On stability and robustness of nonlinear cascaded systems - Application to mechanical systems

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Stabilité et robustesse des cascades non-linéaires et application aux systèmes mécaniques

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Je tiens à exprimer ma profonde reconnaissance à ma famille sur l'amour et la confiance de laquelle j'ai toujours pu compter.

J'adresse une pensée tout particulière à Hélène sans l'éternel soutien de qui je ne me serais probablement pas lancé dans ce projet.

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Preliminary remark

This document synthesizes the research works conducted by A. Chaillet, under the supervision of Dr. A. Loría, in order to obtain the PhD degree from Université Paris Sud. Please note that a French version is also available on request.

For any remark, question or comment, please feel free to contact the author at: `antoinechaillet@yahoo.fr`.

Contents

Preamble

Before introducing, in a more detailed manner, the subject of study of the present thesis, we give, in informal terms, a concise overview of the motivations for this work and its contributions.

Stability. Roughly speaking, stability is the property of a dynamical system that any error signals can be made arbitrarily small provided that the initial errors are sufficiently small. It is a crucial notion from a control point of view as it ensures an acceptable behavior of the plant if its initial configuration is not too far from the nominal one. If, in addition, the error signals eventually tend to zero, we say that this operating point is asymptotically stable. The domain of attraction consists the set of all initial states from which solutions go to zero. We talk about global asymptotic stability when the domain of attraction is the whole state-space.

Obstacles. Through intuitive examples, we expose some of the reasons that may prevent the error signals from converging to zero, as, for instance, the presence of an external perturbation, measurement imprecision, friction, *etc.* In the same way, we show that the domain of attraction may be restricted to a compact neighborhood of the origin, notably in the case of neglected high order nonlinearities. In these situations, most of existing tools fail at ensuring better than ultimate boundedness (convergence of solutions to some neighborhood of the origin) or to local stability (restriction of the domain of attraction).

Semiglobal and practical stability. Such a degradation of performance is not acceptable in many concrete applications, as this may result in a too little operating bandwidth or a too large imprecision. Nevertheless, for controlled systems, the domain of attraction can often be arbitrarily enlarged provided sufficiently large gains. We refer to this property as semiglobal asymptotic stability. For a given system, semiglobal asymptotic stability ensures much more interesting properties than simply local properties, since it establishes that no theoretical obstacle prevents from including any given finite set of initial conditions to the domain of attraction.

In the same way, the steady-state errors can often be diminished at will under a similar tuning of the gains: we call this property practical asymptotic stability. Again, this concept should be seen as a far stronger property than the simple ultimate boundedness of solutions. Indeed, practical asymptotic stability imposes that the precision, after the transients, can be made as fine as desired. In addition, as we will see in more details in the sequel, it suggests a "reasonable" behavior of the transient dynamics which is not the case, in general, of ultimate boundedness.

When it follows from a limitation of the performances of the system, semiglobal practical asymptotic stability constitutes an interesting measure of the robustness of a system to

external perturbations and model uncertainty. But, as we will see, this stability property does not arise exclusively from a degradation of global asymptotic properties, but may also be established by existing results in the literature, such as averaging techniques.

A first goal of the present thesis is to provide a rigorous framework for the study of semiglobal and/or practical asymptotic stability properties. To this end, we provide sufficient conditions, expressed in terms of Lyapunov functions, that guarantee this stability properties. As we will see, these conditions allow that the Lyapunov function depends on the tuning parameter. Indeed, such a choice allows to include in our analysis a much wider scope of applications than if the Lyapunov function was assumed uniform in the tuning parameter. For instance, for mechanical systems, it is common to choose the energy of the system as a Lyapunov function, in which case the control gains, then playing the role of the tuning parameter, naturally appear in the Lyapunov function.

Due to this non-uniformity, compared to classical results, an additional assumption is required that links the upper and lower bounds on the Lyapunov function. We infer the necessity of such an additional condition through an example.

We stress that the stability concept that we will use along the document makes use of two measures: the distance to the ball for which we want attractivity and the Euclidean norm. Here also, this choice is motivated by simplicity and generality reasons. Indeed, if we had chosen to use one single measure, the corresponding Lyapunov function would then have had to vanish on a whole neighborhood of the origin, which would have prevented the use of the Lyapunov function associated to the nominal system. On the opposite, with this choice, most of the semiglobal or practical stability properties that result from a degradation of an "ideal" system due to perturbations can be inferred by using the Lyapunov function of the unperturbed system.

Hence, we propose tools that allow to establish powerful stability properties, *i.e.* semiglobal and/or practical stability, which usually do not require much more conservative assumptions than those needed for the (weaker but certainly more classical) properties of ultimate boundedness and local stability.

We also present a so-called "converse" theorem for semiglobal practical asymptotic stability, *i.e.* a result that guarantees the existence of a Lyapunov function under the assumption of such a stability property. The generality of the concept that we use requires specific precautions compared to the properties that would be uniform in the tuning parameter. We will see that, as this result generates an autonomous bound on the gradient of the generated Lyapunov function, it will be of great help in lightening the assumptions in our results on cascades.

Cascades. In order to simplify the study of a complex system, it is common, in stability analysis, to divide it into smaller interconnected subsystems. In this way, the difficulty of the analysis is often reduced. A particular type of such interconnection is the cascade structure. In this situation, the subsystems are interconnected in a unilateral way, *i.e.* the output of a driving subsystem is the input of a driven subsystem. The modularity offered by this so-called cascade approach gave rise to powerful results, both in analysis and control design.

However, most of the existing results in this domain only treat local or global stability of the origin. Hence, they do not apply to the concepts, although common and powerful, of semiglobal and/or practical stability.

As a second objective, we provide sufficient conditions under which semiglobal and/or practical asymptotic stability is preserved by the cascade interconnection. Roughly speak-

ing, we show that this is the case provided that we explicitly know a Lyapunov function for the driven subsystem and that the solutions are uniformly bounded. In the case of global practical asymptotic stability, we provide a structural criterion to ensure this uniform boundedness of solutions, therefore yielding an easy-to-check condition to guarantee global practical asymptotic stability of the cascade. An illustration of these results consists in stabilizing, by a bounded output feedback, the double integrator affected by a persistently exciting signal. As another application, we rigorously show that smoothing a feedback control law may result in practical stability. Furthermore, we provide a converse Lyapunov result for semiglobal practical asymptotic stability that permits us to relax the requirement of explicitly knowing a Lyapunov function for the driven subsystem. As illustrated by an example, this latter feature is particularly useful when stability is established based on averaging techniques.

Set-stability. The generality offered by set-stability makes it, as we further develop upon, another interesting tool for the stability and robustness analysis of perturbed systems. Indeed, this notion includes, as particular cases, the stability of a single operating point, of a trajectory or even a more complex domain according to the set that is considered. Moreover, as the latter is not assumed to be compact, it is also possible to include to the study the partial stability, which refers to the situation when the behavior of only a part of the state is constrained. We will see that the latter appears very useful when dealing with adaptive control.

The third objective of this work is to provide sufficient conditions for the preservation of the set-stability for cascaded systems. The requirement is first given as a global boundedness of the solutions of the overall cascade. We establish that, in some situations, this can be relaxed to just forward completeness provided a growth restriction on the interconnection term. As an illustrative application, we propose a proof for a recently established result in marine control.

ISS and iISS. So far, we have discussed Lyapunov stability for systems without inputs. A field of stability analysis, regrouped under the paradigm of input to state stability (ISS), is especially concerned by the impact of external signals on the performance of the system. Without going into details, this property imposes that the norm of the current state be bounded by a function of the amplitude of the perturbing signal plus a fading term depending on initial conditions. A relaxed extension of this property is called integral input to state stability (iISS). Instead of the amplitude of the external signal, this property takes into account the "energy" that the latter feeds to the system. The iISS property is very general in stability analysis and provides interesting information about the system. For instance, if the input energy is finite, then the state converges to zero. In this sense, iISS (as well as ISS) therefore constitutes another powerful measure of the robustness of a system to external perturbations.

A fourth part of this text is devoted to the behavior of iISS systems when placed in cascades. We provide elementary conditions under which the cascade composed of an iISS system driven by a globally asymptotically stable one remains globally asymptotically stable. These conditions are expressed in terms of the Lyapunov functions associated to each subsystem, thus generalizing existing trajectory-based results. Under mildly conservative additional assumptions, we establish that the cascade of two iISS subsystems is itself iISS. This latter result is firstly expressed in terms of Lyapunov functions, and then in terms of estimates of the solutions of each subsystem.

Applications. Many of the results presented in this document have been applied in practice, and we expose some of these results in a fifth step.

We study the robustness of PID-controlled robot manipulators to frictions, external disturbances, model uncertainty and taking into account the dynamics of actuators. We prove that, under these environmental constraints, the system is semiglobally practically asymptotically stable. This is confirmed by experimental results.

In an other domain, we show that the leader-follower strategy adopted for the control of spacecraft formations yields global asymptotic stability when all measurements are available. However, in practice, some information on the leader's position may not be available. We show that, provided that these signals are bounded, global practical asymptotic stability can be concluded.

Finally, in the context of underway ship replenishment, where the control of the supply vessel aims at preserving a constant distance from the main ship during the operation, the only measurements available for the main ship are position and heading. No information on its model is at disposal. Under this constraints, we show that a virtual vehicle approach ensures global practical asymptotic stability of the system.

We eventually stress that, although some of the results presented here impose relatively heavy notations for the sake of rigor, the document also aims at giving intuitive explanations of the utilized concepts. In this direction, we give several simple examples to illustrate the purpose and, when possible, provide simplified corollaries that are less general but easier to use in practice.

Also, even though the results presented along this document concern more stability analysis than stabilization, in the sense that no explicit design of control law is presented, they still constitute a prescriptive framework on which one could base control design strategies, as illustrated by the concrete applications of Chapter 6.

Contribution of this thesis

We briefly summarize the main results of this thesis, chapter by chapter, and cite related publications. Labels correspond to the list of publications presented in p. 17.

- Chapter 2: We present new tools for the study of semiglobal and practical stability of nonlinear time-varying systems. Some sufficient conditions, in terms of Lyapunov functions, are proposed. Compared to classical Lyapunov conditions, an additional condition appears, that takes into account the non-uniformity of the Lyapunov function in the tuning parameter. We underline the necessity of such a requirement through an example. Conversely, we prove that such a Lyapunov function can be derived provided sufficient regularity of the right-hand side of the ordinary differential equation.

  This chapter formed the subject of the following publications with A. Loría:

  [(i), (ii), (iv), (viii), (x), (xiii), (xvii)].

- Chapter 3: We extend the results of Chapter 2 to nonlinear time-varying systems presenting a cascade structure. We prove that, under a boundedness condition on the solutions of the overall system, both semiglobal and practical stability properties are preserved by the cascade interconnection of two subsystems. We also give some sufficient conditions to ensure the boundedness condition on the solutions, which are particularly easy to use in the case of global practical stability. Illustrative examples are provided in each context.

  These results were originally presented in the following publications with A. Loría:

  [(i), (ii), (iv), (viii), (x), (xiii), (xv), (xvii)].

- Chapter 4: We analyze the behavior of nonlinear systems that are globally asymptotically stable with respect to a (non necessarily compact) set, when placed in cascade. We provide sufficient conditions under which set-stability, defined with respect to two measures, is preserved by the cascade interconnection.

  These works correspond to the collaboration [(xiv)] with E. Panteley. An extension was proposed with the same coauthor, J. Tsønnàs and T.A. Johansen in [(xix)].

- Chapter 5: We study the preservation of the integral input to state stability property of nonlinear time-invariant systems in cascade. We give some sufficient conditions for the cascade composed of an iISS subsystem driven by a globally asymptotically stable (GAS) subsystem to be GAS. These conditions are expressed in terms of the Lyapunov function associated to each subsystem, thus generalizing existing similar

trajectory-based results. We also provide conditions under which two iISS systems placed in cascade remain iISS. These sufficient conditions are first expressed based on the Lyapunov function for each of the two subsystems, and then on the estimate of their solutions.

These results were prepared with D. Angeli in [(iii), (xii)].

- Chapter 6: We present concrete applications of our main theoretical findings in stabilization problems of mechanical systems. We show that, when taking into account external perturbations (such as friction, torque ripping, *etc.*) and the dynamics of the actuator, PID-controlled robot manipulators are semiglobally practically stable, backed up with experimental results.

  On the other hand, a control for underway fuel replenishment of vessels is designed, using a virtual ship approach, which requires neither *a priori* model knowledge nor velocity measurement for the ship to be replenished. Global practical asymptotic stability is obtained.

  A third application concerns the control of a spacecraft formation, when taking into account bounded external disturbances. According to the assumed level of knowledge we have on the orbital parameters of the leader, various stability properties are derived.

  These applications were the object of the following joint publications with R. Kelly, E. Kyrkjebø, R. Kristiansen, A. Loría, E. Panteley, K. Pettersen and P. J. Nicklasson: [(iv), (vi), (vii), (xvi), (xviii)].

Although not presented in this document, these three years of PhD gave rise to other fruitful collaborations:

- The publication [(ix)] is a joint work with J. de León Morales, A. Loría and G. Besançon where we proposed an adaptive observer for systems that can be put in the so-called output feedback form, based on a convenient persistency of excitation property.

- With A. Loría, G. Besançon and Y. Chitour, we have posed open problems for stabilization of persistently excited systems, and partially solve them in the case of the double integrator, *cf.* [(xi)].

- In [(xx)], with M. Sigalotti, P. Mason, Y. Chitour and A. Loría, this latter problem was further extended and solved with a linear time invariant feedback, with gains uniform in the persistently exciting signal.

Publications

The following is an exhaustive list of publications written during the past three years, that are either published, accepted for publication, or still under review. It contains but is not restricted to the contents of this document.

Journal Papers

i/ A. Chaillet and A. Loría, *A converse theorem for uniform semiglobal practical asymptotic stability: application to cascaded systems*, Accepted for Automatica, 2006.

ii/ A. Chaillet and A. Loría, *Uniform semiglobal practical asymptotic stability for non-autonomous cascaded systems and applications*, Accepted for Autmatica, 2006.

iii/ A. Chaillet and D. Angeli, *Integral input to state stable systems in cascade*, Submitted to Systems and Control Letters, 2005.

iv/ A. Chaillet and A. Loría, *Uniform global practical asymptotic stability for non-autonomous cascaded systems*, Submitted to European Journal of Control, 2006.

A. Chaillet, A. Loría and R. Kelly, *Robustness of PID-controlled manipulators vis a vis actuator dynamics and external disturbances*, Submitted to European Journal of Control, 2006.

Book chapters

vi/ R. Kristiansen, A. Loría, A. Chaillet, and P. J. Nicklasson, *Output feedback control of relative translation in a leader-follower spacecraft formation*, In Proceedings of the Workshop on Group Coordination and Cooperative Control, Lecture Notes in Control and Information Sciences, Springer Verlag, Tromsø, Norway, 2006.

vii/ E. Kyrkjebø, E. Panteley, A. Chaillet, and K. Pettersen, *A virtual vehicle approach to underway replenishment*, In Proceedings of the Workshop on Group Coordination and Cooperative Control, Lecture Notes in Control and Information Sciences, Springer Verlag, Tromsø, Norway, 2006.

viii/ A. Chaillet, *Tools for semiglobal practical stability analysis of cascaded systems and applications*, In Proceedings of the CTS–HYCON Workshop, International Scientific and Technical Encyclopedia, Paris, France, July 2006.

Conference papers

ix/ J. de León Morales, A. Chaillet, A. Loría, and G. Besançon, *Output feedback control via adaptive observers with persistency of excitation*, In IFAC World Congress, Praha, Tcheck Republic, July 2005.

x/ A. Chaillet and A. Loría, *Uniform semiglobal practical asymptotic stability for non-linear time-varying systems in cascade*, In IFAC World Congress, Praha, Tcheck Republic, July 2005.

xi/ A. Loría, A. Chaillet, G. Besançon, and Y. Chitour, *On the PE stabilization of time-varying systems: open questions and preliminary answers*, In CDC 2005, pages 6847–6852, Sevilla, Spain, December 2005.

xii/ A. Chaillet and D. Angeli, *Integral input to state stability for cascaded systems*, In MTNS 2006, Kyoto, Japan, July 2006.

xiii/ A. Chaillet and A. Loría, *Uniform global practical stability for non-autonomous cascaded systems*, In MTNS 2006, Kyoto, Japan, July 2006.

xiv/ A. Chaillet and E. Panteley, *Stability of sets for nonlinear systems in cascade*, In MTNS 2006, Kyoto, Japan, July 2006.

xv/ A. Chaillet, *Estimée du domaine d'attraction pour la cascade de systèmes uniformément asymptotiquement stables*, In CIFA 2006, Bordeaux, France, June 2006.

xvi/ A. Chaillet, A. Loría, and R. Kelly, *Robustness of PID controlled manipulators with respect to external disturbances*, In CDC 2006, San Diego, USA, December 2006.

xvii/ A. Chaillet and A. Loría, *A converse Lyapunov theorem for semiglobal practical asymptotic stability and application to cascades-based control*, In CDC 2006, San Diego, USA, December 2006.

xviii/ R. Kristiansen, A. Loría, A. Chaillet, and P. J. Nicklasson, *Adaptive output feedback control of spacecraft relative translation*, In CDC 2006, San Diego, USA, December 2006.

xix/ J. Tsønnàs, A. Chaillet, E. Panteley, and T. A. Johansen, *Cascaded lemma for set-stable systems*, Submitted to CDC 2006, San Diego, USA, December 2006.

Other reports

xx/ M. Sigalotti, A. Chaillet, P. Mason, Y. Chitour, and A. Loría, *Linear time-invariant stabilization with persistency of excitation*, Internal report, 2006.

xxi/ A. Chaillet and A. Loría, *Uniform semiglobal practical asymptotic stability for non-autonomous cascaded systems and applications*, 2006, http://arxiv.org/PS_cache/math/pdf/0503/0503039.pdf.

Notation

All properties such as "positive", "greater", "increasing", *etc.* are to be understood in the *strict* sense.

$\mathbb{N}$ and $\mathbb{R}$ denote the sets of all nonnegative integers and all real numbers respectively. $\mathbb{N}_{\leq N}$ contains all the nonnegative integers less than or equal to $N \in \mathbb{N}$. In the same way, $\mathbb{R}_{\geq 0}$ is composed of all nonnegative real numbers.

I denotes the identity matrix of appropriate dimension.

A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class* $\mathcal{K}$ ($\alpha \in \mathcal{K}$), if it is increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if, in addition, $\alpha(s) \to \infty$ as $s \to \infty$. A continuous function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{L}$ ($\sigma \in \mathcal{L}$) if it is decreasing and tends to zero as its argument tends to infinity. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be a *class* $\mathcal{KL}$ *function* if $\beta(\cdot, t) \in \mathcal{K}$ for any $t \in \mathbb{R}_{\geq 0}$, and $\beta(s, \cdot) \in \mathcal{L}$ for any $s \in \mathbb{R}_{\geq 0}$.

We denote by $\phi(\cdot, t_0, x_0)$ the solutions of the differential equation $\dot{x} = f(t, x)$ with initial condition $\phi(t_0, t_0, x_0) = x_0$.

We use $|\cdot|$ for the Euclidean norm of vectors and the induced L_2 norm of matrices. We use $\|\cdot\|$ for the essential supremum norm, *i.e.*, for a signal $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$, $\|u\| := \text{ess sup}_{t \geq 0} |u(t)|$.

We denote by $\mathcal{B}_\delta$ the *closed* ball in $\mathbb{R}^n$ of radius δ centered at the origin, *i.e.* $\mathcal{B}_\delta := \{x \in \mathbb{R}^n \ : \ |x| \leq \delta\}$. We use the notation $\mathcal{H}(\delta, \Delta) := \{x \in \mathbb{R}^n \ : \ \delta \leq |x| \leq \Delta\}$. By an abuse of notation, $\mathcal{B}_0 = \mathcal{H}(0, 0) = \{0\}$ and $\mathcal{B}_\infty = \mathcal{H}(0, \infty) = \mathbb{R}^n$.

δ being a nonnegative constant, we define $|x|_\delta := \inf_{z \in \mathcal{B}_\delta} |x - z|$. More generally, for a closed set $\mathcal{A}$, $|\cdot|_{\mathcal{A}}$ represents the distance to this set: $|x|_{\mathcal{A}} := \inf_{z \in \mathcal{A}} |x - z|$.

For a given set E of $\mathbb{R}^n$, $\overset{\circ}{E}$ denotes its interior.

Let $a \in \{0, +\infty\}$ and q_1 and q_2 be class $\mathcal{K}$ functions. We say that $q_2(s) = \mathcal{O}(q_1(s))$ as s tends to a if there exists a nonnegative constant k such that $\limsup_{s \to a} q_2(s)/q_1(s) \leq k$. We say that $q_2(s) = o(q_1(s)))$ if k can be taken to be zero, and that $q_1(s) \sim q_2(s)$ if $\lim_{s \to a} q_2(s)/q_1(s) = 1$.

We say that $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the *Carathéodory conditions* if $f(\cdot, x)$ is measurable for each fixed $x \in \mathbb{R}^n$, $f(t, \cdot)$ is continuous for each fixed $t \in \mathbb{R}_{\geq 0}$ and, for each compact U of $\mathbb{R}_{\geq 0} \times \mathbb{R}^n$, there exists a integrable function $m_U : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $|f(t, x)| \leq m_U(t)$ for all all $(t, x) \in U$.

A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be *locally Lipschitz* if, for any compact U of $\mathbb{R}^n$, there exists a nonnegative constant k_U such that $|f(x) - f(y)| \leq k_U |x - y|$ for all $(x, y) \in U^2$.

When the context is sufficiently explicit, we may omit the arguments of a function.

Chapter 1

Definitions

Lyapunov stability. The works presented in this document appeal to many different types of stability properties. Stability should be understood in the Lyapunov sense. Generally speaking, it refers to the property of a point, a set or a trajectory that any solution starting sufficiently near remains arbitrarily close at all time. It constitutes a crucial feature in control of dynamical systems, as it ensures an acceptable behavior of the plant provided that its initial conditions are sufficiently close to the nominal ones.

The notion of stability may easily be grasped in the context of mechanical systems. Considering a ball on a non flat surface, an equilibrium position is *stable* if, after any sufficiently small perturbations on the position of the ball, it remains for ever arbitrarily near to it. The equilibrium is said to be *asymptotically stable* if, in addition, the ball approaches it asymptotically. This is illustrated by the drawings of Figure 1.1.

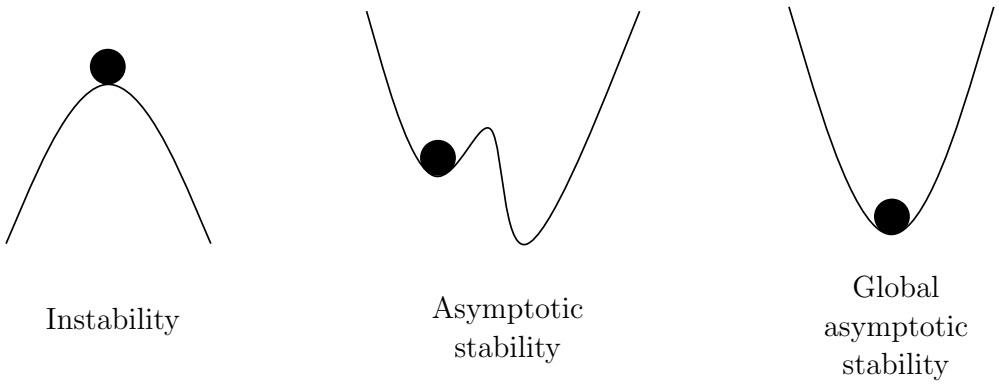


Figure 1.1: Illustration of different types of stability.

In some situations, it is interesting to know how far from the asymptotically stable equilibrium the ball can start and finally return to it. The region of the state space that lead asymptotic convergence is referred to as *domain of attraction*. If the domain of attraction is the whole space, then the equilibrium under consideration is called *globally* asymptotically stable.

While very intuitive in the context of mechanical systems, Lyapunov stability is far from being confined to this area. Generally speaking, the systems considered throughout the document are represented as a finite dimensional differential equation of the form

$$\dot{x} = f(t, x)\,, \tag{1.1}$$

where $t \in \mathbb{R}_{\geq 0}$ represents the time, $x \in \mathbb{R}^n$ is the state and $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to satisfy Carathéodory conditions (*cf.* p. 19) and to be locally Lipschitz in x. More precisely, for each compact U of $\mathbb{R}_{\geq 0} \times \mathbb{R}^n$, we assume that there exists an integrable function $k_U : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that, for all $(t, x) \in U$ and all $(t, y) \in U$,

$$|f(t, x) - f(t, y)| \leq k_U(t) \, |x - y| \ .$$

By virtue of [Hal69, Theorem 5.3], these combined conditions ensure both existence and uniqueness of the solutions of (1.1).

It is worth pointing out the wide variety of systems that can be described by such an equation. To cite a few, it covers a very large number of control problems in mechanics, electrical systems, biology, electronics, electro-magnetics, *etc.*, *cf. e.g.* [OLNSR98, KSV91, Son05b].

Why time-varying systems ? The fact that the right-hand side term of the considered differential equation is time-dependent allows to include in the study many problems of trajectory tracking. This aims at designing a control u in such a way that the solution of the the dynamical system $\dot{x} = f(x, u)$ follows asymptotically a prescribed reference $x_d(t)$. As the adopted control law depends on the time-varying reference trajectory $x_d(t)$, the system in closed-loop, although originally time-invariant, is of the form $\dot{\tilde{x}} = g(t, \tilde{x})$, where $\tilde{x} := x - x_d$. The so stated tracking control problem applies to many physical systems, as for instance in the area of control of mechanical and electromechanical systems (*cf. e.g.* [OLNSR98] and references therein).

Another typical situation in which explicit time-dependence of the dynamical system occurs is that of regulations problems (that is, stabilization of fixed operating point) that do not satisfy Brockett's condition [Bro83] or the more conservative condition presented by Coron in [Cor90]. In this case, the open-loop plant is not stabilizable by any continuously differentiable time-invariant feedback. A time-varying controller is then conceivable. For instance, it was shown by Coron in [Cor92] that any completely controllable smooth system without drift (including nonholonomic mechanical systems) can be stabilized by means of a smooth periodic time-varying state feedback. Also, it was shown in [KT03] that, if a system can be stabilized by a continuous state-feedback, then it is stabilizable (although possibly in a non-uniform way) by a *smooth* time-varying feedback, which may constitute an interesting feature for some applications.

However, this explicit dependence in time of (1.1) can also be of interest from an analysis point of view. Some techniques in the literature, see for instance [Kha96, Lor04], consist in simplifying a complex nonlinear system into a more simple *time-varying* one by considering part of the state as a simple function of time.

Although the results presented along this document concern more stability analysis than stabilization, in the sense that no explicit design of control law is presented, they still constitute a prescriptive framework on which one could base control design strategies.

1.1 Stability of the origin

Many control applications can be formulated as a stabilization problem of the origin of a dynamical system. Typically, one requires that the error between the desired behavior of the system and the actual one converges to zero, leading to the notion of *attractivity* of the origin. In addition, it is usually required that, provided sufficiently small initial errors,

the difference between the desired behavior and the actual one remains arbitrarily small at all time: in other words, *stability* of the origin is desired.

We briefly recall these notions in a nonlinear time-varying context. In this section, $\mathcal{I}$ denote a closed (but not necessarily bounded) subset of $\mathbb{R}^n$ that contains the origin. Please refer to the Notation part (p. 19) for a definition of the mathematical concepts used here.

We start with the notion of uniform boundedness of solutions.

Definition 1.1 (UB/UGB) *Let $\mathcal{I}$ be a closed subset of $\mathbb{R}^n$. The solutions of (1.1) are said to be* uniformly bounded on $\mathcal{I}$ *if, for any nonnegative constant r, there exists a nonnegative $c(r)$ such that, for all $t_0 \in \mathbb{R}_{\geq 0}$, they satisfy*

$$x_0 \in \mathcal{I} \cap \mathcal{B}_r \quad \Rightarrow \quad |\phi(t, t_0, x_0)| \leq c\,, \qquad \forall t \geq t_0\,.$$

If $\mathcal{I} = \mathbb{R}^n$, then the solutions are uniformly globally bounded.

Based on this, we can introduce a precise definition of the stability concept that will be used throughout the document.

Definition 1.2 (US/UGS) *Let $\mathcal{I}$ be a closed subset of $\mathbb{R}^n$. The origin of (1.1) is said to be* uniformly stable on $\mathcal{I}$ *if its solutions are uniformly bounded on $\mathcal{I}$ and, given any positive constant ε, there exists a positive $\delta(\varepsilon)$ such that, for all $t_0 \in \mathbb{R}_{\geq 0}$, the solution of (1.1) satisfies*

$$|x_0| \leq \delta \quad \Rightarrow \quad |\phi(t, t_0, x_0)| \leq \varepsilon\,, \qquad \forall t \geq t_0\,. \tag{1.2}$$

If $\mathcal{I} = \mathbb{R}^n$, then the origin is uniformly globally stable.

Strictly speaking, stability of the origin is a purely local concept which is summarized by (1.2). In many applications, it is also interesting to know a domain from which solutions remain bounded, which explains why a boundedness requirement is imposed in the above definition.

Next, we recall the notion of attractivity of the origin.

Definition 1.3 (UA/UGA) *Let $\mathcal{I}$ be a closed subset of $\mathbb{R}^n$. The origin of (1.1) is said to be* uniformly attractive on $\mathcal{I}$ *if, for all positive numbers r and ε, there exists a positive time $T(r, \varepsilon)$ such that, for all $x_0 \in \mathcal{B}_r \cap \mathcal{I}$ and all $t_0 \in \mathbb{R}_{\geq 0}$, the solution of (1.1) satisfies*

$$|\phi(t, t_0, x_0)| < \varepsilon\,, \qquad \forall t \geq t_0 + T\,.$$

If $\mathcal{I} = \mathbb{R}^n$, then the origin is uniformly globally attractive.

When the two latter properties are combined, the resulting property is called uniform asymptotic stability.

Definition 1.4 (UAS/UGAS) *The origin of (1.1) is said to be* uniformly globally asymptotically stable on $\mathcal{I}$ *if it is both uniformly stable and uniformly attractive on $\mathcal{I}$. If $\mathcal{I} = \mathbb{R}^n$, then the origin is* uniformly globally asymptotically stable.

The "uniformity" requirement in the above definitions refers to the initial time. It corresponds to the independence of δ and T in t_0. In other words: no matter at what time the system's trajectories start, convergence-rate to zero and overshoot remain unchanged.

The importance of uniformity. Uniformity is a crucial property of time-varying systems, as it provides a certain robustness with respect to external disturbances. More precisely, as more detailed in [LLLP05, Section 2.1], it can be shown that the uniform asymptotic stability of the origin of a zero-input system $\dot{x} = f(t, x, 0)$ ensures "stability with respect to constantly acting disturbances" of $\dot{x} = f(t, x, u)$, where u denotes an external signal[1], provided that the $f(t, x, u)$ is locally Lipschitz in x uniformly in t. This concept, also known as *total stability*, was introduced by Malkin, *cf. e.g.* [Mal58]. It states that the trajectories remain arbitrarily small at all time if the initial state and the input signal are sufficiently small. More precisely, it is defined as follows.

Definition 1.5 (Total stability) *The origin of $\dot{x} = f(t, x, 0)$ is said to be* totally stable *if, for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that, for all $t_0 \in \mathbb{R}_{\geq 0}$, the solution of $\dot{x} = f(t, x, u)$ satisfies*

$$\max\{|x_0|, \|u\|\} \leq \delta \quad \Rightarrow \quad |\phi(t, t_0, x_0, u)| \leq \varepsilon\,, \qquad \forall t \geq t_0\,.$$

In a nutshell, by establishing *uniform* asymptotic stability, we guarantee that the behavior of the system is not too much altered by the presence of sufficiently small external disturbances. This robustness property does *not* hold for non-uniform properties, as illustrated by [LLLP05, Example 2.1, p. 28] in which a simple scalar time-varying system is exhibited with the following properties: $\dot{x} = f(t, x, 0)$ is globally asymptotically stable (but not uniformly), nevertheless one can design an arbitrarily small perturbation u in such a way that $\dot{x} = f(t, x, u)$ generates unbounded solutions.

Larger perturbations. While uniform asymptotic stability thus ensures a natural robustness to small external disturbances, it provides no information on the behavior of the system subject to larger perturbations. In stability analysis, it is classical to observe that the presence of a bounded non-vanishing disturbance impedes asymptotic stability, yielding instead the convergence to a (possibly large) neighborhood of the operating point. This property is referred to as *ultimate boundedness*, *cf. e.g.* [Kha01, Yos66].

Definition 1.6 (Ultimate boundedness) *The solutions of $\dot{x} = f(t, x)$ are said to be* uniformly ultimately bounded *if there exist positive constants Δ_0 and c such that, for every $\Delta \in (0; \Delta_0)$, there exists a positive constant $T(\Delta)$ such that, for all $x_0 \in \mathcal{B}_\Delta$ and all $t_0 \in \mathbb{R}_{\geq 0}$, they satisfy*

$$|\phi(t, t_0, x_0)| \leq c\,, \qquad \forall t \geq t_0 + T\,.$$

If this holds for arbitrarily large Δ, then the solutions are globally uniformly ultimately bounded.

In many situations, this property is not enough to ensure correct performances. Indeed, we see that uniform ultimate boundedness is only concerned with the behavior of the system after a sufficiently long time and, hence, does not take into account the transient dynamics. In addition, the domain to which solutions converge may be large, then preventing a good precision.

The aim of the following sections is to introduce stability properties that may help guaranteeing stronger features to perturbed systems.

[1] $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ may consist in any measurable locally essentially bounded function.

1.2 Stability of sets

In the above definition of ultimate boundedness, solutions are required to converge to some ball, of radius c, centered at the origin. It is natural to extend this property to more general sets. In addition, it is interesting to constrain the behavior of the system during the transients, in order to avoid disproportioned overshoots. This motivated the introduction of set-stability [Zub57, HP73].

A general concept. The analysis of set-stability is very general and consequently very common in control practice. This ensues from the fact that the set under consideration may consist in a single operating point (then corresponding to Definitions 1.2, 1.3 and 1.4), a path, or a more complex, possibly unbounded, region of the state-space.

As it will appear more clearly in the following definitions, stability (and, similarly, attractivity) of a single operating point x^* is obtained by considering the set $\{x^*\}$. In this respect, we always consider that the reference point x^* is the origin. This can be assumed without loss of generality, since, if x^* is an equilibrium for (1.1), then 0 is an equilibrium for $\dot{z} = g(t,z) := f(t, z + x^*)$ with the coordinate change $z := x - x^*$.

In the same way, stability of a path may be considered by choosing the set containing all the points of this path.

In the case when the application does not require convergence to the origin but just to a small neighborhood of it, it is appropriate to consider the set as a ball of small radius centered at zero. This allows to define a rigorous formulation of the problems for which a steady-state error is tolerated, and is also at the basis of practical stability as we will see in the next section.

The set may also be decomposed as $\mathbb{R}^{n'} \times \{0\}$, with $n' \in \mathbb{N}_{<n}$, when only part of the state is required to be stable. We refer to this property as *partial stability*, *cf.* [Vor98]. Many applications indeed require the convergence of a reduced number of variables to operate correctly. This concept has also proved useful in presence of superfluous states, or when the plant is inherently unstable with respect to part of the states. See Chapter 4 for details.

The following stability definitions should therefore be seen as general statements, from which all these "particular" cases may be derived.

When dealing with set stability, special attention has to be paid to the existence of solutions for all positive time. $\mathcal{A}$ and $\mathcal{I}$ denoting two closed (but not necessarily bounded) sets of $\mathbb{R}^n$ that contain the origin[2], we therefore start by recalling the notion of forward completeness. Please see [AS99] for a Lyapunov characterization of this property.

Definition 1.7 (Forward completeness) *The system (1.1) is said to be* forward complete *on $\mathcal{I}$ if, for all $x_0 \in \mathcal{I}$ and all $t_0 \in \mathbb{R}_{\geq 0}$, its solution $\phi(t, t_0, x_0)$ est définie pour tout $t \geq t_0$.*

Based on this, we can extend Definition 1.1 to the case when we are not interested in a boundedness of the distance of the solutions from the origin, but from a given closed (not necessarily compact) set $\mathcal{A}$.

[2]This assumption, which can be made without loss of generality, is imposed in order to ensure that $|\cdot|_{\mathcal{A}} \leq |\cdot|$.

Definition 1.8 (UB/UGB with respect to a set) *The solutions of (1.1) are said to be* uniformly bounded on $\mathcal{I}$ with respect to $\mathcal{A}$ *if (1.1) is forward complete on $\mathcal{I}$ and, for any nonnegative constant r, there exists a nonnegative $c(r)$ such that, for all $t_0 \in \mathbb{R}_{\geq 0}$, they satisfy*

$$x_0 \in \mathcal{I} \cap \mathcal{B}_r \quad \Rightarrow \quad |\phi(t, t_0, x_0)|_{\mathcal{A}} \leq c\,, \qquad \forall t \geq t_0\,.$$

If $\mathcal{I} = \mathbb{R}^n$, then the solutions are uniformly globally bounded with respect to $\mathcal{A}$. *Furthermore, for the case that $\mathcal{A} = \{0\}$ and $\mathcal{I} = \mathbb{R}^n$ we simply say, with a slight abuse of terminology, that the solutions of (1.1) are* uniformly globally bounded.

In the case when $\mathcal{A} = \{0\}$, we recover uniform boundedness as introduced in Definition 1.1. We see that an additional requirement, namely forward completeness, is imposed in the above definition. As the set $\mathcal{A}$ may be unbounded, trajectories may explode in finite time while the quantity $|\phi(t, t_0, x_0)|_{\mathcal{A}}$ remains bounded at all time. Assuming forward completeness excludes this possibility. It should be stressed that, in the case when $\mathcal{A}$ is a compact set, this additional requirement is not needed anymore. These remarks hold as well for the next three definitions.

Definition 1.9 (US/UGS of a set) *Assume that (1.1) is forward complete on $\mathcal{I}$. The set $\mathcal{A}$ is said to be* uniformly stable on $\mathcal{I}$ *for (1.1) if the solutions of the latter are uniformly bounded on $\mathcal{I}$ with respect to $\mathcal{A}$ and, given any positive constant ε, there exists a positive $\delta(\varepsilon)$ such that, for all $t_0 \in \mathbb{R}_{\geq 0}$, the solution of (1.1) satisfies*

$$|x_0| \leq \delta \quad \Rightarrow \quad |\phi(t, t_0, x_0)|_{\mathcal{A}} \leq \varepsilon\,, \qquad \forall t \geq t_0\,.$$

If $\mathcal{I} = \mathbb{R}^n$, then the set $\mathcal{A}$ is uniformly globally stable.

Definition 1.10 (UA/UGA of a set) *Assume that (1.1) is forward complete on $\mathcal{I}$. The set $\mathcal{A}$ is said to be* uniformly attractive on $\mathcal{I}$ *for (1.1) if, for all positive numbers r and ε, there exists a positive time $T(r, \varepsilon)$ such that, for all $x_0 \in \mathcal{B}_r \cap \mathcal{I}$ and all $t_0 \in \mathbb{R}_{\geq 0}$, the solution of (1.1) satisfies*

$$|\phi(t, t_0, x_0)|_{\mathcal{A}} < \varepsilon\,, \qquad \forall t \geq t_0 + T\,.$$

If $\mathcal{I} = \mathbb{R}^n$, then the set $\mathcal{A}$ is uniformly globally attractive.

Definition 1.11 (UAS/UGAS of a set) *Assume that (1.1) is forward complete on $\mathcal{I}$. The set $\mathcal{A}$ is said to be* uniformly globally asymptotically stable on $\mathcal{I}$ *for (1.1) if it is both uniformly stable and uniformly attractive on $\mathcal{I}$. If $\mathcal{I} = \mathbb{R}^n$, then the set $\mathcal{A}$ is* Uniformly Globally Asymptotically Stable.

Two measures. It is worth pointing out that these definitions care special cases of stability *with respect to two measures*, *cf.* [Mov60, LL93]. This concept is very general and includes, as we have seen, stability of a single point, of a compact set, of a prescribed trajectory as well as partial stability [Vor98, Vor02]. It was used in *e.g.* [LS76, TP00, Lee04]. Here, the first measure is the distance to the set under consideration $|\cdot|_{\mathcal{A}}$, while the second is the Euclidean norm $|\cdot|$. As we will see later (see Section 2.1), for perturbed systems or when dealing with adaptive control, this choice allows, in many situations, to use the *same* Lyapunov function as the nominal system, which makes this stability property much easier to establish and to use.

In this respect, we stress that the term *uniform* used in the above definitions concerns only the dependence in the initial time. More precisely, the constants c, δ and T in Definitions 1.8, 1.9 and 1.10 are all required to be independent of t_0. Other existing results in the literature (*e.g.* [Yos66, LSW96, TPL02]) use this terminology to underline that the set-stability is defined with the same measure, notably implying that the set $\mathcal{A}$ is positively invariant, which is not the case here.

As in the spirit of Hahn's formulations [Hah63] of stability in terms of $\mathcal{K}$ and $\mathcal{KL}$ estimates (see also [Son98a]), the properties defined above can be written in the following precise way.

Proposition 1.12 ($\mathcal{K}$ characterization of UB/UGB) *Assume that (1.1) is forward complete on $\mathcal{I}$. The solutions of (1.1) are uniformly bounded on $\mathcal{I}$ (resp. uniformly globally bounded) with respect to $\mathcal{A}$ if and only if there exists a class $\mathcal{K}$ function η and a nonnegative constant μ such that, for any $x_0 \in \mathcal{I}$ (resp. $x_0 \in \mathbb{R}^n$) and any $t_0 \in \mathbb{R}_{\geq 0}$, the solution of (1.1) satisfies*

$$|\phi(t, t_0, x_0)|_{\mathcal{A}} \leq \eta(|x_0|) + \mu\,, \qquad \forall t \geq t_0\,.$$

Proposition 1.13 ($\mathcal{K}$ characterization of US/UGS) *Assume that (1.1) is forward complete on $\mathcal{I}$. A closed set $\mathcal{A}$ is uniformly stable on $\mathcal{I}$ (resp. uniformly globally stable) for (1.1) if and only if there exists a class $\mathcal{K}$ function γ such that, for any $x_0 \in \mathcal{I}$ (resp. $x_0 \in \mathbb{R}^n$) and any $t_0 \in \mathbb{R}_{\geq 0}$, the solution of (1.1) satisfies*

$$|\phi(t, t_0, x_0)|_{\mathcal{A}} \leq \gamma(|x_0|)\,, \qquad \forall t \geq t_0\,.$$

Proposition 1.14 ($\mathcal{KL}$ characterization of UAS/UGAS) *Assume that (1.1) is forward complete on $\mathcal{I}$. A closed set $\mathcal{A}$ is uniformly asymptotically stable on $\mathcal{I}$ (resp. uniformly globally asymptotically stable) if and only if there exists a class $\mathcal{KL}$ function β such that, for all $x_0 \in \mathcal{I}$ (resp. $x_0 \in \mathbb{R}^n$) and all $t_0 \in \mathbb{R}_{\geq 0}$, the solution of (1.1) satisfies*

$$|\phi(t, t_0, x_0)|_{\mathcal{A}} \leq \beta(|x_0|\,, t - t_0)\,, \qquad \forall t \geq t_0\,.$$

The proof of these characterizations follows along the same lines as [Vid93, Theorems 53 and 61], we therefore do not recall them here.

When the convergence rate to the set $\mathcal{A}$ is exponential and the dependence in the initial state is linear, the stability is said to be exponential.

Definition 1.15 (UES/UGES of a set) *If, in Proposition 1.14, the class $\mathcal{KL}$ function can be picked as*

$$\beta(s, t) = k_1 s e^{-k_2 t}\,, \qquad \forall s, t \in \mathbb{R}_{\geq 0}$$

for some positive constants k_1 and k_2, then the set $\mathcal{A}$ is said to be uniformly exponentially stable on $\mathcal{I}$ *(resp.* uniformly globally exponentially stable*) with parameters (k_1, k_2).*

For the study of the alteration of a stability property under the influence disturbances, a noteworthy particular case of the above definitions is when the sets under consideration are closed balls. It is indeed at the basis of all the definitions of semiglobal and practical stability properties introduced next. The following proposition, that follows from Propositions 1.12 and 1.14, establishes a strong link of this concept with the $(\sigma \to \rho)$-stability originally introduced in [TPA99] and recalled in Definition A.1 (p. 172).

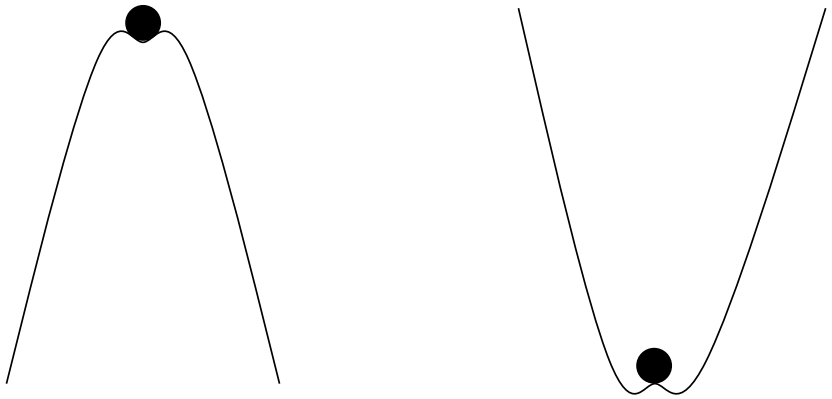
Proposition 1.16 (UAS and $\sigma \to \rho$ stability) *Let $\Delta > \delta > 0$. Then the following implications hold:*

- *If $\mathcal{B}_\delta$ is UAS on $\mathcal{B}_\Delta$, then (1.1) is $(\Delta \to \delta)$-stable;*
- *If (1.1) is $(\Delta \to \delta)$-stable, then $\mathcal{B}_\delta$ is UAS on $\mathcal{B}_{\Delta'}$, for all $\Delta' \in (\delta, \Delta)$.*

The proof of this proposition is detailed in Section A.2. We can notice that no forward completeness assumption is needed anymore as the set under consideration, namely $\mathcal{B}_\delta$, is compact. In this case, uniform asymptotic stability naturally ensures the existence of solutions for all forward time.

1.3 Semiglobal and practical asymptotic stability

The need of a finer analysis. As already pointed out by Hahn in [Hah63] and by La Salle and Lefschetz in [SL61], practical considerations should be taken into account when studying the asymptotic stability of the equilibrium of a given plant. To quote an example of the latter reference, the asymptotic stability of an electrical system operating at 110 V ensures that small variations will be cancelled out. However, if the amplitude of these tolerated variations is tool small, say of some millivolts, the system may not operate correctly. On the opposite, the operating point of a given system may be mathematically unstable, thus generating small oscillations around it, but still guarantee a sufficient precision for an acceptable behavior. Using the intuitive illustration, already used in Figure 1.1, of a ball on a non-flat surface, these would correspond to the following situations:



Asymptotic stability with a small domain of attraction

Instability with a small steady-state error

Figure 1.2: Practical considerations about stability.

A tighter analysis is then capital.

Steady-state errors and restricted domain of attraction. As already noted, non-vanishing perturbations acting on the plant or measurement imprecisions may impede the convergence to the origin by yielding a steady-state error. In the same way, it is often the case that some neglected high-order nonlinearity in the dynamics prevent *global* stability, generating instead an unbounded basin of attraction. In each of these situations, can we expect more than *local* stability and ultimate boundedness ?

In the stability analysis of closed-loop systems, but also in some contexts that are developed later (such as averaging techniques or output feedback control; see Chapter 2), the tuning of some free parameters (typically control gains) often allow to arbitrarily enlarge the domain of attraction, or to diminish at will the magnitude of the steady-state errors. These properties are respectively referred to as *semiglobal* and *practical* stability.

In more formal terms, semiglobal and practical stability properties pertain to parameterized nonlinear time-varying systems of the form

$$\dot{x} = f(t, x, \theta)\,, \tag{1.3}$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}_{\geq 0}$, $\theta \in \mathbb{R}^m$ is a constant parameter and $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz in x and satisfies Carathéodory conditions for any parameter θ under consideration.

Definition 1.17 (USAS) *Let $\Theta \subset \mathbb{R}^m$ be a set of parameters. The system (1.3) is said to be* uniformly semiglobally asymptotically stable on Θ *if, given any $\Delta > 0$, there exists $\theta^\star(\Delta) \in \Theta$ such that the origin is uniformly asymptotically stable on $\mathcal{B}_\Delta$ for the system $\dot{x} = f(t, x, \theta^\star)$.*

Definition 1.18 (UGPAS) *Let $\Theta \subset \mathbb{R}^m$ be a set of parameters. The system (1.3) is said to be* uniformly globally practically asymptotically stable on Θ *if, given any $\delta > 0$, there exists $\theta^\star(\delta) \in \Theta$ such that the ball $\mathcal{B}_\delta$ is uniformly globally asymptotically stable for the system $\dot{x} = f(t, x, \theta^\star)$.*

Definition 1.19 (USPAS) *Let $\Theta \subset \mathbb{R}^m$ be a set of parameters. The system (1.3) is said to be* uniformly semiglobally practically asymptotically stable on Θ *if, given any $\Delta > \delta > 0$, there exists $\theta^\star(\delta, \Delta) \in \Theta$ such that the ball $\mathcal{B}_\delta$ is uniformly asymptotically stable on $\mathcal{B}_\Delta$ for the system $\dot{x} = f(t, x, \theta^\star)$.*

In the above definitions, θ represents the tuning parameter, *e.g.* control gains or any free design parameter. Θ is the set of allowed tuning parameters, which may be bounded due to physical constraints such as limitation of the output of actuators. Δ can be seen as the radius of the estimate of the domain of attraction; in most applications, a larger Δ induces better performance since the operating bandwidth is enlarged. In contrast, δ represents the radius of the ball to which solutions ultimately converge; therefore it is typically required to be small, in order to reduce the steady-state error as much as possible.

Practical stability and ultimate boundedness. As it is further discussed in the sequel (see Chapter 2), practical stability shares similarities with the classical *ultimate boundedness* property (*cf.* Definition 1.6), in the sense that solutions eventually reach a neighborhood of the operating point. It should however be clear to the reader that the above Definitions 1.18 and 1.19 are usually more interesting in practice, as they require the size of this neighborhood to be reducible at will by an adequate tuning and as they require the ball $\mathcal{B}_\delta$ not only to be attractive but also stable (in the sense of Definition **??**).

We also stress that Definitions 1.18 and 1.19 do not require the origin to be an equilibrium for the system (1.3). This indeed fails for many practically stable systems as, for instance, Examples 2.2 and 2.8 given below.

In view of Proposition 1.14, USPAS can be expressed in terms of $\mathcal{KL}$ estimates.

Proposition 1.20 ($\mathcal{KL}$ characterization of USPAS) *The system $\dot{x} = f(t, x, \theta)$ introduced in (1.3) is uniformly semiglobally asymptotically practically stable if and only if, for all positive constants δ and Δ such that $\Delta > \delta$, there exists a parameter $\theta^\star(\delta, \Delta) \in \Theta$ and a class $\mathcal{KL}$ function $\beta_{\delta,\Delta}$ such that, for all $x_0 \in \mathcal{B}_\Delta$ and all $t_0 \in \mathbb{R}_{\geq 0}$, the solution of $\dot{x} = f(t, x, \theta^\star)$ satisfies*

$$|\phi(t, t_0, x_0, \theta^\star)|_\delta \leq \beta_{\delta,\Delta}(|x_0|, t - t_0), \qquad \forall t \geq t_0 .$$

A changing $\mathcal{KL}$ estimate. We stress that the function β is not required to be independent of δ and Δ. Typically, this dependence in the the size of the domain of attraction and the size of the ball to which solutions converge steps through the dependence in the tuning parameter θ. In order to reach some given δ and Δ, it is indeed usually necessary to choose a convenient parameter θ. However, as the dynamics of the system depends on θ, it may happen that the convergence rate as well as the dependence in the initial state (in a word, the $\mathcal{KL}$ estimate β) is affected accordingly. For instance, in mechanical systems, it is a classical phenomenon that, for a given initial condition (t_0, x_0), one observes a larger overshoot when enlarging the control gains in order to diminish the steady-state error. See Example 2.2 for a simple illustration of this phenomenon.

Many definitions of semiglobal and/or practical stability existing in the literature do impose that the $\mathcal{KL}$ estimate be uniform in δ and Δ. See for instance [TPA99, NL04, ST03][3]. As it will be more detailed in the sequel (see Chapter 2), this non-uniformity constitutes a crucial difference with those references: while the above (less conservative) definitions allow to treat a much wider class of systems, they impose a more involved Lyapunov stability analysis. Of course, the natural counterpart of this generality of this notion is that the latter is weaker than the above cited definitions. Yet, it guarantees interesting properties to the system in terms of overshoot, convergence, robustness and precision.

Frequent properties. Semiglobal and/or practical stability properties appear in various situations. An intuitive one is the degraded functioning of a plant due to neglected dynamics, external perturbations, inadequacy with the model, *etc.* A controlled system for which the operating point is uniformly *globally* asymptotically stable typically presents a property of uniform *semiglobal* asymptotic stability in presence of high order nonlinearities, or may be uniformly globally *practically* asymptotically stable if some external non-vanishing perturbations act on it. These common situations are illustrated by elementary examples in Chapter 2.

Although these do not constitute the only occasions to observe semiglobal and/or practical stability (we also could have cited averaging techniques, output feedback control or discrete-time systems; see Chapter 2 for a more exhaustive description), they show that the properties of UGPAS, USAS and USPAS can be seen as measures of the *robustness* of a nonlinear time-varying system to model approximations, imprecisions, external disturbances, *etc.*

[3] Although [TPA99, Definition 3] does not impose this uniformity, the main result in that reference does ensure such a feature.

1.4 Input to State Stability

ISS. For the specific study of robustness with respect to disturbances, a particularly fitting framework is input to state stability (ISS). In informal terms, the ISS property introduced by Sontag in [Son89a] (see also [Son95, Son05a] for surveys on this notion) imposes that the norm of the state at the current time be bounded by a function of the amplitude of the external input plus a fading term in the initial state. This paradigm allows to take into account two phenomena: 1) the state will eventually converge to any arbitrarily small neighborhood of the origin provided that the amplitude of the input is sufficiently small, 2) if the input is null at all time, then the origin of the system is globally asymptotically stable.

iISS. Even though this property has been widely used both in analysis and design, *cf. e.g.* [Son98a, KKK95, JM97, PW96], ISS happens to be too strong a requirement in several cases. This motivated the introduction of Integral Input to State Stability (iISS) [Son98b], which turns out to be a much weaker property. Instead of linking the state to the supremum of the input, it involves a measure of the energy that inputs feed into the system. Similarly to ISS, it ensures global asymptotic stability for the zero-input system and guarantees some robustness to the system with respect to external inputs. For instance, it is shown in [Son98b] that, when the energy of the perturbing input is finite, the asymptotic behavior of the solutions of an iISS system is not affected.

Both the ISS and iISS properties were originally introduced in a time-invariant context, *cf.* [Son89a] and [Son98b] respectively. Although some extensions were made to generalize to time-varying systems (see *e.g.* [Lin96, ELW00, LWC05, MM05]), most of the existing tools that help guaranteeing ISS, and especially iISS, remain limited to autonomous systems. For this reason, the following definitions related to iISS are given by considering systems of the form:

$$\dot{x} = f(x, u) \tag{1.4}$$

where $x \in \mathbb{R}^n$ denotes the state and $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ denotes a locally Lipschitz function. Input signals $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$ may consist in any measurable locally essentially bounded functions.

We give the precise definitions of ISS and iISS below.

Definition 1.21 (ISS) *We say that (1.4) is* input to state stable *if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}_\infty$ function γ such that, for all $x_0 \in \mathbb{R}^n$ and any admissible input u, the solution of (1.4) satisfies*

$$|\phi(t, x_0, u)| \leq \beta(|x_0|, t) + \gamma(\|u\|) , \qquad \forall t \in \mathbb{R}_{\geq 0} . \tag{1.5}$$

The function γ is then referred to as an ISS gain *for (1.4).*

Definition 1.22 (iISS) *We say that (1.4) is* integral input to state stable *if there exist a class $\mathcal{KL}$ function β and class $\mathcal{K}_\infty$ functions γ and μ such that for all $x_0 \in \mathbb{R}^n$ and any admissible input u, the solution of (1.4) satisfies*

$$|\phi(t, x_0, u)| \leq \beta(|x_0|, t) + \gamma\left(\int_0^t \mu(|u(\tau)|) d\tau\right) , \qquad \forall t \in \mathbb{R}_{\geq 0} .$$

The function μ is then referred to as an iISS gain *for (1.4).*

Based on these two formulations, it is easy to see that any ISS or iISS system is globally asymptotically stable when the input is zero at all time. This zero-input system can be viewed as the "nominal" system, and the above properties then give a clear measure of its robustness with respect to external disturbances (their amplitude or their energy according to the case).

ISS and practical stability. Compared to the practical stability property introduced in the previous section, it is worth underlining that both ISS and iISS ensure that a ball, whose radius is directly related to the input, is globally asymptotically stable. However, for a given input signal, this ball is not required to be reducible at will by a convenient tuning of some parameter. For the same reason, we stress that the notion of input to state *practical* stability (ISpS) introduced in [JTP94] should not be understood in the sense of the term "practical" of Definitions 1.18 and 1.19. ISpS differs from ISS in that, instead of the origin itself, a ball centered at it is required to be globally asymptotically stable. More precisely, it corresponds to Definition 1.21 where (1.5) is replaced by

$$|\phi(t, x_0, u)| \leq \beta(|x_0|, t) + \gamma(\|u\|) + \delta\,, \qquad \forall t \in \mathbb{R}_{\geq 0}\,,$$

δ denoting a nonnegative constant. In this context, δ is fixed an not reducible at will by tuning the system's parameters.

In the same way, note that the concept of *semiglobal* integral input to state stability introduced in [ASW00b], where the $\mathcal{KL}$ estimate and the iISS gain are not required to hold over $\mathbb{R}^n$ but only on arbitrarily large compact sets, is different from the "semiglobality" of Definitions 1.17 and 1.19. See Chapters 2 and 5 for a more detailed comparison of these notions.

Chapter 2

Semiglobal and practical asymptotic stability

Natural stability properties. As evoked in Chapter 1, the opportunities of encountering semiglobal and/or practical stability are numerous in control practice. Before presenting necessary and sufficient conditions for it to hold, in terms of Lyapunov functions, we would like to go back on the motivations for this study.

To this end, we start by introducing the following elementary example, which will illustrate the topic along the chapter and should give a more intuitive understanding of the semiglobal practical stability property, in its full generality, and the ways to establish it. Although simple, this example illustrates many properties and difficulties involved in such a stability analysis.

Example 2.1 *Consider the second-order scalar dynamical system described by*

$$\ddot{q} = -\theta_1 q - \theta_2 \dot{q} + b(t, q, \dot{q}) + c(t, q, \dot{q})\dot{q}^2 \tag{2.1}$$

where θ_1 and θ_2 are free control gains, and b and c are locally Lipschitz functions satisfying, for all $q, \dot{q} \in \mathbb{R}$ and all $t \in \mathbb{R}_{\geq 0}$,

$$|b(t, q, \dot{q})| \leq \bar{b}\,, \qquad |c(t, q, \dot{q})| \leq \bar{c}$$

with some nonnegative constants $\bar{b}$ and $\bar{c}$. In the case that b and c are identically zero, the system reduces to a Hurwitz linear system and global exponential stability follows for any positive choice of θ_1 and θ_2. However, in presence of the term b, we see that if the state $(q, \dot{q})$ is small, then the dynamics are predominantly dictated by b. This may yield to undesirable behavior as solutions approach the origin. Similarly, if c is non zero, then the term $c(t, q, \dot{q})\dot{q}^2$ prevails when $\dot{q}$ is large, potentially yielding a restriction of the domain of attraction. Intuitively, we can expect that, by enlarging the gains θ_1 and θ_2, we limit the domination of the b term to very small values of the state and the domination of c to very large values. Accordingly, we would then obtain that the magnitude of the steady-state errors can be reduced at will by choosing sufficiently large gains and, in the same way, that we can arbitrarily enlarge the domain of attraction. This is respectively what is meant by practical *and* semiglobal *stability.* □

Through this intuitive example, we see that perturbations may degrade the performance of a system, especially by reducing the operating bandwidth and generating a steady-state

error. But it also suggests that, in some situations, the effects of these degradations can be made acceptable when some freedom is available on a parameter.

Perturbations of this type may have different sources. Most commonly, asymptotic stability may yield practical stability in presence of a non-vanishing external signal, such as noise. For instance, this may be the result of imperfections in the measurement or in the actuation. This would correspond to the time-dependence of the b term in the above example. The convergence to the origin may also be impeded by the use of saturated control (*cf.* p. 85), the presence of delay [LZ01], *etc.* Neglected dynamics, high-order nonlinearities, model uncertainty, bad knowledge of some parameters, physical constraints, *etc.* may also prevent global asymptotic stability while still allowing semiglobal and/or practical stability.

However, degradation of nominal performance does not constitute the only occasion of encountering uniform semiglobal practical asymptotic stability (USPAS). It may also be concluded from stability analysis tools existing in the literature. USPAS of a nonlinear time-varying system follows for instance from averaging techniques. In [TPA99], it is shown that, if the averaged system of $\dot{x} = f(t, x)$ is globally asymptotically stable, then the system $\dot{x} = f(t/\theta, x)$ is USPAS on the parameter set $\mathbb{R}_{>0}$, implying that, from any compact set of initial condition, it suffices to pick θ small enough for the solutions to converge to any arbitrarily small neighborhood of the origin.

Existing variants. We stress that the term "practical stability" has many variants in the literature of control theory. In many situations, the ball to which solutions converge is not required to be arbitrarily reducible. This is the case in [MP72, Kap73, ZM03], where it is only imposed that any solution starting in a ball never leaves another ball. The notion of input to state practical stability originally introduced in [JTP94] imposes a fading dependence in the initial state, but does not require the attractive neighborhood to be reducible at will.

On the other hand, this term may also denote more conservative properties than that of Definition 1.18, as they require that the $\mathcal{KL}$ estimate, or at least its dependence in the initial state, be the same for all parameters[1] $\theta \in \Theta$. While the latter property is satisfied in many contexts (see *e.g.* [MA00, TPA99, NL04, ST03, TNM05]), it may fail when dealing with perturbed systems: see the example below. In this respect, we stress that, in Definitions 1.17, 1.18 and 1.19, "uniform" refers only to the initial conditions, and not to the tuning parameter.

Similarly to practical stability, the names given in the literature to what we call here semiglobal stability properties vary a lot. It is, for instance, referred to as *potentially global stabilizability* in [Bac86]. Some authors, as for instance [BI91, Hu96], also use the denomination *on compacta stabilizability*.

Stability and stabilizability. It should be underlined that many authors use the terminology "stabilizability" instead of "stability" when dealing with practical or semiglobal properties, see *e.g.* [Bac86, Sus90, TP95, JB96, MS03]. In general terms, these references deal with the problem of finding a control input that makes the solutions converge, in a stable way, to an arbitrarily small neighborhood of the operating point from an arbitrarily large given set of initial states. The system under consideration is consequently a *controlled* system, and not a parameterized system as (1.3). The difference between these two

[1] In other words, the overshoot may depend on the chosen parameter θ, but not the convergence rate.

concepts is slight. They may even coincide on some occasions, as for instance in [Bac86], where the control is *a priori* sought in the form $u = Kx$. It is therefore a stabilizability problem, but the gain matrix K can also be seen as a tuning parameter for the closed-loop system, allowing to address the question in terms of practical *stability*.

The occasions of guaranteeing semiglobal stabilizability are numerous. It is shown in [TP94] that smooth global stabilizability and complete uniform observability (meaning, roughly, that the state may be reconstructed based on the instantaneous knowledge of the input, the output, and a finite number of their derivatives) imply semiglobal stabilizability by *output feedback*. In robust control with respect to model uncertainty, semiglobal stabilizability in ensured for a class of systems by the approach of [Isi97]. Semiglobal asymptotic stability may also arise by the use saturated controls: see [ARKC03] for an example in robotics.

The following example provides a rigorous proof of the intuitive reasoning proposed in Example 2.1. In particular, we give a $\mathcal{KL}$ estimate which is *not* uniform in the tuning parameter θ.

Example 2.2 *Inspired by the peaking phenomenon example in [SK91], we consider the system (2.1) of Example 2.1, with $\theta_1 = \theta^2$, $\theta_2 = 2\theta$, $b(t, q, \dot{q}) = 1$ and $c(t, q, \dot{q}) = 0$ for all $t \in \mathbb{R}_{\geq 0}$ and all $q, \dot{q} \in \mathbb{R}$, θ denoting a positive free parameter. In other words:*

$$\begin{pmatrix} \dot{q} \\ \ddot{q} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\theta^2 & -2\theta \end{pmatrix} \begin{pmatrix} q \\ \dot{q} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{2.2}$$

Let $x := (q, \dot{q})^\top$. Given any initial conditions $x_0 = (q_0, \dot{q}_0)^\top$, the solution of this linear time-invariant system is

$$\phi(t, x_0, \theta) := \begin{pmatrix} q(t, x_0, \theta) \\ \dot{q}(t, x_0, \theta) \end{pmatrix} = \begin{pmatrix} \left[q_0 - \frac{1}{\theta^2} + \left(\theta q_0 + \dot{q}_0 - \frac{1}{\theta}\right) t\right] e^{-\theta t} + \frac{1}{\theta^2} \\ \left[\dot{q}_0 + \left(1 - \theta^2 q_0 - \theta \dot{q}_0\right) t\right] e^{-\theta t} \end{pmatrix} . \tag{2.3}$$

Using that $te^{-t} \leq e^{-t/2}$ for all $t \in \mathbb{R}_{\geq 0}$, it can be seen that

$$|\phi(t, x_0, \theta)| \leq 2\left(1 + \theta + \theta^2\right) |x_0| \, e^{-\theta t/2} + \frac{2 + \theta}{\theta^2} .$$

Let δ be any given positive constant and $\theta^\star(\delta)$ be any positive number satisfying $(2 + \theta^\star)/\theta^{\star 2} \leq \delta$. Then, defining $\beta_\delta(s, t) := 2(1 + \theta^\star + \theta^{\star 2}) s e^{-\theta^\star t/2}$ for all $s, t \in \mathbb{R}_{\geq 0}$, we get that

$$|\phi(t, x_0, \theta)| \leq \beta_\delta(|x_0|, t) + \delta , \qquad \forall t \in \mathbb{R}_{\geq 0} .$$

Noticing that β_δ is a $\mathcal{KL}$ function for any positive δ, we conclude, in view of Definition 1.18, that (2.2) is (U)GPAS[2] on the parameter set $\Theta = \mathbb{R}_{>0}$.

Note that the resulting $\mathcal{KL}$ estimate of the solutions depends on the chosen parameter $\theta^\star$ and so, indirectly, on the chosen tolerance δ. Furthermore, it is impossible to find a function β that is the same for all δ. This stems from the term $\theta^2 q_0 t e^{-t}$ in the expression of $\dot{q}(t, x_0, \theta)$ (cf. Equation (2.3)) which, for any positive t and q_0 diverges as θ tends to infinity. Hence, by choosing a smaller δ, we can expect a larger overshoot. This is confirmed by the following plots representing $|\phi(\cdot, x_0, \theta)|$, with $x_0 = (1, 0)^\top$, for $\theta = 2$ (dots), $\theta = 3$ (dashes) and $\theta = 5$ (plain).

□

[2] The initial "U", standing for "uniform", is groundless as the system under consideration is time-invariant.

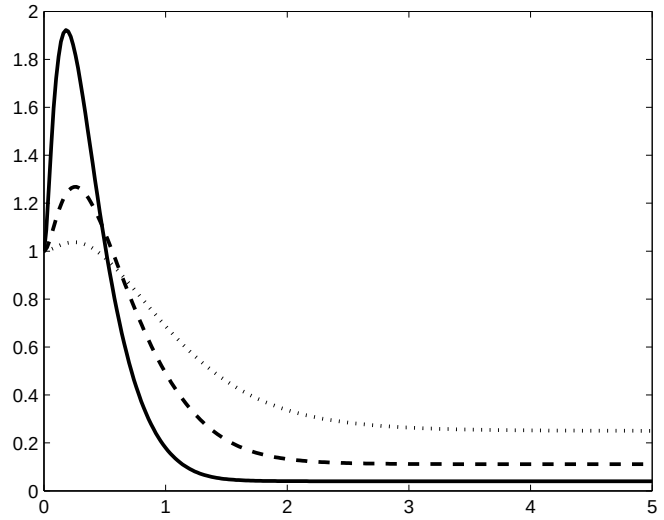


Figure 2.1: Parameter dependence of the $\mathcal{KL}$ estimate.

In the literature, many definitions of semiglobal and/or practical stability impose that the tuning parameter be a (small) positive constant, as for instance [MA00, NL04, TNM05, TPA99]. According to the case, it may for instance consist in a time re-scaling or in the sampling period. In these situations, it is additionally imposed that there exists a value of the parameter such that *every* smaller choice yields to the asymptotic stability of the same ball, with the same $\mathcal{KL}$ estimate. This requirement naturally induces a tuning procedure: in order to converge to a smaller ball, it suffices to pick a smaller parameter. Since our definitions of UGPAS, USAS, and USPAS rely on a possibly non-scalar parameter, no such tuning procedure is offered *in general*. However, in practice, many applications do provide conditions on the parameter to reach a given attractive neighborhood of the origin, or to ensure a given radius of attraction, and therefore suggest a procedure to tune parameters; see Chapter 6 for concrete examples. For instance, in the above example, given a tolerance $\delta > 0$, any parameter θ_1 greater than $\theta^\star(\delta)^2$ and θ_2 greater than $2\theta^\star(\delta)$ ensures that the ball $\mathcal{B}_\delta$ is globally asymptotically stable. So, we directly know that, in order to reach a smaller δ, one should enlarge these two gains.

In a nutshell, the good compromise between generality and strength offered by Definitions 1.18, 1.17 and 1.19 motivated their use. We are next presenting tools that guarantee them in Lyapunov terms.

2.1 Sufficient conditions

Lyapunov's direct method, originally presented in [Lya92], is based on the study of a positive definite continuously differentiable function. If its total derivative along the solutions of the system is non-positive, then stability follows. If this total derivative is negative definite, we conclude asymptotic stability. If, in addition, the Lyapunov function is radially unbounded, then the equilibrium is globally asymptotically stable. This Lyapunov condition happens to be also necessary for global asymptotic stability. To put in perspective

the main results of this chapter, we recall the following classical result from [BK54] –see also [Hah, Chapter 2].

Theorem 2.3 (Lyapunov characterization of UGAS) *Suppose that the function f of (1.1) is locally Lipschitz. Then the origin of $\dot{x} = f(t,x)$ is uniformly globally asymptotically stable if and only if there exists a continuously differentiable function $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and class $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$ and α such that, for all $x \in \mathbb{R}^n$ and all $t \in \mathbb{R}_{\geq 0}$,*

$$\underline{\alpha}(|x|) \leq V(t,x) \leq \overline{\alpha}(|x|) \tag{2.4}$$

$$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x) f(t,x) \leq -\alpha(|x|)\,. \tag{2.5}$$

Let us illustrate its use through the previous example. Although the argument may appear trivial to the reader, we detail the computations for further developments.

Example 2.4 *Suppose that none of the perturbations b and c affects the system introduced in Example 2.1:*

$$\ddot{q} = -\theta_1 q - \theta_2 \dot{q}\,. \tag{2.6}$$

Letting $x := (q, \dot{q})^\top$ and $\theta := (\theta_1, \theta_2)^\top$, the state representation of (2.6) is

$$\dot{x} = A(\theta)x\,, \qquad \text{with} \quad A(\theta) := \begin{pmatrix} 0 & 1 \\ -\theta_1 & -\theta_2 \end{pmatrix}. \tag{2.7}$$

Consider the smooth Lyapunov function candidate

$$V(x) := \frac{\theta_1}{2} q^2 + \frac{1}{2}\dot{q}^2 + \varepsilon q \dot{q}\,,$$

where ε is a positive constant. Using the fact that $|ab| \leq (a^2+b^2)/2$ for all real a and b, we see that

$$\frac{1}{2}(\theta_1 - \varepsilon)q^2 + \frac{1}{2}(1-\varepsilon)\dot{q}^2 \leq V(x) \leq \frac{1}{2}(\theta_1 + \varepsilon)q^2 + \frac{1}{2}(1+\varepsilon)\dot{q}^2\,. \tag{2.8}$$

Hence, by picking any $\varepsilon < \min\{\theta_1; 1\}$, we see that (2.4) holds with

$$\underline{\alpha}(s) := \frac{1}{2}\min\{\theta_1 - \varepsilon; 1-\varepsilon\}s^2\,, \quad \overline{\alpha}(s) := \frac{1}{2}\max\{\theta_1 + \varepsilon; 1+\varepsilon\}s^2\,, \qquad \forall s \in \mathbb{R}_{\geq 0}\,. \tag{2.9}$$

In addition, for all $x \in \mathbb{R}^2$, we have that

$$\begin{aligned} \frac{\partial V}{\partial x}(x)A(\theta)x &\leq -\varepsilon\theta_1 q^2 - (\theta_2 - \varepsilon)\dot{q}^2 + \varepsilon\theta_2 q\dot{q} \\ &\leq -\varepsilon\left(\theta_1 - \frac{\theta_2}{2}\right)q^2 - \left(\theta_2 - \varepsilon - \frac{\varepsilon\theta_2}{2}\right)\dot{q}^2\,. \end{aligned}$$

So, by choosing ε and θ_1 in such a way that $\varepsilon < 2\theta_2/(2+\theta_2)$ and $\theta_1 > \theta_2/2$, we see that (2.5) holds with

$$\alpha(s) := \min\left\{\varepsilon\left(\theta_1 - \frac{\theta_2}{2}\right)\,;\, \theta_2 - \varepsilon - \frac{\varepsilon\theta_2}{2}\right\}s^2\,, \qquad \forall s \in \mathbb{R}_{\geq 0}\,,$$

which is indeed a class $\mathcal{K}_\infty$ function. We conclude from Theorem 2.3 that the origin of (2.6) is globally asymptotically stable. □

When a perturbation is added to a nominal UGAS system, we can expect the negativity of the Lyapunov function's derivative (2.5) to be altered. If the perturbation "acts" around the origin, then one may loose this negativity for small values of the state. In the same way, high order nonlinearities may cause problem for large values of the state. Hence the region of the state-space on which the total derivative of V is negative may be limited by these phenomena. If, nevertheless, a wise choice of some parameter allows to extend this region at will, then we can intuitively expect UGPAS, USAS or USPAS according to the case.

We show in the next sections that this is indeed the case, provided an additional assumption on the parameter-dependence of the functions $\underline{\alpha}$ and $\overline{\alpha}$ that bound the Lyapunov function.

To the best of our knowledge, no such sufficient Lyapunov conditions for the semiglobal and/or practical stability properties introduced in Chapter 1, *i.e.* with a parameter-dependent $\mathcal{KL}$ estimate, has been reported in the literature. In most existing references, in order to establish the semiglobal asymptotic stability property of a given system, the reasoning consists in explicitly estimating the domain of attraction and showing that it can be arbitrarily enlarged by a convenient parameter tuning, *cf. e.g.* [OLK95, ARKC03]. Similarly, practical stability is commonly established by estimating an asymptotically stable compact set, and to show that its size may be reduced at will. The topic of next sections is to provide a precise Lyapunov framework for these notions.

2.1.1 Global practical stability

The following result gives a sufficient condition, in terms of a Lyapunov function defined out of a ball centered at the origin, for the dynamical parameterized system (1.3) to be uniformly globally practically asymptotically stable on a given set of parameters.

Theorem 2.5 (Lyapunov sufficient condition for UGPAS) *Let Θ be a subset of $\mathbb{R}^m$ and suppose that, given any $\delta > 0$, there exist a parameter $\theta^\star(\delta) \in \Theta$, a continuously differentiable Lyapunov function $V_\delta : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and class $\mathcal{K}_\infty$ functions $\underline{\alpha}_\delta$, $\overline{\alpha}_\delta$, α_δ such that, for all $x \in \mathbb{R}^n \setminus \mathcal{B}_\delta$ and all $t \in \mathbb{R}_{\geq 0}$,*

$$\underline{\alpha}_\delta(|x|) \leq V_\delta(t,x) \leq \overline{\alpha}_\delta(|x|) \tag{2.10}$$

$$\frac{\partial V_\delta}{\partial t}(t,x) + \frac{\partial V_\delta}{\partial x}(t,x) f(t,x,\theta^\star) \leq -\alpha_\delta(|x|) \tag{2.11}$$

$$\lim_{\delta \to 0} \underline{\alpha}_\delta^{-1} \circ \overline{\alpha}_\delta(\delta) = 0\,. \tag{2.12}$$

Then the system $\dot{x} = f(t,x,\theta)$ introduced in (1.3) is UGPAS on the parameter set Θ.

An additional requirement. Compared to classical results for Lyapunov stability such as the one recalled in Theorem 2.3, conditions (2.10) and (2.11) are natural. For perturbed systems, (2.10) is notably satisfied by the Lyapunov function associated to the UGAS of the origin of the corresponding nominal systems. (2.11) is similar to the Lyapunov sufficient condition for global ultimate boundedness (*cf. e.g.* [Kha01]). Intuitively, one may expect that these two requirements, when valid for any arbitrarily small δ, suffice to conclude UGPAS. However, we see that an additional assumption (2.12) is required that links the bounds on the Lyapunov function. Indeed, as opposed to previously cited definitions of

practical stability, the Lyapunov function may depend on the tuning parameter θ, and consequently on the radius δ. As clearly shown by Sepulchre in [Sep], this parametrization of the Lyapunov function may induce unexpected behaviors. Such a phenomenon is already described in [KM86] by Kokotović and Marino. In that reference, the authors exhibit a control system in closed-loop with a feedback that makes the domain of attraction shrink to zero when one aims at rejecting the effect of high-order nonlinearities.

To the best of our knowledge, the condition (2.12) constitutes the first one in the literature of practical stability that allows to cope with this parametrization of the bounds on the Lyapunov function. It guarantees that the Lyapunov function is sufficiently *representative* of the norm of the state. In Section 2.1.3 (*cf.* Example 2.19), we will see how crucial this requirement is by recalling the example, originally presented in [KM86], of a system for which all the conditions of Theorem 2.5 are fulfilled except (2.12) and whose non-trivial solutions grow unbounded (in particular, it is not UGPAS)[3].

Before establishing the proof of Theorem 2.5, we introduce the following result, which is a direct adaptation of [PW96, Proposition 13] and allows V_δ to be transformed into a more convenient form.

Lemma 2.6 *Let δ be a positive constant and X be a subset of $\mathbb{R}^n \setminus \mathring{\mathcal{B}}_\delta$. Suppose that there exist a continuously differentiable function $V : \mathbb{R}_{\geq 0} \times X \to \mathbb{R}_{\geq 0}$ and class $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$, α such that, for all $x \in X$ and all $t \in \mathbb{R}_{\geq 0}$,*

$$\underline{\alpha}(|x|) \leq V(t,x) \leq \overline{\alpha}(|x|) \tag{2.13}$$

$$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x) f(t,x) \leq -\alpha(|x|)\,. \tag{2.14}$$

Then, for any positive k, there exists a continuously differentiable function $\mathcal{V} : \mathbb{R}_{\geq 0} \times X \to \mathbb{R}_{\geq 0}$ and class $\mathcal{K}_\infty$ functions $\tilde{\underline{\alpha}}$, $\tilde{\overline{\alpha}}$ such that, for all $x \in X$ and all $t \in \mathbb{R}_{\geq 0}$,

$$\tilde{\underline{\alpha}}(|x|) \leq \mathcal{V}(t,x) \leq \tilde{\overline{\alpha}}(|x|) \tag{2.15}$$

$$\frac{\partial \mathcal{V}}{\partial t}(t,x) + \frac{\partial \mathcal{V}}{\partial x}(t,x) f(t,x) \leq -k\mathcal{V}(t,x)\,, \tag{2.16}$$

and, for any $s \in \mathbb{R}_{\geq 0}$, it holds that

$$\tilde{\underline{\alpha}}^{-1} \circ \tilde{\overline{\alpha}}(s) = \underline{\alpha}^{-1} \circ \overline{\alpha}(s)\,.$$

If, in addition, there exists a continuous nondecreasing function $c : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that, for all $x \in X$ and all $t \in \mathbb{R}_{\geq 0}$,

$$\left| \frac{\partial V}{\partial x}(t,x) \right| \leq c(|x|)\,,$$

then there exists a continuous nondecreasing function $\tilde{c} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that, for all $x \in X$ and all $t \in \mathbb{R}_{\geq 0}$,

$$\left| \frac{\partial \mathcal{V}}{\partial x}(t,x) \right| \leq \tilde{c}(|x|)\,. \tag{2.17}$$

□

[3] This example is given in a USAS context, but a similar argument holds for UGPAS.

Proof of Lemma 2.6. Following the proof lines of [PW96, Proposition 13], we see that the function $\mathcal{V}$ can be defined as $\rho \circ V$ where

$$\begin{cases} \rho(s) &= \exp\left(\int_1^s \frac{2dq}{a(q)}\right), \quad \forall s > 0 \\ \rho(0) &= 0, \end{cases}$$

and a is any class $\mathcal{K}$ function satisfying

$$a(s) \leq \min\left\{s, \frac{2}{k}\alpha \circ \underline{\alpha}^{-1}(s)\right\}, \quad \forall s \in \mathbb{R}_{\geq 0} \qquad \text{and} \qquad \frac{da}{ds}(0) = 0.$$

The bound (2.16) can be established following the same reasoning as in the proof of [PW96, Proposition 13]. More precisely, we have from (2.13) and (2.14) that

$$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x) f(t,x) \leq -\alpha \circ \underline{\alpha}^{-1}(V(t,x)).$$

Hence, from the definition of a,

$$\begin{aligned} \frac{\partial \mathcal{V}}{\partial t}(t,x) + \frac{\partial \mathcal{V}}{\partial x}(t,x) f(t,x) &= \frac{2}{a(V(t,x))}\mathcal{V}(t,x)\left(\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x) f(t,x)\right) \\ &\leq -k\mathcal{V}(t,x). \end{aligned}$$

Furthermore, as $\rho \in \mathcal{K}_\infty$, (2.15) can be satisfied with $\tilde{\underline{\alpha}} := \rho \circ \underline{\alpha}$ and $\tilde{\overline{\alpha}} := \rho \circ \overline{\alpha}$, and we therefore have that

$$\tilde{\underline{\alpha}}^{-1} \circ \tilde{\overline{\alpha}}(s) = (\rho \circ \underline{\alpha})^{-1} \circ (\rho \circ \overline{\alpha})(s) = \left(\underline{\alpha}^{-1} \circ \rho^{-1}\right) \circ (\rho \circ \overline{\alpha})(s) = \underline{\alpha}^{-1} \circ \overline{\alpha}(s).$$

Concerning the bound on the gradient, we have that, for all $x \in X$ and all $t \in \mathbb{R}_{\geq 0}$,

$$\left|\frac{\partial \mathcal{V}}{\partial x}(t,x)\right| \leq \frac{2\mathcal{V}(x)}{a(V(x))}\left|\frac{\partial V}{\partial x}(t,x)\right| \leq \frac{2\tilde{\overline{\alpha}}(|x|)}{a(\underline{\alpha}(|x|))} c(|x|) \leq \tilde{c}(|x|),$$

where[4] $\tilde{c}(s) := \frac{2\tilde{\overline{\alpha}}(s)}{a(\underline{\alpha}(\delta))} c(s)$, which establishes the result. ∎

We are now ready to present the proof of the main result of this section.

Proof of Theorem 2.5. Let $k = 1$, $X = \mathbb{R}^n \setminus \mathcal{B}_\delta$ and V_δ generate, via Lemma 2.6, a continuously differentiable function $\mathcal{V}_\delta$ such that, for all $x \in \mathbb{R}^n \setminus \mathcal{B}_\delta$ and all $t \in \mathbb{R}_{\geq 0}$,

$$\begin{aligned} \tilde{\underline{\alpha}}_\delta(|x|) \leq \mathcal{V}_\delta(t,x) \leq \tilde{\overline{\alpha}}_\delta(|x|) \\ \frac{\partial \mathcal{V}_\delta}{\partial t}(t,x) + \frac{\partial \mathcal{V}_\delta}{\partial x}(t,x) f(t,x,\theta^\star) \leq -\mathcal{V}_\delta(t,x) \end{aligned} \tag{2.18}$$

hold with class $\mathcal{K}_\infty$ functions $\tilde{\underline{\alpha}}_\delta$, $\tilde{\overline{\alpha}}_\delta$ and $\tilde{\alpha}_\delta$, satisfying

$$\tilde{\underline{\alpha}}_\delta^{-1} \circ \tilde{\overline{\alpha}}_\delta(s) = \underline{\alpha}_\delta^{-1} \circ \overline{\alpha}_\delta(s), \qquad \forall s \in \mathbb{R}_{\geq 0}.$$

From the latter and (2.12), we have

$$\lim_{\delta \to 0} \tilde{\underline{\alpha}}_\delta^{-1} \circ \tilde{\overline{\alpha}}_\delta(\delta) = 0. \tag{2.19}$$

[4] This is where the requirement $\delta > 0$ is needed.

Furthermore, from (2.18), we get that

$$|\phi(t,t_0,x_0,\theta^\star)| \geq \delta \quad \Rightarrow \quad \dot{\mathcal{V}}_\delta(t,\phi(t,t_0,x_0,\theta^\star)) \leq -\mathcal{V}_\delta(t,\phi(t,t_0,x_0,\theta^\star))\,. \tag{2.20}$$

Before going further, we introduce the following result which is also of interest for the next sections. It provides a $\mathcal{KL}$ estimate of solutions, based on a differential inequality that holds only out from a neighborhood of the origin. The proof is given in Section A.1.

Lemma 2.7 (Integration lemma) *Let δ be a nonnegative constant and X be a subset of $\mathbb{R}^n \backslash \mathring{\mathcal{B}}_\delta$. Assume that there exist a continuously differentiable function $V : \mathbb{R}_{\geq 0} \times X \to \mathbb{R}_{\geq 0}$, class $\mathcal{K}_\infty$ functions $\underline{\alpha}$ and $\overline{\alpha}$, a non zero real constant k and a function $c : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that, for all $x \in X$ and all $t \in \mathbb{R}_{\geq 0}$,*

$$\underline{\alpha}(|x|) \leq V(t,x) \leq \overline{\alpha}(|x|)$$

and, for all $x_0 \in \mathbb{R}^n$ and all $t_0 \in \mathbb{R}_{\geq 0}$, the solution of $\dot{x} = f(t,x)$ satisfies

$$\phi(t,t_0,x_0) \in X \quad \Rightarrow \quad \dot{V}(t,\phi(t,t_0,x_0)) \leq -kV(t,\phi(t,t_0,x_0)) + c(|x_0|)\,.$$

Then, for all $x_0 \in \mathbb{R}^n$ and all $t_0 \in \mathbb{R}_{\geq 0}$ such that $\phi(t,t_0,x_0) \in X\ \forall t \geq t_0$, we have that

$$|\phi(t,t_0,x_0)| \leq \underline{\alpha}^{-1}\left(\overline{\alpha}(\delta) + \frac{c(|x_0|)}{|k|}\right) + \underline{\alpha}^{-1}\left(\overline{\alpha}(|x_0|)e^{-k(t-t_0)} + \frac{c(|x_0|)}{|k|}\right), \qquad \forall t \geq t_0\,.$$

□

Back to the proof of Theorem 2.5, we obtain via Lemma 2.7 (with $X = \mathbb{R}^n \backslash \mathring{\mathcal{B}}_\delta$, $c(\cdot) \equiv 0$ and $k = 1$), we deduce from (2.20) that, for all $t \geq t_0$,

$$|\phi(t,t_0,x_0,\theta^\star)| \leq \tilde{\underline{\alpha}}_\delta^{-1} \circ \tilde{\overline{\alpha}}_\delta(\delta) + \tilde{\underline{\alpha}}_\delta^{-1}\left(\tilde{\overline{\alpha}}_\delta(|x_0|)e^{-(t-t_0)}\right).$$

Define $\tilde{\delta} := \tilde{\underline{\alpha}}_\delta^{-1} \circ \tilde{\overline{\alpha}}_\delta(\delta)$ and, for all $s,t \in \mathbb{R}_{\geq 0}$,

$$\beta_\delta(s,t) := \tilde{\underline{\alpha}}_\delta^{-1}\left(\tilde{\overline{\alpha}}_\delta(s)e^{-t}\right).$$

Then we have, for all $x_0 \in \mathbb{R}^n$ and all $t_0 \in \mathbb{R}_{\geq 0}$,

$$|\phi(t,t_0,x_0,\theta^\star)|_{\tilde{\delta}} \leq \beta_\delta(|x_0|, t-t_0)\,, \qquad \forall t \geq t_0\,,$$

and it is easy to see that β_δ is a $\mathcal{KL}$ function for all positive δ. Again, we stress that the dependence of β in δ is not in contradiction with Definition 1.18. Furthermore, it follows from (2.19) that $\tilde{\delta}$ can be made arbitrarily small by picking a parameter $\theta^\star(\delta) \in \Theta$ corresponding to a sufficiently small δ. UGPAS of $\dot{x} = f(t,x,\theta)$ follows. ■

The bounds on V. It is worth mentioning that, for perturbed systems, conditions (2.10) and (2.11) may often be satisfied with the Lyapunov function that serves in establishing UGAS of the nominal system. This would have not necessarily been the case if global practical stability was defined based on set-stability with respect to the same measure $|\cdot|_\delta$. Indeed, for this stronger property, it would then have been required that the corresponding Lyapunov be bounded in the following manner:

$$\underline{\alpha}_\delta(|x|_\delta) \leq V_\delta(t,x) \leq \overline{\alpha}_\delta(|x|_\delta)$$

instead of (2.10). It would notably have been required that V vanish on the whole ball $\mathcal{B}_\delta$, which cannot be the case of the original Lyapunov function for the unperturbed UGAS system. Although it is possible, in some situations, to derive a Lyapunov function with such a property based on the original Lyapunov function, this procedure remains far less direct than the approach we propose here.

Based on the same example as before, we show how to apply Theorem 2.5 and illustrate the above remark.

Example 2.8 *Reconsider the system from Example 2.4 and assume that a perturbation $b(t,q,\dot q)$, bounded by a nonnegative constant $\bar b$, affects the system (2.6):*

$$\ddot q = -\theta_1 q - \theta_2 \dot q + b(t,q,\dot q)\,. \tag{2.21}$$

Defining $x := (q,\dot q)^\top$ and $\theta := (\theta_1,\theta_2)^\top$, this can be written as

$$\dot x = A(\theta)x + B(t,x)\,, \qquad \text{where} \qquad B(t,x) := \begin{pmatrix} 0 \\ b(t,q,\dot q) \end{pmatrix} \tag{2.22}$$

and $A(\theta)$ is defined in (2.7). From Example 2.4, we already know that the following function

$$V_\theta(x) := \frac{\theta_1}{2}q^2 + \frac{1}{2}\dot q^2 + \varepsilon q\dot q\,,$$

where ε is a positive constant, is positive definite and radially unbounded provided that $\varepsilon < \min\{\theta_1;1\}$. We also know that, for all $x\in\mathbb{R}^2$,

$$\frac{\partial V_\theta}{\partial x}(x)A(\theta)x \le -\min\left\{\varepsilon\left(\theta_1 - \frac{\theta_2}{2}\right)\,;\,\theta_2 - \varepsilon - \frac{\varepsilon\theta_2}{2}\right\}|x|^2\,.$$

In addition, using the assumed bound on $b(t,q,\dot q)$,

$$\frac{\partial V_\theta}{\partial x}(x)B(t,x) = (\varepsilon q + \dot q)b(t,q,\dot q) \le (\varepsilon+1)\bar b\,|x|$$

Thus, along the solutions of (2.21), the total time derivative of V_θ satisfies

$$\dot V_\theta(t,x) \le -\min\left\{\varepsilon\left(\theta_1 - \frac{\theta_2}{2}\right) - \frac{(\varepsilon+1)\bar b}{|x|}\,;\,\theta_2 - \varepsilon - \frac{\varepsilon\theta_2}{2} - \frac{(\varepsilon+1)\bar b}{|x|}\right\}|x|^2\,.$$

Let δ be any given positive constant, and choose $\varepsilon = 1/2$. Then, provided that $\theta_1 \ge 1$, we satisfy the above requirement $\varepsilon < \min\{1,\theta_1\}$ and we get that, for all $|x|\ge\delta$,

$$\dot V_\theta(t,x) \le -\min\left\{\frac{1}{2}\left(\theta_1 - \frac{\theta_2}{2}\right) - \frac{3\bar b}{2\delta}\,;\,\frac{3\theta_2}{4} - \frac{1}{2} - \frac{3\bar b}{2\delta}\right\}|x|^2\,.$$

So, by choosing[5]

$$\theta_1^\star(\delta) = \frac{10}{3} + \frac{4\bar b}{\delta} \quad \text{and} \quad \theta_2^\star(\delta) = \frac{8}{3} + \frac{2\bar b}{\delta}\,, \tag{2.23}$$

we see that

$$|x|\ge\delta \quad \Rightarrow \quad \dot V_{\theta^\star}(t,x) \le -|x|^2\,, \tag{2.24}$$

[5] As required, this choice ensures notably that $\theta_1^\star(\delta)\ge 1$.

and (2.11) follows. Furthermore, with this choice of parameter, we get from (2.8) and (2.9) that (2.10) holds with

$$\underline{\alpha}_\delta(s) := \frac{1}{2}\min\left\{\theta_1^\star(\delta) - \frac{1}{2}; \frac{1}{2}\right\} s^2\,, \quad \overline{\alpha}_\delta(s) := \frac{1}{2}\max\left\{\theta_1^\star(\delta) + \frac{1}{2}; \frac{3}{2}\right\} s^2\,, \qquad \forall s \in \mathbb{R}_{\geq 0}\,.$$

This, in turn, illustrates clearly the indirect δ-dependence of the bounds on V through the parameter θ. Finally,

$$\lim_{\delta\to 0} \underline{\alpha}_\delta^{-1} \circ \overline{\alpha}_\delta(\delta) = \lim_{\delta\to 0} \sqrt{\frac{\max\left\{\theta_1^\star(\delta) + \frac{1}{2}; \frac{3}{2}\right\}\delta^2}{\min\left\{\theta_1^\star(\delta) - \frac{1}{2}; \frac{1}{2}\right\}}} = \lim_{\delta\to 0} \sqrt{2\left(\frac{10}{3} + \frac{4\bar{b}}{\delta} + \frac{1}{2}\right)\delta^2} = 0\,.$$

which establishes (2.12). UGPAS then follows from Theorem 2.5. □

In this example, the requirement (2.12) that links the lower and upper $\mathcal{K}_\infty$ bounds on the Lyapunov function is fulfilled in view of three properties: these bounds are affine in the tuning parameters, they are polynomial functions of the same degree, and the parameters are affine in the inverse of the radius δ of the attractive ball. As these three conditions arise quite often in control practice, notably in the control of electro-mechanical systems, we state now a result that especially fits to this situation. Although less general, it is more easily applicable. See Chapter 6 for applications of this corollary in control of a spacecraft formation and for the automatic positioning of ships for tasks such as underway replenishment.

Corollary 2.9 (Simplified Lyapunov condition for UGPAS) *Let Θ be a subset of $\mathbb{R}^n$. Assume that there exist a positive number p, real constants $\underline{a}_i$, $\overline{a}_i$, $\underline{b}_i$, $\overline{b}_i$, $i \in \{1, \ldots, n\}$ and, for any $\theta \in \Theta$, a continuously differentiable Lyapunov function V_θ satisfying, for all $x \in \mathbb{R}^n$ and all $t \in \mathbb{R}_{\geq 0}$,*

$$\sum_{i=1}^n (\underline{a}_i + \underline{b}_i\theta_i)\,|x_i|^p \leq V_\theta(t,x) \leq \sum_{i=1}^n (\overline{a}_i + \overline{b}_i\theta_i)\,|x_i|^p \tag{2.25}$$

where, for all $i \in \{1, \ldots, n\}$ and all $\theta \in \Theta$, $\underline{a}_i + \underline{b}_i\theta_i > 0$ and $\bar{a}_i + \overline{b}_i\theta_i > 0$. Suppose further that, given any $\delta > 0$, there exist a parameter $\theta^\star(\delta) \in \Theta$ and a class $\mathcal{K}_\infty$ function α_δ such that, for all x such that $\delta \leq |x|$ and all $t \in \mathbb{R}_{\geq 0}$,

$$\frac{\partial V_{\theta^\star}}{\partial t}(t,x) + \frac{\partial V_{\theta^\star}}{\partial x}(t,x) f(t,x,\theta^\star) \leq -\alpha_\delta(|x|)\,. \tag{2.26}$$

If, furthermore, for all $i \in \{1, \ldots, n\}$, we have

$$\lim_{\delta\to 0} \underline{a}_i + \underline{b}_i\theta_i^\star(\delta) > 0\,, \tag{2.27}$$

$$\overline{b}_i \neq 0 \qquad \Rightarrow \qquad \lim_{\delta\to 0} \theta_i^\star(\delta)\delta^p = 0\,, \tag{2.28}$$

then the system $\dot{x} = f(t,x,\theta)$ is uniformly globally practically asymptotically stable on the parameter set Θ.

A typical application of this corollary concerns the case of systems in closed loop affected by a non-vanishing perturbation. As this will be underlined through concrete examples in Chapter 6, in these situation, the tuning parameter typically consists in the

control gains. As intuitively expected, these need usually to be enlarged in order to achieve a better precision. For this reason, the requirement (2.27) should be trivially satisfied in most cases.

Also, for simplicity of the statement, we have assumed that the tuning parameter and the state have the same dimension n. In most practical applications, the dimension m of the former is not greater than that of the latter, in which case it suffices to complete θ by $n-m$ additional (virtual) components. However, if needed, a similar result may easily be derived for the case $m \geq n$.

Proof of Corollary 2.9. With the notation of Theorem 2.18, we have that, for all $s \in \mathbb{R}_{\geq 0}$,

$$\underline{\alpha}_\delta(s) = \min_{i\in\{1,\dots,n\}} \left\{\underline{a}_i + \underline{b}_i \theta_i^\star(\delta)\right\} s^p \,, \qquad \overline{\alpha}_\delta(s) = \max_{i\in\{1,\dots,n\}} \left\{\overline{a}_i + \overline{b}_i \theta_i^\star(\delta)\right\} s^p \,.$$

It follows that

$$\lim_{\delta\to 0} \underline{\alpha}_\delta^{-1} \circ \overline{\alpha}_\delta(\delta) = \lim_{\delta\to 0} \left(\frac{\max_{i\in\{1,\dots,n\}} \left\{\overline{a}_i + \overline{b}_i \theta_i^\star(\delta)\right\} \delta^p}{\min_{i\in\{1,\dots,n\}} \left\{\underline{a}_i + \underline{b}_i \theta_i^\star(\delta)\right\}} \right)^{1/p} .$$

In addition, (2.27) ensures that

$$\lim_{\delta\to 0} \left(\min_{i\in\{1,\dots,n\}} \left\{\underline{a}_i + \underline{b}_i \theta_i^\star(\delta)\right\} \right) > 0 \,.$$

In addition, from (2.28) we get that

$$\lim_{\delta\to 0} \left(\max_{i\in\{1,\dots,n\}} \left\{\underline{a}_i + \underline{b}_i \theta_i^\star(\delta)\right\} \delta^p \right) = 0 \,.$$

From these three observations, we conclude that

$$\lim_{\delta\to 0} \underline{\alpha}_\delta^{-1} \circ \overline{\alpha}_\delta(\delta) = 0 \,.$$

The assumptions of Theorem 2.18 are then all fulfilled and the conclusion follows. ■

Example 2.10 *Back to the system in Example 2.8, we see from (2.8) that the Lyapunov function*

$$V_\theta(x) := \frac{\theta_1}{2} q^2 + \frac{1}{2}\dot{q}^2 + \varepsilon q \dot{q} \,,$$

satisfies (2.25) with $n = p = 2$, $\underline{a}_1 = -\varepsilon/2$, $\underline{b}_1 = \overline{b}_1 = 1/2$, $\underline{a}_2 = (1-\varepsilon)/2$, $\underline{b}_2 = \overline{b}_2 = 0$, $\overline{a}_1 = \varepsilon/2$ *and* $\overline{a}_2 = (1+\varepsilon)/2$. *Choosing* $\varepsilon = 1/2$ *and assuming that* $\theta_1 \geq 1$, *we have that* $\underline{a}_i + \underline{b}_i \theta_i > 0$ *and* $\overline{a}_i + \overline{b}_i \theta_i > 0$ *for all* $i \in \{1,2\}$. *In addition, we recall from (2.23) and (2.24) that, given any positive constant* δ, *the parameters choice*

$$\theta_1^\star(\delta) = \frac{10}{3} + \frac{4\overline{b}}{\delta} \qquad \textit{and} \qquad \theta_2^\star(\delta) = \frac{8}{3} + \frac{2\overline{b}}{\delta} \,,$$

yields

$$|x| \geq \delta \qquad \Rightarrow \qquad \dot{V}_{\theta^\star}(x) \leq -|x|^2 \,.$$

Hence, the requirements (2.26), (2.27) and (2.28) hold, and UGPAS can be concluded using Corollary 2.9, without needing to compute explicitly $\lim_{\delta\to 0} \underline{\alpha}_\delta^{-1} \circ \overline{\alpha}_\delta(\delta)$ *as required by Theorem 2.5.*

□

2.1.2 Semiglobal practical stability

We recall that semiglobal practical asymptotic stability pertains to the case when one can prove that, by tuning certain parameter of the control system, the estimate of the domain of attraction can be arbitrarily enlarged and the ball to which solutions converge, in a stable way, can be reduced at will.

The following result gives a sufficient condition, in terms of a Lyapunov function, for the dynamical parameterized system (1.3) to be uniformly semiglobally practically asymptotically stable on a given set of parameters.

Theorem 2.11 (Lyapunov sufficient condition for USPAS) *Suppose that, given any $\Delta > \delta > 0$, there exist a parameter $\theta^\star(\delta, \Delta) \in \Theta$, a continuously differentiable Lyapunov function $V_{\delta,\Delta} : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, and class $\mathcal{K}_\infty$ functions $\underline{\alpha}_{\delta,\Delta}$, $\overline{\alpha}_{\delta,\Delta}$, $\alpha_{\delta,\Delta}$ such that, for all $x \in \mathcal{H}(\delta, \Delta)$ and all $t \in \mathbb{R}_{\geq 0}$,*

$$\underline{\alpha}_{\delta,\Delta}(|x|) \leq V_{\delta,\Delta}(t,x) \leq \overline{\alpha}_{\delta,\Delta}(|x|) \tag{2.29}$$

$$\frac{\partial V_{\delta,\Delta}}{\partial t}(t,x) + \frac{\partial V_{\delta,\Delta}}{\partial x}(t,x) f(t,x,\theta^\star) \leq -\alpha_{\delta,\Delta}(|x|) \tag{2.30}$$

Assume further that, for any $\Delta_\star > \delta_\star > 0$, there exist $\Delta > \delta > 0$ such that

$$\underline{\alpha}_{\delta,\Delta}^{-1} \circ \overline{\alpha}_{\delta,\Delta}(\delta) \leq \delta_\star \tag{2.31}$$

$$\overline{\alpha}_{\delta,\Delta}^{-1} \circ \underline{\alpha}_{\delta,\Delta}(\Delta) \geq \Delta_\star . \tag{2.32}$$

Then, the system $\dot{x} = f(t,x,\theta)$ introduced in (1.3) is USPAS on the parameter set Θ.

Two additional requirements. It is worth mentioning that the condition (2.29) often holds in the analysis of control systems. In particular, it holds for systems with additive bounded disturbances when USPAS may be inferred using a Lyapunov function for UGAS of the corresponding unperturbed system. Condition (2.30) also appears naturally in the context of stability of perturbed systems. See Chapter 6 for various examples in the control of electro-mechanical systems. The last two conditions, (2.31) and (2.32), need to be imposed due to the fact that the $\mathcal{K}_\infty$ bounds on the Lyapunov function V are *not* required to be the same for all δ and all Δ.

The reader is invited to refer to Example 2.19 for an illustration of the necessity to impose additional requirements as (2.31) and (2.32). This example is presented in a USAS context but may easily be adapted to the USPAS case.

A less conservative lower bound. It is also worth pointing out that the original result published in [CL06a] uses the following (less conservative) bounds on the Lyapunov function:

$$\underline{\alpha}_{\delta,\Delta}(|x|_\delta) \leq V_{\delta,\Delta}(t,x) \leq \overline{\alpha}_{\delta,\Delta}(|x|)$$

instead of (2.29). Theorem 2.11 remains indeed valid under this less conservative requirement, and corresponds more clearly to the Lyapunov characterization of asymptotic stability with respect to two measures, *cf. e.g.* [Mov60, LL93, TP00]. We have however decided to present the results using Euclidean norms on both sides for the sake of consistency with global results (some of the tools used for UGPAS require bounds as (2.29)), and as this constraint is usually satisfied in practice: see for instance Chapter 6.

Remark 2.12 *By noticing that the uniform asymptotic stability (UAS) of $\mathcal{B}_\delta$ on $\mathcal{B}_\Delta$ implies the UAS of $\mathcal{B}_{\delta'}$ on $\mathcal{B}_{\Delta'}$ for any δ' and Δ' satisfying $\delta \leq \delta' < \Delta' \leq \Delta$, the conclusion of Theorem 2.11 remains valid if (2.29) and (2.30) hold for all δ* small enough *and all Δ* large enough. *This relaxed assumption, which can also be derived for Theorems 2.5 and 2.18, may be useful in control practice.*

We now give the proof of Theorem 2.11.

Proof of Theorem 2.11. Let Δ be any positive constant and choose δ small enough that

$$\underline{\alpha}_{\delta,\Delta}^{-1} \circ \overline{\alpha}_{\delta,\Delta}(\delta) < \Delta\,, \tag{2.33}$$

which is always possible in view of (2.31). Note that this, in turn, ensures $\delta < \Delta$. Let $V_{\delta,\Delta}$ and $\theta^\star(\delta,\Delta)$ then be generated by the assumptions. Applying Lemma 2.6 to $V_{\delta,\Delta}$ with $X = \mathcal{H}(\delta,\Delta)$ and $k = 1$ ensures the existence of a continuously differentiable function $\mathcal{V}_{\delta,\Delta}$ such that, for all $x \in \mathcal{H}(\delta,\Delta)$ and all $t \in \mathbb{R}_{\geq 0}$,

$$\begin{gathered}\tilde{\underline{\alpha}}_{\delta,\Delta}(|x|_\delta) \leq \mathcal{V}_{\delta,\Delta}(t,x) \leq \tilde{\overline{\alpha}}_{\delta,\Delta}(|x|) \\ \frac{\partial \mathcal{V}_{\delta,\Delta}}{\partial t}(t,x) + \frac{\partial \mathcal{V}_{\delta,\Delta}}{\partial x}(t,x) f(t,x,\theta^\star) \leq -\mathcal{V}_{\delta,\Delta}(t,x)\end{gathered} \tag{2.34}$$

hold with class $\mathcal{K}_\infty$ functions $\tilde{\underline{\alpha}}_{\delta,\Delta}$ and $\tilde{\overline{\alpha}}_{\delta,\Delta}$ satisfying

$$\tilde{\underline{\alpha}}_{\delta,\Delta}^{-1} \circ \tilde{\overline{\alpha}}_{\delta,\Delta}(s) = \underline{\alpha}_{\delta,\Delta}^{-1} \circ \overline{\alpha}_{\delta,\Delta}(s)\,, \qquad \forall s \in \mathbb{R}_{\geq 0}\,.$$

Inverting the two sides of this inequality yields:

$$\tilde{\overline{\alpha}}_{\delta,\Delta}^{-1} \circ \tilde{\underline{\alpha}}_{\delta,\Delta}(s) = \overline{\alpha}_{\delta,\Delta}^{-1} \circ \underline{\alpha}_{\delta,\Delta}(s)\,, \qquad \forall s \in \mathbb{R}_{\geq 0}\,.$$

It then follows from (2.33) that

$$\tilde{\underline{\alpha}}_{\delta,\Delta}^{-1} \circ \tilde{\overline{\alpha}}_{\delta,\Delta}(\delta) < \Delta\,. \tag{2.35}$$

In addition, in view of (2.31) and (2.32), we have that, for all $\Delta > 0$,

$$\lim_{\delta \to 0} \tilde{\underline{\alpha}}_{\delta,\Delta}^{-1} \circ \tilde{\overline{\alpha}}_{\delta,\Delta}(\delta) = 0 \tag{2.36}$$

and, for all $\delta > 0$,

$$\lim_{\Delta \to \infty} \tilde{\overline{\alpha}}_{\delta,\Delta}^{-1} \circ \tilde{\underline{\alpha}}_{\delta,\Delta}(\Delta) = \infty\,. \tag{2.37}$$

Before continuing, we present the following result which establishes uniform boundedness of the solutions based on the non-positivity of the derivative of a Lyapunov function on a sufficiently large domain.

Proposition 2.13 (Sufficient condition for UB) *Let b be a positive constant. Suppose that there exists a continuously differentiable function V and two class $\mathcal{K}_\infty$ functions $\underline{\alpha}$ and $\overline{\alpha}$ such that, for all $t \in \mathbb{R}_{\geq 0}$ and all $x \in \mathbb{R}^n$,*

$$\underline{\alpha}(|x|) \leq V(t,x) \leq \overline{\alpha}(|x|) \tag{2.38}$$

$$x \in \mathcal{H}(a,b) \quad \Rightarrow \quad \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x) f(t,x) \leq 0\,, \tag{2.39}$$

where a denotes a positive number such that $\overline{\alpha}(a) < \underline{\alpha}(b)$. Then, for all $t_0 \in \mathbb{R}_{\geq 0}$, the solutions of (1.1) satisfy

$$|x_0| \leq \overline{\alpha}^{-1} \circ \underline{\alpha}(b) \quad \Rightarrow \quad |\phi(t,t_0,x_0)| \leq b\,, \qquad \forall t \geq t_0\,.$$

Proof of Proposition 2.13. We claim that, whenever $V(t,x) = \underline{\alpha}(b)$, its derivative along the trajectories of (1.1), which we denote by $\dot{V}$, is non positive. To this end, notice that (2.38) implies that, if $V(t,x) = \overline{\alpha}(b)$, then $x \in \mathcal{H}(\overline{\alpha}^{-1} \circ \underline{\alpha}(b), b)$, which is nonempty (since $\underline{\alpha}(b) \leq \overline{\alpha}(b)$) and included in $\mathcal{H}(a,b)$ (since it is assumed that $\overline{\alpha}(a) < \underline{\alpha}(b)$). Hence, the claim is proved in view of (2.39). For any $t_0 \in \mathbb{R}_{\geq 0}$ and any $x_0 \in \mathbb{R}^n$, by defining $v(t) := V(t, \phi(t, t_0, x_0))$, we therefore get that, for all $t \geq t_0$,

$$V(t, \phi(t, t_0, x_0)) = \underline{\alpha}(b) \quad \Rightarrow \quad \dot{V}(t, \phi(t, t_0, x_0)) \leq 0\,,$$

which ensures in its turn, by the continuity of $V(\cdot, \phi(\cdot, t_0, x_0))$, that

$$V(t_0, x_0) \leq \underline{\alpha}(b) \quad \Rightarrow \quad V(t, \phi(t, t_0, x_0)) \leq \underline{\alpha}(b)\,, \qquad \forall t \geq t_0\,.$$

The conclusion follows by noticing that, from (2.38),

$$\begin{aligned} |x_0| \leq \overline{\alpha}^{-1} \circ \underline{\alpha}(b) \qquad &\Rightarrow \qquad V(t_0, x_0) \leq \underline{\alpha}(b) \\ V(t, \phi(t, t_0, x_0)) \leq \underline{\alpha}(b) \qquad &\Rightarrow \qquad |\phi(t, t_0, x_0)| \leq b\,. \end{aligned}$$

■

We now come back to the proof of Theorem 2.11. In view of (2.34) and (2.35), the requirements of Proposition 2.13 are fulfilled with $a = \delta$ and $b = \Delta$ and we get that

$$|x_0| \leq \tilde{\Delta} \quad \Rightarrow \quad |\phi(t, t_0, x_0, \theta^\star)| \leq \Delta\,, \qquad \forall t \geq t_0\,,$$

where

$$\tilde{\Delta} := \tilde{\overline{\alpha}}_{\delta,\Delta}^{-1} \circ \tilde{\underline{\alpha}}_{\delta,\Delta}(\Delta)\,. \tag{2.40}$$

Now that we have exhibited a ball $\mathcal{B}_{\tilde{\Delta}}$ of initial states from which solutions never escapes $\mathcal{B}_\Delta$, we can apply Lemma 2.7 (with[6] $X = \mathcal{H}(\delta, \tilde{\Delta})$, $c(\cdot) \equiv 0$ and $k = 1$) to (2.34) and conclude that, for any $x_0 \in \mathcal{B}_{\tilde{\Delta}}$ and all $t_0 \in \mathbb{R}_{\geq 0}$,

$$|\phi(t, t_0, x_0, \theta^\star)| \leq \tilde{\underline{\alpha}}_{\delta,\Delta}^{-1} \circ \tilde{\overline{\alpha}}_{\delta,\Delta}(\delta) + \tilde{\underline{\alpha}}_{\delta,\Delta}^{-1}\left(\tilde{\overline{\alpha}}_{\delta,\Delta}(|x_0|)e^{-(t-t_0)}\right)\,, \qquad \forall t \geq t_0\,.$$

Defining

$$\tilde{\delta} := \tilde{\underline{\alpha}}_{\delta,\Delta}^{-1} \circ \tilde{\overline{\alpha}}_{\delta,\Delta}(\delta) \tag{2.41}$$

and

$$\beta_{\delta,\Delta}(s,t) := \tilde{\underline{\alpha}}_{\delta,\Delta}^{-1}\left(\tilde{\overline{\alpha}}_{\delta,\Delta}(s)e^{-t}\right)\,, \qquad \forall s, t \in \mathbb{R}_{\geq 0}\,,$$

we thus obtain that, for all $x_0 \in \mathcal{B}_{\tilde{\Delta}}$ and all $t_0 \in \mathbb{R}_{\geq 0}$,

$$|\phi(t, t_0, x_0, \theta^\star)|_{\tilde{\delta}} \leq \beta_{\delta,\Delta}(|x_0|, t - t_0)\,, \qquad \forall t \geq t_0\,.$$

The conclusion follows by observing that $\beta_{\delta,\Delta}$ is a $\mathcal{KL}$ function for all positive δ and Δ and that, in view of (2.36), (2.37), (2.40) and (2.41), $\tilde{\Delta}$ and $\tilde{\delta}$ can be, at the same time, arbitrarily enlarged and diminished respectively by originally conveniently choosing Δ and δ. ■

[6] Note that $\mathcal{H}(\delta, \tilde{\Delta}) \neq \emptyset$ in view of (2.35).

Remark 2.14 *We underline the slight difference between the requirements (2.31)–(2.32) and their homologous for UGPAS (2.12) and for USAS (2.57). This is due to the possible conflict that may arise from the simultaneous enlargement of the domain of attraction* and *reduction of the size of the attractive ball*[7]*. To see this more clearly, imagine a Lyapunov function bounded in the following manner:*

$$\delta\,|x| \leq V_{\delta,\Delta}(t,x) \leq \Delta\,|x|\ ,$$

then, for all $\Delta > 0$ *we have that*

$$\lim_{\delta\to 0} \underline{\alpha}^{-1}_{\delta,\Delta} \circ \overline{\alpha}_{\delta,\Delta}(\delta) = \lim_{\delta\to 0} \sqrt{\delta\Delta} = 0$$

and similarly, for all $\delta > 0$*,*

$$\lim_{\Delta\to\infty} \overline{\alpha}^{-1}_{\delta,\Delta} \circ \underline{\alpha}_{\delta,\Delta}(\Delta) = \lim_{\Delta\to\infty} \sqrt{\delta\Delta} = \infty\,,$$

which make (2.12) and (2.57) fulfilled. However, if we go back to the above proof of Theorem 2.11, we see from (2.40) and (2.41) that both $\tilde{\Delta}$ *and* $\tilde{\delta}$ *equal to* $\sqrt{\delta\Delta}$*. Hence we cannot, at the same time, enlarge* $\tilde{\Delta}$ *and diminish* $\tilde{\delta}$*. The more restrictive assumptions (2.31)-(2.32) dismiss this possibility.*

We show how to apply this result through the following elementary example. We recall that non-academic illustrations of the presented results are developed in Chapter 6.

Example 2.15 *We assume that a non-vanishing perturbation* $b(t,q,\dot{q})$ *and a higher order nonlinearity* $c(t,q,\dot{q})\dot{q}^2$*, with* $|b(t,q,\dot{q})| \leq \bar{b}$ *and* $|c(t,q,\dot{q})| \leq \bar{c}$*, now affect the system originally presented in (2.1):*

$$\ddot{q} = -\theta_1 q - \theta_2 \dot{q} + b(t,q,\dot{q}) + c(t,q,\dot{q})\dot{q}^2\,. \tag{2.42}$$

Defining $x := (q,\dot{q})^\top$ *and* $\theta := (\theta_1,\theta_2)^\top$*, this system can be rewritten as*

$$\dot{x} = A(\theta)x + B(t,x) + C(t,x)\,, \qquad \text{where} \quad C(t,x) := \begin{pmatrix} 0 \\ c(t,q,\dot{q})\dot{q}^2 \end{pmatrix} \tag{2.43}$$

and $A(\theta)$ *and* $B(t,x)$ *are respectively defined in (2.7) and (2.22). As proved in Examples 2.4, the function*

$$V_\theta(x) := \frac{\theta_1}{2}q^2 + \frac{1}{2}\dot{q}^2 + \varepsilon q\dot{q}\,,$$

where ε *is a positive constant, is positive definite and radially unbounded provided that* $\varepsilon < \min\{\theta_1;1\}$*. In the sequel, we assume* $\theta_1 \geq 1$ *and* $\varepsilon = 1/2$*. For all* $x \in \mathbb{R}^2$*, we also know, from Example 2.8, that*

$$\frac{\partial V_\theta}{\partial x}(x)\,(A(\theta)x + B(t,x)) \leq -\frac{1}{2}\min\left\{\theta_1 - \frac{\theta_2}{2} - \frac{3\bar{b}}{|x|}\,;\ \frac{3}{2}\theta_2 - 1 - \frac{3\bar{b}}{|x|}\right\}|x|^2\ .$$

Furthermore, it holds that

$$\frac{\partial V_\theta}{\partial x}(x)C(t,x) = \left(\frac{1}{2}q + \dot{q}\right)c(t,q,\dot{q})\dot{q}^2 \leq \frac{3}{2}\bar{c}\,|x|^3\ .$$

[7]The author sincerely thanks Prof. A. R. Teel for his valuable comments in this direction.

Therefore, the total time derivative of V_θ along the solutions of (2.42) satisfies

$$\dot{V}_\theta(t,x) \leq -\frac{1}{2}\min\left\{\theta_1 - \frac{\theta_2}{2} - \frac{3\bar{b}}{|x|} - 3\bar{c}\,|x|\;;\; \frac{3}{2}\theta_2 - 1 - \frac{3\bar{b}}{|x|} - 3\bar{c}\,|x|\right\}|x|^2\;.$$

Let δ and Δ be any given positive constants such that $\delta < \Delta$. Then, for all $x \in \mathcal{H}(\delta,\Delta)$, it holds that

$$\dot{V}_\theta(t,x) \leq -\frac{1}{2}\min\left\{\theta_1 - \frac{\theta_2}{2} - \frac{3\bar{b}}{\delta} - 3\bar{c}\Delta\,;\, \frac{3\theta_2}{2} - \frac{3\bar{b}}{\delta} - 3\bar{c}\Delta\right\}|x|^2\;.$$

Accordingly, by choosing[8]

$$\theta_1^\star(\delta,\Delta) = \frac{10}{3} + \frac{4\bar{b}}{\delta} + 4\bar{c}\Delta \quad \textit{and} \quad \theta_2^\star(\delta,\Delta) = \frac{8}{3} + \frac{2\bar{b}}{\delta} + 2\bar{c}\Delta\,, \tag{2.44}$$

we see that

$$x \in \mathcal{H}(\delta,\Delta) \quad \Rightarrow \quad \dot{V}_{\theta^\star}(t,x) \leq -|x|^2\;, \tag{2.45}$$

and (2.30) follows. Furthermore, since $\theta_1^\star \geq 1$, we get from (2.8), (2.9) and (2.44) that (2.10) holds with the functions defined, for all $s \in \mathbb{R}_{\geq 0}$, by

$$\begin{aligned}
\underline{\alpha}_{\delta,\Delta}(s) &:= \frac{1}{2}\min\left\{\theta_1^\star(\delta,\Delta) - \frac{1}{2}\;;\; \frac{1}{2}\right\}s^2 = \frac{1}{4}s^2\\
\overline{\alpha}_{\delta,\Delta}(s) &:= \frac{1}{2}\max\left\{\theta_1^\star(\delta,\Delta) + \frac{1}{2}\;;\; \frac{3}{2}\right\}s^2 = \left(\frac{23}{12} + \frac{2\bar{b}}{\delta} + 2\bar{c}\Delta\right)s^2\,.
\end{aligned}$$

Therefore, the requirement (2.31)-(2.32) imposes that, given any $\Delta_\star > \delta_\star > 0$, one can find $\Delta > \delta > 0$ such that

$$\begin{aligned}
\delta\sqrt{\frac{23}{3} + \frac{8\bar{b}}{\delta} + 8\bar{c}\Delta} &\leq \delta_\star\\
\frac{\Delta}{\sqrt{\frac{23}{3} + \frac{8\bar{b}}{\delta} + 8\bar{c}\Delta}} &\geq \Delta_\star\,.
\end{aligned}$$

This is in particular fulfilled if

$$\begin{aligned}
\delta^2\left(\frac{23}{3} + \frac{8\bar{b}}{\delta} + 8\bar{c}\Delta\right) &= \delta_\star^2\\
\frac{\Delta^2}{\frac{23}{3} + \frac{8\bar{b}}{\delta} + 8\bar{c}\Delta} &= \Delta_\star^2\,.
\end{aligned} \tag{2.46}$$

Multiplying these equations yields

$$\delta\Delta = \delta_\star\Delta_\star\,. \tag{2.47}$$

Injecting this into (2.46) and rearranging terms, we obtain the following second-order equation in Δ:

$$\Delta^2 - 4\left(\frac{\bar{b}}{2\delta_\star} + \bar{c}\right)\Delta_\star^2\Delta - \frac{23\Delta_\star^2}{3} = 0\,,$$

which clearly admits a positive solution. The corresponding δ can then be obtained by (2.47) and USPAS follows from Theorem 2.11. □

[8] As required, this choice ensures notably that $\theta_1^\star(\delta,\Delta) \geq 1$.

Again, we can see that the requirements (2.31) and (2.32) follow from three combined properties: the bounds on the Lyapunov function are affine in the tuning parameters, they are quadratic functions, and the parameters are affine in $1/\delta$ and in Δ. As illustrated by the analysis of robustness of PID-controlled robot manipulators to external disturbances, model imperfection and actuators'dynamics (*cf.* Section 6.1), these three conditions arise quite often in practice. This justifies the following more restrictive but easier-to-apply statement.

Corollary 2.16 (Simplified Lyapunov condition for USPAS) *Let Θ be a subset of $\mathbb{R}^n$. Assume that there exist real constants $\underline{a}_i$, $\overline{a}_i$, $\underline{b}_i$, $\overline{b}_i$, $i \in \{1, \ldots, n\}$ and, for any $\theta \in \Theta$, a continuously differentiable Lyapunov function V_θ satisfying, for all $x \in \mathbb{R}^n$ and all $t \in \mathbb{R}_{\geq 0}$,*

$$\sum_{i=1}^{n} (\underline{a}_i + \underline{b}_i \theta_i) \, |x_i|^2 \leq V_\theta(t,x) \leq \sum_{i=1}^{n} (\overline{a}_i + \overline{b}_i \theta_i) \, |x_i|^2 \; . \tag{2.48}$$

Suppose further that, given any positive δ and Δ such that $\delta < \Delta$, there exist a parameter $\theta^\star(\delta, \Delta) \in \Theta$, nonnegative constants c_i, d_i, e_i, $i \in \{1, \ldots, n\}$ and a class $\mathcal{K}_\infty$ function $\alpha_{\delta,\Delta}$ such that, for all $x \in \mathcal{H}(\delta, \Delta)$ and all $t \in \mathbb{R}_{\geq 0}$,

$$\frac{\partial V_{\theta^\star}}{\partial t}(t,x) + \frac{\partial V_{\theta^\star}}{\partial x}(t,x) f(t,x,\theta^\star) \leq -\alpha_{\delta,\Delta}(|x|) \, , \tag{2.49}$$

$$\theta_i^\star(\delta, \Delta) = c_i + \frac{d_i}{\delta} + e_i \Delta \, , \qquad \forall i \in \{1, \ldots, n\} \, . \tag{2.50}$$

Then, under the condition that

$$\underline{a}_i + \underline{b}_i c_i > 0 \, , \qquad \forall i \in \{1, \ldots, n\} \, , \tag{2.51}$$

the system $\dot{x} = f(t,x,\theta)$ introduced in (1.3) is uniformly semiglobally practically asymptotically stable on the parameter set Θ.

As to what regards the dimension of θ, a similar remark as that made for Corollary 2.9 holds. Please refer to Chapter 6 for concrete applications in control of mechanical systems.

Proof of Corollary 2.16. Consider the two following constants:

$$\underline{c} := \min_{i \in \{1,\ldots,n\}} \{\underline{a}_i + \underline{b}_i c_i\} \quad \text{and} \quad \overline{c} := \max_{i \in \{1,\ldots,n\}} \max \left\{ \overline{a}_i + \overline{b}_i c_i ; \overline{b}_i d_i ; \overline{b}_i e_i \right\} \; .$$

Using the notations of Theorem 2.11, we can pick, for all $s \in \mathbb{R}_{\geq 0}$,

$$\underline{\alpha}_{\delta,\Delta}(s) = \underline{c} s^2 \, , \qquad \overline{\alpha}_{\delta,\Delta}(s) = \overline{c} \left(1 + \Delta + \frac{1}{\delta} \right) s^2 \, .$$

In view of (2.51), $\underline{c}$ is a positive constant so both $\underline{\alpha}_{\delta,\Delta}$ and $\overline{\alpha}_{\delta,\Delta}$ are class $\mathcal{K}_\infty$ functions. In addition, the requirement (2.31)-(2.32) becomes:

$$\begin{aligned} \left(\frac{\overline{c}(1 + \Delta + 1/\delta)}{\underline{c}} \right)^{1/2} \delta &\leq \delta_\star \\ \left(\frac{\underline{c}}{\overline{c}(1 + \Delta + 1/\delta)} \right)^{1/2} \Delta &\geq \Delta_\star \, . \end{aligned}$$

For this, it is sufficient that

$$\frac{(1+\Delta+1/\delta)\delta^2}{\varepsilon} = \delta_\star^2 \tag{2.52}$$

$$\frac{\varepsilon}{(1+\Delta+1/\delta)\Delta^2} = \Delta_\star^2\,, \tag{2.53}$$

where $\varepsilon := \underline{c}/\overline{c}$. Multiplying (2.52) and (2.53), we get that

$$\delta\Delta = \delta_\star\Delta_\star\,. \tag{2.54}$$

From this and (2.52), we obtain the following second order equation in Δ:

$$\varepsilon\Delta^2 - \Delta_\star^2\left(1+\frac{1}{\delta_\star\Delta_\star}\right)\Delta - \Delta_\star^2 = 0\,,$$

which admits a positive solution for all $\delta_\star, \Delta_\star > 0$. The corresponding δ is then obtained by injecting this solution in (2.54). ■

Example 2.17 *Back to the system in Example 2.15, we see that the Lyapunov function*

$$V_\theta(x) := \frac{\theta_1}{2}q^2 + \frac{1}{2}\dot{q}^2 + \varepsilon q\dot{q}\,,$$

satisfies (2.48) with $n = p = 2$, $\underline{a}_1 = -\varepsilon/2$, $\underline{b}_1 = \overline{b}_1 = 1/2$, $\underline{a}_2 = (1-\varepsilon)/2$, $\underline{b}_2 = \overline{b}_2 = 0$, $\overline{a}_1 = \varepsilon/2$ *and* $\overline{a}_2 = (1+\varepsilon)/2$. *Choosing* $\varepsilon = 1/2$ *and* $\theta_1 \geq 1$, *we then have that* $\underline{a}_i + \underline{b}_i\theta_i > 0$ *and* $\overline{a}_i + \overline{b}_i\theta_i > 0$ *for all* $i \in \{1,2\}$. *In addition, we recall from (2.44) and (2.45) that, given any positive constants* $0 < \delta < \Delta$, *the following implication*

$$x \in \mathcal{H}(\delta,\Delta) \quad \Rightarrow \quad \dot{V}_{\theta^\star}(x) \leq -|x|^2$$

can be obtained with the following choice of parameters:

$$\theta_1^\star(\delta,\Delta) = \frac{10}{3} + \frac{4\bar{b}}{\delta} + 4\bar{c}\Delta \quad \textit{and} \quad \theta_2^\star(\delta,\Delta) = \frac{8}{3} + \frac{2\bar{b}}{\delta} + 2\bar{c}\Delta\,.$$

Hence, the requirements (2.49) holds, and USPAS can be directly concluded using Corollary 2.9.

□

2.1.3 Semiglobal asymptotic stability

In the presence of high order nonlinearities or imperfections in the model, solutions may asymptotically converge, in a stable way[9], to the origin itself, but on finite sets of initial conditions only. If the basin of attraction can be arbitrarily enlarged by a convenient choice of some tuning parameters, then we refer to this property as uniform semiglobal asymptotic stability (USAS, see Definition 1.17). USAS therefore constitutes a stronger property than the USPAS discussed in the previous section.

Similarly to UGPAS and USPAS, USAS can be established by studying the sign of a Lyapunov function and its derivative in a restricted region of the state-space. We give this result below.

[9] By this, we mean that the origin is asymptotically stable on the considered set of initial states.

Theorem 2.18 (Lyapunov sufficient condition for USAS) *Suppose that, given any $\Delta > 0$, there exist a parameter $\theta^\star(\Delta) \in \Theta$, a continuously differentiable Lyapunov function $V_\Delta : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, and class $\mathcal{K}_\infty$ functions $\underline{\alpha}_\Delta$, $\overline{\alpha}_\Delta$, α_Δ such that, for all $x \in \mathcal{B}_\Delta$ and all $t \in \mathbb{R}_{\geq 0}$,*

$$\underline{\alpha}_\Delta(|x|) \leq V_\Delta(t,x) \leq \overline{\alpha}_\Delta(|x|) \tag{2.55}$$

$$\frac{\partial V_\Delta}{\partial t}(t,x) + \frac{\partial V_\Delta}{\partial x}(t,x) f(t,x,\theta^\star) \leq -\alpha_\Delta(|x|)\,. \tag{2.56}$$

Assume further that

$$\lim_{\Delta \to \infty} \overline{\alpha}_\Delta^{-1} \circ \underline{\alpha}_\Delta(\Delta) = \infty\,. \tag{2.57}$$

Then the system $\dot{x} = f(t,x,\theta)$ introduced in (1.3) is uniformly semiglobally asymptotically stable on the parameter set Θ.

This result shares many similarities with Theorems 2.5 and 2.11, so we invite the reader to refer to these statements for comments on the bounds (2.55), (2.56) and (2.57). We stress once again that the condition (2.57) is needed due to the possible parametrization of the $\mathcal{K}_\infty$ bounds on the Lyapunov function by the radius Δ. With the help of the following example, we show that it cannot be removed from the above statement.

Example 2.19 (The importance of (2.57)) *Consider the second-order nonlinear system, originally presented by Kokotović and Marino in [KM86]:*

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\theta^2 x_1 - \theta x_2 + \frac{1}{3}x_2^3\,, \end{aligned}$$

where θ denotes a positive gain. We claim that this system, together with the function

$$V_\theta(x) = \theta^2 x_1^2 + x_2^2 + x_2 tanh\, x_1\,,$$

satisfies all the requirements of Theorem 2.18, except (2.57), but is not USAS. Indeed, its total derivative yields

$$\dot{V}_\theta(x) = -2\theta x_2^2 + \frac{2}{3}x_2^4 + x_2^2 sech^2 x_1 + tanh\, x_1 \left(-\theta^2 x_1 - \theta x_2 + \frac{1}{3}x_2^3\right)\,,$$

from which we easily get that

$$\dot{V}_\theta(x) \leq -\frac{\theta^2}{2} x_1 tanh\, x_1 - \theta x_2^2 - \frac{\theta}{2}(\theta - 1)\, tanh^2 x_1 - \left(\frac{\theta}{2} - 1 + \frac{|x_2|}{3} + \frac{x_2^2}{3}\right) x_2^2\,.$$

Thus, given any positive Δ, it suffices to choose

$$\theta = \theta^\star(\Delta) := \frac{2}{3}\left(3 + \Delta + \Delta^2\right) \tag{2.58}$$

to obtain that, for all $x \in \mathcal{B}_\Delta$,

$$\dot{V}_{\theta^\star}(x) \leq -x_1 tanh\, x_1 - x_2^2 \leq -x_1 tanh\, x_1 - x_2 tanh\, x_2\,.$$

Using the fact that $(a+b)tanh(a+b) \leq 2a tanh(2a) + 2b tanh(2b)$ for all $a, b \in \mathbb{R}_{\geq 0}$, we conclude that (2.56) holds with

$$\alpha_\Delta(s) := \frac{s}{2} tanh\left(\frac{s}{2}\right)\,, \qquad \forall s \in \mathbb{R}_{\geq 0}\,.$$

In addition, with the parameter choice (2.58), the condition (2.55) is fulfilled on the whole state-space $\mathbb{R}^2$ *with the following* $\mathcal{K}_\infty$ *functions:*

$$\begin{aligned}\underline{\alpha}_\Delta(s) &:= \min\left\{\theta^\star(\Delta)^2 - \frac{1}{2}; \frac{1}{2}\right\}\frac{s}{2}tanh\left(\frac{s}{2}\right) = \frac{s}{4}tanh\left(\frac{s}{2}\right)\\ \overline{\alpha}_\Delta(s) &:= \max\left\{\theta^\star(\Delta)^2 + \frac{1}{2}; 1\right\}s^2 = \left(\theta^\star(\Delta)^2 + \frac{1}{2}\right)s^2\,.\end{aligned}$$

To sum up, the considered Lyapunov function is positive definite and radially unbounded, and its total derivative along the solutions of the system is negative definite on a domain that can be made arbitrarily large by enlarging the control gain. However, quite surprisingly, the domain of attraction of this system cannot be arbitrarily enlarged. Actually, it was shown in [KM86] that it even vanishes as the gain θ *tends to infinity. More precisely, it is shown in that reference that the set of initial conditions that generate unbounded trajectories contains the set*

$$\left\{(x_1, x_2) \in \mathbb{R}^2 \,:\, \theta x_1^2 + \frac{1}{\theta}x_2^2 > 3^3\right\}\,.$$

Notice that the boundaries of this region cross the axes $x_1 = 0$ *and* $x_2 = 0$ *at* $x_2 = \pm 3\sqrt{\theta}$ *and* $x_1 = \pm 3/\sqrt{\theta}$ *respectively. Consequently, no matter the parameter choice, the radius of the largest ball contained in the domain of attraction cannot exceed* 3*, which contradicts the property of USAS. In accordance with Theorem 2.18, we see that, indeed, the additional requirement (2.57) is violated:*

$$\lim_{\Delta\to\infty} \overline{\alpha}_\Delta^{-1} \circ \underline{\alpha}_\Delta(\Delta) = \lim_{\Delta\to\infty}\sqrt{\frac{\Delta tanh(\Delta/2)/4}{\theta^\star(\Delta)^2 + 1/2}} = 0 \neq \infty\,.$$

□

Proof of Theorem 2.18. Let Δ be any given positive constant and let V_Δ and $\theta^\star(\Delta)$ be generated by the assumptions. The proof is based on similar arguments as the above cited results, especially Theorem 2.11. The main difference stands in the fact that Lemma 2.6 does not apply as it imposes to work out of a neighborhood $\mathcal{B}_\delta$, with $\delta > 0$, of the origin. This prevents V_Δ to be transformed into a more easily integrable function. To overcome this apparent difficulty, we invoke the following result. The result was presented in [Son89a, Lemma 6.1] in the case that α is smooth. The locally Lipschitz case is a direct consequence of [Kha96, Lemma 3.4] and of the comparison theorem (*cf. e.g.* [Kha96, Lemma 2.5]).

Lemma 2.20 *Let* α *be a locally Lipschitz function of class* $\mathcal{K}$*. Then there exists a class* $\mathcal{KL}$ *function* β *such that any solution*[10] *of the differential inequality*

$$\dot{y} \leq -\alpha(y) \tag{2.59}$$

satisfies

$$|y(t, y_0)| \leq \beta(|y_0|, t)\,, \qquad \forall t \in \mathbb{R}_{\geq 0}\,.$$

□

[10] By this we mean that $y(\cdot, y_0)$ satisfies $\dot{y}(t, y_0) \leq -\alpha(y(t, y_0))$ for all $t \in \mathbb{R}_{\geq 0}$.

First of all, we apply Proposition 2.13 with $a = 0$ and $b = \Delta$ to get that

$$x_0 \in \mathcal{B}_{\tilde{\Delta}} \quad \Rightarrow \quad \phi(t, t_0, x_0, \theta^\star) \in \mathcal{B}_\Delta \,, \qquad \forall t \geq t_0 \,.$$

where

$$\tilde{\Delta} := \overline{\alpha}_\Delta^{-1} \circ \underline{\alpha}_\Delta(\Delta) \,. \tag{2.60}$$

It follows that, for any $x_0 \in \mathcal{B}_{\tilde{\Delta}}$, $V_\Delta(\cdot, \phi(\cdot, t_0, x_0, \theta^\star))$ is a solution of the differential inequality $\dot{y} \leq -\alpha_\Delta(y)$. From Lemma 2.20, we get that there exists a $\mathcal{KL}$ function β_Δ such that, for all $x_0 \in \mathcal{B}_{\tilde{\Delta}}$ and all $t_0 \in \mathbb{R}_{\geq 0}$,

$$V_\Delta(t, \phi(t, t_0, x_0, \theta^\star)) \leq \beta_\Delta(V_\Delta(t_0, x_0), t - t_0) \,, \qquad \forall t \geq t_0 \,.$$

Using the bounds provided by (2.55), we conclude that

$$|\phi(t, t_0, x_0, \theta^\star)| \leq \tilde{\beta}_\Delta(|x_0|, t - t_0) \,, \qquad \forall t \geq t_0 \,,$$

where $\tilde{\beta}_\Delta(s, t) := \underline{\alpha}_\Delta^{-1} \circ \overline{\alpha}_\Delta \circ \beta_\Delta(s, t)$ for all $s, t \in \mathbb{R}_{\geq 0}$. USAS then follows by observing that $\tilde{\beta}_\Delta$ is a $\mathcal{KL}$ function and that, in view of (2.57) and (2.60), $\tilde{\Delta}$ can be made arbitrarily large by originally picking Δ large enough. ■

Through the following example, we next illustrate the utilization of Theorem 2.18.

Example 2.21 *We now assume that the perturbation $b(t, q, \dot{q})$ is identically zero, but the higher order nonlinearity $c(t, q, \dot{q})\dot{q}^2$, with $|c(t, q, \dot{q})| \leq \bar{c}$, still affects the system originally presented in (2.1):*

$$\ddot{q} = -\theta_1 q - \theta_2 \dot{q} + c(t, q, \dot{q})\dot{q}^2 \,. \tag{2.61}$$

Using x as $(q, \dot{q})^\top$, this can be written

$$\dot{x} = A(\theta)x + C(t, x) \,,$$

where $A(\theta)$ is defined in (2.7) and $C(t, x)$ is introduced in (2.43). As proved in Examples 2.4, the function

$$V_\theta(x) := \frac{\theta_1}{2} q^2 + \frac{1}{2}\dot{q}^2 + \varepsilon q\dot{q} \,,$$

where ε is a positive constant, is positive definite and radially unbounded provided that $\varepsilon < \min\{\theta_1; 1\}$. In the sequel, we assume $\theta_1 \geq 1$ and $\varepsilon = 1/2$. Following a similar reasoning as in Example 2.15 by picking $\bar{b} = 0$, it easily follows that, for any $\Delta > 0$, the total derivative of V satisfies

$$|x| \leq \Delta \quad \Rightarrow \quad \dot{V}_{\theta^\star}(t, x) \leq -|x|^2 \,, \tag{2.62}$$

if the tuning parameters are chosen as

$$\theta_1^\star(\Delta) = \frac{10}{3} + 4\bar{c}\Delta \quad \textit{and} \quad \theta_2^\star(\Delta) = \frac{8}{3} + 2\bar{c}\Delta \,. \tag{2.63}$$

Similarly to Example 2.15, the additional requirement (2.57) can be established and USAS then follows from Theorem 2.18. □

Again, a simplified version of this result can be proposed in the context of polynomial bounds on the Lyapunov function.

Corollary 2.22 (Simplified Lyapunov condition for USAS) *Let Θ be a subset of $\mathbb{R}^n$. Assume that there exist a positive number p, real constants $\underline{a}_i$, $\overline{a}_i$, $\underline{b}_i$, $\overline{b}_i$, $i \in \{1, \ldots, n\}$ and, for any $\theta \in \Theta$, a continuously differentiable Lyapunov function V_θ satisfying, for all $x \in \mathbb{R}^n$ and all $t \in \mathbb{R}_{\geq 0}$,*

$$\sum_{i=1}^{n} (\underline{a}_i + \underline{b}_i \theta_i) \, |x_i|^p \leq V_\theta(t,x) \leq \sum_{i=1}^{n} (\overline{a}_i + \overline{b}_i \theta_i) \, |x_i|^p \tag{2.64}$$

where, for all $i \in \{1, \ldots, n\}$ and all $\theta \in \Theta$, $\underline{a}_i + \underline{b}_i \theta_i > 0$ and $\overline{a}_i + \overline{b}_i \theta_i > 0$. Suppose further that, given any $\Delta > 0$, there exist a parameter $\theta^\star(\Delta) \in \Theta$, a class $\mathcal{K}_\infty$ function α_Δ such that, for all $x \in \mathcal{B}_\Delta$ and all $t \in \mathbb{R}_{\geq 0}$,

$$\frac{\partial V_{\theta^\star}}{\partial t}(t,x) + \frac{\partial V_{\theta^\star}}{\partial x}(t,x) f(t,x,\theta^\star) \leq -\alpha_\delta(|x|) \, . \tag{2.65}$$

If furthermore, for all $i \in \{1, \ldots, n\}$, it holds that

$$\lim_{\Delta \to \infty} \underline{a}_i + \underline{b}_i \theta_i^\star(\Delta) > 0 \, , \tag{2.66}$$

$$\overline{b}_i \neq 0 \qquad \Rightarrow \qquad \lim_{\Delta \to \infty} \frac{\Delta^p}{\theta_i^\star(\Delta)} = \infty \, , \tag{2.67}$$

then the system $\dot{x} = f(t,x,\theta)$ introduced in (1.3) is uniformly semiglobally asymptotically stable on the parameter set Θ.

The proof of this result is voluntarily omitted as it consists in the same arguments as for Corollary 2.9.

Example 2.23 *Back to the system in Example 2.21, the Lyapunov function*

$$V_\theta(x) := \frac{\theta_1}{2} q^2 + \frac{1}{2} \dot{q}^2 + \varepsilon q \dot{q} \, ,$$

satisfies (2.64) with $n = 2$, $\underline{a}_1 = -\varepsilon/2$, $\underline{b}_1 = \overline{b}_1 = 1/2$, $\underline{a}_2 = (1-\varepsilon)/2$, $\underline{b}_2 = \overline{b}_2 = 0$, $\overline{a}_1 = \varepsilon/2$ and $\overline{a}_2 = (1+\varepsilon)/2$. Choosing $\varepsilon = 1/2$ and assuming that $\theta_1 \geq 1$, we have that $\underline{a}_i + \underline{b}_i \theta_i > 0$ and $\overline{a}_i + \overline{b}_i \theta_i > 0$ for all $i \in \{1,2\}$. Moreover, we recall from (2.62) and (2.63) that, given any positive constant Δ, the parameters choice

$$\theta_1^\star(\Delta) = \frac{10}{3} + 4\bar{c}\Delta \quad \textit{and} \quad \theta_2^\star(\Delta) = \frac{8}{3} + 2\bar{c}\Delta \, ,$$

yields

$$x \in \mathcal{B}_\Delta \qquad \Rightarrow \qquad \dot{V}_{\theta^\star}(x) \leq -|x|^2 \, .$$

Hence, the requirements (2.65)–(2.67) hold, and USAS follows from Corollary 2.22. □

2.2 Converse results

The crucial role played by Lyapunov functions in the study of the stability of dynamical systems suffers no objection. In the previous section, we have shown that this mathematical object notably constitutes a powerful tool for the particular framework of semiglobal and practical stability. In informal terms, we have shown that, according to the case, it is

sufficient that the total derivative of such a function be negative out of a small neighborhood of the origin, or inside a large domain of the state-space, to ensure practical stability or semiglobal stability respectively, provided an additional requirement on its lower and upper bounds (*cf.* Theorems 2.5, 2.11 and 2.18).

A long story. While this type of sufficient Lyapunov conditions for stability initially mainly motivated the control community, the converse question, *i.e.* whether stability implies the existence of a Lyapunov function, was already posed in 1892 by Lyapunov himself in his fundamental work [Lya92] and the first necessary and sufficient conditions for uniform stability were already proposed in [Per37].

Among the most significative contributors to the field, one can cite Massera who proposed in [Mas49] a Lyapunov function for continuously differentiable time-invariant systems with an asymptotically stable origin, based on a lemma that now bears his name (*cf. e.g.* [Kha96, Lemma A.1]). Malkin proposed in [Mal54] an extension to this result for time-varying systems, provided that the stability property and the differentiability of the right-hand side term both are uniform in time. The first theorem of this type for uniform global asymptotic stability of the origin was proposed by Barbashin and Krasovskĭi, *cf.* [BK54][11]. Kurzweil then managed to relax the regularity assumption made on the right-hand side term to simply continuity, and gave the first results in the case when stability is not defined through the Euclidean norm but through a more general measure, opening the door to set-stability and partial stability, *cf.* [Wil69]. These results have been adapted to stability with respect to two measures by Lakshmikantham and Salvadori in [LS76], including, as a particular case, the set-stability introduced in Definitions 1.9, 1.10 and 1.11.

Since then, many extensions of these results have been made to differential inclusions (*cf.* [TP00] and references therein) and to the formalism of input to state stability: [SW96, ASW00a, SW01, KSW01], but this escapes the scope of the present section (please refer to Chapter 5 for an application of the Lyapunov characterization of integral input to state stability). See also [TP00] for a more extensive review on converse theorems.

A converse result for USPAS ? In what follows, we derive a necessary Lyapunov condition based on the assumption that the system under consideration is USPAS or USAS. This study is not only of mathematical interest. Indeed, as already said, the latter stability properties can be established through various means, and not all of them provide an associated Lyapunov function. This is notably the case with averaging techniques: in [TPA99], Teel *et al.* show that, if its averaged is globally asymptotically stable, then the original system is USPAS. These notions will be more detailed in the sequel, and be at the basis of an illustrative example (see Section 3.1.2).

We also present in the sequel (*cf.* Chapter 3) tools that establish USPAS of systems in *cascade*, based notably on the USPAS of each subsystem when disconnected *and* on the knowledge of a Lyapunov function for the driven subsystem. Providing the existence of a Lyapunov function for a certain class of dynamical system, based on the single fact that the system is USPAS, can therefore be of great help. Such a result should therefore provide a positive function, with negative derivative, in a restricted region of the state-space. The generated Lyapunov function and its $\mathcal{K}_\infty$ should also fit the additional requirements (2.31)–(2.32) or (2.57) according to the case.

[11] In this reference, the mistaken English translation "in the large" actually refers to "in the whole", *i.e.* global. Please see [LLLP06] for clarifications about these notions.

As it is further explained in the next chapter, the gradient of the Lyapunov function associated to the stability property of the driven subsystem plays an important role in the context of stability analysis of cascades. Additionally to the classical features, the converse tool we present below therefore needs to provide a time-invariant bound on the gradient of the Lyapunov-like function it generates.

These two latter features constitute the main difference with other similar results available in the literature, specifically [TP00, LSW96]. We emphasize that, even though the construction from [TP00] also allows to construct bounds on the gradient of the Lyapunov function, this seemingly requires to impose the unnecessary restrictive assumption that the function f in (1.3) be locally Lipschitz in time.

2.2.1 Semiglobal practical stability

In what follows, we present a converse theorem for uniform semiglobal practical asymptotic stability which is tailored for cascaded systems. We start by constructing a Lyapunov function for systems for which a given ball is uniformly asymptotically stable. Compared to the sufficiency result proposed in the previous sections, it requires additional smoothness of the right-hand side term. More precisely, we assume the following.

Assumption 2.24 (Regularity) *There exists a nonnegative constant f_0 and a continuous nondecreasing function L such that*

$$|f(t,0)| \leq f_0\,, \qquad \forall t \in \mathbb{R}_{\geq 0}\,, \tag{2.68}$$

$$\left|\frac{\partial f}{\partial x}(t,x)\right| \leq L(|x|)\,, \qquad \forall t \in \mathbb{R}_{\geq 0}\,, \quad \forall x \in \mathbb{R}^n\,. \tag{2.69}$$

It is worth mentioning that the origin is not required to be an equilibrium for the system under consideration. This indeed fails in many situations when studying practical stability. See Example 2.8 for instance. In this respect, see [BCI05] for an analysis of the local (more precisely, in the asymptotically stable ball $\mathcal{B}_\delta$) behavior of the trajectories generated by a particular class of practically stable systems.

Lemma 2.25 (Converse Lyapunov function for UAS of a ball) *Let $\delta \geq 0$ and $\Delta > 0$ be some given constants such that $\Delta > \delta$. If $\mathcal{B}_\delta$ is uniformly asymptotically stable on $\mathcal{B}_\Delta$ for the dynamical system $\dot{x} = f(t,x)$ introduced in (1.1) then, under Assumption 2.24, for any $\delta' \in (\delta;\Delta)$, there exist: a continuously differentiable function $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, class $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$, α, and a continuous nondecreasing function $c : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that, for all $x \in \mathcal{B}_\Delta$ and all $t \in \mathbb{R}_{\geq 0}$, the following holds:*

$$\underline{\alpha}(|x|_\delta) \leq V(t,x) \leq \overline{\alpha}(|x|) \tag{2.70}$$

$$|x| \geq \delta' \quad \Rightarrow \quad \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x) f(t,x) \leq -\alpha(V(t,x)) \tag{2.71}$$

$$\left|\frac{\partial V}{\partial x}(t,x)\right| \leq c(|x|)\,. \tag{2.72}$$

In addition, if $\delta = 0$, then δ' can be picked as 0 too. Furthermore, if (1.1) is time invariant (i.e. $\dot{x} = f(x)$), then V can be picked time-invariant as well. □

Remark 2.26 *It is worth pointing out that, for any fixed δ, the bounds (2.70) imply the bounds (2.29) required in Theorem 2.11. Indeed, we have that, for all $x \in \mathcal{H}(\delta', \Delta)$ and all $t \in \mathbb{R}_{\geq 0}$,*

$$\hat{\underline{\alpha}}(|x|) \leq V(t,x) \leq \overline{\alpha}(|x|)$$

where $\hat{\underline{\alpha}}$ is defined as

$$\hat{\underline{\alpha}}(s) := \begin{cases} \alpha(s-\delta) & \text{if} \quad s \geq \delta' \\ \alpha_0(s) & \text{if} \quad s \in [0; \delta')\,, \end{cases}$$

α_0 designating any class $\mathcal{K}$ function satisfying $\alpha_0(\delta') = \alpha(\delta' - \delta)$.

Proof of Lemma 2.25. Let the assumption of UAS of $\mathcal{B}_\delta$ on $\mathcal{B}_\Delta$ generate $\beta \in \mathcal{KL}$ such that, for any $x \in \mathcal{B}_\Delta$ and any $t \in \mathbb{R}_{\geq 0}$, the trajectories of (1.1) satisfy

$$|\phi(\tau, t, x)|_\delta \leq \beta(|x|\,, \tau - t)\,, \qquad \forall \tau \geq t\,. \tag{2.73}$$

Note that, without loss of generality, β can be considered as a "class $\mathcal{K}_\infty\mathcal{L}$" function, meaning that it is of class $\mathcal{K}_\infty$ in its first argument[12]. By Massera's lemma (see [Mas49] or *e.g.* [Kha01, Lemma C.1]), there exists a class $\mathcal{K}_\infty$ function η, with $\eta' \in \mathcal{K}$, such that

$$\int_0^\infty \eta \circ \beta(\Delta, \tau) d\tau < \infty \tag{2.74}$$

$$\int_0^\infty \eta'(\beta(\Delta, \tau)) e^{L(\beta(\Delta,0)+\Delta)\tau} d\tau < \infty\,. \tag{2.75}$$

Inspired by Massera-Malkin's constructions, we show that Lemma 2.25 can be established with the function satisfying, for all $x \in \mathcal{B}_\Delta$ and all $t \in \mathbb{R}_{\geq 0}$,

$$V(t,x) := \int_t^\infty \eta(|\phi(\tau, t, x)|_\delta) d\tau\,. \tag{2.76}$$

In the case of a time-invariant system $\dot{x} = f(x)$, we have that $\phi(\tau, t, x) = \phi(\tau, 0, x)$, which already shows that V can be picked as a time-invariant function.

Proof of bound (2.70): In view of (2.73), V can be bounded in the following way:

$$V(t,x) \leq \int_t^\infty \eta \circ \beta(|x|\,, \tau - t) d\tau = \int_0^\infty \eta \circ \beta(|x|\,, \tau) d\tau =: \tilde{\overline{\alpha}}(|x|)\,. \tag{2.77}$$

From (2.74), we see that $\tilde{\overline{\alpha}}(|x|)$ is finite for all $x \in \mathcal{B}_\Delta$. In addition, it clearly vanishes at zero. Furthermore, we claim that it is continuous at zero. To see this, consider any sequence $\{x_i\}_{i\in\mathbb{N}}$ of $\mathcal{B}_\Delta$ that converges to 0 as i tends to infinity. Then, for all $\tau \in \mathbb{R}_{\geq 0}$, $\lim_{i\to\infty} \eta \circ \beta(|x_i|\,, \tau) = 0$ since $\eta \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$. In addition, for all $i \in \mathbb{N}$, $\eta \circ \beta(|x_i|\,, \tau) \leq \eta \circ \beta(\Delta, \tau)$ which, in view of (2.74), is integrable. It follows from Lebesgue's dominated convergence theorem (*cf. e.g.* [KF70, p. 303]) that $\lim_{i\to\infty} \tilde{\overline{\alpha}}(|x_i|) = 0$, which establishes the claim. In turn, since $\tilde{\overline{\alpha}}$ is continuous at zero and finite everywhere else, it can be upper bounded by a class $\mathcal{K}_\infty$ function $\overline{\alpha}$.

Moreover, since V is nonnegative, the bound $V(t,x) \geq \underline{\alpha}(|x|_\delta)$ is trivially satisfied for all $x \in \mathcal{B}_\delta$ and any class $\mathcal{K}_\infty$ function $\underline{\alpha}$. So, from now on, we consider that $|x| > \delta$.

[12] If β is of class $\mathcal{K} \setminus \mathcal{K}_\infty$ in its first argument, we replace it by, for instance, $\tilde{\beta}(s,t) := \beta(s,t)$ for $s \leq \Delta$ and $\beta(s,t) := \beta(\Delta, t)s/\Delta$ for $s > \Delta$.

In this situation, we have that $|x|_\delta = |x| - \delta$. Using (2.73) and the continuity of the solution $\phi(\cdot, t, x)$, there exists a positive time t_1 such that $|\phi(t+t_1, t, x)|_\delta = |x|_\delta /2$ and $|\phi(\tau, t, x)|_\delta \geq |x|_\delta /2$ for all $\tau \in [t; t+t_1]$. We show that this time t_1 cannot be too small. To that end, notice that the mean value theorem ensures the existence of a time $t_2 \in]0; t_1[$ such that

$$\frac{|\phi(t+t_1, t, x)|_\delta - |\phi(t, t, x)|_\delta}{t_1} = \left[\frac{\partial}{\partial \tau}\big(\,|\phi(\tau, t, x)|_\delta\,\big)\right]_{\tau=t+t_2} \geq -\left|\frac{\partial}{\partial \tau}\big(\,|\phi(\tau, t, x)|_\delta\,\big)\right|_{\tau=t+t_2} .$$

It follows that

$$t_1 \left(\sup_{\tau \in]t; t+t_1[} \left|\frac{\partial}{\partial \tau}\big(\,|\phi(\tau, t, x)|_\delta\,\big)\right| \right) \geq |\phi(t, t, x)|_\delta - |\phi(t+t_1, t, x)|_\delta = \frac{|x|_\delta}{2} . \tag{2.78}$$

Since $|\phi(\tau, t, x)| > \delta$ for all $\tau \in]t; t+t_1[$, we have that

$$\frac{\partial}{\partial \tau}\big(\,|\phi(\tau, t, x)|_\delta\,\big) = \frac{\partial}{\partial \tau}\big(\,|\phi(\tau, t, x)| - \delta\big) = \frac{f(\tau, \phi(\tau, t, x))^\top \phi(\tau, t, x)}{|\phi(\tau, t, x)|} . \tag{2.79}$$

In addition, from Assumption 2.24 and using again the mean value theorem,

$$|f(t, x)| \leq f_0 + L(|x|)\,|x| \,, \qquad \forall x \in \mathbb{R}^n, \, \forall t \in \mathbb{R}_{\geq 0} .$$

From this and (2.73) it follows that

$$|f(\tau, \phi(\tau, t, x))| \leq f_0 + L(\beta(|x|\,, \tau - t) + \delta)\big(\beta(|x|\,, \tau - t) + \delta\big) .$$

Thus, injecting this bound into (2.79), it holds that, for all $x \in \mathcal{B}_\Delta \setminus \mathcal{B}_\delta$ and all $\tau \in]t; t+t_1[$,

$$\begin{aligned} \left|\frac{\partial}{\partial \tau}\big(\,|\phi(\tau, t, x)|_\delta\,\big)\right| &\leq f_0 + L(\beta(\Delta, 0) + \delta)\big(\beta(\Delta, 0) + \delta\big) \\ &\leq f_0 + L(\beta(\Delta, 0) + \Delta)\big(\beta(\Delta, 0) + \Delta\big) =: f_M(\Delta) \end{aligned} \tag{2.80}$$

since Δ is assumed greater than δ. In addition, this together with inequality (2.78) implies

$$t_1 \geq \frac{|x|_\delta}{2 f_M(\Delta)} .$$

From this, we can exhibit a class $\mathcal{K}_\infty$ lower bound on V as follows:

$$V(t, x) = \int_t^\infty \eta(|\phi(\tau, t, x)|_\delta) d\tau \geq \int_t^{t+t_1} \eta(|\phi(\tau, t, x)|_\delta) d\tau .$$

Since, by construction, $|\phi(\tau, t, x)|_\delta \geq |x|_\delta /2$ over the interval $[t\,;\, t+t_1]$, it follows that

$$V(t, x) \geq \eta\left(\frac{|x|_\delta}{2}\right) t_1 \geq \eta\left(\frac{|x|_\delta}{2}\right) \frac{|x|_\delta}{2 f_M(\Delta)} =: \underline{\alpha}(|x|_\delta) . \tag{2.81}$$

Note that $\underline{\alpha}$ is, as required, a class $\mathcal{K}_\infty$ function.

Proof of bound (2.71): The total time derivative of V is given by

$$\begin{aligned} \dot{V}(t, x) = \frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) f(t, x) &= -\eta(|\phi(t, t, x)|_\delta) + \int_t^\infty \frac{\partial}{\partial t}(\eta(|\phi(\tau, t, x)|_\delta)) d\tau \\ &+ \int_t^\infty \frac{\partial}{\partial x}(\eta(|\phi(\tau, t, x)|_\delta)) d\tau f(t, x) . \end{aligned} \tag{2.82}$$

Let us introduce the following notations:

$$\phi_t(\tau,t,x) := \frac{\partial}{\partial t}(\phi(\tau,t,x))\,, \qquad \phi_x(\tau,t,x) := \frac{\partial}{\partial x}(\phi(\tau,t,x))\,.$$

Proceeding as in the proof of bound (2.70) we obtain that, for all $\tau \geq t \geq 0$ and $x \in \mathcal{H}(\delta,\Delta)$, such that $|\phi(\tau,t,x)| \geq \delta$,

$$\frac{\partial}{\partial t}(\eta(|\phi(\tau,t,x)|_\delta)) = \phi(\tau,t,x)^\top \phi_t(\tau,t,x)\eta'(|\phi(\tau,t,x)|_\delta)\,, \tag{2.83}$$

$$\frac{\partial}{\partial x}(\eta(|\phi(\tau,t,x)|_\delta)) = \phi(\tau,t,x)^\top \phi_x(\tau,t,x)\eta'(|\phi(\tau,t,x)|_\delta)\,. \tag{2.84}$$

Define $\Gamma(t,x) := \{\tau \geq t \ : \ |\phi(\tau,t,x)| \geq \delta\}$. Since $f(t,x)$ is assumed to be locally Lipschitz in x uniformly in t, the solution $\phi(\tau,t,x)$ is continuous with respect to each of its three arguments (see *e.g.* [Kha01, Theorem 3.5]). In particular, given $\tau \geq 0$ and $x \in \mathbb{R}^n$, if we have that $|\phi(\tau,t^\star,x)| < \delta$ for some $t^\star \geq 0$, then there exists an interval $[t^\star; t^\star + t_\varepsilon]$, with $t_\varepsilon > 0$, on which $|\phi(\tau,t^\star,x)| < \delta$. Hence, we have that $\eta(|\phi(\tau,t,x)|_\delta) = 0$ for all $t \in [t^\star; t^\star + t_\varepsilon]$ and, consequently,

$$\frac{\partial}{\partial t}(\eta(|\phi(\tau,t,x)|_\delta)) = 0\,, \qquad \forall \tau \in \mathbb{R}_{\geq t} \setminus \Gamma\,,$$

which in its turn implies that

$$\int_t^\infty \frac{\partial}{\partial t}(\eta(|\phi(\tau,t,x)|_\delta))d\tau = \int_\Gamma \frac{\partial}{\partial t}(\eta(|\phi(\tau,t,x)|_\delta))d\tau\,. \tag{2.85}$$

From (2.83) and (2.85), we get that

$$\int_t^\infty \frac{\partial}{\partial t}(\eta(|\phi(\tau,t,x)|_\delta))d\tau = \int_\Gamma \phi(\tau,t,x)^\top \phi_t(\tau,t,x)\eta'(|\phi(\tau,t,x)|_\delta)d\tau\,,$$

and, similarly,

$$\int_t^\infty \frac{\partial}{\partial x}(\eta(|\phi(\tau,t,x)|_\delta))d\tau = \int_\Gamma \phi(\tau,t,x)^\top \phi_x(\tau,t,x)\eta'(|\phi(\tau,t,x)|_\delta)d\tau\,. \tag{2.86}$$

Thus, in view of (2.82), we obtain that

$$\dot V = -\eta(|x|_\delta) + \int_\Gamma \phi(\tau,t,x)^\top \Big(\phi_t(\tau,t,x) + \phi_x(\tau,t,x)f(t,x)\Big)\eta'(|\phi(\tau,t,x)|_\delta)d\tau\,.$$

Proceeding as in [Kha01, Exercise 3.30, p.110], we see that $\phi_t(\cdot,t,x) + \phi_x(\cdot,t,x)f(t,x) \equiv 0$, so

$$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)f(t,x) = -\eta(|x|_\delta)\,.$$

Now, consider any $\delta' > \delta$. Then, for all $|x| \geq \delta'$, it holds that $|x|_\delta = |x| - \delta$. Therefore

$$|x|_\delta = \frac{\delta'-\delta}{\delta'}\,|x| + \frac{\delta}{\delta'}\,|x| - \delta \geq \frac{\delta'-\delta}{\delta'}\,|x| \geq \varepsilon\,|x|\,,$$

where $\varepsilon := \frac{\delta'/\delta - 1}{\delta'/\delta}$ is a constant depending only on the ratio δ'/δ. Thus, using the previously established upper bound on V, we obtain that, for all $x \in \mathcal{B}_\Delta \setminus \mathcal{B}_\delta$,

$$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial t}(t,x)f(t,x) \leq -\alpha(V(t,x))\,,$$

where α is the class $\mathcal{K}_\infty$ function defined as

$$\alpha(s) := \eta(\varepsilon\overline{\alpha}^{-1}(s))\,. \tag{2.87}$$

Proof of bound (2.72): According to (2.86), we have that

$$\int_t^\infty \frac{\partial}{\partial x}(\eta(|\phi(\tau,t,x)|_\delta))d\tau = \int_\Gamma \phi(\tau,t,x)^\top \phi_x(\tau,t,x)\eta'(|\phi(\tau,t,x)|_\delta)d\tau\,. \tag{2.88}$$

Notice that ϕ_x is a solution of

$$\frac{\partial \phi_x}{\partial \tau}(\tau,t,x) = \frac{\partial f}{\partial x}\big(\tau,\phi(\tau,t,x)\big)\phi_x(\tau,t,x)\,,\quad \phi_x(t,t,x) = I\,.$$

Integrating from t to $\tau \geq t$ on both sides of the first equality above and using (2.73), we obtain that

$$\phi_x(\tau,t,x) - I \leq L(\beta(|x|\,,0)+\delta)\int_t^\tau \phi_x(s,t,x)ds\,.$$

Hence, applying Gronwall-Bellman's inequality,

$$|\phi_x(\tau,t,x)| \leq e^{L(\beta(|x|,0)+\delta)(\tau-t)} \leq e^{L(\beta(|x|,0)+\delta)}\,.$$

From this and (2.88), we see that

$$\begin{aligned}
\left|\frac{\partial V}{\partial x}(t,x)\right| &\leq \int_\Gamma |\phi(\tau,t,x)|\, e^{L(\beta(|x|,0)+\delta)(\tau-t)}\eta'(|\phi(\tau,t,x)|_\delta)d\tau \\
&\leq \int_\Gamma (\beta(|x|\,,\tau-t)+\delta)\, e^{L(\beta(|x|,0)+\delta)(\tau-t)}\eta'\left(\beta(|x|\,,\tau-t)\right)d\tau \\
&\leq \int_t^\infty (\beta(|x|\,,\tau-t)+\delta)\, e^{L(\beta(\Delta,0)+\delta)(\tau-t)}\eta'\circ\beta(\Delta,\tau-t)d\tau \\
&\leq \int_0^\infty (\beta(|x|\,,\tau)+\Delta)\, e^{L(\beta(\Delta,0)+\Delta)\tau}\eta'\circ\beta(\Delta,\tau)d\tau\,.
\end{aligned}$$

In view of (2.75), this integral is finite for any $x \in \mathcal{B}_\Delta$. Hence, it can be upper bounded by a continuous increasing function $c(|x|)$, which completes the proof. ■

Based on Lemma 2.25 and Remark 2.26, we are now ready to introduce a necessary condition for uniform semiglobal practical asymptotic stability in terms of a Lyapunov-like function.

Theorem 2.27 (Converse Lyapunov function for USPAS) *Assume that the system $\dot{x} = f(t,x,\theta)$ introduced in (1.3) is USPAS on $\Theta \subset \mathbb{R}^m$ and that, for any $\theta \in \Theta$, there exist a nonnegative constant f_θ and a continuous nondecreasing function L_θ such that, for all $x \in \mathbb{R}^n$, Inequalities (2.68) and (2.69) hold for $f_\theta(t,x) := f(t,x,\theta)$. Then, for any $\Delta > \delta > 0$, there exists a parameter $\theta^\star(\delta,\Delta) \in \Theta$, a continuously differentiable function $V_{\delta,\Delta} : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, class $\mathcal{K}_\infty$ functions $\underline{\alpha}_{\delta,\Delta}$, $\overline{\alpha}_{\delta,\Delta}$, $\alpha_{\delta,\Delta}$ and a continuous positive nondecreasing function $c_{\delta,\Delta}$ such that, for all $x \in \mathcal{H}(\delta,\Delta)$ and all $t \in \mathbb{R}_{\geq 0}$,*

$$\underline{\alpha}_{\delta,\Delta}(|x|) \leq V_{\delta,\Delta}(t,x) \leq \overline{\alpha}_{\delta,\Delta}(|x|) \tag{2.89}$$

$$\frac{\partial V_{\delta,\Delta}}{\partial t}(t,x) + \frac{\partial V_{\delta,\Delta}}{\partial x}(t,x)f(t,x,\theta^\star) \leq -\alpha_{\delta,\Delta}(V_{\delta,\Delta}(t,x)) \tag{2.90}$$

$$\left|\frac{\partial V_{\delta,\Delta}}{\partial x}(t,x)\right| \leq c_{\delta,\Delta}(|x|)\,. \tag{2.91}$$

In addition, if (1.3) is time-invariant (i.e. $\dot{x} = f(x,\theta)$), then $V_{\delta,\Delta}$ can be picked as a time-invariant function too.

Proof of Theorem 2.27. By assumption, for any $\Delta > \delta > 0$, there exists a parameter $\theta^\star(\delta,\Delta) \in \Theta$ such that $\mathcal{B}_\delta$ is UAS on B_Δ for the system $\dot{x} = f_\star(t,x) := f(t,x,\theta^\star)$. The conclusion follows from Lemma 2.25 and Remark 2.26. ■

Note that, Theorem 2.27 cannot be rigorously considered as a converse of Theorem 2.11 since the requirements (2.31) and (2.32) are missing: the above result does not provide any information on the dependence of the upper and lower $\mathcal{K}_\infty$ bounds on the generated Lyapunov function $V_{\delta,\Delta}$ in the radii δ and Δ.

A careful inspection of the above construction of $\underline{\alpha}_{\delta,\Delta}$ and $\overline{\alpha}_{\delta,\Delta}$ convinces that, even by assuming uniformity of the $\mathcal{KL}$ estimate in the radius Δ, we cannot ensure (2.32) in general.

Nevertheless, we may expect these bounds on $V_{\delta,\Delta}$ to be independent of the radius δ provided a similar assumption on the $\mathcal{KL}$ estimate of the solutions. We refer to the latter property as δ-USPAS. As we will see in Chapter 3, this class of USPAS systems deserves a particular interest when dealing with cascaded systems.

Definition 2.28 (δ-USPAS) *The system (1.3) is said to be* δ-USPAS *if, given any $\Delta > 0$, there exists a $\mathcal{KL}$ function β_Δ such that, for any $0 < \delta < \Delta$, there exists a parameter $\theta^\star(\delta,\Delta) \in \Theta$ such that the solutions of (1.3) satisfy, for all $t_0 \in \mathbb{R}_{\geq 0}$ and all $x_0 \in \mathcal{B}_\Delta$,*

$$|\phi(t,t_0,x_0)|_\delta \leq \beta_\Delta(|x_0|\,, t-t_0)\,, \qquad \forall t \geq t_0\,.$$

A quick comparison between Definitions 1.19 and 2.28 shows that δ-USPAS is a stronger notion than USPAS. We stress that many definitions of practical stability existing in the literature do impose that the $\mathcal{KL}$ estimate be uniform in δ. This is for instance the case of [NL04, ST03] but, overall, of the semiglobal practical stability property guaranteed by the main result in [TPA99], which studies the USPAS of systems based on averaging techniques. As this fact constitutes the main motivation for the converse result below, we state it as an independent proposition. Its proof follows from Proposition 1.16 and a careful reading of the argument of the main result in [TPA99].

Proposition 2.29 (Averaging induces δ-USPAS) *If the origin of the averaged system (in the sense of [TPA99, Definition 2]) of the system $\dot{x} = f(t,x)$ introduced in (1.1) is globally asymptotically stable, then $\dot{x} = f(t/\theta,x)$ is δ-USPAS on the parameter set $\mathbb{R}_{>0}$.*

Under the stronger assumption of δ-USPAS, we show that the bounds on the generated Lyapunov function, on its derivative and on its gradient may be chosen disregarding the size of the ball to which solutions are required to converge. This fact will be used in the next chapter for the study of cascaded δ-USPAS systems.

Corollary 2.30 (Converse Lyapunov function for δ-USPAS) *Assume that the system $\dot{x} = f(t,x,\theta)$ introduced in (1.3) is δ-USPAS on $\Theta \subset \mathbb{R}^m$ and that there exist a nonnegative constant f_0 and a continuous nondecreasing function L such that (2.68) and (2.69) hold for all $\theta \in \Theta$, all $x \in \mathbb{R}^n$ and all $t \in \mathbb{R}_{\geq 0}$. Then, for any $\Delta > 0$, there exist*

class $\mathcal{K}_\infty$ functions $\underline{\alpha}_\Delta$, $\overline{\alpha}_\Delta$, α_Δ and a continuous positive nondecreasing function c_Δ and, for any $\delta \in (0;\Delta)$, there exist a parameter $\theta^\star(\delta,\Delta) \in \Theta$ and a continuously differentiable function $V_{\delta,\Delta} : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that, for all $x \in \mathcal{H}(\delta,\Delta)$ and all $t \in \mathbb{R}_{\geq 0}$,

$$\underline{\alpha}_\Delta(|x|) \leq V_{\delta,\Delta}(t,x) \leq \overline{\alpha}_\Delta(|x|) \tag{2.92}$$

$$\frac{\partial V_{\delta,\Delta}}{\partial t}(t,x) + \frac{\partial V_{\delta,\Delta}}{\partial x}(t,x) f(t,x,\theta^\star) \leq -\alpha_\Delta(V_{\delta,\Delta}(t,x))\,. \tag{2.93}$$

$$\left|\frac{\partial V_{\delta,\Delta}}{\partial x}(t,x)\right| \leq c_\Delta(|x|)\,. \tag{2.94}$$

In addition, if (1.3) is time-invariant (i.e. $\dot{x} = f(x,\theta)$), then $V_{\delta,\Delta}$ can be picked as a time-invariant function too.

With this construction, we see that the requirement (2.31) of Theorem 2.11, linking the δ-dependence of the $\mathcal{K}_\infty$ bounds on the Lyapunov function, is trivially satisfied. Indeed, due to the independence of $\underline{\alpha}_\Delta$ and $\overline{\alpha}_\Delta$ in δ, and their belonging to class $\mathcal{K}_\infty$, it suffices, for any $\delta_\star > 0$, to pick δ small enough in order to ensure $\underline{\alpha}_\Delta^{-1} \circ \overline{\alpha}_\Delta(\delta) \leq \delta_\star$. See Theorem 3.12 in the next chapter for an application of this result to the stability analysis of cascaded systems.

Proof of Corollary 2.30. In the proof of Lemma 2.25, we see that, if the $\mathcal{KL}$ estimate β is independent of δ, then so is η (see (2.74) and (2.75)). In the same way, the independence of f_0 and L in θ makes f_M independent of δ (*cf.* (2.80)). Hence, in view of (2.77) and (2.81), it is also the case for the functions $\underline{\alpha}$ and $\overline{\alpha}$. Moreover, by picking $\delta' = 2\delta$ (and, in accordance, $\Delta > 2\delta$), we get that $\varepsilon = 1/2$ (see "Proof of bound (2.71)"), which shows with (2.87) that α can be chosen independently of δ as well. Notice finally that this is also the case for the function c (*cf.* "Proof of bound (2.72)"). ■

A converse Lyapunov function for parameterized nonlinear time-varying systems has already been proposed in [Hop66], based on an assumption of asymptotic stability of the origin uniformly in both the initial conditions and the parameters. We stress that the results presented below do not fit into this framework. Theorems 2.27 and 2.31 indeed deal with the original definitions of USPAS and USAS (*cf.* Definitions 1.17 and 1.19), which, as already seen, do not require uniformity in the tuning parameter θ. Concerning Corollary 2.30, even though it requires more uniformity (the $\mathcal{KL}$ estimate of the solutions needs to be the same disregarding the ball to which solutions converge), this requirement remains less conservative. In addition, contrarily to Theorem 2.27 and Corollary 2.30, the main result in [Hop66] only addresses stability of a single point. Finally, the latter reference does not provide any bound on the gradient of the generated Lyapunov function.

It should also be pointed out that a converse result for "practical" stability was proposed by Kaplan in [Kap73]. However, this result fundamentally differs from the results presented here as this notion of "practicality" was only concerned with a *fixed* compact neighborhood of the origin. No possibility of reducing this ball at will, through the tuning of some parameter, was assumed for this stability property.

2.2.2 Semiglobal asymptotic stability

For the sake of completeness, and as it directly follows from the results presented in the previous section, we state here a converse result for uniform semiglobal asymptotic stability.

Theorem 2.31 (Converse Lyapunov function for USAS) *Assume that the system $\dot{x} = f(t, x, \theta)$ introduced in (1.3) is USAS on Θ and that, for any $\theta \in \Theta$, there exist a nonnegative constant f_θ and a continuous nondecreasing function L_θ such that, for all $x \in \mathbb{R}^n$, Inequalities (2.68) and (2.69) hold for $f_\star(t, x) := f(t, x, \theta^\star)$. Then, for any $\Delta > 0$, there exists a parameter $\theta^\star(\Delta) \in \Theta$, a continuously differentiable function $V_\Delta : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, class $\mathcal{K}_\infty$ functions $\underline{\alpha}_\Delta$, $\overline{\alpha}_\Delta$, α_Δ and a continuous positive nondecreasing function c_Δ such that, for all $x \in \mathcal{B}_\Delta$ and all $t \in \mathbb{R}_{\geq 0}$,*

$$\underline{\alpha}_\Delta(|x|) \leq V_\Delta(t, x) \leq \overline{\alpha}_\Delta(|x|)$$

$$\frac{\partial V_\Delta}{\partial t}(t, x) + \frac{\partial V_\Delta}{\partial x}(t, x) f(t, x, \theta^\star) \leq -\alpha_\Delta(V_\Delta(t, x))$$

$$\left| \frac{\partial V_\Delta}{\partial x}(t, x) \right| \leq c_\Delta(|x|) \, .$$

In addition, if (1.3) is time-invariant (i.e. $\dot{x} = f(x, \theta)$), then V_Δ is also a time-invariant function.

No such converse result is provided for uniform global practical stability in this document. There are two main reasons for this. The first one is a theoretical obstacle: the above proof techniques extensively make use of the fact that the state is constrained into a compact region of the state space, which is not the case when considering *global* properties. The second reason stands in the fact that, in most cases, the easiest way to guarantee UGPAS is to apply Theorem 2.5, which is based on Lyapunov arguments. This observation makes the utility of such an extension questionable.

Conclusion. This chapter proposes tools related to semiglobal and/or practical stability properties. Due to the wide generality of the used concepts, and mostly the possible non-uniformity of the estimate of solutions in the tuning parameter, a specific attention needs to be paid in the Lyapunov analysis. Compared to existing results, the sufficient conditions for USPAS, USAS and UGPAS presented here impose indeed an additional assumption that links the bounds on the Lyapunov function. Furthermore, the necessity of imposing such an additional condition is shown through a counter-example. Converse results, for USPAS and USAS, are also presented. The use of most of these results is illustrated by means of academic examples.

Chapter 3

Stability of nonlinear time-varying cascaded systems

Divide to reign. The stability analysis by Lyapunov's second method requires the construction of a strict Lyapunov function, as for instance Theorems 2.3, 2.5, 2.11 and 2.18. This direct approach may be particularly hard for complex or large-scale nonlinear time-varying systems. A natural way of simplifying this problem consists in dividing the system into simpler interconnected subsystems, and to analyze each subsystem separately.

Many tools in the literature provide stability properties of interconnected subsystems based on some information of each component taken individually. Probably the most fundamental result for interconnected system is the small gain theorem for input-output stability, originally introduced by Zames [Zam60, Zam66a, Zam66b] and Sandberg [San63], which studies the general interconnection of two input-to-output stable systems:

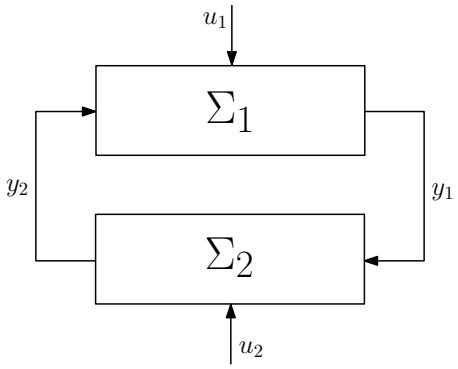


Figure 3.1: General interconnection of two dynamical systems.

A more modern perspective of input-output stability, notably involving the system's state (thus leaving behind Zames-Sandberg's "black-box" approach) has been taken in [JTP94, CTP95, JMW96, Tee96, LWC05]. While providing a strong tool for the stability analysis of interconnected nonlinear systems, it is also at the basis of many control designs, *cf. e.g.* [PW96, JM97].

A particular interconnection: the cascade. In many applications, the system may be represented as a *unidirectional* interconnection of dynamical subsystems, as represented

by Figure 3.2.

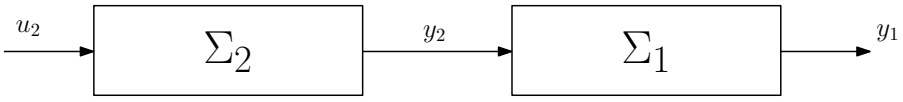


Figure 3.2: Cascade interconnection of two dynamical systems.

The wide occurrence of such a structure in control practice stems from many facts.

It was shown in [Vid80], based on graph-theoretic decomposition techniques, that any interconnected system can be rewritten as a cascade, modulo some renumbering and, overall, aggregation of the state variables. While this result may be surprising at first sight, it results from elementary manipulations and illustrates the generality of the cascade concept in systems theory. Of course, such decomposition is all the more efficient as the original system is less interconnected.

In addition, the cascade structure sometimes arises naturally, due to the physical structure of the system. For instance, in Chapter 6, we study the PID control of robot manipulators. The approach consists in studying the mechanical (*i.e.* the robot arm) and the electrical (*i.e.* actuators) parts separately. This approach is similar to that adopted in [PO96]. As one could intuitively expect it, the overall system consists in a cascade interconnection of these two parts. Please see [PLS99, Lef00, LFP00, dNC00] for other examples of concrete applications.

But this decomposition into cascades can also be done artificially by designing a control law that puts the system in a cascade structure, *cf.* [KKK95, LLLP05, Lef00, SL03] for example. Such an approach is referred to as *cascades-based design*.

A wide literature. In any of these situations, in order to decompose a complex problem into simpler ones using theorems for cascaded systems, it is crucial to answer the question whether the stability properties of both subsystems taken separately remain valid for their cascade interconnection.

From a theoretical point of view, this problem is not trivial. It has attracted the interest of the control community since [MMT78], where graph theory was used to ensure local and global stability properties of the cascade, based on the assumption that the interconnection terms are all "stability preserving mappings". In [Vid80], Vidyasagar made use of converse Lyapunov results to show that uniform *local* asymptotic stability is naturally preserved by the cascade structure.

Nevertheless, the global case presents harder difficulties. Intuitively, we could expect that, in order to preserve the global asymptotic stability of the cascade $\Sigma_1 - \Sigma_2$, it would suffice that the convergence rate of the driving subsystem Σ_2 be sufficiently high. This intuition is wrong in general, as proved in [SK91] through an elementary example involving a linear driving subsystem which yields a stronger peaking of the transients as the convergence is made faster. This transient peaking is enough to destabilize the cascade. Similarly, as shown in [SK03, TH04], neither integrability nor even exponential decay of the solutions of the driving subsystem is sufficient to preserve global asymptotic stability.

Beyond these obstacles, some sufficient conditions for the preservation of global asymptotic stability under the cascade interconnection have been proposed in the literature. In general terms, a fundamental result for the analysis of global stability for nonlinear systems

states that the cascade of uniformly globally asymptotically stable systems (UGAS) remain UGAS if and only if its solutions are uniformly globally bounded. See [SS90a, Son89b] for the proof of this statement in the case of autonomous systems and [PL01] for the case of time-varying systems.

Some work has also been done in order to advantageously replace the requirement of (uniform) global boundedness by more easily checkable conditions. In [SS90a], these take the form of a robustness Lyapunov condition on the driven subsystem that needs to hold for large values of the states. In [PL98], uniform global boundedness of solutions is replaced by the requirements that the interconnection be affine in the state of the driven subsystem, that the solutions of the driving subsystem be integrable and that a Lyapunov function, with a convenient bound on its gradient be known for the driven subsystem. In [PL01], other sufficient conditions were provided, expressed as dominance relationships involving the bounds on the Lyapunov function and on its gradient together with the interconnection term. In [AAS02], an elegant reformulation of the integrability condition posed in [PL01] was established in terms of integral input to state stability. More precisely, it is imposed that the driven subsystem be integral input to state stable and that a condition linking the dissipation rate of the driving subsystem to the iISS gain of the driven one holds (please see Chapter 5 for details).

The obvious drawback of these results is that they impose *global* properties on each subsystem which, as seen in the previous chapters, are impossible to achieve in many situations. Also, the solutions may only asymptotically reach a *neighborhood* of the origin instead of the the origin itself. Below, we extend these results to uniform semiglobal and/or practical asymptotic stability[1].

A word on stabilization. Although this does not constitute the subject of this discussion, it is worth stressing that a considerable work in the literature has been devoted to the *stabilization* of cascaded systems. The problem is well documented for local properties (*cf. e.g.* [Vid80, BI91, CP91]). Nevertheless, in [Sus90], an example illustrates that global and semiglobal stabilizability intrinsically generate additional obstacles compared to local results. It was further shown in [SKS90] that, given any strictly non-minimum phase linear driving subsystem, there exists a globally asymptotically stable driven subsystem such that their cascade is *not* globally stabilizable.

To face this problem, several stabilization approaches were adopted. In [Ort91], Ortega follows a passivity approach to ensure global asymptotic stabilizability of the cascade. In [JSK96, MSJ97], a Lyapunov function is explicitly designed for a wide class of cascaded systems, based on the knowledge of a Lyapunov function for each subsystem and the assumption of a linear growth of the interconnection. An extension of this, allowing a polynomial interconnection, was later proposed in [MSJ99]. See also [SJK97] an references therein.

Global and semiglobal stabilization of cascades via *output* feedback also gave rise to challenging problems, as stressed through examples in [MPD94]. Please refer to [MT91, KKM92, TP94, SAT02] for various results in this area.

[1] Set-stability will be discussed in a cascade context in Chapter 4.

3.1 Semiglobal practical asymptotic stability of cascaded systems

In this section, we address the stability analysis problem for cascades of time-varying systems that are uniformly semiglobally practically asymptotically stable (USPAS). We establish that, under a uniform semiglobal boundedness condition on its solutions, the cascade of two USPAS systems remains USPAS. More precisely, we consider cascaded systems of the form

$$\dot{x}_1 = f_1(t, x_1, \theta_1) + g(t, x, \theta) \tag{3.1a}$$
$$\dot{x}_2 = f_2(t, x_2, \theta_2) \tag{3.1b}$$

where $x := (x_1^\top, x_2^\top)^\top \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, $\theta := (\theta_1^\top, \theta_2^\top)^\top \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$, $t \in \mathbb{R}_{\geq 0}$, f_1, f_2 and g are locally Lipschitz in state and satisfy the Carathéodory conditions.

The subsystems $\dot{x}_1 = f_1(t, x_1, \theta_1)$ and $\dot{x}_2 = f_2(t, x_2, \theta_2)$ are respectively referred to as *driven* and *driving* subsystems, whereas $g(t, x, \theta)$ denotes the *interconnection term*.

In order to simplify the statement of our results, we first introduce the following notation.

Definition 3.1 ($\mathcal{D}$-set) *For any $\Delta > \delta \geq 0$, the* $\mathcal{D}$-set *of the dynamical system $\dot{x} = f(t, x, \theta)$, introduced in (1.3), is defined as*

$$\mathcal{D}_f(\delta, \Delta) := \left\{ \theta \in \mathbb{R}^m \ : \ \mathcal{B}_\delta \text{ is UAS on } \mathcal{B}_\Delta \text{ for (1.3)} \right\}.$$

In other words, given two constants $\delta \geq 0$ and $\Delta > 0$, the set $\mathcal{D}_f(\delta, \Delta)$ contains all the values of the tuning parameters that make $\mathcal{B}_\delta$ uniformly asymptotically stable on $\mathcal{B}_\Delta$ for the system (1.3). Notably, if $\mathcal{D}_f(\delta, \Delta) \cap \Theta \neq \emptyset$ for all $\Delta \geq \delta > 0$, then (1.3) is USPAS on Θ. Note that we have the property that, if $\delta' \leq \delta$ and $\Delta' \geq \Delta$, then $\mathcal{D}_f(\delta', \Delta') \subset \mathcal{D}_f(\delta, \Delta)$. However, in Definition 1.18, nothing is assumed concerning the structure of $\mathcal{D}_f(\delta, \Delta)$. For instance, it may even consist in a single parameter for each pair (δ, Δ). This is worth mentioning, since, as stated in Chapter 2, other definitions of semiglobal practical stability in the literature (see *e.g.* [TPA99, MA00, NL04, TNM05]) do impose that all the parameters in a neighborhood of zero generate the same radii Δ of the ball of attraction and δ of the ball to which solutions converge, implying notably that $\mathcal{D}_f(\delta, \Delta)$ contains an open neighborhood of zero.

All the results presented in this chapter assume the following structure for the interconnection term g.

Assumption 3.2 (Boundedness of the interconnection term) *The function g is uniformly bounded both in time and in θ_2 and vanishes with x_2, i.e., for any $\theta_1 \in \Theta_1$, there exists a nondecreasing function G_{θ_1} and a class $\mathcal{K}$ function Ψ_{θ_1} such that, for all $\theta_2 \in \Theta_2$, all $x \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and all $t \in \mathbb{R}_{\geq 0}$,*

$$|g(t, x, \theta)| \leq G_{\theta_1}(|x|)\Psi_{\theta_1}(|x_2|).$$

3.1.1 With a Lyapunov function for the driven subsystem

The following result provides sufficient conditions for the preservation of USPAS under cascade interconnection, based on the knowledge of a Lyapunov function for the x_1-subsystem.

Theorem 3.3 (Lyapunov USPAS + USPAS + UB ⇒ USPAS) *Under Assumptions 3.2, 3.4, 3.5 and 3.6, the cascaded system (3.1) is USPAS on $\Theta_1 \times \Theta_2$.*

Assumption 3.4 (USPAS of the driving subsystem) *The system (3.1b) is USPAS on Θ_2.*

Assumption 3.5 (Lyapunov USPAS of the driven subsystem) *Given any $\Delta_1 > \delta_1 > 0$, there exist a parameter $\theta_1^\star(\delta_1, \Delta_1) \in \Theta_1$, a continuously differentiable Lyapunov function V_{δ_1,Δ_1}, class $\mathcal{K}_\infty$ functions $\underline{\alpha}_{\delta_1,\Delta_1}$, $\overline{\alpha}_{\delta_1,\Delta_1}$, $\alpha_{\delta_1,\Delta_1}$ and a continuous positive nondecreasing function c_{δ_1,Δ_1} such that, for all $x_1 \in \mathcal{H}(\delta_1, \Delta_1)$ and all $t \in \mathbb{R}_{\geq 0}$,*

$$\underline{\alpha}_{\delta_1,\Delta_1}(|x_1|) \leq V_{\delta_1,\Delta_1}(t, x_1) \leq \overline{\alpha}_{\delta_1,\Delta_1}(|x_1|) \tag{3.2}$$

$$\frac{\partial V_{\delta_1,\Delta_1}}{\partial t}(t, x_1) + \frac{\partial V_{\delta_1,\Delta_1}}{\partial x_1}(t, x_1) f_1(t, x_1, \theta_1^\star) \leq -\alpha_{\delta_1,\Delta_1}(|x_1|) \tag{3.3}$$

$$\left| \frac{\partial V_1}{\partial x_1}(t, x_1) \right| \leq c_{\delta_1,\Delta_1}(|x_1|) \tag{3.4}$$

Assumption 3.6 (Boundedness of solutions) *There exists a positive constant Δ_0 such that, for any given positive numbers δ_1, Δ_1, δ_2, Δ_2, satisfying $\Delta_1 > \max\{\delta_1; \Delta_0\}$ and $\Delta_2 > \delta_2$, and for the parameter $\theta_1^\star(\delta_1, \Delta_1)$ as defined in Assumption 3.5, there exists a parameter $\theta_2^\star \in \mathcal{D}_{f_2}(\delta_2, \Delta_2) \cap \Theta_2$ (cf. Definition 3.1) and a positive number $\gamma(\delta_1, \delta_2, \Delta_1, \Delta_2)$ such that the trajectories of (3.1) with $\theta = \theta^\star$ satisfy*

$$|x_0| \leq \gamma(\delta_1, \delta_2, \Delta_1, \Delta_2) \qquad \Rightarrow \qquad |\phi(t, t_0, x_0, \theta^\star)| \leq \Delta_1 \,, \qquad \forall t \geq t_0 \,.$$

In addition, given any $\Delta_\star > \delta_\star > 0$, there exist some positive δ_1, Δ_1 and Δ_2, with $\Delta_1 > \delta_1$, such that, for all $\delta_2 \in (0; \Delta_2)$,

$$\min\{\Delta_1; \Delta_2; \gamma(\delta_1, \delta_2, \Delta_1, \Delta_2)\} \;\geq\; \Delta_\star \tag{3.5}$$

$$\max\left\{\delta_2; \underline{\alpha}_{\delta_1,\Delta_1}^{-1} \circ \overline{\alpha}_{\delta_1,\Delta_1}(\delta_1)\right\} \;\leq\; \delta_\star \,. \tag{3.6}$$

An application of this result is presented in Section 6.1: we establish that the PID control of robot manipulators is USPAS in presence of external disturbances, model imperfections and when taking into account the dynamics of the actuators.

In view of Theorem 2.11, Assumption 3.5 corresponds to the Lyapunov sufficient condition for USPAS of the zero-input x_1-subsystem, with the additional condition of a bound on the gradient of V_{δ_1,Δ_1}. In turn, we stress that the requirement corresponding to (2.32) is no longer needed under Assumption 3.6. We state the above result under the more restrictive assumption than simply "USPAS" since our proof relies on the explicit knowledge of the Lyapunov function V_{δ_1,Δ_1}. However, as we show in the next section, Assumption 3.5 can be relaxed based on the converse result presented in Section 2.2 provided that f_1 is sufficiently smooth and that a stronger notion of USPAS (namely, δ-USPAS) is assumed on the driven subsystem.

Roughly speaking, Assumption 3.6 imposes that the solutions be uniformly bounded by Δ_1 for all initial conditions in a domain that can be arbitrarily enlarged by conveniently picking δ_1, Δ_1, δ_2 and Δ_2. Note that Proposition 2.13 provides an efficient tool to check Assumption 3.6 in specific control applications.

Remark 3.7 *Assumption 3.6 may be relaxed to uniform boundedness on $\mathcal{B}_{\Delta_1} \times \mathcal{B}_{\Delta_2}$ provided that it holds* uniformly *in Δ_1 and Δ_2,* i.e.*, provided that the class $\mathcal{K}$ function η and the constant μ in Proposition 1.12 are independent of Δ_1 and Δ_2.*

Remark 3.8 *We stress that, similarly to Theorem 2.11, Theorem 3.3 still holds when (3.2) is replaced by the less conservative requirement:*

$$\underline{\alpha}_{\delta_1,\Delta_1}(|x_1|_{\delta_1}) \leq V_{\delta_1,\Delta_1}(t,x_1) \leq \overline{\alpha}_{\delta_1,\Delta_1}(|x_1|)\,.$$

Please refer to [CL06a, CL06b] for details. Furthermore, in view of Remark 2.12, it is in fact sufficient that the requirements of Assumption 3.5 hold for all small *δ_1 and all* large *Δ_1.*

Remark 3.9 *For clarity, we stated the result by assuming that the bounds on the interconnection term are independent of the parameter θ_2. We stress that, if needed in specific applications, Theorem 3.3 still holds when Assumption 3.2 is replaced by the following: for any $\theta \in \Theta_1 \times \Theta_2$, there exists a nondecreasing function G_θ and a class $\mathcal{K}$ function Ψ_θ such that, for all all $x \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and all $t \in \mathbb{R}_{\geq 0}$,*

$$|g(t,x,\theta)| \leq G_\theta(|x|)\Psi_\theta(|x_2|)\,,$$

and, for all $\Delta_1 > \delta_1 > 0$, all $\Delta_2 > 0$, and for the parameter $\theta_1^\star(\delta_1,\Delta_1)$ introduced in Assumption 3.5, it holds that

$$\lim_{\delta_2 \to 0} \Big[\sup \big\{ G_\theta(\Delta_1)\Psi_\theta(2\delta_2) \,:\, \theta \in \{\theta_1^\star(\delta_1,\Delta_1)\} \times \mathcal{D}_{f_2}(\delta_2,\Delta_2) \big\} \Big] = 0\,. \tag{3.7}$$

This appears more clearly along the proof below.

Proof of Theorem 3.3. The argument consists in constructing some balls $\mathcal{B}_\delta$ and $\mathcal{B}_\Delta$ and a $\mathcal{KL}$ estimate for the solutions of the cascaded system, based on the respective balls for the x_1 and the x_2 subsystems.

For any given positive δ_1, Δ_1, δ_2 and Δ_2 satisfying $\Delta_1 > \max\{\delta_1, \Delta_0\}$ and $\Delta_2 > \delta_2$, let $\gamma(\delta_1,\delta_2,\Delta_1,\Delta_2)$ be generated by Assumption 3.6 and define

$$\Delta := \min\{\Delta_1\,;\,\Delta_2\,;\,\gamma(\delta_1,\delta_2,\Delta_1,\Delta_2)\}\,. \tag{3.8}$$

Next, choose any $\theta_1^\star(\delta_1,\Delta_1) \in \Theta_1$ satisfying Assumption 3.5 and any $\theta_2^\star(\delta_2,\Delta_2)$ in the intersection of $\mathcal{D}_{f_2}(\delta_2,\Delta_2)$ and Θ_2 given by Assumption 3.6. We show that, provided that δ_1, δ_2 are sufficiently small and that Δ_1, Δ_2 are large enough, there exists $\delta \in (0;\Delta)$ such that $\mathcal{B}_\delta$ is uniformly asymptotically stable on $\mathcal{B}_\Delta$ for the system (3.1) with $\theta^\star = (\theta_1^{\star\top}, \theta_2^{\star\top})^\top$. To that end, we first show that there exists a positive δ_3 such that the ball $\mathcal{B}_{\delta_3}$ is uniformly stable. More precisely, we construct $\eta_{\Delta_1,\Delta_2}^{\delta_1,\delta_2} \in \mathcal{K}_\infty$ and $\delta_3 > 0$ such that, for all $x_0 \in \mathcal{B}_\Delta$,

$$|\phi_1(t,t_0,x_0,\theta^\star)|_{\delta_3} \leq \eta_{\Delta_1,\Delta_2}^{\delta_1,\delta_2}(|x_0|)\,. \tag{3.9}$$

Then, we use this property to prove that a ball, larger than $\mathcal{B}_{\delta_3}$, is uniformly attractive on $\mathcal{B}_\Delta$ and we construct a $\mathcal{KL}$ estimate for the solutions. Finally, we show that the estimates of the domain of attraction $\mathcal{B}_\Delta$ and of the ball $\mathcal{B}_\delta$ to which solutions converge can be arbitrarily enlarged and diminished respectively.

Proof of uniform stability: Consider the function V_{δ_1,Δ_1} generated by Assumption 3.5 and let Lemma 2.6, with $X = \mathcal{H}(\delta_1,\Delta_1)$ and $k = 1$, generate a function $\mathcal{V}_{\delta_1,\Delta_1}$, class $\mathcal{K}_\infty$ functions $\underline{\tilde{\alpha}}_{\delta_1,\Delta_1}$, $\tilde{\overline{\alpha}}_{\delta_1,\Delta_1}$ and a continuous nondecreasing function $\tilde{c}_{\delta_1,\Delta_1}$ such that, for all $x_1 \in \mathcal{H}(\delta_1,\Delta_1)$ and all $t \in \mathbb{R}_{\geq 0}$,

$$\underline{\tilde{\alpha}}_{\delta_1,\Delta_1}(|x_1|) \leq \mathcal{V}_{\delta_1,\Delta_1}(t,x_1) \leq \tilde{\overline{\alpha}}_{\delta_1,\Delta_1}(|x_1|)$$

$$\frac{\partial \mathcal{V}_{\delta_1,\Delta_1}}{\partial t}(t,x_1) + \frac{\partial \mathcal{V}_{\delta_1,\Delta_1}}{\partial x_1}(t,x_1) f_1(t,x_1,\theta_1^\star) \leq -\mathcal{V}_{\delta_1,\Delta_1}(t,x_1)$$

$$\left|\frac{\partial \mathcal{V}_{\delta_1,\Delta_1}}{\partial x_1}(t,x_1)\right| \leq \tilde{c}_{\delta_1,\Delta_1}(|x_1|)\,,$$

and, for any $s \in \mathbb{R}_{\geq 0}$,

$$\underline{\tilde{\alpha}}^{-1}_{\delta_1,\Delta_1} \circ \tilde{\overline{\alpha}}_{\delta_1,\Delta_1}(s) = \underline{\alpha}^{-1}_{\delta_1,\Delta_1} \circ \overline{\alpha}_{\delta_1,\Delta_1}(s)\,. \tag{3.10}$$

The total time derivative of $\mathcal{V}_{\delta_1,\Delta_1}$ along the trajectories of (3.1) with $\theta = \theta^\star$ yields

$$\dot{\mathcal{V}}_{\delta_1,\Delta_1}(t,x_1) = \frac{\partial \mathcal{V}_{\delta_1,\Delta_1}}{\partial t}(t,x_1) + \frac{\partial \mathcal{V}_{\delta_1,\Delta_1}}{\partial x_1}(t,x_1)\big(f_1(t,x_1,\theta_1^\star) + g(t,x,\theta^\star)\big).$$

Therefore, from Assumption 3.2, it holds that, for all $x_1 \in \mathcal{H}(\delta_1,\Delta_1)$ and all $t \in \mathbb{R}_{\geq 0}$,

$$\dot{\mathcal{V}}_{\delta_1,\Delta_1}(t,x_1) \quad \leq \quad -\mathcal{V}_{\delta_1,\Delta_1}(t,x_1) + \tilde{c}_{\delta_1,\Delta_1}(|x_1|) G_{\theta_1^\star}(|x|) \Psi_{\theta_1^\star}(|x_2|)\,.$$

Let Assumption 3.4 generate a $\mathcal{KL}$ function $\beta_{\delta_2,\Delta_2}$ such that, for all $x_{20} \in \mathcal{B}_{\Delta_2}$ and all $t_0 \in \mathbb{R}_{\geq 0}$,

$$|\phi_2(t,t_0,x_{20},\theta_2^\star)|_{\delta_2} \leq \beta_{\delta_2,\Delta_2}(|x_{20}|\,, t-t_0)\,, \qquad \forall t \geq t_0\,.$$

Using the shorthand notation $\phi_1(t)$ for $\phi_1(t,t_0,x_0,\theta^\star)$ and $v_1(t) := \mathcal{V}_{\delta_1,\Delta_1}(t,\phi_1(t))$ we get, in view of (3.8) and Assumption 3.6, that, for all $x_0 \in \mathcal{B}_\Delta$ and all $t \geq t_0$ such that $\phi_1(t) \in \mathcal{H}(\delta_1,\Delta_1)$,

$$\dot{v}_1(t) \leq -v_1(t) + \tilde{c}_{\delta_1,\Delta_1}(\Delta_1) G_{\theta_1^\star}(\Delta_1) \Psi_{\theta_1^\star}\big(\beta_{\delta_2,\Delta_2}(|x_{20}|\,, t-t_0) + \delta_2\big)\,. \tag{3.11}$$

Thus, we have

$$\phi_1(t) \in \mathcal{H}(\delta_1,\Delta_1) \quad \Rightarrow \quad \dot{v}_1(t) \leq -v_1(t) + c^{\delta_1,\delta_2}_{\Delta_1,\Delta_2}(|x_0|)\,, \tag{3.12}$$

with

$$c^{\delta_1,\delta_2}_{\Delta_1,\Delta_2}(s) := \tilde{c}_{\delta_1,\Delta_1}(\Delta_1) G_{\theta_1^\star}(\Delta_1) \Psi_{\theta_1^\star}(\beta_{\delta_2,\Delta_2}(s,0) + \delta_2)\,, \qquad \forall s \in \mathbb{R}_{\geq 0}\,.$$

The rest of the proof of uniform stability consists in integrating (3.12) in order to construct a bound like (3.9). To this end, we apply Lemma 2.7 to (3.12) with $V = \mathcal{V}_{\delta_1,\Delta_1}$, $k = 1$, $c = c^{\delta_1,\delta_2}_{\Delta_1,\Delta_2}$ and $X = \mathcal{H}(\delta,\Delta)$, to get, in view of (3.8) and Assumption 3.6, that, for all $x_0 \in \mathcal{B}_\Delta$ and all $t \geq t_0 \geq 0$,

$$|\phi_1(t)| \leq \underline{\tilde{\alpha}}^{-1}_{\delta_1,\Delta_1}\left(\tilde{\overline{\alpha}}_{\delta_1,\Delta_1}(\delta_1) + c^{\delta_1,\delta_2}_{\Delta_1,\Delta_2}(|x_0|)\right) + \underline{\tilde{\alpha}}^{-1}_{\delta_1,\Delta_1}\left(\tilde{\overline{\alpha}}_{\delta_1,\Delta_1}(|x_0|) + c^{\delta_1,\delta_2}_{\Delta_1,\Delta_2}(|x_0|)\right)\,.$$

Define the following:

$$\begin{aligned}
\delta_3 \quad &:= \quad \underline{\tilde{\alpha}}^{-1}_{\delta_1,\Delta_1}\Big(\tilde{\overline{\alpha}}_{\delta_1,\Delta_1}(\delta_1) + c^{\delta_1,\delta_2}_{\Delta_1,\Delta_2}(0)\Big) + \underline{\tilde{\alpha}}^{-1}_{\delta_1,\Delta_1}\Big(c^{\delta_1,\delta_2}_{\Delta_1,\Delta_2}(0)\Big) \\
&= \quad \underline{\tilde{\alpha}}^{-1}_{\delta_1,\Delta_1}\Big(\tilde{\overline{\alpha}}_{\delta_1,\Delta_1}(\delta_1) + \tilde{c}_{\delta_1,\Delta_1}(\Delta_1) G_{\theta_1^\star}(\Delta_1) \Psi_{\theta_1^\star}(\delta_2)\Big) \\
&\quad + \quad \underline{\tilde{\alpha}}^{-1}_{\delta_1,\Delta_1}\Big(\tilde{c}_{\delta_1,\Delta_1}(\Delta_1) G_{\theta_1^\star}(\Delta_1) \Psi_{\theta_1^\star}(\delta_2)\Big)
\end{aligned} \tag{3.13}$$

and, for all $s \in \mathbb{R}_{\geq 0}$,

$$\begin{aligned}\eta^{\delta_1,\delta_2}_{\Delta_1,\Delta_2}(s) \;&:=\; \underline{\tilde{\alpha}}^{-1}_{\delta_1,\Delta_1}\Big(\overline{\alpha}_{\delta_1,\Delta_1}(\delta_1) + c^{\delta_1,\delta_2}_{\Delta_1,\Delta_2}(s)\Big) + \underline{\tilde{\alpha}}^{-1}_{\delta_1,\Delta_1}\Big(\overline{\alpha}_{\delta_1,\Delta_1}(s) + c^{\delta_1,\delta_2}_{\Delta_1,\Delta_2}(s)\Big)\\ &\quad - \;\underline{\tilde{\alpha}}^{-1}_{\delta_1,\Delta_1}\Big(\overline{\alpha}_{\delta_1,\Delta_1}(\delta_1) + c^{\delta_1,\delta_2}_{\Delta_1,\Delta_2}(0)\Big) - \underline{\tilde{\alpha}}^{-1}_{\delta_1,\Delta_1}\Big(c^{\delta_1,\delta_2}_{\Delta_1,\Delta_2}(0)\Big)\,.\end{aligned}$$

Then we conclude that, for any $x_0 \in \mathcal{B}_\Delta$ and all $t_0 \in \mathbb{R}_{\geq 0}$, it holds that

$$|\phi_1(t)|_{\delta_3} \leq \eta^{\delta_1,\delta_2}_{\Delta_1,\Delta_2}(|x_0|)\,, \qquad \forall t \geq t_0\,. \tag{3.14}$$

Uniform stability of $\mathcal{B}_{\delta_3}$ on $\mathcal{B}_\Delta$ follows by noticing that $\eta^{\delta_1,\delta_2}_{\Delta_1,\Delta_2}$ is a class $\mathcal{K}$ function. This can be seen by recalling that $c^{\delta_1,\delta_2}_{\Delta_1,\Delta_2}$ is a continuous increasing function.

Proof of uniform attractivity: Consider again (3.11). Since $\beta_{\delta_2,\Delta_2}$ is a $\mathcal{KL}$ function, there is a time $t_1 \geq 0$, independent of t_0 and x_0, such that

$$\beta_{\delta_2,\Delta_2}(\Delta, t - t_0) \leq \delta_2\,, \qquad \forall\, t \geq t_0 + t_1\,.$$

Hence (3.11) implies that, for all $x_0 \in \mathcal{B}_\Delta$,

$$\phi_1(t) \in \mathcal{H}(\delta_1, \Delta_1) \quad \Rightarrow \quad \dot{v}_1(t) \leq -v_1(t) + \tilde{c}_{\delta_1,\Delta_1}(\Delta_1) G_{\theta_1^\star}(\Delta_1) \Psi_{\theta_1^\star}(2\delta_2)\,.$$

Applying again Lemma 2.7 and recalling that, from Assumption 3.6, $|\phi_1(t_0 + t_1)| \leq \Delta_1$, it follows that, for all $x_0 \in \mathcal{B}_\Delta$, all $t_0 \in \mathbb{R}_{\geq 0}$ and all $t \geq t_0 + t_1$,

$$\begin{aligned}|\phi(t)|_{\delta_1} \;&\leq\; \underline{\tilde{\alpha}}^{-1}_{\delta_1,\Delta_1}\left(\tilde{\overline{\alpha}}_{\delta_1,\Delta_1}(\delta_1) + \tilde{c}_{\delta_1,\Delta_1}(\Delta_1) G_{\theta_1^\star}(\Delta_1)\Psi_{\theta_1^\star}(2\delta_2)\right)\\ &\quad + \;\underline{\tilde{\alpha}}^{-1}_{\delta_1,\Delta_1}\left(\tilde{\overline{\alpha}}_{\delta_1,\Delta_1}(\Delta_1) e^{-(t-t_0-t_1)} + \tilde{c}_{\delta_1,\Delta_1}(\Delta_1) G_{\theta_1^\star}(\Delta_1)\Psi_{\theta_1^\star}(2\delta_2)\right)\,.\end{aligned}$$

Defining

$$t_2 := t_1 + \ln\left(\frac{\tilde{\overline{\alpha}}_{\delta_1,\Delta_1}(\Delta_1)}{\tilde{\overline{\alpha}}_{\delta_1,\Delta_1}(\delta_1)}\right)\,,$$

we see that, for all $t \geq t_0 + t_2$,

$$|\phi_1(t)| \leq \delta_4 := 2\underline{\tilde{\alpha}}^{-1}_{\delta_1,\Delta_1}\left(\tilde{\overline{\alpha}}_{\delta_1,\Delta_1}(\delta_1) + \tilde{c}_{\delta_1,\Delta_1}(\Delta_1) G_{\theta_1^\star}(\Delta_1)\Psi_{\theta_1^\star}(2\delta_2)\right)\,. \tag{3.15}$$

In other words, we have, for all $x_0 \in \mathcal{B}_\Delta$,

$$|\phi_1(t)|_{\delta_4} = 0\,, \qquad \forall t \geq t_0 + t_2\,.$$

Finally, let

$$\delta := \max\{\delta_2\,;\,\delta_3\,;\,\delta_4\}\,. \tag{3.16}$$

Then we see that (3.14) implies that $|\phi_1(t)|_\delta \leq \eta(|x_0|)$ for all $t \geq t_0$. From this and what precedes it is not hard to see that, for all $x_0 \in \mathcal{B}_\Delta$,

$$|\phi_1(t)|_\delta \leq \eta^{\delta_1,\delta_2}_{\Delta_1,\Delta_2}(|x_0|) e^{-(t-t_0-t_2)}\,, \qquad \forall t \geq t_0\,.$$

Thus, recalling that t_2 depends neither on t_0 nor on x_0, and defining

$$\beta^{\delta_1,\delta_2}_{\Delta_1,\Delta_2}(s,t) := \sqrt{\eta^{\delta_1,\delta_2}_{\Delta_1,\Delta_2}(s)^2 e^{-2(t-t_2)} + \beta_{\delta_2,\Delta_2}(s,t)^2}\,, \qquad \forall s,t \in \mathbb{R}_{\geq 0}\,,$$

we conclude that, for all $x_0 \in \mathcal{B}_\Delta$,

$$|\phi(t, t_0, x_0, \theta^\star)|_\delta \leq \beta^{\delta_1,\delta_2}_{\Delta_1,\Delta_2}(|x_0|, t - t_0), \qquad \forall t \geq t_0 .$$

Uniform asymptotic stability of $\mathcal{B}_\delta$ on $\mathcal{B}_\Delta$ follows by noticing that $\beta^{\delta_1,\delta_2}_{\Delta_1,\Delta_2}$ is a class $\mathcal{KL}$ function. We stress that the dependence of the $\mathcal{KL}$ estimate in δ and Δ (through Δ_1, Δ_2, δ_1 and δ_2) is not in contradiction with Definition 1.19.

Proof of "semiglobal practical": It is only left to show that δ and Δ can be arbitrarily reduced and enlarged respectively. To this end, consider any desired $\Delta_\star > \delta_\star > 0$ and choose $\delta_1, \Delta_1, \Delta_2$ in such a way that, for all $\delta_2 \in (0; \Delta_2)$,

$$\min\{\Delta_1; \Delta_2; \gamma(\delta_1, \delta_2, \Delta_1, \Delta_2)\} \geq \Delta_\star , \qquad \underline{\alpha}^{-1}_{\delta_1,\Delta_1} \circ \overline{\alpha}_{\delta_1,\Delta_1}(\delta_1) = \underline{\tilde{\alpha}}^{-1}_{\delta_1,\Delta_1} \circ \tilde{\overline{\alpha}}_{\delta_1,\Delta_1}(\delta_1) \leq \frac{\delta_\star}{4} ,$$

as ensured by Assumption 3.6 (see also (3.10)). Then, we get directly from (3.8) that Δ can be made greater than or equal to $\Delta_\star$. Furthermore, it is possible to choose δ_2 small enough that

$$\underline{\tilde{\alpha}}^{-1}_{\delta_1,\Delta_1}\big(\tilde{\overline{\alpha}}_{\delta_1,\Delta_1}(\delta_1) + \tilde{c}_{\delta_1,\Delta_1}(\Delta_1) G_{\theta_1^\star}(\Delta_1) \Psi_{\theta_1^\star}(2\delta_2)\big) \leq \frac{\delta_\star}{2}$$

since $\underline{\tilde{\alpha}}^{-1}_{\delta_1,\Delta_1}$ and $\Psi_{\theta_1^\star}$ both belong to class $\mathcal{K}_\infty$ and none of the above involved functions depend on δ_2. In other words, we can ensure that $\delta_4 \leq \delta_\star$ (*cf.* (3.15)). Invoking again Assumption 3.6 and noticing that $\delta_3 \leq \delta_4$, we conclude in view of (3.16) that we can achieve $\delta \leq \delta_\star$.

In the case that the functions G and Ψ depend on θ_2, and provided that (3.7) holds, we see from (3.15) and (3.16) that δ can still be made arbitrarily small by picking δ_2 small enough. This justifies Remark 3.9. ■

3.1.2 Without a Lyapunov function for the driven subsystem

The main result of the above section provides a tool for the stability analysis of uniformly semiglobally practically asymptotically stable systems placed in cascade (*cf.* (3.1)):

$$\begin{aligned} \dot{x}_1 &= f_1(t, x_1, \theta_1) + g(t, x, \theta) \\ \dot{x}_2 &= f_2(t, x_2, \theta_2) . \end{aligned}$$

However, one may argue that it requires the explicit knowledge of a Lyapunov function for the driven subsystem, which may not be direct when the stability property under consideration is established by other means as, for instance, averaging techniques [TPA99].

Based on Corollary 2.30, we may simplify the statement of Theorem 3.3 in this direction, provided that function f_1 involved in (3.1) is sufficiently smooth and that the x_1-dynamics enjoys the stronger property of δ-USPAS (*cf.* Definition 2.28). Roughly this means that the $\mathcal{KL}$ estimate of its solutions is the same disregarding the amplitude of the steady-state error we want to impose. More precisely, we assume the following.

Assumption 3.10 (USPAS of each subsystems) *The subsystems $\dot{x}_1 = f_1(t, x_1, \theta_1)$ and $\dot{x}_2 = f_2(t, x_2, \theta_2)$ in (3.1) are USPAS on Θ_1 and Θ_2 respectively.*

Assumption 3.11 (Regularity of f_1) *There exist a nonnegative constant f_0 and a continuous nondecreasing function L such that, for all $\theta_1 \in \Theta$, all $x_1 \in \mathbb{R}^{n_1}$ and all $t \in \mathbb{R}_{\geq 0}$,*

$$|f_1(t,0,\theta_1)| \leq f_0\,, \qquad \left|\frac{\partial f_1}{\partial x_1}(t,x_1,\theta_1)\right| \leq L(|x_1|)\,.$$

Theorem 3.12 (USPAS + USPAS + UB ⇒ USPAS) *Under Assumptions 3.2, 3.6, 3.10 and 3.11 the cascaded system (3.1) is USPAS on the parameter set $\Theta_1 \times \Theta_2$.*

Proof of Theorem 3.12. The proof follows directly from Theorem 3.3 by observing that Assumptions 3.10 and 3.11 imply, via Theorem 2.27, Assumption 3.5. ■

Assumption 3.6 may remain hard to check in practice. The following result states that, if the driven subsystem is δ-USPAS, then this assumption is no longer required in the case when solutions of the cascade (3.1) are uniformly globally bounded *uniformly in θ*.

Assumption 3.13 (δ-USPAS and USPAS) *The subsystems $\dot{x}_1 = f_1(t,x_1,\theta_1)$ and $\dot{x}_2 = f_2(t,x_2,\theta_2)$ in (3.1) are respectively δ-USPAS on Θ_1 and USPAS on Θ_2.*

Assumption 3.14 (UGB, uniformly in θ) *There exist $\eta \in \mathcal{K}_\infty$ and $\mu > 0$ such that, for all $x_0 \in \mathbb{R}^n$, all $t_0 \in \mathbb{R}_{\geq 0}$ and all $\theta \in \Theta$, the trajectory of (3.1) satisfies*

$$|\phi(t,t_0,x_0,\theta)| \leq \eta(|x_0|) + \mu\,, \qquad \forall t \geq t_0\,.$$

Corollary 3.15 (δ-USPAS + USPAS + UGB ⇒ USPAS) *Under Assumptions 3.2, 3.11, 3.13 and 3.14, the cascaded system (3.1) is USPAS on the parameter set $\Theta_1 \times \Theta_2$.*

Proof of Corollary 3.15. In the statement of Theorem 3.3, pick Δ_0 as μ and $\gamma(\delta_1,\delta_2,\Delta_1,\Delta_2)$ as $\eta^{-1}(\Delta_1 - \mu)$. Then (3.5) can be fulfilled by simply picking Δ_1 and Δ_2 large enough. In the same way, due to the δ-USPAS assumption on the driven subsystem, the functions $\underline{\alpha}_{\delta_1,\Delta_1}$ and $\overline{\alpha}_{\delta_1,\Delta_1}$ generated by Corollary 2.30 are independent of δ_1. Hence, it suffices to pick δ_1 small enough in order to fulfill (3.6). ■

Corollary 3.15 is particularly useful in applications where USPAS of the driven subsystem is obtained without a Lyapunov function, *e.g.* via averaging: we illustrate this through the following example.

Bounded control of the persistently excited double integrator

Consider the output feedback stabilization of the following double integrator:

$$\dot{x}_1 = x_2 \tag{3.17a}$$
$$\dot{x}_2 = -p(t/\theta)^2 u \tag{3.17b}$$

where u is the control input and θ is a free positive parameter, under the following conditions and assumptions:

- given u_{max}, it is required that $|u(\cdot)| \leq u_{max}$;
- only x_1 is measured;

- p is piecewise continuous and $|p(t)| \leq p_M$, with $p_M > 0$, for all $t \in \mathbb{R}_{\geq 0}$;
- $p(t)^2$ admits an finite average, *i.e.* the following quantity exists and is finite[2]:

$$p_{av}^2 := \lim_{T \to +\infty} \frac{1}{T} \int_0^T p(\tau)^2 d\tau \,; \tag{3.18}$$

For the case that $p(\cdot) \equiv 1$, the control literature provides bounded state feedbacks $u^*(x)$ that globally asymptotically stabilize the origin of the system (3.17), *cf. e.g.* [SS90b, SY91, Tee92]. The output feedback control of this system is also considered in [Tee96, Theorem 7]. In this context, we pose the following problem: provided that we design an observer for (3.17) such that the estimated state, $\hat{x}$, tends to x exponentially, does $u^*(\hat{x})$ stabilize the origin of (3.17)? We give a positive answer to this question by showing, via Theorem 3.12, that $u^*(\hat{x})$ stabilizes (3.17) in the semiglobal practical sense. For clarity of exposition we prove the result for a specific $u^*(x)$ but it holds for other bounded smooth controls.

We stress that the double integrator (3.17) is a particular case of systems of the form $\dot{x} = f(t,x) + g(t)u$ where the "input gain" $g(t)$ is not invertible (since $p(t)$ may vanish for some values of t). This situation may appear in physical systems: see for instance [AL02]. Another motivation is to control by feedback linearization: if $g(t)$ were invertible, we might use $u = g(t)^{-1}[-x + f(t,x)]$; since it is not, but instead persistently exciting, can we use $u = g(t)[-x + f(t,x)]$? For further motivations see [LCBC05]. Please note also that the above problem was solved in [SCM+06], with a linear control, by assuming that the whole state is measured. However, it involves a deeply different approach than the one presented here.

Proposition 3.16 *Let p be a bounded piecewise continuous function satisfying*

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T p(s)^2 ds = p_{av}^2 \,,$$

where p_{av} denotes a finite constant. Then the system (3.17) in closed loop with $u^*(\hat{x}) := -k_1 tanh(\hat{x}_1) - k_2 tanh(\hat{x}_2)$ with k_1, $k_2 > 0$ and the observer

$$\dot{\hat{x}}_1 = \hat{x}_2 - \ell_1 \bar{x}_1 \tag{3.19a}$$
$$\dot{\hat{x}}_2 = -p(t/\theta)^2 u^*(\hat{x}) - \ell_2 \bar{x}_1 \tag{3.19b}$$

where $\ell_i > 0$ and $\bar{x}_i := \hat{x}_i - x_i$ for all $i \in \{1,2\}$, is USPAS on the parameter set $\mathbb{R}_{>0}$. □

The above result provides a separation principle for the bounded output feedback control of (3.17). If u^* is a bounded *state* feedback that stabilizes the nominal system (*i.e.* when considering $p(\cdot) = 1$), then it suffices to evaluate it based on the state *estimates* to achieve the control objective in presence of a persistently exciting signal $p(\cdot)$. This is at the price of a slight degradation of performance since the global asymptotic stability enjoyed by the *nominal state*-feedback controlled yields uniform semiglobal practical stability for the *output*-feedback controlled *perturbed* system.

We stress that an alternative to the proof presented below can be derived from [Tee96, Theorem 7] and [TPA99]. Roughly speaking, the former reference ensures the stabilizability of (3.17) by output feedback when p is replaced by its average. The latter reference can

[2]Note that p_{av}^2 is strictly positive due to the persistency of excitation assumption on $p(t)$.

then be invoked to conclude USPAS of the original system. The arguments presented below however illustrate better the results of this section, which mainly motivated their exposition.

Proof of Proposition 3.16. We start by exhibiting the cascaded structure of the closed-loop system. To that end, let $f(t,x,\theta)$ denote the right hand side of (3.17) with $u = u^*(x)$ and define

$$g(t,x,\bar{x},\theta) := p(t/\theta)^2 \begin{pmatrix} 0 \\ g_1(x_1,\bar{x}_1) + g_2(x_2,\bar{x}_2) \end{pmatrix},$$

where $g_i(x_i,\bar{x}_i) := -k_i(\tanh(\bar{x}_i + x_i) - \tanh(x_i))$ for all $i \in \{1,2\}$. Then, the closed loop system takes the form

$$\dot{x} = f(t,x,\theta) + g(t,x,\bar{x},\theta) \tag{3.20a}$$

$$\dot{\bar{x}} = A\bar{x}\,, \qquad A := \begin{pmatrix} -\ell_1 & 1 \\ -\ell_2 & 0 \end{pmatrix}. \tag{3.20b}$$

First notice that Assumption 3.2 is fulfilled with

$$G(s) := 2p_M^2(k_1+k_2) \qquad \text{and} \qquad \Psi(s) = s\,.$$

This can be seen by observing that $|\tanh(a+b) - \tanh(b)\,| \leq 2\,|a|$ for all $a,b \in \mathbb{R}$. In addition, A is clearly Hurwitz for any positive values of ℓ_1 and ℓ_2, which shows that (3.20b) is globally exponentially stable (and *a fortiori* USPAS). To show USPAS of $\dot{x} = f(t,x,\theta)$ we proceed as follows. Define the following function:

$$f_{av}(x) = \begin{bmatrix} x_2 \\ -p_{av}^2 u^*(x) \end{bmatrix}.$$

Then, for each positive T, we have

$$\left| f_{av}(x) - \frac{1}{T}\int_0^T f(t,x,1)dt \right| \leq \left| p_{av}^2 - \frac{1}{T}\int_0^T p(\tau)^2 d\tau \right| (k_1+k_2)\,|x|\,.$$

From (3.18), it follows that there exists a class $\mathcal{L}$ function σ such that

$$\left| f_{av}(x) - \frac{1}{T}\int_0^T f(t,x,1)dt \right| \leq (k_1+k_2)\,|x|\,\sigma(T)\,. \tag{3.21}$$

Consequently, f_{av} constitutes an average of f in the sense of [TPA99]. Under the assumptions made on (3.17), $\dot{x} = f_{av}(x)$ is globally asymptotically stable. From [TPA99] and Proposition 2.29, it follows that (3.17) with $u = u^*(x)$ is δ-USPAS on the parameter set $\mathbb{R}_{>0}$. This establishes Assumption 3.10.

Uniform boundedness may be shown as follows. Direct computations show that, provided that $\varepsilon > 0$ is picked sufficiently small, the following function is positive definite and radially unbounded

$$V(x) = \frac{1}{2}\,|x_2|^2 + k_1 p_{av}^2 \ln(\cosh(x_1)\,) + \varepsilon \tanh(x_1)\tanh(x_2)\,,$$

i.e. there exist class $\mathcal{K}_\infty$ functions $\underline{\alpha}$ and $\overline{\alpha}$ such that $\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|)$. In addition, V is independent of the tuning parameter θ. Furthermore, V has a negative definite

derivative along the trajectories of $\dot{x} = f_{av}(x)$. More precisely, for sufficiently small $\varepsilon > 0$ there exist q_1, $q_2 > 0$ such that

$$\frac{\partial V}{\partial x}(x) f_{av}(x) \leq -q_1 \mathrm{sech}^2(x_2)\tanh^2(x_1) - q_2 x_2 \tanh(x_2) =: -W(x)\,.$$

Note that W is a continuous positive definite function independent of θ as well. In addition, noticing that $|\tanh(a+b) - \tanh(b)\,| \leq 2\,|a|$ for all $a, b \in \mathbb{R}$, it holds that

$$\left|\frac{\partial V}{\partial x}(x) g(t, x, \bar{x}, \theta)\right| \leq \left|x_2 + \varepsilon \tanh(x_1)\mathrm{sech}^2(x_2)\right| 2p_M^2(k_1 + k_2)\,.$$

It follows that there exists a nonnegative constant λ such that, for all $\bar{x} \in \mathbb{R}^2$, all $t \in \mathbb{R}_{\geq 0}$ and all $\theta \in \mathbb{R}_{>0}$,

$$|x| \geq 1 \quad \Rightarrow \quad \left|\frac{\partial V}{\partial x} g(t, x, \bar{x}, \theta)\right| \leq \lambda W(x)\,.$$

Observing that forward completeness of (3.20) follows from the boundedness of $p(\cdot)$, and recalling that both V and W are independent of θ, we conclude, proceeding as in the proof of [PL01, Theorem 4], that the solutions of the overall system are UGB *uniformly* in θ. The result follows from Corollary 3.15. ■

3.2 Semiglobal asymptotic stability of cascaded systems

This short section exhibits the intrinsic similarities between USPAS and USAS. We show that almost all the results presented above can be directly adapted to uniform semiglobal asymptotic stability modulo little additional requirements due to a technical obstacle in the proof.

3.2.1 With a Lyapunov function for the driven subsystem

Theorem 3.17 (Lyapunov USAS + USAS + UB ⇒ USAS) *Under Assumptions 3.2, 3.18, 3.19 and 3.20, the cascaded system (3.1) is uniformly semiglobally asymptotically stable on the parameter set $\Theta_1 \times \Theta_2$.*

Assumption 3.18 (USAS of the driving subsystem) *The system (3.1b) is USAS on Θ_2.*

Assumption 3.19 (Lyapunov USAS of the driven subsystem) *Given any $\Delta_1 > 0$, there exist a parameter $\theta_1^\star(\Delta_1) \in \Theta_1$, a continuously differentiable Lyapunov function $V_{\Delta_1} : \mathbb{R}_{\geq 0} \times \mathbb{R}^{n_1} \to \mathbb{R}_{\geq 0}$, class $\mathcal{K}_\infty$ functions $\underline{\alpha}_{\Delta_1}$, $\overline{\alpha}_{\Delta_1}$, a positive constant k_{Δ_1} and a continuous positive nondecreasing function c_{Δ_1} such that, for all $x_1 \in \mathcal{B}_{\Delta_1}$ and all $t \in \mathbb{R}_{\geq 0}$,*

$$\begin{gathered}
\underline{\alpha}_{\Delta_1}(|x_1|) \leq V_{\Delta_1}(t, x_1) \leq \overline{\alpha}_{\Delta_1}(|x_1|) \\
\frac{\partial V_{\Delta_1}}{\partial t}(t, x_1) + \frac{\partial V_{\Delta_1}}{\partial x_1}(t, x_1) f_1(t, x_1, \theta_1^\star) \leq -k_{\Delta_1} V_{\Delta_1}(t, x_1) \\
\left|\frac{\partial V_1}{\partial x_1}(t, x_1)\right| \leq c_{\Delta_1}(|x_1|)\,.
\end{gathered} \tag{3.22}$$

Assumption 3.20 (Boundedness of solutions) *There exists a positive constant Δ_0 such that, for any given positive numbers Δ_1, Δ_2, satisfying $\Delta_1 > \Delta_0$, and for the parameter $\theta_1^\star(\Delta_1)$ as defined in Assumption 3.19, there exists a parameter $\theta_2^\star \in \mathcal{D}_{f_2}(0,\Delta_2) \cap \Theta_2$ (cf. Definition 3.1) and a nonnegative constant $\gamma(\Delta_1,\Delta_2)$ such that*

$$\lim_{\Delta_1,\Delta_2 \to +\infty} \gamma(\Delta_1,\Delta_2) = +\infty\,,$$

and the trajectories of (3.1) with $\theta = \theta^\star$ satisfy

$$|x_0| \leq \gamma(\Delta_1,\Delta_2) \qquad \Rightarrow \qquad |\phi(t,t_0,x_0,\theta^\star)| \leq \Delta_1\,, \qquad \forall t \geq t_0\,.$$

The proof of Theorem 3.17 is not detailed as it follows along the same lines as the proof of Theorem 3.3. Roughly speaking, it suffices to pick $\delta_1 = \delta_2 = 0$ in the latter to get the result. However, from a technical point of view, Lemma 2.6 does not apply to this situation, which prevents V_{Δ_1} to be transformed in a more convenient form. This explains why the total derivative of V_{Δ_1} is assumed to satisfy the more conservative bound (3.22).

Yet, by observing that the requirement $\delta > 0$ in Lemma 2.6 serves only to establish a time-invariant bound on the gradient of the transformed Lyapunov function, and as this fact follows as an immediate consequence of the time invariance of the original function V_{Δ_1}, we can show that the following statement holds. It particularly fits (but is not restricted to) the situation when the driven subsystem is time-invariant.

Assumption 3.21 (Lyapunov USAS of the x_1-subsystem) *Given any $\Delta_1 > 0$, there exist a parameter $\theta_1^\star(\Delta_1) \in \Theta_1$, a continuously differentiable Lyapunov function $V_{\Delta_1} : \mathbb{R}^{n_1} \to \mathbb{R}_{\geq 0}$ and class $\mathcal{K}_\infty$ functions $\underline{\alpha}_{\Delta_1}$, $\overline{\alpha}_{\Delta_1}$, α_{Δ_1} such that, for all $x_1 \in \mathcal{B}_{\Delta_1}$ and all $t \in \mathbb{R}_{\geq 0}$,*

$$\underline{\alpha}_{\Delta_1}(|x_1|) \leq V_{\Delta_1}(x_1) \leq \overline{\alpha}_{\Delta_1}(|x_1|)$$

$$\frac{\partial V_{\Delta_1}}{\partial x_1}(x_1) f_1(t,x_1,\theta_1^\star) \leq -\alpha_{\Delta_1}(|x_1|)\,.$$

Corollary 3.22 *Under Assumptions 3.2, 3.18, 3.20 and 3.21, the cascaded system (3.1) is uniformly semiglobally asymptotically stable on $\Theta_1 \times \Theta_2$.*

We stress that, in this situation, the bound on the gradient of the Lyapunov function follows trivially from the time-invariance and continuous differentiability of the latter.

3.2.2 Without a Lyapunov function for the driven subsystem

For the sake of completeness, we present below a result that allows to establish USAS of a cascaded system, without requiring the knowledge of any explicit Lyapunov function. Similarly to USPAS, it requires additional smoothness of the driven subsystem's dynamics.

However, due to the technical obstacle underlined above, we further need to assume that the driven subsystem is time-invariant. In other words, we consider cascaded systems of the form

$$\dot{x}_1 = f_1(x_1,\theta_1) + g(t,x,\theta) \tag{3.23a}$$
$$\dot{x}_2 = f_2(t,x_2,\theta_2)\,. \tag{3.23b}$$

Assumption 3.23 (Regularity of f_1) *There exist a nonnegative constant f_0 and a continuous nondecreasing function L such that, for all $\theta_1 \in \Theta$ and all $x_1 \in \mathbb{R}^{n_1}$,*

$$|f_1(0,\theta_1)| \leq f_0\,, \qquad \left|\frac{\partial f_1}{\partial x_1}(x_1,\theta_1)\right| \leq L(|x_1|)\,.$$

Assumption 3.24 (USAS of each subsystem) *The subsystems $\dot{x}_1 = f_1(x_1,\theta_1)$ and (3.23b) are USAS respectively on Θ_1 and Θ_2.*

Theorem 3.25 (USAS + USAS + UB ⇒ USAS) *Under Assumptions 3.2, 3.20, 3.23 and 3.24, the cascaded system (3.23) is USAS on $\Theta_1 \times \Theta_2$.*

The proof follows along the same lines as the one of Theorem 3.12 by picking $\delta_1 = \delta_2 = 0$. The main difference stands in the fact the time-invariant bound on the gradient of the Lyapunov function generated by Lemma 2.6 cannot be ensured. This difficulty is overpassed by the fact that the driven subsystem is time-invariant, which makes the Lyapunov function provided by Theorem 2.31 time-invariant too. The autonomous bound on its gradient then follows directly from its continuous differentiability.

3.3 Global practical asymptotic stability of cascaded systems

We now present theorems for UGPAS of systems in cascade. While the proof of the stability analysis of cascaded systems for USPAS and USAS have a lot in common, the results below shows that *global* properties have to be treated in a slightly different way. Although they make the problem of the existence of a converse Lyapunov function harder (as pointed out in Section 2.2.2), they allow to provide very efficient tools to establish uniform boundedness of the overall cascade.

We recall that we consider cascaded systems of the form

$$\begin{aligned} \dot{x}_1 &= f_1(t,x_1,\theta_1) + g(t,x,\theta) \\ \dot{x}_2 &= f_2(t,x_2,\theta_2)\,, \end{aligned}$$

originally introduced in (3.1).

Theorem 3.26 (UGPAS + UGPAS + growth restriction ⇒ UGPAS) *Under Assumptions 3.2, 3.28 and 3.27, the cascaded system (3.1) is uniformly globally practically asymptotically stable (UGPAS) on the parameter set $\Theta_1 \times \Theta_2$.*

Assumption 3.27 (UGPAS of the driving subsystem) *The system (3.1b) is UGPAS on Θ_2.*

Assumption 3.28 (Lyapunov UGPAS of the driven subsystem) *Given any $\delta_1 > 0$, there exist a parameter $\theta_1^\star(\delta_1) \in \Theta_1$, a continuously differentiable Lyapunov function V_{δ_1}, class $\mathcal{K}_\infty$ functions $\underline{\alpha}_{\delta_1}$, $\overline{\alpha}_{\delta_1}$, α_{δ_1} and a continuous positive nondecreasing function c_{δ_1} such that, for all $x_1 \in \mathbb{R}^n \setminus \mathcal{B}_{\delta_1}$ and all $t \in \mathbb{R}_{\geq 0}$,*

$$\underline{\alpha}_{\delta_1}(|x_1|) \leq V_{\delta_1}(t,x_1) \leq \overline{\alpha}_{\delta_1}(|x_1|) \tag{3.24}$$

$$\frac{\partial V_{\delta_1}}{\partial t}(t,x_1) + \frac{\partial V_{\delta_1}}{\partial x_1}(t,x_1) f_1(t,x_1,\theta_1^\star) \leq -\alpha_{\delta_1}(|x_1|) \tag{3.25}$$

$$\left|\frac{\partial V_{\delta_1}}{\partial x}(t,x_1)\right| \leq c_{\delta_1}(|x_1|) \tag{3.26}$$

$$\lim_{\delta_1 \to 0} \underline{\alpha}_{\delta_1}^{-1} \circ \overline{\alpha}_{\delta_1}(\delta_1) = 0\,. \tag{3.27}$$

In addition, for the function G_{θ_1} of Assumption 3.2, it holds that, for all $\delta_1 > 0$ and as s tends to $+\infty$,

$$c_{\delta_1}(s) G_{\theta_1^\star}(s) = \mathcal{O}(\alpha_{\delta_1} \circ \overline{\alpha}_{\delta_1}^{-1} \circ \underline{\alpha}_{\delta_1}(s)) \tag{3.28a}$$

$$\alpha_{\delta_1}(s) = \mathcal{O}(\overline{\alpha}_{\delta_1}(s))\,. \tag{3.28b}$$

In view of Theorem 2.5, it is clear that Assumption 3.28 implies that the zero-input x_1-subsystem is UGPAS on Θ_1. We state the above result under the assumption that we know a Lyapunov function V_{δ_1} satisfying the above conditions as the latter implicitly impose uniform global boundedness of the solutions of the cascaded system.

In this respect, we stress that, compared to Theorems 3.3 and 3.17, Theorem 3.26 does *not* require the boundedness of solutions *a priori*. This constitutes a very enjoyable feature in applications, as the corresponding requirement may easily be checked based on the simple inspection of the interconnection term and the bounds on the assumed Lyapunov function, its derivative and its gradient, for the driven subsystem.

Regarding Assumption 3.2, it is worth emphasizing that, at the exception of few articles, as [SAT02, JSK96], it is typically required that the dependence of the interconnection term in x_1 be at most linear (*i.e.* G_{θ_1} affine); see *e.g.* [SJK97]. In fact, such a behavior of g is implicitly imposed by (3.28), but only when $|x_1|$ tends to infinity. In this respect, we underline the similarity existing between the requirements (3.28) and [PL01, Assumption 4] (which borrows from [SC64]): in the particular case that, for all $\delta_1 > 0$, $\lim_{s\to\infty} \overline{\alpha}_{\delta_1}(s)/\underline{\alpha}_{\delta_1}(s) < \infty$, they are even equivalent.

Remark 3.29 *For clarity, Theorem 3.26 is stated under the assumption that the bound on the interconnection term is independent of θ_2. For the case that this does not hold, as it appears more clearly along the proof, it is sufficient to additionally impose that, for all $\theta_1 \in \Theta_1$,*

$$\lim_{\delta_2 \to 0} \Big(\sup \{ \Psi_{\theta_1,\theta_2}(\delta_2) \ : \ \theta_2 \in \mathcal{D}_{f_2}(\delta_2,\infty) \cap \Theta_2 \} \Big) = 0\,.$$

The proof of Theorem 3.26 relies on the following result that also has interest at its own. Similarly to Lemma 2.6, it provides sufficient conditions to transform a "classical" Lyapunov function into another one that presents useful properties of its gradient and its total derivative.

Lemma 3.30 *Let $\delta > 0$ be some given constant and let X be a subset of $\mathbb{R}^n \setminus \mathring{\mathcal{B}}_\delta$. Suppose that there exist a continuously differentiable function $V : \mathbb{R}_{\geq 0} \times X \to \mathbb{R}_{\geq 0}$ and class $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$, α and continuous functions $c, \mu : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, with c nondecreasing, such that, for all $x \in X$ and all $t \in \mathbb{R}_{\geq 0}$,*

$$\underline{\alpha}(|x|) \leq V(t,x) \leq \overline{\alpha}(|x|)$$

$$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x) f(t,x) \leq -\alpha(|x|)\,.$$

$$\left|\frac{\partial V}{\partial x}(t,x)\right| \leq c(|x_1|) \tag{3.29}$$

and, as s tends to $+\infty$,

$$c(s)\mu(s) = \mathcal{O}(\alpha \circ \overline{\alpha}^{-1} \circ \underline{\alpha}(s)) \tag{3.30}$$
$$\alpha(s) = \mathcal{O}(\overline{\alpha}(s))\,, \tag{3.31}$$

Then, for any positive k, there exists a continuously differentiable function $\mathcal{V} : \mathbb{R}_{\geq 0} \times X \to \mathbb{R}_{\geq 0}$, class $\mathcal{K}_\infty$ functions $\tilde{\underline{\alpha}}$ and $\tilde{\overline{\alpha}}$ and a nonnegative constant η such that, for all $x \in X$ and all $t \in \mathbb{R}_{\geq 0}$,

$$\tilde{\underline{\alpha}}(|x|) \leq \mathcal{V}(t,x) \leq \tilde{\overline{\alpha}}(|x|) \tag{3.32}$$

$$\frac{\partial \mathcal{V}}{\partial t}(t,x) + \frac{\partial \mathcal{V}}{\partial x}(t,x) f(t,x) \leq -k\mathcal{V}(t,x) \tag{3.33}$$

$$\left|\frac{\partial \mathcal{V}}{\partial x}(t,x)\right| \mu(|x|) \leq \eta \mathcal{V}(t,x)\,. \tag{3.34}$$

and it holds that

$$\tilde{\underline{\alpha}}^{-1} \circ \tilde{\overline{\alpha}}(s) = \underline{\alpha}^{-1} \circ \overline{\alpha}(s)\,, \qquad \forall s \in \mathbb{R}_{\geq 0}\,. \tag{3.35}$$

□

The above result is very similar to Lemma 2.6 that was mainly used to establish semiglobal properties, at the difference that a stronger bound (3.34) on the gradient is provided for the generated Lyapunov function. This additional feature allows us to get rid of the assumption of uniform boundedness of solutions. Please refer to the proof of Theorem 3.26 below for further details. Before, we give the proof of the above lemma.

Proof of Lemma 3.30. The proof is inspired by [PW96, Proposition 13], originally presented in [LL69]. Let a be a class $\mathcal{K}$ function with the following properties:

- $a(s) = \frac{1}{k}\alpha \circ \overline{\alpha}^{-1}(s)$ for all $s \geq \underline{\alpha}(\delta)$,
- $a(s) \leq s$ for all $s \leq \alpha(\delta)/2$,
- $a'(0) = 0$,

and define ρ as the following function

$$\rho(s) = \exp\left(\int_1^s \frac{d\tau}{a(\tau)}\right)\,, \qquad \forall s \in \mathbb{R}_{\geq 0}\,.$$

Firstly observe that, in view of (3.31), the integral in the exponential diverges. In the same way, since $a(s) \leq s$ in a neighborhood of zero, the integral tends to $-\infty$ when s tends to zero. It can also be seen that ρ is continuous and increasing, which makes it a class $\mathcal{K}_\infty$ function. Also, based on [PW96, Lemma 12], ρ is continuously differentiable too. Hence, by operating the transformation $\mathcal{V} := \rho \circ V$, we see that $\mathcal{V}$ is continuously differentiable as well, and that (3.32) can be established with the following class $\mathcal{K}_\infty$ functions: $\tilde{\underline{\alpha}} := \rho \circ \underline{\alpha}$ and $\tilde{\overline{\alpha}} := \rho \circ \overline{\alpha}$. In turn, we have that, for all $s \in \mathbb{R}_{\geq 0}$,

$$\tilde{\underline{\alpha}}^{-1} \circ \tilde{\overline{\alpha}}(s) = (\underline{\alpha}^{-1} \circ \rho^{-1}) \circ (\rho \circ \overline{\alpha})(s) = \underline{\alpha}^{-1} \circ \overline{\alpha}(s)\,.$$

In addition, for all $x \in X$, $V(t,x) \geq \underline{\alpha}(\delta)$, hence

$$\begin{aligned}\frac{\partial \mathcal{V}}{\partial t}(t,x) + \frac{\partial \mathcal{V}}{\partial x}(t,x) f(t,x) &\leq -\rho'(V(t,x))\alpha(|x|) \\ &\leq -\frac{\mathcal{V}(t,x)}{a(V(t,x))}\alpha \circ \overline{\alpha}^{-1}(V(t,x)) \\ &\leq -k\mathcal{V}(t,x)\,.\end{aligned}$$

Furthermore, for all $x \in X$, we have that

$$\left|\frac{\partial \mathcal{V}}{\partial x}(t,x)\right| \mu(|x|) \leq \rho'(V(t,x))c(|x|)\mu(|x|) \leq \frac{\mathcal{V}(t,x)}{a(V(t,x))}c(|x|)\mu(|x|)\,.$$

The bound (3.34) follows by recalling that $X \subset \mathbb{R}^n \setminus \overset{\circ}{\mathcal{B}}_\delta$ and by noticing that (3.30) together with the continuity of μ ensures the existence of a nonnegative constant η such that

$$\sup_{|x|\geq\delta} \frac{c(|x|)\mu(|x|)}{a(V(t,x))} \leq \sup_{|x|\geq\delta} \frac{kc(|x|)\mu(|x|)}{\alpha \circ \overline{\alpha}^{-1} \circ \underline{\alpha}(|x|)} \leq \eta\,.$$

■

Remark 3.31 *If, in the statement of Theorem 3.26, the Lyapunov function V_{δ_1} (directly) satisfies (3.32), (3.33) and (3.34), then it does not require (3.28) anymore. This follows by noticing that the proof of Theorem 3.26 starts by transforming the original Lyapunov function into another one satisfying (3.32), (3.33) and (3.34) thanks to the previous result.*

We are now ready to give the proof of Theorem 3.26, that guarantees UGPAS for cascaded systems.

Proof of Theorem 3.26. The argument consists in constructing a ball $\mathcal{B}_\delta$ and a $\mathcal{KL}$ estimate for the solutions of the cascaded system, based on the respective balls for the x_1 (*i.e.* (3.1a) with $x_2 \equiv 0$) and the x_2 subsystems, and to show that δ can be arbitrarily reduced by a convenient choice of the parameters.

For any positive number δ_1, let V_{δ_1} and $\theta_1^\star(\delta_1) \in \Theta_1$ be generated by Assumption 3.28. Then, apply Lemma 3.30 to V_{δ_1} on the set $X = \mathbb{R}^{n_1} \setminus \mathcal{B}_{\delta_1}$ with $\mu = G_{\theta_1}$ and $k = 2$. It follows that there exist a function $\mathcal{V}_{\delta_1}$, class $\mathcal{K}_\infty$ functions $\underline{\tilde{\alpha}}_{\delta_1}$, $\tilde{\overline{\alpha}}_{\delta_1}$, and a nonnegative constant η_{δ_1} such that, for all $x_1 \in \mathbb{R}^{n_1} \setminus \mathcal{B}_{\delta_1}$ and all $t \in \mathbb{R}_{\geq 0}$,

$$\begin{aligned}\underline{\tilde{\alpha}}_{\delta_1}(|x_1|) \leq \mathcal{V}_{\delta_1}(t,x_1) \leq \tilde{\overline{\alpha}}_{\delta_1}(|x_1|) \\ \frac{\partial \mathcal{V}_{\delta_1}}{\partial t}(t,x_1) + \frac{\partial \mathcal{V}_{\delta_1}}{\partial x_1}(t,x_1) f_1(t,x_1,\theta_1^\star) \leq -2\mathcal{V}_{\delta_1}(t,x_1) \\ \left|\frac{\partial \mathcal{V}_{\delta_1}}{\partial x_1}(t,x_1)\right| G_{\theta_1}(|x_1|) \leq \eta_{\delta_1}\mathcal{V}_{\delta_1}(t,x_1)\,,\end{aligned} \tag{3.36}$$

with the property that :

$$\lim_{\delta_1 \to 0} \tilde{\underline{\alpha}}_{\delta_1}^{-1} \circ \tilde{\overline{\alpha}}_{\delta_1}(\delta_1) = \lim_{\delta_1 \to 0} \underline{\alpha}_{\delta_1}^{-1} \circ \overline{\alpha}_{\delta_1}(\delta_1) = 0\,. \tag{3.37}$$

Next, let $\Psi_{\theta_1^\star}$ be given by Assumption 3.2 and choose δ_2 small enough that

$$\Psi_{\theta_1^\star}(\delta_2) \leq \frac{1}{\eta_{\delta_1}}\,, \tag{3.38}$$

which is always possible since $\Psi_{\theta_1^\star}$ is a class $\mathcal{K}$ function and neither[3] $\Psi_{\theta_1^\star}$ nor η_{δ_1} depend on δ_2. Finally, let $\theta_2^\star$ be any parameter in $\mathcal{D}_{f_2}(\delta_2,\infty)\cap\Theta_2$.

We proceed in four steps. We first show that, for this choice of $\theta^\star=(\theta_1^\star,\theta_2^\star)$, the cascade (3.1) is forward complete. We then use this property to prove that it is uniformly globally stable with respect to a ball $\mathcal{B}_\delta$, with δ defined based on δ_1 and δ_2, and then that this ball is also uniformly globally attractive. We finally show that the size of this ball $\mathcal{B}_\delta$ can be arbitrarily diminished by a convenient choice of the parameter.

Proof of forward completeness: The total time derivative of $\mathcal{V}_{\delta_1}$ along (3.1) yields

$$\dot{\mathcal{V}}_{\delta_1}(t,x_1)=\frac{\partial\mathcal{V}_{\delta_1}}{\partial t}(t,x_1)+\frac{\partial\mathcal{V}_{\delta_1}}{\partial x_1}(t,x_1)\big(f_1(t,x_1,\theta_1^\star)+g(t,x,\theta^\star)\big).$$

Therefore, in view of Assumption 3.2 and (3.36), it holds that, for all $x_1\in\mathbb{R}^{n_1}\setminus\mathcal{B}_{\delta_1}$ and all $t\in\mathbb{R}_{\geq 0}$,

$$\begin{aligned}\dot{\mathcal{V}}_{\delta_1}(t,x_1) &\leq -2\mathcal{V}_{\delta_1}(t,x_1)+\left|\frac{\partial\mathcal{V}_1}{\partial x_1}(t,x_1)\right||g(t,x,\theta^\star)|\\ &\leq -2\mathcal{V}_{\delta_1}(t,x_1)+\left|\frac{\partial\mathcal{V}_1}{\partial x_1}(t,x_1)\right|G_{\theta_1}(|x_1|)\Psi_{\theta_1^\star}(|x_2|)\\ &\leq -\big(2-\eta_{\delta_1}\Psi_{\theta_1^\star}(|x_2|)\big)\,\mathcal{V}_{\delta_1}(t,x_1)\,. \end{aligned}\tag{3.39}$$

Let Assumption 3.27 generate a class $\mathcal{KL}$ function β_{δ_2} such that, for any $x_{20}\in\mathbb{R}^{n_2}$ and any $t\geq t_0$,

$$|\phi_2(t,t_0,x_{20},\theta_2^\star)|\leq\beta_{\delta_2}(|x_{20}|\,,t-t_0)+\delta_2\,.\tag{3.40}$$

Let $x_0=(x_{10}^\top,x_{20}^\top)^\top\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ be any given initial state and $t_0\in\mathbb{R}_{\geq 0}$ be any given initial time. In order to simplify the notations, we refer to $\phi(\cdot,t_0,x_0,\theta^\star)$ as simply $\phi(\cdot)$ and we define $v_1(\cdot):=\mathcal{V}_{\delta_1}(\cdot,\phi_1(\cdot))$. It follows from (3.39) that

$$|\phi_1(t)|>\delta_1\quad\Rightarrow\quad\dot{v}_1(t)\leq\eta_{\delta_1}\Psi_{\theta_1^\star}(\beta_{\delta_2}(|x_{20}|\,,0))v_1(t)\,.$$

Hence, with Lemma 2.7, we conclude that, for all $t\geq t_0$,

$$|\phi_1(t)|\leq\underline{\tilde{\alpha}}_{\delta_1}^{-1}\circ\tilde{\overline{\alpha}}_{\delta_1}(\delta_1)+\underline{\tilde{\alpha}}_{\delta_1}^{-1}\Big(\tilde{\overline{\alpha}}_{\delta_1}(|x_0|)\exp\big(\eta_{\delta_1}\Psi_{\theta_1^\star}(\beta_{\delta_2}(|x_{20}|\,,0))(t-t_0)\big)\Big)\,.$$

Thus, defining

$$\delta_3:=\underline{\tilde{\alpha}}_{\delta_1}^{-1}\circ\tilde{\overline{\alpha}}_{\delta_1}(\delta_1)\,,\tag{3.41}$$

and, for all $s,t\in\mathbb{R}_{\geq 0}$,

$$\rho_{\delta_2}^{\delta_1}(s,t):=\underline{\tilde{\alpha}}_{\delta_1}^{-1}\Big(\tilde{\overline{\alpha}}_{\delta_1}(s)\exp\big(\eta_{\delta_1}\Psi_{\theta_1^\star}(\beta_{\delta_2}(s,0))t\big)\Big)\,,$$

we obtain that, for all $x_0\in\mathbb{R}^n$ and all $t_0\in\mathbb{R}_{\geq 0}$,

$$|\phi_1(t)|\leq\delta_3+\rho_{\delta_2}^{\delta_1}(|x_0|\,,t-t_0)\,,\qquad\forall t\geq t_0\,.\tag{3.42}$$

[3]Note that, for the case that G depends on θ_2 (and therefore on δ_2), (3.38) remains achievable for δ_2 small enough under the additional condition of Remark 3.29.

Notice that $\rho_{\delta_2}^{\delta_1}(\cdot,t) \in \mathcal{K}_\infty$ for all $t \in \mathbb{R}_{\geq 0}$ and that $\rho_{\delta_2}^{\delta_1}(s,\cdot)$ is a continuous nondecreasing function for all $s \in \mathbb{R}_{\geq 0}$. This ensures forward completeness of (3.1a), and consequently of the cascade (3.1). More precisely, defining

$$\tilde{\rho}_{\delta_2}^{\delta_1}(s,t) := \sqrt{\rho_{\delta_2}^{\delta_1}(s,t)^2 + \beta_{\delta_2}(s,0)^2}\,, \qquad \forall s,t \in \mathbb{R}_{\geq 0}\,,$$

$\tilde{\rho}_{\delta_2}^{\delta_1}$ inherits the same properties as $\rho_{\delta_2}^{\delta_1}$, and we obtain that, for all $x_0 \in \mathbb{R}^n$ and all $t_0 \in \mathbb{R}_{\geq 0}$,

$$|\phi(t)| \leq \delta + \tilde{\rho}_{\delta_2}^{\delta_1}(|x_0|\,, t - t_0)\,, \qquad \forall t \geq t_0\,, \tag{3.43}$$

with

$$\delta := \max\{\delta_2\,;\,\delta_3\}\,. \tag{3.44}$$

Proof of global stability: For all $x_0 \in \mathbb{R}^n$, consider the time $T_{\delta_2}^{\delta_1}(|x_{20}|)$ such that[4]

$$\eta_{\delta_1}\,\Psi_{\theta_1^\star}\Big(\beta_{\delta_2}\big(\,|x_{20}|\,, T_{\delta_2}^{\delta_1}(|x_{20}|)\big) + \delta_2\Big) = 1\,.$$

Note that, in view of (3.38), $T_{\delta_2}^{\delta_1}(|x_{20}|)$ is finite and nonnegative for all $x_{20} \in \mathbb{R}^{n_2}$. Also, $T_{\delta_2}^{\delta_1}(\cdot)$ can be picked as a nondecreasing function. In view of (3.39) and (3.40), we have that, for all $t \geq t_0 + T_{\delta_2}^{\delta_1}(|x_{20}|)$,

$$|\phi_1(t)| > \delta_1 \quad \Rightarrow \quad \dot{v}_1(t) \leq -v_1(t)\,.$$

Invoking again Lemma 2.7, we get that, for all $t \geq t_0 + T_{\delta_2}^{\delta_1}(|x_{20}|)$,

$$|\phi_1(t)| \leq \underline{\tilde{\alpha}}_{\delta_1}^{-1} \circ \tilde{\overline{\alpha}}_{\delta_1}(\delta_1) + \underline{\tilde{\alpha}}_{\delta_1}^{-1}\left(\tilde{\overline{\alpha}}_{\delta_1}\left(\left|\phi\left(t_0 + T_{\delta_2}^{\delta_1}(|x_{20}|)\right)\right|\right) e^{-(t - T_{\delta_2}^{\delta_1}(|x_{20}|) - t_0)}\right)\,.$$

In view of (3.41) and (3.43), and recalling that $T_{\delta_2}^{\delta_1}(\cdot)$ and $\tilde{\rho}_{\delta_2}^{\delta_1}(s,\cdot)$ are nondecreasing, it follows that

$$|\phi_1(t)| \leq \delta_3 + \underline{\tilde{\alpha}}_{\delta_1}^{-1}\left(\tilde{\overline{\alpha}}_{\delta_1}\left(\delta + \tilde{\rho}_{\delta_2}^{\delta_1}\left(|x_0|\,, T_{\delta_2}^{\delta_1}(|x_0|)\right)\right) e^{-(t - T_{\delta_2}^{\delta_1}(|x_0|) - t_0)}\right)\,. \tag{3.45}$$

Next, let $\tau_{\delta_1,\delta_2}(|x_0|)$ be any time instant large enough that

$$\underline{\tilde{\alpha}}_{\delta_1,\delta_2}^{-1}\left(\tilde{\overline{\alpha}}_{\delta_1,\delta_2}\left(\tilde{\rho}_{\delta_2}^{\delta_1}\left(|x_0|\,, T_{\delta_2}^{\delta_1}(|x_0|)\right) + \delta\right) e^{T_{\delta_2}^{\delta_1}(|x_0|)} e^{-(\tau_{\delta_1,\delta_2}(|x_0|) - t_0)}\right) \leq \rho_{\delta_2}^{\delta_1}\left(|x_0|\,, T_{\delta_2}^{\delta_1}(|x_0|)\right)\,.$$

Define further

$$\bar{T}_{\delta_2}^{\delta_1}(\cdot) := \max\left\{T_{\delta_2}^{\delta_1}(\cdot)\,;\,\tau_{\delta_1,\delta_2}(\cdot)\right\}\,.$$

Then, we obtain that

$$|\phi_1(t)| \leq \delta_3 + \rho_{\delta_2}^{\delta_1}(|x_0|\,, T_{\delta_2}^{\delta_1}(|x_0|)) \leq \delta_3 + \rho_{\delta_2}^{\delta_1}(|x_0|\,, \bar{T}_{\delta_2}^{\delta_1}(|x_0|))\,, \qquad \forall t \geq t_0 + \bar{T}_{\delta_2}^{\delta_1}(|x_0|)\,.$$

Noticing finally that (3.42) implies that this relation also holds over the time interval $[t_0; t_0 + \bar{T}_{\delta_2}^{\delta_1}(|x_0|)]$, we conclude that

$$|\phi_1(t)|_{\delta_3} \leq \nu_{\delta_1,\delta_2}(|x_0|)\,, \qquad \forall t \geq t_0\,,$$

[4] $T_{\delta_2}^{\delta_1}(|x_{20}|)$ is taken as zero if $\eta_{\delta_1}\,\Psi_{\theta_1^\star}\,(\beta_{\delta_2}(|x_{20}|\,,0) + \delta_2) \leq 1$.

where

$$\nu_{\delta_1,\delta_2}(\cdot) := \rho_{\delta_2}^{\delta_1}\left(\cdot, T_{\delta_2}^{\delta_1}(\cdot)\right) .$$

Uniform global stability of $\mathcal{B}_\delta$ then follows from Assumption 3.27 and the fact that ν_{δ_1,δ_2} is a class $\mathcal{K}_\infty$ function.

Proof of global attractivity: Reconsider (3.45). For any positive r and ε, define

$$\tilde{T}_{\delta_1}^{\delta_2}(\varepsilon, r) := T_{\delta_1}^{\delta_2}(r) + \ln\left(\frac{\tilde{\overline{\alpha}}_{\delta_1}\left(\delta + \tilde{\rho}_{\delta_2}^{\delta_1}\left(r, T_{\delta_2}^{\delta_1}(r)\right)\right)}{\tilde{\underline{\alpha}}_{\delta_1,\delta_2}(\varepsilon)}\right) .$$

Then it can be seen that, for all $x_0 \in \mathcal{B}_r$ and all $t_0 \in \mathbb{R}_{\geq 0}$,

$$t \geq t_0 + \tilde{T}_{\delta_1}^{\delta_2}(\varepsilon, r) \quad \Rightarrow \quad |\phi_1(t)|_{\delta_3} \leq \varepsilon .$$

This shows the uniform attractivity of $\mathcal{B}_{\delta_3}$ for (3.1a), *cf.* Definition 1.10. The attractivity of $\mathcal{B}_\delta$ follows from Assumption 3.27.

Proof of UGPAS: It is only left to show that δ can be arbitrarily reduced. In view of (3.37) and (3.41), δ_3 can be picked arbitrarily small by choosing δ_1 small enough. It follows from (3.44) that δ can be made arbitrarily small by taking both δ_1 and δ_2 small enough. Thus, it suffices to pick the parameters $\theta_1^\star$ and $\theta_2^\star$ generated by these chosen δ_1 and δ_2, to conclude that, for any $\delta > 0$, there exist some parameters $\theta_1^\star \in \Theta_1$ and $\theta_2^\star \in \Theta_2$ such that $\mathcal{B}_\delta$ is uniformly globally asymptotically stable for the cascade (3.1) with $\theta = \theta^\star$. ■

Smooth rejection of disturbances

We illustrate the efficiency of Theorem 3.26 through the following example. Consider a control system affected by a non-vanishing perturbation, *i.e.*,

$$\dot{x}_1 = f_1(t, x_1) + h(t, x_1)[u + d(t, x_1)] \tag{3.46}$$

where d is a bounded function satisfying Carathédory conditions which is locally Lipschitz in x_1. In general, we have $d(t, 0) \not\equiv 0$, which justifies the denomination "non-vanishing perturbation" –*cf.* [Kha01]. Consider the control problem of finding a control $u(t, x_1)$ such that the closed-loop system is uniformly globally asymptotically stable within the following setting.

Let $u^\star(t, x_1)$ be such that the closed-loop system that makes (3.46) uniformly globally asymptotically stable (UGAS) provided that $d \equiv 0$. Let V_1 be a strict Lyapunov function for this nominal closed-loop system, that is, assume that there exist $\underline{\alpha}_1, \overline{\alpha}_1, \alpha_1 \in \mathcal{K}_\infty$ such that, for all $t \in \mathbb{R}_{\geq 0}$ and $x_1 \in \mathbb{R}^{n_1}$,

$$\underline{\alpha}_1(|x_1|) \leq V_1(t, x_1) \leq \overline{\alpha}_1(|x_1|) \tag{3.47}$$

$$\frac{\partial V_1}{\partial t}(t, x_1) + \frac{\partial V_1}{\partial x_1}(t, x_1)(f_1(t, x_1) + h_1(t, x_1)u^\star(t, x_1)) \leq -\alpha_1(|x_1|) . \tag{3.48}$$

Then, it is well-accepted that the controller $u^\star(t, x_1)$ completed by convenient discontinuous terms of the state (roughly of the same size as the perturbation) still achieves UGAS and, in

certain cases, finite-time stabilization –*cf.* [Utk99, EKNN92]. More precisely, the system (3.46) may be rendered UGAS via the *discontinuous* feedback

$$u(t,x_1) = u^\star(t,x_1) - d_M \mathrm{sign}\left(\frac{\partial V_1}{\partial x_1}(t,x_1)h_1(t,x_1)\right). \tag{3.49}$$

Indeed, a straightforward calculation yields

$$\frac{\partial V_1}{\partial t}(t,x_1) + \frac{\partial V_1}{\partial x_1}(t,x_1)\big(f_1(t,x_1) + h_1(t,x_1)(u(t,x_1)+d(t,x))\big) \quad \leq \quad -\alpha_1(|x_1|)\,. \tag{3.50}$$

Under appropriate regularity properties on the control, the disturbance and the Lyapunov function, and embedding the differential equation (3.46) in a Fillippov differential inclusion, we can conclude from (3.47) and (3.50), that the closed-loop system is UGAS.

We stress that, even though when placed in the right theoretical setting, one may show that the UGAS property of the nominal system is conserved in presence of non-vanishing disturbances, this is at the price of an infinite-gain controller that induces undesirable phenomena such as chattering.

A common remedy adopted in control practice is to replace $\mathrm{sign}(\cdot)$ by a saturation function $\mathrm{sat}(\cdot)$ with "high slope" near zeo, *e.g.* the function $\mathrm{sat}(\sigma x) := \sigma x$ for all $|x| \leq 1/\sigma$, $\mathrm{sat}(\sigma x) := \mathrm{sign}(x)$ for all $|x| > 1/\sigma$, with a sufficiently large σ. In more general terms, we can define a saturation function as follows.

Definition 3.32 (Saturation function) *A function* $sat : \mathbb{R} \to [-1;1]$ *is said to be a* saturation *if it is locally Lipschitz, nondecreasing and satisfies*

$$\lim_{|s|\to\infty} |sat(s)| = 1\,, \qquad \textit{and} \qquad sat(s)s > 0\,, \quad \forall s \neq 0\,.$$

Typical examples of saturation functions are $\tanh(s)$, $\arctan(s)$, $s/(1+s^2)$, $\mathrm{sign}(s)\min\{1;|s|\}$.

For a number of specific applications, for instance mechanical systems with friction, it may be observed in simulations that the use of $\mathrm{sat}(\sigma\,\cdot)$ in place of $\mathrm{sign}(\cdot)$ in (3.49) as an *approximation* of the ideal discontinuous term impedes the asymptotic convergence of the trajectories to the origin. Instead, a steady-state error is commonly observed.

Consider further the case when the system (3.46) is interconnected in cascade with a second subsystem:

$$\dot{x}_1 = f_1(t,x_1) + h(t,x_1)(u+d(t,x)) + g(t,x) \tag{3.51a}$$
$$\dot{x}_2 = f_2(t,x_2)\,. \tag{3.51b}$$

where $u \in \mathbb{R}^m$ is the control, $d : \mathbb{R}_{\geq 0}\times\mathbb{R}^n \to \mathbb{R}^m$ is a non-measured perturbation satisfying $|d(t,x)| \leq d_M$ for all $x \in \mathbb{R}^n$ and all $t \in \mathbb{R}_{\geq 0}$, and f_1, f_2, h, d and g satisfy the Carathéodory conditions and are all locally Lipschitz in x.

Such a situation may arise due to a cascaded-based design (*cf.* [SJK97, LP04]), or from the physical structure of the plant. For instance, one could think of an electro-mechanical system: the x_1 dynamics may be thought of as that of a mechanical system, the perturbation d may represent external disturbances, actuator deficiency, *etc.*, the subsystem (3.51b) represents that of the *closed-loop* dynamics of the actuators which may in turn include disturbances.

As we remarked earlier, UGAS may be achievable for each subsystem of the cascade (3.51), using discontinuous functions of the state. However, the classical theorems for

cascades of UGAS systems (*e.g.* [SS90a, AAS02, PL01]) do not apply as they rely on the assumption that the right-hand side term is sufficiently smooth. In addition, when using smoothening techniques exposed above (*i.e.* sat instead of sign), UGAS is lost. Hence, we rely on Theorem 3.26 to show that, with a smooth approximation of the nominal control law obtained by replacing the sign function by a sufficiently stiff saturation, the cascade is UGPAS. The stability analysis follows a cascades-based reasoning: it consists in showing that each subsystem in the cascade is UGPAS (*i.e.* when $g \equiv 0$) and, then, that the cascaded interconnection does not destroy stability.

Proposition 3.33 (Smooth approximation of sign$(\cdot)$ for cascades) *Let V_1 be any smooth Lyapunov function for the UGAS nominal system $\dot{x}_1 = f_1(t,x_1) + h_1(t,x_1)u^\star(t,x_1)$, i.e., for all $x_1 \in \mathbb{R}^{n_1}$ and all $t \in \mathbb{R}_{\geq 0}$, (3.47), (3.48) hold and*

$$\left|\frac{\partial V_1}{\partial x_1}(t,x_1)\right| \leq c_1(|x_1|)\,,$$

where $c_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a continuous nondecreasing function. Assume further that there exists $\psi \in \mathcal{K}$ and a continuous function $G : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$|g(t,x)| \leq G(|x_1|)\psi(|x_2|)\,, \qquad \forall (x_1,x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \quad \forall t \in \mathbb{R}_{\geq 0}\,, \tag{3.52}$$

and that, as s tends to $+\infty$,

$$\begin{aligned} G(s)c_1(s) &= \mathcal{O}(\alpha_1 \circ \overline{\alpha}_1^{-1} \circ \underline{\alpha}_1(s)) && (3.53)\\ \alpha_1(s) &= \mathcal{O}(\overline{\alpha}_1(s))\,. && (3.54) \end{aligned}$$

Assume finally that (3.51b) is UGAS. Then, for any saturation function sat and any positive constant ε, the overall system (3.51) in closed loop with

$$u(t,x_1) := u^\star(t,x_1) - (1+\varepsilon)d_M sat\left(\theta \frac{\partial V_1}{\partial x_1}(t,x_1)h_1(t,x_1)\right)\,, \tag{3.55}$$

is UGPAS on $\Theta := \mathbb{R}_{>0}$, with θ as tuning parameter.

In particular, if sat is chosen as a smooth function, then the control u inherits the same regularity properties as $u^\star$. Notice also that, for the case of an autonomous system and if $u^\star$ is a state feedback, then u is independent of time as well. Furthermore, the magnitude of the additional control law is only required to be strictly greater than d_M; in particular, if $u^\star$ can be designed as a bounded control, then u is bounded too.

Proof of Proposition 3.33. For all $x_1 \in \mathbb{R}^{n_1}$ and all $t \in \mathbb{R}_{\geq 0}$, let

$$L_{h_1}V_1(t,x_1) := \frac{\partial V_1}{\partial x_1}(t,x_1)h_1(t,x_1)\,.$$

When considering $g(t,x) \equiv 0$, the system (3.51a) in closed loop with (3.55) is

$$\dot{x}_1 = f_1(t,x_1) + h_1(t,x_1)\Big[u^\star(t,x_1) - (1+\varepsilon)d_M \mathrm{sat}\left(\theta L_{h_1}V_1(t,x_1)\right) + d(t,x)\Big]\,.$$

Using (3.48), the assumed properties of sat and the boundedness of the perturbation, the derivative of V_1 along the trajectories of (3.51a) when disconnected yields

$$\begin{aligned} \dot{V}_1(t,x_1) &\leq -\alpha_1(|x_1|) - (1+\varepsilon)d_M L_{h_1}V_1(t,x_1)\mathrm{sat}\left(\theta L_{h_1}V_1(t,x_1)\right) + L_{h_1}V_1(t,x_1)d(t,x)\\ &\leq -\alpha_1(|x_1|) - d_M\left|L_{h_1}V_1(t,x_1)\right|\Big[(1+\varepsilon)\left|\mathrm{sat}\left(\theta L_{h_1}V_1(t,x_1)\right)\right| - 1\Big]\,. \end{aligned} \tag{3.56}$$

Consider any arbitrary $\delta_1 > 0$, and choose $\theta^\star(\delta_1)$ large enough that

$$\text{sat}\left(\frac{\theta^\star \alpha_1(\delta_1)}{2d_M}\right) \geq \frac{1}{1+\varepsilon}\,, \tag{3.57}$$

which is always possible since α_1 is independent of θ and since sat is continuous and tends to 1 as its argument tends to $+\infty$. We claim that, with this choice of parameter,

$$|x_1| \geq \delta_1 \quad \Rightarrow \quad \dot{V}_1(t,x_1) \leq -\frac{1}{2}\alpha_1(|x_1|)\,. \tag{3.58}$$

To see this, assume that $|x_1| \geq \delta_1$ and distinguish the following two cases:

- Case 1: $|L_{h_1}V_1(t,x_1)| \leq \alpha_1(\delta_1)/2d_M$: we then get from (3.56) that

$$\dot{V}_1(t,x_1) \leq -\alpha_1(|x_1|) + \varepsilon d_M\,|L_{h_1}V_1(t,x_1)| \leq -\alpha_1(|x_1|) + \frac{\alpha_1(\delta_1)}{2}\,,$$

  and (3.58) follows.

- Case 2: $|L_{h_1}V_1(t,x_1)| > \alpha_1(\delta_1)/2d_M$: it then follows from (3.57) that

$$|\text{sat}\,(\theta^\star L_{h_1}V_1(t,x_1))| \geq \frac{1}{1+\varepsilon}\,,$$

  and (3.58) directly follows from (3.56).

In view of (3.47) and (3.58), and noticing that the functions $\underline{\alpha}_1$ and $\overline{\alpha}_1$ are independent of δ_1 (which makes (3.27) trivial), we conclude with (3.52), (3.53) and (3.54) that Assumption 3.28 holds, and the conclusion follows applying Theorem 3.26. ■

3.4 Asymptotic stability in the large of cascaded systems

The above-presented results provide sufficient conditions to establish semiglobal and/or practical stability properties of a cascaded system based on the the assumption of similar properties for each subsystem taken separately. It is notably interesting to see that, provided a uniform boundedness of the solutions of the overall system, uniform semiglobal asymptotic stability is preserved, meaning that the size of the basin of attraction can be arbitrarily enlarged.

In practice, control applications typically specify a minimum operating bandwidth. In this respect, the information of USPAS or USAS of the system ensure that any of these specifications can be reached by a convenient choice of some parameters. But a natural question then arises: how to tune these parameters in order to obtain a given region of attraction for the cascade ?

To the best of our knowledge, the first works in the literature of stability analysis that took into account such a specified domain instead of an infinitesimal neighborhood of the operating point, referred to this property as *asymptotic stability in the large*[5].

Stability in the large is the best one can hope for systems with multiple equilibria when no free tuning parameter is available. The use of bounded control inputs may also lead to

[5]We take this opportunity to stress that, as observed in [LLLP06], asymptotic stability in the large has wrongly been confused with global asymptotic stability.

a restrained domain of attraction. Other applications arise in output feedback control (see e.g. [LdLM03]).

Please note that, in the sequel, we refer to stability in the large as uniform asymptotic stability on $\mathcal{B}_\Delta$, where $\mathcal{B}_\Delta$ then constitutes an explicitly specified estimate of the basin of attraction.

The fact that two UAS systems in cascade yields a UAS system is well established (see *e.g.* [Vid80]), but this result does not estimate the domain of attraction of the resulting cascade. In [Son03], it is shown that the driven subsystem keeps its domain of attraction provided that the perturbation induced by the driving subsystem does not make the driven state leave its domain of attraction. Here, we use this idea to provide an explicit expression of an estimate of the domain of attraction of the cascade, based on those corresponding to the subsystems.

The typical standing assumption in the stability analysis of cascaded systems is the (uniform) boundedness of the solutions, see for instance [Son89b, SS90a]. In what follows, we use a similar assumption to show that, providing little restrictive properties of the structure of the dynamical system, the cascade composed of two UAS systems is UAS. Moreover, and overall, an explicit estimate of the domain of attraction of the cascade is provided based on those of the two subsystems, the convergence rate of their solutions and the uniform bound on the solutions of the cascade. We shall consider the stability of

$$\begin{cases} \dot{x}_1 = f_1(t, x_1) + g(t, x) \\ \dot{x}_2 = f_2(t, x_2)\,, \end{cases}$$

as introduced in (3.1), under the following standing assumption.

Assumption 3.34 (Bound on g) *The interconnection term g is uniformly bounded in time and there exists a nondecreasing function $G : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that, for all $x = (x_1^\top, x_2^\top)^\top \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and all $t \in \mathbb{R}_{\geq 0}$,*

$$|g(t, x)| \leq G(|x|)\,|x_2|\ .$$

In a first time, we assume that the driven subsystem is UAS on a given ball and that an associated Lyapunov function is explicitly known.

Assumption 3.35 (Lyapunov UAS of the driven subsystem) *There exist a positive number Δ_1, a continuously differentiable function $V_1 : \mathbb{R}_{\geq 0} \times \mathcal{B}_{\Delta_1} \to \mathbb{R}_{\geq 0}$, some $\mathcal{K}_\infty$ functions $\underline{\alpha}_1$ and $\overline{\alpha}_1$, a positive constant k_1 and a continuous nondecreasing function c_1 such that, for all $x_1 \in \mathcal{B}_{\Delta_1}$ and all $t \in \mathbb{R}_{\geq 0}$,*

$$\underline{\alpha}_1(|x_1|) \leq V_1(t, x_1) \leq \overline{\alpha}_1(|x_1|)$$

$$\frac{\partial V_1}{\partial t}(t, x_1) + \frac{\partial V_1}{\partial x_1}(t, x_1) f_1(t, x_1) \leq -k_1 V_1(t, x_1)$$

$$\left| \frac{\partial V_1}{\partial x_1}(t, x_1) \right| \leq c_1(|x_1|)\,.$$

We also assume that the driving subsystem is UAS on a given ball $\mathcal{B}_{\Delta_2}$, with a $\mathcal{KL}$ estimate on its solutions.

Assumption 3.36 (UAS of the driving system) *The driving subsystem $\dot{x}_2 = f_2(t, x_2)$ is uniformly asymptotically stable on a ball $\mathcal{B}_{\Delta_2}$, $\Delta_2 > 0$, i.e. there exists a $\mathcal{KL}$ function β_2 such that, for all $x_{20} \in \mathcal{B}_{\Delta_2}$ and all $t_0 \in \mathbb{R}_{\geq 0}$,*

$$|x_2(t, t_0, x_{20})| \leq \beta_2(|x_{20}|, t - t_0), \qquad \forall t \geq t_0 .$$

Finally, we impose the following uniform boundedness on the solutions of the overall system.

Assumption 3.37 (Boundedness of solutions) *The solutions of (3.1) are uniformly bounded on a ball $\mathcal{B}_b$, $b > 0$, i.e. there exist $B > 0$ such that they satisfy, for all $x_0 \in \mathcal{B}_b$ and all $t_0 \in \mathbb{R}_{\geq 0}$,*

$$|x(t, t_0, x_0)| \leq B, \qquad \forall t \geq t_0 .$$

Based on these assumptions, we are now ready to present a first result that provides an estimate of the basin of attraction for the overall cascade (3.1).

Theorem 3.38 (UAS + UAS + UB ⇒ UAS) *Under Assumptions 3.34–3.37, the cascaded system (3.1) is UAS on $\mathcal{B}_\Delta$ where*

$$\Delta = \min\left\{ \overline{\alpha}_1^{-1}(\lambda \underline{\alpha}_1(\Delta_1)) \ ; \ \beta_{20}^{-1}\left(\frac{(1-\lambda)\underline{\alpha}_1(\Delta_1)}{c_1(B)G(B)} \right) \ ; \ b \ ; \ \Delta_2 \right\}, \tag{3.59}$$

$\lambda \in (0;1)$ being a free design parameter and $\beta_{20}(\cdot) := \beta_2(\cdot, 0)$.

The proof of this result is largely inspired from the one of [PL01, Lemma 2]. We propose it in Section A.3.

Note that systems with a Lyapunov function as in Assumption 3.35 are fairly common in practice. More precisely, it was shown in [TP99] that, for a nonlinear time-varying system defined by a locally Lipschitz right-hand side, uniform asymptotic stability of the origin is equivalent to the existence of a smooth function satisfying the first two bounds of Assumption 3.35. The bound on the gradient is also little conservative; it is notably trivially satisfied for time-invariants systems.

In the case when such a Lyapunov is nevertheless not provided, we can use the following result. The price to pay is that the driven subsystem should then be uniformly exponentially stable. In addition, a more conservative regularity condition on f_1 is required.

Assumption 3.39 (UES of the driven subsystem) *The driven subsystem $\dot{x}_1 = f_1(t, x_1)$ is uniformly exponentially stable on $\mathcal{B}_{\Delta_1}$ with parameters (k_1, γ_1) (cf. Definition 1.15), f_1 is continuously differentiable and there exist positive constants j and L such that, for all $x_1 \in \mathcal{B}_j$ and all $t_0 \in \mathbb{R}_{\geq 0}$,*

$$\left| \frac{\partial f_1}{\partial x_1}(t, x_1) \right| \leq L, \quad \forall t \geq t_0 .$$

Corollary 3.40 (UAS + UES + UB ⇒ UAS) *Under Assumptions 3.34, 3.36, 3.37 and 3.39, the cascaded system (3.1) is UAS on $\mathcal{B}_\Delta$ where*

$$\Delta = \min\left\{ \frac{\sqrt{\lambda}\underline{k}_1 \tilde{\Delta}_1}{\overline{k}_1} \ ; \ \beta_{20}^{-1}\left(\frac{(1-\lambda)\underline{k}_1 \tilde{\Delta}_1}{c_1 B G(B)} \right) \ ; \ \tilde{\Delta} \right\}, \tag{3.60}$$

with any $\lambda \in (0;1)$ *and*

$$\underline{k}_1 := \frac{1-e^{-2LT}}{2L} \tag{3.61a}$$

$$\overline{k}_1 := \frac{k_1^2(1-e^{-2LT})}{2\gamma_1} \tag{3.61b}$$

$$c_1 := \frac{2k_1(1-e^{-(\gamma_1-L)T})}{\gamma_1-L} \tag{3.61c}$$

$$T := \frac{ln(2k_1^2)}{2\gamma_1} \tag{3.61d}$$

$$\beta_{20}(\cdot) := \beta_2(\cdot,0) \tag{3.61e}$$

$$\tilde{\Delta} := \min\left\{\tilde{\Delta}_1 \ ; \ \Delta_2 \ ; \ b\right\}, \tag{3.61f}$$

and $\tilde{\Delta}_1$ *being any positive number such that*

$$\tilde{\Delta}_1 \leq \Delta_1 \quad \textit{and} \quad \tilde{\Delta}_1 < \frac{j}{k_1}. \tag{3.62}$$

Please refer to Section A.4 for the proof.

A noteworthy particular case of the two previous results is when both subsystems are UES. In this situation, we show that the cascaded system (3.1) is UES as well. We state this fact in the following result.

Assumption 3.41 (UES of the driving subsystem) *The subsystem* $\dot{x}_2 = f_2(t,x_2)$ *is uniformly exponentially stable on a ball* $\mathcal{B}_{\Delta_2}$ *with parameters* (k_2,γ_2).

Theorem 3.42 (UES+UES+UB ⇒ UES) *Under Assumptions 3.34, 3.37, 3.39 and 3.41, the cascaded system (3.1) is UES on* $\mathcal{B}_\Delta$ *where*

$$\Delta = \min\left\{\frac{\underline{k}_1\tilde{\Delta}_1\sqrt{\lambda}}{\overline{k}_1} \ ; \ \frac{(1-\lambda)\underline{k}_1\tilde{\Delta}_1}{k_2c_1BG(B)} \ ; \ \tilde{\Delta}\right\},$$

$\underline{k}_1$, $\overline{k}_1$, k_1, $\tilde{\Delta}_1$ *and* $\tilde{\Delta}$ *being given in (3.61)-(3.62) and* $\lambda \in (0;1)$ *being a free pararameter.*

The proof is given in Section A.4. We underline that the parameters (k,γ) for the UES of (3.1) are explicitly constructed based on the information we have about the two subsystems. This constitutes a noteworthy additional result. For more clarity, they are however not given in the previous statement, but can easily be found along the lines of the proof.

Again, if the bound on the gradient of f_1 in Assumption 3.39 is not fulfilled, one can use the following. This last result assumes instead the existence of a convenient Lyapunov function for the driven subsystem. In this situation, the expression of the estimate of the domain of attraction of the whole cascade is considerably simplified.

Assumption 3.43 *There exist a positive number* Δ_1, *a continuously differentiable function* $V_1 : \mathbb{R}_{\geq 0} \times \mathcal{B}_{\Delta_1}$, *and some positive numbers* $\underline{k}_1$, $\overline{k}_1$, k_1 *and* c_1 *such that, for all* $x_{10} \in \mathcal{B}_{\Delta_1}$ *and all* $t \in \mathbb{R}_{\geq 0}$,

$$\underline{k}_1 |x_1|^2 \leq V_1(t,x_1) \leq \overline{k}_1 |x_1|^2$$

$$\frac{\partial V_1}{\partial t}(t,x_1)+\frac{\partial V_1}{\partial x_1}(t,x_1)f_1(t,x_1)\leq -k_1\left|x_1\right|^2$$
$$\left|\frac{\partial V_1}{\partial x_1}(t,x_1)\right|\leq c_1\left|x_1\right|\,.$$

Corollary 3.44 *Under Assumptions 3.34, 3.37, 3.41 and 3.43 the cascaded system (3.1) is UES on $\mathcal{B}_\Delta$ where*

$$\Delta=\min\left\{\frac{\underline{k}_1\Delta_1}{2\overline{k}_1}\;;\;\underline{k}_1\frac{\Delta_1}{2k_2c_1BG(B)}\;;\;\Delta_2\right\}\,.$$

Proof of Corollary 3.44. The proof is straightforward by noticing that, in Theorem 3.42, Assumption 3.39 is only used to ensure the existence of a Lyapunov function V_1 as in Assumption 3.43, and so we have that $\tilde{\Delta}_1=\Delta_1$. ■

The results presented in this section extend the possibilities of the cascades approach for the analysis of nonlinear time-varying systems. Let us now apply them on the following illustrative example.

Example 3.45 *We consider the following two-dimensional system*

$$\dot{x}_1 = -x_1+x_1^2-\frac{1}{3}\big(1-(x_1^2+x_2^2)\big)x_1^3x_2^2 \tag{3.63a}$$
$$\dot{x}_2 = -x_2\,. \tag{3.63b}$$

It can be put in the cascade form (3.1) by letting

$$f_1(t,x_1)=-x_1(1-x_1)\,,\qquad g(t,x)=-\frac{1}{3}(1-x_1^2-x_2^2)x_1^3x_2\,,\qquad f_2(t,x_2)=-x_2\,.$$

Assumption 3.34 then follows directly with

$$G(s)=\frac{1}{3}\max\{1;s^2\}s^4\,,\qquad \forall s\in\mathbb{R}_{\geq 0}\,.$$

Moreover, considering the function $V_1(x_1)=x_1^2/2$, it holds that

$$\frac{dV_1}{dx_1}(x_1)f_1(t,x_1)\leq -(1-|x_1|)x_1^2\,.$$

Therefore, for any positive $\Delta_1<1$ and all $x_1\in\mathcal{B}_{\Delta_1}$,

$$\frac{dV_1}{dx_1}f_1(t,x_1)\leq -(1-\Delta_1)x_1^2=-2(1-\Delta_1)V_1\,.$$

Assumption 3.43 is then satisfied with

$$\underline{k}_1=\overline{k}_1=\frac{1}{2}\,,\qquad k_1=2(1-\Delta_1)\,,\qquad \textit{and}\qquad c_1=1\,.$$

Furthermore, its derivative along the trajectories of (3.63) yields

$$\dot{V}_1\leq -\big(1-|x|+\frac{1}{3}(1-|x|^2)x_1^2x_2^2\big)x_1^2\,.$$

Notice that, for each $B \leq 1$, $|x| \leq B$ implies $\dot{V}_1 \leq 0$. The conditions of Proposition 2.13 are thus fulfilled with $a = 0$. It follows that, for all $|x_0| \leq B$, $|x(t, t_0, x_0)| \leq B$ for all $t \geq t_0$. In other words, Assumption 3.37 is satisfied with any $b = B \leq 1$. Finally, the x_2 subsystem is clearly UGES with parameters $(k_2, \gamma_2) = (1, 1)$, which establishes Assumption 3.41 for any $\Delta_2 > 0$. Applying Corollary 3.44 shows that the cascade (3.63) is uniformly asymptotically stable on $\mathcal{B}_\Delta$, with

$$\Delta = \min\left\{\Delta_1\sqrt{\lambda}\,;\, \frac{(1-\lambda)\Delta_1^2}{BG(B)}\,;\, B\,;\, \Delta_2\right\}.$$

For instance, for $B = \Delta_1 = \Delta_2 = 0.9$ and $\lambda = 2/3$, we obtain that $\Delta = 0.68$. The size of the basin of attraction of the overall cascade is therefore of the same order as those of the subsystems taken separately. We stress that this estimate is moreover representative of the actual size of the domain of attraction since, for instance, the initial condition $(1, 0)$ (which is an equilibrium) yields non-converging solutions. The largest ball of initial conditions on which the origin is uniformly asymptotically stable is consequently necessarily smaller than the unit ball. Figure 3.3 represents the vector field of the system (3.63) in the state space together with the balls of radius 0.68 and 1 respectively. Note that all solutions starting in the region $\mathbb{R}_{>1} \times \mathbb{R}$ diverge.

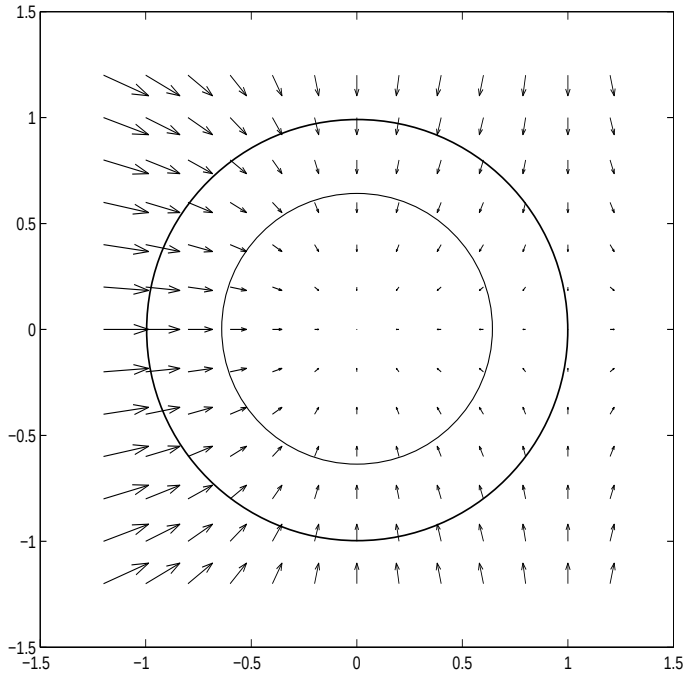


Figure 3.3: Vector field in the state-space.

□

Conclusion. This chapter presents tools that aim at simplifying the study of semiglobal and/or practical stability properties for complex systems. In general terms, they establish USPAS, USAS or UGPAS of cascaded systems based on the assumption of a similar property on each of the interconnected subsystems. In a first step, these results are established under an assumption of boundedness of the solutions of the overall cascade and the explicit knowledge of a Lyapunov function for the driven subsystem. It is worth pointing out that, for the UGPAS case, the solutions' boundedness requirement is replaced by a simple growth order comparison, which makes its use much simpler. Moreover, thanks to the converse theorem established in Chapter 1, we can get rid of the assumption of explicitly knowing a Lyapunov function, modulo a stronger regularity assumption on the right-hand side term of the driven subsystem. Examples are proposed along the chapter to illustrate our purpose.

Chapter 4

Set-stability

In the previous chapter, we have presented some tools that provide semiglobal and practical stability properties for a cascaded system, based on the stability property satisfied by each subsystem taken separately. We have shown that the semiglobal and/or practical stability constitutes a good robustness measure for a globally asymptotically stable system subjected to non-vanishing external disturbances, model imperfection, *etc.*

Although, as it is further underlined through concrete examples in Chapter 6, these types of stability property arise very often in control practice, they might constitute too strong a requirement on some occasions. For instance, control limitations may impede to *arbitrarily* reduce the size of the attractive ball. The following basic example illustrates this fact.

Example 4.1 (From UGPAS to set-stability) *Consider the simple scalar system*

$$\dot{x} = -\theta x + 1\,,$$

where θ is a free positive gain. In view of the results presented in Chapter 2, or simply by integrating directly this differential equation, uniform global practical asymptotic stability (UGPAS) can easily be established. More precisely, given any positive δ, any choice $\theta^\star(\delta) > 1/\delta$ ensures that the ball $\mathcal{B}_\delta$ is globally asymptotically stable.

However, we can imagine that physical constraints prevent to choose an arbitrarily large parameter. If, for instance, θ is restrained to the interval $[0; 10]$, then the smallest achievable steady-state error is $1/10$. Said differently, we lose UGPAS and only conclude global asymptotic stability of any ball of radius larger *than $1/10$.* □

A wide range of applications. Uniform stability of balls centered at the origin indeed constitutes a particular case of the set-stability analysis we wish to conduct in this chapter (*cf.* Section 1.2 for definitions), but the motivations for this study are much wider, as the set under consideration is *not* required to be compact.

For instance, set-stability includes as a special case the analysis of partial stability, *cf.* [Vor98]. Decomposing the state x as $(y^\top, z^\top)^\top$, with $y = (y_1, \ldots, y_p)^\top$ and $z = (z_1, \ldots, z_{n-p})^\top$, with $p \in \mathbb{N}_{\geq 1}$ and $n \in \mathbb{N}_{\geq 2}$, this concept requires that only the y-part of the state be (asymptotically) stable, while the behavior of the remaining variables z is not constrained. More precisely, partial stability can be defined as the following adaptation of [Vor98, Definition 0.3.1][1].

[1] The original definition is given in "$\varepsilon - \delta$" terms, while we have here preferred a $\mathcal{KL}$ formulation.

Definition 4.2 (Partial UAS/UGAS) *The origin of the system $\dot{x} = f(t,x)$ introduced in (1.1) is said to be* partially uniformly asymptotically stable on a closed subset $\mathcal{I}$ of $\mathbb{R}^n$ *if (1.1) is forward complete on $\mathcal{I}$ and, for all $x_0 \in \mathcal{I}$ and all $t_0 \in \mathbb{R}_{\geq 0}$, its solution $\phi(\cdot, t_0, x_0) =: (y(\cdot, t_0, x_0)^\top, z(\cdot, t_0, x_0)^\top)^\top$ satisfies*

$$|y(t, t_0, x_0)| \leq \beta(|x_0|, t - t_0), \qquad \forall t \geq t_0 .$$

If $\mathcal{I} = \mathbb{R}^n$, the origin is said to be partially uniformly globally asymptotically stable, *or simply y-UGAS.*

It can be observed that this definition precisely corresponds to the definition of uniform (global) asymptotic stability of the set $\mathcal{A} = \{0\} \times \mathbb{R}^{n-p}$ (see Definition 1.11).

The concrete developments based on this concept are numerous and concern fields as various as mechanical systems, particle control in electromagnetic fields, ecological systems *etc.*, *cf.* [Vor98] and references therein.

For instance, friction in mechanical systems is often modelled as an exogenous dynamical subsystem (*cf. e.g.* [COaL95, SAGP00]), thus generating an additional "superfluous" state on which no prescribed behavior is imposed.

From a stability analysis point of view, another application of partial stability concerns adaptive control. We may indeed consider as an extended state x the actual state y plus the adaptation error variables z. In many cases, one desires that the 'real' state y presents a convenient asymptotic stability property, while the convergence of the parameters estimation z is often not required.

In view of this generality, it appears interesting and natural to derive sufficient conditions under which stability of (non necessarily compact) sets is preserved by cascade interconnections.

One or two measures ? As it has already been underlined in Chapter 1, the set-stability definition we consider in this document is defined based on two measures, *cf.* Proposition 1.14. This concept, originally introduced by Movchan in [Mov60] is less conservative than its natural one-measure counterpart earlier proposed by Barbashin in [Bar51]. With the same notations as in Proposition 1.14, the latter would correspond to

$$|\phi(t, t_0, x_0)|_{\mathcal{A}} \leq \beta(|x_0|_{\mathcal{A}}, t - t_0), \qquad \forall t \geq t_0 .$$

This type of set-stability has been extensively studied, in both cases of compact and non-compact sets, and has given rise to powerful stability results. See for instance [Wil69, LSW96, TPL02]. See also [Lin96] for a concept of input-to-state stability with respect to non necessarily compact sets.

However, the fact that the distance of the solution from the set $\mathcal{A}$ be only determined by the the distance of initial states from this very set is a strong requirement, notably when $\mathcal{A}$ is unbounded. To take up again the example of adaptive control, although one does not necessarily require that the estimation error converges to zero, the influence of the initial values of the parameters on the behavior of state-solutions is not negligible in most situations, preventing a set-stability approach with respect to *one* measure.

In addition, the Lyapunov characterization of the set-stability defined with one measure, *cf.* [LSW96], requires that the Lyapunov function vanishes on the whole set $\mathcal{A}$, which, as seen in Chapter 1 (p. 41), is an important constraint in practice, even when $\mathcal{A}$ is compact. For the stability analysis of perturbed system, this requirement indeed prevents to use the Lyapunov function associated to the nominal system.

The previous observations mainly motivated the use the set-stability with respect to two measures, as the one given by Definition 1.9. Although not included in the present document, please note that we have provided similar results for the one-measure case in [TCPJ06].

Set-stability for cascades. In this chapter, we study this stability property for cascaded systems by assuming that, for each subsystem taken separately, a given set is globally asymptotically stable. More precisely, we show that, if all the solutions of the cascade are globally bounded (with respect to the origin), then the cross product of the two original sets is globally asymptotically stable for the overall cascade. We also show that the requirement of boundedness of solutions may be relaxed to just global boundedness *with respect to a set* for a certain class of cascaded systems. Furthermore, we give a sufficient condition, in terms of growth rate restrictions, that allows to relax this assumption to just forward completeness and consequently makes our tool easy to use in many applications.

We stress that, for the particular case when the considered sets are balls centered at the origin, we retrieve a direct consequence of [JTP94, Proposition 3.2], which establishes that the input to state practical[2] stability is preserved by the cascade composition.

The results we provide in this chapter rely on similar arguments as the well-known Barbălat's lemma [Bar59] which holds only for time-invariant systems. For this reason, we restrict our attention to autonomous cascades. More precisely, we consider

$$\dot{x}_1 = f_1(x_1) + g(x_1, x_2) \tag{4.1a}$$
$$\dot{x}_2 = f_2(x_2)\,, \tag{4.1b}$$

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$ and all functions are assumed locally Lipschitz.

Please note that the exclusion of time-varying systems from the scope of this study is only apparent as set-stability *includes*, as a special case, the stability analysis of time-varying system. However, as a significative drawback, *uniformity* in the initial time t_0 (and the robustness this naturally induces (*cf.* p. 23)) cannot be guaranteed with a set-stability defined based on two measures as the one we have decided to consider for the reasons exposed above. Let us illustrate this with the following elementary example.

Example 4.3 (Set-stability and time-varying systems) *Consider the following scalar time-varying dynamical system:*

$$\dot{x} = -(1+t)x\,. \tag{4.2}$$

Defining $\xi_1 := t$, $\xi_2 := x$ *and* $\xi := (\xi_1, \xi_2)^\top$, *it can be represented as the two-dimensional time-invariant system*

$$\dot{\xi} = f(\xi)\,, \qquad \textit{where} \qquad f(\xi) := \begin{pmatrix} 1 \\ -(1+\xi_1)\xi_2 \end{pmatrix}. \tag{4.3}$$

Let $\mathcal{A} := \mathbb{R}_{\geq 0} \times \{0\}$. *Due to the fact that* $|\xi|_{\mathcal{A}} = x$, *saying that the origin of (4.2) is globally asymptotically stable is equivalent to saying that the set* $\mathcal{A}$ *is globally asymptotically stable for (4.3), cf. Proposition 1.14. More precisely, it can be seen that, for any* $\xi_0 \in \mathbb{R}^2$, *its solution satisfies*

$$\xi(t, \xi_0) = \begin{pmatrix} t + \xi_{10} \\ \xi_{20} e^{\xi_{10}(1+\xi_{10}/2)} e^{-t(1+t/2)} \end{pmatrix}, \qquad \forall t \in \mathbb{R}_{\geq 0}\,, \tag{4.4}$$

[2] Here, "practical" should not be understood in the sense of Chapter 1, as the attractive ball is fixed and not required to be arbitrarily reducible by a convenient tuning.

and, consequently,

$$|\xi(t,\xi_0)|_{\mathcal{A}} = |\xi_2(t,\xi_0)| \leq \beta(|\xi_0|,t) := |\xi_0|\, e^{|\xi_0|(1+|\xi_0|/2)} e^{-t(1+t/2)}\,, \qquad \forall t \in \mathbb{R}_{\geq 0}\,.$$

To see more clearly why uniformity *of the induced stability property cannot be provided by this approach, notice that (4.4) can be written in the "time-varying" context as*

$$\phi(t,t_0,\xi_0) = x_0 e^{t_0(1+t_0/2)} e^{-t(1+t/2)} = x_0 e^{-(t-t_0)(1+t-t_0)} e^{2t_0(t+t_0)}\,, \qquad \forall t \geq t_0 \geq 0\,,$$

and the corresponding $\mathcal{KL}$ estimate is therefore non-uniform in the initial time t_0.

4.1 Preliminary definitions and tools

We first recall some definitions related to set-stability for nonlinear autonomous systems of the form

$$\dot{x} = f(x) \tag{4.5}$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz function. In the sequel, $\mathcal{A}$ refers to a closed (but not necessarily bounded) set of $\mathbb{R}^n$ that contains the origin. Assuming that $0 \in \mathcal{A}$ allows indeed to guarantee that $|\cdot|_{\mathcal{A}} \leq |\cdot|$. Nevertheless, this is absolutely not restrictive as a simple change of variables always permits to reach this situation.

The following property is an adaptation of [TPL02, Definition 5] to the case when the stability properties are defined through two different measures[3] (namely $|\cdot|_{\mathcal{A}}$ and $|\cdot|$).

Definition 4.4 (GSTS) *The closed set $\mathcal{A}$ is said to be* globally sliding time stable *for (4.5) if there exists class $\mathcal{K}_\infty$ functions T and ρ such that, for all $x_0 \in \mathbb{R}^n$, the solution of (4.5) satisfies*

$$|\phi(t,x_0)|_{\mathcal{A}} \leq \rho(|x_0|)\,, \qquad \forall t \in [0; T(|x_0|)]\,.$$

For the case when $\mathcal{A} = \{0\}$, we say, with a slight abuse of terminology, that (4.5) is GSTS.

Remark 4.5 *This property is little restrictive. For instance, it can be shown that (4.5) is GSTS in each of these cases:*

- *the function f in (4.5) is globally Lipschitz,*
- *the solutions of (4.5) are globally bounded (see Definition 1.8).*

Based on this definition, we present a simple integral criterion for the global asymptotic stability of a given closed (but not necessarily bounded) set. It can be seen as an extension of [TPL02, Theorem 1] and [POM02, Lemma 2.1] to the case of stability with respect to two measures.

Lemma 4.6 (Integral lemma for GAS of a set) *Assume that a given closed subset $\mathcal{A}$ of $\mathbb{R}^n$ is GSTS for (4.5) and that there exists a class $\mathcal{K}$ function σ and a class $\mathcal{K}_\infty$ function δ such that, for all $x_0 \in \mathbb{R}^n$, the solution of (4.5) satisfies*

$$\int_0^\infty \sigma(|\phi(t,x_0)|_{\mathcal{A}})dt \leq \delta(|x_0|)\,. \tag{4.6}$$

[3] In the original reference, the same measure $|\cdot|_{\mathcal{A}}$ was used.

Then $\mathcal{A}$ is globally stable[4]*. If, in addition, the solutions of (4.5) are globally bounded*[5]*, then $\mathcal{A}$ is globally asymptotically stable for (4.5).* □

Proof of Lemma 4.6. The proof is composed of two steps. We first show that $\mathcal{A}$ is globally stable if it is GSTS and (4.6) holds. Next, we establish its global attractiveness under the assumption of global boundedness of the solutions of (4.5).

Proof of GS: The proof of the global stability of $\mathcal{A}$ is inspired by that of [TPL02, Theorem 1]. Let T and ρ be generated by the GSTS of $\mathcal{A}$ and let κ be any $\mathcal{K}_\infty$ function satisfying

$$\kappa^{-1}(s) \leq \min\left\{ s \ ; \ \delta^{-1}\left(\frac{1}{2}T(s)\sigma(s)\right)\right\}, \qquad \forall s \in \mathbb{R}_{\geq 0}\,.$$

The existence of such a function is ensured by the fact that $\sigma \in \mathcal{K}$ and $T, \rho \in \mathcal{K}_\infty$. Notice that $\kappa(s) \geq s$ for all $s \in \mathbb{R}_{\geq 0}$ and that the following property holds:

$$\delta(s) \leq \frac{1}{2}T \circ \kappa(s)\sigma \circ \kappa(s)\,, \qquad \forall s \in \mathbb{R}_{\geq 0}\,. \tag{4.7}$$

We claim that, for any $x_0 \in \mathbb{R}^n$,

$$|\phi(t, x_0)|_{\mathcal{A}} \leq \rho \circ \kappa(|x_0|)\,, \qquad \forall t \in \mathbb{R}_{\geq 0}\,. \tag{4.8}$$

First observe that this holds for $x_0 = 0$ due to (4.6) and the continuity of $\phi(\cdot, x_0)$. For $|x_0| > 0$, we proceed by contradiction. Assume that the property (4.8) does not hold. Then, there exists a time $t_1 \in \mathbb{R}_{\geq 0}$ such that

$$|\phi(t_1, x_0)|_{\mathcal{A}} > \rho \circ \kappa(|x_0|)\,. \tag{4.9}$$

Note that, without loss of generality, ρ can be assumed to satisfy $\rho(s) \geq s$ for all $s \in \mathbb{R}_{\geq 0}$. Therefore

$$|x_0|_{\mathcal{A}} \leq |x_0| \leq \kappa(|x_0|) \leq \rho \circ \kappa(|x_0|)\,.$$

So, invoking again the continuity of $\phi(\cdot, x_0)$, there exists a time $t_2 \in [0, t_1)$ such that

$$|\phi(t_2, x_0)|_{\mathcal{A}} = \kappa(|x_0|) \tag{4.10}$$

$$|\phi(t, x_0)|_{\mathcal{A}} \geq \kappa(|x_0|)\,, \quad \forall t \in (t_2; t_1)\,. \tag{4.11}$$

Furthermore, the GSTS of $\mathcal{A}$ combined with (4.9) and (4.10) implies that $t_1 > t_2 + T(\kappa(|x_0|))$. From (4.6) and (4.11) it follows that, on one hand,

$$\int_{t_2}^{t_1} \sigma(|\phi(t, x_0)|_{\mathcal{A}})dt \geq \int_{t_2}^{t_2+T(\kappa(|x_0|))} \sigma \circ \kappa(|x_0|)dt = T(\kappa(|x_0|))\sigma \circ \kappa(|x_0|)\,,$$

and, on the other hand,

$$\int_{t_2}^{t_1} \sigma(|\phi(t, x_0)|_{\mathcal{A}})dt \leq \int_0^{\infty} \sigma(|\phi(t, x_0)|_{\mathcal{A}})dt \leq \delta(|x_0|)\,,$$

Combining these two bounds, we obtain that

$$T(\kappa(|x_0|))\sigma \circ \kappa(|x_0|) \leq \delta(|x_0|)\,,$$

[4]Global stability and global asymptotic stability of $\mathcal{A}$ are to be understood in the sense of Definitions 1.2 and 1.4. "Uniformity" is pointless in this section since only time-invariant systems are considered.

[5]which implies that $\mathcal{A}$ is GSTS in view of Remark 4.5 since $0 \in \mathcal{A}$.

which contradicts (4.7).

Proof of GA: This second step follows along the same proof-lines of Barbălat's lemma originally presented in [Bar59]; see also [Tao97], [Tee99] and [POM02, Lemma 2.1] for similar approaches.

We proceed by contradiction. Assume that $\lim_{t\to\infty} |\phi(t,x_0)|_{\mathcal{A}} \neq 0$ for some $x_0 \in \mathbb{R}^n$. Then there exist a positive ε and a sequence $\{t_i\}_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty} t_i = +\infty$ and

$$|\phi(t_i,x_0)|_{\mathcal{A}} > \varepsilon\,, \qquad \forall i \in \mathbb{N}\,. \tag{4.12}$$

Notice that the sequence $\{t_i\}_{i\in\mathbb{N}}$ can be picked in such a way that

$$t_{i+1} \geq t_i + T_m\,, \qquad \forall i \in \mathbb{N}\,, \tag{4.13}$$

T_m designating a positive constant. Due to the global boundedness of solutions and the continuity of f, we can see that $\left|\dot{\phi}_1(\cdot,x_0)\right|$ is bounded, which implies that $\phi(\cdot,x_0)$ is uniformly continuous. This means that, given any positive c, there exists a positive T such that, for all $t \in \mathbb{R}_{\geq 0}$ and all $\tau \in [0,T]$, $|\phi(t+\tau,x_0) - \phi(t,x_0)| < c$. Hence, letting

$$\tilde{\sigma}(s) := \begin{cases} \sigma(s) & \text{if} \quad s \in \mathbb{R}_{\geq 0} \\ -\sigma(-s) & \text{if} \quad s \in \mathbb{R}_{<0}\,, \end{cases}$$

and picking[6] c as $\tilde{\sigma}^{-1}(\tilde{\sigma}(\varepsilon/2)/2)$, there exists a positive T such that, for all $t \in \mathbb{R}_{\geq 0}$ and all $\tau \in [0,T]$,

$$|\phi(t+\tau,x_0) - \phi(t,x_0)| < \tilde{\sigma}^{-1}\left(\frac{1}{2}\tilde{\sigma}\left(\frac{\varepsilon}{2}\right)\right)\,. \tag{4.14}$$

Using the properties that $|y+z| \geq |y| - |z|$ for all $y,z \in \mathbb{R}^n$, and $\tilde{\sigma}(a-b) \geq \tilde{\sigma}(a/2) - \tilde{\sigma}(b)$ for all $a,b \in \mathbb{R}$, it follows in view of (4.12) and (4.14) that, for all $t \in [t_i, t_i+T]$

$$\begin{aligned} \tilde{\sigma}(|\phi(t,x_0)|_{\mathcal{A}}) &\geq \tilde{\sigma}\left(|\phi(t_i,x_0)|_{\mathcal{A}} - |\phi(t,x_0) - \phi(t_i,x_0)|_{\mathcal{A}}\right) \\ &\geq \tilde{\sigma}\left(\frac{1}{2}|\phi(t_i,x_0)|_{\mathcal{A}}\right) - \tilde{\sigma}\left(|\phi(t,x_0) - \phi(t_i,x_0)|_{\mathcal{A}}\right) \\ &\geq \tilde{\sigma}\left(\frac{\varepsilon}{2}\right) - \frac{1}{2}\tilde{\sigma}\left(\frac{\varepsilon}{2}\right) = \frac{1}{2}\tilde{\sigma}\left(\frac{\varepsilon}{2}\right)\,. \end{aligned}$$

Based on what precedes and (4.13), we then have that

$$\begin{aligned} \int_0^\infty \sigma(|\phi(t,x_0)|_{\mathcal{A}})dt &= \int_0^\infty \tilde{\sigma}(|\phi(t,x_0)|_{\mathcal{A}})dt \\ &\geq \sum_{i\in\mathbb{N}} \int_{t_i}^{t_{i+1}} \tilde{\sigma}(|\phi(t,x_0)|_{\mathcal{A}})dt \\ &\geq \sum_{i\in\mathbb{N}} \int_{t_i}^{t_i+\min\{T_m;T\}} \tilde{\sigma}(|\phi(t,x_0)|_{\mathcal{A}})dt \\ &\geq \sum_{i\in\mathbb{N}} \frac{1}{2}\tilde{\sigma}\left(\frac{\varepsilon}{2}\right)\min\{T_m;T\} = +\infty\,, \end{aligned}$$

which establishes the contradiction.

■

[6] Note that, even though σ may not be a class $\mathcal{K}_\infty$ function, $\tilde{\sigma}(\varepsilon/2)/2$ necessarily belongs to the domain of invertibility of $\tilde{\sigma}$ by construction.

4.2 On set-stability of cascaded systems

Our first main result in the context of set-stability for cascaded systems states that the cascade of two globally set-stable systems is itself globally set-stable provided that its solutions are globally bounded.

Theorem 4.7 (GAS + GAS + GB ⇒ GAS) *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be closed sets of $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ respectively. Under the Assumptions 4.8–4.11 below, the set $\mathcal{A} := \mathcal{A}_1 \times \mathcal{A}_2$ is globally asymptotically stable for the cascade (4.1).*

Assumption 4.8 (GAS of $\mathcal{A}_1$) *$\mathcal{A}_1$ is globally asymptotically stable for $\dot{x}_1 = f_1(x_1)$.*

Assumption 4.9 (GAS of $\mathcal{A}_2$) *$\mathcal{A}_2$ is globally asymptotically stable for (4.1b).*

Assumption 4.10 (Bound on the interconnection) *There exist a continuous function $g_1 : \mathbb{R}^n \to \mathbb{R}^{n_1}$ and a class $\mathcal{K}_\infty$ function g_2 such that, for all $x = (x_1^\top, x_2^\top)^\top \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$,*

$$|g(x_1, x_2)| \leq |g_1(x)|\, g_2(|x_2|_{\mathcal{A}_2})\,.$$

Assumption 4.11 (GB) *The solutions of (4.1) are globally bounded.*

Proof of Theorem 4.7. We start by invoking [TP00, Corollary 1] to generate a Lyapunov function for each of the two subsystems, based on Assumptions 4.8 and 4.9. More precisely, for each $i \in \{1, 2\}$, there exist a smooth function $V_i : \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$ and class $\mathcal{K}_\infty$ functions $\underline{\alpha}_i$ and $\overline{\alpha}_i$ such that, for all $x_i \in \mathbb{R}^{n_i}$,

$$\underline{\alpha}_i(|x_i|_{\mathcal{A}_i}) \leq V_i(x_i) \leq \overline{\alpha}_i(|x_i|) \tag{4.15}$$

$$\frac{\partial V_i}{\partial x_i}(x_i) f_i(x_i) \leq -V_i(x_i) \leq -\underline{\alpha}(|x_i|_{\mathcal{A}_i})\,. \tag{4.16}$$

In view of Assumption 4.10, the derivative of V_1 along the trajectories of (4.1) yields, for all $x \in \mathbb{R}^n$,

$$\dot{V}_1(x_1) \leq -\underline{\alpha}_1(|x_1|_{\mathcal{A}_1}) + \left|\frac{\partial V_1}{\partial x_1}(x_1)\right| |g_1(x)|\, g_2(|x_2|_{\mathcal{A}_2})\,.$$

Let $c_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be the function defined as

$$c_1(s) := \max_{|x| \leq s} \left|\frac{\partial V_1}{\partial x_1}(x_1)\right| |g_1(x)|\,, \qquad \forall s \in \mathbb{R}_{\geq 0}\,.$$

Due to the smoothness of V_1 and the continuity of g_1, it can be seen that c_1 is a continuous nondecreasing function, and we have that, for all $t \in \mathbb{R}_{\geq 0}$,

$$\dot{V}_1(\phi_1(t, x_0)) \leq -\underline{\alpha}_1(|\phi_1(t, x_0)|_{\mathcal{A}_1}) + c_1(|\phi(t, x_0)|) g_2(|\phi_2(t, x_0)|_{\mathcal{A}_2})\,.$$

From Assumption 4.11 and Proposition 1.12, there exists a class $\mathcal{K}_\infty$ function η and a nonnegative constant μ such that $|\phi(t, x_0)| \leq \eta(|x_0|) + \mu$, so we obtain that

$$\dot{V}_1(\phi_1(t, x_0)) \leq -\underline{\alpha}_1(|\phi_1(t, x_0)|_{\mathcal{A}_1}) + \tilde{c}_1(|x_0|) g_2(|\phi_2(t, x_{20})|_{\mathcal{A}_2})\,, \tag{4.17}$$

where $\tilde{c}_1(\cdot) := c_1(\eta(\cdot) + \mu)$.

Let a_1 and a_2 be positive numbers such that $\underline{\alpha}_2(a_1) < g_2(a_2)$ and let $\tilde{g}_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be any continuous increasing function satisfying, for all $s \in \mathbb{R}_{\geq 0}$,

$$\tilde{g}_2(s) = \begin{cases} \underline{\alpha}_2(s) & \text{if} \quad s \in [0, a_1] \\ g_2(s) & \text{if} \quad s \geq a_2 \,. \end{cases}$$

Note that $\tilde{g}_2$ can always be completed on the interval (a_1, a_2) in order to be an increasing function since $\underline{\alpha}_2(a_1) < g_2(a_2)$ and both $\underline{\alpha}_2$ and g_2 are class $\mathcal{K}_\infty$ functions. Then, it can be seen that $\tilde{g}_2$ is a class $\mathcal{K}_\infty$ function that satisfies

$$\tilde{g}_2(s) = \mathcal{O}(\underline{\alpha}_2(s)) \quad \text{as} \quad s \to 0^+ \tag{4.18a}$$
$$g_2(s) = \mathcal{O}(\tilde{g}_2(s)) \quad \text{as} \quad s \to +\infty \,. \tag{4.18b}$$

Next, we need the following "changing supply rate" result, reminiscent of [ST95].

Proposition 4.12 *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two given closed sets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Let c be a nonnegative constant and $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a continuously differentiable function satisfying, for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$,*

$$\underline{\alpha}(|x|_{\mathcal{A}_1}) \leq V(x) \leq \overline{\alpha}(|x|)$$

$$\frac{\partial V}{\partial x}(x) f(x,u) \leq -\alpha(|x|_{\mathcal{A}_1}) + c\gamma(|u|_{\mathcal{A}_2}) \,,$$

where $\underline{\alpha}$, $\overline{\alpha}$, α and γ are class $\mathcal{K}_\infty$ functions. Let $\tilde{\alpha}$ (resp. $\tilde{\gamma}$) be a class $\mathcal{K}_\infty$ function satisfying

$$\tilde{\alpha}(s) = \mathcal{O}(\alpha(s)) \quad \textit{as} \quad s \to 0^+$$
$$\big(\textit{resp.}\ \ \gamma(s) = \mathcal{O}(\tilde{\gamma}(s)) \quad \textit{as} \quad s \to +\infty\big) \,.$$

If $\underline{\alpha}$, $\overline{\alpha}$, α, γ and V are independent of c, there exist a continuously differentiable $\tilde{V}$ and class $\mathcal{K}_\infty$ functions $\tilde{\gamma}$ (resp. $\tilde{\alpha}$), $\underline{\tilde{\alpha}}$ and $\tilde{\overline{\alpha}}$, independent of c, such that, for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$,

$$\underline{\tilde{\alpha}}(|x|_{\mathcal{A}_1}) \leq \tilde{V}(x) \leq \tilde{\overline{\alpha}}(|x|)$$

$$\frac{\partial \tilde{V}}{\partial x}(x) f(x,u) \leq -\tilde{\alpha}(|x|_{\mathcal{A}_1}) + c\tilde{\gamma}(|u|_{\mathcal{A}_2}) \,.$$

The proof of this result follows from minor modifications that of [ST95, Theorem 2]. The only difference stands in the fact that the measures involved are not necessarily Euclidean, and that the generated functions are shown to be independent of c.

In view of (4.18b) and noticing that $|x_1|_{\mathcal{A}_1} \leq |x|$, we can apply Proposition 4.12 to V_1 with $\tilde{\gamma} = \tilde{g}_2$ and $c = \tilde{c}_1(|x_0|)$ to obtain that there exists a continuously differentiable function $\tilde{V}_1$ such that, for all $x_1 \in \mathbb{R}^{n_1}$,

$$\underline{\tilde{\alpha}}_1(|x_1|_{\mathcal{A}_1}) \leq \tilde{V}_1(x_1) \leq \tilde{\overline{\alpha}}_1(|x_1|) \tag{4.19}$$

and, for all $t \in \mathbb{R}_{\geq 0}$,

$$\dot{\tilde{V}}_1(\phi_1(t, x_0)) \leq -\tilde{\alpha}_1(|\phi_1(t, x_0)|_{\mathcal{A}_1}) + \tilde{c}_1(|x_0|)\tilde{g}_2(|\phi_2(t, x_{20})|_{\mathcal{A}_2}) \,,$$

where $\underline{\tilde{\alpha}}_1, \overline{\tilde{\alpha}}_1 \in \mathcal{K}_\infty$. In addition, the functions $\underline{\tilde{\alpha}}_1$, $\overline{\tilde{\alpha}}_1$, $\tilde{\alpha}_1$ and $\tilde{V}_1$ are all independent of $\tilde{c}_1(|x_0|)$ and consequently of x_0. Integrating the previous differential inequality, we obtain that

$$\int_0^\infty \tilde{\alpha}_1(|\phi_1(t,x_0)|_{\mathcal{A}_1})dt \leq \tilde{V}_1(x_{10}) + \tilde{c}_1(|x_0|)\int_0^\infty \tilde{g}_2(|\phi_2(t,x_{20})|_{\mathcal{A}_2})dt\,.$$

Hence, in view of (4.19), we have that

$$\int_0^\infty \tilde{\alpha}_1(|\phi_1(t,x_0)|_{\mathcal{A}_1})dt \leq \overline{\tilde{\alpha}}_1(|x_{10}|) + \tilde{c}_1(|x_0|)\int_0^\infty \tilde{g}_2(|\phi_2(t,x_{20})|_{\mathcal{A}_2})dt\,. \tag{4.20}$$

In order to upper bound the integral in the right-hand side term of the previous bound, we follow a similar procedure. Based on (4.15), (4.16) and (4.18a), we apply Proposition 4.12 to V_2 with $\tilde{\alpha} = \tilde{g}_2$ and $c = 0$. We obtain that there exists a continuously differentiable function $\tilde{V}_2$ such that, for all $x_2 \in \mathbb{R}^{n_2}$,

$$\underline{\tilde{\alpha}}_2(|x_2|_{\mathcal{A}_2}) \leq \tilde{V}_2(x_2) \leq \overline{\tilde{\alpha}}_2(|x_2|)$$

$$\frac{\partial \tilde{V}_2}{\partial x_2}(x_2)f_2(x_2) \leq -\tilde{g}_2(|x_2|_{\mathcal{A}_2})\,,$$

where $\underline{\tilde{\alpha}}_2$ and $\overline{\tilde{\alpha}}_2$ are class $\mathcal{K}_\infty$ functions. Integrating the last inequality, we obtain that

$$\int_0^\infty \tilde{g}_2(|\phi_2(t,x_{20})|_{\mathcal{A}_2})dt \leq \overline{\tilde{\alpha}}_2(|x_{20}|)\,. \tag{4.21}$$

Substituting this bound into (4.20), we obtain

$$\int_0^\infty \tilde{\alpha}_1(|\phi_1(t,x_0)|_{\mathcal{A}_1})dt \leq \overline{\tilde{\alpha}}_1(|x_{10}|) + \tilde{c}_1(|x_0|)\overline{\tilde{\alpha}}_2(|x_{20}|)\,. \tag{4.22}$$

Thus, defining the following class $\mathcal{K}_\infty$ function

$$\sigma(s) := \min\left\{\tilde{\alpha}_1\left(\sqrt{s/2}\right)\ ;\ \tilde{g}_2\left(\sqrt{s/2}\right)\right\}\,,$$

we get from (4.21) and (4.22) that

$$\int_0^\infty \left[\sigma\left(2\,|\phi_1(t,x_0)|^2_{\mathcal{A}_1}\right) + \sigma\left(2\,|\phi_2(t,x_{20})|^2_{\mathcal{A}_2}\right)\right]dt \leq \overline{\tilde{\alpha}}_1(|x_{10}|) + \tilde{c}_1(|x_0|)\overline{\tilde{\alpha}}_2(|x_{20}|)\,.$$

Since σ is an increasing function, we have that $\sigma(a+b) \leq \sigma(2a) + \sigma(2b)$ for all $a, b \in \mathbb{R}_{\geq 0}$. Therefore, using the fact that $|x|^2_{\mathcal{A}} = |x_1|^2_{\mathcal{A}_1} + |x_2|^2_{\mathcal{A}_2}$, we get that

$$\int_0^\infty \sigma\left(|\phi(t,x_0)|^2_{\mathcal{A}}\right)dt \leq \overline{\tilde{\alpha}}_1(|x_0|) + \tilde{c}_1(|x_0|)\overline{\tilde{\alpha}}_2(|x_0|)\,.$$

The global asymptotic stability of $\mathcal{A}$ then follows from Lemma 4.6. ■

It is worth noting that no Lyapunov function needs to be explicitly known for this first result. However, the assumption of boundedness of solutions (with respect to the origin) is strong. In the case when the solutions are only bounded with respect to the set $\mathcal{A}$, the result still holds provided a stronger requirement on the interconnection term and on the gradient of the (supposedly known) Lyapunov function of the driven subsystem.

Corollary 4.13 *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two given closed subsets of $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ respectively. Under Assumptions 4.9, 4.14, 4.15 and 4.16, the set $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ is globally asymptotically stable for the cascade (4.1).*

Assumption 4.14 (Lyapunov GAS of $\mathcal{A}_1$) *There exist a continuously differentiable function $V_1 : \mathbb{R}_{\geq 0} \times \mathbb{R}^{n_1} \to \mathbb{R}_{\geq 0}$ and class $\mathcal{K}_\infty$ functions $\underline{\alpha}_1$, $\overline{\alpha}_1$, α_1 such that, for all $x_1 \in \mathbb{R}^{n_1}$,*

$$\underline{\alpha}_1(|x_1|_{\mathcal{A}_1}) \leq V_1(x_1) \leq \overline{\alpha}_1(|x_1|) \tag{4.23}$$

$$\frac{\partial V_1}{\partial x_1}(x_1) f_1(x_1) \leq -\alpha_1(|x_1|_{\mathcal{A}_1}) \,. \tag{4.24}$$

Assumption 4.15 (Bound on $L_g V_1$) *There exists a continuous nondecreasing function $g_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and a class $\mathcal{K}_\infty$ function g_2 such that, for all $x = (x_1^\top, x_2^\top)^\top \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$,*

$$\left| \frac{\partial V_1}{\partial x_1}(x_1) g(x) \right| \leq g_1(|x|_{\mathcal{A}}) g_2(|x_2|_{\mathcal{A}2}) \,.$$

Assumption 4.16 (GB w.r.t. $\mathcal{A}$) *The solutions of (4.1) are globally bounded with respect to $\mathcal{A}$.*

We stress that global asymptotic stability of $\mathcal{A}_1$ for the driven subsystem $\dot{x}_1 = f_1(x_1)$ guarantees the existence of a function V_1 satisfying Assumption 4.14, *cf. e.g.* [TP00]. However, the explicit knowledge V_1 is required since we need its gradient to also satisfy Assumption 4.15.

Proof of Corollary 4.13. This result follows directly from Theorem 4.7 by noticing that, in the proof of the latter, $c_1(|x|)$ can then be replaced by $g_1(|x|_{\mathcal{A}})$. Hence, based on Assumption 4.16, we see that (4.17) remains valid and the rest of the proof is exactly the same. ■

In the two previous results, the most difficult requirement to verify is often the global boundedness of the solutions of (4.1) (with respect to the origin or to the set $\mathcal{A}$ according to the case). Similarly to the approach adopted for uniform global practical asymptotic stability of cascades (*cf.* Section 3.3), we now present a result which relaxes this assumption to just forward completeness of (4.1), provided a growth rate restriction of the x_1-dependency of the interconnection term with respect to the dissipation function of the driven subsystem.

Corollary 4.17 (GAS + GAS + FC + growth restriction ⇒ GAS) *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be given closed subsets of $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ respectively. Under Assumptions 4.9, 4.14, 4.18 and 4.19, the set $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ is globally asymptotically stable for the cascade (4.1).*

Assumption 4.18 (Bound on $L_g V_1$) *There exists a continuous nondecreasing function $g_{11} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and a class $\mathcal{K}_\infty$ function g_2 such that, for all $x = (x_1^\top, x_2^\top)^\top \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$,*

$$\left| \frac{\partial V_1}{\partial x_1}(x_1) g(x) \right| \leq g_{11}(|x_1|_{\mathcal{A}_1}) g_2(|x_2|_{\mathcal{A}2}) \,.$$

Assumption 4.19 (FC + growth restriction) *The system (4.1) is forward complete and it holds that*

$$g_{11}(s) = \mathcal{O}(\alpha_1(s)) \,, \qquad \text{as} \quad s \to +\infty \,.$$

Proof of Corollary 4.17. In view of Theorem 4.13, and noticing that Assumption 4.18 implies Assumption 4.15, it is enough to show that solutions are globally bounded with respect to $\mathcal{A}$. The proof is based on similar arguments as the one of [PL01, Theorem 3]. First, from the forward completeness assumption, there exists a continuous nondecreasing function $\vartheta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that, for all $x_0 \in \mathbb{R}^n$, the solution of (4.1) satisfies

$$|\phi(t, x_0)|_{\mathcal{A}} \leq |\phi(t, x_0)| \leq \vartheta(|x_0|, t), \qquad \forall t \in \mathbb{R}_{\geq 0}. \tag{4.25}$$

Next, in view of Assumptions 4.14 and 4.18, the derivative of V_1 along the trajectories of (4.1) satisfies, for all $x =\in \mathbb{R}^n$,

$$\dot{V}_1 \leq -\alpha_1(|x_1|_{\mathcal{A}_1}) + g_{11}(|x_1|_{\mathcal{A}_1}) g_2(|x_2|_{\mathcal{A}_2}). \tag{4.26}$$

In addition, we know from Assumption 4.19 that there exist positive constants s_0 and λ such that

$$g_{11}(s) \leq \lambda \alpha_1(s), \qquad \forall s \geq s_0. \tag{4.27}$$

Furthermore, Assumption 4.9 ensures that there exists a $\mathcal{KL}$ function β_2 such that, for all $x_{20} \in \mathbb{R}^{n_2}$,

$$|\phi_2(t, x_{20})|_{\mathcal{A}_2} \leq \beta_2(|x_{20}|, t), \qquad \forall t \in \mathbb{R}_{\geq 0}. \tag{4.28}$$

Using the fact that g_2 is a class $\mathcal{K}_\infty$ function, we get that, for any $x_{20} \in \mathbb{R}^{n_2}$, there exists a nonnegative time $T(|x_{20}|)$ such that

$$g_2(|\phi_2(t, x_{20})|_{\mathcal{A}_2}) \leq \frac{1}{\lambda}, \qquad \forall t \geq T(|x_{20}|).$$

Note that, without loss of generality, $T(\cdot)$ can be picked as a continuous nondecreasing function. From (4.26), (4.27) and the previous inequality, we obtain that, for all $t \geq T(|x_{20}|)$,

$$|\phi_1(t, x_0)|_{\mathcal{A}_1} \geq s_0 \quad \Rightarrow \quad \dot{V}_1(\phi_1(t, x_0)) \leq 0.$$

Using a direct extension of [Yos66, Theorem 10.2], the previous implication ensures the boundedness of $|\phi_1(t, x_0)|_{\mathcal{A}_1}$ (and consequently, in view of (4.28), of $|\phi(t, x_0)|_{\mathcal{A}}$) for all $t \geq T(|x_{20}|)$. In other words, there exists $\eta \in \mathcal{K}_\infty$ and $\mu > 0$ such that, for all $x_0 \in \mathbb{R}^n$,

$$|\phi(t, x_0)|_{\mathcal{A}} \leq \eta(|x_0|) + \mu, \qquad \forall t \geq T(|x_{20}|).$$

Thus, in view of (4.25) and recalling that $T(\cdot)$ is continuous and increasing, we obtain that

$$|\phi(t, x_0)|_{\mathcal{A}} \leq \tilde{\eta}(|x_0|) + \tilde{\mu}, \qquad \forall t \in \mathbb{R}_{\geq 0},$$

where, for all $s \in \mathbb{R}_{\geq 0}$,

$$\begin{aligned} \tilde{\eta}(s) &:= \eta(s) + \vartheta(s, T(s)) - \vartheta(0, T(0)) \\ \tilde{\mu} &:= \mu + \vartheta(0, T(0)). \end{aligned}$$

The conclusion follows from Corollary 4.13 by observing that $\tilde{\eta}$ is a class $\mathcal{K}_\infty$ function. ■

4.3 Example: cross-track formation control of underactuated surface vessels

In order to illustrate one of the possible uses of the results of this chapter, we present an alternative stability proof for a recent result for the formation control of multiple underactuated surface vessels along a straight line, with a prescribed velocity, *cf.* [BPP06]. The interested reader is invited to refer to this reference for a more detailed description of the motivations and challenges related to this objective. For simplicity of exposition, we consider here the case of two boats, extension to a larger number of boats being straightforward.

We stress that the control objectives considered in Section 6.3 may appear similar to those presented here, since the motion of both ships are required to be synchronized. However, the strategies are fundamentally different. A master-slave approach is adopted there, while here the two boats act at the same hierarchical level. Moreover, it is here assumed that the path to be followed is a straight line and that full information on both ships is available.

The dynamics of the considered underactuated surface vessels is described by

$$\dot{\eta} = R(\psi)\nu \tag{4.29}$$

$$\dot{\nu} + M^{-1}C(\nu)\nu + M^{-1}D\nu = B\tau\,, \tag{4.30}$$

where $\eta := (\xi, \zeta, \psi)^\top$ is composed of the Cartesian coordinates of the boat in a Earth-fixed frame and the yaw angle, $\nu := (u, v, r)^\top$ contains the surge and sway velocities, $\tau = (\tau_u, \tau_v)^\top$ are the surge thrust and the yaw torque considered here as control inputs and

$$R(\psi) = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In the sequel, except when explicitly stated, indexes 1 and 2 refer to vessels 1 and 2 respectively. When no index is specified, the corresponding relationship implicitly holds for both vessels. For simplicity, we assume that the vessels are identical and that

$$M = \begin{pmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & m_{23} \\ 0 & m_{23} & m_{33} \end{pmatrix}, \qquad D = \begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & d_{23} \\ 0 & d_{32} & d_{33} \end{pmatrix},$$

$$C(\nu) = \begin{pmatrix} 0 & 0 & -m_{22}v - m_{23}r \\ 0 & 0 & m_{11}u \\ m_{22}v + m_{23}r & -m_{11}u & 0 \end{pmatrix}, \qquad B := \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The task to achieve is twofold. First, we want that each ship i follows a straight path given by a distance $d_i \in \mathbb{R}$ from an agreed origin. This first goal can be summarized by

$$\lim_{t\to\infty} \zeta_i(t) = d_i\,, \qquad \text{and} \qquad \lim_{t\to\infty} \psi(t) = 0\,. \tag{4.31}$$

Second, we want the vessels to be synchronized, in order to move at the same constant prescribed velocity u_d along the ξ direction. This can be formulated as

$$\lim_{t\to\infty} \xi_1(t) - \xi_2(t) = 0\,, \qquad \text{and} \qquad \lim_{t\to\infty} u_1(t) = \lim_{t\to\infty} u_2(t) = u_d\,. \tag{4.32}$$

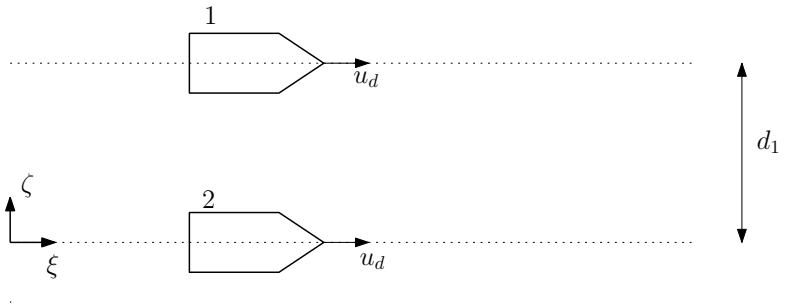


Figure 4.1: Cross-track formation control goals.

The ideal configuration we want to achieve is summarized by Figure 4.1. For simplicity, we have considered the origin of the fixed frame on the path to be followed by the second vessel (hence $d_2 = 0$).

In the sequel, we design a controller that makes the surge velocity u bounded by a positive constant U_M. Following the authors of [BPP06], we adopt the assumption that

$$|v| \leq \min\{U_M \, ; \, c\,|r|\}\,, \tag{4.33}$$

where c denotes a positive constant. We assume as well that the reference velocity u_d of the formation satisfies

$$U_m + a < |u_d| < U_M - a\,, \tag{4.34}$$

where a and U_m are positive constants. From (4.29), the kinematic equations for each boat can be written as

$$\dot{\xi} = u\cos\psi - v\sin\psi \tag{4.35}$$
$$\dot{\zeta} = u\sin\psi + v\cos\psi \tag{4.36}$$
$$\dot{\psi} = r\,. \tag{4.37}$$

In view of (4.31), we define: $\tilde{\zeta} := \zeta - d$. The ζ-error dynamics is then given by

$$\dot{\tilde{\zeta}} = u\sin\psi + v\cos\psi\,. \tag{4.38}$$

Next, we let

$$\psi_d := -\arctan\left(\tilde{\zeta}/\ell\right)\,, \tag{4.39}$$

where ℓ denotes a positive constant satisfying $\ell > c$, *cf* (4.33). As this appears more clearly in the sequel, this yaw angle is chosen in such a way that the ζ-error tends to zero. This choice of ψ_d comes from so-called *lign of sight guidance* algorithm; see [BPP06] for details. We further define $\tilde{\psi} := \psi - \psi_d$. Using (4.37), the ψ-error dynamics is then given by

$$\dot{\tilde{\psi}} = r - \frac{\ell}{\ell^2 + \tilde{\zeta}^2}(u\sin\psi + v\cos\psi)\,.$$

Next, define

$$r_d := \frac{\ell}{\tilde{\zeta}^2 + \ell^2}(u\sin\psi + v\cos\psi) - k_\psi\tilde{\psi} = \frac{\ell}{\tilde{\zeta}^2 + \ell^2}\dot{\tilde{\zeta}} - k_\psi\tilde{\psi}\,, \tag{4.40}$$

where k_ψ is a positive constant. Then the $\tilde{\psi}$ dynamics becomes

$$\dot{\tilde{\psi}} = -k_\psi \tilde{\psi} + \tilde{r}\,, \tag{4.41}$$

where $\tilde{r} := r - r_d$. In view of (4.29), and letting[7] $\Omega := m_{22}m_{33} - m_{23}^2$, it can be seen from (4.30) that

$$\dot{\tilde{r}} = \frac{m_{23}}{\Omega}(m_{11}ur + d_{22}v + d_{23}r) + \frac{m_{22}}{\Omega}\left(\tau_r - (m_{22}v + m_{23}r)u + m_{11}uv - d_{32}v - d_{33}r\right) - \dot{r}_d\,.$$

The following control

$$\begin{aligned}\tau_r &= (m_{22}v + m_{23}r)u - m_{11}uv + d_{32}v + d_{33}r \\ &\quad - \frac{m_{23}}{m_{22}}(m_{11}ur + d_{22}v + d_{23}r) + \frac{\Omega}{m_{22}}\left(\dot{r}_d - k_r\tilde{r}\right)\,,\end{aligned} \tag{4.42}$$

where k_r denotes a positive gain, then yields

$$\dot{\tilde{r}} = -k_r\tilde{r}\,. \tag{4.43}$$

In the same way, the control law

$$\tau_u = (m_{22}v + m_{23}r)r + m_{11}\left(\dot{u}_c - k_u(u - u_c)\right)\,, \tag{4.44}$$

where $t \mapsto u_c(t)$ denotes any given speed reference and k_u is a positive control gain, yields the following dynamics

$$\dot{\tilde{u}} = -k_u\tilde{u}\,, \tag{4.45}$$

where $\tilde{u} := u - u_c$ is the surge velocity tracking error. These observations, together with the cascade approach we follow, allows to consider ψ, r and u as control inputs in the sequel. The ζ-error dynamics (4.38) can trivially be written

$$\dot{\tilde{\zeta}} = u\sin\psi_d + v\cos\psi_d + \gamma(\psi, \psi_d, u, v)\tilde{\psi}\,, \tag{4.46}$$

where

$$\gamma(\psi, \psi_d, u, v) := \frac{u\sin\psi + v\cos\psi - u\sin\psi_d - v\cos\psi_d}{\tilde{\psi}}\,.$$

In view of (4.33) and provided that

$$U_m \leq |u| \leq U_M\,, \tag{4.47}$$

for the positive constants U_m and U_M involved in (4.34), it holds that

$$|\gamma(\psi, \psi_d, u, v)| \leq |u| + |v| \leq 2U_M\,.$$

Thus, injecting the expression of ψ_d into (4.46), we can see that the (r, ψ, ζ, u)-error dynamics possesses the cascade structure composed of (4.41), (4.43), (4.45) and

$$\dot{\tilde{\zeta}} = -u\frac{\tilde{\zeta}}{\sqrt{\tilde{\zeta}^2 + \ell^2}} + v\frac{\ell}{\sqrt{\tilde{\zeta}^2 + \ell^2}} + \gamma(\psi, \psi_d, u, v)\tilde{\psi}\,. \tag{4.48}$$

[7] Ω is a positive constant since M is positive definite.

Claim 4.20 *If (4.47) holds, then the origin of the cascaded system (4.41), (4.43), (4.45) and (4.48) is globally asymptotically stable.*

Proof of Claim 4.20. First notice that the origin of the driving subsystem of this cascade, namely (4.41), (4.43) and (4.45), is globally asymptotically stable for any positive choice of the gains k_ψ, k_r and k_u. This establishes Assumption 4.9. In order to study the driven system taken separately (*i.e.* (4.48) with $\tilde{\psi} = 0$), consider the function $V_\zeta(\tilde{\zeta}) := \tilde{\zeta}^2/2$. Then, it can be seen that, along its trajectories,

$$\dot{V}_\zeta(\tilde{\zeta}) \leq -\frac{u\tilde{\zeta}^2}{\sqrt{\tilde{\zeta}^2+\ell^2}} + \left|\tilde{\zeta}\right| |v| \, .$$

But, using (4.40), (4.33) and the fact that $|r| \leq |r_d| + |\tilde{r}|$, it can be seen that the following bound holds

$$|v| \leq c\left(\left|\dot{\tilde{\zeta}}\right|/\ell + k_\psi \left|\tilde{\psi}\right| + |\tilde{r}|\right) \, .$$

Thus, since we consider the isolated $\tilde{\zeta}$-subsystem (*i.e.* $\tilde{\psi} = \tilde{r} = 0$), we obtain

$$\dot{V}_\zeta(\tilde{\zeta}) \leq -\frac{u\left|\tilde{\zeta}\right|^2}{\sqrt{\tilde{\zeta}^2+\ell^2}} + \frac{c}{\ell}\left|\tilde{\zeta}\right|\left|\dot{\tilde{\zeta}}\right| \leq -\frac{u\left|\tilde{\zeta}\right|^2}{\sqrt{\tilde{\zeta}^2+\ell^2}} + \frac{c}{\ell}\dot{V}_\zeta(\tilde{\zeta}) \, .$$

We conclude in view of (4.47) that, along the trajectories of the driven subsystem,

$$\left(1-\frac{c}{\ell}\right)\dot{V}_\zeta(\tilde{\zeta}) \leq -\frac{U_m\left|\tilde{\zeta}\right|^2}{\sqrt{\tilde{\zeta}^2+\ell^2}} \, .$$

Recalling that $c < \ell$, Assumption 4.14 is then fulfilled. Finally, we have that

$$\left|\frac{\partial V_\zeta}{\partial \tilde{\zeta}}(\tilde{\zeta})\gamma(\psi,\psi_d,u,v)\right| \leq 2U_M\left|\tilde{\zeta}\right| \, ,$$

which establishes Assumption 4.18 with $g_{11}(\cdot) = 2U_M$ and Assumption 4.14 follows trivially due to the boundedness of g_{11}. The claim follows by applying Corollary 4.17 and recalling that forward completeness is not needed since the sets under consideration (the origin) are compact. ■

In order to guarantee (4.47) while stabilizing the ξ-error dynamics, we choose as speed references

$$u_{c,1} = u_d - a\,\mathrm{sat}(\xi_1 - \xi_2) \, , \qquad \text{and} \qquad u_{c,2} = u_d - a\,\mathrm{sat}(\xi_2 - \xi_1) \, , \tag{4.49}$$

where sat denotes any odd saturation function (*cf.* Definition 3.32) and a is the positive constant involved in (4.34).

Proposition 4.21 (Formation control of vessels) *Assume that (4.33) and (4.34) hold. Then, the control law (4.42), (4.39), (4.40), (4.44), (4.49) achives the control objectives (4.31) and (4.32). More precisely, the set*

$$\mathcal{A} := \left\{(\eta_1,\eta_2,\nu_1,\nu_2) \in \mathbb{R}^{12} \, : \, \xi_1 = \xi_2 \text{ and } \zeta_i = d_i \, , \, \psi_i = 0 \, , \, u_i = u_d \, , \, v_i = 0 \, , \, \forall i \in \{1,2\}\right\}$$

is globally asymptotically stable.

Proof of Proposition 4.33. In view of (4.39), (4.35) can be written

$$\begin{aligned}\dot{\xi} &= u\cos\psi_d - (\cos\psi_d - \cos\psi)u - v\sin\psi \\ &= \frac{\ell}{\sqrt{\ell^2+\tilde{\zeta}^2}}u - (\cos\psi_d - \cos\psi)u - v\sin\psi \\ &= u_c + \tilde{u} + \left(\frac{\ell}{\sqrt{\ell^2+\tilde{\zeta}^2}} - 1\right)u - (\cos\psi_d - \cos\psi)u - v\sin\psi\,.\end{aligned}$$

From this and (4.49), we obtain

$$\begin{aligned}\dot{\xi}_1 &= u_d - a\,\mathrm{sat}(\xi_1-\xi_2) + \left(\frac{\ell}{\sqrt{\ell^2+\tilde{\zeta}_1^2}} - 1\right)u_1 - (\cos\psi_{d,1} - \cos\psi_1)u_1 - v_1\sin\psi_1 \\ \dot{\xi}_2 &= u_d - a\,\mathrm{sat}(\xi_2-\xi_1) + \left(\frac{\ell}{\sqrt{\ell^2+\tilde{\zeta}_2^2}} - 1\right)u_2 - (\cos\psi_{d,2} - \cos\psi_2)u_2 - v_2\sin\psi_2\,.\end{aligned}$$

Defining $\tilde{\xi} := \xi_1 - \xi_2$ and considering the variables[8]

$$x_1 := (\xi_1, \tilde{\xi})^\top \qquad \text{and} \qquad x_2 := (\tilde{r}_1, \tilde{r}_2, \tilde{\psi}_1, \tilde{\psi}_2, \tilde{\zeta}_1, \tilde{\zeta}_2, \tilde{u}_1, \tilde{u}_2)^\top\,,$$

the formation objective $\lim_{t\to\infty}\xi_1(t) - \xi_2(t) = 0$ can be summarize by the convergence of $\tilde{\xi}(t)$ to zero and the overall system can be put in the cascade form (4.1) where $f_2(x_2)$ designates the right-hand side of (4.41), (4.43), (4.45), (4.48) and

$$f_1(x_1) = \begin{pmatrix} u_d - a\,\mathrm{sat}(\tilde{\xi}) \\ -2a\,\mathrm{sat}(\tilde{\xi}) \end{pmatrix},$$

$$g(x_1,x_2) = \begin{pmatrix} g_0(\tilde{\psi}_1, \psi_{d,1}, \tilde{\zeta}_1, \tilde{u}_1, u_{c_1}, v_1) \\ g_0(\tilde{\psi}_1, \psi_{d,1}, \tilde{\zeta}_1, \tilde{u}_1, u_{c_1}, v_1) - g_0(\tilde{\psi}_2, \psi_{d,2}, \tilde{\zeta}_2, \tilde{u}_2, u_{c_2}, v_1) \end{pmatrix}, \tag{4.50}$$

where

$$g_0(\tilde{\psi}, \psi_d, \tilde{\zeta}, \tilde{u}, u_c, v) := \left(\frac{\ell}{\sqrt{\ell^2+\tilde{\zeta}^2}} - 1\right)(\tilde{u}+u_c) - (\cos\psi_d - \cos\psi)(\tilde{u}+u_c) - v\sin\psi\,. \tag{4.51}$$

Considering the function $V_1(x_1) := \tilde{\xi}^2/2$, the requirements (4.23) and (4.24) are fulfilled with $\underline{\alpha}_1(s) = \overline{\alpha}_1(s) = s^2/2$, $\alpha_1(s) := 2a\,s\mathrm{sat}(s)$ for all $s \in \mathbb{R}_{\geq 0}$ and

$$\mathcal{A}_1 := \left\{x_1 = (\xi_1, \tilde{\xi})^\top \in \mathbb{R}^2 \,:\, \xi_1 \in \mathbb{R} \text{ and } \tilde{\xi} = 0\right\}.$$

This establishes Assumption 4.14. Note that, when x_1 belongs to $\mathcal{A}_1$, the formation goal $\xi_1 = \xi_2$ is achieved. In addition, it can be seen that

$$v\sin\psi = v\frac{\sin\psi}{\psi}\psi = v\frac{\sin\psi}{\psi}(\psi_d + \tilde{\psi}) = v\frac{\sin\psi}{\psi}\left(-\arctan\left(\tilde{\zeta}/\ell\right) + \tilde{\psi}\right).$$

[8] Indexes 1 and 2 of x do not refer to the vessels, but to the notation used in Corollary 4.17.

This observation coupled with (4.50) and (4.51) and the boundedness of u and v ensures that g vanishes whenever x_2 equals zero, which is enough to conclude the existence of a class $\mathcal{K}_\infty$ function g_2 such that $|g(x_1, x_2)| \leq g_2(|x_2|)$. Assumption 4.18 then follows with $g_{11}(\cdot) = 1$. In turn, we have that $g_{11}(s) = \mathcal{O}(\alpha_1(s))$ as s tends to infinity, which establishes Assumption 4.19. Finally, (4.45) guarantees an exponential convergence of $\tilde{u}$ to zero *without* overshoot. So the fact that u_c lies in the interval (U_m, U_M) ensures (4.47). Assumption 4.9 then follows from Claim 4.20 with the set $\mathcal{A}_2 = \{0\}$. Thus, we can apply Corollary 4.17 to conclude that the set $\mathcal{A}_1 \times \mathcal{A}_2$, which coincides with $\mathcal{A}$, is GAS for the overall system.

■

Conclusion. This chapter presents three results for establishing the set-stability of cascaded systems. The first one does not require the knowledge of any Lyapunov function, but is based on a conservative assumption of boundedness of solutions of the overall system. The second assumes the boundedness of solutions only with respect to the set under consideration but, as a counterpart, requires the knowledge of an explicit Lyapunov function for the driven subsystem. Finally, the third one is more easily applicable as it only requires a growth comparison between the functions involved. The range of applications of such a stability concept is wide as it notably includes partial stability, stability of a given trajectory, of a given ball *etc.*

Chapter 5

Integral input to state stability for cascaded systems

In the previous chapters, we proposed some sufficient conditions to guarantee the preservation of stability properties for cascaded systems. Such tools can be particularly useful when studying the robustness of controlled nonlinear systems to model uncertainty or to external signals. In this chapter, we consider the cascade interconnection of systems with inputs.

The ISS paradigm. A branch of the control theory is especially dedicated to the evaluation of the impact of external signals on the stability of a dynamical system. It concerns systems of the form

$$\dot{x} = f(x, u)$$

as introduced in (1.4), where u denotes the external input[1]. The key notion in this field is the input to state stability (ISS), originally introduced by Sontag in [Son89a], and which we recalled in Definition 1.21. As evoked in Chapter 1, this property links the norm of the current state to the infinity norm of the applied input, through a nonlinear inequality which also takes into account a fading term due to initial conditions.

This way of formulating external and internal stability notions is particularly well adapted to analyze stability of cascades. A well known result, see *e.g.* [ST95], states that the ISS property is preserved under cascade interconnections. Since ISS implies global asymptotic stability (GAS) when the input is identically zero, it follows that the cascade composed of an ISS subsystem driven by a GAS subsystem is GAS.

Besides, ISS is easily checkable based on the study of a Lyapunov-like function. More precisely, we have the following characterization.

Theorem 5.1 (Lyapunov characterization of ISS, [SW95]) *The dynamical system $\dot{x} = f(x, u)$ as defined in (1.4) is input to state stable if and only if there exists a smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and class $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$, α and γ such that, for all $x \in \mathbb{R}^n$ and all admissible input u,*

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|) \tag{5.1}$$

$$\frac{\partial V}{\partial x}(x) f(x, u) \leq -\alpha(|x|) + \gamma(|u|)\,. \tag{5.2}$$

[1] $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$ may consist in any measurable locally essentially bounded signal.

In the sequel, the function α is referred to as *dissipation rate* while *supply rate* denotes the function γ.

The iISS property. Even though this characterization has been widely used both in analysis and design, ISS happens to be too strong a requirement in several cases. This motivated the introduction of integral input to state stability (iISS) [Son98b], which is a more general property. Instead of linking the state to the supremum of the input, it involves a measure of the energy that inputs feed into the system, *cf.* Definition 1.22.

Similarly to ISS, it ensures the global asymptotic stability of the zero-input system and guarantees some robustness to the system with respect to external signals. For instance, if the supplied energy is finite, then solutions converge to zero, *i.e.* the asymptotic behavior of an iISS system is not altered by the presence of an input with finite energy. iISS is furthermore characterized by similar Lyapunov-like conditions as for ISS. The result below was established by Angeli *et al.* in [ASW00a].

Theorem 5.2 (Lyapunov characterization of iISS, [ASW00a]) *The dynamical system $\dot{x} = f(x,u)$ as defined in (1.4) is integral input to state stable if and only if there exists a smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and class $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$, γ and a continuous positive definite*[2] *function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that (5.1) and (5.2) hold for all $x \in \mathbb{R}^n$ and all admissible input u.*

As established in [ASW00a], iISS is more conservative than asking that the zero-input $\dot{x} = f(x,0)$ be GAS and that $\dot{x} = f(x,u)$ be forward complete. Yet, it holds very often in specific applications for subsystems involved in cascades. In this respect, please refer to [LSW02] where Liberzon and coauthors proposed a control design that makes the system iISS with respect to disturbances. In the same reference, it is also shown that the cascade of an ISS subsystem driven by a iISS subsystem is itself iISS.

iISS and cascades. It is therefore of interest to know whether similar properties –as those that hold in the ISS case– are actually true for iISS systems. For instance: Is the cascade of two iISS systems iISS ? Is the cascade of an iISS system driven by a GAS system GAS ?

The following counter-example, originally[3] proposed in [PL01, AAS02], shows that the answer to the second question is negative in general.

Example 5.3 (GAS + iISS $\not\Rightarrow$ GAS) *Consider the cascaded system*

$$\dot{x}_1 = -sat(x_1) + x_1 x_2 \tag{5.3a}$$
$$\dot{x}_2 = -x_2^3, \tag{5.3b}$$

where $sat(s) := sign(s)\min\{1; |s|\}$. It was proved in [AAS02] that, although (5.3a) is iISS with respect to x_2 and (5.3b) is globally asymptotically stable, the overall cascade (5.3) generates unbounded solutions for all initial states in $\mathbb{R}_{\geq 3} \times \{1\}$. Indeed, for $x_{20} = 1$, the solution of (5.3b) is $1/\sqrt{1+2t}$. Hence, as long as $\phi_1(t, x_{10}) \geq 1$, the first differential equation is

$$\dot{x}_1 = -1 + \frac{x_1}{\sqrt{1+2t}},$$

[2] Instead of class $\mathcal{K}_\infty$ in Theorem 5.1.

[3] We have, from A. Loría, that this example was originally stated to the authors of [PL01] by L. Praly.

therefore

$$\phi_1(t, x_{10}) = e^{\sqrt{1+2t}-1}\left(x_{10} - \int_0^t e^{1-\sqrt{1+2\tau}}d\tau\right).$$

But, considering the change of variable $s = -1 + \sqrt{1+2\tau}$*, we get that*

$$\int_0^t e^{1-\sqrt{1+2\tau}}d\tau \leq \int_0^\infty e^{1-\sqrt{1+2\tau}}d\tau = \int_0^\infty e^{-s}(s+1)ds = 2\,.$$

Thus, if $x_{10} \geq 3$*, then it holds that* $\phi_1(t, x_{10}) \geq e^{\sqrt{1+2t}-1} \geq 1$ *at all time, and so* $\lim_{t\to\infty}\phi_1(t, x_{10}) = \infty$*.* □

In [AAS02], an additional sufficient condition is proposed to restrict the iISS gain of the driven subsystem in relation to the convergence rate of the state trajectories generated by the driving subsystem. Roughly speaking, the decay rate of the driving subsystem's state has to be large enough with respect to the iISS gain of the driven one.

A way to ensure a sufficient speed of convergence of the input is requiring that it is integrable. Such a sufficient condition was originally established in [PL98] for time-varying systems. In that paper, Panteley and Loría proved UGAS of the cascade under growth-order restrictions on the interconnection term and, most importantly, the condition that the state trajectories of the driving system be uniformly globally integrable. See below for a more precise comparison between these two results.

Thus, while the second question posed above has been studied and partially answered, the first question (*i.e.* whether the cascade of iISS systems is itself iISS) is still open. A direct extension of Example 5.3 gives a negative answer in general.

Example 5.4 (iISS + iISS $\not\Rightarrow$ iISS) *Consider the cascaded system*

$$\dot{x}_1 = -sat(x_1) + x_1x_2 \tag{5.4a}$$
$$\dot{x}_2 = -x_2^3 + u\,. \tag{5.4b}$$

Applying Theorem 5.1 with the function $V(x_2) = x_2^2$*, it follows immediately that (5.4b) is ISS, and a fortiori iISS. In addition, as shown in Example 5.3, the driven subsystem (5.4a) is also iISS. Nevertheless, if the cascade (5.4) were iISS, then it would be globally asymptotically stable when* u *is identically zero, which, as seen in Example 5.3, is not the case.* □

In what follows, we provide relatively mild additional conditions which are sufficient for the iISS property to be conserved by the cascade structure. These are firstly given in the case when an explicit iISS Lyapunov-like function is known for each of the two subsystems, *cf.* Section 5.2. Roughly, it suffices that the dissipation term of the driving subsystem dominates the supply function of the driven subsystem in a neighborhood of the origin. The second step, exposed in Section 5.3, consists in stating this condition in terms of the estimates of the trajectories of the two subsystems when disconnected. More precisely, in the case of a continuously differentiable zero-input driving subsystem, we recover the sufficient condition derived from [AAS02] that the driven subsystem has a locally Lipschitz iISS gain and that the driving one be 0-LES (see Definition 5.6 below).

In addition, in Section 5.1, we complete the main result in [AAS02] by giving a sufficient condition for the cascade composed of an iISS subsystem together with a GAS one to remain GAS in the case when explicit Lyapunov functions are known. Roughly, it is again required

that the dissipation term of the GAS subsystem dominates the supply function of the iISS one around zero. This result may be useful in practice since the iISS and GAS properties are commonly established through Lyapunov arguments.

For a better understanding of these results and their proof, we recall the following definitions.

Definition 5.5 (0-GAS) *The origin of the system $\dot{x} = f(x,u)$ as defined in (1.4) is said to be* 0-GAS[4] *if the origin of $\dot{x} = f(x,0)$ is globally asymptotically stable (cf. Definition 1.10).*

Definition 5.6 (0-LES) *The origin of the system $\dot{x} = f(x,u)$ as defined in (1.4) is said to be* 0-LES *if there exists a positive constant Δ such that the origin of $\dot{x} = f(x,0)$ is exponentially stable on $\mathcal{B}_\Delta$ (cf. Definition 1.15).*

Definition 5.7 (BEFBS) *The system $\dot{x} = f(x,u)$ as defined in (1.4) is said to satisfy the* bounded energy frequently bounded state *property if there exists a class $\mathcal{K}_\infty$ function σ such that, for all $x_0 \in \mathbb{R}^n$, its solutions satisfy*

$$\int_0^\infty \sigma(|u(\tau)|)d\tau < \infty \quad \Rightarrow \quad \liminf_{t\to+\infty} |\phi(t,x_0,u)| < \infty\,.$$

5.1 Global asymptotic stability for cascades, Lyapunov-based

The approach based on trajectories. As explained, the main result in [AAS02] studies the cascade connection of an iISS system driven by a globally asymptotically stable system:

$$\dot{x}_1 = f_1(x_1,x_2) \tag{5.5a}$$
$$\dot{x}_2 = f_2(x_2)\,, \tag{5.5b}$$

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, f_1 and f_2 are locally Lipschitz and satisfy $f_1(0,0) = 0$ and $f_2(0) = 0$. In that work, Arcak *et al.* proposed a sufficient condition for (5.5) to be globally asymptotically stable. More precisely, the authors established the following:

Theorem 5.8 (GAS + iISS $\Rightarrow$ GAS, trajectory based, [AAS02]) *Assume that (5.5a) is iISS with an iISS gain μ_1 (cf. Definition 1.22) and that there exists $\eta_2, \nu_2 \in \mathcal{K}_\infty$ such that, for all $x_{20} \in \mathbb{R}^{n_2}$, the solutions of (5.5b) satisfy*

$$|\phi_2(t,x_{20})| \le \eta_2\left(\nu_2(|x_{20}|)e^{-t}\right)\,, \qquad \forall t \in \mathbb{R}_{\ge 0}\,.$$

Then, under the condition that

$$\int_0^1 \frac{\mu_1 \circ \eta_2(s)}{s} ds < \infty\,, \tag{5.6}$$

the origin of the cascade (5.5) is globally asymptotically stable.

[4] We may also say, with a slight abuse of terminology, that the system $\dot{x} = f(x,u)$ is 0-GAS.

It is worth stressing that the sufficient conditions provided by Panteley and Loría in [PL98, PL01], who addressed the same question in a time-varying context, share strong similarities with (5.6) and the sufficient conditions they derive impose similar requirements on the subsystems' behavior.

The Lyapunov alternative. The above result requires the knowledge of an explicit estimate of the trajectories of each subsystem, as we need to know the iISS gain μ_1 of the driven subsystem and the convergence rate η_2 of the driving one in order to check condition (5.6). It is often the case that such estimates are obtained using the integration of a convenient Lyapunov function for each subsystem. In addition, we have seen that the existence of a Lyapunov-like function as in Theorem 5.2 is equivalent to the iISS property. It therefore appears natural to derive a similar sufficient condition based directly on the information provided by such Lyapunov functions. This is the object of the following result, which hence can be seen as a natural counter-part of [AAS02, Theorem 1] for the case when the global asymptotic stability of the driving subsystem and the iISS of the driven one are not established through an explicit estimate of their solutions, but instead in terms of Lyapunov functions.

Theorem 5.9 (GAS + iISS ⇒ GAS, Lyapunov-based) *Let V_1 and V_2 be two continuous positive definite radially unbounded functions, differentiable on $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}\setminus\{0\}$ respectively and satisfying, for all $x_1\in\mathbb{R}^{n_1}$ and all $x_2\in\mathbb{R}^{n_2}\setminus\{0\}$,*

$$\frac{\partial V_1}{\partial x_1}(x_1)f_1(x_1,x_2)\leq -\alpha_1(|x_1|)+\gamma_1(|x_2|) \tag{5.7}$$

$$\frac{\partial V_2}{\partial x_2}(x_2)f_2(x_2)\leq -\alpha_2(|x_2|)\,, \tag{5.8}$$

where α_1 and α_2 are continuous positive definite functions and γ_1 is of class $\mathcal{K}$. Then, under the condition that $\gamma_1(s)=\mathcal{O}(\alpha_2(s))$ when s tends to zero, the origin of the cascade (5.5) is globally asymptotically stable.

It is worth noting that, in view of Theorem 5.2, the existence of a function V_1 satisfying (5.7) is equivalent to saying that the driven subsystem (5.5a) is iISS with respect to x_2. Similarly, the condition (5.8) is equivalent to the global asymptotic stability of (5.5b).

Proof of Theorem 5.9. Let us first consider the particular case when the initial condition of the driving subsystem is $x_{20}=0$. Since 0 is an equilibrium of this subsystem, we then have that $\phi_2(t,0)=0$ for all $t\in\mathbb{R}_{\geq 0}$, and consequently convergence to zero of solutions follows directly from (5.7) (see *e.g.* [Kha96, Corollary 3.3]).

In the sequel, we therefore consider that $x_{20}\in\mathbb{R}^{n_2}\setminus\{0\}$. We underline the fact that, due to the regularity condition imposed on f_2, it then holds that $|\phi_2(t,x_{20})|\neq 0$ for all $t\in\mathbb{R}_{\geq 0}$. To see this more clearly, notice that, since f_2 is assumed to be locally Lipschitz and $f_2(0)=0$, we have that $|f_2(x_2)|\leq L(|x_2|)\,|x_2|$ for some continuous nondecreasing function L. In addition, (5.8) ensures that the trajectories of the x_2-subsystem (5.5b) are bounded, so $m_2(x_{20}):=\sup_{t\geq 0}|\phi_2(t,x_{20})|$ is finite and positive for all $x_{20}\neq 0$. From these observations, it holds that

$$\begin{aligned}\frac{\partial}{\partial t}(|\phi_2(t,x_{20})|^2) &= 2f_2(\phi_2(t,x_{20}))^\top\phi_2(t,x_{20})\\ &\geq -2L(|\phi_2(t,x_{20})|)\,|\phi_2(t,x_{20})|^2\geq -a(x_{20})\,|\phi_2(t,x_{20})|^2\,,\end{aligned}$$

where $a(x_{20}) := 2L(m_2(x_{20}))$ is a positive constant since $x_{20} \neq 0$. In other words, $|\phi_2(\cdot, x_{20})|^2$ satisfies the differential inequality $\dot{y} \geq -a(x_{20})y$. From the comparison lemma, we conclude that $|\phi_2(t, x_{20})|^2 \geq |x_{20}|^2 e^{-a(x_{20})t}$, which is, as claimed, positive at all time.

The proof is mainly based on the following "changing dissipation rate" result.

Proposition 5.10 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a locally Lipschitz function such that $f(0) = 0$ and $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a continuous positive definite radially unbounded function, differentiable on $\mathbb{R}^n \setminus \{0\}$, and satisfying, for all $x \in \mathbb{R}^n \setminus \{0\}$,*

$$\frac{\partial V}{\partial x}(x) f(x) \leq -\alpha(|x|)\,,$$

where α is a continuous positive definite function. If $\tilde{\alpha}$ is a continuous positive definite function satisfying $\tilde{\alpha}(s) = \mathcal{O}(\alpha(s))$ as s tends to zero, then there exists a continuous positive definite radially unbounded function $\tilde{V}$, differentiable on $\mathbb{R}^n \setminus \{0\}$ and such that, for all $x \in \mathbb{R}^n \setminus \{0\}$,

$$\frac{\partial \tilde{V}}{\partial x}(x) f(x) \leq -\tilde{\alpha}(|x|)\,.$$

Proof of Proposition 5.10. The proof follows along the lines of the main result in [ST95] proposed by Sontag and Teel. We define the new Lyapunov-like function as

$$\tilde{V} := \int_0^V q(s) ds\,,$$

where $q : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, to be defined later, denotes a nondecreasing continuous function satisfying $q(s) > 0$ for all $s > 0$. This transformation preserves the properties of V, that is: continuity, differentiability out of the origin, positive definiteness and radial unboundedness. In addition, notice that the continuity, positive definiteness and radial unboundedness of V ensures the existence of a class $\mathcal{K}_\infty$ function $\underline{\alpha}$ such that $V(x) \geq \underline{\alpha}(|x|)$ for all $x \in \mathbb{R}^n$ (see [Kha96, Lemma 3.5]). Hence, the total derivative of $\tilde{V}$ satisfies, for all $x \in \mathbb{R}^n \setminus \{0\}$,

$$\frac{\partial \tilde{V}}{\partial x}(x) f(x) \leq q(V(x)) \frac{\partial V}{\partial x}(x) f(x) \leq -q \circ \underline{\alpha}(|x|)\alpha(|x|)\,.$$

Note that, by the assumption on the local relative behavior of α and $\tilde{\alpha}$, the function $r \mapsto \alpha \circ \underline{\alpha}^{-1}(r)/\tilde{\alpha} \circ \underline{\alpha}^{-1}(r)$ is upper bounded on any interval $(0, r_0]$, $r_0 > 0$. Hence, the function defined by

$$\tilde{q}(r) := \sup_{0 < t \leq r} \frac{\alpha \circ \underline{\alpha}^{-1}(t)}{\tilde{\alpha} \circ \underline{\alpha}^{-1}(t)}\,, \qquad \forall r > 0\,,$$

is a well defined nondecreasing function. Pick q as any nondecreasing continuous function satisfying $q(r) > \tilde{q}(r)$ for all $r > 0$, and let $r = \underline{\alpha}(|x|)$, for $x \in \mathbb{R}^n \setminus \{0\}$. We finally get, as desired, that $q \circ \underline{\alpha}(|x|)\alpha(|x|) \geq \tilde{\alpha}(|x|)$. ■

Notice that, by assumption, it holds that $2\gamma_1(s) = \mathcal{O}(\alpha_2(s))$ in a neighborhood of 0. Apply Proposition 5.10 to the dissipation inequality (5.8) with the function $\tilde{\alpha}(\cdot) = 2\gamma_1(\cdot)$. Then there exists a continuous positive definite radially unbounded function $\tilde{V}_2$, differentiable out of zero, and satisfying, for all $x_2 \neq 0$,

$$\frac{\partial \tilde{V}_2}{\partial x_2}(x_2) f_2(x_2) \leq -2\gamma_1(|x_2|)\,.$$

By summing this inequality with (5.7), we get that, for all $(x_1, x_2) \in \mathbb{R}^{n_1} \times (\mathbb{R}^{n_2} \setminus \{0\})$,

$$\frac{\partial \mathcal{V}}{\partial x_1}(x) f_1(x_1, x_2) + \frac{\partial \mathcal{V}}{\partial x_2}(x) f_2(x_2) \leq -\alpha_1(|x_1|) - \gamma_1(|x_2|) =: -\psi(x)\,,$$

where, $x := (x_1^\top, x_2^\top)^\top$ and $\mathcal{V}(x) := V_1(x_1) + \tilde{V}_2(x_2)$. Notice that $\mathcal{V}_1$ inherits the properties of V_1 and $\tilde{V}_2$: it is continuous positive definite and radially unbounded. Therefore, (see e.g. [Kha96, Lemma 3.5]) there exist two class $\mathcal{K}_\infty$ functions $\underline{\alpha}$ and $\overline{\alpha}$ such that

$$\underline{\alpha}(|x|) \leq \mathcal{V}(x) \leq \overline{\alpha}(|x|)\,. \tag{5.9}$$

Moreover, let $\varphi(s) := \inf_{|x| \geq s} \psi(x)$ for all $s \in \mathbb{R}_{\geq 0}$. Then φ is a continuous positive definite function and we have that, for all $x \in \mathbb{R}^{n_1} \times (\mathbb{R}^{n_2} \setminus \{0\})$,

$$\frac{\partial \mathcal{V}}{\partial x_1}(x) f_1(x_1, x_2) + \frac{\partial \mathcal{V}}{\partial x_2}(x_1, x_2) f_2(x_2) \leq -\varphi(|x|) \leq -\varphi \circ \overline{\alpha}^{-1}(\mathcal{V}(x)),\,.$$

Since, as previously shown, $\phi_2(t, x_{20}) \neq 0$, we finally get that, for all $t \in \mathbb{R}_{\geq 0}$,

$$\dot{\mathcal{V}}(\phi(\cdot, x_0)) \leq -\varphi \circ \overline{\alpha}^{-1}\left(\mathcal{V}(\phi(t, x_0))\right)\,.$$

Noticing that $\varphi \circ \overline{\alpha}^{-1}$ is a continuous positive definite function, we can apply [ASW00a, Corollary IV.3] to establish the existence of a class $\mathcal{KL}$ function $\tilde{\beta}$ such that

$$\mathcal{V}(\phi(t, x_0)) \leq \tilde{\beta}(\mathcal{V}(0), t)\,, \qquad \forall t \in \mathbb{R}_{\geq 0}\,.$$

In view of (5.9), we then get that, for all $x_{20} \neq 0$,

$$|\phi(t, x_0)| \leq \beta(|x_0|\,, t)\,, \qquad \forall t \in \mathbb{R}_{\geq 0}\,,$$

where

$$\beta(s, t) := \underline{\alpha}^{-1} \circ \tilde{\beta}(\overline{\alpha}(s), t)\,, \qquad \forall s, t \in \mathbb{R}_{\geq 0}\,,$$

and the conclusion follows by noticing that $\beta(s, \cdot)$ is a $\mathcal{KL}$ function. ■

The above result can be seen as a corollary of Theorem 5.13 presented below. We have however decided to present it separately as its proof involves an approach (the change of supply rates) which cannot be followed in the context of Theorem 5.13.

Remark 5.11 *It is worth mentioning that, if an upper bound on V_2 of the form $V_2(x_2) \leq \overline{\alpha}_2(|x_2|)$ is explicitly known, where $\overline{\alpha}_2$ denotes a $\mathcal{K}_\infty$ function, the condition in Theorem 5.9 (namely $\gamma_1(s) = \mathcal{O}(\alpha_2(s))$ as $s \to 0$) can be considerably relaxed. More precisely, it suffices that there exists a constant $q \in [0, 1)$ such that*

$$\gamma_1(s) = \mathcal{O}\left(\frac{\alpha_2(s)}{\overline{\alpha}_2(s)^q}\right)\,, \qquad \textit{and} \qquad \alpha_2(s) = o(\overline{\alpha}_2(s)^q)\,, \qquad \textit{as} \quad s \to 0\,. \tag{5.10}$$

Indeed, consider the function $\mathcal{V}_2(\cdot) := V_2(\cdot)^{1-q}$. Then $\mathcal{V}_2$ is a positive definite function, differentiable out of the origin, and we get from (5.8) that

$$\frac{\partial \mathcal{V}_2}{\partial x_2}(x_2) f_2(x_2) \leq -(1-q)\,\alpha_2(|x_2|) V_2^{-q} \leq -(1-q)\,\frac{\alpha_2(|x_2|)}{\overline{\alpha}(|x_2|)^q} =: -\tilde{\alpha}_2(|x_2|)\,.$$

In view of (5.10), $\tilde{\alpha}_2$ is a continuous positive definite function. Hence Theorem 5.9 applies with the new Lyapunov function $\mathcal{V}_2$, and establishes that (5.5) is globally asymptotically stable. In this respect, notice that allowing V_2 to be non-differentiable at the origin is useful, as further illustrated by the following example.

Example 5.12 *Consider the following two dimensional cascaded system:*

$$\begin{aligned}\dot{x}_1 &= -sat(x_1) + x_1 x_2 \\ \dot{x}_2 &= -\frac{x_2}{1+x_2^2}\,,\end{aligned}$$

where $sat(s) := sign(s)\min\{|s|, 1\}$. *In order to study the global asymptotic stability of the origin of ths cascade, we will make use of the following functions:*

$$V_1(x_1) = \frac{1}{2} ln(1+x_1^2) \qquad and \qquad V_2(x_2) = \frac{1}{2}x_2^2\,.$$

First, notice that

$$\frac{\partial V_1}{\partial x_1}(x_1)(-sat(x_1) + x_1 x_2) = -\frac{x_1 sat(x_1)}{1+x_1^2} + \frac{x_1^2 x_2}{1+x_1^2} \leq -\frac{x_1 sat(x_1)}{1+x_1^2} + |x_2|\,.$$

In the same way,

$$\frac{\partial V_2}{\partial x_2}(x_2)\left(-\frac{x_2}{1+x_2^2}\right) = -\frac{x_2^2}{1+x_2^2}\,.$$

Please note that GAS of the overall cascade does not directly follow from the study of the Lyapunov function $V_1 + V_2$. *Indeed, the unbounded term* $|x_2|$ *generated by* $\dot{V}_1$ *cannot be compensated by the bounded term* $x_2^2/(1+x_2^2)$ *provided by* $\dot{V}_2$. *Nevertheless, this property can easily be inferred by Theorem 5.9 and Remark 5.11. To see this, first notice that* V_1 *and* V_2 *satisfy (5.7) and (5.8) with*

$$\alpha_1(s) = \frac{s\, sats}{1+s^2}\,, \qquad \gamma_1(s) = s \qquad and \qquad \alpha_2(s) = \frac{s^2}{1+s^2}\,, \qquad \forall s \in \mathbb{R}_{\geq 0}\,.$$

Since the requirement $\gamma_1(s) = \mathcal{O}(\alpha_2(s))$ *as* s *tends to zero does not hold, it is not possible to apply Theorem 5.9 directly. Nevertheless, it is possible to conclude using the previous remark with* $q = 1/2$. *Indeed, an upper bound on* V_2 *is* $\overline{\alpha}_2(|x_2|) := |x_2|^2/2$. *We can further notice that the function* $\overline{\alpha}_2(s)^q = s/\sqrt{2}$ *strictly dominates* $\alpha_2(s)$ *around zero, and that* $\alpha_2(s)/\overline{\alpha}(s)^q = s\sqrt{2}/(1+s^2)$ *dominates* $\gamma_1(s)$, *which are enough to conclude.* □

5.2 Integral input to state stability for cascades, Lyapunov-based

Our second result in the context of integral input to state stability (iISS) concerns the cascade connection of two iISS systems, in the case when an iISS-Lyapunov function is explicitly known for each of them. For the sake of generality, it is allowed that the driven subsystem depends also on the input of the driving one. We therefore deal with dynamical systems of the following form:

$$\dot{x}_1 = f_1(x_1, x_2, u) \tag{5.11a}$$
$$\dot{x}_2 = f_2(x_2, u) \tag{5.11b}$$

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$ is a measurable locally essentially bounded function, f_1 and f_2 are locally Lipschitz and satisfy $f_1(0,0,0) = 0$ and $f_2(0,0) = 0$. Such system can be represented by Figure 5.1.

For this type of cascaded interconnections, we have the following.

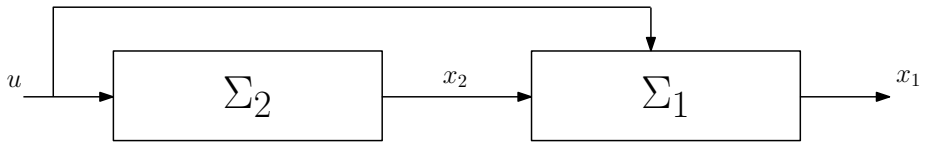


Figure 5.1: Cascade with direct feeding of the driven subsystem.

Theorem 5.13 (iISS + iISS ⇒ iISS, Lyapunov-based) *Let $V_1 : \mathbb{R}^{n_1} \to \mathbb{R}_{\geq 0}$ be a differentiable function and $V_2 : \mathbb{R}^{n_2} \to \mathbb{R}_{\geq 0}$ be a continuous function, differentiable out of the origin. Suppose that there exist a class $\mathcal{K}$ function ν_1 and, for all $i \in \{1,2\}$, a continuous positive definite function α_i, a class $\mathcal{K}$ function γ_i, and two class $\mathcal{K}_\infty$ functions $\underline{\alpha}_i$ and $\overline{\alpha}_i$ such that, for all $x_i \in \mathbb{R}^{n_i}$ and all $u \in \mathbb{R}^p$,*

$$\underline{\alpha}_i(|x_i|) \leq V_i(x_i) \leq \overline{\alpha}_i(|x_i|) \tag{5.12}$$

$$\frac{\partial V_1}{\partial x_1}(x_1) f_1(x_1, x_2, u) \leq -\alpha_1(|x_1|) + \gamma_1(|x_2|) + \nu_1(|u|) \tag{5.13}$$

$$x_2 \neq 0 \qquad \Rightarrow \qquad \frac{\partial V_2}{\partial x_2}(x_2) f_2(x_2, u) \leq -\alpha_2(|x_2|) + \gamma_2(|u|)\,. \tag{5.14}$$

If, in addition, $\gamma_1(s) = \mathcal{O}(\alpha_2(s))$ as s tends to zero, then the cascade (5.11) is iISS.

We stress that (5.13) is equivalent to saying that the driven subsystem (5.11a) is iISS with respect to x_2 and u. This can be rigorously established based on Theorem 5.2 by taking $\gamma(\cdot)$ as $\gamma_1(\cdot) + \nu_1(\cdot)$. Furthermore, in view of the same result, condition (5.14) consists in an (apparent) slight relaxation of the Lyapunov characterization of iISS for (5.11b), as V_2 is not required to be differentiable at zero.

The above result proposes an easy-to-check sufficient condition to guarantee the preservation of the iISS property under a cascade interconnection. It is expressed as a local domination of the driving system's dissipation rate on the supply rate of the driven one.

This does not constitute the first attempt to guarantee iISS for cascaded iISS subsystems. In [Ito04, Ito05], Ito provides sufficient conditions for the preservation of iISS under cascade. Similarly to Theorem 5.13, these conditions are expressed with respect to the supply and dissipation rates of the Lyapunov function associated to each subsystem. However, Ito implicitly assumes that one of the two subsystems is ISS and, contrarily to Theorem 5.13, the requirement in that reference involves also the upper and lower bounds on the Lyapunov functions and the dominance is imposed on the whole[5] $\mathbb{R}_{\geq 0}$ (and not just in a neighborhood of zero). These features make the above tool more general and, often, easier to apply. This is illustrated by the following example.

Example 5.14 *Consider the following two-dimensional cascaded system:*

$$\dot{x}_1 = -x_1(1 - x_2^2 - u) \tag{5.15a}$$

$$\dot{x}_2 = -sat(x_2) + x_2 u\,. \tag{5.15b}$$

We use the Lyapunov functions

$$V_i(x_i) := ln(1 + x_i^2)\,, \qquad i \in \{1,2\}\,.$$

[5] In that reference, a *local* requirement is proposed only in the case of ISS driving subsystem.

Direct computations then show that

$$\begin{aligned}\dot{V}_1(x_1) &\leq -\frac{x_1^2}{1+x_1^2} + |x_2|^2 + |u| \\ \dot{V}_2(x_2) &\leq -\frac{x_2 sat(x_2)}{1+x_2^2} + |u|\,.\end{aligned}$$

Using the notation of Theorem 5.13, we have that, for all $s \in \mathbb{R}_{\geq 0}$,

$$\alpha_1(s) = \frac{s^2}{1+s^2}\,, \qquad \gamma_1(s) = s^2\,, \qquad \nu_1(s) = \gamma_2(s) = s\,, \qquad \alpha_2(s) = \frac{sat(s)s}{1+s^2}\,,$$

so $\gamma_1(s) = \mathcal{O}(\alpha_2(s)$ *near* 0 *and we conclude that the cascade (5.15) is iISS. However, the condition imposed by [Ito05, Corollary 2] to reach such a conclusion is the existence of positive constants* c *and* q*, with* $q \geq 1$*, such that*

$$\left(\gamma_1 \circ \underline{\alpha}_2^{-1}(s)\right)^q \leq c\,\alpha_2 \circ \overline{\alpha}_2^{-1}(s)\,, \qquad \forall s \in \mathbb{R}_{\geq 0}\,,$$

where $\underline{\alpha}_2$ *and* $\overline{\alpha}_2$ *are class* $\mathcal{K}_\infty$ *functions satisfying*

$$\underline{\alpha}_2(|x_2|) \leq V(x_2) \leq \overline{\alpha}_2(|x_2|)\,.$$

Even though these bounds can be chosen very tight, i.e.

$$\underline{\alpha}_2(s) = \overline{\alpha}_2(s) = ln(1+s^2)\,, \qquad \forall s \in \mathbb{R}_{\geq 0}\,,$$

the above condition does not hold. This follows from the observation that, for all $q \geq 1$*,* $\gamma_1^q \in \mathcal{K}_\infty$ *while* α_2 *is a bounded function. This makes [Ito05, Corollary 2] inapplicable to this case.* □

We next formally establish the above result.

Proof of Theorem 5.13. To the best of our knowledge, no changing supply rate result, as the one in [ST95], exists for iISS systems. The approach adopted here is therefore different from the proof of Theorem 5.9. The proof consists in showing separately that (5.11) is 0-GAS and satisfies the BEFBS property and then applying Theorem [AISW04, Theorem 3] which shows equivalence between iISS and the combination of the above two properties. The first step is actually straightforward in view of Theorem 5.9, by picking in the latter $f_1(\cdot,\cdot)$ as $f_1(\cdot,\cdot,0)$ and $f_2(\cdot)$ as $f_2(\cdot,0)$. To establish the second one, we introduce the following result.

Lemma 5.15 *Let* $\omega : \mathbb{R}_{\geq 0} \to \mathbb{R}$ *be a continuous function. Suppose that* $y : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *is a locally Lipschitz function satisfying, for almost all* $t \in \mathbb{R}_{\geq 0}$*,*

$$y(t) > 0 \quad \Rightarrow \quad \dot{y}(t) \leq \omega(t) \tag{5.16}$$

and that

$$y(t) = 0 \quad \Rightarrow \quad \omega(t) \geq 0\,. \tag{5.17}$$

Then $\dot{y}(t) \leq \omega(t)$ *actually holds for almost all* $t \in \mathbb{R}_{\geq 0}$*.* □

Proof of Lemma 5.15. Let $\chi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be defined as follows:

$$\chi(y) = \begin{cases} 1 & \text{if} \quad y = 0 \\ 0 & \text{if} \quad y > 0\,, \end{cases}$$

and, for all $y \in \mathbb{R}_{\geq 0}$, let $\mathcal{G}(y)$ denote the following set:

$$\mathcal{G}(y) := \{t \in \mathbb{R}_{\geq 0} \,:\, y(t) = y \quad \text{and} \quad \dot{y}(t) \quad \text{exists}\}\,.$$

Then, by the area formula for locally Lipschitz functions , we have:

$$\int_0^{+\infty} \chi(y(t))|\dot{y}(t)|dt = \int_{-\infty}^{+\infty} \chi(y)\,\mathrm{card}\,(\mathcal{G}(y))\,dy = 0\,.$$

Let K denote the set $\{t \in \mathbb{R}_{\geq 0} : y(t) = 0\}$. The above argument shows that the set $\{t \in K : \dot{y}(t) \neq 0\}$ has zero-measure. Hence, for almost all t in K, we have $\dot{y}(t) = 0$ and consequently, by virtue of (5.17), for almost all $t \in K$ it holds $\dot{y}(t) \leq 0 \leq \omega(t)$. Since, by the assumption (5.16) the inequality holds for almost all $t \notin K$, the claim follows. ■

Let us go back to the proof of Theorem 5.13. Consider any initial state $(x_{10}, x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and any admissible input u. Notice that, in view of (5.12), $\alpha_2(|\phi_2(\cdot, x_{20}, u)|)$ vanishes whenever $V(\phi_2(\cdot, x_{20}, u)) = 0$. By considering $\omega(\cdot) = -\alpha_2(|\phi_2(\cdot, x_{20}, u)|) + \gamma_2(|u(\cdot)|)$ in Lemma 5.15, it therefore holds that, for almost all $t \in \mathbb{R}_{\geq 0}$,

$$\dot{V}_2(\phi_2(t, x_{20}, u)) \leq -\alpha_2(|\phi_2(t, x_{20}, u)|) + \gamma_2(|u(t)|)\,. \tag{5.18}$$

We establish the BEFBS property under the following "Bounded Energy" assumption:

$$\int_0^{\infty} \gamma(|u(\tau)|)d\tau \leq c\,, \tag{5.19}$$

where $\gamma(s) := \max\{\gamma_2(s); \nu_1(s)\}$ and c is a positive constant. Integrating Inequality (5.18) indeed yields, for all $t \in \mathbb{R}_{\geq 0}$,

$$\begin{aligned} \int_0^t \alpha_2(|\phi_2(\tau, x_{20}, u)|)d\tau &\leq V_2(x_{20}) - V_2(\phi_2(t, x_{20}, u)) + \int_0^t \gamma_2(|u(\tau)|)d\tau \\ &\leq V_2(x_{20}) + c\,. \end{aligned}$$

Moreover, in view of (5.12) for $i = 2$, Inequality (5.18) implies that, for almost all $t \in \mathbb{R}_{\geq 0}$,

$$\dot{V}_2(\phi_2(t, x_{20}, u)) \leq -\alpha_2 \circ \overline{\alpha}_2^{-1}(V_2(\phi_2(t, x_{20}, u))) + \gamma_2(|u(t)|)\,.$$

Since $\alpha_2 \circ \overline{\alpha}_2^{-1}$ is a continuous positive definite function, [ASW00a, Corollary IV.3] establishes the existence of a class $\mathcal{KL}$ function β_2 such that

$$V_2(\phi_2(t, x_{20}, u)) \leq \beta_2(V_2(x_{20}), t) + 2\int_0^t \gamma_2(|u(\tau)|)d\tau\,.$$

Using again the bounds on V_2 provided by (5.12)[6], it follows that

$$|\phi_2(t, x_{20}, u)| \leq \underline{\alpha}_2^{-1}\Big(2\beta_2\big(\overline{\alpha}_2(|x_{20}|), t\big)\Big) + \underline{\alpha}_2^{-1}\left(4\int_0^t \gamma_2(|u(\tau)|)d\tau\right)\,. \tag{5.20}$$

[6] And the "weak triangular inequality": $\alpha(a + b) \leq \alpha(2a) + \alpha(2b)$ for any nonnegative a and b, if $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is nondecreasing.

By the way, as we will need it later, notice that a similar reasoning based on (5.13) leads to the following bound on the trajectories of (5.11a), where β_1 denotes a $\mathcal{KL}$ function:

$$|\phi_1(t,x_{10},x_2)| \leq \underline{\alpha}_1^{-1}\Big(2\beta_1\big(\overline{\alpha}_1(|x_{10}|),t\big)\Big)+\underline{\alpha}_1^{-1}\left(4\int_0^t\Big[\gamma_1(|\phi_2(\tau,x_{20},u)|)+\nu_1(|u(\tau)|)\Big]d\tau\right). \tag{5.21}$$

In view of (5.19) and (5.20), [Son98b, Proposition 6] ensures that $\lim_{t\to\infty}|\phi_2(t,x_{20},u)|=0$. Notably, there exists a finite time $T\geq 0$ such that $|\phi_2(t,x_{20},u)|\leq 1$ for all $t\geq T$. Furthermore, since $\gamma_1(s)=\mathcal{O}(\alpha_2(s))$ in a neighborhood of zero and both these functions are continuous, there exists a positive constant k such that

$$\gamma_1(s)\leq k\alpha_2(s)\,,\quad \forall s\in[0;1]\,. \tag{5.22}$$

Using (5.19), (5.20) and (5.22), we can achieve the following computation:

$$\begin{aligned}\int_0^\infty\gamma_1(|\phi_2(\tau,x_{20},u)|)d\tau &\leq \int_0^T\gamma_1(|\phi_2(\tau,x_{20},u)|)d\tau+\int_T^\infty\gamma_1(|\phi_2(\tau,x_{20},u)|)d\tau\\ &\leq \int_0^T\gamma_1(|\phi_2(\tau,x_{20},u)|)d\tau+k\int_T^\infty\alpha_2\left(|\phi_2(\tau,x_{20},u)|\right)d\tau\\ &\leq \int_0^T\gamma_1(|\phi_2(\tau,x_{20},u)|)d\tau+k(V_2(x_{20})+c)\,.\end{aligned}$$

Since T is finite, this shows that, under the bounded energy condition (5.19), the integral $\int_0^\infty\gamma_1(|\phi_2(\tau,x_{20},u)|)d\tau$ is bounded as well. Finally, notice that, since β_1 is a $\mathcal{KL}$ function, (5.19) and (5.21) imply that

$$\begin{aligned}\limsup_{t\to\infty}|\phi_1(t,x_{10},x_2)| &\leq \underline{\alpha}_1^{-1}\left(8\int_0^\infty\gamma_1(|\phi_2(\tau,x_{20},u)|)d\tau\right)+\underline{\alpha}_1^{-1}\left(8\int_0^\infty\nu_1(|u(\tau)|)d\tau\right)\\ &\leq \underline{\alpha}_1^{-1}\left(8\int_0^\infty\gamma_1(|\phi_2(\tau,x_{20},u)|)d\tau\right)+\underline{\alpha}_1^{-1}\left(8c\right)\,.\end{aligned}$$

In a nutshell, under the bounded energy assumption (5.19), the upper limit (and, *a fortiori*, its lower limit) of the norm of the trajectories of (5.11a), as t goes to infinity, is finite. This establishes the BEFBS property for the whole cascade (5.11). As evoked in the beginning of the proof, the conclusion follows from [AISW04, Theorem 3].

■

A direct consequence of Theorem 5.13, which is of notable interest in stability analysis, concerns the case when the driven subsystem does not depend on the input u. The system then takes the more classical cascade form

$$\dot{x}_1 = f_1(x_1,x_2) \tag{5.23a}$$
$$\dot{x}_2 = f_2(x_2,u)\,. \tag{5.23b}$$

Corollary 5.16 (iISS + iISS ⇒ iISS, Lyapunov-based) *Let V_1 be a differentiable function and V_2 be a continuous function differentiable out of the origin. Suppose that, for all $i\in\{1,2\}$, there exist: a continuous positive definite function α_i, a class $\mathcal{K}$ function γ_i, and some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_i$ and $\overline{\alpha}_i$ such that, for all $(x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ and all $u\in\mathbb{R}^p$, (5.12) and (5.14) hold and*

$$\frac{\partial V_1}{\partial x_1}(x_1)f_1(x_1,x_2,u)\leq-\alpha_1(|x_1|)+\gamma_1(|x_2|)\,.$$

Then, under the condition that $\gamma_1(s) = \mathcal{O}(\alpha_2(s))$ as s tends to 0, the cascade (5.23) is iISS.

Proof of Corollary 5.16. It suffices, with a slight abuse, to pick ν_1 as the zero function in the proof of Theorem 5.13. ■

Intuitively, one could expect that the cascade keeps the same iISS gain as its driving subsystem. This is however not the case in general, as shown by the following counter-example. This example also illustrates how the fact that V_2 is not required to be differentiable at the origin can be profitable in some situations.

Example 5.17 *Consider the following two-dimensional cascaded system:*

$$\begin{aligned} \dot{x}_1 &= -sat(x_1) + x_1 x_2 \\ \dot{x}_2 &= -x_2 + u\,. \end{aligned}$$

First, we show that this cascade is iISS. To this end, let $V_1(x) = ln(1+x_1^2)/2$ and $V_2(x_2) = |x_2|$. Using the same notations as in Corollary 5.16, and referring to the computations detailed in Example 5.12, we see that their derivatives satisfy the following upper bounds:

$$\frac{dV_1}{dx_1}(x_1) f_1(x_1, x_2) \leq -\frac{x_1 sat x_1}{1+x_1^2} + |x_2|$$

$$x_2 \neq 0 \qquad \Rightarrow \qquad \frac{dV_2}{dx_2}(x_2) f_2(x_2, u) \leq -|x_2| + |u|\,.$$

The previous result easily applies and establishes that the cascade is iISS. Next, we exhibit an iISS gain for the driving subsystem. Since it is linear and time-invariant, it is direct to see that its solutions satisfy

$$|\phi_2(t, x_{20}, u)| \leq |x_{20}|\, e^{-t} + \int_0^t \left| e^{-(t-\tau)} u(\tau) \right| d\tau = |x_{20}|\, e^{-t} + \int_0^t \left| \left(e^{-(t-\tau)/2} \right) \left(e^{-(t-\tau)/2} u(\tau) \right) \right| d\tau\,.$$

The two functions in brackets in the latter integral are in L_4 (and actually in L_p for all positive p). Hence, we can apply Holder's inequality to get that

$$\begin{aligned} |\phi_2(t, x_{20}, u)| &\leq |x_{20}|\, e^{-t} + \left(\int_0^t e^{-2(t-\tau)/3} d\tau \right)^{3/4} \left(\int_0^t e^{-2(t-\tau)} \left| u(\tau) \right|^4 d\tau \right)^{1/4} \\ &\leq |x_{20}|\, e^{-t} + \frac{3}{2} \left(\int_0^t u(\tau)^4 d\tau \right)^{1/4}\,. \end{aligned}$$

This shows that an admissible iISS gain for the driving subsystem is the function $\mu(s) = s^4$. However, if it were an iISS gain for the whole cascade as well, then [Son98b, Proposition 6] would notably ensure that, if the integral $\int_0^\infty u(\tau)^4 d\tau$ is finite, then $\limsup_{t\to\infty} |\phi(t, x_0, u)| = 0$. We show that this is not the case. Consider indeed the feedback input $u = x_2 - x_2^3$. The closed-loop cascaded system then becomes

$$\begin{aligned} \dot{x}_1 &= -sat(x_1) + x_1 x_2 \\ \dot{x}_2 &= -x_2^3\,. \end{aligned}$$

As seen in Example 5.3, for $x_{20} = 1$ the solution of the x_2-subsystem is $1/\sqrt{1+2t}$, which ensures that $\int_0^\infty u(\tau)^4 d\tau < \infty$ whereas, for any $x_{10} \geq 3$, the corresponding trajectory of this system grows unbounded. □

5.3 Integral input to state stability for cascades, trajectory-based

In this section, we address the same problem as above, *i.e.* deriving sufficient conditions for the preservation of the iISS property under cascade interconnection, but without requiring the knowledge of any Lyapunov function. Instead, greater stability properties are required for each subsystem. It is indeed imposed that the driving subsystem be 0-LES, and that the iISS gain of the driven subsystem be locally Lipschitz.

Theorem 5.18 (iISS + iISS ⇒ iISS, trajectory-based) *Assume that the system (5.11a) is iISS with respect to (x_2, u) with an iISS gain μ_1, and that the system (5.5b) is iISS and 0-LES (cf. Definition 5.6). Assume also that $f_2(\cdot, 0)$ is continuously differentiable. Then, under the condition that μ_1 is locally Lipschitz, the cascade (5.11) is iISS.*

To the best of our knowledge, this constitutes the first result that proposes trajectory-based sufficient conditions for the preservation of the iISS property under cascade interconnection.

It is interesting to see that the obtained sufficient condition is hardly more conservative than the one in [AAS02, Corollary 2], while ensuring a more interesting property to the overall cascade (that is, iISS instead of GAS). More precisely, the latter reference imposes that

$$\int_0^1 \frac{\mu_1(s)}{s} ds < \infty \,,$$

which is fulfilled when μ_1 is locally Lipschitz.

Also, similarly to Theorem 5.13, note that this result applies to cascaded systems like (5.23), *i.e.* when the driven subsystem does not depend on the input u.

Proof of Theorem 5.18. The proof consists in designing a Lyapunov-like function for the driving subsystem, in order to follow a similar reasoning as in the proof of Theorem 5.13. Namely, we will show 0-GAS and BEFBS. We point out that any locally Lipschitz function of class $\mathcal{K}$ can be upper bounded by a differentiable function of class $\mathcal{K}$. Based on this observation, we will consider without loss of generality that μ_1 is differentiable. We start by introducing the following lemma, similar to a result in [SJK97], which establishes a local Lipschitz property for the estimate of the trajectories of a GAS and LES system.

Lemma 5.19 ($\mathcal{KL}$ estimate for GAS and LES systems) *A system $\dot{x} = f(x)$, with $f : \mathbb{R}^n \to \mathbb{R}^n$ locally Lipschitz, is GAS and LES if and only if there exists a locally Lipschitz function η of class $\mathcal{K}$ and a positive constant k such that, for all initial conditions $x_0 \in \mathbb{R}^n$, the corresponding solution satisfies*

$$|\phi(t, x_0)| \leq \eta(|x_0|) e^{-kt} \,, \qquad \forall t \in \mathbb{R}_{\geq 0} \,.$$

□

Proof of Lemma 5.19. One direction of the implication is straightforward. Indeed, suppose that $|\phi(t, x_0)| \leq \eta(|x_0|) e^{-kt}$ for all $x_0 \in \mathbb{R}^n$ and all $t \in \mathbb{R}_{\geq 0}$. Then the system is GAS. In addition, since η is a locally Lipschitz function of class $\mathcal{K}$, there exists a nonnegative constant λ such that $\eta(s) \leq \lambda s$ for all $s \in [0; 1]$. Thus, for all $|x_0| \leq 1$, it follows that $|\phi(t, x_0)| \leq \lambda |x_0| e^{-kt}$, which establishes LES.

The converse is proved using similar arguments as for [AAS02, Lemma 4]. Since the system is LES, there exist some positive constants k_1, k and ε such that

$$|x_0| \leq \varepsilon \quad \Rightarrow \quad |\phi(t, x_0)| \leq k_1 |x_0| e^{-kt}, \qquad \forall t \in \mathbb{R}_{\geq 0}. \tag{5.24}$$

Also, since it is GAS, for any $x_0 \in \mathbb{R}^n$ there exists a time $T \geq 0$, depending on $|x_0|$, such that $|\phi(T(|x_0|), x_0)| \leq \varepsilon$. Hence, it holds that

$$|\phi(t, x_0)| \leq k_1 |\phi(T(|x_0|), x_0)| e^{-k(t-T(|x_0|))} \leq k_1 \varepsilon e^{kT(|x_0|)} e^{-kt}, \qquad \forall t \geq T(|x_0|). \tag{5.25}$$

Moreover, the GAS assumption also establishes the existence of a $\mathcal{KL}$ function such that, for all x_0, $|\phi(t, x_0)| \leq \beta(|x_0|, t)$. It follows that, for all $t \in [0; T(|x_0|)]$,

$$|\phi(t, x_0)| \leq \beta(|x_0|, 0) e^{kt} e^{-kt} \leq \beta(|x_0|, 0) e^{kT(|x_0|)} e^{-kt}.$$

This inequality together with (5.25) shows that, for all $t \in \mathbb{R}_{\geq 0}$,

$$|\phi(t, x_0)| \leq e^{-kt} \max\{k_1 \varepsilon \ ; \ \beta(|x_0|, 0)\} e^{kT(|x_0|)}.$$

From the previous inequality and (5.24), it is possible to see that $|\phi(t, x_0)| \leq \tilde{\eta}(|x_0|) e^{-kt}$, where

$$\tilde{\eta}(s) := \begin{cases} k_1 s & \text{if} \quad 0 \leq s \leq \varepsilon/2 \\ \sigma(s) & \text{if} \quad \varepsilon/2 < s \leq \varepsilon \\ \max\{k_1 \varepsilon \, ; \, \beta(s, 0)\} e^{kT(s)} & \text{if} \quad s > \varepsilon, \end{cases}$$

where σ is any continuous increasing function such that

$$\sigma(\varepsilon/2) = k_1 \varepsilon/2 \qquad \text{and} \qquad \sigma(\varepsilon) = \max\{k_1 \varepsilon \ ; \ \beta(\varepsilon, 0)\} e^{kT(\varepsilon)}.$$

Since the function $T(\cdot)$ can be chosen continuous and nondecreasing, $\tilde{\eta}$ is a class $\mathcal{K}$ function. Note in addition that $\tilde{\eta}$ is differentiable over $[0; \varepsilon/2]$, and is consequently locally Lipschitz around zero. This shows that it can be upper bounded on all $\mathbb{R}_{\geq 0}$ by a locally Lipschitz function η of class $\mathcal{K}$, which establishes the result. ■

We can now continue the proof of Theorem 5.18. Since the driving subsystem is iISS, it is 0-GAS. Hence, from the previous lemma, we see that there exists a locally Lipschitz class $\mathcal{K}$ function η_2 and a positive constant k_2 such that the trajectories of the zero input driving subsystem satisfy, for any $x_{20} \in \mathbb{R}^{n_2}$,

$$|\phi_2(t, x_{20}, 0)| \leq \eta_2(|x_{20}|) e^{-k_2 t}, \qquad \forall t \in \mathbb{R}_{\geq 0},$$

which means, using the terminology of [AAS02], that $\dot{x}_2 = f_2(x_2, 0)$ is GAS(α_2) with $\alpha_2(s) := s$. In addition, since μ_1 is locally Lipschitz and positive definite, there exist a positive constant λ such that $\mu_1(s) \leq \lambda s$ for all $s \in [0; 1]$. Consequently, we have that

$$\int_0^1 \frac{\mu_1 \circ \alpha_2(s)}{s} ds \leq \int_0^1 \frac{\lambda^2 s}{s} ds \leq \lambda^2.$$

The 0-GAS of the cascade (5.11) then follows from [AAS02, Theorem 1].

The proof of the BEFBS property is based on the following two lemmas. The first one ensures the existence of a converse Lyapunov-like function for GAS and LES systems, with a prescribed dissipation rate.

Lemma 5.20 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable function such that the system $\dot{x} = f(x)$ is GAS and LES. Let μ be a given differentiable function of class $\mathcal{K}_\infty$. Then there exists a continuous function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ differentiable over $\mathbb{R}^n \backslash \{0\}$, class $\mathcal{K}_\infty$ functions $\underline{\alpha}$ and $\overline{\alpha}$, and a continuous function $c : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that, for all $x \in \mathbb{R}^n \setminus \{0\}$,*

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|)$$

$$\frac{\partial V}{\partial x}(x) f(x) \leq -\mu(|x|) \tag{5.26}$$

$$\left| \frac{\partial V}{\partial x}(x) \right| \leq c(x) \,. \tag{5.27}$$

□

Proof of Lemma 5.20. The result is inspired by the converse theorems based on Massera's lemma (see [Mas49] or, *e.g.*, [Kha96, Theorem 3.14]). The novelty consists in allowing to assign a prescribed dissipation term to the constructed Lyapunov-like function. We define

$$V(x) := \int_0^\infty \mu(|\phi(\tau, x)|) d\tau \,.$$

Upper bound on V: Since the system is assumed to be GAS and LES, Lemma 5.19 ensures the existence of a positive constant k and a locally Lipschitz $\mathcal{K}_\infty$ function η such that $|\phi(\tau, x)| \leq \eta(|x|) e^{-k\tau}$. Based on this observation, we get that

$$V(x) \leq \int_0^\infty \mu\left(\eta(|x|) e^{-k\tau}\right) d\tau =: \overline{\alpha}(|x|) \,. \tag{5.28}$$

We claim that $\overline{\alpha}$ is of class $\mathcal{K}$. Indeed, it is clear that $\overline{\alpha}(0) = 0$. For any $x \in \mathbb{R}^n \setminus \{0\}$, consider the change of variable $s = \eta(|x|) e^{-k\tau}$. Then we can see that

$$\overline{\alpha}(|x|) = \int_0^{\eta(|x|)} \frac{\mu(s)}{ks} ds \,. \tag{5.29}$$

However, since μ is differentiable, it is locally Lipschitz, so there exists a nonnegative L_x such that $\mu(s) = \mu(s) - \mu(0) \leq L_x s$ for all $s \in [0; \eta(|x|)]$. This shows that the previous integral is finite, and therefore that $\overline{\alpha}$ is finite over $\mathbb{R}_{>0}$. Moreover, it can easily be seen from (5.29) that $\overline{\alpha}$ is continuous and increasing, which finishes to establish the claim.

Lower bound on V: The lower bound on V is obtained as follows. Notice first that, since f is continuously differentiable, it is locally Lipschitz, so there exists a continuous nondecreasing function L such that, for all $x \in \mathbb{R}^n$, $|f(x)| \leq L(|x|)\,|x|$. Hence

$$\frac{\partial}{\partial \tau}(|\phi(\tau, x)|^2) = 2 f(\phi(\tau, x))^\top \phi(\tau, x) \geq -2L(|\phi(\tau, x)|)\,|\phi(\tau, x)|^2 \geq -b(x)\,|\phi(\tau, x)|^2 \,,$$

where $b(x) := 2L(\sup_{\tau \geq 0} |\phi(\tau, x)| + 1)$ is a positive constant which is finite since the system is assumed to be GAS. Thus, $|\phi(\cdot, x)|^2$ satisfies the differential inequality $\dot{y} \geq -b(x) y$. From the comparison lemma, we conclude that $|\phi(\tau, x)|^2 \geq |x|^2 e^{-b(x)\tau}$. Therefore

$$V(x) \geq \int_0^\infty \mu\left(|x|\, e^{-b(x)\tau/2}\right) d\tau \geq \int_0^{1/b(x)} \mu\left(|x|\, e^{-b(x)\tau/2}\right) d\tau \geq \mu\left(|x|^2 e^{-1}\right) .$$

Hence, the choice $\underline{\alpha}(s) := \mu\left(s^2 e^{-1}\right)$ is an appropriate class $\mathcal{K}_\infty$ lower bound[7] for V.

Gradient of V: The next point consists in showing that V is differentiable. To this end, notice that, for any $x \in \mathbb{R}^n$, the solution of $\dot{x} = f(x)$ satisfies

$$\phi(t,x) = x + \int_0^t f(\phi(\tau,x))d\tau\,.$$

We introduce the notation

$$\phi_x(t,x) := \frac{\partial(\phi(t,x))}{\partial x}\,,$$

and differentiate the previous equality with respect to x to get that

$$\phi_x(t,x) = 1 + \int_0^t \frac{\partial f}{\partial x}(\phi(\tau,x))\phi_x(\tau,x)d\tau\,.$$

Differentiating next with respect to t, we obtain that $\phi_x(\cdot,x)$ is solution of the differential equation

$$\frac{\partial}{\partial t}(\phi_x(t,x)) = A(t,x)\phi_x(t,x)\,, \qquad \phi_x(0,x) = 1\,, \tag{5.30}$$

where

$$A(t,x) := \frac{\partial f}{\partial x}(\phi(t,x))\,.$$

We define

$$A_\infty := \lim_{t\to\infty} A(t,x)\,.$$

Since the trajectory $\phi(t,x)$ tends to 0 and f is continuously differentiable, we can see that

$$A_\infty = \frac{\partial f}{\partial x}(0)\,,$$

which shows that A_∞ is independent of x. Also, since the system is assumed to be LES, it follows from [Kha96, Theorem 3.13] that A_∞ is a Hurwitz matrix. Hence, for any positive definite symmetric matrix Q, there exists a positive definite symmetric matrix P such that $A_\infty^\top P + PA_\infty = -Q$. Consider the Lyapunov function

$$\mathcal{V}(\phi_x) := \phi_x^\top P \phi_x\,. \tag{5.31}$$

Then its derivative along the solution of (5.30) yields

$$\begin{aligned}
\dot{\mathcal{V}}(\phi_x) &= \phi_x^\top \Big(A(t,x)^\top P + PA(t,x)\Big)\phi_x \\
&= -\phi_x^\top Q\phi_x + \phi_x^\top \Big[(A(t,x) - A_\infty)^\top P + P(A(t,x) - A_\infty)\Big]\phi_x \\
&\leq -q_m\,|\phi_x|^2 + |\mathcal{A}(t,x)|\,|\phi_x|^2\,,
\end{aligned} \tag{5.32}$$

where $q_m > 0$ is the minimum eigenvalue of Q and $\mathcal{A}(t,x) := (A(t,x) - A_\infty)^\top P + P(A(t,x) - A_\infty)$. We can see that $\lim_{t\to\infty} |\mathcal{A}(t,x)| = 0$. Hence, for all $x \in \mathbb{R}^n$, there exists a finite time $T(x)$ such that $|\mathcal{A}(t,x)| \leq q_m/2$, and consequently

$$\dot{\mathcal{V}}(\phi_x(t,x)) \leq -\frac{q_m}{2}\,|\phi_x(t,x)|^2\,, \qquad \forall t \geq T(x)\,.$$

[7] Note that, in view of (5.28), this establishes in turn that $\overline{\alpha}$ is a class $\mathcal{K}_\infty$ function as well.

From this and (5.31), we conclude that there exist two positive constants k_1 and k_2 such that

$$|\phi_x(t,x)| \leq k_1 \, |\phi_x(T(x),x)| \, e^{-k_2(t-T(x))} \,, \qquad \forall t \geq T(x) \,. \tag{5.33}$$

It is worth mentioning that the forward completeness is ensured by (5.32) and the fact that $|\mathcal{A}(\cdot,x)|$ is bounded (since it is continuous and has a finite limit), *cf.* [AS99]. Therefore, the function defined as

$$c_1(x) := \sup_{t\in[0,T(x)]} |\phi_x(t,x)| \,, \qquad \forall x \in \mathbb{R}^n$$

is well defined over $\mathbb{R}^n$. Recalling that

$$\sup_{t\geq 0} |\phi(t,x)| \leq \eta(|x|)$$

and using also (5.33), it holds that, for all $x \in \mathbb{R}^n$,

$$\int_0^\infty |\phi_x(\tau,x)| \, d\tau = \int_0^{T(x)} |\phi_x(\tau,x)| \, d\tau + \int_{T(x)}^\infty |\phi_x(\tau,x)| \, d\tau \leq c_1(x)T(x) + \frac{k_1}{k_2} \, |\phi_x(T(x),x)| \,.$$

Since, as proved in the beginning of the proof, $\phi(t,x) \neq 0$ for all $x \neq 0$ and all $t \in \mathbb{R}_{\geq 0}$, it follows that

$$\begin{aligned}
\left| \int_0^\infty \mu'(|\phi(\tau,x)|) \frac{\phi(\tau,x)}{|\phi(\tau,x)|} \phi_x(\tau,x) d\tau \right| &\leq \int_0^\infty \left|\mu'(|\phi(\tau,x)|)\right| |\phi_x(\tau,x)| \, d\tau \\
&\leq \sup_{s\in[0;\eta(|x|)]} |\mu'(s)| \int_0^\infty |\phi_x(\tau,x)| \, d\tau \\
&\leq \sup_{s\in[0;\eta(|x|)]} |\mu'(s)| \left(c_1(x)T(x) + \frac{k_1}{k_2} \, |\phi_x(T(x),x)| \right) .
\end{aligned}$$

Thus, the left-hand side of the previous inequality exists and is finite for all $x \in \mathbb{R}^n \setminus \{0\}$. However, the norm of this very integral also satisfies

$$\left| \int_0^\infty \mu'(|\phi(\tau,x)|) \frac{\phi(\tau,x)}{|\phi(\tau,x)|} \phi_x(\tau,x) d\tau \right| = \left| \int_0^\infty \frac{\partial}{\partial x} \big(\mu(|\phi(\tau,x)|)\big) d\tau \right| = \left| \frac{\partial V}{\partial x}(x) \right| .$$

This establishes that V is differentiable over $\mathbb{R}^n \setminus \{0\}$ and, in turn, provides the bound (5.27) with any continuous function $c : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying

$$c(x) \geq \sup_{s\in[0;\eta(|x|)]} |\mu'(s)| \left(c_1(x)T(x) + \frac{k_1}{k_2} \, |\phi_x(T(x),x)| \right) , \qquad \forall x \in \mathbb{R}^n \,.$$

Upper bound on $\dot{V}$: Finally, we exhibit the bound on the total derivative of V along the trajectories. Let x be any vector of $\mathbb{R}^n$ and $t \in \mathbb{R}_{\geq 0}$, we then have that

$$\begin{aligned}
V(x) &= \int_0^\infty \mu(|\phi(\tau,x)|) d\tau \\
&= \int_0^t \mu(|\phi(\tau,x)|) d\tau + \int_t^\infty \mu(|\phi(\tau,x)|) d\tau \\
&= \int_0^t \mu(|\phi(\tau,x)|) d\tau + \int_t^\infty \mu\big(\, |\phi(\tau-t,\phi(t,x))| \,\big) d\tau \\
&= \int_0^t \mu(|\phi(\tau,x)|) d\tau + \int_0^\infty \mu(|\phi(\tau,\phi(t,x))|) d\tau \,.
\end{aligned}$$

We thus get that, for all $x \in \mathbb{R}^n$ and all $t \in \mathbb{R}_{\geq 0}$,

$$V(\phi(t,x)) - V(x) = -\int_0^t \mu(|\phi(\tau,x)|)d\tau\,.$$

The bound (5.26) follows by differentiating this equality with respect to t. ■

The second lemma we need is an extension of [ASW00a, Proposition II.5] and creates a bridge between the notions of 0-GAS and iISS in terms of a (not necessarily radially unbounded) Lyapunov-like function. The novelty here consists in explicitly specifying the behavior of the dissipation term around the origin.

Lemma 5.21 *Let $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ be a locally Lipschitz function. Suppose that there exists a continuous function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ differentiable out of the origin such that, for all $x \in \mathbb{R}^n \setminus \{0\}$,*

$$\begin{aligned} \underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|) \\ \frac{\partial V}{\partial x}(x) f(x,0) \leq -\mu(|x|) \\ \left|\frac{\partial V}{\partial x}(x)\right| \leq c(x)\,, \end{aligned}$$

where $\underline{\alpha}$, $\overline{\alpha}$ denote $\mathcal{K}_\infty$ functions, $\mu \in \mathcal{K}$, and $c : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a continuous function. Then there exists a continuous positive definite function $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ differentiable on $\mathbb{R}^n \setminus \{0\}$ such that, for all $x \in \mathbb{R}^n \setminus \{0\}$ and all $u \in \mathbb{R}^p$, it holds that

$$\frac{\partial W}{\partial x}(x) f(x,u) \leq -\rho(|x|) + \delta(|u|)\,,$$

where δ is of class $\mathcal{K}$ and ρ is a continuous positive definite function satisfying $\rho(s) \sim \mu(s)$ in a neighborhood of zero. □

Proof of Lemma 5.21. The proof we present here consists in slight modifications of the one of [ASW00a, Proposition II.5]. We first establish the following result, which should be seen as an adaptation of [ASW00a, Lemma IV.10].

Proposition 5.22 *Under the assumptions of Lemma 5.21, the function V is such that, for all $x \in \mathbb{R}^n \setminus \{0\}$ and all $u \in \mathbb{R}^p$,*

$$\frac{\partial V}{\partial x}(x) f(x,u) \leq -\mu(|x|) + \nu(|x|)\delta(|u|)\,,$$

where δ is a class $\mathcal{K}$ function and ν is a positive continuous increasing function.

Proof of Proposition 5.22. Consider $x \neq 0$ and compute the total derivative of V along the trajectories of the system with input u:

$$\begin{aligned} \frac{\partial V}{\partial x}(x) f(x,u) &= \frac{\partial V}{\partial x}(x) f(x,0) + \frac{\partial V}{\partial x}(x)\big[f(x,u) - f(x,0) - f(0,u)\big] + \frac{\partial V}{\partial x}(x) f(0,u) \\ &\leq -\mu(|x|) + \left|\frac{\partial V}{\partial x}(x)\right| |f(x,u) - f(x,0) - f(0,u)| + \left|\frac{\partial V}{\partial x}(x)\right| |f(0,u)|\,. \end{aligned}$$

Define the following function

$$\gamma(r,s) := r + s + \max_{|x|\leq r, |u|\leq s} |f(x,u) - f(x,0) - f(0,u)| \,, \qquad \forall r,s \in \mathbb{R}_{\geq 0} \,.$$

Then, γ is of class $\mathcal{K}$ in each of its two arguments. So, by [ASW00a, Corollary IV.5], there exists a class $\mathcal{K}$ function σ such that $\gamma(r,s) \leq \sigma(r)\sigma(s)$. It follows that

$$\frac{\partial V}{\partial x}(x) f(x,u) \leq -\mu(|x|) + \left|\frac{\partial V}{\partial x}(x)\right| \sigma(|x|)\sigma(|u|) + \left|\frac{\partial V}{\partial x}(x)\right| |f(0,u)| \,.$$

Define next, for all $r > 0$,

$$\kappa(r) := r + \sup_{0<|x|\leq r} \left|\frac{\partial V}{\partial x}(x)\right| \,.$$

Note that κ is well defined for all positive r since V is differentiable over $\mathbb{R}^n \setminus \{0\}$ and

$$\limsup_{|x|\to 0} \left|\frac{\partial V}{\partial x}(x)\right| \leq c(0) < \infty \,.$$

If, in addition, we let $\kappa(0) := 0$, then κ is a positive definite nondecreasing function, continuous on $\mathbb{R}_{>0}$. Hence, there exists a continuous increasing function $\tilde{\kappa}$ such that $\tilde{\kappa}(\cdot) \geq \kappa(\cdot)$. Thus we get that, for all $x \neq 0$,

$$\frac{\partial V}{\partial x}(x) f(x,u) \leq -\mu(|x|) + \tilde{\kappa}(|x|)\sigma(|x|)\sigma(|u|) + \tilde{\kappa}(|x|)\,|f(0,u)| \,.$$

By the local Lipschitz continuity of f, there exists a class $\mathcal{K}$ function χ such that $|f(0,u)| \leq \chi(|u|)$. This final observation establishes the result with the functions $\nu(\cdot) = \tilde{\kappa}(\cdot)(1+\sigma(\cdot))$ and $\delta(\cdot) := \sigma(\cdot) + \chi(\cdot)$. ■

Let's now go back to the proof of Lemma 5.21. Define the following function

$$\pi(r) := \int_0^r \frac{ds}{1 + \nu \circ \underline{\alpha}^{-1}(s)} \,,$$

where ν is the positive continuous increasing function generated by the previous proposition. Notice that, since $\nu \circ \underline{\alpha}^{-1}$ is a nonnegative function, π belongs to class $\mathcal{K}$. Letting $W := \pi \circ V$, it follows that W is positive definite and differentiable out of the origin and, for all $x \neq 0$, we have that

$$\frac{\partial W}{\partial x}(x) f(x,u) = \frac{\partial V}{\partial x}(x) f(x,u) \frac{1}{1 + \nu \circ \underline{\alpha}^{-1}(V(x))} \leq \frac{-\mu(|x|)}{1 + \nu \circ \underline{\alpha}^{-1} \circ \overline{\alpha}(|x|)} + \frac{\nu(|x|)\delta(|u|)}{1 + \nu(|x|)} \,.$$

Define

$$\rho(s) := \frac{\mu(s)}{1 + \nu \circ \underline{\alpha}^{-1} \circ \overline{\alpha}(s)} \,, \qquad \forall s \in \mathbb{R}_{\geq 0} \,.$$

Then ρ is a continuous positive definite function. In addition,

$$\lim_{s\to 0} \frac{\mu(s)}{\rho(s)} = \lim_{s\to 0} 1 + \nu \circ \underline{\alpha}^{-1} \circ \overline{\alpha}(s) = 1 \,,$$

which establishes the result.

■

Based on these two lemmas, we now complete the proof of Theorem 5.18. As already seen, the driving zero-input subsystem $\dot{x}_2 = f_2(x_2, 0)$ is GAS and LES. Apply Lemma 5.20 to it with the function $\mu(s) := \mu_1(2s)$, where $\mu_1 \in \mathcal{K}_\infty$ is the iISS gain of the driven subsystem[8]. The conditions of Lemma 5.21 are then also fulfilled, and we conclude the existence of a continuous function W, differentiable out of the origin, such that, for all $x_2 \neq 0$ and all u

$$\frac{\partial W}{\partial x_2}(x_2) f_2(x_2, u) \leq -\rho(|x_2|) + \sigma(|u|)\,,$$

where $\sigma \in \mathcal{K}$ and ρ is a continuous positive definite function such that $\rho(s) \sim \mu_1(2s)$ around zero. By letting $\omega(t) = -\rho(|\phi_2(t, x_{20}, u)|) + \sigma(|u(t)|)$ in Lemma 5.15, it follows that, for almost all $t \in \mathbb{R}_{\geq 0}$,

$$\dot{W}(\phi_2(t, x_{20}, u)) \leq -\rho(|\phi_2(t, x_{20}, u)|) + \sigma(|u(t)|)\,.$$

Integrating this inequality yields

$$\int_0^\infty \rho(|\phi_2(\tau, x_{20}, u)|) d\tau \leq W(x_{20}) + \int_0^\infty \sigma(|u(\tau)|) d\tau\,. \tag{5.34}$$

On the other hand, by the assumption of iISS on the driving subsystem, there exists $\beta_2 \in \mathcal{KL}$, and $\gamma_2, \mu_2 \in \mathcal{K}$ such that, for all $x_{20} \in \mathbb{R}^{n_2}$ and all admissible u,

$$|\phi_2(t, x_{20}, u)| \leq \beta_2(|x_{20}|, t) + \gamma_2\left(\int_0^t \mu_2(|u(\tau)|) d\tau\right)\,. \tag{5.35}$$

We show that the state is "frequently bounded" under the following bounded energy assumption:

$$\int_0^\infty \mu_M(|u(\tau)|) d\tau < \infty\,, \tag{5.36}$$

where

$$\mu_M(s) := \max\{\mu_1(2s)\,;\, \mu_2(s)\,;\, \sigma(s)\}\,, \qquad \forall s \in \mathbb{R}_{\geq 0}\,.$$

To this end, first notice that, in virtue of [Son98b, Proposition 6], this assumption together with (5.35) ensures that

$$\lim_{t\to\infty} |\phi_2(t, x_{20}, u)| = 0\,. \tag{5.37}$$

Moreover, provided (5.36), the integral of the left-hand side of (5.34) is finite. Hence, since Lemma 5.21 ensures that $\rho(s) \sim \mu_1(2s)$ as s tends to zero, we have that

$$\int_0^\infty \mu_1(2\,|\phi_2(\tau, x_{20}, u)|) d\tau < \infty\,.$$

Finally, since μ_1 is the iISS gain of the driven subsystem, there exists $\beta_1 \in \mathcal{KL}$ and $\gamma_1 \in \mathcal{K}$ such that the trajectories of (5.11a) satisfy

$$\begin{aligned}
|\phi_1(t, x_{10}, x_2)| &\leq \beta_1(|x_{10}|, t) + \gamma_1\left(\int_0^t \mu_1\left(\left|\left(\phi_2(\tau, x_{20}, u), u(\tau)\right)\right|\right) d\tau\right) \\
&\leq \beta_1(|x_{10}|, t) + \gamma_1\left(\int_0^t \mu_1\left(|\phi_2(\tau, x_{20}, u)| + |u(\tau)|\right) d\tau\right) \\
&\leq \beta_1(|x_{10}|, t) + \gamma_1\left(\int_0^t \mu_1\left(2\,|\phi_2(\tau, x_{20}, u)|\right) d\tau + \int_0^t \mu_1\left(2\,|u(\tau)|\right) d\tau\right)
\end{aligned}$$

[8] If $\mu_1 \notin \mathcal{K}_\infty$, the whole reasoning can be done with any locally Lipschitz $\tilde{\mu}_1 \in \mathcal{K}_\infty$ such that $\tilde{\mu}_1(s) = \mu_1(s)$ for all $s \in [0; 1]$ and $\tilde{\mu}_1(s) \geq \mu_1(s)$ for all $s \geq 1$.

Thus, under the bounded energy condition (5.36), we get that $\limsup_{t\to\infty} |\phi_1(t, x_{10}, x_2)|$ is finite, and consequently, with (5.37), that the cascade (5.11) satisfies the BEFBS property. We may therefore conclude iISS by virtue of [AISW04, Theorem 3].

■

Conclusion. In this chapter are exposed two results that guarantee the preservation of the iISS property under cascade interconnection. One is based on the Lyapunov functions associated to each subsystem while the other relies on their solutions' estimates. As a corollary, a Lyapunov condition that ensures that the cascade composed of an iISS subsystem driven by a GAS one is itself GAS. The simplicity of the obtained conditions is worth being underlined. Academic examples are also proposed to illustrate the purpose.

Chapter 6

Application to mechanical systems

The aim of this chapter is to present concrete applications of the results presented in this document. These applications concern the control of mechanical systems. More precisely, we start by studying the robustness of PID controlled robot manipulators to external disturbances, actuators'dynamics and model uncertainty. We then provide some results for the control of a spacecraft formation by a leader-follower approach. The last section is devoted to the control of a supply ship in the underway replenishment operation.

6.1 PID control of robot manipulators

The main goal of the section is to study the robustness of PID-controlled robot manipulators to uncertainty and disturbances. In general terms, uncertainty may stem from imprecision on numerical values of certain parameters, inadequacy of the dynamic model, neglected dynamics, approximation of unavailable measurements, *etc.* Disturbances may take the form of noise in the measurements or external forces from physical interaction with the environment such as friction and,in general, all forces that depend on time, position and velocity and which affect the motion.

Friction. In the literature of mechanical systems, special focus is given to friction effects. These are phenomena that depend on multiple factors such as nature of the materials in contact, lubrication, temperature, *etc.* They are therefore highly complex from a modelling viewpoint. For this reason, only approximate models of friction forces and torques are available –*cf.* [Dah68, COaL95, SAGP00]. We distinguish two families of friction models: the static models, in which the friction force or torque depends on the *instantaneous* relative velocity between bodies (such as viscous friction), and dynamic models which depend on the past values of the relative velocity –*cf.* [COaL95, SAGP00]. The former are adequate to approximately model friction in relatively high-velocity motion tasks while the latter are more appropriate to model friction effects at low velocities. Whether dynamic or static, friction effects may be modelled as an input force that depends, in general, on time and state.

Neglected dynamics. Performances of controlled robot manipulators may also be degraded due to the influence of neglected dynamics. A particularly important situation in which dynamics are neglected is non-model-based control of manipulators. That is, when the robot is considered as a black-box system that has input torques and measurable out-

puts (positions and, often, velocities). Not only Proportional Integral Derivative (PID) control of robot manipulators is a clear example of control with neglected dynamics, but it is the most common in industrial manipulators. Indeed, manipulators used in most production lines are PID-controlled, in which case control design boils down to gain-tuning of the PID gains. This fact and the considerable amount of papers on PID control of robots confirm the relevance of this controller –*cf.* [Kel95, PC96, Roc96, CC04] and references in the latter to cite a few.

In model-based control, *i.e.* where the dynamics of the robot are considered in the control law, neglected dynamics correspond, for instance, to unmodelled phenomena, dynamics of actuators and possibly of sensors. The most common actuators are Direct-Current drives, which may be modelled by a linear differential equation –*cf.* [SHV05, KSL05]. However, one may also consider robots driven by Alternate-Current motors –*cf.* [PO97]. In either case, neglecting their dynamics in the control loop may considerably hamper performance. In the problem treated in [PO97], the motor dynamics is highly nonlinear and global asymptotic stability of the closed-loop system with the corresponding ideal control input is obtained. See also [ALLG97] for a result on control of robots taking into account the DC motors'dynamics under ideal conditions (*i.e.* without disturbances).

Robust control and robustness analysis. Robust control of manipulators with respect to disturbances has also been extensively studied, especially in the context of robots with friction. However, they involve highly nonlinear, and possibly discontinuous, controllers– *cf. e.g.* [Tom00, POG98, LC00]. In spite of the rich literature on PID control of robots, we are not aware of a formal study of robustness of PID-controlled manipulators, with respect to unmodelled dynamics and general additive disturbances.

In this section, under reasonable assumptions, we establish that PID-controlled robot manipulators are uniformly semiglobally practically asymptotically stable. In other words, the robot may be operated from arbitrarily large initial conditions and brought to a any given admissible configuration with a steady-state error that may be arbitrarily reduced by enlarging the control gains. Performance is limited only by the physical constraints imposed on the size of the control gains.

It may be argued that modifications of PID control to achieve global asymptotic stability, such as introducing nonlinear terms (*cf.* [Ari94]) or making the integrator time varying –*cf.* [LNL00], may yield better performance. We precise that the subject of this section is not to propose a new robust controller for robot manipulators, but to analyze the robustness of the classical linear time-invariant PID control.

Firstly, we consider robots under PID control and external disturbances. In a second step, we analyze the same scenario taking into account the actuator dynamics. For simplicity, we concentrate on DC actuators described by first order linear differential equations, but the approach may be generalized to more complex dynamics. We establish that PID control is robust to external disturbances, model uncertainty and neglected actuator dynamics (considered themselves under input disturbances). Our stability proofs are constructive, *i.e.*, in contrast to others –*cf.* [OLK95], we do not appeal to La Salle's invariance principle [LaS60]. As for instance in [Roc96, QD91], we prove stability with a Lyapunov function that is positive definite, radially unbounded and has negative definite total derivative. However, in contrast to [Roc96] where the system is regarded as linear with sector-bound non-linearities (which may yield conservative bounds), we use an energy-like Lyapunov function and, as a byproduct, we provide a tuning procedure that

takes into account the size of the desired set of initial conditions and the desired tolerance on the steady-state error.

6.1.1 Robustness with respect to external disturbances

The robot model. We consider the problem of set-point control of a rigid-joint robot manipulator under PID control and in presence of disturbances. In this context, the Lagrangian dynamics of a robot manipulator with n rigid-joints is given by

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u + p_1(t,q,\dot{q})\,, \tag{6.1}$$

where q contains the position of the joints, $D(q) \in \mathbb{R}^{n\times n}$ is symmetric positive definite for all $q \in \mathbb{R}^n$, $N(q,\dot{q}) := \dot{D}(q) - 2C(q,\dot{q})$ is skew-symmetric for all $(q,\, \dot{q}) \in \mathbb{R}^n \times \mathbb{R}^n$ and $u \in \mathbb{R}^n$ corresponds to the torques. The disturbance p_1 represents external forces acting on the robot. As most common in the literature of robot control, we restrict our attention to systems satisfying the following.

Assumption 6.1 *The functions $D(\cdot)$, $C(\cdot,\cdot)$, $g(\cdot)$ are twice continuously differentiable and the partial derivatives of their elements are over-bounded by nondecreasing functions of $|q|$ and $|\dot{q}|$. Furthermore, we assume that there exist positive constants d_m, d_M, k_c and k_g such that[1] for all q and $\dot{q}$ of $\mathbb{R}^n$,*

$$d_m \le |D(q)| \le d_M\,, \qquad |C(q,\dot{q})| \le k_c\,|\dot{q}|\,, \qquad \left|\frac{\partial g(q)}{\partial q}\right| \le k_g\,.$$

As a fist step, our control problem is to design u so that the robot manipulator stabilizes around a desired constant set-point ($q = q_*$, $\dot{q} = 0$). It is further imposed that control be of the PID type. That is, the input torques that achieve the control objective are given by

$$u^* = -K_p\tilde{q} - K_d\dot{q} + \nu \tag{6.2a}$$
$$\dot{\nu} = -K_i\tilde{q}\,, \qquad \nu(0) := \hat{g}(q_*) \tag{6.2b}$$

where $\hat{g}(q_*)$ is an initial guess of the *unknown* constant pre-computed gravitational forces vector, $\tilde{q} := q - q_*$ and K_p, K_d and K_i are symmetric positive definite matrices representing control gains.

We stress that the above setting is fairly common in practice of robot control: not only PID control is probably the most popular control technique but, often, industrial manipulators come with a black-box controller of PID type, meaning that control design for the user of an industrial robot boils down to gain-tuning for the built-in PID.

Disturbances. We establish now our results for the perturbed system (6.1). We assume that the perturbations may be modelled by a function $p_1 : \mathbb{R}_{\ge 0}\times\mathbb{R}^n\times\mathbb{R}^n$ that is continuous in all arguments. However, if needed in particular situations, we can relax this hypothesis to assuming that p_1 is such that the right hand side of (6.1) satisfies the Carathéodory conditions for existence of solutions. This assumption is not of pure theoretical interest. A typical example where the usual local Lipschitz assumption (even continuity) does not

[1] This is true for instance for open kinematic chains with only revolute or only prismatic joints. See *e.g.* [SV89, SS96].

hold is when dealing with systems with Coulomb friction, *i.e.* in the case that p_1 contains terms including $\operatorname{sign}(\dot{q})$ which is discontinuous at $\dot{q} = 0$. For such cases, we shall rely on [KH99] where it is proved that Carathéodory solutions exist for systems with Coulomb friction.

In this setting, we establish the following result, that quantifies the robustness of PID controlled manipulators when actuators'dynamics is neglected.

Proposition 6.2 (Robustness of the mechanical part) *Consider the system (6.1) in closed-loop with (6.2) and under Assumption 6.1. Assume that there exists non-negative numbers p_{10}, p_{11} and p_{12} such that, for all $t \in \mathbb{R}_{\geq 0}$ and all $(q, \dot{q}) \in \mathbb{R}^{2n}$,*

$$|p_1(t, q, \dot{q})| \quad \leq \quad p_{10} + p_{11} \left|(\tilde{q}, \dot{q})\right| + p_{12} \left|(\tilde{q}, \dot{q})\right|^2 \tag{6.3}$$

Then, the closed-loop system is uniformly semiglobally practically asymptotically stable with K_p, K_i and K_d as tuning parameters.

Remark 6.3 *For commodity, we bound the disturbance p_1 by a function of $|(\tilde{q}, \dot{q})|$ instead of $|(q, \dot{q})|$ as may be more natural. However, if*

$$|p_1(t, q, \dot{q})| \leq p'_{10} + p'_{11} \left|(q, \dot{q})\right| + p'_{12} \left|(q, \dot{q})\right|^2 , \tag{6.4}$$

where p'_{10}, p'_{11} and p'_{12} are positive constants, then there exist p_{10}, p_{11} and $p_{12} > 0$ such that (6.3) holds. As a matter of fact, a simple calculation using (6.4) yields $p_{10} := p'_{10} + p'_{11} |q_*| + p_{12} |q_*|^2$, $p_{11} := p'_{11}$ and $p_{12} := p'_{12}$.

Proof of Proposition 6.2. The proof is constructive, *i.e.* we provide a strict Lyapunov function which, moreover, helps to establish a tuning procedure, *cf.* Claim 6.5. For analytical purposes, let $\varepsilon_1 > 0$ be sufficiently small and define the variable

$$s := \frac{1}{\varepsilon_1} \tilde{q} + K_i^{-1} (g(q_*) - \nu) .$$

We decompose K_p in the following manner:

$$K_p = K'_p + \frac{1}{\varepsilon_1} K_i .$$

Notice that K'_p can be made symmetric positive definite by a convenient choice of K_p. The closed-loop system can then be written

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) - g(q_*) + K'_p \tilde{q} + K_d \dot{q} - K_i s \quad = \quad p_1(t, q, \dot{q}) \tag{6.5a}$$

$$\dot{s} \quad = \quad \tilde{q} + \frac{1}{\varepsilon_1} \dot{q} . \tag{6.5b}$$

To formally study the stability of the closed-loop system we rewrite it in the state-space form, *i.e.* defining $x_1^\top := (\dot{q}^\top, \tilde{q}^\top, s^\top)^\top \in \mathbb{R}^{3n}$ and $\theta_1 := (K_d, K'_p, K_i) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}$, the equations (6.5) become $\dot{x}_1 = f_1(t, x_1, \theta_1)$ where

$$f_1(t, x_1, \theta_1) \quad := \quad \begin{pmatrix} \dot{q} \\ -D(q)^{-1} \left[C(q, \dot{q})\dot{q} + g(q) - g(q_*) + K'_p \tilde{q} + K_d \dot{q} - K_i s - p_1(t, q, \dot{q}) \right] \\ \tilde{q} + \frac{1}{\varepsilon_1} \dot{q} \end{pmatrix} .$$

In the sequel, we consider that the tuning parameters are[2]:

$$\theta_{11} := \lambda_m(K_d)\,, \qquad \theta_{12} := \lambda_m(K'_p)\,, \qquad \theta_{13} := \lambda_m(K_i)\,, \tag{6.6}$$

instead of the matrices gains K_d, K'_p, K_i as this makes clearer the proof of Proposition 6.2. Notice that this is only a notation convention since, for each choice of θ_1, one can design corresponding gain matrices K'_p, K_d and K_i.

The rest of the proof consists in verifying the conditions of Corollary 2.16. For this, let Δ_1 and δ_1 be any positive constants satisfying $\delta_1 < \Delta_1$. Following the literature on control of robots with uncertainties (see for instance [OLK95, Kel95, CG95]), we let[3]

$$V_1 := V_{11} + V_{12} + V_{13} \tag{6.7}$$

where

$$V_{11} := \frac{1}{2}\dot{q}^\top D(q)\dot{q} + \frac{1}{2}\tilde{q}^\top K'_p\tilde{q} + U(q) - U(q_*) - \tilde{q}^\top g(q_*) \tag{6.8a}$$

$$V_{12} := \frac{\varepsilon_1}{2} s^\top K_i s + \varepsilon_1 \tilde{q}^\top D(q)\dot{q} \tag{6.8b}$$

$$V_{13} := \varepsilon_2 s^\top D(q)\dot{q}\,, \tag{6.8c}$$

and ε_1 and ε_2 are (small) positive constants. Notice that the function V_{11} corresponds to the energy function of [TA81], the second term in the definition of V_{12} corresponds to the cross-term introduced in [Kod88]. Roughly speaking, we need to show that V_1 is positive definite and its total derivative is negative definite for all $t \in \mathbb{R}_{\geq 0}$ and all x_1 such that $\delta_1 \leq |x_1| \leq \Delta_1$. To that end, without much loss of generality, let the control gains matrices K'_p, K_d and K_i be such that

$$\lambda_M(K_d) \leq \ell\lambda_m(K_d)\,, \qquad \lambda_M(K'_p) \leq \ell\lambda_m(K'_p)\,, \qquad \text{and} \qquad \lambda_M(K_i) \leq \ell\lambda_m(K_i)\,, \tag{6.9}$$

where ℓ designates a positive constant. Then, the following two claims hold true (see Appendix A.6 and A.7 for their respective proofs).

Claim 6.4 (Bounds on V_1) *The Lyapunov function V_1 satisfies*

$$\underline{a}_1 |\dot{q}|^2 + \underline{b}_2\theta_{12} |\tilde{q}|^2 + \underline{b}_3\theta_{13} |s|^2 \leq V_1(x_1) \leq \overline{a}_1 |\dot{q}|^2 + (\overline{a}_2 + \overline{b}_2\theta_{12}) |\tilde{q}|^2 + (\overline{a}_3 + \overline{b}_3\theta_{13}) |s|^2\,, \tag{6.10}$$

where $\underline{a}_1$, $\underline{b}_2$, $\underline{b}_3$, $\overline{a}_1$, $\overline{a}_2$, $\underline{b}_2$, $\overline{a}_3$ and $\overline{b}_3$ are positive numbers independent of the gain θ_1.

Claim 6.5 (Tuning procedure) *Assume that $\varepsilon_1 + \varepsilon_2 \leq 1$ and choose the gains as*

$$\theta_{11} = \theta^\star_{11}(\delta_1, \Delta_1) := a_d + \frac{b_d}{\delta_1} + c_d\Delta_1 \tag{6.11a}$$

$$\theta_{12} = \theta^\star_{12}(\delta_1, \Delta_1) := a_p + \frac{b_p}{\delta_1} + c_p\Delta_1 \tag{6.11b}$$

$$\theta_{13} = \theta^\star_{13}(\delta_1, \Delta_1) := a_i + \frac{b_i}{\delta_1} + c_i\Delta_1\,, \tag{6.11c}$$

where a_d, b_d, c_d, a_p, b_p, c_p, a_i, b_i and c_i the positive constants, independent of δ_1 and Δ_1, given in Section A.7. Then, the total derivative of V_1 satisfies

$$\frac{\partial V_1}{\partial x_1} f_1(t, x_1, \theta_1) \leq -\frac{\varepsilon_1\theta_{12}}{2} |\tilde{q}|^2 - \frac{\theta_{11}}{2} |\dot{q}|^2 - \frac{\varepsilon_2\theta_{13}}{2} |s|^2\,, \qquad \forall x_1 \in \mathcal{H}(\delta_1, \Delta_1)\,. \tag{6.12}$$

[2]We recall that $\lambda_m(A)$ and $\lambda_M(A)$ denote respectively the smallest and largest eigenvalues of A.

[3]It should be understood that V depends on the tuning parameter θ_1. We omit to explicitly write this dependency for clarity.

In view of Claim 6.4, the requirement (2.48) of Corollary 2.16 holds for V_1. In addition, Claim 6.5 ensures that (2.49) holds. Furthermore, proceeding as in Example 2.17, a simple inspection of (6.11) shows that (2.51) are also satisfied. In virtue of Corollary 2.16, this ends the proof of the proposition. ■

Several interesting corollaries stem from Proposition 6.2. The first one concerns the case of a vanishing perturbation. Its proof follows along the same lines of that of Proposition 6.2, by invoking Corollary 2.16.

Corollary 6.6 (Vanishing perturbation) *In the case that the assumptions of Proposition 6.2 hold with $p_{10} = 0$, the equilibrium point $(q, \dot{q}) = (q_*, 0)$, for the PID-controlled robot, is semiglobally asymptotically stable.*

It worth pointing out that, under vanishing perturbations (that may include high order nonlinearities), we are left with the same stability property as for the PI^{2}D controller proposed in [OLK95] and the saturated PID controller of [ARKC03], where no disturbance is taken into account.

Another interesting case that is included in Proposition 6.2 is that of *motion control* under PID control.

Corollary 6.7 (Trajectory tracking) *Let $t \mapsto q_*(t)$ denote a bounded reference trajectory with $\dot{q}_*(\cdot)$ and $\ddot{q}_*(\cdot)$ continuous and bounded. Then, under the assumptions of Proposition 6.2, the system (6.1) in closed loop with the PID control law*

$$u^* = -K_p\tilde{q} - K_d\dot{\tilde{q}} + \nu \tag{6.13a}$$

$$\dot{\nu} = -K_i\tilde{q}\,, \qquad \nu(0) := \hat{g}(q_*(0))\,, \tag{6.13b}$$

where $\tilde{q} := q - q_*$, is uniformly semiglobally practically asymptotically stable.

Sketch of proof of Corollary 6.7. The closed-loop system is given by (6.5b) and

$$D(q)\ddot{\tilde{q}} + C(q,\dot{q})\dot{\tilde{q}} + g(q) - g(q_*) + K_p'\tilde{q} + K_d\dot{q} - K_i s = p_1(t,q,\dot{q}) - C(q,\dot{q})\dot{q}_* - D(q)\ddot{q}_*\,. \tag{6.14}$$

Let σ_* be a positive constant satisfying

$$\max\{|q_*(t)|\,,\,|\dot{q}_*(t)|\,,\,|\ddot{q}_*(t)|\} \le \sigma_*\,, \qquad \forall\, t \in \mathbb{R}_{\ge 0}\,.$$

Notice that $C(q,\dot{q})\dot{q}_* - D(q)\ddot{q}_*$ then satisfies

$$|C(q,\dot{q})\dot{q}_* - D(q)\ddot{q}_*| \le k_c\sigma_*\,|\dot{q}| + d_M\sigma_*\,.$$

Therefore, redefining p_1 as the right hand side of (6.14), we see that it still satisfies the required condition (6.3). Thus, the result may be obtained exactly as for Proposition 6.2, with the Lyapunov function V_1 defined in (6.7) by replacing $\dot{q}$ with $\dot{\tilde{q}}$. ■

It is worth remarking that the proof for the case of motion control follows as the proof of Proposition 6.2 since we do not appeal to La Salle's invariance principle but we provide a strict Lyapunov function, *i.e.* whose total derivative is negative definite along closed-loop trajectories in the absence of disturbances.

Discontinuous friction. In the case when p_1 contains perturbations due to friction, discontinuous functions of the state may be introduced. In general term, it may take the following form:

$$p_1(t,q,\dot{q}) := F_1\dot{q} + F_2\mathrm{Sign}(\dot{q}) + F_3 z(t) + p_1'(t,q,\dot{q}) \tag{6.15}$$

where F_1 is a non-negative matrix of appropriate dimensions, $\mathrm{Sign}(\dot{q})$ denotes the vector $(\mathrm{sign}(\dot{q}_1),\ldots,\mathrm{sign}(\dot{q}_n))^\top$, $z(t)$ is the *bounded* solution of a dynamic friction model –*cf.* [COaL95, SAGP00], F_2 and F_3 are bounded matrices of appropriate dimensions and p_1' represents additional disturbances[4]. In view of the regularity assumptions imposed on p_1 and the results established in [KH99] , we see that the theoretical result contained in Proposition 6.2 remains valid.

6.1.2 PID control considering actuators'dynamics with disturbances

We consider now the regulation problem when actuator dynamics are taken into account. The input torques $u \in \mathbb{R}^n$ are delivered by Direct-Current (DC) motors, whose dynamics are given by

$$L\dot{i} + Ri + K_b\dot{q} = v + p_2(t,i)\,, \tag{6.16}$$

where $i \in \mathbb{R}^n$ is the vector of rotor currents, L and R are the $n \times n$ matrices of the rotors'inductances and resistances respectively, $K_b\dot{q}$ represents the back electromotive force in the motors and v is the vector of input voltages, *i.e.* the control inputs.

We assume that each motor produces an ideal output torque, *i.e.* $u_j = k_{tj} i_j$ with $k_{tj} > 0$ for each $j \in \{1,\ldots,n\}$. We define $K_t := \mathrm{diag}\{k_{tj} : j = 1,\ldots,n\}$. We stress that this assumption is not conservative since, in the case that phenomena such as torque ripple and backlash are present, they may be modelled by continuous bounded functions $d_j(t,i)$ and the actual torque takes the form $u_j = k_{tj} i_j + d_j(t,i)$. In this case, the corresponding disturbances $d_j(t,i)$ may be accounted for in the term p_1 defined in Proposition 6.2.

The term p_2 represents additional external disturbances acting on the rotor. We pursue the same control objective as above, *i.e.* to stabilize the robot around the set-point q_* with zero velocity. Our control objective is achieved via cascaded-based control; *i.e.* the approach consists in designing a reference $i^* := K_t^{-1}u^*$ (so that, when $\tilde{i} := i - i^* = 0$, we have $u = u^*$) and building a control law v that makes that, ideally, $\tilde{i}$ goes to zero; hence $u \to u^*$. However, in view of the disturbances, a steady-state error is to be expected. Relying on Corollary 2.16, we show that the PID-controlled manipulator, including the motors'dynamics, is uniformly semiglobally practically asymptotically stable.

Proposition 6.8 (Robustness of PID controlled manipulators) *Consider the system (6.1), (6.16) in closed-loop with (6.2) and*

$$v := R'\tilde{i} + Ri^* + K_b\dot{q} + L\dot{i}^*\,, \quad i^* = K_t^{-1}u^*\,.$$

Let $\tilde{q} := q - q_*$ and $\tilde{i} := i - i^*$. Assume that the conditions of Proposition 6.2 hold and the disturbances p_2 are bounded as

$$|p_2(t,i)| \le p_{20} + \rho_2(|\tilde{i}|)\,|\tilde{i}|\,, \qquad \forall\, t \in \mathbb{R}_{\ge 0}\,, \quad \forall i \in \mathbb{R}^{3n}\,, \tag{6.17}$$

where p_{20} is a nonnegative constant and $\rho_2 : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is a continuous function. Then the closed-loop system is uniformly semiglobally practically asymptotically stable. In addition, if $p_{10} = p_{20} = 0$ then the cascade is uniformly semiglobally asymptotically stable.

[4] *e.g.*, in the motion control problem, terms depending on a time-varying reference trajectory.

We stress that the bound (6.17) holds for any function of i uniformly bounded in t.

Proposition 6.8 establishes that, if one knows how to globally asymptotically stabilize a robot using PID control when neglecting the DC drive dynamics and in the absence of external disturbances, then semiglobal practical asymptotic stability can be established in the presence of a wide class of non-dissipative forces and taking into account the actuators'dynamics. In other words, given any tolerance on the steady-state error with respect to the operating point and any domain of initial errors, one can always find control gains such that the closed loop system is uniformly asymptotically stable on this set of initial conditions modulo the tolerance given. Moreover, the tuning procedure given in Claim 6.5 is still valid under the influence of the actuators'dynamics.

It is also important to observe that the result contained in Proposition 6.8 remains valid in the contexts of motion control. More precisely, Corollary 6.6 extends to the case of systems with DC dynamics under the conditions from Proposition 6.8.

Proof of Proposition 6.8. The closed-loop system, including the actuator dynamics, consists in Equations (6.5) and

$$L\dot{\tilde{i}} + (R + R')\tilde{i} \;=\; p_2(t, i)\,. \tag{6.18}$$

Notice that it has a cascaded structure:

$$\dot{x}_1 \;=\; f_1(t, x_1, \theta_1) + K_t x_2 \tag{6.19a}$$
$$\dot{x}_2 \;=\; f_2(t, x_2, \theta_2)\,, \tag{6.19b}$$

where x_1 and f_1 have been previously defined, $x_2 := \tilde{i}$, $\theta_2 = R'$ and

$$f_2(t, x_2, \theta_2) \;:=\; -L^{-1}(R + R')\tilde{i} + L^{-1}p_2(t, i)$$

The proof of the proposition is constructed with the aim at verifying the conditions of Corollary 2.16. For this we use the result from Proposition 6.2 and prove further that the motor closed-loop system (6.18) $\dot{x}_2 = f_2(t, x_2, \theta_2)$ is uniformly semiglobally practically asymptotically stable and that the PID-controlled robot system, taking into account the interconnection term $K_t\tilde{i}$, remains uniformly semiglobally practically asymptotically stable.

USPAS of the motor closed-loop dynamics: While this property may appear intuitively clear in view of the linearity of the motor dynamics, we present the proof of uniform semiglobal practical asymptotic stability of the motor for further development. In particular, this analysis helps to show that the tuning procedure that stems from the proof of Proposition 6.2 remains valid even in spite of the actuator dynamics.

With the notation $x_2 = \tilde{i}$, we consider the Lyapunov function

$$V_2(x_2) := \frac{1}{2}x_2^2\,. \tag{6.20}$$

In view of (6.17), its derivative along the solutions of (6.18) satisfies the following upper bound:

$$\frac{\partial V_2}{\partial x_2}(x_2) f_2(t, x_2, \theta_2) \leq -\frac{\lambda_m(R) + \theta_2}{\lambda_M(L)}\,|x_2|^2 + \frac{p_{20} + \rho_2\left(|x_2|\right)|x_2|}{\lambda_m(L)}\,|x_2|\,.$$

Hence, given any $\Delta_2 > \delta_2 > 0$, it holds that, for all $x_2 \in \mathcal{H}(\delta_2, \Delta_2)$,

$$\frac{\partial V_2}{\partial x_2}(x_2) f_2(t, x_2, \theta_2) \leq -\left(\frac{\lambda_m(R) + \theta_2}{\lambda_M(L)} - \frac{p_{20}}{\lambda_m(L)\delta_2} - \frac{\rho_2(\Delta_2)}{\lambda_m(L)} \right) |x_2|^2 \,. \tag{6.21}$$

By choosing any $\theta_2 = \lambda_m(R') \geq \theta_2^\star(\delta_2, \Delta_2)$ where

$$\theta_2^\star(\delta_2, \Delta_2) := \frac{\lambda_M(L)}{\lambda_m(L)} \left(\frac{p_{20}}{\delta_2} + \rho_2(\Delta_2) \right) , \tag{6.22}$$

we obtain

$$\frac{\partial V_2}{\partial x_2}(x_2) f_2(t, x_2, \theta_2^\star) \leq -\frac{\lambda_m(R)}{\lambda_M(L)} |x_2|^2 = -\frac{2\lambda_m(R)}{\lambda_M(L)} V_2(x_2) \,, \qquad \forall x_2 \in \mathcal{H}(\delta_2, \Delta_2) \,.$$

It is easy to see that the requirements (2.29) and (2.30) of Theorem 2.11 hold. Also, the upper an lower-bounds on V_2 can both be picked as $s \mapsto s^2/2$. This makes the third requirement (2.31)-(2.32) trivially satisfied. Thus, uniform semiglobal practical asymptotic stability follows from Theorem 2.11.

In the case when $p_{20} = 0$, uniform semiglobal asymptotic stability follows similarly from Corollary 2.22.

USPAS of the PID-controlled robot with actuator dynamics: We use now the Lyapunov function

$$V(x_1, x_2) := V_1(x_1) + V_2(x_2) \,.$$

From (6.10) and (6.20) we see that V satisfies the requirement (2.48) of Corollary 2.16. We now compute the total derivative of V along the trajectories of the closed-loop system (6.19). To that end, we first observe that, in view of (6.7)–(6.9) and (6.20), there exists a positive constant c_{δ_1,Δ_1} such that, with the parameter choice proposed in (6.11) and for all $x_1 \in \mathbb{R}^{3n}$,

$$\left| \frac{\partial V_1}{\partial x_1}(x_1) \right| \leq c_{\delta_1,\Delta_1} |x_1| \,, \tag{6.23}$$

In view of (6.12), (6.21) and (6.23), we see that the derivative of V along the trajectories of the system (6.19) yields, for all $x = (x_1^\top, x_2^\top)^\top \in \mathcal{H}(\delta_1, \Delta_1) \times \mathcal{H}(\delta_2, \Delta_2)$,

$$\dot{V} \leq -\alpha_{\delta_1,\Delta_1} |x_1|^2 + c_{\delta_1,\Delta_1} \lambda_M(K_t) |x_1| |x_2| - \left(\frac{\lambda_m(R) + \theta_2}{\lambda_M(L)} - \frac{p_{20}}{\lambda_m(L)\delta_2} - \frac{\rho_2(\Delta_2)}{\lambda_m(L)} \right) |x_2|^2$$

where

$$\alpha_{\delta_1,\Delta_1} := \frac{1}{2} \min \left\{ \varepsilon_1 \theta_{12}^\star(\delta_1, \Delta_1) \;;\; \theta_{11}^\star(\delta_1, \Delta_1) \;;\; \varepsilon_2 \theta_{13}^\star(\delta_1, \Delta_1) \right\} . \tag{6.24}$$

Noticing that, for any positive ε_3, it holds that

$$c_{\delta_1,\Delta_1} \lambda_M(K_t) |x_1| |x_2| \leq \frac{1}{2} \left(\varepsilon_3 |x_1^2| + \frac{c_{\delta_1,\Delta_1}^2 \lambda_M(K_t)^2}{\varepsilon_3} |x_2|^2 \right) ,$$

we get that, for all $x \in \mathcal{H}(\delta_1, \Delta_1) \times \mathcal{H}(\delta_2, \Delta_2)$,

$$\dot{V} \leq -\alpha_{\delta_1,\Delta_1} |x_1|^2 + \frac{\varepsilon_3}{2} |x_1|^2 - \left(\frac{\lambda_m(R) + \theta_2}{\lambda_M(L)} - \frac{p_{20}}{\lambda_m(L)\delta_2} - \frac{\rho_2(\Delta_2)}{\lambda_m(L)} - \frac{c_{\delta_1,\Delta_1}^2 \lambda_M(K_t)^2}{2\varepsilon_3} \right) |x_2|^2 \,.$$

Hence, by picking

$$\varepsilon_3 = \frac{1}{2}\min\left\{\varepsilon_1\left(a_p + \frac{b_p}{\delta_1} + c_p\Delta_1\right); a_d + \frac{b_d}{\delta_1} + c_d\Delta_1; \varepsilon_2\left(a_i + \frac{b_i}{\delta_1} + c_i\Delta_1\right)\right\}$$

and R' in such a way that

$$\theta_2 = \lambda_M(L)\left(\frac{p_{20}}{\lambda_m(L)\delta_2} + \frac{\rho(\Delta_2)}{\lambda_m(L)} + \frac{c^2_{\delta_1,\Delta_1}\lambda_M(K_t)^2}{2\varepsilon_3}\right),$$

we see with (6.24) that

$$\dot{V} \leq -\frac{\alpha_{\delta_1,\Delta_1}}{2}|x_1|^2 - \frac{\lambda_m(R')}{\lambda_M(L)}|x_2|^2$$

for all $x \in \mathcal{H}(\delta,\Delta)$, where[5] $\delta := \max\{\delta_1,\delta_2\}$ and $\Delta := \min\{\Delta_1,\Delta_2\}$. Furthermore, the requirements (2.51) are fulfilled in view of (6.11). The result follows invoking Corollary 2.16. Similarly, in the case that $p_{10} = p_{12} = 0$, uniform semiglobal asymptotic stability follows from Corollary 2.22. ■

6.1.3 Experimental results

We now present some experimental results obtained with the "Pelicano" manipulator. This robot is composed of two rotational links q_1 and q_2. Its model parameters are given below.

$$\begin{aligned}
D(q) &= \begin{pmatrix} 0.3353 + 0.0244\cos(q_2) & 0.0127 + 0.0122\cos(q_2) \\ 0.0127 + 0.0122\cos(q_2) & 0.0127 \end{pmatrix} \mathrm{N\,m\,s^2\,rad^{-1}} \\
C(q,\dot{q}) &= \begin{pmatrix} -0.0122\sin(q_2)\dot{q}_2 & -0.0122\sin(q_2)(\dot{q}_1+\dot{q}_2) \\ 0.0122\sin(q_2)\dot{q}_1 & 0 \end{pmatrix} \mathrm{N\,m\,s\,rad^{-1}} \\
g(q) &= \begin{pmatrix} 11.5081\sin(q_1) + 0.4596\sin(q_1+q_2) \\ 0.4596\sin(q_1+q_2) \end{pmatrix} \mathrm{N\,m}\,.
\end{aligned}$$

The torque developed by the actuators is limited to 15 Nm for the first joint, and to 4 Nm for the second one. The following experimental results are obtained with the gains:

$$K_p' = \begin{pmatrix} 4.54 & 0 \\ 0 & 4.54 \end{pmatrix}, \qquad K_d = \begin{pmatrix} 0.7 & 0 \\ 0 & 0.7 \end{pmatrix}, \qquad K_i = \begin{pmatrix} 3.51 & 0 \\ 0 & 3.51 \end{pmatrix},$$

from initial conditions $q_{10} = 3.14$ rad and $q_{20} = 1.1$ rad, with zero initial velocity.

These results are presented in Figure 6.1. The position, velocity and applied torque is plotted for each joint. This data shows an acceptable behavior of the manipulator with input torques that remain acceptable for the considered actuators. A zoom of these curves show the predicted steady-state error that arises from friction and other perturbations, *cf.* Figure 6.1.

[5] $\mathcal{H}(\delta,\Delta) \neq \emptyset$ if δ_1 and δ_2 are originally chosen small enough (or Δ_1 and Δ_2 large enough).

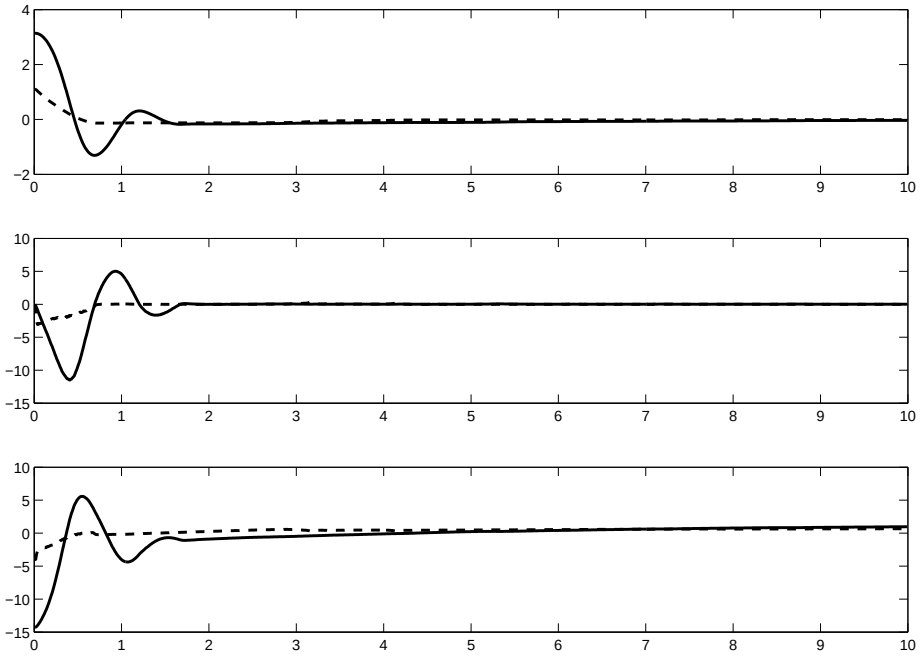


Figure 6.1: 1- q_1 (plain), q_2 (dashed); 2- $\dot{q}_1$ (plain), $\dot{q}_2$ (dashed); 3- u_1 (plain), u_2 (dashed).

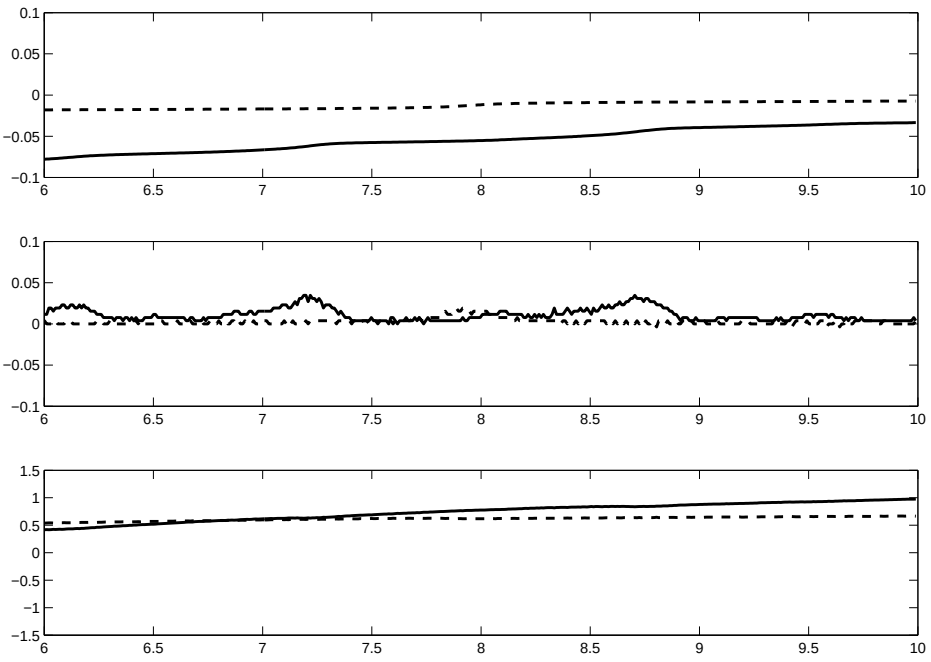


Figure 6.2: Zoom of the previous curves for large t.

6.2 Spacecraft formation

First of all, we stress that all the results presented in this section are fruits of a collaboration with R. Kristiansen, A. Loría and P. J. Nicklasson, and will be part of the topic of the upcoming PhD thesis of R. Kristiansen[6]. We strongly invite the interested reader to consult this reference for a more detailed treatment of the questions raised below.

The object of this section is the control of a spacecraft formation in a leader-follower configuration. This interaction between multiple spacecrafts is indeed revolutionizing the way of performing space-based operations, and brings out several advantages in space mission accomplishment, as well as new opportunities and applications for such missions. Replacing large and complex spacecraft by an array of simpler micro-satellites introduces a multitude of advantages regarding mission cost and performance. However, the advantages of using spacecraft formations come at a cost of increased complexity and technological challenges.

Specific constraints. Formation flying introduces requires a detailed knowledge and a tight control of relative distance and velocity for each spacecraft. As in other applications of cooperative control, the control problem for the follower simplifies as the knowledge about the leader and its orbit increases. However, complete knowledge of the leader is hard to achieve in practice.

Another challenge is that the spacecraft parameters change during its lifetime, by fuel consumption and body deformations. The orbital parameters must often be changed to achieve mission goals, both as planned changes in orbit acquisition and unexpected necessary changes during the operation. Such changes lead to modifications in the system parameters, which can be hard to communicate to the follower. In addition, equipment for determining position and velocity is costly, heavy and computationally demanding, and therefore the follower spacecraft must often rely on measurements of the position of the leader spacecraft only. Hence, the challenge lies in synchronized control of the formation, with as little exchange of information between the spacecraft as possible.

Existing control strategies. Position feedback control of leader-follower spacecraft formations has received some attention during the last years. The first solution to this control problem was presented in [dQYYK99], and the use of a nonlinear control law results in global uniform ultimate boundedness of position and velocity tracking errors. The solution includes a filtering scheme to estimate the relative velocity. A similar result was also presented in [YYKdQ00], providing the same stability properties to the closed-loop system. Nonlinear adaptive tracking control was developed in [dQKY00] and ensures global asymptotic position tracking errors. This latter result was however based on a circular orbit assumption. Later, in [PK01], a nonlinear tracking controller for both translation and rotation was presented, including an adaptation law to account for unknown mass and inertia parameters of the spacecraft. The controller ensures global asymptotic stability of position and velocity errors. Based on the latter two references, semiglobal asymptotic convergence of relative translation errors was proved in [WKS02] for an adaptive output feedback controller using relative position only, with a similar filtering scheme as in [YYKdQ00]. This result was extended to a similar result for both relative translation and

[6] Narvic University College, Norway.

rotation in [WPK05], tracing the steps of [PK01].

Little information. The purpose of this section is to provide a solution to the spacecraft formation control problem with as little knowledge about the leader spacecraft as possible. This relieves the necessity for communication between the spacecraft, and the leader spacecraft can change its orbital parameters without communicating such changes to every other spacecraft in the formation. This is desirable especially for largely populated formation, to diminish the overall communication load. We present a solution to the problem of tracking relative translation in a leader-follower spacecraft formation using feedback from relative position only. The controller design is performed for two different levels of knowledge about the leader spacecraft and its orbit. The first controller assumes perfect knowledge of the leader and its orbital parameters and that the orbital perturbations working on the follower are known. It renders the equilibrium point of the closed-loop system uniformly globally asymptotically stable, using measurements of relative position only. A filter, similar to the one in [Kel93], is included, using the method of approximate differentiation to provide sufficient knowledge about the relative velocity to solve the control problem. The second controller uses the framework of the first to render the closed-loop system uniformly globally practically asymptotically stable, with knowledge of *bounds* on orbital parameters, orbital perturbations and leader control force only.

6.2.1 Problem formulation

Let us here formulate the satellite formation problem that we study in this section. The general orbit equation for two point masses m_1 and m_2 (cf. [Bat99])

$$\ddot{r} + \frac{\mu}{|r|^3} r = 0\,, \tag{6.25}$$

where r is the relative position of masses and $\mu = G\,(m_1 + m_2)$, G being the universal constant of gravity, is the equation describing the uncontrolled orbit dynamics for a spacecraft under ideal conditions. This equation can be generalized to include force terms due to control input vectors from onboard actuators, aerodynamic disturbances, gravitational forces from other bodies, solar radiation, magnetic fields and so on. Accordingly, (6.25) can be expressed for the leader and follower spacecrafts as

$$\begin{aligned} \ddot{r}_l &= -\frac{\mu}{|r_l|^3} r_l + \frac{f_{dl}}{m_l} + \frac{u_l}{m_l} \\ \ddot{r}_f &= -\frac{\mu}{|r_f|^3} r_f + \frac{f_{df}}{m_f} + \frac{u_f}{m_f} \end{aligned}$$

where $f_{dl},\ f_{df} \in \mathbb{R}^3$ are the disturbance force terms due to external perturbation effects and $u_l,\ u_f \in \mathbb{R}^3$ are the actuator forces of the leader and follower respectively. In addition, spacecrafts'masses are assumed to be small relative to the mass of the Earth M_e, so $\mu = GM_e$. Taking the second order derivative of the relative position vector $q = r_f - r_l$, and using the true anomaly $\nu(t)$ of the leader, which is the orbit plane angle measured in the center of the Earth between the orbit perigee point and the leader spacecraft center of mass, the relative position dynamics can be written as (cf. [KGNG05])

$$m_f \ddot{q} + C(\dot{\nu})\dot{q} + D(\dot{\nu}, \ddot{\nu}, |r_f|) + \sigma(|r_l|\,, |r_f|) = U + F_d\,, \tag{6.26}$$

where

$$C(\dot{\nu}) = 2m_f \begin{pmatrix} 0 & -\dot{\nu} & 0 \\ \dot{\nu} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$D(\dot{\nu}, \ddot{\nu}, |r_f|) = m_f \begin{pmatrix} \frac{\mu}{|r_f|^3} - \dot{\nu}^2 & -\ddot{\nu} & 0 \\ \ddot{\nu} & \frac{\mu}{|r_f|^3} - \dot{\nu}^2 & 0 \\ 0 & 0 & \frac{\mu}{|r_f|^3} \end{pmatrix},$$

$$\sigma(|r_l|, |r_f|) = m_f \mu \left(\frac{|r_l|}{|r_f|^3} - \frac{1}{|r_l|^2},\ 0,\ 0 \right)^\top.$$

The composite disturbance force F_d is given by

$$F_d = f_{df} - \frac{m_f}{m_l} f_{dl}$$

and the relative control force U is

$$U = u_f - \frac{m_f}{m_l} u_l \,.$$

For control design, we introduce the more convenient notation

$$C(\dot{\nu}) = 2m_f \dot{\nu} \bar{C} \tag{6.27}$$

$$D(\dot{\nu}, \ddot{\nu}, |r_f|) = m_f \frac{\mu}{|r_f|^3} I + m_f \dot{\nu}^2 \bar{D} + m_f \ddot{\nu} \bar{C} \tag{6.28}$$

where

$$\bar{C} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \bar{D} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The rate of the true anomaly of the leader spacecraft is given by

$$\dot{\nu}(t) = \frac{n_l \left(1 + e_l \cos \nu(t)\right)^2}{\left(1 - e_l^2\right)^{3/2}}, \tag{6.29}$$

where $n_l = \sqrt{\mu / a_l^3}$ is the mean motion of the leader, a_l being the semimajor axis of the leader orbit, and e_l its orbit eccentricity. Differentiation of (6.29) results in the rate of change of the true anomaly:

$$\ddot{\nu}(t) = \frac{-2n_l^2 e_l \left(1 + e_l \cos \nu(t)\right)^3 \sin \nu(t)}{\left(1 - e_l^2\right)^3}.$$

Based on these expressions, we see that, when the leader spacecraft is revolving the Earth in an elliptical orbit, the true anomaly rate $\dot{\nu}(t)$ and true anomaly rate of change $\ddot{\nu}(t)$ are bounded by constants. We therefore assume the following.

Assumption 6.9 (Bound on true anomaly) *There exist positive constants $\overline{\omega}_{\dot{\nu}}$ and $\overline{\omega}_{\ddot{\nu}}$ such that, for all $t \in \mathbb{R}_{\geq 0}$,*

$$|\dot{\nu}(t)| \leq \overline{\omega}_{\dot{\nu}} \qquad \text{and} \qquad |\ddot{\nu}(t)| \leq \overline{\omega}_{\ddot{\nu}} \,.$$

We establish two results that are presented in order of increasing complexity. First, we assume that the leader's true anomaly $\nu(t)$, true anomaly rate $\dot{\nu}(t)$ and the orbital perturbations F_d are known and, in a second step, we relax these hypotheses by assuming that only *bounds* on $\nu(t)$, $\dot{\nu}(t)$ and F_d are known. We make it clear that the property of asymptotic stability is lost (and replaced by practical asymptotic stability) due to the lack of measurements.

6.2.2 Measurements available

Under the assumptions that the leader spacecraft is controlled to overcome external disturbances in an elliptic orbit, and the follower spacecraft has available measurements of relative position q, leader true anomaly rate $\dot{\nu}(t)$, true anomaly rate of change $\ddot{\nu}(t)$ and orbital perturbations f_{df}, we have the following.

Proposition 6.10 (Measurements available: UGAS) *Assuming that the desired relative position $q_*(t)$, desired relative velocity $\dot{q}_*(t)$ and desired relative acceleration $\ddot{q}_*(t)$ are all bounded functions and that Assumption 6.9 holds, the origin of the system (6.26), in closed loop with the control law*

$$u_f = -k_p\tilde{q} - k_d\vartheta + \sigma - f_{df} + D(\dot{\nu}, \ddot{\nu}, |r_f|) + C(\dot{\nu})\,\dot{q}_* + m_f\ddot{q}_* \tag{6.30}$$

$$\dot{q}_c = -a\vartheta \tag{6.31}$$

$$\vartheta = q_c + b\tilde{q} \tag{6.32}$$

where $\tilde{q} := q - q_*$, is uniformly globally asymptotically stable for some convenient choice of the control gains k_p, k_d, a and b.

Proof of Proposition 6.10. Denoting the state vector as

$$x := \left(\tilde{q}^\top, \dot{\tilde{q}}^\top, \vartheta^\top\right)^\top ,$$

the closed-loop dynamics of the system in (6.26) and the controller (6.30)-(6.32) are

$$m_f\ddot{\tilde{q}} = A(t, x) \tag{6.33}$$

where

$$A(t, x) := -C(\dot{\nu})\dot{\tilde{q}} - k_p\tilde{q} - k_d\vartheta \,. \tag{6.34}$$

Differentiating (6.32) and inserting (6.31) results in

$$\dot{\vartheta} = \dot{q}_c + b\dot{\tilde{q}} = -a\vartheta + b\dot{\tilde{q}} \,. \tag{6.35}$$

To prove UGAS of the origin $\left(\tilde{q}, \dot{\tilde{q}}, \vartheta\right) = (0, 0, 0)$ of the closed-loop system, the Lyapunov function candidate

$$V(x) = \frac{1}{2}x^\top P_1 x \tag{6.36}$$

is used, where

$$P_1 := \begin{pmatrix} k_p & \varepsilon_1 m_f & 0 \\ \varepsilon_1 m_f & m_f & -\varepsilon_1 m_f \\ 0 & -\varepsilon_1 m_f & k_d/b \end{pmatrix}$$

with ε_1 as a positive design variable. Evaluating the eigenvalues of the matrix P_1 we obtain that $V(x)$ is positive definite if

$$\varepsilon_1^2 \leq \min\left\{\frac{k_p}{2m_f},\ \frac{k_d}{2bm_f}\right\}. \tag{6.37}$$

Under this condition, V can then be bounded in the following way

$$p_m |x|^2 \leq V(x) \leq p_M |x|^2, \tag{6.38}$$

for some positive constants p_m and p_M[7]. The derivative of $V(x)$ along the trajectories of (6.33) and (6.35) is

$$\dot{V}_1(x) = \left(\dot{\tilde{q}} + \varepsilon_1 \tilde{q} - \varepsilon_1 \vartheta\right)^\top m_f \ddot{\tilde{q}} + \tilde{q}^\top k_p \dot{\tilde{q}} + \vartheta^\top \frac{k_d}{b} \dot{\vartheta} + \varepsilon_1 m_f \dot{\tilde{q}}^\top \left(\dot{\tilde{q}} - \dot{\vartheta}\right)$$

and insertion of (6.33) and (6.35) results in

$$\dot{V}_1(x) = -\frac{1}{2} x^\top Q_1\left(\dot{\nu}(t)\right) x$$

where

$$Q_1(\dot{\nu}) := \begin{pmatrix} 2\varepsilon_1 k_p I & \varepsilon_1 C(\dot{\nu}) & \varepsilon_1 \left[k_d - k_p\right] I \\ -\varepsilon_1 C(\dot{\nu}) & 2\varepsilon_1 m_f \left(b-1\right) I & \varepsilon_1 C(\dot{\nu}) - \varepsilon_1 m_f a I \\ -\varepsilon_1 \left[k_d - k_p\right] I & -\varepsilon_1 C(\dot{\nu}) - \varepsilon_1 m_f a I & 2\left[\frac{a}{b} - \varepsilon_1\right] k_d I \end{pmatrix}.$$

Using Assumption 6.9, the skew-symmetry property of $C(\dot{\nu})$ and Schur's complement on the submatrices in $Q_1(\dot{\nu})$, we obtain that the latter is positive definite when

$$k_p \left(b-1\right) \geq 4 m_f \overline{\omega}_{\dot{\nu}}^2 \tag{6.39}$$

$$k_d \left(\frac{a}{b} - \varepsilon_1\right)\left(b-1\right) \geq \varepsilon_1 \left(4 m_f \overline{\omega}_{\dot{\nu}}^2 + m_f a^2\right) \tag{6.40}$$

$$\frac{a}{b} k_p k_d \geq \varepsilon_1 \left[\left(k_d - k_p\right)^2 + k_p k_d\right]. \tag{6.41}$$

These conditions can, in their turn, be fulfilled by picking k_p and k_d large enough that (6.39) and (6.40) hold and, then, by picking ε_1 small enough in order that (6.41) holds. We conclude with the classical Theorem 2.3 that the closed-loop system is uniformly globally asymptotically stable.

■

[7] Even though p_m and p_M depend on the gains k_p and k_d, this dependency is not a crucial issue here as we aim to establish uniform global asymptotic stability.

6.2.3 When only bounds are known

We now relax the assumption that the instantaneous values of $\dot{\nu}(t)$ and $\ddot{\nu}(t)$ are available to measurement, and rather assume that we know the values of $\overline{\omega}_{\dot{\nu}}$ and $\overline{\omega}_{\ddot{\nu}}$ on the leader true anomaly as given by Assumption 6.9. In addition, we relax the assumptions that orbital perturbations f_{df} are known, and instead assume that the perturbation term is bounded as $|f_{df}| \leq \overline{\omega}_f$. Similarly, we relax the requirement on leader spacecraft control, and assume that the sum of forces working on the leader due to control thrust and external perturbations are bounded, such that $|f_{dl} + u_l| \leq \overline{\omega}_l$. Finally, we also assume that the follower spacecraft has available measurements of relative position q only. For these assumptions, we have the following.

Proposition 6.11 (Known bounds: UGPAS) *Under the above assumptions and assuming further that Assumption 6.9 holds and that the desired relative position q_*, desired relative velocity $\dot{q}_*$ and desired relative acceleration $\ddot{q}_*$ are all bounded functions, the system (6.26), in closed loop with the control law given by (6.31), (6.32) and*

$$u_f = -k_p\tilde{q} - k_d\vartheta + \sigma + m_f\left(\frac{\mu}{|r_f|^3}I + \overline{\omega}_{\dot{\nu}}^2\bar{D} + \overline{\omega}_{\ddot{\nu}}\bar{C}\right)q + 2m_f\overline{\omega}_{\dot{\nu}}\bar{C}\dot{q}_* + m_f\ddot{q}_* \tag{6.42}$$

where $\tilde{q} = q - q_*$, is uniformly globally practically asymptotically stable on the parameter set $\mathbb{R}^3_{>0}$ with k_p, k_d and b as tuning parameters.

The comparison between Propositions 6.10 and 6.11 clearly illustrates the imprecision that may arise from a lack of measurements and the presence of non-measured external disturbances. The steady-state error resulting from these phenomena can however be reduced at will by a convenient tuning of the control gains, which induces a good performance of the above control law as confirmed by simulations (see below).

Proof of Proposition 6.11. The closed-loop dynamics of the system in (6.26) and the controller (6.31), (6.32) and (6.42) are

$$m_f\ddot{\tilde{q}} - m_f\left(\tilde{\omega}_{\dot{\nu}^2}\bar{D} + \tilde{\omega}_{\ddot{\nu}}\bar{C}\right)(\tilde{q} + q_*) + C(\dot{\nu})\dot{\tilde{q}} - 2m_f\tilde{\omega}_{\dot{\nu}}\bar{C}\dot{q}_* + \frac{m_f}{m_l}(f_{dl} + u_l) + k_p\tilde{q} + k_d\vartheta = f_{df} \tag{6.43}$$

where the denotations $\tilde{\omega}_{\dot{\nu}} = \overline{\omega}_{\dot{\nu}} - \dot{\nu}$, $\tilde{\omega}_{\ddot{\nu}} = \overline{\omega}_{\ddot{\nu}} - \ddot{\nu}$ and $\tilde{\omega}_{\dot{\nu}^2} = \overline{\omega}_{\dot{\nu}}^2 - \dot{\nu}^2$ have been used. Note that this closed-loop system is the same as (6.33) with an additional perturbation term

$$G(t, \tilde{q}) := G_1(\tilde{q}) + G_2(t, f_{dl}, f_{df}, u_l) \tag{6.44}$$

consisting of the vanishing and non-vanishing perturbations, G_1 and G_2, given by

$$\begin{aligned} G_1(\tilde{q}) &:= m_f\left(\tilde{\omega}_{\dot{\nu}^2}\bar{D} + \tilde{\omega}_{\ddot{\nu}}\bar{C}\right)\tilde{q} \\ G_2(t, f_{dl}, f_{df}, u_l) &:= m_f\left(\tilde{\omega}_{\dot{\nu}^2}\bar{D} + \tilde{\omega}_{\ddot{\nu}}\bar{C}\right)q_*(t) + 2m_f\tilde{\omega}_{\dot{\nu}}\bar{C}\dot{q}_*(t) - \frac{m_f}{m_l}(f_{dl}+u_l) + f_{df}\,. \end{aligned}$$

Accordingly, the closed-loop system can be written as

$$m_f\ddot{\tilde{q}} = A(t, x) + G(t, \tilde{q})\ . \tag{6.45}$$

By assumption, the desired relative position q_*, relative velocity $\dot{q}_*$, follower orbital perturbations f_{df} and leader forces $f_{dl}+u_l$ are all bounded. More precisely, using that $|q_*| \leq \overline{\omega}_{q_*}$,

$|\dot{q}_*| \leq \overline{\omega}_{\dot{q}_*}$, $|f_{df}| \leq \overline{\omega}_f$, $|f_{dl} + u_l| \leq \overline{\omega}_l$, and Assumption 6.9, we find that $|G| \leq \overline{\omega}_{G_1} |\tilde{q}| + \overline{\omega}_{G_2}$, where

$$\overline{\omega}_{G_1} = m_f \left(\overline{\omega}_{\dot{\nu}^2} + 2\overline{\omega}_{\ddot{\nu}}\right) \tag{6.46}$$

$$\overline{\omega}_{G_2} = m_f \left(\overline{\omega}_{\dot{\nu}^2} + 2\overline{\omega}_{\ddot{\nu}}\right) \overline{\omega}_{q_*} + 2m_f \overline{\omega}_{\dot{\nu}} \overline{\omega}_{\dot{q}_*} + \frac{m_f}{m_l} \overline{\omega}_l + \overline{\omega}_f \,. \tag{6.47}$$

To analyze the stability of the closed-loop system (6.45) we use the same Lyapunov function as before (*cf.* (6.36)). The total derivative of $V(x)$ along the trajectories of (6.35) and (6.43) yields

$$\dot{V}_2(x) = -\frac{1}{2} x^\top Q_1 \left(\dot{\nu}\right) x + \frac{\partial V}{\partial x} G\left(t, \tilde{q}\right) \leq -\frac{1}{2} x^\top Q_2(\dot{\nu}) x + Q_0 \left(\dot{\nu}, q_*, \dot{q}_*\right) x \,,$$

where $Q_2(\dot{\nu}) := \left[q_{ij}\right]_{i,j \in \{1,2,3\}}$ with submatrices given by

$$q_{11} = 2\varepsilon_1 \left(k_p I - m_f \tilde{\omega}_{\dot{\nu}^2} \bar{D}\right) \qquad q_{12} = q_{21}^\top = \varepsilon_1 C(\dot{\nu}) - m_f \left(\tilde{\omega}_{\dot{\nu}^2} \bar{D} - \tilde{\omega}_{\ddot{\nu}} \bar{C}\right) \tag{6.48}$$

$$q_{22} = 2\varepsilon_1 m_f \left(b - 1\right) I \qquad q_{13} = q_{31}^\top = \varepsilon_1 \left[\left(k_d - k_p\right) I + m_f \left(\tilde{\omega}_{\dot{\nu}^2} \bar{D} - \tilde{\omega}_{\ddot{\nu}} \bar{C}\right)\right] \tag{6.49}$$

$$q_{33} = 2\left(\frac{a}{b} - \varepsilon_1\right) k_d I \qquad q_{23} = q_{32}^\top = -\varepsilon_1 C(\dot{\nu}) - \varepsilon_1 m_f a I \tag{6.50}$$

and

$$Q_0 \left(\dot{\nu}, q_*, \dot{q}_*\right) = \begin{pmatrix} \varepsilon_1 m_f \left[\left(\tilde{\omega}_{\dot{\nu}^2} + \tilde{\omega}_{\ddot{\nu}}\right) q_* + 2\tilde{\omega}_{\dot{\nu}} \dot{q}_*\right] \\ m_f \left[\left(\tilde{\omega}_{\dot{\nu}^2} + \tilde{\omega}_{\ddot{\nu}}\right) q_* + 2\tilde{\omega}_{\dot{\nu}} \dot{q}_*\right] - \frac{m_f}{m_l} \left(f_{dl} + u_l\right) + f_{df} \\ -\varepsilon_1 m_f \left[\left(\tilde{\omega}_{\dot{\nu}^2} + \tilde{\omega}_{\ddot{\nu}}\right) q_* + 2\tilde{\omega}_{\dot{\nu}} \dot{q}_*\right] \end{pmatrix}^\top . \tag{6.51}$$

In view of (6.46), (6.47) and the assumptions on the external forces f_{dl}, u_l and f_{df} it follows that there exists $q_0 > 0$ independent of the control gains and the states, such that $|Q_0(\dot{\nu}(t), q_*(t), \dot{q}_*(t))| \leq q_0$. On the other hand, the conditions for positive definiteness of Q_2 are (6.40),

$$\varepsilon_1^2 \left(b - 1\right) \left(k_p + m_f \tilde{\omega}_{\dot{\nu}^2}\right) \geq 4\varepsilon_1^2 m_f \overline{\omega}_{\dot{\nu}}^2 + 4\varepsilon_1 m_f \overline{\omega}_{\dot{\nu}} \tilde{\omega}_{\ddot{\nu}} + m_f \left(\tilde{\omega}_{\dot{\nu}^2}^2 + \tilde{\omega}_{\ddot{\nu}}^2\right) , \tag{6.52}$$

and

$$\left(\frac{a}{b} - \varepsilon_1\right) k_d \left(k_p + m_f \tilde{\omega}_{\dot{\nu}^2}\right) \geq \varepsilon_1 \left[m_f^2 \tilde{\omega}_{\ddot{\nu}}^2 + \left(k_d - k_p + m_f \tilde{\omega}_{\dot{\nu}^2}\right)^2\right] . \tag{6.53}$$

Thus, $\dot{V}_2(x)$ is negative definite if (6.37), (6.40), (6.52) and (6.53) hold and

$$|x| \geq 2 \frac{q_0}{q_{2,m}} \tag{6.54}$$

where $q_0 \geq |Q_0 \left(\dot{\nu}(t), q_*(t), \dot{q}_*(t)\right)|$ and $q_{2,m} \leq |Q_2 \left(\dot{\nu}(t)\right)|$ for all $t \in \mathbb{R}_{\geq 0}$.

Furthermore, to verify the conditions of Corollary 2.9, we exhibit a quadratic upper-bound on $-x^\top Q_2 x$. To that end, we use the formula $2|ab| \leq a^2 + b^2$ for any $a, b \in \mathbb{R}$, to obtain

$$\begin{aligned} x^\top Q_2 x \geq & \left(\lambda_m(q_{11}) - \lambda_M(q_{12}) - \lambda_M(q_{13})\right) |\tilde{q}|^2 + \left(\lambda_m(q_{22}) - \lambda_M(q_{12}) - \lambda_M(q_{23})\right) \left|\dot{\tilde{q}}\right|^2 \\ & + \left(\lambda_m(q_{33}) - \lambda_M(q_{13}) - \lambda_M(q_{23})\right) |\vartheta|^2 \,, \end{aligned}$$

where $\lambda_m(A)$ and $\lambda_M(A)$ denote, respectively, the minimum and maximum eigenvalue of the matrix A. Due to the structure of the sub-matrices q_{ij}, it is always possible to choose the gains k_p, k_d and b large enough that

$$\begin{aligned}\lambda_m(q_{11}) &\geq 2(\lambda_M(q_{12}) + \lambda_M(q_{13}))\\ \lambda_m(q_{22}) &\geq 2(\lambda_M(q_{12}) + \lambda_M(q_{23}))\\ \lambda_m(q_{33}) &\geq 2(\lambda_M(q_{13}) + \lambda_M(q_{23}))\,,\end{aligned}$$

which results in

$$x^\top Q_2 x \geq \frac{1}{2}\left(\lambda_m(q_{11})\,|\tilde{q}|^2 + \lambda_m(q_{22})\,\left|\dot{\tilde{q}}\right|^2 + \lambda_m(q_{33})\,|\vartheta|^2\right)\,.$$

That is, we can choose $q_{2,m} \geq \frac{1}{2}\min\{\lambda_m(q_{11})\,,\,\lambda_m(q_{22})\,,\,\lambda_m(q_{22})\}$. Note that each of these terms can be arbitrarily enlarged by an appropriate choice of k_p, k_d and b. Thus, $q_{2,m}$ can be enlarged accordingly. Given any positive δ, it can notably be picked as $q_{2,m} = 2q_0/\delta$. Moreover, it can be seen that the resulting $q_{2,m}$ consists in a linear combinations of the gains. This ensures that a convenient choice of the gain can be picked affine in $1/\delta$, which ensures conditions (2.27) and (2.28).

Finally, in view of (6.38) the Lyapunov function $V(x)$ also satisfies (2.25) and (2.26). We conclude with Corollary 2.9 that the system (6.26) in closed loop with the control law (6.31), (6.32) and (6.42) is uniformly globally practically asymptotically stable with k_p, k_d and b as tuning parameters.

■

6.2.4 Simulation results

To conclude this work on leader-follower spacecraft formation control and illustrate the performance of the presented control laws, we now present simulation results. The leader spacecraft is assumed to be following an elliptic orbit with eccentricity $e_l = 0.6$. Both spacecraft have mass $m_l = m_f = 100$ kg. The follower spacecraft is assumed to have available continuous thrust in all directions, limited to 27 N. The follower has initial values $q_0 = (20, 10, -20)^\top$, and is further commanded to track sinusoidal trajectories around the leader, given as

$$q_*\,(t) = \left(-10\cos\left(\frac{3\pi}{T_o}t\right),\ 10\sin\left(\frac{4\pi}{T_o}t\right),\ 5\cos\left(\frac{5\pi}{T_o}t\right)\right)^\top\,,$$

where T_o is the orbital period of the leader spacecraft. A possible scenario for this motion is in-orbit inspection, where the follower moves in orbit around the leader. In all simulations performed, we used the controller gains $k_p = 3$, $k_d = 5$, $a = 1$ and $b = 5$. Orbital perturbation forces due to gravitational perturbations and aerodynamic drag are included in the simulations.

The result from simulating the system (6.26) in closed loop with the controller (6.30)-(6.32) is shown in Figure 6.3.

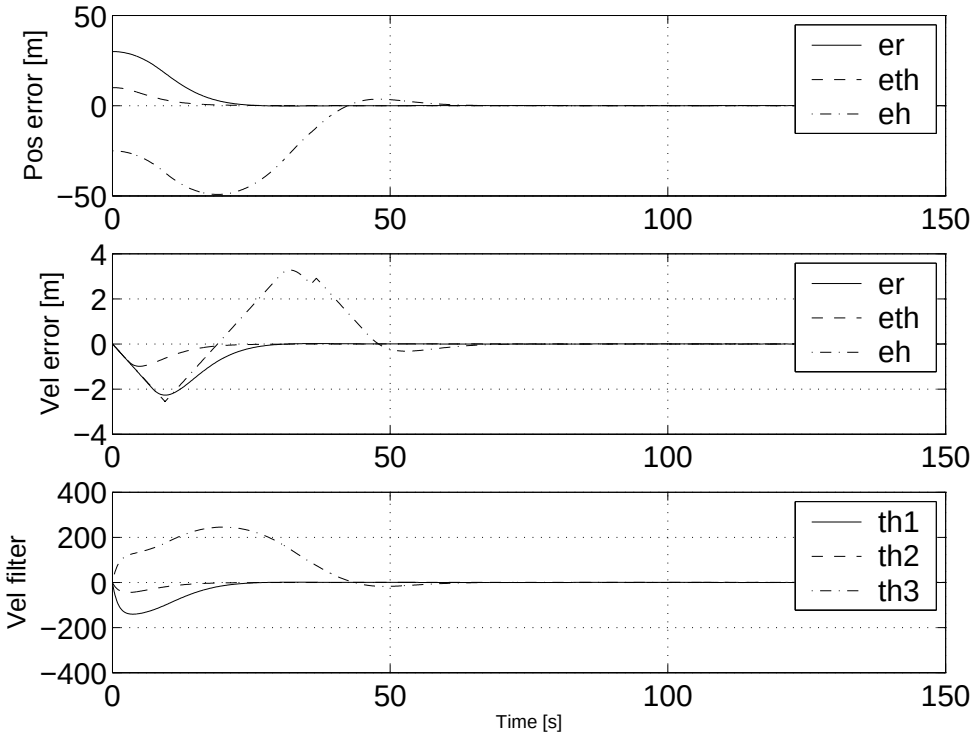


Figure 6.3: Known perturbations, leader true anomaly rate and rate of change: position, velocity and velocity filter output.

This is the case where the leader spacecraft true anomaly and rate of change are known to the follower spacecraft. The follower settles and tracks the desired trajectory without errors in relative position and relative velocity.

The results for the case where only bounds on the perturbations, leader true anomaly rate and rate of change are known are presented in Figure 6.4.

The UGPAS property of the closed-loop system is seen in the figure as persistent oscillations around the origin (see the zoomed parts on the right-hand side). In conformity with our theoretical predictions, the magnitude of the oscillations can be arbitrarily diminished by increasing the controller gains. It is worth noting that a good precision can be reached without requiring a too large thrust amplitude.

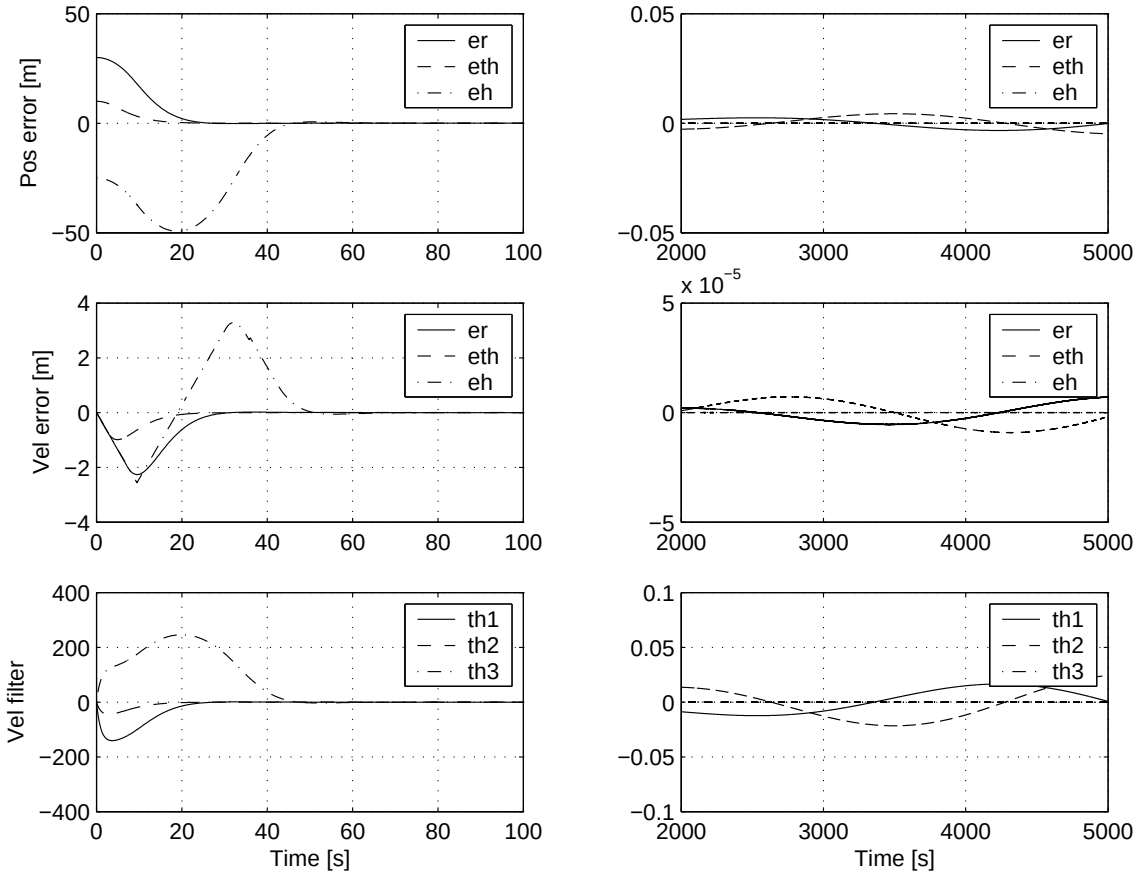


Figure 6.4: Position error, velocity error and velocity filter output for the case when only the bounds on the leader true anomaly rate and rate of change are known

6.3 Underway ship replenishment

The results presented below stem from a collaboration with E. Kyrkjebø, E. Panteley and K. Pettersen. They will constitute part of the subject of the upcoming PhD thesis of R. Kyrkjebø[8]. The interested reader should therefore refer to this work for a more detailed presentation of the topic.

A challenging problem. Underway ship replenishment consists, for a supply vessel, to closely follow a vessel in order to transfer fuel. It requires a close coordination of two vessels. Up to now, the underway ship replenishment manoeuvre has been conducted using manual control together with control flags to exchange instructions between the vessels.

Recent advances in control theory and measurement systems, in particular the introduction of the Global Positioning System (GPS) and the Automatic Identification System (AIS), now allow automatic control approaches for replenishment purposes. These autopilots are faced with the goal of suppressing effects of external disturbances due to wind, waves and currents, while achieving the accuracy demands of the operation using a reduced set of measurements. The introduction of autopilots expand the range of operating conditions for safe replenishment in terms of increased manoeuvrability in close waters or in

[8]NTNU, Trondheim, Norway.

the proximity of other vessels, and in the robustness towards environmental disturbances.

Control approaches used in [FCY04] and [SIF03] are based on the assumption that a complete mathematical model of both vessels is available, and thus autopilots for both vessels can be designed to suppress the effects of external disturbances. However, in a practical leader-follower replenishment operation, the follower may have limited access to information of the control input, model and states of the leader. Therefore, in order to lighten the information requirements on the leader vessel, we propose bellow a virtual vehicle approach where the only information available from the main (leader) ship are position and heading measurements, *cf.* [KP03].

Synchronization can be seen as a type of state cooperation among two or more systems. It was introduced in a control context by [Ble71], and has since received an increasing attention in the control community (*cf. e.g.* [NRA03, FNP00]). Synchronization has been utilized in maritime application by [KP03, SIF03, EP01]. The two latter references expanded on traditional tracking methods with predefined paths, and introduced a synchronization feedback from the actual position of a vessel (subject to disturbances) to the other vessels through a path parametrization variable. All vessels have predefined paths with individual tracking controllers requiring knowledge of model parameters and control inputs for all vessels, and the synchronization is in terms of progression along the path. Based on the results of [NRA03] for synchronization of mechanical systems, [KP03] proposed a leader-follower synchronization observer-controller scheme for underway replenishment. Experimental results on this scheme were presented in [KWPN04] addressing practical tuning issues and performance. No predefined path with known derivatives or model parameter information for the leader vessel is required anymore, and the coordination of the vessels is achieved using a controller that synchronizes the position and velocity of the follower to the leader based on position measurements only, through the design of state observers.

In this section, we propose a virtual vehicle approach to the underway replenishment problem to impose a cascaded structure of the systems, as opposed to the controller-observer approach proposed in [KP03] where the observers and controller are closely interconnected. The virtual vehicle is designed to follow the behaviour of the leader based on position feedback, and provides a velocity output through the controller design. The states of the virtual vehicle can thus be used in a synchronization controller to control the follower to the virtual vehicle.

We have made the additional assumption on the problem of [KP03] in that the velocity of the follower is assumed to be known to focus our treatment on the interplay between the virtual vehicle and the follower. This reasonable assumption enables to extend the stability results from semi-global uniform ultimate boundedness of the closed-loop errors in [KP03] to uniform global practical asymptotic stability.

6.3.1 Preliminaries

Vehicles definitions and reference frames. In the development of the underway replenishment control scheme, several reference frames, intermediate vehicles and dynamic and kinematic models are used. A brief introduction to these concepts is given here. See [Fos02] for a more elaborate discussion.

The control problem studied is as follows: Given the position (x_1, x_2) and heading angle ψ of a leader vessel, we want the follower vessel to follow the leader with its position shifted by a distance d at an angle γ_m relative to the leader. For this purpose we will utilize the concepts of a reference vehicle and a virtual vehicle, and we designate the following

vehicles as illustrated by Figure 6.5.

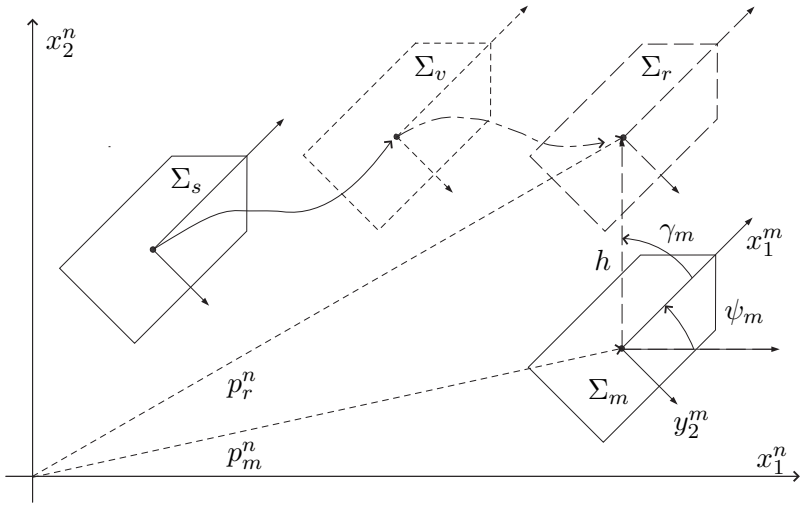


Figure 6.5: Vehicles and coordinate frames

- Σ_m: leader (main) vessel. The position $x_m = (x_1, x_2, \psi)^\top$ is measured.

- Σ_r: reference vehicle shifted a distance h in the direction given by the angle γ_m relative to the position of the leader vehicle.

- Σ_v: virtual vehicle controlled to track the reference vehicle Σ_r through a kinematic model approach.

- Σ_s: follower (supply) vessel synchronizing to the leader vessel. The position x and velocity $\dot{x}$ is available for control design, and the parameters of its model are known.

Note that the only physical vehicles in the control scheme are the leader vessel Σ_m and the follower vessel Σ_s synchronizing to the leader. Through the use of a virtual vehicle as an intermediate control vehicle in the scheme, we can control the physical follower vessel to the leader using the known velocity of the virtual reference. Note also that, although we derive the control scheme for one follower vessel, it can be easily extended to any number of followers providing the introduction of a collision avoidance scheme.

Vessel kinematics and dynamics can be expressed in different reference frames, and we define the two essential reference frames used in this text as (*cf.* [Fos02] for details):

- *NED*: fixed reference frame defined relative to the Earth's reference ellipsoid, where the ${x_1}^n$-axis points toward true North, the ${x_2}^n$-axis toward East, and the ${x_3}^n$-axis points downwards normal to the Earth's surface.

- $BODY_p$: body-fixed moving reference frame where the origin is chosen in the center of gravity of the vehicle p, and the axes $(x_{1,p}^b, x_{2,p}^b, x_{3,p}^b)$ coincide with the principal axes of inertia. Due to vessel symmetry, we can choose the $x_{1,p}^b$-axis along the axis of inertia in the forward direction of the vessel, the $x_{2,p}^b$-axis directed to the right and the $x_{3,p}^b$-axis to complete the right-handed coordinate system pointing downwards.

In the case considered here, the vector of vessel generalized coordinates $x^n = (x_1, x_2, \psi)^\top$ is defined in the *NED* frame, where (x_1, x_2) is the position with respect to the ${x_1}^n$- and ${x_2}^n$-axis, and ψ is the heading angle of the vessel about the ${x_3}^n$-axis. The velocities $\nu_p^b = (u, v, r)^\top$ in the surge, sway and yaw directions are defined in the $BODY_p$ frame of the vehicle p. Superscripts n and b will be dropped from the notation when the reference frame is evident from the context. Subscripts $p \in \{m, r, v, s\}$ on these vectors will indicate their vehicle of origin (**m**ain, **r**eference, **v**irtual, **s**upply).

The marine vessel equations of motions can be written in vectorial form in the *BODY* frame of the vessel as ([Fos02])

$$\dot{x} = J(x)\nu \tag{6.55}$$

$$M_\nu \dot{\nu} + C_\nu(\nu)\,\nu + D_\nu(\nu)\,\nu + g_\nu(x) = \tau_\nu\,, \tag{6.56}$$

where M_ν is a constant positive definite inertia matrix including added mass effects, $C_\nu(\nu)$ is a skew-symmetric matrix of Coriolis and centripetal forces (satisfying $C_\nu(\nu) + C_\nu^\top(\nu) = 0$), $D_\nu(\nu)$ is a non-symmetric damping matrix, and gravitational/buoyancy forces in $g_\nu(x)$ can be ignored for surface vessels. $J(x)$ is a Jacobian-like transformation matrix from the *BODY* frame to the *NED* frame, and in a 3-degrees of freedom surface application where pitch and roll motion are negligible, the matrix $J(x)$ reduces to a simple rotation matrix around the ${x_3}^n$-axis as

$$J(x) = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.57}$$

Inserting the kinematic equation (6.55) and its derivative in the dynamics (6.56) yields the dynamic model in the *NED* frame

$$M(x)\ddot{x} + C(x,\dot{x})\dot{x} + D(x,\dot{x})\dot{x} + g(x) = \tau\,, \tag{6.58}$$

where the inertia matrix $M(x)$ is positive definite but no longer constant[9]. The dynamical model (6.58) in the NED frame satisfies a number of properties similar to those of robotics systems (*cf.* Section 6.1). Notably, the inertia matrix $M(x)$ is differentiable, there exists positive constants M_m and M_M such that

$$M_m \le M(x) \le M_M\,, \qquad \forall x \in \mathbb{R}^3\,,$$

and the matrix $\dot{M}(x) - 2C(x,\dot{x})$ is skew-symmetric. Similarly to [PE95], we will also assume that the dissipation vector $d(x,\dot{x}) := D(x,\dot{x})\dot{x}$ for a marine vessel is continuously differentiable and satisfies for some positive constant D_m

$$\left|\frac{\partial d(x,\dot{x})}{\partial \dot{x}}\right| \ge D_m\,, \qquad \forall\, x, \dot{x} \in \mathbb{R}^3$$

and, for a continuous function $D_M : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$,

$$\left|\frac{\partial d(x,\dot{x})}{\partial \dot{x}}\right| \le D_M(|\dot{x}|)\,.$$

Reference vehicle kinematics. As a first step in order to assure a safe replenishment operation, we design a reference position for the follower vessel at some distance h from

[9] Please refer to [KPCP06] for a precise expression of M, D, C, g and τ.

the leader in the form of a reference vehicle with a kinematic model. Figure 6.6 concerns the general situation of an arbitrary heading assignment, *i.e.* the heading angle of the reference vehicle ψ_r can be different from the heading angle of the leader vessel ψ_m, *cf.* .

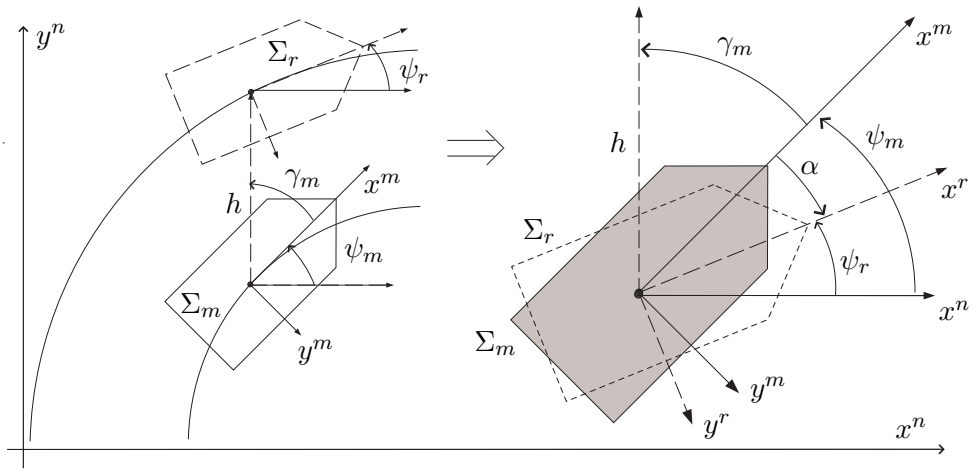


Figure 6.6: Reference vehicle at the distance h and angle γ_m from the leader vessel

The STREAM (Standard Tension Alongside Replenishment Method) is currently the preferred underway replenishment configuration at sea (cf. [MC99]). In the underway replenishment scenario, it is desirable that the reference vehicle is placed at a constant distance h orthogonally off one of the sides of the leader with $\gamma_m = \pm\frac{\pi}{2}$. This configuration corresponds to a replenishment operation where the supply ship moves in parallel with the leader at a fixed distance h and with the same heading angle. In this case, the supply ship is always at a right angle to the replenished ship, and the tension on the replenishment rig is at a minimum. This greatly simplifies the kinematic equations of the reference vehicle. Indeed, the parallel motion suggests that $J(x_r) = J(x_m)$, and the position of the reference vehicle in the *NED* frame becomes

$$x_r = J(x_r)\,\chi_r^r = x_m + J(x_m)\,d_r^m \tag{6.59}$$

Differentiating (6.59) we obtain

$$\dot{x}_r = \dot{x}_m + J(x_m)\,S(r_m)\,d_r^m \tag{6.60}$$

since the vector d_r^m is constant in this particular operation. Taking $\gamma_m = -\frac{\pi}{2}$, we obtain the component form for (6.60) as

$$\begin{aligned} \dot{x}_{1r} &= \dot{x}_{1m} + dr_m \cos\psi_m \\ \dot{x}_{2r} &= \dot{x}_{2m} + dr_m \sin\psi_m \\ \dot{\psi}_r &= r_m \end{aligned} \tag{6.61}$$

Please refer to [KPCP06] and the upcoming PhD thesis of E. Kyrkjebø for details. Defining

$$u_r := u_m + dr_m\,, \qquad v_r := v_m\,, \qquad \text{and} \qquad r_r := r_m\,,$$

it can be seen that the kinematic model of the reference vehicle can be written as

$$\dot{x}_r = J(x_m)\,\nu_r \tag{6.62}$$

where $\nu_r = (u_m + dr_m,\ v_m,\ r_m)^\top$. In this particular case, only the reference forward velocity is changed ($u_r = u_m + dr_m$) with respect to that of the leader vessel. Note that this is necessary for the follower vessel to maintain its position parallel to the leader vessel during turns due to the difference in turn radius.

6.3.2 Virtual vehicle design

The only measurement available from the leader vessel is the position/heading measurement x_m, and since we have no information on the parameters of its mathematical model or of the control input to the leader vessel, we cannot design a model-based observer for the leader states. An alternative approach is to use the time filtered derivatives from the position measurements at the expense of robustness under noisy conditions. In order to reduce noise sensitivity, we propose instead to design a virtual vehicle as an intermediate controlled vessel Σ_v stabilizing to the reference vehicle Σ_r based on position measurement feedback only.

As in [GMP+98], on the first step (kinematic level) we consider the velocities ν_v of the virtual vehicle as control inputs, and design them in such a way that we ensure convergence of the virtual trajectories to the reference trajectories. In a way, we can consider the trajectories x_v and velocities ν_v as estimates of x_r and ν_r, that is, the virtual vehicle is a form of kinematic estimator of the leader states through the position feedback loop.

The virtual vehicle is defined by its kinematic model

$$\dot{x}_v = J(x_v)\,\nu_v\,. \tag{6.63}$$

Based on practical considerations, we assume that the velocity and acceleration of the leader vessel are bounded, and thus the velocity and acceleration of the reference vehicle satisfy

$$\sup_{t\in\mathbb{R}_{\geq 0}} |\nu_r(t)| \leq V_M\,, \tag{6.64}$$

where V_M denotes a positive constant. In view of (6.62), the virtual vehicle tracking errors $e_v := x_v - x_r$ satisfy

$$\dot{e}_v = J(x_v)\,\nu_v - J(x_m)\,\nu_r\,.$$

We propose the following control law for the virtual vehicle

$$\nu_v \;=\; -J(x_v)^{-1}L_1 e_v - J(x_v)^{-1}L_2 z \tag{6.65}$$

where L_1 and L_2 are symmetric positive gain matrices, and

$$\dot{z} \;:=\; e_v\,. \tag{6.66}$$

The closed-loop equations can be written in the following form

$$\dot{e}_v = -L_1 e_v - L_2 z - J(x_m)\,\nu_r\,. \tag{6.67}$$

Consider the following Lyapunov function candidate

$$V_v(z,\ e_v) = \frac{1}{2}\left(e_v^\top e_v + z^\top L_2 z + z^\top e_v\right)\,. \tag{6.68}$$

Differentiating along the closed-loop trajectories we get

$$\dot{V}_v(z, e_v) = -e_v^\top\left(L_1 - \frac{1}{2}I\right)e_v - \frac{1}{2}z^\top L_2 z - \frac{1}{2}z^\top L_1 e_v - \left(e_v^\top + \frac{1}{2}z^\top\right)J(x_m)\,\nu_r\,. \tag{6.69}$$

Using (6.64) and the relation $2|ab| \leq (\lambda a^2 + b^2/\lambda)$ for any real a, b and any positive λ, it follows that

$$\dot{V}_v \leq -\frac{1}{4}\left(2L_{2,m} - \frac{1}{\lambda}L_{1,M} - \frac{3V_M}{|(e,z)|}\right)|z|^2 - \frac{1}{4}\left(4L_{1,m} - 2 - \lambda L_{1,M} - \frac{6V_M}{|(e_v,z)|}\right)|e_v|^2 \,, \tag{6.70}$$

where λ designates any positive constant and $L_{i,m}$ (resp. $L_{i,M}$) designates the minimum (resp. maximum) eigenvalue of L_i, $i \in \{1,2\}$. Similarly to the previous sections, we design the gain matrices L_1 and L_2 in such a way that $L_{i,M} \leq \ell\ L_{i,m}$ for some $\ell > 0$. Then, letting $\lambda = 2/\ell$ and δ_v be any given positive constant, we can see in view of (6.64) that any gain matrices satisfying

$$L_{1,m} = 3 + \frac{3V_M}{\delta_v} \tag{6.71}$$

$$L_{2,m} = 2 + \frac{3\ell^2}{4} + \left(1 + \frac{\ell^2}{4}\right)\frac{3V_M}{2\delta_v} \tag{6.72}$$

generate the following bound of the derivative of V_v:

$$|(e_v,z)| \geq \delta_v \quad \Rightarrow \quad \dot{V}_v(z,e_v) \leq -|e_v|^2 - |z|^2 \,. \tag{6.73}$$

Note that V_v is positive definite and radially unbounded for this choice of gains. More precisely, we have

$$\frac{1}{4}|e_v|^2 + \frac{1}{8}\left(6 + 3\ell^2 + (4+\ell^2)\frac{3V_M}{2\delta_v}\right)|z|^2 \leq V_v \leq \frac{3}{4}|e_v|^2 + \frac{1}{8}\left(10 + 3\ell^2 + (4+\ell^2)\frac{3V_M}{2\delta_v}\right)|z|^2 \,.$$

Due to the linear dependency of $L_{1,m}$ and $L_{2,m}$ in $1/\delta_v$, we conclude with Corollary 2.9 that (6.66)-(6.67) is uniformly globally practically asymptotically stable with $L_{1,m}$ and $L_{2,m}$ as tuning parameters.

6.3.3 Follower vehicle design

Using the velocity information from the virtual vehicle design, we can design a synchronization controller for the follower vessel Σ_s to follow the virtual vehicle Σ_v. Note that the body-fixed velocity ν_v is now known through (6.65), and, with the kinematic relationship of (6.63), we can obtain the velocity $\dot{x}_v$ of the virtual vehicle in the *NED* frame. Furthermore, due to our design of the virtual velocity controller, we can also obtain an expression for the acceleration of the virtual vehicle which will be partly available for control purposes. More precisely, we get from (6.65) and (6.67) that

$$\begin{aligned} \dot{x}_v &= J(x_v)\,\nu_v = -L_1 e_v - L_2 z \\ \ddot{x}_v &= -L_1\dot{e}_v - L_2 e_v = \left(L_1^2 - L_2\right)e_v + L_1 L_2 z + L_1 J(x_m)\,\nu_r \,. \end{aligned} \tag{6.74}$$

In our synchronization approach, we will assume that the velocity of the follower vessel is known. Define the synchronization errors as

$$e = x - x_v, \qquad \dot{e} = \dot{x} - \dot{x}_v, \qquad \ddot{e} = \ddot{x} - \ddot{x}_v \,.$$

Using the sliding surface from [SL87] as a passive filtering of the virtual vehicle states, we can design a virtual reference trajectory as

$$\begin{aligned} \dot{y}_v &= \dot{x}_v - \Lambda e \\ \ddot{y}_v &= \ddot{x}_v - \Lambda\dot{e} \,, \end{aligned}$$

where $\Lambda > 0$. Let us denote

$$\ddot{y}_v' = \left(L_1^2 - L_2\right) e_v + L_1 L_2 z - \Lambda \dot{e}$$

in which case, in view of (6.74),

$$\ddot{y}_v = \ddot{y}_v' + L_1 J\left(x_m\right) \nu_r \,.$$

Notice that, although $\ddot{y}_v$ is not accessible since ν_r is not measured, $\ddot{y}_v'$ is available for control design. Defining

$$s := \dot{x} - \dot{y}_v = \dot{e} + \Lambda e$$

as a measure of tracking, we can rewrite (6.58) as

$$M(x)\dot{s} = -C(x,\dot{x})s - D(x,\dot{x})s + \tau - M(x)\ddot{y}_v - C(x,\dot{x})\dot{y}_v - D(x,\dot{x})\dot{y}_v - g(x)\,. \tag{6.75}$$

We propose the following control law

$$\tau = M(x)\ddot{y}_v' + C(x,\dot{x})\dot{y}_v + D(x,\dot{x})\dot{y}_v + g(x) - K_d s - K_p e\,, \tag{6.76}$$

where K_p and K_d are symmetric positive gain matrices. Consider the following Lyapunov function candidate

$$V_e\left(e,\ s\right) = \frac{1}{2} s^\top M(x) s + \frac{1}{2} e^\top K_p e\,. \tag{6.77}$$

Differentiating along the closed-loop trajectories we get

$$\dot{V}_e\left(e,\ s\right) = -s^\top \left[D(x,\dot{x}) + K_d\right] s - e^\top \Lambda^\top K_p e - s^\top M(x) L_1 J\left(x_m\right) \nu_r \,.$$

Let δ_e be any given positive constant. Then, from (6.64), it holds that, for all $|(e,s)| \geq \delta_e$,

$$\dot{V}_e\left(e,\ s\right) \leq -\left(D_m + K_{d,m} - \frac{1}{\delta_e} M_M L_{1,M} V_M\right) |s|^2 - \left(\Lambda_m K_{p,m} - \frac{1}{\delta_e} M_M L_{1,M} V_M\right) |e|^2 \,. \tag{6.78}$$

Proceeding as for the virtual vehicle, we can invoke Corollary 2.9 by observing that the choice of $K_{d,m}$ and $K_{p,m}$ can be made as an affine function of $1/\delta_e$, and conclude uniform global practical asymptotic stability.

6.3.4 Stability analysis of the overall system

The control law of the follower synchronizes the follower vessel to the virtual vehicle based on a computed virtual reference velocity from the virtual vehicle controller, and the virtual vehicle is in turn stabilized to the reference vehicle parallel to the leader vessel.

Theorem 6.12 *Consider the vessel model (6.58) satisfying Properties 1-3, the virtual vehicle control law (6.65) and the synchronization controller (6.76). Under assumptions (6.64), the overall closed-loop system is uniformly globally practically asymptotically stable.*

Proof of Theorem 6.12. Take as a positive definite Lyapunov function candidate the following composition of the Lyapunov functions (6.68) and (6.77).

$$V\left(\widetilde{\eta}\right) = \frac{1}{2} \widetilde{\eta}^\top P\, \widetilde{\eta}\,,$$

where $\widetilde{\eta} = \left(e^\top, s^\top, z^\top, e_v^\top\right)^\top$ and

$$P = \begin{pmatrix} K_p & 0 & 0 & 0 \\ 0 & M(x) & 0 & 0 \\ 0 & 0 & L_2 & \frac{1}{2}I \\ 0 & 0 & \frac{1}{2}I & \frac{1}{2}I \end{pmatrix}.$$

Differentiating along trajectories yields

$$\dot{V}(\widetilde{\eta}) = -\widetilde{\eta}^\top Q\, \widetilde{\eta} + \sigma(s,\ e_v,\ z,\ \nu_r) \tag{6.79}$$

where

$$Q = \begin{pmatrix} \Lambda^\top K_p & 0 & 0 & 0 \\ 0 & D(x,\dot{x}) + K_d & 0 & 0 \\ 0 & 0 & \frac{1}{2}L_2 & \frac{1}{4}L_1 \\ 0 & 0 & \frac{1}{4}L_1 & L_1 - \frac{1}{2}I \end{pmatrix}$$

and

$$\sigma(s,\ e_v,\ z,\ \nu_r) := -s^\top M(x) L_1 J(x_m)\,\nu_r - \frac{1}{2} z^\top J(x_m)\,\nu_r - e_v^\top J(x_m)\,\nu_r\,.$$

Let δ be any given positive constant. Then we have the following property:

$$|\widetilde{\eta}| \geq \delta \quad \Rightarrow \quad |\sigma(s,\ e_v,\ z,\ \nu_r)| \leq \frac{V_M}{\delta}\left(M_M L_{1,M}\,|s|^2 + \frac{|z|^2}{2} + |e_v|^2\right).$$

Consequently, in view of (6.70) and (6.78), and repeating a similar reasoning while choosing the minimum eigenvalue of the gain matrices K_p, K_d, L_1 and L_2 large enough, it holds that

$$|\widetilde{\eta}| \geq \delta \quad \Rightarrow \quad \dot{V}(\widetilde{\eta}) \leq -|\widetilde{\eta}|^2\,.$$

Since the dependency on the bound on σ (and so on the gain matrices) in $1/\delta$ is again affine, uniform global practical asymptotic stability follows from Corollary 2.9. ■

6.3.5 Simulation study

The underway replenishment scheme presented above is tested in a simulation environment in MATLAB using the surface ship model of Cybership II from [SSF04]. In the simulations, the distance between the ships is $h = 2$ m with $\gamma_m = -\pi/2$, and the model matrices in the body frame were

$$\mathbf{M} = \begin{pmatrix} 25.8 & 0 & 0 \\ 0 & 33.8 & 1.0115 \\ 0 & 1.0115 & 2.76 \end{pmatrix}$$

$$\mathbf{C}(\nu) = \begin{pmatrix} 0 & 0 & -33.8v - 1.0115r \\ 0 & 0 & 25.8u \\ 33.8v + 1.0115r & -25.8u & 0 \end{pmatrix}$$

$$\mathbf{D}(\nu) = \begin{pmatrix} 0.72{+}1.33|u|{+}5.87u^2 & 0 & 0 \\ 0 & 0.8896{+}36.5|v|{+}0.805|r| & 7.25{+}0.845|v|{+}3.45|r| \\ 0 & 0.0313{+}3.96|v|{-}0.130|r| & 1.90{-}0.080|v|{+}0.75|r| \end{pmatrix},$$

where $\nu = (u, v, r)^\top$ are the body fixed velocities in surge, sway and yaw, respectively. Controller gains were chosen as $K_p = \mathrm{diag}(70,\ 140,\ 70)$, $K_d = \mathrm{diag}(100,\ 100,\ 50)$,

$L_1 = \mathrm{diag}(0.8,\ 1.6,\ 1.6)$, $L_2 = \mathrm{diag}(0.55,\ 0.55,\ 0.55)$, and $\Lambda = \mathrm{diag}(0.3,\ 0.3,\ 0.3)$. In the simulations, the leader ship tracks reference trajectory $t \mapsto \sin(\omega t)$ with frequency $\omega = 1/15\mathrm{rad}.s^{-1}$ with heading angle ψ_m along the tangent line. Initial states were chosen as $x(0) = (0,\ 0,\ 0^\top$ for the follower, $x_v(0) = (1,\ 0.5,\ \frac{\pi}{4})\top$ for the virtual vehicle and as $x_m(0) = (2,\ 4,\ 0)^\top$ for the leader ship to illustrate stability in all degrees of freedom as illustrated in the upper plot of Figure 6.7.

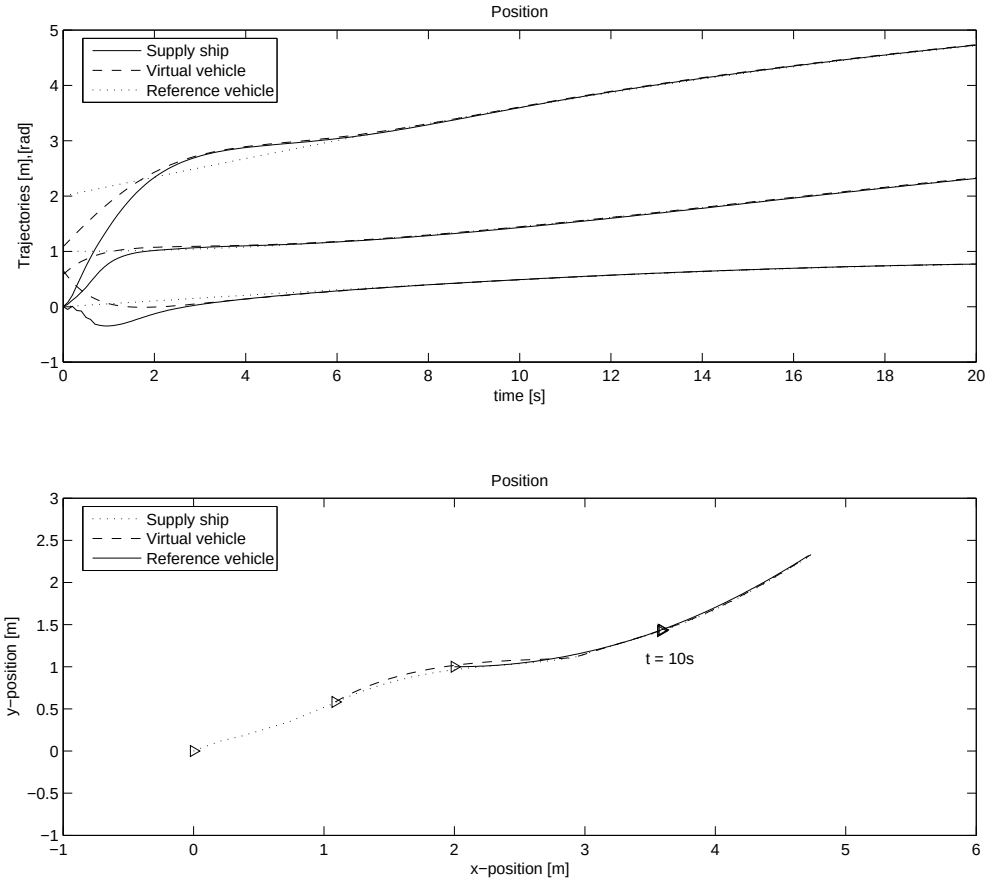


Figure 6.7: Trajectories of the follower x, the virtual vehicle x_v and the reference vehicle x_r in the upper plot, and the planar plot of the vehicles with special marks at initial states and at time $t = 10$s in the lower.

From Figure 6.8 we see that the virtual vehicle control errors $e_\nu = x_v - x_r$, the synchronization errors $e = x - x_v$ and the overall control errors $x - x_r$ confirm our theoretical expectations. We observe small remaining oscillations, especially in the velocity errors, due to the unknown velocity of the leader ship. However, due to the practical stability property of the closed-loop system, the magnitude of these oscillations can be arbitrarily reduced (within control saturation limits) by enlarging the gains.

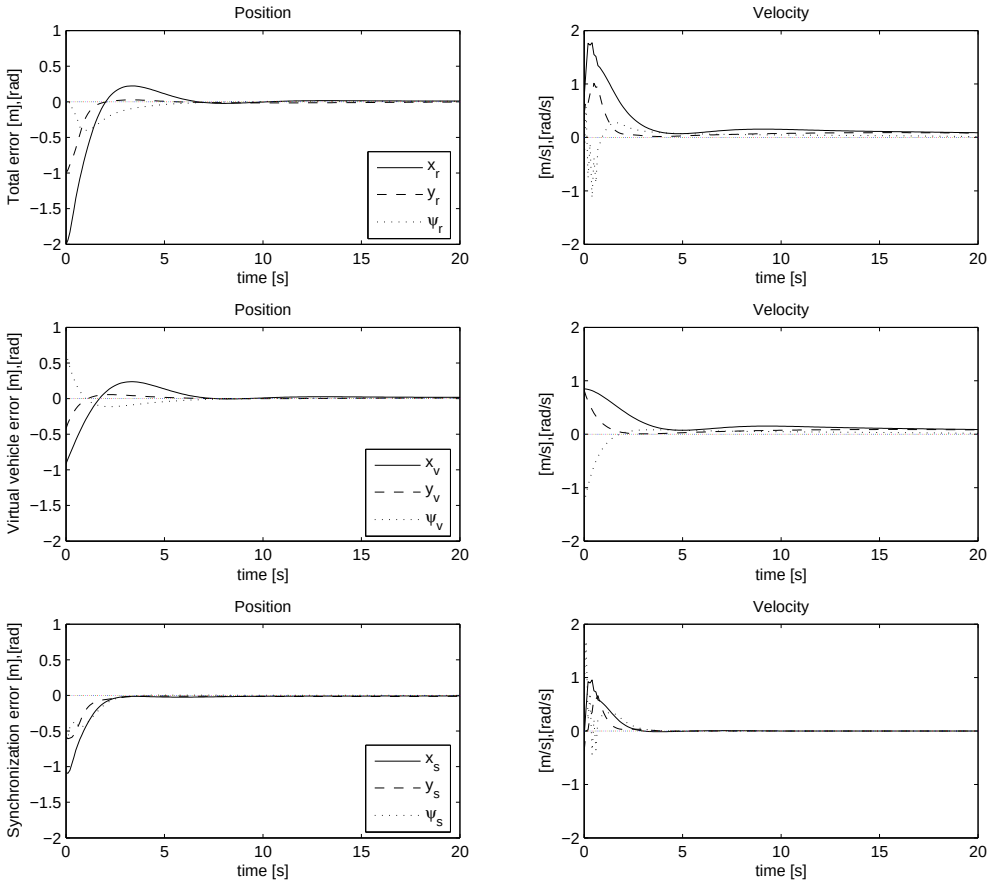


Figure 6.8: The total errors $x - x_r$ in the upper row, the virtual vehicle control errors $x_v - x_r$ in the middle row, and the synchronization errors $x - x_v$ in the lower row, with positions on the left and velocities on the right.

Conclusion. This chapter presents three concrete applications of the results introduced along the document. These applications concern mechanical systems. The first one studies the robustness of PID-controlled robot manipulators to a wide class of perturbations and when taking into account the dynamics of the actuators. USPAS is obtained, meaning that a convenient choice of the PID gains allows an arbitrarily large domain of attraction with an arbitrarily tight precision. The second application aims at controlling the formation of spacecrafts in case of uncertainties in the orbital parameters. The last one deals with surface vessels. The goal is to synchronize two ships in a parallel motion using as little information of the leader vehicle as possible.

Conclusion and further research

This document presents new tools for stability and robustness analysis of nonlinear dynamical systems. Let us summarize our main results.

Sufficient conditions for semiglobal and practical stability. We first provide a rigorous Lyapunov framework for uniform semiglobal and/or practical asymptotic stability. "Semiglobal" refers to the situation when the domain of attraction is not the whole state-space but a set that may be arbitrarily enlarged by a convenient tuning of some parameters. "Practical" concerns the case when an arbitrarily small compact neighborhood of the origin (instead of the origin itself) is asymptotically stable. On some occasions, semiglobal practical stability is an inherent property of the system or results from the control design: global asymptotic stability is impeded by external disturbances, model uncertainty, *etc.* On other situations, it constitutes the best one can prove when no strict Lyapunov function is available, or when using alternative techniques such as averaging. The generality of semiglobal and/or practical stability is further reinforced by a specific feature offered by the stability properties under consideration. Namely, as opposed to many related concepts, they allow the estimate of solutions to depend on the tuning parameter and so, potentially, on the radius of the desired domain of attraction and the amplitude of the tolerated steady-state error. As a counterpart, a more involved attention needs to be paid in the stability analysis to guarantee that the Lyapunov function be sufficiently "representative" of the state-norm. More precisely, compared to classical results for global asymptotic stability, an additional requirement on the $\mathcal{K}_\infty$ bounds on the Lyapunov function is imposed. We illustrate the importance of this condition by showing that, when the latter is violated, even boundedness of solutions is not guaranteed in general. All these results are illustrated by elementary examples.

Necessary condition for USPAS. Semiglobal and/or practical stability can be seen as a measurement of robustness in the sense that it often stems from a degradation of global and/or asymptotic stability due to external disturbances, model imperfection, *etc.* while still guaranteeing interesting performances to the system under consideration. But uniform semiglobal practical asymptotic stability (USPAS) may be inferred by other means than the knowledge of a convenient Lyapunov function as, for instance, via averaging techniques. For this reason, we derive a converse Lyapunov result for the class of USPAS systems whose solutions' estimate is independent of the radius of the attractive ball. This necessary Lyapunov condition is especially designed to fit the context of cascaded systems as it also guarantees a time-invariant bound on the gradient of the Lyapunov function.

Cascades of systems are often encountered, and are at the basis of many control strategies. For this reason, a large part of this report is devoted to the analysis of cascaded

systems.

USPAS, USAS, UGPAS for cascades. With the proposed Lyapunov framework for semiglobal and/or practical asymptotic stability, some tools are presented that ensure the preservation of a given stability property (USPAS, USAS or UGPAS) by cascade interconnection. In general terms, similarly to existing results for global asymptotic stability, it is required that the solutions of the overall cascade be bounded and that a convenient Lyapunov function be explicitly known for the driven subsystem. In view of the converse result evoked above, we may relax this latter requirement in the semiglobal case for a wide class of systems. This is particularly useful when invoking averaging techniques, as illustrated by the output feedback control of the double integrator affected by a persistently exciting signal. Furthermore, in the case of uniform global asymptotic stability, the boundedness assumption on the solutions of the cascade is replaced by growth restriction on the interconnection term. This makes this tool particularly easy to apply in specific control problems. We illustrate its use by quantifying the effect of smoothing a sign function in disturbance rejection.

Set-stability for cascades. In the case when, due to actuators limitation or structural constraints, semiglobal and/or practical asymptotic stability of perturbed or uncertain systems is not achievable, one may be interested in studying the stability of fixed sets. In addition, the generality of the set-stability concept makes it an interesting tool. Among the particular cases it encompasses, let us cite partial stability which has proved useful in many control applications. We show that, if some (non necessarily compact) sets are globally asymptotically stable (GAS) for two subsystems taken separately, then their cross product is GAS for the corresponding cascade provided that its solutions are globally bounded. Again, we show that, on some occasions, this requirement can be replaced by a simple growth-order condition on the interconnection term (plus forward completeness). Applying this result, we provide a concise proof for a recently established result of formation control of surface vessels along a straight path and with a prescribed velocity.

iISS for cascades. We further provide a stability analysis for cascaded systems with inputs. While the concept of input to state stability is known to be preserved by the cascade interconnection, this is not the case for the more general notion of integral input to state stability (iISS). Additional requirements need to be imposed for the cascade composed of two iISS systems to be iISS. These conditions are firstly expressed in terms of Lyapunov functions and then in terms of estimates of the solutions of each subsystem taken separately. The application of these new results is illustrated through academic examples.

Applications. Finally, we apply most of the presented results to specific control applications. This constitutes the purpose of the last chapter. We analyze the robustness of PID-controlled manipulators to friction, model uncertainty, actuators' dynamics and other disturbances. Another application concerns the formation control of spacecrafts. We establish global practical asymptotic stability of the corresponding system in the case that only bounds on the leader's anomaly are available. Finally, we show that a similar stability property can be obtained for the synchronization of two surface vessels with little information on the leader vehicle.

Thus, we believe that our theoretical contributions constitute useful tools for robustness

and stability analysis as well as for control design in specific concrete applications.

Future works and open problems

As far as theory is concerned, the following problems constitute directions in which further research may be carried out.

Cascades control. It would be interesting to derive constructive control strategies based on the cascades results presented in this document. For instance, in the case that the classical backstepping approach does not apply due to structural constraints or presence of disturbances, a semiglobally and/or practically asymptotically stabilizing control may remain achievable.

Partial USPAS. Some specific perturbed or imprecise control systems present the property that part of the state can be made semiglobally and/or practically asymptotically stable, while the behavior of the rest of the variables cannot be properly constrained. For instance, this behavior could be expected when taking into account external disturbances, such as current, wind and waves, in the example of Section 4.3. Such a feature correspond to some kind of "partial semiglobal practical asymptotic stability". To the best of our knowledge, no general work has been done in this direction. We believe that mixing the results presented for semiglobal practical stability (Chapters 2 and 3) and (non-compact) set-stability (Chapter 4) can provide interesting results for this question.

Concerning applications of the presented results, the following problems will soon be addressed.

Surface vessels formation. The main illustrative example of Chapter 4 consists in a proof of a recently developed control for the formation control of underactuated surface vessels along a straight path. So far, this does not take into account disturbances due to wind, current and waves and requires a full knowledge of the model, position and velocity of each ship. Interesting future extensions of this result will aim at relaxing these requirements by allowing more complex prescribed paths and by taking into account these external noises and possible model uncertainty. A collision avoidance scheme may also be implemented.

PD control of ships. Based on our results for semiglobal practical asymptotic stability, we plan to relax some of the requirements in [LFP00] for the PD control of ships, when taking into account external disturbances such as wind, wave and currents.

Appendix A

Proof of auxiliary results

A.1 Proof of of Lemma 2.7

Let $x_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}_{\geq 0}$ be any initial conditions such that $\phi(t, t_0, x_0) \in X$ for all $t \geq t_0$. For simplicity, we write $\phi(\cdot, t_0, x_0)$ as $\phi(\cdot)$ and we define $v(\cdot) := V(\cdot, \phi(\cdot))$. We distinguish two cases: whether the trajectories start from outside or inside $\mathcal{B}_\delta$.

Case 1: $|x_0| > \delta$.

In this case, there exists[1] $T_0 \in (0; \infty]$ such that $|\phi(t)| > \delta$ for all $[t_0; t_0 + T_0)$ and $|\phi(t_0 + T_0)| = \delta$. Hence, using the comparison lemma, we get that

$$v(t) \leq \left(v(t_0) - \frac{c(|x_0|)}{k} \right) e^{-k(t-t_0)} + \frac{c(|x_0|)}{|k|}, \qquad \forall t \in [t_0; t_0 + T_0) .$$

Using the bounds on V, it follows that

$$|\phi(t)| \leq \underline{\alpha}^{-1} \left(\overline{\alpha}(|x_0|) e^{-k(t-t_0)} + \frac{c(|x_0|)}{|k|} \right), \qquad \forall t \in [t_0; t_0 + T_0) .$$

In addition, for each $t \geq t_0 + T_0$, either $|\phi(t)| \leq \delta$ in which case[2]

$$|\phi(t)| \leq \underline{\alpha}^{-1} \left(\overline{\alpha}(\delta) + \frac{c(|x_0|)}{|k|} \right),$$

or $|\phi(t)| > \delta$. In this second case, we can again invoke the continuity of the solution to see that there exists a nonempty time-interval $[\tau; \tau + T]$, with $T \in (0; \infty]$, containing t and such that $|\phi(s)| > \delta$ for all $s \in (\tau; \tau + T]$, with $|\phi(\tau)| = \delta$. Hence, integrating from τ to $t \in [\tau; \tau + T]$, we obtain in the same way as before that, whenever $|\phi(t)| > \delta$, it holds that

$$|\phi(t)| \leq \underline{\alpha}^{-1} \left(\overline{\alpha}(\delta) e^{-k(t-\tau)} + \frac{c(|x_0|)}{|k|} \right) \leq \underline{\alpha}^{-1} \left(\overline{\alpha}(|x_0|) e^{-k(t-t_0)} + \frac{c(|x_0|)}{|k|} \right) . \tag{A.1}$$

To sum up, for all $t \geq t_0$, we have the following:

$$|x_0| > \delta \quad \Rightarrow \quad |\phi(t)| \leq \underline{\alpha}^{-1} \left(\overline{\alpha}(|x_0|) e^{-k(t-t_0)} + \frac{c(|x_0|)}{|k|} \right) . \tag{A.2}$$

[1] If $|\phi(t)| > \delta$ forever after, we consider that $T_0 = \infty$.

[2] This is direct by noticing that $\underline{\alpha}(s) \leq \overline{\alpha}(s)$ for all $s \in \mathbb{R}_{\geq 0}$ and that $c(|x_0|)/k \geq 0$.

Case 2: $|x_0| \leq \delta$.
In this case, as long as $|\phi(t)| \leq \delta$, we trivially[2] have that

$$|\phi(t)| \leq \underline{\alpha}^{-1}\left(\overline{\alpha}(\delta) + \frac{c(|x_0|)}{|k|}\right).$$

If $|\phi(t)| > \delta$ at some instant $t > t_0$, then, again, there exists a nonempty time-interval $[\tau; \tau + T]$, with $T \in (0; \infty]$ and $\tau > t_0$, containing t and such that $|\phi(s)| > \delta$ for all $s \in (\tau; \tau + T]$, with $|\phi(\tau)| = \delta$. Thus, from (A.1), we obtain that

$$|\phi(t)| \leq \underline{\alpha}^{-1}\left(\overline{\alpha}(\delta)e^{-k(t-\tau)} + \frac{c(|x_0|)}{|k|}\right) \leq \underline{\alpha}^{-1}\left(\overline{\alpha}(\delta) + \frac{c(|x_0|)}{|k|}\right).$$

Hence, for all $t \geq t_0$,

$$|x_0| \leq \delta \quad \Rightarrow \quad |\phi(t)| \leq \underline{\alpha}^{-1}\left(\overline{\alpha}(\delta) + \frac{c(|x_0|)}{|k|}\right). \tag{A.3}$$

The conclusion follows from (A.2) and (A.3).

A.2 Proof of Proposition 1.16

We start by recalling the following definition from [TPA99].

Definition A.1 ($(\Delta \to \delta)-$stability) *Given $\Delta > \delta \geq 0$, the origin of the system $\dot{x} = f(t, x)$ is said to be $(\Delta \to \delta)-$stable if*

1. *for each $\epsilon > \delta$, there exists $\eta(\epsilon) > 0$ such that, for all $t_0 \in \mathbb{R}_{\geq 0}$,*

$$|x_0| \leq \eta(\varepsilon) \quad \Rightarrow \quad |\phi(t, t_0, x_0)| \leq \epsilon\,, \qquad \forall t \geq t_0\,. \tag{A.4}$$

2. *for each $r \in (0; \Delta)$, there exists $\nu(r) > 0$ such that*

$$|x_0| \leq r \quad \Rightarrow \quad |\phi(t, t_0, x_0)| \leq \nu(r)\,, \qquad \forall t \geq t_0\,. \tag{A.5}$$

3. *for each $r \in (0; \Delta)$ and each $\epsilon > \delta$, there exists a finite $T(r, \epsilon) > 0$ such that, for all $t_0 \in \mathbb{R}_{\geq 0}$,*

$$|x_0| \leq r \quad \Rightarrow \quad |\phi(t, t_0, x_0)| \leq \epsilon\,, \qquad \forall t \geq t_0 + T(r, \epsilon)\,. \tag{A.6}$$

Assume that the ball $\mathcal{B}_\delta$ is UAS on $\mathcal{B}_\Delta$. Then, in view of Proposition 1.14, there exists $\beta \in \mathcal{KL}$ such that, for all $|x_0| \leq \Delta$, and all $t \geq t_0 \geq 0$,

$$\begin{aligned} |\phi(t, t_0, x_0)|_\delta &\leq \beta(|x_0|\,, 0) =: \alpha(|x_0|) \\ |\phi(t, t_0, x_0)|_\delta &\leq \beta(\Delta, t - t_0) =: \sigma(t - t_0)\,. \end{aligned}$$

Note that the so-defined functions α and σ are of class $\mathcal{K}$ and $\mathcal{L}$ respectively. Given any $\epsilon > \delta$, define $\eta(\epsilon) := \alpha^{-1}(\epsilon - \delta)$. Then we have that

$$|x_0| \leq \eta \quad \Rightarrow \quad |\phi(t, t_0, x_0)| \leq \epsilon\,, \quad \forall t \geq t_0\,,$$

which establishes A.4. Given any positive $r < \Delta$, we have that

$$|x_0| \leq r \quad \Rightarrow \quad |\phi(t, t_0, x_0)| \leq \delta + \alpha(r) =: \nu\,, \quad \forall t \geq t_0\,,$$

which establishes A.5. Let $0 < r < \Delta$ and $\epsilon > \delta$. Then

$$|x_0| \leq r \quad \Rightarrow \quad |\phi(t, t_0, x_0)|_\delta \leq \sigma(t - t_0)\,, \quad \forall t \geq t_0\,.$$

σ is bijective from $[0; \infty[$ to $]0; \sigma(0)]$. If $\epsilon - \delta \leq \sigma(0)$, let $T := \sigma^{-1}(\epsilon - \delta)$. Then we have that

$$|x_0| \leq r \quad \Rightarrow \quad |\phi(t, t_0, x_0)| \leq \epsilon\,, \quad \forall t \geq t_0 + T\,.$$

On the other hand if $\epsilon - \delta > \sigma(0)$, then

$$|x_0| \leq r \quad \Rightarrow \quad |\phi(t, t_0, x_0)|_\delta \leq \sigma(t - t_0) \leq \sigma(0) \leq \epsilon - \delta\,, \quad \forall t \geq t_0\,,$$

which establishes A.6.

Assume now that the system is $(\Delta \to \delta)-$stable. We now follow the prooflines of [Kha96, Lemma 3.3]. First notice than (A.4) and (A.6) can be written

$$\forall \varepsilon > 0,\, \exists \eta(\varepsilon) > 0 : \qquad |x_0| \leq \eta \quad \Rightarrow \quad |\phi(t, t_0, x_0)|_\delta \leq \varepsilon\,, \quad \forall t \geq t_0\,. \tag{A.7}$$

$$\forall 0 < r < \Delta,\, \forall \varepsilon > 0,\, \exists T(r, \varepsilon) > 0 : \;\; |x_0| \leq r \;\; \Rightarrow \;\; |\phi(t, t_0, x_0)|_\delta \leq \varepsilon\,, \;\; \forall t \geq t_0 + T\,. \tag{A.8}$$

Given $\varepsilon > 0$, let $\overline{\eta}(\varepsilon)$ be the supremum of all the η's such that (A.7) holds. Then $\overline{\eta}(\varepsilon)$ is positive nondecreasing, but not necessarily continuous. Let $\zeta \in \mathcal{K}$ be such that

$$\zeta(s) \leq \overline{\eta}(s) \quad \text{and} \quad \lim_{s \to \infty} \zeta(s) = c := \lim_{s \to \infty} \overline{\eta}(s)\,.$$

Let Δ' be any positive constant smaller than Δ. First observe that $\Delta' \leq c$. Indeed, letting $r = \Delta'$ in (A.5), one knows that there exists a positive ν such that, for any $|x_0| \leq \Delta'$, we have that $|\phi(t, t_0, x_0)|_\delta \leq |\phi(t, t_0, x_0)| \leq \nu$ for all $t \geq t_0$. Hence, for any $\varepsilon \geq \nu$, it holds that $|\phi(t, t_0, x_0)|_\delta \leq \varepsilon$. In other words, for ε large enough (i.e. greater that ν), $|x_0| \leq \Delta'$ implies that $|\phi(t, t_0, x_0)|_\delta \leq \varepsilon$ for all $t \geq t_0$. The choice $\eta = \Delta'$ is therefore convenient for (A.7) if ε is large enough. Since, by its definition, $\bar{\eta}(\varepsilon) \geq \eta(\varepsilon)$, we necessarily have that, as claimed, $\lim_{\varepsilon \to \infty} \bar{\eta}(\varepsilon) \geq \Delta'$.

Now, let $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be defined as

$$\alpha(s) := \begin{cases} \zeta^{-1}(s) & \text{if} \quad s \leq \Delta' \\ \zeta^{-1}(\Delta') \frac{s^2}{\Delta'^2} & \text{if} \quad s > \Delta' \end{cases}$$

Note that, in view the previous observation, ζ^{-1} makes sense on the interval $[0; \Delta']$. Furthermore, we can see that α is a class $\mathcal{K}_\infty$ function. For all $|x_0| \leq \Delta'$, let $\varepsilon = \alpha(|x_0|)$. Then, we have that

$$|x_0| = \alpha^{-1}(\varepsilon) = \zeta(\varepsilon) \leq \overline{\eta}(\varepsilon)\,.$$

Thus, by the definition of $\overline{\eta}$, we have that, for all $|x_0| \leq \Delta'$,

$$|\phi(t, t_0, x_0)|_\delta \leq \varepsilon = \alpha(|x_0|)\,, \quad \forall t \geq t_0\,.$$

There is only uniform attractivity left to prove. To that end, given $\varepsilon > 0$, let $\underline{T}(\Delta', \varepsilon)$ be the infimum of all the T's such that (A.8) holds with $r = \Delta'$. We then have that

$$|x_0| \leq \Delta' \qquad \Rightarrow \qquad |\phi(t, t_0, x_0)| \leq \varepsilon + \delta\,, \quad \forall t \geq t_0 + \underline{T}\,, \tag{A.9}$$

and

$$\sup_{t_0 \leq t < t_0 + \underline{T}} |\phi(t, t_0, x_0)| > \varepsilon + \delta\,.$$

The function $\underline{T}(\Delta', \cdot)$ is nonnegative nondecreasing and satisfies $\underline{T}(\Delta', \varepsilon) = 0$ for all $\varepsilon \geq \alpha(\Delta')$. Define now

$$W(\varepsilon) := \frac{2}{\varepsilon} \int_{\varepsilon/2}^{\varepsilon} \underline{T}(\Delta', s) ds + \frac{\Delta'}{\varepsilon} .$$

Then $W :]0; \infty[\to]0; \infty[$ is positive, continuous, decreasing and tends to zero as its argument tends to infinity. Let $\tilde{\sigma}$ denote its invert. Then $\tilde{\sigma}$ satisfies the same properties. Notice that, for all $\varepsilon > 0$, it holds that $\underline{T}(\Delta', \varepsilon) + \Delta'/\varepsilon \leq W(\varepsilon)$. From this, we have that for all $t \geq t_0$,

$$\underline{T}(\Delta', \tilde{\sigma}(t - t_0)) < W(\tilde{\sigma}(t - t_0)) = t - t_0 .$$

Thus, by letting $\varepsilon = \tilde{\sigma}(t - t_0)$ in (A.9), we conclude that, for all $t \geq t_0$, and all $|x_0| \leq \Delta'$,

$$|\phi(t, t_0, x_0)|_\delta \leq \tilde{\sigma}(t - t_0) .$$

However, $\tilde{\sigma}$ does not belong to class $\mathcal{L}$, as it tends to infinity when its argument tends to zero. To overpass this problem, we consider the instant $t_1 > 0$ at which $\tilde{\sigma}(t_1) = \alpha(\Delta')$ and efine

$$\sigma(t) := \begin{cases} -(t - t_1) + \alpha(\Delta') & \text{if} \quad 0 \leq t \leq t_1 \\ \tilde{\sigma}(t) & \text{if} \quad t > t_1 \end{cases}$$

Then σ belongs to class $\mathcal{L}$ and, using the fact that $|\phi(t, t_0, x_0)|_\delta \leq \alpha(\Delta')$ over $[t_0; t_0 + t_1]$, we have, as desired, that $|\phi(t, t_0, x_0)|_\delta \leq \sigma(t - t_0)$ for all $t \geq t_0$.

A.3 Proof of Theorem 3.38

We first show that the system is uniformly stable. More precisely, we explicitly construct $\Delta > 0$ and $\alpha \in \mathcal{K}_\infty$ such that, for all $x_0 \in \mathcal{B}_\Delta$,

$$|x(t, t_0, x_0)| \leq \alpha(|x_0|) . \tag{A.10}$$

We then use this property to prove uniform attractivity of the origin on the same ball $\mathcal{B}_\Delta$, and then deduce a $\mathcal{KL}$ estimate for the solutions of (3.1).

Proof of uniform stability: From Assumptions 3.34 and 3.35, the time derivative of V_1 along the trajectories of (3.1) yields, for any $x_1 \in \mathcal{B}_{\Delta_1}$,

$$\begin{aligned} \dot{V}_1(t, x_1) &= \frac{\partial V_1}{\partial t}(t, x_1) + \frac{\partial V_1}{\partial x_1}(t, x_1)\big(f_1(t, x_1) + g(t, x)\big) \\ &\leq -k_1 V_1(t, x_1) + \left|\frac{\partial V_1}{\partial x_1}(t, x_1)\right| |g(t, x)| \\ &\leq -k_1 V_1(t, x_1) + c_1(|x_1|) G(|x|) g_2(|x_2|) . \end{aligned}$$

Defining

$$\Gamma := \{t \geq t_0 \ : \ x_1(t, t_0, x_{10}) \in \mathcal{B}_{\Delta_1}\} , \tag{A.11}$$

we get that, for any $t \in \Gamma$,

$$\dot{v}_1(t) \leq -k_1 v_1(t) + c_1(|x_1(t)|) G(|x(t)|) \, |x_2(t)| , \tag{A.12}$$

where we used the shorthand notation $x_1(t)$ for $x_1(t, t_0, x_0)$ and $v_1(t) := V_1(t, x_1(t))$. Thus, using Assumptions 3.36 and 3.37, we have in view of (3.59) that, for all $x_0 \in \mathcal{B}_\Delta$ and all $t \in \Gamma$,

$$\dot{v}_1(t) \leq -k_1 v_1(t) + \tilde{\beta}(|x_0| , t - t_0) , \tag{A.13}$$

where $\tilde{\beta} \in \mathcal{KL}$ is defined, for all $s, t \in \mathbb{R}_{\geq 0}$, as

$$\tilde{\beta}(s,t) := c_1(B)G(B)\beta_2(s,t)\,.$$

Notably, we have that, for all $x_0 \in \mathcal{B}_\Delta$ and all $t \in \Gamma$,

$$\dot{v}_1(t) \leq -k_1 v_1(t) + \tilde{\beta}(|x_0|\,,0)\,. \tag{A.14}$$

We now show that, for any $x_0 \in \mathcal{B}_\Delta$, $x_1(t)$ remains in $\mathcal{B}_{\Delta_1}$ forever after (i.e., $\Gamma = \mathbb{R}_{\geq t_0}$). To that end, notice that, since $x_0 \in \mathcal{B}_\Delta$, we have that $t_0 \in \Gamma$. Hence, in view of (A.11) and invoking the continuity of the solutions of (3.1), there exists $T_0 > 0$ such that

$$[t_0; t_0 + T_0] \subset \Gamma\,.$$

Integrating (A.14) from t_0 to any $t \in [t_0; t_0 + T_0]$ we get that, for all $x_0 \in \mathcal{B}_\Delta$ and all $t \in [t_0; t_0 + T_0]$,

$$v_1(t) \leq (v_1(t_0) - \tilde{\beta}(|x_0|\,,0))e^{-k_1(t-t_0)} + \tilde{\beta}(|x_0|\,,0)\,,$$

which gives

$$|x_1(t)| \leq {\underline{\alpha}_1}^{-1}\left(\overline{\alpha}_1(|x_0|) + \tilde{\beta}(|x_0|\,,0)\right)\,. \tag{A.15}$$

With the Δ proposed in (3.59)[3], is then possible to see that, as desired, any solution $x_1(t)$ staring in $\mathcal{B}_\Delta$ remains in $\mathcal{B}_{\Delta_1}$ forever after. In addition, for any $x_0 \in \mathcal{B}_\Delta$, we have that $\Gamma = \mathbb{R}_{\geq t_0}$ or, said differently, $T_0 = \infty$. Therefore, (A.15) holds for all $t \geq t_0$. Thus, we have in turn shown that, for any $x_0 \in \mathcal{B}_\Delta$,

$$|x_1(t)| \leq \alpha_1(|x_0|)\,,$$

where α_1 is the $\mathcal{K}_\infty$ function defined as

$$\alpha_1(s) := {\underline{\alpha}_1}^{-1}\left(\overline{\alpha}_1(s) + \tilde{\beta}(s,0)\right)\,, \quad \forall s \geq 0\,. \tag{A.16}$$

Thus, introducing the following class $\mathcal{K}$ function

$$\alpha(\cdot) := \sqrt{\alpha_1(\cdot)^2 + \beta_2(\cdot,0)^2}\,,$$

we have with Assumption 3.36 that, for all $x_0 \in \mathcal{B}_\Delta$, the solutions of (3.1) satisfy

$$|x(t,t_0,x_0)| \leq \alpha(|x_0|)\,, \quad \forall t \geq t_0\,. \tag{A.17}$$

Proof of uniform attractivity: Assume that $x_0 \in \mathcal{B}_\Delta$. Then, we have that (A.13) holds for all $t \geq t_0$. For any $\varepsilon_1 > 0$, let $T_1 \geq 0$ be the instant, independent of t_0, where[4]

$$\tilde{\beta}(\Delta, T_1) = \varepsilon_1\,.$$

Then, from (A.13), we get that, for all $t \geq t_0 + T_1$,

$$\dot{v}_1(t) \leq -k_1 v_1(t) + \varepsilon_1\,.$$

[3] β_{20} can be assumed to be of class $\mathcal{K}_\infty$ without loss of generality since β_2 is only constrained over $[0; \Delta_2]$. This observation ensures the existence of β_{20}^{-1}.

[4] If $\tilde{\beta}(\Delta, 0) \leq \varepsilon_1$, T_1 is taken as 0.

Therefore, for all $t \geq t_0 + T_1$,

$$v_1(t) \leq (v_1(t_0 + T_1) - \varepsilon_1) e^{-k_1(t-t_0-T_1)} + \varepsilon_1 .$$

But, using the US bound (A.17), we have that

$$v_1(t_0 + T_1) \leq \overline{\alpha}_1 \left(|x_1(t_0 + T_1)| \right) \leq \overline{\alpha}_1 \left(\alpha(|x_0|) \right) .$$

Hence, for all $t \geq t_0 + T_1$,

$$|x_1(t)| \leq \underline{\alpha}_1{}^{-1} \left(\overline{\alpha}_1(\alpha(\Delta)) e^{-k_1(t-t_0-T_1)} + \varepsilon_1 \right) .$$

Notably, we have that

$$|x_1(t)| \leq \underline{\alpha}_1{}^{-1}(2\varepsilon_1) =: \varepsilon , \qquad \forall t \geq t_0 + T_2 ,$$

where

$$T_2 := T_1 + \frac{1}{k_1} \ln \left(\frac{\overline{\alpha}_1(\alpha(\Delta))}{\varepsilon_1} \right) .$$

Observing that ε is arbitrary and that T_2 is independent of t_0, and recalling that $\dot{x}_2 = f_2(t, x_2)$ is ULA on $\mathcal{B}_\Delta$ as well, we conclude the uniform attractivity of the origin for (3.1), with $\mathcal{B}_\Delta$ as an estimate of its domain of attraction.

Construction of the $\mathcal{KL}$ estimate:

Invoking [Vid93, Lemma 57], we conclude from the uniform local attractivity of (3.1) to the existence of $\eta \in \mathcal{L}$ such that, for all $x_0 \in \mathcal{B}_\Delta$,

$$|x(t, t_0, x_0)| \leq \eta(t - t_0) , \qquad \forall t \geq t_0 . \tag{A.18}$$

Multiplying (A.17) and (A.18) gives, for all $x_0 \in \mathcal{B}_\Delta$,

$$|x(t, t_0, x_0)| \leq \beta(|x_0| , t - t_0) , \qquad \forall t \geq t_0 ,$$

where, for all $s, t \in \mathbb{R}_{\geq 0}$,

$$\beta(s, t) := \sqrt{\alpha(s)\eta(t)} .$$

Since α and η and β_2 are respectively of class $\mathcal{K}$ and $\mathcal{L}$, β is clearly a class $\mathcal{KL}$ function, and the conclusion follows.

A.4 Proof of Corollary 3.40

From Assumption 3.39, the solutions of $\dot{x}_1 = f_1(t, x_1)$ satisfy, for all $x_{10} \in \mathcal{B}_{\Delta_1}$ and all $t \geq t_0$,

$$\left| x_1^{f_1}(t, t_0, x_{10}) \right| \leq k_1 |x_{10}| e^{-\gamma_1(t-t_0)} , \quad \forall t \geq t_0 .$$

Let $\tilde{\Delta}_1$ be any positive number satisfying (3.62). Then all the requirements of [Kha96, Theorem 3.14] are fulfilled and we get that there exist a continuously differentiable function $V_1 : \mathbb{R}_{\geq 0} \times \mathcal{B}_{\Delta_1}$ and some positive $\underline{k}_1$, $\overline{k}_1$ and k_1 and c_1 such that, for all $x_1 \in \mathcal{B}_{\tilde{\Delta}_1}$ and all $t \in \mathbb{R}_{\geq 0}$,

$$\underline{k}_1 |x_1|^2 \leq V_1(t, x_1) \leq \overline{k}_1 |x_1|^2$$

$$\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1(t, x_1) \leq -k_1 |x_1|^2$$
$$\left| \frac{\partial V_1}{\partial x_1}(t, x_1) \right| \leq c_1 |x_1| \ .$$

In addition, the proof of [Kha96, Theorem 3.14] is established with the coefficients $\underline{k}_1$, $\overline{k}_1$, c_1 given in (3.61a)-(3.61c) (and $k_1 = 1/2$). Thus, the rest of the proof follows the one of Theorem 3.38, where the functions $\underline{\alpha}_1(s)$, $\overline{\alpha}_1(s)$ and $c_1(s)$ are now respectively $\underline{k}_1 s^2$, $\overline{k}_1 s^2$ and $c_1 s$ and $\tilde{\Delta}_1$ plays the role of Δ_1.

A.5 Proof of Theorem 3.42

The proof is an adaptation of the one of Theorem 3.38, and we therefore use the same notations. Notice that, in this case, $\beta_2(s,t) = k_2 s e^{-\gamma_2 t}$. In view of (A.16), uniform stability of the origin is therefore obtained with $\alpha_1(s) = d_1 s$, where

$$d_1 := \frac{1}{\underline{k}_1}(\overline{k}_1 + c_1 B G(B) k_2) \, .$$

In addition (A.12) implies that, for any $x_0 \in \mathcal{B}_\Delta$,

$$\dot{v}_1(t) \leq -\frac{k_1}{\overline{k}_1} v_1(t) + c_1 d_1 |x_0| \, G(B) k_2 |x_0| \, e^{-\gamma_2 (t - t_0)} \, .$$

If $\gamma_2 \neq \frac{k_1}{\overline{k}_1}$, the integration of this equation yields, with the comparison lemma,

$$v_1(t) \leq v_1(t_0) e^{-\frac{k_1}{\overline{k}_1}(t - t_0)} + a_1 |x_0|^2 \left(e^{-\gamma_2 (t - t_0)} - e^{-\frac{k_1}{\overline{k}_1}(t - t_0)} \right)$$

where

$$a_1 := \frac{c_1 d_1 G(B) k_2}{\frac{k_1}{\overline{k}_1} - \gamma_2} \, .$$

And, if $\gamma_2 = \frac{k_1}{\overline{k}_1}$,

$$v_1(t) = \left(v_1(t_0) + a_2 |x_0|^2 \, (t - t_0) \right) e^{-\gamma_2 (t - t_0)}$$

with $a_2 := c_1 d_1 G(B) k_2$. Thus, defining

$$\begin{aligned} \sigma_1 &:= \min \left\{ \gamma_2 \ ; \ k_1 / \overline{k}_1 \right\} \\ \sigma_2 &:= \max \left\{ a_1 \ ; \ a_2 \right\} \, , \end{aligned}$$

it is possible to see that, in any case,

$$v_1(t) \leq \left(v_1(t_0) + \sigma_2 |x_0|^2 \right) e^{-\sigma_1 (t - t_0)} \, .$$

Therefore, we have that, for all $x_0 \in \mathcal{B}_\Delta$,

$$|x_1(t)| \leq \frac{\sqrt{\overline{k}_1 + \sigma_2}}{\underline{k}_1} |x_0| \, e^{-\frac{\sigma_1}{2}(t - t_0)} \, ,$$

which establishes UES of (3.1) on $\mathcal{B}_\Delta$ with parameters $\left(\frac{\sqrt{\overline{k}_1 + \sigma_2}}{\underline{k}_1}, \frac{\sigma_1}{2} \right)$.

A.6 Proof of Claim 6.4

Using the notation (6.6), it was shown in [Tom91] that V_{11} is positive definite and radially unbounded provided that

$$\theta_{12} > k_g \,. \tag{A.19}$$

In view of (6.11b), this condition holds since $a_p > k_g$. Then, it can be shown in view of (6.9) that there exist some positive constants $\underline{a}$, $\overline{a}$ and $\underline{b}$ such that, for all K_p' satisfying (A.19),

$$\underline{a}\,|\dot{q}|^2 + \underline{b}\theta_{12}\,|\tilde{q}|^2 \le V_{11} \le \overline{a}\,|\dot{q}|^2 + \theta_{12}\,|\tilde{q}|^2 \,.$$

Also, based on (6.9) and (A.19), it can easily be seen that

$$\begin{aligned} \frac{\varepsilon_1}{2}\left(-d_M\,|\dot{q}|^2 - d_M\,|\tilde{q}|^2 + \theta_{13}\,|s|^2\right) &\le V_{12} \le \frac{\varepsilon_1}{2}\left(d_M\,|\dot{q}|^2 + d_M\,|\tilde{q}|^2 + \ell\theta_{13}\,|s|^2\right) \\ -\frac{\varepsilon_2 d_M}{2}\left(|\dot{q}|^2 + |s|^2\right) &\le V_{13} \le \frac{\varepsilon_2 d_M}{2}\left(|\dot{q}|^2 + |s|^2\right) . \end{aligned}$$

Hence, under the condition that

$$\varepsilon_1 + \varepsilon_2 \le \frac{\underline{a}}{d_M} \tag{A.20a}$$

$$\theta_{12} \ge \frac{\varepsilon_1 d_M}{\underline{b}} \tag{A.20b}$$

$$\theta_{13} \ge \frac{2\varepsilon_2 d_M}{\varepsilon_1} \,, \tag{A.20c}$$

the inequalities in (6.10) hold with[5] $\underline{a}_1 := \underline{a}/2$, $\underline{b}_2 := \underline{b}/2$, $\underline{b}_3 := \varepsilon_1/4$, $\overline{a}_1 := \overline{a} + d_M(\varepsilon_1 + \varepsilon_2)/2$, $\overline{a}_2 := \varepsilon_1 d_M/2$, $\bar{b}_2 := 1$, $\overline{a}_3 := \varepsilon_2 d_M/2$ and $\bar{b}_3 := \varepsilon_1\ell/2$. This completes the proof of the claim.

A.7 Proof of Claim 6.5

Direct computations show that:

$$\begin{aligned} \frac{dV_{11}}{dx_1} f_1(t,x_1,\theta_1) &= -\dot{q}^\top K_d \dot{q} - \dot{q}^\top K_i s + \dot{q}^\top p_1(t,q,\dot{q}) \\ \frac{dV_{12}}{dx_1} f_1(t,x_1,\theta_1) &= \varepsilon_1\Big[-\tilde{q}^\top K_p'\tilde{q} + \tilde{q}^\top\left(C(q,\dot{q})^\top - K_d\right)\dot{q} + \dot{q}^\top D(q)\dot{q} \\ &\qquad -\tilde{q}^\top(g(q) - g(q_*)) + \frac{1}{\varepsilon_1} s^\top K_i \dot{q} + \tilde{q}^\top p_1(t,q,\dot{q})\Big] \\ \frac{dV_{13}}{dx_1} f_1(t,x_1,\theta_1) &= \varepsilon_2\Big[-s^\top K_i s + \dot{q}^\top D(q)\tilde{q} + \frac{1}{\varepsilon_1}\dot{q}^\top D(q)\dot{q} + s^\top C(q,\dot{q})^\top \dot{q} \\ &\qquad -s^\top(g(q) - g(q_*)) - s^\top K_p'\tilde{q} - s^\top K_d\dot{q} + s^\top p_1(t,q,\dot{q})\Big] . \end{aligned}$$

Therefore, the derivative of V_1 along the x_1-subsystem yields

$$\begin{aligned} \frac{dV_1}{dx_1} f_1(t,x_1,\theta_1) &= -\dot{q}^\top K_d\dot{q} - \varepsilon_1\tilde{q}^\top K_p'\tilde{q} - \varepsilon_2 s^\top K_i s + \tilde{q}^\top\left(\varepsilon_2 D(q) - \varepsilon_1 K_d + \varepsilon_1 C(q,\dot{q})^\top\right)\dot{q} \\ &\quad + \left(\varepsilon_1 + \frac{\varepsilon_2}{\varepsilon_1}\right)\dot{q}^\top D(q)\dot{q} - \varepsilon_2 s^\top K_p'\dot{q} + \varepsilon_2 s^\top\left(C(q,\dot{q})^\top - K_d\right)\dot{q} \\ &\quad - \left(\varepsilon_1\tilde{q}^\top + \varepsilon_2 s^\top\right)(g(q) - g(q_*)) + \left(\dot{q}^\top + \varepsilon_1\tilde{q}^\top + \varepsilon_2 s^\top\right)p_1(t,q,\dot{q}) \,. \end{aligned}$$

[5] The conditions (A.20b) and (A.20b) hold in view of (6.11a) and (6.11b).

It follows that

$$\begin{aligned}\frac{dV_1}{dx_1}f_1(t,x_1,\theta_1) &\leq -\theta_{11}\left|\dot{q}\right|^2-\varepsilon_1\theta_{12}\left|\tilde{q}\right|^2-\varepsilon_2\theta_{13}\left|s\right|^2\\ &+ \frac{1}{2}\left(\left(2\varepsilon_1+\frac{2\varepsilon_2}{\varepsilon_1}+\varepsilon_2\right)d_M+(\varepsilon_1+\varepsilon_2)\lambda_M(K_d)\right)\left|\dot{q}\right|^2\\ &+ \frac{1}{2}\left(\varepsilon_1\lambda_M(K_d)+(2\varepsilon_1+\varepsilon_2)k_g+\varepsilon_2 d_M+\varepsilon_2\lambda_M(K_p')\right)\left|\tilde{q}\right|^2\\ &+ \frac{\varepsilon_2}{2}\left(\lambda_M(K_p')+\lambda_M(K_d)+k_g\right)\left|s\right|^2\\ &+ \frac{k_c}{3}\left(\varepsilon_1\left|\tilde{q}\right|^3+2(\varepsilon_1+\varepsilon_2)\left|\dot{q}\right|^3+\varepsilon_2\left|s\right|^3\right)+\max\left\{\varepsilon_1;\varepsilon_2;1\right\}\left|x_1\right|\left|p_1(t,q,\dot{q})\right|\,.\end{aligned}$$

Notice, in view of (6.3), that for all x_1 such that $\delta_1 \leq |x_1| \leq \Delta_1$

$$\left|x_1\right|\left|p_1(t,\tilde{q},\dot{q})\right| \leq \left(\frac{p_{10}}{\delta_1}+p_{11}+p_{12}\Delta_1\right)\left|x_1\right|^2\,.$$

Based on the previous observation and the assumption that $\varepsilon_1+\varepsilon_2 \leq 1$ it follows that in order to get (6.12) it is sufficient to have

$$\begin{aligned}\theta_{11} &\geq \left(2\varepsilon_1+\frac{2\varepsilon_2}{\varepsilon_1}+\varepsilon_2\right)d_M+(\varepsilon_1+\varepsilon_2)\lambda_M(K_d)+2p_{11}+\frac{2p_{10}}{\delta_1}+2\left(\frac{2k_c}{3}(\varepsilon_1+\varepsilon_2)+p_{12}\right)\Delta_1\\ \varepsilon_1\theta_{12} &\geq \varepsilon_1\lambda_M(K_d)+(2\varepsilon_1+\varepsilon_2)k_g+\varepsilon_2 d_M+\varepsilon_2\lambda_M(K_p')+2p_{11}+\frac{2p_{10}}{\delta_1}+2\left(\frac{\varepsilon_1 k_c}{3}+p_{12}\right)\Delta_1\\ \theta_{13} &\geq \lambda_M(K_p')+\lambda_M(K_d)+k_g+2p_{11}+\frac{2p_{10}}{\delta_1}+2\left(\frac{2k_c}{3}+p_{12}\right)\Delta_1\,.\end{aligned}$$

The latter is fulfilled provided that the gains are chosen large enough so that

$$\frac{\theta_{11}}{2} \geq \left(2\varepsilon_1+\frac{2\varepsilon_2}{\varepsilon_1}+\varepsilon_2\right)d_M+2p_{11}+\frac{2p_{10}}{\delta_1}+2\left(\frac{2k_c}{3}(\varepsilon_1+\varepsilon_2)+p_{12}\right)\Delta_1 \tag{A.21a}$$

$$\frac{\varepsilon_1\theta_{12}}{2} \geq (2\varepsilon_1+\varepsilon_2)k_g+\varepsilon_2 d_M+2p_{11}+\frac{2p_{10}}{\delta_1}+2\left(\frac{\varepsilon_1 k_c}{3}+p_{12}\right)\Delta_1 \tag{A.21b}$$

$$\frac{\theta_{13}}{2} \geq k_g+2p_{11}+\frac{2p_{10}}{\delta_1}+2\left(\frac{k_c}{3}+p_{12}\right)\Delta_1\,, \tag{A.21c}$$

and, based on (6.9), that the following holds

$$\frac{\varepsilon_1\theta_{12}}{2} \geq \varepsilon_1\ell\theta_{11}+\varepsilon_2\ell\theta_{12} \tag{A.22a}$$

$$\frac{\theta_{11}}{2} \geq (\varepsilon_1+\varepsilon_2)\ell\theta_{11} \tag{A.22b}$$

$$\frac{\theta_{13}}{2} \geq \ell(\theta_{12}+\theta_{11})\,. \tag{A.22c}$$

We can summarize the conditions (A.20a) and (A.22) in the following way: it is sufficient to first choose ε_1 and ε_2 in such a way that[6]

$$\begin{aligned}\varepsilon_2 &\leq \frac{\varepsilon_1}{3\ell}\\ \varepsilon_1+\varepsilon_2 &\leq \min\left\{\frac{1}{2\ell};\frac{a}{d_M}\right\}\,,\end{aligned}$$

[6]Since $\ell \geq 1$ by definition, it is clear that $\varepsilon_1+\varepsilon_2 \leq 1$ as required.

and then, based on this choice, to design the control gains according to (6.11) where a_d, b_d, c_d, a_p, b_p, c_p, a_i, b_i and c_i are the positive constants, independent of δ_1 and Δ_1, given by

$$\begin{aligned}
a_d &:= 2\left(2\varepsilon_1 + \frac{2\varepsilon_2}{\varepsilon_1} + \varepsilon_2\right) d_M + 2p_{11} \\
b_d &:= 4p_{10} \\
c_d &:= 4\left(\frac{2(\varepsilon_1+\varepsilon_2)k_c}{3} + p_{12}\right) \\
a_p &:= 2\max\left\{\left(2 + \frac{\varepsilon_2}{\varepsilon_1}\right) k_g + \frac{\varepsilon_2 d_M}{\varepsilon_1} + \frac{2p_{11}}{3} \;;\; \frac{\varepsilon_1 \ell a_d}{\varepsilon_1 - 2\ell\varepsilon_2} \;;\; \frac{\varepsilon_1 d_M}{2}\right\} \\
b_p &:= 2\max\left\{\frac{2p_{10}}{\varepsilon_1} \;;\; \frac{\varepsilon_1 \ell b_d}{\varepsilon_1 - 2\ell\varepsilon_2}\right\} \\
c_p &:= 2\max\left\{\frac{2k_c}{3} + \frac{2p_{12}}{\varepsilon_1} \;;\; \frac{\varepsilon_1 \ell c_d}{\varepsilon_1 - 2\ell\varepsilon_2}\right\} \\
a_i &:= 2\max\left\{k_g + 2p_{11} \;;\; \ell(a_d + a_p) \;;\; \frac{\varepsilon_2 d_M}{\varepsilon_1}\right\} \\
b_i &:= 2\max\left\{2p_{10} \;;\; \ell(b_d + b_p)\right\} \\
c_i &:= 2\max\left\{\frac{2k_c}{3} + 2p_{12} \;;\; \ell(c_d + c_p)\right\}.
\end{aligned}$$

The proof of the claim follows.

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Abstract. We present new tools for stability and robustness analysis of nonlinear dynamical systems. We provide a precise Lyapunov framework for uniform semiglobal and practical asymptotic stability. "Semiglobal" refers to the situation when the domain of attraction is not the whole state-space but, instead, a compact set that may be arbitrarily enlarged by a convenient tuning of parameters. "Practical" concerns the case when an arbitrarily small compact neighborhood of the origin (instead of the origin itself) is asymptotically stable. As opposed to many related concepts, they allow the estimate of solutions to depend on the tuning parameter and so, potentially, on the radius of the desired domain of attraction and the amplitude of the tolerated steady-state error. Compared to classical results for global asymptotic stability, this feature requires to impose an additional requirement on the bounds on the Lyapunov function. We illustrate the importance of this condition by showing that, when the latter is violated, no stability property is ensured. We also derive a converse Lyapunov result for the class of USPAS systems whose solutions' estimate is independent of the radius of the attractive ball. The generated Lyapunov function is especially designed to fit the context of cascaded systems as its gradient is bounded by a time-invariant function.

With the proposed Lyapunov framework for semiglobal and practical asymptotic stability, some tools are presented that ensure the preservation of these stability properties by cascade interconnection. In general terms, similarly to existing results for global asymptotic stability, it is required that the solutions of the overall cascade be bounded and that a convenient Lyapunov function be explicitly known for the driven subsystem. In view of the converse result we establish, we relax this latter requirement in the semiglobal case for a wide class of systems. This is particularly useful when invoking averaging techniques, as illustrated by the output feedback control of the double integrator affected by a persistently exciting signal. Furthermore, in the case of uniform global practical asymptotic stability, the boundedness assumption on the solutions of the cascade is replaced by growth restriction on the interconnection term. This makes it easy to apply in specific problems. We illustrate its use by quantifying the effect of smoothing sign functions in disturbance rejection.

We show that, if some (non necessarily compact) sets are globally asymptotically stable (GAS) for two subsystems taken separately, then their cross product is GAS for the corresponding cascade provided that its solutions are globally bounded. On some occasions, this requirement can be replaced by a simple growth order condition on the interconnection term (plus forward completeness). This work includes, as a special case, partial stability for cascades. As an illustration, we provide a concise proof for a recently established result of formation control of surface vessels along a straight path and with a prescribed velocity.

We analyze the stability of cascaded systems with inputs by providing sufficient conditions under which integral input to state stability is preserved by cascade interconnection. These conditions are first expressed in Lyapunov terms and then in terms of estimates of the solutions of each subsystem taken separately.

We illustrate the significance of our theoretical findings by solving specific open problems in the field of mechanical systems. We proceed to the robustness analysis of PID-controlled manipulators to friction, model uncertainty, actuators' dynamics, *etc.* Another application concerns the formation control of spacecrafts. We establish global practical asymptotic stability of the corresponding system when only bounds on the leader's anomaly are available. Finally, we show that a similar stability property can be obtained for the synchronization of two surface vessels with little information on the leader vehicle.