Superfluidité et localisation quantique dans les condensats de Bose-Einstein unidimensionnels

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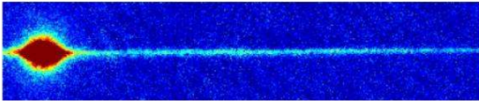
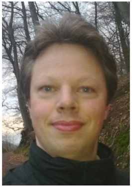


People involved in this work

P. Leboeuf

N. Pavloff

M. Albert



T. Paul

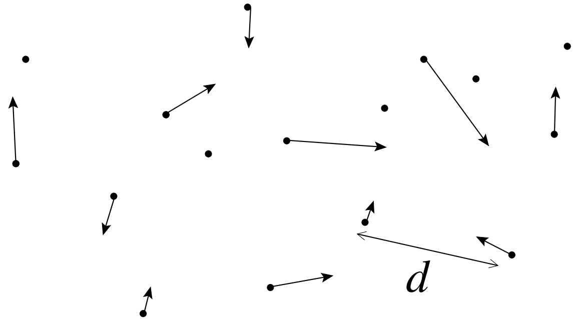
P. Schlagheck

Outline

Outline

The system under considerations

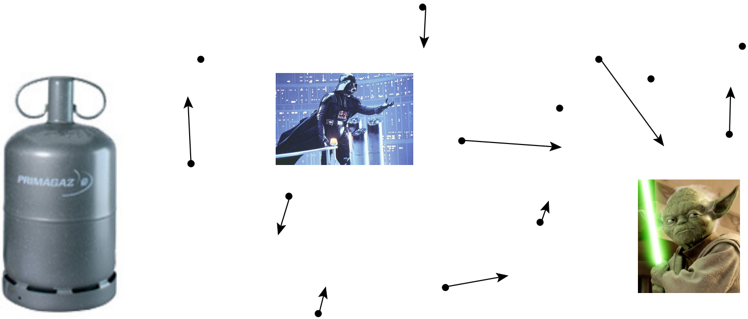
Classical gas at room temperature



- d is the inter-particle distance.
- T is related to the average kinetic energy.

The system under considerations

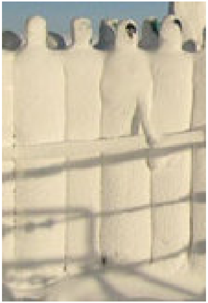
Classical gas at room temperature with interactions

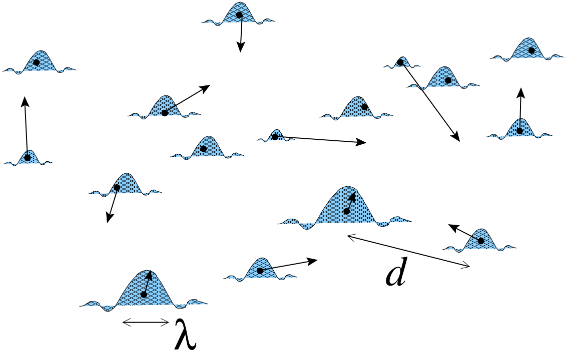


- d is the inter-particle distance.
- T is related to the average kinetic energy.

The system under considerations

Quantum gas at low temperature

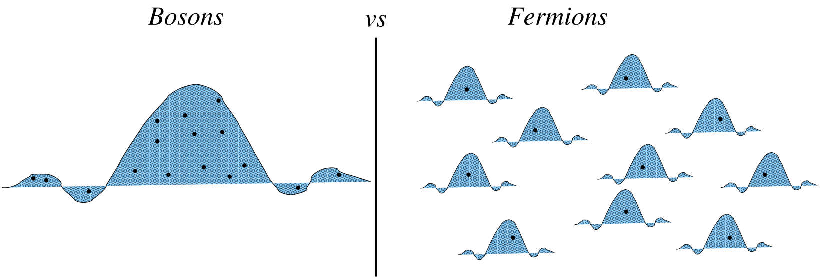




- d is the inter-particle distance.
- λ is the de Broglie wave length.

The system under considerations

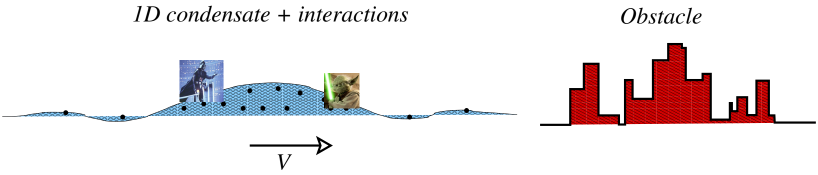
At very low temperature, particles become indistinguishable.



- Bosons like each other $\rightarrow$ condensation in a single quantum state (Bose-Einstein condensation).
- Fermions don't like each other $\rightarrow$ effective "interactions".

The system under considerations

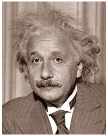
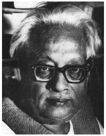
Bosons at $T = 0$, weak repulsive contact interactions, 1D.



- Interaction effects in phase coherent systems.
- Non-linear transport, superfluidity, Anderson localization.

Bose-Einstein Condensation : Basic Concepts

- N Bosons without interaction

When $\lambda_{dB} \sim d$ Bosons condense into a macroscopic wave function

$$\Psi(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N) = \psi(\vec{r}_1)\psi(\vec{r}_2)\cdots\psi(\vec{r}_N)$$

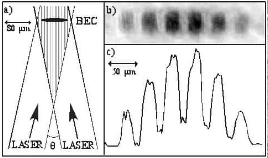
- BEC also exists with interactions (Bogoliubov, Penrose, Onsager, Feynman)

Gross-Pitaevskii equation (weakly interacting bec)

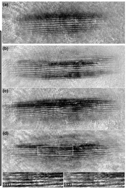
$$\imath\hbar\partial_t\psi(\vec{r},t) = \left(-\frac{\hbar^2}{2m}\Delta + V_{ext}(\vec{r}) + \underbrace{gN|\psi(\vec{r},t)|^2}_{\text{mean field interaction}}\right)\psi(\vec{r},t)$$

Bose-Einstein Condensation : Basic Concepts

- Phase Coherence

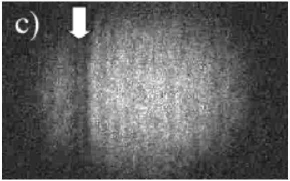


LKB (2004)



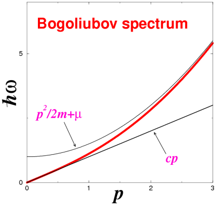
MIT

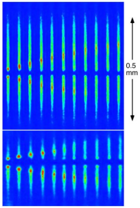
- Superfluidity



Pullman (2007)

- Collective excitations

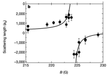




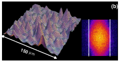
MIT (1997)

- Very clean experiments

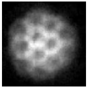
- Interactions are controlled at will



- Disorder is well controlled

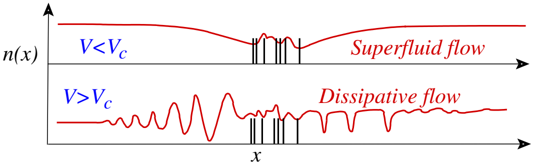


- Seeing the wave function



Superfluidity

- Discovery of Superfluididy in Liquid Helium (1928-1938)

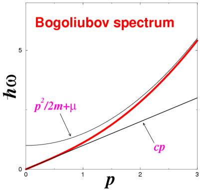


- No dissipation
- Irrotational
- No entropy
- Critical velocity

- Many Body phase coherence $\boxed{v_s = \frac{\hbar}{m}\vec{\nabla}\phi}$

- Landau's criterion for dissipation

$$\boxed{V \geq V_c = \min_p \left(\frac{\varepsilon(p)}{p}\right)}$$



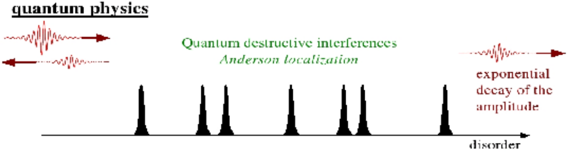
$$\Rightarrow \quad \text{if} \quad \varepsilon(p) = \frac{p^2}{2m} \quad \rightarrow \quad V_c = 0$$

$$\Rightarrow \quad \text{if} \quad \varepsilon(p) = cp \quad \rightarrow \quad V_c = c$$

$c \sim \sqrt{density}$ is the speed of sound.

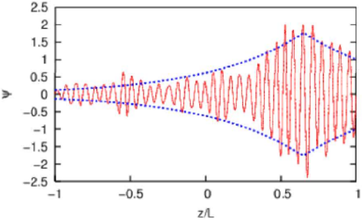
Anderson Localization of non-interacting particles

- Dynamical picture



$$\langle \ln [T(E)] \rangle = -L/L_{loc} \ , \quad P(T,L) = \frac{1}{\sqrt{4\pi}T}\sqrt{\frac{L_{loc}}{L}} \exp\left[-\frac{L_{loc}}{4L}\left(\frac{L}{L_{loc}} + \ln T\right)^2\right]$$

- Static picture

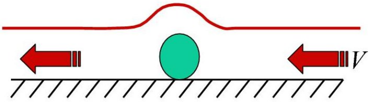


$$\ln|\psi_E(z)| \sim -\frac{|z|}{L_{loc}(E)}$$

- fast oscillations gives E
- exponential decaying envelope

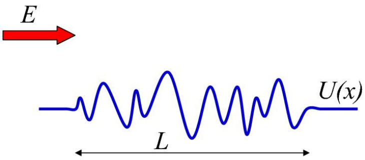
General motivations

Superfluidity



- No dissipation
- No drag force
- Perfect transmission

Anderson Localization



- Exponential decay
- $T \sim \exp\left(-L/L_{loc}\right)$
- Log Normal distribution

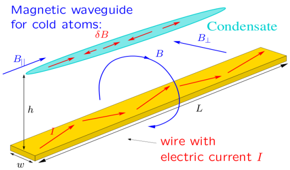
Critical velocity

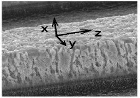
$$v_c = \min(\varepsilon(p)/p) = c$$

$\boxed{\textit{Interaction}} \Leftrightarrow \boxed{\textit{Disorder}}$

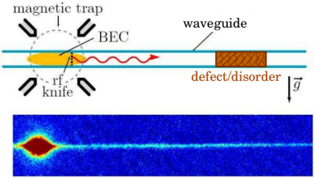
Recent experiments on localization with cold atoms

- Atom laser, atom chip





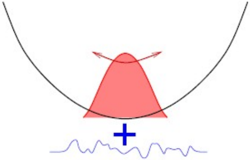
$\Rightarrow$ random variations of the magnetic potential along the waveguide

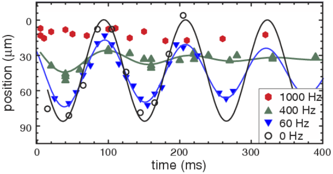


J. Fortagh *et al* PRA 2002, J. Estève *et al* PRA 2004, T. Paul *et al* PRA 2005

Guerin *et al* PRL 2006

- Dipole oscillations in disorder : Firenze, Rice, Hannover, Orsay (2001-)

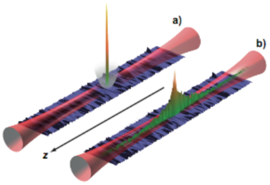


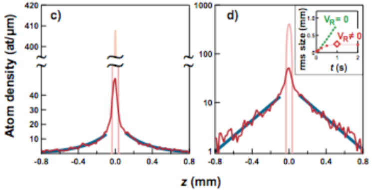


No relation with Anderson localization : M. Albert *et al* PRL 2008

Recent experiments on localization with cold atoms

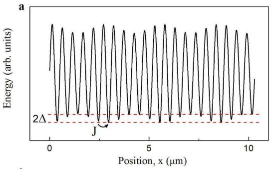
- Orsay : Billy *et al*, *Nature 2008* : Weakly interaction BEC in a speckle disorder

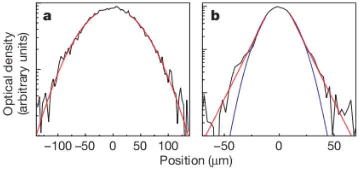




- Firenze : Roati *et al*, *Nature 2008* : Non interacting cold atoms in a quasi-periodic potential

$$\hat{H}_{exp} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_1\cos^2(k_1 x) + V_2\cos^2(k_2 x)$$





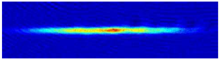
Different from Anderson localization : M. Albert and P. Leboeuf, arXiv :0905.2331.

Outline

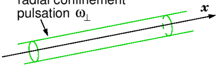
Transport of a 1D condensate

3D Gross-Pitaevskii equation (weakly interacting bec)

$$\imath\hbar\partial_t\psi(\vec{r},t) = \left(-\frac{\hbar^2}{2m}\Delta + V_{ext}(\vec{r}) + \underbrace{gN|\psi(\vec{r},t)|^2}_{\text{mean field interaction}}\right)\psi(\vec{r},t)$$

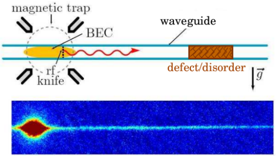


quasi-1D condensate (IOTA)
longitudinal size $\sim 10^2\mu$m
transverse size $\sim 1\mu$m



harmonic radial confinement :

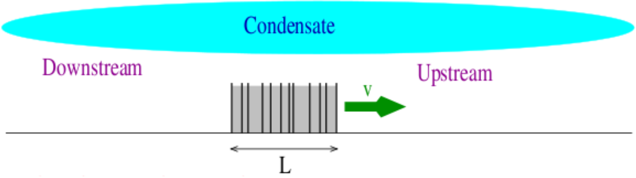
$$V_\perp(\vec{r}_\perp) = \frac{1}{2}m\omega_\perp^2 r_\perp^2 \ .$$

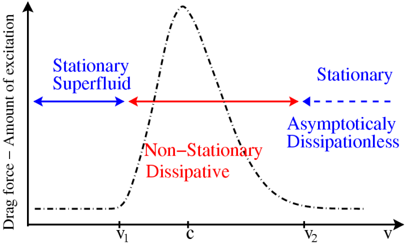


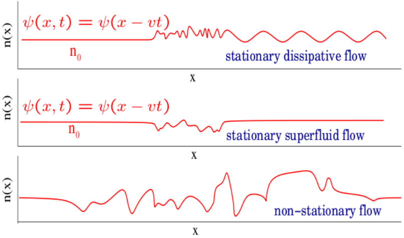
Effective 1d equation

$$\imath\hbar\partial_t\psi(x,t) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_{ext}(x) + g_\nu|\psi(x,t)|^{2\nu}\right)\psi(x,t)$$

Different regimes of transport







- The two regimes below v_2 does not exist without interaction.
- For $v \gg c$ one recovers the ideal gas.

Transport of a 1D BEC in free space

Effective 1d equation

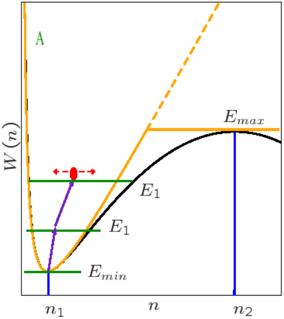
$$\imath\hbar\partial_t\psi(x,t) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_{ext}(x) + g_\nu|\psi(x,t)|^{2\nu}\right)\psi(x,t)$$

Steady flow with current j

$$\begin{cases} \psi(x,t) = A(x)e^{iS(x)}e^{-i\mu t} \\ \frac{1}{2}\left(\frac{dA}{dx}\right)^2 + W(A) = E_{cl} \\ W(A) = \frac{j^2}{2A^4} + \mu A^2 - \frac{g_\nu}{\nu+1}A^{2(\nu+1)} \end{cases}$$

- $A(x)$ is the position of a pseudo particle in $W(A)$ at time x.
- $n = A^2$ is the density of the condensate.

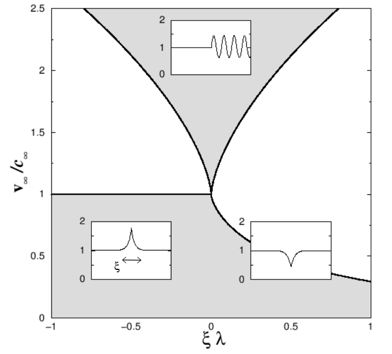
Free modes



Leboeuf et al PRA 64 (2001)

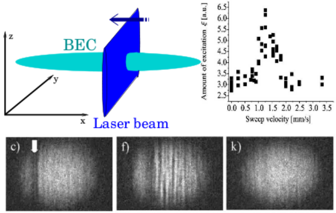
Phase Diagram with a single impurity

We consider a single localized defect $U(x) = \lambda \xi \delta(x)$



Leboeuf, Pavloff PRA 2001

Recent experimental test (Gaussian barrier)

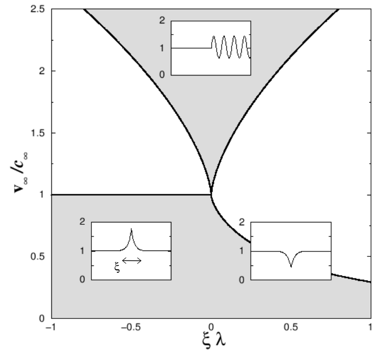


P. Engels and C. Atherton, Phys. Rev. Lett. **99**, 160405 (2007)

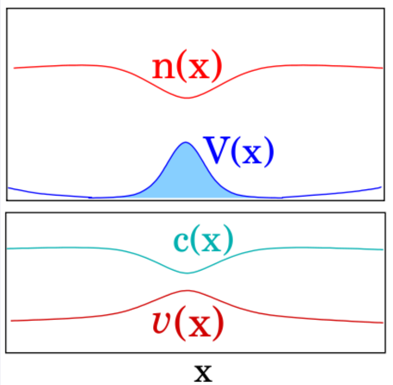
$$\xi = \hbar / \sqrt{m\mu}$$

Phase Diagram with a single impurity

We consider a single localized defect $U(x) = \lambda\xi\delta(x)$



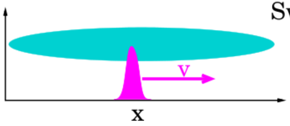
Leboeuf, Pavloff PRA 2001

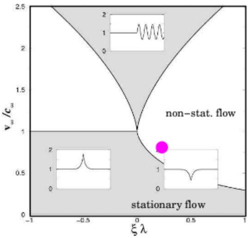


Local Landau's criterion : Superfluidity breaks down if $v(x) = c(x)$.

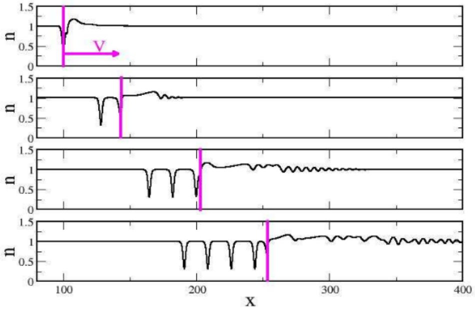
$c(x) \sim n^{\nu/2}$

Soliton emission past an impurity, numerical simulation

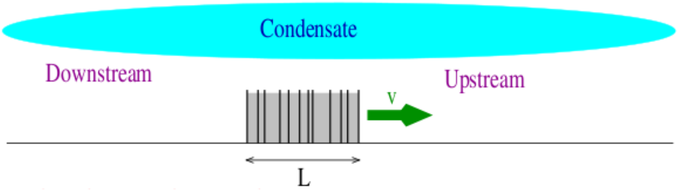


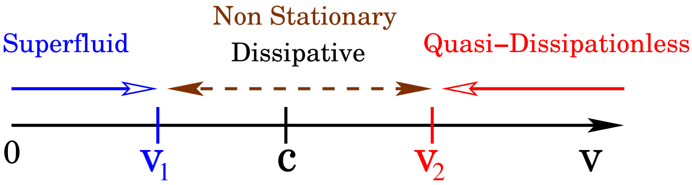


Sweeping an impurity through the condensate



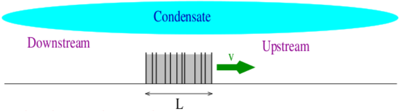
Transport properties in disorder





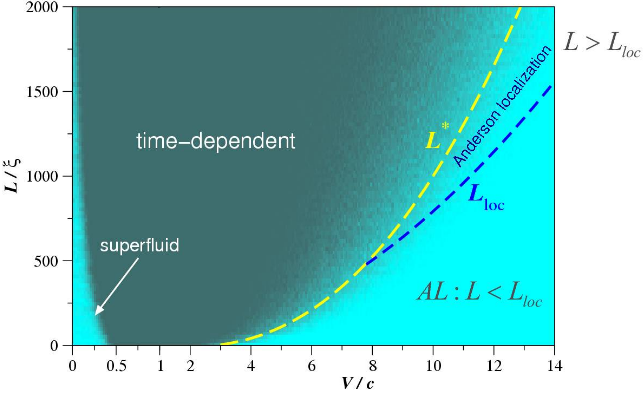
- How do the critical velocities depend on disorder ?
- Effects of interactions on Anderson localization ?

Global picture



$$V(x) = \lambda\xi \sum_{i=1}^{N} \delta(x-x_i)\,,\ \langle V(x)V(0)\rangle - V^2 = \sigma\delta(x)$$

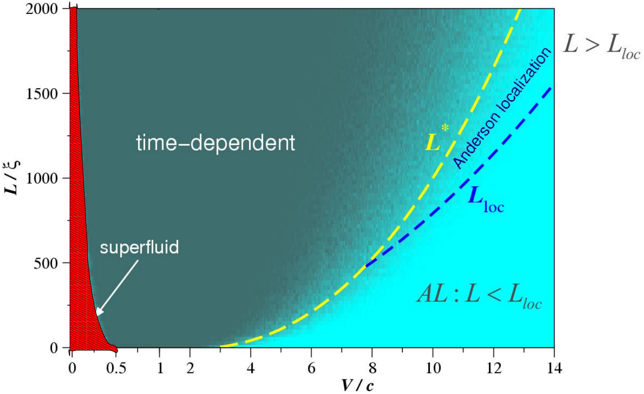
Paul et al, PRL 98, 210602 (2007)



$$\lambda = \xi\, n_i = 0.5$$

Superfluid Regime

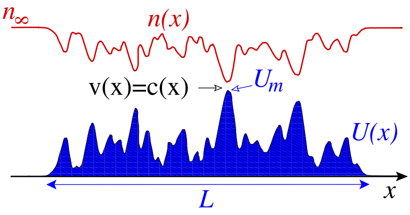
Breakdown of superfluidity and extreme value statistics in a disordered BEC
M. Albert, T. Paul, N. Pavloff and P. Leboeuf *to be submitted*

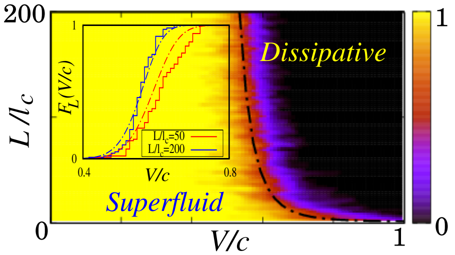


$$\lambda = \xi\, n_i = 0.5$$

Distribution of the critical velocity

For smooth disorder one can apply the Landau's criterion localy $v(x) = c(x)$.





Superfluidity breaks down where the disorder is maximal.

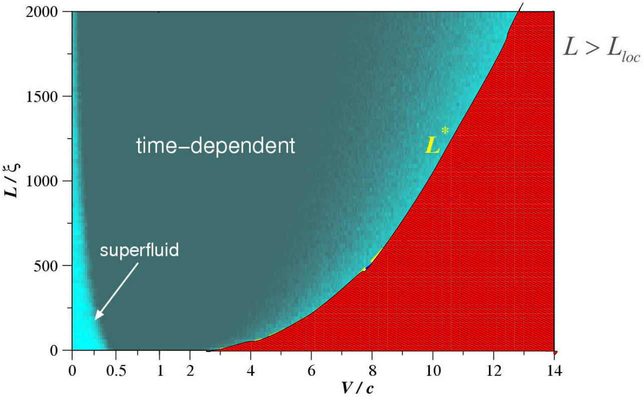
- Landau's criterion relates U_m to v_c
- Extreme value statistics $\longrightarrow$ $\mathcal{P}_L(U_m)$

$\Rightarrow$ $P_L(v_c)$ as a function of L, $\langle U \rangle$, l_c, etc ...

M. Albert, T. Paul, N. Pavloff, P. Leboeuf, *to be submitted*.

Supersonic stationary regime

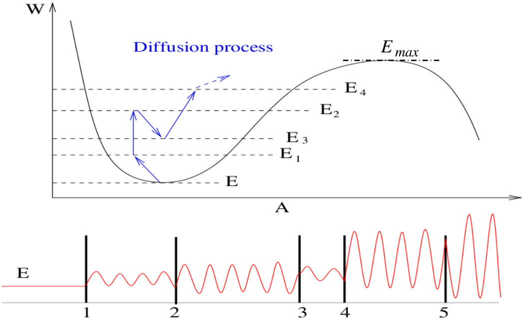
Anderson localization of a weakly interacting one dimensional Bose gas
T. Paul, M. Albert, N. Pavloff, P. Schlagheck and P. Leboeuf *PRA **80** (2009)*



$$\lambda = \xi \, n_i = 0.5$$

Supersonic regime : diffusion model

Disorder induces a random walk in energy space



$$\lambda = \frac{mE_{cl}}{2\hbar\kappa^2}\ ,\ \kappa = \frac{m}{\hbar}\sqrt{v^2 - c^2}$$

Transmission $T = \dfrac{1}{1+\lambda}$

$$\frac{\partial P}{\partial x} = \frac{\partial}{\partial \lambda}\left[\lambda(1+\lambda)\frac{\partial P}{\partial \lambda}\right]$$

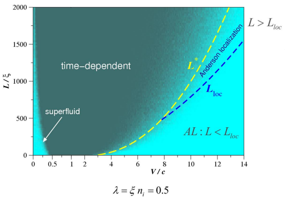
DMPK equation (1982-1988)

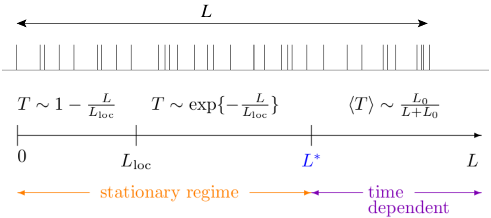
$x = L/L_{loc}(\kappa)$

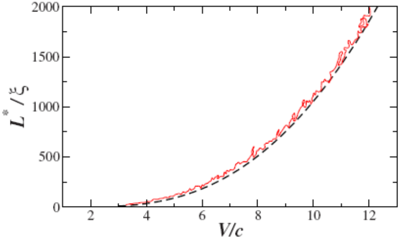
$$P(\lambda, x) = \frac{e^{-x/4}}{\sqrt{2\pi x^3}} \int_{\mathrm{acosh}(1+2\lambda)}^{\infty} \frac{u\, e^{-u^2/(4x)}}{\sqrt{\cosh(u) - 1 - 2\lambda}}\, du$$

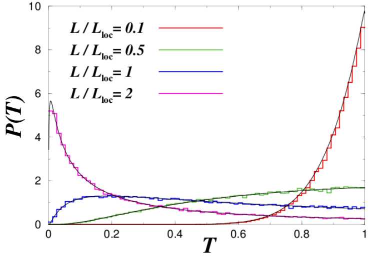
When $\langle E_{cl}\rangle(L) = E_{max}$ the number of stationary solutions decreases drastically. This gives a very good estimate for the supersonic boundary.

Anderson Localization + interactions



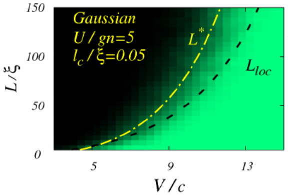


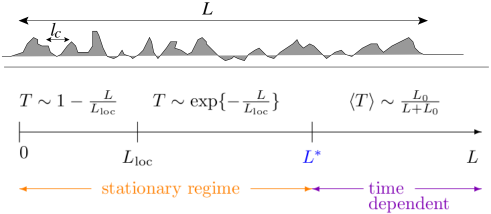




- Renormalization of L_{loc} by interactions $k \to \kappa = \frac{m}{\hbar}\sqrt{v^2 - c^2}$
- Destruction of localization for $L > L^*$ due to time dependent fluctuations.

T. Paul *et al* PRL **98** (2007), T. Paul, *et al* PRA **80** (2009).



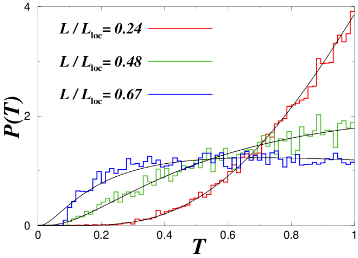


Correlation function $\quad (t = L/L_{loc})$

$$\langle U(x)U(x')\rangle \sim \exp\left[-\frac{1}{2}\left(\frac{x-x'}{\ell_c}\right)^2\right]$$

No analytical proof but DMPK eq works.

Similar results for other short range correlated disorders.

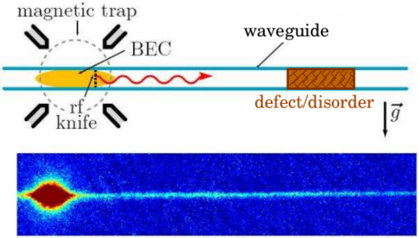


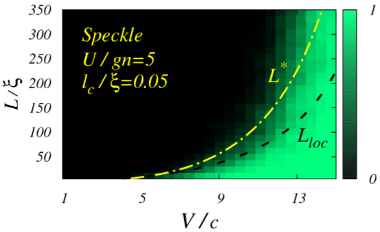
- Renormalization of L_{loc} by interactions $k \rightarrow \kappa = \frac{m}{\hbar}\sqrt{v^2 - c^2}$
- Destruction of localization for $L > L^*$ due to time dependent fluctuations.

T. Paul *et al* PRL **98** (2007), T. Paul, M. A. *et al* PRA **80** (2009).

Orsay experiment PRL 97 (2006)

Laser speckle $\langle V(x)V(x')\rangle = V_R^2\left[\frac{\sin[(x-x')/\ell_c]}{(x-x')/\ell_c}\right]^2, \quad L_{loc} = \frac{\hbar^4\kappa^2}{\pi m V_R^2\,\ell_c(1-\kappa\ell_c)}$



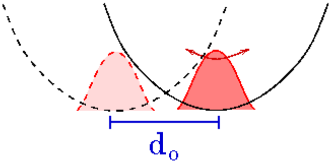


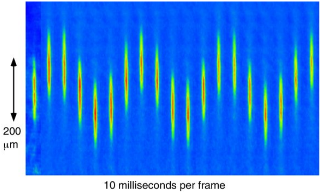
$\ell_c \simeq 0.26\,\mu m$, $v \simeq 1.6\, mm/s$, $c \simeq 0.9\, mm/s$, $V_R = 34\, Hz \Rightarrow$ $L_{loc} \simeq 0.25\, mm$

Reachable experimentally !

Outline

Dipole oscillations in a harmonic trap

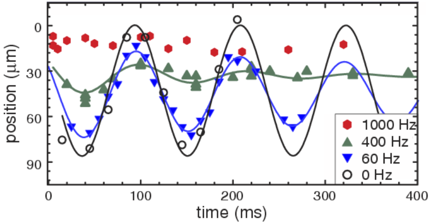




MIT 1997

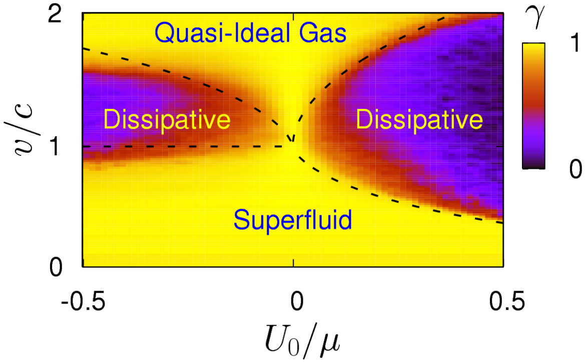
Many groups are studying the damping of collective modes

- Firenze : periodic and speckle potential (PRL **86** (2001), PRL **95** (2005))
- IOTA Orsay : speckle (not published)
- Rice (USA) : speckle PRA **75** (2008)
- Hannover : Bloch oscillations in quasi-periodic potential NJP **10** (2008))

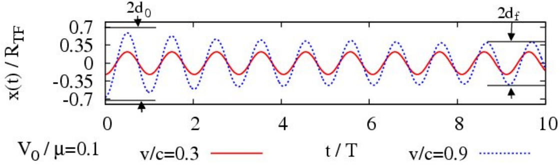


Damping and Frequency shift of collective modes.

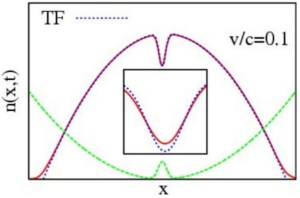
Global picture with a single peak

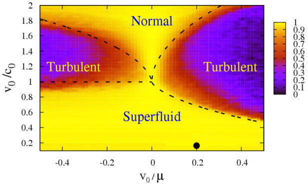


- yellow = undampded motion
- Existence of 3 regimes
  - Superfluid
  - Dissipative
  - Normal/quasi-dissipationless
- lower dashed lines = local Landau criterion
- upper dashed lines = existence of quasi-stationary states



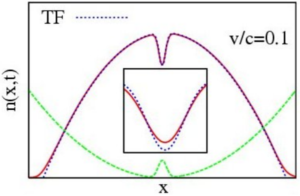
Condensate density during the first oscillation

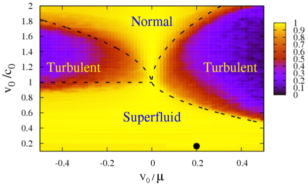




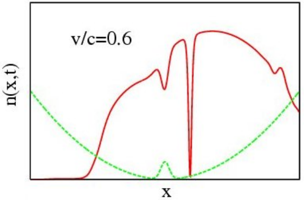
- Superfluid regime
- No damping

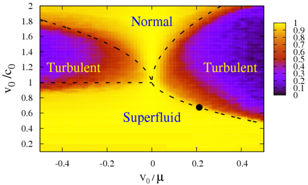
Different regimes of transport





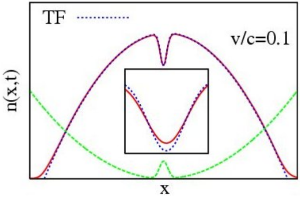
- Superfluid regime
- No damping

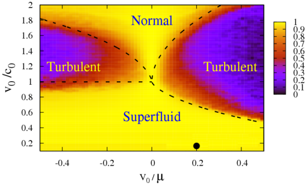




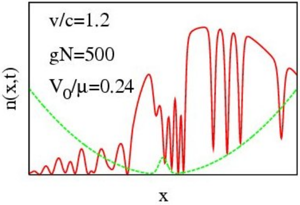
- One soliton is emitted
- Damping

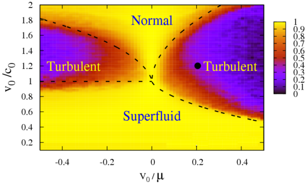
Different regimes of transport





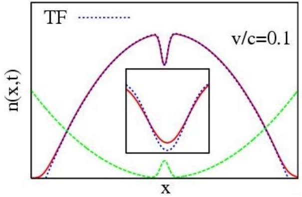
- Superfluid regime
- No damping

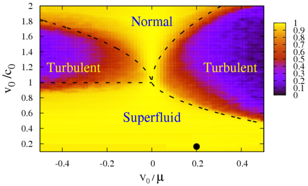




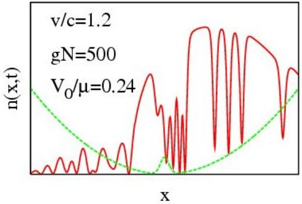
- Strongly time dependent
- Oscillations are destroyed

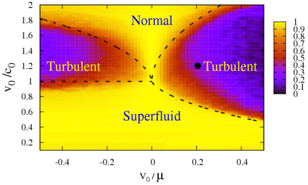
Different regimes of transport





- Superfluid regime
- No damping





- Strongly time dependent
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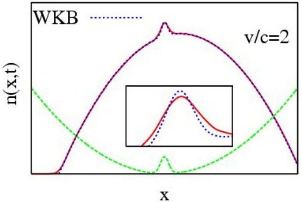
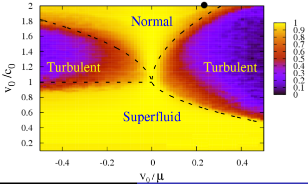
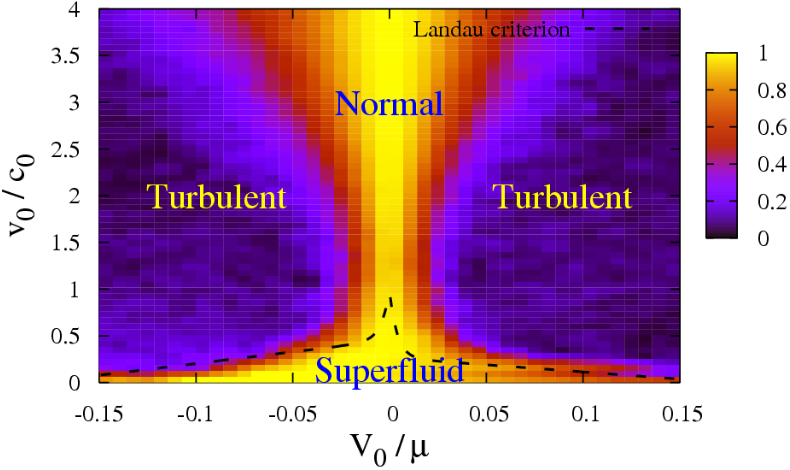




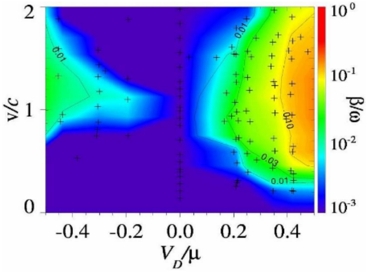
- Quasi-dissipationless
- No damping

Dipole oscillations in disorder

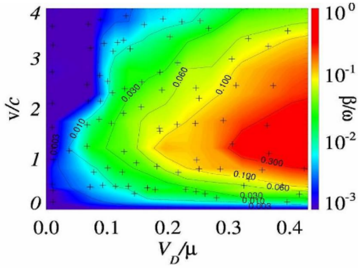


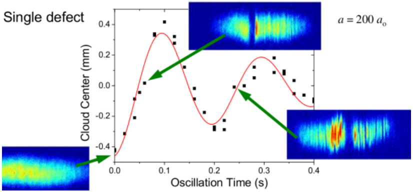
New experimental results at Rice (Hulet's group)

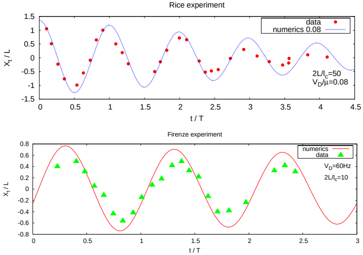
Single defect



Speckle

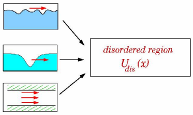


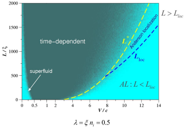


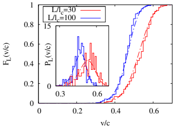


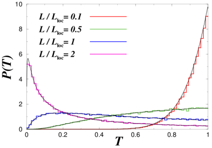
R. Hulet, http ://www.sit.it/SIF/resources/public/files/va2009/hulet_0624.pdf

Conclusion and outlook



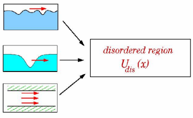


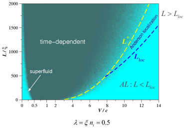


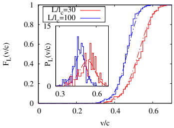


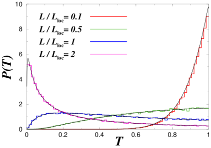
- General picture of transport : Superfluid, Dissipative, Quasi-Ideal Gas regimes.
- The superfluid critical velocity is related to extreme events.
- Effects of interaction on localization
  - DMPK eq., Renormalization of L_{loc}, $k \rightarrow \kappa = m\sqrt{v^2 - c^2}/\hbar$.
  - Destruction of localization for $L > L^*$ due to time dependent fluctuations.
- The localization transition in bichromatic potentials is not related to Anderson localization.
- Open questions : time dependent regime, link between stationary states and wave packets, higher dimension, strongly correlated regime, Fermi systems ...

Conclusion and outlook





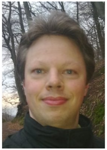




P. Leboeuf

N. Pavloff

M. Albert



T. Paul

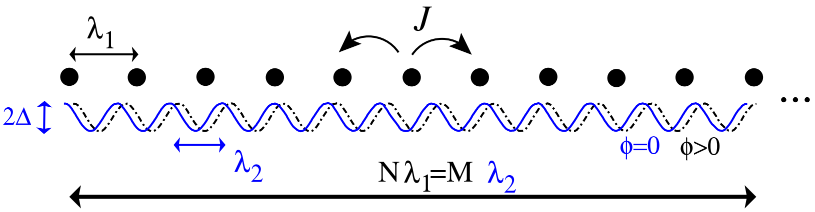
P. Schlagheck

Thank you for your attention !

For further reading see

- Paul *et al* PRA **80** 033615 (2009)
- Albert *et al* PRL **100** 250405 (2008)
- Albert and Leboeuf arXiv :0905.2331 (2009)
- Albert *et al, to be submitted*

1d Aubry-André-Harper Hamiltonian



1d Aubry-André Hamiltonian

$$\hat{H}_{AA} = J\sum_n |n\rangle\langle n+1| + |n+1\rangle\langle n| + \Delta\sum_n \cos\left(2\pi\beta n + \phi\right)|n\rangle\langle n|$$

$$\hat{T}_{\lambda_1}|n\rangle = \exp(-i\hat{p}\,\lambda_1/\hbar)|n\rangle = |n+1\rangle \quad \text{and} \quad \hat{q}|n\rangle = q_n|n\rangle = n\lambda_1|n\rangle$$

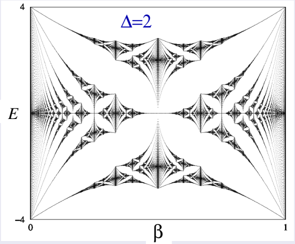
1d Harper Hamiltonian

$$\hat{H}_H = 2J\cos\left(2\pi\frac{\hat{p}}{P}\right) + \Delta\cos\left(2\pi\frac{\hat{q}}{Q} + \phi\right) \;,\; Q = \lambda_2 \;,\; P = 2\pi\hbar/\lambda_1$$

Phenomenology

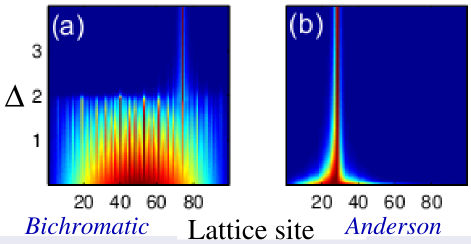
Hofstadter's Butterfly

- $\beta = p/q$: band structure
- β irrational : Cantor ensemble

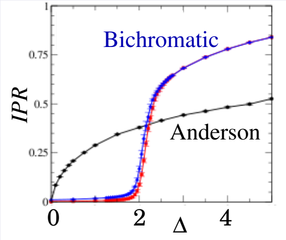


Localization transition at $\Delta = 2J$

$\neq$ Anderson : localization $\forall\,\Delta$ in 1d



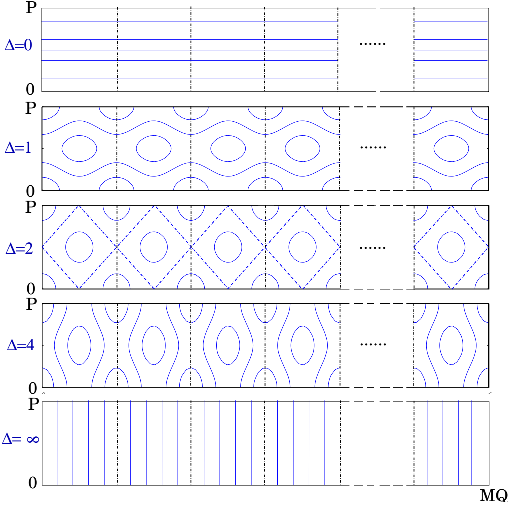
Lahini *et al* cond-mat 2008



Roux *et al* PRA 2008

Underlying classical dynamics in phase space

$$\mathcal{H} = 2J\cos(2\pi p/P) + \Delta\cos(2\pi q/Q + \phi)$$



- All classical trajectories are delocalized
- The last extended trajectory disappears
- All classical trajectories are localized

Firenze Experiment

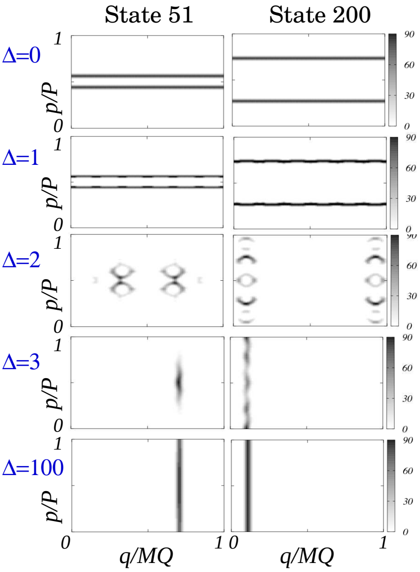
$\lambda_1 = 1032\ nm$ and $\lambda_2 = 862\ nm$

$$\frac{\lambda_1}{\lambda_2} = \frac{1032}{862} = \frac{512}{431} \simeq 1.19$$

$$\beta(= \frac{M}{N}) = \frac{512}{431} - 1 = \frac{85}{431} \simeq 0.19$$

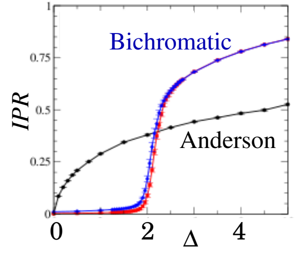
Even if the semi-classical limit is not achieved the classical dynamics is mostly responsible for the localization transition.

$\Rightarrow$ Localization is not related to Anderson localization



Anderson Model

Inverse Participation Ratio



Roux *et al* PRA 2008

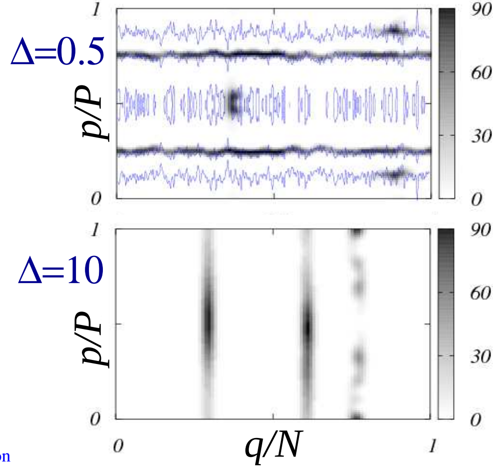
Non trivial Localization for $\Delta < 2J$.

Bichromatic

delocalized | classical localization

quantum localization | $2J$ | classical localization | Δ

Anderson



$N = 401$, $\phi = 0$. States 21, 201 and 301.

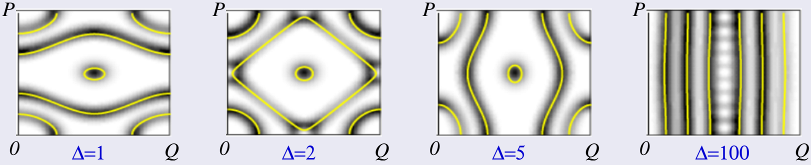
Phase Space representation and Semi-classical limit

- Diagonalization of $\hat{H} \to N$ eigenstates $|\psi_k\rangle$, E_k
- Projection on coherents states $|q,p\rangle$ (Gaussian wave packets of width $\sim h = \beta$).
- Husimi distribution $W_{\psi_k}(q,p) \sim |\langle q,p|\psi_k\rangle|^2$.

Semi-classical limit $h = \beta \to 0$

W_ψ concentrates on classical trajectories

(M=1) *Leboeuf-Saraceno/Voros 1989* $\quad W_\psi \sim \exp\left(-\frac{1}{h}\frac{(E-\mathcal{H}(q,p))^2}{\dot{q}^2+\dot{p}^2}\right)$

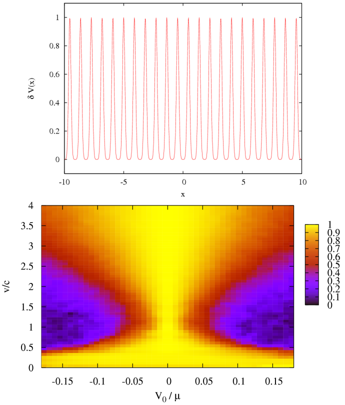


Bohr-Sommerfeld quantization rule $\quad S = \int_{tr} p dq = (n + \left[\frac{1}{2}\right])h$

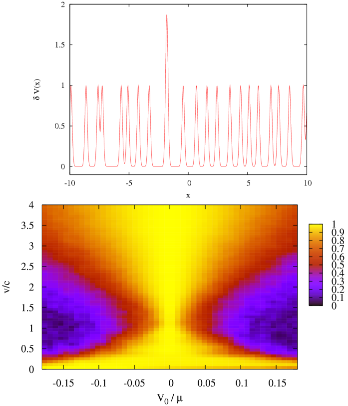
1 quantum state $\leftrightarrow$ 1 classical trajectory

Anderson Localization ?

Without disorder



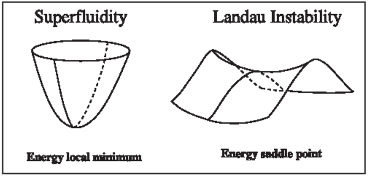
With disorder



Not visible in the range of parameters reachable in current experiments !

Landau and Dynamical instabilities of the superflow

Landau's instability

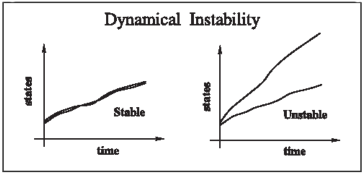


$$\psi(x) = \phi_0(x) + \delta\phi(x)$$

$$\Rightarrow \delta E = \int dx(\delta\phi, \delta\phi^*)\,\mathcal{M}\begin{pmatrix}\delta\phi\\ \delta\phi^*\end{pmatrix}$$

- Eigenvalue of $\mathcal{M} < 0$
- Equivalent to Landau's criterion for small potential

Dynamical instability

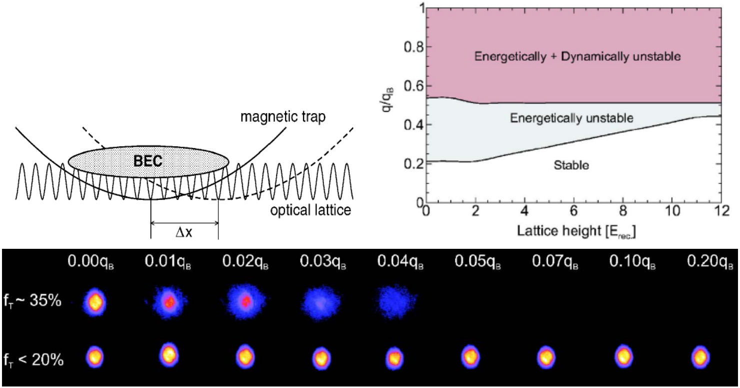


$$\psi(x,t) = \phi_0(x) + \delta\phi(x,t)$$

$$\Rightarrow i\frac{\partial}{\partial t}\begin{pmatrix}\delta\phi\\ \delta\phi^*\end{pmatrix} = \sigma_z\mathcal{M}\begin{pmatrix}\delta\phi\\ \delta\phi^*\end{pmatrix}$$

- Eigenvalue of $\mathcal{M} \in \mathbb{C}$.
- Only with periodic potential.

Experimental evidence of dynamical instability

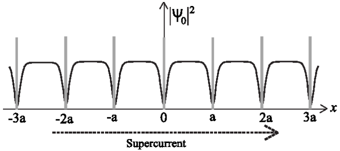


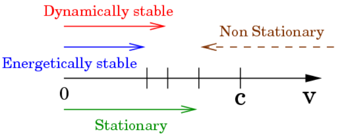
De Sarlo et al, PRA 72, 013603 (2005). Wu and Niu, New J. Phys., 5, 104, (2003).

- Energetical instability needs a “dissipative” effect to be activated.
- Dynamical instability occurs on a shorter time scale.

Criterion for the breakdown of superfluidity ?

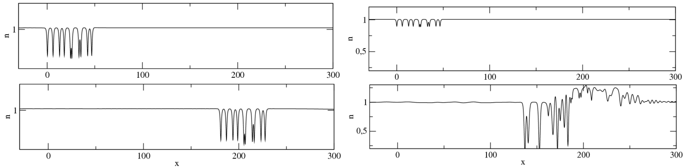
- The periodic case is pathological. Danshita et al, PRA 75 (2007).





but for a single delta peak all the criteria converge.

- Numerical studies for finite size obstacle and disorder (T. Paul not published).

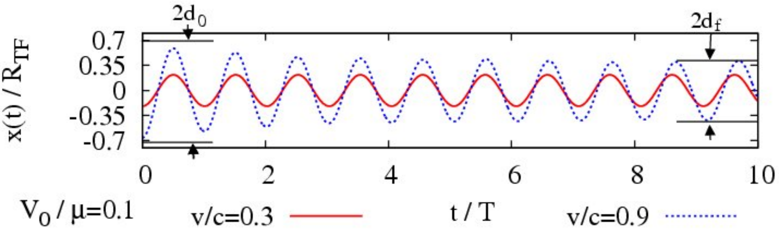


- Conjecture : Existence of subsonic stationary states ⇔ Energetic criterion.
- For smooth potential : local Landau's criterion.

Frequency shift in the superfluid regime

Center of mass motion = harmonic oscillator with frequency shift

$$\frac{d^2}{dt^2}\langle x\rangle = -\omega_0^2\langle x\rangle + \frac{1}{m}\int_{-\infty}^{+\infty} dx\, n(x-\langle x\rangle)\frac{dV(x)}{dx}$$



Frequency shift in the superfluid regime

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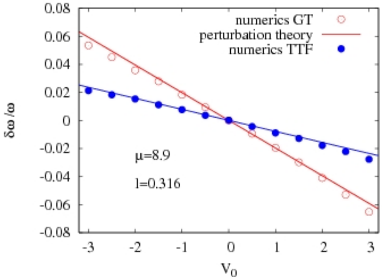
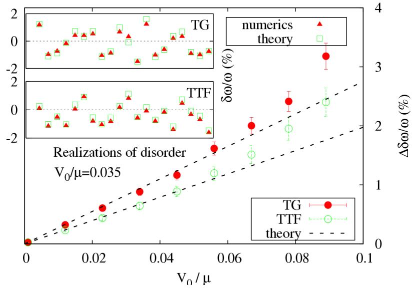
Single gaussian peak



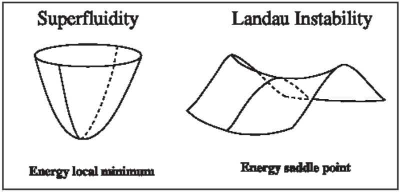
Speckle potential



This perturbative treatment is equivalent to sum rules approach (Modugno PRA 73 013606) and hydrodynamic theory (Albert not published).

Landau's (energetical) instability of the superflow

Is the Superfluid state the ground state of the system ? $\Rightarrow$ Linear stability analysis by expanding the energy functionnal arround ϕ_0.



$$\psi(x) = \phi_0(x) + \delta\phi(x)$$

$$\Rightarrow \delta E = \int dx(\delta\phi, \delta\phi^*)\,\mathcal{M}\begin{pmatrix} \delta\phi \\ \delta\phi^* \end{pmatrix}$$

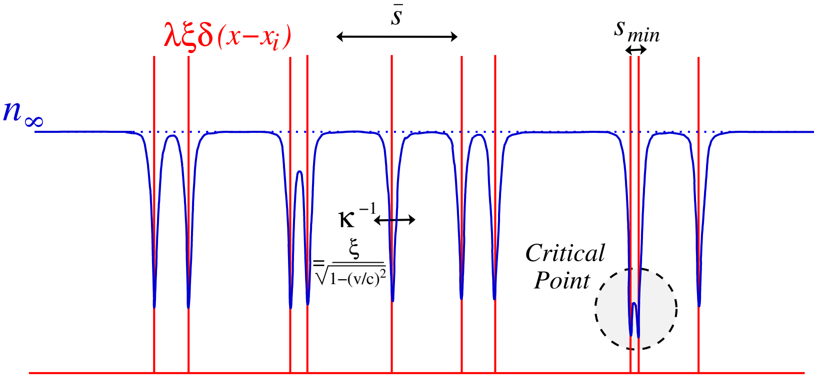
- Eigenvalues of $\mathcal{M}$ < 0 ?

- First step : find the (stationary) superfluid state (not easy !).
- Second step : diagonalize $\mathcal{M}$ for different v.

In the homogeneous case ($V(x) = 0$) : Landau's criterion $v_c = c = \sqrt{gn_0/m}$.

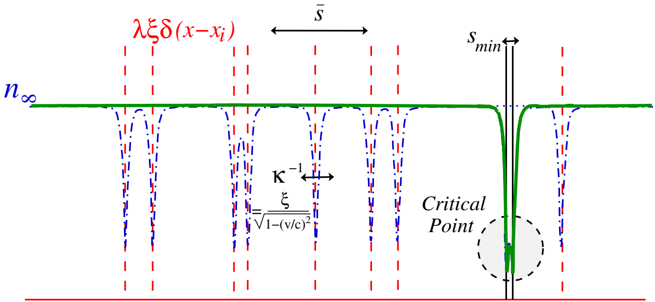
This is also possible to solve this problem for a few potentials (δ, double δ, array of δ and square barrier) but in general this is a very difficult problem (even numerically).

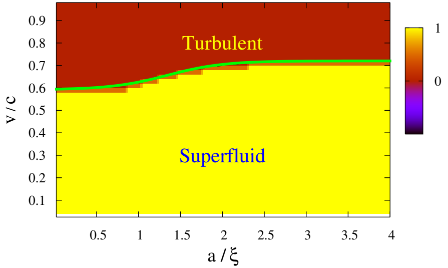
Distribution of the critical velocity with delta peaks



One cannot apply the Landau's criterion because the density is not smooth.

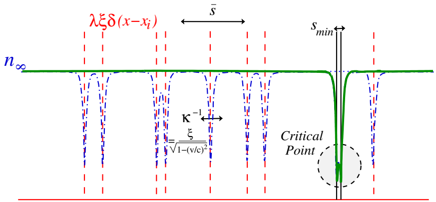
First Idea : take the two closest peaks

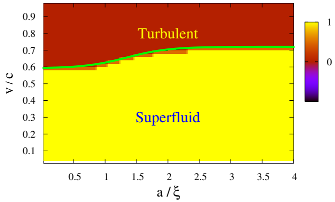


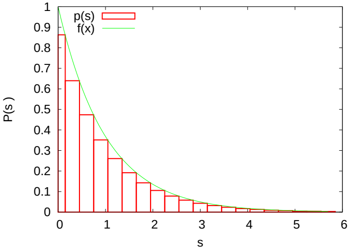


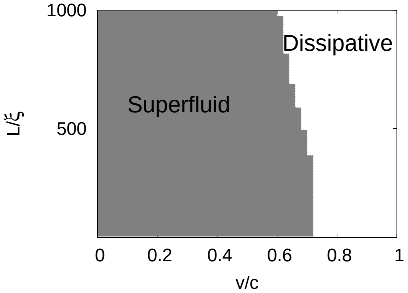
Sols and Zapata PRB 53 6693 (1995).

First Idea : take the two closest peaks

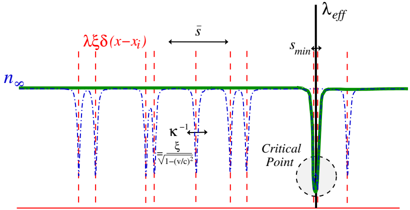


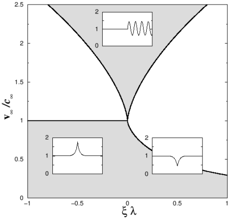


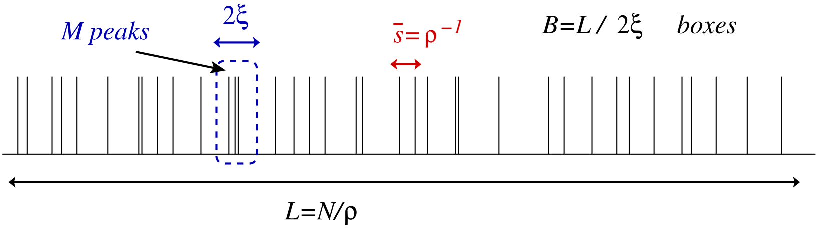




Second Idea : effective single impurity







Playing with extreme value statistics yields $\lambda_{eff} = \lambda\langle M_{max}\rangle(L)$. Then using the single impurity criterion one gets $v_c(L)$.

Second Idea : effective single impurity