

On stability and robustness of nonlinear systems

Application to cascades

Antoine Chaillet

Université Paris Sud – L2S

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- 1 Semiglobal and practical stability
- 2 Semiglobal and/or practical stability of cascades
- 3 Set-stability
- 4 Integral input to state stability
- 5 Conclusion and future work

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 - Lyapunov sufficient conditions
 - Lyapunov necessary conditions
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Motivations and definitions

Introductory example

Consider the scalar controlled system

$$\dot{x} = u.$$

- Control law: $u = -\theta x$, $\theta > 0 \Rightarrow$ global exponential stability.
- Additive perturbation:

$$\dot{x} = -\theta x + 1$$

\Rightarrow No more convergence to zero (ultimate boundedness):

$$x(t) = \left(x_0 - \frac{1}{\theta}\right) e^{-\theta t} + \frac{1}{\theta}.$$

Instead, the ball of radius $1/\theta$ is globally asymptotically stable.

\rightsquigarrow Practical stability.

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- High-order non-linearity:

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⇒ Domain of attraction is restricted to $\mathcal{B}_{\sqrt{\theta}}$.

↪ Semiglobal stability.

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Causes for semiglobal and/or practical stability

Semiglobal and practical stability may arise from the **inherent** structure of the plant, or result from a **degradation** of global asymptotic stability:

- Non-vanishing perturbations
- Model imprecision
- Lack of measurement
- Use of saturated control.

It may also be derived by stability analysis **tools**:

- Averaging techniques [Teel, Peuteman, Aeyels, 99]
- Controllability results.

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Motivations and definitions

Precise definitions

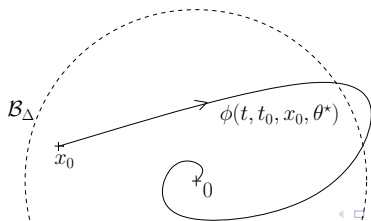
We consider parameterized systems of the form:

$$\dot{x} = f(t, x, \theta). \quad (1)$$

Definition (USAS)

The system is said to be **uniformly semiglobally asymptotically stable** on Θ if, for all $\Delta > 0$, there exist $\theta^*(\Delta) \in \Theta$ and a \mathcal{KL} function β_Δ such that, for all $x_0 \in \mathcal{B}_\Delta$ and all $t_0 \geq 0$,

$$|\phi(t, t_0, x_0, \theta^*)| \leq \beta_\Delta(|x_0|, t - t_0), \quad \forall t \geq t_0.$$



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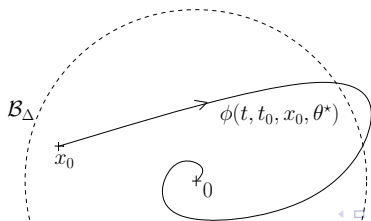
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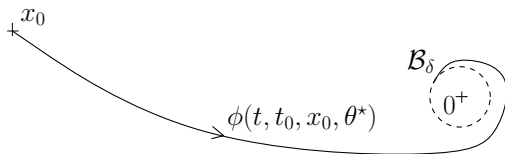
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Definition (UGPAS)

The system is said to be **uniformly globally practically asymptotically stable on Θ** if, for all $\delta > 0$, there exist $\theta^*(\delta) \in \Theta$ and a \mathcal{KL} function β_δ such that, for all $x_0 \in \mathbb{R}^n$ and all $t_0 \geq 0$,

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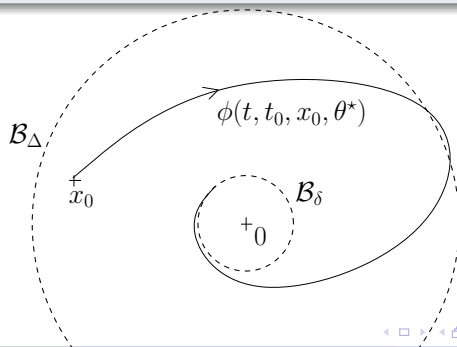
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Definition (USPAS)

The system is said to be **uniformly semiglobally practically asymptotically stable on Θ** if, for all $\Delta > \delta > 0$, there exist $\theta^*(\delta, \Delta) \in \Theta$ and a \mathcal{KL} function $\beta_{\delta, \Delta}$ such that, for all $x_0 \in \mathcal{B}_\Delta$ and all $t_0 \geq 0$,

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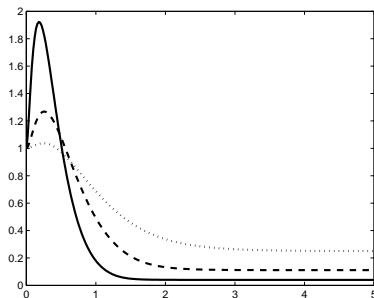
Motivations and definitions

Comparison with related concepts

These definitions of USPAS, UGPAS and USAS differ from others existing in the literature in that:

- They allow the \mathcal{KL} estimate to depend on δ and/or Δ , as opposed to e.g. [Nesić, Loría, 04], [Tan, Nesić, Mareels, 05]. Example:

$$\ddot{q} = -\theta^2 q - 2\theta\dot{q} + 1.$$



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Comparison with related concepts

These definitions of USPAS, UGPAS and USAS differ from others existing in the literature in that:

- They allow the \mathcal{KL} estimate to depend on δ and/or Δ .
- They are defined with respect to **two** measures, as opposed to e.g. [Lin, Sontag, Wang, 96], [Teel, Panteley, Loría, 02]:

$$|\phi(t, t_0, \mathbf{x}_0)|_\delta \leq \beta_\delta(|\mathbf{x}_0|, t - t_0).$$

- \mathcal{B}_δ is not necessarily invariant.
- Weaker (more general) concept, while still guaranteeing interesting properties.
- Lyapunov condition easier to check.

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- \mathcal{B}_δ is not necessarily invariant.
- Weaker (more general) concept, while still guaranteeing interesting properties.
- Lyapunov condition easier to check.
- No prescribed tuning procedure, as opposed to e.g. [Teel, Peuteman, Aeyels, 99], [Tan, Nesić, Mareels, 05]:
 - The tuning parameter may not be scalar.
 - However: tuning procedure often evident from the analysis.

For semiglobal asymptotic stability:

Theorem (Lyapunov sufficient condition for USAS)

Suppose that, given any $\Delta > 0$, there exist $\theta^*(\Delta) \in \Theta$, a C^1 function V_Δ and class \mathcal{K}_∞ functions $\underline{\alpha}_\Delta$, $\bar{\alpha}_\Delta$, α_Δ such that, for all $|\mathbf{x}| \leq \Delta$ and all $t \in \mathbb{R}_{\geq 0}$,

$$\begin{aligned}\underline{\alpha}_\Delta(|\mathbf{x}|) &\leq V_\Delta(t, \mathbf{x}) \leq \bar{\alpha}_\Delta(|\mathbf{x}|) \\ \frac{\partial V_\Delta}{\partial t}(t, \mathbf{x}) + \frac{\partial V_\Delta}{\partial \mathbf{x}}(t, \mathbf{x})f(t, \mathbf{x}, \theta^*) &\leq -\alpha_\Delta(|\mathbf{x}|)\end{aligned}$$

$$\lim_{\Delta \rightarrow \infty} \bar{\alpha}_\Delta^{-1} \circ \underline{\alpha}_\Delta(\Delta) = \infty. \quad (2)$$

Then $\dot{\mathbf{x}} = f(t, \mathbf{x}, \theta)$ is USAS on the parameter set Θ .

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Lyapunov sufficient conditions

Importance of the additional condition

Condition (2) cannot be removed from the above statement. For instance, consider the system from [Kokotović, Marino, 86]:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\theta^2 x_1 - \theta x_2 + \frac{1}{3} x_2^3,\end{aligned}$$

with the function

$$V_\theta(x) = \theta^2 x_1^2 + x_2^2 + x_2 \tanh x_1.$$

Given any $\Delta > 0$, the choice $\theta^*(\Delta) := \frac{2}{3} (3 + \Delta + \Delta^2)$ ensures

$$\dot{V}_{\theta^*}(x) \leq -\frac{|x|}{2} \tanh\left(\frac{|x|}{2}\right), \quad \forall |x| \leq \Delta.$$

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In addition, for all $x \in \mathbb{R}^2$,

$$\frac{|x|}{4} \tanh\left(\frac{|x|}{2}\right) \leq V_{\theta^*}(x) \leq \left(\theta^*(\Delta)^2 + \frac{1}{2}\right) |x|^2.$$

However, the additional condition is violated:

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And, indeed, any solution starting with $|x_{10}| \geq 3/\sqrt{\theta}$, $|x_{20}| \geq 3\sqrt{\theta}$ grows unbounded.

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Similarly, for global practical stability, we have that:

Theorem (Lyapunov sufficient condition for UGPAS)

Suppose that, given any $\delta > 0$, there exist $\theta^*(\delta) \in \Theta$, a C^1 function V_δ and class \mathcal{K}_∞ functions $\underline{\alpha}_\delta, \bar{\alpha}_\delta, \alpha_\delta$ such that, for all $|\mathbf{x}| \geq \delta$ and all $t \in \mathbb{R}_{\geq 0}$,

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USPAS

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Assume further that, $\forall \Delta_* > \delta_* > 0$, $\exists \Delta > \delta > 0$ such that

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Lyapunov necessary conditions

Converse Lyapunov function for USPAS/USAS

USPAS may be derived by other means (e.g. averaging): we propose a converse result tailored for cascades applications.

Definition (δ -USPAS)

The system $\dot{x} = f(t, x, \theta)$ is said to be δ -USPAS if, given any $\Delta > 0$, there exists a \mathcal{KL} function β_Δ such that, for any $0 < \delta < \Delta$, there exists a parameter $\theta^*(\delta, \Delta) \in \Theta$ such that, for all $t_0 \in \mathbb{R}_{\geq 0}$ and all $x_0 \in \mathcal{B}_\Delta$,

$$|\phi(t, t_0, x_0)|_\delta \leq \beta_\Delta(|x_0|, t - t_0), \quad \forall t \geq t_0.$$

Proposition (Averaging [Teel, Peuteman, Aeyels, 99] induces δ -USPAS)

If the origin of the average of the system $\dot{x} = f(t, x)$ is globally asymptotically stable, then $\dot{x} = f(t/\theta, x)$ is δ -USPAS on the parameter set $\mathbb{R}_{>0}$.

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Theorem (Converse Lyapunov function for δ -USPAS)

Assume that the system $\dot{\mathbf{x}} = f(t, \mathbf{x}, \theta)$ is δ -USPAS on $\Theta \subset \mathbb{R}^m$ and that there exist $f_0 \geq 0$ and a continuous nondecreasing function L such that

$$|f(t, 0, \theta)| \leq f_0, \quad \left| \frac{\partial f}{\partial \mathbf{x}}(t, \mathbf{x}, \theta) \right| \leq L(|\mathbf{x}|), \quad \forall t \in \mathbb{R}_{\geq 0}, \forall \mathbf{x} \in \mathbb{R}^n, \forall \theta \in \Theta.$$

Then, for any $\Delta > \delta > 0$, there exist a C^1 function $V_{\delta, \Delta} : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ such that, for all $\delta \leq |\mathbf{x}| \leq \Delta$ and all $t \in \mathbb{R}_{\geq 0}$,

$$\begin{aligned} \underline{\alpha}_{\Delta}(|\mathbf{x}|) &\leq V_{\delta, \Delta}(t, \mathbf{x}) \leq \bar{\alpha}_{\Delta}(|\mathbf{x}|) \\ \frac{\partial V_{\delta, \Delta}}{\partial t}(t, \mathbf{x}) + \frac{\partial V_{\delta, \Delta}}{\partial \mathbf{x}}(t, \mathbf{x})f(t, \mathbf{x}, \theta^*) &\leq -\alpha_{\Delta}(V_{\delta, \Delta}(t, \mathbf{x})) \\ \left| \frac{\partial V_{\delta, \Delta}}{\partial \mathbf{x}}(t, \mathbf{x}) \right| &\leq c_{\Delta}(|\mathbf{x}|). \end{aligned}$$

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Theorem (Converse Lyapunov function for δ -USPAS)

Assume that the system $\dot{\mathbf{x}} = f(t, \mathbf{x}, \theta)$ is δ -USPAS on $\Theta \subset \mathbb{R}^m$ and that there exist $f_0 \geq 0$ and a continuous nondecreasing function L such that

$$|f(t, \mathbf{0}, \theta)| \leq f_0, \quad \left| \frac{\partial f}{\partial \mathbf{x}}(t, \mathbf{x}, \theta) \right| \leq L(|\mathbf{x}|), \quad \forall t \in \mathbb{R}_{\geq 0}, \forall \mathbf{x} \in \mathbb{R}^n, \forall \theta \in \Theta.$$

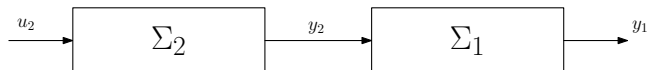
Then, for any $\Delta > \delta > 0$, there exist a C^1 function $V_{\delta, \Delta} : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ such that, for all $\delta \leq |\mathbf{x}| \leq \Delta$ and all $t \in \mathbb{R}_{\geq 0}$,

$$\begin{aligned} \underline{\alpha}_{\Delta}(|\mathbf{x}|) &\leq V_{\delta, \Delta}(t, \mathbf{x}) \leq \bar{\alpha}_{\Delta}(|\mathbf{x}|) \\ \frac{\partial V_{\delta, \Delta}}{\partial t}(t, \mathbf{x}) + \frac{\partial V_{\delta, \Delta}}{\partial \mathbf{x}}(t, \mathbf{x})f(t, \mathbf{x}, \theta^*) &\leq -\alpha_{\Delta}(V_{\delta, \Delta}(t, \mathbf{x})) \\ \left| \frac{\partial V_{\delta, \Delta}}{\partial \mathbf{x}}(t, \mathbf{x}) \right| &\leq c_{\Delta}(|\mathbf{x}|). \end{aligned}$$

- 1 Semiglobal and practical stability
- 2 Semiglobal and/or practical stability of cascades**
 - Motivations for the study of cascades
 - USPAS/USAS systems in cascade
 - UGPAS systems in cascade
 - Applications
- 3 Set-stability
- 4 Integral input to state stability
- 5 Conclusion and future work

Semiglobal and practical stability of cascades

Why studying cascades ?

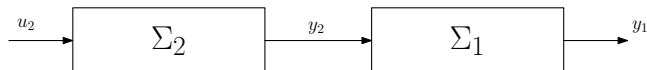


- Cascades may appear in various contexts:
 - Natural structure of the plant
 - Control law designed in order to put the system in a cascade form
 - Decompose a complex problem into simpler ones.
- Widely studied structure: Coron, Kokotović, Ortega, Praly, Sepulchre, Sontag, Sussmann, Teel, Vidyasagar, . . .
- Various approaches: stability analysis, state/output-feedback stabilizability, ISS, Lyapunov construction, . . .
- Many analysis and design tools based on [Seibert, Suarez, 90], [Sontag, 89], [Panteley, Loría, 98]:

$$\text{UGAS} + \text{UGAS} + \text{UGB} \Rightarrow \text{UGAS}.$$

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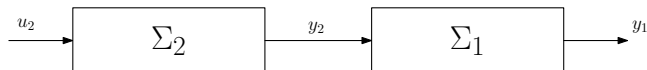


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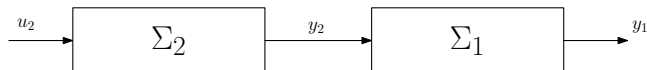


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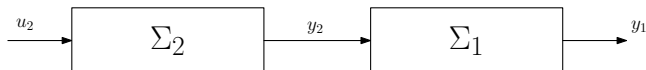


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Semiglobal and practical stability of cascades

USPAS cascaded systems

Considered cascades:

$$\dot{x}_1 = f_1(t, x_1, \theta_1) + g(t, x, \theta) \quad (3a)$$

$$\dot{x}_2 = f_2(t, x_2, \theta_2) \quad (3b)$$

Semiglobal and practical stability of cascades

USPAS cascaded systems

Theorem (USPAS + USPAS + UB \Rightarrow USPAS)

- A1 - Given any $\Delta_1 > \delta_1 > 0$, there exist a parameter $\theta_1^* \in \Theta_1$, a function $V_{\delta_1, \Delta_1} \in C^1$, class \mathcal{K}_∞ functions $\underline{\alpha}_{\delta_1, \Delta_1}$, $\bar{\alpha}_{\delta_1, \Delta_1}$, $\alpha_{\delta_1, \Delta_1}$ and a continuous positive nondecreasing function $\mathbf{c}_{\delta_1, \Delta_1}$ such that, for all $\delta_1 \leq |\mathbf{x}_1| \leq \Delta_1$ and all $t \in \mathbb{R}_{\geq 0}$,

$$\underline{\alpha}_{\delta_1, \Delta_1}(|\mathbf{x}_1|) \leq V_{\delta_1, \Delta_1}(t, \mathbf{x}_1) \leq \bar{\alpha}_{\delta_1, \Delta_1}(|\mathbf{x}_1|)$$

$$\frac{\partial V_{\delta_1, \Delta_1}}{\partial t}(t, \mathbf{x}_1) + \frac{\partial V_{\delta_1, \Delta_1}}{\partial \mathbf{x}_1}(t, \mathbf{x}_1) f_1(t, \mathbf{x}_1, \theta_1^*) \leq -\alpha_{\delta_1, \Delta_1}(|\mathbf{x}_1|)$$

$$\left| \frac{\partial V_1}{\partial \mathbf{x}_1}(t, \mathbf{x}_1) \right| \leq \mathbf{c}_{\delta_1, \Delta_1}(|\mathbf{x}_1|).$$

- A2 - For any $\theta_1 \in \Theta_1$, there exists a nondecreasing function G_{θ_1} and a class \mathcal{K} function Ψ_{θ_1} such that, for all $\theta_2 \in \Theta_2$, all $\mathbf{x} \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and all $t \in \mathbb{R}_{\geq 0}$, $|g(t, \mathbf{x}, \theta)| \leq G_{\theta_1}(|\mathbf{x}_1|) \Psi_{\theta_1}(|\mathbf{x}_2|)$.

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Semiglobal and practical stability of cascades

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Theorem (USPAS + USPAS + UB \Rightarrow USPAS), cont'd

- A3 - The system $\dot{x}_2 = f_2(t, x_2, \theta_2)$ is USPAS on Θ_2 .
- A4 - There exists $\Delta_0 > 0$ such that, for any given $\delta_1, \Delta_1, \delta_2, \Delta_2 > 0$, satisfying $\Delta_1 > \max\{\delta_1; \Delta_0\}$ and $\Delta_2 > \delta_2$, and for the parameter $\theta_1^*(\delta_1, \Delta_1)$, there exists $\theta_2^* \in \mathcal{D}_{f_2}(\delta_2, \Delta_2) \cap \Theta_2$ and $\gamma(\delta_1, \delta_2, \Delta_1, \Delta_2) > 0$ such that

$$|x_0| \leq \gamma(\delta_1, \delta_2, \Delta_1, \Delta_2) \quad \Rightarrow \quad |\phi(t, t_0, x_0, \theta^*)| \leq \Delta_1, \quad \forall t \geq t_0.$$

In addition, given any $\Delta_* > \delta_* > 0$, there exist $\delta_1, \Delta_1, \Delta_2 > 0$, with $\Delta_1 > \delta_1$, such that, for all $\delta_2 \in (0; \Delta_2)$,

$$\min \{ \Delta_1; \Delta_2; \gamma(\delta_1, \delta_2, \Delta_1, \Delta_2) \} \geq \Delta_*$$

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Under A1-A4, the cascade is USPAS on $\Theta_1 \times \Theta_2$.

Semiglobal and practical stability of cascades

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Semiglobal and practical stability of cascades

Application

Robustness analysis of PID-controlled robots:

The ideas of this result were used to show that PID-controlled robot manipulators are **robust** to external disturbances (e.g. friction, noise), high order non-linearities, and actuators' dynamics:

- The resulting system is USPAS
- Driven subsystem: mechanical part (robot arm)
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Semiglobal and practical stability of cascades

USPAS/USAS cascaded systems, without a Lyapunov function

Theorem (USPAS + USPAS + UGB \Rightarrow USPAS)

- A1 - $\dot{x}_1 = f_1(t, x_1, \theta_1)$ is δ -USPAS on Θ_1
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Semiglobal and practical stability of cascades

UGPAS cascaded systems

$$\dot{x}_1 = f_1(t, x_1, \theta_1) + g(t, x, \theta)$$

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Semiglobal and practical stability of cascades

UGPAS cascaded systems

Theorem (UGPAS + UGPAS + growth restriction \Rightarrow UGPAS), cont'd

A4 - For the functions of A5 and A6, for all $\delta_1 > 0$ and as $s \rightarrow +\infty$,

$$\begin{aligned}c_{\delta_1}(s)G_{\theta_1^*}(s) &= \mathcal{O}(\alpha_{\delta_1} \circ \bar{\alpha}_{\delta_1}^{-1} \circ \underline{\alpha}_{\delta_1}(s)) \\ \alpha_{\delta_1}(s) &= \mathcal{O}(\bar{\alpha}_{\delta_1}(s)).\end{aligned}$$

Under A1-A4, the cascade is UGPAS on $\Theta_1 \times \Theta_2$.

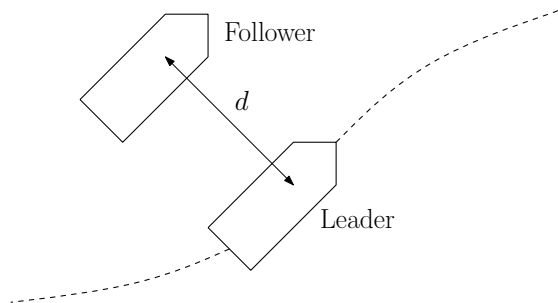
Semiglobal and practical stability of cascades

Application

Synchronization for replenishment procedure:

This result was used to synchronize surface vessels with a virtual ship approach:

- Leader-follower strategy
- No information needed on the main ship except position.



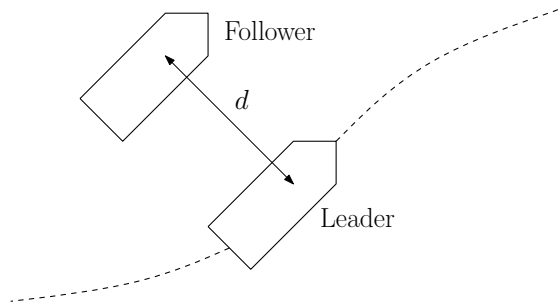
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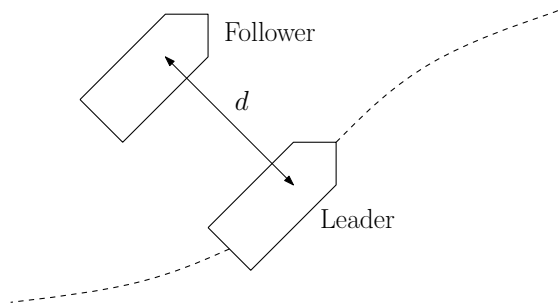
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- 1 Semiglobal and practical stability
- 2 Semiglobal and/or practical stability of cascades
- 3 Set-stability**
 - Motivations and definitions
 - Set-stability of cascades
 - Illustrative example
- 4 Integral input to state stability
- 5 Conclusion and future work

Set-stability

Definition

We deal with **autonomous** systems:

$$\dot{x} = f(x).$$

Definition (GAS of a set)

Assume that $\dot{x} = f(x)$ is forward complete on \mathbb{R}^n . The closed set \mathcal{A} is said to be **globally asymptotically stable** if there exists a \mathcal{KL} function β such that, for all $x_0 \in \mathbb{R}^n$,

$$|\phi(t, t_0, x_0)|_{\mathcal{A}} \leq \beta(|x_0|, t), \quad \forall t \geq 0.$$

- Stability with respect to **two** measures
- \mathcal{A} is not necessarily compact.

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General concept, it includes as special cases:

- Stability of the **origin**: $\mathcal{A} = \{0\}$
- Stability of a **path**: $\mathcal{A} = \{x_d(t) : t \geq 0\}$
- Ultimate boundedness, stability of a **fixed ball**: $\mathcal{A} = \mathcal{B}_r$
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Set-stability for cascades

Under a UGB assumption

We consider autonomous cascades:

$$\dot{x}_1 = f_1(x_1) + g(x_1, x_2) \quad (4a)$$

$$\dot{x}_2 = f_2(x_2), \quad (4b)$$

Theorem (GAS + GAS + GB \Rightarrow GAS)

- A1 - \mathcal{A}_1 is GAS for $\dot{x}_1 = f_1(x_1)$.
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- A3 - There exist a continuous function g_1 and a class \mathcal{K}_∞ function g_2 such that, for all $x = (x_1^T, x_2^T)^T \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$,

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- A4 - The solutions of (4) are globally bounded.

Under A1-A4, the set $\mathcal{A} := \mathcal{A}_1 \times \mathcal{A}_2$ is GAS for the cascade (4).

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Set-stability for cascades

Using growth order restrictions

Theorem (GAS + GAS + growth restriction \Rightarrow GAS)

- A1-bis - There exist a C^1 function V_1 , class \mathcal{K}_∞ functions $\underline{\alpha}_1, \bar{\alpha}_1, \alpha_1, g_2$ and a continuous function g_{11} such that, for all $x_1 \in \mathbb{R}^{n_1}$,

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- A4-bis - The cascade (4) is forward complete and $g_{11}(s) = \mathcal{O}(\alpha_1(s))$ as $s \rightarrow +\infty$.

Under A1-bis, A2, A3 and A4-bis, the set $\mathcal{A}_1 \times \mathcal{A}_2$ is GAS for (4).

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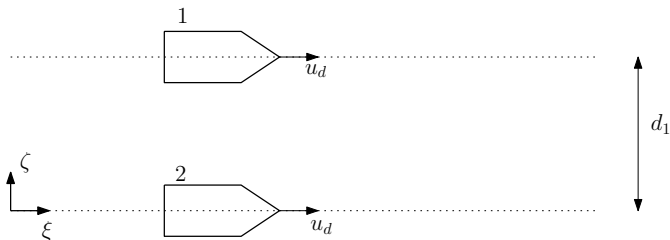
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Application

Control of ships in formation:

Formation control of surface vessels along a straight path, with a prescribed velocity (an alternative proof for [Børhaug, Pavlov, Pettersen, 06]):

- Same hierarchical level for all ships
- The set to be stabilized contains $\{(x_1, x_2) : \xi_1 = \xi_2\}$ (unbounded)
- Cascaded structure arises from the analysis: we use velocities as “virtual controls”.



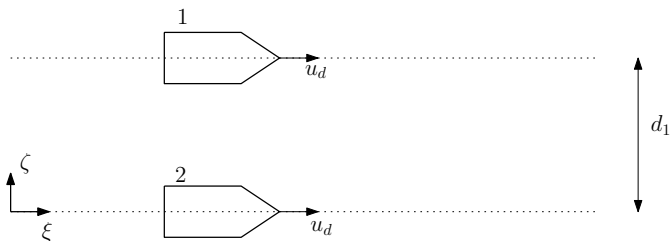
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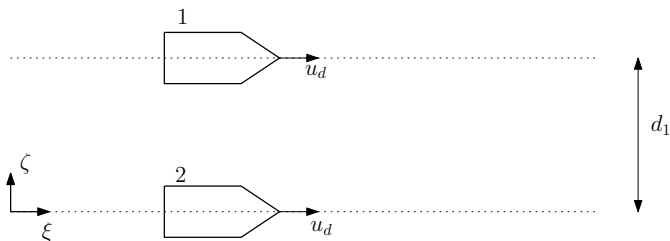
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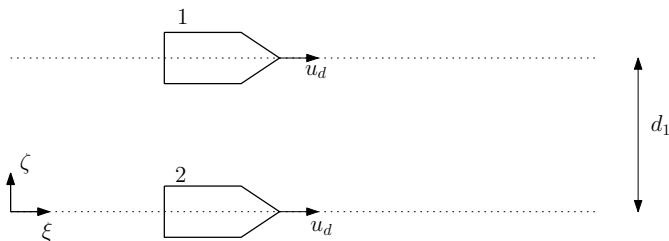
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- 4 Integral input to state stability**
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Integral input to state stability

Definitions

$$\dot{x} = f(x, u). \quad (5)$$

Definition (ISS, [Sontag, 89])

We say that (5) is **input to state stable** if there exist a class \mathcal{KL} function β and a class \mathcal{K}_∞ function γ such that, for all x_0 and all u ,

$$|\phi(t, x_0, u)| \leq \beta(|x_0|, t) + \gamma(\|u\|), \quad \forall t \geq 0.$$

Definition (iISS, [Sontag, 98])

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Motivations

- **iISS is much more general than ISS**
- iISS guarantees **robustness** to the system:
 - Stability of a ball of radius proportional to the energy of the input
 - Finite supplied energy \Rightarrow convergence to zero
- In [Panteley, Loría, 98]: preservation of the UGAS property under, among other, integrability of the x_2 -solutions and growth restrictions.
- In [Arcak, Angeli, Sontag, 02], it was shown that:
$$\text{GAS} + \text{iISS} + \text{trajectory-based condition} \Rightarrow \text{GAS}$$
- What about Lyapunov-based conditions ?
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Lyapunov characterization

Theorem (Lyapunov characterization of iISS, [Angeli, Sontag, Wang, 00])

The system $\dot{x} = f(x, u)$ is iISS if and only if there exist a C^∞ function V , class \mathcal{K}_∞ functions $\underline{\alpha}$, $\bar{\alpha}$, γ and a **continuous positive definite** function α such that, for all x and all u ,

$$\underline{\alpha}(|x|) \leq V(x) \leq \bar{\alpha}(|x|)$$

$$\frac{\partial V}{\partial x}(x)f(x, u) \leq -\alpha(|x|) + \gamma(|u|).$$

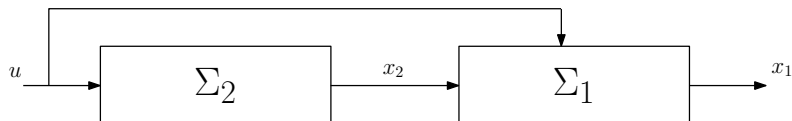
Integral input to state stability

iISS for cascades

Cascades under consideration:

$$\dot{x}_1 = f_1(x_1, x_2, u) \quad (6a)$$

$$\dot{x}_2 = f_2(x_2, u) \quad (6b)$$



Integral input to state stability

iISS for cascades, Lyapunov-based

Theorem (iISS + iISS \Rightarrow iISS, Lyapunov-based)

Let $V_1 : \mathbb{R}^{n_1} \rightarrow \mathbb{R}_{\geq 0}$ be a differentiable function and $V_2 : \mathbb{R}^{n_2} \rightarrow \mathbb{R}_{\geq 0}$ be a continuous function, differentiable out of the origin. Suppose that there exist a class \mathcal{K} function ν_1 and, for all $i \in \{1, 2\}$, a continuous positive definite function α_i , a class \mathcal{K} function γ_i , and two class \mathcal{K}_∞ functions $\underline{\alpha}_i$ and $\bar{\alpha}_i$ such that, for all $x_i \in \mathbb{R}^{n_i}$ and all $u \in \mathbb{R}^p$,

$$\underline{\alpha}_i(|x_i|) \leq V_i(x_i) \leq \bar{\alpha}_i(|x_i|)$$

$$\frac{\partial V_1}{\partial x_1}(x_1) f_1(x_1, x_2, u) \leq -\alpha_1(|x_1|) + \gamma_1(|x_2|) + \nu_1(|u|)$$

$$x_2 \neq 0 \quad \Rightarrow \quad \frac{\partial V_2}{\partial x_2}(x_2) f_2(x_2, u) \leq -\alpha_2(|x_2|) + \gamma_2(|u|).$$

If, in addition, $\gamma_1(s) = \mathcal{O}(\alpha_2(s))$ as $s \rightarrow 0$, then the cascade is iISS.

Integral input to state stability

iISS for cascades, Lyapunov-based

- V_2 does not need to be differentiable at zero
- Condition easy to check
- Direct corollary:

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Integral input to state stability

iISS for cascades, trajectory-based

Theorem (iISS + iISS \Rightarrow iISS, trajectory-based)

Assume that

- $\dot{x}_1 = f_1(x_1, x_2, u)$ is iISS with a **locally Lipschitz iISS gain**
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Then the cascade is iISS.

Integral input to state stability

iISS for cascades, trajectory-based

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- 1 Semiglobal and practical stability
- 2 Semiglobal and/or practical stability of cascades
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- 4 Integral input to state stability
- 5 Conclusion and future work**

- Main results:

- Rigorous Lyapunov framework for semiglobal and practical stability
- Stability and robustness analysis of cascaded systems:
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 - Set-stability
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- Applications to control of mechanical systems:
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- Future research:

- Constructive control design based on these results, e.g. backstepping
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List of publications



Journal papers:

- 1- A. Chaillet and A. Loria, *A converse theorem for uniform semiglobal practical asymptotic stability: application to cascaded systems*, Accepted for Automatica, 2006.
- 2- A. Chaillet and A. Loria, *Uniform semiglobal practical asymptotic stability for non-autonomous cascaded systems and applications*, Submitted to Systems and Control Letters, 2005.
- 3- A. Chaillet and D. Angeli, *Integral input to state stable systems in cascade*, Submitted to Systems and Control Letters, 2005.
- 4- A. Chaillet and A. Loria, *Uniform global practical asymptotic stability for non-autonomous cascaded systems*, Submitted to European Journal of Control, 2006.
- 5- A. Chaillet, A. Loria and R. Kelly, *Robustness of PID-controlled manipulators vis a vis actuator dynamics and external disturbances*, Submitted to European Journal of Control, 2006.



Book chapters:

- 6- R. Kristiansen, A. Loria, A. Chaillet, and P. J. Nicklasson, *Output feedback control of relative translation in a leader-follower spacecraft formation*, In Proceedings of the Workshop on Group Coordination and Cooperative Control, Lecture Notes in Control and Information Sciences, Springer Verlag, Tromsø, Norway, 2006.
- 7- E. Kyrkjebø, E. Panteley, A. Chaillet, and K. Pettersen, *A virtual vehicle approach to underway replenishment*, In Proceedings of the Workshop on Group Coordination and Cooperative Control, Lecture Notes in Control and Information Sciences, Springer Verlag, Tromsø, Norway, 2006.
- 8- A. Chaillet, *Tools for semiglobal practical stability analysis of cascaded systems and applications*, In Proceedings of the CTS-HYCON Workshop, International Scientific and Technical Encyclopedia, Paris, France, July 2006.



Conference papers:

- 9- A. Chaillet and A. Loria, *Uniform semiglobal practical asymptotic stability for nonlinear time-varying systems in cascade*, In Proc. 16th. IFAC World Congress, Praha, Tcheck Republic, July 2005.
- 10- A. Chaillet and D. Angeli, *Integral input to state stability for cascaded systems*, In MTNS 2006, Kyoto, Japan, July 2006.
- 11- A. Chaillet and A. Loria, *Uniform global practical stability for non-autonomous cascaded systems*, In MTNS 2006, Kyoto, Japan, July 2006.
- 12- A. Chaillet and E. Panteley, *Stability of sets for nonlinear systems in cascade*, In MTNS 2006, Kyoto, Japan, July 2006.
- 13- A. Chaillet, *Estimée du domaine d'attraction pour la cascade de systèmes uniformément asymptotiquement stables*, In CIFA 2006, Bordeaux, France, June 2006.
- 14- A. Chaillet, A. Loria, and R. Kelly, *Robustness of PID controlled manipulators with respect to external disturbances*, Submitted to CDC 2006, San Diego, USA, December 2006.
- 15- A. Chaillet and A. Loria, *A converse Lyapunov theorem for semiglobal practical asymptotic stability and application to cascades-based control*, Submitted to CDC 2006, San Diego, USA, December 2006.
- 16- R. Kristiansen, A. Loria, A. Chaillet, and P. J. Nicklasson, *Adaptive output feedback control of spacecraft relative translation*, Submitted to CDC 2006, San Diego, USA, December 2006.
- 17- J. Tsønnås, A. Chaillet, E. Panteley, and T. A. Johansen, *Cascaded lemma for set-stable systems*, Submitted to CDC 2006, San Diego, USA, December 2006.