

Contribution à la résolution numérique des problèmes de Helmholtz

Magdalena Grigoroscuta-Strugaru

INRIA Magique-3D and LMA, University of Pau

Advisors:

Mohamed Amara (INRIA, LMA)

Henri Calandra (TOTAL)

Rabia Djellouli (CSUN, INRIA)



Outline

- Motivation and Context

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- Solution methodologies for Helmholtz problems

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 - A **new** DG method

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 - An **improved** modified DG method

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 - A **new** DG method
 - A **modified** DG method
 - An **improved** modified DG method
- Summary and perspectives

Motivation and Context

Applications

Applications

- **Geophysical exploration**

Applications

- **Geophysical exploration**
- **Radar**

Applications

- **Geophysical exploration**
- **Radar**
- **Sonar**

Applications

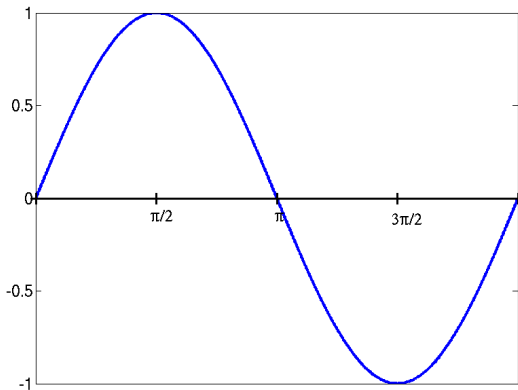
- **Geophysical exploration**
- Radar
- Sonar
- Medical imaging

Applications

- **Geophysical exploration**
- Radar
- Sonar
- Medical imaging
- Nondestructive testing

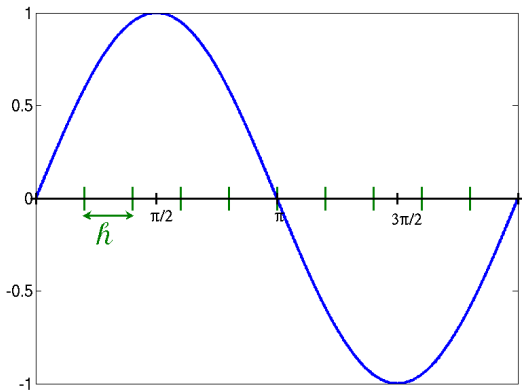
Numerical Difficulties

Numerical Difficulties



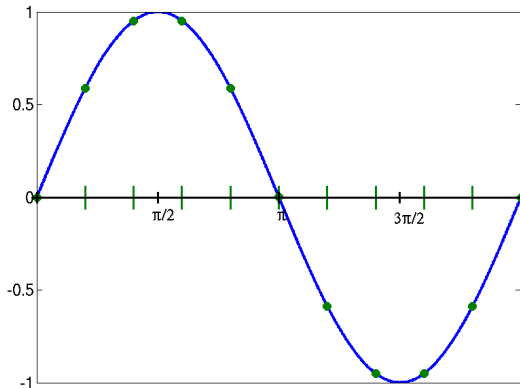
$$ka = 1$$

Numerical Difficulties



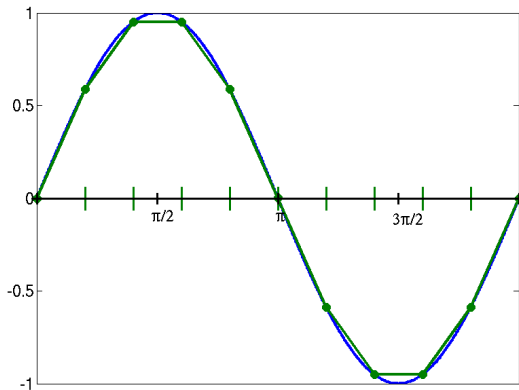
$$ka = 1, \quad \frac{h}{a} = \frac{1}{10}$$

Numerical Difficulties



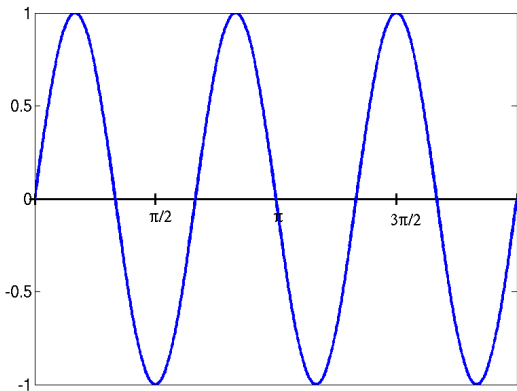
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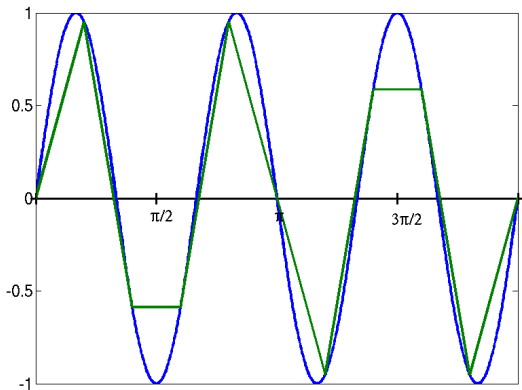
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Numerical Difficulties



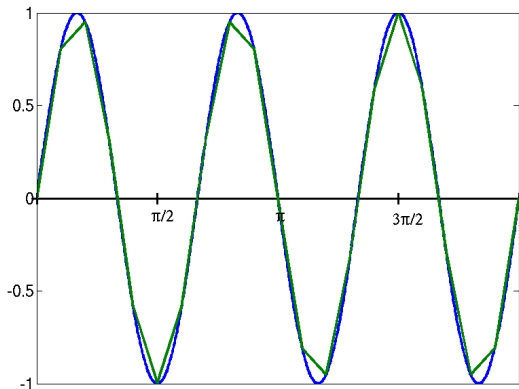
$$ka = 3$$

Numerical Difficulties



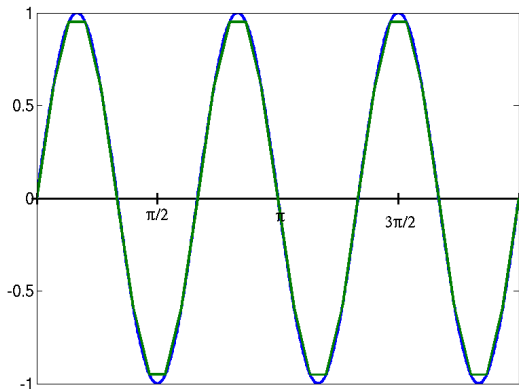
$$ka = 3, \quad \frac{h}{a} = \frac{1}{10}$$

Numerical Difficulties



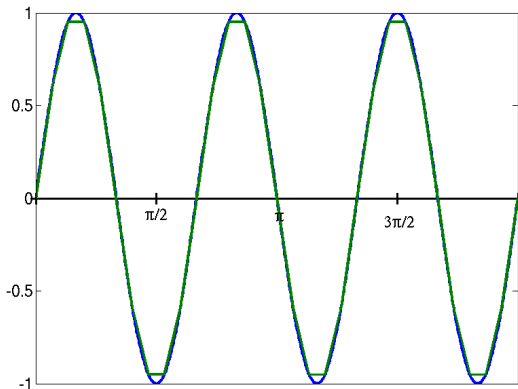
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Numerical Difficulties



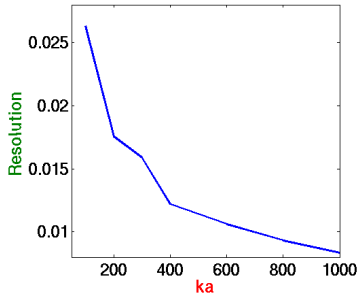
$$ka = 3, \quad \frac{h}{a} = \frac{1}{30}$$

Numerical Difficulties



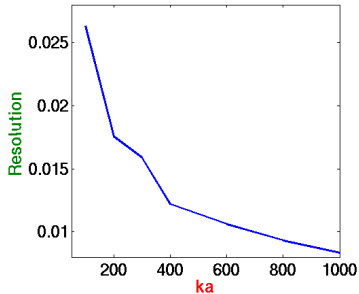
$$ka = 3, \quad \frac{h}{a} = \frac{1}{30} \implies kh = \frac{1}{10}$$

Numerical Difficulties



Resolution necessary to achieve 10% on the relative error

Numerical Difficulties



Resolution necessary to achieve 10% on the relative error



$kh \neq \text{constant}$

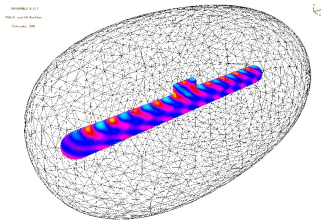
Numerical Difficulties

$$\frac{|u - u_h|_1}{|u|_1} \leq C_1 kh + C_2 k^3 h^2; \quad kh < 1$$

(Babuška et al (95, 00))

Numerical Difficulties

“Realistic” simulation (Tezaur *et al* (02))



$$ka = 10$$

System of about **9.6** million complex unknowns

Motivation and Context

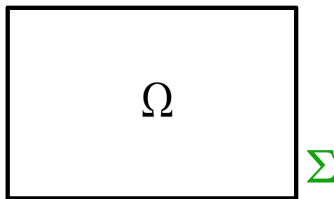
Prominent Plane Waves Based Approaches

- Weak Element Method (**Rose** (75))
- Partition of Unity Method
(**Babuška-Melenk** (97), **Laghrouche-Bettes** (00))
- Ultra-Weak Variational Method
(**Cessenat-Després** (98))
- Least Squares Method (**Monk-Wang** (99))
- Discontinuous Galerkin Method: DGM
(**Farhat et al** (01, 03, 04, 05))

DGM Formulation (Farhat *et al*)

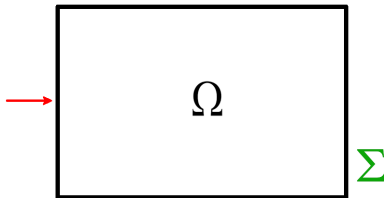
DGM Formulation (Farhat *et al*)

Mathematical model



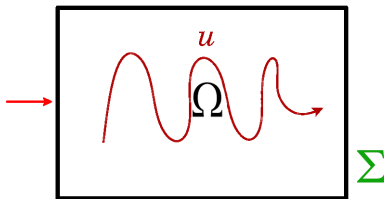
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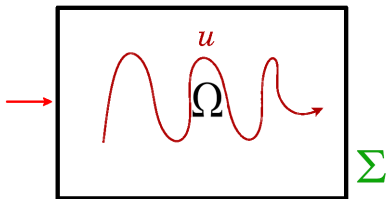
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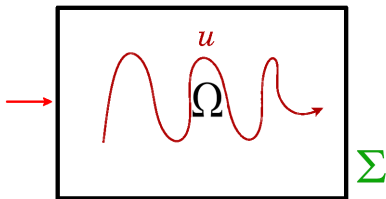
Mathematical model



$$\Delta u + k^2 u = 0 \quad \text{in } \Omega$$

DGM Formulation (Farhat *et al*)

Mathematical model



$$\Delta u + k^2 u = 0 \quad \text{in } \Omega$$
$$\partial_n u = iku + g \quad \text{on } \Sigma$$

DGM Formulation (Farhat *et al*)

Approximation

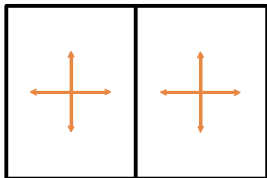


DGM Formulation (Farhat *et al*)

Approximation



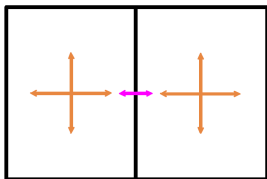
DGM Formulation (Farhat *et al*) Approximation



$$u \equiv \sum_{l=1}^4 u_l^K \phi_l^K \text{ in } K$$
$$\phi_l^K = e^{i\mathbf{k} \cdot \mathbf{d}_l}$$

DGM Formulation (Farhat *et al*)

Approximation



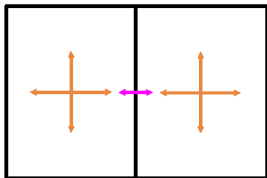
$$u \equiv \sum_{l=1}^4 u_l^K \phi_l^K \text{ in } K$$

$$\phi_l^K = e^{i\mathbf{k}x \cdot \mathbf{d}_l}$$

$$\lambda = \lambda^K \in \mathbb{C} \text{ on } \partial K \cap \partial K'$$

DGM Formulation (Farhat *et al*)

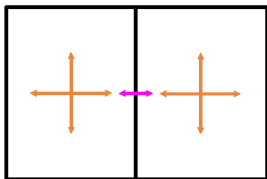
Variational Formulation



$$\begin{cases} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, \boldsymbol{\lambda}) = F(\mathbf{v}) \\ b(\mathbf{u}, \boldsymbol{\mu}) = 0 \end{cases}$$

DGM Formulation (Farhat *et al*)

Variational Formulation

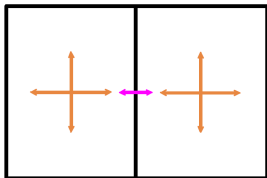


$$\begin{cases} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, \boldsymbol{\lambda}) = F(\mathbf{v}) \\ b(\mathbf{u}, \boldsymbol{\mu}) = 0 \end{cases}$$

$$a(\mathbf{u}, \mathbf{v}) = \sum_{\mathbf{K} \in \tau_h} \left(\int_{\mathbf{K}} (\nabla \mathbf{u} \cdot \nabla \bar{\mathbf{v}} - k^2 \mathbf{u} \bar{\mathbf{v}}) dx - i k \int_{\partial \mathbf{K} \cap \Sigma} \mathbf{u} \bar{\mathbf{v}} ds \right)$$

DGM Formulation (Farhat *et al*)

Variational Formulation

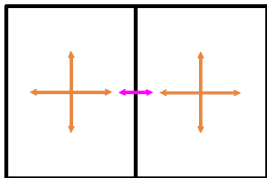


$$\begin{cases} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, \boldsymbol{\lambda}) = F(\mathbf{v}) \\ b(\mathbf{u}, \boldsymbol{\mu}) = 0 \end{cases}$$

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DGM Formulation (Farhat *et al*)

Variational Formulation

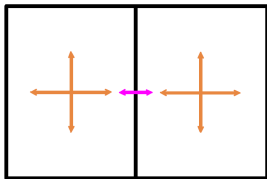


$$\begin{cases} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, \boldsymbol{\lambda}) = F(\mathbf{v}) \\ b(\mathbf{u}, \boldsymbol{\mu}) = 0 \end{cases}$$

$$F(\mathbf{v}) = \sum_{\mathbf{K} \in \tau_h} \int_{\partial \mathbf{K} \cap \Sigma} g \bar{v} ds$$

DGM Formulation (Farhat *et al*)

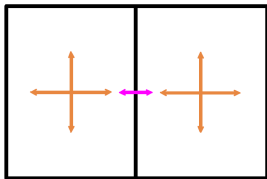
Algebraic Formulation



$$\begin{cases} Au + B\lambda = f \\ B^T u = 0 \end{cases}$$

DGM Formulation (Farhat *et al*)

Algebraic Formulation



$$\begin{cases} Au + B\lambda = f \\ B^T u = 0 \end{cases}$$

$$\Rightarrow B^T A^{-1} B \lambda = B^T A^{-1} f$$

DGM Formulation (Farhat *et al*)

Main Features

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- **Plane waves** for **local** approximation of the field

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- Global system: **symmetric** and **sparse**

DGM Formulation (Farhat *et al*)

Main Features

- **Plane waves** for **local** approximation of the field
- **Lagrange multipliers** to enforce continuity
- **Analytical** evaluation of the **elementary** matrices
- Global system: **symmetric** and **sparse**
- **Size** of the global system $\equiv \#$ Lagrange multipliers

DGM Formulation (Farhat *et al*)

Performance

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DGM **outperforms** high order FE methods:

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- **R-4-1, R-8-2** require 5 to 7 times fewer dof than **Q2**

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Performance

DGM **outperforms** high order FE methods:

- **R-4-1, R-8-2** require 5 to 7 times fewer dof than **Q2**
- **Q-16-4** requires 6 times fewer dof than **Q4**

DGM Formulation (Farhat *et al*)

Performance

DGM **outperforms** high order FE methods:

- **R-4-1, R-8-2** require 5 to 7 times fewer dof than **Q2**
- **Q-16-4** requires 6 times fewer dof than **Q4**
- **Q-32-8** requires 25 times fewer dof than **Q4**

DGM Formulation (Farhat *et al*)

Issues

DGM Formulation (Farhat *et al*)

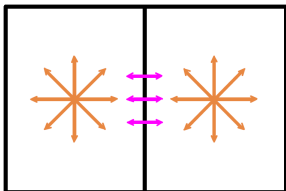
Issues

Inf-Sup condition: Discrete spaces
compatibility

DGM Formulation (Farhat *et al*)

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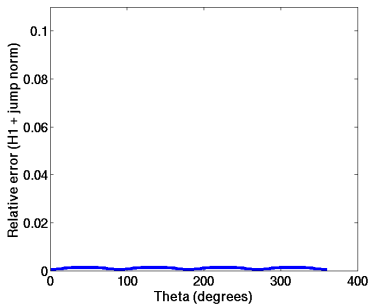
plane waves *vs.* # Lagrange multipliers

?

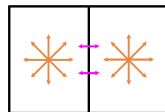
DGM Formulation (Farhat *et al*)

Issues

Inf-Sup condition: Discrete spaces
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Relative error, $ka=10$

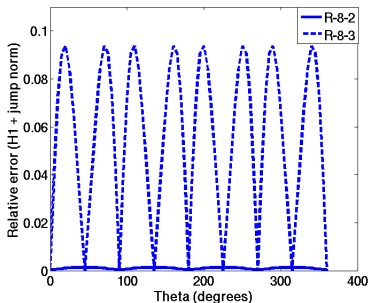


R-8-2 element

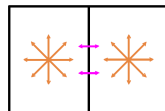
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Issues

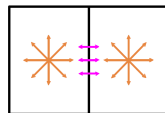
Inf-Sup condition: Discrete spaces
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Relative error, $ka=10$



R-8-2 element



R-8-3 element

DGM Formulation (Farhat *et al*)

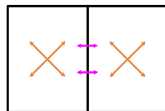
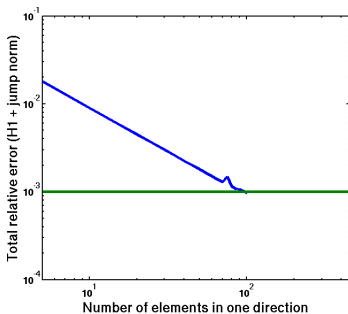
Issues

Inf-Sup condition: Numerical **instabilities**

DGM Formulation (Farhat *et al*)

Issues

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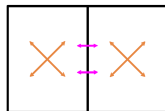
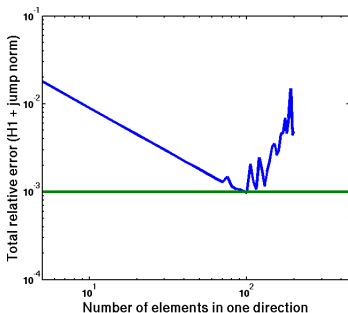
R-4-2 element

Total relative error, **ka=1**

DGM Formulation (Farhat *et al*)

Issues

Inf-Sup condition: Numerical **instabilities**



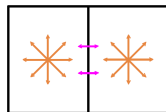
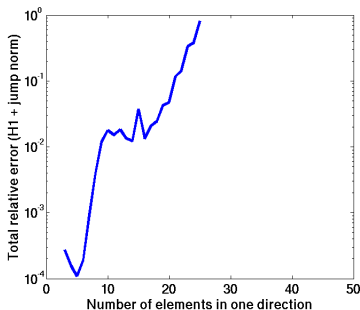
R-4-2 element

Total relative error, **ka=1**

DGM Formulation (Farhat *et al*)

Issues

Inf-Sup condition: Numerical **instabilities**



R-8-2 element

Total relative error, **ka=1**

Our objective: build on top of DGM

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- **Preserve** the good features of DGM

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- **Preserve** the good features of DGM
- **Overcome** the numerical instabilities

DGM Formulation: Another point of view

$$\begin{cases} \Delta \mathbf{u} + k^2 \mathbf{u} = 0 & \text{in } \Omega \\ \partial_n \mathbf{u} = i k \mathbf{u} + g & \text{on } \Sigma \end{cases}$$

DGM Formulation: Another point of view

$$\begin{cases} \Delta \mathbf{u} + k^2 \mathbf{u} = 0 & \text{in } \Omega \\ \partial_n \mathbf{u} = i k \mathbf{u} + g & \text{on } \Sigma \end{cases}$$

- Split the solution \mathbf{u} :

$$\mathbf{u} = \varphi + \Phi(\lambda)$$

DGM Formulation: Another point of view

$$\left\{ \begin{array}{l} \Delta \varphi^K + k^2 \varphi^K = 0 \\ \end{array} \right. \quad \text{in } K$$

$$\left\{ \begin{array}{l} \Delta \Phi^K(\lambda) + k^2 \Phi^K(\lambda) = 0 \\ \end{array} \right. \quad \text{in } K$$

DGM Formulation: Another point of view

$$\left\{ \begin{array}{ll} \Delta \varphi^K + k^2 \varphi^K = 0 & \text{in } K \\ \partial_n \varphi^K = ik \varphi^K + g & \text{on } \partial K \cap \Sigma \end{array} \right.$$

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$$\lambda^{K'} = -\lambda^K \text{ on } \partial K \cap \partial K'$$

DGM Formulation: Another point of view

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$$\lambda^{K'} = -\lambda^K \text{ on } \partial K \cap \partial K'$$

$$[u] = [\varphi + \Phi(\lambda)] = 0 \text{ on each interior edge}$$

DGM Formulation: Another point of view

- Solve **local** variational problems in each K :

$$\int_{\partial K} (\partial_n v - i k v \chi_\Sigma) \bar{w} ds = L(w)$$

DGM Formulation: Another point of view

- Solve **local** variational problems in each K :

$$\int_{\partial K} (\partial_n v - i k v \chi_\Sigma) \bar{w} ds = L(w)$$

- Solve one **global** variational problem:

$$\sum_{e \subset \dot{\Omega}} \frac{1}{|e|} \int_e [\Phi(\lambda)] \bar{\mu} = - \sum_{e \subset \dot{\Omega}} \frac{1}{|e|} \int_e [\varphi] \bar{\mu}$$

New solution methodologies for Helmholtz problems

A new DGM

A new DGM: first idea



Restore the weak continuity of the field in the least-squares sense:

A new DGM: first idea



Restore the weak continuity of the field in the least-squares sense:

$$\sum_{e \in \mathring{\Omega}} \frac{1}{|e|} \int_e [\Phi(\lambda)] [\overline{\Phi(\mu)}] = - \sum_{e \in \mathring{\Omega}} \frac{1}{|e|} \int_e [\varphi] [\overline{\Phi(\mu)}]$$

A new DGM: first idea

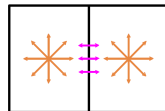
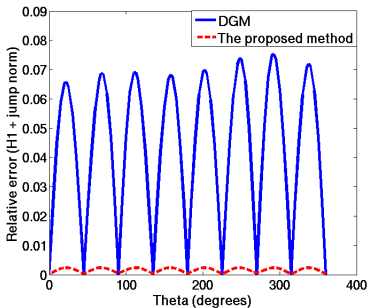


Restore the weak continuity of the field in the least-squares sense:

$$\sum_{e \in \mathring{\Omega}} \frac{1}{|e|} \int_e [\Phi(\lambda)] [\overline{\Phi(\mu)}] = - \sum_{e \in \mathring{\Omega}} \frac{1}{|e|} \int_e [\varphi] [\overline{\Phi(\mu)}]$$

Gain: Hermitian and **positive semi-definite** global matrix

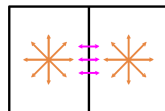
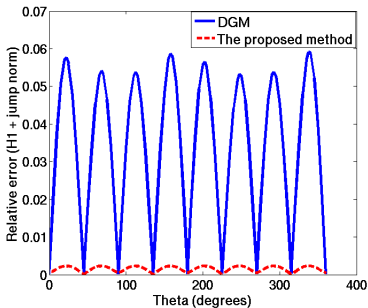
A new DGM: Illustrative example of the improvement for a fixed resolution



R-8-3 element

Relative error, $ka=10$

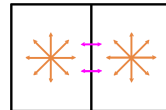
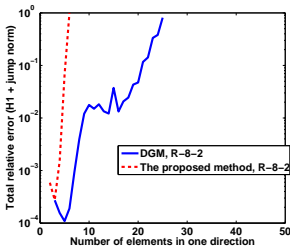
A new DGM: Illustrative example of the improvement for a fixed resolution



R-8-3 element

Relative error, $ka=30$

A new DGM: Persistence of the numerical instabilities



R-8-2 element

Total relative error, $ka=1$

A new DGM: second idea



Reformulate the local variational problem:

A new DGM: second idea



Reformulate the local variational problem:

$$\int_{\partial K} (\partial_n v - i k v \chi_\Sigma)(\partial_n \bar{w} + i k \bar{w} \chi_\Sigma) = L(w)$$

A new DGM: second idea

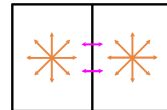
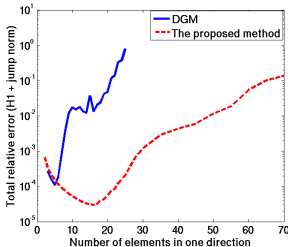


Reformulate the local variational problem:

$$\int_{\partial K} (\partial_n v - i k v \chi_\Sigma)(\partial_n \bar{w} + i k \bar{w} \chi_\Sigma) = L(w)$$

Gain: Hermitian and **positive definite** local matrix

A new DGM: Illustrative example of the improvement for a fixed frequency



R-8-2 element

Total relative error, $ka=1$

A new DGM: Solution strategy

- Solve **local** variational problems in each K :

$$a_K(v, w) = \int_{\partial K} (\partial_n v - i k v \chi_\Sigma)(\partial_n \bar{w} + i k \bar{w} \chi_\Sigma)$$

A new DGM: Solution strategy

- Solve **local** variational problems in each K :

$$a_K(v, w) = \int_{\partial K} (\partial_n v - i k v \chi_\Sigma)(\partial_n \bar{w} + i k \bar{w} \chi_\Sigma)$$

- Solve one **global** variational problem:

$$A(\lambda, \mu) = \sum_{e \in \dot{\Omega}} (\beta_e \int_e [\Phi(\lambda)][\overline{\Phi(\mu)}] + \gamma_e \int_e [[\partial_n \Phi(\lambda)]] [[\partial_n \overline{\Phi(\mu)}]])$$

A new DGM: Accomplishment

- A priori error estimates
- Application in geophysical exploration

A new DGM: A priori error estimates

$$\|u - u_h\|_{0,\Omega} \leq \hat{C} \inf_{\mu_h, v_h} \left[\sum_{K \in \tau_h} \frac{1}{k^4 h_K^3} \left(\|\lambda - \mu_h\|_{0,\partial K \cap \dot{\Omega}}^2 + \|\lambda - \partial_n v_h\|_{0,\partial K \cap \dot{\Omega}}^2 \right) \right]^{1/2}$$

A new DGM: A priori error estimates. Application

\tilde{R} - m - n element ($m \geq 2N + 1$):

$$\|u - u_h\|_{0,\Omega} \leq \frac{\hat{C}}{k^2} \left[h^n |\Phi(\lambda)|_{n+2,\Omega}$$

$$+ h^{n-1} |\Phi(\lambda)|_{n+1,\Omega}$$

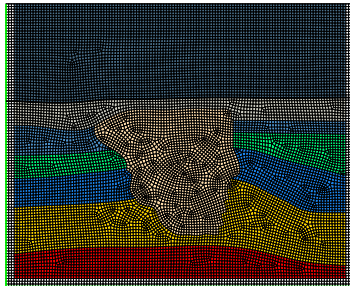
$$+ h^{N-2} \left(\sum_{l=0}^N k^{N+1-l} |\Phi(\lambda)|_{l,\Omega} \right.$$

$$\left. + |\Phi(\lambda)|_{N+1,\Omega} + h |\Phi(\lambda)|_{N+2,\Omega} \right]$$

A new DGM: Application in geophysical exploration

A new DGM: Application in geophysical exploration

- **Objective:** produce images of the subsurface from tomography measurements



A new DGM: Application in geophysical exploration

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A new DGM: Application in geophysical exploration

- **Objective:** produce images of the subsurface from tomography measurements
- Wave propagation in **time** domain:
Discrete Fourier Transform

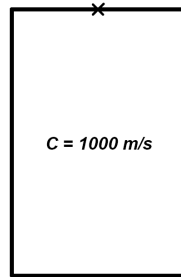
A new DGM: Application in geophysical exploration

- **Objective:** produce images of the subsurface from tomography measurements
- Wave propagation in **time** domain:
Discrete Fourier Transform
- Solve Helmholtz equation (reduced wave equation)

A new DGM: Application in geophysical exploration

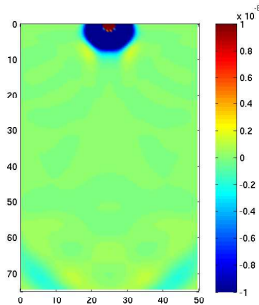
- **Objective:** produce images of the subsurface from tomography measurements
- Wave propagation in **time** domain:
Discrete Fourier Transform
- Solve Helmholtz equation (reduced wave equation)
- Build the solution in **time** domain:
Inverse Discrete Fourier Transform

A new DGM: Application in geophysical exploration



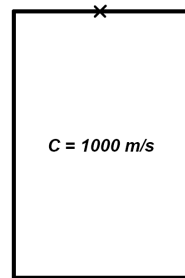
Homogeneous medium

A new DGM: Application in geophysical exploration



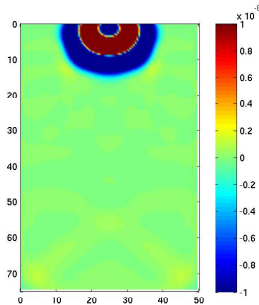
R-4-1 element

$$\frac{1}{30} \leq kh \leq 2$$



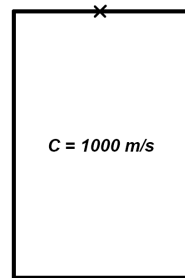
Homogeneous medium

A new DGM: Application in geophysical exploration



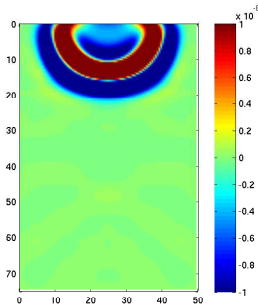
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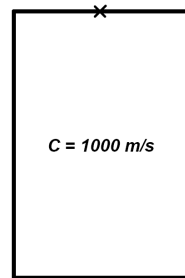
Homogeneous medium

A new DGM: Application in geophysical exploration



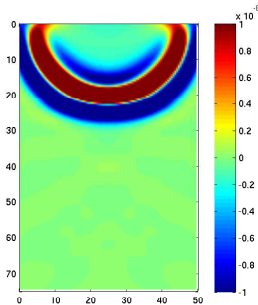
R-4-1 element

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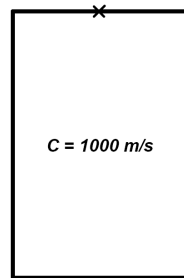
Homogeneous medium

A new DGM: Application in geophysical exploration



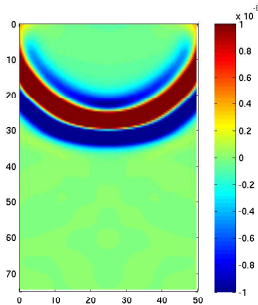
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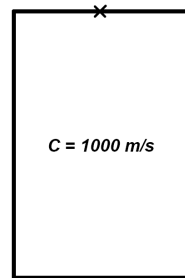
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A new DGM: Application in geophysical exploration



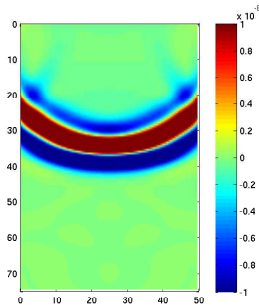
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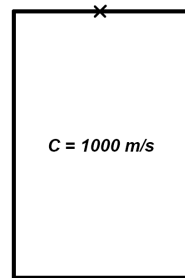
Homogeneous medium

A new DGM: Application in geophysical exploration



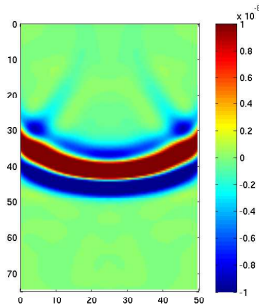
R-4-1 element

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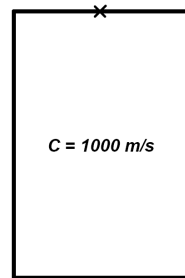
Homogeneous medium

A new DGM: Application in geophysical exploration



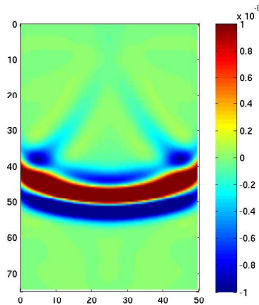
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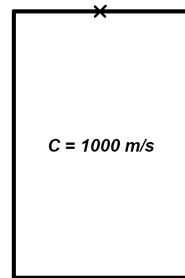
Homogeneous medium

A new DGM: Application in geophysical exploration



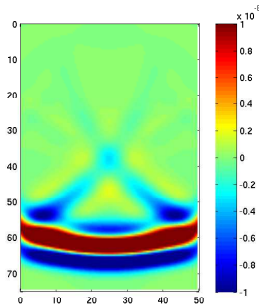
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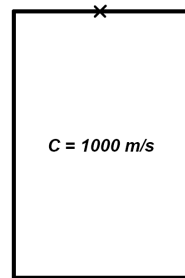
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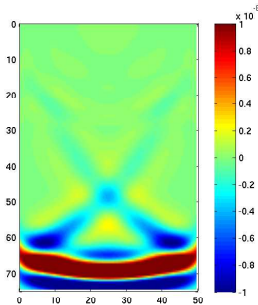
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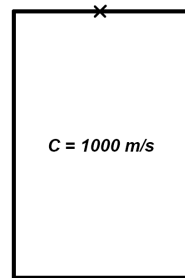
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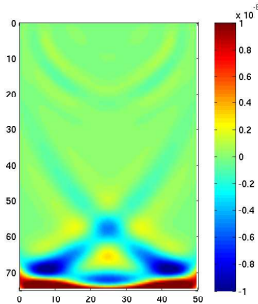
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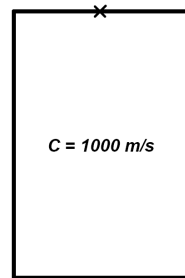
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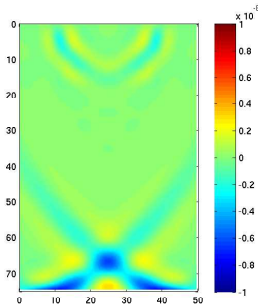
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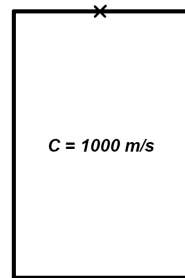
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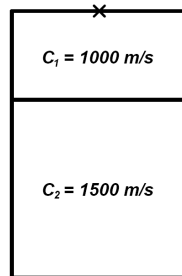
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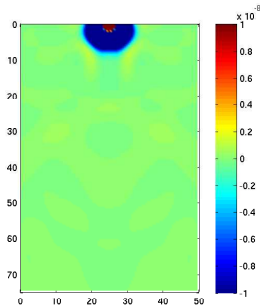
Homogeneous medium

A new DGM: Application in geophysical exploration



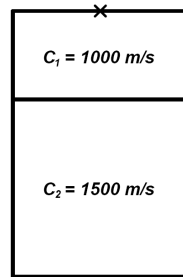
Stratified medium

A new DGM: Application in geophysical exploration



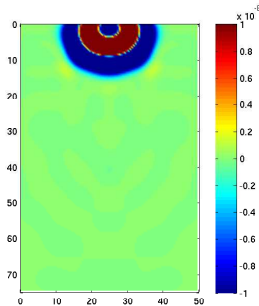
R-4-1 element

$$\frac{1}{50} \leq kh \leq 2$$



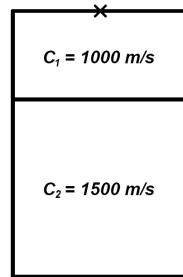
Stratified medium

A new DGM: Application in geophysical exploration



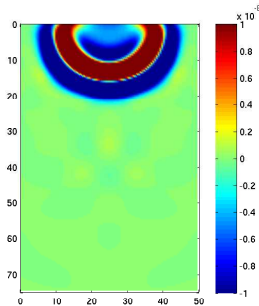
R-4-1 element

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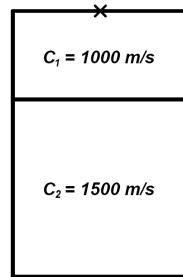
Stratified medium

A new DGM: Application in geophysical exploration



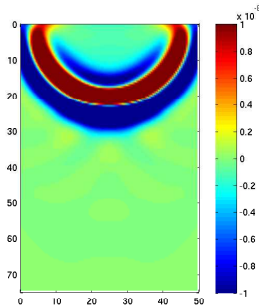
R-4-1 element

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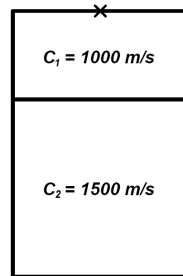
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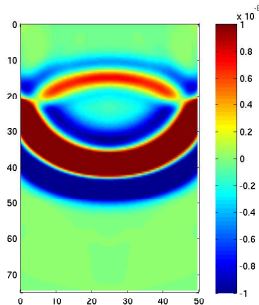
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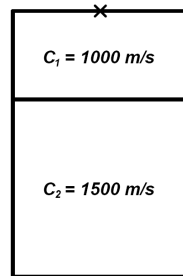
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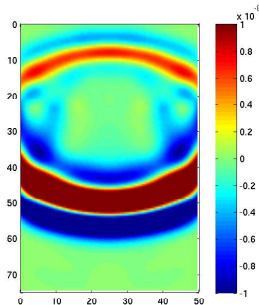
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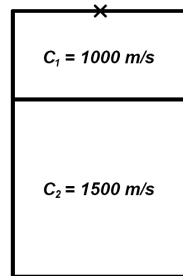
Stratified medium

A new DGM: Application in geophysical exploration



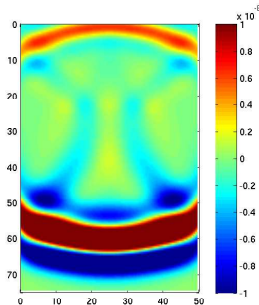
R-4-1 element

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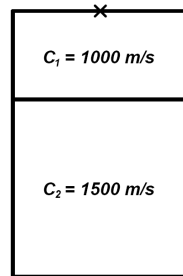
Stratified medium

A new DGM: Application in geophysical exploration



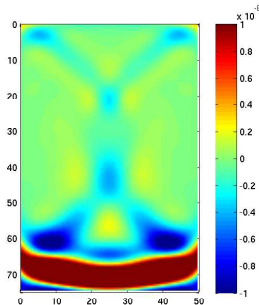
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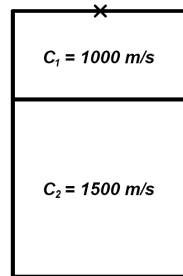
Stratified medium

A new DGM: Application in geophysical exploration



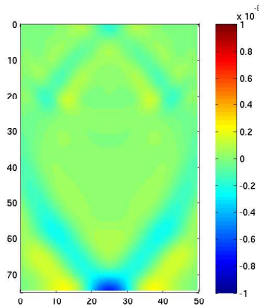
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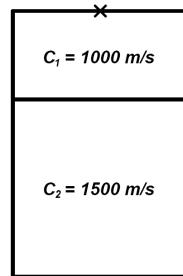
Stratified medium

A new DGM: Application in geophysical exploration



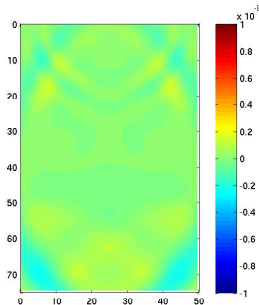
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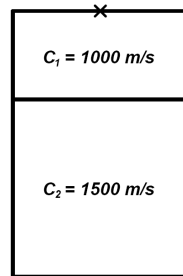
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A new DGM: Application in geophysical exploration



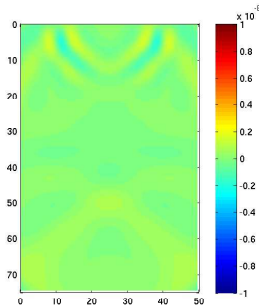
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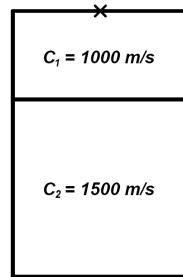
Stratified medium

A new DGM: Application in geophysical exploration



R-4-1 element

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Stratified medium

A new DGM: Application in geophysical exploration

- Multi-frequency solver

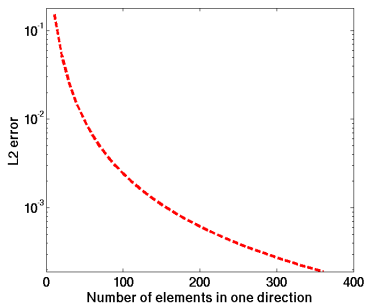
A new DGM: Application in geophysical exploration

- Multi-frequency solver
- Resolution: 3 to 300 elements per wavelength

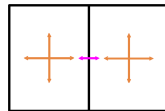
A new DGM: Application in geophysical exploration

- Multi-frequency solver
- Resolution: 3 to 300 elements per wavelength
- R-4-1 element: accurate and stable element

A new DGM: Stability of the R-4-1 element



L^2 error, $ka=10$



R-4-1 element

A new DGM: Summary

DGM:

- Solve **local** variational problems in each K :

$$a_K(\mathbf{v}, w) = \int_{\partial K} (\partial_n \mathbf{v} - i \mathbf{k} \mathbf{v} \chi_\Sigma) \bar{w} ds$$

- Solve **global** variational problem:

$$A(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \sum_{e \in \mathcal{C}\dot{\Omega}} \frac{1}{|e|} \int_e [\Phi(\boldsymbol{\lambda})] \bar{\boldsymbol{\mu}}$$

A new DGM: Summary

A new DGM:

- **NEW local** variational problems in each K :

$$a_K(\mathbf{v}, w) = \int_{\partial K} (\partial_n \mathbf{v} - i \mathbf{k} \mathbf{v} \chi_\Sigma)(\partial_n \bar{w} + i \mathbf{k} \bar{w} \chi_\Sigma) ds$$

- Solve **global** variational problem:

$$A(\lambda, \mu) = \sum_{e \in \mathcal{C}\Omega} \frac{1}{|e|} \int_e [\Phi(\lambda)] \bar{\mu}$$

A new DGM: Summary

A new DGM:

- **NEW local** variational problems in each K :

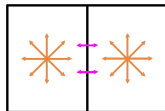
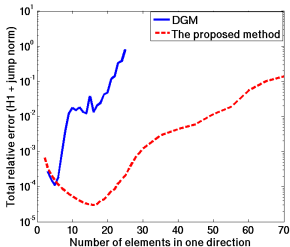
$$a_K(\mathbf{v}, w) = \int_{\partial K} (\partial_n \mathbf{v} - i \mathbf{k} \mathbf{v} \chi_\Sigma)(\partial_n \bar{w} + i \mathbf{k} \bar{w} \chi_\Sigma) ds$$

- **NEW global** variational problem:

$$A(\lambda, \mu) = \sum_{e \in \hat{\Omega}} \left(\beta_e \int_e [\Phi(\lambda)][\overline{\Phi(\mu)}] + \right. \\ \left. \gamma_e \int_e [[\partial_n \Phi(\lambda)]] [[\partial_n \overline{\Phi(\mu)}]] \right)$$

A new DGM: Summary

- Numerical instabilities: improvement, but not enough

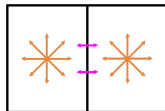
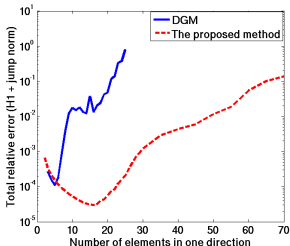


R-8-2 element

Total relative error, $ka=1$

A new DGM: Summary

- Numerical instabilities: improvement, but not enough



R-8-2 element

Total relative error, $ka=1$

- Source of the instabilities: local problems **nearly singular**

New solution methodologies for Helmholtz problems

A modified DGM (mDGM)

mDGM: main idea



Modify the local problems:

$$\begin{cases} \Delta \varphi^K + k^2 \varphi^K = 0 & \text{in } K \\ \partial_n \varphi^K - ik \varphi^K = g & \text{on } \partial K \cap \Sigma \\ \partial_n \varphi^K = 0 & \text{on } \partial K \cap \dot{\Omega} \end{cases}$$

$$\begin{cases} \Delta \Phi^K(\lambda) + k^2 \Phi^K(\lambda) = 0 & \text{in } K \\ \partial_n \Phi^K(\lambda) - ik \Phi^K(\lambda) = 0 & \text{on } \partial K \cap \Sigma \\ \partial_n \Phi^K(\lambda) = \lambda & \text{on } \partial K \cap \dot{\Omega} \end{cases}$$

mDGM: main idea



Modify the local problems:

$$\left\{ \begin{array}{l} \Delta \varphi^K + k^2 \varphi^K = 0 \quad \text{in } K \\ \partial_n \varphi^K - ik \varphi^K = g \quad \text{on } \partial K \cap \Sigma \\ \partial_n \varphi^K - \underline{i \alpha \varphi^K} = 0 \quad \text{on } \partial K \cap \mathring{\Omega} \end{array} \right.$$

$$\left\{ \begin{array}{l} \Delta \Phi^K(\lambda) + k^2 \Phi^K(\lambda) = 0 \quad \text{in } K \\ \partial_n \Phi^K(\lambda) - ik \Phi^K(\lambda) = 0 \quad \text{on } \partial K \cap \Sigma \\ \partial_n \Phi^K(\lambda) - \underline{i \alpha \Phi^K(\lambda)} = \lambda \quad \text{on } \partial K \cap \mathring{\Omega} \end{array} \right.$$

mDGM: main idea



Modify the local problems:

$$\left\{ \begin{array}{l} \Delta \varphi^K + k^2 \varphi^K = 0 \quad \text{in } K \\ \partial_n \varphi^K - ik \varphi^K = g \quad \text{on } \partial K \cap \Sigma \\ \partial_n \varphi^K - i \alpha \varphi^K = 0 \quad \text{on } \partial K \cap \mathring{\Omega} \end{array} \right.$$

$$\left\{ \begin{array}{l} \Delta \Phi^K(\lambda) + k^2 \Phi^K(\lambda) = 0 \quad \text{in } K \\ \partial_n \Phi^K(\lambda) - ik \Phi^K(\lambda) = 0 \quad \text{on } \partial K \cap \Sigma \\ \partial_n \Phi^K(\lambda) - i \alpha \Phi^K(\lambda) = \lambda \quad \text{on } \partial K \cap \mathring{\Omega} \end{array} \right.$$

Gain: Well-posed local problems for $\alpha \in \mathbb{R}_+^*$

mDGM: main idea



Modify the local problems:

$$\left\{ \begin{array}{l} \Delta \varphi^K + k^2 \varphi^K = 0 \text{ in } K \\ \partial_n \varphi^K - i k \varphi^K = g \text{ on } \partial K \cap \Sigma \\ \partial_n \varphi^K - i \alpha \varphi^K = 0 \text{ on } \partial K \cap \mathring{\Omega} \end{array} \right.$$

$$\left\{ \begin{array}{l} \Delta \Phi^K(\lambda) + k^2 \Phi^K(\lambda) = 0 \text{ in } K \\ \partial_n \Phi^K(\lambda) - i k \Phi^K(\lambda) = 0 \text{ on } \partial K \cap \Sigma \\ \partial_n \Phi^K(\lambda) - i \alpha \Phi^K(\lambda) = \lambda \text{ on } \partial K \cap \mathring{\Omega} \end{array} \right.$$

Gain: Well-posed local problems for $\alpha \in \mathbb{R}_+^*$

Price: Both u and $\partial_n u$ are discontinuous

mDGM: Solution strategy

- Solve **local** variational problems in each K :

$$\int_{\partial K} (\partial_n v - i k v) \bar{w} ds = L(w)$$

mDGM: Solution strategy

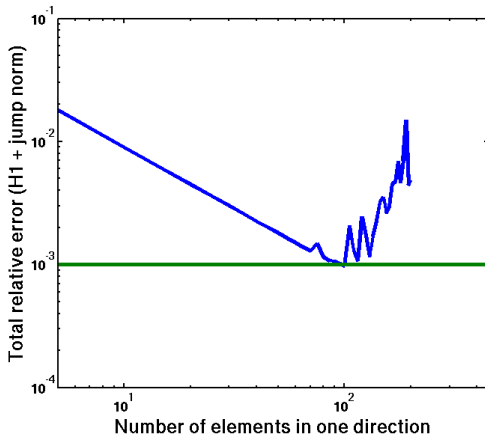
- Solve **local** variational problems in each K :

$$\int_{\partial K} (\partial_n v - i k v) \bar{w} ds = L(w)$$

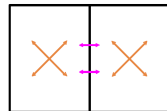
- Solve one **global** variational problem:

$$\sum_{e \in \mathcal{C}\hat{\Omega}} \left(\beta_e \int_e [\Phi(\lambda)] [\overline{\Phi(\mu)}] + \gamma_e \int_e [[\partial_n \Phi(\lambda)]] [[\partial_n \overline{\Phi(\mu)}]] \right) =$$
$$- \sum_{e \in \mathcal{C}\hat{\Omega}} \left(\beta_e \int_e [\varphi] [\overline{\Phi(\mu)}] + \gamma_e \int_e [[\partial_n \varphi]] [[\partial_n \overline{\Phi(\mu)}]] \right)$$

mDGM: Performance assessment

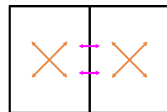
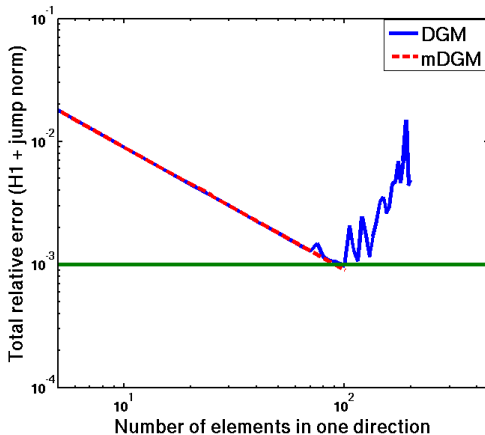


Total relative error, $ka=1$



R-4-2 element

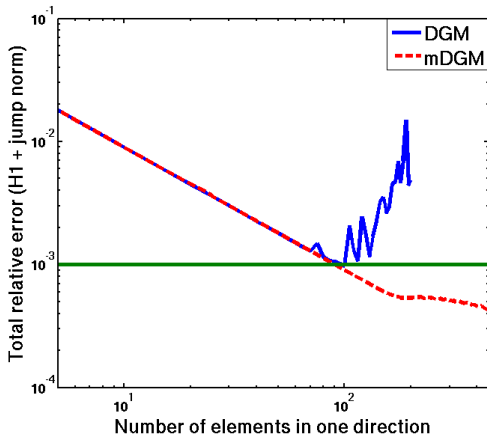
mDGM: Performance assessment



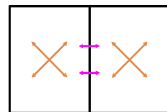
R-4-2 element

Total relative error, $ka=1$

mDGM: Performance assessment

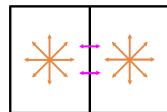
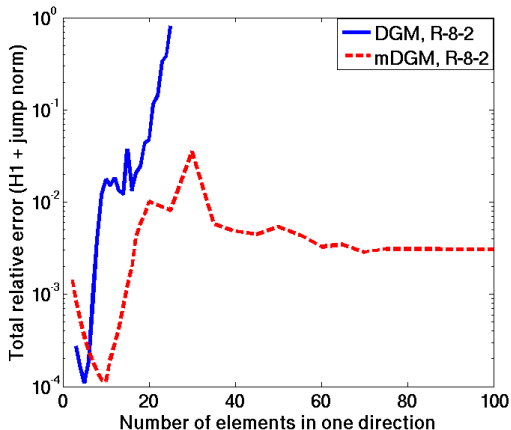


Total relative error, $ka=1$



R-4-2 element

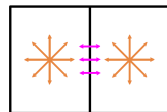
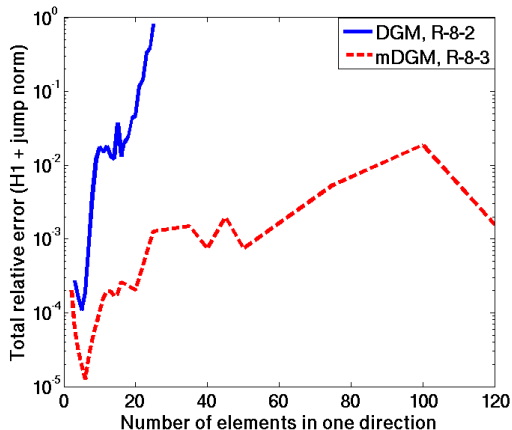
mDGM: Performance assessment



R-8-2 element

Total relative error, $ka=1$

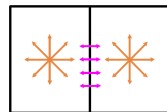
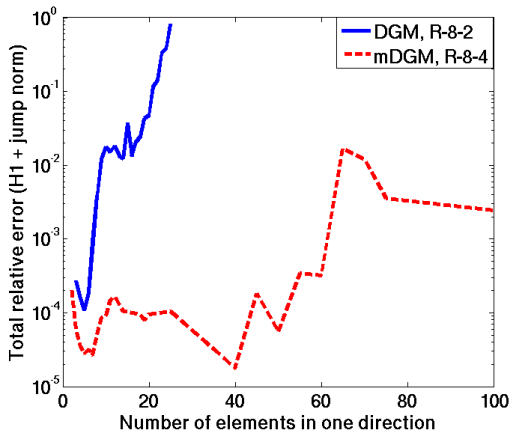
mDGM: Performance assessment



R-8-3 element

Total relative error, $ka=1$

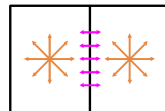
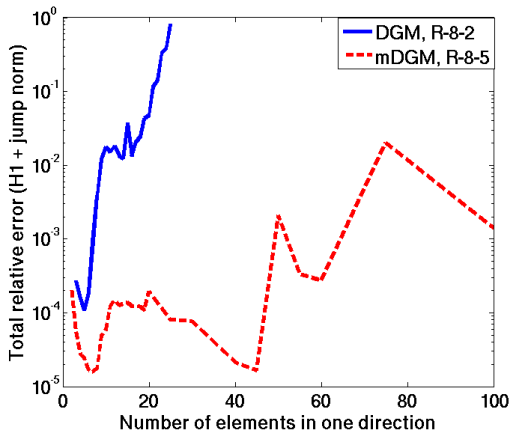
mDGM: Performance assessment



R-8-4 element

Total relative error, $ka=1$

mDGM: Performance assessment



R-8-5 element

Total relative error, $ka=1$

mDGM: Implementation issue

$$F_{jl} = \sum_{e \in \mathring{\Omega}} \left(\beta_e \int_e [\Phi_h(\mu_l)] [\overline{\Phi_h(\mu_j)}] \right. \\ \left. + \gamma_e \int_e [[\partial_n \Phi_h(\mu_l)]] [[\partial_n \overline{\Phi_h(\mu_j)}]] \right)$$

F is Hermitian

mDGM: Implementation issue

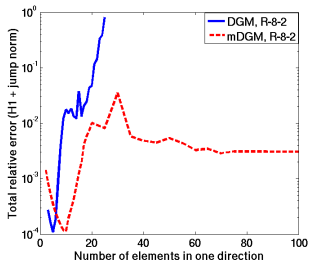
$$\begin{aligned} F_{jl} = & \sum_{e \in \mathring{\Omega}} \left(\beta_e \int_e [\Phi_h(\mu_l)] [\overline{\Phi_h(\mu_j)}] \right. \\ & \left. + \gamma_e \int_e [[\partial_n \Phi_h(\mu_l)]] [[\partial_n \overline{\Phi_h(\mu_j)}]] \right) \end{aligned}$$



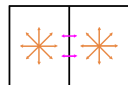
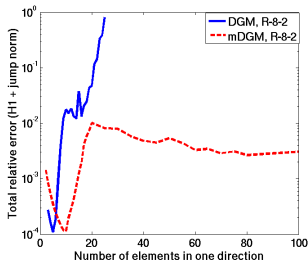
F is Hermitian, but **NOT** in practice

mDGM: Implementation issue

Before



After

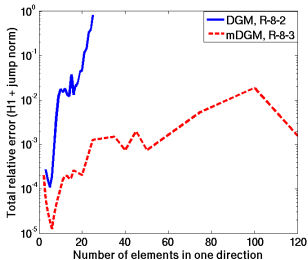


R-8-2
element

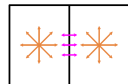
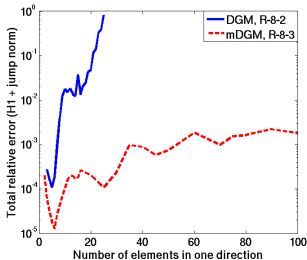
Total relative error, $ka=1$

mDGM: Implementation issue

Before



After

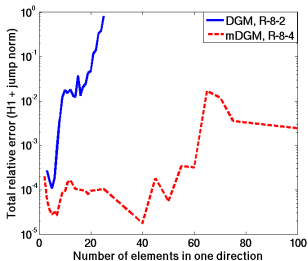


R-8-3
element

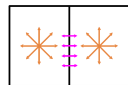
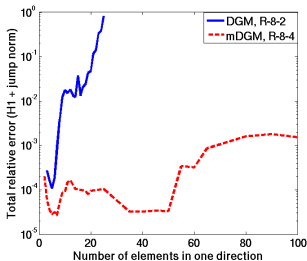
Total relative error, $ka=1$

mDGM: Implementation issue

Before



After

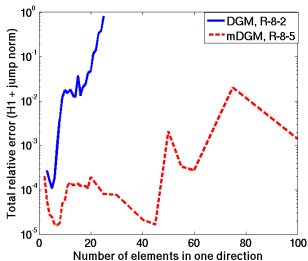


R-8-4
element

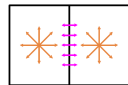
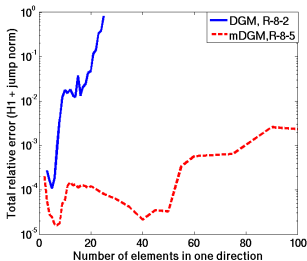
Total relative error, $ka=1$

mDGM: Implementation issue

Before



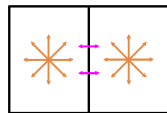
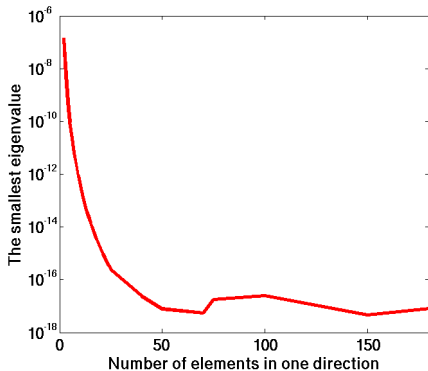
After



R-8-5
element

Total relative error, $ka=1$

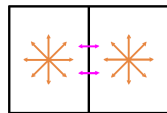
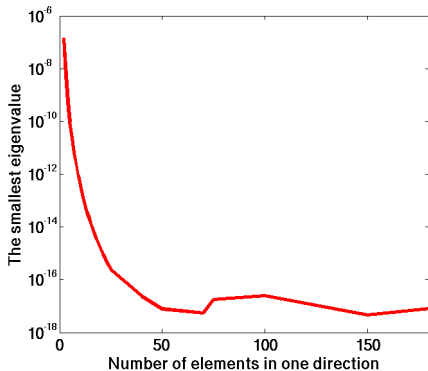
mDGM: Numerical issue



R-8-2 element

The smallest eigenvalue, $ka=1$

mDGM: Numerical issue



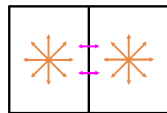
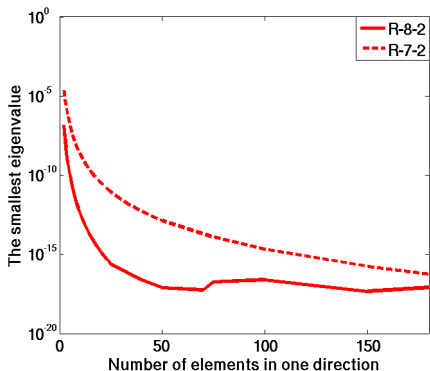
R-8-2 element

The smallest eigenvalue, $ka=1$

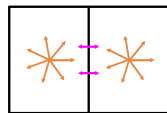


Loss of the linear independence

mDGM: Numerical issue



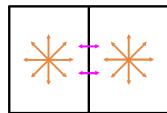
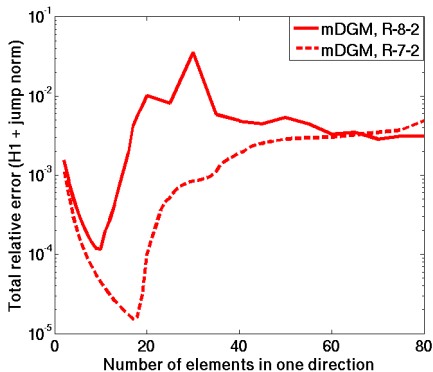
R-8-2 element



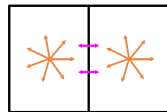
R-7-2 element

The smallest eigenvalue, $ka=1$

mDGM: Numerical issue



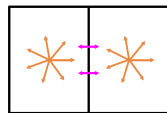
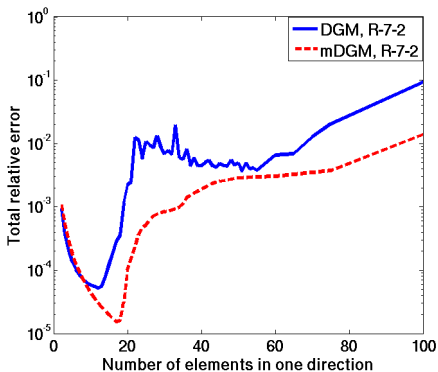
R-8-2 element



R-7-2 element

Total relative error, **ka**=1

mDGM: Numerical issue



R-7-2 element

Total relative error, $ka=1$

mDGM: Summary

Local problems:

- **DGM:**

$$a_K(v, w) = \int_{\partial K} (\partial_n v - i k v \chi_\Sigma) \bar{w} ds$$

- **mDGM:**

$$a_K(v, w) = \int_{\partial K} (\partial_n v - i k v) \bar{w} ds$$

mDGM: Summary

Global problems:

- **DGM:**

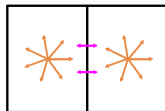
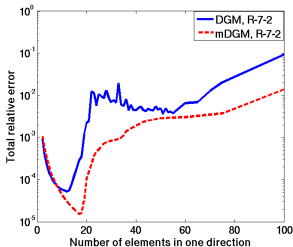
$$A(\lambda, \mu) = \sum_{e \in \mathring{\Omega}} \frac{1}{|e|} \int_e [\Phi(\lambda)] \bar{\mu}$$

- **mDGM:**

$$A(\lambda, \mu) = \sum_{e \in \mathring{\Omega}} \left(\beta_e \int_e [\Phi(\lambda)] [\overline{\Phi(\mu)}] \right. \\ \left. + \gamma_e \int_e [[\partial_n \Phi(\lambda)]] [[\partial_n \overline{\Phi(\mu)}]] \right)$$

mDGM: Summary

- Numerical instabilities: improvement, but not enough

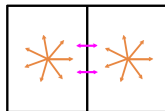
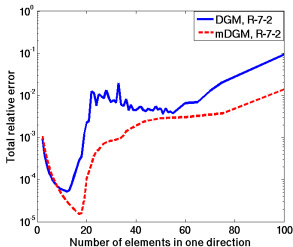


R-7-2 element

Total relative error, $ka=1$

mDGM: Summary

- Numerical instabilities: improvement, but not enough



R-7-2 element

Total relative error, $ka=1$

- Source of the instabilities: **loss** of the linear independence

New solution methodologies for Helmholtz problems

An improved modified DGM (imDGM)

imDGM: main idea



Reformulate the local variational problems:

imDGM: main idea



Reformulate the local variational problems:

$$\int_{\partial K} (\partial_n v - i k v)(\partial_n \bar{w} + i k \bar{w}) ds = L(w)$$

imDGM: main idea

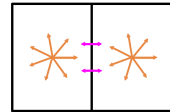
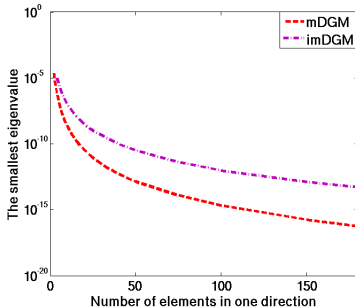


Reformulate the local variational problems:

$$\int_{\partial K} (\partial_n v - i k v)(\partial_n \bar{w} + i k \bar{w}) ds = L(w)$$

Gain: Hermitian and **positive definite** local matrix

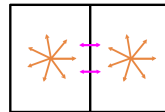
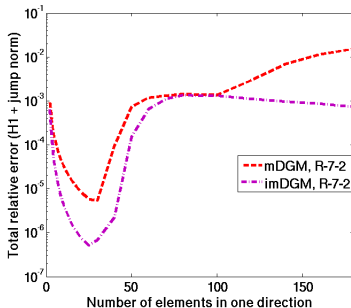
imDGM: Performance assessment



R-7-2 element

The smallest eigenvalue, $ka=1$

imDGM: Performance assessment



R-7-2 element

Total relative error, $ka=1$

imDGM: Comparison with DGM and LSM (Monk-Wang, 1999)

imDGM: Comparison with DGM and LSM (Monk-Wang, 1999)

- An improved modified DGM (imDGM)

$$A(\lambda, \mu) = \sum_{e \in \mathring{\Omega}} \left(\beta_e \int_e [\Phi(\lambda)] [\overline{\Phi(\mu)}] + \gamma_e \int_e [[\partial_n \Phi(\lambda)]] [[\partial_n \overline{\Phi(\mu)}]] \right)$$

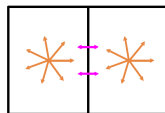
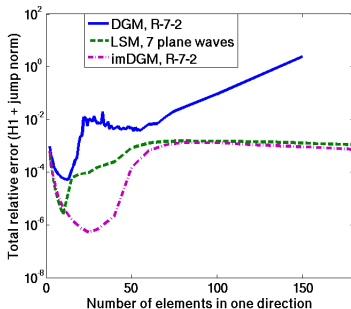
- Least-Squares Method (LSM)

$$A(u, v) = \sum_{e \in \mathring{\Omega}} \left(k^2 \int_e [u] [\overline{v}] + \int_e [[\partial_n u]] [[\partial_n \overline{v}]] \right)$$

- Discontinuous Galerkin Method (DGM)

$$A(\lambda, \mu) = \sum_{e \in \mathring{\Omega}} \left(\beta_e \int_e [\Phi(\lambda)] \overline{\mu} \right)$$

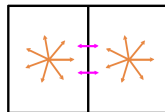
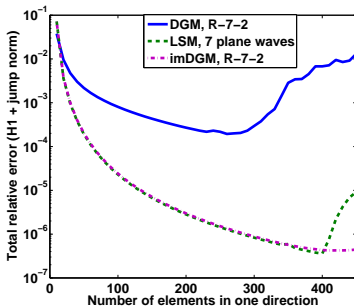
imDGM: Comparison with DGM and LSM



R-7-2 element

Total relative error, $ka=1$

imDGM: Comparison with DGM and LSM

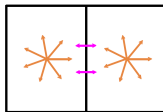


R-7-2 element

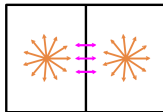
Total relative error, $ka=20$

imDGM: Performance assessment for a fixed resolution: $kh=2$

ka R-7-2 R-11-3



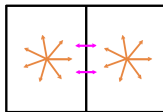
R-7-2 element



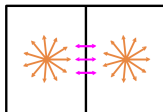
R-11-3 element

imDGM: Performance assessment for a fixed resolution: $kh=2$

ka	R-7-2	R-11-3
50	28%	0.05%



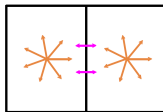
R-7-2 element



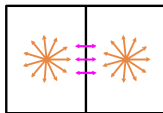
R-11-3 element

imDGM: Performance assessment for a fixed resolution: $kh=2$

ka	R-7-2	R-11-3
50	28%	0.05%
100	51%	0.07%



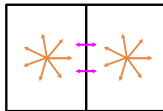
R-7-2 element



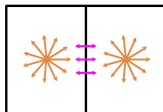
R-11-3 element

imDGM: Performance assessment for a fixed resolution: $kh=2$

ka	R-7-2	R-11-3
50	28%	0.05%
100	51%	0.07%
200	69%	0.20%



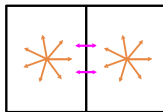
R-7-2 element



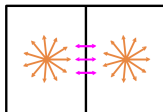
R-11-3 element

imDGM: Performance assessment for a fixed resolution: $kh=2$

ka	R-7-2	R-11-3
50	28%	0.05%
100	51%	0.07%
200	69%	0.20%



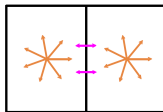
R-7-2 element



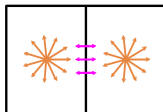
R-11-3 element

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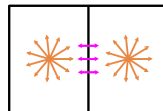
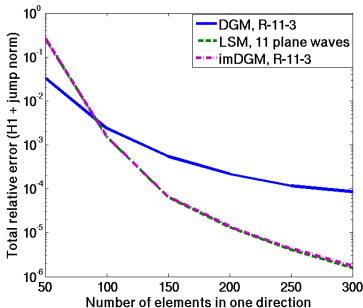
R-7-2 element



R-11-3 element

Computational cost increased by **50%**
Gain of more than **two orders** of magnitude

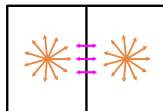
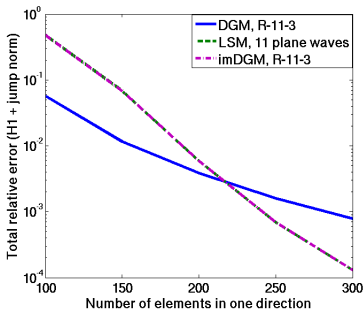
imDGM: Comparison with DGM and LSM for a fixed frequency



R-11-3 element

Total relative error, $ka=200$

imDGM: Comparison with DGM and LSM for a fixed frequency



R-11-3 element

Total relative error, $ka=400$

Summary and Perspectives

Accomplishment

Accomplishment

- Small systems: Hermitian and **positive definite**

Accomplishment

- Small systems: Hermitian and **positive definite**
- Global system: Hermitian, sparse and **positive semi-definite**

Accomplishment

- Small systems: Hermitian and **positive definite**
- Global system: Hermitian, sparse and **positive semi-definite**
- Performance: **outperforms** DGM (accuracy and stability)

Short-term goals

- Assess the performance of higher-order elements

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- Assess the performance of higher-order elements
- Experiment other shape functions

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- Assess the performance of higher-order elements
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- Unstructured mesh

Short-term goals

- Assess the performance of higher-order elements
- Experiment other shape functions
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- Apply iterative methods for the global system

Short-term goals

- Assess the performance of higher-order elements
- Experiment other shape functions
- Unstructured mesh
- Apply iterative methods for the global system
- Extend the mathematical analysis to imDGM

Mid-term goals

- Three-dimensional acoustic scattering problems

Mid-term goals

- Three-dimensional acoustic scattering problems
- Elasto-acoustic scattering problems

Thank you for your attention!