.3.3 Well-posedness for the variable bottom KdV equation

We prove now the well-posedness of the equations (1.2.17), (1.2.18). We consider the following general class of equations

$$\left| \begin{array}{l} u_t + cu_x + kc_x u + \varepsilon g u u_x + \frac{1}{6}\mu c^5 u_{xxx} = 0 \\ u_{|_{t=0}} = u^0, \end{array} \right. \tag{1.3.23}$$

where $k \in \mathbb{R}$, $g = \frac{3}{2}$ for (1.2.18) and $g = \frac{3}{2c}$ for (1.2.17). This class of equations is not included in the family of equations stated in §3.1 because $m = 0$ (here there is no $\partial_x^2 \partial_t u$ term).

Theorem 1.3.2. *Let $s > \frac{3}{2}$ and $b \in H^\infty(\mathbb{R})$. Let also $\wp'$ be a family of parameters $\theta = (\varepsilon, \beta, \alpha, \mu)$ satisfying (1.2.15). Assume morever that*

$$\exists c_0 > 0, \quad \forall\, \theta \in \wp', \qquad c(x) = \sqrt{1 - \beta b^{(\alpha)}(x)} \geq c_0.$$

Then for all $u^0 \in H^{s+1}(\mathbb{R})$, there exists $T > 0$ and a unique family of solutions $(u^\theta)_{\theta \in \wp}$ to (1.3.23) bounded in $C([0, \frac{T}{\varepsilon}]; H^{s+1}(\mathbb{R})) \cap C^1([0, \frac{T}{\varepsilon}]; H^s(\mathbb{R}))$.

Proof. As in the proof of Theorem 1.3.1, for all v smooth enough, let us define the "linearized" operator $\mathcal{L}(v, \partial)$ as:

$$\mathcal{L}(v, \partial) \;\; = \;\; \partial_t + c\partial_x + kc_x + \varepsilon g v \partial_x + \frac{1}{6}\mu c^5 \partial_x^3.$$

We define now the initial value problem as:

$$\left\{ \begin{array}{l} \mathcal{L}(v, \partial)u = \varepsilon f, \\ u_{|_{t=0}} = u^0. \end{array} \right. \tag{1.3.24}$$

Equation (1.3.24) is a linear equation which can be solved in any interval of time in which the coefficients are defined. We establish some precise energy estimates on the solution. First remark that when $m = 0$, the energy norm $|\cdot|_{X^s}$ defined in the proof of Theorem 1.3.1 does not allow to control the term $g u_{xxx}$ (for instance). Indeed, Lemma 1.3.3 requires $m \neq 0$ to be true. We show that a control of this term is however possible if we use an adequate weight function to defined the energy and use the dispersive properties of the equation. More precisely, inspired by [24] let us define the "energy" norm for all $s \geq 0$ as:

$$E^s(u)^2 = |w \Lambda^s u|_2^2$$

where the weight function w will be determined later. For the moment, we just require that there exists two positive numbers w_1, w_2 such that $\forall\, x \in \mathbb{R}$

$$w_1 \leq w(x) \leq w_2,$$

so that $E^s(u)$ is uniformly equivalent to the standard H^s-norm. Differentiating $\frac{1}{2}e^{-\varepsilon\lambda t}E^s(u)^2$ with respect to time, one gets using (1.3.24)

$$\begin{aligned}\frac{1}{2}e^{\varepsilon\lambda t}\partial_t(e^{-\varepsilon\lambda t}E^s(u)^2) &= -\frac{\varepsilon\lambda}{2}E^s(u)^2 - \big([\Lambda^s, c]\partial_x u, w\Lambda^s u\big) - (c\partial\Lambda^s u, w\Lambda^s u\big)\\ &-k(\Lambda^s(u\partial_x c), w\Lambda^s u) - \varepsilon\big([\Lambda^s, gv]\partial_x u, w\Lambda^s u) - \varepsilon\big(gv\partial\Lambda^s u, w\Lambda^s u\big)\\ &-\frac{1}{6}\mu\big([\Lambda^s, c^5]\partial_x^3 u, w\Lambda^s u\big) - \frac{1}{6}\mu\big(c^5\partial^3\Lambda^s u, w\Lambda^s u\big) + \varepsilon\big(\Lambda^s f, w\Lambda^s u\big).\end{aligned}$$

It is clear, by a simple integration by parts that

$$|\varepsilon\big(gv\partial\Lambda^s u, w\Lambda^s u\big)| \leq \varepsilon C(|v|_{W^{1,\infty}}, |w|_{W^{1,\infty}})E^s(u)^2. \tag{1.3.25}$$

Let us now focus on the seventh and eighth terms of the right hand side of the previous identity. In order to get an adequate control of the seventh term, we have to write explicitly the commutator $[\Lambda^s, c^5]$:

$$[\Lambda^s, c^5]\partial_x^3 u = \{\Lambda^s, c^5\}_2\partial_x^3 u + Q_1\partial_x^3 u,$$

where $\{\cdot,\cdot\}_2$ stands for the second order Poisson brackets,

$$\{\Lambda^s, c^5\}_2 = -s\partial_x(c^5)\Lambda^{s-2}\partial_x + \frac{1}{2}[s\partial_x^2(c^5)\Lambda^{s-2} - s(s-2)\partial_x^2(c^5)\Lambda^{s-4}\partial_x^2]$$

and Q_1 is an operator of order $s-3$ that can be controlled by the general commutator estimates of (see [51]). We thus get

$$|\big(Q_1\partial_x^3 u, w\Lambda^s u\big)| \leq C\varepsilon E^s(u)^2.$$

We now use the identity $\Lambda^2 = 1 - \partial_x^2$ and the fact that $\alpha\beta = O(\varepsilon)$, to get, as in (1.3.25),

$$|\big([s\partial_x^2(c^5)\Lambda^{s-2} - s(s-2)\partial_x^2(c^5)\Lambda^{s-4}\partial_x^2]\partial_x^3 u, w\Lambda^s u\big)| \leq \varepsilon C(|w|_{W^{1,\infty}})E^s(u)^2.$$

This leads to the expression

$$\frac{1}{6}\mu\big([\Lambda^s, c^5]\partial_x^3 u, w\Lambda^s u\big) = \frac{s}{6}\mu\big(\partial c^5\Lambda^s\partial^2 u, w\Lambda^s u\big) + Q_2,$$

where $|Q_2| \leq \varepsilon C(|w|_{W^{1,\infty}})E^s(u)^2$. Remarking now that:

$$\frac{s}{6}\mu\big(\partial c^5\Lambda^s\partial^2 u, w\Lambda^s u\big) = \\ -\frac{s}{6}\mu\big(\partial(\partial(c^5)w)\Lambda^s\partial u, \Lambda^s u\big) - \frac{s}{6}\mu\big(\partial(c^5)w, (\Lambda^s\partial u)^2\big).$$

The control of the eighth term comes in the same way:

$$\frac{1}{6}\mu\big(c^5\partial^3\Lambda^s u, w\Lambda^s u\big) = -\frac{1}{12}\mu\big(\partial^3(c^5 w)\Lambda^s u, \Lambda^s u\big) \\ -\frac{1}{4}\mu\big(\partial^2(wc^5)\Lambda^s\partial u, \Lambda^s u\big) - \frac{1}{4}\mu\big(\partial(wc^5)\Lambda^s u, \Lambda^s\partial^2 u\big)$$

similarly:

$$-\frac{1}{4}\mu\big(\partial(wc^5)\Lambda^s u, \Lambda^s\partial^2 u\big) = \\ \frac{1}{4}\mu\big(\partial^2(c^5 w)\Lambda^s\partial u, \Lambda^s u\big) + \frac{1}{4}\mu\big(\partial(c^5 w), (\Lambda^s\partial u)^2\big).$$

We choice here w so that

$$-\frac{s}{6}\mu\big(\partial(c^5)w, (\Lambda^s\partial u)^2\big) + \frac{1}{4}\mu\big(\partial(c^5 w), (\Lambda^s\partial u)^2\big) = 0 \tag{1.3.26}$$

therefore if one take $w = c^{5(\frac{2s}{3}-1)}$ we get easily (1.3.26). Finally, one has

$$\frac{1}{6}\mu\big([\Lambda^s, c^5]\partial_x^3 u, w\Lambda^s u\big) + \frac{1}{6}\mu\big(c^5\partial^3\Lambda^s u, w\Lambda^s u\big) \\ = Q_2 - \frac{s}{6}\mu\big(\partial(\partial(c^5)w)\Lambda^s\partial u, \Lambda^s u\big) - \frac{1}{12}\mu\big(\partial^3(c^5 w)\Lambda^s u, \Lambda^s u\big) \\ -\frac{1}{4}\mu\big(\partial^2(c^5 w)\Lambda^s\partial u, \Lambda^s u\big) + \frac{1}{4}\mu\big(\partial^2(c^5 w)\Lambda^s\partial u, \Lambda^s u\big),$$

using again the fact that $\alpha\beta = O(\varepsilon)$ one can deduce

$$e^{\varepsilon\lambda t}\partial_t(e^{-\varepsilon\lambda t}E^s(u)^2) \leq \varepsilon\big(C(E^s(v)) - \lambda\big)E^s(u)^2 + 4\varepsilon E^s(f)E^s(u).$$

This inequality, together with end of the proof of Theorem 1.3.1, easily yields the result. □

1.4 Rigorous justification of the variable bottom equations

1.4.1 Rigorous justification of the variable bottom CH equation

We restrict here our attention to parameters ε, β, α and μ linked by the relations

$$\varepsilon = O(\sqrt{\mu}),\ \beta\alpha = O(\varepsilon),\ \beta\alpha = O(\mu^2). \tag{1.4.1}$$

These conditions are stronger than (1.2.2), and this allows us to control the secular effects that prevented us from proving an H^s-consistency (and *a fortiori* a full justification) for the variable bottom equations of §1.2. The notion of H^s-consistency is defined below:

Definition 1.4.1. *Let $\wp_1$ be a family of parameters $\theta = (\varepsilon, \beta, \alpha, \mu)$ satisfying (4.2.3). A family $(\zeta^\theta, u^\theta)_{\theta\in\wp_1}$ is H^s-consistent of order $s \geq 0$ and on $[0, \frac{T}{\varepsilon}]$ with the GN equations (1.2.1), if for all $\theta \in \wp_1$, (and denoting $h^\theta = 1 + \varepsilon\zeta^\theta - \beta b^{(\alpha)}$),*

$$\begin{cases} \zeta_t^\theta + [h^\theta u]_x = \mu^2 r_1^\theta \\ u_t^\theta + \zeta_x^\theta + \varepsilon u^\theta u_x^\theta = \frac{\mu}{3}\frac{1}{h^\theta}[(h^\theta)^3(u_{xt}^\theta + \varepsilon u^\theta u_{xx}^\theta - \varepsilon(u_x^\theta)^2)]_x + \mu^2 r_2^\theta \end{cases}$$

with $(r_1^\theta, r_2^\theta)_{\theta\in\wp_1}$ bounded in $L^\infty([0, \frac{T}{\varepsilon}], H^s(\mathbb{R})^2)$.

Remark 1.4.1. *Since $H^s(\mathbb{R})$ is continously embedded in $L^\infty(\mathbb{R})$ for $s > 1/2$, the H^s-consistency implies the L^∞-consistency when $s > 1/2$.*

Proposition 1.4.1. *Let $b \in H^\infty(\mathbb{R})$ and $p \in \mathbb{R}$. Assume that*

$$A = p, \quad B = p - \frac{1}{6} \quad E = -\frac{3}{2}p - \frac{1}{6}, \quad F = -\frac{9}{2}p - \frac{23}{24}.$$

Then:

- *For any family $\wp_1$ of parameters satisfying (4.2.3),*
- *For all $s \geq 0$ large enough and $T > 0$,*
- *For any bounded family $(u^\theta)_{\theta\in\wp_1} \in C([0, \frac{T}{\varepsilon}], H^s(\mathbb{R}))$ solving (1.1.12),*

the familly $(\zeta^\theta, u^\theta)_{\theta\in\wp_1}$ with (omitting the index θ)

$$\zeta := cu + \frac{\varepsilon}{4}u^2 + \frac{\mu}{6}c^4 u_{xt} - \varepsilon\mu c^4[\frac{1}{6}uu_{xx} + \frac{5}{48}u_x^2], \tag{1.4.2}$$

is H^s-consistent on $[0, \frac{T}{\varepsilon}]$ with the GN equations (1.2.1).

Proof. This is clear by using the same arguments of the proof of Proposition 1.2.1 if we remark that now the term $c_x u$ (responsible of the secular growth effects) is of order $O(\mu^2)$ in $L^\infty([0, \frac{T}{\varepsilon}], H^s(\mathbb{R}))$. Therefore this quantity is not required in the identity (1.2.7) that defines v. In the proof of Proposition 1.2.1 there is therefore no $c_x u$ term and the $O_{L^\infty}(\mu^2)$ terms in Proposition 1.2.1 are now of order $O(\mu^2)$. ∎

In Proposition 1.4.1 , we constructed a family (u^θ, ζ^θ) *consistent* with the Green-Naghdi equations in the sense of Definition 1.4.1. A consequence of the following theorem is a stronger result: this family provides a good approximation of the exact solutions $(\underline{u}^\theta, \underline{\zeta}^\theta)$ of the Green-Naghdi equations with same initial data in the sense that $(\underline{u}^\theta, \underline{\zeta}^\theta) = (u^\theta, \zeta^\theta) + O(\mu^2 t)$ for times $O(1/\varepsilon)$.

Theorem 1.4.1. *Let $b \in H^\infty(\mathbb{R})$, $s \geq 0$ and $\wp_1$ be a family of parameters satisfying (4.2.3) with $\beta = O(\varepsilon)$. Let also A, B, E and F be as in Proposition 1.4.1. If $B < 0$ then there exists $D > 0$, $P > D$ and $T > 0$ such that for all $u_0^\theta \in H^{s+P}(\mathbb{R})$:*

- *There is a unique family $(u^\theta, \zeta^\theta)_{\theta \in \wp_1} \in C([0, \dfrac{T}{\varepsilon}]; H^{s+P}(\mathbb{R}) \times H^{s+P-2})$ given by the resolution of (1.1.12) with initial condition u_0^θ and formula (1.4.2);*
- *There is a unique family $(\underline{u}^\theta, \underline{\zeta}^\theta)_{\theta \in \wp_1} \in C([0, \dfrac{T}{\varepsilon}]; H^{s+D}(\mathbb{R})^2)$ solving the Green-Naghdi equations (1.2.1) with initial condition $(u_0^\theta, \zeta^\theta_{|t=0})$.*

Moreover, one has for all $\theta \in \wp_1$,

$$\forall t \in [0, \frac{T}{\varepsilon}], \qquad |\underline{u}^\theta - u^\theta|_{L^\infty([0,t]\times\mathbb{R})} + |\underline{\zeta}^\theta - \zeta^\theta|_{L^\infty([0,t]\times\mathbb{R})} \leq \mathrm{Cst}\ \mu^2 t.$$

Remark 1.4.2. *It is known (see [3]) that the Green-Naghdi equations give, under the scaling (1.1.9) with $\beta = O(\varepsilon)$ a correct approximation of the exact solutions of the full water waves equations (with a precision $O(\mu^2 t)$ and over a time scale $O(1/\varepsilon)$). It follows that the unidirectional approximation discussed above approximates the solution of the water waves equations with the same accuracy.*

Remark 1.4.3. *We used the unidirectional equations derived on the velocity as the basis for the approximation justified in the Theorem 1.4.1. One could of course use instead the unidirectional approximation (1.2.10) derived on the surface elevation.*

Proof. The first point of the theorem is a direct consequence of Theorem 1.3.1. Thanks to Proposition 1.4.1, we know that $(u^\theta, \zeta^\theta)_{\theta \in \wp_1}$ is H^s-consistent with the Green-Naghdi equations (1.2.1), so that the second point of the theorem and the error estimate follow at once from the well-posedness and stability of the Green-Naghdi equations under the present scaling (see Theorem 4.10 of [4]). $\square$

1.4.2 Rigorous justification of the variable bottom KdV-top equation

In this subsection the parameters ε, β, α and μ are assumed to satisfy

$$\varepsilon = O(\mu), \ \beta\alpha = O(\varepsilon^2). \tag{1.4.3}$$

We give first a Proposition regarding the H^s-consistency result for the KdV-top equation.

$$\zeta_t + c\zeta_x + \frac{3}{2c}\varepsilon\zeta\zeta_x + \frac{1}{6}\mu c^5\zeta_{xxx} + \frac{1}{2}c_x\zeta = 0, \tag{1.4.4}$$

Proposition 1.4.2. *Let $b \in H^\infty(\mathbb{R})$. Then:*

- *For any family $\wp_1'$ of parameters satisfying (4.2.5),*
- *For all $s \geq 0$ large enough and $T > 0$,*
- *For any bounded family $(\zeta^\theta)_{\theta\in\wp_1'} \in C([0, \frac{T}{\varepsilon}], H^s(\mathbb{R}))$ solving (1.4.4)*

the familly $(\zeta^\theta, u^\theta)_{\theta\in\wp'}$ with (omitting the index θ)

$$u := \frac{1}{c}\Big(\zeta - \frac{\varepsilon}{4c^2}\zeta^2 + \mu\frac{1}{6}c^4\zeta_{xx}\Big) \tag{1.4.5}$$

is H^s-consistent on $[0, \frac{T}{\varepsilon}]$ with the equations (1.2.1).

Remark 1.4.4. *Similarly, one can prove that the solution of the following KdV-top equations*

$$u_t + cu_x + \frac{3}{2}\varepsilon u u_x + \frac{1}{6}\mu c^5 u_{xxx} + \frac{3}{2}c_x u = 0, \tag{1.4.6}$$

on the velocity, with

$$\zeta := cu + \frac{\varepsilon}{4}u^2 - \frac{\mu}{6}c^5 u_{xx} \tag{1.4.7}$$

is H^s-consistent with the equations Green-Naghdi equations (1.2.1).

Proof. One can adapt the proof of Proposition 1.2.3 in the same way as we adapted the proof of Proposition 1.2.1 to establish Proposition 1.4.1 (we also use the fact that if a family is consistent with the Boussinesq equations (4.1.7), it is also consistent with the Green-Naghdi equations (1.2.1) under the present scaling). $\square$

The following Theorem deals with the rigorous justification of the KdV variable bottom equation:

Theorem 1.4.2. *Let $b \in H^\infty(\mathbb{R})$, $s \geq 0$ and $\wp_1'$ be a family of parameters satisfying (4.2.5) with $\beta = O(\varepsilon)$. If $B < 0$ then there exist $D > 0$, $P > D$ and $T > 0$ such that for all $\zeta_0^\theta \in H^{s+P}(\mathbb{R})$:*

- *There is a unique family $(\zeta^\theta, u^\theta)_{\theta\in\wp_1'} \in C([0, \frac{T}{\varepsilon}]; H^{s+P-2}(\mathbb{R}) \times H^{s+P})$ given by the resolution of (1.4.4) with initial condition ζ_0^θ and formula (1.4.5);*

- *There is a unique family* $(\underline{\zeta}^\theta, \underline{u}^\theta)_{\theta\in\wp_1'} \in C([0,\frac{T}{\varepsilon}]; H^{s+D}(\mathbb{R})^2)$ *solving the Green-Naghdi equations (1.2.1) with initial condition* $(\zeta_0^\theta, u^\theta_{|t=0})$.

Moreover, one has for all $\theta \in \wp_1'$,

$$\forall t \in [0,\frac{T}{\varepsilon}], \qquad |\underline{u}^\theta - u^\theta|_{L^\infty([0,t]\times\mathbb{R})} + |\underline{\zeta}^\theta - \zeta^\theta|_{L^\infty([0,t]\times\mathbb{R})} \leq \mathrm{Cst}\, \varepsilon^2 t.$$

Remark 1.4.5. *We used the unidirectional equation derived for the free surface elevation as the basis for the approximation justified in the Theorem 1.4.2. One could of course use instead the unidirectional approximation (1.4.6) derived on velocity.*

Remark 1.4.6. *It is known (see [10, 67, 36]) that in the KdV scaling* $\varepsilon = O(\mu)$ *and for flat bottoms, any initial condition for the Boussinesq (or Green-Naghdi) equations splits up into two counter propagating waves, each of which evolving according to a KdV equation. It is possible to choose the initial condition in order to cancel the left going wave (for instance); one thus gets a purely rightgoing wave. In the more nonlinear regime* $\varepsilon = O(\sqrt{\mu})$ *investigated in [20] and in the present paper, the initial condition under consideration is of this kind in the sense that it leads to a right going wave (up to small corrector terms). It is not clear however that any general initial condition splits up into two counter propagating waves, each of which evolving according to an equation of the form (1.1.12). Indeed, the nonlinear terms are more important in this regime and the nonlinear interaction of the counter propagating components may grow secularly (see [52] for a description of this phenomenon). The presence of topography terms makes such a general decoupling even more difficult to establish since they also induce secular terms in the backscattered component (this is the reason why we can only prove* L^∞*-consistency in Proposition 1.2.1). If such a decoupling holds, it will therefore be with a weaker precision than in the KdV, flat bottom, regime.*

Proof. The first point of the theorem is a direct consequence of Theorem 1.3.2. Thanks to Proposition 1.4.2, we know that $(u^\theta, \zeta^\theta)_{\theta\in\wp_1'}$ is H^s-consistent with the Green-Naghdi equations (1.2.1), so that the second point follows as in Theorem 1.4.1. □

PARTIE II

Green-Naghdi equations

Large Time existence For $1D$ Green-Naghdi equations

We consider in this chapter the $1D$ Green-Naghdi equations that are commonly used in coastal oceanography to describe the propagation of large amplitude surface waves. We show that the solution of the Green-Naghdi equations can be constructed by a standard Picard iterative scheme so that there is no loss of regularity of the solution with respect to the initial condition.

2.1 Introduction

2.1.1 Presentation of the problem

The water-waves problem for an ideal liquid consists of describing the motion of the free surface and the evolution of the velocity field of a layer of perfect, incompressible, irrotational fluid under the influence of gravity. This motion is described by the free surface Euler equations that are known to be well-posed after the works of Nalimov [60], Yasihara [79], Craig [22], Wu [74, 75] and Lannes [50]. But, because of the complexity of these equations, they are often replaced for pratical purposes by approximate asymptotic systems. The most prominent examples are the Green-Naghdi equations (GN) – which is a widely used model in coastal oceanography ([29, 9, 30] and, for instance, [40, 48])–, the Shallow-Water equations, and the Boussinesq systems; their range of validity depends on the physical characteristics of the flow under consideration. In other words, they depend on certain assumptions made on the dimensionless parameters ε, μ defined as:

$$\varepsilon = \frac{a}{h_0}, \quad \mu = \frac{h_0^2}{\lambda^2};$$

where a is the order of amplitude of the waves and the bottom variations; λ is the wave-length of the waves and the bottom variations; h_0 is the reference depth. The parameter ε is often called nonlinearity parameter; while μ is the shallowness parameter. In the shallow-water scaling ($\mu \ll 1$), and without smallness assumption on ε one can derive the so-called Green-Naghdi equations (see [29, 53] for a derivation and [3] for a rigorous justification) also called Serre or fully nonlinear Boussinesq equations [54].
In nondimensionalized variables, denoting by $u(t,x)$ the parameterization of the vertically averaged horizontal component of the velocity at time t, the equations read

$$\begin{cases} \partial_t \zeta + \nabla \cdot (hu) = 0, \\ (h + \mu h \mathcal{T}[h, \varepsilon b])\partial_t u + h\nabla\zeta + \varepsilon h(u \cdot \nabla)u \\ \;+ \mu\varepsilon\big\{ -\dfrac{1}{3}\nabla[(h^3((u\cdot\nabla)(\nabla\cdot u) - (\nabla\cdot u)^2)] + h\Re[h,\varepsilon b]u \big\} = 0, \end{cases} \tag{2.1.1}$$

where $h = 1 + \varepsilon(\zeta - b)$ and

$$\mathcal{T}[h,\varepsilon b]W = -\frac{1}{3h}\nabla(h^3\nabla\cdot W) + \frac{\varepsilon}{2h}[\nabla(h^2\nabla b\cdot W) - h^2\nabla b\nabla\cdot W] + \varepsilon^2\nabla b\nabla b\cdot W,$$

while the purely topographical term $\Re[h,\varepsilon b]u$ is defined as:

$$\begin{aligned} \Re[h,\varepsilon b]u \;&=\; \frac{\varepsilon}{2h}[\nabla(h^2(u\cdot\nabla)^2 b) - h^2((u\cdot\nabla)(\nabla\cdot u) - (\nabla\cdot u)^2)\nabla b] \\ &\quad + \varepsilon^2((u\cdot\nabla)^2 b)\nabla b. \end{aligned}$$

This model is often used in coastal oceanography because it takes into account the dispersive effects neglected by the shallow-water and it is more nonlinear than the Boussinesq equations. A recent rigorous justification of the GN model was given by Li [55] in $1D$ and for flat bottoms, and by B. Alvarez-Samaniego and D. Lannes [3] in 2008 in the general case. This latter reference relies on well-posedness results for these equations given in [4] and based on general well-posedness results for evolution equations using a Nash-Moser scheme. The result of [4] covers both the case of $1D$ and $2D$ surfaces, and allows for non flat bottoms. The reason why a Nash-Moser scheme is used there is because the estimates on the linearized equations exhibit losses of derivatives. However, in the $1D$ case with flat bottoms, such losses do not occur and it is possible to construct a solution with a standard Picard iterative scheme as in [55]. Our goal here is to show that it is also possible to use such a simple scheme in the $1D$ case with non flat bottoms, thanks to a careful analysis of the linearized equations.

2.1.2 Organization of the chapter

We start by giving some preliminary results in Section 2.2.1; the main theorem is then stated in Section 2.2.2 and proved in Section 2.2.3. Finally, in Section 2.3, we give the existence and uniqueness of a solution to the linear Cauchy problem associated to the Green-Naghdi equations. The proof of the energy conservation, stated in the main theorem, is given in Section 2.4.

2.1.3 Notation

We denote by $C(\lambda_1, \lambda_2, ...)$ a constant depending on the parameters λ_1, λ_2, ... and *whose dependence on the* λ_j *is always assumed to be nondecreasing.*
The notation $a \lesssim b$ means that $a \leq Cb$, for some nonegative constant C whose exact expression is of no importance (*in particular, it is independent of the small parameters involved*).
Let p be any constant with $1 \leq p < \infty$ and denote $L^p = L^p(\mathbb{R})$ the space of all Lebesgue-measurable functions f with the standard norm

$$|f|_{L^p} = \Big(\int_{\mathbb{R}} |f(x)|^p dx\Big)^{1/p} < \infty.$$

When $p = 2$, we denote the norm $|\cdot|_{L^2}$ simply by $|\cdot|_2$. The inner product of any functions f_1 and f_2 in the Hilbert space $L^2(\mathbb{R})$ is denoted by

$$(f_1, f_2) = \int_{\mathbb{R}} f_1(x) f_2(x) dx.$$

The space $L^\infty = L^\infty(\mathbb{R})$ consists of all essentially bounded, Lebesgue-measurable functions f with the norm

$$|f|_{L^\infty} = \operatorname{ess\,sup} |f(x)| < \infty.$$

We denote by $W^{1,\infty} = W^{1,\infty}(\mathbb{R}) = \{f \in L^\infty, \partial_x f \in L^\infty\}$ endowed with its canonical norm.
For any real constant s, $H^s = H^s(\mathbb{R})$ denotes the Sobolev space of all tempered distributions f with the norm $|f|_{H^s} = |\Lambda^s f|_2 < \infty$, where Λ is the pseudo-differential operator $\Lambda = (1 - \partial_x^2)^{1/2}$.
For any functions $u = u(x,t)$ and $v(x,t)$ defined on $\mathbb{R} \times [0,T)$ with $T > 0$, we denote the inner product, the L^p-norm and especially the L^2-norm, as well as the Sobolev norm, with respect to the spatial variable x, by $(u,v) = (u(\cdot,t), v(\cdot,t))$, $|u|_{L^p} = |u(\cdot,t)|_{L^p}$, $|u|_{L^2} = |u(\cdot,t)|_{L^2}$, and $|u|_{H^s} = |u(\cdot,t)|_{H^s}$, respectively.
Let $C^k(\mathbb{R})$ denote the space of k-times continuously differentiable functions and $C_0^\infty(\mathbb{R})$ denote the space of infinitely differentiable functions, with compact support in $\mathbb{R}$; we also denote by $C_b^\infty(\mathbb{R})$ the space of infinitely differentiable functions that are bounded together with all their derivatives.
Let f be a function of the independent variables x_1, x_2,...,x_m; its partial derivative with respect to x_k is denoted by $\partial_{x_k} f = f_{x_k}$ for $1 \leq k \leq m$.

For any closed operator T defined on a Banach space X of functions, the commutator $[T,f]$ is defined by $[T,f]g = T(fg) - fT(g)$ with f, g and fg belonging to the domain of T.

2.2 Well-posedness of the Green-Naghdi equations in $1D$

For one dimensional surfaces, the Green-Naghdi equations (2.1.1) can be simplified, after some computations, into

$$\begin{cases} \partial_t \zeta + \partial_x(hu) = 0, \\ (h + \mu h \mathcal{T}[h, \varepsilon b])[\partial_t u + \varepsilon u \partial_x u] + h\partial_x \zeta + \varepsilon \mu h Q[h, \varepsilon b](u) = 0 \end{cases} \tag{2.2.1}$$

where $h = 1 + \varepsilon(\zeta - b)$ and

$$\mathcal{T}[h, \varepsilon b]w = -\frac{1}{3h}\partial_x(h^3 w_x) + \frac{\varepsilon}{2h}[\partial_x(h^2 b_x w) - h^2 b_x w_x] + \varepsilon^2 b_x^2 w,$$

$$Q[h, \varepsilon b](w) = \frac{2}{3h}\partial_x(h^3 w_x^2) + \varepsilon h w_x^2 b_x + \varepsilon \frac{1}{2h}\partial_x(h^2 w^2 b_{xx}) + \varepsilon^2 w^2 b_{xx} b_x.$$

Remark 2.2.1. *The interest of the formulation (2.2.1) of the Green-Naghdi equation is that all the third order derivatives of u have been factorized by $(h + \mu h \mathcal{T}[h, \varepsilon b])$. Indeed, $Q[h, \varepsilon b]$ is a second order differential operator. This was used in [55] in the case of flat bottoms ($b = 0$).*

2.2.1 Preliminary results

For the sake of simplicity, we write

$$\mathfrak{T} = h + \mu h \mathcal{T}[h, \varepsilon b].$$

We always assume that the nonzero depth condition

$$\exists\, h_0 > 0, \quad \inf_{x \in \mathbb{R}} h \geq h_0, \quad h = 1 + \varepsilon(\zeta - b) \tag{2.2.2}$$

is valid initially, which is a necessary condition for the GN system (2.2.1) to be physically valid. We shall demonstrate that the operator $\mathfrak{T}$ plays an important role in the energy estimate and the local well-posedness of the GN system (2.2.1). Therefore, we give here some of its properties.
The following lemma gives an important invertibility result on $\mathfrak{T}$.

Lemma 2.2.1. *Let $b \in C_b^\infty(\mathbb{R})$ and $\zeta \in W^{1,\infty}(\mathbb{R})$ be such that (2.2.2) is satisfied. Then the operator*

$$\mathfrak{T} : H^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$$

is well defined, one-to-one and onto.

Remark 2.2.2. *Here and throughout the rest of this chapter, and for the sake of simplicity, we do not try to give some optimal regularity assumption on the bottom parameterization b. This could easily be done, but is of no interest for our present purpose. Consequently, we ommit to write the dependance on b of the different quantities that appear in the proof.*

Proof. In order to prove the invertibility of $\mathfrak{T}$, let us first remark that the quantity $|v|_*^2$ defined as

$$|v|_*^2 = |v|_2^2 + \mu|\partial_x v|_2^2,$$

is equivalent to the $H^1(\mathbb{R})$-norm but not uniformly with respect to $\mu \in (0,1)$. We define by $H^1_*(\mathbb{R})$ the space $H^1(\mathbb{R})$ endowed with this norm. The bilinear form:

$$a(u,v) = (hu,v) + \mu\big(h\big(\frac{h}{\sqrt{3}}u_x - \frac{\sqrt{3}}{2}\varepsilon b_x u\big), \frac{h}{\sqrt{3}}v_x - \frac{\sqrt{3}}{2}\varepsilon b_x v\big) + \frac{\mu\varepsilon^2}{4}(hb_x u, b_x v).$$

is obviously continous on $H^1_*(\mathbb{R}) \times H^1_*(\mathbb{R})$. Remarking that

$$\begin{aligned} a(v,v) = (\mathfrak{T}v, v) \;\; = \;\; & (hv,v) \\ & +\mu\big(h\big(\frac{h}{\sqrt{3}}v_x - \frac{\sqrt{3}}{2}\varepsilon b_x v\big), \frac{h}{\sqrt{3}}v_x - \frac{\sqrt{3}}{2}\varepsilon b_x v\big) + \frac{\mu\varepsilon^2}{4}(hb_x v, b_x v), \end{aligned}$$

we have

$$\begin{aligned} |v|_*^2 \;\; & \leq \;\; |v|_2^2 + \frac{3\mu}{h_0^2}|\frac{h}{\sqrt{3}}v_x|_2^2 \\ & \leq \;\; |v|_2^2 + \frac{6\mu}{h_0^2}\Big(|\frac{h}{\sqrt{3}}v_x - \frac{\sqrt{3}}{2}\varepsilon b_x v|_2^2 + \frac{3\varepsilon^2}{4}|b_x v|_2^2\Big). \end{aligned}$$

One deduces that

$$\max\Big\{1, \frac{18}{h_0^2}\Big\}\Big(|v|_2^2 + \mu|\frac{h}{\sqrt{3}}v_x - \frac{\sqrt{3}}{2}\varepsilon b_x v|_2^2 + \frac{\mu\varepsilon^2}{4}|b_x v|_2^2\Big) \geq |v|_*^2.$$

Since from (2.2.2) we also get

$$a(v,v) \;\; \geq \;\; h_0|v|_2^2 + \mu h_0\Big(|\frac{h}{\sqrt{3}}v_x - \frac{\sqrt{3}}{2}\varepsilon b_x v|_2^2 + \frac{\mu\varepsilon^2}{4}|b_x v|_2^2\Big),$$

it is easy to deduce that

$$a(v,v) \geq \frac{h_0}{\max\left\{1, \frac{18}{h_0^2}\right\}}|v|_*^2. \tag{2.2.3}$$

In particular, a is coercive on H^1_*. Using Lax-Milgram lemma, for all $f \in L^2(\mathbb{R})$, there exists unique $u \in H^1_*(\mathbb{R})$ such that, for all $v \in H^1_*(\mathbb{R})$

$$a(u,v) = (f,v);$$

equivalently, there is a unique variational solution to the equation

$$\mathfrak{T}u = f. \tag{2.2.4}$$

We then get from the definition of $\mathfrak{T}$ that

$$\partial_x^2 u = \frac{hu + \frac{\varepsilon\mu}{2}\partial_x(h^2 b_x)u + \varepsilon^2\mu h b_x^2 u - \frac{\mu}{3}\partial_x h^3 \partial_x u - f}{\frac{\mu h^3}{3}};$$

since $u \in H^1(\mathbb{R})$ and $f \in L^2(\mathbb{R})$, we get $\partial_x^2 u \in L^2(\mathbb{R})$ and thus $u \in H^2(\mathbb{R})$. $\square$

The following lemma then gives some properties of the inverse operator $\mathfrak{T}^{-1}$.

Lemma 2.2.2. *Let $b \in C_b^\infty(\mathbb{R})$, $t_0 > 1/2$ and $\zeta \in H^{t_0+1}(\mathbb{R})$ be such that (2.2.2) is satisfied. Then:*

(i) $\forall 0 \leq s \leq t_0+1, \quad |\mathfrak{T}^{-1}f|_{H^s} + \sqrt{\mu}|\partial_x \mathfrak{T}^{-1} f|_{H^s} \leq C(\frac{1}{h_0}, |h-1|_{H^{t_0+1}})|f|_{H^s}$;

(ii) $\forall 0 \leq s \leq t_0+1, \quad \sqrt{\mu}|\mathfrak{T}^{-1}\partial_x g|_{H^s} \leq C(\frac{1}{h_0}, |h-1|_{H^{t_0+1}})|g|_{H^s}$;

(iii) If $s \geq t_0+1$ and $\zeta \in H^s(\mathbb{R})$ then:

$$\| \mathfrak{T}^{-1} \|_{H^s(\mathbb{R}) \to H^s(\mathbb{R})} + \sqrt{\mu} \| \mathfrak{T}^{-1}\partial_x \|_{H^s(\mathbb{R}) \to H^s(\mathbb{R})} \leq c_s,$$

where c_s is a constant depending on $\frac{1}{h_0}$, $|h-1|_{H^s}$ and independent of $(\mu,\varepsilon) \in (0,1)^2$.

Proof. **Step 1**. We prove that if $u \in H^1_*(\mathbb{R})$ solves

$$\mathfrak{T}u = f + \sqrt{\mu}\partial_x g$$

for $f, g \in L^2(\mathbb{R})$, then one has

$$|u|_{H^1_*} \leq C\big(\frac{1}{h_0}\big)\big(|f|_2 + |g|_2\big).$$

Indeed, multiplying the equation by u and integrating by parts, one gets, with the notations used in the proof of lemma 2.2.1

$$a(u,u) \leq (f,u) - (g, \sqrt{\mu}\partial_x u).$$

We thus get from the proof of Lemma 2.2.1 and Cauchy-Schwarz inequality that

$$\frac{h_0}{\max\{1, \frac{18}{h_0^2}\}}|u|^2_{H^1_*} \leq |f|_2|u|_2 + |g|_2|u|_{H^1_*},$$

and the result follows easily.

Step 2. We prove here that $|\mathfrak{T}^{-1}f|_{H^s} + \sqrt{\mu}|\partial_x\mathfrak{T}^{-1}f|_{H^s} \leq C(\frac{1}{h_0}, |h-1|_{H^{t_0+1}})|f|_{H^s}$. Indeed, if $f \in H^s$ and $u = \mathfrak{T}^{-1}f$ then $\mathfrak{T}u = f$. Applying Λ^s to this identity, we get

$$\begin{array}{rcl} \mathfrak{T}(\Lambda^s u) &=& \Lambda^s f + [\mathfrak{T}, \Lambda^s]u \\ &=& \tilde{f} + \sqrt{\mu}\partial_x\tilde{g}, \end{array}$$

with,

$$\tilde{f} = \Lambda^s f - [\Lambda^s, h]u + \frac{\varepsilon\mu}{2}[\Lambda^s, h^2 b_x]u_x - \varepsilon^2\mu[\Lambda^s, h^2 b_x]u,$$

and

$$\tilde{g} = \frac{\sqrt{\mu}}{3}[\Lambda^s, h^3]u_x - \frac{\varepsilon\sqrt{\mu}}{2}[\Lambda^s, h^2 b_x]u.$$

Now, one can deduce from the commutator estimate (see e.g Lemma 4.6 of [4])

$$|[\Lambda^s, F]G|_2 \lesssim |\nabla F|_{H^{t_0}}|G|_{H^{s-1}} \tag{2.2.5}$$

that

$$|\tilde{f}|_2 + |\tilde{g}|_2 \quad \leq \quad |f|_{H^s} + C\big(\frac{1}{h_0}, |h-1|_{H^{t_0+1}}\big)\big(|u|_{H^{s-1}} + \sqrt{\mu}|\partial_x u|_{H^{s-1}}\big).$$

One can use Step 1 and a continuous induction on s to show that the inequality (i) holds for $0 \leq s \leq t_0 + 1$.

Step 3. We prove here that $\sqrt{\mu}|\mathfrak{T}^{-1}\partial_x g|_{H^s} \leq C(\frac{1}{h_0}, |h-1|_{H^{t_0+1}})|g|_{H^s}$. Indeed, if $g \in H^s$ and $u = \sqrt{\mu}\mathfrak{T}^{-1}\partial_x g$ then $\mathfrak{T}u = \sqrt{\mu}\partial_x g$ and thus

$$\mathfrak{T}(\Lambda^s u) = \tilde{f} + \sqrt{\mu}\partial_x\tilde{g},$$

with,

$$\tilde{f} = -[\Lambda^s, h]u + \frac{\varepsilon\mu}{2}[\Lambda^s, h^2 b_x]u_x - \varepsilon^2\mu[\Lambda^s, h^2 b_x]u,$$

and

$$\tilde{g} = \Lambda^s g + \frac{\sqrt{\mu}}{3}[\Lambda^s, h^3]u_x - \frac{\varepsilon\sqrt{\mu}}{2}[\Lambda^s, h^2 b_x]u.$$

Proceeding now as for the Step 2, one can deduce (ii).
Step 4. If $s \geq t_0 + 1$ then one can prove (iii) proceeding as in Step 2 and 3 above, but replacing the commutator estimate (2.2.5) by the following one

$$|[\Lambda^s, F]G|_2 \lesssim |\nabla F|_{H^{s-1}} |G|_{H^{s-1}}. \tag{2.2.6}$$

$\square$

2.2.2 Linear analysis

In order to rewrite the GN equations (2.2.1) in a condensed form, let us decompose $Q[h, \varepsilon b](u)$ as

$$\varepsilon\mu h Q[h, \varepsilon b](u) = Q_1[U]u_x + q_2(U)$$

where $U = (\zeta, u)^T$ and

$$\begin{aligned} Q_1[U]f &= \frac{2}{3}\varepsilon\mu\partial_x(h^3 u_x f) + \varepsilon^2\mu h^2 b_x u_x f + \varepsilon^2 \mu h^2 b_{xx} u f \\ q_2(U) &= \varepsilon^3 \mu h b_{xx} b_x u^2 + \frac{1}{2}\varepsilon^2\mu\partial_x(h^2 b_{xx})u^2. \end{aligned} \tag{2.2.7}$$

The Green-Naghdi equations (2.2.1) can be written after applying $\mathfrak{T}^{-1}$ to both sides of the second equation in (2.2.1) as

$$\partial_t U + A[U]\partial_x U + B(U) = 0, \tag{2.2.8}$$

with $U = (\zeta, u)^T$ and where

$$A[U] = \begin{pmatrix} \varepsilon u & h \\ \mathfrak{T}^{-1}(h\cdot) & \varepsilon u + \mathfrak{T}^{-1}Q_1[U] \end{pmatrix} \tag{2.2.9}$$

and

$$B(U) = \begin{pmatrix} \varepsilon b_x u \\ \mathfrak{T}^{-1} q_2(U) \end{pmatrix}. \tag{2.2.10}$$

This subsection is devoted to the proof of energy estimates for the following initial value problem around some reference state $\underline{U} = (\underline{\zeta}, \underline{u})^T$:

$$\begin{cases} \partial_t U + A[\underline{U}]\partial_x U + B(\underline{U}) = 0; \\ U_{|t=0} = U_0. \end{cases} \tag{2.2.11}$$

We define now the X^s spaces, which are the energy spaces for this problem.

Definition 2.2.1. *For all $s \geq 0$ and $T > 0$, we denote by X^s the vector space $H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R})$ endowed with the norm*

$$\forall\, U = (\zeta, u) \in X^s, \quad |U|^2_{X^s} := |\zeta|^2_{H^s} + |u|^2_{H^s} + \mu |\partial_x u|^2_{H^s},$$

while X^s_T stands for $C([0, \frac{T}{\varepsilon}]; X^s)$ endowed with its canonical norm.

First remark that a symmetrizer for $A[\underline{U}]$ is given by

$$S = \begin{pmatrix} 1 & 0 \\ 0 & \underline{\mathfrak{T}} \end{pmatrix}, \tag{2.2.12}$$

with $\underline{h} = 1 + \varepsilon(\underline{\zeta} - b)$ and $\underline{\mathfrak{T}} = \underline{h} + \mu \underline{h} \mathcal{T}[\underline{h}, \varepsilon b]$. A natural energy for the IVP (2.2.11) is given by

$$E^s(U)^2 = (\Lambda^s U, S \Lambda^s U). \tag{2.2.13}$$

The link between $E^s(U)$ and the X^s-norm is investigated in the following Lemma.

Lemma 2.2.3. *Let $b \in C^\infty_b(\mathbb{R})$, $s \geq 0$ and $\underline{\zeta} \in W^{1,\infty}(\mathbb{R})$. Under the condition (2.2.2), $E^s(U)$ is uniformly equivalent to the $|\cdot|_{X^s}$-norm with respect to $(\mu, \varepsilon) \in (0,1)^2$:*

$$E^s(U) \leq C\big(|\underline{h}|_{L^\infty}, |\underline{h}_x|_{L^\infty}\big) |U|_{X^s},$$

and

$$|U|_{X^s} \leq C\big(\frac{1}{h_0}\big) E^s(U).$$

Proof. Notice first that

$$E^s(U)^2 = |\Lambda^s \zeta|^2_2 + (\Lambda^s u, \underline{\mathfrak{T}} \Lambda^s u),$$

one gets the first estimate using the explicit expression of $\underline{\mathfrak{T}}$, integration by parts and Cauchy-Schwarz inequality.
The other inequality can be proved by using that $\inf_{x \in \mathbb{R}} h \geq h_0 > 0$ and proceeding as in the proof of Lemma 2.2.1. □

We prove now the energy estimates in the following proposition:

Proposition 2.2.1. *Let $b \in C^\infty_b(\mathbb{R})$, $t_0 > 1/2$, $s \geq t_0 + 1$. Let also $\underline{U} = (\underline{\zeta}, \underline{u})^T \in X^s_T$ be such that $\partial_t \underline{U} \in X^{s-1}_T$ and satisfying the condition (2.2.2) on $[0, \frac{T}{\varepsilon}]$. Then for all $U_0 \in X^s$ there exists a unique solution $U = (\zeta, u)^T \in X^s_T$ to (2.2.11) and for all $0 \leq t \leq \frac{T}{\varepsilon}$*

$$E^s(U(t)) \leq e^{\varepsilon \lambda_T t} E^s(U_0) + \varepsilon \int_0^t e^{\varepsilon \lambda_T (t - t')} C(E^s(\underline{U})(t')) dt'.$$

For some $\lambda_T = \lambda_T\big(\sup_{0 \leq t \leq T/\varepsilon} E^s(\underline{U}(t)), \sup_{0 \leq t \leq T/\varepsilon} |\partial_t \underline{h}(t)|_{L^\infty}\big)$.

Proof. Existence and uniqueness of a solution to the IVP (2.2.11) is achieved in Section 2.3 and we thus focus our attention on the proof of the energy estimate. For any $\lambda \in \mathbb{R}$, we compute

$$e^{\varepsilon\lambda t}\partial_t(e^{-\varepsilon\lambda t}E^s(U)^2) = -\varepsilon\lambda E^s(U)^2 + \partial_t(E^s(U)^2).$$

Since

$$E^s(U)^2 = (\Lambda^s U, S\Lambda^s U),$$

we have

$$\partial_t(E^s(U)^2) = 2(\Lambda^s\zeta, \Lambda^s\zeta_t) + 2(\Lambda^s u, \underline{\mathfrak{T}}\Lambda^s u_t) + (\Lambda^s u, [\partial_t, \underline{\mathfrak{T}}]\Lambda^s u). \tag{2.2.14}$$

One gets using the equations (2.2.11) and integrating by parts,

$$\begin{aligned}\frac{1}{2}e^{\varepsilon\lambda t}\partial_t(e^{-\varepsilon\lambda t}E^s(U)^2) &= -\frac{\varepsilon\lambda}{2}E^s(U)^2 - (SA[\underline{U}]\Lambda^s\partial_x U, \Lambda^s U)\\ &\quad -\big(\big[\Lambda^s, A[\underline{U}]\big]\partial_x U, S\Lambda^s U\big) - (\Lambda^s B(\underline{U}), S\Lambda^s U)\\ &\quad +\frac{1}{2}(\Lambda^s u, [\partial_t, \underline{\mathfrak{T}}]\Lambda^s u). \end{aligned}\tag{2.2.15}$$

We now turn to bound from above the different components of the r.h.s of (2.2.15).
• Estimate of $(SA[\underline{U}]\Lambda^s\partial_x U, \Lambda^s U)$. Remarking that

$$SA[\underline{U}] = \begin{pmatrix} \varepsilon\underline{u} & \underline{h} \\ \underline{h} & \underline{\mathfrak{T}}(\varepsilon\underline{u}\cdot) + Q_1[\underline{U}] \end{pmatrix},$$

we get

$$\begin{aligned}(SA[\underline{U}]\Lambda^s\partial_x U, \Lambda^s U) &= (\varepsilon\underline{u}\Lambda^s\zeta_x, \Lambda^s\zeta) + (\underline{h}\Lambda^s u_x, \Lambda^s\zeta)\\ &\quad +(\underline{h}\Lambda^s\zeta_x, \Lambda^s u) + \big((\underline{\mathfrak{T}}(\varepsilon\underline{u}\cdot) + Q_1[\underline{U}])\Lambda^s u_x, \Lambda^s u\big)\\ &=: A_1 + A_2 + A_3 + A_4.\end{aligned}$$

We now focus to control $(A_j)_{1\le j\le 4}$.
− Control of A_1. Integrating by parts, one obtains

$$A_1 = (\varepsilon\underline{u}\Lambda^s\zeta, \Lambda^s\zeta_x) = -\frac{1}{2}(\varepsilon\underline{u}_x\Lambda^s\zeta, \Lambda^s\zeta)$$

one can conclude by Cauchy-Schwarz inequality that

$$|A_1| \le \varepsilon C(|\underline{u}_x|_{L^\infty})E^s(U)^2.$$

− Control of $A_2 + A_3$. First remark that

$$|A_2 + A_3| = |(\underline{h}_x \Lambda^s u, \Lambda^s \zeta)| \leq |\underline{h}_x|_{L^\infty} E^s(U)^2;$$

we get,

$$|A_2 + A_3| \leq \varepsilon C(|\underline{h}_x|_{L^\infty}) E^s(U)^2.$$

− Control of A_4. One computes,

$$\begin{aligned} A_4 &= \varepsilon\big(\underline{\mathfrak{T}}(\underline{u}\Lambda^s u_x), \Lambda^s u\big) + (Q_1[\underline{U}]\Lambda^s u_x, \Lambda^s u) \\ &=: A_{41} + A_{42}. \end{aligned}$$

Note that

$$\begin{aligned} A_{41} &= \varepsilon(\underline{h}\ \underline{u}\Lambda^s u_x, \Lambda^s u) + \frac{\varepsilon\mu}{3}(\underline{h}^3\ (\underline{u}\Lambda^s u_x)_x, \Lambda^s u_x) \\ &\quad -\frac{\varepsilon^2\mu}{2}(\underline{h}^2\ b_x(\underline{u}\Lambda^s u_x)_x, \Lambda^s u) - \frac{\varepsilon^2\mu}{2}(\underline{h}^2 b_x \underline{u}\Lambda^s u_x, \Lambda^s u_x) \\ &\quad +\varepsilon^3\mu(\underline{h}\ \underline{u} b_x^2 \Lambda^s u_x, \Lambda^s u); \end{aligned}$$

since

$$(\underline{h}^3\ (\underline{u}\Lambda^s u_x)_x, \Lambda^s u_x) = \frac{1}{2}\big(-(\underline{h}_x^3\ \underline{u}\Lambda^s u_x, \Lambda^s u_x) + (\underline{h}^3\ \underline{u}_x \Lambda^s u_x, \Lambda^s u_x)\big),$$

by using successively integration by parts and the Cauchy-Schwarz inequality, one obtains directly:

$$|A_{41}| \leq \varepsilon C(|\underline{u}|_{W^{1,\infty}}, |\underline{\zeta}|_{W^{1,\infty}}) E^s(U)^2.$$

For A_{42}, remark that

$$\begin{aligned} |A_{42}| &= |(Q_1[\underline{U}]\Lambda^s u_x, \Lambda^s u)| \\ &= \big| -\frac{2}{3}\varepsilon\mu(\underline{h}^3 \underline{u}_x \Lambda^s u_x, \Lambda^s u_x) + \varepsilon^2\mu(\underline{h}^2\ \underline{u}_x b_x \Lambda^s u_x, \Lambda^s u) \\ &\quad +\varepsilon^2\mu(\underline{h}^2\ \underline{u} b_{xx} \Lambda^s u_x, \Lambda^s u)\big| \end{aligned}$$

therefore

$$|A_{42}| \leq \varepsilon C(|\underline{u}|_{W^{1,\infty}}, |\underline{\zeta}|_{W^{1,\infty}}) E^s(U)^2.$$

This shows that

$$|A_4| \leq \varepsilon C(|\underline{u}|_{W^{1,\infty}}, |\underline{\zeta}|_{W^{1,\infty}}) E^s(U)^2.$$

• Estimate of $\big([\Lambda^s, A[\underline{U}]]\partial_x U, S\Lambda^s U\big)$. Remark first that

$$\begin{aligned}\big([\Lambda^s, A[\underline{U}]]\partial_x U, S\Lambda^s U\big) &= ([\Lambda^s, \varepsilon\underline{u}]\zeta_x, \Lambda^s\zeta) + ([\Lambda^s, \underline{h}]u_x, \Lambda^s\zeta)\\ &\quad +([\Lambda^s, \underline{\mathfrak{T}}^{-1}\ \underline{h}]\zeta_x, \underline{\mathfrak{T}}\Lambda^s u) + ([\Lambda^s, \varepsilon\underline{u}]u_x, \underline{\mathfrak{T}}\Lambda^s u)\\ &\quad +\big([\Lambda^s, \underline{\mathfrak{T}}^{-1}Q_1[\underline{U}]]u_x, \underline{\mathfrak{T}}\Lambda^s u\big)\\ &=: B_1 + B_2 + B_3 + B_4 + B_5.\end{aligned}$$

– Control of $B_1 + B_2 = ([\Lambda^s, \varepsilon\underline{u}]\zeta_x, \Lambda^s\zeta) + ([\Lambda^s, \underline{h}]u_x, \Lambda^s\zeta)$. Since $s \geq t_0 + 1$, we can use the commutator estimate (3.3.11) to get

$$|B_1 + B_2| \leq \varepsilon C(E^s(\underline{U}))E^s(U)^2.$$

– Control of $B_4 = ([\Lambda^s, \varepsilon\underline{u}]u_x, \underline{\mathfrak{T}}\Lambda^s u)$. By using the explicit expression of $\underline{\mathfrak{T}}$ we get

$$\begin{aligned}B_4 &= ([\Lambda^s, \varepsilon\underline{u}]u_x, \underline{h}\Lambda^s u) + \frac{\mu}{3}(\partial_x[\Lambda^s, \varepsilon\underline{u}]u_x, \underline{h}^3\ \Lambda^s u_x)\\ &\quad -\frac{\varepsilon\mu}{2}([\Lambda^s, \varepsilon\underline{u}]u_x, \underline{h}^2\ b_x\Lambda^s u_x) + \frac{\varepsilon\mu}{2}([\Lambda^s, \varepsilon\underline{u}]u_x, \partial_x(\underline{h}^2\ b_x\Lambda^s u))\\ &\quad +\varepsilon^2\mu([\Lambda^s, \varepsilon\underline{u}]u_x, \underline{h}\ b_x^2\Lambda^s u),\end{aligned}$$

using the Cauchy-Schwarz inequality and the fact that

$$\partial_x[\Lambda^s, f]g = [\Lambda^s, f_x]g + [\Lambda^s, f]g_x$$

one obtains directly:

$$|B_4| \leq \varepsilon C(E^s(\underline{U}))E^s(U)^2.$$

– Control of $B_3 = ([\Lambda^s, \underline{\mathfrak{T}}^{-1}\ \underline{h}]\zeta_x, \underline{\mathfrak{T}}\Lambda^s u)$. Remark first that

$$\underline{\mathfrak{T}}[\Lambda^s, \underline{\mathfrak{T}}^{-1}]\underline{h}\zeta_x = \underline{\mathfrak{T}}[\Lambda^s, \underline{\mathfrak{T}}^{-1}\ \underline{h}]\zeta_x - [\Lambda^s, \underline{h}]\zeta_x;$$

morever, since $[\Lambda^s, \underline{\mathfrak{T}}^{-1}] = -\underline{\mathfrak{T}}^{-1}[\Lambda^s, \underline{\mathfrak{T}}]\underline{\mathfrak{T}}^{-1}$, one gets

$$\underline{\mathfrak{T}}[\Lambda^s, \underline{\mathfrak{T}}^{-1}\ \underline{h}]\zeta_x = -[\Lambda^s, \underline{\mathfrak{T}}]\underline{\mathfrak{T}}^{-1}\underline{h}\zeta_x + [\Lambda^s, \underline{h}]\zeta_x,$$

and one can check by using the explicit expression of $\underline{\mathfrak{T}}$ that

$$\begin{aligned}\underline{\mathfrak{T}}[\Lambda^s, \underline{\mathfrak{T}}^{-1}\ \underline{h}]\zeta_x &= -[\Lambda^s, \underline{h}]\underline{\mathfrak{T}}^{-1}\underline{h}\zeta_x + \frac{\mu}{3}\partial_x\{[\Lambda^s, \underline{h}^3]\partial_x(\underline{\mathfrak{T}}^{-1}\underline{h}\zeta_x)\}\\ &\quad -\frac{\varepsilon\mu}{2}\partial_x[\Lambda^s, \underline{h}^2 b_x]\underline{\mathfrak{T}}^{-1}\underline{h}\zeta_x + \frac{\varepsilon\mu}{2}[\Lambda^s, \underline{h}^2 b_x]\partial_x\underline{\mathfrak{T}}^{-1}\underline{h}\zeta_x\\ &\quad -\varepsilon^2\mu[\Lambda^s, \underline{h}b_x^2]\underline{\mathfrak{T}}^{-1}\underline{h}\zeta_x + [\Lambda^s, \underline{h}]\zeta_x.\end{aligned}$$

One deduces directly from Lemma 2.2.2, an integration by parts, and Cauchy-Schwarz inequality that

$$\begin{aligned}|B_3| &\leq C\big(\frac{1}{h_0},|\underline{h}-1|_{H^s}\big)\Big\{\Big(\big(|\underline{h}_x|_{H^{s-1}}+\frac{\varepsilon\mu}{2}|\underline{h}^2 b_x|_{H^s}+\varepsilon^2\mu|\underline{h}b_x^2|_{H^s}\big)|\underline{h}\zeta_x|_{H^{s-1}}\\ &\quad+\Big(\frac{1}{3}|\underline{h}_x^3|_{H^{s-1}}+\frac{\varepsilon\sqrt{\mu}}{2}|\underline{h}^2 b_x|_{H^s}\Big)|\underline{h}\zeta_x|_{H^{s-1}}+|\underline{h}_x|_{H^{s-1}}|\zeta_x|_{H^{s-1}}\Big)\Big\}|\Lambda^s u|_{H^1_*}.\end{aligned}$$

Finally, since

$$|\underline{h}\zeta_x|_{H^{s-1}} \leq C(E^s(\underline{U}))E^s(U) \quad \text{and} \quad |\underline{h}b_x^2|_{H^s}+|\underline{h}^2 b_x|_{H^s} \leq C(E^s(\underline{U})),$$

we deduce

$$|B_3| \leq \varepsilon C(E^s(\underline{U}))E^s(U)^2.$$

$-$ Control of $B_5=\big(\big[\Lambda^s,\underline{\mathfrak{T}}^{-1}Q_1[\underline{U}]\big]u_x,\underline{\mathfrak{T}}\Lambda^s u\big)$. Let us first write

$$\underline{\mathfrak{T}}\big[\Lambda^s,\underline{\mathfrak{T}}^{-1}Q_1[\underline{U}]\big]u_x=-[\Lambda^s,\underline{\mathfrak{T}}]\underline{\mathfrak{T}}^{-1}Q_1[\underline{U}]u_x+\big[\Lambda^s,Q_1[\underline{U}]\big]u_x$$

so, that

$$\begin{aligned}\underline{\mathfrak{T}}\big[\Lambda^s,\underline{\mathfrak{T}}^{-1}Q_1[\underline{U}]\big]u_x &= -[\Lambda^s,\underline{h}]\underline{\mathfrak{T}}^{-1}Q_1[\underline{U}]u_x+\frac{\mu}{3}\partial_x\{[\Lambda^s,\underline{h}^3]\partial_x(\underline{\mathfrak{T}}^{-1}Q_1[\underline{U}]u_x)\}\\ &\quad-\frac{\varepsilon\mu}{2}\partial_x\{[\Lambda^s,\underline{h}^2 b_x]\underline{\mathfrak{T}}^{-1}Q_1[\underline{U}]u_x\}+\frac{\varepsilon\mu}{2}[\Lambda^s,\underline{h}^2 b_x]\partial_x(\underline{\mathfrak{T}}^{-1}Q_1[\underline{U}]u_x)\\ &\quad-\varepsilon^2\mu[\Lambda^s,\underline{h}b_x^2]\underline{\mathfrak{T}}^{-1}Q_1[\underline{U}]u_x+[\Lambda^s,Q_1[\underline{U}]]u_x.\end{aligned}$$

To control the term $\big(\big[\Lambda^s,Q_1[\underline{U}]\big]u_x,\Lambda^s u\big)$ we use the explicit expression of $Q_1[\underline{U}]$:

$$Q_1[\underline{U}]f=\frac{2}{3}\varepsilon\mu\partial(\underline{h}^3\underline{u}_x f)+\varepsilon^2\mu\underline{h}^2 b_x\underline{u}_x f+\varepsilon^2\mu\underline{h}^2 b_{xx}\underline{u}f,$$

and the fact that

$$\partial_x[\Lambda^s,f]g=[\Lambda^s,\partial_x(f\cdot)]g.$$

Similarly to control the term $\big(\partial_x\{[\Lambda^s,\underline{h}^3]\partial_x(\underline{\mathfrak{T}}^{-1}Q_1[\underline{U}]u_x)\},\Lambda^s u\big)$ we use the explicit expression of $Q_1[\underline{U}]$, the commutator estimate (3.3.11) and Lemma 2.2.2. Indeed,

$$\begin{aligned}\big(\partial_x\{[\Lambda^s,\underline{h}^3]\partial_x(\underline{\mathfrak{T}}^{-1}Q_1[\underline{U}]u_x)\},\Lambda^s u\big) &= -\frac{2}{3}\varepsilon\mu\big([\Lambda^s,\underline{h}^3]\partial_x(\underline{\mathfrak{T}}^{-1}\partial_x(\underline{h}^3\underline{u}_x u_x)),\Lambda^s u_x\big)\\ &\quad-\varepsilon^2\mu\big([\Lambda^s,\underline{h}^3]\partial_x(\underline{\mathfrak{T}}^{-1}(\underline{h}^2 b_x\underline{u}_x u_x)),\Lambda^s u_x\big)\\ &\quad-\varepsilon^2\mu\big([\Lambda^s,\underline{h}^3]\partial_x(\underline{\mathfrak{T}}^{-1}(\underline{h}^2 b_{xx}\underline{u}u_x)),\Lambda^s u_x\big).\end{aligned}$$

and thus, after remarking that

$$\begin{aligned} |\partial_x(\underline{\mathfrak{T}}^{-1}\partial_x(\underline{h}^3\underline{u}_x u_x))|_{H^{s-1}} &\leq |\underline{\mathfrak{T}}^{-1}\partial_x(\underline{h}^3\underline{u}_x u_x)|_{H^s} \\ &\leq \| \mathfrak{T}^{-1}\partial_x \|_{H^s(\mathbb{R})\to H^s(\mathbb{R})} |\underline{h}^3\underline{u}_x u_x|_{H^s}. \end{aligned}$$

we can proceed as for the control of B_3 to get

$$|B_5| \leq \varepsilon C(E^s(\underline{U}))E^s(U)^2.$$

• Estimate of $(\Lambda^s B(\underline{U}), S\Lambda^s U)$. Note first that

$$B(\underline{U}) = \begin{pmatrix} \varepsilon b_x \underline{u} \\ \\ \underline{\mathfrak{T}}^{-1} q_2(\underline{U}) \end{pmatrix}$$

where, $q_2(\cdot)$ as in (3.3.2), so that

$$\begin{aligned} (\Lambda^s B(\underline{U}), S\Lambda^s U) &= (\Lambda^s(\varepsilon b_x\underline{u}), \Lambda^s\zeta) + (\Lambda^s(\underline{\mathfrak{T}}^{-1}q_2(\underline{U})), \underline{\mathfrak{T}}\Lambda^s u) \\ &= (\Lambda^s(\varepsilon b_x\underline{u}), \Lambda^s\zeta) - \big([\Lambda^s, \underline{\mathfrak{T}}]\underline{\mathfrak{T}}^{-1}q_2(\underline{U}), \Lambda^s u\big) \\ &\quad +\big(\Lambda^s q_2(\underline{U}), \Lambda^s u\big). \end{aligned}$$

Using again here the explicit expressions of $\underline{\mathfrak{T}}$, $q_2(\underline{U})$ and Lemma 2.2.2, we get

$$(\Lambda^s B(\underline{U}), S\Lambda^s U) \leq \varepsilon C(E^s(\underline{U}))E^s(U).$$

• Estimate of $(\Lambda^s u, [\partial_t, \underline{\mathfrak{T}}]\Lambda^s u)$. We have that

$$\begin{aligned} (\Lambda^s u, [\partial_t, \underline{\mathfrak{T}}]\Lambda^s u) &= (\Lambda^s u, \partial_t\underline{h}\Lambda^s u) + \frac{\mu}{3}(\Lambda^s u_x, \partial_t\underline{h}^3\Lambda^s u_x) \\ &\quad -\frac{\varepsilon\mu}{2}(\Lambda^s u, \partial_t\underline{h}^2 b_x\Lambda^s u_x) - \frac{\varepsilon\mu}{2}(\Lambda^s u_x, \partial_t\underline{h}^2 b_x\Lambda^s u) \\ &\quad +\varepsilon^2\mu(\Lambda^s u, \partial_t\underline{h} b_x^2\Lambda^s u). \end{aligned}$$

Controlling these terms by $\varepsilon C(E^s(\underline{U}), |\partial_t\underline{h}|_{L^\infty})E^s(U)^2$ follows directly from a Cauchy-Schwarz inequality and an integration by parts.

Gathering the informations provided by the above estimates and using the fact that $H^s(\mathbb{R}) \subset W^{1,\infty}$, we get

$$e^{\varepsilon\lambda t}\partial_t(e^{-\varepsilon\lambda t}E^s(U)^2) \leq \varepsilon\big(C(E^s(\underline{U}), |\partial_t\underline{h}|_{L^\infty}) - \lambda\big)E^s(U)^2 + \varepsilon C(E^s(\underline{U}))E^s(U).$$

Taking $\lambda = \lambda_T$ large enough (how large depending on $\sup_{t\in[0,\frac{T}{\varepsilon}]} C(E^s(\underline{U}), |\partial_t \underline{h}|_{L^\infty})$ to have the first term of the right hand side negative for all $t \in [0, \frac{T}{\varepsilon}]$, one deduces

$$\forall t \in [0, \frac{T}{\varepsilon}], \qquad e^{\varepsilon\lambda t}\partial_t(e^{-\varepsilon\lambda t}E^s(U)^2) \le \varepsilon C(E^s(\underline{U}))E^s(U).$$

Integrating this differential inequality yields therefore

$$\forall t \in [0, \frac{T}{\varepsilon}], \qquad E^s(U) \le e^{\varepsilon\lambda_T t}E^s(U_0) + \varepsilon\int_0^t e^{\varepsilon\lambda_T(t-t')}C(E^s(\underline{U})(t'))dt'.$$

$\square$

2.2.3 Main result

In this subsection we prove the main result of this chapter, which shows well-posedness of the Green-Naghdi equations over large times.

Theorem 2.2.1. *Let $b \in C_b^\infty(\mathbb{R})$, $t_0 > 1/2$, $s \ge t_0 + 1$. Let also the initial condition $U_0 = (\zeta_0, u_0)^T \in X^s$, and satisfy (2.2.2). Then there exists a maximal $T_{max} > 0$, uniformly bounded from below with respect to $\varepsilon, \mu \in (0,1)$, such that the Green-Naghdi equations (2.2.1) admit a unique solution $U = (\zeta, u)^T \in X^s_{T_{max}}$ with the initial condition $(\zeta_0, u_0)^T$ and preserving the nonvanishing depth condition (2.2.2) for any $t \in [0, \frac{T_{max}}{\varepsilon})$. In particular if $T_{max} < \infty$ one has*

$$|U(t,\cdot)|_{X^s} \longrightarrow \infty \quad as \quad t \longrightarrow \frac{T_{max}}{\varepsilon},$$

or

$$\inf_{\mathbb{R}} h(t,\cdot) = \inf_{\mathbb{R}} 1 + \varepsilon(\zeta(t,\cdot) - b(\cdot)) \longrightarrow 0 \quad as \quad t \longrightarrow \frac{T_{max}}{\varepsilon}.$$

Morever, the following conservation of energy property holds

$$\partial_t\Big(|\zeta|_2^2 + (hu, u) + \mu(h\mathcal{T}u, u)\Big) = 0,$$

where $\mathcal{T} = \mathcal{T}[h, \varepsilon b]$.

Remark 2.2.3. *For 2D surface waves, non flat bottoms, B. A. Samaniego and D. Lannes [4] proved a well-posedness result to the Green-Naghdi using a Nash-Moser scheme. Our result only use a standard Picard iterative and there is therefore no loss of regularity of the solution with respect to the initial condition. In the one-dimensional case and for flat bottoms, our result coincides with the one proved by Li in [55].*

Remark 2.2.4. *Our approach does not admit a straightforward generalization to the 2D case. The main reason is that the natural energy norm X^s is then given by*

$$|U|_{X^s}^2 = |\zeta|_{H^s}^2 + |u|_{H^s}^2 + \mu|\nabla \cdot u|_{H^s}^2,$$

which does not *control the $H^1(\mathbb{R}^2)$ norm of u (since u takes its values in $\mathbb{R}^2$, the information on the rotational of u is missing).*

Remark 2.2.5. *No smallness assumption on ε nor μ is required in the theorem. The fact that T_{max} is uniformly bounded from below with respect to these parameters allows us to say that if some smallness assumption is made on ε, then the existence time becomes larger, namely of order $O(1/\varepsilon)$. This is consistent with the existence time obtained for the (simpler) physical models derived under some smallness assumption on ε, like the Boussinesq models. In fact, such models can be derived from the Green-Naghdi equations [53]. The present theorem also has some direct implication for the justification of variable-bottom Camassa-Holm equations [37].*

Proof. We want to construct a sequence of approximate solution $(U^n = (\zeta^n, u^n))_{n\geq 0}$ by the induction relation

$$U^0 = U_0, \qquad \text{and} \qquad \forall n \in \mathbb{N}, \quad \begin{cases} \partial_t U^{n+1} + A[U^n]\partial_x U^{n+1} + B(U^n) = 0; \\ U^{n+1}_{|t=0} = U_0. \end{cases} \tag{2.2.16}$$

By Proposition 2.2.1, we know that there is a unique solution $U^{n+1} \in C([0,\infty); X^s)$ to (2.2.16) if $U^n \in C([0,\infty); X^s)$ and U^n satisfies (2.2.2) for all times. Let $R > 0$ be such that $E^s(U^0) \leq R/2$, it follows from Proposition 2.2.1 that U^{n+1} satisfies the following inequality

$$E^s(U^{n+1}(t)) \;\; \leq \;\; e^{\varepsilon\lambda_T t}E^s(U^0) + \varepsilon\int_0^t e^{\varepsilon\lambda_T(t-t')}C(E^s(U^n(t'))dt',$$

we suppose now that

$$\sup_{t\in[0,\frac{T}{\varepsilon}]} E^s(U^n(t)) \leq R,$$

therefore

$$E^s(U^{n+1}(t)) \leq R/2 + (e^{\varepsilon\lambda_T t} - 1)(R/2 + \frac{C(R)}{\lambda_T}).$$

Hence, there is $T > 0$ small enough such that

$$\sup_{t\in[0,\frac{T}{\varepsilon}]} E^s(U^{n+1}(t)) \leq R.$$

Using now the link between $E^s(U)$ and $|U|_{X^s}$ given by Lemma 3.3.2 we get

$$\sup_{t\in[0,\frac{T}{\varepsilon}]} |U^{n+1}(t)|_{X^s} \leq C\big(\frac{1}{h_0}\big)R.$$

We also know from the equations that

$$\partial_t \zeta^{n+1} = -h^n u_x^{n+1} - \varepsilon \zeta_x^n u^{n+1} + \varepsilon b_x u^{n+1}.$$

Hence, one gets

$$|\partial_t h^{n+1}|_{L^\infty} = \varepsilon |\partial_t \zeta^{n+1}|_{L^\infty} \leq \varepsilon C\big(\frac{1}{h_0}\big) R. \tag{2.2.17}$$

Since moreover

$$h^{n+1} = h^{n+1}_{t=0} + \int_0^t \partial_t \zeta^{n+1},$$

we can deduce from (2.2.17) and the fact that $h^{n+1}_{t=0} = 1 + \varepsilon(\zeta_0 - b) \geq h_0$ that it is possible to choose T small enough for U^{n+1} to satisfy (2.2.2) on $[0, \dfrac{T}{\varepsilon}]$, with h_0 replaced by $h_0/2$.
Finally, we deduce that the Cauchy problem

$$\left\{ \begin{array}{l} \partial_t U^{n+1} + A[U^n]\partial_x U^{n+1} + B(U^n) = 0; \\ U^{n+1}_{|t=0} = U_0 \end{array} \right.$$

has a unique solution U^{n+1} satisfing (2.2.2) and the inequality

$$E^s(U^{n+1}) \quad \leq \quad e^{\varepsilon \lambda_T t} E^s(U_0) + \varepsilon \int_0^t e^{\varepsilon \lambda_T (t-t')} C(E^s(U^n)(t'))dt',$$

when $0 \leq t \leq \dfrac{T}{\varepsilon}$ and λ_T depending only on $\sup\limits_{t \in [0, \frac{T}{\varepsilon}]} E^s(U^n)$. Thanks to this energy estimate, one can conclude classically (see e.g. [2]) to the existence of

$$T_{max} = T(E^s(U_0)) > 0,$$

and of a unique solution $U \in X^s_{T_{max}}$ to (2.2.1) preserving the inequality (2.2.2) for any $t \in [0, \dfrac{T_{max}}{\varepsilon}]$ as a limit of the iterative scheme

$$U^0 = U_0, \qquad \text{and} \qquad \forall n \in \mathbb{N}, \qquad \left\{ \begin{array}{l} \partial_t U^{n+1} + A[U^n]\partial_x U^{n+1} + B(U^n) = 0; \\ U^{n+1}_{|t=0} = U_0. \end{array} \right.$$

The fact that T_{max} is bounded from below by some $T > 0$ independent of $\varepsilon, \mu \in (0,1)$ follows from the analysis above, while the behavior of the solution as $t \to T_{max}/\varepsilon$ if $T_{max} < \infty$ follows from standard continuation arguments.
Though the conservation of the energy can be found in some references (e.g. [18]), we reproduce it in Section 2.4 for the sake of completeness.

□

2.3 Existence of solutions for the linearized equations

In this section we examine existence, uniqueness, and regularity for solutions to the following system of equations:

$$\left\{\begin{array}{l}\partial_t U + A[\underline{U}]\partial_x U = f;\\ U_{|_{t=0}} = U_0,\end{array}\right. \tag{2.3.1}$$

where $\underline{U} = (\underline{\zeta}, \underline{u})^T \in X_T^s$ is such that $\partial_t \underline{U} \in X_T^{s-1}$ and satisfy the condition (2.2.2) on $[0, \dfrac{T}{\varepsilon}]$. We begin the proof by the following lemma (see for instance [72]):

Lemma 2.3.1. *Let $\varphi \in C_0^\infty(\mathbb{R})$, such that $\varphi(r) = 1$ for $|r| \le 1$. Let also*

$$J^\delta = \varphi(\delta|D|), \quad \delta > 0.$$

Then:
(i) $\forall s, s' \in \mathbb{R}$, J^δ: $H^s(\mathbb{R}) \longmapsto H^{s'}(\mathbb{R})$ is a bounded linear operator.
(ii) J^δ commutes with Λ^s and is self-adjoint operator.
(iii) $\forall f \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $v \in L^2(\mathbb{R})$ there exists C independent of δ such that

$$|[f, J^\delta]v|_{H^1} \le C|f|_{C^1}|v|_{L^2}.$$

(iv) $\forall f \in H^s(\mathbb{R})$, $s > \dfrac{1}{2}$, $J^\delta f \in L^\infty(\mathbb{R})$ with

$$|J^\delta f|_{L^\infty} \le C|f|_{L^\infty}$$

where C is a constant independent of δ.

Our strategy will be to obtain a solution to (2.3.1) as a limit of solutions U_δ to

$$\left\{\begin{array}{l}\partial_t U_\delta + J^\delta A[\underline{U}]J^\delta\partial_x U_\delta = f;\\ U_{\delta_{|_{t=0}}} = U_0.\end{array}\right. \tag{2.3.2}$$

For any $\delta > 0$, $A^\delta = J^\delta A[\underline{U}]J^\delta$ is a bounded linear operator on each X^s, and $F^\delta = A^\delta\partial_x + B(\underline{U}) \in C^1(X^s)$ so by Cauchy-Lipschitz the ODE (2.3.2) has a unique solution, $U_\delta \in C([0, T/\varepsilon], X^s)$. Our task will be to obtain estimates on U_δ, independent of $\delta \in (0,1)$ and to show that the solution U_δ has a limit as $\delta \searrow 0$ solving (2.3.1). To do this, we remark that

$$\begin{aligned}\frac{1}{2}\partial_t E^s(U_\delta)^2 \;=\;& -(SA[\underline{U}]\Lambda^s\partial_x J^\delta U_\delta, \Lambda^s J^\delta U_\delta) - \big(\big[\Lambda^s, A[\underline{U}]\big]\partial_x J^\delta U_\delta, S\Lambda^s J^\delta U_\delta\big)\\ &+(\Lambda^s f, S\Lambda^s U_\delta) + \frac{1}{2}(\Lambda^s u_\delta, [\partial_t, \underline{\mathfrak{T}}]\Lambda^s u_\delta)\\ &+\big(\big[S, J^\delta\big]\Lambda^s U_\delta, A[\underline{U}]\Lambda^s\partial_x J^\delta U_\delta\big) + \big(\big[S, J^\delta\big]\Lambda^s U_\delta, \big[\Lambda^s, A[\underline{U}]\big]\partial_x J^\delta U_\delta\big).\end{aligned} \tag{2.3.3}$$

Note that we do not give any details for the control of the components of the r.h.s (2.3.3) other than the last two terms because the others can be handled exactly as in Proposition 2.2.1. To estimate the last two terms of the r.h.s (2.3.3), we have that

$$\begin{aligned}\big(\big[S,J^\delta\big]\Lambda^s U_\delta, A[\underline{U}]\Lambda^s\partial_x J^\delta U_\delta\big) &= \big(\big[\underline{\mathfrak{T}},J^\delta\big]\Lambda^s u_\delta, \underline{\mathfrak{T}}^{-1}(\underline{h}\Lambda^s\partial_x J^\delta u_\delta)\big)\\ &+\big(\big[\underline{\mathfrak{T}},J^\delta\big]\Lambda^s u_\delta, \varepsilon\underline{u}\Lambda^s\partial_x J^\delta u_\delta\big)\\ &+\big(\big[\underline{\mathfrak{T}},J^\delta\big]\Lambda^s u_\delta, \underline{\mathfrak{T}}^{-1}Q_1[\underline{U}]\Lambda^s\partial_x J^\delta u_\delta\big).\end{aligned}$$

One can check by using the explicit expression of $\underline{\mathfrak{T}}$ that

$$\begin{aligned}\big[\underline{\mathfrak{T}},J^\delta\big] &= \big[\underline{h},J^\delta\big]-\frac{\mu}{3}\partial_x\big[\underline{h}^3,J^\delta\big]\partial_x-\frac{\varepsilon\mu}{2}\big[\underline{h}^2 b_x,J^\delta\big]\partial_x\\ &+\frac{\varepsilon\mu}{2}\partial_x\big[\underline{h}^2 b_x,J^\delta\big]+\varepsilon^2\mu\big[\underline{h}b_x^2,J^\delta\big].\end{aligned}$$

One deduces directly from Lemma 2.3.1, an integration by parts, the Cauchy-Schwarz inequality and the explicit expression of $Q_1[\underline{U}]$ that

$$\big(\big[S,J^\delta\big]\Lambda^s U_\delta, A[\underline{U}]\Lambda^s\partial_x J^\delta U_\delta\big)\leq CE^s(U_\delta)^2,$$

similarly, one can conclude

$$\big(\big[S,J^\delta\big]\Lambda^s U_\delta, \big[\Lambda^s, A[\underline{U}]\big]\partial_x J^\delta U_\delta\big)\leq CE^s(U_\delta)^2,$$

where C is a constant independent of δ. By the Proposition 2.2.1, we have

$$\begin{aligned}&-(SA[\underline{U}]\Lambda^s\partial_x J^\delta U_\delta,\Lambda^s J^\delta U_\delta)-\big(\big[\Lambda^s,A[\underline{U}]\big]\partial_x J^\delta U_\delta, S\Lambda^s J^\delta U_\delta\big)\\ &+(\Lambda^s f, S\Lambda^s U_\delta)+\frac{1}{2}(\Lambda^s u_\delta,[\partial_t,\underline{\mathfrak{T}}]\Lambda^s u_\delta)\\ &\leq CE^s(U_\delta)^2+CE^s(f)^2.\end{aligned}$$

Consequently, we obtain an estimate of the form

$$\frac{d}{dt}E^s(U_\delta)^2\leq CE^s(U_\delta)^2+CE^s(f)^2. \qquad (2.3.4)$$

Thus Gronwall's inequality yields an estimate

$$E^s(U_\delta)^2\leq C(t)\big[E^s(U_0)^2+\sup_{[0,t]}E^s(f)^2\big], \qquad (2.3.5)$$

independent of $\delta\in(0,1)$. Thanks to this energy estimate, one can conclude classically (see e.g. [72]) to the existence of a unique solution $U\in C([0,T],X^s)$ to (2.3.1).

2.4 Conservation of the energy

In order to prove that

$$\partial_t\Big(|\zeta|_2^2 + (hu,u) + \mu(h\mathcal{T}u,u)\Big) = 0,$$

we multiply the first equation of (2.2.1) by ζ and the second by u, integrateon $\mathbb{R}$, and sum both equations to find

$$\frac{1}{2}\partial_t|\zeta|_2^2+(\partial_x(hu),\zeta)+(\partial_t u,hu)+\mu(h\mathcal{T}\partial_t u,u)+(\partial_x\zeta,hu)+\varepsilon(u\partial_x u,hu)+\mu\varepsilon(\mathcal{Q}u,u) = 0.$$

Therefore

$$\frac{1}{2}\partial_t|\zeta|_2^2 + \frac{1}{2}\partial_t(hu,u) - \frac{1}{2}(\partial_t h,u^2) + \varepsilon(u\partial_x u,hu) + \mu(h\mathcal{T}\partial_t u,u) + \mu\varepsilon(\mathcal{Q}u,u) = 0,$$

where the term $\mathcal{Q}u$ is defined as:

$$\begin{aligned}\mathcal{Q}u \quad &= \quad -\frac{1}{3}\partial_x[(h^3(u\partial_x^2 u - (\partial_x u)^2) + \frac{\varepsilon}{2}[\partial_x(h^2 u\partial_x(u\partial_x b) - h^2\partial_x b(u\partial_x^2 u - (\partial_x u)^2)] \\ &\quad\quad +\varepsilon^2 h\partial_x b(u\partial_x(u\partial_x b)).\end{aligned}$$

Using now the fact that $h = 1+\varepsilon(\zeta-b)$ and the first equation of (2.2.1), we get

$$-\frac{1}{2}(\partial_t h,u^2) + \varepsilon(u\partial_x u,hu) = -\frac{\varepsilon}{2}(\partial_x(hu),u^2) + \varepsilon(u\partial_x u,hu) = 0.$$

Thus,

$$\frac{1}{2}\partial_t|\zeta|_2^2 + \frac{1}{2}\partial_t(hu,u) + \mu(h\mathcal{T}\partial_t u,u) + \mu\varepsilon(\mathcal{Q}u,u) = 0. \tag{2.4.1}$$

Regarding now the term $\mu(h\mathcal{T}\partial_t u,u)$, we remark as in [18] that

$$\begin{aligned}\mu(h\mathcal{T}\partial_t u,u) \quad &= \quad \mu(\mathcal{T}_1^* h\mathcal{T}_1\partial_t u,u) + \mu(\mathcal{T}_2^* h\mathcal{T}_2\partial_t u,u), \\ &= \quad \mu(h\mathcal{T}_1\partial_t u,\mathcal{T}_1 u) + \mu(h\mathcal{T}_2\partial_t u,\mathcal{T}_2 u), \\ &= \quad \mu(h(\partial_t(\mathcal{T}_1 u) - \partial_t\mathcal{T}_1 u),\mathcal{T}_1 u) + \mu(h\partial_t(\mathcal{T}_2 u),\mathcal{T}_2 u),\end{aligned}$$

with $\mathcal{T}_j^*(j=1,2)$ denoting the adjoint of the operators $\mathcal{T}_j$ given by

$$\mathcal{T}_1 u = \frac{h}{\sqrt{3}}\partial_x u - \varepsilon\frac{\sqrt{3}}{2}\partial_x b u, \quad \text{and} \quad \mathcal{T}_2 u = \frac{\varepsilon}{2}\partial_x b u.$$

It comes:

$$\begin{aligned}\mu(h\mathcal{T}\partial_t u,u) \quad &= \quad \frac{\mu}{2}\partial_t(h\mathcal{T}_1 u,\mathcal{T}_1 u) - \frac{\mu}{2}(\partial_t h,(\mathcal{T}_1 u)^2) - \mu(h(\partial_t\mathcal{T}_1 u,\mathcal{T}_1 u) \\ &\quad\quad +\frac{\mu}{2}\partial_t(h\mathcal{T}_2 u,\mathcal{T}_2 u) - \frac{\mu}{2}(\partial_t h,(\mathcal{T}_2 u)^2), \\ &= \quad \frac{\mu}{2}\partial_t(h\mathcal{T}_1 u,\mathcal{T}_1 u) + \frac{\mu}{2}\partial_t(h\mathcal{T}_2 u,\mathcal{T}_2 u) - \frac{\mu}{2}(\partial_t h,(\mathcal{T}_1 u)^2 + (\mathcal{T}_2 u)^2) \\ &\quad\quad -\mu(h\partial_t\mathcal{T}_1 u,\mathcal{T}_1 u).\end{aligned}$$

Inject this result in (2.4.1) to get:

$$\begin{aligned}\frac{1}{2}\partial_t\Big(|\zeta|_2^2+(hu,u)+\mu(h\mathcal{T}u,u)\Big) &= \frac{\mu}{2}(\partial_t h,(\mathcal{T}_1u)^2+(\mathcal{T}_2u)^2)\\ &\quad+\mu(h\partial_t\mathcal{T}_1u,\mathcal{T}_1u)-\mu\varepsilon(\mathcal{Q}u,u).\end{aligned}$$

Noting that $h\partial_t\mathcal{T}_1u=\partial_t h(\mathcal{T}_1u+\sqrt{3}\mathcal{T}_2u)$, it comes:

$$\begin{aligned}\frac{1}{2}(\partial_t h,(\mathcal{T}_1u)^2+(\mathcal{T}_2u)^2)+(h\partial_t\mathcal{T}_1u,\mathcal{T}_1u) &= \frac{1}{2}\Big(\partial_t h,3(\mathcal{T}_1u)^2+(\mathcal{T}_2u)^2+2\sqrt{3}\mathcal{T}_1u\mathcal{T}_2u\Big),\\ &= \frac{\mu}{2}\Big(\partial_t h,(\sqrt{3}\mathcal{T}_1u+\mathcal{T}_2u)^2\Big),\\ &= \varepsilon\Big(u,\frac{h}{2}\partial_x(\sqrt{3}\mathcal{T}_1u+\mathcal{T}_2u)^2\Big),\end{aligned}$$

where we used here the first equation of (2.2.1). Finally, we get:

$$\frac{1}{2}\partial_t\Big(|\zeta|_2^2+(hu,u)+\mu(h\mathcal{T}u,u)\Big) = \mu\varepsilon\Big(\frac{h}{2}\partial_x(\sqrt{3}\mathcal{T}_1u+\mathcal{T}_2u)^2-\mathcal{Q}u,u\Big).$$

One can easily show that $\Big(\frac{h}{2}\partial_x(\sqrt{3}\mathcal{T}_1u+\mathcal{T}_2u)^2-\mathcal{Q}u,u\Big)=0$, which implies easily the result.

Derivation and analysis of a new $2D$ Green-Naghdi system

In this chapter we derive a variant of the $2D$ Green-Naghdi equations that model the propagation of two-directional, nonlinear dispersive waves in shallow water. This new model has the same accuracy as the standard $2D$ Green-Naghdi equations. Its mathematical interest is that it allows a control of the rotational part of the (vertically averaged) horizontal velocity, which is not the case for the usual Green-Naghdi equations. Using this property, we show that the solution of these new equations can be constructed by a standard Picard iterative scheme so that there is no loss of regularity of the solution with respect to the initial condition. Finally, we prove that the new Green-Naghdi equations conserve the almost irrotationality of the vertically averaged horizontal component of the velocity.

3.1 Introduction

3.1.1 General setting

The water-waves problem, consists in studying the motion of the free surface and the evolution of the velocity field of a layer of fluid under the following assumptions: the fluid is ideal, incompressible, irrotationnal, and under the only influence of gravity. Many works have set a good theoretical background for this problem. Its local well-posedness has been discussed among others by Nalimov [60], Yosihara [79], Craig [22], Wu [74, 75] and Lannes [50]. Since we are interested here in the asymptotic behavior of the solutions, it is convenient to work with a non-dimensionalized version of the equations. In this framework, (see for instance [3] for details), the free surface is parametrized by $z = \varepsilon\zeta(t, X)$ (with $X = (x, y) \in \mathbb{R}^2$) and the bottom by $z = -1 + \beta b(X)$. Here, ε and β are dimensionless parameters defined as

$$\varepsilon = \frac{a_{surf}}{h_0}, \quad \beta = \frac{b_{bott}}{h_0};$$

where a_{surf} is the typical amplitude of the waves and b_{bott} is the typical amplitude of the bottom deformations, while h_0 is the depth. One can use the incompressibility and irrotationality conditions to write the non-dimensionalized water-waves equations under Bernoulli's formulation, in terms of a velocity potential φ associated to the flow, where $\varphi(t, .)$ is defined on $\Omega_t = \{(X, z), -1 + \beta b(x) < z < \varepsilon\zeta(t, X)\}$ (i.e. the velocity field is given by $U = \nabla_{X,z}\varphi$) :

$$\begin{cases} \mu\partial_x^2\varphi + \mu\partial_y^2\varphi + \partial_z^2\varphi = 0, & \text{at} \quad -1 + \beta b < z < \varepsilon\zeta, \\ \partial_z\varphi - \mu\beta\nabla b \cdot \nabla\varphi = 0 & \text{at} \quad z = -1 + \beta b, \\ \partial_t\zeta - \dfrac{1}{\mu}(-\mu\varepsilon\nabla\zeta \cdot \nabla\varphi + \partial_z\varphi) = 0, & \text{at} \quad z = \varepsilon\zeta, \\ \partial_t\varphi + \dfrac{1}{2}(\varepsilon|\nabla\varphi|^2 + \dfrac{\varepsilon}{\mu}(\partial_z\varphi)^2) + \zeta = 0 & \text{at} \quad z = \varepsilon\zeta, \end{cases} \tag{3.1.1}$$

where $\nabla = (\partial_x, \partial_y)^T = (\partial_1, \partial_2)^T$. The *shallowness* parameter μ appearing in this set of equations is defined as

$$\mu = \frac{h_0^2}{\lambda^2},$$

where, λ is the typical wave-length of the waves. Making assumptions on the size of ε, β, and μ one is led to derive (simpler) asymptotic models from (3.1.1). In the shallow-water scaling ($\mu \ll 1$), one can derive (when no smallness assumption is made on ε and β) the standard Green-Naghdi equations (see §3.2.1 below for a derivation and [3] for a rigorous justification). For two-dimensional surfaces and over uneven bottoms these equations couple the free surface elevation ζ to the vertically averaged horizontal component of the velocity,

$$v(t, X) = \frac{1}{1 + \varepsilon\zeta - \beta b} \int_{-1+\beta b}^{\varepsilon\zeta} \nabla\varphi(t, X, z)dz, \tag{3.1.2}$$

and can be written as:

$$\begin{cases} \partial_t\zeta + \nabla \cdot (hv) = 0, \\ \big(h + \mu\mathcal{T}[h, \beta b]\big)\partial_t v + h\nabla\zeta + \varepsilon\big(h + \mu\mathcal{T}[h, \beta b]\big)(v \cdot \nabla)v \\ \quad + \mu\varepsilon\big\{\dfrac{2}{3}\nabla[(h^3(\partial_1 v \cdot \partial_2 v^\perp + (\nabla \cdot v)^2)] + \Re[h, \beta b](v)\big\} = 0, \end{cases} \tag{3.1.3}$$

where $h = 1 + \varepsilon\zeta - \beta b$, $v = (V_1, V_2)^T$, $v^\perp = (-V_2, V_1)^T$, and

$$\begin{aligned} \mathcal{T}[h, \beta b]W \quad = \quad & -\frac{1}{3}\nabla(h^3\nabla \cdot W) + \frac{\beta}{2}[\nabla(h^2\nabla b \cdot W) - h^2\nabla b\nabla \cdot W] \\ & +\beta^2 h\nabla b\nabla b \cdot W, \end{aligned} \tag{3.1.4}$$

while the purely topographic term $\Re[h,\beta b](v)$ is defined as:

$$\begin{aligned}\Re[h,\beta b](v) &= \frac{\beta}{2}\nabla\big(h^2(V_1^2\partial_1^2 b + 2V_1V_2\partial_1\partial_2 b + V_2^2\partial_2^2 b)\big)\\ &\quad +\beta h^2\big(\partial_1 v\cdot\partial_2 v^\perp + (\nabla\cdot v)^2\big)\nabla b\\ &\quad +\beta^2 h\big(V_1^2\partial_1^2 b + 2V_1V_2\partial_1\partial_2 b + V_2^2\partial_2^2 b\big)\nabla b.\end{aligned} \tag{3.1.5}$$

A rigorous justification of the standard GN model was given by Li [55] in $1D$ and for flat bottoms, and by B. Alvarez-Samaniego and D. Lannes [3] in the general case. This latter reference relies on well-posedness results for these equations given in [4] and based on general well-posedness results for evolution equations using a Nash-Moser scheme. The result of [4] covers both the case of $1D$ and $2D$ surfaces, and allows for non flat bottoms. The reason why a Nash-Moser scheme is used there is because the estimates on the linearized equations exhibit losses of derivatives. However, in the $1D$ case, such losses do not occur and it is possible to construct a solution with a standard Picard iterative scheme as in [55, 38] with flat and non flat bottoms respectively. But, this is not the case in $2D$, since for instance, the term $\partial_1 v\cdot\partial_2 v^\perp$ is not controled by the energy norm $|\cdot|_{Y^s}$ naturally associated to (3.1.3) (see [4]),

$$|(\zeta,v)|_{Y^s}^2 = |\zeta|_{H^s}^2 + |v|_{H^s}^2 + \mu|\nabla\cdot v|_{H^s}^2.$$

This is the motivation for the present derivation of a new variant of the $2D$ Green-Naghdi equations (3.1.3). This variant has the same accuracy as the standard $2D$ Green-Naghdi equations (3.1.3) (see §3.2.2 below for a derivation) and can be written under the form:

$$\begin{cases}\partial_t\zeta + \nabla\cdot(hv) = 0,\\ \Big(h+\mu\big(\mathcal{T}[h,\beta b] - \nabla^\perp\mathrm{curl}\,\big)\Big)\partial_t v + h\nabla\zeta\\ \quad +\varepsilon\Big(h+\mu\big(\mathcal{T}[h,\beta b] - \nabla^\perp\mathrm{curl}\,\big)\Big)(v\cdot\nabla)v\\ \quad +\mu\varepsilon\Big\{\frac{2}{3}\nabla[h^3(\partial_1 v\cdot\partial_2 v^\perp + (\nabla\cdot v)^2)] + \Re[h,\beta b](v)\Big\} = 0,\end{cases} \tag{3.1.6}$$

where $\nabla^\perp = (-\partial_y,\partial_x)^T$, $\mathrm{curl}\, v = \partial_1 V_2 - \partial_2 V_1$, while the linear operators $\mathcal{T}[h,\varepsilon b]$ and $\Re[h,\beta b]$ are defined in (3.1.4) and (3.1.5).
The reason why the new terms (involving $\nabla^\perp$curl) do not affect the precision of the model is because the solutions to (3.1.3) are nearly irrotational (in the sense that $\mathrm{curl}\, v$ is small). This property is of course satisfied also by our new model (3.1.6). The presence of these new terms allows the definition of a new energy norm that controls also the rotational part of v. Consequently, we show that it is possible to use a standard Picard iterative scheme to prove the well-posedness of (3.1.6), so that there is no loss of regularity of the solution with respect to the initial condition.

3.1.2 Organization of the chapter

We first recall the derivation of the standard $2D$ Green-Naghdi equations in Section 3.2.1 while in Section 3.2.2 we derive the new model (3.1.6). We give some preliminary results in Section 3.3.1; the main theorem which proves the well-posedness of this new Green-Naghdi system is then stated in Section 3.3.2 and proved in Section 3.3.3. Finally, in Section 3.3.4 we prove that (3.1.6) conserves the almost irrotationality of v.

3.1.3 Notation

We denote by $C(\lambda_1, \lambda_2, ...)$ a constant depending on the parameters λ_1, λ_2, ... and *whose dependence on the λ_j is always assumed to be nondecreasing.*
The notation $a \lesssim b$ means that $a \leq Cb$, for some nonegative constant C whose exact expression is of no importance (*in particular, it is independent of the small parameters involved*).
Let p be any constant with $1 \leq p < \infty$ and denote $L^p = L^p(\mathbb{R}^2)$ the space of all Lebesgue-measurable functions f with the standard norm

$$|f|_{L^p} = \big(\int_{\mathbb{R}^2} |f(X)|^p dX\big)^{1/p} < \infty.$$

When $p = 2$, we denote the norm $|\cdot|_{L^2}$ simply by $|\cdot|_2$. The inner product of any functions f_1 and f_2 in the Hilbert space $L^2(\mathbb{R}^2)$ is denoted by

$$(f_1, f_2) = \int_{\mathbb{R}^2} f_1(X) f_2(X) dX.$$

The space $L^\infty = L^\infty(\mathbb{R}^2)$ consists of all essentially bounded, Lebesgue-measurable functions f with the norm

$$|f|_{L^\infty} = \operatorname{ess\,sup} |f(X)| < \infty.$$

We denote by $W^{1,\infty} = W^{1,\infty}(\mathbb{R}^2) = \{f \in L^\infty, \nabla f \in (L^\infty)^2\}$ endowed with its canonical norm.
For any real constant s, $H^s = H^s(\mathbb{R}^2)$ denotes the Sobolev space of all tempered distributions f with the norm $|f|_{H^s} = |\Lambda^s f|_2 < \infty$, where Λ is the pseudo-differential operator $\Lambda = (1 - \Delta)^{1/2}$.
For any functions $u = u(x,t)$ and $v(x,t)$ defined on $\mathbb{R}^2 \times [0,T)$ with $T > 0$, we denote the inner product, the L^p-norm and especially the L^2-norm, as well as the Sobolev norm, with respect to the spatial variable X, by $(u, v) = (u(\cdot,t), v(\cdot,t))$, $|u|_{L^p} = |u(\cdot,t)|_{L^p}$, $|u|_{L^2} = |u(\cdot,t)|_{L^2}$, and $|u|_{H^s} = |u(\cdot,t)|_{H^s}$, respectively.
Let $C^k(\mathbb{R}^2)$ denote the space of k-times continuously differentiable functions and $C_0^\infty(\mathbb{R}^2)$ denote the space of infinitely differentiable functions, with compact support

in $\mathbb{R}^2$; we also denote by $C_b^\infty(\mathbb{R}^2)$ the space of infinitely differentiable functions that are bounded together with all their derivatives.
For any closed operator T defined on a Banach space X of functions, the commutator $[T, f]$ is defined by $[T, f]g = T(fg) - fT(g)$ with f, g and fg belonging to the domain of T.
We denote $v^\perp = (-V_2, V_1)^T$ and $\operatorname{curl} v = \partial_1 V_2 - \partial_2 V_1$ where $v = (V_1, V_2)^T$.

3.2 Derivation of the new Green-Naghdi model

This section is devoted to the derivation of a new Green-Naghdi asymptotic model for the water-waves equations in the shallow water ($\mu \ll 1$) of the same accuracy as the standard $2D$ Green-Naghdi equations (3.1.3).

3.2.1 Derivation of the standard Green-Naghdi equations (3.1.3)

We recall here the main steps of [53] for the derivation of the standard $2D$ Green-Naghdi equations (3.1.3). In order to reduce the model (3.1.1) into a model of two equations we introduce the trace of the velocity potential at the free surface, defined as

$$\psi = \varphi_{|_{z=\varepsilon\zeta}},$$

and the Dirichlet-Neumann operator $\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\cdot$ as

$$\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi = -\mu\varepsilon\nabla\zeta \cdot \nabla\varphi_{|_{z=\varepsilon\zeta}} + \partial_z\varphi_{|_{z=\varepsilon\zeta}},$$

with φ solving the boundary value problem

$$\begin{cases} \mu\partial_x^2\varphi + \mu\partial_y^2\varphi + \partial_z^2\varphi = 0, \\ \partial_n\varphi_{|_{z=-1+\beta b}} = 0, \\ \varphi_{|_{z=\varepsilon\zeta}} = \psi. \end{cases} \tag{3.2.1}$$

As remarked in [82, 23], the equations (3.1.1) are equivalent to a set of two equations on the free surface parametrization ζ and the trace of the velocity potential at the surface $\psi = \varphi_{|_{z=\varepsilon\zeta}}$ involving the Dirichlet-Neumann operator. Namely

$$\begin{cases} \partial_t\zeta + \dfrac{1}{\mu}\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi = 0, \\ \partial_t\psi + \zeta + \dfrac{\varepsilon}{2}|\nabla\psi|^2 - \varepsilon\mu\dfrac{(\frac{1}{\mu}\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi + \nabla(\varepsilon\zeta)\cdot\nabla\psi)^2}{2(1+\varepsilon^2\mu|\nabla\zeta|^2)} = 0. \end{cases} \tag{3.2.2}$$

It is a straightforward consequence of Green's identity that

$$\frac{1}{\mu}\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi = -\nabla\cdot(hv),$$

with $h = 1+\varepsilon\zeta-\beta b$ and $v = \dfrac{1}{1+\varepsilon\zeta-\beta b}\displaystyle\int_{-1+\beta b}^{\varepsilon\zeta}\nabla\varphi(t,X,z)dz$. Therefore, the first equation of (3.2.2) exactly coincides with the first equation of (3.1.3). In order to derive the evolution equation on v, the key point is to obtain an asymptotic expansion of $\nabla\psi$ with respect to μ and in terms of v and ζ. As in [53], we look for an asymptotic expansion of φ under the form

$$\varphi_{app} = \sum_{j=0}^{N}\mu^j\varphi_j. \tag{3.2.3}$$

Plugging this expression into the boundary value problem (3.2.1) one can cancel the residual up to the order $O(\mu^{N+1})$ provided that

$$\forall j = 0,...,N, \qquad \partial_z^2\varphi_j = -\partial_x^2\varphi_{j-1} - \partial_y^2\varphi_{j-1} \tag{3.2.4}$$

(with the convention that $\varphi_{-1}=0$), together with the boundary conditions

$$\forall j = 0,...,N, \qquad \begin{cases} \varphi_{j_{|z=\varepsilon\zeta}} = \delta_{0,j}\psi, \\ \partial_z\varphi_j = \beta\nabla b\cdot\nabla\varphi_{j-1_{|z=-1+\beta b}} \end{cases} \tag{3.2.5}$$

(where $\delta_{0,j}=1$ if $j=0$ and 0 otherwise).
By solving the ODE (3.2.4) with the boundary conditions (3.2.5), one finds (see [53])

$$\varphi_0 = \psi, \tag{3.2.6}$$
$$\varphi_1 = (z-\varepsilon\zeta)\big(-\frac{1}{2}(z+\varepsilon\zeta)-1+\beta b\big)\Delta\psi + \beta(z-\varepsilon\zeta)\nabla b.\nabla\psi. \tag{3.2.7}$$

According to formulae (3.2.6), (3.2.7), the horizontal component of the velocity in the fluid domain is given by

$$V(z) = \nabla\varphi_0(z) + \nabla\varphi_1(z) + O(\mu^2).$$

The averaged velocity is thus given by

$$v = \frac{1}{h}\int_{-1+\beta b}^{\varepsilon\zeta}\left(\nabla\varphi_0(z)+\nabla\varphi_1(z)\right)dz + O(\mu^2),$$

or equivalently

$$v = \nabla\psi - \mu\frac{1}{h}\mathcal{T}[h,\beta b]\nabla\psi + O(\mu^2),$$

and thus

$$\nabla\psi = v + \mu\frac{1}{h}\mathcal{T}[h,\beta b]\nabla\psi + O(\mu^2), \tag{3.2.8}$$

where $\mathcal{T}[h,\beta b]$ is as in (3.1.4). As in [53], taking the gradient of the second equation of (3.2.2), replacing $\nabla\psi$ by its expression (3.2.8) and $\frac{1}{\mu}\mathcal{G}_\mu[\varepsilon\zeta,\beta b]\psi$ by $-\nabla\cdot(hv)$ in the resulting equation, gives the standard Green-Naghdi equations (after dropping the $O(\mu^2)$ terms),

$$\begin{cases} \partial_t\zeta + \nabla\cdot(hv) = 0, \\ \big(h+\mu\mathcal{T}[h,\beta b]\big)\partial_t v + h\nabla\zeta + \varepsilon\big(h+\mu\mathcal{T}[h,\beta b]\big)(v\cdot\nabla)v \\ \quad + \mu\varepsilon\big\{\frac{2}{3}\nabla[h^3(\partial_1 v\cdot\partial_2 v^\perp + (\nabla\cdot v)^2)] + \Re[h,\beta b](v)\big\} = 0, \end{cases}$$

where $h = 1+\varepsilon\zeta-\beta b$, $v=(V_1,V_2)^T$, $v^\perp = (-V_2,V_1)^T$, and the linear operators $\mathcal{T}[h,\varepsilon b]$ and $\Re[h,\beta b]$ being defined in (3.1.4) and (3.1.5).

3.2.2 Derivation of the new Green-Naghdi system (3.1.6)

It is quite common in the literature to work with variants of the Green-Naghdi equations (3.1.6) that differ only up to terms of order $O(\mu^2)$ in order to improve the dispersive properties of the model (see for instance [73, 16]) or to change its mathematical properties [56]. Our approach here is in the same spirit since our goal is to derive a new model with better energy estimates.
In order to obtain the new Green-Naghdi system (3.1.6), let us remark that, from the expression of v one has

$$\nabla\psi = v + \frac{\mu}{h}\mathcal{T}[h,\beta b]\nabla\psi + O(\mu^2);$$

and since $v = \nabla\psi + O(\mu)$, one gets the following Lemma:

Lemma 3.2.1. *Let v be the vertically averaged horizontal component of the velocity given above, one obtains*

$$\begin{aligned} \operatorname{curl}\partial_t v &= O(\mu), \\ \operatorname{curl}(v\cdot\nabla)v &= O(\mu). \end{aligned} \tag{3.2.9}$$

Remark 3.2.1. *For the sake of simplicity, we denote by $O(\mu)$ any family of functions $(f_\mu)_{0<\mu<1}$ such that $\big(\frac{1}{\mu}f_\mu\big)_{0<\mu<1}$ remains bounded in $L^\infty([0,\frac{T}{\varepsilon}],H^r(\mathbb{R}^2))$, for some r large enough.*

Proof. By applying the operator $(\operatorname{curl}\partial_t)(\cdot)$ to the identity $v=\nabla\psi+O(\mu)$ one gets

$$\operatorname{curl}\partial_t v = O(\mu).$$

In order to prove the second identity of (3.2.9), replace $v=\nabla\psi+O(\mu)$ in $(v\cdot\nabla)v$ and apply the operator $\operatorname{curl}(\cdot)$ to deduce

$$\operatorname{curl}(v\cdot\nabla)v = O(\mu).$$

□

Using Lemma 3.2.1, the quantities $\mu\nabla^\perp \mathrm{curl}\, \partial_t v$ and $\mu\nabla^\perp \mathrm{curl}\, \varepsilon(v\cdot\nabla)v$ are of size $O(\mu^2)$, which is the precision of the GN equations (3.1.3). We can thus include these new terms in the second equation of (3.1.3) to get

$$\begin{cases} \partial_t\zeta + \nabla\cdot(hv) = 0, \\ \Big(h + \mu\big(\mathcal{T}[h,\beta b] - \nabla^\perp \mathrm{curl}\,\big)\Big)\partial_t v + h\nabla\zeta \\ +\,\varepsilon\Big(h + \mu\big(\mathcal{T}[h,\beta b] - \nabla^\perp \mathrm{curl}\,\big)\Big)(v\cdot\nabla)v \\ +\,\mu\varepsilon\Big\{\frac{2}{3}\nabla[h^3(\partial_1 v\cdot\partial_2 v^\perp + (\nabla\cdot v)^2)] + \Re[h,\beta b](v)\Big\} = O(\mu^2), \end{cases}$$

(the linear operators $\mathcal{T}[h,\varepsilon b]$ and $\Re[h,\beta b]$ being defined in (3.1.4) and (3.1.5)).

Remark 3.2.2. *We added the quantity $\mu\nabla^\perp \mathrm{curl}\,\partial_t v = O(\mu^2)$ to the standard GN equations (3.1.3) to obtain an energy norm $|\cdot|_{X^s}$:*

$$|(\zeta,v)|^2_{X^s} = |\zeta|^2_{H^s} + |v|^2_{H^s} + \mu|\nabla\cdot v|^2_{H^s} + \mu|\mathrm{curl}\, v|^2_{H^s}.$$

The last term is absent from the energy $|\cdot|_{Y^s}$ associated to the standard GN equations (3.1.3) (see [4]). We will see in the next section that it allows a control of the term $\partial_1 v\cdot\partial_2 v^\perp$.

Remark 3.2.3. *The bilinear operator $\Re[h,\beta b](v)$ only involves second order derivatives of v while third order derivatives of v have been factorized by $h+\mu\big(\mathcal{T}[h,\varepsilon b] - \nabla^\perp \mathit{curl}\,\big)$. The fact that the operator $\Re[h,\beta b](v)$ does not involve third order derivatives is of great interest for the well-posedness of the new Green-Naghdi model.*

3.3 Mathematical analysis of the new Green-Naghdi model

3.3.1 Preliminary results

For the sake of simplicity, we take here and throughout the rest of this chapter $(\beta=\varepsilon)$ and we write

$$\mathfrak{T} = h + \mu\big(\mathcal{T}[h,\varepsilon b] - \nabla^\perp \mathrm{curl}\,\big).$$

We always assume that the nonzero depth condition

$$\exists\, h_{min} > 0, \quad \inf_{X\in\mathbb{R}^2} h \geq h_{min}, \quad h = 1 + \varepsilon(\zeta - b) \tag{3.3.1}$$

is valid initially, which is a necessary condition for the new GN type system (3.1.6) to be physically valid. We shall demonstrate that the operator $\mathfrak{T}$ plays an important

role in the energy estimate and the local well-posedness of the GN type system (3.1.6). Therefore, we give here some of its properties.
The following lemma gives an important invertibility result on $\mathfrak{T}$ and some properties of the inverse operator $\mathfrak{T}^{-1}$.

Lemma 3.3.1. *Let $b \in C_b^\infty(\mathbb{R}^2)$, $t_0 > 1$ and $\zeta \in H^{t_0+1}(\mathbb{R}^2)$ be such that (3.3.1) is satisfied. Then, the operator $\mathfrak{T}$ has a bounded inverse on $(L^2(\mathbb{R}^2))^2$, and*
(i) For all $0 \leq s \leq t_0 + 1$, one has

$$|\mathfrak{T}^{-1} f|_{H^s} + \sqrt{\mu}|\nabla \cdot \mathfrak{T}^{-1} f|_{H^s} + \sqrt{\mu}|\mathrm{curl}\, \mathfrak{T}^{-1} f|_{H^s} \leq C(\frac{1}{h_{min}}, |h-1|_{H^{t_0+1}})|f|_{H^s},$$

and

$$\sqrt{\mu}|\mathfrak{T}^{-1}\nabla g|_{H^s} + \sqrt{\mu}|\mathfrak{T}^{-1}\nabla^\perp g|_{H^s} \leq C(\frac{1}{h_{min}}, |h-1|_{H^{t_0+1}})|g|_{H^s}.$$

(ii) If $s \geq t_0 + 1$ and $\zeta \in H^s(\mathbb{R}^2)$ then

$$\| \mathfrak{T}^{-1} \|_{(H^s)^2 \to (H^s)^2} + \sqrt{\mu} \| \mathfrak{T}^{-1}\nabla \|_{(H^s)^2 \to (H^s)^2} + \sqrt{\mu} \| \mathfrak{T}^{-1}\nabla^\perp \|_{(H^s)^2 \to (H^s)^2} \leq c_s,$$

and

$$\sqrt{\mu} \| \nabla \cdot \mathfrak{T}^{-1} \|_{(H^s)^2 \to (H^s)^2} + \sqrt{\mu} \| \mathrm{curl}\, \mathfrak{T}^{-1} \|_{(H^s)^2 \to (H^s)^2} \leq c_s,$$

where c_s is a constant depending on $\dfrac{1}{h_{min}}$, $|h-1|_{H^s}$ and independent of $(\mu,\varepsilon) \in (0,1)^2$.

Remark 3.3.1. *Here and throughout the rest of this chapter, and for the sake of simplicity, we do not try to give some optimal regularity assumption on the bottom parametrization b. This could easily be done, but is of no interest for our present purpose. Consequently, we omit to write the dependence on b of the different quantities that appear in the proof.*

Proof. It can be remarked that the operator $\mathfrak{T}$ is L^2 self-adjoint; since, one has

$$\begin{aligned}
&\Big(h + \mu\big(\mathcal{T}[h, \varepsilon b] - \nabla^\perp \mathrm{curl}\ \big)v, v\Big) = (hv, v) \\
&+\mu\Big(h\Big(\frac{h}{\sqrt{3}}\nabla \cdot v - \varepsilon\frac{\sqrt{3}}{2}\nabla b \cdot v\Big), \frac{h}{\sqrt{3}}\nabla \cdot v - \varepsilon\frac{\sqrt{3}}{2}\nabla b \cdot v\Big) + \frac{\mu\varepsilon^2}{4}(h\nabla b \cdot v, \nabla b \cdot v) \\
&+\mu(\mathrm{curl}\, v, \mathrm{curl}\, v),
\end{aligned}$$

and using the fact that $\inf_{\mathbb{R}^2} h \geq h_{min}$, one deduces that

$$(\mathfrak{T}v, v) \geq E(\varepsilon b, v),$$

with

$$E(\varepsilon b, v) := h_{min}|v|_2^2 + \mu h_{min}|\frac{h}{\sqrt{3}}\nabla \cdot v - \varepsilon\frac{\sqrt{3}}{2}\nabla b \cdot v|_2^2 + \frac{\mu\varepsilon^2 h_{min}}{4}|\nabla b \cdot v|_2^2 + \mu|\mathrm{curl}\, v|_2^2,$$

proceeding exactly as in the proof of the Lemma 1 of [38] it follows that $\mathfrak{T}$ has an inverse bounded on $(L^2(\mathbb{R}^2))^2$.
For the rest of the proof, One can proceed as in the proof of Lemma 4.7 of [4], to get the result. $\square$

3.3.2 Linear analysis of (3.1.6)

In order to rewrite the new GN type equations (3.1.6) in a condensed form, let us write $U = (\zeta, v^T)^T$, $v = (V_1, V_2)^T$ and

$$Q = h^3(-\partial_1 v \cdot \partial_2 v^\perp - (\nabla \cdot v)^2).$$

We decompose now the $O(\varepsilon\mu)$ term of the second equation of (3.1.6) under the form

$$-\frac{2}{3}\nabla Q + R[h, \varepsilon b](v) = R_1[U]v + r_2(U),$$

with for all $f = (F_1, F_2)^T$

$$\begin{aligned} R_1[U]f &= -\frac{2}{3}\nabla Q[U]f + \frac{\varepsilon}{2}\nabla(h^2(V_1F_1\partial_1^2 b + 2V_1F_2\partial_1\partial_2 b + V_2F_2\partial_2^2 b)) \\ &\quad -\varepsilon h^2(-\partial_1 v \cdot \partial_2 f^\perp - (\nabla \cdot v)(\nabla \cdot f))\nabla b; \\ r_2(U) &= \varepsilon^2 h(V_1^2\partial_1^2 b + 2V_1V_2\partial_1\partial_2 b + V_2^2\partial_2^2 b)\nabla b, \end{aligned}$$

where

$$Q[U]f = h^3(-\partial_1 v \cdot \partial_2 f^\perp - (\nabla \cdot v)(\nabla \cdot f)),$$

(in particular, $Q = Q[U]v$). The new Green-Naghdi equations (3.1.6) can thus be written after applying $\mathfrak{T}^{-1}$ to both sides of the second equation in (3.1.6) as

$$\partial_t U + A[U]U + B(U) = 0, \tag{3.3.2}$$

with $U = (\zeta, V_1, V_2)^T$, $v = (V_1, V_2)^T$ and where

$$A[U] = \begin{pmatrix} \varepsilon v \cdot \nabla & h\nabla\cdot \\ \mathfrak{T}^{-1}(h\nabla) & \varepsilon(v\cdot\nabla) + \varepsilon\mu\mathfrak{T}^{-1}R_1[U] \end{pmatrix} \tag{3.3.3}$$

and

$$B(U) = \begin{pmatrix} \varepsilon\nabla b \cdot v \\ \varepsilon\mu\mathfrak{T}^{-1}r_2(U) \end{pmatrix}. \tag{3.3.4}$$

This subsection is devoted to the proof of energy estimates for the following initial value problem around some reference state $\underline{U} = (\underline{\zeta}, \underline{V_1}, \underline{V_2})^T$:

$$\begin{cases} \partial_t U + A[\underline{U}]U + B(\underline{U}) = 0; \\ U_{|t=0} = U_0. \end{cases} \tag{3.3.5}$$

We define first the X^s spaces, which are the energy spaces for this problem.

Definition 3.3.1. *For all $s \geq 0$ and $T > 0$, we denote by X^s the vector space $H^s(\mathbb{R}^2) \times (H^{s+1}(\mathbb{R}^2))^2$ endowed with the norm*

$$\forall\, U = (\zeta, v) \in X^s, \quad |U|^2_{X^s} := |\zeta|^2_{H^s} + |v|^2_{(H^s)^2} + \mu|\nabla \cdot v|^2_{H^s} + \mu|\mathrm{curl}\, v|^2_{H^s},$$

while X^s_T stands for $C([0, \frac{T}{\varepsilon}]; X^s)$ endowed with its canonical norm.

We define the matrix S as

$$S = \begin{pmatrix} 1 & 0 \\ 0 & \underline{\mathfrak{T}} \end{pmatrix}, \tag{3.3.6}$$

with $\underline{h} = 1 + \varepsilon(\underline{\zeta} - b)$ and $\underline{\mathfrak{T}} = \underline{h} + \mu(\mathcal{T}[\underline{h}, \varepsilon b] - \nabla^\perp(\mathrm{curl}\,))$. A natural energy for the IVP (3.3.5) is given by

$$E^s(U)^2 = (\Lambda^s U, S\Lambda^s U). \tag{3.3.7}$$

The link between $E^s(U)$ and the X^s-norm is investigated in the following Lemma.

Lemma 3.3.2. *Let $b \in C_b^\infty(\mathbb{R}^2)$, $s \geq 0$ and $\underline{\zeta} \in W^{1,\infty}(\mathbb{R}^2)$. Under the condition (3.3.1), $E^s(U)$ is uniformly equivalent to the $|\cdot|_{X^s}$-norm with respect to $(\mu, \varepsilon) \in (0,1)^2$:*

$$E^s(U) \leq C\big(|\underline{h}|_{W^{1,\infty}}\big)|U|_{X^s},$$

and

$$|U|_{X^s} \leq C\big(\frac{1}{h_{min}}\big)E^s(U).$$

Proof. Notice first that

$$E^s(U)^2 = |\Lambda^s\zeta|_2^2 + (\Lambda^s v, \underline{\mathfrak{T}}\Lambda^s v),$$

one gets the first estimate using the explicit expression of $\underline{\mathfrak{T}}$, integration by parts and Cauchy-Schwarz inequality.
The other inequality can be proved by using that $\inf_{x\in\mathbb{R}^2} h \geq h_{min} > 0$ and proceeding as in the proof of Lemma 3.3.1. $\square$

We prove now the energy estimates in the following proposition. It is worth insisting on the fact that these estimates are uniform with respect to $\varepsilon, \mu \in (0,1)$; since the control of the $s+1$ order derivatives by the X^s-norm disappears as $\mu \to 0$, the uniformity with respect to μ requires particular care (see the control of B_{46} in the proof for instance), but is very important for the application since $\mu \ll 1$.

Proposition 3.3.1. *Let $b \in C_b^\infty(\mathbb{R}^2)$, $t_0 > 1$, $s \geq t_0 + 1$. Let also $\underline{U} = (\underline{\zeta}, \underline{V_1}, \underline{V_2})^T \in X_T^s$ be such that $\partial_t \underline{U} \in X_T^{s-1}$ and satisfying the condition (3.3.1) on $[0, \frac{T}{\varepsilon}]$. Then for all $U_0 \in X^s$ there exists a unique solution $U = (\zeta, V_1, V_2)^T \in X_T^s$ to (3.3.5) and for all $0 \leq t \leq \frac{T}{\varepsilon}$*

$$E^s(U(t)) \leq e^{\varepsilon \lambda_T t} E^s(U_0) + \varepsilon \int_0^t e^{\varepsilon \lambda_T (t-t')} C(E^s(\underline{U})(t'))dt',$$

for some $\lambda_T = \lambda_T \big(\sup_{0 \leq t \leq T/\varepsilon} E^s(\underline{U}(t)), \sup_{0 \leq t \leq T/\varepsilon} |\partial_t \underline{h}(t)|_{L^\infty} \big)$.

Proof. Existence and uniqueness of a solution to the IVP (3.3.5) is achieved by using classical methods as in appendix A of [38] for the standard $1D$ Green-Naghdi equations and we thus focus our attention on the proof of the energy estimate. For any $\lambda \in \mathbb{R}$, we compute

$$e^{\varepsilon \lambda t} \partial_t (e^{-\varepsilon \lambda t} E^s(U)^2) = -\varepsilon \lambda E^s(U)^2 + \partial_t (E^s(U)^2).$$

Since

$$E^s(U)^2 = (\Lambda^s U, S \Lambda^s U),$$

and $U = (\zeta, V_1, V_2)^T$, $v = (V_1, V_2)^T$, we have

$$\partial_t (E^s(U)^2) = 2(\Lambda^s \zeta, \Lambda^s \zeta_t) + 2(\Lambda^s v, \underline{\mathfrak{T}} \Lambda^s v_t) + (\Lambda^s v, [\partial_t, \underline{\mathfrak{T}}] \Lambda^s v). \tag{3.3.8}$$

One gets using the equations (3.3.5) and integrating by parts,

$$\begin{aligned} \frac{1}{2} e^{\varepsilon \lambda t} \partial_t (e^{-\varepsilon \lambda t} E^s(U)^2) &= -\frac{\varepsilon \lambda}{2} E^s(U)^2 - (SA[\underline{U}] \Lambda^s U, \Lambda^s U) \\ &- \big(\big[\Lambda^s, A[\underline{U}] \big] U, S \Lambda^s U \big) - (\Lambda^s B(\underline{U}), S \Lambda^s U) + \frac{1}{2} (\Lambda^s v, [\partial_t, \underline{\mathfrak{T}}] \Lambda^s v). \end{aligned} \tag{3.3.9}$$

We now turn to bound from above the different components of the r.h.s of (3.3.9).

• Estimate of $(SA[\underline{U}] \Lambda^s U, \Lambda^s U)$. Remarking that

$$SA[\underline{U}] = \begin{pmatrix} \varepsilon \underline{v} \cdot \nabla & \underline{h} \nabla \cdot \\ \underline{h} \nabla & \underline{\mathfrak{T}}(\varepsilon \underline{v} \cdot \nabla) + \varepsilon \mu R_1[\underline{U}] \end{pmatrix},$$

we get

$$\begin{aligned} (SA[\underline{U}] \Lambda^s U, \Lambda^s U) &= (\varepsilon \underline{v} \cdot \nabla \Lambda^s \zeta, \Lambda^s \zeta) + (\underline{h} \nabla \cdot \Lambda^s v, \Lambda^s \zeta) \\ &\quad + (\underline{h} \nabla \Lambda^s \zeta, \Lambda^s v) + \big((\underline{\mathfrak{T}}(\varepsilon \underline{v} \cdot \nabla) + \varepsilon \mu R_1[\underline{U}]) \Lambda^s v, \Lambda^s v \big) \\ &=: A_1 + A_2 + A_3 + A_4. \end{aligned}$$

We now focus on the control of $(A_j)_{1\leq j\leq 4}$.
− Control of A_1. Integrating by parts, one obtains

$$A_1 = (\varepsilon\underline{v}\cdot\nabla\Lambda^s\zeta, \Lambda^s\zeta) = -\frac{1}{2}(\varepsilon\nabla\cdot\underline{v}\Lambda^s\zeta, \Lambda^s\zeta),$$

and one can conclude by Cauchy-Schwarz inequality that

$$|A_1| \leq \varepsilon C(|\nabla\cdot\underline{v}|_{L^\infty})E^s(U)^2.$$

− Control of $A_2 + A_3$. First remark that

$$|A_2 + A_3| = |(\nabla\underline{h}\cdot\Lambda^s v, \Lambda^s\zeta)| \leq |\nabla\underline{h}|_{(L^\infty)^2}E^s(U)^2;$$

we get,

$$|A_2 + A_3| \leq \varepsilon C(|\nabla\underline{h}|_{(L^\infty)^2}E^s(U)^2.$$

− Control of A_4. One computes,

$$\begin{aligned} A_4 &= \varepsilon\big(\underline{\mathfrak{T}}((\underline{v}\cdot\nabla)\,\Lambda^s v), \Lambda^s v\big) + (\varepsilon\mu R_1[\underline{U}]\Lambda^s v, \Lambda^s v) \\ &=: A_{41} + A_{42}. \end{aligned}$$

Note first that

$$\begin{aligned} A_{41} &= \varepsilon(\underline{h}\,(\underline{v}\cdot\nabla)\Lambda^s v, \Lambda^s v) + \frac{\varepsilon\mu}{3}(\underline{h}^3\,\nabla\cdot(\underline{v}\cdot\nabla)\Lambda^s v), \Lambda^s\nabla\cdot v) \\ &\quad -\frac{\varepsilon^2\mu}{2}(\underline{h}^2\,\nabla b\cdot(\underline{v}\cdot\nabla)\Lambda^s v, \Lambda^s\nabla\cdot v) - \frac{\varepsilon^2\mu}{2}(\underline{h}^2\nabla b\nabla\cdot(\underline{v}\cdot\nabla)\Lambda^s v, \Lambda^s v) \\ &\quad +\varepsilon^3\mu(\underline{h}\,\nabla b\nabla b\cdot(\underline{v}\cdot\nabla)\Lambda^s v, \Lambda^s v) + \varepsilon\mu(\mathrm{curl}\,(\underline{v}\cdot\nabla)\Lambda^s v), \Lambda^s\mathrm{curl}\,v); \end{aligned}$$

remark also that

$$\begin{aligned} (\mathrm{curl}\,(\underline{v}\cdot\nabla)\Lambda^s v), \Lambda^s\mathrm{curl}\,v) &= -\frac{1}{2}\Big((\mathrm{curl}\,\Lambda^s v, \partial_1\underline{V_1}\mathrm{curl}\,\Lambda^s v) + (\nabla\wedge\Lambda^s v, \partial_2\underline{V_2}\nabla\wedge\Lambda^s v)\Big) \\ &\quad +(\mathrm{curl}\,\Lambda^s v, \partial_1\underline{v}\cdot\nabla\Lambda^s V_2) - (\mathrm{curl}\,\Lambda^s v, \partial_2\underline{v}\cdot\nabla\Lambda^s V_1), \end{aligned}$$

and that, for all F and G smooth enough, one has

$$((G\cdot\nabla)v, F) = -(v, F\,\nabla\cdot G) - (v, (G\cdot\nabla)F).$$

By using successively the above identities, the following relation (3.3.10)

$$|\nabla F_1|_2^2 + |\nabla F_2|_2^2 = |\nabla\cdot F|_2^2 + |\mathrm{curl}\,F|_2^2, \tag{3.3.10}$$

integration by parts and the Cauchy-Schwarz inequality, one obtains directly:

$$|A_{41}| \leq \varepsilon C(|\underline{v}|_{(W^{1,\infty})^2}, |\underline{\zeta}|_{W^{1,\infty}})E^s(U)^2.$$

For A_{42}, remark that

$$\begin{aligned}
|A_{42}| &= |(\varepsilon\mu R_1[\underline{U}]\Lambda^s v, \Lambda^s v)| \\
&= \Big| + \frac{2}{3}\varepsilon\mu(Q[\underline{U}]\Lambda^s v, \Lambda^s \nabla\cdot v) - \mu\varepsilon^2\Big(\underline{h}^2(-\partial_1\underline{v}\cdot\partial_2\Lambda^s v^{\perp} - (\nabla\cdot\underline{v})(\nabla\cdot\Lambda^s v))\nabla b, \Lambda^s v\Big) \\
&\quad -\frac{\mu\varepsilon^2}{2}\Big(\underline{h}^2(\underline{V}_1\Lambda^s V_1\partial_1^2 b + 2\underline{V}_1\Lambda^s V_2\partial_1\partial_2 b + \underline{V}_2\Lambda^s V_2\partial_2^2 b), \Lambda^s\nabla\cdot v\Big)\Big|,
\end{aligned}$$

where

$$Q[\underline{U}]f = \underline{h}^3(-\partial_1\underline{v}\cdot\partial_2 f^{\perp} - (\nabla\cdot\underline{v})(\nabla\cdot f)).$$

we deduce that

$$|A_{42}| \leq \varepsilon C(|\underline{v}|_{(W^{1,\infty})^2}, |\underline{\zeta}|_{W^{1,\infty}})E^s(U)^2.$$

The estimates proved in A_{41} and A_{42} show that

$$|A_4| \leq \varepsilon C(|\underline{v}|_{(W^{1,\infty})^2}, |\underline{\zeta}|_{W^{1,\infty}})E^s(U)^2.$$

• Estimate of $\big(\big[\Lambda^s, A[\underline{U}]\big]U, S\Lambda^s U\big)$. Remark first that

$$\begin{aligned}
&\big(\big[\Lambda^s, A[\underline{U}]\big]U, S\Lambda^s U\big) = ([\Lambda^s, \varepsilon\underline{v}]\cdot\nabla\zeta, \Lambda^s\zeta) + ([\Lambda^s, \underline{h}]\nabla\cdot v, \Lambda^s\zeta) \\
&+ \quad ([\Lambda^s, \underline{\mathfrak{T}}^{-1}\ \underline{h}]\nabla\zeta, \underline{\mathfrak{T}}\Lambda^s v) + ([\Lambda^s, \varepsilon(\underline{v}\cdot\nabla)]v, \underline{\mathfrak{T}}\Lambda^s v) + \varepsilon\mu\big(\big[\Lambda^s, \underline{\mathfrak{T}}^{-1}R_1[\underline{U}]\big]v, \underline{\mathfrak{T}}\Lambda^s v\big) \\
&=: \quad B_1 + B_2 + B_3 + B_4 + B_5.
\end{aligned}$$

− Control of $B_1 + B_2 = ([\Lambda^s, \varepsilon\underline{v}]\cdot\nabla\zeta, \Lambda^s\zeta) + ([\Lambda^s, \underline{h}]\nabla\cdot v, \Lambda^s\zeta)$.
Since $s \geq t_0 + 1$, we can use the following commutator estimate (3.3.11) (see e.g [51])

$$|[\Lambda^s, F]G|_2 \lesssim |\nabla F|_{H^{s-1}}|G|_{H^{s-1}}. \tag{3.3.11}$$

to get

$$|B_1 + B_2| \leq \varepsilon C(E^s(\underline{U}))E^s(U)^2.$$

− Control of $B_4 = ([\Lambda^s, \varepsilon(\underline{v}\cdot\nabla)]v, \underline{\mathfrak{T}}\Lambda^s v)$. By using the explicit expression of $\underline{\mathfrak{T}}$ we get

$$\begin{aligned}
B_4 &= ([\Lambda^s, \varepsilon\underline{v}\cdot\nabla]v, \underline{h}\Lambda^s v) + \frac{\mu}{3}(\nabla\cdot[\Lambda^s, \varepsilon(\underline{v}\cdot\nabla)]v, \underline{h}^3\ \Lambda^s\nabla\cdot v) \\
&\quad -\frac{\varepsilon\mu}{2}([\Lambda^s, \varepsilon(\underline{v}\cdot\nabla)]v, \underline{h}^2\ \nabla b\Lambda^s\nabla\cdot v) + \frac{\varepsilon\mu}{2}([\Lambda^s, \varepsilon(\underline{v}\cdot\nabla)]v, \nabla(\underline{h}^2\ \nabla b\cdot\Lambda^s v)) \\
&\quad +\varepsilon^2\mu([\Lambda^s, \varepsilon(\underline{v}\cdot\nabla)]v, \underline{h}\ \nabla b\nabla b\cdot\Lambda^s v) + \mu(\mathrm{curl}\,[\Lambda^s, \varepsilon(\underline{v}\cdot\nabla)]v, \Lambda^s\mathrm{curl}\, v) \\
&:= B_{41} + B_{42} + B_{43} + B_{44} + B_{45} + B_{46}.
\end{aligned}$$

One obtains as for the control of B_1 and B_2 above that

$$|B_{4j}| \leq \varepsilon C(E^s(\underline{U}))E^s(U)^2, \quad j \in \{1, 3, 4, 5\}.$$

The control of B_{42} and B_{46} is more delicate because of the dependence on μ (recall that the energy $E^s(U)$ controls $s+1$ derivatives of v, but with a small coefficient $\sqrt{\mu}$ in front of the derivatives of order $O(\sqrt{\mu})$). For B_{46}, we thus proceed as follows: we first write

$$\begin{aligned}
B_{46} &= \mu(\mathrm{curl}\,[\Lambda^s, \varepsilon(\underline{v}\cdot\nabla)]v, \Lambda^s \mathrm{curl}\, v)\\
&= \mu(\partial_1[\Lambda^s, \varepsilon\underline{V_1}\partial_1]V_2, \Lambda^s\mathrm{curl}\, v) + \mu(\partial_1[\Lambda^s, \varepsilon\underline{V_2}\partial_2]V_2, \Lambda^s\mathrm{curl}\, v)\\
&\quad -\mu(\partial_2[\Lambda^s, \varepsilon\underline{V_1}\partial_1]V_1, \Lambda^s\mathrm{curl}\, v) - \mu(\partial_2[\Lambda^s, \varepsilon\underline{V_2}\partial_2]V_1, \Lambda^s\mathrm{curl}\, v)\\
&:= B_{461} + B_{462} + B_{463} + B_{464}.
\end{aligned}$$

Remarking that for all $j \in \{1,2\}$ we have

$$\partial_j[\Lambda^s, f]g = [\Lambda^s, \partial_j f]g + [\Lambda^s, f]\partial_j g,$$

and

$$[\Lambda^s, f\partial_j]g = [\Lambda^s, f]\partial_j g,$$

we can rewrite B_{461} under the form

$$\begin{aligned}
B_{461} &= \mu(\partial_1[\Lambda^s, \varepsilon\underline{V_1}\partial_1]V_2, \Lambda^s\mathrm{curl}\, v)\\
&= \mu([\Lambda^s, \varepsilon\partial_1\underline{V_1}]\partial_1 V_2, \Lambda^s\mathrm{curl}\, v) + \mu([\Lambda^s, \varepsilon\underline{V_1}]\partial_1^2 V_2, \Lambda^s\mathrm{curl}\, v).
\end{aligned}$$

It is then easy to use the commutator estimate (3.3.11) in order to obtain

$$|B_{46j}| \le \varepsilon C(E^s(\underline{U}))E^s(U)^2, \quad j \in \{1,2,3,4\}.$$

Similarly, B_{42} is controled by $\varepsilon C(E^s(\underline{U}))E^s(U)^2$. The estimates proved in $(B_{4j})_{j\in\{1,2,3,4,5,6\}}$ show that

$$|B_4| \le \varepsilon C(E^s(\underline{U}))E^s(U)^2.$$

− Control of $B_3 = ([\Lambda^s, \underline{\mathfrak{T}}^{-1}\,\underline{h}]\nabla\zeta, \underline{\mathfrak{T}}\Lambda^s v)$. Remark first that

$$\underline{\mathfrak{T}}[\Lambda^s, \underline{\mathfrak{T}}^{-1}]\underline{h}\nabla\zeta = \underline{\mathfrak{T}}[\Lambda^s, \underline{\mathfrak{T}}^{-1}\,\underline{h}]\nabla\zeta - [\Lambda^s, \underline{h}]\nabla\zeta;$$

moreover, since $[\Lambda^s, \underline{\mathfrak{T}}^{-1}] = -\underline{\mathfrak{T}}^{-1}[\Lambda^s, \underline{\mathfrak{T}}]\underline{\mathfrak{T}}^{-1}$, one gets

$$\underline{\mathfrak{T}}[\Lambda^s, \underline{\mathfrak{T}}^{-1}\,\underline{h}]\nabla\zeta = -[\Lambda^s, \underline{\mathfrak{T}}]\underline{\mathfrak{T}}^{-1}\underline{h}\nabla\zeta + [\Lambda^s, \underline{h}]\nabla\zeta,$$

and one can check by using the explicit expression of $\underline{\mathfrak{T}}$ that

$$\begin{aligned}
\underline{\mathfrak{T}}[\Lambda^s, \underline{\mathfrak{T}}^{-1}\,\underline{h}]\nabla\zeta &= -[\Lambda^s, \underline{h}]\underline{\mathfrak{T}}^{-1}\underline{h}\nabla\zeta + \frac{\mu}{3}\nabla\{[\Lambda^s, \underline{h}^3]\nabla\cdot(\underline{\mathfrak{T}}^{-1}\underline{h}\nabla\zeta)\}\\
&\quad -\frac{\varepsilon\mu}{2}\nabla[\Lambda^s, \underline{h}^2\nabla b]\cdot\underline{\mathfrak{T}}^{-1}\underline{h}\nabla\zeta + \frac{\varepsilon\mu}{2}[\Lambda^s, \underline{h}^2\nabla b]\nabla\cdot(\underline{\mathfrak{T}}^{-1}\underline{h}\nabla\zeta)\\
&\quad -\varepsilon^2\mu[\Lambda^s, \underline{h}\nabla b\nabla b^T]\underline{\mathfrak{T}}^{-1}\underline{h}\nabla\zeta + [\Lambda^s, \underline{h}]\nabla\zeta.
\end{aligned}$$

Integrating by parts yields

$$\begin{aligned}
B_3 &= \big(\underline{\mathfrak{T}}[\Lambda^s, \underline{\mathfrak{T}}^{-1}\ \underline{h}]\nabla\zeta, \Lambda^s v\big)\\
&= -\big([\Lambda^s, \underline{h}]\underline{\mathfrak{T}}^{-1}\underline{h}\nabla\zeta, \Lambda^s v\big) - \frac{\mu}{3}\big(\{[\Lambda^s, \underline{h}^3]\nabla\cdot(\underline{\mathfrak{T}}^{-1}\underline{h}\nabla\zeta)\}, \Lambda^s\nabla\cdot v\big)\\
&\quad +\frac{\varepsilon\mu}{2}\big([\Lambda^s, \underline{h}^2\nabla b]\cdot\underline{\mathfrak{T}}^{-1}\underline{h}\nabla\zeta, \Lambda^s\nabla\cdot v\big) + \frac{\varepsilon\mu}{2}\big([\Lambda^s, \underline{h}^2\nabla b]\nabla\cdot(\underline{\mathfrak{T}}^{-1}\underline{h}\nabla\zeta), \Lambda^s v\big)\\
&\quad -\varepsilon^2\mu\big([\Lambda^s, \underline{h}\nabla b\nabla b^T]\underline{\mathfrak{T}}^{-1}\underline{h}\nabla\zeta, \Lambda^s v\big) + \big([\Lambda^s, \underline{h}]\nabla\zeta, \Lambda^s v\big).
\end{aligned}$$

One deduces directly from Lemma 3.3.1, the commutator estimate (3.3.11), and Cauchy-Schwarz inequality that

$$\begin{aligned}
|B_3| \ \leq\ & C\big(\frac{1}{h_{min}}, |\underline{h}-1|_{H^s}\big)\Big\{\Big(|\nabla\underline{h}|_{H^{s-1}} + \varepsilon^2\mu|\underline{h}^2\nabla b\nabla b^T|_{H^s} + \frac{\varepsilon\mu}{2}|\underline{h}\nabla b|_{H^s}\Big)|\underline{h}\nabla\zeta|_{H^{s-1}}\\
&+\Big(\frac{1}{3}|\nabla\underline{h}^3|_{H^{s-1}} + \frac{\varepsilon\sqrt{\mu}}{2}|\underline{h}^2\nabla b|_{H^s}\Big)|\underline{h}\nabla\zeta|_{H^{s-1}} + |\nabla\underline{h}|_{H^{s-1}}|\nabla\zeta|_{H^{s-1}}\Big)\Big\}|v|_{X^s}.
\end{aligned}$$

Finally, we deduce

$$|B_3| \leq \varepsilon C(E^s(\underline{U}))E^s(U)^2.$$

− Control of $B_5 = \varepsilon\mu\big(\big[\Lambda^s, \underline{\mathfrak{T}}^{-1}R_1[\underline{U}]\big]v, \underline{\mathfrak{T}}\Lambda^s v\big)$. Let us first write

$$\underline{\mathfrak{T}}\big[\Lambda^s, \underline{\mathfrak{T}}^{-1}R_1[\underline{U}]\big]v = -[\Lambda^s, \underline{\mathfrak{T}}]\underline{\mathfrak{T}}^{-1}R_1[\underline{U}]v + \big[\Lambda^s, R_1[\underline{U}]\big]v$$

so, that

$$\begin{aligned}
\underline{\mathfrak{T}}\big[\Lambda^s, \underline{\mathfrak{T}}^{-1}R_1[\underline{U}]\big]v \ =\ & -[\Lambda^s, \underline{h}]\underline{\mathfrak{T}}^{-1}R_1[\underline{U}]v + \frac{\mu}{3}\nabla\{[\Lambda^s, \underline{h}^3]\nabla\cdot(\underline{\mathfrak{T}}^{-1}R_1[\underline{U}]v)\}\\
&-\frac{\varepsilon\mu}{2}\nabla\{[\Lambda^s, \underline{h}^2\nabla b]\cdot\underline{\mathfrak{T}}^{-1}R_1[\underline{U}]v\} + \frac{\varepsilon\mu}{2}[\Lambda^s, \underline{h}^2\nabla b]\nabla\cdot(\underline{\mathfrak{T}}^{-1}R_1[\underline{U}]v)\\
&-\varepsilon^2\mu[\Lambda^s, \underline{h}\nabla b\nabla b^T]\underline{\mathfrak{T}}^{-1}R_1[\underline{U}]v + [\Lambda^s, R_1[\underline{U}]]v.
\end{aligned}$$

To control the term $\big(\big[\Lambda^s, R_1[\underline{U}]\big]v, \Lambda^s v\big)$ we use the explicit expression of $R_1[\underline{U}]$:

$$\begin{aligned}
R_1[\underline{U}]f \ =\ & -\frac{2}{3}\nabla Q[\underline{U}]f + \frac{\varepsilon}{2}\nabla(\underline{h}^2(\underline{V_1}F_1\partial_1^2 b + 2\underline{V_1}F_2\partial_1\partial_2 b + \underline{V_2}F_2\partial_2^2 b))\\
&-\varepsilon h^2(-\partial_1\underline{v}\cdot\partial_2 f^{\perp} - (\nabla\cdot\underline{v})(\nabla\cdot f))\nabla b,
\end{aligned}$$

where,

$$Q[\underline{U}]f = \underline{h}^3(-\partial_1\underline{v}\partial_2 f^{\perp} - (\nabla\cdot\underline{v})(\nabla\cdot f)).$$

As for the control of the term $\big(\nabla\{[\Lambda^s, \underline{h}^3]\nabla\cdot(\underline{\mathfrak{T}}^{-1}R_1[\underline{U}]v)\}, \Lambda^s v\big)$ we use the explicit expression of $R_1[\underline{U}]$, the relation (3.3.10), the commutator estimate (3.3.11) and

Lemma 3.3.1. Indeed,

$$
\begin{aligned}
\big(\nabla\{[\Lambda^s,\underline{h}^3]\nabla\cdot(\underline{\mathfrak{T}}^{-1}R_1[\underline{U}]v)\},\Lambda^s v\big) &= -\frac{2}{3}\big([\Lambda^s,\underline{h}^3]\nabla\cdot(\underline{\mathfrak{T}}^{-1}\nabla Q[\underline{U}]v),\Lambda^s\nabla\cdot v\big)\\
&- \frac{\varepsilon}{2}\big([\Lambda^s,\underline{h}^3]\nabla\cdot\underline{\mathfrak{T}}^{-1}\nabla(\underline{h}^2(\underline{V_1}V_1\partial_1^2 b+2\underline{V_1}V_2\partial_1\partial_2 b+\underline{V_2}V_2\partial_2^2 b)),\Lambda^s\nabla\cdot v\big)\\
&- \varepsilon\big([\Lambda^s,\underline{h}^3]\nabla\cdot\underline{\mathfrak{T}}^{-1}(h^2(-\partial_1\underline{v}\cdot\partial_2\Lambda^s v^\perp-(\nabla\cdot\underline{v})(\nabla\cdot v))\nabla b),\Lambda^s\nabla\cdot v\big).
\end{aligned}
$$

and thus, after remarking that

$$
|\nabla\cdot(\underline{\mathfrak{T}}^{-1}\nabla Q[\underline{U}]v)|_{H^{s-1}} \quad\leq\quad |\underline{\mathfrak{T}}^{-1}\nabla Q[\underline{U}]v|_{H^s},
$$

we can proceed as for the control of B_3 to get

$$
|B_5|\leq \varepsilon C(E^s(\underline{U}))E^s(U)^2.
$$

• Estimate of $(\Lambda^s B(\underline{U}),S\Lambda^s U)$. Note first that

$$
B(\underline{U})=\begin{pmatrix}\varepsilon\nabla b\cdot\underline{v}\\ \varepsilon\mu\underline{\mathfrak{T}}^{-1}r_2(\underline{U})\end{pmatrix}
$$

so that

$$
\begin{aligned}
(\Lambda^s B(\underline{U}),S\Lambda^s U) &= (\Lambda^s(\varepsilon\nabla b\cdot\underline{v}),\Lambda^s\zeta)+(\Lambda^s(\underline{\mathfrak{T}}^{-1}r_2(\underline{U})),\underline{\mathfrak{T}}\Lambda^s v)\\
&= (\Lambda^s(\varepsilon\nabla b\cdot\underline{v}),\Lambda^s\zeta)-\varepsilon\mu\big([\Lambda^s,\underline{\mathfrak{T}}]\underline{\mathfrak{T}}^{-1}r_2(\underline{U}),\Lambda^s v\big)\\
&\quad+\varepsilon\mu\big(\Lambda^s r_2(\underline{U}),\Lambda^s v\big).
\end{aligned}
$$

Using again here the explicit expressions of $\underline{\mathfrak{T}}$, $r_2(\underline{U})$ and Lemma 3.3.1, we get

$$
(\Lambda^s B(\underline{U}),S\Lambda^s U)\leq\varepsilon C(E^s(\underline{U}))E^s(U).
$$

• Estimate of $(\Lambda^s v,[\partial_t,\underline{\mathfrak{T}}]\Lambda^s v)$. We have that

$$
\begin{aligned}
(\Lambda^s v,[\partial_t,\underline{\mathfrak{T}}]\Lambda^s v) &= (\Lambda^s v,\partial_t\underline{h}\Lambda^s v)+\frac{\mu}{3}(\Lambda^s\nabla\cdot v,\partial_t\underline{h}^3\Lambda^s\nabla\cdot v)\\
&\quad-\frac{\varepsilon\mu}{2}(\Lambda^s v,\partial_t\underline{h}^2\nabla b\Lambda^s\nabla\cdot v)-\frac{\varepsilon\mu}{2}(\Lambda^s\nabla\cdot v,\partial_t\underline{h}^2\nabla b\cdot\Lambda^s v)\\
&\quad+\varepsilon^2\mu(\Lambda^s v,\partial_t\underline{h}\nabla b\nabla b^T\Lambda^s v).
\end{aligned}
$$

Controlling these terms by $\varepsilon C(E^s(\underline{U}),|\partial_t\underline{h}|_{L^\infty})E^s(U)^2$ follows directly from a Cauchy-Schwarz inequality and an integration by parts.

Gathering the informations provided by the above estimates and using the fact that the embedding $H^s(\mathbb{R}^2)\subset W^{1,\infty}(\mathbb{R}^2)$ is continuous, we get

$$
e^{\varepsilon\lambda t}\partial_t(e^{-\varepsilon\lambda t}E^s(U)^2)\leq\varepsilon\big(C(E^s(\underline{U}),|\partial_t\underline{h}|_{L^\infty})-\lambda\big)E^s(U)^2+\varepsilon C(E^s(\underline{U}))E^s(U).
$$

Taking $\lambda = \lambda_T$ large enough (how large depending on $\sup_{t\in[0,\frac{T}{\varepsilon}]} C(E^s(\underline{U}), |\partial_t \underline{h}|_{L^\infty})$ to have the first term of the right hand side negative for all $t \in [0, \frac{T}{\varepsilon}]$, one deduces

$$\forall t \in [0, \frac{T}{\varepsilon}], \qquad e^{\varepsilon\lambda t}\partial_t(e^{-\varepsilon\lambda t}E^s(U)^2) \leq \varepsilon C(E^s(\underline{U}))E^s(U).$$

Integrating this differential inequality yields therefore

$$\forall t \in [0, \frac{T}{\varepsilon}], \qquad E^s(U) \leq e^{\varepsilon\lambda_T t}E^s(U_0) + \varepsilon \int_0^t e^{\varepsilon\lambda_T(t-t')}C(E^s(\underline{U})(t'))dt',$$

which is the desired result. □

3.3.3 Main result

In this subsection we prove the main result of this chapter, which shows the well-posedness of the new Green-Naghdi equations (3.1.6) for times of order $O(\frac{1}{\varepsilon})$.

Theorem 3.3.1. *Let $b \in C_b^\infty(\mathbb{R}^2)$, $t_0 > 1$, $s \geq t_0 + 1$. Let also $U_0 = (\zeta_0, v_0^T)^T \in X^s$ be such that (3.3.1) is satisfied. Then there exists a maximal $T_{max} > 0$, uniformly bounded from below with respect to $\varepsilon, \mu \in (0,1)$, such that the new Green-Naghdi equations (3.1.6) admit a unique solution $U = (\zeta, v^T)^T \in X^s_{T_{max}}$ with the initial condition $(\zeta_0, v_0^T)^T$ and preserving the nonvanishing depth condition (3.3.1) for any $t \in [0, \frac{T_{max}}{\varepsilon})$. In particular if $T_{max} < \infty$ one has*

$$|U(t,\cdot)|_{X^s} \longrightarrow \infty \quad as \quad t \longrightarrow \frac{T_{max}}{\varepsilon},$$

or

$$\inf_{\mathbb{R}^2} h(t,\cdot) = \inf_{\mathbb{R}^2} 1 + \varepsilon(\zeta(t,\cdot) - b(\cdot)) \longrightarrow 0 \quad as \quad t \longrightarrow \frac{T_{max}}{\varepsilon}.$$

Remark 3.3.2. *For 2D surface waves, non flat bottoms, B. A. Samaniego and D. Lannes [4] proved a well-posedness result for the standard 2D Green-Naghdi equations using a Nash-Moser scheme. Our result only uses a standard Picard iterative and there is therefore no loss of regularity of the solution of the new 2D Green-Naghdi equations with respect to the initial condition.*

Remark 3.3.3. *No smallness assumption on ε nor μ is required in the theorem. The fact that T_{max} is uniformly bounded from below with respect to these parameters allows us to say that if some smallness assumption is made on ε, then the existence time becomes larger, namely of order $O(1/\varepsilon)$.*

Proof. Using the energy estimate of Proposition 3.3.1, one proves the result following the same lines as in the proof of Theorem 1 in [38], which is itself an adaptation of the standard proof of well-posedness of hyperbolic systems (e.g [2, 72]). □

3.3.4 Conservation of the almost irrotationality of v

To obtain the new $2D$ Green-Naghdi model (3.1.6) we used the fact that $\mathrm{curl}\, v = O(\mu)$, we prove in the following Theorem that the new model (3.1.6) conserves of course this property.

Theorem 3.3.2. *Let $t_0 > 1$, $s \geq t_0 + 1$, $U_0 = (\zeta_0, v_0^T)^T \in X^{s+4}$ with $|\mathrm{curl}\, v_0|_{H^s} \leq \mu C(|U_0|_{X^s})$. Then, the solution $U = (\zeta, v^T)^T \in X^{s+4}_{T_{max}}$ of the new Green-Naghdi equations (3.1.6) with the initial condition $(\zeta_0, v_0^T)^T$ satisfies*

$$\forall\, 0 < T < T_{max}, \quad |\mathrm{curl}\, v|_{L^\infty([0,\frac{T}{\varepsilon}],H^s)} \leq \mu\, C(T, |U_0|_{X^{s+4}}).$$

Proof. Applying the operator $\mathrm{curl}\,(\cdot)$ to the second equation of the model (3.1.6) after multiplying it by $\dfrac{1}{h}$ yields

$$\partial_t w + \varepsilon \nabla \cdot (vw) = \varepsilon\mu f_1 + g,$$

where

$$\begin{aligned} f_1 \;\; = \;\; & -\mathrm{curl}\, \big(\frac{1}{h}\mathcal{T}[h,\varepsilon b] - \frac{1}{h}\nabla^\perp \mathrm{curl}\,\big) v \cdot \nabla v \\ & -\mathrm{curl}\, \frac{1}{h}\Big\{ \frac{2}{3}\nabla[h^3(\partial_1 v \cdot \partial_2 v^\perp + (\nabla \cdot v)^2)] + \Re[h,\varepsilon b](v) \Big\} \end{aligned}$$

and

$$g = -\mu \mathrm{curl}\, \big(\frac{1}{h}\mathcal{T}[h,\varepsilon b] - \frac{1}{h}\nabla^\perp \mathrm{curl}\,\big)\partial_t v.$$

From the identity $\mathrm{curl}(\frac{1}{h}W) = -\varepsilon\frac{1}{h^2}\nabla^\perp(\zeta - b)\cdot W + \frac{1}{h}\mathrm{curl}\ W$, we deduce that g can be put under the form

$$g = \varepsilon\mu f_2 + \mu \mathrm{curl}\,(\frac{1}{h}\nabla^\perp \partial_t w),$$

with

$$f_2 = \frac{1}{h^2}\nabla^\perp(\zeta - b)\cdot \mathcal{T}[h,\varepsilon b]\partial_t v + \frac{1}{2h}\mathrm{curl}(h^2 \nabla b\, \partial_t \nabla \cdot v) - \frac{\varepsilon}{h}\mathrm{curl}(h\nabla b \nabla b \cdot \partial_t v).$$

We have thus shown that w solves

$$\Big(I - \mu \mathrm{curl}\,\big(\frac{1}{h}\nabla^\perp\,\big)\Big)\partial_t w + \varepsilon\nabla\cdot(vw) = \varepsilon\mu f_1 + \varepsilon\mu f_2.$$

Energy estimates on this equation then show that for all $0 < T < T_{max}$, one has

$$|w|_{L^\infty([0,T/\varepsilon],H^s)} \leq \mu C(T, |U|_{X^s_T})\big(|f_1|_{L^\infty([0,T/\varepsilon],H^s)} + |f_2|_{L^\infty([0,T/\varepsilon],H^s)}\big).$$

Now, one deduces from the explicit expression of f_j $(j = 1,2)$ that $|f_j|_{L^\infty([0,T/\varepsilon],H^s)} \leq C(|U|_{X_T^{s+4}})$; with Theorem 3.3.1, we deduce that $|f_j|_{L^\infty([0,T/\varepsilon],H^s)} \leq C(T, |U_0|_{X_T^{s+4}})$, and the result follows. ∎

PARTIE III

Numerical simulation

A numerical study of variable depth KdV equations and generalizations of Camassa-Holm-like equations

In this present chapter we study numerically the KdV-top equation derived in [37], and compare it with the Boussinesq equations over uneven bottom. We use here a finite-difference scheme that conserves a discrete energy for the fully discrete scheme. We also compare this the approach with discontinuous Galerkin method of [76]. For the equations derived in the same reference [37], in the case of stronger nonlinearities and related to the Camassa-Holm equation we find many finite difference schemes that conserve a discrete energy for the fully discrete scheme. We compare this approach with the discontinuous Galerkin method of [77].

4.1 Introduction

4.1.1 General setting

This chapter is devoted to the numerical comparison of different asymptotic models for the water waves problem for uneven bottoms. These equations describe the motion of the free surface and the evolution of the velocity field of a layer of fluid under the following assumptions: the fluid is ideal, incompressible, irrotationnal, and under the influence of gravity. The solutions of these equations are very difficult to describe, because of their complexity. We thus look for approximate models and hence for approximate solutions. The main asymptotical models used in coastal oceanography, including shallow-water equations, Boussinesq systems, Green-Naghdi equations (GN) have been rigorously justified in [3]. Some of these models capture the existence of solitary water waves and the associated phenomenon of soliton manifestation [41]. The Korteweg-de Vries (KdV) equation originally derived over flat bottoms [49] is an approximation of the Boussinesq equations, and this relation has been rigorously justified in [22, 67, 10, 36]. When the bottom is uneven, various generalizations of the KdV equation with non constant coefficients (called KdV-top) have been proposed [47, 70, 27, 59, 32, 78, 64, 31, 43, 66], and rigorously justified in [37]. One of the aims of this chapter is to study numerically these KdV-top equations, and to compare them with the Boussinesq equations over uneven bottom. The KdV equation on flat bottom can be numerically solved by using finite difference schemes [65, 81], or discontinuous Galerkin schemes [76]. It is treated with finite differences in [18] by using a Crank-Nicolson relaxation method in time introduced by Besse-Bruneau in [6] and justified by Besse in [7]. Our finite-difference scheme is inspired from these earlier works. We propose a modification so that the numerical scheme conserves a discrete energy for the fully discrete scheme (in space and time). We also compare this approach with the discontinuous Galerkin method of [76].

The generalization of the KdV-top equation to more nonlinear regimes (related to Camassa-Holm [15] and Degasperis-Procesi [26] equations) contains higher order nonlinear dispersive/nonlocal balances not present in the KdV and BBM equations. In 2008 Constantin and Lannes [20], rigorously justified these generalizations of the KdV equation in the case of flat bottoms . They proved that these equations can be used to furnish approximations to the governing equations for water waves. These Camassa-Holm (CH) equations on flat bottom can be numerically studied by using finite difference schemes [34, 35, 20, 25], or discontinous Galerkin schemes [77]. In 2009 S. Israwi [37], investigated the case of variable bottoms in the same scaling as in [20]. He derived a new variable coefficients class of equations which takes into account topographic effects and generalizes the CH-like equations of Constantin-Lannes [20]. In the present article, we find many finite difference schemes for these new models, so that the numerical scheme conserves a discrete energy for the fully discrete scheme (in space and time). We also compare this approach with the discontinuous Galerkin method [77].

4.1.2 Presentation of two-ways models : Boussinesq and Green-Naghdi equations

Parameterizing the free surface by $z = \varepsilon\zeta(t,x)$ (with $x \in \mathbb{R}$) and the bottom by $z = -1 + \beta b^{(\alpha)}(x)$ (with $b^{(\alpha)}(x) = b(\alpha x)$), one can use the incompressibility and irrotationality conditions to write the classical adimensionalized water waves in terms of a velocity potential φ associated with the flow, and where $\varphi(t,.)$ is defined on $\Omega_t = \{(x,z), -1 + \beta b^{(\alpha)}(x) < z < \varepsilon\zeta(t,x)\}$ (i.e. the velocity field is given by

$v = \nabla_{x,z}\varphi$):

$$
\begin{cases}
\mu\partial_x^2\varphi + \partial_z^2\varphi = 0, & \text{at} \quad -1+\beta b^{(\alpha)} < z < \varepsilon\zeta, \\
\partial_z\varphi - \mu\beta\alpha\partial_x b^{(\alpha)}\partial_x\varphi = 0 & \text{at} \quad z = -1+\beta b^{(\alpha)}, \\
\partial_t\zeta - \dfrac{1}{\mu}(-\mu\varepsilon\partial_x\zeta\partial_x\varphi + \partial_z\varphi) = 0, & \text{at} \quad z = \varepsilon\zeta, \\
\partial_t\varphi + \dfrac{1}{2}(\varepsilon(\partial_x\varphi)^2 + \dfrac{\varepsilon}{\mu}(\partial_z\varphi)^2) + \zeta = 0 & \text{at} \quad z = \varepsilon\zeta.
\end{cases}
\tag{4.1.1}
$$

The dimensionless parameters are defined as :

$$
\varepsilon = \frac{a}{h_0}, \quad \mu = \frac{h_0^2}{\lambda^2}, \quad \beta = \frac{b_0}{h_0};
$$

where a is a typical amplitude of the waves; λ is the wavelength, b_0 is the order of amplitude of the variations of the bottom topography; λ/α is the wavelength of the bottom variations; h_0 is the reference depth. We also recall that $b^{(\alpha)}(x) = b(\alpha x)$.
The parameter ε is often called nonlinearity parameter; while μ is the shallowness parameter. Asymptotic models from (1.1.3) are derived by making assumptions on the size of ε, β, α, and μ. In the shallow-water scaling ($\mu \ll 1$), one can derive (ε, β and α do not need to be small) the so-called Green-Naghdi equations (see [29, 53] for a derivation and [3, 38] for a rigorous justification). For one-dimensional surfaces and over uneven bottoms these equations couple the free surface elevation ζ to the vertically averaged horizontal component of the velocity,

$$
u(t,x) = \frac{1}{1+\varepsilon\zeta-\beta b^{(\alpha)}}\int_{-1+\beta b^{(\alpha)}}^{\varepsilon\zeta} \partial_x\varphi(t,x,z)dz \tag{4.1.2}
$$

and can be written as:

$$
\begin{cases}
\partial_t\zeta + \partial_x(hu) = 0, \\
(1+\dfrac{\mu}{h}\mathcal{T}[h,\beta b^{(\alpha)}])\partial_t u + \partial_x\zeta + \varepsilon u\partial_x u \\
\quad + \mu\varepsilon\big\{ -\dfrac{1}{3h}\partial_x(h^3(u\partial_x^2 u) - (\partial_x u)^2) + \Im[h,\beta b^{(\alpha)}]u\big\} = 0
\end{cases}
\tag{4.1.3}
$$

where $h = 1+\varepsilon\zeta-\beta b^{(\alpha)}$ and

$$
\mathcal{T}[h,\beta b^{(\alpha)}]W = -\frac{1}{3}\partial_x(h^3\partial_x W) + \frac{\beta}{2}\partial_x(h^2\partial_x b^{(\alpha)})W + \beta^2 h(\partial_x b^{(\alpha)})^2 W,
$$

while the purely topographical term $\Im[h,\beta b^{(\alpha)}]u$ is defined as:

$$
\begin{aligned}
\Im[h,\beta b^{(\alpha)}]u \quad = \quad & \frac{\beta}{2h}[\partial_x(h^2(u\partial_x)^2 b^{(\alpha)}) - h^2((u\partial_x^2 u) - (\partial_x u)^2)\partial_x b^{(\alpha)}] \\
& +\beta^2((u\partial_x)^2 b^{(\alpha)})\partial_x b^{(\alpha)}.
\end{aligned}
$$

We remark that the Green-Naghdi equations can then be simplified over uneven bottoms into

$$\begin{cases} \zeta_t + [hu]_x = 0 \\ u_t + \zeta_x + \varepsilon u u_x = \dfrac{\mu}{3h}[h^3(u_{xt} + \varepsilon u u_{xx} - \varepsilon u_x^2)]_x, \end{cases} \tag{4.1.4}$$

where $O(\mu^2)$ terms have been discarded, and provided that the parameters satisfy $\theta = (\alpha, \beta, \varepsilon, \mu) \in \wp$, where the set $\wp$ is defined as

$$\begin{aligned} \wp \;=\; & \{(\alpha, \beta, \varepsilon, \mu) \text{ such that } \varepsilon = O(\sqrt{\mu}),\ \beta\alpha = O(\mu),\ \beta\alpha = O(\varepsilon), \\ & \beta\alpha^{3/2} = O(\mu^2),\ \beta\alpha\varepsilon = O(\mu^2)\}. \end{aligned} \tag{4.1.5}$$

In order to obtain the KdV equation (called KdV-top) originally derived in [47, 70, 27], stronger assumptions on ε, β, α and μ must be made namely that the paramters belong to the subset $\wp' \subset \wp$ defined as:

$$\wp' = \{(\alpha, \beta, \varepsilon, \mu) \text{ such that } \varepsilon = O(\mu), \quad \alpha\beta = O(\varepsilon), \quad \alpha^{3/2}\beta = O(\varepsilon^2)\}. \tag{4.1.6}$$

Neglecting the $O(\mu^2)$ terms, one obtains from (1.2.1) the following Boussinesq system:

$$\begin{cases} \zeta_t + [hu]_x = 0 \\ u_t + \zeta_x + \varepsilon u u_x = \dfrac{\mu}{3}(c^4 u_{xt})_x, \end{cases} \tag{4.1.7}$$

where $c = \sqrt{1 - \beta b^{(\alpha)}}$.

4.2 Numerical scheme for the Kdv-top equations

In this section, attention is given to the regime of slow variations of the bottom topography under the long-wave scaling $\varepsilon = O(\mu)$. We investigate several situations satisfying the condition (4.1.6) on the parameters ε, β, α and μ.

4.2.1 The continuous case

The KDV-top (or original) model

The model studied in this section is the following (ζ is the elevation)

$$\zeta_t + \Gamma_1\zeta + \frac{3}{2c}\varepsilon\zeta\zeta_x + \frac{1}{6}\mu c^5 \zeta_{xxx} = 0, \tag{4.2.1}$$

and

$$\Gamma_1\zeta = \frac{1}{2}(c\,\zeta_x + \partial_x(c\,\zeta)),$$

where $c = \sqrt{1-\beta b^{(\alpha)}}$. We assume that (4.1.6) is satisfied without any further assumptions; to this regime corresponds the so-called KDV-top (or original) model (4.2.1). It is related to the Boussinesq equations in the meaning of consistency (see below) and it was originally derived in [47, 70, 27]. We list here some of the properties of this model. The proof of all the results below can be found in [37]. Let us first define two different kinds of consistency, namely, L^∞ and H^s consistency.

Definition 4.2.1. *Let $\wp_0 \subset \wp$ be a family of parameters (with $\wp$ as in (4.1.5)). A family $(\zeta^\theta, u^\theta)_{\theta\in\wp_0}$ is L^∞-consistent on $[0, \frac{T}{\varepsilon}]$ with the GN equations (4.1.4), if for all $\theta \in \wp_0$ (and denoting $h^\theta = 1 + \varepsilon\zeta^\theta - \beta b^{(\alpha)}$),*

$$\begin{cases} \zeta^\theta_t + [h^\theta u^\theta]_x = \mu^2 r_1^\theta \\ u^\theta_t + \zeta^\theta_x + \varepsilon u^\theta u^\theta_x = \dfrac{\mu}{3h^\theta}[(h^\theta)^3(u^\theta_{xt} + \varepsilon u^\theta u^\theta_{xx} - \varepsilon(u^\theta_x)^2)]_x + \mu^2 r_2^\theta \end{cases}$$

with $(r_1^\theta, r_2^\theta)_{\theta\in\wp_0}$ bounded in $L^\infty([0, \frac{T}{\varepsilon}] \times \mathbb{R})$.

When the residual is bounded in H^s and not in L^∞, we speak of H^s-consistency. When $s > 1/2$, this H^s-consistency is obviously stronger than the L^∞-consistency.

Definition 4.2.2. *Let $\wp_0 \subset \wp$ be a family of parameters (with $\wp$ as in (4.1.5)). A family $(\zeta^\theta, u^\theta)_{\theta\in\wp_0}$ is H^s-consistent of order $s \geq 0$ and on $[0, \frac{T}{\varepsilon}]$ with the GN equations (4.1.4), if for all $\theta \in \wp_0$, (and denoting $h^\theta = 1 + \varepsilon\zeta^\theta - \beta b^{(\alpha)}$),*

$$\begin{cases} \zeta^\theta_t + [h^\theta u]_x = \mu^2 r_1^\theta \\ u^\theta_t + \zeta^\theta_x + \varepsilon u^\theta u^\theta_x = \dfrac{\mu}{3}\dfrac{1}{h^\theta}[(h^\theta)^3(u^\theta_{xt} + \varepsilon u^\theta u^\theta_{xx} - \varepsilon(u^\theta_x)^2)]_x + \mu^2 r_2^\theta \end{cases}$$

with $(r_1^\theta, r_2^\theta)_{\theta\in\wp_0}$ bounded in $L^\infty([0, \frac{T}{\varepsilon}], H^s(\mathbb{R})^2)$.

Remark 4.2.1. *The definitions can be adapted to define L^∞ and H^s consistency with the Boussinesq equations (4.1.7) rather then the GN equations (4.1.4).*

For the KdV-top model (4.2.1), H^s-consistency cannot be established, but L^∞-consistency holds as shown below:

Theorem 4.2.1. *Let $s > \frac{3}{2}$, $b \in H^\infty(\mathbb{R})$ and $\zeta_0 \in H^{s+1}(\mathbb{R})$. For all $\theta \in \wp'$*

$$\wp' = \{(\alpha, \beta, \varepsilon, \mu) \ \text{such that} \ \varepsilon = O(\mu),\ \alpha\beta = O(\varepsilon),\ \alpha^{3/2}\beta = O(\varepsilon^2)\},$$

we obtain the following properties :

- *there exists* $T > 0$ *and a unique family of solutions* $(\zeta^\theta)_{\theta\in\wp'}$ *to (4.2.1) bounded in* $C([0, \frac{T}{\varepsilon}]; H^{s+1}(\mathbb{R}))$ *with initial condition* ζ_0 *;*
- *the familly* $(\zeta^\theta, u^\theta)_{\theta\in\wp'}$ *with (omitting the index* θ*)*

$$u := \frac{1}{c}\Big(\zeta - \frac{1}{2}\int_{-\infty}^{x} \frac{c_x}{c}\zeta\, ds - \frac{\varepsilon}{4c^2}\zeta^2 + \mu\frac{1}{6}c^4\zeta_{xx}\Big) \tag{4.2.2}$$

is L^∞*-consistent on* $[0, \frac{T}{\varepsilon}]$ *with the equations (4.1.7).*

Remark 4.2.2. *The term* $\int_{-\infty}^{x} \frac{c_x}{c}\zeta\, ds$ *does not necessarily decay at infinity, and this the reason why* H^s*-consistency does not hold in general. The problem of the* conver- gence *of the solution of (4.2.1) to the solution of (4.1.7) remains open; numerical simulations are performed in §2.2 to being some insight on this matter.*

The gentle model

In a first stage, we restrict here our attention to parameters ε, β, α and μ such that

$$\varepsilon = \mu, \quad \beta = O(\varepsilon), \quad \alpha = O(\varepsilon). \tag{4.2.3}$$

These conditions are stronger than $(\alpha, \beta, \varepsilon, \mu) \in \wp'$; we remark in particular that under the condition (4.2.3), the model (4.2.1) can be written after neglecting the $O(\mu^2)$ terms as :

$$\zeta_t + c\zeta_x + \frac{3}{2}\varepsilon\zeta\zeta_x + \frac{1}{6}\varepsilon\zeta_{xxx} + \frac{1}{2}c_x\zeta = 0, \tag{4.2.4}$$

we keep here the term $\frac{1}{2}c_x\zeta$ which is of order $O(\mu^2)$, to obtain a conservative scheme, and in that case, we are able to deduce an energy preserved by this model. This model (4.2.4) will be called gentle model since it is only able to handle gentle variations of bottom topography.

Proposition 4.2.1. *Let* b *and* ζ_0 *be given by the above theorem and* ζ *solve (4.2.4). Then, for all* $t \in [0, \frac{T}{\varepsilon}]$*,*

$$\int_{\mathbb{R}} |\zeta(t)|^2\, dx = \int_{\mathbb{R}} |\zeta_0|^2\, dx.$$

Remark 4.2.3. *With the choice of parameters (4.2.3), the model (4.2.4), is* H^s*-consistent on* $[0, \frac{T}{\varepsilon}]$ *with the equations (4.1.7), and a full justification (convergence) can given for this model (see [37]).*

The strong model

We consider here stronger variations of the topography, i.e. :

$$\mu = \varepsilon, \quad \beta = O(1), \quad \alpha = O(\varepsilon^{4/3}). \tag{4.2.5}$$

In order to obtain model with better properties, we add terms of order $O(\mu^2)$, so that we get equation (4.2.6) :

$$\zeta_t + \Gamma_1\zeta + \varepsilon\frac{3}{2}\Big(\frac{1}{c}\Big)^{2/3}\zeta\left(\Big(\frac{1}{c}\Big)^{1/3}\zeta\right)_x + \frac{\mu}{6}\Gamma_3\zeta = 0, \tag{4.2.6}$$

where, the skew-symmetric operators Γ_1 and Γ_3 are defined as

$$\Gamma_1\zeta = \frac{1}{2}(c\,\zeta_x + \partial_x(c\,\zeta)),$$

and

$$\Gamma_3\zeta = c^5\,\zeta_{xxx} + \frac{3}{2}(c^5)_x\,\zeta_{xx} + \frac{3}{4}(c^5)_{xx}\,\zeta_x + \frac{1}{8}(c^5)_{xxx}\,\zeta.$$

It is remarked that (4.2.6) differ from (4.2.1) only up to terms of order $O(\mu^2)$ under the condition (4.2.5), indeed

$$\begin{aligned}
\varepsilon\frac{3}{2c}\zeta\zeta_x &= \varepsilon\frac{3}{2}\Big(\frac{1}{c}\Big)^{2/3}\zeta\left(\Big(\frac{1}{c}\Big)^{1/3}\zeta\right)_x + O(\mu^2),\\
\frac{\mu}{6}c^5\zeta_{xxx} &= \frac{\mu}{6}\Gamma_3\zeta + O(\mu^2).
\end{aligned}$$

The interest of this formulation of the nonlinear and of the dispersive term is that it allows for exact conservation of the energy.

Proposition 4.2.2. *Let b and ζ_0 be given by the above theorem and ζ solve (4.2.6). Then, for all $t \in [0, \frac{T}{\varepsilon}]$,*

$$\int_{\mathbb{R}} |\zeta(t)|^2\,dx = \int_{\mathbb{R}} |\zeta_0|^2\,dx.$$

4.2.2 The numerical case

For any function f, let us denote by $f^n(x)$ the approximation of $f(t,x)$ with $t = n\Delta t$, $f^{n+1/2}(x)$ the one of $f(t,x)$ with $t = (n+1/2)\Delta t$ and by $f_i(t)$ the approximation of $f(t,x)$ with $x = i\Delta x$, $f_{i+1/2}(t)$ the one of $f(t,x)$ with $x = (i+1/2)\Delta x$.

L^2 conservative finite-difference schemes

We derive in the Lemma below a spatial discretization for the following nonlinear terms

$$u_t + u^p u_x \ \text{ and } \ u_t + 2f[u]u_x + f[u]_x u;$$

where, $p \in \mathbb{N}^*$, $f[u] = u_{xx}$ and $f[u]_x = u_{xxx}$ so that the finite difference schemes conserve the discrete L^2 norm.

Lemma 4.2.1. *The following schemes for $u_t + u^p u_x$ and $u_t + 2f[u]u_x + f[u]_x u$:*

$$\frac{u^{n+1} - u^n}{\Delta t} + \frac{1}{p+2}\Big(D_1\big(\frac{u^{n+1} + u^n}{2}\big)_i (u^{n+1/2})_i^p + D_1\big((u^{n+1/2})^p \frac{u^{n+1} + u^n}{2}\big)_i\Big),$$

and

$$\frac{u^{n+1} - u^n}{\Delta t} + D_1\big(\frac{u^{n+1} + u^n}{2}\big)_i f[u^{n+1/2})]_i + D_1\big(f[u^{n+1/2}]\frac{u^{n+1} + u^n}{2}\big)_i,$$

respectively are conservatives, that is to say we have the equality :

$$\sum_i (u_i^n)^2 = \sum_i (u_i^0)^2,$$

where, the matrix D_1 is the classical centered discretizations of the derivative ∂_x.

Proof. Taking in the above schemes the inner product with $\dfrac{u_i^{n+1} + u_i^n}{2}$, using the fact that for all $v, w \in \mathbb{R}^m$

$$(D_1 v, w) = -(v, D_1 w),$$

where $m = \dim(D_1)$, one easily obtains :

$$\sum_i (u_i^{n+1})^2 = \sum_i (u_i^n)^2.$$

$\square$

The numerical scheme of the gentle model

We choose here a spatial discretization for the gentle model (4.2.4) so that the discrete L^2-norm is preserved by the fully discrete scheme. Lemma 1 shows how to discretize the nonlinear term $\frac{3}{2}\varepsilon\zeta\zeta_x$ in a conservative way, and the third order term $\frac{\mu}{6}\zeta_{xxx}$ does not raise any difficulties. For the variable coefficients linear terms

$\Gamma_1\zeta = \dfrac{1}{2}(c\,\zeta_x + \partial_x(c\,\zeta))$, the situation is more delicate, we propose a special conservative discretization that allows a discrete version of Proposition 4.2.1, which gives the final discretization of (4.2.4):

$$\begin{aligned}&\frac{\zeta_i^{n+1}-\zeta_i^n}{\Delta t} + \Big(D_1^v\frac{\zeta^{n+1}+\zeta^n}{2}\Big)_i + \varepsilon\Big[\frac{1}{2}\Big(\zeta_i^{n+\frac12} + \frac{\zeta_{i+1}^{n+\frac12}+\zeta_{i-1}^{n+\frac12}}{2}\Big)\Big(D_1\frac{\zeta^{n+1}+\zeta^n}{2}\Big)_i \\ &+\frac{1}{2}\frac{\zeta_{i+1}^{n+1}+\zeta_{i+1}^n+\zeta_{i-1}^{n+1}+\zeta_{i-1}^n}{4}\Big(D_1\zeta^{n+\frac12}\Big)_i + \frac{1}{6}\Big(D_3\frac{\zeta^{n+1}+\zeta^n}{2}\Big)_i\Big] = 0,\end{aligned} \tag{4.2.7}$$

where the matrices D_1 and D_3 are the classical centered discretizations of the derivatives ∂_x and ∂_x^3, while the skew-symmetric matrix D_1^v is as follows:

$$\Big(D_1^v\zeta^n\Big)_i = \frac{c_{i+1/2}\,\zeta_{i+1}^n - c_{i-1/2}\,\zeta_{i-1}^n}{2\Delta x},$$

(the index v stands for "variable coefficients", if $c=1$ one has $D_1^v = D_1$.) Throughout this section, we will denote by $(\zeta^n)_{n\in\mathbb{N}}$ the unique sequence which solves (4.2.7) for all $n\in\mathbb{N}$. We obtain the conservation of a discrete energy (whose continuous version is stated in Propostion 4.2.1).

Theorem 4.2.2. *The L^2-norm of ζ^n is conserved, that is,*

$$\forall n\in\mathbb{N},\quad \sum_i(\zeta_i^n)^2 = \sum_i(\zeta_i^0)^2.$$

Therefore, the finite-difference scheme (4.2.7) is stable.

Proof. Taking in (4.2.7) the inner product with $\dfrac{\zeta_i^{n+1}+\zeta_i^n}{2}$, using the fact that D_1, D_3 and D_1^v are skew-symmetric matrices, we obtain

$$\begin{aligned}&\sum_i\Big(\frac{\zeta_i^{n+1}-\zeta_i^n}{\Delta t}\frac{\zeta_i^{n+1}+\zeta_i^n}{2}\Big) + \varepsilon\sum_i\Big(\Big[\frac{1}{2}\Big(\zeta_i^{n+\frac12}+\frac{\zeta_{i+1}^{n+\frac12}+\zeta_{i-1}^{n+\frac12}}{2}\Big)\Big(D_1\frac{\zeta^{n+1}+\zeta^n}{2}\Big)_i \\ &+\frac{1}{2}\frac{\zeta_{i+1}^{n+1}+\zeta_{i+1}^n+\zeta_{i-1}^{n+1}+\zeta_{i-1}^n}{4}\Big(D_1\zeta^{n+\frac12}\Big)_i\Big]\frac{\zeta_i^{n+1}+\zeta_i^n}{2}\Big) = S_1(\zeta)+\varepsilon S_2(\zeta) = 0.\end{aligned}$$

We show first that $S_2 = 0$. In order to do this, we remark that

$$\begin{aligned}
S_2(\zeta) &= \frac{1}{2}\sum_i \zeta_i^{n+\frac{1}{2}}\Big(\frac{\zeta_{i+1}^{n+1}-\zeta_{i-1}^{n+1}+\zeta_{i+1}^{n}-\zeta_{i-1}^{n}}{4\Delta x}\Big)\frac{\zeta_i^{n+1}+\zeta_i^n}{2} \\
&\quad+\frac{1}{2}\sum_i \frac{\zeta_{i+1}^{n+\frac{1}{2}}+\zeta_{i-1}^{n+\frac{1}{2}}}{2}\Big(\frac{\zeta_{i+1}^{n+1}-\zeta_{i-1}^{n+1}+\zeta_{i+1}^{n}-\zeta_{i-1}^{n}}{4\Delta x}\Big)\frac{\zeta_i^{n+1}+\zeta_i^n}{2} \\
&\quad+\frac{1}{2}\sum_i \frac{\zeta_{i+1}^{n+1}+\zeta_{i+1}^{n}+\zeta_{i-1}^{n+1}+\zeta_{i-1}^{n}}{4}\Big(D_1\zeta^{n+\frac{1}{2}}\Big)_i\frac{\zeta_i^{n+1}+\zeta_i^n}{2} \\
&= S_{21}+S_{22}+S_{23};
\end{aligned}$$

with a change of subscripts in S_{21}, we get

$$\begin{aligned}
S_{21} &= \frac{1}{8\Delta x}\sum_i \zeta_{i-1}^{n+\frac{1}{2}}\zeta_i^{n+1}\frac{\zeta_{i-1}^{n+1}+\zeta_{i-1}^{n}}{2}-\zeta_{i+1}^{n+\frac{1}{2}}\zeta_i^{n+1}\frac{\zeta_{i+1}^{n+1}+\zeta_{i+1}^{n}}{2} \\
&\quad+\zeta_{i-1}^{n+\frac{1}{2}}\zeta_i^{n}\frac{\zeta_{i-1}^{n+1}+\zeta_{i-1}^{n}}{2}-\zeta_{i+1}^{n+\frac{1}{2}}\zeta_i^{n}\frac{\zeta_{i+1}^{n+1}+\zeta_{i+1}^{n}}{2} \\
&= \frac{1}{16\Delta x}\sum_i \frac{\zeta_i^{n+1}+\zeta_i^n}{2}\Big[2\zeta_{i-1}^{n+\frac{1}{2}}\big(\zeta_{i-1}^{n+1}+\zeta_{i-1}^{n}\big)-2\zeta_{i+1}^{n+\frac{1}{2}}\big(\zeta_{i+1}^{n+1}+\zeta_{i+1}^{n}\big)\Big],
\end{aligned}$$

and we remind that

$$\begin{aligned}
S_{22} &= \frac{1}{16\Delta x}\sum_i \Big[\zeta_{i-1}^{n+\frac{1}{2}}\big(\zeta_{i+1}^{n+1}+\zeta_{i+1}^{n}-\zeta_{i-1}^{n+1}-\zeta_{i-1}^{n}\big)+\zeta_{i+1}^{n+\frac{1}{2}}\big(\zeta_{i+1}^{n+1}+\zeta_{i+1}^{n}-\zeta_{i-1}^{n+1} \\
&\qquad -\zeta_{i-1}^{n}\big)\Big]\frac{\zeta_i^{n+1}+\zeta_i^n}{2}.
\end{aligned}$$

Summing S_{21} and S_{22}, we get

$$\begin{aligned}
S_{21}+S_{22} &= \frac{1}{16\Delta x}\sum_i \Big[(\zeta_{i-1}^{n+\frac{1}{2}}-\zeta_{i+1}^{n+\frac{1}{2}})(\zeta_{i+1}^{n+1}+\zeta_{i+1}^{n}+\zeta_{i-1}^{n+1}+\zeta_{i-1}^{n})\Big]\frac{\zeta_i^{n+1}+\zeta_i^n}{2} \\
&= -S_{23}.
\end{aligned}$$

Finally, we deduce that $S_2(\zeta) = 0$. It follows now that $S_1(\zeta) = 0$, which implies easily the result.

$\square$

The numerical scheme of the strong model

We recall that

$$\Gamma_1 \zeta \;=\; \frac{1}{2}(c\,\zeta_x \,+\, \partial_x(c\,\zeta)),$$

and

$$\Gamma_3 \zeta \;=\; c^5\,\zeta_{xxx} \,+\, \frac{3}{2}(c^5)_x\,\zeta_{xx} \,+\, \frac{3}{4}\,(c^5)_{xx}\,\zeta_x \,+\, \frac{1}{8}(c^5)_{xxx}\,\zeta.$$

These two operators are discretized by matrices D_1^v and D_3^v :

$$\Big(D_1^v \zeta^n\Big)_i \;=\; \frac{c_{i+1/2}\;\zeta_{i+1}^n - c_{i-1/2}\;\zeta_{i-1}^n}{2\Delta x}.$$

$$\Big(D_3^v \zeta^n\Big)_i = \frac{c_{i+1}^5\,\zeta_{i+2}^n \;-2\,c_{i+1/2}^5\;\zeta_{i+1}^n \;+\; 2\,c_{i-1/2}^5\;\zeta_{i-1}^n - c_{i-1}^5\,\zeta_{i-2}^n}{2\Delta x^3}.$$

These two matrices are skew-symmetric and do not modify the stability of the scheme. We choose for the strong model (4.2.6) a fully discrete scheme, similar to the previous section :

$$\begin{aligned}
&\frac{\zeta_i^{n+1}-\zeta_i^n}{\Delta t} + \Big(D_1^v \frac{\zeta^{n+1}+\zeta^n}{2}\Big)_i \,+\, \varepsilon\Big(\frac{1}{c}\Big)_i^{1/3}\Big[\frac{1}{2}\Big(\Big(\frac{1}{c}\Big)_i^{1/3}\zeta_i^{n+\frac{1}{2}} \\
&+\frac{\big(\frac{1}{c}\big)_{i+1}^{1/3}\zeta_{i+1}^{n+\frac{1}{2}} + \big(\frac{1}{c}\big)_{i-1}^{1/3}\zeta_{i-1}^{n+\frac{1}{2}}}{2}\Big)\Big(D_1\Big(\frac{1}{c}\Big)^{1/3}\frac{\zeta^{n+1}+\zeta^n}{2}\Big)_i \\
&+\frac{1}{2}\frac{\big(\frac{1}{c}\big)_{i+1}^{1/3}\zeta_{i+1}^{n+1} + \big(\frac{1}{c}\big)_{i+1}^{1/3}\zeta_{i+1}^{n} + \big(\frac{1}{c}\big)_{i-1}^{1/3}\zeta_{i-1}^{n+1} + \big(\frac{1}{c}\big)_{i-1}^{1/3}\zeta_{i-1}^{n}}{4}\times \\
&\Big(D_1\Big(\frac{1}{c}\Big)^{1/3}\zeta^{n+\frac{1}{2}}\Big)_i\Big] + \frac{\mu}{6}\Big(D_3^v\frac{\zeta^{n+1}+\zeta^n}{2}\Big)_i = 0.
\end{aligned} \tag{4.2.8}$$

Theorem 4.2.3. *The discrete scheme (4.2.8) of the model (4.2.6) conserves an energy i.e*

$$\forall n \in \mathbb{N}, \quad \sum_i (\zeta_i^n)^2 = \sum_i (\zeta_i^0)^2.$$

Proof. Taking in (4.2.8) the inner product with $\dfrac{\zeta_i^{n+1}+\zeta_i^n}{2}$, using the fact that D_1, D_3^v and D_1^v are skew-symmetric matrix, we obtain

$$\begin{aligned}
&\sum_i\Big(\frac{\zeta_i^{n+1}-\zeta_i^n}{\Delta t}\frac{\zeta_i^{n+1}+\zeta_i^n}{2}\Big) + \varepsilon\sum_i\Big(\Big[\frac{1}{2}\Big(v_i^{n+\frac{1}{2}} + \frac{v_{i+1}^{n+\frac{1}{2}}+v_{i-1}^{n+\frac{1}{2}}}{2}\Big)\Big(D_1\frac{v^{n+1}+v^n}{2}\Big)_i \\
&+\frac{1}{2}\frac{v_{i+1}^{n+1}+v_{i+1}^n+v_{i-1}^{n+1}+v_{i-1}^n}{4}\Big(D_1 v^{n+\frac{1}{2}}\Big)_i\Big]\frac{v_i^{n+1}+v_i^n}{2}\Big) = S_1(\zeta)+\varepsilon S_2(v) = 0,
\end{aligned}$$

where

$$v = \left(\frac{1}{c}\right)^{1/3} \zeta.$$

□

Numerical validation

Since there does not exist explicit solutions of (4.2.4) or (4.2.6), we have chosen a high-order Local Discontinuous Galerkin (LDG) in space and high-order Gauss-Runge-Kutta scheme in time (see [33]), in order to obtain very accurate of implementation results, that can be used as reference solutions to validate the finite difference method. The main advantage of finite difference schemes are their simplicity and quickness. It also gives very good conservation of energy. Details about the LDG method can be found in [76] (for a flat bottom), the extension to variable bottom does not raise important difficulties. In the case of flat bottoms, analytical solutions are well-known, and consist of solitary-waves. Let us consider the following initial condition parameterized by c_1

$$\zeta_0(x) \;=\; 2\, c_1 \,\mathrm{sech}^2\Big(\sqrt{\frac{3 c_1 \varepsilon}{2\mu}}\, x\Big),$$

Therefore, the analytical solution for a flat bottom of (4.2.1) is equal to

$$\zeta(x,t) \;=\; \zeta_0(x - c'\, t),$$

with a real velocity c'

$$c' \;=\; 1 + \varepsilon c_1.$$

We can wonder what is the influence of the bottom for this solitary-wave. To this aim, we consider a sinusoidal bottom

$$b(x) \;=\; \sin(2\pi\alpha x).$$

In the figure 4.2.1, we have displayed the solution for this initial condition ($c_1 \;=\; 0.5$) for a sinusoidal bottom and with different models.
In the tables 4.1, 4.2, 4.3, the L^2 error has been computed for the LDG method and the finite-difference method for a flat bottom and a sinusoidal bottom for the two models (4.2.4), (4.2.6) presented. In these tables, the time step Δt has been chosen small enough so that there is no error due to time-discretization, the error comes only because of space discretization.
In figure 4.2.2, we displayed the variation of the L^2-norm for the different proposed schemes. In this figure, we observe that the discrete energy of (4.2.7) and (4.2.8) is conserved (the magnitude of the variations is 10^{-15} due to machine precision), whereas the energy of LDG scheme is decreasing (in the figure, we see that the variation of energy is increasing, so that the total energy is strictly decreasing).

	LDG, order 1		LDG, order 3		LDG, order 7		Finite Difference	
N	Error	Order	Error	Order	Error	Order	Error	Order
160	1.083e-1	-	1.889e-3	-	2.168e-4	-	2.129e-1	-
320	2.859e-2	1.92	1.489e-4	3.67	8.367e-7	8.02	6.195e-2	1.78
640	5.049e-3	2.50	6.231e-6	4.58	1.882e-9	8.80	1.634e-2	1.92
1280	8.414e-4	2.59	4.469e-7	3.80	1.328e-11	7.15	4.392e-3	1.90
2560	1.545e-4	2.46	2.606e-8	4.10	7.580e-14	7.45	1.100e-3	1.997

Table 4.1: L^2 errors for the solitary-wave and flat bottom between the numerical solution and the analytical one for $t = 13.333$. N denotes here the number of degrees of freedom. ($c_1 = 0.5 \quad \varepsilon = \mu = 0.1$)

	LDG, order 1		LDG, order 3		LDG, order 7		Finite Difference	
N	Error	Order	Error	Order	Error	Order	Error	Order
160	1.350e-1	-	1.770e-2	-	9.192e-3	-	2.395e-1	-
320	4.469e-2	1.59	2.124e-3	3.06	3.247e-5	8.15	6.804e-2	1.82
640	1.119e-2	2.00	4.347e-5	5.61	1.785e-7	7.51	1.743e-2	1.97
1280	1.950e-3	2.52	2.460e-6	4.14	7.071e-10	7.98	4.367e-3	1.997
2560	2.908e-4	2.75	1.600e-7	3.95	2.926e-12	7.92	1.092e-3	2.00

Table 4.2: L^2 errors for the solitary-wave and sinusoidal bottom between the numerical solution and a reference solution for $t = 13.33$ and for the gentle model (4.2.4). N denotes here the number of degrees of freedom. ($c_1 = 0.5 \quad \beta = 0.5, \quad \varepsilon = \mu = 0.1 \quad \alpha = 0.05$)

	LDG, order 1		LDG, order 3		LDG, order 7		Finite Difference	
N	Error	Order	Error	Order	Error	Order	Error	Order
160	2.419e-1	-	8.4212e-2	-	2.263e-2	-	4.263e-1	-
320	1.312e-1	0.88	8.751e-3	3.27	3.780e-4	5.90	2.202e-1	0.95
640	4.514e-2	1.54	2.718e-4	5.01	1.819e-6	7.70	7.546e-2	1.55
1280	8.523e-3	2.41	8.208e-6	5.05	4.460e-9	8.67	2.294e-2	1.72
2560	1.218e-3	2.81	5.246e-7	3.97	4.131e-11	6.75	8.924e-3	1.36

Table 4.3: L^2 errors for the solitary-wave and sinusoidal bottom between the numerical solution and a reference solution for $t = 13.33$ and for the strong model (4.2.6). N denotes here the number of degrees of freedom. ($c_1 = 0.5 \quad \beta = 0.5, \quad \varepsilon = \mu = 0.1 \quad \alpha = 0.05$)

As a final test, we propose to check numerically the accuracy of the approximation provided by the KdV-top models (4.2.4), and the strong model (4.2.6) (we recall here that the strong model (4.2.6) is not fully justified mathematically, and that we are only able to get L^∞ consistency) in comparison to the Boussinesq equations (4.1.7). The initial condition for ζ is the solitary wave (the same as previously, but centered at $x = -10$), and the expression of initial condition for u in the Boussinesq

equations is given by (4.2.2). The computational domain is $[-100, 100]$ and we have computed the relative error for $T = 50$. For a flat bottom (see Fig. 4.2.3), we see that the solutions of the KdV and Boussinesq equations differ from $O(\varepsilon^2)$. For an uneven bottom (see Fig. 4.2.4) the relative error seems to be in $O(\varepsilon^2)$ when $\alpha = \beta = O(\varepsilon)$, whereas it seems to be in $O(\varepsilon)$ when $\beta = O(1)$. We can see that the strong model gives almost the same solutions as the original model (4.2.1).
We have displayed the solution obtained for $\beta = 0.5$, $\varepsilon = 0.018$ and $T = 50$ on Fig. 4.2.5. We can see that the solution of the Boussinesq equations is non-null for a large range of x, on the interval $[-60, 40]$, whereas solutions of KdV models are non-null for a smaller range $[-20, 40]$. And we clearly see that in this case, the strong and original model give much better solution (relative L^2 error is respectively equal to 6.4 % and 7.4%) than gentle model (relative L^2 error is equal to 62.2 %).

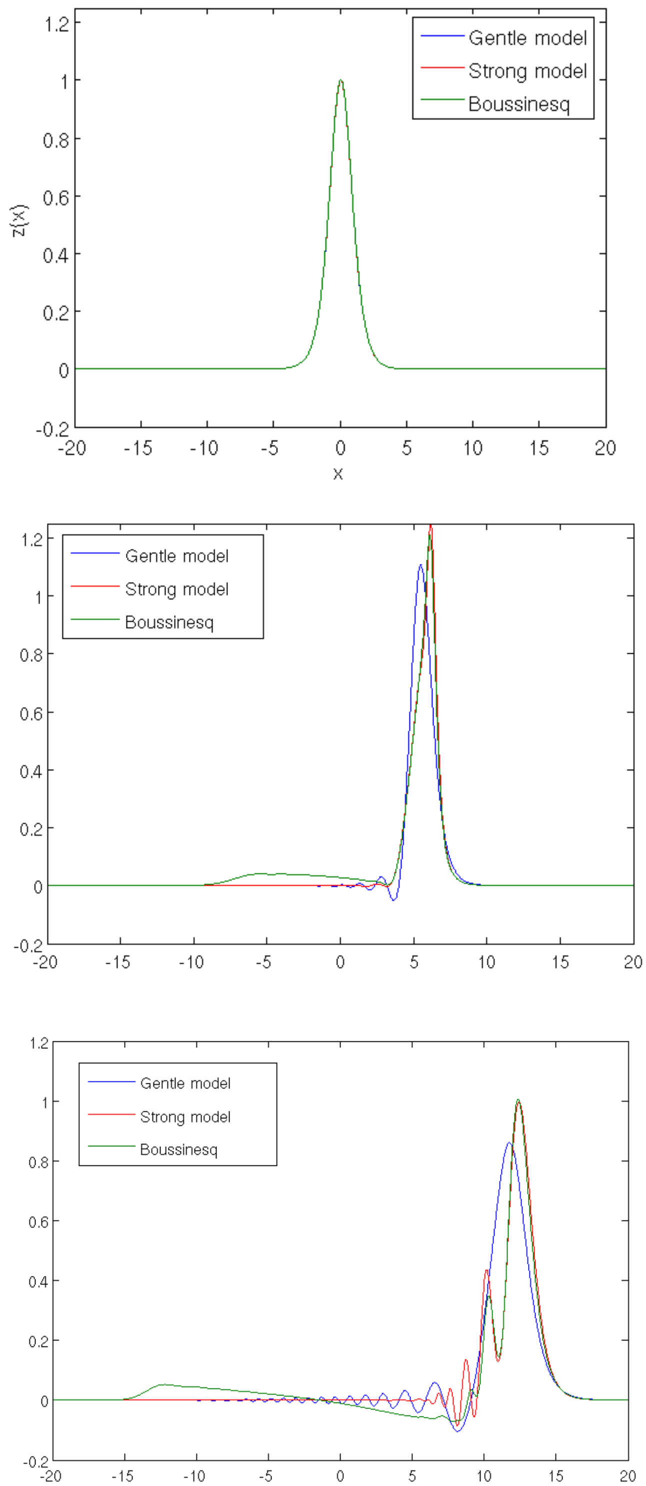

Figure 4.2.1: Solution of KdV equation for a sinusoidal bottom, with gentle, strong model and Boussinesq model, for $t = 0, \quad t = 6.67, \quad t = 13.33$. ($c_1 = 0.5 \quad \beta = 0.5, \quad \varepsilon = \mu = 0.1 \quad \alpha = 0.05$)

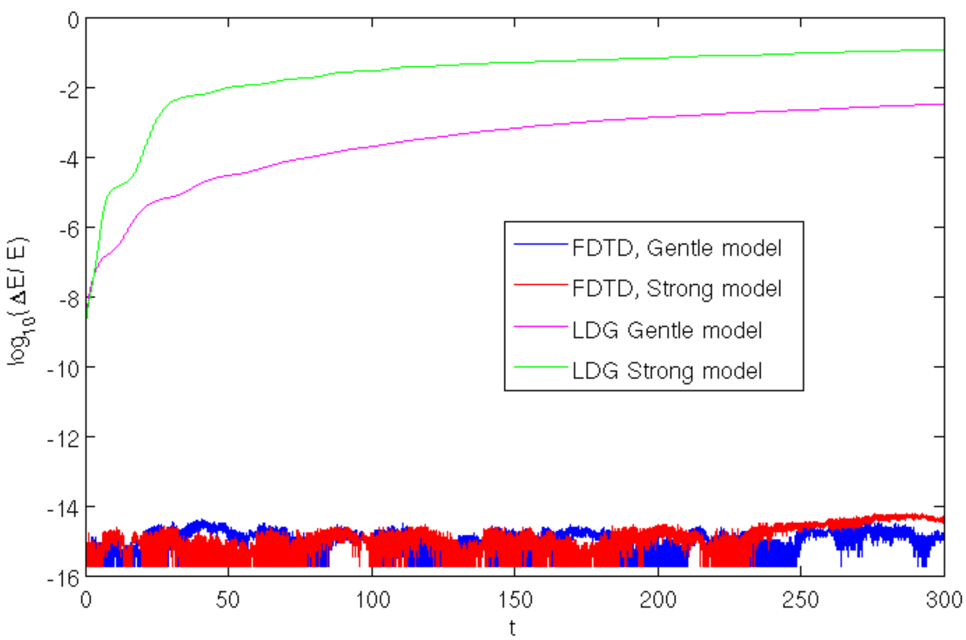

Figure 4.2.2: Logarithm of variation of energy versus time for gentle scheme (4.2.7) and strong scheme (4.2.8) for solitary-wave and sinusoidal bottom ($c_1 = 0.5 \quad \beta = 0.5, \quad \varepsilon = \mu = 0.1 \quad \alpha = 0.05$). Finite difference and third order LDG with 640 degrees of freedom.

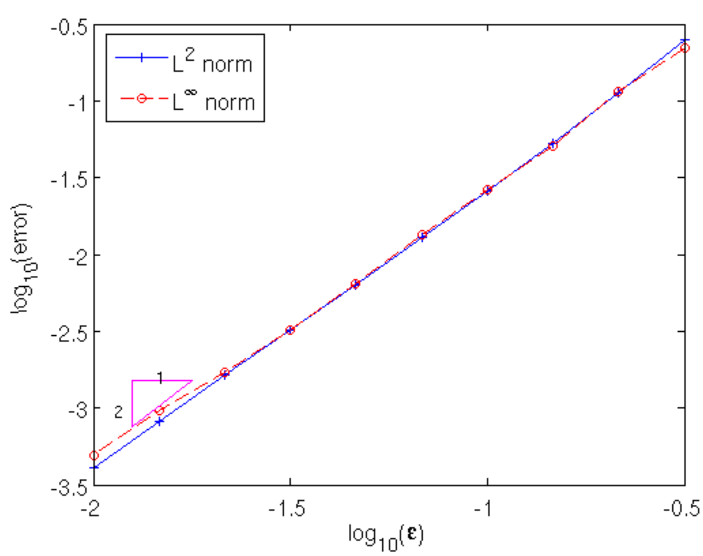

Figure 4.2.3: Relative error between solutions of Boussinesq equations and KdV for a flat bottom (Log-log scale)

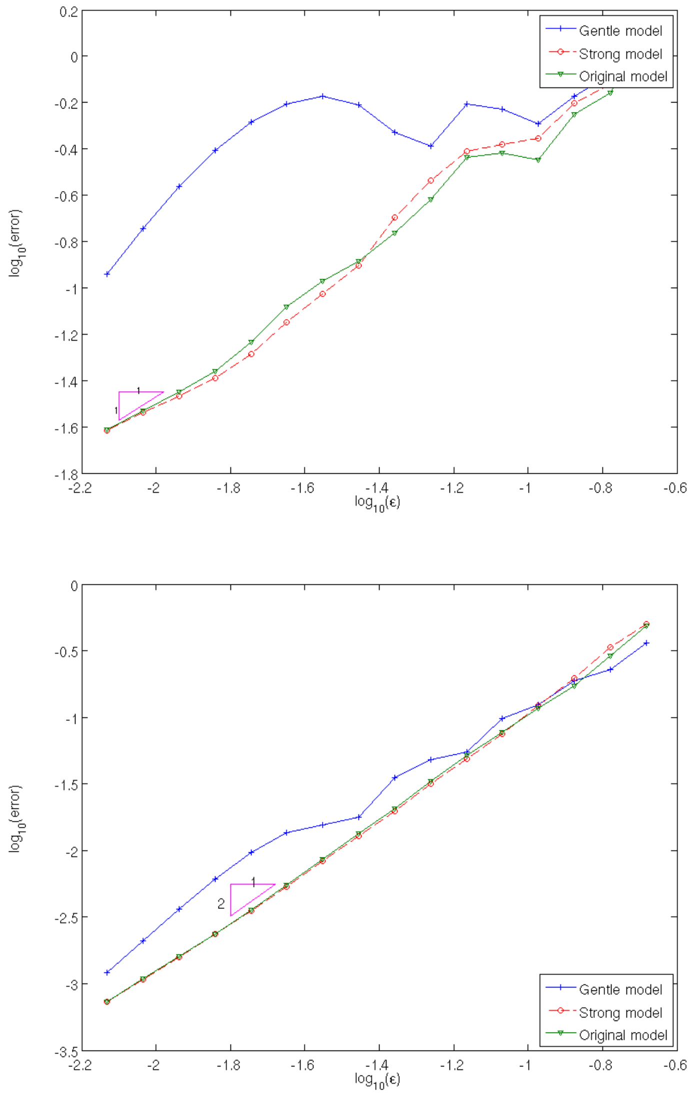

Figure 4.2.4: Relative error between solutions of Boussinesq equations and KdV for a sinusoidal bottom (Log-log scale). We have considered gentle model (4.2.4), strong model (4.2.6) and original model (4.2.1). On above $\alpha = 0.5\,\varepsilon$, $\beta = 0.5$, on below $\alpha = 0.5\varepsilon$, $\beta = \varepsilon$.

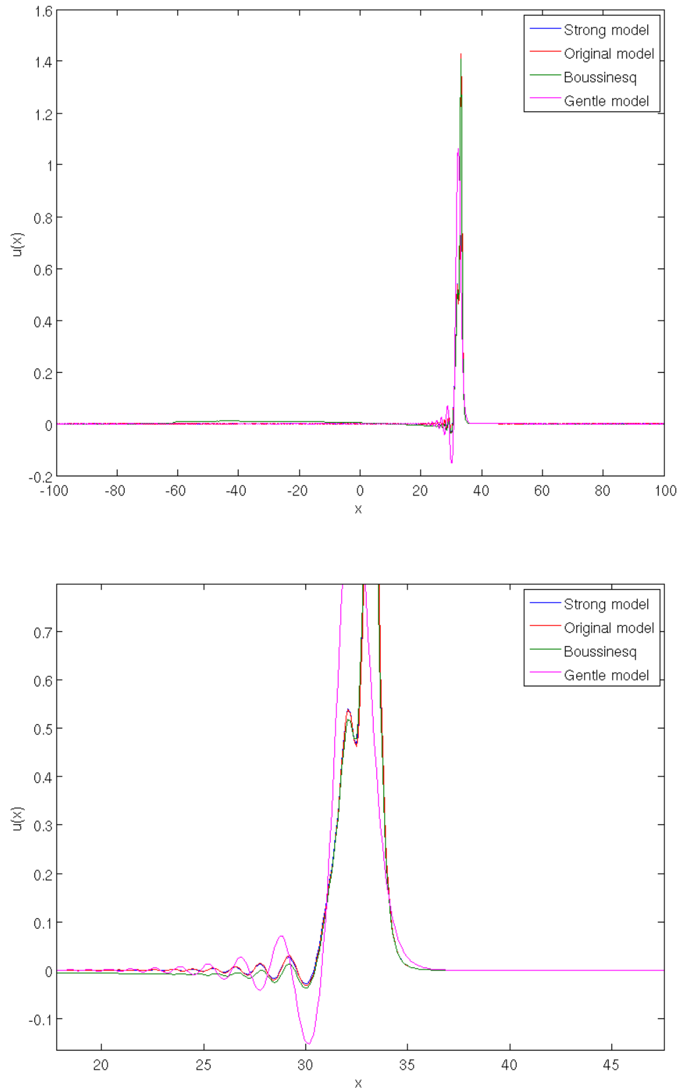

Figure 4.2.5: Solution obtained for a solitary-wave with a sinusoidal bottom for $\beta = 0.5$, $\varepsilon = 0.018$, $\alpha = 0.5\varepsilon$, and $T = 50$. On bottom, you can see the solution on a reduced interval so that you can observe the differences.

4.3 Numerical scheme for the Camassa-Holm-like equations

Now we consider the generalizations to more nonlinear regimes of the KdV-top equation derived in [20] for flat bottoms and [37] for variable bottoms.

4.3.1 The continuous case

The original model

The family of equations on the surface elevation ζ (see [37]) :

$$\zeta_t + c\zeta_x + \frac{1}{2}c_x\zeta + \frac{3}{2c}\varepsilon\zeta\zeta_x - \frac{3}{8c^3}\varepsilon^2\zeta^2\zeta_x + \frac{3}{16c^5}\varepsilon^3\zeta^3\zeta_x$$
$$+\mu(\tilde{A}\zeta_{xxx} + B\zeta_{xxt}) = \varepsilon\mu\tilde{E}\zeta\zeta_{xxx} + \varepsilon\mu\Big(\partial_x(\frac{\tilde{F}}{2}\zeta)\zeta_{xx} + \zeta_x\partial_x^2(\frac{\tilde{F}}{2}\zeta)\Big), \quad (4.3.1)$$

where

$$\begin{aligned}\tilde{A} &= Ac^5 - Bc^5 + Bc\\ \tilde{E} &= Ec^3 - \frac{3}{2}Bc^3 + \frac{3}{2c}B\\ \tilde{F} &= Fc^3 - \frac{9}{2}Bc^3 + \frac{9}{2c}B,\end{aligned}$$

called here original model, can be used to construct an approximate solution consistent with the Green-Naghdi equations.

Theorem 4.3.1. *Let $s > \frac{3}{2}$, $b \in H^\infty(\mathbb{R})$ and $\zeta_0 \in H^{s+1}(\mathbb{R})$. Assume that*

$$A = q, \quad B = q - \frac{1}{6} \quad E = -\frac{3}{2}q - \frac{1}{6}, \quad F = -\frac{9}{2}q - \frac{5}{24}.$$

For all $\theta \in \wp$ such that

$$\begin{aligned}\wp \quad = \quad &\{(\alpha,\beta,\varepsilon,\mu) \text{ such that } \varepsilon = O(\sqrt{\mu}),\ \beta\alpha = O(\mu),\ \beta\alpha = O(\varepsilon),\\ &\beta\alpha^{3/2} = O(\mu^2),\ \beta\alpha\varepsilon = O(\mu^2)\}\end{aligned}$$

we obtain

- *there exists $T > 0$ and a unique family of solutions $(\zeta^\theta)_{\theta\in\wp}$ to (4.3.1) bounded in $C([0, \frac{T}{\varepsilon}]; H^{s+1}(\mathbb{R}))$ with initial condition ζ_0 ;*

- *the familly* $(\zeta^\theta, u^\theta)_{\theta\in\wp}$ *with (omitting the index* θ*)*

$$u := \frac{1}{c}\Big(\zeta + \frac{c^2}{c^2+\varepsilon\zeta}\Big(-\frac{1}{2}\int_{-\infty}^{x}\frac{c_x}{c}\zeta - \frac{\varepsilon}{4c^2}\zeta^2 - \frac{\varepsilon^2}{8c^4}\zeta^3 + \frac{3\varepsilon^3}{64c^6}\zeta^4$$
$$-\mu\frac{1}{6}c^3\zeta_{xt} + \varepsilon\mu c^2\big[\frac{1}{6}\zeta\zeta_{xx} + \frac{1}{48}\zeta_x^2\big]\Big)\Big)$$

is L^∞*-consistent on* $[0, \frac{T}{\varepsilon}]$ *with the GN equations (4.1.4).*

Remark 4.3.1. *If we take* $q = \frac{1}{12}$, $b = 0$, *i.e if we consider a flat bottom, then one can recover the equation (19) of [20]:*

$$\zeta_t + \zeta_x + \frac{3}{2}\varepsilon\zeta\zeta_x - \frac{3}{8}\varepsilon^2\zeta^2\zeta_x + \frac{3}{16}\varepsilon^3\zeta^3\zeta_x$$
$$+\frac{\mu}{12}(\zeta_{xxx} - \zeta_{xxt}) = -\frac{7}{24}\varepsilon\mu(\zeta\zeta_{xxx} + 2\zeta_x\zeta_{xx}). \tag{4.3.2}$$

The ratio 2 : 1 between the coefficients of $\zeta_x\zeta_{xx}$ *and* $\zeta\zeta_{xxx}$ *is crucial in our considerations.*

The gentle model

Choosing $q = \frac{1}{12}$, $\alpha = \varepsilon$ and $\beta = \mu^{3/2}$ the equation (4.3.1) reads after neglecting the $O(\mu^2)$ terms:

$$\zeta_t + c\zeta_x + \frac{1}{2}c_x\zeta + \frac{3}{2}\varepsilon\zeta\zeta_x - \frac{3}{8}\varepsilon^2\zeta^2\zeta_x + \frac{3}{16}\varepsilon^3\zeta^3\zeta_x$$
$$+\frac{\mu}{12}(\zeta_{xxx} - \zeta_{xxt}) = -\frac{7}{24}\varepsilon\mu(\zeta\zeta_{xxx} + 2\zeta_x\zeta_{xx}). \tag{4.3.3}$$

This model (4.3.3) is called gentle model since it is only able to handle gentle variations of bottom topography. It is more advantageous to use the equations (4.3.2) and (4.3.3), to study numerically the Camassa-Holm-like equations. In that case, we are able to deduce in the following Proposition (see [37], in the case of (4.3.3)) an energy preserved by these two models.

Proposition 4.3.1. *Let* b *and* ζ_0 *be given by the above theorem and* ζ *solves (4.3.2) or (4.3.3). Then, for all* $t \in [0, \frac{T}{\varepsilon}]$,

$$\int_{\mathbb{R}} |\zeta|^2 + \frac{\mu}{12}|\zeta_x|^2\,dx = \int_{\mathbb{R}} |\zeta_0|^2 + \frac{\mu}{12}|\zeta_{0x}|^2\,dx.$$

The strong model

We consider here stronger variations of the parameters, i.e. :

$$\varepsilon = \sqrt{\mu}, \quad \beta = O(\varepsilon), \quad \alpha = O(\mu). \tag{4.3.4}$$

In order to obtain a stable model, as in the KdV-scaling we add terms of order $O(\mu^2)$. Choosing $q = \frac{1}{12}$, so that we get equation (4.3.5) after neglecting the $O(\mu^2)$ terms of (4.3.1):

$$\begin{aligned}
&\zeta_t + c\zeta_x + \frac{1}{2}c_x\zeta + \frac{3}{2}\varepsilon\Big(\frac{1}{c}\Big)^{2/3}\zeta\,\Big(\Big(\frac{1}{c}\Big)^{1/3}\zeta\Big)_x \\
&-\frac{3\varepsilon^2}{8}\Big(\frac{1}{c^3}\Big)^{1/4}\Big(\Big(\frac{1}{c^3}\Big)^{1/4}\zeta\Big)^2\,\Big(\Big(\frac{1}{c^3}\Big)^{1/4}\zeta\Big)_x + \frac{3}{16}\varepsilon^3\frac{1}{c}\Big(\frac{1}{c}\zeta\Big)^3\Big(\frac{1}{c}\zeta\Big)_x \\
&+\mu(a_{1/12})^{1/2}\Big((a_{1/12})^{1/2}\zeta\Big)_{xxx} - \mu(b_{1/12})^{1/2}\Big((b_{1/12})^{1/2}\zeta\Big)_{xxx} \\
&-\frac{\mu}{12}\zeta_{xxt} = -\frac{7}{24}\varepsilon\mu(\zeta\zeta_{xxx} + 2\zeta_x\zeta_{xx}).
\end{aligned} \tag{4.3.5}$$

where, $a_{1/12} = \frac{1}{6}c^5$ and $b_{1/12} = \frac{1}{12}c$. This model (4.3.5) is called strong model since it is able to handle strong variations of bottom topography.

Proposition 4.3.2. *Let b and ζ_0 be given by the above theorem and ζ solves (4.3.5). Then, for all $t \in [0, \frac{T}{\varepsilon}]$,*

$$\int_{\mathbb{R}} |\zeta|^2 + \frac{\mu}{12}|\zeta_x|^2\,dx = \int_{\mathbb{R}} |\zeta_0|^2 + \frac{\mu}{12}|\zeta_{0x}|^2\,dx.$$

4.3.2 The numerical case

The numerical scheme of the model (4.3.2)

In this subsection, we propose a numerical scheme such that the discrete version of the scalar product

$$\Big(\big(1 - \frac{\mu}{12}\partial_x^2\big)\zeta, \zeta\Big),$$

is preserved as in Propostion 4.3.1. The numerical scheme used here is a simple finite difference scheme whose final discretized version reads

$$\begin{aligned}
&M\,\frac{\zeta^{n+1} - \zeta^n}{\Delta t} + D_1\frac{\zeta^{n+1} + \zeta^n}{2} + \varepsilon D_{\frac{3}{2}uu_x}(\zeta^{n+1/2}, \frac{\zeta^n + \zeta^{n+1}}{2}) \\
&-\frac{3\varepsilon^2}{8}D_{u^2u_x}(\zeta^{n+1/2}, \frac{\zeta^n + \zeta^{n+1}}{2}) + \frac{3\varepsilon^3}{16}D_{u^3u_x}(\zeta^{n+1/2}, \frac{\zeta^n + \zeta^{n+1}}{2}) \\
&+\frac{\mu}{12}\,D_3\frac{\zeta^{n+1} + \zeta^n}{2} = -\frac{7}{24}\varepsilon\mu\,D_{2u_xu_{xx}+uu_{xxx}}(\zeta^{n+1/2}, \frac{\zeta^n + \zeta^{n+1}}{2})
\end{aligned} \tag{4.3.6}$$

where

$$M = (1 - \frac{\mu}{12} D_2),$$

and (see Lemma 1 in order to justify these choice of $D_{\frac{3}{2}uu_x}$, $D_{u^2u_x}$ and $D_{u^3u_x}$)

$$\begin{aligned}\Big(D_{\frac{3}{2}uu_x}(\zeta^{n+1/2}, \frac{\zeta^n+\zeta^{n+1}}{2})\Big)_i &= \frac{1}{2}\Big(\zeta_i^{n+\frac{1}{2}} + \frac{\zeta_{i+1}^{n+\frac{1}{2}} + \zeta_{i-1}^{n+\frac{1}{2}}}{2}\Big)\Big(D_1\frac{\zeta^{n+1}+\zeta^n}{2}\Big)_i \\ &+\frac{1}{2}\frac{\zeta_{i+1}^{n+1}+\zeta_{i+1}^{n}+\zeta_{i-1}^{n+1}+\zeta_{i-1}^{n}}{4}\Big(D_1\zeta^{n+\frac{1}{2}}\Big)_i,\end{aligned}$$

$$\begin{aligned}\Big(D_{u^2u_x}(\zeta^{n+1/2}, \frac{\zeta^n+\zeta^{n+1}}{2})\Big)_i &= \frac{1}{4}\Big((\zeta_i^{n+\frac{1}{2}})^2 + \frac{(\zeta_{i+1}^{n+\frac{1}{2}})^2 + (\zeta_{i-1}^{n+\frac{1}{2}})^2}{2}\Big)\Big(D_1\frac{\zeta^{n+1}+\zeta^n}{2}\Big)_i \\ &+\frac{1}{2}\frac{\zeta_{i+1}^{n+1}+\zeta_{i+1}^{n}+\zeta_{i-1}^{n+1}+\zeta_{i-1}^{n}}{4}\frac{\zeta_{i+1}^{n+\frac{1}{2}}+\zeta_{i-1}^{n+\frac{1}{2}}}{2} \times \\ &\Big(D_1\zeta^{n+\frac{1}{2}}\Big)_i,\end{aligned}$$

$$\begin{aligned}\Big(D_{u^3u_x}(\zeta^{n+1/2}, \frac{\zeta^n+\zeta^{n+1}}{2})\Big)_i &= \frac{1}{5}\Big((\zeta_i^{n+\frac{1}{2}})^3 + \frac{(\zeta_{i+1}^{n+\frac{1}{2}})^3 + (\zeta_{i-1}^{n+\frac{1}{2}})^3}{2}\Big)\Big(D_1\frac{\zeta^{n+1}+\zeta^n}{2}\Big)_i \\ &+\frac{3}{5}\frac{\zeta_{i+1}^{n+1}+\zeta_{i+1}^{n}+\zeta_{i-1}^{n+1}+\zeta_{i-1}^{n}}{4} \times \\ &\frac{(\zeta_{i+1}^{n+\frac{1}{2}})^2+\zeta_{i+1}^{n+\frac{1}{2}}\zeta_{i-1}^{n+\frac{1}{2}}+(\zeta_{i-1}^{n+\frac{1}{2}})^2}{3}\Big(D_1\zeta^{n+\frac{1}{2}}\Big)_i,\end{aligned}$$

for the term $2\zeta_x\zeta_{xx} + \zeta\zeta_{xxx}$ we propose the following special conservative discretizations

$$\begin{aligned}\Big(D_{2u_xu_{xx}+uu_{xxx}}(\zeta^{n+1/2}, \frac{\zeta^n+\zeta^{n+1}}{2})\Big)_i &= \Big(D_1\frac{\zeta^{n+1}+\zeta^n}{2}\Big)_i(D_2\zeta^{n+\frac{1}{2}})_i \\ &+\Big[D_1\Big(D_2\zeta^{n+\frac{1}{2}}\Big(\frac{\zeta^{n+1}+\zeta^n}{2}\Big)\Big)\Big]_i,\end{aligned}$$

or one can use the Lemma 1 to get a simple conservative discretizations of this term.

The numerical scheme of the gentle model (4.3.3)

We choose here a fully discrete scheme for the model (4.3.3) , similar to the previous scheme but taking into account the topography effects: (we replace the matrix D_1

by D_1^v)

$$
\begin{aligned}
&M\,\frac{\zeta^{n+1}-\zeta^n}{\Delta t}+D_1^v\frac{\zeta^{n+1}+\zeta^n}{2}+\varepsilon D_{\frac{3}{2}uu_x}(\zeta^{n+1/2},\frac{\zeta^n+\zeta^{n+1}}{2})\\
&-\frac{3\varepsilon^2}{8}D_{u^2u_x}(\zeta^{n+1/2},\frac{\zeta^n+\zeta^{n+1}}{2})+\frac{3\varepsilon^3}{16}D_{u^3u_x}(\zeta^{n+1/2},\frac{\zeta^n+\zeta^{n+1}}{2})\\
&+\frac{\mu}{12}\,D_3\frac{\zeta^{n+1}+\zeta^n}{2}=-\frac{7}{24}\varepsilon\mu\,D_{2u_xu_{xx}+uu_{xxx}},(\zeta^{n+1/2},\frac{\zeta^n+\zeta^{n+1}}{2})
\end{aligned}
\tag{4.3.7}
$$

The following theorem proves that the discrete equations of the model (4.3.2) and (4.3.3) are stable.

Theorem 4.3.2. *The inner product*

$$\Big(M\zeta^n,\zeta^n\Big),$$

where ζ^n solves (4.3.6) or (4.3.7), is conserved.

Remark 4.3.2. *We used the discrete stable scheme (4.3.6) for the equation (4.3.2) and (4.3.7) for the equation (4.3.3) , but one can similarly choose a stable discrete scheme using the spatial discretizations for the $\zeta\zeta_{xxx}+2\zeta_x\zeta_{xx}$ found in Lemma 4.2.1. In practice, we chose this solution, since the discrete schemes are simpler to implement in that case.*

Proof. We only prove the theorem for (4.3.7), which is the most difficult one because of the topography effects. Taking in (4.3.7) the inner product with $\dfrac{\zeta_i^{n+1}+\zeta_i^n}{2}$, using the fact that D_1, D_3 and D_1^v are skew-symmetric matrices, we obtain

$$\begin{aligned}
&\sum_i M\Big(\frac{\zeta_i^{n+1}-\zeta_i^n}{\Delta t}\Big)\frac{\zeta_i^{n+1}+\zeta_i^n}{2}+\varepsilon\sum_i\Big(\Big[\frac12\Big(\zeta_i^{n+\frac12}+\frac{\zeta_{i+1}^{n+\frac12}+\zeta_{i-1}^{n+\frac12}}{2}\Big)\Big(D_1\frac{\zeta^{n+1}+\zeta^n}{2}\Big)_i\\
&+\frac12\frac{\zeta_{i+1}^{n+1}+\zeta_{i+1}^n+\zeta_{i-1}^{n+1}+\zeta_{i-1}^n}{4}\Big(D_1\zeta^{n+\frac12}\Big)_i\Big]\frac{\zeta_i^{n+1}+\zeta_i^n}{2}\Big)\\
&-\frac{3\varepsilon^2}{8}\sum_i\Big(\frac14\Big((\zeta_i^{n+\frac12})^2+\frac{(\zeta_{i+1}^{n+\frac12})^2+(\zeta_{i-1}^{n+\frac12})^2}{2}\Big)\Big(D_1\frac{\zeta^{n+1}+\zeta^n}{2}\Big)_i\\
&+\frac12\frac{\zeta_{i+1}^{n+1}+\zeta_{i+1}^n+\zeta_{i-1}^{n+1}+\zeta_{i-1}^n}{4}\frac{\zeta_{i+1}^{n+\frac12}+\zeta_{i-1}^{n+\frac12}}{2}\Big(D_1\zeta^{n+\frac12}\Big)_i\Big)\frac{\zeta_i^{n+1}+\zeta_i^n}{2}\\
&+\frac{3\varepsilon^3}{16}\sum_i\Big(\frac15\Big((\zeta_i^{n+\frac12})^3+\frac{(\zeta_{i+1}^{n+\frac12})^3+(\zeta_{i-1}^{n+\frac12})^3}{2}\Big)\Big(D_1\frac{\zeta^{n+1}+\zeta^n}{2}\Big)_i\\
&+\frac35\frac{\zeta_{i+1}^{n+1}+\zeta_{i+1}^n+\zeta_{i-1}^{n+1}+\zeta_{i-1}^n}{4}\quad\frac{(\zeta_{i+1}^{n+\frac12})^2+\zeta_{i+1}^{n+\frac12}\zeta_{i-1}^{n+\frac12}+(\zeta_{i-1}^{n+\frac12})^2}{3}\Big(D_1\zeta^{n+\frac12}\Big)_i\Big)\frac{\zeta_i^{n+1}+\zeta_i^n}{2}\\
&+\frac{7\varepsilon\mu}{24}\sum_i\Big(\Big(D_1\frac{\zeta^{n+1}+\zeta^n}{2}\Big)_i(D_2\zeta^{n+\frac12})_i+\Big[D_1\Big(D_2\zeta^{n+\frac12}\Big(\frac{\zeta^{n+1}+\zeta^n}{2}\Big)\Big)\Big]_i\Big)\frac{\zeta_i^{n+1}+\zeta_i^n}{2}\\
&=S_1(\zeta)+\varepsilon S_2(\zeta)-\frac{3\varepsilon^2}{8}S_3(\zeta)+\frac{3\varepsilon^3}{16}S_4(\zeta)+\frac{7\varepsilon\mu}{24}S_5(\zeta)=0.
\end{aligned}$$

Proceeding exactly as in the proof of Theorem 4.2.2, one can show that $S_2(\zeta)=S_3(\zeta)=S_4(\zeta)=0$.
Using now the fact that for all $u,v\in\mathbb{R}^m$, one has

$$(D_1v,u)=-(v,D_1u),$$

to obtain $S_5(\zeta) = 0$, and since M is a symmetric matrix one gets easily the result. $\square$

The numerical scheme of the strong model (4.3.5)

Here again, we use numerical scheme for the equation (4.3.5) so that the discrete quantity $\Big(M\zeta^n, \zeta^n\Big)$ is preserved.

$$
\begin{aligned}
&M\,\frac{\zeta^{n+1}-\zeta^n}{\Delta t} + D_1^v\frac{\zeta^{n+1}+\zeta^n}{2}\\
&+\varepsilon\Big(\frac{1}{c}\Big)^{1/3} D_{\frac{3}{2}uu_x}\Big(\Big(\frac{1}{c}\Big)^{1/3}\zeta^{n+1/2}, \Big(\frac{1}{c}\Big)^{1/3}\frac{\zeta^n+\zeta^{n+1}}{2}\Big)\\
&-\frac{3\varepsilon^2}{8}\Big(\frac{1}{c^3}\Big)^{1/4} D_{u^2u_x}\Big(\Big(\frac{1}{c^3}\Big)^{1/4}\zeta^{n+1/2}, \Big(\frac{1}{c^3}\Big)^{1/4}\frac{\zeta^n+\zeta^{n+1}}{2}\Big)\\
&+\frac{3\varepsilon^3}{16}\frac{1}{c}\, D_{u^3u_x}(\frac{1}{c}\zeta^{n+1/2}, \frac{1}{c}\frac{\zeta^n+\zeta^{n+1}}{2})\\
&+\mu\,(a_{1/12})^{1/2}D_3(a_{1/12})^{1/2}\,\frac{\zeta^{n+1}+\zeta^n}{2}\\
&-\mu\,(b_{1/12})^{1/2}D_3(b_{1/12})^{1/2}\,\frac{\zeta^{n+1}+\zeta^n}{2}\\
&= -\frac{7}{24}\varepsilon\mu\, D_{2u_xu_{xx}+uu_{xxx}}(\zeta^{n+1/2}, \frac{\zeta^n+\zeta^{n+1}}{2})
\end{aligned}
\tag{4.3.8}
$$

where, $a_{1/12} = \dfrac{1}{6}c^5$ and $b_{1/12} = \dfrac{1}{12}c$.

Theorem 4.3.3. *The inner product* $\Big(M\zeta^n, \zeta^n\Big)$*, where* ζ^n *solves (4.3.8), is conserved.*

Proof. Taking in (4.3.8) the inner product with $\dfrac{\zeta_i^{n+1}+\zeta_i^n}{2}$, remarking that

$$\Big(D_3(a_{1/12})^{1/2}\,\frac{\zeta^{n+1}+\zeta^n}{2}, (a_{1/12})^{1/2}\,\frac{\zeta^{n+1}+\zeta^n}{2}\Big) = 0,$$

and

$$\Big(D_3(b_{1/12})^{1/2}\,\frac{\zeta^{n+1}+\zeta^n}{2}, (b_{1/12})^{1/2}\,\frac{\zeta^{n+1}+\zeta^n}{2}\Big) = 0.$$

Using the fact that D_1, D_3 and D_1^v are skew-symmetric matrix, we obtain

$$\begin{aligned}
&\sum_i M\Big(\frac{\zeta_i^{n+1}-\zeta_i^n}{\Delta t}\Big)\frac{\zeta_i^{n+1}+\zeta_i^n}{2}+\varepsilon\sum_i\Big(\Big[\frac{1}{2}\Big(v_i^{n+\frac12}+\frac{v_{i+1}^{n+\frac12}+v_{i-1}^{n+\frac12}}{2}\Big)\Big(D_1\frac{v^{n+1}+v^n}{2}\Big)_i\\
&+\frac{1}{2}\frac{v_{i+1}^{n+1}+v_{i+1}^n+v_{i-1}^{n+1}+v_{i-1}^n}{4}\Big(D_1v^{n+\frac12}\Big)_i\Big]\frac{v_i^{n+1}+v_i^n}{2}\Big)\\
&-\frac{3\varepsilon^2}{8}\sum_i\Big(\frac{1}{4}\Big((w_i^{n+\frac12})^2+\frac{(w_{i+1}^{n+\frac12})^2+(w_{i-1}^{n+\frac12})^2}{2}\Big)\Big(D_1\frac{w^{n+1}+w^n}{2}\Big)_i\\
&+\frac{1}{2}\frac{w_{i+1}^{n+1}+w_{i+1}^n+w_{i-1}^{n+1}+w_{i-1}^n}{4}\frac{w_{i+1}^{n+\frac12}+w_{i-1}^{n+\frac12}}{2}\Big(D_1w^{n+\frac12}\Big)_i\Big)\frac{w_i^{n+1}+w_i^n}{2}\\
&+\frac{3\varepsilon^3}{16}\sum_i\Big(\frac{1}{5}\Big((\zeta_i^{n+\frac12})^3+\frac{(\zeta_{i+1}^{n+\frac12})^3+(\zeta_{i-1}^{n+\frac12})^3}{2}\Big)\Big(D_1\frac{\zeta^{n+1}+\zeta^n}{2}\Big)_i\\
&+\frac{3}{5}\frac{\zeta_{i+1}^{n+1}+\zeta_{i+1}^n+\zeta_{i-1}^{n+1}+\zeta_{i-1}^n}{4}\quad\frac{(\zeta_{i+1}^{n+\frac12})^2+\zeta_{i+1}^{n+\frac12}\zeta_{i-1}^{n+\frac12}+(\zeta_{i-1}^{n+\frac12})^2}{3}\Big(D_1\zeta^{n+\frac12}\Big)_i\Big)\frac{\zeta_i^{n+1}+\zeta_i^n}{2}\\
&+\frac{7\varepsilon\mu}{24}\sum_i\Big(\Big(D_1\frac{\zeta^{n+1}+\zeta^n}{2}\Big)_i(D_2\zeta^{n+\frac12})_i+\Big[D_1\Big(D_2\zeta^{n+\frac12}\Big(\frac{\zeta^{n+1}+\zeta^n}{2}\Big)\Big)\Big]_i\Big)\frac{\zeta_i^{n+1}+\zeta_i^n}{2}\\
&=S_1(\zeta)+\varepsilon S_2(v)-\frac{3\varepsilon^2}{8}S_3(w)+\frac{3\varepsilon^3}{16}S_4(\zeta)+\frac{7\varepsilon\mu}{24}S_5(\zeta)=0.
\end{aligned}$$

where,

$$v=\Big(\frac{1}{c}\Big)^{1/3}\zeta,\quad w=\Big(\frac{1}{c^2}\Big)^{1/4}\zeta.$$

□

Numerical validation

We consider the same initial condition as for KdV equation:

$$\zeta_0(x)\;=\;2\,c_1\,\mathrm{sech}^2(\sqrt{\frac{3c_1\varepsilon}{2\mu}}\,x).$$

We will produce the same experiment as for KdV equation with a sinusoidal bottom

$$b(x)\;=\;\sin(2\pi\alpha x).$$

In the figure 4.3.1, we have displayed the solution for this initial condition ($c_1\;=\;0.5$) for a sinusoidal bottom and with different models.
However, the domain of interest of Camassa-Holm equations appears when $\varepsilon\;=\;\sqrt{\mu}$, that's why we will always consider this relation in the sequel. With the same initial

condition, we have computed the solution for $T = 20$ for the different models as shown in Fig. 4.3.2.
We can see that strong and original models give very close solutions while the gentle model provides a different solution. For this problem, we have performed a study of the convergence in order to compare LDG method and the presented finite difference method. However, for LDG method, centered fluxes have been used, inducing a non-optimal convergence for odd orders. As in tables 4.4, 4.5, the convergence of LDG method seems to be in $O(h^{r+1})$ (r being the order of approximation) for even orders, while we observe a convergence of $O(h^{r-1})$ ($h = \Delta x$) for odd orders. For finite-difference code, we have used the following time step :

$$\Delta t = 0.01 \, \frac{320}{N}$$

where N denotes the number of points.

LDG, order 1			LDG, order 3		LDG, order 4		Finite Difference	
N	Error	Order	Error	Order	Error	Order	Error	Order
320	1.05e-1	-	2.74e-2	-	2.32e-2	-	3.33e-1	-
640	5.79e-2	0.86	3.79e-3	2.85	7.97e-4	4.86	1.08e-1	1.62
1280	4.85e-2	0.26	2.55e-4	3.89	1.25e-5	5.99	2.95e-2	1.87
2560	4.54e-2	0.09	7.90e-5	1.69	2.04e-7	5.94	7.41e-3	1.99
5120	4.49e-2	0.02	2.28e-5	1.79	1.39e-8	3.88	1.85e-3	2.00

Table 4.4: L^2 errors for the solitary-wave and sinusoidal bottom between the numerical solution and a reference solution for $t = 20$ and for the gentle model (4.3.3). N denotes here the number of degrees of freedom. ($c_1 = 0.5,\ \mu = 0.05,\ \varepsilon = \sqrt{\mu},\ \alpha = 0.5\,\varepsilon,\ \beta = \varepsilon$)

LDG, order 1			LDG, order 3		LDG, order 4		Finite Difference	
N	Error	Order	Error	Order	Error	Order	Error	Order
320	1.14e-1	-	1.58e-2	-	9.81e-3	-	3.51e-1	-
640	5.59e-2	1.03	3.12e-3	2.34	2.03e-3	2.27	1.12e-1	1.65
1280	4.78e-2	0.23	4.06e-4	2.94	6.40e-5	4.98	2.84e-2	1.98
2560	4.62e-2	0.05	8.23e-5	2.30	2.00e-6	5.00	7.58e-3	1.91
5120	4.59e-2	0.01	2.37e-5	1.80	5.11e-8	5.29	1.90e-3	2.00

Table 4.5: L^2 errors for the solitary-wave and sinusoidal bottom between the numerical solution and a reference solution for $t = 20$ and for the strong model (4.3.5). N denotes here the number of degrees of freedom. ($c_1 = 0.5,\ \mu = 0.05,\ \varepsilon = \sqrt{\mu},\ \alpha = 0.5\,\varepsilon,\ \beta = \varepsilon$)

In figure 4.3.3, we displayed the variation of the L^2-norm for the different proposed schemes. In this figure, we observe that the discrete energy of finite difference schemes is conserved, however the conservation is not as good as for KdV-top equation.

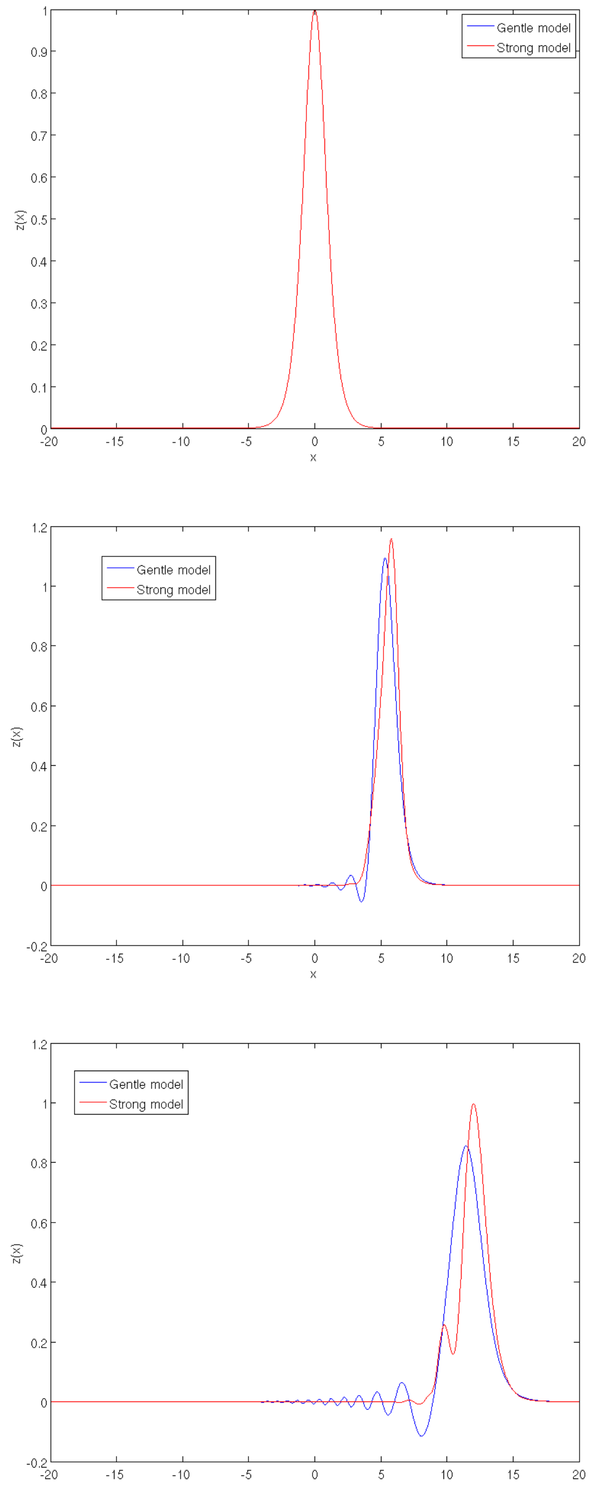

Figure 4.3.1: Solution of Camassa-Holm equation for a sinusoidal bottom, with gentle or strong model for $t = 0, \quad t = 6.67, \quad t = 13.33.$ $(c_1 = 0.5 \quad \beta = 0.5, \quad \varepsilon = \mu = 0.1 \quad \alpha = 0.05)$

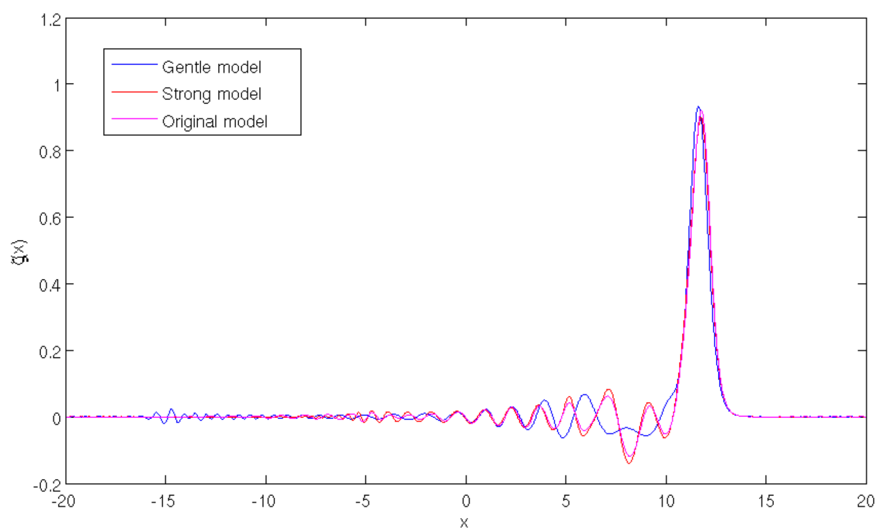

Figure 4.3.2: Solution of Camassa-Holm equation for a sinusoidal bottom, with gentle (4.3.3), strong (4.3.5) and original model (4.3.1) for $t = 20$. ($c_1 = 0.5,\ \mu = 0.05,\ \varepsilon = \sqrt{\mu},\ \alpha = 0.5\,\varepsilon,\ \beta\ =\ \varepsilon$)

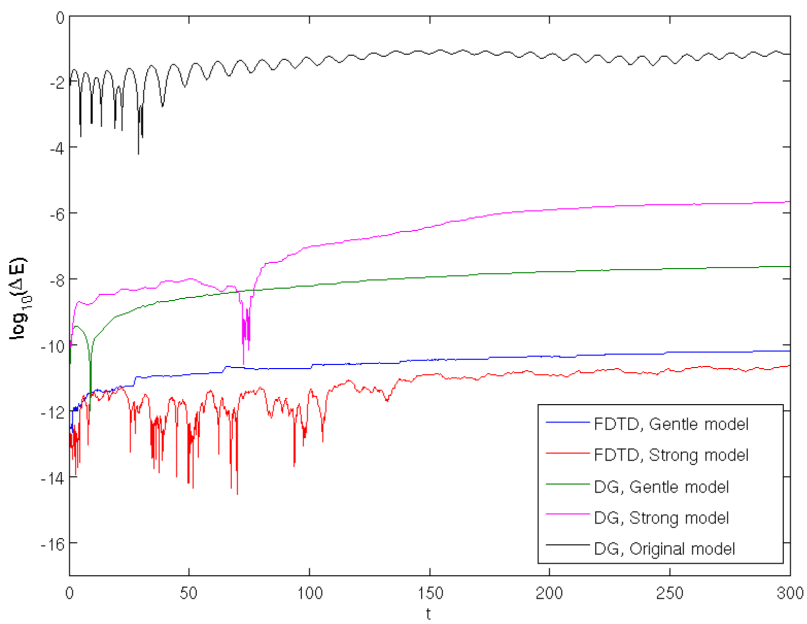

Figure 4.3.3: Logarithm of variation of energy versus time for gentle scheme, strong scheme and original scheme for solitary-wave and sinusoidal bottom ($c_1 = 0.5,\ \mu = 0.05,\ \varepsilon = \sqrt{\mu},\ \alpha = 0.5\,\varepsilon,\ \beta = \varepsilon$) Finite difference and fourth order LDG with 5120 degrees of freedom.

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Abstract

We study here the water-waves problem for uneven bottoms in a highly nonlinear regime where the small amplitude assumption of the KdV equation is enforced. It is known, that for such regimes, a generalization of the KdV equation can be derived and justified when the bottom is flat. We generalize here this result with a new class of equations taking into account variable bottom topographies. We also demonstrate that these new models are well-posed. We then proceed to study them numerically and compare their behavior with the Boussinesq equations over uneven bottoms. Regimes with stronger nonlinearities than the KdV/Boussinesq regime are then investigated. In particular, a variable coefficient generalization of a Camassa-Holm type equation is derived and justified. We also study the Green-Naghdi equations that are commonly used in coastal oceanography to describe the propagation of large amplitude surface waves. We improve previous results on the well posedness of these equations in the case of one dimensional surface waves. In the $2D$ case, we derive and study a new system of the same accuracy as the standard $2D$ Green-Naghdi equations, but with better mathematical behavior.

Keywords : Water-waves, uneven bottoms, shallow water models, Boussinesq models, Korteweg-de Vries approximation, Camassa-Holm approximation, Green-Naghdi equations.