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Enumerative geometry of curves with exceptional secant planes

A dissertation presented

by

Ethan Cotterill

to

The Department of Mathematics

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

in the subject of

Mathematics

Harvard University

Cambridge, Massachusetts

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Contents

1 Introduction: Brill–Noether theory for pairs of linear series **6**

1.1 Acknowledgements, and a note on chronology 8

1.2 Roadmap 9

2 Validative study **14**

3 Quantitative study **27**

3.1 Test families 28

3.2 The value of N_{K_3} 32

3.3 Formulas for A and A', and their significance 38

3.4 The case $r = 1$ 41

3.5 The case $r = s$ 59

4 Divisor class calculations via multiple-point formulas **59**

4.1 Set-up for multiple-point formulas 60

4.2 Evaluation of multiple-point formulas 61

4.3 Validity of multiple-point formulas 62

4.4 The case $r = 1$ 63

4.5 From generating functions to generalized hypergeometric series 66

4.6 Examples 69

5 Secant-plane divisors on $\overline{\mathcal{M}}_g$ **70**

5.1 Recapitulation of Khosla's work 71

5.2 Slope calculations 73

5.3 Nonemptiness of secant-plane divisors with $r = 1$ 74

5.4 Slopes of secant-plane divisors with $r = 1$ 76

6 Boundary coefficients of secant-plane divisors on $\overline{\mathcal{M}}_g$ 79

6.1 Determination of b_1 79

6.2 Determination of b_2 80

7 Planes incident to linear series on a general curve when $\rho = 1$ 83

7.1 The case $r = 1$ 85

Abstract

We study curves with linear series that are exceptional with regard to their secant planes. Working in the framework of an extension of Brill-Noether theory to pairs of linear series, we prove that a general curve of genus g has no exceptional secant planes, in a very precise sense. We also address the problem of computing the number of linear series with exceptional secant planes in a one-parameter family in terms of tautological classes associated with the family. We obtain conjectural generating functions for the tautological coefficients of secant-plane formulas associated to series g_m^{2d-1} that admit d-secant $(d-2)$-planes. We also describe a strategy for computing the classes of divisors associated to exceptional secant plane behavior in the Picard group of the moduli space of curves in a couple of naturally-arising infinite families of cases, and we give a formula for the number of linear series with exceptional secant planes on a general curve equipped with a one-dimensional family of linear series.

1 Introduction: Brill–Noether theory for pairs of linear series

Determining when an abstract curve C comes equipped with a map to $\mathbb{P}^s$ of degree m is central to curve theory. There is a *quantitative* aspect of this study, which involves determining formulas that describe the expected behavior of linear series along a curve. There is also an aspect that we will call *validative*: it involves checking that the expected behavior holds. The Brill–Noether theorem, which is both quantitative and validative, asserts that when the Brill–Noether number $\rho(g, s, m)$ is nonnegative, ρ gives the dimension of the space of series g^s_m on a general curve C of genus g, and that there is an explicit simple formula for the class of the space of linear series $G^s_m(C)$.

In what follows, let $\mathcal{G}^s_m$ denote the moduli stack of curves of genus g with linear series g^s_m [Kh1, Kh2]. Since every linear series without base points determines a map to projective space, it is natural to identify a series with its image. Singularities of the image of a curve under the map defined by a series come about because the series admits certain subseries with base points; abusively, we refer to these subseries as "singularities" of the series itself. Eisenbud and Harris [EH1] showed that a general g^3_m on a general curve of genus g has no *double points*, or equivalently, that no inclusion

$$g^2_{m-2} + p_1 + p_2 \hookrightarrow g^3_m$$

exists, for any pair (p_1, p_2) of points along the curve. They also showed that series with double points sweep out a divisor inside the space of all series g^3_m along curves of genus g.

Generalizing the preceding example, we say that an s-dimensional linear series g^s_m *has*

a d-secant $(d-r-1)$-plane provided an inclusion

$$g^{s-d+r}_{m-d} + p_1 + \cdots + p_d \hookrightarrow g^s_m \tag{1.1}$$

exists. Geometrically, (1.1) means that the image of the g^s_m intersects a $(d-r-1)$-dimensional linear subspace of $\mathbb{P}^s$ in d-points; such a linear subspace is a "d-secant $(d-r-1)$-plane". On the other hand, since our point of view places more emphasis on linear series than on their images, it is convenient to use "d-secant $(d-r-1)$-plane" to mean any inclusion (1.1).

Next, let

$$\mu(d,s,r) := d - r(s+1-d+r).$$

The invariant μ computes the expected dimension of the space of d-secant $(d-r-1)$-planes along a fixed g^s_m. For example, when $\mu(d,s,r) = 0$, we expect that the g^s_m admits finitely many d-secant $(d-r-1)$-planes. An answer to the quantitative question of *how many* was known classically in special cases, and Macdonald [M] gave an essentially complete solution in the nineteen-fifties. The validative question has been addressed by Farkas in his recent preprint [Fa2].

In this work, we study the analogous problem in case the series g^s_m is allowed to move. Namely, we attempt to describe both quantitatively and validatively the behavior of linear series with secant planes in flat families of curves. There are a number of reasons why such a study is of broader interest. The structure of the cone of effective divisor classes on the moduli space of curves plays a fundamental rôle in the birational geometry of the moduli space of curves. A fundamental invariant of an effective divisor class is its *slope* [HM]. Over the past twenty-five years, a variety of effective divisors have been constructed,

but the question of which slopes arise among effective divisors on $\overline{\mathcal{M}}_g$ is far from settled. Whenever $\rho + \mu = -1$, we expect a divisor on $\overline{\mathcal{M}}_g$ associated to curves that admit linear series with exceptional secant plane behavior. Using techniques introduced by Eisenbud and Harris, applied to pairs of series as in (1.1), we prove that the expectation holds. Further, using techniques due to Eisenbud and Harris, Kleiman, and Ran, we determine a conjecturally complete set of relations for the tautological coefficients of the corresponding secant-plane divisor classes whenever $r = 1$ or $r = s$. When $r = 1$, we go further, and determine conjectural generating functions for the tautological coefficients; as a byproduct of this analysis, we are able to write down explicit conjectural formulas for the slope of the corresponding divisors.

1.1 Acknowledgements, and a note on chronology

This work constitutes my doctoral thesis, and it was carried out under the supervision of Joe Harris. I thank Joe, and also Steve Kleiman at MIT, for countless valuable conversations. Special thanks are also due to Izzet Coskun, Noam Elkies, and Ziv Ran for their helpful interventions at critical stages of this project. Thanks are due to Sabin Cautis, Dawei Chen, Deepak Khosla, and Maksym Fedorchuk for further help with geometry and to Emeric Deutsch, Dragos Oprea, Lauren Williams, and Akalu Tefera for help with combinatorics.

After the bulk of this work was written, the preprint [Fa2] appeared. Our Theorem 1 is proved there, via a different argument. Our proof, which was obtained independently, is significantly simpler, if less far-reaching, than Farkas'. Moreover, our argument is used in an essential way to determine the boundary coefficients b_1 and b_2 of secant-plane divisors on $\overline{\mathcal{M}}_g$. All of the results in Section 1 had been obtained by late December 2006, when a

research statement announcing them was circulated.

1.2 Roadmap

The material following this introduction is organized as follows. In the second section, we address the validative problem of determining when a curve possesses linear series with exceptional secant planes. The first two theorems establish that on a general curve, there are no linear series with exceptional secant planes when the expected number of such series is zero. We show:

Theorem 1. *Assume that $\rho = 0$ and $\mu = -1$. Under these conditions, a general curve C of genus g admits no s-dimensional linear series g^s_m with d-secant $(d-r-1)$-planes.*

We prove Theorem 1 by showing that on a certain semistable model of a g-cuspidal rational curve, there are no linear series with exceptional secant planes whenever $\rho = 0$ and $\mu = -1$. Our argument is based on Schubert calculus, together with the theory of limit linear series developed by Eisenbud and Harris, and proceeds along much the same lines as the limit linear series-based proof of the Brill–Noether theorem given in [HM, Ch. 5]. An upshot of Theorem 1 is that the loci inside $\overline{\mathcal{M}}_g$ whose classes we compute in section 2 are indeed divisors. Moreover, a slight elaboration of the argument we use to prove Theorem 1 yields a stronger statement. Namely, we have:

Theorem 2. *If $\rho + \mu = -1$, then a general curve C of genus g admits no s-dimensional linear series g^s_m with d-secant $(d-r-1)$-planes.*

Theorem 2 suggests that there are many more secant-plane divisors on the moduli space worth studying besides those treated in Sections 3 through 6 of this thesis. Finally, we

prove the following theorem, which gives geometric significance to the enumerative study carried out in Section 7:

Theorem 3. *If $\rho = 1$ and $\mu = -1$, then there are finitely many linear series g^s_m with d-secant $(d-r-1)$-planes on a general curve C of genus g.*

In Section 3, we begin our quantitative study of curves with exceptional secant planes. We attempt to solve the problem of computing the expected number of linear series with exceptional secant planes in a given one-parameter family by computing the number of exceptional series along judiciously-chosen "test families". Our general secant-plane formula reads

$$N_d^{d-r-1} = P_\alpha \alpha + P_\beta \beta + P_\gamma \gamma + P_c c + P_{\delta_0} \delta_0,$$

so five relations are needed to determine the tautological coefficients $P_\alpha, P_\beta,\ P_\gamma, P_c$, and P_{δ_0}. Whenever $r = 1$ or $r = s$, we find four out of the five relations needed; in general, we conjecturally obtain four out of five relations, with a fourth relation hinging on a conjecture about secant planes to $K3$ surfaces (Conjecture 1, Section 3.2). Section 3.3 is devoted to establishing the enumerative nature of our two most basic relations among tautological coefficients, which are derived from the study of the enumerative geometry of a *fixed* curve in projective space carried out in [ACGH].

When $r = 1$, our results are strongest, and a key player in the sections to follow enters in Section 3.4; namely, a generating function for the expected number N_d of d-secant $(d-2)$-planes to a g^{2d-2}_m. We show:

Theorem 4.

$$\sum_{d \geq 0} N_d z^d = \left(\frac{2}{(1+4z)^{1/2} + 1} \right)^{2g-2-m} \cdot (1+4z)^{\frac{g-1}{2}}.$$

The work of Lehn [Le] suggests that such a generating function should exist. We first discovered a crude version of this formula experimentally, and a conversation with Dragos Oprea led the author to deduce the "smooth" version given above. As we will see in the proof of Theorem 4, there is an intimate relation between N_d and Catalan numbers, whose generating series is $\frac{2}{(1+4z)^{1/2}+1}$. Unfortunately, to prove Theorem 4, we are forced to rely upon Macdonald's classical formula for d-secant $(d-2)$-planes to a g_m^{2d-2}, as our attempts at a purely combinatorial proof have thus far met with only partial success. We hope to settle this point more satisfactorily in a subsequent paper.

In Section 4, we deduce a conjectural fifth relation among tautological secant-plane divisor coefficients whenever $r = 1$ or $r = s$, by calculating secant-plane formulas in a variety of particular cases. Our computations, carried out in Maple, are based on an application of Kleiman's multiple point formula [Kl] to the projection of an incidence correspondence of curves and secant planes onto a Grassmann bundle of secant planes. In Section 4.4, we use our generating function for N_d obtained in Section 3.4, together with the fifth relation obtained via multiple-point formulas, to (conjecturally) determine generating functions for the tautological coefficients P, whenever $r = 1$. In Section 4.5, we use the generating functions determined in Section 4.4 in order to realize each of the tautological coefficients P as linear combinations of generalized hypergeometric functions. In Section 4.6, we list secant-plane formulas in a number of particular examples.

Sections 5 and 6 are devoted to calculations of secant-plane divisor classes on $\overline{\mathcal{M}}_g$. In Section 5.1, we review Deepak Khosla's computation of the Gysin pushforward from $A^1(\mathcal{G}_m^s)$ to $A^1(\widetilde{M}_{g,1})$, where $\widetilde{M}_{g,1} \subset \overline{\mathcal{M}}_{g,1}$ is a partial compactification of the space of smooth marked curves of genus g. Applying Khosla's result, we compute the coefficients b_λ and b_0 associated

to the Hodge class and the boundary class of irreducible nodal curves, respectively, of secant-plane divisor classes on $\overline{\mathcal{M}}_g$. As a consequence, we deduce in Section 5.2 that the slope of secant-plane divisors is computed by $\frac{b_\lambda}{b_0}$ whenever $r = 1$ or $r = s$ and $g \leq 23$. We then specialize to the case $r = 1$, and use our hypergeometric formulas for tautological coefficients to prove, in Section 5.3, that secant-plane divisors on $\overline{\mathcal{M}}_g$ are nonempty when $r = 1$. The class of each secant-plane divisor depends on the degree of incidence, d, as well as a second parameter, a. In Section 5.4, we determine explicit formulas for the slopes of secant-plane divisors in the case $r = 1$, for small values of a. We also determine the asymptotics of the slope function as d approaches infinity, for arbitrary (fixed) values a. In Section 6, we compute the coefficients b_1 and b_2 (corresponding to boundary classes δ_1 and δ_2, respectively)) of secant-plane divisors on $\overline{\mathcal{M}}_g$, as functions of b_λ and b_0. Our Theorem 6 states that the pullback of any secant-plane divisor class Sec under the map $j_2 : \overline{\mathcal{M}}_{2,1} \to \overline{\mathcal{M}}_g$ given by attaching marked genus-2 curves to a general "broken flag" curve is supported along the locus of curves with marked Weierstrass points.

Finally, in Section 7 we prove an enumerative formula for the number of linear series with exceptional secant planes along a general curve when $\rho = 1$. Namely, we have:

Theorem 7. *Let $\rho = 1, \mu = -1$. The number $N_d'^{,d-r-1}$ of linear series g^s_m with d-secant $(d-r-1)$-planes on a general curve of genus g is given by*

$$\begin{aligned} N_d'^{,d-r-1} = & \frac{(g-1)!1!\cdots s!}{(g-m+s)!\cdots(g-m+2s-1)!(g-m+2s+1)!} \cdot \\ & [(-gm+m^2-3ms+2s^2-m+s-g)A \\ & + (gd+g-md-m+2sd+2s+d+1)A'] \end{aligned}$$

where A and A' compute, respectively, the expected number of d-secant $(d-r)$-planes to a

g_m^{s+1} that intersect a general line, and the expected number of $(d+1)$-secant $(d-r)$-planes to a g_{m+1}^{s+1}. Note that formulas for A and A' were computed by Macdonald in [M].

Subsequently, we specialize to the case $r = 1$, where we obtain a hypergeometric formula for the number $N_d'^{,d-2}$ of $(2d-1)$-dimensional series with d-secant $(d-2)$-planes along a general curve when $\rho = 1$. Using that formula, we prove Theorem 8, which characterizes exactly when $N_d'^{,d-2}$ is positive, and we determine the asymptotics of $N_d'^{,d-2}$ as d approaches infinity.

2 Validative study

We begin by proving the following theorem.

Theorem 1. *Assume that $\rho = 0$ and $\mu = -1$. Under these conditions, a general curve C of genus g admits no s-dimensional linear series g^s_m with d-secant $(d-r-1)$-planes.*

The theorem asserts that on C, there are no *pairs* of series $(g^{s-d+r}_m, g^s_m) \in G^{s-d+r}_m(C) \times G^s_m(C)$ satisfying (1.1) for any choice of d-tuple $(p_1, \ldots, p_d) \in C^d$. To prove it, we specialize C to a broken flag curve $\widetilde{C}$ of the type used in Eisenbud and Harris' proof of the Giesker-Petri theorem [EH2]: $\widetilde{C}$ is a semi-stable curve comprised of a "spine" of rational curves Y_i, some of which are linked via a sequence of rational curves to g elliptic "tails" $E_1, \ldots, E_g$. See Figure 1. It then suffices to show that $\widetilde{C}$ admits no *limit linear series* $g^{s-d+r}_m \hookrightarrow g^s_m$ satisfying (1.1).

Assume for the sake of argument that $\widetilde{C}$ *does* in fact admit a (limit linear) series $g^{s-d+r}_m \hookrightarrow g^s_m$ satisfying (1.1); we will obtain a contradiction by showing that (1.1) is incompatible with basic numerical restrictions obeyed by the vanishing sequences of the g^s_m and g^{s-d+r}_m at intersection points of rational components along the spine of $\widetilde{C}$. Recall [HM, p. 276] that since $\rho(g, s, m) = 0$, the set of vanishing sequences of g^s_m at the points of attachment $p_i = Y_i \cap Y_{i-1}$ is in bijective correspondence with the set of s-dimensional series along C or $\widetilde{C}$.

In what follows, let V_Z denote the aspect of the g^s_m along the component $Z \subset \widetilde{C}$. We will systematically use the following three basic facts from the theory of limit linear series [EH3]:

- **LS1.** At a node p along which components $Y, Z \subset \widetilde{C}$ intersect transversely, the

vanishing sequences $a(V_Y, p)$ and $a(V_Z, p)$ verify

$$a_i(V_Y) + a_{s-i}(V_Z) \geq m$$

for all $0 \leq i \leq s$. Moreover, if $\rho(g, s, m) = 0$, then each of the preceding inequalities is an equality.

- **LS2.** Assume that $\rho(g, s, m) = 0$, and that a set of compatible bases for V along $\widetilde{C}$ has been chosen, in the sense that $V_{Y_i} \subset V_{Y_{i+1}}$, for every i.

  - If Y_i is linked via rational curves to an elliptic tail, then

    $$a_j(V_{Y_{i+1}}, p_{i+1}) = a_j(V_{Y_i}, p_i) + 1$$

    for all $0 \leq j \leq s$ except for a single index j, for which

    $$a_j(V_{Y_i}, p_{i+1}) = a_j(V_{Y_i}, p_i).$$

  - If Y_i is not linked via rational curves to an elliptic tail, then

    $$a(V_{Y_{i+1}}, p_{i+1}) = a(V_{Y_i}, p_i).$$

- **LS3.** The vanishing sequences of a linear series along points in $\mathbb{P}^1$ are in bijection with Schubert cycles in $H^*(\mathbb{G}(s, m), \mathbb{Z})$, the integral cohomology ring of the Grassmannian of s-dimensional subspaces of an m-dimensional projective space. A smooth rational curve $\mathbb{P}^1$ admits a linear series g^s_m with ramification sequences $\alpha_i = \alpha(V, r_i)$ at distinct points of $r_i \in \mathbb{P}^1$ if and only if the product of the corresponding Schubert cycles is nonzero in $H^*(\mathbb{G}(s, m), \mathbb{Z})$.

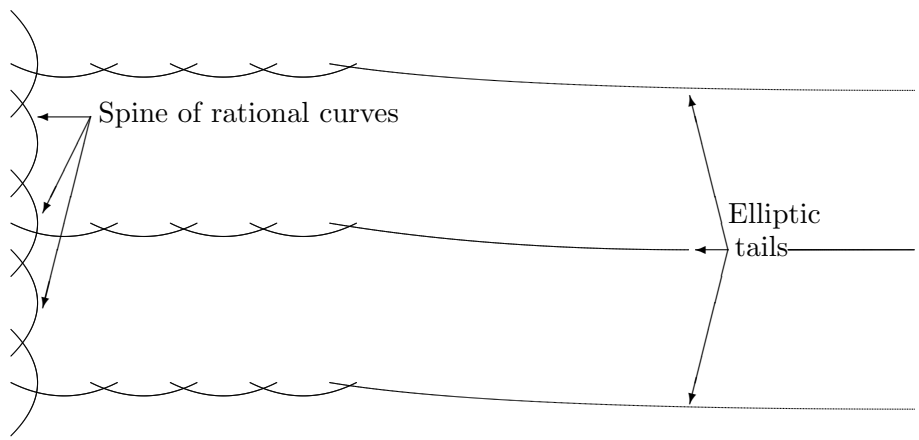

Figure 1: A broken flag curve.

- **LS4.** Let (L, V) denote a linear series along a reducible curve $Y \cup_q Z$. If Z is a smooth and irreducible elliptic curve, then the aspect V_Z of the linear series along Z has a cusp at q, i.e., the ramification sequence $\alpha(V_z, q)$ satisfies

$$\alpha(V_z, q) \geq (0, 1, \ldots, 1).$$

For convenience, we make the following simplifying assumption, which we will remove later.

No q_i lies along an elliptic tail.

Note that, by repeated blowing-up, we are also free to assume that no q_i lies at a point of attachment linking components of $\widetilde{C}$.

Now fix a component Y_i along the spine. If it is interior to the spine, then it has at least two special points $p_i = 0$ and $p_{i+1} = \infty$ corresponding to the intersections with adjacent rational components Y_{i-1} and Y_i, respectively, along the spine. Furthermore, if it is linked to an elliptic tail, then it has an additional special point, call it 1. If Y_i is not interior to the spine, then it has two special points, one of which (either p_i or p_{i+1}, which we label by

0 or ∞, respectively) corresponds to an intersection with an adjacent component along the spine and another, 1, arising from the fact that Y_i is linked to an elliptic tail.

Denote the *vanishing orders* of V_{Y_i} at 0 (resp., ∞) by a_j (resp., b_j), $0 \leq j \leq s$; if V_{Y_i} is spanned by sections $\sigma_j(t), 0 \leq j \leq s$ in a local uniformizing parameter t for which $\mathrm{ord}_t(\sigma_i) < \mathrm{ord}_t(\sigma_j)$ whenever $i < j$, then $a_i := \mathrm{ord}_t(\sigma_i)$. Denote the corresponding vanishing orders of the g_m^{s-d+r} along Y_i by u_j and v_j, respectively. Note that the sequence (u_j) (resp., (v_j)) is a subsequence of (a_j) (resp, (b_j)). Recall that (u_j) and (v_j) correspond to Schubert cycles in $H^*(\mathbb{G}(s-d+r, m), \mathbb{Z})$.

Now assume that a simple base point of our g_m^{s-d+r} lies along Y_i; the existence of the base point imposes restrictions on the Schubert cycles corresponding to (u_j) and (v_j). To make sense of these, we introduce the following terminology: we say that (u_j) and (v_j) are *complementary* if

$$u_j = a_{k(j)} \text{ and } v_j = b_{s-k(s-d+r-j)}$$

for some sequence of nonnegative integers $k(j), j = 0, \ldots, s-d+r$. If the g_m^{s-d+r} along $\widetilde{C}$ has a base point along Y_i, then (u_j) and (v_j) fail to be complementary to one another by a precise amount, as follows.

Lemma 1. *Assume that a base point of the g_m^{s-d+r} lies along Y_i, and that $u_j = a_{k(j)}, j = 0, \ldots, s-d+r$. Then*

$$v_j = b_{s-k(s-d+r-j)-k'(j)}, j = 0, \ldots, s$$

for some sequence of nonnegative integers $k'(j), j = 0, \ldots, s-d+r$, at least $(s-d+r)$ of which are equal to 1 or more.

A similar statement applies to the case where multiple base points of the g_m^{s-d+r} lie

along Y_i:

Lemma 2. *Assume that the g_m^{s-d+r} along Y_i has base points $i_1p_1 + \cdots + i_{d-1}p_d$, where $i_1, \ldots, i_{d-1}$ are nonnegative and $i_1 + \cdots + i_{d-1} \leq d$. Then*

$$v_j = b_{s-k(s-d+r-j)-k'(j)}, j = 0, \ldots, s$$

for some sequence of nonnegative integers $k'(j), j = 0, \ldots, s-d+r$, at least $(s-d+r)$ of which are equal to at least $(i_1 + \cdots + i_{d-1})$. For the remaining index j,

$$k(j) \geq i_1 + \cdots + i_{d-1} - 1.$$

Given (v_j), define the sequence $(u'_j), j = 0, \ldots, s-d+r$ by setting $u'_j := m - v_j$ for every j. If Y_i is interior to $\widetilde{C}$, then, by $LS1$, (u'_j) is a subsequence of the vanishing sequence $a(V_{Y_{i+1},p_{i+1}}) = (a'_0, \ldots, a'_s)$. Letting

$$u'_j = a'_{k''(j)},$$

the first lemma asserts that the sequences $k(j)$ and $k''(j)$ satisfy

$$k''(j) \geq k(j) + 1$$

for at least $(s-d+r)$ values of j. In other words, the base point forces $(s-d+r)$ vanishing order indices $k(j)$ to "shift to the right" by at least one place. More generally, Lemmas 1 and 2 imply that d base points, possibly occurring with multiplicities, force at least $d(s-d+r)$ shifts of vanishing order indices. On the other hand, shifts of vanishing order indices are constrained; namely, each index can shift at most $s-(s-d+r) = d-r$ places. So the maximum possible number of shifts is $(s-d+r+1)(d-r)$.

Now notice that

$$(s-d+r+1)(d-r)-d(s-d+r)=\mu,$$

while $\mu=-1$ by assumption. So provided all base points occur along interior components of the spine of $\widetilde{C}$, we have a contradiction. A trivial modification of the same argument yields a contradiction whenever base points lie along either of the two ends of the spine. So, modulo our simplifying assumption, we have reduced to proving Lemmas 1 and 2.

Proof of Lemmas 1 and 2. Consider first the case where the component Y along which the base point p lies has three special points $0, 1,$ and ∞. Assume, moreover, that p is a simple base point. The Schubert cycle corresponding to p in $H^*(\mathbb{G}(s-d+r,m),\mathbb{Z})$ is $\sigma^{(p)}=\sigma_{1,\ldots,1}$. On the other hand, by $LS4$, the g_m^{s-d+r} along Y has at least a cusp at 1; i.e., the corresponding Schubert cycle $\sigma^{(1)}$ satsifies

$$\sigma^{(1)}\geq\sigma_{1,\ldots,1,0}.$$

Meanwhile, by $LS3$, the intersection

$$\sigma^{(0)}\cdot\sigma^{(1)}\cdot\sigma^{(\infty)}\cdot\sigma^{(p)}\in H^*(\mathbb{G}(s-d+r,m)) \tag{2.1}$$

is necessarily nonzero. Since $\sigma^{(p)}=\sigma_{1,\ldots,1}$, (2.1) is clearly nonzero if and only if the corresponding intersection

$$\sigma^{(0)}\cdot\sigma^{(1)}\cdot\sigma^{(\infty)}$$

is nonzero **in** $\mathbf{H}^*(\mathbb{G}(\mathbf{s}-\mathbf{d}+\mathbf{r},\mathbf{m}-\mathbf{1}),\mathbb{Z})$. In particular, we must have

$$\sigma^{(0)}\cdot\sigma^{(\infty)}\cdot\sigma_{1,\ldots,1,0}\neq 0\in H^*(\mathbb{G}(s-d+r,m-1)).$$

Assume that the vanishing sequence of the g^s_m along Y at 0 is

$$a(V_Y, 0) = (a_0, \ldots, a_s)$$

and that, correspondingly, the vanishing sequence of the g^{s-d+r}_m at 0 is

$$(u_0, \ldots, u_{s-d+r}) = (a_{k(0)}, \ldots, a_{k(s-d+r)}).$$

We then have

$$\sigma^{(0)} = \sigma_{a_{k(s-d+r)}-(s-d+r),\ldots,a_{k(1)}-1,a_{k(0)}}$$

The sequence

$$(v_0, \ldots, v_{s-d+r}) = (b_{s-k(s-d+r)}, \ldots, b_{s-k(0)})$$

is complementary to $(u_0, \ldots, u_{s-d+r})$. Let $\sigma^{(0^\vee)}$ denote the Schubert cycle corresponding to (v_j); then

$$\sigma^{(0^\vee)} = \sigma_{b_{s-k(0)}-(s-d+r),\ldots,b_{s-k(s-d+r)}}.$$

The key observation to make is as follows. Combining $LS1$ and $LS2$, we have

$$b_{s-i} = m - 1 - a_i \tag{2.2}$$

for every i in $\{0, \ldots, s\}$, except for a unique index j for which $b_{s-j} = m - a_j$. It follows that the intersection

$$\sigma^{(0)} \cdot \sigma^{(0^\vee)} \in H^*(\mathbb{G}(s-d+r, m-1))$$

is either 0 or is supported at a point, depending upon whether

$$b_{s-k(j)} = m - 1 - a_{k(j)}$$

for all j in $\{0, \ldots, s-d+r\}$ or not.

Note, on the other hand, that

$$\sigma^{(0)} \cdot \sigma_{1,\ldots,1,0}$$

is supported along a union of Schubert cycles

$$\sigma^{(0')} = \sigma_{a_{k(s-d+r)}-(s-d+r)+k'''(s-d+r),\ldots,a_{k(1)}-1+k'''(1),a_{k(0)}+k'''(0)}$$

for some sequence of nonnegative integers $k'''(j), j = 0, \ldots, s-d+r$, at least each $(s-d+r)$ of which are equal to at least one.

Now write

$$\sigma^{(\infty)} = \sigma_{b_{s-k(0)}-(s-d+r)-k'(0),\ldots,b_{s-k(s-d+r)}-k'(s-d+r)}.$$

If the intersection

$$\sigma^{(0')} \cdot \sigma^{(\infty)} = \sigma_{a_{k(s-d+r)}-(s-d+r)+k'''(s-d+r),\ldots,a_{k(0)}+k'''(0)}$$

$$\cdot\, \sigma_{b_{s-k(0)}-(s-d+r)-k'(0),\ldots,b_{s-k(s-d+r)}-k'(s-d+r)}$$

is nonzero, then the $(s-d+r+1)$ sums of complementary indices

$$a_{k(s-d+r)} - (s-d+r) + k'''(s-d+r) + b_{s-k(s-d+r)} - k'(s-d+r)$$

$$\ldots$$

$$a_{k(0)} + k'''(0) + b_{s-k(0)} - (s-d+r) - k'(0)$$

are each at most $m-1-(s-d+r)$. By (2.2), coupled with the fact that $(s-d+r)$ values of $k'''(j), j = 0, \ldots, s-d+r$ are nonzero, it follows that the same is true of the values of $k'(j), j = 0, \ldots, s-d+r$. The conclusion of the first lemma follows immediately in the case where Y has three special points. The preceding argument also extends immediately to cover those cases where multiple base points $i_1p_1 + \cdots + i_np_n$ lie along Y. Finally, if Y has only two special points instead of three, simply note that by $LS1$ and $LS2$, we have

$$b_{s-i} = m - a_i$$

for every i in $\{0, \ldots, s\}$, and argue as before. $\square$

To complete the proof of Theorem 1, we explain how to remove the simplifying assumption inserted at the beginning. Namely, assume that the g_m^{s-d+r} admits a base point p along an elliptic tail E. Say that E intersects the rational component Z of $\widetilde{C}$ in a node q of $\widetilde{C}$. Note that the vanishing sequence at q of the g_m^{s-d+r} along E is bounded above by

$$(m - s + d - r - 2, \ldots, m - 3, m - 1);$$

otherwise, the subpencil of sections of the g_m^{s-d+r} along E that vanish to maximal order define (upon removal of the $(m-3)$-fold base point $(m-3)q$) a g_1^1, which is absurd. It follows, by $LS1$, that the vanishing sequence at q of the g_m^{s-d+r} along Z is *at least*

$$(1, 3, \ldots, s - d + r + 2),$$

which in turn implies that the same estimate holds for the vanishing sequence of the g_m^{s-d+r} along the rational component Y_i of the spine of $\widetilde{C}$ linked to E at the corresponding node $\widetilde{q}$.

In other words, if the g_m^{s-d+r} has a base point along E, then the g_m^{s-d+r} also has a base point and a cusp along Y_i. So, in effect, we are reduced to the "simplified" setting, and are free to argue as before.

An elaboration of the preceding argument yields the following generalization of Theorem 1.

Theorem 2. *If $\rho + \mu = -1$, then a general curve C of genus g admits no s-dimensional linear series g_m^s with d-secant $(d - r - 1)$-planes.*

To prove Theorem 2, we argue much as before, and assume that a flag curve $\widetilde{C}$ carries an inclusion (1.1). We stipulate that base points of the g_m^{s-d+r} lie along the spine of $\widetilde{C}$, away

from intersections of components of $\widetilde{C}$. Like before, we obtain the theorem by analyzing the vanishing sequences of the g_m^{s-d+r} along a component of $\widetilde{C}$ at the points 0 and ∞, under the assumption that a base point lies along that component. The result follows from the following statement, proved along the lines of Lemmas 1 and 2 above:

Lemma 3. *Assume that ρ is nonnegative. If a flag curve $\widetilde{C}$ carries an inclusion* (1.1)*, then the vanishing order indices of the g_m^{s-d+r} shift at least $d(s-d+r)-\rho$ times.*

As a consequence, whenever $\rho+\mu=-1$, a flag curve $\widetilde{C}$ carries **no** inclusions (1.1), which implies Theorem 2.

Proof. Note that, by the Brill-Noether theorem, $\widetilde{C}$ carries no g_m^s, so ρ is automatically nonnegative. When ρ is positive, a modified version of the conditions on vanishing sequences in the case $\rho=0$ given in $LS2$ applies. Given a choice of compatible bases for the g_m^s along $\widetilde{C}$, we have

$$a(V_{Y_{i+1}}, p_{i+1}) \geq a(V_{Y_i}, p_i),$$

and, moreover, whenever Y_i is linked to an elliptic tail, there is at most one index $j, 0 \leq j \leq s$, for which

$$a_j(V_{Y_i}, p_i) = a_j(V_{Y_{i+1}}, p_{i+1}).$$

On the other hand, it is no longer the case that the other $(s-d+r)$ vanishing orders a_k satisfy

$$a_k(V_{Y_i}, p_i) = a_k(V_{Y_{i+1}}, p_{i+1}) + 1;$$

rather, the total amount by which "jumps" in vanishing orders exceed 1 is at most ρ.

Now say that Y_i is linked to an elliptic tail and that, for some index k,

$$a_k(V_{Y_i}, p_i) = a_k(V_{Y_{i+1}}, p_{i+1}) + \nu(k)$$

for some integer $\nu(k) \in \{1, \ldots, \rho+1\}$. In place of (2.2), we instead deduce that

$$b_{s-k} \leq m - \nu(k) - a_k. \tag{2.3}$$

As before, we study intersections of Schubert cycles

$$\sigma^{(0)} \cdot \sigma^{(\infty)} \cdot \sigma_{1,\ldots,1,0}$$

in $H^*(\mathbb{G}(s-d+r, m-1))$, which are sums of intersections

$$\sigma^{(0')} \cdot \sigma^{(\infty)} = \sigma_{a_{k(s-d+r)}-(s-d+r)+k'''(s-d+r),\ldots,a_{k(0)}+k'''(0)}$$

$$\cdot\, \sigma_{b_{s-k(0)}-(s-d+r)-k'(0),\ldots,b_{s-k(s-d+r)}-k'(s-d+r)}.$$

If such an intersection is nonzero, then, just as before, the $(s-d+r+1)$ sums of complementary indices

$$a_{k(s-d+r)} - (s-d+r) + k'''(s-d+r) + b_{s-k(s-d+r)} - k'(s-d+r)$$

$$\ldots$$

$$a_{k(0)} + k'''(0) + b_{s-k(0)} - (s-d+r) - k'(0)$$

are each at most $m-1-(s-d+r)$. By (2.3), coupled with the fact that $(s-d+r)$ values of $k'''(j), j = 0, \ldots, s-d+r$ are nonzero, it follows that at least $(s-d+r-\tau)$ values of $k'(j), j = 0, \ldots, s-d+r$ are nonzero, where τ is the number of indices k for which $\nu(k)$ is strictly greater than 1. Lemma 3 follows immediately. □

We next prove a finiteness result for linear series with exceptional secant planes on a general curve in the case where $\rho = 1$.

Theorem 3. *If $\rho = 1$ and $\mu = -1$, then there are finitely many linear series g^s_m with d-secant $(d-r-1)$-planes on a general curve C of genus g.*

Proof. Since the space of linear series on a general curve is irreducible whenever ρ is positive, it suffices to show that some linear series without d-secant $(d-r-1)$-planes exists on C. To this end, it suffices to show that some smoothable linear series without d-secant $(d-r-1)$-planes exists on a flag curve $\widetilde{C}$ obtained by specialization from C.

We construct a particular choice of flag curve and linear series as follows. Fix a smooth irreducible elliptic curve $\widetilde{E}$ with general j-invariant, together with a general curve $\widetilde{Y}$ of genus $(g-1)$. Next, specialize $\widetilde{E}$ and $\widetilde{Y}$ to flag curves E and Y. Glue E and Y transversely, letting q denote their intersection. Let

$$C' := Y \cup_q E.$$

Furthermore, let $G^s_m(C')$ denote the space of limit linear series along C', and let

$$G^s_m(C')_{(1,1,\ldots,1,1)}$$

denote the subspace of $G^s_m(C')$ comprising limit linear series V_Y for which

$$\alpha(V_Y, q) \geq (0, 1, \ldots, 1, 2). \tag{2.4}$$

The vanishing sequence corresponding to $(1, 1, \ldots, 1, 1)$ is $(1, 2, 3, \ldots, s, s+1)$; by $LS1$, we deduce that

$$a(V_E, q) \geq (m-s-1, m-s, m-s+1, \ldots, s-3, s-2, s-1),$$

i.e., that

$$\alpha(V_E, q) \geq (m-s-1, \ldots, m-s-1). \tag{2.5}$$

Now let

$$r_Y = (1, \ldots, 1) \text{ and } r_E = (m-s-1, \ldots, m-s-1).$$

The modified Brill-Noether numbers $\rho(Y,(r_Y)_q)$ and $\rho(E,(r_E)_q)$, which compute the expected dimensions of the spaces of limit linear series along Y and E with ramification at q prescribed by (2.4) and (2.5), respectively, are

$$\rho(Y,(r_Y)_q) = \rho(g-1,s,m) - (s+1) = \rho(g,s,m) + s - (s+1) = 0$$

and

$$\rho(E,(r_E)_q) = \rho(1,s,m) - (s+1)(m-s-1) = 1.$$

Since $\widetilde{Y}$ and $\widetilde{E}$ are general, their respective spaces of limit linear series $G^s_m(Y,(r_Y)_q)$ and $G^s_m(E,(r_E)_q)$ are of expected dimension, by Eisenbud and Harris' generalized Brill–Noether theorem [EH3]. It follows immediately that $G^s_m(C')_{(1,\ldots,1)}$ is of expected dimension, so every linear series in $G^s_m(C')_{(1,\ldots,1)}$ smooths, by the Regeneration Theorem [HM, Thm 5.41].

To prove Theorem 3, it now suffices to show that no limit linear series in $G^s_m(C')_{(1,\ldots,1)}$ admits an inclusion (1.1). Note, however, that

$$a(V_Y,q) \geq (1,\ldots,1)$$

implies that along any component of the spine of C', any g^s_m satisfies

$$b_{s-i} \geq m-1-a_i$$

for every index $i \in \{0,\ldots,s\}$. (This is clear along E, where the special points 0 and ∞ have vanishing sequences $(0,1,\ldots,s)$ and $(m-s-1,m-s,\ldots,m-1)$, and along Y it follows from the fact that $\rho(Y,(r_Y)_q) = 0$.) It now follows by the same argument used to prove theorems 1 and 2 that no limit linear series in $G^s_m(C')_{(1,\ldots,1)}$ admits an inclusion (1.1). $\square$

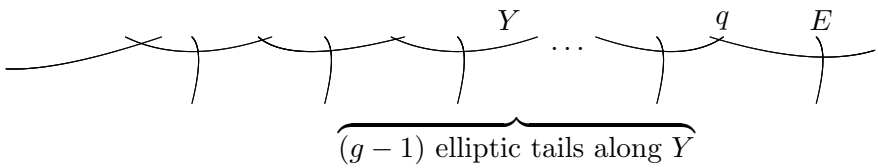

Figure 2: $C' = Y \cup E$. Here $a(V_Y, q) = (1, \ldots, 1)$ and $a(V_E, q) = (m-s-1, \ldots, m-s-1)$.

3 Quantitative study

In this section, we study the following problem. Let $\pi : \mathcal{X} \to B$ denote a one-parameter (flat) family of curves whose generic fiber is smooth, with some finite number of special fibers that are irreducible curves with nodes. We equip each fiber of π with an s-dimensional series g^s_m. That is, $\mathcal{X}$ comes equipped with a line bundle $\mathcal{L}$, and on B there is a vector bundle $\mathcal{V}$ of rank $(s+1)$, such that

$$\mathcal{V} \hookrightarrow \pi_* \mathcal{L}.$$

If $\mu = -1$, we expect finitely many fibers of π to admit linear series with d-secant $(d - r - 1)$-planes. We then ask for a formula for the number of such series, given in terms of "tautological" invariants associated with the family π.

One natural approach to the problem is to view those fibers whose associated linear series admit d-secant $(d-r-1)$-planes as a degeneracy locus for a map of vector bundles over B. This is the point of view adopted by Ziv Ran in his work [R2, R3] on Hilbert schemes of families of nodal curves. Used in tandem with Porteous' formula for the class of a degeneracy locus of a map of vector bundles, Ran's work shows that the number of d-secant $(d-r-1)$-planes is a function N_d^{d-r-1} of tautological invariants of the family π, namely:

$$\alpha := \pi_*(c_1^2(\mathcal{L})), \beta := \pi_*(c_1(\mathcal{L}) \cdot \omega), \gamma := \pi_*(\omega^2), \delta_0, \text{ and } c := c_1(\mathcal{V}) \tag{3.1}$$

where $\omega = c_1(\omega_{\mathcal{X}/B})$ and where δ_0 denotes the locus of points $b \in B$ for which the corresponding fiber $\mathcal{X}_b$ is singular.

In other words, for any fixed choice of s, we have

$$N_d^{d-r-1} = P_\alpha \alpha + P_\beta \beta + P_\gamma \gamma + P_c c + P_{\delta_0} \delta_0 \tag{3.2}$$

where the arguments P are polynomials in m and g with coefficients in $\mathbb{Q}$. Unfortunately, the computational complexity of the calculus developed by Ran to evaluate N_d^{d-r-1} grows exponentially with d. On the other hand, given that a formula (3.2) in tautological invariants exists, the problem of evaluating it reduces to producing sufficiently many relations among the coefficients P.

In fact, the polynomials P satisfy one "obvious" relation, obtained by normalizing $\mathcal{L}$ by a factor from B that trivializes $\mathcal{V}$, and noting that the formula (3.2) is invariant under such normalizations. Namely, we require that

$$\begin{aligned} &P_\alpha \pi_* \left(c_1(\mathcal{L}) - \frac{\pi^* c_1(\mathcal{V})}{s+1} \right)^2 + P_\beta \pi_* \left(c_1 \left(\mathcal{L} - \frac{\pi^* c_1(\mathcal{V})}{s+1} \right) \cdot \omega \right) + P_\gamma \gamma + P_{\delta_0} \delta_0 \\ &= P_\alpha \pi_* (c_1^2(\mathcal{L}) + P_\beta \pi_* (c_1(\mathcal{L}) \cdot \omega) + P_\gamma \gamma + P_{\delta_0} \delta_0 + P_c. \end{aligned}$$

The coefficient of c in the left-hand expression is $-\frac{2m}{s+1} P_\alpha - \frac{2g-2}{s+1} P_\beta$; since the coefficient of c on the right-hand expression is P_c, we deduce that

$$2m P_\alpha + (2g-2) P_\beta + (s+1) P_c = 0. \tag{3.3}$$

3.1 Test families

To find additional relations among the tautological coefficients P, our strategy is to evaluate the formula (3.2) along test families whose secant-plane behavior we understand, and

thereby obtain relations among the coefficients of (3.2) that determine the polynomials P. Our test families are as follows:

1. *Family one.* Projections of a generic curve of degree m in $\mathbb{P}^{s+1}$ from points along a disjoint line.

2. *Family two.* Projections of a generic curve of degree $m+1$ in $\mathbb{P}^{s+1}$ from points along the curve.

3. *Family three.* Generic pencils of curves of class $[C]$ on $K3$ surfaces $X \subset \mathbb{P}^s$ with Picard number two that contain smooth curves of degree m and genus g. (Such surfaces were shown to exist, for a dense set of (d, m, s), in [Kn2, Thm. 1.1].)

Now assume that $\mu(d, s, r) = -1$. Let A denote the expected number of d-secant $(d-r)$-planes to a curve of degree m and genus g in $\mathbb{P}^{s+1}$ that intersect a general line. Let A' denote the expected number of $(d+1)$-secant $(d-r)$-planes to a curve of degree $(m+1)$ and genus g in $\mathbb{P}^{s+1}$. The expected number of fibers of the first (resp., second) family with d-secant $(d-r-1)$-planes equals A (resp., $(d+1)A'$).

Determining those relations among the tautological coefficients induced by the three families requires knowing the values of $\alpha, \beta, \omega, \gamma$, and c along each family $\pi : \mathcal{X} \to B$. These are determined as follows.

- *Family one.* The base and total spaces of our family are $B = \mathbb{P}^1$ and $\mathcal{X} = \mathbb{P}^1 \times C$, respectively. Letting π_1 and π_2 denote, respectively, the projections of $\mathcal{X}$ onto $\mathbb{P}^1$ and C, we have

$$\mathcal{L} = \pi_2^* \mathcal{O}_C(1), \omega_{\mathcal{X}/\mathbb{P}^1} = \pi_2^* \omega_C, \text{ and } \mathcal{V} = \mathcal{O}_{\mathbb{G}}(-1) \otimes \mathcal{O}_{\mathbb{P}^1}$$

where $\mathbb{G} = \mathbb{G}(s, s+1)$ denotes the Grassmannian of hyperplanes in $\mathbb{P}^{s+1}$. Accordingly,

$$\alpha = \beta = \gamma = \delta_0 = 0, \text{ and } c = -1.$$

It follows that

$$P_c = -A.$$

- *Family two.* This time, $\mathcal{X} = C \times C$ and $B = C$. Here

$$\mathcal{L} = \pi_2^* \mathcal{O}_C(1) \otimes \mathcal{O}(-\Delta), \omega_{\mathcal{X}/\mathbb{P}^1} = \pi_2^* \omega_C, \text{ and } \mathcal{V} = \mathcal{O}_{\mathbb{G}}(-1) \otimes \mathcal{O}_C.$$

Consequently, letting $H = c_1(\mathcal{O}_C(1))$, we have

$$\alpha = -2\Delta \cdot \pi_2^*(m+1)\{\mathrm{pt}_C\} + \Delta^2 = -2m - 2g,$$

$$\beta = (\pi_2^* H - \Delta) \cdot \pi_2^* K_C = 2 - 2g,$$

$$c = -m - 1, \text{ and } \gamma = \delta_0 = 0.$$

It follows that

$$(-2m - 2g)P_\alpha + (2 - 2g)P_\beta + (-m - 1)P_c = (d+1)A'.$$

- *Family three.* Let S denote a $K3$ surface in $\mathbb{P}^s$, such that

$$\mathrm{Pic}\ S = \mathbb{Z}H \oplus \mathbb{Z}[C].$$

where H is the class of a hyperplane section, while C is a smooth, irreducible curve of genus g such that $C \cdot H = m$. The base locus of a pencil of curves of class $[C]$ consists of $[C]^2 = (2g-2)$ points. Accordingly, we have

$$\mathcal{X} = \mathrm{Bl}_{2g-2\ \mathrm{pts}} S \text{ and } B = \mathbb{P}^1.$$

Clearly, $c_1(\mathcal{L}) = H$. Likewise, the relative dualizing sheaf of our family is given by

$$\omega_{\mathcal{X}/\mathbb{P}^1} = \omega_{\mathcal{X}} \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(2).$$

Now let f denote the class of a fiber of π, and let $E_i, 1 \leq i \leq 2g-2$, denote the classes of the exceptional divisors of the blow-up $\mathcal{X} \to S$. Then

$$\begin{aligned} \omega &= K_{\mathcal{X}} + 2f \\ &= \sum_{i=1}^{2g-2} E_i + 2\left([C] - \sum_{i=1}^{2g-2} E_i\right) = 2[C] - \sum_{i=1}^{2g-2} E_i. \end{aligned}$$

Whence,

$$\gamma = 4[C]^2 + \sum_{i=1}^{2g-2} E_i^2 = 6g-6, \alpha = H^2 = 2s-2, \text{ and}$$

$$\beta = 2[C] \cdot H = 2m.$$

We compute δ_0 as follows. Let $\mathbb{C}^2 \subset H^0(\mathcal{O}_S(C))$ denote the two-dimensional subspace of sections defining our pencil. Let $\mathcal{X}_2$ denote the fiber product $\mathcal{X} \times_{\mathbb{P}^1} \mathcal{X}$, equipped with projections π_1 and π_2 onto each of its factors. Now let

$$E := (\pi_1)_*(\pi_2^* \mathcal{O}_{\mathcal{X}}(C) \otimes \mathcal{O}_{\mathcal{X}_2}/\mathcal{O}_{\mathcal{X}_2}(-\Delta));$$

over a point $p \in \mathbb{P}^1$, E_p comprises sections of $\mathcal{O}_S(C)$ modulo those vanishing to order 2 at p.

Note that the singular fibers of π comprise the locus where the evaluation map

$$\mathbb{C}^2 \otimes \mathcal{O}_S \xrightarrow{\text{ev}} E$$

fails to be surjective. It follows that $\delta_0 = c_2(E)$. On the other hand, it is not hard to see that there is an exact sequence

$$0 \to \mathcal{O}_S(C) \otimes \mathcal{T}_S^* \to E \to \mathcal{O}_S(C) \to 0;$$

it follows that

$$\begin{aligned} c_t(E) &= c_t(\mathcal{O}_S(C)) \cdot c_t(\mathcal{O}_S(C) \otimes \mathcal{T}_S^*) \\ &= (1 + t[C]) \cdot (1 + t(\alpha_1 + 2[C]) + t^2(\alpha_1[C] + [C]^2 + \alpha_2)) \end{aligned}$$

where $\alpha_i = c_i(\mathcal{T}_S^*)$. We deduce that

$$\delta_0 = 2\alpha_1[C] + 3[C]^2 + \alpha_2.$$

Here $\alpha_1 = c_1(K_S) = \sum_{i=1}^{2g-2} E_i$, while

$$\alpha_2 = \chi(S) = 24.$$

It follows that $\delta_0 = 6g + 18$.

Finally, the vector bundle $\mathcal{V} \to \mathbb{P}^1$ is trivial, since the $\mathbb{P}^s$ to which the fibers of $\mathcal{X} \to \mathbb{P}^1$ map is fixed. So $c = 0$.

Therefore, the third family yields the relation

$$(2s-2)P_\alpha + 2mP_\beta + (6g-6)P_\gamma + (6g+18)P_{\delta_0} = N_{K_3} \tag{3.4}$$

where N_{K_3} denotes the number of fibers of π with exceptional secant-plane behavior.

3.2 The value of N_{K_3}

If $r = 1$, then $\mu = -1$ forces $d = 2s - 1$ and $d - r - 1 = s - 2$, so S admits no d-secant $(d-2)$-planes, by [Kn1, Thm. 1.1]. It follows that $N_{K_3} = 0$ when $r = 1$. At the other extreme, if $r = s$ then the assumption that $\mu = -1$ forces

$$d = 2s - 1 \text{ and } d - r - 1 = s - 2.$$

By Bézout's theorem, the degree-$(2s-2)$ surface S admits no $(s-2)$-planes, so again we have $N_{K_3} = 0$.

For a general choice of (d, m, r, s), the value of N_{K_3} is unclear. However, we conjecture that the following is true.

Conjecture 1. *Let $X \subset \mathbb{P}^s$ be a $K3$ surface with Picard group*

$$Pic\ X = \mathbb{Z}L \oplus \mathbb{Z}\Lambda$$

where

$$L^2 = 2s-2, \Lambda^2 = 2g-2, \text{ and } \Lambda \cdot L = m.$$

If

$$\rho(g, s, m) = 0 \text{ and } \mu(d, s, r) = -1, \tag{3.5}$$

then X admits no d-secant $(d-r-1)$-planes, except possibly when $m = 2s$ and $g = s+1$.

NB: The hypothesis that $\rho(g, s, m) = 0$ implies that

$$m = s(a+1), \text{ and } g = (s+1)a, \tag{3.6}$$

for some positive integer a. When $a = 1$, i.e., when $m = 2s$ and $g = s+1$, the curves of class L on X are canonical curves. As soon as any such curve admits a d-secant $(d-r-1)$-plane, it admits an r-dimensional family of such secant planes. Consequently, those canonical curves with d-secant $(d-r-1)$-planes comprise a locus of codimension at least 2. As a result, the case $a = 1$ has no bearing on our determination (in Sections 4-6) of the classes of secant-plane divisors on $\mathcal{G}^s_m$ or $\overline{\mathcal{M}}_g$.

The following argument supports our conjecture. Assume that some d-secant $(d-r-1)$-plane to X exists. Let $Z \subset X$ denote the intersection of that plane with X. We will obtain

a contradiction, *under the additional assumption that Z be curvilinear.* Provided Z is curvilinear, there is some smooth hyperplane section Y of X that passes through Z.

Note $H^1(X, L) = 0$, because L is globally generated. Whence, the exact sequence defining Z in X

$$0 \to L \otimes \mathcal{I}_{Z/X} \to L \to L \otimes \mathcal{O}_Z \to 0$$

induces an exact sequence

$$0 \to H^0(X, L \otimes \mathcal{I}_{Z/X}) \to H^0(X, L) \xrightarrow{\text{ev}} H^0(X, \mathcal{O}_Z) \to H^1(X, L \otimes \mathcal{I}_{Z/X}) \to 0$$

in cohomology. Here $h^0(X, \mathcal{O}_Z) = d$, and $\text{rk}(\text{ev}) = d - r$ because Z determines a d-secant $(d - r - 1)$-plane to X, by assumption. It follows that

$$h^1(X, L \otimes \mathcal{I}_{Z/X}) = r. \tag{3.7}$$

On the other hand, we clearly have

$$L \otimes \mathcal{I}_{Y/X} \cong \mathcal{O}_X,$$

while the adjunction theorem on X implies $\mathcal{I}_{Z/Y}(K_X + L) \cong \mathcal{O}_Y(K_Y - Z)$, i.e., that

$$L \otimes \mathcal{I}_{Z/Y} \cong \mathcal{O}_Y(K_Y - Z).$$

It follows that the exact sequence of (twisted) ideal sheaves

$$0 \to L \otimes \mathcal{I}_{Y/X} \to L \otimes \mathcal{I}_{Z/X} \to L \otimes \mathcal{I}_{Z/Y} \to 0$$

induces an exact sequence

$$\begin{aligned} H^1(X, \mathcal{O}_X) \to H^1(X, L \otimes \mathcal{I}_{Z/X}) \to H^1(Y, \mathcal{O}_Y(K_Y - Z)) \to H^2(X, \mathcal{O}_X) \\ \to H^2(X, L \otimes \mathcal{I}_{Z/X}) \end{aligned}$$

in cohomology. Here $H^1(X, \mathcal{O}_X) = 0$, while

$$H^2(X, \mathcal{O}_X) \cong H^0(X, K_X)^\vee \cong H^0(X, \mathcal{O}_X)^\vee \cong \mathbb{C},$$

$$H^1(Y, \mathcal{O}_Y(K_Y - Z)) \cong H^0(Y, \mathcal{O}_Y(Z)), \text{ and}$$

$$H^2(X, L \otimes \mathcal{I}_{Z/X}) \cong H^2(X, L) \cong H^0(X, -L)^\vee = 0.$$

By (3.7), it follows that Z defines a g^r_d with $\rho(g, r, d) = -1$ along the canonical curve Y.

Note that Lazarsfeld's theorem [La, Lem. 1.3] states that provided there are no multiple or reducible curves of class L on X, the g^r_d defined by Z is Brill–Noether general, which yields the desired contradiction. More precisely, Lazarsfeld shows that provided a certain vector bundle $\mathcal{F}$ admits no nontrivial endomorphisms, there are no multiple or reducible curves of class L on X. Further, as was pointed out in [FKP], to show that $\mathcal{F}$ admits no nontrivial endomorphisms, it suffices to show that on X there is no decomposition

$$L = M + N \tag{3.8}$$

where M and N are effective and verify $h^0(M) \geq 2, h^0(N) \geq 2$.

To see why, recall that the argument of [La, Lem. 1.3] establishes that if $\mathcal{F}$ admits nontrivial endomorphisms, then $c_1(\mathcal{F}^*) = L$ decomposes nontrivially as a sum of effective classes $M + N$, where

$$M = c_1(\widetilde{M}) \text{ and } N = c_1(\widetilde{N})$$

for suitably chosen coherent sheaf quotients $\widetilde{M}$ and $\widetilde{N}$ of $\mathcal{F}^*$. But Lazarsfeld also shows that $\mathcal{F}^*$ is generated by its global sections, so det $\widetilde{M}$ and det $\widetilde{N}$ are also generated by their global sections (and are nontrivial); it follows that $h^0(M) \geq 2$ and $h^0(N) \geq 2$.

To show that no decomposition (3.8) exists, we assume the opposite and argue for a contradiction. Note that if a decomposition (3.8) exists, then because det $\widetilde{M}$ and det $\widetilde{N}$ are

generated by their global sections, $h^1(M) = h^1(N) = 0$, and the Riemann-Roch formula yields

$$h^0(M) = 2 + \frac{1}{2}M^2 \text{ and } h^0(N) = 2 + \frac{1}{2}N^2.$$

Since $h^0(M) \geq 2, h^0(N) \geq 2$, we have

$$M^2 \geq 0 \text{ and } N^2 \geq 0. \tag{3.9}$$

On the other hand, we also have

$$M \cdot \Lambda \geq 0 \text{ and } N \cdot \Lambda \geq 0. \tag{3.10}$$

Now let

$$M = \alpha L + \beta\Lambda \text{ and } N = (1-\alpha)L - \beta\Lambda.$$

Then

$$\begin{aligned}
M^2 &= (\alpha L + \beta\Lambda)^2 \\
&= \alpha^2(2s-2) + \beta^2(2g-2) + 2\alpha\beta m \geq 0, \\
N^2 &= ((1-\alpha)L - \beta)^2 \\
&= (1-\alpha)^2(2s-2) + \beta^2(2g-2) - 2(1-\alpha)\beta m \geq 0, \\
M \cdot \Lambda &= (\alpha L + \beta\Lambda) \cdot \Lambda \\
&= \alpha m + \beta(2g-2) \geq 0, \text{ and} \\
N \cdot \Lambda &= ((1-\alpha)L - \beta) \cdot \Lambda = (1-\alpha)m - \beta(2g-2) \geq 0.
\end{aligned}$$

Note that the last two inequalities combine to yield

$$0 \leq \alpha m + \beta(2g-2) \leq m. \tag{3.11}$$

There are now two cases to consider, namely: $(\alpha > 0, \beta < 0)$, and $(\alpha < 0, \beta > 0)$. The argument is virtually identical in either case; we present it in the first case.

First, observe that (3.11) implies that

$$-\frac{\beta}{\alpha}(2g-2) \leq m \leq -\frac{\beta}{(\alpha-1)}(2g-2). \tag{3.12}$$

Similarly, the inequality deduced from $M^2 \geq 0$ above implies that

$$m \leq -\frac{\alpha}{\beta}(s-1) - \frac{\beta}{\alpha}(g-1). \tag{3.13}$$

Now let $x = -\frac{\beta}{\alpha} > 0$. Then (3.13) may be rewritten as

$$(g-1)x^2 - mx + (s-1) \geq 0.$$

The left-hand side of (3.12) forces

$$\begin{aligned} x &\leq \frac{m - \sqrt{m^2 - 4(g-1)(s-1)}}{2g-2}, \text{ i.e.,} \\ -\beta &\leq \left(\frac{m - \sqrt{m^2 - 4(g-1)(s-1)}}{2g-2}\right)\alpha. \end{aligned}$$

The right-hand side of (3.12) now forces

$$\begin{aligned} m &\leq (m - \sqrt{m^2 - 4(g-1)(s-1)})\frac{\alpha}{\alpha-1}, \text{ i.e.,} \\ 1 - \frac{1}{\alpha} &\leq 1 - \frac{\sqrt{m^2 - 4(g-1)(s-1)}}{m}, \text{ i.e.,} \\ \alpha &\leq \frac{m}{\sqrt{m^2 - 4(g-1)(s-1)}}. \end{aligned}$$

Next, we apply (3.6), with $a \geq 2$. We deduce that $\alpha \leq 1$ necessarily, except when $a = 2$, when $\alpha = 2$ is also a possibility.

Similarly, if $(\alpha < 0, \beta > 0)$, we conclude that $-\alpha \leq 1$ except possibly when $a = 2$, when $\alpha = -2$ is also a possibility.

We now analyze the possibilities that remain.

- **If** $\alpha = 1$, then the left-hand side of (3.11) yields $-\beta \leq \frac{m}{2g-2} = \frac{s(a+1)}{2(s+1)a-2}$, which forces $\beta = 0$.

- Similarly, **if** $\alpha = 0$, then the right-hand side of (3.11) yields $\beta = 0$.

- **If** $\alpha = -1$, then the right-hand side of (3.11) yields $\beta \leq \frac{m}{g-1} = \frac{s(a+1)}{(s+1)a-1}$, so that $\beta = 0$ or 1. Then (3.11) forces $\beta = 1$. But then $M \cdot L = (-L+\Lambda) \cdot L = m-(2g-2) \geq 0$ forces $m \geq 2g-2$, which contradicts (3.6).

- **If** $a = 2$ **and** $\alpha = -2$, then the right-hand side of (3.11) forces $\beta \leq 2$. So either $(\alpha, \beta) = (-2, 1)$, or $(\alpha, \beta) = (-2, 2)$. But the left-hand side of (3.11) precludes $(\alpha, \beta) = (-2, 1)$, and the condition that $N^2 \geq 0$ precludes $(\alpha, \beta) = (-2, 2)$.

3.3 Formulas for A and A', and their significance

Formulas for A and A' were calculated by Macdonald [M] and [ACGH]. Such formulas are only valid so long as the loci in question are actually zero-dimensional. On the other hand, for the purpose of calculating class formulas for secant-plane divisors on $\overline{\mathcal{M}}_g$, it suffices to verify that Macdonald's formulas are enumerative for a certain "dense" subset of of the set of 4-tuples (d, m, r, s). Namely, it suffices to show that for every fixed triple (d, r, s), Macdonald's formulas are enumerative whenever $m = m(d, r, s)$ is sufficiently large. To do so, we view the curve $C \subset \mathbb{P}^{s+1}$ in question as the image under projection of a *non-special* curve $\widetilde{C}$ in a higher-dimensional ambient space. We then re-interpret the secant behavior of C in terms of the secant behavior of $\widetilde{C}$; the latter, in turn, may be characterized completely because $\widetilde{C}$ is non-special.

Given a curve $\widetilde{C}$, let L be a line bundle of degree $\widetilde{m}$ on $\widetilde{C}$, let $V \subset H^0(\widetilde{C}, L)$; the pair

(L, V) defines a linear series on $\widetilde{C}$. Now let $S^{\widetilde{d}}(L)$ denote the vector bundle

$$S^{\widetilde{d}}(L) = (\pi_{1,\ldots,\widetilde{d}})_*(\pi^*_{\widetilde{d}+1} L \otimes \mathcal{O}_{\mathrm{Sym}^{\widetilde{d}+1}\widetilde{C}} / \mathcal{O}_{\mathrm{Sym}^{\widetilde{d}+1}\widetilde{C}}(-\Delta_{\widetilde{d}+1}))$$

over $\mathrm{Sym}^{\widetilde{d}}\widetilde{C}$, where $\pi_i, i = 1 \ldots \widetilde{d}+1$ denote the $\widetilde{d}+1$ projections of $\mathrm{Sym}^{\widetilde{d}+1}\widetilde{C}$ to $\widetilde{C}$, $\pi_{1,\ldots,\widetilde{d}}$ denotes the product of the first $\widetilde{d}$ projections, and $\Delta_{\widetilde{d}+1} \subset \mathrm{Sym}^{\widetilde{d}+1}\widetilde{C}$ denotes the "big" diagonal of $(\widetilde{d}+1)$-tuples whose ith and $(\widetilde{d}+1)$st coordinates are the same. The bundle $S^{\widetilde{d}}(L)$ has fiber $H^0(L/L(-D))$ over a divisor $D \subset \mathrm{Sym}^{\widetilde{d}}\widetilde{C}$.

Note that the $\widetilde{d}$-secant $(\widetilde{d} - \widetilde{r} - 1)$-planes to the image of $\widetilde{C}$ under (L, V) correspond to the sublocus of $\mathrm{Sym}^{\widetilde{d}}\widetilde{C}$ over which the evaluation map

$$V \xrightarrow{\mathrm{ev}} S^{\widetilde{d}}(L) \tag{3.14}$$

has rank $(\widetilde{d} - \widetilde{r})$.

Moreover, by Serre duality,

$$H^0(\omega_{\widetilde{C}} \otimes L^\vee \otimes \mathcal{O}_{\widetilde{C}}(p_1 + \cdots + p_{\widetilde{d}}))^\vee \cong H^1(L(-p_1 - \cdots - p_{\widetilde{d}})); \tag{3.15}$$

both vector spaces are zero whenever $\omega_{\widetilde{C}} \otimes L^\vee \otimes \mathcal{O}_{\widetilde{C}}(p_1 + \cdots + p_{\widetilde{d}})$ has negative degree. In particular, whenever

$$\widetilde{m} \geq 2g - 1 + \widetilde{d}, \tag{3.16}$$

the vector space on the right-hand side of (3.15) is zero. It follows that the evaluation map (3.14) is surjective for the complete linear series $(L, H^0(\mathcal{O}_{\widetilde{C}}(D))$ whenever $D \subset \widetilde{C}$ is a divisor of degree $\widetilde{m}$ verifying (3.16). Equivalently, whenever (3.16) holds, every $\widetilde{d}$-tuple of points in $\widetilde{C}$ determines a secant plane to the image of $(L, H^0(\mathcal{O}_{\widetilde{C}}(D))$ is of maximal dimension $(\widetilde{d} - 1)$.

Now let $\widetilde{s} := h^0(\mathcal{O}_{\widetilde{C}}(D))$. Somewhat abusively, we will identify $\widetilde{C}$ with its image in $\mathbb{P}^{\widetilde{s}}$. Let C denote the image of $\widetilde{C}$ under projection from an $(\widetilde{s}-s-2)$-dimensional center $\Gamma \subset \mathbb{P}^{\widetilde{s}}$ disjoint from $\widetilde{C}$.

Note that $\widetilde{d}$-secant $(\widetilde{d}-\widetilde{r}-1)$-planes to C are in bijective correspondence with those $\widetilde{d}$-secant $(\widetilde{d}-1)$-planes to $\widetilde{C}$ that have at least $(\widetilde{r}-1)$-dimensional intersections with Γ. These, in turn, comprise a subset $\mathcal{S} \subset \mathbb{G}(\widetilde{d}-1, \widetilde{s})$ defined by

$$\mathcal{S} = \mathcal{V} \cap \sigma_{\underbrace{s-\widetilde{d}+\widetilde{r}+2, \ldots, s-\widetilde{d}+\widetilde{r}+2}_{\widetilde{r} \text{ times}}} \tag{3.17}$$

where $\mathcal{V}$, the image of $\mathrm{Sym}^{\widetilde{d}}\widetilde{C}$ in $\mathbb{G}(\widetilde{d}-1, \widetilde{s})$, is the variety of $\widetilde{d}$-secant $(\widetilde{d}-1)$-planes to $\widetilde{C}$, and the term involving σ denotes the Schubert cycle of $(\widetilde{d}-1)$-planes to $\widetilde{C}$ that have at least $(\widetilde{r}-1)$-dimensional intersections with Γ. For a general choice of projection center Γ, the intersection (3.17) is transverse; it follows that

$$\dim \mathcal{S} = \widetilde{d} - \widetilde{r}(s-\widetilde{d}+\widetilde{r}+2), \tag{3.18}$$

In particular, if $\widetilde{d} = d+1$ and $\widetilde{r} = r$, then $\dim \mathcal{S} = 1 + \mu(d,s,r) = 0$, which shows that for any choice of (d,r,s), the formula for A' is enumerative whenever $m = m(d,r,s)$ is chosen to be sufficiently large.

Similarly, to handle A, note that there is a bijection between $\widetilde{d}$-secant $(\widetilde{d}-\widetilde{r}-1)$-planes to C that intersect a general line and $\widetilde{d}$-secant $(\widetilde{d}-1)$-planes to $\widetilde{C}$ that have at least $(\widetilde{r}-1)$-dimensional intersections with Γ, and which further intersect a general line $l \subset \mathbb{P}^{\widetilde{s}}$. These, in turn, comprise a subset $\mathcal{S}' \subset \mathbb{G}(\widetilde{d}-1, \widetilde{s})$ given by

$$\mathcal{S}' = \mathcal{V} \cap \sigma_{\underbrace{s-\widetilde{d}+\widetilde{r}+2, \ldots, s-\widetilde{d}+\widetilde{r}+2}_{\widetilde{r} \text{ times}}, s-\widetilde{d}+\widetilde{r}+1}. \tag{3.19}$$

For a general choice of projection center Γ and line l, the intersection (3.19) is transverse. In particular, if $\widetilde{d} = d$ and $\widetilde{r} = r - 1$, then $\dim \mathcal{S}' = 0$, which shows that for any choice of (d, r, s), the formula for A is enumerative whenever m is sufficiently large.

Note that the equation $\mu = -\alpha - 1$ may be rewritten in the following form:

$$s = \frac{d + \alpha + 1}{r} + d - 1 - r.$$

As a result, r necessarily divides $(d + \alpha + 1)$, say $d = \gamma r - \alpha - 1$, and correspondingly,

$$s = (\gamma - 1)r + \gamma - \alpha - 2.$$

In particular, whenever $\rho = 0$ and $\mu = -1$, we have $1 \leq r \leq s$. As a result, we will focus mainly on the two "extremal" cases of series where $r = 1$ or $r = s$.

3.4 The case $r = 1$

As a special case of [ACGH, Ch. VIII, Prop. 4.2], the expected number of $(d+1)$-secant $(d-1)$-planes to a curve C of degree $(m+1)$ and genus g in $\mathbb{P}^{2d}$ is

$$A' = \sum_{\alpha=0}^{d+1} (-1)^{\alpha} \binom{g + 2d - (m+1)}{\alpha} \binom{g}{d + 1 - \alpha}. \tag{3.20}$$

In fact, the formula for A in case $r = 1$ is implied by the preceding formula. To see why, note that d-secant $(d-1)$-planes to a curve C of degree m and genus g in $\mathbb{P}^{2d}$ that intersect a disjoint line l are in bijection with d-secant $(d-2)$-planes to a curve C of degree m and genus g in $\mathbb{P}^{2d-2}$ (simply project with center l). It follows that

$$A = \sum_{\alpha=0}^{d} (-1)^{\alpha} \binom{g + 2d - (m+3)}{\alpha} \binom{g}{d - \alpha}.$$

Remark. Denote the generating function for the formulas $A = A(d, g, m)$ in case $r = 1$ by $\sum_{d\geq 0} N_d(g, m)z^d$, where

$$N_d(g, m) := \# \text{ of } d-\text{secant } (d-2)-\text{planes to a } g_m^{2d-2} \text{ on a genus-}g \text{ curve.}$$

(As a matter of convention, we let $N_0(g, m) = 1$, and $N_1(g, m) = c_1(L)$.)

We have the following apparently new generating function for $N_d(g, m)$ (here we view g and m as fixed, and we allow the parameter d to vary).

Theorem 4.

$$\sum_{d\geq 0} N_d(g, m)z^d = \left(\frac{2}{(1+4z)^{1/2}+1}\right)^{2g-2-m} \cdot (1+4z)^{\frac{g-1}{2}}. \tag{3.21}$$

Proof. We will in fact prove that

$$\sum_{d\geq 0} N_d(g, m)z^d = \exp\left(\sum_{n>0} \frac{(-1)^{n-1}}{n}\left[\binom{2n-1}{n-1}m + \left(4^{n-1} - \binom{2n-1}{n-1}\right)(2g-2)\right]z^n\right). \tag{3.22}$$

To see that the formulas (3.22) and (3.21) are equivalent, begin by recalling that the generating function $C(z) = \sum_{n\geq 0} C_n z^n$ for the Catalan numbers $C_n = \frac{\binom{2n}{n}}{n+1}$ is given explicitly by

$$C(z) = \frac{1-\sqrt{1-4z}}{2z}.$$

On the other hand, we have

$$\frac{\binom{2n-1}{n-1}}{n} = \left(2 - \frac{1}{n}\right)C_{n-1};$$

whence, (3.22) may be rewritten as follows:

$$\begin{aligned}\sum_{d\geq 0} N_d(g,m)z^d &= \exp\left[\sum_{n>0}(-1)^{n-1}\left[\left[\left(2-\frac{1}{n}\right)(m-2g+2)C_{n-1}z^n\right]+4^{n-1}\cdot(2g-2)\frac{z^n}{n}\right]\right]\\ &= \exp\Bigg[(2m-4g+4)z\sum_{n>0}(-1)^{n-1}C_{n-1}z^{n-1}\\ &\quad -(m-2g+2)\sum_{n>0}(-1)^{n-1}C_{n-1}\frac{z^n}{n}+(2g-2)\sum_{n>0}(-4)^{n-1}\frac{z^n}{n}\Bigg]\\ &= \exp\Bigg[(2m-4g+4)zC(-z)-(m-2g+2)\int C(-z)dz\\ &\quad +(2g-2)\int\frac{1}{1+4z}dz\Bigg]\end{aligned}$$

where $\int$ denotes integration of formal power series. Here

$$\begin{aligned}-\int C(-z)dz &= \int\frac{1-(1+4z)^{1/2}}{2z}dz\\ &= -(1+4z)^{1/2}-\frac{\ln((1+4z)^{1/2}-1)}{2}+\frac{\ln((1+4z)^{1/2}+1)}{2}+\frac{\ln z}{2}.\end{aligned}$$

We deduce that

$$\sum_{d\geq 0} N_d(g,m)z^d = \exp\left[(2g-2-m)\left(\ln 2-\ln((1+4z)^{1/2}+1)\right)+\frac{(g-1)\ln(1+4z)}{2}\right],$$

and (3.21) follows.

To prove (3.22), proceed as follows. Begin by fixing a positive integer $d>0$, and let C denote the image of a g^{2d-2}_m that is sufficiently "nonspecial" in the sense of the preceding section. Then, as noted in the preceding section, $N_d(g,m)$ computes the degree of the locus of d-tuples in $\mathrm{Sym}^d C$ for which the evaluation map (3.14) has rank $(d-1)$. In fact, we will find it more convenient to work instead on the usual d-tuple product C^d. Clearly, $N_d(g,m)$ computes $\frac{1}{d!}$ times the degree $\widetilde{N_d}(g,m)$ of the locus along which the corresponding evaluation map has rank $(d-1)$, since there are $d!$ permutations of any given d-tuple corresponding to a given d-secant plane.

On the other hand, Porteous' formula implies that $\widetilde{N_d}(g,m)$ is equal to the degree of

the determinant

$$\begin{vmatrix} c_1 & c_2 & \cdots & c_{d-1} & c_d \\ 1 & c_1 & \cdots & c_{d-2} & c_{d-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & c_1 \end{vmatrix} \tag{3.23}$$

where c_i denotes the ith Chern class of the secant bundle $S^d(L)$ over C^d. Note [R1] that the Chern polynomial of $S^d(L)$ is given by

$$c_t(S^d(L)) = (1 + l_1 t) \cdot (1 + (l_2 - \Delta_2)t) \cdots (1 + (l_d - \Delta_d)t)$$

where $l_i, 1 \leq i \leq d$ is the pullback of $c_1(L)$ along the ith projection $C^d \to C$, and $\Delta_j, 2 \leq j \leq d$ is the (first Chern class of the) diagonal defined by

$$\Delta_j = \{(x_1, \ldots, x_d) \in C^d | x_i = x_j \text{ for some } i < j\}.$$

In particular, modulo l_i's, we have

$$c_i = (-1)^i s_i(\Delta_2, \ldots, \Delta_d)$$

where s_i denotes the ith elementary symmetric function. In general, it's not hard to check that if $s_i(x_1, \ldots, x_d)$ is the ith elementary symmetric function in the indeterminates x_i, then

$$\begin{vmatrix} s_1 & s_2 & \cdots & s_{d-1} & s_d \\ 1 & s_1 & \cdots & s_{d-2} & s_{d-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & s_1 \end{vmatrix} = \sum_{\substack{i_1, \ldots, i_d \geq 0 \\ i_1 + \cdots i_d = d}} x_1^{i_1} \cdots x_d^{i_d}.$$

It follows that the term of degree one in $(2g-2)$ and zero in m of the determinant (3.23) is equal to the term of appropriate degree in

$$(-1)^d \sum_{\substack{i_1,\dots,i_{d-1}\geq 0\\ i_1+\cdots i_{d-1}=d}} \Delta_2^{i_1}\cdots\Delta_d^{i_{d-1}}. \tag{3.24}$$

Similarly, the term of degree zero in $(2g-2)$ and one in m of (3.23) is equal to the term of corresponding degree in

$$(-1)^{d-1} \sum_{\substack{i_1,\dots,i_{d-1}\geq 0\\ i_1+\cdots i_{d-1}=d-1}} \sum_{j=1}^{d} a_j l_j \Delta_2^{i_1}\cdots\Delta_d^{i_{d-1}} \tag{3.25}$$

where $a_j = 1$ if $j = 1$ and $a_j = i_j + 1$ whenever $2 \leq j \leq d$.

As an immediate consequence of the way in which the coefficients a_j are defined, the intersection (3.25) pushes down to

$$\begin{aligned}(-1)^{d-1} &\sum_{\substack{i_1,\dots,i_{d-1}\geq 0\\ i_1+\cdots i_{d-1}=d-1}} \left(1+\sum_{j=1}^{d-1}(i_j+1)\right)\Delta_2^{i_1}\cdots\Delta_d^{i_{d-1}}\\ &= (-1)^{d-1}(2d-1)\sum_{\substack{i_1,\dots,i_{d-1}\geq 0\\ i_1+\cdots i_{d-1}=d-1}} \Delta_2^{i_1}\cdots\Delta_d^{i_{d-1}}.\end{aligned} \tag{3.26}$$

Lemma 4. *Up to a sign, the term of degree zero in* $(2g-2)$ *and degree one in* m *in* (3.26) *is equal to*

$$\binom{2d-1}{d-1}(d-1)!\cdot m.$$

Lemma 5. *Up to a sign, the term of degree one in* $(2g-2)$ *and zero in* m *in* (3.24) *is equal to*

$$\left(4^{d-1}-\binom{2d-1}{d-1}\right)(d-1)!\cdot(2g-2).$$

To go further, the following observation will play a crucial rôle. For any $d \geq 1$, let K_d denote the complete graph on d labeled vertices $v_1,\dots,v_d$, whose edges $e_{i,j} = \overline{v_i v_j}$ are each

oriented with arrows pointing towards v_j whenever $i < j$. Very roughly, the degree of our determinant (3.23) computes a sum of monomials involving Δ_i and l_j, where $2 \leq i \leq d$ and $1 \leq j \leq d$, and so may be viewed as a tally $\widetilde{S}$ of (not-necessarily connected) subgraphs of K_d, each counted with the appropriate weights. By the Exponential Formula [St, 5.1.6], the exponential generating function for the latter, as d varies, is equal to e^{E_S}, where E_S is the exponential generating function for the corresponding tally of connected subgraphs, which correspond, in turn, to the intersections described in Lemmas 4 and 5.

More precisely now, fix an integer $d \geq 1$, and consider subgraphs of K_d having some number τ of connected components $G_1, \ldots, G_\tau$. (Strictly speaking, we are not merely interested in subgraphs, but in graphs supported on K_d in which at most one edge appears with multiplicity 2, so our terminology is abusive.) Say that the component subgraph G_i has $n_e(i)$ vertices; we stipulate that either these are connected by $n_e(i)$ edges, or else that G_i has a unique "marked" vertex and $(n_e(i)-1)$ edges. Marked vertices v_j correspond to instances of l_j, while edges $e_{i,j}$ correspond to small diagonals $\Delta_{i,j} = \{(x_1, \ldots, x_d) \in C^d | x_i = x_j\}$ associated to d-tuples whose ith and jth coordinates agree. Note that

$$\Delta_j = \sum_{i=1}^{j-1} \Delta_{i,j} \tag{3.27}$$

for every $2 \leq j \leq d$. In the case where no marked vertex appears, at most one edge $e_{i,j}$ may appear with multiplicity 2, in which case it corresponds to $\Delta_{i,j}^2$.

In the case where G_i has no marked vertices, assign to each vertex v_j in G_i a weight

$$w_{G_i,j} = \binom{\text{indeg}(G_i, j)}{i_1, \ldots, i_{j-1}}$$

where $\text{indeg}(G_i, j)$ is equal to the indegree of v_j in G_i, i.e., the total number of edges of G_i

incident with v_j, counted with their nonnegative multiplicities $i_1, \ldots, i_{j-1}$. Let

$$w_{G_i} = \prod_j w_{G_i,j}$$

where the product is over all vertices v_j appearing in G_i.

Similarly, in the case where G_i contains a marked vertex, assign to each vertex v_j in G_i (including the marked vertex) the weight

$$w_{G_i,j} = (\mathrm{indeg}(G_i, j))!,$$

and let

$$w_{G_i} = (2n(e_i) + 1) \prod_j w_{G_i,j}$$

where the product is over all vertices v_j appearing in G_i.

Now let

$$P_{G_i}^{(1)} := (-1)^{n(e_i)+1} w(G_i)(2g-2), \text{ and } P_{G_i}^{(2)} := (-1)^{n(e_i)} w(G_i) m.$$

Set $P_G^{(k)} := \prod_{i=1}^{\tau} P_{G_i}^{(k)}, k = 1, 2$. Then $P_G := P_G^{(1)} + P_G^{(2)}$ represents the contribution of the intersection product corresponding to G to the degree of the determinant (3.23); P_G is a polynomial in m and $(2g-2)$ with integer coefficients. Here the $P_{G_i}^{(k)}, k = 1, 2$ correspond to monomial intersection products of the forms

$$w(G_i)\Delta_{i_1,i'_1} \cdots \Delta_{i_{n(e_i)},i'_{n(e_i)}}, \text{ and } w(G_i) l_j \Delta_{i_1,i'_1} \cdots \Delta_{i_{n(e_i)-1},i'_{n(e_i)-1}},$$

respectively. The fact that our weights $w(G_i)$ have been appropriately chosen, i.e., that the degree of the determinant (3.23) is computed by $\sum_G P_G$, follows easily from our remarks preceding the statements of Lemmas 4 and 5, together with standard intersection theory on C^d. We use the basic facts that

$$l_j \cdot \Delta_{i,j} = p_i^* m\{\mathrm{pt}_C\},$$

and

$$\Delta_{i,j}^2 = -p_i^* \omega_C \cdot \Delta_{i,j} = -(2g-2) p_i^* \{\mathrm{pt}_C\} \cdot \Delta_{i,j}$$

for every choice of (i, j). Here p_i denotes the projection of C^d to the ith copy of C.

On the other hand, it is not hard to see that given any subset $B \subset \{1, \ldots d\}$, the values of the functions f_1 and f_2 that compute the weighted tallies of all connected subgraphs of the complete graph on B with or without marked vertices, respectively, depend only on the cardinality of B. Let $\widetilde{f}_1$ and $\widetilde{f}_2$ denote the functions that compute the corresponding "disconnected" weighted tallies of subgraphs of K_d. Allowing d to vary, we obtain exponential generating functions E_{f_i} and $E_{\widetilde{f}_i}$ for f_i and $\widetilde{f}_i$, respectively, where $i = 1, 2$. The Exponential Formula implies that E_{f_i} and $E_{\widetilde{f}_i}$ are related by

$$E_{\widetilde{f}_i} = \exp(E_{f_i}),$$

for $i = 1, 2$. Now let

$$\widetilde{f} := \sum_G P_G.$$

Since every subgraph of K_d of interest to us can be realized as the union of a subgraph (possibly disconnected) with marked vertices and a subgraph without marked vertices, the exponential generating function $E_{\widetilde{f}}$ of $\widetilde{f}$ satisfies

$$E_{\widetilde{f}} = E_{\widetilde{f}_1} \cdot E_{\widetilde{f}_2}$$

by [St, Prop. 5.1.3].

Consequently, to prove Theorem 4 it would suffice to prove Lemmas 4 and 5. Unfortunately, thus far we have been unable to push the combinatorics through to obtain complete proofs of Lemmas 4 and 5, though we have partial proofs, which we present at the end of

this section. On the other hand, we can give an easy proof of (3.22) by appealing to (3.20), as follows. Namely, the Exponential Formula implies that

$$\sum_{d\geq 0} N_d(g,m)z^d = \exp(\sum_{n>0}[\phi_1 m + \phi_2(2g-2)]z^n)$$

where ϕ_1 and ϕ_2 are rational functions of n. It suffices to show that

$$\phi_1 = (-1)^{n-1}\frac{\binom{2n-1}{n-1}}{n} \text{ and } \phi_2 = (-1)^{n-1}\frac{4^{n-1}-\binom{2n-1}{n-1}}{n}. \tag{3.28}$$

Now let $\widetilde{\pi} = g-1$. Note that (3.20) implies that

$$N_d(g,m) = \sum_{\alpha=0}^{d}(-1)^{\alpha}\binom{\widetilde{\pi}+2d-1-m}{\alpha}\binom{\widetilde{\pi}+1}{d-\alpha}. \tag{3.29}$$

We view the expression on the right side of (3.29) as a polynomial in m and π with coefficients in $\mathbb{Q}[d]$, whose term of degree 1 in m and degree 0 in $\widetilde{\pi}$ is ϕ_1, and whose term of degree 0 in m and degree 1 in $\widetilde{\pi}$ is ϕ_2. As a matter of notation, given any polynomial Q in m and $\widetilde{\pi}$, we let $[m^{\alpha}\widetilde{\pi}^{\beta}]Q$ denote the coefficient of $m^{\alpha}\widetilde{\pi}^{\beta}$ in Q.

To prove the first identity in (3.28), note that, by (3.29),

$$\begin{aligned}
\phi_1 &= [m]\sum_{\alpha=0}^{n}(-1)^{\alpha}\binom{\widetilde{\pi}+2n-1-m}{\alpha}\binom{\widetilde{\pi}+1}{n-\alpha}\\
&= [m]\sum_{\alpha=0}^{n}(-1)^{\alpha}\binom{2n-1-m}{\alpha}\binom{\widetilde{\pi}+1}{n-\alpha}\\
&= [m]\sum_{\alpha=n-1}^{n}(-1)^{\alpha}\binom{2n-1-m}{\alpha}\binom{\widetilde{\pi}+1}{n-\alpha}\\
&= [m]\left((-1)^{n-1}\binom{2n-1-m}{n-1}+(-1)^{n}\binom{2n-1-m}{n}\right)\\
&= (-1)^{n-1}\binom{2n-1}{n-1}\left(-\sum_{i=n+1}^{2n-1}\frac{1}{i}\right)+(-1)^{n}\binom{2n-1}{n}\left(-\sum_{i=n}^{2n-1}\frac{1}{i}\right)\\
&= (-1)^{n-1}\frac{\binom{2n-1}{n-1}}{n}.
\end{aligned}$$

Similarly, to prove the second identity in (3.28), note that, by (3.29),

$$\begin{aligned}
\phi_2 &= \frac{1}{2}\cdot[\widetilde{\pi}]\sum_{\alpha=0}^{n}(-1)^{\alpha}\binom{\widetilde{\pi}+2n-1-m}{\alpha}\binom{\widetilde{\pi}+1}{n-\alpha}\\
&= \frac{1}{2}\cdot[\widetilde{\pi}]\sum_{\alpha=0}^{n}(-1)^{\alpha}\binom{\widetilde{\pi}+2n-1}{\alpha}\binom{\widetilde{\pi}+1}{n-\alpha}\\
&= \frac{1}{2}\cdot(-1)^n\left(\sum_{i=0}^{n-2}\binom{2n-1}{i}\frac{(n-2-i)!}{(n-i)!}-\binom{2n-1}{n-1}\sum_{i=n+1}^{2n-1}\frac{1}{i}+\binom{2n-1}{n}\sum_{i=n+1}^{2n-1}\frac{1}{i}\right)\\
&= \frac{1}{2}\cdot(-1)^n\left(\sum_{i=0}^{n-2}\binom{2n-1}{i}\left(\frac{1}{n-i-1}-\frac{1}{n-i}\right)-\binom{2n-1}{n-1}\left(1-\frac{1}{n}\right)\right).
\end{aligned}$$

Note that

$$4^{n-1}=2^{2n-1}/2=\frac{1}{2}\cdot\sum_{i=0}^{2n-1}\binom{2n-1}{i}=\sum_{i=0}^{n-1}\binom{2n-1}{i}.$$

Whence, to show that $\phi_2=(-1)^{n-1}\left(\frac{4^{n-1}-\binom{2n-1}{n-1}}{n}\right)$, it suffices to show that

$$(n-1)\binom{2n-1}{n-1}=\sum_{i=0}^{n-2}\left(\frac{n}{n-i-1}-\frac{n}{n-i}+2\right)\binom{2n-1}{i}.$$

We have checked the latter identity with the SumTools package for Maple, which implements the Wilf–Zeilberger algorithm [PWZ] for checking identities involving binomial coefficients.

NB: It is not hard to check that $\frac{2}{(1+4z)^{1/2}+1}=C(-z)$, so (3.21) may be reexpressed in the following more compact form:

$$\sum_{d\geq 0}N_d(g,m)z^d=C(-z)^{2g-2-m}\cdot(1+4z)^{\frac{g-1}{2}}. \tag{3.30}$$

□

Towards a combinatorial proof of Lemma 4. The only terms of relevance (i.e., of degree one in m and zero in $(2g-2)$) in (3.26) correspond to $(d-1)$-tuples $(i_1,\ldots,i_{d-1})$ that satisfy the additional constraint

$$\sum_{k=1}^{j}i_k\leq j,\ \text{for all } 1\leq j\leq d-1. \tag{3.31}$$

Notice that the number of such $(d-1)$-tuples is exactly the $(d-1)$st Catalan number $C(d-1)$.

We now expand (3.26) according to (3.27). The monomials of relevance in the resulting expanded intersection product are exactly those in which no diagonal factor $\Delta_{i,j}$ is repeated.

Accordingly, proving the lemma now transposes into the following combinatorial problem. Let K_d denote the complete graph on d labeled vertices $v_1, \ldots, v_d$, whose edges $e_{i,j}$ are marked as before. Consider the set $\mathcal{T}$ of connected spanning trees on K_d. To each vertex $v_j, 2 \leq j \leq d$ of a graph G in $\mathcal{T}$, assign the weight

$$w_{G,j} = (\text{indeg}(G, j))!.$$

where $\text{indeg}(G, j)$ denotes the total indegree of the vertex v_j in G. Now set $w_G = \prod_{2 \leq j \leq d} w_{G,j}$. In light of (3.26), it then suffices to show that

$$(2d-1) \sum_{G \in \mathcal{T}} w_G = \binom{2d-1}{d-1} (d-1)!,$$

i.e., that

$$\sum_{G \in \mathcal{T}} w_G = \frac{(2d-2)!}{d!}. \tag{3.32}$$

Since $\mathcal{T}$ has $C(d-1)$ elements, (3.39) will follow provided we can show that the average weight w_G over all G in $\mathcal{T}$ equals $(d-1)!$.

To this end, let $a_{i_1,\ldots,i_{d-1}}$ denote the number of connected spanning trees on K_d with indegrees $i_1, \ldots, i_{d-1}$ at vertices $v_2, \ldots, v_d$. Clearly, we have

$$\sum_{G \in \mathcal{T}} w_G = \sum_{i_1,\ldots,i_{d-1}} a_{i_1,\ldots,i_{d-1}} i_1! \ldots i_{d-1}!$$

where the $i_j, 1 \leq j \leq d-1$ are nonnegative integers whose sum equals $(d-1)$, and which satisfy the constraint (3.31). It then suffices to show that for any given choice of $(d-$

1)-tuple $(i_1, \ldots, i_{d-1})$ satisfying our constraints, the average value of all $a_{j_1,\ldots,j_{d-1}}$ arising from permuting $(i_1, \ldots, i_{d-1})$ (while still respecting (3.31)) equals $\frac{(d-1)!}{i_1!\ldots i_{d-1}!}$. As a matter of terminology, let an *admissible* permutation of a given $(d-1)$-tuple $(i_1, \ldots, i_{d-1})$ denote a $(d-1)$-tuple obtained by permuting $(i_1, \ldots, i_{d-1})$ that satisfies (3.31). Let $\phi(i_1, \ldots, i_{d-1})$ denote the number of admissible permutations of a given $(d-1)$-tuple $(i_1, \ldots, i_{d-1})$. Note that $\phi(i_1, \ldots, i_{d-1})$ is exactly the number of Dyck paths associated to the corresponding partition $(\lambda_1^{e_1}, \ldots, \lambda_1^{e_l})$ of $(d-1)$, obtained by discarding every instance of zero in $(i_1, \ldots, i_{d-1})$. Here $\lambda_i^{e_i}$ denotes a sequence of e_i identical terms λ_i. See [De1, Sect. 2] for generalities and terminology related to Dyck paths. We will follow the conventions established there. We then have

$$\phi(i_1, \ldots, i_{d-1}) = \frac{(d-1)!}{(d-k)!e_1!\ldots e_l!} \tag{3.33}$$

where $k = \sum_{i=1}^{l} e_i$, by [St, Thm. 5.3.10]. Accordingly, we have reduced to showing that

$$a_\lambda = \frac{(d-1)!}{(d-k)!e_1!\ldots e_l!} \cdot \frac{(d-1)!}{(\lambda_1!)^{e_1}\ldots(\lambda_l!)^{e_l}} \tag{3.34}$$

where $a_\lambda := \sum_{(j_1,\ldots,j_{d-1})} a_{(j_1,\ldots,j_{d-1})}$ is the total number of connected spanning trees with indegree sequences $(j_1, \ldots, j_{d-1})$ that are admissible permutations of a fixed indegree sequence $(i_1, \ldots, i_{d-1})$ corresponding to the partition $\lambda = (\lambda_1^{e_1}, \ldots, \lambda_l^{e_l})$. At present, we are unable to prove (3.34); however, we have a recursive strategy for computing the sums a_λ that we now describe.

Namely, fix $1 \leq m \leq l$, let $\lambda = (\lambda_1^{e_1}, \ldots, \lambda_l^{e_l})$ where $\lambda_i < \lambda_j$ whenever $i < j$, and consider those connected spanning trees $\mathcal{T}$ whose indegrees correspond to λ, and whose indegree at v_d is λ_m. The indegrees of the remaining vertices correspond to $\widehat{\lambda} = (\lambda_1^{e_1}, \ldots, \lambda_m^{e_m-1}, \ldots, \lambda_l^{e_l})$. If, for a given tree $\mathcal{T}$, we delete all edges passing through v_d, what remains is a subgraph

$\mathcal{G}$ of K_d with $(d-1-\lambda_m)$ edges, whose number of vertices depends on the number $c_{\mathcal{G}}$ of connected components of $\mathcal{G}$. Associated with each such graph is a partition $\widetilde{\lambda}$ of $(d-1-\lambda_m)$ subordinate to $\widehat{\lambda}$, and for a given $\mathcal{G}$, $c_{\mathcal{G}}$ is equal to the number of parts in the corresponding partition. Since each connected component has one more vertex than it has edges, we see immediately that

$$(d-1-\lambda_m)+c_{\mathcal{G}} \leq d-1,$$

i.e., that

$$c_{\mathcal{G}} \leq \lambda_m. \tag{3.35}$$

The key point underlying our recursive strategy for computing a_λ is that writing down the contribution to a_λ of any particular partition $\widetilde{\lambda}$ subordinate to $\widehat{\lambda}$ is a simple matter. To begin with, we have $\binom{d-1}{(d-1-\lambda_m)+c_{\mathcal{G}}}$ choices for the vertex set of $\mathcal{G}$. Next, let

$$\widehat{\lambda} = \coprod_{i=1}^{c_{\mathcal{G}}} \widehat{\lambda}_i \tag{3.36}$$

be the decomposition of $\widehat{\lambda}$ that defines $\widetilde{\lambda}$. For every $1 \leq i \leq c_{\mathcal{G}}$, let

$$\widetilde{\lambda}_i := \sum_{j=1} \operatorname{card}(\widehat{\lambda}_i),$$

so that $\widetilde{\lambda} = (\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_{c_{\mathcal{G}}})$. There are then

$$\binom{d-1-\lambda_m+c_{\mathcal{G}}}{\widetilde{\lambda}} \prod_{i=1}^{c_{\mathcal{G}}} a_{\widehat{\lambda}_i}$$

possible choices for $\mathcal{G}$. Finally, there are $\prod_{i=1}^{c_{\mathcal{G}}}(\widetilde{\lambda}_i+1)$ ways of completing any given $\mathcal{G}$ to a connected spanning tree with indegree λ_m at v_d. So altogether, we see that

$$a_\lambda = \sum_{i=1}^{m} \sum_{\substack{(\coprod_{i=1}^{c_{\mathcal{G}}} \widehat{\lambda}_i, \widetilde{\lambda}) \\ c_{\mathcal{G}} \leq \lambda_m}} \left(\binom{d-1}{d-1-\lambda_m+c_{\mathcal{G}}} \binom{d-1-\lambda_m+c_{\mathcal{G}}}{\widetilde{\lambda}} \prod_{i=1}^{c_{\mathcal{G}}} a_{\widehat{\lambda}_i} \prod_{i=1}^{c_{\mathcal{G}}} (\widetilde{\lambda}_i+1) \right)$$

where the inside sum is taken over all decompositions (3.36) that respect (3.35).

Towards a combinatorial proof of Lemma 5. This time, our lemma reduces to the following combinatorial problem. Consider the set $\mathcal{S}$ of connected spanning graphs supported along K_d, with d edges. To each vertex $v_j, 2 \leq j \leq d$ of a graph G in $\mathcal{S}$, assign the following weight $w_{G,j}$:

$$w_{G,j} = \binom{\text{indeg}(G,j)}{i_1, \ldots, i_{j-1}}.$$

Let $w_G = \prod_{2 \leq i \leq d} w_{G,j}$. We must show that

$$\sum_{G \in \mathcal{S}} w_G = \left(4^{d-1} - \binom{2d-1}{d-1}\right)(d-1)!.$$

Let $\mathcal{S}_1 \subset \mathcal{S}$ (resp. $\mathcal{S}_2 \subset \mathcal{S}$) comprise graphs containing one edge appearing with multiplicity 2 (resp., graphs all of whose edges appear with multiplicity 1). Clearly, $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$. Then, since $4^{d-1} = \sum_{i=0}^{d-1} \binom{2d-1}{i}$, it will suffice to show that

$$\sum_{G \in \mathcal{S}_1} w_G = \binom{2d-1}{d-2} \cdot (d-1)! \tag{3.37}$$

and

$$\sum_{G \in \mathcal{S}_2} w_G = \left(\sum_{i=0}^{d-3} \binom{2d-1}{i}\right) \cdot (d-1)!. \tag{3.38}$$

We attempt to prove (3.37) as follows. As before, let $a_{i_1,\ldots,i_{d-1}}$ denote the number of connected spanning trees on K_d with indegrees $i_1, \ldots, i_{d-1}$ at vertices $v_2, \ldots, v_d$. Our task is to show that

$$\sum_{i_1,\ldots,i_{d-1}} \sum_{j=1}^{d-1} \binom{i_j+1}{2} i_1! \ldots \widehat{(i_j!)} \ldots i_{d-1}! a_{i_1,\ldots,i_{d-1}} = \binom{2d-1}{d-2} \cdot (d-1)!.$$

To this end, we assume the fact, conjectured earlier, that the average value of all $a_{j_1,\ldots,j_{d-1}}$ arising from permuting $(i_1, \ldots, i_{d-1})$ while respecting (3.31) equals $\frac{(d-1)!}{i_1! \ldots i_{d-1}!}$. Letting $\phi(i_1, \ldots, i_{d-1})$ denote the number of permutations of $(i_1, \ldots, i_{d-1})$ that satisfy (3.31), we

are then reduced to showing that

$$\sum_{\substack{i_1,\ldots,i_{d-1}\\ 0\le i_1\le\cdots\le i_{d-1}}} \phi(i_1,\ldots,i_{d-1})\cdot\sum_{j=1}^{d-1}\binom{i_j+1}{2}=\binom{2d-1}{d-2};$$

by (3.33), this amounts to the statement that

$$\sum_{\lambda}\frac{(d-1)!}{(d-k)!e_1!\ldots e_l!}\sum_{i=1}^{l}e_i\binom{\lambda_i+1}{2}=\binom{2d-1}{d-2} \tag{3.39}$$

where $k=\sum_{i=1}^{l}e_i$ and the sum in (3.39) is over all partitions $\lambda=(\lambda_1^{e_1},\ldots,\lambda_l^{e_l})$ of $(d-1)$. Unfortunately, verifying (3.39) also remains out of reach for the moment.

Proving (3.38) seems to be an order of magnitude more difficult than proving the already-unproven (3.37). We limit ourselves to the following observations, which hint at the difficulty of establishing (3.38). Let $a'_{i_1,\ldots,i_{d-1}}$ denote the number of connected subgraphs G with d edges supported along K_d, all of whose edges appear with multiplicity 1, and for which $\mathrm{indeg}(v_j)=i_j$, for all $2\le j\le d$. Note that it is always possible to remove some edge of such a subgraph in order to yield a connected spanning tree. Accordingly, a $(d-1)$-tuple $(i'_1,\ldots,i'_{d-1})$ describes a set of admissible indegrees (i.e., it arises as the set of indegrees of a connected spanning subgraph of K_d on d edges) provided it is obtained from a set of indegrees $(i_1,\ldots,i_{d-1})$ of a connected spanning tree via

$$i'_k=i_k \text{ for all k}\neq j,\ \text{ while } i'_j=i_j+1,$$

for some index $2\le j\le d$.

Lemma 6. *The number of admissible $(d-1)$-tuples $(i'_1,\ldots,i'_{d-1})$ so obtained is given by*

$$4\sum_{i=0}^{d-3}\frac{\binom{2i+3}{i}}{i+4}.$$

Proof. Let

$$\mathcal{I} = \{(i_1, \ldots, i_{d-1}) | i_j \geq 0 \text{ for all } j, \sum_{j=1}^{k} i_j \leq k \text{ for all } k, \text{ and } \sum_{j=1}^{d-1} i_j = d-1\}$$

denote the set of admissible $(d-1)$-tuples of indegrees corresponding to connected spanning trees of K_d, and let

$$\mathcal{I}' = \{(i'_1, \ldots, i'_{d-1}) | 0 \leq i_j \leq j \text{ for all } j, (\underline{i'_j}) = (\underline{i_j}) + e_k \text{ for some } k, (\underline{i_j}) \in \mathcal{I}\}$$

denote the set of admissible $(d-1)$-tuples of indegrees corresponding to connected spanning subgraphs of K_d with d edges. There is a bijection between $\mathcal{I}$ and the set of Dyck paths of semilength $(d-1)$, given by

$$(i_1, \ldots, i_{d-1}) \Longleftrightarrow (U \underbrace{D \ldots D}_{i_1 \text{ times}}) \cdots (U \underbrace{D \ldots D}_{i_{d-1} \text{ times}}). \tag{3.40}$$

Similarly, there is a bijection between $\mathcal{I}'$ and the set of (unordered) pairs

$$\begin{aligned} \mathcal{A}_{\mathcal{I}} = \{&(P, U_i) | P \text{ is a Dyck path of semilength } (d-1) \text{ and } U_i \\ &\text{is the } i\text{th } U \text{ appearing in the second ascent of } P\}. \end{aligned} \tag{3.41}$$

Namely, given any element (P, U_i) in $\mathcal{A}_{\mathcal{I}}$, insert a D directly after the U corresponding to U_i in P, forming $\widetilde{P}$. The descent sequence of $\widetilde{P}$ is then an element of $\mathcal{I}'$. In this way we obtain a map from $\mathcal{A}_{\mathcal{I}}$ to $\mathcal{I}'$, which is easily checked to be one-to-one and surjective. For example, for surjectivity, argue as follows. Given any $(d-1)$-tuple $(i'_1, \ldots, i'_{d-1})$, form a sequence $\widetilde{P}$ of U's and D's via

$$(i'_1, \ldots, i'_{d-1}) \Longleftrightarrow (U \underbrace{D \ldots D}_{i'_1 \text{ times}}) \cdots (U \underbrace{D \ldots D}_{i'_{d-1} \text{ times}}),$$

much as before. Clearly, the path $\widetilde{P}$ has *at least* two nontrivial ascents; deleting a single D appearing in the second descent of $\widetilde{P}$ yields a Dyck path.

On the other hand, as E. Deutsch [De2] has shown, the set $\mathcal{A}_{\mathcal{I}}$ has cardinality $4\sum_{i=0}^{d-3}\frac{\binom{2i+3}{i}}{i+4}$; for the sake of completeness, we sketch Deutsch's generating function argument, which implies the desired result.

As is well known, every nonempty Dyck path P admits a decomposition

$$P = UADB$$

where A and B are (possibly empty) Dyck paths. Our first step will be to determine the bivariate generating function $G(t,z)$ for Dyck paths with given *first* ascent marked by t and semilength marked by z, i.e., the coefficient of $t^j z^k$ in the expansion of $G(t,z)$ equals the number of Dyck paths of semilength k with first ascent length j. Note that

$$G(1,z) = C(z)$$

where $C(z)$ is the generating function for Catalan numbers, which satisfies $C(z) = 1 + zC(z)^2$. Note also that the length of the first ascent of P is one more than the length of the first ascent of A, and is independent of the length of the first ascent of B. It follows that

$$G(t,z) = 1 + tzG(t,z)G(1,z) = 1 + tzG(t,z)C(z),$$

i.e., that

$$G(t,z) = \frac{1}{1 - tzC(z)}. \tag{3.42}$$

Now let $\widetilde{G}(t,z)$ denote the bivariate generating function for Dyck paths with second ascent marked by t and semilength marked by z. Note that if A is of the form

$$A = \underbrace{U\ldots U}_{n \text{ times}}\underbrace{D\ldots D}_{n \text{ times}}$$

for some $n \geq 0$, then the second ascent of P is the first ascent of B; otherwise, the second ascent of P is the second ascent of A. Letting

$$Q(z) = \frac{1}{1-z} \tag{3.43}$$

denote the generating function of the sequence $(\underbrace{U \ldots U}_{n \text{ times}} \underbrace{D \ldots D}_{n \text{ times}})_{n \geq 0}$, we now find that

$$\widetilde{G}(t,z) = 1 + zQ(z)G(t,z) + z(\widetilde{G}(t,z) - Q(z))\widetilde{G}(t,z),$$

which, in view of (3.42) and (3.43), simplfies to

$$\widetilde{G}(t,z) = \frac{1-tz}{(1-z)(1-tzC)}. \tag{3.44}$$

Finally, let $\widehat{G}(z)$ denote the generating function for the total lengths of second ascents of Dyck paths of given semilength. According to [De2, p. 171], we have

$$\begin{aligned} \widehat{G}(z) &= \left.\frac{d\widetilde{G}}{dt}\right|_{t=1} \\ &= \frac{z^2 C(z)^4}{1-z}, \text{ by (3.44).} \end{aligned} \tag{3.45}$$

On the other hand, from [De1, (B.5)], we have that

$$[z^n]C(z)^s = s \cdot \frac{\binom{2n+s}{n}}{2n+s}$$

for every pair of positive integers (n, s), where the left-hand side of the preceding equality denotes the coefficient of z^n in C^s. It follows from (3.45) that

$$[z^n]\widehat{G}(z) = \sum_{j=2}^{\infty}[z^{n-j}]C(z)^4 = \sum_{j=2}^{n}[z^{n-j}]C(z)^4 = 4\sum_{j=2}^{n}\frac{\binom{2n-2j+4}{n-j}}{2n-2j+4},$$

i.e., that

$$[z^n]\widehat{G}(z) = 4\sum_{i=0}^{n-2}\frac{\binom{2i+4}{i}}{2i+4} = 2\sum_{i=0}^{n-2}\frac{\binom{2i+4}{i}}{i+2} = 4\sum_{i=0}^{n-2}\frac{\binom{2i+3}{i}}{i+4}.$$

□

NB: It is conceivable that the methods of [Le] may be used to prove (3.22), though we haven't checked this.

3.5 The case $r = s$

From [ACGH, Ch. VIII, Prop. 4.2], we see that the expected number of $2s$-secant $(s-1)$-planes to a curve C of degree $(m+1)$ and genus g in $\mathbb{P}^{s+1}$ is

$$A' = \frac{(-1)^{\binom{s}{2}}}{2}[((1+t_1)(1+t_2))^{m-g-s}(1+t_1+t_2)^g(t_1-t_2)^2]_{t_1^{s+1}t_2^{s+1}}.$$

Similarly, Macdonald's formula [M, Thm. 4] specializes nicely in the case $r = s$. It implies that the expected number of $(2s-1)$-secant $(s-1)$-planes to a curve C of degree m and genus g in $\mathbb{P}^{s+1}$ that intersect a disjoint line is

$$A = \frac{(-1)^{\binom{s}{2}}}{2}[((1+t_1)(1+t_2))^{m-g-s}(1+t_1+t_2)^g(t_1-t_2)^2(2t_1t_2+t_1+t_2)]_{t_1^{s+1}t_2^{s+1}}.$$

To see this, simply note that the condition imposed by requiring an $(s-1)$-plane to intersect a line in $\mathbb{P}^{s+1}$ defines the Schubert cycle σ_1 in $\mathbb{G}(s-1, s+1)$; the formula for A above follows from Macdonald's by a straightforward calculation.

4 Divisor class calculations via multiple-point formulas

Unfortunately, the secant-plane divisor coefficients P are not uniquely determined by the relations obtained in the preceding section; rather, an additional relation is needed. In this section, we will describe an alternative method for computing secant-plane divisor classes. Using this alternative approach, we are led to the following conjecture, which is borne out in every computable case.

Conjecture 2. *When $r = 1$, the polynomials $P_\alpha, P_\beta, P_\gamma,$ and P_{δ_0} satisfy*

$$2(d-1)P_\alpha + (m-3)P_\beta = (6-3g)(P_\gamma + P_{\delta_0}); \tag{4.1}$$

when $r = s$, the polynomials P satisfy

$$2(s-1)P_\alpha + (2m-3s)P_\beta = (6s-3m)P_\gamma - (15m-30s+12-6g)P_{\delta_0}. \tag{4.2}$$

Key observation: Because every divisor on the stack $\mathcal{G}^s_m$ of curves with linear series (see [Kh1] and [Kh2] for its construction) is determined by its degrees along 1-parameter families of linear series, our general formula (3.2) for N_d^{d-r-1} determines the class of a secant-plane divisor in $\mathcal{G}^s_m$ as a sum involving tautological classes on $\mathcal{G}^s_m$.

The upshot of the latter observation is that the conjectural relations (4.1) and (4.2), coupled with the four relations among tautological coefficients P obtained in Section 3, determine the classes of secant-plane divisors on $\mathcal{G}^s_m$.

4.1 Set-up for multiple-point formulas

Our alternative method for calculating secant-plane formulas is as follows. As before, we let $\pi : \mathcal{X} \to B$ denote a one-parameter family of curves, whose total space $\mathcal{X}$ comes equipped with a line bundle $\mathcal{L}$, and whose base space B comes equipped with a rank-$(s+1)$ vector bundle $\mathcal{V} \hookrightarrow \pi_*\mathcal{L}$. When either $r = 1$ or $r = s$ (and, conjecturally, in general), the fact that $N_{K_3} = 0$ expresses the coefficient P_{δ_0} in terms of the other secant-plane divisor coefficients, none of which depend upon the number of singular fibers in π. Consequently, we will assume that every fiber of π is a smooth curve. As before, the pair $(\mathcal{L}, \mathcal{V})$ defines a map $f : \mathcal{X} \to \mathbb{P}\mathcal{V}^*$ of B-schemes, whose fibers over points in B are maps from curves to s-dimensional projective spaces.

Now let $\mathcal{G}$ denote the Grassmann bundle of $(d-r-1)$-dimensional subspaces of fibers of $\mathbb{P}\mathcal{V}^*$ over B, and let $\mathcal{I}_{\mathcal{X}} \subset \mathcal{X} \times_B \mathcal{G}$ denote the incidence correspondence canonically obtained from f. The secant-plane locus of interest to us is (the pushforward to B of) Kleiman's dth multiple-point locus [Kl] associated with the projection $\rho : \mathcal{I}_{\mathcal{X}} \to \mathcal{G}$. Because every fiber of π is a smooth curve, π is a curvilinear map. Consequently, according to [Kat], the Chow class m_k of the kth multiple point locus of ρ satisfies

$$m_k = \rho^* \rho_* m_{k-1} + \sum_{i=1}^{k} (-1)^i p_i m_{k-i} \tag{4.3}$$

for certain polynomials $p_i, 1 \leq i \leq k$ in the Chern classes of the virtual normal bundle $\mathcal{N}_\rho$ of ρ. More accurately, (4.3) applies provided every multiple-point locus $m_i, 1 \leq i \leq k$ has the expected dimension. We will address this issue, in our particular case, in a moment. Note that [Kaz, bot. p.11] gives an explicit generating series for the polynomials p_i.

4.2 Evaluation of multiple-point formulas

Evaluating the iterative formulas (4.3) requires computing the Chern classes of the virtual normal bundle $\mathcal{N}_\rho$. These we calculate as follows. Letting $\mathcal{Q}_{\mathcal{G}}$ denote the quotient bundle on $\mathcal{G}$, note that because $\mathcal{I}_{\mathcal{X}} \subset \mathcal{X} \times_B \mathcal{G}$ is the zero locus of the natural map of vector bundles

$$\mathcal{L}^* \to \mathcal{Q}_{\mathcal{G}},$$

its normal bundle $\mathcal{N}_{\mathcal{I}_{\mathcal{X}}/\mathcal{X}\times_B\mathcal{G}}$ is simply the pullback of $\mathcal{L} \otimes \mathcal{Q}_{\mathcal{G}}$ to $\mathcal{I}_{\mathcal{X}}$. On the other hand, $\mathcal{N}_{\mathcal{I}_{\mathcal{X}}/\mathcal{X}\times_B\mathcal{G}}$ and $\mathcal{N}_\rho$ fit into an exact sequence

$$0 \to \mathcal{T}_{\mathcal{X}/B} \to \mathcal{N}_{\mathcal{I}_{\mathcal{X}}/\mathcal{X}\times_B\mathcal{G}} \to \mathcal{N}_\rho \to 0,$$

which implies that their Chern polynomials are related by

$$\begin{aligned} c_t(\mathcal{N}_\rho) &= c(-T_{\mathcal{X}/B})c(\mathcal{N}_{\mathcal{I}_\mathcal{X}/\mathcal{X}\times_B\mathcal{G}}) \\ &= (1+t\omega+t^2\omega^2)c(\mathcal{N}_{\mathcal{I}_\mathcal{X}/\mathcal{X}\times_B\mathcal{G}}) \end{aligned}$$

where $\omega = c_1(\omega_{\mathcal{X}/B})$ is the first Chern class of the relative dualizing sheaf of π.

It is now a relatively straightforward matter to write a computer program to compute secant-plane divisor classes; this we have done in Maple. Code is available upon request.

4.3 Validity of multiple-point formulas

To establish the validity of our expressions for secant-plane divisor classes arising from multiple-point formulas, we argue along much the same lines as in Section 3.3. By Theorem 1, every secant-plane locus (in the case where $\rho = 0$ and $\mu = -1$) is a divisor, as expected. So Porteous' formula, applied to a suitable evaluation map of vector bundles

$$\mathcal{V} \to S^d(\mathcal{L})$$

over $\mathcal{X}_B^d$, accurately predicts the class Z of the zero-dimensional subscheme of $\mathcal{X}_B^d$ comprised of d-tuples that define $(d-r-1)$-planes on the image of f. The pushforward of Z to B is the secant-plane locus of interest to us. In order to calculate the pushforward of Z to B, on the other hand, it suffices to calculate the pushforward for *sufficiently large* values of m, the degree of the restriction of $\mathcal{L}$ to any fiber of π.

Now assume $m >> 0$. In order to show that for every $1 \leq i \leq d$, the ith multiple-point locus M_i of ρ has the expected dimension, it suffices, by upper-semicontinuity, to exhibit a special family of linear series g_m^s with multiple-point loci of the expected dimension. An easy modification of the argument in Section 3.3 shows that the family defined by projecting

a sufficiently general curve of degree m in $\mathbb{P}^{s+1}$ from points along a disjoint line has the desired property.

4.4 The case $r = 1$

When $r = 1$, the results of the preceding subsections imply that

$$\begin{aligned} P_\alpha &= \left[\frac{m+1-2d}{2g}\right] N_d(g,m) - \left[\frac{d+1}{2g}\right] N_{d+1}(g,m+1), \\ P_\beta &= \frac{-mP_\alpha + dN_d(g,m)}{g-1}, \text{ and} \\ P_c &= -N_d(g,m). \end{aligned} \tag{4.4}$$

Similarly, assuming the conjectural relation (4.1), we find that

$$\begin{aligned} P_\gamma &= \frac{1}{12}\left[\left(\frac{(2g+6)(d-1)}{2-g} + (s-1)\right)P_\alpha + \left(\frac{(g+3)(m-3)}{2-g} + m\right)P_\beta\right] \\ &= \left[\frac{5(d-1)}{6(2-g)}\right]P_\alpha + \left[\frac{(g+3)(m-3)}{12(2-g)} + \frac{m}{12}\right]P_\beta, \text{ since } s = 2d-1, \text{ and} \\ P_{\delta_0} &= -\frac{1}{12}\left[s - 1 + \frac{2(d-1)(g-1)}{2-g}\right]P_\alpha - \frac{1}{12}\left[m + \frac{(m-3)(g-1)}{2-g}\right]P_\beta \\ &= \left[-\frac{(d-1)}{6(2-g)}\right]P_\alpha - \frac{1}{12}\left[m + \frac{(m-3)(g-1)}{2-g}\right]P_\beta, \text{ since } s = 2d-1. \end{aligned} \tag{4.5}$$

Given our generating function (3.21) for $N_d(g,m)$ (and, thus, for P_c, since $P_c = -N_d(g,m)$), determining generating functions for $P_\alpha = P_\alpha(d,g,m), P_\beta = P_\beta(d,g,m), P_\gamma = P_\gamma(d,g,m)$, and $P_{\delta_0} = P_{\delta_0}(d,g,m)$ is now a purely formal matter. Namely, let

$$Z_{g,m}(z) := \left(\frac{2}{(1+4z)^{1/2}+1}\right)^{2g-2-m} \cdot (1+4z)^{\frac{g-1}{2}}. \tag{4.6}$$

Then, according to (3.21),

$$\sum_{d\geq 0} N_d(g,m) z^d = Z_{g,m}(z).$$

By (4.4), it follows that

$$\begin{aligned}\sum_{d\geq 0} P_\alpha(d,g,m)z^d &= \frac{1}{2g}\cdot\sum_{d\geq 0}[(m+1-2d)N_d(g,m)-(d+1)N_{d+1}(g,m+1)]z^d\\ &= \left(\frac{m+1}{2g}-\frac{z}{g}\frac{d}{dz}\right)Z_{g,m}(z)-\left(\frac{1}{2g}\cdot\frac{d}{dz}\right)Z_{g,m+1}(z)\\ &= Z_{g,m}(z)\left[\frac{1}{2}-\frac{1}{2(1+4z)^{1/2}}\right].\end{aligned}$$

Similarly, we have

$$\begin{aligned}\sum_{d\geq 0} P_\beta(d,g,m)z^d &= -\frac{m}{g-1}\sum_{d\geq 0}P_\alpha(d,g,m)z^d+\frac{z}{g-1}\cdot\frac{d}{dz}Z_{g,m}(z)\\ &= Z_{g,m}(z)\left[\frac{2z}{1+4z}-\frac{4z}{(1+4z)^{1/2}((1+4z)^{1/2}+1)}\right],\end{aligned}$$

and conjecturally also

$$\begin{aligned}\sum_{d\geq 0} P_\gamma(d,g,m)z^d &= -\left(\frac{5}{6(2-g)}\right)\sum_{d\geq 0}P_\alpha(d,g,m)z^d+z\left(\frac{5}{6(2-g)}\right)\frac{d}{dz}\sum_{d\geq 0}P_\alpha(d,g,m)z^d\\ &\quad+\left(\frac{(g+3)(m+3)}{12(2-g)}+\frac{m}{12}\right)\sum_{d\geq 0}P_\beta(d,g,m)z^d\\ &= Z_{g,m}(z)\left[\frac{z(32z^2-7(1+4z)^{3/2}+36z+7)}{6(1+4z)^{5/2}((1+4z)^{1/2}+1)}\right]\text{ and}\\ \sum_{d\geq 0} P_{\delta_0}(d,g,m)z^d &= \left(\frac{1}{6(2-g)}\right)\sum_{d\geq 0}P_\alpha(d,g,m)z^d-z\left(\frac{1}{6(2-g)}\right)\sum_{d\geq 0}P_\alpha(d,g,m)z^d\\ &\quad-\left(\frac{1}{12m}+\frac{(m-3)(g-1)}{12(2-g)}\right)\sum_{d\geq 0}P_\beta(d,g,m)z^d\\ &= Z_{g,m}(z)\left[\frac{z(32z^2-(1+4z)^{3/2}+12z+1)}{6(1+4z)^{5/2}((1+4z)^{1/2}+1)}\right].\end{aligned}$$

Remark: Note that the quotients $\frac{\sum_{d\geq 0}P_\gamma(d,g,m)z^d}{Z_{g,m}(z)}$ and $\frac{\sum_{d\geq 0}P_{\delta_0}(d,g,m)z^d}{Z_{g,m}(z)}$ determine exponential generating functions for the constant terms of $P_\gamma(d,g,m)$ and $P_{\delta_0}(d,g,m)$, respectively, when these are viewed (for fixed choices of d) as polynomials in m and $(2g-2)$. To see why, note that $P_\gamma(d,g,m)$ may be viewed as the outcome of performing a weighted count of $(d+1)$-edged subgraphs $\mathcal{G}$ of the complete labeled graph on d vertices with the following

property. Namely, we require that $\mathcal{G}$ possess a *unique* connected $(\widehat{d}+1)$-edged subgraph on $\widehat{d}$ vertices, for some $\widehat{d} \leq d$, and that the remaining connected components of $\mathcal{G}$ have either d edges, or $(d-1)$ edges and a marked vertex corresponding to some pullback of $c_1(\mathcal{L})$. On the other hand, it is easy to see that the exponential generating function for the constant terms of $P_\gamma(d,g,m)$ exactly counts (up to a weighting scheme, which we will not specify here) connected $(d+1)$-edged connected subgraphs of the complete labeled graph on d vertices. So our assertion about $\frac{\sum_{d\geq 0} P_\gamma(d,g,m)z^d}{Z_{g,m}(z)}$ follows from [St, Prop. 5.1.1]; the corresponding assertion about $\frac{\sum_{d\geq 0} P_{\delta_0}(d,g,m)z^d}{Z_{g,m}(z)}$ is proved similarly.

Now let

$$X(z) := \frac{z(32z^2 - 7(1+4z)^{3/2} + 36z + 7)}{6(1+4z)^{5/2}((1+4z)^{1/2}+1)}, \text{ and } Y(z) := \frac{z(32z^2 - (1+4z)^{3/2} + 12z + 1)}{6(1+4z)^{5/2}((1+4z)^{1/2}+1)}.$$

The function $Y(z)$ has Taylor series

$$\frac{1}{6}(3z^2 - 20z^3 + 105z^4 - 504z^5 + 2310z^6 - 10296z^7 + 45045z^8 - 194480z^9 + \dots).$$

In fact, it is not hard to show that

$$[z^n]Y(z) = \frac{(-1)^{n-2}}{6} \cdot \frac{(2n-1)!}{n!(n-2)!}.$$

Similarly, $X(z)$ has Taylor series

$$\frac{1}{6}(-3z^2 + 28z^3 - 177z^4 + 960z^5 - 4806z^6 + 22920z^7 - 105837z^8 + 477688z^9 - \dots),$$

and we have

$$[z^n]X(z) = (-1)^{n-1}\left(\frac{1}{2}\binom{2n}{n}(n+1) - 2^{2n-1} - \frac{1}{6} \cdot \frac{(2n-1)!}{n!(n-2)!}\right).$$

To conclude that P_γ and P_{δ_0} have the generating functions predicted by (4.5), or equivalently, that the tautological coefficients satisfy the conjectural relation (4.2), it suffices to show that the following.

Conjecture 3. *The exponential generating function for the constant terms of $P_\gamma(d,g,m)$ (resp., P_{δ_0}) is $X(z)$ (resp., $Y(z)$).*

Moreover, in order to verify Conjecture 3, it suffices to verify the statement for *either* P_γ or P_{δ_0}, since each of these is related to the other via the "$K3$" relation (3.4). The relative simplicity of the expression $[z^n]Y(z)$, in particular, suggests that a combinatorial proof of Conjecture 3 should be within grasp. We plan to address this matter in a subsequent paper.

4.5 From generating functions to generalized hypergeometric series

Using the results of the preceding subsection, it is possible to realize P_c, P_α, P_β, P_γ, and P_{δ_0} as linear combinations of generalized hypergeometric series. Namely, we have the following result.

Theorem 5. *When $r=1$, the tautological secant-plane divisor coefficients $P_\alpha = P_\alpha(d,g,m)$, $P_\beta = P_\beta(d,g,m)$, and $P_c = P_c(d,g,m)$ are given by*

$$P_c = -\frac{g!(2g-2-m)!}{(g-2d)!d!(2g-2-m+d)!}\,{}_3F_2\left[\begin{matrix} -\frac{g}{2}+\frac{m}{2}+1-d, & -\frac{g}{2}+\frac{m+3}{2}-d, & -d \\ \frac{g+1}{2}-d, & \frac{g}{2}+1-d & \end{matrix}\middle|1\right],$$

$$P_\alpha = \frac{g!(2g-2-m)!}{2(g-2d)!d!(2g-2-m+d)!}\,{}_3F_2\left[\begin{matrix} -\frac{g}{2}+\frac{m}{2}+1-d, & -\frac{g}{2}+\frac{m+3}{2}-d, & -d \\ \frac{g+1}{2}-d, & \frac{g}{2}+1-d & \end{matrix}\middle|1\right]$$

$$-\frac{(g-1)!(2g-2-m)!}{2(g-2d-1)!d!(2g-2-m+d)!}\,{}_3F_2\left[\begin{matrix} -\frac{g}{2}+\frac{m}{2}+1-d, & -\frac{g}{2}+\frac{m+1}{2}-d, & -d \\ \frac{g+1}{2}-d, & \frac{g}{2}-d & \end{matrix}\middle|1\right],$$

$$P_\beta = \frac{2(g-2)!(2g-2-m)!}{(g-2d)!(d-1)!(2g-3-m+d)!}\,{}_3F_2\left[\begin{matrix} -\frac{g}{2}+\frac{m}{2}+1-d, & -\frac{g}{2}+\frac{m+3}{2}-d, & 1-d \\ \frac{g+1}{2}-d, & \frac{g}{2}+1-d & \end{matrix}\middle|1\right]$$

$$-\frac{2(g-1)!(2g-1-m)!}{(g+1-2d)!(d-1)!(2g-2-m+d)!}{}_3F_2\left[\begin{matrix} -\frac{g}{2}+\frac{m}{2}+1-d, & -\frac{g}{2}+\frac{m+3}{2}-d, & 1-d \\ \frac{g}{2}+1-d, & \frac{g+3}{2}-d & \end{matrix}\middle|1\right],$$

Moreover, when $d \geq 3$, we have, assuming Conjecture 3:

$$P_\gamma = \frac{8(g-5)!(2g-1-m)!}{3(g+1-2d)!(d-3)!(2g-m+d-4)!}{}_3F_2\left[\begin{matrix} -\frac{g}{2}+\frac{m}{2}+1-d, & -\frac{g}{2}+\frac{m+3}{2}-d, & 3-d \\ \frac{g}{2}+1-d, & \frac{g+3}{2}-d & \end{matrix}\middle|1\right]$$

$$-\frac{7(g-2)!(2g-1-m)!}{12(g-2d)!(d-1)!(2g-2-m+d)!}{}_3F_2\left[\begin{matrix} -\frac{g}{2}+\frac{m}{2}+1-d, & -\frac{g}{2}+\frac{m+1}{2}-d, & 1-d \\ \frac{g+1}{2}-d, & \frac{g}{2}+1-d & \end{matrix}\middle|1\right]$$

$$+\frac{3(g-5)!(2g-1-m)!}{(g-1-2d)!(d-2)!(2g-3-m+d)!}{}_3F_2\left[\begin{matrix} -\frac{g}{2}+\frac{m}{2}-d, & -\frac{g}{2}+\frac{m+1}{2}-d, & 2-d \\ \frac{g+1}{2}-d, & \frac{g}{2}-d & \end{matrix}\middle|1\right]$$

$$+\frac{7(g-5)!(2g-1-m)!}{12(g-3-2d)!(d-1)!(2g-2-m+d)!}{}_3F_2\left[\begin{matrix} -\frac{g+1}{2}+\frac{m}{2}-d, & -\frac{g}{2}-1+\frac{m}{2}-d, & 1-d \\ \frac{g}{2}-1-d, & \frac{g-1}{2}-d & \end{matrix}\middle|1\right],$$

and

$$P_{\delta_0} = \frac{8(g-5)!(2g-1-m)!}{3(g+1-2d)!(d-3)!(2g-m+d-4)!}{}_3F_2\left[\begin{matrix} -\frac{g}{2}+\frac{m}{2}+1-d, & -\frac{g}{2}+\frac{m+3}{2}-d, & 3-d \\ \frac{g}{2}+1-d, & \frac{g+3}{2}-d & \end{matrix}\middle|1\right]$$

$$-\frac{(g-2)!(2g-1-m)!}{12(g-2d)!(d-1)!(2g-2-m+d)!}{}_3F_2\left[\begin{matrix} -\frac{g}{2}+\frac{m}{2}+1-d, & -\frac{g}{2}+\frac{m+1}{2}-d, & 1-d \\ \frac{g+1}{2}-d, & \frac{g}{2}+1-d & \end{matrix}\middle|1\right]$$

$$+\frac{(g-5)!(2g-1-m)!}{(g-1-2d)!(d-2)!(2g-3-m+d)!}{}_3F_2\left[\begin{matrix} -\frac{g}{2}+\frac{m}{2}-d, & -\frac{g}{2}+\frac{m+1}{2}-d, & 2-d \\ \frac{g+1}{2}-d, & \frac{g}{2}-d & \end{matrix}\middle|1\right]$$

$$+\frac{(g-5)!(2g-1-m)!}{12(g-3-2d)!(d-1)!(2g-2-m+d)!}{}_3F_2\left[\begin{matrix} -\frac{g+1}{2}+\frac{m}{2}-d, & -\frac{g}{2}-1+\frac{m}{2}-d, & 1-d \\ \frac{g}{2}-1-d, & \frac{g-1}{2}-d & \end{matrix}\middle|1\right].$$

The same formulas for P_γ and P_{δ_0} hold when $d = 2$, except that in each case the first hypergeometric summand should be suppressed. Finally, $P_\gamma(1,g,m) = P_{\delta_0}(1,g,m) = 0$.

Proof. Recall (see, e.g., [PWZ]) that

$$ {}_pF_q\left[\begin{matrix} a_1, & \dots, & a_p \\ b_1, & \dots, & b_q \end{matrix}\middle|\phi\right] = \sum_{k=0}^{\infty} \frac{(a_1)^{(k)}\cdots(a_p)^{(k)}}{(b_1)^{(k)}\cdots(b_q)^{(k)}}\phi^k $$

where $(u)^{(k)} = \frac{\Gamma(u+k)}{\Gamma(u)}$ is the Pochhammer symbol.

Using (3.30), we find that

$$ \begin{aligned} P_c(d,g,m) &= -[z^d](C(-z)^{2g-2-m}\cdot(1+4z)^{\frac{g-1}{2}}) \\ &= -\sum_{k=0}^{d}[z^k]C(-z)^{2g-2-m}\cdot[z^{d-k}](1+4z)^{\frac{g-1}{2}}. \end{aligned} $$

Here

$$ [z^k]C(-z)^{2g-2-m} = (-1)^k\frac{2g-2-m}{k+2g-2-m}\binom{2k+2g-3-m}{k}. $$

It follows that

$$ P_c(d,g,m) = -\sum_{k=0}^{d}(-1)^k\frac{2g-2-m}{k+2g-2-m}\binom{2k+2g-3-m}{k}4^{d-k}\binom{\frac{g-1}{2}}{d-k}. \tag{4.7} $$

Using [Ko, Algorithm 2.2], we check easily that the expression on the right side of (4.7) equals

$$ -4^d\binom{\frac{g-1}{2}}{d}{}_3F_2\left[\begin{matrix} g-1-\frac{m}{2}, & g-\frac{m+1}{2}, & -d \\ 2g-1-m, & \frac{g+1}{2}-d & \end{matrix}\middle|1\right]. $$

On the other hand, for every nonnegative integer n, we have the following equality of hypergeometric series [GR]:

$$ {}_3F_2\left[\begin{matrix} w, & x, & -n \\ y, & z & \end{matrix}\middle|1\right] = \frac{(-w-x+y+z)_n}{(z)_n}{}_3F_2\left[\begin{matrix} -w+y, & -x+y, & -n \\ y, & -w-x+y+z & \end{matrix}\middle|1\right]. $$

Taking $w = g-1-\frac{m}{2}$, $x = g-\frac{m+1}{2}$, $y = \frac{g+1}{2}-d$, $z = 2g-1-m$, and $n = d$, we deduce that

$$ -4^d\binom{\frac{g-1}{2}}{d}{}_3F_2\left[\begin{matrix} g-1-\frac{m}{2}, & g-\frac{m+1}{2}, & -d \\ 2g-1-m, & \frac{g+1}{2}-d & \end{matrix}\middle|1\right] $$

$$= -4^d \binom{\frac{g-1}{2}}{d} \frac{(\frac{g}{2}+1-d)_d}{(2g-1-m)_d} {}_3F_2\left[\begin{matrix} -\frac{g}{2}+\frac{m}{2}+1-d, & -\frac{g}{2}+\frac{m+3}{2}-d, & -d \\ \frac{g+1}{2}-d, & \frac{g}{2}+1-d & \end{matrix}\middle|1\right]$$

Finally, it is elementary to check that

$$4^d \binom{\frac{g-1}{2}}{d} \frac{(\frac{g}{2}+1-d)_d}{(2g-1-m)_d} = \frac{g!(2g-2-m)!}{(g-2d)!d!(2g-2-m+d)!};$$

it follows that

$$P_c = -\frac{g!(2g-2-m)!}{(g-2d)!d!(2g-2-m+d)!} {}_3F_2\left[\begin{matrix} -\frac{g}{2}+\frac{m}{2}+1-d, & -\frac{g}{2}+\frac{m+3}{2}-d, & -d \\ \frac{g+1}{2}-d, & \frac{g}{2}+1-d & \end{matrix}\middle|1\right],$$

as desired. The proofs of the the other equalities are similar. □

4.6 Examples

In this subsection, we record several secant-plane formulas in cases where either $r = 1$ or $r = s$.

- $r = 1, d = 2, s = 3$**.** In this case, $N_d^{d-r-1} = N_2^0$ comprises 3-dimensional linear series with double points. We have

$$2!N_2^0 = (-6+2m)\alpha - 4\beta + (2g-2+3m-m^2)c - \gamma + \delta_0.$$

- $r = 1, d = 3, s = 5$ **(case of 5-dimensional series with trisecant lines).** We have

$$3!N_3^1 = (3m^2-27m-6g+66)\alpha + (72-12m)\beta + (28-3m)\gamma + (3m-20)\delta_0$$
$$+ (24-m^3+9m^2+6mg-26m-24g)c.$$

- $r = 1, d = 4, s = 7$ **(case of 7-dimensional series with 4-secant 2-planes).** We have

$$4!N_4^2 = (-1008+168g-24mg-72m^2+452m+4m^3)\alpha + (360m-1440+48g-24m^2)\beta +$$
$$+ (372g-360+342m-119m^2-m^4+18m^3-12g^2-132mg+12m^2g)c$$
$$+ (12g-720+130m-6m^2)\gamma + (6m^2-98m-12g+432)\delta_0.$$

- $r = 1, d = 5, s = 9$ **(case of 9-dimensional series with 5-secant 3-planes).** We have

$$\begin{aligned}5!N_5^3 &= (1020mg - 60m^2g - 4500g + 60g^2 + 19560 + 5m^4 + 1735m^2 - 150m^3 - 9270m)\alpha \\ &+ (240mg - 2400g + 33600 - 40m^3 - 10160m + 1080m^2)\beta \\ &+ (20000 + 60mg - 800g + 370m^2 - 10m^3 - 4640m)\gamma \\ &+ (20m^3g - 60mg^2 - 420m^2g + 6720 + 480g^2 + 2980mg - 5944m + 30m^4 \\ &- 355m^3 + 2070m^2 - m^5 - 7200g)c \\ &+ (60mg + 640g + 10m^3 + 2960m - 290m^2 - 10720)\delta_0.\end{aligned}$$

- $r = 2, d = 3, s = 2$ **(case of 2-dimensional series with triple points).** We have

$$\begin{aligned}3!N_3^0 &= (3m^2 - 18m - 6g + 30)\alpha + (18 - 3m)\beta + 4\gamma - 2\delta_0 \\ &+ (12m^2 - 2m^3 + 6mg - 22m + 12 - 12g)c.\end{aligned}$$

- $r = 3, d = 5, s = 3$ **(case of 3-dimensional series with 5-secant lines).** We have

$$\begin{aligned}4!N_5^1 &= (10m^4 - 180m^3 + 1250m^2 + 5160 - 60m^2g - 4020m + 600mg + 60g^2 - 1620g)\alpha \\ &+ (360m^2 - 20m^3 + 4800 + 60mg - 2200m - 480g)\beta + (1520 - 450m + 40m^2 - 80g)\gamma \\ &+ (2400 + 2190m^2 + 1940mg - 2640g - 635m^3 - 60mg^2 - 480m^2g \\ &+ 40m^3g - 3680m - 5m^5 + 90m^4 + 240g^2)c + (40g - 20m^2 + 210m - 640)\delta_0.\end{aligned}$$

Note that in each of the above examples, we have realized the class of a divisor on the stack $\mathcal{G}_m^s$ in terms of tautological classes c, α, β, γ, and δ_0.

5 Secant-plane divisors on $\overline{\mathcal{M}}_g$

Thus far, we have seen how to determine the coefficients $P_\alpha, P_\beta, P_\gamma,$ and P_{δ_0} of secant-plane divisors on the space of linear series $\mathcal{G}_m^s$. For the sake of calculation, we have assumed $\rho = 0$;

whenever this is the case, every secant-plane divisor pushes forward to a divisor on $\overline{\mathcal{M}}_g$. Khosla's determination of the Gysin map [Kh1] (see also [Kh2]), which we review now, will allow us to compute the coefficients of the Hodge class λ and of the "irreducible" boundary divisor δ_0, of secant-plane divisors on $\overline{\mathcal{M}}_g$.

5.1 Recapitulation of Khosla's work

Let $\widetilde{\mathcal{M}}_{g,1}$ denote the open substack of $\overline{\mathcal{M}}_{g,1}$ equal to the complement of the closure of the substack swept out by reducible unions of smooth curves intersecting transversely in two points. Let $\widetilde{\pi} : \mathcal{C} \to \widetilde{\mathcal{M}}_{g,1}$ denote the universal curve, with relative dualizing sheaf $\widetilde{\omega}$. Recall that for all $g \geq 3$ [Ha],

$$\mathrm{Pic}(\widetilde{\mathcal{M}}_{g,1}) \otimes \mathbb{Q} = \mathbb{Q}\lambda \oplus \mathbb{Q}\delta_0 \oplus_{i=1}^{g-1} \mathbb{Q}\delta_i \oplus \mathbb{Q}\psi.$$

Here

$$\lambda = c_1(\widetilde{\pi}_*\widetilde{\omega}) \text{ and } \psi = c_1(\omega_{\widetilde{\mathcal{M}}_{g,1}/\widetilde{\mathcal{M}}_g}),$$

while δ_0 corresponds to irreducible nodal curves, and $\delta_i, i \geq 1$ corresponds to reducible unions of curves of genera i and $(g-i)$ marked along the component of genus i.

There is, correspondingly, a Deligne-Mumford stack $\mathcal{G}^s_m$ of curves with linear series. When ρ is nonnegative, a unique component of the stack of linear series, which we denote abusively by $\mathcal{G}^s_m$, dominates the moduli stack. Moreover, the projection $\eta : \mathcal{G}^s_m \to \widetilde{\mathcal{M}}_{g,1}$ is generically smooth with fiber dimension ρ.

Now let $\pi : \mathcal{C}^s_m \to \mathcal{G}^s_m$ denote the universal curve. There is a coherent sheaf $\mathcal{L}$ on $\mathcal{C}^s_m$ with torsion-free fibers, whose degree is m on the marked component of every fiber, and whose degree is zero on unmarked components of fibers. Furthermore, $\mathcal{L}$ is trivialized along

the marked section of π. It is not hard to see that the preceding two properties characterize $\mathcal{L}$ uniquely. Finally, there is a subbundle

$$\mathcal{V} \to \pi_* \mathcal{L}$$

whose fibers are marked aspects of linear series.

In [Kh1, Thm. 3.5], Khosla computes the images under the Gysin pushforward

$$\eta_* : A^1(\mathcal{G}^s_m) \to A^1(\widetilde{\mathcal{M}}_{g,1})$$

of the tautological classes

$$\alpha = \pi_*(c_1^2(\mathcal{L})), \beta = \pi_*(c_1(\mathcal{L}) \cdot c_1(\omega)), \text{ and } c = c_1(\mathcal{V})$$

where $\omega = \omega_{\mathcal{C}^s_m/\mathcal{G}^s_m}$ is the relative dualizing sheaf. Note that α, β, and c are precisely those tautological classes that appear in the basic secant-plane formula (3.2). Moreover, there is no mention of the standard class $\gamma = \pi_*(c_1^2(\omega))$ here; that is because, as explained in [HM],

$$\gamma = 12\lambda - \delta_0. \tag{5.1}$$

For our purposes, the contributions of ψ and of $\delta_i, i \geq 1$ to the pushforwards of the standard classes are immaterial, so we omit them. Khosla's formulas, streamlined in this way, read as follows.

$$\begin{aligned}
\eta_*\alpha &= mN\left[\frac{(gm - 2g^2 + 8m - 8g + 4)}{(g-1)(g-2)}\lambda + \frac{2g^2 - gm + 3g - 4m - 2}{6(g-1)(g-2)}\delta_0\right], \\
\eta_*\beta &= mN\left[\frac{6}{g-1}\lambda - \frac{1}{2(g-1)}\delta_0\right], \text{ and} \\
\eta_* c &= N\left[\frac{-(g+3)\xi + 5s(s+2)}{2(g-1)(g-2)}\lambda + \frac{(g+1)\xi - 3s(s+2)}{12(g-1)(g-2)}\delta_0\right]
\end{aligned} \tag{5.2}$$

where

$$\begin{aligned}
N &= \frac{g! \cdot \prod_{i=1}^s i!}{\prod_{i=0}^s (g - m + s + i)!} \text{ is the degree of the covering } \eta, \text{ and} \\
\xi &= 3(g-1) + \frac{(s-1)(g+s+1)(3g - 2m + s - 3)}{g - m + 2s + 1}.
\end{aligned} \tag{5.3}$$

Using the equations (5.2), (5.1), and (4.4), we can explicitly determine the class Sec of any secant-plane divisor on $\overline{\mathcal{M}}_g$, modulo the boundary classes $\delta_i, i \geq 1$ whenever $r = 1$ or $r = s$. Namely, we have, modulo contributions from boundary divisors corresponding to reducible curves,

$$\begin{aligned} \mathrm{Sec} &= P_\alpha \eta_* \alpha + P_\beta \eta_* \beta + P_c \eta_* c + P_\gamma \cdot N(12\lambda - \delta_0) + N P_{\delta_0} \delta_0 \\ &= b_\lambda \lambda - b_0 \delta_0 \end{aligned} \tag{5.4}$$

where $b_\lambda = b_\lambda(d)$ and $b_0 = b_0(d)$ are explicitly determined rational functions of g and m, for any given choice of d.

5.2 Slope calculations

Recall [HM] that the slope of an effective divisor $D \subset \overline{\mathcal{M}}_g$ with class

$$D = b_\lambda \lambda - b_0 \delta_0 - \sum_{i=1}^{\lfloor \frac{g}{2} \rfloor} b_i \tag{5.5}$$

is defined to be the quantity

$$s(D) = \frac{b_\lambda}{\min_i \{b_i\}}.$$

As explained in [FP, Cor. 1.2], we have $s(D) = \frac{b_\lambda}{b_0}$ whenever $g \leq 23$ and provided

$$\begin{aligned} &\frac{b_\lambda}{b_0} \leq 6 + \frac{11}{\lfloor \frac{g}{2} \rfloor + 1}, \text{ and} \\ &\frac{b_\lambda}{b_0} \leq \frac{88828}{12870} \text{ whenever } 20 \leq g. \end{aligned} \tag{5.6}$$

Using the computer, one checks that the ratio $\frac{b_\lambda}{b_0}$ of the first two coefficients in the expansion (5.5) of Sec in terms of standard classes satisfies the conditions 5.6 whenever $g \leq 23$ and either $r = 1$ or $r = s$. It follows that whenever $r = 1$ or $r = s$,

$$s(\mathrm{Sec}) = \frac{b_\lambda}{b_0} \tag{5.7}$$

for all $g \leq 23$. We expect, moreover, that the equation (5.7) holds for *all* g. In the following table, we compile slopes of some secant-plane divisors in the case $r = 1$.

Genus g	d	s	m	$\frac{b_\lambda}{b_0} - (6 + \frac{12}{g+1})$	$\frac{b_\lambda}{b_0} - (6 + \frac{11}{\lfloor \frac{g}{2} \rfloor + 1})$	$\frac{b_\lambda}{b_0} - \frac{8828}{12870}$
8	2	3	9	0	$-13/15$	N/A
12	2	3	12	$693/12389$	$-3952/6671$	N/A
16	2	3	15	$756/13379$	$-3257/7083$	N/A
20	2	3	18	$1539/30247$	$-1632/4321$	$-7775369/27805635$
12	3	5	15	$308/6539$	$-2117/3521$	N/A
18	3	5	20	$32232/596239$	$-130031/313810$	N/A
16	4	7	16	$2520/46427$	$-11357/24579$	N/A
20	5	9	20	$2508/47159$	$-2529/6737$	$-12023068/43352595$

Note that all entries in the second-to-last column are negative, as are all entries in the last column in all cases where $g \geq 20$. It follows that $\frac{b_\lambda}{b_0}$ computes the slope in every case listed. On the other hand, the fact that

$$\frac{b_\lambda}{b_0} - \left(6 + \frac{12}{g+1}\right) \geq 0$$

in every case shows that in each case, the slope of Sec is at least that of the Brill–Noether divisor on $\overline{\mathcal{M}}_g$. The lone zero at the top of column 5 is explained by the fact that every curve of genus 8 that admits a g^3_9 with nodes also carries a g^2_7, and $\rho(8, 2, 7) = -1$. So the corresponding secant-plane divisor is a Brill–Noether divisor on $\overline{\mathcal{M}}_8$.

5.3 Nonemptiness of secant-plane divisors with $r = 1$

Using Theorem 5, we can verify that secant-plane divisors on $\overline{\mathcal{M}}_g$ are nonempty whenever $\rho = 0$ and $r = 1$. Namely, since the pushforward $A^1(\mathcal{G}^s_m) \to A^1(\overline{\mathcal{M}}_g)$ is finite and nonzero,

it suffices to show that the classes of the corresponding secant-plane divisors on $\mathcal{G}^s_m$ are nonzero. Moreover, in light of the calculation carried out for our first test family in Section 3.1, the desired nonvanishing property will follow from showing that the tautological coefficient $P_c = P_c(d, g, m)$ is nonzero for every specialization

$$g = a(s+1) = 2ad, \text{ and } m = (a+1)s = (2d-1)(a+1) \tag{5.8}$$

where a and d are positive integers, $a \geq 2$. (Note that the equations (5.8) encode the fact that $\rho = 0$. The possibility that $a = 1$ is precluded because in that case the corresponding series g^{2d-1}_m are canonical, and do not determine a divisor in $\mathcal{G}^{2d-1}_m$, essentially because every canonical curve that admits a $(d-2)$-secant plane admits a one-parameter family of such planes.) Moreover, by Theorem 5, we have

$$P_c = P_c(a, d) = -\frac{(2ad)!(2ad-2d+a-1)!}{(2ad-2d)!d!(2ad-d+a-1)!} {}_3F_2\left[\begin{matrix} -\frac{a}{2}+\frac{1}{2}, & -\frac{a}{2}+1, & -d \\ ad+\frac{1}{2}-d, & ad+1-d & \end{matrix}\middle| 1\right]. \tag{5.9}$$

Using (5.9), it is not hard to check that

$$-P_c(a, d) = \frac{(2ad)!}{(2ad-d+a-1)!d!} Q(a, d) \tag{5.10}$$

where

$$Q(a, d) = \sum_{i=0}^{\lfloor \frac{a-1}{2} \rfloor} (-1)^i \frac{((2a-2)d+a-1)!}{((2a-2)d+2i)!} \cdot \frac{d!}{(d-i)!} \cdot \frac{(a-1)!}{(a-1-2i)!} \cdot \frac{1}{i!}. \tag{5.11}$$

Whenever $a \geq 2$ and $d \geq 1$, the ith summand in the sum (5.11) has larger absolute value than the $(i+1)$th summand; consequently, $P_c(a, d)$ is negative for all $a \geq 2$ and $d \geq 1$. Nonemptiness follows immediately.

5.4 Slopes of secant-plane divisors with $r = 1$

Note that for any particular choice of $a \geq 2$, the ith summand in (5.11) is a polynomial of degree $(a-1-i)$. It follows that

$$P_c(a,d) = -\frac{(2ad)!}{(2ad-d+a-1)!d!}\left[\frac{((2a-2)d+a-1)!}{((2a-2)d)!} - \frac{((2a-2)d+a-1)!d}{((2a-2)d+2)!}\cdot(a-1)(a-2) + O(d^{a-3})\right].$$

Similarly, we have

$$\begin{aligned}
P_\alpha(a,d) &= \frac{(2ad)!}{2(2ad-d+a-1)!d!}\sum_{i=0}^{\lfloor\frac{a-1}{2}\rfloor}(-1)^i\frac{((2a-2)d+a-1)!}{((2a-2)d+2i)!}\cdot\frac{d!}{(d-i)!}\cdot\frac{(a-1)!}{(a-1-2i)!}\cdot\frac{1}{i!} \\
&\quad - \frac{(2ad-1)!}{2(2ad-d+a-1)!d!}\sum_{i=0}^{\lfloor\frac{a}{2}\rfloor}(-1)^i\frac{((2a-2)d+a-1)!}{((2a-2)d+2i-1)!}\cdot\frac{d!}{(d-i)!}\cdot\frac{a!}{(a-2i)!}\cdot\frac{1}{i!} \\
&= \frac{(2ad)!}{2(2ad-d+a-1)!d!}\left[\frac{((2a-2)d+a-1)!}{((2a-2)d)!} - \frac{((2a-2)d+a-1)!d}{((2a-2)d+2)!}\cdot(a-1)(a-2) + O(d^{a-3})\right] \\
&\quad - \frac{(2ad-1)!}{2(2ad-d+a-1)!d!}\left[\frac{((2a-2)d+a-1)!}{((2a-2)d-1)!} - \frac{((2a-2)d+a-1)!d}{((2a-2)d+1)!}\cdot a(a-1) + O(d^{a-2})\right],
\end{aligned}$$

$$\begin{aligned}
P_\beta(a,d) &= \frac{2(2ad-2)!}{(2ad-d+a-2)!(d-1)!}\sum_{i=0}^{\lfloor\frac{a-1}{2}\rfloor}(-1)^i\frac{((2a-2)d+a-1)!}{((2a-2)d+2i)!}\cdot\frac{(d-1)!}{(d-1-i)!}\cdot\frac{(a-1)!}{(a-1-2i)!}\cdot\frac{1}{i!} \\
&\quad - \frac{2(2ad-1)!}{(2ad-d+a-1)!(d-1)!}\sum_{i=0}^{\lfloor\frac{a-1}{2}\rfloor}(-1)^i\frac{((2a-2)d+a)!}{((2a-2)d+2i+1)!}\cdot\frac{(d-1)!}{(d-1-i)!}\cdot\frac{(a-1)!}{(a-1-2i)!}\cdot\frac{1}{i!} \\
&= \frac{2(2ad-2)!}{(2ad-d+a-2)!(d-1)!}\left[\frac{((2a-2)d+a-1)!}{((2a-2)d)!} - \frac{((2a-2)d+a-1)!(d-1)}{((2a-2)d+2)!}\cdot(a-1)(a-2) + O(d^{a-3})\right] \\
&\quad - \frac{2(2ad-1)!}{(2ad-d+a-1)!(d-1)!}\left[\frac{((2a-2)d+a)!}{((2a-2)d+1)!} - \frac{((2a-2)d+a)!(d-1)}{((2a-2)d+3)!}\cdot(a-1)(a-2) + O(d^{a-3})\right],
\end{aligned}$$

$$
\begin{aligned}
P_\gamma(a,d) &= \frac{8(2ad-5)!}{3(2ad-d+a-3)!(d-3)!}\sum_{i=0}^{\lfloor\frac{a-1}{2}\rfloor}(-1)^i\frac{((2a-2)d+a)!}{((2a-2)d+2i+1)!}\cdot\frac{(d-3)!}{(d-3-i)!}\cdot\frac{(a-1)!}{(a-1-2i)!}\cdot\frac{1}{i!}\\
&- \frac{7(2ad-2)!}{12(2ad-d+a-1)!(d-1)!}\sum_{i=0}^{\lfloor\frac{a}{2}\rfloor}(-1)^i\frac{((2a-2)d+a)!}{((2a-2)d+2i)!}\cdot\frac{(d-1)!}{(d-1-i)!}\cdot\frac{a!}{(a-2i)!}\cdot\frac{1}{i!}\\
&+ \frac{3(2ad-5)!}{(2ad-d+a-2)!(d-2)!}\sum_{i=0}^{\lfloor\frac{a+1}{2}\rfloor}(-1)^i\frac{((2a-2)d+a)!}{((2a-2)d+2i-1)!}\cdot\frac{(d-2)!}{(d-2-i)!}\cdot\frac{(a+1)!}{(a+1-2i)!}\cdot\frac{1}{i!}\\
&+ \frac{7(2ad-5)!}{12(2ad-d+a-1)!(d-1)!}\sum_{i=0}^{\lfloor\frac{a+3}{2}\rfloor}(-1)^i\frac{((2a-2)d+a)!}{((2a-2)d+2i-3)!}\cdot\frac{(d-1)!}{(d-1-i)!}\cdot\frac{(a+3)!}{(a+3-2i)!}\cdot\frac{1}{i!}\\
&= \frac{8(2ad-5)!}{3(2ad-d+a-3)!(d-3)!}\left[\frac{((2a-2)d+a)!}{((2a-2)d+1)!}-\frac{((2a-2)d+a)!(d-3)}{((2a-2)d+3)!}\cdot(a-1)(a-2)+O(d^{a-3})\right]\\
&- \frac{7(2ad-2)!}{12(2ad-d+a-1)!(d-1)!}\left[\frac{((2a-2)d+a)!}{((2a-2)d)!}-\frac{((2a-2)d+a)!(d-1)}{((2a-2)d+2)!}\cdot a(a-1)+O(d^{a-2})\right]\\
&+ \frac{3(2ad-5)!}{(2ad-d+a-2)!(d-2)!}\left[\frac{((2a-2)d+a)!}{((2a-2)d-1)!}-\frac{((2a-2)d+a)!(d-2)}{((2a-2)d+1)!}\cdot(a+1)a+O(d^{a-1})\right]\\
&+ \frac{7(2ad-5)!}{12(2ad-d+a-1)!(d-1)!}\left[\frac{((2a-2)d+a)!}{((2a-2)d-3)!}-\frac{((2a-2)d+a)!(d-1)}{((2a-2)d-1)!}\cdot(a+3)(a+2)+O(d^{a+1})\right],
\end{aligned}
$$

and

$$
\begin{aligned}
P_{\delta_0}(a,d) &= \frac{8(2ad-5)!}{3(2ad-d+a-3)!(d-3)!}\sum_{i=0}^{\lfloor\frac{a-1}{2}\rfloor}(-1)^i\frac{((2a-2)d+a)!}{((2a-2)d+2i+1)!}\cdot\frac{(d-3)!}{(d-3-i)!}\cdot\frac{(a-1)!}{(a-1-2i)!}\cdot\frac{1}{i!}\\
&- \frac{(2ad-2)!}{12(2ad-d+a-1)!(d-1)!}\sum_{i=0}^{\lfloor\frac{a}{2}\rfloor}(-1)^i\frac{((2a-2)d+a)!}{((2a-2)d+2i)!}\cdot\frac{(d-1)!}{(d-1-i)!}\cdot\frac{a!}{(a-2i)!}\cdot\frac{1}{i!}\\
&+ \frac{(2ad-5)!}{(2ad-d+a-2)!(d-2)!}\sum_{i=0}^{\lfloor\frac{a+1}{2}\rfloor}(-1)^i\frac{((2a-2)d+a)!}{((2a-2)d+2i-1)!}\cdot\frac{(d-2)!}{(d-2-i)!}\cdot\frac{(a+1)!}{(a+1-2i)!}\cdot\frac{1}{i!}\\
&+ \frac{(2ad-5)!}{12(2ad-d+a-1)!(d-1)!}\sum_{i=0}^{\lfloor\frac{a+3}{2}\rfloor}(-1)^i\frac{((2a-2)d+a)!}{((2a-2)d+2i-3)!}\cdot\frac{(d-1)!}{(d-1-i)!}\cdot\frac{(a+3)!}{(a+3-2i)!}\cdot\frac{1}{i!}\\
&= \frac{8(2ad-5)!}{3(2ad-d+a-3)!(d-3)!}\left[\frac{((2a-2)d+a)!}{((2a-2)d+1)!}-\frac{((2a-2)d+a)!(d-3)}{((2a-2)d+3)!}\cdot(a-1)(a-2)+O(d^{a-3})\right]\\
&- \frac{(2ad-2)!}{12(2ad-d+a-1)!(d-1)!}\left[\frac{((2a-2)d+a)!}{((2a-2)d)!}-\frac{((2a-2)d+a)!(d-1)}{((2a-2)d+2)!}\cdot a(a-1)+O(d^{a-2})\right]\\
&+ \frac{(2ad-5)!}{(2ad-d+a-2)!(d-2)!}\left[\frac{((2a-2)d+a)!}{((2a-2)d-1)!}-\frac{((2a-2)d+a)!(d-2)}{((2a-2)d+1)!}\cdot(a+1)a+O(d^{a-1})\right]\\
&+ \frac{(2ad-5)!}{12(2ad-d+a-1)!(d-1)!}\left[\frac{((2a-2)d+a)!}{((2a-2)d-3)!}-\frac{((2a-2)d+a)!(d-1)}{((2a-2)d-1)!}\cdot(a+3)(a+2)+O(d^{a+1})\right].
\end{aligned}
$$

On the other hand, when $r = 1$, Khosla's formulas (5.2) imply that

$$\begin{aligned}
\eta_*\alpha &= -\frac{N(2d-1)(a+1)[(2a^2-2a)d^2+(a^2+a-8)d+(4a+2)]}{(2ad-1)(ad-1)}\lambda \\
&\quad + \frac{N(2d-1)(a+1)[(2a^2-2a)d^2+(a^2-4)d+(2a+1)]}{6(2ad-1)(ad-1)}\delta_0, \\
\eta_*\beta &= \frac{6N(2d-1)(a+1)}{2ad-1}\lambda - \frac{N(2d-1)(a+1)}{2(2ad-1)}\delta_0, \text{ and} \\
\eta_*c &= -\frac{N(2d-1)[(2a^3-2a)d^3+(a^3+6a^2-a-8)d^2+(3a^2+2a-4)d+a]}{(2d+a)(2ad-1)(ad-1)}\lambda \\
&\quad + \frac{N(2d-1)d[(2a^3-2a)d^2+(a^3+4a^2-a-4)d+(2a^2-2)]}{6(2d+a)(2ad-1)(ad-1)}\delta_0.
\end{aligned} \tag{5.12}$$

Using our hypergeometric formulas for tautological coefficients in tandem with the push-forward formulas (5.12) and (5.4), we may write down the "virtual slopes" $\frac{b_\lambda}{b_0}$ of secant-plane divisors with $r = 1$ *for any particular value of* a. In the following table, we record the virtual slopes corresponding to $2 \le a \le 5$.

a	$\frac{b_\lambda}{b_0}$
2	$\frac{2(96d^4+80d^3-110d^2-62d+5)}{32d^4+8d^3-30d^2-8d+1}$
3	$\frac{3(9216d^6+15552d^5+5240d^4-6372d^3-5218d^2-1067d+69)}{4608d^6+6048d^5+772d^4-2780d^3-1609d^2-205d+21}$
4	$\frac{2(25920d^7+45360d^6+24387d^5-6006d^4-12143d^3-5213d^2-790d+38)}{8640d^7+12744d^6+4853d^5-2585d^4-3032d^3-1041d^2-105d+8}$
5	$\frac{2(9830400d^8+18595840d^7+12571776d^6+958200d^5-3620196d^4-2433066d^3-734307d^2-89401d+3285)}{3276800d^8+5488640d^7+3012992d^6-174328d^5-1038520d^4-575170d^3-145032d^2-12207d+720}$

Likewise, our formulas readily yield asymptotics in d for the virtual slopes of secant-plane divisors with $r = 1$. Namely, we find that when $r = 1$, any secant-plane divisor on $\overline{\mathcal{M}}_g = \overline{\mathcal{M}}_{2ad}$ has virtual slope equal to

$$\frac{b_\lambda}{b_0} = \frac{6S_1 d + S_2 + O(d^{-1})}{S_1 d + S_3 + O(d^{-1})}$$

where

$$S_1 = 256a^{10} - 1024a^9 + 1280a^8 - 1280a^6 + 1024a^5 - 256a^4,$$

$$S_2 = 384a^{10} + 384a^9 - 13824a^7 + 768a^8 + 26496a^6 - 18048a^5 + 3072a^4 + 768a^3, \text{ and}$$

$$S_3 = 64a^{10} - 192a^9 - 2944a^7 + 1024a^8 + 3136a^6 - 448a^5 - 1152a^4 + 512a^3.$$

In particular, the difference between the secant-plane divisor (virtual) slope and that of the Brill-Noether divisor on $\overline{\mathcal{M}}_{2ad}$ is equal to

$$\frac{b_\lambda}{b_0} - 6 - \frac{12}{2ad+1} = \frac{3}{ad(a+1)} + O(d^{-2}) = \frac{6}{(a+1)g} + O(g^{-2}).$$

6 Boundary coefficients of secant-plane divisors on $\overline{\mathcal{M}}_g$

Much as in the preceding section, write the class of Sec as an expansion in terms of standard divisor classes on $\overline{\mathcal{M}}_g$:

$$\mathrm{Sec} = a\lambda - b_0\delta_0 - \sum_{i=1}^{\lfloor \frac{g}{2} \rfloor} b_i\delta_i.$$

In this section, we will determine b_1 and b_2.

6.1 Determination of b_1

Consider the curve $Y \approx \mathbb{P}^1 \hookrightarrow \overline{\mathcal{M}}_g$ given by attaching a general pencil of plane cubics to a general genus-g flag curve Y at a general point of Y. By the same argument used to prove Theorem 1, we see that Y avoids every secant-plane divisor. On the other hand, it is well-known (see, e.g., [HM]) that

$$Y \cdot \lambda = 1, Y \cdot \delta_0 = 12, Y \cdot \delta_1 = -1, \text{ and } Y \cdot \delta_i = 0 \text{ for all } i \geq 2.$$

It follows that

$$b_\lambda - 12b_0 + b_1 = 0.$$

6.2 Determination of b_2

Given any integer $\alpha \geq 2$, let

$$j_\alpha : \overline{\mathcal{M}}_{\alpha,1} \to \overline{\mathcal{M}}_g$$

denote the map defined by attaching a fixed flag curve C of genus $(g-\alpha)$ at a fixed general point of C to any genus-α curve Y with a marked point. Much as in [FP, proof of Thm 1.1], whose argument we follow, we have the following result.

Theorem 6. *If $\alpha = 2$, then j_α^*Sec is supported on the Weierstrass locus.*

Recall that the Weierstrass locus comprises curves marked along Weierstrass points, and has class

$$\overline{\mathcal{W}} = -\lambda + \frac{g(g+1)}{2}\delta_0 - \sum_{i=1}^{g-1} \binom{g-i+1}{2} \delta_i$$

according to [Cu].

Proof. Assume, for the sake of argument, that j_α^*Sec is *not* supported on the Weierstrass locus; this means exactly that some curve $C \cup_p Y$, where p is not a Weierstrass point of Y, carries a pair of limit linear series (g_m^s, g_m^{s-d+r}) satisfying (1.1). Moreover, by additivity of the generalized Brill–Noether number, we have

$$\rho(2, s, m; r(Y, p)) + \rho(g-2, s, m; r(C, p)) = \rho(g, s, m) = 0 \tag{6.1}$$

where $r(Y, p)$ and $r(C, p)$ denote the total ramification of the g_m^{s-d+r} along Y and C, respectively. Since (C, p) is Brill–Noether general by assumption, and (Y, p) is Brill–Noether general whenever p is not a Weierstrass point of Y, (6.1) forces

$$\rho(2, s, m; r(Y, p)) = \rho(g-2, s, m; r(C, p)) = 0. \tag{6.2}$$

Because p is not a Weierstrass point of Y, we now deduce that the vanishing sequence at p of the aspect of the g^s_m along Y is either

$$a(V_Y, p) = (m-s-2, m-s-1, \ldots, m-3, m), \text{ or}$$

$$a(V_Y, p) = (m-s-2, m-s-1, \ldots, m-4, m-2, m-1).$$

Now assume that a base points of the included series g^{s-d+r}_m lie along Y. Thus, $(d-a)$ base points lie along C, which in turn forces $(d-a)(s-d+r)$ shifts of vanishing order indices of the g^{s-d+r}_m along C, as in Lemmas 1 and 2 of Section 2. By the basic additivity relation **LS1**, it follows that the total ramification of the g^{s-d+r}_{m-a} along Y obtained by removing the a base points from our g^{s-d+r}_m is at least

$$r = (s-d+r+1)(m-s-2) + (s-d+r)(d-a).$$

But an easy calculation yields

$$\begin{aligned}\rho(2, s-d+r, m-a) - r &= 2 + (r-d-1) - (a-d+r), \text{ since } \mu(d,s,r) = -1 \\ &= 1-a.\end{aligned}$$

Because (Y, p) is Brill–Noether general, it follows that $a \leq 1$. Clearly, the case $a = 0$ is impossible, since this would imply that every base point of the g^{s-d+r}_m lies along C, and, therefore, that C admits a d-secant $(d-r-1)$-plane. On the other hand, the case $a = 1$ is also precluded, because in that situation **LS1** forces the top two vanishing orders at p of the g^{s-d+r}_m along Y to be maximal. This is clearly impossible when

$$a(V_Y, p) = (m-s-2, m-s-1, \ldots, m-3, m);$$

for, if there is a base point along Y, then the order to which the g^{s-d+r}_m along Y vanishes at p must be less than m. Similarly, if

$$a(V_Y, p) = (m-s-2, m-s-1, \ldots, m-4, m-2, m-1),$$

then subtracting the base point from the g_m^{s-d+r} along Y yields a g_{m-1}^{s-d+r} containing a subpencil Γ of sections vanishing to orders $(m-2)$ and $(m-1)$, respectively. Subtracting $(m-2)$ base points from Γ yields a g_1^1 along the genus-2 curve Y, which is absurd. $\square$

As explained in [HM, Thm. 6.65], Theorem 6 implies that

$$b_2 = \frac{5}{2}b_1 - \frac{b_\lambda}{2}.$$

It is natural to ask whether a suitable modification of the argument used to prove Theorem 6 may be used to determine any of the remaining boundary divisor coefficients $b_i, i \geq 3$. The basic question to ask is whether there exist reducible curves $C \cup_p Y$ of genus g, with (C, p) a general pointed flag curve of genus $(g-i)$ and (Y, p) a Brill–Noether general pointed curve of genus i, that admit exceptional secant planes. In other words, we'd like to see how the pullbacks $j_i^*\mathrm{Sec}$ relate to the Brill–Noether divisors on $\overline{\mathcal{M}}_{i,1}$ studied in [EH4].

Question: For which values of $i \geq 3$ and (d, r) is it the case that the pullback of Sec along j_i is supported along a union of Brill–Noether divisors on $\overline{\mathcal{M}}_{i,1}$?

As noted in [FP], if $j_i^*\mathrm{Sec}$ is supported along a union of Brill–Noether divisors on $\overline{\mathcal{M}}_{i,1}$ for every $i \leq j$, then every boundary divisor coefficient $b_i, i \leq j$, because, as is shown in [EH4], the class of every Brill–Noether divisor on $\overline{\mathcal{M}}_{i,1}$ is a linear combination of $\overline{\mathcal{W}}$ and the pullback of the Brill–Noether divisor on $\overline{\mathcal{M}}_i$.

7 Planes incident to linear series on a general curve when $\rho = 1$

In this section, we use the results of the preceding one to deduce a new formula for the number $N_d'^{,d-r-1}$ of linear series with exceptional secant planes on a general curve of genus g, which is applicable whenever $\rho = 1$ and $\mu = -1$. (By Theorem 3, that number is always finite.) Namely, we have the following result.

Theorem 7. *Let $\rho = 1, \mu = -1$. The number $N_d'^{,d-r-1}$ of linear series g^s_m with d-secant $(d-r-1)$-planes on a general curve of genus g is given by*

$$N_d'^{,d-r-1} = \frac{(g-1)!1!\cdots s!}{(g-m+s)!\cdots(g-m+2s-1)!(g-m+2s+1)!}$$

$$[(-gm+2gs+m^2-3ms+2s^2-m+s+g)A+(gd+g-md-m+2sd+2s+d+1)A'].$$

where A and A' are as defined in Section 3.1.

Proof. We use the basic set-up of Section 3.1, as well as the relations among the tautological coefficients P_α, P_β, and P_{δ_0} obtained there, to prove Theorem 7. Namely, let C denote a general curve of genus g such that $\rho(g,s,m) = 1$, and consider the test family $\pi : \mathcal{X} \to B$ with total space $\mathcal{X} = W^s_m(C) \times C$ and base $B = W^s_m(C)$. Let $\mathcal{L}$ denote the pullback of any degree-m Poincaré bundle $\widetilde{\mathcal{L}} \to \mathrm{Pic}^m(C) \times C$ by the inclusion $i \times 1_C : W^s_m(C) \times C \to \mathrm{Pic}^m(C) \times C$. Let θ and η denote the integral cohomology classes of the pullbacks to $\mathrm{Pic}^m(C) \times C$ of the theta divisor on $\mathrm{Pic}^m(C)$ and a point on C, respectively. As explained in [ACGH, Ch. 8], we then have

$$\begin{aligned} c_1(\mathcal{L}) &= (m\eta+\gamma)\cdot\nu^* w^s_m \\ &= \underbrace{\Delta(g-m+s,\ldots,g-m+s)}_{(s+1)\text{ times}}\cdot(m\eta+\gamma)\cdot\nu^*\theta^{g-1} \end{aligned}$$

where $\nu : \mathrm{Pic}^m(C) \times C \to \mathrm{Pic}^m(C)$ is the natural projection, and where $\Delta(a_1, \ldots, a_n)$ denotes the determinant of the $n \times n$ matrix with (i,j)th entry $\frac{1}{(a_i+j-i)!}$ (in our case, $n = s+1$).

It follows immediately that

$$\begin{aligned}\alpha &= \Delta(g-m+s, \ldots, g-m+s) \cdot (m\eta + \gamma)^2 \cdot \nu^*\theta^{g-1} \\ &= \Delta(g-m+s, \ldots, g-m+s) \cdot (-2\eta\theta) \cdot \nu^*\theta^{g-1} \\ &= -2g!\Delta(g-m+s, \ldots, g-m+s)\end{aligned}$$

and, likewise, that

$$\beta = \gamma = \delta = 0,$$

since $\omega = \pi_2^* K_C = \pi_2^*(2g-2)\{\mathrm{pt}_C\}$ in this case.

Finally, let Γ denote any section of $\mathrm{Pic}^d(C) \times C \to \mathrm{Pic}^d(C)$ associated to a divisor of large degree on C. Note that $\mathcal{V}$ is the kernel bundle for the evaluation map

$$\mathcal{E} := \nu_*(\widetilde{\mathcal{L}}(\Gamma)) \xrightarrow{\mathrm{ev}} \nu_*(\widetilde{\mathcal{L}}(\Gamma)/\widetilde{\mathcal{L}}) =: \mathcal{F}$$

of vector bundles over $\mathrm{Pic}^d(C)$, restricted to the locus along which ev has a kernel of rank $(s+1)$. On the other hand, the vector bundle $\mathcal{F}$ has trivial Chern classes. Accordingly, the kernel number formula of [HT] yields

$$\begin{aligned}c &= -\Delta_{g-m+s+1, g-m+s, \ldots, g-m+s}(c_t(-\mathcal{E})) \\ &= -g!\Delta(g-m+s+1, g-m+s, \ldots, g-m+s).\end{aligned}$$

Here $\Delta(a_1, \ldots, a_n)(\mathcal{F})$ denotes the determinant of the $n \times n$ matrix with (i,j)th entry $c_{a_i+j-i}(\mathcal{F})$, for any vector bundle $\mathcal{F}$.

On the other hand, from the results of Section 3.1, we see that

$$P_\alpha = \frac{(m-s)A - (d+1)A'}{2g} \text{ and } P_c = -A.$$

It follows immediately that

$$N_d'^{,d-r-1} = -2g!\Delta(g-m+s,\ldots,g-m+s)\frac{(m-s)A-(d+1)A'}{2g}$$

$$+\, g!\Delta(g-m+s+1,g-m+s,\ldots,g-m+s)A.$$

To simplify the latter expression, we use the well-known fact (see, e.g., [ACGH, p.320]) that

$$\Delta(a_1,\ldots,a_n) = \frac{\prod_{j>i}(a_i-a_j-i+j)}{\prod_{i=1}^{n}(a_i-i+n-1)!}.$$

We deduce that

$$\Delta(g-m+s,\ldots,g-m+s) = \frac{s!\cdots 1!}{(g-m+2s)!\cdots(g-m+s)!} \text{ and}$$

$$\Delta(g-m+s+1,g-m+s,\ldots,g-m+s)$$

$$= \frac{(s+1)!(s-1)!\cdots 1!}{(g-m+2s+1)!(g-m+2s-1)!\cdots(g-m+s)!}.$$

It follows that

$$N_d'^{,d-r-1} = \frac{(g-1)!1!\cdots s!}{(g-m+s)!\cdots(g-m+2s-1)!(g-m+2s+1)!}$$

$$\left[-2g(g-m+2s+1)\left(\frac{(m-s)A}{2g}-\frac{(d+1)A'}{2g}\right)+g(s+1)A\right]$$

$$= \frac{(g-1)!1!\cdots s!}{(g-m+s)!\cdots(g-m+2s-1)!(g-m+2s+1)!}$$

$$[(-gm+2gs+m^2-3ms+2s^2-m+s+g)A+(gd+g-md-m+2sd+2s+d+1)A'].$$

□

7.1 The case $r = 1$

Following our usual practice, we now specialize to the case $r = 1$, so that $N_d'^{,d-r-1}$ counts series with d-secant $(d-2)$-planes. Here we obtain stronger results by applying Theorem 5, which characterizes the tautological secant-plane coefficients $P = P(d,g,m)$ in terms of

hypergeometric series. (Note, as above, that $\beta = \gamma = \delta_0 = 0$, and that the expressions for P_α and P_c are not merely conjectural.) Because $\rho = 1$, we have

$$g = 2ad + 1, \text{ and } m = (a+1)(2d-1) + 1$$

for suitably chosen positive integers a and d (here, as usual, d denotes incidence, and $s = 2d - 1$).

Accordingly, we have

$$\begin{aligned}
N_d'^{,d-2} &= \frac{g!1!\cdots s!}{(g-m+s)!\cdots(g-m+2s-1)!(g-m+2s+1)!}[-(s+1)P_c - 2(g-m+2s+1)P_\alpha] \\
&= \frac{(2ad+1)!1!\cdots(2d-1)!}{a!\cdots(a+2d-2)!(a+2d)!}[-2dP_c(a,d) - 2(2d+a)P_\alpha(a,d)] \\
&= \frac{(2ad+1)!1!\cdots(2d-1)!}{a!\cdots(a+2d-2)!(a+2d)!}[-2dP_c(a,d) - 2(2d+a)(-\frac{1}{2}P_c(a,d) + P_{\alpha,2}(a,d))] \\
&= \frac{(2ad+1)!1!\cdots(2d-1)!}{a!\cdots(a+2d-2)!(a+2d)!}[aP_c(a,d) - (4d+2a)P_{\alpha,2}(a,d)]
\end{aligned} \tag{7.1}$$

where

$$\begin{aligned}
P_c(a,d) &= -\frac{(2ad+1)!}{(2ad-d+a)!d!}\sum_{i=0}^{\lfloor\frac{a-1}{2}\rfloor}(-1)^i\frac{((2a-2)d+a)!}{((2a-2)d+2i+1)!}\cdot\frac{d!}{(d-i)!}\cdot\frac{(a-1)!}{(a-1-2i)!}\cdot\frac{1}{i!}, \text{ and} \\
P_{\alpha,2}(a,d) &= -\frac{(2ad)!}{2(2ad-d+a)!d!}\sum_{i=0}^{\lfloor\frac{a}{2}\rfloor}(-1)^i\frac{((2a-2)d+a)!}{((2a-2)d+2i)!}\cdot\frac{d!}{(d-i)!}\cdot\frac{a!}{(a-2i)!}\cdot\frac{1}{i!}.
\end{aligned} \tag{7.2}$$

We have the following result.

Theorem 8. *The number of series with exceptional secant planes $N_d'^{,d-2}$ is zero when either $a = 1$ or $d = 1$, and is positive whenever $a > 1$ and $d > 1$.*

Proof. First assume that $a = 1$. Note that (7.2) implies that

$$P_c(1,d) = -\frac{(2d+1)!(d+1)}{((d+1)!)^2}, \text{ and } P_{\alpha,2}(1,d) = -\frac{(2d)!(d+1)}{2((d+1)!)^2}.$$

It follows that

$$P_c(1,d) - (4d+2)P_{\alpha,2}(1,d) = 0;$$

and whence, by (7.1), that $N_d'^{,d-2} = 0$.

Similarly, (7.2) implies that

$$P_c(a,1) = -(a+2), \text{ and } P_{\alpha,2}(a,1) = -\frac{a}{2},$$

so that

$$P_c(a,1) - (4d+2)P_{\alpha,2}(a,1) = 0,$$

and, therefore, $N_d'^{,d-2} = 0$.

Now assume that $a > 1$, and $d > 1$. In view of (7.1), we need only show that $(4d + 2a)P_{\alpha,2}(a,d) < aP_c(a,d)$ whenever $a > 1$ and $d > 1$, i.e., that

$$\begin{aligned}&\sum_{i=0}^{\lfloor\frac{a-1}{2}\rfloor}(-1)^i\frac{((2a-2)d+a)!}{((2a-2)d+2i+1)!}\cdot\frac{d!}{(d-i)!}\cdot\frac{(a-1)!}{(a-1-2i)!}\cdot\frac{2ad+1}{i!}\\ &<\sum_{i=0}^{\lfloor\frac{a}{2}\rfloor}(-1)^i\frac{((2a-2)d+a)!}{((2a-2)d+2i)!}\cdot\frac{d!}{(d-i)!}\cdot\frac{a!}{(a-2i)!}\cdot\frac{2d+a}{i!}.\end{aligned}$$

To this end, write

$$\sum_{i=0}^{\lfloor\frac{a}{2}\rfloor}(-1)^i\frac{((2a-2)d+a)!}{((2a-2)d+2i)!}\cdot\frac{d!}{(d-i)!}\cdot\frac{a!}{(a-2i)!}\cdot\frac{2d+a}{i!} = \sum_{i=0}^{\lfloor\frac{a}{2}\rfloor}(-1)^iP_1^{(i)}, \text{ and}$$

$$\sum_{i=0}^{\lfloor\frac{a-1}{2}\rfloor}(-1)^i\frac{((2a-2)d+a)!}{((2a-2)d+2i+1)!}\cdot\frac{d!}{(d-i)!}\cdot\frac{(a-1)!}{(a-1-2i)!}\cdot\frac{2ad+1}{i!} = \sum_{i=0}^{\lfloor\frac{a-1}{2}\rfloor}(-1)^iP_2^{(i)}.$$

Here

$$\frac{P_1^{(i)}}{P_2^{(i)}} = \frac{(2d+a)((2a-2)d+2i+1)}{(a-2i)(2ad+1)}$$

for all $i \le \lfloor\frac{a}{2}\rfloor$. So unless $a = 4k$ for some $k \ge 1$ (a case we will handle separately below), we need only show that the quantity

$$T := \sum_{i=0}^{\lfloor\frac{a-1}{2}\rfloor}(-1)^i\left[\frac{(2d+a)((2a-2)d+2i+1)}{(a-2i)(2ad+1)} - 1\right]P_2^{(i)}$$

is positive. (When $a = 4k$, and only in that case, we have

$$\sum_{i=0}^{\lfloor \frac{a}{2} \rfloor} (-1)^i P_1^{(i)} - \sum_{i=0}^{\lfloor \frac{a-1}{2} \rfloor} (-1)^i P_2^{(i)} < T.)$$

To this end, it suffices, in turn, to show that $\frac{T_i}{T_{i+1}} > 1$, where

$$T_i = \left[\frac{(2d+a)((2a-2)d+2i+1)}{(a-2i)(2ad+1)} - 1 \right] P_2^{(i)}.$$

Using the computer, we find that

$$\frac{T_i}{T_{i+1}} = \frac{2[(2a-2)d^2 + (2ai+2i-a+1)d + (a+1)i][(a-1)d+i+1][(2a-2)d+2i+3](i+1)}{[(2a-2)d^2 + (2ai+2i+a+3)d + (a+1)(i+1)](a-1-2i)(a-2i)(d-i)}.$$

Here

$$\begin{aligned} &\frac{(2a-2)d^2 + (2ai+2i-a+1)d + (a+1)i}{(2a-2)d^2 + (2ai+2i+a+3)d + (a+1)(i+1)} \\ &= 1 - \frac{(2a-2)d + (a+1)}{(2a-2)d^2 + (2ai+2i+a+3)d + (a+1)(i+1)} > 1 - \frac{1}{d}, \end{aligned}$$

while

$$\frac{((2a-2)d+2i+3}{a-1-2i} > 2, \text{ and } \frac{(a-1)d+i+1}{(a-2i)(d-i)} > 1 - \frac{1}{a}.$$

We conclude that $\frac{T_i}{T_{i+1}} > 1$ whenever $a > 1$ and $d > 1$.

It remains to treat the case where $a = 4k$ for some $k \geq 1$. To conclude the proof of our theorem, it will suffice to show that

$$P_1^{(i)} - P_1^{(i+1)} - P_2^{(i)} > 0 \tag{7.3}$$

for $i = \frac{a}{2} - 2$. With the aid of the computer, we calculate

$$P_1^{(i)} - P_1^{(i+1)} - P_2^{(i)} = \frac{((2a-2)d+a)!d!a!}{(a-2i)!(d-i)!(i+1)!((2a-2)d+2i+2)!} \widetilde{Q}(a,d,i)$$

where

$$\widetilde{Q}(a,d,i) = (8a^2i - 16ai + 8a^2 - 16a + 8i + 8)d^3$$
$$+ (-12 + 8a^2i^2 + 4a^2i + 32ai + 8ai^2 + 18a - 32i - 6a^2 - 24i^2)d^2$$
$$+ (-4a + 4 + 8ai^3 - a^3 + 10a^2i - 4ai + 4a^2i^2 + 12i + a^2 + 20i^2 + 16i^3)d$$
$$+ (4i + a^3i - 4a^2i^2 + 4ai + 10ai^2 + 8ai^3 - a^2i + 8i^2 + 4i^3).$$

Taking $i = \frac{a}{2} - 2$, we find that

$$\widetilde{Q}\left(a, d, \frac{a}{2} - 2\right) = (4a^3 - 16a^2 + 20a - 8)d^3 + (2a^4 - 12a^3 + 12a^2 + 18a - 44)d^2$$
$$+ (2a^4 - 14a^3 + 24a^2 + 2a - 68)d + \left(\frac{1}{2}a^4 - \frac{7}{2}a^3 + 12a^2 - 22a - 8\right),$$

which is positive whenever $a \geq 4$ and $d \geq 2$. □

Finally, we calculate the asymptotic behavior of $N_d'^{,d-2}$, using (7.1). To this end, note that (7.1) implies that when $r = 1$,

$$P_c(a,d) = -\frac{(2ad+1)!}{(2ad-d+a)!d!}\left[\frac{((2a-2)d+a)!}{((2a-2)d+1)!} + O(d^{a-2})\right], \text{ and}$$
$$P_{\alpha,2}(a,d) = -\frac{(2ad)!}{2(2ad-d+a)!d!}\left[\frac{((2a-2)d+a)!}{((2a-2)d)!} + O(d^{a-1})\right].$$

It follows that

$$N_d'^{,d-2} = \frac{(2ad+1)!1!\cdots(2d-1)!}{a!\cdots(a+2d-2)!(a+2d)!}[aP_c(a,d) - (4d+2a)P_{\alpha,2}(a,d)]$$
$$= \frac{(2ad+1)!1!\cdots(2d-1)!}{a!\cdots(a+2d-2)!(a+2d)!} \cdot \frac{(2ad)!}{(2ad-d+a)!d!}\left[\frac{2d((2a-2)d+a)!}{((2a-2)d)!}\right.$$
$$\left. - \left(\frac{(2ad+1)a((2a-2)d+a)!}{((2a-2)d+1)!} + \frac{a((2a-2)d+a)!}{((2a-2)d)!}\right) + O(d^{a-1})\right]$$
$$= \frac{(2ad+1)!1!\cdots(2d-1)!}{a!\cdots(a+2d-2)!(a+2d)!} \cdot \frac{(2ad)!}{(2ad-d+a)!d!} \cdot \frac{((2a-2)d+a)!}{((2a-2)d+1)!}$$
$$[(4a-4)d^2 + (-4a^2 + 2a + 2)d + O(1)].$$

NB: Theorem 8 establishes that no series with $a = 1$ and $\rho = 1$ on a general curve C of genus g admits d-secant $(d-2)$-planes. This is easy to explain on geometric grounds.

Namely, provided d is sufficiently large, every such series arises as the image of a canonical curve $\widetilde{C} \subset \mathbb{P}^{2d}$ from a point along a curve. Moreover, d-secant $(d-2)$-planes of our original series are in bijection with $(d+1)$-secant $(d-1)$-planes to $\widetilde{C}$. On the other hand, any $(d+1)$-secant $(d-1)$-plane to $\widetilde{C}$ defines an inclusion of linear series

$$g^d_{3d-1} + p_1 + \cdots + p_{d+1} \hookrightarrow g^{2d}_{4d} \tag{7.4}$$

along C. But in fact $\rho(2d+1, d, 3d-1) < 0$; whence, by the Brill–Noether theorem, no inclusions (7.4) exist.

Similarly, when $d = 1$, $N_d'^{,d-2}$ counts one-dimensional series with base points. Theorem 8 establishes that no such series exist on a general curve of genus g, which also confirms the Brill–Noether theorem in a special case.

References

[ACGH] E. Arbarello, M. Cornalba, P. Griffiths, and J. Harris, "Geometry of algebraic curves", Grundlehren der Math. Wiss. 267, Springer, New York, 1985.

[Cu] F. Cukierman, *Families of Weierstrass points*, Duke Math. J. **58** (1989), 317-346.

[De1] E. Deutsch, *Dyck path enumeration*. Discrete Math. **204** (1999), no. 1-3, 167–202.

[De2] E. Deutsch, private communication on 2/13/07.

[DH] S. Diaz and J. Harris, *Geometry of the Severi variety*, Trans. Amer. Math. Soc. **309** (1988), no. 1, 1-34.

[EH1] D. Eisenbud and J. Harris, *Divisors on general curves and cuspidal rational curves*, Invent. Math. **74** (1983), no. 3, 371-418.

[EH2] D. Eisenbud and J. Harris, *A simpler proof of the Gieseker-Petri theorem on special divisors*. Invent. Math. 74 (1983), no. 2, 269-280.

[EH3] D. Eisenbud and J. Harris, *Limit linear series: basic theory*, Invent. Math. **85** (1986), no. 2, 337-371.

[EH4] D. Eisenbud and J. Harris, *Irreducibility and monodromy of some families of linear series*, Ann. Sci. École Norm. Sup. (4) **20** (1987), no. 1, 65-87.

[Fa1] G. Farkas, *Koszul divisors on moduli spaces of curves*, `http://arXiv.org/math.AG/0607475`.

[Fa2] G. Farkas, *Higher ramification and varieties of secant divisors on the generic curve*, `http://arXiv.org/math.AG/07042513`.

[FKP] F. Flamini, A. Knutsen, and G. Pacienza, *Singular curves on a $K3$ surface and linear series on their normalizations*, to appear in Int. J. Math.

[FP] G. Farkas and M. Popa, *Effective divisors on $\overline{\mathcal{M}}_g$, curves on $K3$ surfaces, and the slope conjecture*, J. Algebraic Geometry **14** (2005), no. 2, 241–267.

[Gr] M. Green, *Koszul cohomology and the geometry of projective varieties*, J. Diff. Geom. **19** (1984), 125–171.

[GL] M. Green and R. Lazarsfeld, *On the projective normality of complete linear series on an algebraic curve*, Invent. Math **83** (1986), 73–90.

[GR] G. Gasper and M. Rahman, "Basic hypergeometric series", Encyclopedia of mathematics and its applications, v.96, Cambridge University Press, New York, 2004.

[Ha] J. Harer, *The second homology group of the mapping class group of an orientable surface*, Invent. Math. **72** (1983), no. 2, 221–239.

[HM] J. Harris and I. Morrison, *Slopes of effective divisors on the moduli space of curves*, Invent. Math. **99** (1990), no. 2, 321-355.

[HT] J. Harris and L. Tu, *Chern numbers of kernel and cokernel bundles*, Invent. Math. **75** (1984), 467-475.

[Kat] S. Katz, *Iteration of multiple point formulas and applications to conics*, in "Algebraic geometry: Sundance, 1986", 147–155, Springer Lecture Notes in Math. **1311**, Springer, Berlin, 1988.

[Kaz] M. Kazarian, *Morin maps and their characteristic classes*, preprint.

[Kh1] D. Khosla, *Moduli spaces of curves with linear series and the slope conjecture*, `http://arXiv.org/math.AG/0608024`.

[Kh2] D. Khosla, *Tautological classes on moduli spaces of curves with linear series, and a push-forward formula when* $\rho = 0$, `http://arXiv.org/math.AG/0704.1340`.

[Kl] S. Kleiman, *Multiple-point formulas. II. The Hilbert scheme*, in "Enumerative geometry (Sitges, 1987)", 101-138, Springer Lecture Notes in Math., 1436, Springer, Berlin, 1990.

[Kn1] A. L. Knutsen, *On kth order embeddings of* $K3$ *surfaces and Enriques surfaces.* Manuscripta Math. **104** (2001), no. 2, 211-237.

[Kn2] A. L. Knutsen, *Smooth curves on projective* $K3$ *surfaces*, Math. Scand. **90** (2002), no. 2, 215-231.

[Ko] W. Koepf, "Hypergeometric summation: An algorithmic apporach to summation and special function identities", Vieweg, Braunschweig/Wiesbaden, 1998.

[La] R. Lazarsfeld, *Brill–Noether-Petri without degenerations*, J. Diff. Geom. **23** (1986), 299-307.

[Le] M. Lehn, *Chern classes of tautological sheaves on Hilbert schemes of points on surfaces*, Invent. Math. **136** (1999), no. 1, 157-207.

[M] I. G. Macdonald, *Some enumerative formulae for algebraic curves*, Proc. Cambridge Philos. Soc. **54** (1958), 399-416.

[PWZ] M. Petkovšek, H. Wilf, and D. Zeilberger, "A=B", A. K. Peters, Wellesley, MA, 1996.

[R1] Z. Ran, *Curvilinear enumerative geometry*, Acta. Math. **155** (1985), no. 1-2, 81–101.

[R2] Z. Ran, *Geometry on nodal curves*. Compos. Math. **141** (2005), no. 5, 1191-1212.

[R3] Z. Ran, *Geometry on nodal curves, II: cycle map and intersection calculus*, `http://arXiv.org/math.AG/0410120`.

[Sl] N.J.A. Sloane, `http://www.research.att.com/~njas/sequences/ol.html`.

[St] R. Stanley, "Enumerative combinatorics", vol. 2, Cambridge Studies in Advanced Math. 49, Cambridge University Press, Cambridge, 1997.

[Z] D. Zeilberger, *The method of creative telescoping*, J. Symbolic Comput. **11** (1991), no. 3, 195–204.