The Lie structure on the Hochschild cohomology d'algèbres monomiales.

Selene Sanchez-Flores

UNIVERSITÉ MONTPELLIER II
SCIENCES ET TECHNIQUES DU LANGUEDOC

Thèse

pour obtenir le grade de

DOCTEUR DE L'UNIVERSITÉ MONTPELLIER II

Discipline: Mathématiques
Ecole Doctorale: Information, Structures et Systèmes.

présentée et soutenue publiquement par

Selene Camelia SANCHEZ-FLORES

15 Juin 2009

La structure de Lie de la cohomologie de Hochschild d'algèbres monomiales.

Jury:

Alain BRUGUIÈRES
Gilles HALBOUT
Christian KASSEL
Andrea SOLOTAR
Bruno VALLETTE
Claude CIBILS Directeur de Thèse

Rapporteurs:

Christof GEISS
Christian KASSEL
Andrea SOLOTAR

Remerciements.

Je souhaite adresser mes remerciements les plus sincères à Claude Cibils pour m'avoir guidée avec patience tout au long de ces années. Ses conseils avisés ainsi que ses connaissances et son expérience en mathématiques ont été essentiels pour ma recherche. Pendant mon doctorat, son soutien et son encouragement ainsi que sa disponibilité ont été importants et je lui en suis très reconnaissante.

Je voudrais exprimer ma gratitude à Andrea Solotar pour avoir rédigé un des rapports de cette thèse. Pendant mon séjour à Buenos Aires, j'ai profité de son temps pour discuter de mon sujet de recherche ainsi que de son accueil chaleureux. Je suis très contente qu'elle fasse partie du jury de ma thèse.

Je suis très honorée que Christian Kassel soit rapporteur de ma thèse et membre de mon jury. J'apprécie beaucoup ses commentaires sur mon travail.

Je tiens à remercier Christof Geiss pour avoir accepté de faire un rapport sur ma thèse ainsi que pour l'intérêt qu'il a porté à ma recherche.

Je remercie également Alain Bruguières, Gilles Halbout et Bruno Vallette pour avoir accepté de faire partie du jury.

J'ai eu l'opportunité de suivre des cours et des exposés de Bernhard Keller à Jussieu et pendant des conférences dès mon arrivée en France, je le remercie pour ses enseignements.

Claire et Yann ont toujours été disponibles pour m'aider dès le début de mes études à Paris, je les en remercie. Ils m'ont aussi transmis leur enthousiasme pour les mathématiques. J'ai la chance d'être leur amie.

Patrick, Benoît et Mariano m'ont écouté avec patience et ils ont répondu avec gentillesse à toutes mes questions, je les en remercie. J'ai trouvé leur compagnie très agréable pendant les conférences.

Un grand merci aux thésards de Jussieu pour leur solidarité envers moi, en particulier à Mairi et Florent pour m'avoir offert un bureau pour travailler et aussi à Esther pour son accord ainsi que les thésards du 7ème étage pour m'avoir accueillie.

J'ai toujours été motivée par le séminaire des "Bourbakettes". Merci beaucoup à tous ceux qui y participent pour que les séances continuent.

Je remercie mes amis Laura, Juliette, Nico B., Maria, Nabil, Nicolas L., Giovanni, Maria Paula, Sarah, Mairi, Athina et José pour leur soutien moral pendant toutes ces années à Cheva.

Je salue très chaleureusement Amadeo, Luca, Pierre W., Farid, Peng, Delphine, Mounir, Elhoim, Myss, Myrjam, Claire C., Ngan, Pierre-Guy, Thomas, Cecilia, Ruchi, François, Jean-François P., ainsi que Emily B. et Florence T. Finalement, je remercie Anne P. de m'avoir aidée avec la correction de mes remerciements.

Agradecimientos.

Agradezco al Consejo Nacional de Ciencia y Tecnología de México por la beca que me otorgo para la realización de esta tesis.

Esta beca la obtuvé gracias al apoyo de mis profesores de la UNAM, a todos ellos les doy las gracias. Quiero hacer una mención especial, al profesor José Ríos Montes del Instituto de Matemáticas. Además de sus enseñanzas, me transmitió un entusiasmo particular por el Álgebra. Le agradezco por sus consejos con respecto a mis estudios de posgrado.

A los profesores Ana Meda Guardiola y Jaime Vázquez Alamilla, les agradezco por el apoyo que me brindaron durante mis estudios incluso cuando ya no estaba en la Facultad de Ciencias. Al profesor Emilio Lluis Riera quiero darle las gracias por su interés en mi carrera.

Al profesor José Antonio de la Peña le doy las gracias por haberme aconsejado hacer una tesis con Claude Cibils. Al profesor Raymundo Bautista le agradezco por darme la oportunidad de dar una plática en Morelia.

Le agradezco al profesor Pascual Jara por la invitación a Granada para exponer mis resultados. Muchas gracias a Javier por las discusiones de mates y por hacer mi estancia agradable.

Le agradezco a la profesora Sonia Natale por haberme invitado a Córdoba. En mi estancia en la Famaf, tuve la oportunidad de discutir sobre mi tema de tesis con el profesor Leandro Cagliero a quien agradezco mucho su ayuda.

A mis papás, Estela y Ernesto, les agradezco su apoyo incondicional a pesar de la distancia. Muchas gracias por su ayuda en todo, por escucharme y por alentarme con sus consejos. Sus experiencias y aventuras me han aportado mucho. Intento siempre seguir su ejemplo en cuanto a perseverancia y optimismo.

A mis hermanas Jacos y Mary les agradezco por motivarme y consentirme y a Chucho, Dana y Kari por contagiarme con su alegría.

Me he favorecido de la ayuda de los papás de Pauch. Muchas gracias a Eva y Roberto por brindarme su apoyo en nuestros planes.

Las conversaciones con mi amigo Cheko por gmail, me ayudaron a despejar mi mente y aclarar mis ideas durante la redacción de la tesis. Muchas gracias. Leo me ayudo mucho con los tramites de la beca, le doy también las gracias.

Me he beneficiado mucho de las experiencias de varios mexicanos que han estudiado matemáticas en Francia, quiero agradecer a Tabo, Mau, Avenilde, Victor, Lili, Arturo y Mauricio.

A Michèle y Elodie les agradezco su paciencia para practicar el francés conmigo, gracias a su ayuda disfruté mucho cosas de Paris.

Le agradezco a Pauch infinitamente por hacerme descubrir y explorar este mundo de las matemáticas. Me ha transmitido ese placer por hacer una demostración, por entender un cálculo y por apreciar la belleza de un razonamiento lógico. Su apoyo constante ha sido pilar fundamental en mis estudios, me ha ayudado a superar desilusiones y frustraciones. Hemos compartido muchas cosas en esta gran aventura, por eso y más gracias.

La structure de Lie de la cohomologie de Hochschild d'algèbres monomiales.

The Lie structure on the Hochschild cohomology of monomial algebras.

Selene Camelia SÁNCHEZ-FLORES

Contents

Introduction en français.

Cette thèse porte sur la structure de Lie de la cohomologie de Hochschild, donnée par le crochet de Gerstenhaber. Plus précisément, nous étudions *la structure d'algèbre de Lie du premier groupe de cohomologie et la structure de module de Lie des groupes de cohomologie de Hochschild* de certaines algèbres monomiales.

Dans cette introduction, nous précisons d'abord le cadre de la thèse. Nous présentons ensuite un aperçu de la recherche réalisée précédemment. Puis nous examinons l'objectif et la motivation de ce travail. Finalement, nous donnons une description détaillée de chaque chapitre de cette thèse. Les résultats de la première section du chapitre 4, de la section 3 du chapitre 5 et les annexes A et B ont fait l'objet de la publication [**SF08**].

Cadre de la thèse. Soit A une k-algèbre associative avec unité où k est un corps commutatif. La *cohomologie de Hochschild en degré* n *de* A, dénotée $\mathsf{HH}^n(\mathsf{A})$, est définie de la manière suivante:

$$\mathsf{HH}^n(\mathsf{A}) = \mathsf{HH}^n(\mathsf{A},\mathsf{A}) = \mathsf{Ext}^n_{\mathsf{A}^e}(\mathsf{A},\mathsf{A})$$

où A^e est l'algèbre enveloppante $\mathsf{A}^{op} \otimes_k \mathsf{A}$ de A. Par exemple, $\mathsf{HH}^0(\mathsf{A})$ est le centre de l'algèbre et $\mathsf{HH}^1(\mathsf{A})$ est l'espace de dérivations extérieures, c'est à dire le quotient des dérivations de l'algèbre modulo les dérivations intérieures. Remarquons que $\mathsf{HH}^1(\mathsf{A})$ a une structure d'algèbre de Lie donnée par le crochet commutateur.

En 1963, Gerstenhaber a introduit deux opérations sur les groupes de cohomologie de Hochschild: le cup-produit

$$\_ \smile \_ : \mathsf{HH}^n(\mathsf{A}) \times \mathsf{HH}^m(\mathsf{A}) \longrightarrow \mathsf{HH}^{n+m}(\mathsf{A}).$$

et le crochet

$$[-\,,-] : \mathsf{HH}^n(\mathsf{A}) \times \mathsf{HH}^m(\mathsf{A}) \longrightarrow \mathsf{HH}^{n+m-1}(\mathsf{A}).$$

Il a montré que la *cohomologie de Hochschild de A*

$$\mathsf{HH}^*(\mathsf{A}) = \bigoplus_{n=0}^{\infty} \mathsf{HH}^n(\mathsf{A})\,,$$

munie du cup-produit est une algèbre commutative graduée. En outre, il a démontré que $\mathsf{HH}^{*+1}(\mathsf{A})$ munie du crochet de Gerstenhaber a une structure d'algèbre de Lie graduée. Par conséquent, $\mathsf{HH}^1(\mathsf{A})$ *est une algèbre de Lie et* $\mathsf{HH}^n(\mathsf{A})$ *est un module de Lie sur* $\mathsf{HH}^1(\mathsf{A})$. En fait, le crochet de Gerstenhaber restreint à $\mathsf{HH}^1(\mathsf{A})$ est le crochet commutateur des dérivations extérieures. En plus, le cup-produit et le crochet de Gerstenhaber munissent $\mathsf{HH}^*(A)$ de la structure d'algèbre de Gerstenhaber.

Intérêt et recherche précédente. Les structures algébriques sur la cohomologie de Hochschild sont importantes dans l'étude de la théorie des représentations et des déformations de l'algèbre. Les deux structures, celle d'algèbre commutative graduée et celle d'algèbre de Lie graduée, sont préservées sous la relation d'équivalence dérivée. Il a été montré d'abord que la structure d'algèbre commutative de $\mathsf{HH}^*(A)$ est invariante sous la relation d'équivalence dérivée [**Hap89, Ric91**]. Puis, dans [**Kel04**], Keller a prouvé que le crochet de Gerstenhaber sur $\mathsf{HH}^{*+1}(A)$ est aussi préservé sous la relation d'équivalence dérivée.

Cependant, la compréhension des deux structures est une tâche difficile car les calculs sont compliqués. Néanmoins, plusieurs résultats ont été obtenus afin de: (1) décrire la structure d'algèbre de la cohomologie de Hochschild pour certaines algèbres, [**Hol96, CS97, Cib98, ES98, EH99, SW00, SA02, EHS02, GA08, Eu07b, FX06**]; (2) étudier la cohomologie de Hochschild modulo nilpotence [**GSS03, GSS06, GS06**]; (3) calculer le crochet de Gerstenhaber [**Bus06, Eu07a, SA07**].

Objectif. Le but de cette thèse est d'étudier la structure de Lie de la cohomologie de Hochschild pour les algèbres monomiales de dimension finie. Une *algèbre monomiale* est définie comme le quotient de l'algèbre de chemins d'un carquois par un idéal bilatère engendré par un ensemble de chemins de longueur au moins deux. Nous utilisons les données combinatoires intrinsèques à de telles algèbres pour étudier la structure de Lie définie sur la cohomologie de Hochschild par le crochet de Gerstenhaber. En fait, nous examinons deux aspects de cette structure algébrique. Le premier est la *relation entre la semi-simplicité du premier groupe de cohomologie de Hochschild et la nullité des groupes de cohomologie de Hochschild*. Dans le second aspect, nous nous concentrons sur la *structure de module de Lie des groupes de cohomologie de Hochschild d'une famille d'algèbres particulière: celles dont le radical de Jacobson au carré est nul.*

Motivation. Une des motivations principales de cette recherche, a été suggérée par Christian Kassel à partir des résultats de Claudia Strametz. Dans [**Str06**], Strametz a étudié la structure d'algèbre de Lie du premier groupe de cohomologie de Hochschild d'une algèbre monomiale. Elle a réussi à décrire le crochet commutateur en termes de la combinatoire du carquois. Une de ses contributions a été de donner des conditions nécessaires et suffisantes aux données combinatoires des algèbres monomiales afin de garantir la semi-simplicité dans le premier groupe de cohomologie. De plus, elle a montré que dans ce cas, l'algèbre de Lie semi-simple obtenue est un produit direct de certaines algèbres de Lie de matrices de trace nulle. Les modules de dimension finie sur ces algèbres de Lie sont classifiés et une question naturelle se pose: *Quelle est la description de la cohomologie de Hochschild en degré n en tant que module de Lie sur le groupe de dérivations extérieures, quand cette dernière est semi-simple?* Nous démontrons dans cette thèse que pour les algèbres monomiales sur un corps de caractéristique zéro, les groupes de cohomologie de Hochschild de degré au moins deux sont nuls, ce qui nous amène à nous poser d'autres questions concernant la structure de la cohomologie. En particulier, nous voulons trouver des exemples où la structure de module de Lie des groupes de cohomologie de Hochschild est non triviale, ce qui amène à considérer le cas où l'algèbre de Lie en degré un est non semi-simple. Nous considérons alors des algèbres monomiales dont le radical de Jacobson au carré est nul. Pour de telles algèbres, Claude Cibils a calculé, dans [**Cib98**], les groupes de cohomologie en utilisant la combinatoire du carquois.

Résumé des chapitres. Cette thèse est divisée en cinq chapitres. Dans le premier et second chapitres, nous présentons des résultats concernant l'algèbre de Lie du premier groupe de cohomologie de Hochschild. Dans le troisième chapitre, nous considérons la relation entre semi-simplicité sur $\mathsf{HH}^1(\mathsf{A})$ où A est monomiale et la nullité des groupes de cohomologie de Hochschild. Les chapitres quatre et cinq traitent de la structure de module de Lie de HH^n des algèbres monomiales dont le radical au carré est nul. Décrivons à présent chaque chapitre en détail.

`Premier chapitre.` Dans ce chapitre, le but est de rappeler la description combinatoire, donnée par Strametz, du crochet commutateur défini sur $\mathsf{HH}^1(\mathsf{A})$. Pour cela, nous rappelons la description de $\mathsf{HH}^1(\mathsf{A})$ donnée en termes de flèches parallèles. Le contenu de ce chapitre est plutôt technique, néanmoins les deux descriptions combinatoires présentées sont les principaux outils pour comprendre la structure d'algèbre de Lie, ce qui est l'objectif du chapitre suivant.

`Second chapitre.` Nous spécifions la structure d'algèbre de Lie de $\mathsf{HH}^1(\mathsf{A})$: lorsque le radical de A au carré est nul d'une part et lorsque A est triangulaire et complètement monomiale d'autre part. Une *algèbre complètement monomiale* est une algèbre monomiale qui vérifie la propriété suivante: tout chemin de longueur au moins deux parallèle à un chemin nul dans l'algèbre, est aussi nul. En particulier, les algèbres monomiales dont le radical au carré est nul sont des algèbres complètement monomiales. Nous étudions deux cas séparément: celui des algèbres complètement monomiales dont le carquois ne contient pas de cycles orientés d'une part et le cas des algèbres de radical carré nul sans restriction sur le carquois d'autre part.

En tenant compte du théorème de décomposition de Levi, nous calculons d'abord le radical résoluble de $\mathsf{HH}^1(\mathsf{A})$ pour ensuite obtenir la partie semi-simple. Cette dernière est précisément le quotient par le radical résoluble. Dans ce chapitre nous assumons que le corps k est algébriquement clos de caractéristique zéro.

Pour les algèbres monomiales de radical carré nul, nous montrons que le premier groupe de cohomologie de Hochschild est une algèbre de Lie réductive, c'est à dire qu'elle est la somme directe d'une algèbre de Lie semi-simple et une algèbre de Lie abélienne.

Etant donné un carquois Q, nous dénotons $\overline{Q}$, le carquois obtenu en identifiant les flèches parallèles, i.e. les flèches parallèles multiples dans Q sont vues comme une seule flèche dans $\overline{Q}$. Nous dénotons S l'ensemble des flèches de $\overline{Q}$ qui correspondent à plus qu'une flèche dans Q. Nous avons la proposition suivante.

Proposition. *Soit* $A = kQ/ < Q_2 >$ *une algèbre monomiale dont le radical au carré est zéro où le corps* k *est algébriquement clos de caractéristique zéro et où le carquois* Q *est fini et connexe. Alors*

$$HH^1(A) \cong \prod_{\alpha \in S} sl_{|\alpha|}(k) \ \times \ k^{\chi(\overline{Q})}$$

où $\chi(\overline{Q}) = |\overline{Q}_1| - |\overline{Q}_0| + 1$ *est la caractéristique d'Euler de* $\overline{Q}$. *En particulier,* $HH^1(A)$ *est réductive.*

En particulier lorsque le carquois est un cycle orienté, l'ensemble S est vide. De plus $\chi(\overline{Q}) = 1$, ainsi dans ce cas $HH^1(A)$ est l'algèbre de Lie abélienne de dimension un.

Nous appliquons la proposition ci-dessus au carquois à boucles multiples. Par définition, le *carquois à boucles multiples* est le carquois donné par un seul sommet et au moins deux boucles. L'algèbre monomiale dont le radical au carré est zéro associée au carquois à boucles multiples est $k[x_1, \dots x_r]/ < x_i x_j >_{i,j=1}^{r}$ où r est le nombre de boucles. Suite à la proposition ci-dessus, le premier groupe de cohomologie de Hochschild est $gl_r(k)$, l'algèbre de Lie des matrices carrées de taille r.

Ensuite, nous considérons les algèbres complètement monomiales sans cycles orientés dans leur carquois. La partie semi-simple de leur premier groupe de cohomologie de Hochschild peut être exprimé dans les mêmes termes que dans le cas précédent. Le radical résoluble n'est plus abélien, cependant nous donnons une description de celui-ci.

`Troisième chapitre.` Notre principal objectif dans ce chapitre est de montrer que sous l'hypothèse de semi-simplicité du premier groupe de cohomologie de Hochschild d'une algèbre monomiale, les groupes de cohomologie de Hochschild sont nuls à partir du deuxième degré. Précisément, nous montrons le théorème suivant.

Théorème. *Soit* Q *un carquois fini et connexe et soit* Z *un ensemble minimal de chemins qui est stable par chemins parallèles. Considérons l'algèbre monomiale de dimension finie* $A = kQ/ < Z >$ *où le corps* k *est algébriquement clos et de caractéristique zéro. Si la graphe sous-jacent à* $\overline{Q}$ *est un arbre alors*

- $HH^0(A) = k$
- $HH^1(A) = \prod_{\alpha \in S} sl_{|\alpha|}(k)$ *et*
- $HH^n(A) = 0$ *pour tout* $n \geq 2$.

Pour prouver le théorème précédent, nous utilisons la résolution projective de Happel-Bardzell [**Bar97**] qui est une résolution projective de A comme A^e-module à gauche.

Corollaire. *Soit* $\mathsf{A} = \mathsf{kQ}/ < \mathsf{Z} >$ *une algèbre monomiale de dimension finie où le corps* k *est algébriquement clos et de caractéristique zéro. Si* $\mathsf{HH}^1(\mathsf{A})$ *est semi-simple alors* $\mathsf{HH}^n(\mathsf{A}) = 0$ *pour toute* $n \geq 2$*.*

Pour démontrer le corollaire, nous utilisons les conditions pour la semi-simplicité données par Strametz. Au début de ce chapitre, nous rappelons et examinons ces conditions pour énoncer son résultat.

Proposition ([**Str06**]). *Soit* Q *un carquois fini et connexe et soit* Z *un ensemble minimal de chemins. Soit* $\mathsf{A} = \mathsf{kQ}/ < \mathsf{Z} >$ *l' algèbre monomiale de dimension finie où le corps* k *est algébriquement clos et de caractéristique zéro. Les assertions suivantes sont équivalentes:*

(1) $\mathsf{HH}^1(\mathsf{A})$ *est semi-simple.*
(2) La graphe sous-jacent du carquois $\overline{\mathsf{Q}}$ *est un arbre,* Z *est stable par chemins parallèles et l'ensemble* S *est non vide.*
(3) $\mathsf{HH}^1(\mathsf{A})$ *est isomorphe au produit direct non trivial suivant:*

$$\prod_{\alpha \in \mathsf{S}} \mathsf{sl}_{|\alpha|}(\mathsf{k}).$$

Nous présentons une autre démonstration du théorème de Strametz.

`Quatrième chapitre.` Nous supposons, tout au long du chapitre, que A est une algèbre monomiale dont le radical carré est zéro. Nous étudions la structure de module de Lie des groupes de cohomologie de Hochschild, induite par le crochet de Gerstenhaber

$$\mathsf{HH}^1(\mathsf{A}) \times \mathsf{HH}^n(\mathsf{A}) \longrightarrow \mathsf{HH}^n(\mathsf{A}).$$

L'étude dépend de trois cas selon le carquois:

1. le carquois est une boucle,
2. le carquois est un cycle orienté mais pas une boucle, et
3. le carquois n'est pas un cycle orienté.

L'outil principal pour comprendre la structure de module de Lie est la description combinatoire du groupe de cohomologie de Hochschild et du crochet de Gerstenhaber. Dans la première section, nous présenterons le complexe combinatoire donné dans [**Cib98**] qui calcule la cohomologie de Hochschild. Ensuite, nous rappelons la formulation du crochet de Gerstenhaber donné dans [**SF08**] pour la réalisation des groupes de cohomologie de Hochschild. Cette section rassemble quelques résultats de mon article [**SF08**] pour lesquels les preuves sont données dans l'appendice de cette thèse. Dans le reste du chapitre nous explorons la structure de module de Lie pour les cas mentionnés ci-dessus.

Dans le premier cas, l'algèbre que nous considérons est en fait l'algèbre de nombres duaux. Les groupes de cohomologie de Hochschild de degré ≥ 1 de cette algèbre sont des espaces vectoriels de dimension un. Dans la proposition suivante, nous donnons une base de $\mathsf{HH}^n(\mathsf{A})$.

Proposition. *Soit* $A = k[x]/ < x^2 >$ *où* k *est un corps de caractéristique zéro. Pour* $n \geq 1$*, considérons l'application* $\varphi_n : A^{\otimes n} \to A$ *donnée par*

$$\varphi_n(f_1 \otimes \cdots \otimes f_i \otimes \cdots \otimes f_n) = \begin{cases} \prod_{i=1}^{n} \lambda(f_i, x) & \text{si } n \text{ est pair} \\ \prod_{i=1}^{n} \lambda(f_i, x)x & \text{si } n \text{ est impair.} \end{cases}$$

où $f_i = \lambda(f_i, x)x + \lambda(f_i, 1)$ *pour* $i = 1, \cdots, n$*. Alors* $HH^n(A) \cong k\, \varphi_n$*.*

Les modules de Lie de dimension un sur une algèbre de Lie abélienne sont donnés par la multiplication par un scalaire dans le corps. Nous précisons le scalaire qui détermine le module de Lie $HH^n(A)$.

Proposition. *Soit* $A = k[x]/ < x^2 >$ *où* k *est un corps de caractéristique zéro. Pour* $n \geq 1$*, la structure de module de Lie des groupes de cohomologie de Hochschild, donné par le crochet de Gerstenhaber,*

$$HH^1(A) \times HH^n(A) \longrightarrow HH^n(A)\,,$$

est donnée par

$$\varphi_1.\varphi_n = \begin{cases} -n\, \varphi_n & \text{si } n \text{ est pair} \\ (1-n)\, \varphi_n & \text{si } n \text{ est impair.} \end{cases}$$

Alors

$$HH^{2n}(A) \cong HH^{2n+1}(A)$$

en tant que modules de Lie.

Maintenant dénotons la cohomologie en degrés impairs par:

$$HH^{\mathrm{odd}}(A) = \bigoplus_{n=0}^{\infty} HH^{2n+1}(A).$$

Il est clair que $HH^{\mathrm{odd}}(A)$ muni du crochet de Gerstenhaber est une algèbre de Lie. Nous décrivons cette algèbre de Lie. Soit $\mathcal{W}$ l'algèbre de Lie des dérivations de $k[x]$, i.e. $\mathcal{W} = \mathrm{Der}(k[x], k[x])$. Elle admet la graduation suivante:

$$\mathcal{W} = \bigoplus_{n=0}^{\infty} W_n$$

où W_n est l'espace vectoriel engendré par la dérivation $\phi_n : k[x] \to k[x]$ définie par $\phi_n(x^i) = ix^{n+i-1}$. En fait, toute dérivation est une combinaison linéaire des dérivations ϕ_n. De plus, le crochet commutateur est donné par la formule suivante:

$$[\phi_n, \phi_m] = (n-m)\phi_{n+m-1}.$$

Clairement, le crochet est gradué si nous considérons les éléments de W_n comme étant de degré $n-1$. D'autre part, nous dénotons

$$\mathcal{W}^{\mathrm{odd}} = \bigoplus_{n=0}^{\infty} W_{2n+1}$$

la sous-algèbre de Lie de $\mathcal{W}$.

Proposition. *Soit* k *un corps de caractéristique zéro et* A *l'algèbre des nombres duaux, c'est à dire* $A = k[x]/ < x^2 >$. *L'algèbre de Lie* $HH^{odd}(A)$ *est isomorphe à l'algèbre de Lie de dimension infinie* $\mathcal{W}^{odd}$.

Dans le deuxième cas, soit Q un cycle orienté de longueur N où $N \geq 2$. Un élément f dans $A = kQ/ < Q_2 >$ est donné par une combinaison linéaire

$$f = \sum_{i=1}^{N} \lambda(f, e_i)e_i + \lambda(f, a_i)a_i$$

où $e_1, \ldots, e_N$ sont les sommets du carquois et $a_1, \ldots, a_N$ sont les flèches. Nous donnons une base de l'espace vectoriel $HH^n(A)$. Voici quelques notations.

Pour $i = 1, \ldots, N$ et pour $cN > 0$ un multiple positif de N, nous dénotons

$$\sigma_i : \{1, \ldots, cN\} \to \{1, \cdots, N\}$$

la fonction périodique de période N (i.e. $\sigma_i(j) = \sigma_i(j + N)$) telle que σ_i restreint à l'ensemble $\{1, \ldots, N\}$ est la permutation cyclique suivante:

- si $i = 1$ alors $\sigma_1(j) = j$ pour $j = 1, \ldots, N$;
- si $i = N$ alors $\sigma_N(1) = N$ et $\sigma_N(j) = j - 1$ pour $j = 2, \ldots N$,
- si $1 < i < N$ alors $\sigma_i(j) = i + (j - 1)$ pour $j = 1, \ldots, N - i + 1$ et $\sigma_i(j) = (j - 1) - (N - i)$ pour $j = N - i + 2, \ldots N$.

Nous dénotons

$$\pi_i : A^{\otimes cN} \to k$$

l'application linéaire donnée par

$$\pi_i(f_1 \otimes \cdots \otimes f_{cN}) = \prod_{j=1}^{cN} \lambda(f_j, a_{\sigma_i(j)}).$$

Nous démontrons le résultat suivant:

Proposition. *Soit* $A = kQ/ < Q_2 >$ *où le carquois* Q *est le cycle orienté de longueur* N *où* $N \geq 2$ *et le corps* k *est de caractéristique zéro. Considérons l'application linéaire* $\varphi_1 : A \to A$ *donnée par*

$$\varphi_1(f) = \lambda(f, a_1)a_1.$$

Alors $HH^1(A) \cong k\,\varphi_1$.

Pour $n \geq 1$, *un multiple de* N, *considérons l'application* $\varphi_n : A^{\otimes n} \to A$ *donnée par*

$$\varphi_n(f_1 \otimes \cdots \otimes f_n) = \sum_{i=1}^{N} \pi_i(f_1 \otimes \cdots \otimes f_n)e_i$$

et l'application $\varphi_{n+1} : A^{\otimes n+1} \to A$ *donnée par*

$$\varphi_{n+1}(f_1 \otimes \cdots \otimes f_{n+1}) = \pi_1(f_1 \otimes \cdots \otimes f_n)\lambda(f_{n+1}, a_1)a_1\,.$$

Alors $HH^n(A) \cong k\,\varphi_n$ *et* $HH^{n+1}(A) \cong k\,\varphi_{n+1}$ *si* n *est pair et* $HH^n(A)$ *est nul autrement.*

D'après ce résultat, $\mathsf{HH}^n(A)$ est un espace vectoriel nul ou de dimension un. En particulier pour le cas où cet espace est de dimension un, la structure de module de Lie de $\mathsf{HH}^n(A)$ est donnée par la multiplication par un scalaire non nul. Nous précisons ce scalaire dans la proposition suivante.

PROPOSITION. *Soit* $A = kQ/ < Q_2 >$ *où le carquois* Q *est le cycle orienté de longueur* N *où* $N \geq 2$ *et le corps* k *est de caractéristique zéro. Pour* $n \geq 1$*, la structure de module de Lie des groupes de cohomologie de Hochschild, donné par le crochet de Gerstenhaber,*

$$\mathsf{HH}^1(A) \times \mathsf{HH}^n(A) \longrightarrow \mathsf{HH}^n(A)$$

est donné par

$$\varphi_1.\varphi_n = -c\,\varphi_n$$

où c *est an entier tel que* cN *est pair et* $n = cN$ *ou* $n = cN+1$*, et* c *est zéro autrement.*

Alors, pour tout entier positif c*,* $\mathsf{HH}^{cN}(A) \cong \mathsf{HH}^{cN+1}(A)$ *en tant que modules de Lie.*

Comme auparavant, nous décrivons $\mathsf{HH}^{odd}(A)$. Pour cela, soit $\mathcal{W}^*$ la sous-algèbre de $\mathcal{W}$ suivante:

$$\mathcal{W}^* = \bigoplus_{n=0}^{\infty} W_{n+1}.$$

La proposition suivante décrit la structure d'algèbre de Lie de $\mathsf{HH}^{odd}(A)$ dans le cas du cycle orienté.

PROPOSITION. *Soit* $A = kQ/ < Q_2 >$ *où* k *est un corps de caractéristique zéro, où* Q *est le cycle orienté de longueur* N *et* $N \geq 2$*. L'algèbre de Lie* $\mathsf{HH}^{odd}(A)$ *est isomorphe à* $\mathcal{W}^*$*.*

Dans le dernier cas, nous supposons que le carquois n'est pas un cycle orienté, i.e. Q peut admettre des cycles orientés mais Q ne peut pas être réduit à un cycle orienté. Dans ce cas, la description de la structure de module de Lie est donnée en termes du carquois de ces raccourcis et des ses cycles orientés. Voici d'abord quelques définitions et quelques notations.

Soient X et Y des ensembles des chemins de Q, alors $X \parallel Y$ dénote l'ensemble de couples de chemins (α, β) dans $X \times Y$ que sont *parallèles*, c'est à dire qui partagent le même sommet d'origine et le même sommet d'arrivée. Nous dénotons $k(X \parallel Y)$ le k-espace vectoriel dont la base est l'ensemble $X \parallel Y$. On appelle $k(Q_n \parallel Q_1)$ l'espace de *raccourcis* et $k(Q_n \parallel Q_0)$ l'espace des *cycles orientés pointés.*

La description des groupes de cohomologie de Hochschild comme des quotients de l'espace de raccourcis est donné dans [**Cib98**]. L'énoncé précis est le suivant.

Théorème ([**Cib98**]). *Soit* $A = kQ/ < Q_2 >$ *où* Q *n'est pas un cycle orienté. Alors, si* $n \geq 1$

$$HH^n(A) \cong \frac{k(Q_n \parallel Q_1)}{\operatorname{Im} D_{n-1}}$$

où

$$D_{n-1} : k(Q_{n-1} \parallel Q_0) \longrightarrow k(Q_n \parallel Q_1)$$

est l'application linéaire suivante

$$D_{n-1}(\gamma^{n-1}, e) = \sum_{a \in Q_1 e} (a\gamma^{n-1}, a) + (-1)^n \sum_{a \in eQ_1} (\gamma^{n-1}a, a)\,.$$

De plus, si $n > 1$*, alors*

$$\dim_k HH^n(A) = |Q_n \parallel Q_1| - |Q_{n-1} \parallel Q_0|.$$

En utilisant la réalisation ci-dessus, nous décrivons la structure de module de Lie. Nous démontrons le théorème suivant.

Théorème. *Soit* $A = kQ/ < Q_2 >$ *où* Q *est un carquois fini. Si* Q *n'est pas un cycle orienté alors la structure de module de Lie des groupes de cohomologie de Hochschild donnée par le crochet de Gerstenhaber*

$$HH^1(A) \times HH^n(A) \longrightarrow HH^n(A)$$

est induite par une fonction bilinéaire

$$k(Q_1 \parallel Q_1) \times k(Q_n \parallel Q_1) \longrightarrow k(Q_n \parallel Q_1)$$

donnée par les formules suivantes:

$$(a, x).(\alpha, y) = \delta_y^a \cdot (\alpha, x) - \sum_{i=1}^{n} \delta_x^{a_i} \cdot (\alpha \underset{i}{\diamond} a, y)$$

où δ *est le symbole de Kronecker et* $\alpha = a_1 \cdots a_i \cdots a_n$ *est un chemin de longueur* n *constitué par les flèches* a_i*. Pour* $a_i = x$ *le chemin* $\alpha \underset{i}{\diamond} a$ *est obtenu en remplaçant* a_i *par* a*.*

`Cinquième chapitre.` Dans ce chapitre, nous considérons deux types de carquois: ceux sans cycles orientés et les carquois à boucles multiples. Nous relions la structure de module de Lie des groupes de cohomologie de Hochschild aux modules de Lie sur l'algèbre de Lie des matrices carrées de trace zéro $\mathfrak{sl}_r(k)$, en utilisant le théorème ci-dessus.

Dans le premier cas nous considérons des algèbres monomiales triangulaires dont le radical carré est zéro. Introduisons quelques notations pour énoncer le résultat. Étant donné une flèche α dans $\overline{Q}_1$, nous dénotons V_α le k-espace vectoriel dont la base est l'ensemble α. L'espace vectoriel V_α a une structure de module de Lie sur $\prod_{\alpha' \in \overline{Q}_1} \operatorname{End}_k(V_{\alpha'})$ donné par:

$$(f_{\alpha'})_{\alpha' \in \overline{Q}_1} . \nu = f_\alpha(\nu).$$

De plus, étant donné un raccourci T dans $\overline{Q}$, i.e. T est un élément de $\overline{Q}_n \parallel \overline{Q}_1$, nous écrivons $T = (\alpha_1 \cdots \alpha_i \cdots \alpha_n, \gamma)$ où α_i et γ sont des flèches dans $\overline{Q}_1$.

THÉORÈME. *Soit* $A = kQ/ < Q_2 >$ *où* Q *est un carquois fini sans cycles orientés et* k *est un corps algébriquement clos de caractéristique zéro. Alors pour* $n > 1$ *la structure de module de Lie de* $HH^n(A)$ *sur* $HH^1(A)$ *induite par le crochet de Gerstenhaber est donnée de la manière suivante:*

$$HH^n(A) \cong \bigoplus_{T \in \overline{Q}_n \| \overline{Q}_1} HH^n_T(Q)$$

où

$$HH^n_T(Q) = V^*_{\alpha_1} \underset{k}{\otimes} \cdots \underset{k}{\otimes} V^*_{\alpha_i} \underset{k}{\otimes} \cdots \underset{k}{\otimes} V^*_{\alpha_n} \underset{k}{\otimes} V_\gamma$$

De plus, $HH^n_T(Q)$ *est irréductible.*

Le deuxième cas est le carquois à boucles multiples. Soit A l'algèbre monomiale dont le radical au carré est zéro, associée à ce carquois. Nous avons déjà mentionné que le premier groupe de cohomologie de Hochschild est isomorphe à $\mathfrak{gl}_r(k)$. Pour cette algèbre, nous relions la structure de module de Lie à un produit tensoriel de deux $\mathfrak{gl}_r(k)$-module de Lie. L'énoncé précis est le suivant:

THÉORÈME. *Soit* $A = kQ/ < Q_2 >$ *où* Q *est le carquois à boucles multiples et* k *est un corps algébriquement clos de caractéristique zéro. Alors,* $HH^1(A) \cong \mathfrak{gl}_r(k)$ *en tant qu'algèbres de Lie, où* r *est le nombre de boucles. La structure de module de Lie (induite par le crochet de Gerstenhaber) de* $HH^n(A)$ *sur* $HH^1(A) \cong \mathfrak{gl}_r(k)$ *est donnée par l'isomorphisme suivant*

$$HH^n(A) \cong V^{*\otimes n-1} \otimes \mathfrak{sl}_r(k)$$

où V *est le* $\mathfrak{gl}_r(k)$*-module standard et* $\mathfrak{sl}_r(k)$ *est le* $\mathfrak{gl}_r(k)$*-module canonique (i.e. donné par la restriction du module adjoint).*

Une observation simple nous donne le résultat suivant.

COROLLAIRE. *Pour* A *comme ci-dessus,* $HH^2(A) = V^* \otimes \mathfrak{sl}_r(k)$ *et pour* $n > 2$,

$$HH^n(A) \cong V^* \otimes HH^{n-1}(A).$$

Supposons que le corps de base est algébriquement clos de caractéristique zéro de façon à étudier $HH^n(A)$ comme module sur $\mathfrak{sl}_r(k)$. Rappelons deux résultats classiques de la théorie de Lie, (voir par exemple [**EW06, FH91**]).

(1) Tout module de Lie de dimension finie sur $\mathfrak{sl}_r(k)$ a une décomposition en somme directe de modules irréductibles.
(2) Les modules irréductibles sur $\mathfrak{sl}_r(k)$ sont déterminés de manière unique par leur vecteur de plus haut poids. Nous dénotons Γ_λ le module irréductible sur $\mathfrak{sl}_r(k)$ de plus haut poids λ.

Pour le carquois à deux boucles, nous donnons de manière explicite la décomposition en somme directe de modules irréductibles des groupes de cohomologie de Hochschild, considérés comme des modules sur $\mathfrak{sl}_2(k)$. Le résultat est le suivant:

Proposition. *Soit* k *un corps algébriquement clos de caractéristique zéro, soit* Q *le carquois à deux boucles et* $A = kQ/ < Q_2 >$. *Pour* $n \geq 1$ *soit*

$$h(n) = \max\{\, l \,|\, n + 1 - 2l \geq 0 \,\}.$$

Pour $l = 0, \ldots, h(n)$ *soit* $q(n,l)$ *l'entier suivant:*

$$q(n,l) = \begin{cases} \binom{n-1}{l} & \text{si } l = 0,1 \\ \binom{n+1}{l} - \binom{n+1}{l-1} - \binom{n-1}{l-1} + \binom{n-1}{l-2} & \text{si } l \geq 2 \end{cases}$$

La décomposition de $HH^n(A)$ *en somme directe de modules de Lie irréductibles sur* $sl_2(k)$ *est donnée par*

$$HH^n(A) \cong \bigoplus_{l=0}^{h(n)} \Gamma_{n+1-2l}^{q(n,l)}$$

où Γ_t^q *dénote la somme directe de* q *copies de* Γ_t*, qui est l'unique* $sl_2(k)$*-module irréductible de dimension* $t+1$.

Dans le cas général du carquois à boucles multiples, nous obtenons la décomposition en somme directe de modules irréductibles du deuxième groupe de la cohomologie de Hochschild, considéré comme module sur $sl_r k$. Soit $\Gamma_{(a_1,a_2,\ldots,a_{r-1})}$ l'unique module irréductible sur $sl_r(k)$ qui a comme plus haut poids $a_1 w_1 + a_2 w_2 + \cdots a_{r-1} w_r$ où les w_i sont les poids fondamentaux sur $sl_r(k)$.

Proposition. *Soit* $A = kQ/ < Q_2 >$ *où* Q *est le carquois à* r *boucles. La décomposition de* $HH^2(A)$ *en somme directe de modules irréductibles sur* $sl_r(k)$ *est donnée par:*

$$HH^2(A) = \begin{cases} \Gamma_3 \oplus \Gamma_1 & \textit{if } r = 2 \\ \Gamma_{(1,2)} \oplus \Gamma_{(0,2)} \oplus \Gamma_{(0,1)} & \textit{if } r = 3 \\ \Gamma_{(1,0,\ldots,0,2)} \oplus \Gamma_{(1,0,\ldots,1,0)} \oplus \Gamma_{(0,\ldots,0,1)} & \textit{if } r > 3 \end{cases}$$

Rappelons que la règle de Littlewood-Richardson est utilisée afin d'obtenir la décomposition en somme directe de modules irréductibles du produit tensorielle de deux modules irréductibles sur $sl_r(k)$. Un cas spécial est donné par la proposition suivante:

Proposition. *(Clebsch-Gordon) Pour* $sl_2(k)$ *et* $a \geq 1$

$$V^* \otimes \Gamma_a = \Gamma_{a+1} \oplus \Gamma_{a-1}.$$

Si $a = 0$ *alors* $V^* \otimes \Gamma_0 = V^*$.

Si $r > 2$ nous avons la règle de Littlewood-Richardson:

Proposition. *(Règle de Littlewood-Richardson) Pour $\mathfrak{sl}_r(k)$ où $r \geq 3$, la décomposition en somme directe de modules irréductibles de $V^* \otimes \Gamma_{(a_1,a_2,\ldots,a_{r-1})}$ est*

$$\Gamma_{(a_1,a_2,\ldots,a_{r-1}+1)} \oplus \bigoplus_{a_{i+1}\geq 1} \Gamma_{(a_1,\ldots,a_i+1,a_{i+1}-1,\ldots,a_{r-1})} \oplus \Gamma_{(a_1-1,a_2,\ldots,a_{r-1})}$$

En utilisant les propositions ci-dessus, nous décrivons un algorithme qui nous permet de trouver la décomposition en somme directe de modules irréductibles sur $\mathfrak{sl}_n(k)$ des groupes de cohomologie de Hochschild.

Algorithme. Soit $A = kQ/ < Q_2 >$ où Q est le carquois à r boucles. Nous avons mentionné que $HH^1(A)$ est isomorphe à $\mathfrak{gl}_r(k)$ et nous considérons $HH^n(A)$ comme module de Lie sur $\mathfrak{sl}_r(k)$. Notre but est d'expliquer l'algorithme que calcule la décomposition en somme directe de modules irréductibles sur $\mathfrak{sl}_r(k)$ de $HH^n(A)$. Le premier pas est donné par la proposition ci-dessus qui donne la décomposition voulue de $HH^2(A)$. Pour $n \geq 2$ supposons que nous avons la décomposition suivante:

$$HH^n(A) = \bigoplus_{\mathbf{a}} \Gamma_{\mathbf{a}}.$$

Afin de calculer la décomposition de $HH^{n+1}(A)$ nous utilisons que $HH^{n+1}(A) = V^* \otimes HH^n(A)$. Comme les sommes directes et les produits tensoriel commutent, le pas suivant est de calculer la décomposition de $V^* \otimes \Gamma_{\mathbf{a}}$ pour chaque $\Gamma_{\mathbf{a}}$ qui apparaît dans la décomposition de $HH^n(A)$. Pour cela, nous appliquons la règle de Littlewood-Richardson, et de cette manière nous trouvons la décomposition de $HH^{n+1}(A)$.

Dans le cas $r = 2$, remarquons que nous obtenons la règle de Pascal tronquée. Le tableau suivant donne la décomposition des groupes de cohomologie de Hochschild pour les degrés 2 à 7

n	Γ_0	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8
$HH^2(A)$		1		1					
$HH^3(A)$	1		2		1				
$HH^4(A)$		3		3		1			
$HH^5(A)$	3		6		4		1		
$HH^6(A)$		9		10		5		1	
$HH^7(A)$	9		19		15		6		1

Pour $\mathfrak{sl}_r(k)$ nous obtenons une règle "tronquée" de Pascal généralisée.

Introduction in English.

This thesis is about the Lie structure on the Hochschild cohomology, given by the Gerstenhaber bracket. More precisely, we study the *Lie algebra structure of the first Hochschild cohomology group and the Lie module structure of the Hochschild cohomology groups* of some monomial algebras.

In this introduction, we specify first the framework. Next, we present an overview of the research realized previously. Then we discuss the objective and the motivation for this work. Finally there is a detailed description of each chapter of this thesis. The results of the first section of the chapter 4, the third section of chapter 5 and the annexes A and B were published in [**SF08**].

Framework. Let A be an associative unital k-algebra where k is a field. The *Hochschild cohomology group in degree* n *of* A, denoted $HH^n(A)$, refers to

$$HH^n(A) = HH^n(A, A) = \mathrm{Ext}^n_{A^e}(A, A)$$

where A^e is the enveloping algebra $A^{op} \otimes_k A$ of A. For instance, $HH^0(A)$ is the center of A and the first Hochschild cohomology group $HH^1(A)$ is the vector space of the outer derivations, this is the quotient of the derivations by the interior derivations. Note that $HH^1(A)$ has a Lie algebra structure given by the commutator bracket.

In [**Ger63**], Gerstenhaber introduced two operations on the Hochschild cohomology groups: the cup product

$$\_ \smile \_ : HH^n(A) \times HH^m(A) \longrightarrow HH^{n+m}(A).$$

and the bracket

$$[-, -] : HH^n(A) \times HH^m(A) \longrightarrow HH^{n+m-1}(A).$$

He proved that the *Hochschild cohomology of* A,

$$HH^*(A) = \bigoplus_{n=0}^{\infty} HH^n(A),$$

provided with the cup product is a graded commutative algebra. Furthermore, he demonstrated that $HH^{*+1}(A)$ endowed with the Gerstenhaber bracket has a graded Lie algebra structure. Consequently, $HH^1(A)$ *is a Lie algebra and* $HH^n(A)$ *is a Lie module over* $HH^1(A)$. As a matter of fact, the Gerstenhaber bracket restricted to $HH^1(A)$ is the commutator bracket of the outer derivations. In addition, the cup product and the Gerstenhaber bracket endow $HH^*(A)$ with the Gerstenhaber algebra structure.

Interest and previous research. The algebraic structures on the Hochschild cohomology are important in the study of the representation and deformation theory of the algebra.

Both structures, the graded commutative algebra and the graded Lie algebra, are preserved under derived equivalence. First, it was shown that the commutative algebra structure of $\mathsf{HH}^*(\mathsf{A})$ is invariant under derived equivalence [**Hap89, Ric91**]. Then, in [**Kel04**], Keller proved that the Gerstenhaber bracket on $\mathsf{HH}^{*+1}(\mathsf{A})$ is also preserved under derived equivalence.

However, understanding both structures is a difficult assignment since the computations are complicated. Nevertheless, several results have been obtained in order to: (1) describe the Hochschild cohomology algebra (or ring) for some algebras, [**Hol96, CS97, Cib98, ES98, EH99, SW00, SA02, EHS02, GA08, Eu07b, FX06**]; (2) study the Hochschild cohomology ring modulo nilpotence, [**GSS03, GSS06, GS06**] and (3) compute the Gerstenhaber bracket [**Bus06, Eu07a, SA07**].

Objective. The aim of this thesis is to study the Lie structure on the Hochschild cohomology of finite dimensional monomial algebras. A *monomial algebra* is defined as the quotient of the path algebra of a quiver by a two-sided ideal generated by a set of paths of length at least two. We use the intrinsic combinatorial data of such algebras to study the Lie structure defined on the Hochschild cohomology by the Gerstenhaber bracket. Actually, we discuss two aspects of such algebraic structure. The first one is the *relationship between semisimplicity on the first Hochschild cohomology groups and the vanishing of the Hochschild cohomology groups*. In the second one, we center our attention to the *Lie module structure of the Hochschild cohomology groups of a particular family of monomial algebras: those whose Jacobson radical square is zero.*

Motivation. One of the principal motivation of this research was suggested by Christian Kassel from the results of Claudia Strametz. In [**Str06**], Strametz studied the Lie algebra structure of the first Hochschild cohomology group of monomial algebras. She succeed to describe the commutator bracket in terms of the combinatorics of the quiver. One of her contributions was to provide sufficient and necessarily conditions to the combinatorial data of the monomial algebra in order to guarantee the semisimplicity on the first Hochschild cohomology group. Moreover, she showed that in this case, the semisimple Lie algebra obtained is isomorphic to a direct product of some Lie algebras of trace zero square matrices. Finite dimensional modules over these Lie algebras are classified so a natural question arise: *What is the description of the Hochschild cohomology group in degree* n *as a Lie module over the Lie algebra of outer derivations, when this one is semisimple?* We show in this thesis that for monomial algebras over a field of characteristic zero, the Hochschild cohomology groups of degree at least two are zero, however this answer brings out some other questions concerning the structure of the cohomology. For instance, the pursuit of examples where the Lie module structure of the Hochschild cohomology groups is not trivial, this lead to consider the case where the Lie algebra in the first degree is not semisimple.

Hence, we consider monomial algebras whose radical square is zero. For these algebras, Claude Cibils has computed, in [**Cib98**], the Hochschild cohomology groups using the combinatorics of the quiver.

Contents of chapters. This thesis is divided into five chapters. In Chapters 1 and 2, we present results concerning the Lie algebra structure of the first Hochschild cohomology group. In chapter 3, we consider the relationship between semisimplicity on $\mathsf{HH}^1(A)$ where A is monomial and the vanishing of the Hochschild cohomology groups. Chapters 4 and 5 are concerned with the Lie module structure of $\mathsf{HH}^n(A)$ where A is a monomial algebra of radical square zero. We will describe next the contents of each chapter more in detail.

`First chapter.` In this chapter, the aim is to recall combinatorial description, given by Strametz, of the commutator bracket defined on $\mathsf{HH}^1(A)$. To do so, we will remind the description of $\mathsf{HH}^1(A)$ given in terms of parallel arrows. The contents of this chapter is rather technical, nevertheless both combinatorial descriptions presented here are the principal tools to understand the Lie algebra structure, which is the objective of the subsequent chapter.

`Second chapter.` We specify the Lie algebra structure of $\mathsf{HH}^1(A)$: when A has radical square zero in one hand and when A is a triangular complete monomial algebra in the other hand. A *complete monomial algebra* is a monomial algebra that verifies the following property: every path of length at least two parallel to a path which is zero in the algebra is also zero. For instance, the radical square zero monomial algebras are complete monomial. We study both cases separately: the complete monomial algebras whose quiver contains no oriented cycles in one hand and radical square zero monomial algebras without any restriction on the quiver in the other hand.

Keeping in mind Levi's decomposition theorem, we compute first the solvable radical of $\mathsf{HH}^1(A)$ in order to obtain then the semisimple part. Recall that the semisimple part is precisely the quotient by its solvable radical. In this chapter we assume that the field k is algebraically closed of characteristic zero.

For the radical square zero monomial algebras, we show that the first Hochschild cohomology groups is a reductive Lie algebra, this means that it is the direct sum of a semisimple Lie algebra and an abelian Lie algebra.

Given a quiver Q, we denote $\overline{Q}$ the quiver obtained by identifying parallel arrows, i.e. multiple parallel arrows in Q are seen as only one arrow in $\overline{Q}$. Denote S the set of arrows in $\overline{Q}$ that correspond to more than one arrow in Q. We have the following proposition.

PROPOSITION. *Let* $A = kQ/ < Q_2 >$ *be a monomial algebra of radical square zero where* k *is an algebraically closed field of characteristic zero, and* Q *is a finite connected quiver. Then*

$$\mathsf{HH}^1(A) \cong \prod_{\alpha \in S} \mathfrak{sl}_{|\alpha|}(k) \times k^{\chi(\overline{Q})}$$

where $\chi(\overline{Q}) = |\overline{Q}_1| - |\overline{Q}_0| + 1$ *is the Euler characteristic of* $\overline{Q}$*. Therefore,* $\mathsf{HH}^1(A)$ *is reductive.*

In particular, when the quiver is an oriented cycle the set S is empty. Moreover $\chi(\overline{Q}) = 1$, hence in this case $HH^1(A)$ is the one dimensional abelian Lie algebra.

We apply the above proposition to the multiple loops quiver. By definition, the *multiple-loops quiver* is the quiver with one vertex and at least two loops. The monomial algebra of radical square zero associated to the multiple-loops quiver is $k[x_1, \dots x_r]/ < x_i x_j >_{i,j=1}^{r}$ where r is the number of loops. As a consequence of the above proposition, the first Hochschild cohomology group is $\mathfrak{gl}_r(\,k\,)$, the Lie algebra of square matrices of size r.

Next we consider triangular complete monomial algebras. The semisimple part of the first Hochschild cohomology group can be expressed in the same terms as in the previous case. The solvable radical is not longer abelian however we give a description of it.

`Third chapter.` Our principal objective in this chapter is to show that under the hypothesis of semisimplicity on the first Hochschild cohomology group of a monomial algebra, the Hochschild cohomology groups vanish from the second degree. More precisely, we prove the following theorem.

THEOREM. *Let* Q *be a finite connected quiver, and* Z *a minimal set of paths closed by parallel paths. Consider the finite dimensional monomial algebra* $A = kQ/ < Z >$*, where* k *is an algebraically closed field of characteristic zero. If the underlined graph of* $\overline{Q}$ *is a tree then*

- $HH^0(A) = k$
- $HH^1(A) = \prod_{\alpha \in S} \mathfrak{sl}_{|\alpha|}(k)$ *and*
- $HH^n(A) = 0$ *for all* $n \geq 2$.

To prove the above theorem, we use as a tool the Happel-Bardzell projective resolution [**Bar97**] which is a projective resolution of A as a left A^e-module.

COROLLARY. *Let* $A = kQ/ < Z >$ *be a finite dimensional monomial algebra where* k *is an algebraically closed field of characteristic zero. If* $HH^1(A)$ *is semisimple then* $HH^n(A) = 0$ *for all* $n \geq 2$.

In order to demonstrate the corollary, we used the conditions for semisimplicity given by Strametz. At the beginning of this chapter, we recall and discuss those conditions in order to restate her theorem.

PROPOSITION ([**Str06**]). *Let* Q *be a quiver and* Z *a minimal set of paths. Let* $A = kQ/ < Z >$ *be the finite dimensional monomial algebra where* k *is an algebraically closed field of characteristic zero. The following conditions are equivalent:*

(1) $HH^1(A)$ *is semisimple.*
(2) The underlined graph of the quiver $\overline{Q}$ *is a tree,* Z *is closed by parallel arrows and the set* S *is not empty.*

(3) $\mathsf{HH}^1(A)$ *is isomorphic to the following non trivial direct product of Lie algebras:*

$$\prod_{\alpha \in S} \mathfrak{sl}_{|\alpha|}(k).$$

We present another proof of Strametz' theorem.

`Chapter four.` We will assume, throughout this chapter, that A is a monomial algebra with radical square zero. In this chapter, we will study the Lie module structure on the Hochschild cohomology groups, induced by the Gerstenhaber bracket

$$\mathsf{HH}^1(A) \times \mathsf{HH}^n(A) \longrightarrow \mathsf{HH}^n(A).$$

The study is in three cases, depending on the quiver:

1. the quiver is a loop,
2. the quiver is an oriented cycle but not the loop, and
3. the quiver is not an oriented cycle.

The principal tool to understand the Lie module structure is the combinatorial description of both the Hochschild cohomology group and the Gerstenhaber bracket. In the first section, we will present the combinatorial complex given in [**Cib98**] that computes Hochschild cohomology. Next, we recall the formulation of the Gerstenhaber bracket given in [**SF08**] for Cibils' realization of the Hochschild cohomology groups. This section collects some results of my paper [**SF08**], where proofs are given in the appendix of this thesis. In the rest of the chapter we explore the Lie module structure for the cases above mentioned.

In the first case, the algebra that we are considering is in fact the algebra of the dual numbers. The Hochschild cohomology vector spaces of degree ≥ 1 of this algebra are one dimensional. In the following proposition, we provide a basis of $\mathsf{HH}^n(A)$.

Proposition. *Let* $A = k[x]/ < x^2 >$ *where* k *is of characteristic zero. For* $n \geq 1$*, consider the map* $\varphi_n : A^{\otimes n} \to A$ *given by:*

$$\varphi_n(f_1 \otimes \cdots \otimes f_i \otimes \cdots \otimes f_n) = \begin{cases} \prod_{i=1}^n \lambda(f_i, x) & \text{if } n \text{ is even} \\ \prod_{i=1}^n \lambda(f_i, x) x & \text{if } n \text{ is odd.} \end{cases}$$

where $f_i = \lambda(f_i, x)x + \lambda(f_i, 1)$ *for* $i = 1, \cdots, n$*. Then* $\mathsf{HH}^n(A) \cong k\,\varphi_n$

The one dimensional Lie modules over an abelian Lie algebra are given by the multiplication by some scalar. We precise the scalar determines the Lie module $\mathsf{HH}^n(A)$.

Proposition. *Let* $A = k[x]/ < x^2 >$ *where* k *is of characteristic zero. For* $n \geq 1$*, the Lie module structure on the Hochschild cohomology groups given by Gerstenhaber bracket,*

$$\mathsf{HH}^1(A) \times \mathsf{HH}^n(A) \longrightarrow \mathsf{HH}^n(A),$$

is given by:

$$\varphi_1 . \varphi_n = \begin{cases} -n\,\varphi_n & \text{if } n \text{ is even} \\ (1-n)\,\varphi_n & \text{if } n \text{ is odd.} \end{cases}$$

Therefore,

$$HH^{2n}(A) \cong HH^{2n+1}(A)$$

considered as Lie modules.

Now we denote the cohomology in odd degrees by

$$HH^{odd}(A) = \bigoplus_{n=0}^{\infty} HH^{2n+1}(A).$$

It is clear that the Gerstenhaber bracket endows $HH^{odd}(A)$ with a Lie algebra structure. We will describe such Lie algebra. Let $\mathcal{W}$ be the Lie algebra of derivations of $k[x]$, i.e. $\mathcal{W} = Der(k[x], k[x])$. Such Lie algebra admits the following graduation:

$$\mathcal{W} = \bigoplus_{n=0}^{\infty} W_n$$

where W_n is the vector space generated by the derivation $\phi_n : k[x] \to k[x]$ defined by $\phi_n(x^i) = ix^{n+i-1}$. In fact, any derivation is linear combination of the ϕ_n's. Moreover, the commutator is given by the following formula:

$$[\phi_n, \phi_m] = (n - m)\phi_{n+m-1}.$$

Clearly, the bracket is graded if we consider the elements of W_n as of degree $n - 1$. Beside, we will denote

$$\mathcal{W}^{odd} = \bigoplus_{n=0}^{\infty} W_{2n+1}$$

the Lie subalgebra of $\mathcal{W}$.

Proposition. *Let* k *be a field of characteristic zero and* A *the algebra of the dual numbers, this is* $A = k[x]/ < x^2 >$*. The Lie algebra* $HH^{odd}(A)$ *is isomorphic to the infinite dimensional Lie algebra* $\mathcal{W}^{odd}$*.*

For the second case, let Q be an oriented cycle of length N where $N \geq 2$. An element f in $A = kQ/ < Q_2 >$ is given by a linear combination

$$f = \sum_{i=1}^{N} \lambda(f, e_i)e_i + \lambda(f, a_i)a_i$$

where $e_1, \dots, e_N$ are the vertices of the quiver and $a_1, \dots, a_N$ are the arrows. We give a basis of the vector space $HH^n(A)$. Let us give some notations.

For $i = 1, \cdots, N$ and for $cN > 0$ a positive multiple of N, we denote

$$\sigma_i : \{1, \dots, cN\} \to \{1, \cdots, N\}$$

the periodic function with period N (i.e. $\sigma_i(j) = \sigma_i(j+N)$) such that σ_i restricted to the set $\{1, \dots, N\}$ is the following cyclic permutation:

- if $i = 1$ then $\sigma_1(j) = j$ for $j = 1, \dots, N$;
- if $i = N$ then $\sigma_N(1) = N$ and $\sigma_N(j) = j - 1$ for $j = 2, \dots N$,
- if $1 < i < N$ then $\sigma_i(j) = i + (j - 1)$ for $j = 1, \dots, N - i + 1$ and $\sigma_i(j) = (j - 1) - (N - i)$ for $j = N - i + 2, \dots N$.

We denote
$$\pi_i : A^{\otimes cN} \longrightarrow k$$
the linear map given by
$$\pi_i(f_1 \otimes \cdots \otimes f_{cN}) = \prod_{j=1}^{cN} \lambda(f_j, a_{\sigma_i(j)}).$$
We show the following result.

PROPOSITION. *Let* $A = kQ/ < Q_2 >$ *where* k *is a field of characteristic zero and* Q *is the oriented cycle of length* N *with* $N \geq 2$. *Consider the map* $\varphi_1 : A \longrightarrow A$ *given by*
$$\varphi_1(f) = \lambda(f, a_1)a_1$$
Then $HH^1(A) \cong k\,\varphi_1$.

For $n \geq 1$, *a multiple of* N, *consider the map* $\varphi_n : A^{\otimes n} \longrightarrow A$ *given by*
$$\varphi_n(f_1 \otimes \cdots \otimes f_n) = \sum_{i=1}^{N} \pi_i(f_1 \otimes \cdots \otimes f_n)e_i$$
and the map $\varphi_{n+1} : A^{\otimes n+1} \longrightarrow A$ *given by*
$$\varphi_{n+1}(f_1 \otimes \cdots \otimes f_{n+1}) = \pi_1(f_1 \otimes \cdots \otimes f_n)\lambda(f_{n+1}, a_1)a_1 .$$
Then $HH^n(A) \cong k\,\varphi_n$ *and* $HH^{n+1}(A) \cong k\,\varphi_{n+1}$ *if* n *is even and* $HH^n(A)$ *is zero otherwise.*

According to this result, $HH^n(A)$ is either zero or a one dimensional vector space. In the particular case of dimension one, the Lie module structure of $HH^n(A)$ is given by the multiplication by a non-zero scalar. We precise this scalar in the following proposition.

PROPOSITION. *Let* $A = kQ/ < Q_2 >$ *where* Q *is an oriented cycle of length* N *with* $N \geq 2$ *and* k *is a field of characteristic zero. For* $n \geq 1$, *the Lie module structure on the Hochschild cohomology groups, given by the Gerstenhaber bracket,*
$$HH^1(A) \times HH^n(A) \longrightarrow HH^n(A)$$
is given as follows
$$\varphi_1.\varphi_n = -\,c\,\varphi_n$$
where c *is an integer such that* cN *is even and either* $n = cN$ *or* $n = cN + 1$ *and* c *is zero otherwise.*

Therefore, for all positive integer c, $HH^{cN}(A) \cong HH^{cN+1}(A)$ *as Lie modules.*

As before, we describe $HH^{odd}(A)$. To do so let $\mathcal{W}^*$ be the Lie subalgebra of $\mathcal{W}$ given as follows:
$$\mathcal{W}^* = \bigoplus_{n=0}^{\infty} W_{n+1}.$$
The following proposition describes the Lie algebra structure of $HH^{odd}(A)$ in the oriented cycle case.

PROPOSITION. *Let* $A = kQ/ < Q_2 >$ *where* Q *is the oriented cycle of length* N *with* $N \geq 2$*, and* k *is a field of characteristic zero. The Lie algebra* $HH^{odd}(A)$ *is isomorphic to* $\mathcal{W}^*$*.*

In the last case we suppose that the quiver is not an oriented cycle, i.e. Q can admit cycles but Q cannot be reduced to an oriented cycle. For this case, the description of the Lie module structure is given in terms of the quiver: of its shortcuts and oriented cycles. Here is some definitions notations.

Let X and Y be sets consisting of paths of Q, then $X \parallel Y$ denotes the set of all couples of paths (α, β) in $X \times Y$ that are *parallels*, this means that they share the same source and the same target. We denote $k(X \parallel Y)$ the k-vector space whose basis is the set $X \parallel Y$. We call $k(Q_n \parallel Q_1)$, the space of *shortcuts* and $k(Q_n \parallel Q_0)$, the space of *pointed oriented cycles*.

The description of the Hochschild cohomology groups as quotients of the space of shortcuts is given in [**Cib98**]. The precise statement is the following.

THEOREM ([**Cib98**]). *Let* $A = kQ/ < Q_2 >$ *where* Q *is not an oriented cycle. Then, if* $n \geq 1$

$$HH^n(A) \cong \frac{k(Q_n \parallel Q_1)}{\operatorname{Im} D_{n-1}}$$

where

$$D_{n-1} : k(Q_{n-1} \parallel Q_0) \longrightarrow k(Q_n \parallel Q_1)$$

is the following linear map

$$D_{n-1}(\gamma^{n-1}, e) = \sum_{a \in Q_1 e} (a\gamma^{n-1}, a) + (-1)^n \sum_{a \in eQ_1} (\gamma^{n-1}a, a)\,.$$

Moreover, if $n > 1$ *then*

$$\dim_k HH^n(A) = |Q_n \parallel Q_1| - |Q_{n-1} \parallel Q_0|.$$

Using the above realization, we describe the Lie module structure. We prove the following theorem.

THEOREM. *Let* $A = kQ/ < Q_2 >$ *where* Q *is a finite quiver. If* Q *is not an oriented cycle then the Lie module structure of the Hochschild cohomology groups given by the Gerstenhaber bracket*

$$HH^1(A) \times HH^n(A) \longrightarrow HH^n(A)$$

is induced by the following bilinear map:

$$k(Q_1 \parallel Q_1) \times k(Q_n \parallel Q_1) \longrightarrow k(Q_n \parallel Q_1)$$

given as follows

$$(a, x).(\alpha, y) = \delta^a_y \cdot (\alpha, x) - \sum_{i=1}^{n} \delta^{a_i}_x \cdot (\alpha \underset{i}{\diamond} a, y)$$

where δ *is the Kronecker symbol and* $\alpha = a_1 \cdots a_i \cdots a_n$ *is a path of length* n *constituted of arrows* a_i*. For* $a_i = x$ *the path* $\alpha \underset{i}{\diamond} a$ *is obtained by replacing* a_i *with* a*.*

`Chapter five.` In this chapter we consider two kinds of quivers: those without oriented cycles and the multiple-loops. We relate the Lie module structure of the Hochschild cohomology groups with the Lie modules over the Lie algebra of trace zero square matrices $\mathfrak{sl}_r(k)$, using the above theorem.

In the first case, we consider triangular monomial algebra of radical square zero. Let us introduce some notation in order to state the result. Given an arrow α in $\overline{Q}_1$, we denote V_α the k-vector space whose basis is the set α. The vector space V_α has a Lie module structure over $\prod_{\alpha' \in \overline{Q}_1} \mathrm{End}_k(V_{\alpha'})$ given by:

$$(f_{\alpha'})_{\alpha' \in \overline{Q}_1}.\nu = f_\alpha(\nu)$$

Moreover, given a shortcut T in $\overline{Q}$, i.e. T is an element of $\overline{Q}_n \parallel \overline{Q}_1$, we write $T = (\alpha_1 \cdots \alpha_i \cdots \alpha_n, \gamma)$ where α_i and γ are arrows in $\overline{Q}_1$.

Theorem. *Let $A = kQ/ < Q_2 >$ where Q is a finite quiver without oriented cycles and k is an algebraically closed field of characteristic zero. Then for $n > 1$ the Lie module structure of $HH^n(A)$ over $HH^1(A)$ induced by the Gerstenhaber bracket is given as follows:*

$$HH^n(A) \cong \bigoplus_{T \in \overline{Q}_n \parallel \overline{Q}_1} HH^n_T(Q)$$

where

$$HH^n_T(Q) = V^*_{\alpha_1} \underset{k}{\otimes} \cdots \underset{k}{\otimes} V^*_{\alpha_i} \underset{k}{\otimes} \cdots \underset{k}{\otimes} V^*_{\alpha_n} \underset{k}{\otimes} V_\gamma$$

Moreover, $HH^n_T(Q)$ is irreducible.

The second case is the multiple-loops quiver. Let A be the monomial algebra of radical square zero, associated to this quiver. We have mentioned that the first Hochschild cohomology group is isomorphic to $\mathfrak{gl}_r(k)$. For this algebra, we relate the Lie module structure with a tensor product of two well known modules of $\mathfrak{gl}_r(k)$. The exact statement is as follows:

Theorem. *Let $A = kQ/ < Q_2 >$ where Q is a multiple-loops quiver and k is an algebraically closed field of characteristic zero. Then $HH^1(A) \cong \mathfrak{gl}_r(k)$ as Lie algebras, where r is the number of loops. The Lie module structure (induced by the Gerstenhaber bracket) of $HH^n(A)$ over $HH^1(A) \cong \mathfrak{gl}_r(k)$ is given by the following isomorphism*

$$HH^n(A) \cong V^{*\otimes n-1} \otimes \mathfrak{sl}_r(k)$$

where V is the standard $\mathfrak{gl}_r(k)$-module and $\mathfrak{sl}_r(k)$ is the usual $\mathfrak{gl}_r(k)$-module (i.e. given by the restriction of the adjoint module).

A simple observation gives the following:

Corollary. *For an algebra A as above, $HH^2(A) = V^* \otimes \mathfrak{sl}_r(k)$ and for $n > 2$ we have*

$$HH^n(A) \cong V^* \otimes HH^{n-1}(A).$$

Assume that the ground field is algebraically closed and of characteristic zero, in order to study $HH^n(A)$ as a module over $sl_r(k)$. Let us recall two classical Lie theory results, (see for instance [**EW06, FH91**]).

(1) Every (finite dimensional) $sl_r(k)$-module has a decomposition into direct sum of irreducible modules
(2) The irreducible modules over $sl_r(k)$ are uniquely determined by their vector of highest weight. We denote Γ_λ the irreducible $sl_r(k)$ module of highest weight λ.

For the two loop quiver, we provide explicitly the decomposition into direct sum of irreducible modules of the Hochschild cohomology groups, considered as modules over $sl_2(k)$. The result is as follows:

PROPOSITION. *Let* k *be an algebraically closed field of characteristic zero,* Q *be the two-loops quiver and* $A = kQ/ < Q_2 >$. *For* $n \geq 1$ *let*

$$h(n) = max\{\, l \,|\, n + 1 - 2l \geq 0\, \}.$$

For $l = 0, \dots, h(n)$ *let* $q(n, l)$ *be the following number:*

$$q(n,l) = \begin{cases} \begin{pmatrix} n-1 \\ l \end{pmatrix} & \text{if } l = 0,1 \\ \\ \begin{pmatrix} n+1 \\ l \end{pmatrix} - \begin{pmatrix} n+1 \\ l-1 \end{pmatrix} - \begin{pmatrix} n-1 \\ l-1 \end{pmatrix} + \begin{pmatrix} n-1 \\ l-2 \end{pmatrix} & \text{if } l \geq 2 \end{cases}$$

The decomposition of $HH^n(A)$ *into a direct sum of irreducible Lie modules over* $sl_2(k)$ *is given by*

$$HH^n(A) \cong \bigoplus_{l=0}^{h(n)} \Gamma_{n+1-2l}^{q(n,l)}$$

where Γ_t^q *denotes the direct sum of* q *copies of* Γ_t *that is the unique irreducible* $sl_2(k)$*-module of dimension* $t + 1$.

In the general case of multiple-loops quiver, we obtain the decomposition into direct sum of irreducibles modules of the second Hochschild cohomology groups, considered as a module over $sl_r(k)$. Let $\Gamma_{(a_1,a_2,\dots,a_{r-1})}$ be the unique irreducible module over $sl_r(k)$ of highest weight $a_1w_1 + a_2w_2 + \cdots a_{r-1}w_r$ where w_i are the fundamental weights of $sl_r(k)$.

PROPOSITION. *Let* $A = kQ/ < Q_2 >$ *where* Q *is a multi-loop quiver with* r *loops. The decomposition of* $HH^2(A)$ *into direct sum of irreducible modules as a module over* $sl_r(k)$ *is given by:*

$$HH^2(A) = \begin{cases} \Gamma_3 \oplus \Gamma_1 & \text{if } r = 2 \\ \Gamma_{(1,2)} \oplus \Gamma_{(0,2)} \oplus \Gamma_{(0,1)} & \text{if } r = 3 \\ \Gamma_{(1,0,\dots,0,2)} \oplus \Gamma_{(1,0,\dots,1,0)} \oplus \Gamma_{(0,\dots,0,1)} & \text{if } r > 3 \end{cases}$$

Recall that the Littlewood Richardson rule is used to find the decomposition into direct sum of irreducibles of the tensor product of two irreducible modules of $\mathfrak{sl}_r(k)$. A special case is given in the next proposition.

PROPOSITION. *(Clebsch-Gordon) For* $\mathfrak{sl}_2(k)$ *and* $a \geq 1$ *the following holds*

$$V^* \otimes \Gamma_a = \Gamma_{a+1} \oplus \Gamma_{a-1}.$$

If $a = 0$ *then* $V^* \otimes \Gamma_0 = V^*$.

For $r > 2$, we have the Littlewood-Richardson rule:

PROPOSITION. *(Littlewood-Richardson rule) For* $\mathfrak{sl}_r(k)$ *with* $r \geq 3$*, the decomposition into direct sum of irreducible modules of* $V^* \otimes \Gamma_{(a_1,a_2,\ldots,a_{r-1})}$ *is*

$$\Gamma_{(a_1,a_2,\ldots,a_{r-1}+1)} \oplus \bigoplus_{a_{i+1}\geq 1} \Gamma_{(a_1,\ldots,a_i+1,a_{i+1}-1,\ldots,a_{r-1})} \oplus \Gamma_{(a_1-1,a_2,\ldots,a_{r-1})}$$

Using the above propositions, we provide an algorithm that allows us to find the decomposition into direct sum of irreducible modules over $\mathfrak{sl}_n(k)$ of the Hochschild cohomology groups.

ALGORITHM. Let $A = kQ/ < Q_2 >$ where Q is a multi-loop quiver with r loops. We have mentioned that $HH^1(A)$ is isomorphic to $\mathfrak{gl}_r(k)$. We consider $HH^n(A)$ as $\mathfrak{sl}_r(k)$-modules. Our aim is to explain an algorithm to calculate the decomposition into direct sum of irreducible $\mathfrak{sl}_r(k)$-modules of $HH^n(A)$. The first step is given by the above proposition which gives the wanted decomposition for $HH^2(A)$. For $n \geq 2$, let us suppose that we have the following decomposition:

$$HH^n(A) = \bigoplus_{\mathbf{a}} \Gamma_{\mathbf{a}}\,.$$

In order to calculate de decomposition of $HH^{n+1}(A)$ we used that $HH^{n+1}(A) = V^* \otimes HH^n(A)$, which is a consequence of the above theorem. Since direct sums and tensor products of Lie modules commute, the next step is to calculate the decomposition of $V^* \otimes \Gamma_{\mathbf{a}}$ for each $\Gamma_{\mathbf{a}}$ that appears in the decomposition of $HH^n(A)$. To do so, we apply the Littlewood-Richardson rule, and in this way we find the decomposition of $HH^{n+1}(A)$.

In fact, for $r = 2$ notice that we obtain a "truncated" Pascal rule. The following table gives the decomposition for the Hochschild cohomology groups of degrees between 2 and 7.

n	Γ_0	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8
$HH^2(A)$		1		1					
$HH^3(A)$	1		2		1				
$HH^4(A)$		3		3		1			
$HH^5(A)$	3		6		4		1		
$HH^6(A)$		9		10		5		1	
$HH^7(A)$	9		19		15		6		1

For $\mathfrak{sl}_r(k)$ we obtain a "truncated" generalized Pascal rule.

CHAPTER 1

The space of outer derivations of a monomial algebra.

It is well known that the first Hochschild cohomology group is in fact the vector space of outer derivations. Clearly, it is endowed with a Lie algebra structure given by the commutator bracket. Such structure was studied by C. Strametz for the case of finite dimensional monomial algebras, using combinatorial tools.

In this chapter, we will recall the description of the space of outer derivations using the combinatorial data of a finite-dimensional algebra. Such description is obtained from the complex induced by the Happel-Bardzell projective resolution. Then we will remind the description of the commutator bracket given by Strametz in the same terms as the combinatorial realization of the first Hochschild cohomology group of a monomial algebra.

We will begin recalling the definition of a monomial algebra, and we will fix the notation that we will be using all along this thesis.

1.1. Monomial Algebras.

Peter Gabriel's theorem states that any basic, finite dimensional algebra over an algebraically closed field is isomorphic to a quotient of a path algebra by an admissible ideal. Monomial algebras are certain quotients of a path algebra, they play a special role in the study of finite dimensional algebras. In this section we will recall some generalities about monomial algebras: we will begin with the definitions of a quiver and its path algebra.

A *quiver* Q is a directed graph. This means that a quiver is given by the following data: a set Q_0 called the set of *vertices* and a set Q_1 called the set of *arrows*, together with two applications which are called the *source*, denoted s, and the *target*, denoted t, both defined from the set of arrows to the set of vertices.

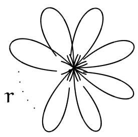
For example, the *loop* quiver is given by one vertex and one arrow. The source and the target functions are the same:

$$\begin{tikzcd} \cdot \arrow[loop above] \end{tikzcd}$$

Another example is the *Kronecker quiver*, given by a double arrow, this means that $Q_0 = \{e_1, e_2\}$, $Q_1 = \{a, a'\}$ and for the source and target functions, $s(a) = s(a') = e_1$ and $t(a) = t(a') = e_2$.

$$\begin{tikzcd} e_1 \arrow[r, shift left, "a"] \arrow[r, shift right, "a'"'] & e_2 \end{tikzcd}$$

The *multiple loops* quiver is given by one vertex and several loops, like in the following figure:



where r is the number of loops.

The *multiple arrows* quiver is given by the following data: $Q_0 = \{e_1, e_2\}$, $Q_1 = \{a_1, \dots a_r\}$, $s(a_i) = e_1$ and $t(a_i) = e_2$ for $i = 1, \dots, r$. We will denote it by

$$e_1 \xrightarrow{r} e_2 \qquad = \qquad e_1 \;\begin{array}{c} \xrightarrow{a_1} \\ \vdots \\ \xrightarrow[a_r]{} \end{array}\; e_2$$

Given a quiver Q, one constructs *paths* by concatenating arrows as follows. Let $a_1, \dots, a_n$ be arrows such that $s(a_i) = t(a_{i+1})$ for $i = 1, \dots, n-1$, the expression $a_1 a_2 \cdots a_n$ is a path denoted p.

$$\cdot \xrightarrow{a_n} \cdot \xrightarrow{a_{n-1}} \cdot \cdots\cdots \cdot \xrightarrow{a_1} \cdot$$

Let us remark that the source map and the target map can also be defined for paths as follows: $s(p) = s(a_n)$ and $t(p) = t(a_1)$. Let p and q be two paths, we say that they are *composable* if and only if $s(p) = t(q)$, in this case we write pq for the path obtained after concatenation. Besides, a path c such that $s(c) = t(c)$ is called an *oriented cycle*. For example, the arrow in the loop quiver is an oriented cycle. Moreover, the *length* of a path is the number of arrows used in its expression as concatenation of arrows. We will denote Q_n the set of all paths of Q of length n. The set of vertices Q_0, which is the set of paths of length zero, will be considered as the set of *trivial paths*.

Let p and q be two paths. We say that q *divides* p if there exist paths x and y such that $p = xqy$ where x and y are paths. Moreover, the *underlying graph* of the quiver is the graph obtained when the orientations of the arrows are ignored.

Let k be a field. We define the *path algebra* of a quiver Q as follows: its underlying vector space has a basis given by all paths of the quiver, and the multiplication is given by the concatenation of paths whenever they are composable and zero otherwise. Let us remark that kQ_0 has an algebra structure and kQ_1 becomes a $kQ_0 - kQ_0$ bimodule. Then the path algebra can be equivalently defined as the tensor algebra of kQ_1 over kQ_0, i.e.

$$kQ = T_{kQ_0}(kQ_1) = \bigoplus_n kQ_1^{\otimes^n_{kQ_0}}.$$

For instance, the path algebra of the loop quiver is the polynomial algebra in one variable.

Definition. An *admissible ideal* I is an ideal of a path algebra of a quiver Q such that

$$< Q_n > \subseteq \; I \subseteq < Q_2 >$$

for some n, where $< Q_i >$ is the two-sided ideal generated by Q_i.

Example. The two-sided ideal generated by Q_n with $n \geq 2$ is an admissible ideal. Another example is the zero ideal which is admissible if and only if the quiver has no oriented cycles.

Definition (Monomial algebra). A *monomial algebra* is a quotient of the path algebra of a quiver Q by a two-sided ideal generated by a set of paths of length at least two, which we will denote Z.

Assumptions. In this thesis, we are concerned with finite dimensional monomial algebras, according to P. Gabriel theorem quoted above, this means

$$A = kQ/ < Z > \text{ where } < Z > \text{ is an admissible ideal.}$$

In the sequel, we will assume that

- Q_1 and Q_0 are non-empty sets.
- Q is finite, i.e. Q_0 and Q_1 are finite sets.
- Q is a connected, i.e. the underlying graph of Q is connected.
- Z is minimal, this means that for all path p in Z and for all path $q \neq p$ that strictly divides the path p, q does not belong to the set Z.

This last assumption is not restrictive since we can always extract from a set of paths, a minimal subset such that both sets generate the same ideal.

Let B be the set of paths of Q which are not divided by any element of Z. It is clear that the elements of B form a basis of the monomial algebra A. Moreover, the Jacobson radical (i.e. the intersection of all left maximal ideals) of a monomial algebra denoted $r = \operatorname{rad} A$ is given by

$$r = \frac{< Q_1 >}{< Z >}$$

where $< Q_1 >$ is the two-sided ideal generated by Q_1 (see for instance [**Cib90**]). Furthermore, $E = kQ_0$ is isomorphic to A/r. Moreover $A \cong E \oplus r$, as predicted by Wedderburn-Malcev theorem.

1.2. A description of the first Hochschild cohomology group.

For any algebra A, the Hochschild cohomology groups are computed as the cohomology of the complex obtained after applying the functor $\operatorname{Hom}_{A^e}(-, A)$ to any projective resolution of A as a left A^e-module. The Hochschild cohomology groups computation for a monomial algebra has been done using the Happel-Bardzell projective resolution. Once we have such resolution, a complex is induced as we have just explained, after applying the functor $\operatorname{Hom}_{A^e}(-, A)$. Then such complex is simplified: the space of cochains are expressed in terms of parallel paths and the differential maps are expressed in terms of an operation that replaces parallel arrows in a path. Let us first introduce both tools: parallel paths and such operation.

Definition (Parallel paths). Given a quiver, we say that two paths α and β are *parallels* if and only if they have the same source and the same target. If α and β are parallels we write $\alpha \parallel \beta$.

Notation. Let X and Y be sets consisting of paths of Q, then $X \parallel Y$ denotes the set of all couples of paths (α, β) in $X \times Y$ that are parallels. We denote $k(X \parallel Y)$ the k-vector space generated by $X \parallel Y$.

Now, we introduce the operation that replaces parallel arrows in a path, the operation $\underset{i}{\diamond}$ where i is a natural number between 1 and the length of the path.

Definition (Operation $\underset{i}{\diamond}$). Given a path α in Q_n with $n \geq 1$ we will suppose its expression in arrows is as follows: $\alpha = a_1 \cdots a_i \cdots a_n$. Fix a natural number i from 1 to n. Now, let β be a non trivial path in Q_m such that $a_i \parallel \beta$. From such data, a path in Q_{n+m-1} can be obtained by replacing the arrow a_i by the path β:

$$a_1 \cdots a_{i-1} \beta a_{i+1} \cdots a_n.$$

The following picture illustrate such replacing:

$$\begin{tikzcd} \cdot \arrow[r, "a_n"] & \cdot \arrow[r, dotted, dash] & \cdot \arrow[r, "a_{i-1}"] & \cdot \arrow[r, dotted, "a_i"'] \arrow[r, squiggly, bend left=60, "\beta"] & \cdot \arrow[r, "a_{i+1}"] & \cdot \arrow[r, dotted, dash] & \cdot \arrow[r, "a_1"] & \cdot \end{tikzcd}$$

Let us denote $k(Q_n)$ the vector space generated by all paths of length n, the operation $\underset{i}{\diamond}$ is given by:

$$\begin{array}{rlll} \underset{i}{\diamond}: & k(Q_n) \times k(Q_m) & \to & k(Q_{n+m-1}) \\ & (\alpha, \beta) & \mapsto & \alpha \underset{i}{\diamond} \beta = \begin{cases} a_1 \cdots a_{i-1} \beta a_{i+1} \cdots a_n & \text{if } a_i \parallel \beta \\ 0 & \text{otherwise.} \end{cases} \end{array}$$

Notation. Let α be a path in Q and (a, γ) in $Q_1 \parallel B$. Following Strametz we denote $\alpha^{(a,\gamma)}$ the element in A given by the sum of all nonzero paths (i.e. paths in B) obtained by replacing each appearance of the arrow a in α by γ. If the path α does not contain the arrow a or if every replacement of a in α is not a path in B, then $\alpha^{(a,\gamma)} = 0$. For example, let $\alpha = aba$ be a path, $\alpha^{(a,\gamma)} = ab\gamma + \gamma ba$ in case $ab\gamma$ and γba are paths in B. In general, if the expression in arrows of α is $a_1 \cdots a_i \cdots a_n$, then $\alpha^{(a,\gamma)}$ is the element of A given by

$$\alpha^{(a,\gamma)} = \sum_{i=1}^{n} \delta_{a_i}^{a} \, \chi_B(\alpha \underset{i}{\diamond} \gamma) \, \alpha \underset{i}{\diamond} \gamma$$

where $\delta_{a_i}^{a}$ is the Kronecker symbol and χ_B is the characteristic function. It is clear that α is parallel to $\alpha \underset{i}{\diamond} \gamma$ for all i. Now, let us suppose α is in a certain set of paths X. We denote $(\alpha, \alpha^{(a,\gamma)})$ the element in $k(X \parallel B)$ given by the following sum:

$$(\alpha, \alpha^{(a,\gamma)}) := \sum_{i=1}^{n} \delta_{a_i}^{a} \, \chi_B(\alpha \underset{i}{\diamond} \gamma) \, (\alpha, \alpha \underset{i}{\diamond} \gamma).$$

Now, we are able to state the proposition that gives a combinatorial description of the vector space of outer derivations $HH^1(A)$ when A is monomial.

Proposition ([**Str06**]). *Let* $A = kQ/ < Z >$ *be a monomial algebra and let* B *denote the basis of* A*, described before. The beginning of the complex induced by the Happel-Bardzell resolution can be described in the following way:*

$$0 \longrightarrow k(Q_0 \parallel B) \xrightarrow{\psi_0} k(Q_1 \parallel B) \xrightarrow{\psi_1} k(Z \parallel B) \to \cdots$$

The maps ψ_0 *and* ψ_1 *are given by*

$$\psi_0(e,\gamma) = \sum_{a \mid s(a)=e} (a, a\gamma) - \sum_{a \mid t(a)=e} (a, \gamma a) \tag{1}$$

$$\psi_1(a,\gamma) = \sum_{p \in Z} (p, p^{(a,\gamma)}) \tag{2}$$

Therefore,

$$HH^1(A) \cong \frac{\operatorname{Ker}\psi_1}{\operatorname{Im}\psi_0} \quad .$$

We will give a sketch of the proof of the above proposition. As we have wrote, the principal tool used in the computation of HH^1 is the Happel-Bardzell resolution. Since we are interested in the computation of the first Hochschild cohomology group, let us only recall the beginning of such resolution. The next proposition gives the beginning of a resolution of A as a left A^e-module where $A = kQ/ < Z >$ is any finite-dimensional monomial algebra.

Proposition ([**Bar97**]). *Let* Q *be a finite connected quiver and* $< Z >$ *be an admissible ideal of the path algebra* kQ*. Suppose* Z *is a minimal set of paths. Let* $E = kQ_0$ *be the semisimple commutative algebra generated by* Q_0*. Consider the sequence*

$$A \underset{E}{\otimes} kZ \underset{E}{\otimes} A \xrightarrow{\delta_1} A \underset{E}{\otimes} kQ_1 \underset{E}{\otimes} A \xrightarrow{\delta_0} A \underset{E}{\otimes} A \xrightarrow{\mu} A \to 0$$

given by the maps:

$$\begin{aligned} \mu(x \underset{E}{\otimes} y) &= xy \\ \delta_0(x \underset{E}{\otimes} a \underset{E}{\otimes} y) &= xa \underset{E}{\otimes} y - x \underset{E}{\otimes} ay \\ \delta_1(x \underset{E}{\otimes} p \underset{E}{\otimes} y) &= \sum_{p \in Z} xp_1 \cdots p_{i-1} \underset{E}{\otimes} p_i \underset{E}{\otimes} p_{i+1} \cdots p_n y \ . \end{aligned}$$

where x*,* y *are in* A*,* a *is in* Q_1 *and* p *is in* Z *which has an expression in arrows* $p = p_1 \cdots p_i \cdots p_n$*.*

Then the above sequence is exact at $A \underset{E}{\otimes} kQ_1 \underset{E}{\otimes} A$ *and at* $A \underset{E}{\otimes} kQ_0 \underset{E}{\otimes} A$*.*

After applying the functor $\operatorname{Hom}_{A^e}(-, A)$ to the above resolution, a complex is obtained. The cochains of this complex, $\operatorname{Hom}_{A^e}(A \underset{E}{\otimes} kX \underset{E}{\otimes} A, A)$, can be identified with $\operatorname{Hom}_{E^e}(X, A)$ using to the following lemma:

Lemma. *Let* E *be any subalgebra of* A*. Let* M *be an* $E - E$ *bimodule and* N *be an* $A - A$ *bimodule. Then*

$$\operatorname{Hom}_{A^e}(A \underset{E}{\otimes} M \underset{E}{\otimes} A, N) \cong \operatorname{Hom}_{E^e}(M, N).$$

The above lemma, which we use to simplify the complex induced by the Happel-Bardzell resolution, provides an isomorphic complex. The beginning of such complex computes the zero and the first Hochschild cohomology group.

Remark. The first Hochschild cohomology of A is the quotient of the kernel of ψ_1 by the image of ψ_0 from the complex:

$$0 \to \mathrm{Hom}_{E^e}(kQ_0, A) \xrightarrow{\psi_0} \mathrm{Hom}_{E^e}(kQ_1, A) \xrightarrow{\psi_1} \mathrm{Hom}_{E^e}(kZ, A)$$

given by the maps

$$\begin{array}{rccl} \psi_0 f: & kQ_1 & \to & A \\ & a & \mapsto & af(s(a)) - f(t(a))a \end{array} \tag{3}$$

where f is in $\mathrm{Hom}_{E^e}(kQ_0, A)$.

$$\begin{array}{rccl} \psi_1 g: & kZ & \to & A \\ & p & \mapsto & \displaystyle\sum_{i=1}^{l(p)} p_1 \cdots p_{i-1} g(p_i) p_{i+1} \cdots p_{l(p)} y \end{array} \tag{4}$$

where g is in $\mathrm{Hom}_{E^e}(kQ_1, A)$

In order to obtain the complex given by Strametz we simplify the above complex. For a monomial algebra, we identify $\mathrm{Hom}_{E^e}(kX, A)$ with $k(X \parallel B)$ where X is Q_0, Q_1 or Z. Such identification is based on the following lemma.

Lemma. *Let X and Y be sets of paths. Then*

$$\mathrm{Hom}_{E^e}(kX, kY) \cong k(X \parallel Y).$$

We provide the explicit isomorphism used to prove the lemma. Given f in $\mathrm{Hom}_{E^e}(kX, kY)$ and α a path in X, $f(\alpha)$ is a linear combination of paths of Y, i.e.

$$f(\alpha) = \sum_{\beta \in Y} \lambda_f(\alpha, \beta)\, \beta$$

where the sum is over all paths β in Y. Since f is an E-E bimodule map, then $f(\alpha) = f(t(\alpha)\alpha s(\alpha)) = t(\alpha) f(\alpha) s(\alpha)$. Hence $\lambda_f(\alpha, \beta) \neq 0$ for a path β then $\alpha \parallel \beta$. Therefore, we have the following well-defined morphism:

$$\begin{array}{rcl} \mathrm{Hom}_{E^e}(kX, kY) & \to & k(X \parallel Y) \\ f & \mapsto & \displaystyle\sum_{\alpha \in X} \sum_{\beta \in Y} \lambda_f(\alpha, \beta)\, (\alpha, \beta). \end{array}$$

Conversely, we have the following morphism:

$$\begin{array}{rcl} k(X \parallel Y) & \to & \mathrm{Hom}_{E^e}(kX, kY) \\ (\alpha, \beta) & \mapsto & \begin{array}{rccl} f_{(\alpha,\beta)}: & kX & \to & kY \\ & x & \mapsto & \delta_x^\alpha\, \beta \end{array} \end{array}$$

where δ_x^α is the Kronecker symbol and x is a path in X. The above isomorphisms enable to transcribe morphisms of $E - E$ modules into parallel arrows and vice versa. Finally we proceed to translate the differentials into the operations of replacement of arrows using the previous identifications.

1.3. Combinatorial commutator bracket.

The Lie algebra structure of the outer derivations is given by the commutator bracket. In order to study such structure using the above combinatorial presentation we translate the commutator bracket in terms of the combinatorial description of $\mathsf{HH}^1(A)$ by Strametz. In order to translate the commutator bracket she gave maps from $C^1(A,A)$ to $k(Q_1 \parallel B)$ and vice versa. Those maps induce inverse linear isomorphisms at the cohomological level, i.e. between $\mathsf{HH}^1(A)$. In fact, they are induced from some comparison maps, which are written explicitly for the first degree. This procedure enables us to give a Lie algebra structure to the cochain complex $k(Q_1 \parallel B)$.

In this section we will present the quasi-isomorphism maps and the combinatorial commutator bracket.

Given f in $\mathrm{Hom}_k(A,A)$ and a an arrow, $f(a)$ is a linear combination of paths of B, i.e.

$$f(a) = \sum_{\gamma \in B} \lambda_f(a,\gamma)\,\gamma.$$

The map from $C^1(A,A)$ to $k(Q_1 \parallel B)$ is induced by:

$$\begin{array}{ccl} C^1(A,A) & \to & k(Q_1 \parallel B) \\ f & \mapsto & \displaystyle\sum_{a \in Q_1} \sum_{(a,\gamma) \in (Q_1 \parallel B)} \lambda_f(a,\gamma)(a,\gamma). \end{array}$$

The inverse map is induced by

$$\begin{array}{ccl} k(Q_1 \parallel B) & \to & C^1(A,A) \\ & & \\ (a,\gamma) & \mapsto & F_{(a,\gamma)} : \begin{array}{ccl} A & \to & A \\ \alpha & \mapsto & \displaystyle\sum_{i=1}^{n} \delta_{a_i}^{a}\, \alpha \underset{i}{\diamond} \gamma \end{array} \end{array}$$

where $\alpha = a_1 \cdots a_n$ is an element of B. The induced isomorphisms on $\mathsf{HH}^1(A)$ given by the above maps are useful to prove the following proposition.

Theorem ([**Str06**]). *Let* $A = kQ/ < Z >$ *be a monomial algebra and let* B *be the corresponding basis of* A*. Consider the bracket*

$$[-\,,-]_S : k(Q_1 \parallel B) \times k(Q_1 \parallel B) \longrightarrow k(Q_1 \parallel B)$$

given by

$$
\begin{aligned}
[(a,\alpha)\,,\,(b,\beta)]_S &= (b,\beta^{(a,\alpha)}) - (a,\alpha^{(b,\beta)}) \\
&= \sum_{i=1}^{m} \delta^{a}_{b_i}\,\chi_B(\beta \underset{i}{\diamond} \alpha)\,(b,\beta \underset{i}{\diamond} \alpha) \\
&- \sum_{i=1}^{n} \delta^{b}_{a_i}\,\chi_B(\alpha \underset{i}{\diamond} \beta)\,(a,\alpha \underset{i}{\diamond} \beta)
\end{aligned} \tag{5}
$$

where the decomposition in arrows of α *and* β *are* $\alpha = a_1 \cdots a_n$ *and* $\beta = b_i \cdots b_m$*; the functions* δ^y_x *and* χ_B *are the Kronecker symbol and the characteristic function respectively.*

The above bracket induces a Lie algebra structure on the first Hochschild cohomology group $\mathsf{HH}^1(A) \cong \mathsf{Ker}\,\psi_1/\mathrm{Im}\,\psi_0$ *which is a Lie algebra isomorphic to* $\mathsf{HH}^1(A)$ *with the commutator bracket.*

Next we describe the derivations and inner derivations of A as an $E-E$ bimodule (see [**Str06**]). Let us introduce notation in order to state the result. We denote

$$\mathsf{Der}_{E^e}(A,A) = \mathsf{Der}_k(A,A) \cap \mathsf{Hom}_{E^e}(A,A)$$

where $\mathsf{Der}_k(A,A)$ is the Lie algebra of derivations of A. Denote

$$\mathsf{Ad}_{E^e}(A,A) = \mathsf{Ad}_k(A,A) \cap \mathsf{Hom}_{E^e}(A,A)$$

where $\mathsf{Ad}_k(A,A)$ is the Lie ideal of inner derivations of A . We have the following description of $\mathsf{Der}_{E^e}(A,A)$ and $\mathsf{Ad}_{E^e}(A,A)$.

PROPOSITION ([**Str06**]). *The Lie algebra* $\mathsf{Ker}\,\psi_1$ *(with the bracket described in the preceding theorem) and* $\mathsf{Der}_{E^e}(A,A)$ *(with the canonical bracket) are isomorphic.*

COROLLARY ([**Str06**]). *The Lie ideal* $\mathrm{Im}\,\psi_0$ *of* $\mathsf{Ker}\,\psi_1$ *and the Lie ideal* $\mathsf{Ad}_{E^e}(A,A)$ *of* $\mathsf{Der}_{E^e}(A,A)$ *are isomorphic.*

CHAPTER 2

Lie algebra structure.

In this chapter we are concerned about the Lie algebra structure of the first Hochschild cohomology group of a monomial algebra. We will determine its radical and its semisimple part in two cases. The first one is when the monomial algebra is of radical square zero. In the second case, we consider triangular and complete monomial algebras. Then, using Levi's decomposition theorem, we obtain a complete description of the Lie algebra structure of the first Hochschild cohomology of such algebras. We will assume, throughout this chapter, that the field is algebraically closed of characteristic zero. We will begin by some technical results.

2.1. Parallel arrows.

The combinatorial commutator bracket described in the chapter before induces a Lie algebra structure on the cochains $k(Q_1 \parallel B)$. Let us remark that $k(Q_1 \parallel Q_1)$ becomes a Lie subalgebra with the combinatorial commutator bracket

$$[-\,,-]_S : k(Q_1 \parallel Q_1) \times k(Q_1 \parallel Q_1) \to k(Q_1 \parallel Q_1)$$

given by

$$[\,(a, a')\,,\,(b, b')\,]_S = \delta^a_{b'}\,(b, a') - \delta^b_{a'}\,(a, b')\,.$$

We call $k(Q_1 \parallel Q_1)$ the Lie algebra of *parallel arrows.* In this section we will study both Lie algebras: $k(Q_1 \parallel Q_1)$ and $k(Q_1 \parallel B)$. The results from this section will allow us to compute the semisimple part and the radical of $HH^1(A)$ in the two cases mentioned before.

Given a quiver Q we have that $\parallel$ is an equivalence relation on the set of arrows Q_1. We denote $\overline{Q_1}$ the set of equivalence classes. It is clear that the maps source s and target t are well defined on $\overline{Q}_1$. The quiver which has Q_0 as vertices and $\overline{Q_1}$ as set of arrows, together with the maps s and t will be denoted $\overline{Q}$. Note that in the quiver $\overline{Q}$, all multiple parallel arrows of Q are identified.

For example

$$Q \qquad \rightsquigarrow \qquad \overline{Q}$$

$$\begin{tikzcd} & \cdot \arrow[dr, "n_1"] & \\ \cdot \arrow[ur, "n_2"] \arrow[rr, "n"'] & & \cdot \end{tikzcd} \qquad \begin{tikzcd} & \cdot \arrow[dr, "\alpha"] & \\ \cdot \arrow[ur, "\beta"] \arrow[rr, "\gamma"'] & & \cdot \end{tikzcd}$$

where $\alpha = \{a^{(1)}, \dots, a^{(n_1)}\}$, $\beta = \{b^{(1)}, \dots, b^{(n_2)}\}$ and $\gamma = \{c^{(1)}, \dots, c^{(n)}\}$.

We will show that the Lie algebra $k(Q_1 \parallel Q_1)$ can be expressed as a direct product of endomorphism Lie algebras $\mathfrak{gl}_\alpha$ where α is an arrow of $\overline{Q}$. Let us introduce such Lie algebras.

NOTATION. Given α in $\overline{Q}_1$ we denote

$$\mathfrak{gl}_\alpha = \bigoplus_{a,a' \in \alpha} k\,(a, a')$$

and

$$I_\alpha = \sum_{a \in \alpha} (a, a).$$

Clearly, the vector space $\mathfrak{gl}_\alpha$ together with the above bracket is a Lie subalgebra of $k(Q_1 \parallel Q_1)$. Let us show that these are endomorphism Lie algebras. Denote V_α the vector space whose basis is the set α, so $\mathrm{dim}_k(V_\alpha) = |\alpha|$. Consider $\mathrm{End}_k\,(V_\alpha)$, the Lie algebra of endomorphism of V_α with the commutator bracket. We will show that $\mathrm{End}_k\,(V_\alpha)$ and $\mathfrak{gl}_\alpha$ are isomorphic. Given $a, a' \in \alpha$, denote $f_{(a,a')} : V_\alpha \to V_\alpha$ the linear morphism given by:

$$f_{(a,a')}(\sum_{x \in \alpha} \lambda_x x) = \lambda_a a'.$$

The inverse map is given by the following:

$$\begin{array}{ccc} \mathfrak{gl}_\alpha & \to & \mathrm{End}_k\,(V_\alpha) \\ (a, a') & \mapsto & -f_{(a,a')} \end{array}$$

The minus sign in the right side is needed in order to guarantee this map to be an homomorphism of Lie algebras. Therefore, $\mathfrak{gl}_\alpha$ is isomorphic to the Lie algebra of endomorphism of V_α. We will use such Lie algebra to describe the Lie algebra structure of the parallel arrows. The description of the Lie algebra $k(Q_1 \parallel Q_1)$ and its radical is given by the following lemma.

LEMMA 2.1.1.

$$k(Q_1 \parallel Q_1) = \prod_{\alpha \in \overline{Q}_1} \mathfrak{gl}_\alpha$$

as Lie algebras. Moreover,

$$\mathrm{rad}\, k(Q_1 \parallel Q_1) = \prod_{\alpha \in \overline{Q}_1} k\, I_\alpha.$$

PROOF. If $\alpha \neq \beta$ then $\mathfrak{gl}_\alpha \cap \mathfrak{gl}_\beta = 0$ and

$$[(a, a'), (b, b')]_S = 0$$

for all (a, a') in $\mathfrak{gl}_\alpha$ and (b, b') in $\mathfrak{gl}_\beta$. Then it is easy to conclude that $k(Q_1 \parallel Q_1)$ is the product of all $\mathfrak{gl}_\alpha$. Now, recall that the radical of Lie algebras commutes with finite products, so the radical of $k(Q_1 \parallel Q_1)$ is the product of the radicals of the $\mathfrak{gl}_\alpha$'s. Since the radical of $\mathfrak{gl}_\alpha$ is $k\, I_\alpha$, we obtain that the radical of $k(Q_1 \parallel Q_1)$

is the direct product of $k\,I_\alpha$'s. For the last statement, it is enough to compute $[\,(a,a')\,,\,I_\alpha]_S$ where a, a' are in α, and $\alpha \in \overline{Q}_1$.

$$\begin{aligned} [\,(a,a')\,,\,I_\alpha]_S &= \textstyle\sum_{x\in\alpha}[\,(a,a')\,,\,(x,x)\,]_S \\ &= \textstyle\sum_{x\in\alpha}\delta^x_a(a,a') - \delta^x_{a'}(a,a') \\ &= (a,a') - (a,a') = 0 \end{aligned}$$

□

Now we are ready to study the Lie algebra $k(Q_1 \parallel B)$. Let us remark that

$$k(Q_1 \parallel B) = k(Q_1 \parallel Q_0) \oplus k(Q_1 \parallel Q_1) \oplus \bigoplus_{i=2}^{N} k(Q_1 \parallel B \cap Q_i)$$

where N is the maximum length of non-zero paths in A. If we set that the elements of $k(Q_1 \parallel B \cap Q_i)$ are of degree $i-1$, the combinatorial commutator bracket is graded. Let

$$R = \bigoplus_{i=2}^{N} k(Q_1 \parallel B \cap Q_i).$$

Lemma 2.1.2. *Let Q be a quiver without loops and consider the Lie algebra $k(Q_1 \parallel B)$. Then R is a solvable ideal. Since $k(Q_1 \parallel Q_1)$ is a subalgebra, the following decomposition of $k(Q_1 \parallel B)$ holds*

$$k(Q_1 \parallel B) = k(Q_1 \parallel Q_1) \oplus R.$$

Proof. First we prove that R is an ideal, to do so let (x,γ^n) be in $k(Q_1 \parallel B)$ and let (y,γ^m) be in R (i.e. $m \geq 2$). Using the definition of the combinatorial commutator bracket, $[(x,\gamma^n),(y,\gamma^m)]_S$ is in $k(Q_1 \parallel B \cap Q_{n+m-1})$ where $n+m-1 \geq 2$. So it is clear that R is an ideal. Let us prove that R is solvable, i.e. that its derived series $\mathscr{D}^l(R)$ vanishes for some l. Recall that if $\mathfrak{g}$ is a Lie algebra then its derived series is the sequence defined by $\mathscr{D}^0(\mathfrak{g}) = \mathfrak{g}$ and $\mathscr{D}^{l+1}(\mathfrak{g}) = [\mathscr{D}^l(\mathfrak{g})\,,\,\mathscr{D}^l(\mathfrak{g})]$. In order to prove that R is solvable, let us remark the following:

$$\mathscr{D}^1(R) = [\,R\,,\,R]_S \subseteq \bigoplus_{i=3}^{N} k(Q_1 \parallel B \cap Q_i).$$

Moreover, the derived series of R satisfies

$$\mathscr{D}^{k+1}(R) = [\mathscr{D}^k(R)\,,\,\mathscr{D}^k(R)]_S \subseteq \bigoplus_{i=i_{k+1}}^{N} k(Q_1 \parallel B \cap Q_i)$$

where $i_k \leq i_{k+1} \leq N$. Then it is clear that R is solvable since there exists k such that $B \cap Q_i$ is empty for $i > k$. Therefore $\mathscr{D}^{k+1}(R) = 0$. □

The following lemma allows to calculate the radical of $k(Q_1 \parallel B)$.

Lemma 2.1.3. *Let Q be a quiver without loops. Consider the Lie algebra $k(Q_1 \parallel B)$. Then $\mathrm{rad}\,k(Q_1 \parallel Q_1) \oplus R$ is a solvable ideal. Therefore, it belongs to $\mathrm{rad}\,k(Q_1 \parallel B)$.*

Proof. Let $I = \mathfrak{rad}\, k(Q_1 \parallel Q_1) \oplus R = \prod_{\alpha \in \overline{Q}_1} k\, I_\alpha \oplus R$. First, we will show that I is an ideal. Since we have shown that R is an ideal, it is enough to prove that $[\, k(Q_1 \parallel B)\, ,\, \prod_{\alpha \in \overline{Q}_1} k\, I_\alpha]_S$ belongs to I. Let (x, γ^n) be in $k(Q_1 \parallel B)$, we will calculate $[\,(x,\gamma^n)\, ,\, I_\alpha]_S$ for all α in $\overline{Q}_1$. For $n = 1$, this means that γ^1 is an arrow which is parallel to x. So, if $x \notin \alpha$ then it is clear that $[\,(x,\gamma^1)\, ,\, I_\alpha]_S = 0$. Now, if $x \in \alpha$ then $[\,(x,\gamma^1)\, ,\, I_\alpha]_S = (x,\gamma^1) - (x,\gamma^1) = 0$. Therefore, for all $\alpha \in \overline{Q}_1$, $[\,(x,\gamma^1)\, ,\, I_\alpha]_S = 0$. If $n \geq 2$ then we will show that $[\,(x,\gamma^n)\, ,\, I_\alpha]_S$ belongs to R. Using the combinatorial bracket definition, $[\,(x,\gamma^n)\, ,\, I_\alpha]_S$ is a multiple of (x,γ^n). Since $n \geq 2$, $[\,(x,\gamma^n)\, ,\, I_\alpha]_S$ is in R for all α in $\overline{Q}_1$. We conclude that $[\, k(Q_1 \parallel B)\, ,\, I\,]_S \subseteq R$. From this inclusion we infer that I is an ideal. Moreover, $[\, I\, ,\, I\,]_S \subseteq R$, then I is solvable since R is solvable. $\square$

Lemma 2.1.4. *Let Q be a quiver without loops. Consider the Lie algebra $k(Q_1 \parallel B)$. Then*

$$\mathfrak{rad}\, k(Q_1 \parallel B) = \mathfrak{rad}\, k(Q_1 \parallel Q_1) \oplus R$$

Proof. The above lemma gives the inclusion:

$$\mathfrak{rad}\, k(Q_1 \parallel B) \supseteq \mathfrak{rad}\, k(Q_1 \parallel Q_1) \oplus R.$$

From the Lemma (2.1.2) we know that $k(Q_1 \parallel B) = k(Q_1 \parallel Q_1) \oplus R$. Let x be in $k(Q_1 \parallel B)$, there exists y in $k(Q_1 \parallel Q_1)$ and z in R such that $x = y + z$. In order to prove the other inclusion ($\subseteq$), let us show the following: if x is in $\mathfrak{rad}\, k(Q_1 \parallel B)$ then y is in $k(Q_1 \parallel Q_1)$. Consider the projection map:

$$p : k(Q_1 \parallel B) \to k(Q_1 \parallel Q_1).$$

It is clear that p is a Lie algebra epimorphism, therefore

$$p(\mathfrak{rad}\, k(Q_1 \parallel B)) \subseteq \mathfrak{rad}\, k(Q_1 \parallel Q_1).$$

Therefore, if x is in $\mathfrak{rad}\, k(Q_1 \parallel B)$ then $y = p(x)$ is in $\mathfrak{rad}\, k(Q_1 \parallel Q_1)$. $\square$

2.2. Radical square zero.

Now, we deal with a particular case of monomial algebras: those of radical square zero. In this case, Z is the set of all paths of length two, i.e. $Z = Q_2$. The set of vertices and arrows form a basis, i.e. $B = Q_0 \cup Q_1$. In [**Cib98**], Cibils describe the Hochschild cohomology groups for such algebras using a complex which coincides with the complex induced by the Happel-Bardzell resolution.

In the next paragraph, we recall the computation of the first Hochschild cohomology group for monomial algebras of radical square zero. Let us remark that the chain complex $k(Q_0 \parallel Q_0 \cup Q_1)$ is isomorphic as a vector space to $k(Q_0 \parallel Q_0) \oplus k(Q_0 \parallel Q_1)$. Therefore, the beginning of the Happel-Bardzell complex becomes

$$0 \to k(Q_0 \parallel Q_0) \oplus k(Q_0 \parallel Q_1) \xrightarrow{\psi_0} k(Q_1 \parallel Q_0) \oplus k(Q_1 \parallel Q_1) \xrightarrow{\psi_1} k(Q_2 \parallel Q_0) \oplus k(Q_2 \parallel Q_1).$$

Moreover, for any couple (e, a) in $Q_0 \parallel Q_1$, $\psi_0(e, a) = 0$ and for any couple (a, a') in $Q_1 \parallel Q_1$, $\psi_1(a, a') = 0$. So, the differential can be restated as follows:

$$\psi_0 = \begin{pmatrix} 0 & 0 \\ D_0 & 0 \end{pmatrix}$$

$$\psi_1 = \begin{pmatrix} 0 & 0 \\ D_1 & 0 \end{pmatrix}$$

where

$$\begin{array}{rccl} D_0 : & k(Q_0 \parallel Q_0) & \to & k(Q_1 \parallel Q_1) \\ & (e, e) & \mapsto & \displaystyle\sum_{a \in Q_1 e} (a, a) - \sum_{a \in eQ_1} (a, a) \end{array} \tag{6}$$

and

$$\begin{array}{rccl} D_1 : & k(Q_1 \parallel Q_0) & \to & k(Q_2 \parallel Q_1) \\ & (a, e) & \mapsto & \displaystyle\sum_{b \in Q_1 e} (ba, b) + \sum_{b \in eQ_1} (ab, b). \end{array}$$

Remark. We already know that $HH^0(A)$ is the center of A. From the above complex, we deduce that the center of A is $\ker \psi_0$, which is equal to $k(Q_0 \parallel Q_1) \oplus \ker D_0$. Let us remark that $D_0(\sum_{e \in Q_0}(e, e)) = 0$. Then the element $\sum_{e \in Q_0}(e, e)$, which is the unit of A, is in $\ker D_0$, which is not surprising since the unity of A is always in its center.

Lemma 2.2.1. *Let Q be a quiver. The dimension of* $\operatorname{Im} D_0$ *is* $|Q_0| - 1$.

Proof. Suppose Q is a loop or a multiple loops quiver then $D_0 = 0$ and $\operatorname{Im} D_0$ has dimension 0. Moreover in this case, we obtain the result since $|Q_0| - 1 = 0$. Now, suppose Q is not a loop nor a multiple loops quiver, then we will show that $\ker D_0$ is one dimensional. Let x be in $\ker D_0$, we write $x = \sum_{e \in Q_0} \lambda_e(e, e)$ with λ_e in k. Since $D_0(x) = 0$,

$$\sum_{e \in Q_0} \lambda_e \Big(\sum_{a \in Q_1 e} (a, a) - \sum_{a \in eQ_1} (a, a) \Big)$$

is zero. Let a be an arrow such that $s(a) \neq t(a)$. Notice that the element (a, a) appears in the above linear combination with coefficient $\lambda_{s(a)} - \lambda_{t(a)}$. Therefore $\lambda_{s(a)} = \lambda_{t(a)}$ for all a such that $s(a) \neq t(a)$. We conclude that $\lambda_e = \lambda'_e$ for all e, e' in Q_0 since Q is connected. We infer that $\ker D_0$ is the vector space generated by $\sum_{e \in Q_0} e$, the unit of A. Notice that the following is an exact sequence of vector space:

$$0 \to \ker D_0 \to k(Q_0 \parallel Q_0) \xrightarrow{D_0} \operatorname{Im} D_0 \to 0$$

Finally, the vector space $\operatorname{Im} D_0$ has dimension $|Q_0| - 1$ since $\ker D_0$ is one dimensional. $\square$

In view of the above complex, in order to compute $HH^1(A)$, we have to determine the kernel of ψ_1 and the image of ψ_0. Such computation is done in three separated cases: first for the loop, then for the oriented cycle of length greater or equal two and finally for quivers that are not just an oriented cycle.

For the **loop**, let e be the vertex and a be the arrow. It is clear that $D_0 = 0$ and $D_1(a, e) = 2(a^2, a)$. Now, if $\mathsf{char} k = 2$ then $D_1 = 0$ and therefore we conclude

$$\mathsf{HH}^1(A, A) = k(a, e) \oplus k(a, a).$$

Moreover, if $\mathsf{char} k \neq 2$ the map D_1 is clearly injective so $\mathsf{Ker}\, D_1 = 0$. So $\mathsf{Ker}\, \psi_1 = k(a, a)$. Since $\mathrm{Im}\, \psi_0$ is zero,

$$\mathsf{HH}^1(A) = k(a, a).$$

For the **oriented cycle**, let $e_1, \dots e_N$ be the vertices and $a_1, \dots a_N$ be the arrows. We will suppose $N \geq 2$ and $s(a_i) = e_i$. Then $D_1 = 0$, so $\mathsf{HH}^1(A)$ is isomorphic to the quotient of $k(Q_1 \parallel Q_1)$ by $\mathrm{Im}\, D_0$. Therefore,

$$\mathsf{HH}^1(A) = \frac{\oplus_{i=1}^N k(a_i, a_i)}{< (a_i, a_i) - (a_{i-1}, a_{i-1}) >_{i=2,\dots,N}}$$

For a quiver that is **not an oriented cycle**, the map D_1 is injective, this was proven by Cibils in [**Cib98**]. Therefore,

$$\mathsf{HH}^1(A) = \frac{k(Q_1 \parallel Q_1)}{\mathrm{Im}\, D_0}.$$

Since we have obtained the explicit computation of the first Hochschild cohomology group of a monomial algebra of radical square zero, we are able to study its Lie algebra structure. First, we give some notation and some technical results.

NOTATION. Denote $\chi(Q)$ the Euler characteristic of the underlying graph of the quiver Q, i.e.

$$\chi(Q) = |Q_1| - |Q_0| + 1.$$

Let us remark that if the underlying graph of Q is a tree then $\chi(Q) = 0$.

LEMMA 2.2.2. *Let Q be a quiver that is not an oriented cycle. Consider the Lie algebra $k(Q_1 \parallel Q_1)$. Then $\mathrm{Im}\, D_0$ is an abelian ideal of $k(Q_1 \parallel Q_1)$. Therefore,*

$$\mathrm{Im}\, D_0 \subseteq \mathrm{rad}\, k(Q_1 \parallel Q_1).$$

Moreover, if the underlying graph of $\overline{Q}$ is a tree then

$$\mathrm{Im}\, D_0 = \mathrm{rad}\, k(Q_1 \parallel Q_1).$$

PROOF. Let us remark that

$$D_0(e, e) = \sum_{\alpha \in \overline{Q}_1 e} I_\alpha - \sum_{\alpha \in e\overline{Q}_1} I_\alpha.$$

In order to show that $\mathrm{Im}\, D_0$ is an ideal of $k(Q_1 \parallel Q_1)$, let us compute first $[I_\alpha\, , (x, x')]_S$. If $x \notin \alpha$ then it is clear that $[I_\alpha\, , (x, x')]_S = 0$. If $x \in \alpha$ then $[I_\alpha\, , (x, x')]_S = (x, x') - (x, x') = 0$. We conclude that $[D_0(e, e)\, , (x, x')]_S = 0$ for all $e \in Q_0$ and for all $(x, x') \in k(Q_1 \parallel Q_1)$. Therefore, $[D_0(e, e)\, , w]_S = 0$ for all $w \in k(Q_1 \parallel Q_1)$. From this computation, it is clear that $\mathrm{Im}\, D_0$ is an abelian ideal. For the last statement, recall that $\mathrm{rad}\, k(Q_1 \parallel Q_1)$ is equal to $\prod_{\alpha\in\overline{Q}} I_\alpha$, so its dimension is $|\overline{Q}_1|$. From the above remark $\mathrm{Im}\, D_0$ has dimension $|Q_0|-1$. Since $\overline{Q}$ is a tree, $|\overline{Q}_1| = |Q_0|-1$ and therefore both ideals $\mathrm{Im} D_0$ and $\mathrm{rad}\, k(Q_1 \parallel Q_1)$, are equal. □

The following lemma is a well known result from Lie algebras. We will use it as a tool to compute the radical of the Lie algebras that we are dealing with.

LEMMA 2.2.3. *Let* $\mathfrak{g}$ *be a Lie algebra and* I *be a solvable ideal. Then*

$$\mathrm{rad}\,\frac{\mathfrak{g}}{I} = \frac{\mathrm{rad}\,\mathfrak{g}}{I} \quad .$$

PROOF. Consider the canonical projection $p : \mathfrak{g} \twoheadrightarrow \mathfrak{g}/I$, therefore

$$p(\mathrm{rad}\,\mathfrak{g}) \subseteq \mathrm{rad}\,\frac{\mathfrak{g}}{I}$$

since the image of a solvable ideal is solvable. Since I belongs to $\mathrm{rad}\,\mathfrak{g}$, the image of $\mathrm{rad}\,\mathfrak{g}$ under p is:

$$p(\mathrm{rad}\,\mathfrak{g}) = \frac{\mathrm{rad}\,\mathfrak{g}}{I}$$

We conclude that

$$\frac{\mathrm{rad}\,\mathfrak{g}}{I} \subseteq \mathrm{rad}\,\frac{\mathfrak{g}}{I}.$$

In order to prove the lemma we have to prove the equality. To do so, we use the bijective correspondence between the ideals of the quotient $\mathfrak{g}/I$ and the ideals of $\mathfrak{g}$ that contain I. Let us suppose J is an ideal of $\mathfrak{g}$ which contains I such that

$$\mathrm{rad}\,\frac{\mathfrak{g}}{I} = \frac{J}{I}.$$

Clearly, J contains $\mathrm{rad}\,\mathfrak{g}$ using the above inclusion We will prove that $J = \mathrm{rad}\,\mathfrak{g}$. It is enough to see that J is solvable, since if this is true then J belongs to $\mathrm{rad}\,\mathfrak{g}$ and we obtain the result. We know that J/I is a solvable ideal of $\mathfrak{g}/I$, so $\mathscr{D}^l(J/I) = 0$ for certain l. Let us also notice that

$$\mathscr{D}^l(J/I) = \frac{\mathscr{D}^l(J) + I}{I}.$$

Then $\mathscr{D}^l(J) \subseteq I$. This implies

$$[\mathscr{D}^l(J)\, ,\, \mathscr{D}^l(J)] \subseteq [I\, ,\, I]$$

and therefore we infer that $\mathscr{D}^{l+l'}(J) = 0$ for some l'. □

REMARK. Let Q be a quiver that is not an oriented cycle. We apply the above lemma to $\mathfrak{g} = k(Q_1 \parallel Q_1)$ and to $I = \mathrm{Im}\, D_0$. Then

$$\mathrm{rad}\,\frac{k(Q_1 \parallel Q_1)}{\mathrm{Im}\, D_0} = \frac{\mathrm{rad}\, k(Q_1 \parallel Q_1)}{\mathrm{Im} D_0} \quad .$$

NOTATION. We will denote S the set of non-trivial equivalence classes:

$$S = \{\alpha \in \overline{Q}_1 \mid |\alpha| > 1\}$$

Remark. If α is in S denote $\mathfrak{sl}_{|\alpha|}(k)$ the simple Lie algebra of $|\alpha| \times |\alpha|$ matrices of trace zero. It is clear that $\mathfrak{gl}_\alpha / k\, I_\alpha$ is isomorphic as a Lie algebra to $\mathfrak{sl}_{|\alpha|}(k)$ if the characteristic of the field is zero.

Corollary 2.2.4. *Assume that the field k is algebraically closed and of characteristic zero. Let Q be a quiver which is not an oriented cycle. Then,*

$$\frac{k(Q_1 \parallel Q_1)}{\operatorname{Im} D_0} \cong \prod_{\alpha \in S} \mathfrak{sl}_{|\alpha|}(k) \times k^{\chi(\overline{Q})}$$

as Lie algebras.

Proof. First, we will compute the semisimple part. To do so we have to compute the quotient of $k(Q_1 \parallel Q_1)/\operatorname{Im} D_0$ by its radical. The radical is given by the quotient $\operatorname{rad} k(Q_1 \parallel Q_1)/\operatorname{Im} D_0$ using the above lemma. Therefore, it is clear that the semisimple part is $k(Q_1 \parallel Q_1)/\operatorname{rad} k(Q_1 \parallel Q_1)$ which is isomorphic to $\prod_{\alpha \in S} \mathfrak{sl}_{|\alpha|}(k)$. The quotient $\operatorname{rad} k(Q_1 \parallel Q_1)/\operatorname{Im} D_0$ is isomorphic to the quotient of $\prod_{\alpha \in \overline{Q}_1} k\, I_\alpha$ by $\operatorname{Im} D_0$ which is isomorphic $k^{\chi(\overline{Q})}$. The last assertion follows from the fact that $\prod_{\alpha \in \overline{Q}_1} k\, I_\alpha$ is abelian of dimension $|\overline{Q}_1|$ and $\operatorname{Im} D_0$ is of dimension $|Q_0| - 1$. Once we have the semisimple part and the radical, we apply Levi's theorem which gives us the decomposition. We will show that the product is direct. Let $\overline{x}$ be in $k(Q_1 \parallel Q_1)/\operatorname{Im} D_0$ and $\overline{y}$ be in its radical where x is in $k(Q_1 \parallel Q_1)$ and y is in $\prod_{\alpha \in \overline{Q}_1} k\, I_\alpha$. Using Lemma 2.1.1, $[x, y]_S = 0$, therefore $[\overline{x}, \overline{y}]_S = 0$. $\square$

The computation of $HH^1(A)$ and the study of the Lie algebra structure of $k(Q_1 \parallel Q_1)$ by $\operatorname{Im} D_0$ provides the following result:

Proposition 2.2.5. *Let $A = kQ/ < Q_2 >$ be a monomial algebra of radical square zero where k is an algebraically closed field of characteristic zero and Q is a finite connected quiver. Then*

$$HH^1(A) \cong \prod_{\alpha \in S} \mathfrak{sl}_{|\alpha|}(k) \times k^{\chi(\overline{Q})} \quad .$$

Therefore, $HH^1(A)$ is reductive.

Proof. First, if Q is the loop we have shown that $HH^1(A) = k(a, a)$, which is clearly isomorphic to k. Moreover, since $\chi(\overline{Q}) = 1$ and $S = \emptyset$ we obtain the above isomorphism. Second, if Q is an oriented cycle of length ≥ 2, we have proved that $HH^1(A)$ is the quotient of $\oplus_{i=1}^N k(a_i, a_i)$ by the two sided ideal generated by elements of the form $(a_i, a_i) - (a_{i-1}, a_{i-1})$, which is the image of D_0. The numerator is abelian of dimension N (where N is the length of the oriented cycle) and the denominator is of dimension $N - 1$ (see Lemma 2.2.1). Then $HH^1(A)$ is one dimensional, therefore isomorphic to k. Moreover, since $\chi(\overline{Q}) = 1$ and $S = \emptyset$ we obtain the above isomorphism for the oriented cycle. Finally, if Q is not the oriented cycle, then $HH^1(A)$ is the quotient of $k(Q_1 \parallel Q_1)$ by $\operatorname{Im} D_0$. We apply the corollary 2.2.4 to obtain the above isomorphism. $\square$

The next corollary gives Strametz' conditions for semisimplicity. We will study those conditions in the next chapter.

COROLLARY 2.2.6. *Let* $A = kQ/ < Q_2 >$ *be a monomial algebra of radical square zero. Then* $HH^1(A)$ *is semisimple if and only if* $S \neq \emptyset$ *and the quiver* $\overline{Q}$ *is a tree.*

Next, we illustrate two examples we will use later: the first example is when the quiver is one loop and the second is when the quiver is the multi-loops quiver.

EXAMPLE. The above proposition implies that $HH^1(k[x]/ < x^2 >) = k$, which is the one dimensional abelian Lie algebra.

EXAMPLE. Let Q be the multi-loops quiver: this is the quiver which has one vertex and several loops. Assume that the number of loops is greater or equal two, $\overline{Q}$ is the one loop quiver. If $A = kQ/ < Q_2 >$, the above proposition implies that $HH^1(A)$ is isomorphic to $\mathfrak{sl}_r(k) \times k \cong \mathfrak{gl}_r(k)$ where r is the number of loops and k is a field of characteristic zero and algebraically closed. Denote $HH^1(A)_{ss}$ the semisimple part of a Lie algebra, this is the quotient by its solvable radical. Then $HH^1(A)_{ss} = \mathfrak{sl}_r(k)$

2.3. Triangular complete monomial.

In this section, we will study the Lie algebra structure of the first Hochschild cohomology group of a a triangular complete monomial algebra. Let us begin by the definitions of a triangular algebra and a complete monomial algebra. Let $A = kQ/I$ be any finite dimensional algebra, so I is an admissible ideal of the path algebra kQ.

DEFINITION (Triangular algebra). If Q has no oriented cycles, we say A is *triangular algebra.*

DEFINITION (Complete monomial algebra). Let $A = kQ/ < Z >$ be a monomial algebra. We say that A is *complete* if and only if any path of length at least two which is parallel to a path in $< Z >$ is also in $< Z >$.

EXAMPLE. Radical square zero monomial algebras are complete monomial algebras.

REMARK. In [**Hap89**], Happel called "semi-commutative monomial algebras" what we call "complete monomial algebras".

We compute the first Hochschild cohomology group of triangular complete monomial algebras using the complex induced from the Bardzell-Happel resolution. Since A is triangular, the set $Q_0 \parallel B$ is in fact $Q_0 \parallel Q_0$. Since A is complete monomial then for all p in Z and (a, γ) in $Q_1 \parallel B$, $p^{(a,\gamma)}$ is in $< Z >$ by definition. Consider the map $\psi_1 : k(Q_1 \parallel B) \to k(Z \parallel B)$ from the complex induced by the Bardzell-Happel resolution, we obtain that $\psi_1(a, \gamma) = 0$ for all

(a,γ) in $Q_1 \parallel B$. Moreover, $Z \parallel B = Z \parallel Q_1$. Therefore, the complex induced by the Bardzell-Happel projective resolution is:

$$0 \longrightarrow k(Q_0 \parallel Q_0) \xrightarrow{\psi_0} k(Q_1 \parallel B) \xrightarrow{0} k(Z \parallel Q_1)$$

where the map ψ_0 is in fact the map D_0. Therefore,

$$\mathsf{HH}^1(A) = \frac{k(Q_1 \parallel B)}{\operatorname{Im} D_0}.$$

The following lemma is analogue to the corresponding lemma for radical square zero algebras. We show that the denominator of the above quotient is in the radical of the numerator.

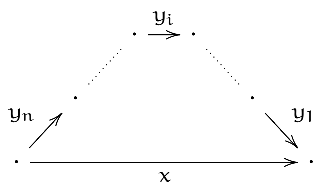
Lemma 2.3.1. *Let Q be a quiver without oriented cycles. Consider the Lie algebra $k(Q_1 \parallel B)$. Then $\operatorname{Im} D_0$ is an abelian ideal, therefore*

$$\operatorname{Im} D_0 \subseteq \operatorname{rad} k(Q_1 \parallel B).$$

Proof. Let e be in Q_0 and let (x,γ) be in $Q_1 \parallel B$, i.e the arrow x is parallel to γ, a path in B of length n. We will show that $[\,D_0(e,e)\,,\,(x,\gamma)\,]_S = 0$. Since

$$D_0(e,e) = \sum_{\alpha \in \overline{Q}_1 e} I_\alpha - \sum_{\alpha \in e\overline{Q}_1} I_\alpha.$$

we will compute $[\,I_\alpha\,,\,(x,\gamma)]_S$ for α in $\overline{Q}_1$ such that $s(\alpha) = e$ or $t(\alpha) = e$. Assume $\gamma^n = y_1 \cdots y_i \cdots y_n$. Since Q has no oriented cycles (x,γ) can be drawn as follows:



Let us denote $e_0 = t(y_1) = t(x)$, $e_i = s(y_i) = t(y_{i+1})$ for $i = 1, \ldots, n-1$ and $e_n = s(y_n) = s(x)$. It is clear that all e_i for $i = 0, \ldots, n$ are different. First, suppose e is a vertex with $e \neq e_i$ for all $i = 0, \cdots, n$. Let a be an arrow such that $s(a) = e$ (or $t(a) = e$), $[\,(a,a)\,,\,(x,\gamma)\,]_S = 0$ since a is neither x nor any y_i. Therefore $[\,I_\alpha\,,\,(x,\gamma)\,]_S = 0$ for all α in $\overline{Q}_1 e$ and for all α in $e\overline{Q}_1$. We conclude that if $e \neq e_i$ then $[\,D_0(e,e)\,,\,(x,\gamma)\,]_S = 0$. Suppose now that $e = e_i$ for some $i = 1, \ldots, n-1$. Let us remark that y_i which is in the decomposition of γ, is an arrow whose source is e. Notice also that the arrow y_{i+1} which is also in the decomposition of γ, is an arrow whose target is e. A simple calculation gives us that:

$$\begin{array}{rcl} [\,(y_i, y_i)\,,\,(x,\gamma)\,]_S & = & (x,\gamma) \\ [\,(y_{i+1}\,,\,y_{i+1}), (x,\gamma)\,]_S & = & (x,\gamma). \end{array}$$

Denote α_i the equivalence class of y_i and denote also by α_{i+1} the equivalence class of y_{i+1}. Observe that α_i is in $\overline{Q}_1 e$ while α_{i+1} is in $e\overline{Q}_1$. It is clear that $[\,I_{\alpha_i}\,,\,(x,\gamma)\,]_S = [\,I_{\alpha_{i+1}}\,,\,(x,\gamma)\,]_S = (x,\gamma)$. Now, let α be in $\overline{Q}_1$ such that $s(\alpha) = e$ (resp. $t(\alpha) = e$), but different from α_i (resp. α_{i+1}). Then $[\,I_\alpha\,,\,(x,\gamma)\,]_S = 0$

since for all a in α, a is neither x nor any y_i. We can conclude now that if $e = e_i$ for some $i = 1, \ldots, n-1$, then

$$[D_0(e,e), (x,\gamma)]_S = [I_{\alpha_i}, (x,\gamma)]_S - [I_{\alpha_{i+1}}, (x,\gamma)]_S = (x,\gamma) - (x,\gamma) = 0.$$

Finally, suppose $e = e_0$, both arrows y_n and x have source e. A simple calculation give us that:

$$\begin{aligned} [(y_n,y_n), (x,\gamma)]_S &= (x,\gamma) \\ [(x,x), (x,\gamma)]_S &= -(x,\gamma). \end{aligned}$$

Denote α_n the equivalence class of y_n and denote α_x the equivalence class of x. Both α_n and α_x are in $\overline{Q}_1 e$. It is clear that $[I_{\alpha_n}, (x,\gamma)]_S = (x,\gamma)$ and that $[I_{\alpha_x}, (x,\gamma)]_S = -(x,\gamma)$. As for the case before, for all α in $\overline{Q}_1$ such that $s(\alpha) = e$ (resp. $t(\alpha) = e$), but different from α_n and from α_x, we infer $[I_\alpha, (x,\gamma)]_S = 0$. We can conclude now that if $e = e_0$,

$$[D_0(e,e), (x,\gamma)]_S = [I_{\alpha_n}, (x,\gamma)]_S + [I_{\alpha_x}, (x,\gamma)]_S = (x,\gamma) - (x,\gamma) = 0.$$

A similar argument, give us that for $e = e_n$

$$[D_0(e,e), (x,\gamma)]_S = -[I_{\alpha_0}, (x,\gamma)]_S - [I_{\alpha_x}, (x,\gamma)]_S = -(x,\gamma) + (x,\gamma) = 0$$

where α_0 is the equivalence class of y_1. Both α_0 and α_x are in $e\overline{Q}_1$. □

LEMMA 2.3.2. *Let* Q *be a quiver without oriented cycles. Consider the Lie algebra* $k(Q_1 \parallel B)$. *Then*

$$\operatorname{rad} \frac{k(Q_1 \parallel B)}{\operatorname{Im} D_0} = \frac{\operatorname{rad} k(Q_1 \parallel B)}{\operatorname{Im} D_0} \quad .$$

PROOF. We apply Lemma 2.2.3. □

Recall that R is the solvable ideal of $k(Q_1 \parallel B)$ given by the direct sum of all $k(Q_1 \parallel Q_i \cap B)$ where i goes from 2 to N, where N is the maximum of all length of non-zero paths in A.

PROPOSITION 2.3.3. *Let* $A = kQ/ < Z >$ *be a triangular complete monomial algebra where* k *is an algebraically closed field of characteristic zero. Then*

$$HH^1(A) \cong \prod_{\alpha \in S} sl_{|\alpha|}(k) \ltimes \left(k^{\,\chi(\overline{Q})} \oplus R\right).$$

Moreover $[sl_{|\alpha|}(k), k^{\chi(\overline{Q})} \oplus R] \subseteq R$.

PROOF. As we did for the radical square case, first we compute the semisimple part. So we have to compute the quotient of $k(Q_1 \parallel B)/\operatorname{Im} D_0$ by its radical, which is $\operatorname{rad} k(Q_1 \parallel B)/\operatorname{Im} D_0$ using the above lemma. Clearly, the semisimple part is isomorphic to the quotient of $k(Q_1 \parallel B)$ by $\operatorname{rad} k(Q_1 \parallel B)$. Recall that $k(Q_1 \parallel B)$ is equal to $k(Q_1 \parallel Q_1) \oplus R$ and $\operatorname{rad} k(Q_1 \parallel B)$ is equal to $\operatorname{rad} k(Q_1 \parallel Q_1) \oplus R$. Therefore, the semisimple part of $HH^1(A)$ is $\prod_{\alpha \in S} sl_{|\alpha|}(k)$. To compute the radical of $HH^1(A)$, we have to compute the quotient of $\operatorname{rad} k(Q_1 \parallel Q_1) \oplus R$ by $\operatorname{Im} D_0$. Since $\operatorname{Im} D_0$ is an abelian ideal of $k(Q_1 \parallel Q_1)$, then the radical is precisely $k^{\chi(\overline{Q})} \oplus R$. Using the Levi decomposition theorem we obtain the result. □

Corollary 2.3.4. *Let* A *be a triangular complete monomial algebra. Then* $HH^1(A)$ *is semisimple if and only if* $\overline{Q}$ *is a tree and* S *is not an empty set.*

Definition. A is a *schurian algebra* if for any two vertices e, e' in the quiver, $\dim_k eAe' \leq 1$.

As a consequence of Proposition 2.3.3, we describe the Lie algebra structure of HH^1 when the algebra is triangular complete monomial and schurian as well.

Corollary 2.3.5. *If* A *is a triangular, schurian and complete monomial algebra,* $HH^1(A)$ *is an abelian Lie algebra of dimension* $\chi(Q)$*.*

Proof. We apply Proposition 2.3.3. Note that S is empty. Remark that $Q_1 \parallel B \cap Q_i$ is empty for $i \geq 2$ so $R = 0$. Therefore, we can conclude. $\square$

As a consequence of the above proposition, we deduce a result given in [**Cib00**]. This paper collects some computations of the dimension of HH^1 of certain quotients of path algebras. In [**Cib00**], Cibils computes the dimension of HH^1 of the quotient of a path algebra of a narrow quiver by an admissible ideal. Recall that a quiver Q is said to be *narrow* if and only if for any two vertices e and e' there is at most one path from e to e'.

More precisely, let Q be a narrow quiver without oriented cycles and let Z be a set of paths. Then $HH^1(kQ/ < Z >)$ is the abelian Lie algebra given by the direct product of $\chi(Q)$ copies of the field.

CHAPTER 3

Semisimplicity and vanishing Hochschild cohomology.

In [**Str06**], Strametz gave necessarily and sufficient conditions for the semisimplicity of $\mathsf{HH}^1(A)$ when A is a monomial algebra. If we assume that the field is algebraically closed and of characteristic zero, her theorem states that $\mathsf{HH}^1(A)$ is semisimple if and only if the following conditions are satisfied:

- the underlying graph of the quiver $\overline{Q}$ is a tree,
- there exists a non trivial class in $\overline{Q}_1$ and
- the ideal $< Z >$ is completely saturated.

The aim of this chapter is to prove the following result: let A be a monomial algebra with $\mathsf{HH}^1(A)$ semisimple. Then $\mathsf{HH}^n(A) = 0$ for all $n \geq 2$. To do so we will use the above stated conditions for semisimplicity. We begin recalling the definition of completely saturated ideal, and we prove that under the assumption that the underlying graph of the quiver $\overline{Q}$ is a tree, the completely saturated condition is equivalent to being closed under parallel paths. Therefore, we are able to restate Strametz's conditions. In the second section of this chapter, we provide another proof of Strametz's theorem. Then in the third section we prove the main result of the chapter. We will proceed as follows: we assume the above conditions in order to compute the complex from the Bardzell-Happel resolution. Then we show that the Hochschild cohomology groups are zero from degree two.

3.1. Completely saturated condition.

Definition (Completely Saturated). Let $a \parallel b$ be two parallel arrows. We say a and b are equivalent if $p^{(a,b)} = 0 = p^{(b,a)}$ for all p in Z. The ideal $< Z >$ is called *completely saturated* if all parallel arrows are equivalent.

The next lemma gives technical condition to determine whether an ideal generated by paths is completely saturated.

Lemma 3.1.1. *An ideal $< Z >$ is completely saturated if and only if for any path p in Z the following condition is verified: for each arrow p_i in the expression of p and for any arrow a parallel to p_i, the path $p \underset{i}{\diamond} a$ is in $< Z >$.*

Proof. Let $p = p_1 \cdots p_i \cdots p_n$ be a path in Z and a, b two parallel arrows. Recall that the element $p^{(a,b)}$ in A is given as follows:

$$p^{(a,b)} = \sum_{i=1}^{n} \delta_{p_i}^{a} \chi_B(p \underset{i}{\diamond} b) p \underset{i}{\diamond} b$$

where $\delta_{p_i}^{a}$ is the Kronecker delta and χ_B is the characteristic function. Remark that $p^(a, a) = 0$ for all a in Q_1. Moreover, suppose a is not parallel to any arrow

p_i then $p \underset{i}{\diamond} a = 0$ for all i and therefore $p^{(a,-)} = 0$. Now, suppose $a \neq b$ and that a is parallel to some arrow that composes the path p. In this case, we will show that all paths which are summands of $p^{(a,b)}$ are different. Let i, j be two natural numbers from 1 to n such that $p_i = a = p_j$. If $p \underset{i}{\diamond} b = p_1 \cdots a \cdots p_n = p \underset{j}{\diamond} a$ then $i = j$ since $a \neq b$. Therefore all paths $p \underset{i}{\diamond} b$ in the above sum are different.

($\Rightarrow$) Let a be an arrow such that $a \parallel p_i$ for some i. If $a = p_i$ then $p \underset{i}{\diamond} a = p$, which is clearly in $< Z >$. Suppose $a \neq p_i$. By hypothesis, all couples of parallel arrows are equivalent, in particular (p_i, a). So

$$0 = p^{(p_i,a)} = \sum_{j=1}^{n} \delta_{p_j}^{p_i} \chi_B(p \underset{j}{\diamond} a) p \underset{j}{\diamond} a.$$

We have shown that all paths in the right side of the formula are different. Therefore since $\chi_B(p \underset{j}{\diamond} a)\, p \underset{j}{\diamond} a = 0$ for all j. We conclude that $p \underset{i}{\diamond} a$ is in $< Z >$.

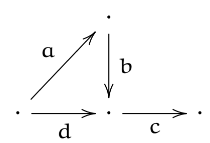
($\Leftarrow$) Let $a \parallel b$, let us show that they are equivalent, i.e. $p^{(a,b)} = 0 = p^{(b,a)}$. At the above formula, it is clear that we are only interested in arrows p_i which are equal to a or b. Consider these p_i's, evidently p_i is parallel to a and to b. By hypothesis, the paths $p \underset{i}{\diamond} b$ and $p \underset{i}{\diamond} a$ are in $< Z >$ for all p_i that is equal to a or to b. Then $\chi_B(p \underset{i}{\diamond} a) = 0 = \chi_B(p \underset{i}{\diamond} b)$. Using the above formula we deduce that $p^{(a,b)} = 0$ and $p^{(b,a)} = 0$. □

Lemma 3.1.2. *Let Q be a quiver and Z be a minimal set of paths of length at least two. If $A = kQ/ < Z >$ is a complete monomial algebra then $< Z >$ is completely saturated.*

Proof. Let p be in Z, p_i be an arrow in the expression of p, and a be an arrow parallel to p_i. Since A is a complete monomial algebra and $p \parallel p \diamond a_i$, $p \diamond a_i$ is in $< Z >$. By the above Lemma we conclude that $< Z >$ is completely saturated. □

Definition (Closed under parallel paths). A set of paths Z is called *closed under parallel paths* if and only if for any path p in Z the following condition is verified: if q is a parallel path to p then q is in Z.

Example. Let Q be the following quiver:



1) If $Z = \{cb\}$ then Z is clearly closed under parallel paths. The algebra $A = kQ/ < Z >$ is not complete monomial since cba is in $< Z >$ but cd which is parallel to cba is not in $< Z >$.
2) If $Z = Q_2$ then $A = kQ/ < Z >$ is a complete monomial algebra. The set Z is not closed under parallel paths since $d \parallel ba$ but d is not in Z.

3) If $Z = \{cd, cba\}$ then Z is closed under parallel paths and $A = kQ/ < Z >$ is a complete monomial algebra.
4) If $Z = \{ba\}$ then $A = kQ/ < Z >$ is not complete monomial since cba is a path in $< Z >$ parallel to cd which is not in $< Z >$. The set Z is not closed under parallel paths.

The objective in this section is to prove that in Strametz's theorem, we can replace "completely saturated" by "closed under parallel paths". To do so, let us first show the following:

LEMMA 3.1.3. *Let Z be a minimal set of paths of length at least two. If the set Z is closed under parallel paths, then the ideal $< Z >$ is completely saturated.*

PROOF. Let p be in Z whose expression in arrows is $p_1 \cdots p_i \cdots p_n$. If an arrow a is parallel to some p_i then the path p is clearly parallel to the path obtained by replacing p_i with a, which is $p \underset{i}{\diamond} a$. Since Z is closed under parallel paths, then $p \underset{i}{\diamond} a$ is in $< Z >$ for all i. □

The converse of the above implication is not always true. For example, let $Z = Q_n$ where $n > 1$, the ideal $< Z >$ is completely saturated but Z is not necessarily closed under parallel paths. For instance, let Q be the quiver of the above example and let $Z = Q_2$. Notice that $< Z >$ is completely saturated and Z is not closed under parallel paths. If the underlying graph of $\overline{Q}$ is a tree then the converse holds. Indeed, under this assumption, parallel paths are as follows.

LEMMA 3.1.4. *Assume that the underlying graph of $\overline{Q}$ is a tree. Let $\alpha = a_1 \cdots a_n$ and $\beta = b_1 \cdots b_m$ be parallel paths. Then $n = m$ and $a_i \parallel b_i$.*

PROOF. A pair of parallel paths in Q provides a pair of parallel paths in $\overline{Q}$. Since $\overline{Q}$ is a tree, the former are equal, hence the original paths only differ by parallel arrows. □

Finally, the following proposition allows us to restate Strametz's theorem as we have explained above.

PROPOSITION 3.1.5. *Let Q be a quiver and Z be a minimal set of paths of length at least two. Assume that the underlying graph of $\overline{Q}$ is a tree. If $< Z >$ is completely saturated ideal then Z is closed under parallel paths.*

PROOF. Let $p = p_1 \cdots p_n$ be a path in Z and q be a parallel path to p. Using the above lemma, $q = q_1 \cdots q_n$ where $q_i \parallel p_i$ for all i. Since we suppose $< Z >$ is completely saturated, for each i from 1 to n and a parallel to p_i, the path $p \underset{i}{\diamond} a$ is in the ideal $< Z >$ (Lemma 3.1.1). Let us notice that it is enough to prove that for any path p in Z, the path $p \underset{i}{\diamond} a$ is actually in Z. Once we have shown this, we can set $p(0) = p$ and for $i = 1, \ldots, n$ we can set $p(i) = p(i-1) \underset{i}{\diamond} q_i$. Then each $p(i)$ will be in Z and in particular $p(n) = q$. Then we would have shown what we wanted. So let us prove that for any path p in Z, and for all $a \parallel p_i$, the path $p \underset{i}{\diamond} a$ is actually in Z: by induction on $l(p)$, the length of the path p. Let us suppose $l(p) = 2$, then $p = p_1p_2$, and let $a \parallel p_i$. Since $< Z >$ is completely saturated, $p \underset{i}{\diamond} a$ is in $< Z >$. This means that $p \underset{i}{\diamond} a = \alpha p'\beta$ where p' is in Z, α and β are paths. Then it is easy to see that α and β are trivial paths otherwise p' is in Q_1, which is not possible since $Z \cap Q_1 = \emptyset$. Now, let us suppose $l(p) = n > 2$. Let $p = p_1 \cdots p_n$ be in Z and let $a \parallel p_i$. Since $< Z >$ is completely saturated, then $p \underset{i}{\diamond} a$ is in $< Z >$, therefore $p \underset{i}{\diamond} a = \alpha p'\beta$ where p' is in Z. We have that p' is parallel to $p_{j_1} \cdots p_{j_2}$ since the underlying graph of $\overline{Q}$ is a tree. If α or β are non trivial paths, then $l(p') < n$. By the induction hypothesis $p_{j_1} \cdots p_{j_2}$ is in Z since it is parallel to p' which is in Z. Since Z is minimal this is a contradiction. Therefore, α and β are trivial paths. We obtain that $p \underset{i}{\diamond} a$ is in Z. □

COROLLARY 3.1.6. *Let Q be a quiver and Z be a minimal set of paths of length at least two. Assume that the underlying graph of $\overline{Q}$ is a tree. The following statements are equivalent:*

(1) $A = kQ/ < Z >$ *is a complete monomial algebra.*
(2) $< Z >$ *is a completely saturated ideal.*
(3) Z *is closed under parallel paths.*

PROOF. ($1 \Rightarrow 2$) see Lemma 3.1.2. ($2 \Rightarrow 3$) see the above Proposition.

($3 \Rightarrow 1$) Let p be a path in $< Z >$ and q be a path parallel to p. Assume $p = \alpha p'\beta$ where p' is in Z and α and β are paths. Since the underlying graph of $\overline{Q}$ is a tree, $q = \alpha' q' \beta'$ where $\alpha \parallel \alpha'$, $\beta \parallel \beta'$ and $p' \parallel q'$. By hypothesis, q' is in Z, therefore q is in $< Z >$. □

3.2. Semisimplicity of the first Hochschild cohomology group.

In this section, we will provide another proof of the result given by Strametz in [**Str06**] about the sufficient and necessarily conditions for the semisimplicity of $HH^1(A)$. Let us begin proving that if $HH^1(A)$ is semsimple then necessarily the underlying graph of $\overline{Q}$ is a tree.

PROPOSITION 3.2.1. *Let $A = kQ/ < Z >$ be a finite dimensional monomial algebra where k is an algebraically closed field of characteristic zero. If $HH^1(A)$ is semisimple then the underlying graph of the quiver $\overline{Q}$ is a tree.*

PROOF. Recall that $HH^1(A)$ is the quotient of the kernel of the map ψ_1 by the image of the map ψ_0. So we have a surjective Lie map: $\ker\psi_1 \twoheadrightarrow HH^1(A)$ and the left side is a Lie subalgebra of $k(Q_1 \parallel B)$. Then $\operatorname{rad}\ker\psi_1$ belongs to $\operatorname{rad} HH^1(A)$, which is zero since it is semisimple. Therefore,

$$\operatorname{rad}\ker\psi_1 \subseteq \operatorname{Im}\psi_0.$$

On the other hand, let us remark that for all a in Q_1 and p in Z, $p^{(a,a)}$ is zero. So $\psi_1(a,a) = 0$, i.e (a,a) is in $\ker\psi_1$ for all a in Q_1. Therefore, for all $\alpha \in \overline{Q}_1$, the element $I_\alpha = \sum_{a\in\alpha}(a,a)$ is also in $\ker\psi_1$. Furthermore, since

$\operatorname{rad} k(Q_1 \parallel Q_1)$ belongs to $\operatorname{rad} k(Q_1 \parallel B)$, I_α is in $\operatorname{rad} k(Q_1 \parallel B)$. Hence the set of vectors $\{I_\alpha\}_{\alpha\in\overline{Q}_1}$ belongs to $\operatorname{rad} k(Q_1 \parallel B) \cap \ker\psi_1$. Recall that

$$\operatorname{rad}\ker\psi_1 = \ker\psi_1 \cap \operatorname{rad} k(Q_1 \parallel B).$$

Then we have shown that $\{I_\alpha\}$ belongs to $\operatorname{rad}\ker\psi_1$, which belongs to $\operatorname{Im}\psi_0$. Since $\{I_\alpha\}_{\alpha\in\overline{Q}_1}$ belongs to the image of ψ_0, we deduce that $\{I_\alpha\}_{\alpha\in\overline{Q}_1}$ belongs to $\operatorname{Im} D_0$. Besides, let us remark that the set $\{I_\alpha\}_{\alpha\in\overline{Q}}$ is clearly linearly independent in the vector space $k(Q_1 \parallel B)$, therefore it is linearly independent in $\operatorname{Im} D_0$ whose dimension is $|Q_0| - 1$. For this reason, $|\overline{Q}_1| \leq |Q_0| - 1$. This means that $\chi(\overline{Q}) = |\overline{Q}_1| - |\overline{Q}_0| + 1 = 0$, so the underlying graph of the quiver $\overline{Q}$ is a tree. □

Remark. Let $M_n(k)$ be the vector space of all square matrices of size n. We denote $\operatorname{tr}(N)$ the trace of a matrix N. Consider the following exact sequence

$$0 \to k \xrightarrow{\iota} M_n(k) \xrightarrow{\rho} sl_n(k) \to 0\,.$$

where the maps are $\iota(\lambda) = \lambda\,\mathrm{Id}_n$ and $\rho(N) = N - \operatorname{tr}(N)/n\,\mathrm{Id}_n$ and Id_n is the identity matrix. Notice that ι and ρ are Lie maps. Moreover, this exact sequence of Lie algebras splits where the section is given by the inclusion.

Since $\mathfrak{gl}_\alpha \cong \operatorname{End}_k(V_\alpha)$ for all α in S and I_α corresponds to the identity in $\operatorname{End}_k(V_\alpha)$, we deduce that the following exact sequence of Lie algebras splits

$$0 \to k\,I_\alpha \to \mathfrak{gl}_\alpha \to sl_{|\alpha|}(k) \to 0\,.$$

Hence the following exact sequence of Lie algebras splits

$$0 \to \operatorname{rad} k(Q_1 \parallel Q_1) \to k(Q_1 \parallel Q_1) \to \prod_{\alpha\in S} sl_\alpha(k) \to 0\,.$$

Denote $\mathfrak{s}$ the isomorphic copy of $\prod_{\alpha\in S} sl_\alpha(k)$ in $k(Q_1 \parallel Q_1)$. For α in S, denote $\mathfrak{h}_\alpha = \left\{\sum_{a\in\alpha}\lambda_a(a,a) \mid \sum_{a\in\alpha}\lambda_a = 0\right\}$ the Lie subalgebra of $k(Q_1 \parallel Q_1)$. Clearly, it is isomorphic to the Cartan subalgebra of all diagonal matrices of trace zero of $sl_{|\alpha|}(k)$. Denote $\mathfrak{h} = \prod_{\alpha\in S}\mathfrak{h}_\alpha$. Notice that $\mathfrak{h}$ is a Cartan subalgebra of the semisimple Lie algebra $\mathfrak{s}$.

Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{g}'$ a semisimple subalgebra of $\mathfrak{g}$. If $\mathfrak{g}'$ contains a Cartan subalgebra of $\mathfrak{g}$, we called $\mathfrak{g}'$ *regular*.

Proposition 3.2.2. *Let $A = kQ/ < Z >$ be a finite dimensional monomial algebra where k is an algebraically closed field of characteristic zero. If $HH^1(A)$ is semisimple then*

$$HH^1(A) \hookrightarrow \prod_{\alpha\in S} sl_\alpha(k)$$

and $HH^1(A)$ contains the subalgebra $\mathfrak{h}$. Therefore, $HH^1(A)$ is isomorphic to a regular subalgebra of $\prod_{\alpha\in S} sl_\alpha(k)$.

Proof. First, let us remark that the complex obtained from the Bardzell-Happel resolution becomes:

$$0 \longrightarrow k(Q_0 \parallel Q_0) \xrightarrow{D_0} k(Q_1 \parallel Q_1) \xrightarrow{\psi_1} k(Z \parallel B)$$

So $\mathsf{HH}^1(A) = \ker\psi_1/\operatorname{Im} D_0$. We have that $\operatorname{Im} D_0$ is an abelian ideal of $k(Q_1 \parallel Q_1)$, therefore it is an abelian ideal of $\ker\psi_1$. Using the remark of the Lemma 2.2.3, we obtain

$$\operatorname{rad}\mathsf{HH}^1(A) = \frac{\operatorname{rad}\ker\psi_1}{\operatorname{Im} D_0}.$$

The fact that $\mathsf{HH}^1(A)$ is semisimple implies $\operatorname{rad}\mathsf{HH}^1(A) = 0$. Therefore, $\operatorname{rad}\ker\psi_1 = \operatorname{Im} D_0$. Moreover, since $\overline{Q}$ is a tree $\operatorname{Im} D_0$, which is the radical of $\ker\psi_1$, is isomorphic to $\operatorname{rad} k(Q_1 \parallel Q_1)$, using Lemma 2.2.2. Then we have the following diagram:

(7)
$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Im} D_0 \arrow[r] \arrow[d, equal, "\wr"] & \ker\psi_1 \arrow[r, "p"] \arrow[d, hook] & \mathsf{HH}^1(A) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \operatorname{rad} k(Q_1 \parallel Q_1) \arrow[r] & k(Q_1 \parallel Q_1) \arrow[r] & \prod_{\alpha\in S} \mathfrak{sl}_{|\alpha|}(k) \arrow[r] & 0
\end{tikzcd}$$

that is clearly commutative. Moreover notice that (a, a) is in $\ker\psi_1$ for all a in Q_1. Then $\mathfrak{h}_\alpha \subseteq \ker\psi_1$ for all α in $\overline{Q}_1$. Using the above commutative diagram, it is easy to see that $p(\mathfrak{h}_\alpha) = 0$ if and only if $|\alpha| = 1$. Let α in S, we conclude $\mathfrak{h}_\alpha$ is a non zero subalgebra of $\mathsf{HH}^1(A)$. Therefore $\mathfrak{h}$ is also a non trivial subalgebra. □

Remark. Let Δ be the root system associated to $\mathfrak{s}$. There is a bijection between the roots of Δ and the couples (a, a') in $Q_1 \parallel Q_1$ with $a \neq a'$ given as follows. For every α in $\overline{Q}_1$ and every couple (a, a') of different arrows in α, we associate the following linear map:

$$\begin{array}{rccc} r^\alpha_{(a,a')}: & \mathfrak{h}^* & \to & k \\ & h & \mapsto & \lambda_{(a',a')} - \lambda_{(a,a)} \end{array}$$

where $h = \sum_{x\in Q_1} \lambda_{(x,x)}(x, x)$ with $\sum_{x\in Q_1} \lambda_x = 0$. A simple computation gives us that

$$h.(a, a') = (\lambda_{(a',a')} - \lambda_{(a,a)})(a, a') = r^\alpha_{(a,a')}(a, a')$$

for all h in $\mathfrak{h}$, therefore $r^\alpha_{(a,a')}$ is a root of $\mathfrak{s}$ and $\mathfrak{s}_{r^\alpha_{(a,a')}} = k(a, a')$ denotes the root space of $\mathfrak{s}$. Hence, the Cartan decomposition of $\mathfrak{s}$ is

$$\mathfrak{s} = \mathfrak{h} \oplus \bigoplus_{r\in\Delta} \mathfrak{s}_r$$

In order to describe completely $\mathsf{HH}^1(A)$, we will study $\ker\psi_1$.

Lemma 3.2.3. *Let $A = kQ/ < Z >$ be a finite dimensional monomial algebra where k is an algebraically closed field of characteristic zero. If $\mathsf{HH}^1(A)$ is semisimple then the following conditions on $\ker\psi_1$ hold:*

(1) *If (a, a') is in $\ker\psi_1$ then (a', a) is in $\ker\psi_1$.*
(2) *If (a, a') and (a', a'') are in $\ker\psi_1$ then (a, a'') is in $\ker\psi_1$.*

Proof. Since $\mathsf{HH}^1(A)$ is a semisimple regular subalgebra of $\mathfrak{s}$, following Dynkin (see [**Dyn52**]), we have the following decomposition

$$\mathsf{HH}^1(A) = \mathfrak{h}' \oplus \bigoplus_{r\in\Delta'} \mathfrak{s}_r$$

where $\mathfrak{h}' \subseteq \mathfrak{h}$ and Δ' is a subsystem of Δ with the following properties:

(i) If r_1 is in Δ' then $-r_1$ is in Δ'.

(ii) If r_1 and r_2 are in Δ' such that $r_1 + r_2$ is in Δ then $r_1 + r_2$ is in Δ'.

Besides, using diagram (7) notice that for all (a, a') in $\ker \psi_1$ with $a \neq a'$, the couple (a, a') is a non trivial element of $\mathsf{HH}^1(A)$. Then the above decomposition of $\mathsf{HH}^1(A)$ implies that (a, a') in $\ker \psi_1$ if and only if $r^{\alpha}_{(a,a')}$ is in Δ'. Finally, it is easy to show that condition (i) becomes (1) and (ii) becomes (2). □

LEMMA 3.2.4. *Assume that the underlying graph of* $\overline{Q}$ *is a tree. Let*

$$w = \sum_{\gamma \in \overline{Q}_1} \sum_{x,y \in \gamma} \lambda_{(x,y)}(x, y)$$

be in $\ker \psi_1$. *Then* (x, y) *is in* $\ker \psi_1$ *if* $\lambda_{(x,y)} \neq 0$.

PROOF. Since w is in $\ker \psi_1$,

$$0 = \sum_{\gamma \in \overline{Q}_1} \sum_{x,y \in \gamma} \lambda_{(x,y)} \sum_{p \in Z} (p, p^{(x,y)}).$$

Let p, q be two paths in Z and (x, y) and (x', y') parallel arrows in $Q_1 \parallel Q_1$. Remark that $(p, p^{(x,y)}) = (q, q^{(x',y')})$ if and only if $p = q$ and $(x, y) = (x', y')$. From this remark, we conclude that if $\lambda_{x,y} \neq 0$ then $p^{(x,y)} = 0$ for all p in Z. Therefore $\psi_1\,(x, y) = 0$ □

PROPOSITION 3.2.5. *Let* $A = kQ/ < Z >$ *be a finite dimensional monomial algebra where* k *is an algebraically closed field of characteristic zero.* $\mathsf{HH}^1(A)$ *is semisimple if and only if*

$$\mathsf{HH}^1(A) \cong \prod_{\alpha \in S} sl_{\alpha}(k)$$

PROOF. Using the above commutative diagram (7), notice that if we prove that $\ker \psi_1 = k(Q_1 \parallel Q_1)$ then we obtain the result. Fix α in S. Let (a, a') be in $Q_1 \parallel Q_1$, with $a \neq a'$. We will show that if (a, a') is not in $\ker \psi_1$, there exists a non trivial abelian ideal of $\mathsf{HH}^1(A)$, which is a contradiction to the fact that $\mathsf{HH}^1(A)$ is semisimple. Suppose then that (a, a') is not in $\ker \psi_1$, by property (1) of the above lemma the element (a', a) is neither in $\ker \psi_1$. Denote

$$\alpha_a = \{x \in \alpha \mid (x, a) \in \ker \psi_1\},$$

$$\alpha_{a'} = \{x \in \alpha \mid (x, a') \in \ker \psi_1\}$$

and $\alpha_c = \alpha - \{\alpha_a \cup \alpha_{a'}\}$. Clearly a belongs to α_a and a' belongs to $\alpha_{a'}$. Notice that if x is in α_a or $\alpha_{a'}$ then either (a, x) is in $\ker \psi_1$ or (a', x) is in $\ker \psi_1$ by property (1) of Lemma (3.2.3). Then, by property (2) of the same lemma, the set $\alpha_a \cap \alpha_{a'}$ is empty. Moreover, let x be in α_a and let y be in $\alpha_{a'}$ then neither (x, y) nor (y, x) is in $\ker \psi_1$, otherwise (a, a') is in $\ker \psi_1$. Therefore, if

$$w = \sum_{\gamma \in \overline{Q}_1} \sum_{x,y \in \gamma} \lambda_{(x,y)}(x, y)$$

is in $\ker\psi_1$ then $\lambda_{(x,y)} = 0 = \lambda_{(y,x)}$ for all $x \in \alpha_a$ and $y \in \alpha_{a'}$ since of the above lemma. Denote

$$I_{\alpha_a} = \sum_{x\in\alpha_a}(x,x) \quad\text{and}\quad I_{\alpha_{a'}} = \sum_{x\in\alpha_{a'}}(x,x)$$

Clearly I_{α_a} and $I_{\alpha_{a'}}$ is in $\ker\psi_1$. Define

$$J = \{\lambda_a I_\alpha + \lambda_{a'} I_{\alpha_{a'}} \mid \lambda_a, \lambda_{a'} \in k\}.$$

We will prove that J is an ideal of $\ker\psi_1$. Let $z = \lambda_a I_a + \lambda_{a'} I_{a'}$ be in J. We compute $[w,z]_S$ by computing $[w, I_{\alpha_a}]_S$ and $[w, I_{\alpha_{a'}}]_S$. So,

$$\begin{array}{rcl} [w\,,\, I_\alpha]_S & = & \sum_{x,y\in\alpha}\lambda_{(x,y)}\,[\,(x,y)\,,\, I_{\alpha_a}]_S \\ & = & \sum_{x,y\in\alpha_a}\lambda_{(x,y)}[\,(x,y)\,,\, I_{\alpha_a}]_S + \sum_{x,y\in\alpha_{a'}}\lambda_{(x,y)}[\,(x,y)\,,\, I_{\alpha_a}]_S \\ & + & \sum_{x,y\in\alpha_c}\lambda_{(x,y)}[\,(x,y)\,,\, I_{\alpha_a}]_S \\ & = & \sum_{x,y\in\alpha_a}\lambda_{(x,y)}(x,y) - (y,x) = 0. \end{array}$$

A similar computation gives $[\,w\,,\, I_{\alpha_{a'}}\,] = 0$. Therefore J is an abelian ideal of $\ker\psi_1$. Using the above commutative diagram (7), $p(J) \neq 0$ otherwise $p(J) \subseteq k\,I_\alpha$ which is not possible since $p(J)$ is a vector space of dimension two. Therefore $p(J)$ is a non trivial abelian ideal of $\mathsf{HH}^1(A)$. This contradiction comes from the assumption that (a,a') is not in $\ker\psi_1$. Therefore we obtain that $k(Q_1 \parallel Q_1) = \ker\psi_1$ and we infer the result. $\square$

Lemma 3.2.6. *Assume that the underlying graph of $\overline{Q}$ is a tree. Let Z be any minimal set of paths of length at least two. The set Z is closed under parallel paths if and only if the map $\psi_1 = 0$.*

Proof. ($\Rightarrow$) Let (a,a') be in $k(Q_1 \parallel Q_1)$, We assert that $\psi_1(a,a') = 0$. Let $p = p_1 \cdots p_n$ be in Z. Since Z is closed under parallel paths, for all $a' \parallel p_i$, $p \underset{i}{\diamond} a'$ is in Z. Then we conclude that $\delta^a_{p_i}\chi_B(p \underset{i}{\diamond} a')(p, p \underset{i}{\diamond} a')$ is zero for all i. Therefore $p^{(a,a')} = 0$ for all p, so $\psi_1(a,a') = 0$ for all (a,a').

($\Leftarrow$) Let $p = p_1 \cdots p_n$ be in Z and let a be an arrow parallel to p_i. First, let us remark the following: since $\psi_1(p_i, a_i) = 0$, for any q in Z, $q^{(p_i,a_i)} = 0$. In the particular case of $q = p$, $p^{(p_i,a_i)} = 0$ which implies that $p \underset{i}{\diamond} a_i$ is in $< Z >$. Therefore, $< Z >$ is completely saturated since the condition of Lemma 3.1.1 is satisfied. Since the underlying graph of $\overline{Q}$ is a tree and Z is completely saturated, Z is closed under parallel paths using the Proposition 3.1.5. $\square$

Next, we will give another proof of Strametz's theorem.

Proposition ([**Str06**]). *Let Q be a quiver and Z a minimal set of paths. Let $A = kQ/ < Z >$ be a finite dimensional monomial algebra where k is an algebraically closed field of characteristic zero. The following conditions are equivalent:*

(1) $\mathsf{HH}^1(A)$ *is semisimple.*
(2) The underlying graph of the quiver $\overline{Q}$ is a tree, Z is closed under parallel paths and the set S is not empty.

(3) $\mathsf{HH}^1(A)$ *is isomorphic to the following non trivial product of Lie algebras*

$$\prod_{\alpha \in S} sl_{|\alpha|}(k).$$

PROOF. (1) $\Leftrightarrow$ (3) See above proposition. (3) $\Rightarrow$ (2) It is clear that S is not empty and using Proposition 3.2.1, the underlying graph of $\overline{Q}$ is a tree. So, it is enough to prove that Z is closed under parallel paths. Using the above commutative diagram (7) we obtain by hypothesis that

$$\frac{\ker \psi_1}{\operatorname{Im} D_0} \cong \frac{k(Q_1 \parallel Q_1)}{\operatorname{rad} k(Q_1 \parallel Q_1)}.$$

Again, since the underlying graph of $\overline{Q}$ is a tree, then both denominators are isomorphic, i.e $\operatorname{Im} D_0 \cong \operatorname{rad} k(Q_1 \parallel Q_1)$. Therefore, both numerators $\ker \psi_1$ and $k(Q_1 \parallel Q_1)$ have the same dimension. Since $\ker \psi_1$ belongs to $k(Q_1 \parallel Q_1)$, $\ker \psi_1 = k(Q_1 \parallel Q_1)$, then $\psi_1 = 0$. Hence, by Lemma 3.2.6, Z is closed under parallel paths.
(2) $\Rightarrow$ (3). Since the underlying graph of $\overline{Q}$ is a tree, A is triangular. We use Proposition 2.3.3 that describes $\mathsf{HH}^1(A)$ in the case where A is a triangular complete monomial algebra. Notice that $\chi(\overline{Q}) = 0$ and that $Q_1 \parallel B \cap Q_i$ is empty for $i \geq 2$ since $\overline{Q}$ is a tree. □

3.3. Vanishing of the Hochschild cohomology.

In the sequel, we will assume that the characteristic of the field is zero. Let A be a finite dimensional monomial algebra. In this section, we prove that if $\mathsf{HH}^1(A)$ is semisimple then the Hochschild cohomology groups vanish in higher degrees. In fact, we prove directly that if A is complete monomial and the underlying graph of $\overline{Q}$ is a tree then $\mathsf{HH}^n(A) = 0$ for $n \geq 2$. The principal tool is the Happel-Bardzell projective resolution. Let us recall some facts about this resolution.

In [**Hap89**], Happel provides the projectives for a minimal projective resolution of a finite dimensional k-algebra over its enveloping algebra. Then in [**Bar97**], Bardzell describes the projective modules for monomial algebras in terms of the combinatorics of A and he describes the morphisms of the resolution.

NOTATION. The Happel-Bardzell minimal projective resolution for monomial algebras given in [**Bar97**] is denoted by:

$$\mathbf{B} = \quad \cdots \to P_{n+1} \to P_n \to \cdots P_2 \to P_1 \to P_0 \xrightarrow{\mu} A \to 0.$$

The projective modules and morphisms are given explicitly in terms of the quiver and the set of paths Z. The construction is rather technical, we provide a sketch of it:

$$\begin{array}{rcl} P_0 & = & A \underset{kQ_0}{\otimes} A \\ P_1 & = & A \underset{kQ_0}{\otimes} kQ_1 \underset{kQ_0}{\otimes} A \\ P_2 & = & A \underset{kQ_0}{\otimes} kZ \underset{kQ_0}{\otimes} A \\ P_n & = & A \underset{kQ_0}{\otimes} kAP_n \underset{kQ_0}{\otimes} A \end{array}$$

where AP_n is a set of paths constructed by induction: for $n > 2$ and for each path p in Z, an *associated sequence*, $(p, r_2, \cdots, r_n)$, of $n-1$ paths in Z is given, then to each associated sequence a certain path is defined with $s(p)$ as source and $t(r_n)$ as target. Then AP_n is the collection of all those paths. The definition of "associated sequence" and the construction of paths from this associated sequences is given in [**Bar97**]. For the purpose of this thesis we just need the following property.

Lemma ([**Bar97**]). *Let p^n in AP_n. The set*

$$\mathrm{Sub}(p^n) = \{\, p^{n-1} \in AP_{n-1} \mid p^{n-1} \textit{ divides } p^n \}$$

contains two paths p_o^{n-1} and p_t^{n-1} where $s(p_o^{n-1}) = s(p^n)$ and $t(p_t^{n-1}) = t(p^n)$. Furthermore, if n is odd then $\mathrm{Sub}(p^n) = \{p_o^{n-1}, p_t^{n-1}\}$

The above lemma will enable to compute the complex obtained from the Happel-Bardzell projective resolution for monomial algebras whose first Hochschild cohomology group is semisimple.

Lemma 3.3.1. *Let $A = kQ/ < Z >$ be a monomial algebra over k a field of characteristic zero. Assume Z is closed under parallel paths and that the underlying graph of $\overline{Q}$ is a tree. Let p^n be a path in AP_n. Then any path parallel to p^n is in the ideal $< Z >$.*

Proof. The proof is by induction. For $n = 2$ the statement is true since Z is closed under parallel paths. Now, let us suppose $n > 2$. Let p^n be a path in AP_n. By the lemma of Bardzell, $p^n = Lp_o^{n-1}$ where p_o^{n-1} is in AP_{n-1}. If α is a parallel path to p^n then $\alpha = L'p'$ where $L' \parallel L$ and $p' \parallel p_o^{n-1}$ since α is obtained by replacing parallel arrows in p^n since $\overline{Q}$ is a tree. By the induction hypothesis, p' is in $< Z >$ since it is parallel to a path in AP_{n-1}. Therefore α is in $< Z >$. □

Theorem 3.3.2. *Let $A = kQ/ < Z >$ be a finite dimensional complete monomial algebra where k is a field of characteristic zero. If the underlying graph of $\overline{Q}$ is a tree then*

- $HH^0(A) = k$
- $HH^1(A) = \prod_{\alpha \in S} sl_{|\alpha|}(k)$ *and*
- $HH^n(A) = 0$ *for all* $n \geq 2$.

Proof. For A a complete monomial algebra which the underlying graph of $\overline{Q}$ is a tree, let us remark that $Z \parallel B$ is empty since Z is closed under parallel paths and the elements of B form a basis of A. In general $AP_n \parallel B$ is empty for

$n \geq 2$ using the above lemma. The complex obtained after applying the functor $\mathrm{Hom}_{A^e}(-, A)$ to the Happel-Bardzell resolution is isomorphic to the following complex:

$$0 \to k(Q_0 \parallel Q_0) \xrightarrow{\psi_0} k(Q_1 \parallel Q_1) \xrightarrow{\psi_1} 0 \to 0 \to \cdots \to 0 \to \cdots$$

We deduce that $HH^0(A) = k$ and $HH^n(A) = 0$ for $n \geq 2$. We use Proposition 2.3.3 to describe the first Hochschild cohomology group. □

Notice that under the hypothesis of the above result, if $\overline{Q} = Q$ then Q is a tree and S is clearly empty. Therefore $HH^1(A)$ is zero, which is a result from Bardzell and Marcos [**BM98**].

Corollary 3.3.3. *Let* $A = kQ/ < Z >$ *be a finite dimensional monomial algebra where* k *is a field of characteristic zero. If* $HH^1(A)$ *is semisimple then* $HH^n(A) = 0$ *for all* $n \geq 2$.

CHAPTER 4

Hochschild cohomology groups as Lie modules.

Let A be a finite dimensional monomial algebra of radical square zero, i.e. $A = kQ/ < Q_2 >$ where Q is a finite connected quiver. We will study the Lie module structure on the Hochschild cohomology groups of these algebras. Such Lie module structure is the induced by the Gerstenhaber bracket,

$$[- \, , \, -] : HH^1(A) \times HH^n(A) \to HH^n(A) \, ,$$

defined in [**Ger63**]. The principal tools in our research are the description of the Lie algebra of $HH^1(A)$ together with the combinatorial description of the Hochschild cohomology groups and the Gerstenhaber bracket. The description of the Lie algebra structure on the first Hochschild cohomology group of such algebras, has been provided in chapter two.

In the present chapter, we begin recalling the combinatorial description of the Hochschild cohomology groups and the Gerstenhaber bracket. Then we present results concerning the Lie module structure. We divide our study in three cases: the first one is when the quiver is just a loop. The second case is when the quiver is an oriented cycle but is not reduced to a loop. The last case is when the quiver is not an oriented cycle.

4.1. Combinatorial Gerstenhaber bracket.

In this section, we recall the computations of the Hochschild cohomology groups given by Cibils in [**Cib98**] and the description of the Gerstenhaber bracket given in [**SF08**]. Both descriptions are given in terms of the quiver.

The Hochschild cohomology groups have been computed from a combinatorial complex. Such complex is in fact isomorphic to the reduced complex, which is the complex induced from the reduced projective resolution. We refer the reader to the appendix A for the formulation of both. In [**Cib98**], the Hochschild cohomology groups of a radical square zero algebra are obtained from the following complex, which we denote $C^\bullet(Q)$:

$$0 \to k(Q_0 \parallel Q_0) \oplus k(Q_0 \parallel Q_1) \xrightarrow{\begin{pmatrix} 0 & 0 \\ D_0 & 0 \end{pmatrix}} k(Q_1 \parallel Q_0) \oplus k(Q_1 \parallel Q_1) \xrightarrow{\begin{pmatrix} 0 & 0 \\ D_1 & 0 \end{pmatrix}} \cdots$$

$$\cdots \, k(Q_n \parallel Q_0) \oplus k(Q_n \parallel Q_1) \xrightarrow{\begin{pmatrix} 0 & 0 \\ D_n & 0 \end{pmatrix}} k(Q_{n+1} \parallel Q_0) \oplus k(Q_{n+1} \parallel Q_1) \, \cdots$$

where the map

$$D_n : k(Q_n \parallel Q_0) \to k(Q_{n+1} \parallel Q_1)$$

is defined as follows

$$D_n(\gamma, e) = \sum_{a \in Q_1 e} (a\gamma, a) + (-1)^{n+1} \sum_{a \in eQ_1} (\gamma a, a) \tag{8}$$

where the path γ, of length n, is parallel to the vertex e, in other words γ is a cycle at vertex e.

Before we continue, let us remark that the Jacobson radical r of A is kQ_1. Moreover, the Wedderburn-Malcev decomposition of these algebras is $A = E \oplus r$ where $E = kQ_0$.

LEMMA ([**Cib98**]). *Let* A *be a monomial algebra of radical square zero. The cochain of the reduced complex,* $C^n_E(r, A) = \mathrm{Hom}_{E^e}(r^{\otimes^n_E}, A)$*, is isomorphic as a vector space to*

$$k(Q_n \parallel Q_0 \cup Q_1) = k(Q_n \parallel Q_0) \oplus k(Q_n \parallel Q_1).$$

The isomorphism is explicit. Then the differentials are translated in order to obtain $C^\bullet(Q)$ which is isomorphic to the reduced complex.

In order to compute the Hochschild cohomology groups using the complex $C^\bullet(Q)$, it is enough to compute the kernel and the image of the maps D_n. Such computation has been done in three separated cases:

- when the quiver is a loop
- when the quiver is an oriented cycle but not a loop and
- when the quiver is not an oriented cyle.

The statement of the result of such computation will be given later. Now, we will proceed to compute the Gerstenhaber bracket using this combinatorial complex. Notice that the Gerstenhaber bracket is defined on the Hochschild cohomology groups using the Hochschild complex (see apendix B for details). So we need to translate the Gerstenhaber bracket into the combinatorial complex $C^\bullet(Q)$. Since this combinatorial complex is isomorphic to the reduced complex, it is enough to compute the reduced bracket.

The *reduced bracket* is defined in [**SF08**], using the reduced complex. The exact formulation of the reduced bracket can be found in the appendix B of this thesis. In the same appendix, we show that the reduced bracket endows

$$C^{*+1}_E(r, A) = \bigoplus_{n=1}^{\infty} C^n_E(r, A)$$

with the structure of a graded Lie algebra, see Proposition B.2.3. The proof is based on the construction of two maps of complexes between the Hochschild complex and the reduced complex. The precise construction of such maps of complexes can be found in appendix A of this thesis. Furthermore, the Gerstenhaber bracket and the reduced bracket provide the same graded Lie algebra structure on $HH^{*+1}(A)$ (see Proposition B.2.5). In view of this result and since the reduced complex is isomorphic to $C^\bullet(Q)$, in order to compute the Gerstenhaber bracket we must compute the reduced bracket using the combinatorial interpretation given by the above lemma.

Definition. Let Q be a finite quiver. Let α and β be paths of length n and m respectively, given by

$$\alpha = a_1 \cdots a_i \cdots a_n \quad \text{and} \quad \beta = b_1 \cdots b_i \cdots b_m$$

where the a_i and b_j are arrows. The bilinear map

$$[-,-]_Q : k(Q_n \parallel Q_0 \cup Q_1) \times k(Q_m \parallel Q_0 \cup Q_1) \longrightarrow k(Q_{n+m-1} \parallel Q_0 \cup Q_1)$$

is defined as follows

$$\begin{aligned} [(\alpha, x), (\beta, y)]_Q &= \sum_{i=1}^{n} (-1)^{(i-1)(m-1)} (\alpha, x) \underset{i}{\circ} (\beta, y) \\ &\quad -(-1)^{(n-1)(m-1)} \sum_{i=1}^{m} (-1)^{(i-1)(n-1)} (\beta, y) \underset{i}{\circ} (\alpha, x). \end{aligned}$$

where

$$(\alpha, x) \underset{i}{\circ} (\beta, y) = \delta^{y}_{a_i} (\alpha \underset{i}{\diamond} \beta, x)$$

and

$$(\beta, y) \underset{i}{\circ} (\alpha, x) = \delta^{x}_{b_i} (\beta \underset{i}{\diamond} \alpha, y).$$

The expression δ stands for the Kronecker symbol. Recall that $\alpha \underset{i}{\diamond} \beta$ is the path obtained by replacing the arrow a_i by the path β. In the same way $\beta \underset{i}{\diamond} \alpha$ is the path obtained by replacing b_i by the path α. For more details see the definition of operation $\underset{i}{\diamond}$, given in chapter one.

Remark. If $n = m = 1$, the above formula coincides with the combinatorial commutator bracket given in [**Str06**].

We obtain the following result.

Theorem 4.1.1. *Let Q be a finite quiver. The vector space $C^{*+1}(Q)$ with the bracket $[-,-]_Q$ is a graded Lie algebra.*

Moreover, if $A = kQ/ < Q_2 >$ the graded Lie algebra $C^{*+1}_E(r, A)$ endowed with the reduced bracket is isomorphic to $C^{*+1}(Q)$ endowed with the bracket $[-,-]_Q$.

Proof. Given a quiver Q, let $A = kQ/ < Q_2 >$. Let us remark that $C^{*+1}(Q)$ is isomorphic as a vector space to $C^{*+1}(r, A)$ using the above lemma. A straightforward verification shows that the bracket $[-,-]_Q$ is the combinatorial translation of the reduced bracket. The definition of the reduced bracket can be found in the appendix B. Since $C^{*+1}(r, A)$ with the reduced bracket is a graded Lie algebra (see Proposition B.2.3), we infer that $C^{*+1}(Q)$ with $[-,-]_Q$ is a graded Lie algebra. The isomorphisms defined by Cibils induce an isomorphism of graded Lie algebras. □

The combinatorial bracket endows a graded Lie algebra structure on

$$C^{*+1}(Q) = \bigoplus_{n=1}^{\infty} k(Q_n \parallel Q_0) \oplus k(Q_n \parallel Q_1).$$

Next, we will show that the combinatorial bracket on the cohomology of the complex $\mathsf{C}^\bullet(\mathsf{Q})$ is well defined. In order to do so, we need more notation.

NOTATION. Consider the combinatorial complex $\mathsf{C}^\bullet(\mathsf{Q})$. Let $\mathsf{Z}^n(\mathsf{Q})$ be the kernel of the differential $\begin{pmatrix} 0 & 0 \\ D_n & 0 \end{pmatrix}$ and let $\mathsf{B}^{n+1}(\mathsf{Q})$ be the image of the differential $\begin{pmatrix} 0 & 0 \\ D_n & 0 \end{pmatrix}$.

LEMMA 4.1.2. *Let* $\mathsf{A} = k\mathsf{Q}/ < \mathsf{Q}_2 >$. *The maps*

$$[-\,,\,-]_{\mathsf{Q}} : \mathsf{Z}^n(\mathsf{Q}) \times \mathsf{Z}^m(\mathsf{Q}) \to \mathsf{Z}^{n+m-1}(\mathsf{Q})$$

and

$$[-\,,\,-]_{\mathsf{Q}} : \mathsf{B}^n(\mathsf{Q}) \times \mathsf{Z}^m(\mathsf{Q}) \to \mathsf{B}^{n+m-1}(\mathsf{Q})$$

are well defined. Hence we have a well defined bracket on the Hochschild cohomology groups:

$$[-\,,\,-]_{\mathsf{Q}} : \mathsf{HH}^n(\mathsf{A}) \times \mathsf{HH}^m(\mathsf{A}) \longrightarrow \mathsf{HH}^{n+m-1}(\mathsf{A})\,.$$

PROOF. Let us denote δ the differential $\begin{pmatrix} 0 & 0 \\ D_n & 0 \end{pmatrix}$. Using Lemma B.2.4, we obtain

$$\delta[\xi, \xi']_{\mathsf{Q}} = [\,\xi\,,\,\delta\xi',]_{\mathsf{Q}} + (-1)^{m-1}[\,\delta\xi\,,\,\xi]_{\mathsf{Q}}$$

where ξ and ξ' are elements of the cochains of $\mathsf{C}^\bullet(\mathsf{Q})$, since the combinatorial bracket is the translation of the reduced bracket. From this formula, clearly we infer that ξ and ξ' are two cocycles then $[\,\xi\,,\,\xi'\,]$ is a cocyle. Moreover if $\delta\xi$ is a coboundary and ξ' is a cocyle, the formula gives that $[\delta\xi\,,\,\xi'\,]$ is a co-boundary too. Therefore the bracket $[-\,,\,-]_{\mathsf{Q}}$ is well defined at the cohomology level of $\mathsf{C}^\bullet(\mathsf{Q})$, i.e. in $\mathsf{HH}^n(\mathsf{A})$. Notice that we denote $[-\,,\,-]_{\mathsf{Q}}$ the combinatorial bracket defined on the Hochschild cohomology groups. ☐

Moreover, the combinatorial bracket endows the same Lie graded algebra on the Hochschild cohomology as the Gerstenhaber bracket.

COROLLARY 4.1.3. *Let* $\mathsf{A} = k\mathsf{Q}/ < \mathsf{Q}_2 >$ *where* Q *is a finite quiver. The graded Lie algebra structure on* $\mathsf{HH}^{*+1}(\mathsf{A})$ *given by the Gerstenhaber bracket is isomorphic to the the graded Lie algebra structure induced on the cohomology of the complex* $\mathsf{C}^{*+1}(\mathsf{Q})$ *given by* $[-\,,\,-]_{\mathsf{Q}}$.

PROOF. Proposition B.2.5 from appendix B states that the graded Lie algebra $\mathsf{HH}^{*+1}(\mathsf{A})$ endowed with the Gerstenhaber bracket is isomorphic to $\mathsf{HH}^{*+1}(\mathsf{A})$ endowed with the reduced bracket. Since the graded Lie algebra $\mathsf{C}^{*+1}(\mathsf{Q})$ endowed with $[-\,,\,-]_{\mathsf{Q}}$ is isomorphic to the graded Lie algebra $\mathsf{C}^{*+1}(r, \mathsf{A})$ together with the reduced bracket, we obtain the result. ☐

The above corollary provides a combinatorial tool to study the graded Lie algebra $\mathsf{HH}^{*+1}(\mathsf{A})$ where $\mathsf{A} = k\mathsf{Q}/ < \mathsf{Q}_2 >$. In the next sections, we will study closely the Lie module structure on $\mathsf{HH}^n(\mathsf{A})$ using the above results as a principal tool.

4.2. Algebra of dual numbers.

In this section, we will assume that the quiver Q is a loop and the characteristic of the field is zero. Then, the radical square zero monomial algebra that we consider is $A = k[x]/ < x^2 >$, the algebra of the dual numbers.

We already know, from Proposition 2.2.5 of chapter two, that its first Hochschild cohomology is the field, therefore as a Lie algebra it has to be the one dimensional abelian Lie algebra. We will show that the Hochschild cohomology vector spaces of degree $n \geq 1$ are one dimensional vector spaces, and we will provide a basis of $HH^n(A)$. We know that one dimensional Lie modules over an abelian Lie algebra are given by the multiplication by some scalar in the field. We will precise this scalar for the Lie module $HH^n(A)$. To do so we will first state the result which computes the dimension of the Hochschild cohomology vector spaces.

PROPOSITION (see for instance [**Cib98**]). *Let* $A = k[x]/ < x^2 >$ *where* k *is a field of characteristic zero. Then,*

$$\begin{aligned} HH^0(A) &\cong A \\ HH^n(A) &\cong k \quad \textit{for } n \geq 1 \end{aligned}$$

The proof is based on the computation of the kernel and the image of the maps D_n, given by the equation (8), of the complex $C^\bullet(Q)$. Let us sketch the proof in [**Cib98**]. If Q is the loop quiver, we denote e the only vertex and by a the only arrow. Then, for the loop quiver, the combinatorial complex $C^\bullet(Q)$ is as follows:

$$0 \to k(e,e) \oplus k(e,a) \xrightarrow{0} k(a,e) \oplus k(a,a) \xrightarrow{\begin{pmatrix} 0 & 0 \\ D_1 & 0 \end{pmatrix}} \cdots$$

$$\cdots\ k(a^n,e) \oplus k(a^n,a) \xrightarrow{\begin{pmatrix} 0 & 0 \\ D_n & 0 \end{pmatrix}} k(a^{n+1},e) \oplus k(a^{n+1},a) \cdots$$

A short computation gives that the map $D_n = 0$ for n even and $D_n(a^n, e) = 2(a^{n+1}, a)$ for n odd. Since $\operatorname{char} k = 0$, we infer that D_n injective for n odd. Therefore for $n = 0$,

$$HH^0(A) = k(e,e) \oplus k(e,a) \cong A$$

If n is odd,

$$HH^n(A) = k(a^n, a) \cong k.$$

If $n > 0$ is even

$$HH^n(A) = \frac{k(a^n,e) \oplus k(a^n,a)}{k(a^n,a)} = k(a^n,e) \cong k.$$

PROPOSITION 4.2.1. *Let* $A = k[x]/ < x^2 >$ *where* k *is of characteristic zero. For* $n \geq 1$*, consider the map* $\varphi_n : A^{\otimes n} \to A$ *given by:*

$$\varphi_n(f_1 \otimes \cdots \otimes f_i \otimes \cdots \otimes f_n) = \begin{cases} \prod_{i=1}^n \lambda(f_i, x) & \textit{if } n \textit{ is even} \\ \prod_{i=1}^n \lambda(f_i, x)x & \textit{if } n \textit{ is odd.} \end{cases}$$

where $f_i = \lambda(f_i, x)x + \lambda(f_i, 1)$ *for* $i = 1, \cdots, n$*. Then* $HH^n(A) \cong k\,\varphi_n$

PROOF. Let Q be the loop with vertex e and arrow a, and let $A = kQ/ < Q_2 >$. Using the above description of $HH^n(A)$ for $n \geq 1$ we obtain that $HH^n(A) \cong k\,(a^n, e)$ if n is even and $HH^n(A) \cong k\,(a^n, a)$ if n is odd. Since $k(Q_n \parallel Q_0 \cup Q_1) \cong \mathrm{Hom}_{E^e}(r^{\otimes n}, A)$ and using the induced quasi-isomorphism $\mathrm{Hom}_{E^e}(r^{\otimes n}, A) \to \mathrm{Hom}_k(A^{\otimes n}, A)$ from the Appendix A, we infer that (a^n, e) corresponds to the map φ_n if n is even and that (a^n, a) corresponds to map φ_n if n is odd. $\square$

Before continue, let us remark the following.

REMARK. Let $\mathfrak{g}$ be the one dimensional abelian Lie algebra, (i.e. $\mathfrak{g} = k$). The one dimensional Lie modules of $\mathfrak{g}$ are determined by some scalar. Given c in k, denote $k_c = k$ the Lie module given by:

$$\begin{gathered} \mathfrak{g} \times k_c \to k_c \\ \lambda.\mu = c\lambda\mu \end{gathered}$$

where λ and μ are in k. Notice that $k_c \cong k_{c'}$ as Lie modules over $\mathfrak{g}$ if and only if $c = c'$. Moreover, if V is a one dimensional Lie module over $\mathfrak{g}$ then $V \cong k_c$ where c is obtained as the result of the action of the element 1 in $\mathfrak{g}$ over the basis of V. We have the following bijection

$$\begin{array}{ccc} k & \xrightarrow{1:1} & \{\text{one dimensional Lie modules over } \mathfrak{g}\}/\sim \\ c & \mapsto & k_c \end{array}$$

Since $HH^n(A)$ is a one dimensional vector space, its Lie module structure is clearly given as in the above remark. We will precise the scalar in the field that determines the Lie module structure by a simple computation of $[-\,,-]_Q$.

PROPOSITION 4.2.2. *Let $A = k[x]/ < x^2 >$ where k is of characteristic zero. For $n \geq 1$, the Lie module structure on the Hochschild cohomology groups given by Gerstenhaber bracket,*

$$HH^1(A) \times HH^n(A) \longrightarrow HH^n(A)\,,$$

is given by:

$$\varphi_1.\varphi_n = \begin{cases} -n\,\varphi_n & \text{if } n \text{ is even} \\ (1-n)\,\varphi_n & \text{if } n \text{ is odd.} \end{cases}$$

Therefore,

$$HH^{2n}(A) \cong HH^{2n+1}(A)$$

considered as Lie modules.

PROOF. Let Q be the loop with vertex e and arrow a, and let $A = kQ/ < Q_2 >$. As a consequence of corollary 4.1.3 and using the combinatorial description of $HH^n(A)$ and the bracket $[-\,,-]_Q$, we deduce that the Lie module structure on the Hochschild cohomology groups given by the Gerstenhaber bracket $HH^1(A) \times HH^n(A) \longrightarrow HH^n(A)$ is induced by the following morphisms. If n is even, $k(Q_1 \parallel Q_1) \times k(Q_n \parallel Q_0) \longrightarrow k(Q_n \parallel Q_0)$ is given as follows: $(a,a).(a^n,e) = [\,(a,a)\,,\,(a^n,e)\,]_Q = -n\,(a^n,e)$. If n is odd, $k(Q_1 \parallel Q_1) \times k(Q_n \parallel Q_1) \longrightarrow k(Q_n \parallel Q_1)$ is given as follows: $(a,a).(a^n,a) =$

$[\,(a,a)\,,\,(a^n,a)\,]_Q = (1-n)\,(a^n,a)$. Using the above proposition we know that (a^n,e) corresponds to φ_n if n is even and that (a^n,a) corresponds to φ_n if n is odd. We obtain the result. □

Denote

$$\mathsf{HH}^{\mathrm{odd}}(A) = \bigoplus_{n=0}^{\infty} \mathsf{HH}^{2n+1}(A).$$

It is clear that the Gerstenhaber bracket endows $\mathsf{HH}^{\mathrm{odd}}(A)$ with a Lie algebra structure. To describe this Lie algebra, let us introduce some notation. Denote $\mathcal{W}$ the Lie algebra of derivations of $k[x]$, i.e. $\mathcal{W} = \mathsf{Der}(k[x], k[x])$. For n a positive integer, let $\phi_n : k[x] \to k[x]$ be the derivation defined as follows: $\phi_n(x^i) = ix^{n+i-1}$ where i is a positive integer. The commutator bracket os such derivations is given by

$$[\phi_n, \phi_m] = (n-m)\phi_{n+m-1}.$$

Now, it is easy to see that any derivation on $k[x]$ is a linear combinations of ϕ_n's. Denote by W_n the vector space generated by ϕ_n. Then

$$\mathcal{W} = \bigoplus_{n=0}^{\infty} W_n\,.$$

Clearly, the commutator bracket is graded if we consider elements of W_n of degree $n-1$. We will denote

$$\mathcal{W}^{\mathrm{odd}} = \bigoplus_{n=0}^{\infty} W_{2n+1}$$

the Lie subalgebra of $\mathcal{W}$.

Proposition 4.2.3. *Let* k *be a field of characteristic zero and* $A = k[x]/ < x^2 >$ *the algebra of the dual numbers. The Lie algebra* $\mathsf{HH}^{\mathrm{odd}}(A)$ *is isomorphic to te Lie algebra* $\mathcal{W}^{\mathrm{odd}}$.

Proof. Using the formula for the bracket, we have

$$[\,(a^n,a)\,,\,(a^m,a)\,]_Q = \ (n-m)\,(a^{n+m-1},a)\,.$$

Using Proposition 4.2.1 we deduce that $[\varphi_n, \varphi_m] = (n-m)\varphi_{n+m-1}$. □

4.3. Oriented cycle.

Assume that the quiver Q is an oriented cycle of length $N \geq 2$ and the characteristic of the field is zero. In this section, we will determine the Lie module structure on $\mathsf{HH}^n(A)$ where $A = kQ/ < Q_2 >$. Using Proposition 2.2.5 of chapter two, we know that the first Hochschild cohomology is the one dimensional abelian Lie algebra. Moreover, we will show that $\mathsf{HH}^n(A)$ is either zero or a one dimensional vector space and we will provide a basis of $\mathsf{HH}^n(A)$ when it is not trivial. The Lie module structure on the non-trivial Hochschild cohomology vector spaces is determined by a scalar on the field. We will precise this scalar. As we did in previous section, we begin recalling the computation of the Hochschild cohomology vector spaces.

PROPOSITION ([**Cib98**]). *Let* $A = kQ/ < Q_2 >$ *where* Q *is the oriented cycle of length* N *with* $N \geq 2$ *and* k *is a field of characteristic zero. If* n *or* $n-1$ *is an even multiple of* N

$$HH^n(A) \cong k\,,$$

otherwise $HH^n(A)$ *is zero.*

If Q is an oriented cycle of length N, let $\{e_1, \ldots, e_N\}$ be the set of vertices and $\{a_1, \ldots, a_N\}$ be the set of arrows such that $s(a_i) = e_i$, $t(a_i) = e_{i+1}$ for $i = 1, \ldots, N-1$ and $t(a_N) = e_1 = s(a_1)$. The combinatorial complex in this case is as follows:

$$0 \longrightarrow k(Q_0 \parallel Q_0) \xrightarrow{D_0} k(Q_1 \parallel Q_1) \xrightarrow{0} \cdots$$

$$\cdots \xrightarrow{0} k(Q_N \parallel Q_0) \xrightarrow{D_N} k(Q_{N+1} \parallel Q_1) \xrightarrow{0} \cdots$$

$$\cdots \xrightarrow{0} k(Q_{cN} \parallel Q_0) \xrightarrow{D_{cN}} k(Q_{cN+1} \parallel Q_1) \xrightarrow{0} \cdots$$

where $c \geq 0$ is a positive integer. Let us notice that if $n \neq cN$ and $n \neq cN+1$, clearly $HH^n(A) = 0$ since $Q_n \parallel Q_0$ and $Q_{n+1} \parallel Q_1$ are empty sets. Now, the others cohomology groups are:

$$HH^{cN}(A) = \ker D_{cN} \quad \text{and} \quad HH^{cN+1}(A) = \frac{k(Q_{cN+1} \parallel Q_1)}{\operatorname{Im} D_{cN}}$$

where c is a positive integer. In order to compute $HH^{cN}(A)$ and $HH^{cN+1}(A)$ it is enough to compute the kernel and the image of D_{cN}. In [**Cib98**], the following is proved:

- if cN is odd then D_{cN} is injective, therefore both cohomology groups $HH^{cN}(A)$ and $HH^{cN+1}(A)$ are zero.
- if cN is even then D_{cN} has a one dimensional kernel, therefore both cohomology groups $HH^{cN}(A)$ and $HH^{cN+1}(A)$ are one dimensional.

We introduce the following notation in order to state the computation of the kernel and the image of the map D_{cN}.

NOTATION. For $i = 1, \cdots, N$ we denote γ_{e_i} the only oriented cycle of length N that starts and ends at vertex e_i and and $\gamma_{e_i}^c$ denotes the composition of the path γ_{e_i} with itself c times. If $c = 0$ we set $\gamma_{e_i}^c = e_i$.

LEMMA 4.3.1. *Let* Q *be the oriented cycle of length* N *with* $N \geq 2$ *and* k *is a field of characteristic zero. For* $c \geq 0$ *consider the map*

$$D_{cN} : k(Q_{cN} \parallel Q_0) \longrightarrow k(Q_{cN+1} \parallel Q_1)$$

given by (8). If cN *is an even multiple of* N *then*

$$\ker D_{cN} = k \sum_{i=1}^{N} (\gamma_{e_i}^c, e_i) \quad \textit{and} \quad \frac{k(Q_{cN+1} \parallel Q_1)}{\operatorname{Im} D_N} \cong k(\gamma_{e_2}^c a_1, a_1).$$

Proof. Notice that $Q_{cN} \parallel Q_0$ consists of all $(\gamma^c_{e_i}, e_i)$ and $Q_{cN+1} \parallel Q_1$ consists of all $(\gamma^c_{t(a_i)}a_i, a_i)$. For $i = 1, \cdots, N$, we denote b_i the the only arrow such that $t(b_i) = e_i$, so $s(a_i) = e_i = t(b_i)$. If cN is even then D_{cN} is given by

$$D_{cN}(\gamma^c_{e_i}, e_i) = (a_i\gamma^c_{e_i}, a_i) - (\gamma^c_{e_i} b_i, b_i).$$

Let us remark that $(\gamma^c_{e_i} b_i, b_i) = (b_i\gamma^c_{s(b_i)}, b_i)$. Let x be in $\ker D_{cN}$, suppose $x = \sum_{i=1}^N \lambda_{e_i}(\gamma^c_{e_i}, e_i)$. Since $D_{cN}(x) = 0$, the linear combination given by

$$\sum_{i=1}^{N} \lambda_{e_i} \Big((a_i\gamma_{e_i}, a_i) - (b_i\gamma^c_{s(b_i)}, b_i) \Big)$$

is zero. Notice that the element $(b_i\gamma^c_{s(b_i)}, b_i)$ appears in the above linear combination with coefficient $\lambda_{s(b_i)} - \lambda_{e_i}$, which is zero. Therefore for all arrow b_i, $\lambda_{s(b_i)} = \lambda_{t(b_i)}$. Since Q is connected, we deduce that $\lambda_{e_i} = \lambda_{e_j}$ for all i, j. We obtain that $\ker D_{cN}$ is generated by the sum of all $(\gamma^c_{e_i}, e_i)$. Now, the vector space $\operatorname{Im} D_{cN}$ has dimension $|Q_{cN+1} \parallel Q_1| - 1 = |Q_1| - 1 = |Q_0| - 1$. Notice that if w is an element in $k(Q_{cN+1} \parallel Q_1)$, then it can be written as follows:

$$w = \sum_{i=1}^{N} \lambda_{a_i}(\gamma^c_{t(a_i)}a_i, a_i) = \operatorname{tr}_1(w)(\gamma^c_{e_2}a_1, a_1) + \sum_{i=2}^{N} \operatorname{tr}_i(w) D_{cN}(\gamma^c_{e_i}, e_i)$$

where $\operatorname{tr}_i(w) = \sum_{j=i}^N \lambda_{a_j}$. We deduce that $\{D_{cN}(\gamma^c_{e_i}, e_i)\}_{i=2}^N$ is a basis of $\operatorname{Im} D_{cN}$ and in order to complete a basis for $k(Q_{cN+1} \parallel Q_1)$ we include $\{(\gamma^c_2 a_1, a_1)\}$. Therefore we obtain the last statement. $\square$

The above lemma provides a basis of $HH^n(A)$ in terms of the combinatorics of the quiver. We will give the linear map in $\operatorname{Hom}_k(A^{\otimes}, A)$ that corresponds to this basis. To do so, let us introduce some notation.

Notation. For $i = 1, \cdots, N$ and $cN > 0$ a positive multiple of N, we denote

$$\sigma_i : \{1, \ldots, cN\} \to \{1, \cdots, N\}$$

the periodic function with period N (i.e. $\sigma_i(j) = \sigma_i(j+N)$) such that σ_i restricted to $\{1, \ldots, N\}$ is the following cyclic permutation:

- if $i = 1$ then $\sigma_1(j) = j$ for $j = 1, \ldots, N$;
- if $i = N$ then $\sigma_N(1) = N$ and $\sigma_N(j) = j - 1$ for $j = 2, \ldots N$,
- if $1 < i < N$ then $\sigma_i(j) = i + (j - 1)$ for $j = 1, \ldots, N - i + 1$ and $\sigma_i(j) = (j - 1) - (N - i)$ for $j = N - i + 2, \ldots N$.

Example. If $N = 2$ then for $j = 0, \ldots, c - 1$:

$$\sigma_1(2j + 1) = 1\,,\ \sigma_1(2j + 2) = 2\,,\ \sigma_2(2j + 1) = 2\,,\ \sigma_2(2j + 2) = 1.$$

If $N = 3$ then $\sigma_1|_{\{1,2,3\}} = (1)$, $\sigma_2|_{\{1,2,3\}} = (123)$ and $\sigma_3|_{\{1,2,3\}} = (132)$. Therefore,

$$\begin{aligned}
&\sigma_1(3j + 1) = 1, \quad \sigma_1(3j + 2) = 2, \quad \sigma_1(3j + 3) = 3,\\
&\sigma_2(3j + 1) = 2, \quad \sigma_2(3j + 2) = 3, \quad \sigma_2(3j + 3) = 1,\\
&\sigma_3(3j + 1) = 3, \quad \sigma_3(3j + 2) = 1, \quad \sigma_3(3j + 3) = 2,
\end{aligned}$$

for $j = 0, \ldots, c - 1$.

An element f in A is written as a linear combination

$$f = \sum_{i=1}^{N} \lambda(f, e_i)e_i + \lambda(f, a_i)a_i$$

where $e_1, \ldots, e_N$ are the vertices and $a_1, \ldots, a_N$ are the arrows of Q.

NOTATION. For $i = 1, \cdots, N$ and $cN > 0$ a positive multiple of N we denote

$$\pi_i : A^{\otimes cN} \to k$$

the linear map given by

$$\pi_i(f_1 \otimes \cdots \otimes f_{cN}) = \prod_{j=1}^{cN} \lambda(f_j, a_{\sigma_i(j)}).$$

We show the following result.

PROPOSITION 4.3.2. *Let* $A = kQ/ < Q_2 >$ *where* k *is a field of characteristic zero and* Q *is the oriented cycle of length* N *with* $N \geq 2$. *Consider the map* $\varphi_1 : A \to A$ *given by*

$$\varphi_1(f) = \lambda(f, a_1)a_1$$

Then $HH^1(A) \cong k\,\varphi_1$.

For $n \geq 1$, *a multiple of* N, *consider the map* $\varphi_n : A^{\otimes n} \to A$ *given by*

$$\varphi_n(f_1 \otimes \cdots \otimes f_n) = \sum_{i=1}^{N} \pi_i(f_1 \otimes \cdots \otimes f_n)e_i$$

and the map $\varphi_{n+1} : A^{\otimes n+1} \to A$ *given by*

$$\varphi_{n+1}(f_1 \otimes \cdots \otimes f_{n+1}) = \pi_1(f_1 \otimes \cdots \otimes f_n)\lambda(f_{n+1}, a_1)a_1\,.$$

Then $HH^n(A) \cong k\,\varphi_n$ *and* $HH^{n+1}(A) \cong k\,\varphi_{n+1}$ *if* n *is even and* $HH^n(A)$ *is zero otherwise.*

PROOF. The vector space $HH^n(A)$ is trivial except if either n or $n-1$ is a multiple of N and this multiple is even. We use the above lemma to obtain a basis of $HH^{cN}(A)$ and $HH^{cN+1}(A)$ when cN is even. The basis is given in terms of the quiver. Using the isomorphism $k(Q_{cN} \parallel Q_0) \cong \mathrm{Hom}_{E^e}(r^{\otimes cN}, A)$, the element $(\gamma^c_{e_i}, e_i)$ corresponds to the linear map $\pi'_i : r^{\otimes cN} \to A$ given by $\pi'_i(f_1 \otimes \cdots \otimes f_{cN}) = \pi_i(f_1 \otimes \cdots \otimes f_{cN})e_i$. Using the above lemma, the basis of $HH^{cN}(A)$ is given by the sum of all π'_i's. Using the induced quasi-isomorphism from the Appendix A we infer that such sum corresponds to φ_{cN}. Besides, recall that $k(Q_{cN+1} \parallel Q_1) \cong \mathrm{Hom}_{E^e}(r^{\otimes cN+1}, A)$. The element $(\gamma^c_{e_i}a_i, a_i)$ corresponds to the linear map $\pi''_i : r^{\otimes cN+1} \to A$ given by $\pi''_i(f_1 \otimes \cdots \otimes f_{cN} \otimes f_{cN+1}) = \pi_i(f_1 \otimes \cdots \otimes f_{cN})\lambda(f_{cN+1}, a_i)a_i$. Using the above lemma, the basis of $HH^{cN+1}(A)$ is given by $(\gamma^c_2 a_1, a_1)$ which corresponds to the map π''_1 in $\mathrm{Hom}_{E^e}(r^{\otimes cN+1}, A)$. Using the induced quasi-isomorphism from the Appendix A we infer that π''_1 corresponds φ_{cN}. Using the description of the Hochschild cohomology groups we obtain the last statement. $\square$

We conclude that $\mathsf{HH}^n(A)$ is either zero or a one dimensional vector space. For $\mathsf{HH}^n(A)$ which is not zero, we point out that the Lie module structure on $\mathsf{HH}^n(A)$ is indeed given by the multiplication by some scalar in the field. For each n, we precise this scalar in the following proposition.

PROPOSITION 4.3.3. *Let $A = kQ/ < Q_2 >$ where Q is an oriented cycle of length N with $N \geq 2$ and k is a field of characteristic zero. For $n \geq 1$, the Lie module structure on the Hochschild cohomology groups, given by the Gerstenhaber bracket,*

$$\mathsf{HH}^1(A) \times \mathsf{HH}^n(A) \longrightarrow \mathsf{HH}^n(A)$$

is given as follows

$$\varphi_1.\varphi_n = -c\,\varphi_n$$

where c is a positive integer such that cN is even and either $n = cN$ or $n = cN+1$ and c is zero otherwise.

Therefore, for all positive integer c, $\mathsf{HH}^{cN}(A) \cong \mathsf{HH}^{cN+1}(A)$ as Lie modules.

PROOF. From corollary 4.1.3, in order to determine the Lie module structure, we use the combinatorial description of $\mathsf{HH}^n(A)$ and we compute the bracket $[-,-]_Q$. Suppose $cN > 0$ is an even multiple of N. The Lie module structure on $\mathsf{HH}^{cN}(A)$ given by the Gerstenhaber bracket is induced by the following morphism $k(Q_1 \parallel Q_1) \times k(Q_{cN} \parallel Q_0) \longrightarrow k(Q_{cN} \parallel Q_0)$ given as follows: $(a_i, a_i).(\gamma^c_{e_j}, e_j) = [(a_i, a_i), (\gamma^c_{e_j}, e_j)]_Q$. A simple computation of the combinatorial bracket gives $[(a_i, a_i), (\gamma^c_{e_j}, e_j)]_Q = -c(\gamma^c_{e_j}, e_j)$. Using the above proposition, we know that φ_1 corresponds to (a_1, a_1) and φ_{cN} to the sum of all $(\gamma^c_{e_j}, e_j)$, then we obtain that $[\varphi_1, \varphi_{cN}] = -c\,\varphi_{cN}$. We obtain the result for $\mathsf{HH}^{cN}(A)$. Likewise, the Lie module structure on $\mathsf{HH}^{cN+1}(A)$ is induced by the morphism $k(Q_1 \parallel Q_1) \times k(Q_n \parallel Q_1) \longrightarrow k(Q_n \parallel Q_1)$ given as follows: $(a_i, a_i).(\gamma^c_{t(a_j)}a_j, a_j) = [(a_i, a_i), (\gamma^c_{t(a_j)}a_j, a_j)]_Q$. As we did before, a simple computation of the combinatorial bracket gives: $[(a_i, a_i), (\gamma^c_{t(a_j)}a_j, a_j)]_Q = -c(\gamma^c_{t(a_j)}a_j, a_j)$. Using the above proposition (a_1, a_1) corresponds to φ_1 and $(\gamma^c_2 a_1, a_1)$ to φ_{cN+1} and then we obtain that $[\varphi_1, \varphi_{cN+1}] = -c\,\varphi_{cN+1}$. □

Recall that we denote $\mathcal{W}$ the Lie algebra $\mathsf{Der}(k[x], k[x])$, see the previous section. We denote $\mathcal{W}^*$ the Lie subalgebra of $\mathcal{W}$ given as follows:

$$\mathcal{W}^* = \bigoplus_{n=0}^{\infty} W_{n+1}.$$

We have the following proposition:

PROPOSITION 4.3.4. *Let $A = kQ/ < Q_2 >$ where Q is the oriented cycle of length N with $N \geq 2$ and k is a field of characteristic zero. The Lie algebra $\mathsf{HH}^{\mathrm{odd}}(A)$ is isomorphic to $\mathcal{W}^*$.*

PROOF. If N is even,

$$\mathsf{HH}^{\mathrm{odd}}(A) = \bigoplus_{c=0}^{\infty} \mathsf{HH}^{cN+1}(A)$$

and if N is odd,

$$\mathrm{HH}^{\mathrm{odd}}(A) = \bigoplus_{c=0}^{\infty} \mathrm{HH}^{2cN+1}(A).$$

Define $L_c = \mathrm{HH}^{cN+1}(A)$ if N is even and $L_c = \mathrm{HH}^{2cN+1}(A)$ if N is odd. L_c is a one dimensional vector space and $[L_c, L'_c]_Q \subseteq L_{c+c'}$. We assume that N is even without lost of generality. To determine the Lie algebra structure on $\mathrm{HH}^{\mathrm{odd}}(A)$, we need to compute the Gerstenhaber bracket

$$k(Q_{cN+1} \parallel Q_1) \times k(Q_{c'N+1} \parallel Q_1) \longrightarrow k(Q_{(c+c')N+1} \parallel Q_1).$$

We have that $[\,(\gamma^{c}_{t(a_j)} a_j, a_j)\,,\,(\gamma^{c'}_{t(a_i)} a_i, a_i)\,]_Q = \,c\,(\gamma^{c+c'}_{t(a_j)} a_j, a_j) - c'\,(\gamma^{c+c'}_{t(a_i)} a_i, a_i)$. Let us remark that if $i = j$ then

$$[\,(\gamma^{c}_{t(a_j)} a_i, a_i)\,,\,(\gamma^{c'}_{t(a_i)} a_i, a_i)\,]_Q = \,(c - c')\,(\gamma^{c+c'}_{t(a_j)} a_i, a_i)\,.$$

Using Proposition 4.3.2 we deduce the statement. □

4.4. Quivers different from an oriented cycle.

In this section, we consider $A = kQ/ < Q_2 >$ where Q is not an oriented cycle. We will like to emphasize that the quiver can have oriented cyles but it cannot be reduced to an oriented cycle. Let us recall the description of the Hochschild cohomology groups of such algebras. In [**Cib98**], the description is given using the combinatorial complex $C^{\bullet}(Q)$. The map D_n, given by (8), is injective for $n \geq 1$ when Q is not an oriented cycle. Therefore, for $n > 1$ the kernel of the differential

$$k(Q_n \parallel Q_0) \oplus k(Q_n \parallel Q_1) \xrightarrow{\begin{pmatrix} 0 & 0 \\ D_n & 0 \end{pmatrix}} k(Q_{n+1} \parallel Q_0) \oplus k(Q_{n+1} \parallel Q_1)$$

is actually $k(Q_n \parallel Q_1)$ which is called the *space of shortcuts*. Moreover, $\mathrm{Im}\, D_{n-1}$ is isomorphic to $k(Q_{n-1} \parallel Q_0)$ *the space of pointed oriented cycles*. The statement is as follows:

Theorem ([**Cib98**]). *Let $A = kQ/ < Q_2 >$ where Q is not an oriented cycle. Then, if $n \geq 1$*

$$\mathrm{HH}^n(A) \cong \frac{k(Q_n \parallel Q_1)}{\mathrm{Im}\, D_{n-1}}$$

where

$$D_{n-1} : k(Q_{n-1} \parallel Q_0) \longrightarrow k(Q_n \parallel Q_1)$$

is the linear map given by (8). Moreover, if $n > 1$

$$\dim_k \mathrm{HH}^n(A) = |Q_n \parallel Q_1| - |Q_{n-1} \parallel Q_0|.$$

The combinatorial description of both the Hochschild cohomology groups and the Gerstenhaber bracket provides the following result:

THEOREM 4.4.1. *Let* $A = kQ/ < Q_2 >$ *where* Q *is a finite quiver. If* Q *is not an oriented cycle then the Lie module structure on the Hochschild cohomology groups given by the Gerstenhaber bracket*

$$HH^1(A) \times HH^n(A) \longrightarrow HH^n(A)$$

is induced by the following bilinear map:

$$k(Q_1 \parallel Q_1) \times k(Q_n \parallel Q_1) \longrightarrow k(Q_n \parallel Q_1)$$

given as follows

$$(a, x).(\alpha, y) = \delta_y^a \cdot (\alpha, x) - \sum_{i=1}^{n} \delta_x^{a_i} \cdot (\alpha \underset{i}{\diamond} a, y)$$

where δ *is the Kronecker symbol and* $\alpha = a_1 \cdots a_i \cdots a_n$ *is a path of length* n *constituted of arrows* a_i. *For* $a_i = x$ *the path* $\alpha \underset{i}{\diamond} a$ *is obtained by replacing* a_i *with* a.

PROOF. Using the above proposition, we obtain that $Z^n(Q)$, the space of cocyles of $C^\bullet(Q)$, is equal to $k(Q_n \parallel Q_1)$. Moreover, $B^n(Q)$, the space of coboundaries, is $\operatorname{Im} D_{n-1}$. Then we compute the combinatorial bracket $[-, -]_Q$ on the space of shortcuts and we obtain the above formula. Using corollary 4.1.2, we know that $[-, -]_Q$ is well defined in the quotient of $k(Q_n \parallel Q_1)$ by $\operatorname{Im} D_{n-1}$. Finally using 4.1.3 we obtain the result. $\square$

The above theorem gives a combinatorial description of the Lie module structure on $HH^n(A)$. In the next chapter, we will apply this theorem in two cases: when the quiver has no oriented cycles and when it is the multiple-loops quiver.

CHAPTER 5

Triangular and multiple-loops quiver.

In this chapter we consider monomial algebras A with radical square zero associated to a quiver without oriented cycles or to the multiple-loops quiver. In the previous chapter, we have described the Lie module structure induced by the Gerstenhaber bracket on $\mathsf{HH}^n(A)$ for such algebras. In fact, we have considered a more general case: when the quiver is not an oriented cycle. The description in this case has been given in terms of the combinatorics of the quiver. In the present chapter, we will relate such Lie module structure with the Lie modules over the Lie algebra of square matrices of trace zero $\mathfrak{sl}_r(k)$. We will use as a tool the combinatorial description of such Lie modules.

5.1. Triangular

In this section we consider triangular monomial algebras with radical square zero, i.e. $A = kQ/ < Q_2 >$ where Q is a quiver without oriented cycles. We will prove that the Lie module structure on $\mathsf{HH}^n(A)$ is isomorphic to a direct sum of tensor products of "standard modules" over $\mathfrak{gl}_n(k)$.

Let us begin with the following observation. Since Q has no oriented cycles the combinatorial complex is the following:

$$C^\bullet(Q) = \quad 0 \longrightarrow k(Q_0 \parallel Q_0) \xrightarrow{D_0} k(Q_1 \parallel Q_1) \xrightarrow{0} \cdots$$

$$\cdots \xrightarrow{0} k(Q_n \parallel Q_1) \xrightarrow{0} k(Q_{n+1} \parallel Q_1) \xrightarrow{0} \cdots$$

It is clear that for $n > 1$, the differentials are zero since $Q_n \parallel Q_0$ is an empty set. Therefore $\mathsf{HH}^n(A) = k(Q_n \parallel Q_1)$ for $n > 1$. In view of this, we will study $k(Q_n \parallel Q_1)$ as a Lie module in a more general setting that we explain in the next paragraph.

Remark. In the previous chapter, Theorem 4.1.1 states that

$$C^{*+1}(Q) = \bigoplus_{n=1}^{\infty} k(Q_n \parallel Q_0) \oplus k(Q_n \parallel Q_1)$$

together with the combinatorial bracket is a graded Lie algebra. Therefore, given any quiver Q, the graded vector space of *shortcuts*, i.e.

$$\bigoplus_{n=1}^{\infty} k(Q_n \parallel Q_1)\,,$$

equipped with the combinatorial bracket $[-,-]_Q$ is also a graded Lie algebra since the bracket of two shortcuts is a shortcut. As a consequence $k(Q_n \parallel Q_1)$ is a Lie module over the Lie algebra $k(Q_1 \parallel Q_1)$.

Recall that $k(Q_1 \parallel Q_1)$, the Lie algebra of parallel arrows, has been already studied in the second chapter of this thesis. We will study the Lie module structure on $k(Q_n \parallel Q_1)$ over this Lie algebra. To do so, we need to introduce some notation.

NOTATION. Define an equivalence relation on the set $Q_n \parallel Q_1$ as follows. Let $\alpha = a_1 \cdots a_i \cdots a_n$ and $\beta = b_1 \cdots b_i \cdots b_n$ be paths in Q_n. Then

$$(\alpha, x) \sim (\beta, y) \text{ if and only if } x \parallel y \text{ and } a_i \parallel b_i \text{ for all } i.$$

Denote $\mathcal{T}_n$ the set of equivalence classes $(Q_n \parallel Q_1)/\sim$.

REMARK. The following map is a bijection between $\mathcal{T}_n$ and $\overline{Q}_n \parallel \overline{Q}_1$, the set of shortcuts of the quiver $\overline{Q}$:

$$\begin{array}{ccc} \mathcal{T}_n & \to & \overline{Q}_n \parallel \overline{Q}_1 \\ [t] = [\,(a_1 \cdots a_i \cdots a_n, x)\,] & \mapsto & (\alpha_1 \cdots \alpha_i \cdots \alpha_n, \gamma) \end{array}$$

where

- [t] denotes the class of t,
- $t = (a_1 \cdots a_i \cdots a_n, x)$ is a shortcut in Q,
- a_i belongs to α_i and x to γ,
- α_i is in $\overline{Q}_1$ for all i, as well as γ and
- $(\alpha_1 \cdots \alpha_i \cdots \alpha_n, \gamma)$ is a shortcut in $\overline{Q}$.

NOTATION. Let $T = (\alpha_1 \cdots \alpha_i \cdots \alpha_n, \gamma)$ be a shortcut in $\overline{Q}_n \parallel \overline{Q}_1$. We write

$$"t \in T"$$

when $t = (a_1 \cdots a_i \cdots a_n, x)$ is a shortcut in $Q_n \parallel Q_1$ such that its class $[t]$ correspond to T through the above bijection. We also write "t belongs to T".

Moreover, given T in $\overline{Q}_n \parallel \overline{Q}_1$, we denote

$$kT = \bigoplus_{t \in T} k\,t$$

which is a subvector space of $k(Q_n \parallel Q_1)$.

LEMMA 5.1.1. kT *is a submodule of* $k(Q_n \parallel Q_1)$ *over the Lie algebra* $k(Q_1 \parallel Q_1)$. *Hence*

$$k(Q_n \parallel Q_1) = \bigoplus_{T_n \in \overline{Q}_n \parallel \overline{Q}_1} kT$$

as Lie modules over $k(Q_1 \parallel Q_1)$.

PROOF. To show that kT is submodule of $k(Q_n \parallel Q_1)$, consider $t \in T$ and (a, a') in $k(Q_1 \parallel Q_1)$. Let $t = (a_1 \cdots a_i \cdots a_n, x)$. A simple computation gives

$$\begin{aligned} (a, a').t &= [(a, a'), (a_1 \cdots a_i \cdots a_n, x)]_Q \\ &= \delta_a^x (a_1 \cdots a_i \cdots a_n, a') \\ &- \textstyle\sum_{i=1}^n \delta_{a'}^{a_i} (a_1 \cdots a \cdots a_n, x) \end{aligned} \tag{9}$$

where the shortcuts that appear in the right side of the equation are in the class of t. So $(a,a').t$ is a linear combination of shortcuts belonging to T. This implies that kT is a submodule of $k(Q_n \parallel Q_1)$ over $k(Q_1 \parallel Q_1)$. Moreover, let T and T' be in $\overline{Q}_n \parallel \overline{Q}_1$. Clearly $kT \cap kT' = 0$ whenever $T \neq T'$. Finally, $k(Q_n \parallel Q_1)$ is the direct sum of the submodules kT for all T in $\overline{Q}_n \parallel \overline{Q}_1$. □

Since $k(Q_n \parallel Q_1)$ is a direct sum of Lie modules kT, we will investigate in more detail such modules. In fact, we will relate tensor products of "standard modules" over the Lie algebra of endomorphism with kT. We begin fixing notation about standard modules.

Remark. We have shown in Lemma 2.1.1 that

$$k(Q_1 \parallel Q_1) = \prod_{\alpha \in \overline{Q}_1} \mathfrak{gl}_\alpha \cong \prod_{\alpha \in \overline{Q}_1} \mathrm{End}_k(V_\alpha)$$

as a Lie algebra where V_α is the vector space with basis the set α. Notice that for every α in $\overline{Q}_1$, we can consider V_α as a module over $k(Q_1 \parallel Q_1)$.

We will write explicitly the Lie module structure on V_α over $k(Q_1 \parallel Q_1)$ using the following map:

$$k(Q_1 \parallel Q_1) \times V_\alpha \longrightarrow V_\alpha$$
$$(a,a').x \;=\; \delta_x^a\, a'$$

where a and a' are parallel arrows, x is an arrow in α and δ is the Kronecker symbol. We write explicitly the Lie module structure of the dual V^* using the following map:

$$k(Q_1 \parallel Q_1) \times V_\alpha^* \longrightarrow V_\alpha^*$$
$$(a,a').x^* \;=\; -\,\delta_x^{a'}\, a^*\,.$$

We will use V_α and its dual in order to describe the Lie module kT. In fact, kT is given by a tensor product of those Lie modules. Let us recall the tensor product of Lie modules in a general setting: let $\mathfrak{g}$ be a Lie algebra and M and N be two Lie modules over $\mathfrak{g}$. The tensor product $M \underset{k}{\otimes} N$ is a Lie module over $\mathfrak{g}$ given by the map:

$$\mathfrak{g} \times M \underset{k}{\otimes} N \to M \underset{k}{\otimes} N$$

given by

$$g.m \otimes n = g.m \otimes n + m \otimes g.n\,.$$

Lemma 5.1.2. *Let $T = (\alpha_1 \cdots \alpha_i \cdots \alpha_n, \gamma)$ be a shortcut in $\overline{Q}_n \parallel \overline{Q}_1$, then*

$$kT \cong V_{\alpha_1}^* \underset{k}{\otimes} \cdots \underset{k}{\otimes} V_{\alpha_i}^* \underset{k}{\otimes} \cdots V_{\alpha_n}^* \underset{k}{\otimes} V_\gamma$$

as Lie modules over $k(Q_1 \parallel Q_1)$.

Proof. Let $T = (\alpha_1 \cdots \alpha_i \cdots \alpha_n, \gamma)$ be in $\overline{Q}_n \parallel \overline{Q}_1$ and let $t = (a_1 \cdots a_n, x)$ be in $Q_n \parallel Q_1$ such that $t \in T$. Consider the map

$$\psi : kT \to V_{\alpha_1}^* \underset{k}{\otimes} \cdots \underset{k}{\otimes} V_{\alpha_i}^* \underset{k}{\otimes} \cdots V_{\alpha_n}^* \underset{k}{\otimes} V_\gamma$$

given by

$$\psi(t) = a_1^* \otimes \cdots \otimes a_i^* \otimes \cdots \otimes a_n^* \otimes x.$$

We will prove that ψ is a morphism of Lie modules, i.e. $\psi((a,a').t) = (a,a').\psi(t)$. In view of (9) in the proof of the above lemma, we remark that

$$\begin{aligned} \psi((a,a').t) &= \delta_a^x\, a_1^* \otimes \cdots \otimes a_i^* \cdots \otimes a_n^* \otimes a' \\ &- \sum_{i=1}^{n} \delta_{a'}^{a_i}\, a_1^* \otimes \cdots \otimes a^* \cdots \otimes a_n^* \otimes x. \end{aligned}$$

Now, let us compute $(a,a').\psi(t)$:

$$\begin{aligned} (a,a').\psi(t) &= \sum_{i=1}^{n} a_1^* \otimes \cdots \otimes (a,a').a_i^* \otimes \cdots \otimes a_n^* \otimes x \\ &+ a_1^* \otimes \cdots \otimes a_i^* \otimes \cdots \otimes a_n^* \otimes (a,a').x \\ &= \sum_{i=1}^{n} -\delta_{a'}^{a_i}\, a_1^* \cdots \otimes \cdots \otimes a^* \otimes \cdots a_n^* \otimes x \\ &+ \delta_a^x\, a_1^* \otimes \cdots \otimes a_i^* \otimes \cdots \otimes a_n^* \otimes a'. \end{aligned}$$

Then ψ is a morphism of Lie module and is clearly bijective. Therefore we have an isomorphism of Lie modules. □

In the next lemma, we prove that the Lie module kT is generated by any $t \in T$ whenever T is constituted by different arrows.

LEMMA 5.1.3. *Let* $T = (\alpha_1 \ldots \alpha_n, \chi)$ *be a shortcut in* $\overline{Q}_n \parallel \overline{Q}_1$ *such that* $\alpha_i \neq \chi$ *for any* i *and* $\alpha_i \neq \alpha_j$ *for any* $i \neq j$. *Then* kT *is generated by any* $t \in T$ *as Lie module over* $k(Q_1 \parallel Q_1)$.

PROOF. Let $t = (a_1 \cdots a_n, x)$ and $t' = (a'_1 \cdots a'_n, x)$ be shortcuts that belong to T, so $a_i \parallel a'_i$ and $x \parallel x'$. We will show that t' is in the Lie module generated by t. Denote $t^{(0)} = (x,x').t$, notice that $t^{(0)}$ belongs to the Lie module generated by t. Since $x' \neq a_i$ for all i,

$$t^{(0)} = [\,(x,x')\,,\,(a_1 \cdots a_i, x)\,]_Q = (a_1 \cdots a_i, x').$$

For $i = 1, \ldots, n$, define recursively $t^{(i)}$ as follows:

$$t^{(i)} = (a'_i, a_i).t^{(i-1)} = [\,(a'_i, a_i)\,,\,t]_Q.$$

Notice first that each $t^{(i)}$ is in the Lie module generated by t. Again as a consequence of the hypothesis, $x' \neq a_i$, $a_i \neq a'_j$ for $j < i$ and $a_i \neq a_j$ for $i < j$. A simple calculation of the above bracket shows

$$\begin{aligned} t^{(i)} &= (a'_i, a_i).(a'_1 \cdots a'_{i-1} a_i a_{i+1} \cdots a_n, x') \\ &= (a'_1 \cdots a'_{i-1} a'_i a_{i+1} \cdots a_n, x'). \end{aligned}$$

Thus we deduce that $t' = t^{(n)}$ which is in the Lie module generated by t. □

LEMMA 5.1.4. *Let* $T = (\alpha_1 \ldots \alpha_n, \chi)$ *be a shortcut in* $\overline{Q}_n \parallel \overline{Q}_1$ *such that* $\alpha_i \neq \chi$ *for all* i *and* $\alpha_i \neq \alpha_j$ *for all* $i \neq j$. *Then the Lie module* kT *is generated by any of its nonzero elements. Therefore,* kT *is an irreducible Lie module.*

In particular, if Q *has no oriented cycles then* kT *is an irreducible Lie module for any shortcut* T.

Proof. Let

$$w = \sum_{t\in T} \lambda_t t$$

be a non-zero element of kT, we will prove it generates kT. There exists a $t \in T$ that appears in the linear combination of w with non-zero coefficient. Fix $t = (a_1 \cdots a_n, x)$ a shortcut whose coefficient λ_t is not zero. By hypothesis, $x \neq a_i$ for all i. Denote W the Lie module generated by w. Set

$$w^{(0)} = (x, x).w = [\,(x, x)\,,\, w\,]_Q.$$

Clearly $w^{(0)}$ is an element of W, a simple computation of the bracket gives:

$$w^{(0)} = \sum_{j=1}^{n} \sum_{a'_j\in\alpha_j} \lambda_{(a'_1\cdots a'_n, x)}(a'_1 \cdots a'_n, x)$$

For $i = 1, \dots, n$, we set:

$$w^{(i)} = (a_i, a_i).w^{(i-1)} = [\,(a_i, a_i)\,,\, w^{(i-1)}\,]_Q.$$

It is clear that every element $w^{(i)}$ is in W. Moreover, for $i < n$ we have that $w^{(i)}$ is equal to the following linear combination:

$$(-1)^i \sum_{j=i+1}^{n} \sum_{a'_j\in\alpha_j} \lambda_{(a_1\cdots a_i a'_{i+1}\cdots a'_n, x)}(a_1 \cdots a_i a'_{i+1} \cdots a'_n, x).$$

Then $w^{(n)} = \pm\lambda_t t$ which is in W and using the above lemma W must be the Lie module kT. Clearly kT is irreducible if it is generated by any of its non-zero elements. Finally, if Q has no oriented cycles then any shortcut T satisfies the hypothesis of the statement. □

We have studied $k(Q_n \parallel Q_1)$ as a Lie module over $k(Q_1 \parallel Q_1)$ using the decomposition in direct sum of submodules kT. We have also seen that if the quiver has no oriented cycles then kT is irreducible.

Next, we will consider $k(Q_n \parallel Q_1)$ as a Lie module over the quotient of the Lie algebra $k(Q_1 \parallel Q_1)$ by $\operatorname{Im} D_0$. Moreover, we will also obtain the same decomposition in direct sum of submodules kT.

Lemma 5.1.5. *Let* Q *be a quiver without oriented cycles. Then* $k(Q_n \parallel Q_1)$ *is a Lie module over the quotient* $k(Q_1 \parallel Q_1)$ *by* $\operatorname{Im} D_0$ *with the following map*

$$\frac{k(Q_1 \parallel Q_1)}{\operatorname{Im} D_0} \times k(Q_n \parallel Q_1) \to k(Q_n \parallel Q_1)$$

$$\overline{(a, a')}.t \;=\; (a, a').t = [\,(a, a')\,,\, t\,]_Q$$

where $\overline{(a, a')}$ *is the class of an element in* $Q_1 \parallel Q_1$ *and* t *is a shortcut. Moreover,*

$$k(Q_n \parallel Q_1) \;=\; \bigoplus_{T\in\overline{Q}_n\parallel\overline{Q}_1} kT$$

as a Lie module over the quotient of $k(Q_1 \parallel Q_1)$ *by* $\operatorname{Im} D_0$.

PROOF. From Lemma 4.1.2 of the previous chapter, the combinatorial bracket of a coboundary and a cocyle is a coboundary. Using the description of $C^\bullet(Q)$ when Q has no oriented cycles given at the beginning of this chapter, $B^1(Q) = \operatorname{Im} D_0$, $Z^n(Q) = k(Q_n \parallel Q_1)$, and $B^n(Q) = 0$ for $n > 1$. Therefore

$$[\operatorname{Im} D_0\,,\, k(Q_n \parallel Q_1)]_Q = 0$$

for $n > 1$. Therefore, a Lie module structure on $k(Q_n \parallel Q_1)$ over the quotient $k(Q_1 \parallel Q_1)$ by $\operatorname{Im} D_0$ is induced by the above formula. We use Lemma 5.1.1 to obtain the decomposition of $k(Q_n \parallel Q_1)$ as a direct sum of kT, considered as modules over the quotient. □

In this paragraph, we return to the Hochschild cohomology of triangular monomial algebras of radical square zero. Let $A = kQ/ < Q_2 >$ where Q has no oriented cycles and k is a field of characteristic zero. Recall that $HH^1(A)$ is the quotient of the Lie algebra $k(Q_1 \parallel Q_1)$ by the image of D_0 and that for $n > 1$, $HH^n(A)$ is the space of shortcuts $k(Q_n \parallel Q_1)$. As before we will use tensor products of standard modules to describe $HH^n(A)$. We need the following remark in order to understand the Lie module structure over $HH^1(A)$ on those tensor products.

REMARK. Given a quiver Q without oriented cycles, let $T = (\alpha_1 \dots \alpha_n, \chi)$ be a shortcut in $\overline{Q}_n \parallel \overline{Q}_1$. Recall that kT is isomorphic to the tensor product

$$V^*_{\alpha_1} \underset{k}{\otimes} \cdots \underset{k}{\otimes} V^*_{\alpha_i} \underset{k}{\otimes} \cdots \underset{k}{\otimes} V^*_{\alpha_n} \underset{k}{\otimes} V_\gamma$$

as Lie modules over $k(Q_1 \parallel Q_1)$. Since kT is a Lie module over the quotient of the Lie algebra $k(Q_1 \parallel Q_1)$ by $\operatorname{Im} D_0$ (Lemma 5.1.5), we endow the above tensor product with a Lie module structure over the quotient via the isomorphism.

NOTATION. Let Q be a quiver without oriented cycles and T be a shortcut in $\overline{Q}$. We denote the above tensor product $HH^n_T(Q)$ when we consider it as a Lie module over $HH^1(A)$ as explained above.

In order to state the result, let us recall some facts about $HH^1(A)$. In chapter two, we proved that $HH^1(A)$ is reductive, i.e. it is a direct product of its semisimple part $HH^1(A)_{ss}$ and its radical $HH^1(A)_{ab}$ which is an abelian Lie algebra. Moreover,

$$HH^1(A)_{ss} \cong \prod_{\alpha \in S} sl_{|\alpha|}(k) \quad \text{and} \quad HH^1(A)_{ab} \cong \frac{\prod_{\alpha \in \overline{Q}_1} I_\alpha}{\operatorname{Im} D_0} \cong k^{\chi(\overline{Q})}$$

where

$$S = \{\alpha \in \overline{Q}_1 \text{ such that } |\alpha| > 1\}$$

and

$$\chi(\overline{Q}) = |\overline{Q}_1| - |\overline{Q}_0| + 1.$$

The next statement describes explicitly the Lie module structure on $HH^n(A)$ over $HH^1(A)_{ss}$ and over $HH^1(A)_{ab}$.

THEOREM 5.1.6. *Let $A = kQ/ < Q_2 >$ where Q is a finite quiver without oriented cycles and k a field of characteristic zero. Then for $n > 1$ the Lie module structure on $HH^n(A)$ over $HH^1(A)$ induced by the Gerstenhaber bracket is as follows:*

$$HH^n(A) \cong \bigoplus_{T \in \overline{Q}_n \| \overline{Q}_1} HH^n_T(Q) \underset{k}{\otimes} k$$

where

$$HH^n_T(Q) = V^*_{\alpha_1} \underset{k}{\otimes} \cdots \underset{k}{\otimes} V^*_{\alpha_i} \underset{k}{\otimes} \cdots \underset{k}{\otimes} V^*_{\alpha_n} \underset{k}{\otimes} V_\gamma$$

Moreover, $HH^n_T(Q)$ is irreducible.

More precisely, the Lie module structure on $HH^n_T(A)$ over $HH^1(A)_{ss}$ is given by

$$(f_\alpha)_{\alpha \in S} . x_{\alpha_1} \otimes \cdots \otimes x_{\alpha_i} \otimes \cdots x_{\alpha_n} \otimes x_\gamma \otimes \lambda = \sum_{\alpha_i \in S} x_{\alpha_1} \otimes \cdots \otimes f_{\alpha_i}(x_{\alpha_i}) \otimes \cdots x_{\alpha_n} \otimes x_\gamma \otimes \lambda$$
$$- \chi_S(\gamma)\, x_{\alpha_1} \otimes \cdots \otimes x_{\alpha_i} \otimes \cdots x_{\alpha_n} \otimes f^t_\gamma(x_\gamma) \otimes \lambda$$

The Lie module structure on $HH^n_T(A)$ over $HH^1(A)_{ab}$ is given by

$$(\lambda_\alpha)_{\alpha \in \overline{Q}_1} . x_{\alpha_1} \otimes \cdots \otimes x_{\alpha_i} \otimes \cdots x_{\alpha_n} \otimes x_\gamma \otimes \lambda = x_{\alpha_1} \otimes \cdots \otimes x_{\alpha_i} \otimes \cdots x_{\alpha_n} \otimes x_\gamma \otimes \lambda(\lambda_\gamma - \textstyle\sum_i^n \lambda_i)$$

PROOF. Since Q has no oriented cycles, $HH^n(A) = k(Q_n \parallel Q_1)$. By Lemma 5.1.5, $HH^n(A)$ as a Lie module over $HH^1(A)$, is equal to the direct sum of kT where the sum runs over all shortcuts T in $\overline{Q}$. Since we have endowed $HH^n_T(Q)$ with a Lie module structure over $HH^1(A)$ in such a way that kT is isomorphic to $HH^n_T(Q)$ as Lie modules over $HH^1(A)$, we obtain the above decomposition. To prove that $HH^n_T(Q)$ is irreducible, the same proofs of Lemmas 5.1.3 and 5.1.4 can be considered using the way the Lie module over $HH^1(A)$ is induced. Finally, we consider the isomorphism between $HH^1(A)$ and the direct product of $HH^1(A)_{ss}$ and $HH^1(A)_{ab}$ given in chapter two. A straightforward computation of the combinatorial bracket via such isomorphism gives the last part of the statement. □

5.2. Multiple-loops quiver.

In this section we deal with the monomial algebra of radical square zero given by the multiple-loop quiver which is not reduced to a single loop. We denote r the number of loops, $r \geq 2$. The monomial algebra we consider is $A = k[x_1, \ldots, x_r]/ < \{x_i x_j\}_{i,j=1}^r >$. Since Q is not an oriented cycle, $HH^1(A) = k(Q_1 \parallel Q_1)$, which is in fact $\mathfrak{gl}_r(k)$. Moreover, for $n > 1$,

$$HH^n(A) \cong \frac{k(Q_n \parallel Q_1)}{\operatorname{Im} D_{n-1}}$$

where the map D_{n-1} is defined by (8). Recall that the map D_n is injective, therefore we have

$$\dim_k HH^n(A) = r^{n+1} - r^{n-1}.$$

In order to study the Lie module structure on $HH^n(A)$ over $\mathfrak{gl}_r(k)$ induced by the Gerstenhaber bracket, we study the numerator and the denominator of the above quotient. We relate this Lie module structure with the known Lie modules over $\mathfrak{gl}_r(k)$. We begin introducing some notation.

NOTATION. Let Q be a finite quiver. Recall that the vector space generated by the set of arrows Q_1 is denoted kQ_1. Then kQ_1 has a Lie module structure over $k(Q_1 \parallel Q_1)$ through the following map:

$$k(Q_1 \parallel Q_1) \times kQ_1 \longrightarrow kQ_1$$

$$(a, a').x = \delta_x^a a'.$$

A straightforward verification gives

$$[(a, a'), (b, b')].x = (a, a').(b, b').x - (b, b').(a, a').x$$

so the above map provides a Lie module structure on kQ_1 over $k(Q_1 \parallel Q_1)$.

Denote $V = kQ_1$ this Lie module over $k(Q_1 \parallel Q_1)$. Moreover, the dual Lie module structure V^* is given by the following map:

$$k(Q_1 \parallel Q_1) \times kQ_1^* \longrightarrow kQ_1^*$$

$$(a, a').x = -\delta_x^{a'} a$$

REMARK. If Q is the multiple-loops quiver, we know that $k(Q_1 \parallel Q_1)$ is isomorphic to $\mathfrak{gl}_r(k)$ where r is the number of loops. The Lie module V described above is isomorphic to the standard module k^r.

In the previous section, we have study the Lie module structure on the shortcut space $k(Q_n \parallel Q_1)$ over $k(Q_1 \parallel Q_1)$. Such Lie module structure is induced by the combinatorial bracket $[-, -]_Q$. The results that we have obtained where stated in a general setting. We are going to apply some of them in the case of the multiple-loops quiver.

PROPOSITION 5.2.1. *Let Q be the multi-loop quiver where r, the number of loops, is greater or equal two. Then*

$$k(Q_n \parallel Q_1) \cong V^{*\otimes n} \otimes V$$

as Lie modules over $\mathfrak{gl}_r(k)$, where V is the standard module, i.e. isomorphic to k^r as Lie module over $\mathfrak{gl}_r(k)$.

PROOF. As a consequence of Lemma 5.1.1, $k(Q_n \parallel Q_1)$ has a decomposition into a direct sum of Lie modules kT where the sum runs over all shortcuts T in $\overline{Q}$. Moreover by Lemma 5.1.2 the Lie modules kT are isomorphic to a tensor product of standard modules. For the multiple loops quiver, there is only one shortcut $T = (\alpha^n, \alpha)$ where α is the only arrow in $\overline{Q}$. Therefore, $k(Q_n \parallel Q_1) = kT \cong V_\alpha^{*\otimes n} \otimes V_\alpha$. Since the Lie module V_α is equal to V we obtain the result. □

Next, we investigate the denominator of the quotient that computes $\mathsf{HH}^n(A)$, this means the image of the map D_{n-1}. We study its Lie module structure over $\mathsf{HH}^1(A)$. To do so we work in a more general setting.

For quivers that are not oriented cycles, the map

$$D_n : k(Q_n \parallel Q_0) \to k(Q_n \parallel Q_1)$$

is injective for $n \geq 1$. Therefore, within those cases we can identify $k(Q_n \parallel Q_0)$, the space of *pointed oriented cycles*, with the image of D_n. We are going to study both the space of pointed oriented cycles and the image of the map D_n as Lie modules over $k(Q_1 \parallel Q_1)$.

The following two lemmas describes explicitly the Lie module structure of $k(Q_n \parallel Q_0)$ and of $\operatorname{Im} D_n$.

LEMMA 5.2.2. *Let* Q *be any finite quiver. Given a path* α*, let us suppose* $\alpha = a_1 \cdots a_i \cdots a_n$ *where the* a_i*'s are arrows. The combinatorial bracket* $[-,-]_Q$ *endows* $k(Q_n \parallel Q_0)$ *with a Lie module structure over* $k(Q_1 \parallel Q_1)$ *as follows:*

$$(a, a').(\alpha, e) = [\,(a, a')\,,\,(\alpha, e)\,]_Q = -\sum_{i=1}^{n} \delta_{a_i}^{a'} (\alpha \underset{i}{\diamond} a, e)$$

where $\alpha \underset{i}{\diamond} a$ *is the path obtained by replacing* a_i *by* a *if* $a' = a_i$*.*

PROOF. Recall that Theorem 4.1.1 states that $C^{*+1}(Q)$ equipped with the combinatorial bracket is a graded Lie algebra. The bracket $[-,-]_Q$ endows $k(Q_n \parallel Q_0) \oplus k(Q_n \parallel Q_1)$ with a Lie module structure over $k(Q_1 \parallel Q_0) \oplus k(Q_1 \parallel Q_1)$. In particular, $k(Q_n \parallel Q_0) \oplus k(Q_n \parallel Q_1)$ becomes a Lie module over $k(Q_1 \parallel Q_1)$. Moreover $k(Q_n \parallel Q_0)$ is a submodule since the combinatorial bracket of a parallel arrow (a, a') and a pointed oriented cycle (α, e) is indeed a sum of pointed oriented cycles in $k(Q_n \parallel Q_0)$. □

LEMMA 5.2.3. *Let* Q *be a finite quiver that is not an oriented cycle. The image of the linear map* D_{n-1} *is a Lie submodule of* $k(Q_n \parallel Q_1)$ *over* $k(Q_1 \parallel Q_1)$*.*

PROOF. Recall that the combinatorial bracket endows $k(Q_n \parallel Q_1)$ with a Lie module structure over $k(Q_1 \parallel Q_1)$ using Theorem 4.1.1. Besides, $Z^n(Q) = k(Q_n \parallel Q_1)$ and $B^n(Q) = \operatorname{Im} D_{n-1} \subseteq Z^n(Q)$. Moreover

$$[\,Z^1(Q)\,,\,Z^n(Q)\,]_Q \subseteq Z^n(Q) \;\text{ and }\; [\,Z^1(Q)\,,\,B^n(Q)\,]_Q \subseteq B^n(Q)$$

(see Lemma 4.1.2). Therefore $B^n(Q)$ is a submodule of $Z^n(Q)$ over the Lie algebra $k(Q_1 \parallel Q_1)$. □

Since we identify $k(Q_n \parallel Q_0)$ with $\operatorname{Im} D_n$, we need to prove that they are isomorphic as Lie modules.

LEMMA 5.2.4. *Let* Q *be a finite quiver that is not an oriented cycle. The map* $D_n : k(Q_n \parallel Q_0) \to k(Q_{n+1} \parallel Q_1)$ *is a morphism of Lie modules.*

PROOF. We will show that

$$D_n((a,a').(\alpha,e)) = (a,a').D_n(\alpha,e)$$

for a and a' two parallel arrows and α an oriented cycle such that $s(\alpha) = t(\alpha) = e$. Recall that $(a,a').(\alpha,e) = \delta_{a_i}^{a'}(\alpha \underset{i}{\diamond} a, e)$. Denote $\alpha_i = \alpha \underset{i}{\diamond} a$ whenever $a' = a_i$. Let us begin computing the left side.

$$\begin{aligned} D_n((a,a').(\alpha,e)) &= D_n(\sum_{i=1}^{n}(\alpha_i,e)) \\ &= \sum_{i=1}^{n}\sum_{x\,|\,s(x)=e}(\alpha_i x, x) + (-1)^{n+1}\sum_{i=1}^{n}\sum_{x\,|\,t(x)=e}(x\alpha_i, x). \end{aligned}$$

In order to compute the right side, let us begin computing $(a,a').(\alpha x, x)$ and $(a,a').(x\alpha, x)$ where x is an arrow such that $s(x) = e$ or $t(x) = e$ respectively:

$$\begin{aligned} (a,a').(\alpha x, x) &= \delta_x^a(\alpha a, a') - \sum_{i=1}^{n}(\alpha_i x, x) - \delta_x^{a'}(\alpha a, a') \quad \text{or} \\ (a,a').(x\alpha, x) &= \delta_x^a(a\alpha, a') - \sum_{i=1}^{n}(x\alpha_i, x) - \delta_x^{a'}(a\alpha, a') \quad . \end{aligned}$$

Let suppose first that a is an arrow such that $s(a)$ and $t(a)$ is different from e. Then

$$\begin{aligned} (a,a').D_n(\alpha,e) &= \sum_{x\,|\,s(x)=e}(a.a').(\alpha x, x) + (-1)^{n+1}\sum_{x\,|\,t(x)=e}(a.a').(x\alpha, x) \\ &= \sum_{x\,|\,s(x)=e}\sum_{i=1}^{n}(\alpha_i x, x) + (-1)^{n+1}\sum_{x\,|\,t(x)=e}\sum_{i=1}^{n}(x\alpha_i, x) \end{aligned}$$

since $a \neq x$ for any x such that $t(x) = e$ and $s(x) = e$. Therefore, we obtain the result in this case. Now we consider the other case, when a is an arrow such that $s(a)$ or $t(a)$ is equal to e. Without lost of generality let us suppose $s(a) = e$, obviously $s(a') = e$. Then

$$\begin{aligned} \sum_{x\,|\,s(x)=e}(a.a').(\alpha x, x) &= \sum_{x\,|\,s(x)=e}\delta_x^a(\alpha a, a') - \sum_{x\,|\,s(x)=e}\sum_{i=1}^{n}(\alpha_i x, x) \\ &\quad - \sum_{x\,|\,s(x)=e}\delta_x^{a'}(\alpha a, a') \\ &= (\alpha a, a') - \sum_{x\,|\,s(x)=e}\sum_{i=1}^{n}(\alpha_i x, x) - (\alpha a, a') \\ &= \sum_{x\,|\,s(x)=e}\sum_{i=1}^{n}(\alpha_i x, x). \end{aligned}$$

We deduce $(a,a').D_n(\alpha,e) = D_n((a,a').(\alpha,e))$. □

PROPOSITION 5.2.5. *Let Q be the multi-loop quiver where r, the number of loops, is greater or equal two. Then*

$$\operatorname{Im} D_{n-1} \cong V^{*\otimes n-1}$$

as Lie modules over $\mathfrak{gl}_r(k)$ *where* V *is the the standard module, i.e. isomorphic to* k^r *as Lie module over* $\mathfrak{gl}_r(k)$.

PROOF. From Lemma (5.2.4), $\operatorname{Im} D_{n-1}$ is isomorphic to the Lie module $k(Q_{n-1} \parallel Q_0)$ over $k(Q_1 \parallel Q_1)$. Now, let us consider the following map:

$$\phi : k(Q_{n-1} \parallel Q_0) \to V^{*\otimes n-1}$$

$$\phi(\alpha, e) = a_1 \otimes \cdots \otimes a_i \otimes \cdots \otimes a_n$$

where $\alpha = a_1 \cdots a_i \cdots a_n$ is an oriented cycle. From Lemma (5.2.2) we have have that $\phi((a, a').(\alpha, e)) = (a, a')\phi(\alpha, e)$. Therefore ϕ is a Lie morphism and it is clearly bijective. □

THEOREM 5.2.6. *Let* $A = kQ/ < Q_2 >$ *where* Q *is a multiple-loops quiver and* k *is an algebraically closed field of characteristic zero. Then* $HH^1(A) \cong \mathfrak{gl}_r(k)$ *where* r *is the number of loops. The Lie module structure (induced by the Gerstenhaber bracket) of* $HH^n(A)$ *over* $HH^1(A) \cong \mathfrak{gl}_r(k)$ *is given as follows:*

$$HH^n(A) \cong V^{*\otimes n-1} \otimes \mathfrak{sl}_r(k)$$

where V *is the standard* $\mathfrak{gl}_r(k)$*-module and* $\mathfrak{sl}_r(k)$ *is the usual* $\mathfrak{gl}_r(k)$*-module (i.e. given by the restriction of the adjoint module).*

PROOF. For the multiple loops quiver, we have shown that the Lie module $k(Q_n \parallel Q_1)$ is isomorphic to $V^{*\otimes n} \otimes V$ (see Proposition 5.2.1) and $k(Q_{n-1} \parallel Q_0)$ is isomorphic to $V^{*\otimes n-1}$ (see Proposition 5.2.5). Now, we compute the quotient that gives Hochschild cohomology:

$$HH^n(A) \cong \frac{V^{*\otimes n} \otimes V}{V^{*\otimes n-1}} \cong \frac{V^{*\otimes n-1} \otimes \mathfrak{gl}_r(k)}{V^{*\otimes n-1} \otimes k} \cong V^{*\otimes n-1} \otimes \mathfrak{sl}_r(k)\,.$$

The last equality comes from the following fact. Recall that the following exact sequence of Lie modules over $\mathfrak{gl}_r(k)$ splits:

$$0 \to k \to \mathfrak{gl}_r(k) \to \mathfrak{sl}_r(k) \to 0.$$

where the section is the inclusion. Therefore

$$0 \to V^{*\otimes n-1} \to V^{*\otimes n-1} \otimes \mathfrak{gl}_r(k) \to V^{*\otimes n-1} \otimes \mathfrak{sl}_r(k) \to 0$$

is an exact sequence. □

A simple observation gives the following result:

COROLLARY 5.2.7. *For an algebra* A *as above,* $HH^2(A) = V^* \otimes \mathfrak{sl}_r(k)$ *and for* $n > 2$ *we have*

$$HH^n(A) \cong V^* \otimes HH^{n-1}(A)$$

From now on we will assume that the ground field is algebraically closed and of characteristic zero. The above theorem determines completely the Lie module $HH^n(A)$ over $HH^1(A)$. Now, we will study $HH^n(A)$ as a Lie module over $\mathfrak{sl}_r(k)$ as we will explain. Let us recall two classical Lie theory results, (see for instance [**EW06, FH91**])

(1) Every (finite dimensional) $\mathfrak{sl}_r(k)$-module has a decomposition into direct sum of irreducible modules
(2) The irreducible modules over $\mathfrak{sl}_r(k)$ are uniquely determined by their vector of highest weight. We denote Γ_λ the irreducible $\mathfrak{sl}_r(k)$ module of highest weight λ.

Then $HH^n(A)$ has a decomposition as a direct sum of irreducible modules over $\mathfrak{sl}_r k$ as follows:

$$HH^n(A) = \bigoplus_\lambda \Gamma_\lambda^{q_\lambda}$$

Next we provide an algorithm to determine each q_λ using the usual tools of classical Lie theory.

NOTATION. Let $\Gamma_{(a_1,a_2,\dots,a_{r-1})}$ be the unique irreducible module over $\mathfrak{sl}_r(k)$ of weight $a_1w_1 + a_2w_2 + \cdots a_{r-1}w_r$ where w_i are the fundamental weights. Let us notice that for $\mathfrak{sl}_2 k$, the unique irreducible module of dimension $a+1$ is Γ_a.

REMARK. Let $A = kQ/ < Q_2 >$ where Q is a multi-loop quiver. The above result gives an algorithm to calculate the Hochschild cohomology groups considered as modules over $\mathfrak{sl}_r(k)$ where r is the number of loops. First, let us remark that $HH^2(A) \cong V^* \otimes \mathfrak{sl}_r(k)$ where V^* is the dual of the standard module of $\mathfrak{sl}_r(k)$. For $n > 2$ we have

$$HH^{n+1}(A) = V^* \otimes HH^n(A) \quad .$$

The Littlewood Richardson rule is used to find the decomposition into direct sum of irreducibles of the tensor product of two irreducible modules of $\mathfrak{sl}_r(k)$. A special case is given in the next proposition. A proof of this result can be found in the book "Representation theory" of Fulton and Harris [**FH91**]. For $r = 2$, we have the Clebsch-Gordon theorem:

PROPOSITION. *(Clebsch-Gordon) For $\mathfrak{sl}_2(k)$ and for $a \geq 1$ the following holds:*

$$V^* \otimes \Gamma_a = \Gamma_{a+1} \oplus \Gamma_{a-1}.$$

For $a = 0$, $V^* \otimes \Gamma_0 = V^*$.

PROPOSITION. *(Littlewood-Richardson rule) For $\mathfrak{sl}_r(k)$ with $r \geq 3$, the decomposition into direct sum of irreducible modules of $V^* \otimes \Gamma_{(a_1,a_2,\dots,a_{r-1})}$ is*

$$\Gamma_{(a_1,a_2,\dots,a_{r-1}+1)} \oplus \bigoplus_{a_{i+1}\geq 1} \Gamma_{(a_1,\dots,a_i+1,a_{i+1}-1,\dots,a_{r-1})} \oplus \Gamma_{(a_1-1,a_2,\dots,a_{r-1})}$$

EXAMPLE. For $\mathfrak{sl}_3(k)$

$$V^* \otimes \Gamma_{(a,b)} = \Gamma_{(a,b+1)} \oplus \Gamma_{(a+1,b-1)} \oplus \Gamma_{(a-1,b)}$$

if $a, b \geq 1$. For $b = 0$ and $a \geq 1$,

$$V^* \otimes \Gamma_{(a,0)} = \Gamma_{(a,b+1)} \oplus \Gamma_{(a-1,b)}.$$

Now, for $a = 0$ and $b \geq 1$

$$V^* \otimes \Gamma_{(0,b)} = \Gamma_{(a,b+1)} \oplus \Gamma_{(a-1,b)}.$$

Finally

$$V^* \otimes \Gamma_{(0,0)} = \Gamma_{(0,1)}.$$

PROPOSITION 5.2.8. *Let* $A = kQ/ < Q_2 >$ *where* Q *is a multi-loop quiver with* r *loops. The decomposition of* $HH^2(A)$ *into direct sum of irreducible modules as a module over* $sl_r(k)$ *is given by*

$$HH^2(A) = \begin{cases} \Gamma_3 \oplus \Gamma_1 & \textit{if } r = 2 \\ \Gamma_{(1,2)} \oplus \Gamma_{(0,2)} \oplus \Gamma_{(0,1)} & \textit{if } r = 3 \\ \Gamma_{(1,0,\dots,0,2)} \oplus \Gamma_{(1,0,\dots,1,0)} \oplus \Gamma_{(0,\dots,0,1)} & \textit{if } r > 3 \end{cases}$$

PROOF. We know that $HH^2(A) = V^* \oplus sl_r(k)$. In order to find the decomposition into direct sum of irreducible modules, we apply the above proposition. For $r = 2$, let us remark that $sl_2(k)$ is Γ_2. For $r = 3$ we have that $sl_3(k)$ is $\Gamma_{(1,1)}$. For $r > 3$ we have that $sl_r(k)$ is $\Gamma_{(1,0,\dots,0,1)}$. □

ALGORITHM. In this paragraph, our aim is to explain an algorithm to calculate the decomposition into direct sum of irreducible $sl_r(k)$-modules of $HH^n(A)$. The first step is given by the above proposition. For $n \geq 2$, let us suppose we have found the decomposition of $HH^n(A)$:

$$HH^n(A) = \bigoplus_{\mathbf{a}} \Gamma_{\mathbf{a}}\,.$$

In order to calculate de decomposition of $HH^{n+1}(A)$ we use Corollary 5.2.7 which says that $HH^{n+1}(A) = V^* \otimes HH^n(A)$. Now, recall that direct sums and tensor products of Lie modules commute so the next step is to calculate the decomposition of $V^* \otimes \Gamma_{\mathbf{a}}$ for each $\Gamma_{\mathbf{a}}$ that appears in the decomposition of $HH^n(A)$. To do so, we apply the Littlewood-Richardson rule which is stated above for this case.

REMARK. We find the same algorithm described in [**SF08**] for $r = 2$

5.3. Two-loops quiver.

In the previous section we considered the multiple loops quiver: we found an algorithm that calculates the decomposition into direct sum of irreducible Lie modules of $HH^n(A)$ as a module of $sl_r(k)$ where r is the number of loops. In this section, we consider the two loops quiver (i.e. $r = 2$): we give the explicit decomposition into direct sum of irreducible Lie modules of $HH^n(A)$ as a module over $sl_2(k)$. We will begin providing a copy of the Lie algebra $sl_2(k)$ in $HH^1(A)$.

PROPOSITION 5.3.1. *Assume that* Q *is the two loops quiver where* e *is the vertex and the loops are denoted* a *and* b. *Let* $A = kQ/ < Q_2 >$. *Then the* *elements*

$$\begin{array}{rcl} \mathsf{H} &=& (b,b)-(a,a)\\ \mathsf{E} &=& (a,b)\\ \mathsf{F} &=& (b,a) \end{array}$$

generate a copy of the Lie algebra $\mathsf{sl}_2(k)$ *in* $\mathsf{HH}^1(A)$*. Moreover, the Lie algebra* $\mathsf{HH}^1(A)$ *is isomorphic to* $\mathsf{sl}_2(k) \times k$.

Proof. First notice that $\mathsf{HH}^1(A) \cong k(Q_1 \parallel Q_1)$ and that the elements H, E, F and $I = (a,a)+(b,b)$ form a basis of $\mathsf{HH}^1(A)$. A straightforward verification of the following relations

$$[\mathsf{H}\,,\,\mathsf{E}]_Q = 2\mathsf{E}, \quad [\mathsf{H}\,,\,\mathsf{F}]_Q = -2\mathsf{F}, \quad [\mathsf{E}\,,\,\mathsf{F}]_Q = \mathsf{H}$$

proves that $\mathsf{HH}^1(A)$ contains a copy of $\mathsf{sl}_2 k$. Finally, it is easy to see that

$$[I\,,\,\mathsf{H}]_Q = 0, \quad [I\,,\,\mathsf{E}]_Q = 0, \quad [I\,,\,\mathsf{F}]_Q = 0,$$

□

We begin describing the eigenvector spaces of H as an endomorphism of $k(Q_n \parallel Q_0)$ and $\operatorname{Im} D_{n-1}$. Given a path γ^n in Q_n we denote $a(\gamma^n)$ and $b(\gamma^n)$ the number of occurrences of the arrow a and the arrow b in the decomposition of γ^n, respectively.

Map (ν). Define ν as the map given by:

$$\begin{array}{rccl} \nu_n : & Q_n & \to & \mathbb{Z}\\ & \gamma^n & \mapsto & a(\gamma^n) - b(\gamma^n) \end{array}$$

Lemma 5.3.2. *For all* γ^n *in* Q_n*,*

$$\begin{array}{rcl} \mathsf{H}.(\gamma^n, b) &=& (\nu_n(\gamma^n)+1)\,(\gamma^n, a)\\ \mathsf{H}.(\gamma^n, a) &=& (\nu_n(\gamma^n)-1)\,(\gamma^n, b) \end{array}$$

and for all γ^{n-1} *in* Q_{n-1}*,*

$$\mathsf{H}.D_{n-1}(\gamma^{n-1}, e) = \nu_{n-1}(\gamma^{n-1})\, D_{n-1}(\gamma^{n-1}, e)\,.$$

Proof. It is easy to check through a straightforward verification of the combinatorial bracket. □

Proposition 5.3.3. *Assume that* $\operatorname{char} k = 0$*.*

(1) Consider H *as an endomorphism of* $k(Q_n \parallel Q_1)$*. The eigenvalues of* H *are* $n+1-2l$ *where* $l = 0, \ldots n+1$*. Denote* $W(\lambda)$ *the eigenspace of* H *of the eigenvalue* λ*.*

$$\dim_k W(n+1-2l) = \binom{n+1}{l}.$$

(2) Consider H *as an endomorphism of* $\operatorname{Im} D_{n-1}$ *The eigenvalues of* H *restricted to* $\operatorname{Im} D_{n-1}$ *are* $n-1-2l$ *where* $l = 0, \ldots n-1$*. As above, denote* $W(\lambda)$ *the eigenspace of* H *of eigenvalue* λ*.*

$$\dim_k W(n-1-2l) = \binom{n-1}{l}.$$

PROOF. (i) From the above lemma, it is clear that the set

$$\{(\gamma^n, a) \mid \gamma^n \in Q_n\} \cup \{(\gamma^n, a) \mid \gamma^n \in Q_n\}$$

is a basis of $k(Q_n \parallel Q_1)$ consisting of eigenvectors. We also have that (γ^n, a) and (γ^n, b) are eigenvectors of eigenvalue $\nu(\gamma^n) + 1$ and $\nu(\gamma^n) - 1$ respectively. Since $a(\gamma^n) + b(\gamma^n) = n$ for all paths γ^n, $\nu(\gamma^n) = n - 2b(\gamma^n)$ where $b(\gamma^n)$ varies from 0 to n. Then $\nu(\gamma^n) \pm 1$ is of the form $n + 1 - 2l(\gamma^n)$ where $l = 0 \ldots, n+1$. Let us remark the following:

– (a^n, b) is the only eigenvector of value $n + 1$
– (b^n, a) is the only eigenvector of value $-(n + 1)$
– If $0 < l < n + 1$,
  • (γ^n, a) is an eigenvector of eigenvalue $n + 1 - 2l$ iff $l = b(\gamma^n)$
  • (γ^n, b) is an eigenvector of eigenvalue $n + 1 - 2l$ iff $l - 1 = b(\gamma^n)$

On the other hand, if $0 < l < n+1$, we know that there are $\binom{n}{l}$ paths γ^n such that $b(\gamma^n) = l$ and $\binom{n}{l-1}$ paths γ^n such that $b(\gamma^n) = l - 1$. Therefore, there are

$$\binom{n}{l} + \binom{n}{l-1} = \binom{n+1}{l}$$

eigenvectors (γ^n, x) of eigenvalue $n + 1 - 2l$.
(ii) From the above lemma, it is clear that the set

$$\{D_{n-1}(\gamma^{n-1}, e) \mid \gamma^{n-1} \in Q_{n-1}\}$$

is a basis of $\operatorname{Im} D_{n-1}$ consisting of eigenvectors. We also have that $D_{n-1}(\gamma^{n-1}, e)$ is an eigenvector of eigenvalue $\nu(\gamma^{n-1})$. Since $a(\gamma^{n-1}) + b(\gamma^{n-1}) = n - 1$ for all paths γ^{n-1}, $\nu(\gamma^{n-1}) = n - 1 - 2b(\gamma^{n-1})$ where $b(\gamma^n)$ varies from 0 to $n - 1$. Therefore the eigenvalues are of the form $n - 1 - 2l$ where l varies from 0 to $n - 1$ and there are $\binom{n-1}{l}$ eigenvectors of eigenvalue $n + 1 - 2l$. $\square$

Recall the following result from Lie theory:

LEMMA 5.3.4 (General Multiplicity Formula [**BH06**]). *Let V a finite dimensional $sl_2(k)$-module. For every integer t, let V_t be the eigenspace of H of eigenvalue t. Then for any nonnegative integer t, the number of copies of Γ_t that appear in the decomposition into direct sum of irreducibles is* $\dim V_{t-2} - \dim V_t$

A consequence of the above lemma is the following result:

LEMMA 5.3.5. *Let k be an algebraically closed field of characteristic zero, Q be a quiver and $A = kQ/ < Q_2 >$. For $n \geq 1$,*

$$h(n) = \max\{l \mid n + 1 - 2l \geq 0\}.$$

For $l = 0, \dots, h(n)$,

$$p(n,l) = \begin{cases} \begin{pmatrix} n \\ l \end{pmatrix} & \text{if } l = 0 \\ \\ \begin{pmatrix} n \\ l \end{pmatrix} - \begin{pmatrix} n \\ l-1 \end{pmatrix} & \text{if } l \geq 1 \end{cases}$$

Then

(1) *the decomposition into direct sum of irreducibles of* $k(Q_n \parallel Q_1)$ *as* $sl_2(k)$ *Lie module is given by*

$$k(Q_n \parallel Q_1) \cong \bigoplus_{l=0}^{h(n)} \Gamma_{n+1-2l}^{p(n+1,l)}.$$

(2) *the decomposition into direct sum of irreducibles of* $\operatorname{Im} D_{n-1}$ *as* $sl_2(k)$ *Lie module is given by*

$$\operatorname{Im} D_{n-1} \cong \bigoplus_{l=0}^{h(n)-1} \Gamma_{n-1-2k}^{p(n-1,l)}.$$

Proposition 5.3.6. *Let* k *be an algebraically closed field of characteristic zero,* Q *be the two-loops quiver and* $A = kQ/ < Q_2 >$. *For* $n \geq 1$ *and* $l = 0, \dots, h(n)$,

$$q(n,l) = \begin{cases} \begin{pmatrix} n-1 \\ l \end{pmatrix} & \text{if } l = 0,1 \\ \\ \begin{pmatrix} n+1 \\ l \end{pmatrix} - \begin{pmatrix} n+1 \\ l-1 \end{pmatrix} - \begin{pmatrix} n-1 \\ l-1 \end{pmatrix} + \begin{pmatrix} n-1 \\ l-2 \end{pmatrix} & \text{if } l \geq 2 \end{cases}$$

The decomposition of $HH^n(A)$ *into a direct sum of irreducible Lie modules over* $sl_2(k)$ *is given by*

$$HH^n(A) \cong \bigoplus_{l=0}^{h(n)} \Gamma_{n+1-2l}^{q(n,l)}.$$

where Γ_t^q *denotes the direct sum of* q *copies of* Γ_t *that is the unique irreducible* $sl_2(k)$*-module of dimension* $t+1$.

Algorithm. There is an algorithm that give us the decomposition of $HH^n(A)$ into direct sum of irreducible modules, which is described in the previous section. We will explain it again in this paragraph for the case of the two-loops quiver. We use the following table to write such decomposition:

n	Γ_0	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	$\cdots$
$HH^2(A)$		1		1					
$\vdots$									
$HH^n(A)$	q_0	q_1	q_2	q_3	q_4	q_5	q_6	q_7	$\cdots$

In the above table, at the row $\mathsf{HH}^n(A)$, the number that appears in the column Γ_t stands for the number of copies of the irreducible module Γ_t that appears in the decomposition of $\mathsf{HH}^n(A)$. We leave a blank space if no Γ_t appears in the decomposition of $\mathsf{HH}^n(A)$. We fix the first row of the table with the decomposition of $\mathsf{HH}^2(A)$. Now, given the entries of the row $\mathsf{HH}^n(A)$, we can fill out the coefficients of the next row, this is for $\mathsf{HH}^{n+1}(A)$, in the following manner:

(1) Add a column just before the column Γ_0, consisting of zeros.
(2) Write the coefficients of the next row using the rule from Pascal's triangle: add the number directly above and to the left with the number directly above and to the right.
(3) If n is even the number of copies of Γ_1 that appear in the decomposition of HH^{n+1} is equal to the number of copies of Γ_0 that appear in the decomposition of $\mathsf{HH}^n(A)$

	$(-)$	Γ_0	Γ_1	$\cdots$	Γ_{t-1}	Γ_t	Γ_{t+1}	$\cdots$
$\mathsf{HH}^n(A)$	0	q_0	q_1	$\cdots$	q_{t-1}	q_t	q_{t+1}	$\cdots$
$\mathsf{HH}^{n+1}(A)$	0	q_1	$\cdots$	$\cdots$	$\cdots$	$q_{t-1}+q_{t+1}$	$\cdots$	$\cdots$

LEMMA 5.3.7. *(1) If n is even then $q(n,h(n)) = q(n+1,h(n+1))$.*
(2) If $n \geq 2$ then $q(n,l)+q(n,l+1) = q(n+1,l+1)$.

PROOF. For the first equality, we verify by a direct computation for $n = 2$ and $n = 4$. For $n \geq 6$, we use that if n is even

$$\binom{n+1}{n/2} = \binom{n+1}{n/2+1}.$$

For the second equality, we verify by a direct computation for $l = 0$ and $l = 1$. For $l \geq 2$, we use Pascal triangle's rule:

$$\binom{n}{l} + \binom{n}{l+1} = \binom{n+1}{l+1}.$$

□

REMARK. Moreover,

$$q(n,2) = \binom{n-1}{2}.$$

Finally, once we have the decomposition of $\mathsf{HH}^n(A)$ into a direct sum of irreducible modules over $\mathfrak{sl}_2 k$, we return to study $\mathsf{HH}^n(A)$ as a $\mathsf{HH}^1(A)$-module.

COROLLARY 5.3.8.

$$\mathsf{HH}^n(A) \cong \bigoplus_{l=0}^{h(n)} \Gamma_{n+1-2l}^{q(n,l)} \otimes k.$$

as Lie modules over $\mathsf{HH}^1(A)$.

PROOF. Notice that

$$I.(\gamma^n, x) = (1 - a(\gamma^n) - b(\gamma^n))(\gamma^n, x) = (1 - n)(\gamma^n, x)$$

$\square$

APPENDIX A

A comparison map between the bar projective resolution and the reduced bar projective resolution.

In this appendix, we deal with finite dimensional k-algebras whose semisimple part (i.e the quotient by its Jacobson radical) is isomorphic to a finite number of copies of the field. Monomial algebras are a particular case of these algebras.

A.1. Two projective resolutions.

The usual A^e-projective resolution of A used to calculate the Hochschild cohomology groups is the standard bar resolution. The *standard bar resolution*, that we will denote **S**, is given by the following exact sequence:

$$\mathbf{S} = \qquad \cdots \to A^{\otimes_k^{n+1}} \xrightarrow{\delta} A^{\otimes_k^n} \xrightarrow{\delta} \cdots \xrightarrow{\delta} A^{\otimes_k^3} \xrightarrow{\delta} A \underset{k}{\otimes} A \xrightarrow{\mu} A \to 0$$

where μ is the multiplication and the A^e-morphisms δ are given by

$$\delta(x_1 \otimes \cdots \otimes x_{n+1}) = \sum_{i=1}^{n} (-1)^{i+1} x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1}$$

where $x_i \in A$ and $\otimes$ means $\underset{k}{\otimes}$.

Now, the A^e-projective resolution of A used in [**Cib98**] to compute the Hochschild cohomology groups of a monomial radical square zero is the *reduced bar resolution.* It is defined for a finite dimensional k-algebra A whose Wedderburn-Malcev decomposition is given by the direct sum $A = E \oplus r$ where r is the Jacobson radical of A and $E \cong A/r \cong k \times k \cdots \times k$. In the sequel A denotes an algebra verifying those conditions. Let us denote **R** the reduced bar resolution. It is given by the following exact sequence:

$$\mathbf{R} := \cdots \to A \underset{E}{\otimes} r^{\otimes_E^{n+1}} \underset{E}{\otimes} A \xrightarrow{\delta} A \underset{E}{\otimes} r^{\otimes_E^{n}} \underset{E}{\otimes} A \xrightarrow{\delta} \cdots \xrightarrow{\delta} A \underset{E}{\otimes} r \underset{E}{\otimes} A \xrightarrow{\delta} A \underset{E}{\otimes} A \xrightarrow{\mu} A \to 0$$

where μ is the multiplication and the A^e-morphisms δ are given by

$$\begin{aligned}\delta(a \otimes x_1 \otimes \cdots \otimes x_{n+1} \otimes b) &= a x_1 \otimes x_2 \otimes \cdots \otimes x_{n+1} \otimes b \\ &+ \textstyle\sum_{i=1}^{n} (-1)^i a \otimes x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes b \\ &+ (-1)^{n+1}\, a \otimes x_1 \otimes \cdots \otimes x_n \otimes x_{n+1} b\end{aligned}$$

where $a, b \in A$, $x_i \in r$ and $\otimes$ means $\underset{E}{\otimes}$. The proof that this sequence is a projective resolution can be found in [**Cib90**].

A.2. Comparison maps.

Theorically, a comparison map exists between these two projective resolutions. The objective of this section is to give an explicit comparison map between the projective resolutions **S** and **R** in both directions. Such comparison map will induce some quasi-isomorphisms between the Hochschild cochain complex and the complex induced by the reduced bar resolution. The explicit calculations of these quasi-isomorphisms, enables to reformulate the Gerstenhaber bracket.

In this paragraph, we are going to give two maps of complexes:

$$p : \mathbf{S} \to \mathbf{R} \text{ and } s : \mathbf{R} \to \mathbf{S}.$$

This means we will define maps (p_n) and (s_n) such that the following diagrams (10)

$$\begin{tikzcd}
\cdots A \underset{k}{\otimes} A^{\otimes_k^{n+1}} \underset{k}{\otimes} A \arrow[r, "\delta"] \arrow[d, "p_{n+1}"'] & A \underset{k}{\otimes} A^{\otimes_k^{n}} \underset{k}{\otimes} A \cdots \arrow[d, "p_n"'] & A \underset{k}{\otimes} A \arrow[r, "\mu"] \arrow[d, "p_0"'] & A \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \\
\cdots A \underset{E}{\otimes} r^{\otimes_E^{n+1}} \underset{E}{\otimes} A \arrow[r, "\delta"] \arrow[d, "s_{n+1}"'] & A \underset{E}{\otimes} r^{\otimes_E^{n}} \underset{E}{\otimes} A \cdots \arrow[d, "s_n"'] & A \underset{E}{\otimes} A \arrow[r, "\mu"] \arrow[d, "s_0"'] & A \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \\
\cdots A \underset{k}{\otimes} A^{\otimes_k^{n+1}} \underset{k}{\otimes} A \arrow[r, "\delta"] & A \underset{k}{\otimes} A^{\otimes_k^{n}} \underset{k}{\otimes} A \cdots & A \underset{k}{\otimes} A \arrow[r, "\mu"] & A \arrow[r] & 0
\end{tikzcd}$$

commute.

Map (p_n). We define p_0 as the linear map given by

$$\begin{array}{rccc} p_0 : & A \underset{k}{\otimes} A & \to & A \underset{E}{\otimes} A \\ & a \underset{k}{\otimes} b & \mapsto & a \underset{E}{\otimes} b \end{array} .$$

Now, let $n \geq 1$. Define

$$p_n : A \underset{k}{\otimes} A^{\otimes_k^n} \underset{k}{\otimes} A \to A \underset{E}{\otimes} r^{\otimes_E^n} \underset{E}{\otimes} A$$

as the linear map given by

$$a \underset{k}{\otimes} x_1 \underset{k}{\otimes} \cdots \underset{k}{\otimes} x_i \underset{k}{\otimes} \cdots \underset{k}{\otimes} x_{n+1} \underset{k}{\otimes} b \mapsto a \underset{E}{\otimes} \pi(x_1) \underset{E}{\otimes} \cdots \underset{E}{\otimes} \pi(x_i) \underset{E}{\otimes} \cdots \underset{E}{\otimes} \pi(x_{n+1}) \underset{E}{\otimes} b.$$

where π denotes the projection map from A to the Jacobson radical. Notice that p_n is an A^e-morphism for all n.

In order to define the maps (s_n) we introduce some notation. In the sequel, let E_0 denote a complete system of orthogonal, idempotents and primitives of E. Note that the set E_0 is finite.

Remark. Now, consider elements of $A \underset{E}{\otimes} r^{\otimes_E^n} \underset{E}{\otimes} A$ of the form

$$ae_{j_1} \underset{E}{\otimes} \cdots \underset{E}{\otimes} e_{j_{i-1}} x_{i-1} e_{j_i} \underset{E}{\otimes} e_{j_i} x_i e_{j_{i+1}} \underset{E}{\otimes} e_{j_{i+1}} x_{i+1} e_{j_{i+2}} \underset{E}{\otimes} \cdots \underset{E}{\otimes} e_{j_{n+1}} b$$

where each e_{j_i} is in E_0, a, b are in A and x_i in r. It is not difficult to see that those elements generate the vector space $A \underset{E}{\otimes} r^{\otimes_E^n} \underset{E}{\otimes} A$. Indeed,

$$a \underset{E}{\otimes} x_1 \underset{E}{\otimes} \cdots \underset{E}{\otimes} x_i \underset{E}{\otimes} \cdots \underset{E}{\otimes} x_n \underset{E}{\otimes} b =$$

$$\sum_{j_1,\dots,j_{n+1}} ae_{j_1} \underset{E}{\otimes} \cdots \underset{E}{\otimes} e_{j_{i-1}} x_{i-1} e_{j_i} \underset{E}{\otimes} e_{j_i} x_i e_{j_{i+1}} \underset{E}{\otimes} e_{j_{i+1}} x_{i+1} e_{j_{i+2}} \underset{E}{\otimes} \cdots \underset{E}{\otimes} e_{j_{n+1}} b$$

where the sum is over all $(n+1)$-tuples $(e_{j_1}, \dots, e_{j_i}, \dots, e_{j_{n+1}})$ of elements of E_0.

MAP (s_n). Define s_0 as the linear map given by

$$\begin{array}{rccc} s_0 : & A \underset{E}{\otimes} A & \rightarrow & A \underset{k}{\otimes} A \\ & ae \underset{E}{\otimes} eb & \mapsto & ae \underset{k}{\otimes} eb \end{array} .$$

So we have

$$s_0(a \underset{E}{\otimes} b) = \sum_{e \in E_0} ae \underset{k}{\otimes} eb \,.$$

This map is well defined since $s_0(ae \underset{E}{\otimes} b) = ae \underset{k}{\otimes} eb = s_0(a \underset{E}{\otimes} eb)$ for all $e \in E$. Now, let $n \geq 1$. Define

$$s_n : A \underset{E}{\otimes} r^{\otimes_E^n} \underset{E}{\otimes} A \rightarrow A \underset{k}{\otimes} A^{\otimes_k^n} \underset{k}{\otimes} A$$

as the linear map given by

$$\begin{array}{l} ae_{j_1} \underset{E}{\otimes} \cdots \underset{E}{\otimes} e_{j_{i-1}} x_{i-1} e_{j_i} \underset{E}{\otimes} e_{j_i} x_i e_{j_{i+1}} \underset{E}{\otimes} e_{j_{i+1}} x_{i+1} e_{j_{i+2}} \underset{E}{\otimes} \cdots \underset{E}{\otimes} e_{j_{n+1}} b \mapsto \\ ae_{j_1} \underset{k}{\otimes} \cdots \underset{k}{\otimes} e_{j_{i-1}} x_{i-1} e_{j_i} \underset{k}{\otimes} e_{j_i} x_i e_{j_{i+1}} \underset{k}{\otimes} e_{j_{i+1}} x_{i+1} e_{j_{i+2}} \underset{k}{\otimes} \cdots \underset{k}{\otimes} e_{j_{n+1}} b \end{array}$$

where each e_{j_i} is in E_0. So we have that

$$s_n(a \underset{E}{\otimes} x_1 \underset{E}{\otimes} \cdots \underset{E}{\otimes} x_i \underset{E}{\otimes} \cdots \underset{E}{\otimes} x_n \underset{E}{\otimes} b) =$$

$$\sum_{j_1,\dots,j_{n+1}} ae_{j_1} \underset{k}{\otimes} \cdots \underset{k}{\otimes} e_{j_{i-1}} x_{i-1} e_{j_i} \underset{k}{\otimes} e_{j_i} x_i e_{j_{i+1}} \underset{k}{\otimes} e_{j_{i+1}} x_{i+1} e_{j_{i+2}} \underset{k}{\otimes} \cdots \underset{k}{\otimes} e_{j_{n+1}} b$$

where the sum is over all $(n+1)$-tuples $(e_{j_1}, \dots, e_{j_i}, \dots, e_{j_{n+1}})$ of elements of E_0. Notice that s_n is an A^e-morphism.

REMARK. It is clear that $p_n s_n = \mathrm{id}_{A \underset{E}{\otimes} r^{\otimes_E^n} \underset{E}{\otimes} A}$.

LEMMA A.2.1. *The maps*

$$p : \mathbf{S} \rightarrow \mathbf{R} \text{ and } s : \mathbf{R} \rightarrow \mathbf{S}$$

defined above are maps of complexes.

PROOF. A straightforward verification shows that the diagram (10) is commutative. $\square$

A.3. Induced quasi-isomorphism.

We will denote the *Hochschild cochain complex* by $\mathbf{C}^\bullet(\mathbf{A},\mathbf{A})$. Recall that it is defined by the complex,

$$\begin{aligned} 0 \to A \xrightarrow{\delta} \mathsf{Hom}_k(A,A) \xrightarrow{\delta} \quad \cdots \\ \cdots \longrightarrow \quad \mathsf{Hom}_k(A^{\otimes_k^n},A) \xrightarrow{\delta} \mathsf{Hom}_k(A^{\otimes_k^{n+1}},A) \quad \cdots \end{aligned}$$

where $\delta(a)(x) = xa - ax$ for a in A and

$$\begin{aligned} \delta f(x_1 \otimes \cdots \otimes x_n \otimes x_{n+1}) = \quad & x_1 f(x_2 \otimes \cdots \otimes x_{n+1}) + \\ & \textstyle\sum_{i=1}^n (-1)^i f(x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1}) + \\ & (-1)^{n+1} f(x_1 \otimes \cdots \otimes x_n) x_{n+1} \end{aligned}$$

for f in $\mathsf{Hom}_k(A^{\otimes_k^n},A)$. Notice that after applying the functor $\mathsf{Hom}_{A^e}(-,A)$ to the standard bar resolution, the Hochschild cochain complex is obtained by identifying $\mathsf{Hom}_{A^e}(A \otimes_k A^{\otimes_k^n} \otimes_k A, A)$ to $\mathsf{Hom}_k(A^{\otimes_k^n},A)$. The *reduced complex* is obtained from the reduced bar resolution in a similar way. First we apply $\mathsf{Hom}_{A^e}(-,A)$ to the reduced bar resolution, then we identify the vector space $\mathsf{Hom}_{A^e}(A \otimes_E r^{\otimes_E^n} \otimes_E A, A)$ to $\mathsf{Hom}_{E^e}(r^{\otimes_E^n},A)$. Therefore, the reduced bar complex that we denote $\mathbf{R}^\bullet(\mathbf{A},\mathbf{A})$ is given by

$$\begin{aligned} 0 \to A^E \xrightarrow{\delta} \mathsf{Hom}_{E^e}(r,A) \xrightarrow{\delta} \quad \cdots \\ \cdots \longrightarrow \quad \mathsf{Hom}_{E^e}(r^{\otimes_E^n},A) \xrightarrow{\delta} \mathsf{Hom}_{E^e}(r^{\otimes_E^{n+1}},A) \quad \cdots \end{aligned}$$

where A^E is the subalgebra of A defined as follows:

$$A^E = \{a \in A \mid ae = ea \text{ for all } e \in E\}.$$

The differentials for the reduced complex are given through the above formulas.

In this paragraph, we will compute the quasi-isomorphisms between the Hochschild cochain complex and the reduced complex, induced by the comparison maps p and s. We will denote them by

$$p^\bullet : \mathbf{R}^\bullet(\mathbf{A},\mathbf{A}) \to \mathbf{C}^\bullet(\mathbf{A},\mathbf{A}) \quad \text{and} \quad s^\bullet : \mathbf{C}^\bullet(\mathbf{A},\mathbf{A}) \to \mathbf{R}^\bullet(\mathbf{A},\mathbf{A}).$$

Map ($p^\bullet$). In degree zero, $p_0 : A^E \to A$ is the inclusion map. For $n \geq 1$,

$$p^n : \mathsf{Hom}_{E^e}(r^{\otimes_E^n},A) \longrightarrow \mathsf{Hom}_k(A^{\otimes_k^n},A)$$

is given by

$$p^n f(x_1 \underset{k}{\otimes} \cdots \underset{k}{\otimes} x_n) = f(\pi(x_1) \underset{E}{\otimes} \cdots \underset{E}{\otimes} \pi(x_n))$$

where f is in $\mathsf{Hom}_{E^e}(r^{\otimes_E^n},A)$ and $x_i \in r$.

Map ($s^\bullet$). In degree zero, $s^0 : A \to A^E$ is given by

$$s^0(x) = \sum_{e \in E_0} exe$$

where $x \in A$. For $n \geq 1$,

$$s^n : \mathsf{Hom}_k(A^{\otimes_k^n},A) \longrightarrow \mathsf{Hom}_{E^e}(r^{\otimes_E^n},A)$$

is given by

$$s^n f(x_1 \underset{E}{\otimes} \cdots \underset{E}{\otimes} x_n) = \sum_{j_0,\ldots,j_n} e_{j_0} f(e_{j_0} x_1 e_{j_1} \underset{k}{\otimes} \cdots \underset{k}{\otimes} e_{j_{i-1}} x_i e_{j_i} \underset{k}{\otimes} \cdots \underset{k}{\otimes} e_{j_{n-1}} x_n e_{j_n}) e_{j_n}$$

where the sum is over all $(n+1)$-tuples $(e_{j_0}, \ldots, e_{j_i}, \ldots, e_{j_n})$ of elements of E_0, f is in $\mathrm{Hom}_k(A^{\otimes_k^n}, A)$ and x_i is in r.

Remark. $s^\bullet p^\bullet = \mathrm{id}_{\mathbf{R}^\bullet(\mathbf{A},\mathbf{A})}$.

APPENDIX B

Gerstenhaber and reduced bracket.

The Gerstenhaber bracket is defined on the Hochschild cohomology groups using the Hochschild complex. In this chapter we will define the reduced bracket using the reduced complex. We show that the Gerstenhaber bracket and the reduced bracket provides the same graded Lie algebra structure on $HH^{*+1}(A)$. We begin recalling the Gerstenhaber bracket in order to fix notation.

B.1. Gerstenhaber bracket.

Set $C^0(A, A) := A$ and for $n \geq 1$, we will denote the space of Hochschild cochains by

$$C^n(A, A) = \mathrm{Hom}_k(A^{\otimes_k^n}, A).$$

In [**Ger63**], Gerstenhaber defined a right pre-Lie system $\{C^n(A, A), \circ_i\}$ where elements of $C^n(A, A)$ are declared to have degree $n - 1$. The operation $\circ_i$ is given as follows.

Definition. Given $n \geq 1$, let us fix $i = 1, \dots, n$. The bilinear map

$$\circ_i : C^n(A, A) \times C^m(A, A) \longrightarrow C^{n+m-1}(A, A)$$

is given by the following formula:

$$f^n \circ_i g^m(x_1 \otimes \cdots \otimes x_{n+m-1}) := f^n(x_1 \otimes \cdots \otimes g^m(x_i \otimes \cdots \otimes x_{i+m-1}) \otimes \cdots \otimes x_{n+m-1})$$

where f^n is in $C^n(A, A)$ and g^m is in $C^m(A, A)$.

Then he proved that such pre-Lie system induces a graded pre-Lie algebra structure on

$$C^{*+1}(A, A) := \bigoplus_{n=1}^{\infty} C^n(A, A)$$

by defining an operation $\circ$ as follows.

Definition. Let f^n be in $C^n(A, A)$ and g^m be in $C^m(A, A)$. The bilinear map

$$\circ : C^n(A, A) \times C^m(A, A) \longrightarrow C^{n+m-1}(A, A)$$

is given by

$$f^n \circ g^m = \sum_{i=1}^{n} (-1)^{(i-1)(m-1)} f^n \circ_i g^m.$$

Finally, $C^{*+1}(A, A)$ becomes a graded Lie algebra by defining the bracket as the graded commutator of $\circ$. So we have the following definition:

Definition. The *Gerstenhaber bracket* is the bilinear map

$$[-\,,\,-]: C^n(A,A)\times C^m(A,A)\longrightarrow C^{n+m-1}(A,A)$$

given as follows:

$$[f^n\,,\,g^m] := f^n\circ g^m - (-1)^{(n-1)(m-1)}g^m\circ f^n.$$

Clearly, the Gerstenhaber bracket restricted to $C^1(A,A)$ is the usual Lie commutator bracket.

Theorem ([**Ger63**]). *Let A be an associative k-algebra with unit. The Gerstenhaber bracket endows $C^{*+1}(A,A)$ with a graded Lie algebra structure, i.e. the bracket satisfied the following conditions:*

(1) $[f^n,g^m] = -(-1)^{(n-1)(m-1)}[g^m,f^n]$
(2) $(-1)^{(n-1)(p-1)}[[f^n,g^m],h^p]+$
$(-1)^{(p-1)(m-1)}[[h^p,f^n],g^m]+$
$(-1)^{(m-1)(n-1)}[g^m,h^p],f^n]=0$

where f^n, g^m and h^p are in $C^n(A,A)$, $C^m(A,A)$ and $C^p(A,A)$ respectively.

Proposition ([**Ger63**]). *The Gerstenhaber bracket satisfies:*

$$\delta[f^n\,,\,g^m] = [f^n\,,\,\delta g^m] + (-1)^{m-1}[\delta f^n\,,\,g^m]$$

where δ is the differential of the Hochschild cochain complex.

This result implies that the bracket of two cocycles is a cocyle and that the bracket of a cocyle and coboundary is a coboundary. Therefore, the bilinear map:

$$[-\,,\,-]: HH^n(A)\times HH^m(A)\longrightarrow HH^{n+m-1}(A)$$

is well defined.

Corollary ([**Ger63**]). *Let A be an associative k-algebra with unit. Then $HH^{*+1}(A)$ endowed with the induced Gerstenhaber bracket is a graded Lie algebra.*

B.2. Reduced Bracket.

In order to define the reduced bracket, we proceed in the same way as Gerstenhaber did. We will define the reduced bracket as the graded commutator of an operation $\underset{R}{\circ}$. Such operation will be given by $\underset{i}{\circ}$. Denote $C^n_E(r,A)$ the cochain space of the reduced complex, this is

$$C^n_E(r,A) = \mathrm{Hom}_{E^e}(r^{\otimes^n_E},A).$$

Definition. Let $n\geq 1$ and fix $i=1,\dots,n$. The bilinear map

$$\underset{i}{\circ}: C^n_E(r,A)\times C^m_E(r,A)\to C^{n+m-1}_E(r,A)$$

is given by the following formula:

$$f^n\underset{i}{\circ}g^m(x_1\underset{E}{\otimes}\cdots\underset{E}{\otimes}x_{n+m-1}) := f^n(x_1\underset{E}{\otimes}\cdots\underset{E}{\otimes}\pi g^m(x_i\underset{E}{\otimes}\cdots\underset{E}{\otimes}x_{i+m-1})\underset{E}{\otimes}\cdots\underset{E}{\otimes}x_{n+m-1})$$

where f^n is in $C^n_E(r, A)$ and g^m is in $C^m_E(r, A)$ and $x_1, \ldots, x_{n+m-1}$ are in r. Let us remark that the image of g^m does not necessarily belong to the radical but the image of πg^m clearly does. Therefore $f^n \underset{i}{\circ} g^m$ is well defined.

Then we can define $\underset{R}{\circ}$ on

$$C^{*+1}_E(r, A) = \bigoplus_{n=1}^{\infty} C^n_E(r, A)$$

as above but using $\underset{i}{\circ}$ instead of $\circ_i$. This means that

$$f^n \underset{R}{\circ} g^m = \sum_{i=1}^{n} (-1)^{(i-1)(m-1)} f^n \underset{i}{\circ} g^m$$

Let us remark that $\underset{R}{\circ}$ is a graded operation on $C^{*+1}_E(r, A)$ by declaring elements of $C^n_E(r, A)$ to have degree $n - 1$.

Definition. We call the *reduced bracket*, denoted $[-\,,\,-]_R$, to be the graded commutator bracket of $\underset{R}{\circ}$:

$$[-\,,\,-]_R : C^n_E(r, A) \times C^m_E(r, A) \longrightarrow C^{n+m-1}_E(r, A)$$

given by

$$[f^n\,,\,g^m]_R = f^n \underset{R}{\circ} g^m - (-1)^{(n-1)(m-1)} g^m \underset{R}{\circ} f^n.$$

The following lemmas will relate the Gerstenhaber bracket and the reduced bracket.

Lemma B.2.1. *We have the following formula:*

$$[f^n\,,\,g^m]_R = s^{n+m-1}[\,p^n f^n\,,\,p^m g^m\,].$$

Proof. A straightforward verification shows that

$$f^n \underset{i}{\circ} g^m = s^{n+m-1}(\,p^n f^n \circ_i p^m g^m\,).$$

Since s^{n+m-1} is a linear application we have the formula wanted. □

Lemma B.2.2. *We have the following formula:*

$$p^{n+m-1}[\,f^n\,,\,g^m\,]_R = [\,p^n f^n\,,\,p^m g^m\,]$$

Proof. Since p^{n+m-1} is a complex morphism, we prove that

$$p^{n+m-1}(f^n \underset{i}{\circ} g^m) = p^n f^n \circ_i p^m g^m$$

by a direct computation. □

We will write p^* for the morphism

$$p^* : C^{*+1}_E(r, A) \longrightarrow C^{*+1}(A, A)$$

induced by $p^\bullet$. We have the following proposition as a consequence of the above lemmas relating both brackets.

Proposition B.2.3. *The graded product* $[-\,,\,-]_R$ *endows* $C^*_E(r, A)$ *with the structure of a graded Lie algebra. We also have that* p^* *is a morphism of graded Lie algebras.*

Proof. Using Lemma B.2.1, it is easy to see that the reduced bracket satisfies the graded antisymmetric property as a consequence of the fact that the Gerstenhaber bracket satisfies the same condition. For the graded Jacobi identity, we proceed in the same way. First, we write a formula that relates both brackets, using Lemma B.2.1 and Lemma B.2.2

$$\begin{aligned}[[f^n\,,\,g^m]_R\,,\,h^l]_R &= s^{n+m+p-2}[p^{n+m-1}[f^n\,,\,g^m]_R\,,\,p^l h^l] \\ &= s^{n+m+p-2}[[p^n f^n\,,\,p^m g^m]\,,\,p^l h^l]\end{aligned}$$

Then, using the linearity of $s^{n+m+p-2}$ and the fact that the Gerstenhaber bracket satisfies the graded Jacobi identity, we have proved that $[-\,,\,-]_R$ satisfies the two conditions of the definition of graded Lie algebra. Finally, p^* becomes a Lie graded morphism using Lemma B.2.2. □

Lemma B.2.4. *Let* δ *be the differential of the Hochschild cocomplex. Then*

$$\delta[f^n\,,\,g^m]_R = [f^n\,,\,\delta g^m]_R + (-1)^{m-1}[\delta f^n\,,\,g^m]_R.$$

Hence we have a well defined bracket on the Hochschild cohomology vector spaces:

$$[-\,,\,-]_R : HH^n(A) \times HH^m(A) \longrightarrow HH^{n+m-1}(A)\,.$$

Proof. We have that

$$\begin{aligned}\delta[f^n\,,\,g^m]_R &= \delta s^{n+m-1}[p^n f^n\,,\,p^m g^m] \\ &= s^{n+m-1}\delta[p^n f^n\,,\,p^m g^m] \\ &= s^{n+m-1}[p^n f^n\,,\,\delta p^m g^m] + (-1)^{m-1}s^{n+m-1}[\delta p^n f^n\,,\,p^m g^m] \\ &= s^{n+m-1}[p^n f^n\,,\,p^m \delta g^m] + (-1)^{m-1}s^{n+m-1}[p^n \delta f^n\,,\,p^m g^m] \\ &= [f^n\,,\,\delta g^m]_R + (-1)^{m-1}[\delta f^n\,,\,g^m]_R\end{aligned}$$

□

We have equipped $HH^{*+1}(A)$ with a graded Lie algebra structure induced by the reduced bracket. We know that $HH^{*+1}(A)$ is already a graded Lie algebra and this structure is given by the Gerstenhaber bracket. Then we have the following theorem.

Theorem B.2.5. *The graded Lie algebra* $HH^{*+1}(A)$ *endowed with the Gerstenhaber bracket is isomorphic to* $HH^{*+1}(A)$ *endowed with the reduced bracket.*

Proof. We continue to write $\overline{p^*}$ for the automorphism of $HH^{*+1}(A)$ given by the family of morphisms $(\overline{p^n})$. A direct consequence of the above proposition is that $\overline{p^*}$ becomes an isomorphism of graded Lie algebras. □

Bibliography

[Bar97] M.J. Bardzell, *The alternating syzygy behavior of monomial algebras*, J. Algebra **188** (1997), no. 1, 69–89.

[BH06] M.R. Bremner and I.R. Hentzel, *Alternating triple systems with simple Lie algebras of derivations*, Non-associative algebra and its applications, Lect. Notes Pure Appl. Math., vol. 246, Chapman & Hall/CRC, Boca Raton, FL, 2006, pp. 55–82.

[BM98] M.J. Bardzell and E.N. Marcos, *Induced boundary maps for the cohomology of monomial and Auslander algebras*, Algebras and modules, II (Geiranger, 1996), CMS Conf. Proc., vol. 24, Amer. Math. Soc., Providence, RI, 1998, pp. 47–54.

[Bus06] J.C. Bustamante, *The cohomology structure of string algebras*, J. Pure Appl. Algebra **204** (2006), no. 3, 616–626.

[Cib90] C. Cibils, *Rigidity of truncated quiver algebras*, Adv. Math. **79** (1990), no. 1, 18–42.

[Cib98] ———, *Hochschild cohomology algebra of radical square zero algebras*, Algebras and modules, II (Geiranger, 1996), CMS Conf. Proc., vol. 24, Amer. Math. Soc., Providence, RI, 1998, pp. 93–101.

[Cib00] ———, *On* H^1 *of finite dimensional algebras*, Bol. Acad. Nac. Cienc. (Córdoba) **65** (2000), 73–80, Colloquium on Homology and Representation Theory (Spanish) (Vaquerías, 1998).

[CS97] C. Cibils and A. Solotar, *Hochschild cohomology algebra of abelian groups*, Arch. Math. (Basel) **68** (1997), no. 1, 17–21.

[Dyn52] E. B. Dynkin, *Semisimple subalgebras of semisimple Lie algebras*, Mat. Sbornik N.S. **30(72)** (1952), 349–462.

[EH99] K. Erdmann and T. Holm, *Twisted bimodules and Hochschild cohomology for self-injective algebras of class* A_n, Forum Math. **11** (1999), no. 2, 177–201.

[EHS02] K. Erdmann, T. Holm, and N. Snashall, *Twisted bimodules and Hochschild cohomology for self-injective algebras of class* A_n*. II*, Algebr. Represent. Theory **5** (2002), no. 5, 457–482.

[ES98] K. Erdmann and N. Snashall, *On Hochschild cohomology of preprojective algebras. I, II*, J. Algebra **205** (1998), no. 2, 391–412, 413–434.

[Eu07a] Ching-Hwa Eu, *The calculus structure of the Hochschild homology/cohomology of preprojective algebras of Dynkin quivers*, arXiv:0706.2418v2[math.RT] (2007).

[Eu07b] ———, *The product in the Hochschild cohomology ring of preprojective algebras of Dynkin quivers*, arXiv:math/0703568v2[math.RT] (2007).

[EW06] K. Erdmann and M.J. Wildon, *Introduction to Lie algebras*, Springer Undergraduate Mathematics Series, Springer-Verlag London Ltd., London, 2006.

[FH91] W. Fulton and J. Harris, *Representation theory*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991, A first course, Readings in Mathematics.

[FX06] Jinmei Fan and Yunge Xu, *On Hochschild cohomology rings of Fibonacci algebras*, Front. Math. China **1** (2006), no. 4, 526–537.

[GA08] P. Tirao G. Ames, L. Cagliero, *Comparison morphisms and the Hochschild cohomology ring of truncated quiver algebras*, arXiv:math/0603056v2[math.KT] (2008).

[Ger63] M. Gerstenhaber, *The cohomology structure of an associative ring*, Ann. of Math. (2) **78** (1963), 267–288.

[GS06] E.L. Green and N. Snashall, *The Hochschild cohomology ring modulo nilpotence of a stacked monomial algebra*, Colloq. Math. **105** (2006), no. 2, 233–258.

[GSS03] E.L. Green, N. Snashall, and Ø. Solberg, *The Hochschild cohomology ring of a self-injective algebra of finite representation type*, Proc. Amer. Math. Soc. **131** (2003), no. 11, 3387–3393 (electronic).

[GSS06] ______, *The Hochschild cohomology ring modulo nilpotence of a monomial algebra*, J. Algebra Appl. **5** (2006), no. 2, 153–192.

[Hap89] D. Happel, *Hochschild cohomology of finite-dimensional algebras*, Séminaire d'Algèbre Paul Dubreil et Marie-Paul Malliavin, 39ème Année (Paris, 1987/1988), Lecture Notes in Math., vol. 1404, Springer, Berlin, 1989, pp. 108–126.

[Hol96] T. Holm, *The Hochschild cohomology ring of a modular group algebra: the commutative case*, Comm. Algebra **24** (1996), no. 6, 1957–1969.

[Kel04] B. Keller, *Hochschild cohomology and derived Picard groups*, J. Pure Appl. Algebra **190** (2004), no. 1-3, 177–196.

[Ric91] J. Rickard, *Derived equivalences as derived functors*, J. London Math. Soc. (2) **43** (1991), no. 1, 37–48.

[SA02] M. Suarez-Alvarez, *Algebra structure on the Hochschild cohomology of the ring of invariants of a Weyl algebra under a finite group*, J. Algebra **248** (2002), no. 1, 291–306.

[SA07] ______, *Applications of the change-of-rings spectral sequence to the computation of Hochschild cohomology*, arXiv:math/0707.3210 (2007).

[SF08] S. Sanchez-Flores, *The Lie module structure on the the Hochschild cohomology groups of monomial of algebras of radical square zero*, J. Algebra **320** (2008), 4249–4269.

[Str06] C. Strametz, *The Lie algebra structure on the first Hochschild cohomology group of a monomial algebra*, J. Algebra Appl. **5** (2006), no. 3, 245–270.

[SW00] S.F. Siegel and S.J. Witherspoon, *The Hochschild cohomology ring of a cyclic block*, Proc. Amer. Math. Soc. **128** (2000), no. 5, 1263–1268.

Résumé.

La structure de Lie de la cohomologie de Hochschild d'algèbres monomiales.
Cette thèse porte sur la structure de Lie de la cohomologie de Hochschild, donnée par le crochet de Gerstenhaber. Plus précisément, on étudie *la structure d'algèbre de Lie du premier groupe de cohomologie et la structure de module de Lie des groupes de cohomologie de Hochschild* de certaines algèbres monomiales. Une *algèbre monomiale* est définie comme le quotient de l'algèbre de chemins d'un carquois par un idéal bilatère admissible engendré par un ensemble de chemins de longueur au moins deux. On utilise les données combinatoires intrinsèques à telles algèbres pour étudier la structure de Lie définie sur la cohomologie de Hochschild. En fait, on examine deux aspects de cette structure algébrique. Le premier est la *relation entre la semi-simplicité du premier groupe de cohomologie de Hochschild et la nullité des groupes de cohomologie de Hochschild.* Dans le second aspect, on se concentre sur la *structure de module de Lie des groupes de cohomologie de Hochschild d'une famille d'algèbres particulière: celles dont le radical de Jacobson au carré est nul.*

Abstract.

The Lie structure on the Hochschild cohomology d'algèbres monomiales.
This thesis is about the Lie structure on the Hochschild cohomology, given by the Gerstenhaber bracket. More precisely, we study the *Lie algebra structure of the first Hochschild cohomology group and the Lie module structure of the Hochschild cohomology groups* of some monomial algebras. The aim of this thesis is to study the Lie structure on the Hochschild cohomology of finite-dimensional monomial algebras. A *monomial algebra* is defined as the quotient of the path algebra of a quiver by a two-sided admissible ideal generated by a set of paths of length at least two. We use the intrinsic combinatorial data of such algebras to study the Lie structure defined on the Hochschild cohomology by the Gerstenhaber bracket. Actually, we discuss two aspects of such algebraic structure. The first one is the *relationship between semi-simplicity on the first Hochschild cohomology group and the vanishing of the Hochschild cohomology groups.* In the second one, we center our attention to the *Lie module structure of the Hochschild cohomology groups of a particular family of monomial algebras: those whose Jacobson radical square is zero.*

Mots-clés: algèbre monomiale, cohomologie de Hochschild, crochet de Gerstenhaber.

Discipline: Mathématiques.

Lieu de préparation de la thèse:
Institut de Mathématiques et de Modélisation de Montpellier.
Université de Montpellier II.
Case Courrier 51.
Place Eugène Bataillon.
F-34095 Montpellier Cedex 5.
France.