Black holes in string theory: towards an understanding of quantum gravity

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UNIVERSITÉ PARIS Sud 11

et

INSTITUT DE PHYSIQUE THÉORIQUE - CEA/SACLAY

Thèse de doctorat

Spécialité **Physique Théorique**

Black holes in string theory: towards an understanding of quantum gravity

Trous noirs en théorie des cordes : vers une compréhension de la gravité quantique

présentée par **Clément RUEF**

pour obtenir le grade de

Docteur de l'Université Paris 11

Soutenue le 18 juin 2010 devant le jury composé de

Emilian Dudas	Président du Jury
Costas Bachas	Examinateur
Ruben Minasian	Examinateur
Iosif Bena	Directeur de thèse
Ashoke Sen	Rapporteur
Henning Samtleben	Rapporteur

Abstract

In this thesis I present the work I did during my PhD at the Institute of Theoretical Physical (IPhT), CEA Saclay, under the supervision of Iosif Bena. The framework I have been working in is string theory, and more precisely supergravities in ten and eleven dimensions, as low energy limits of string theory. The first part of the thesis deals with the study of supersymmetric three-charge black holes and black hole microstates: Using supersymmetric D-branes called supertubes, we have performed a probe analysis of supergravity solutions, and showed how this approach *exactly captures*, in all known cases, the physical properties of the complete supergravity solution. We also found that when the supertube is in a magnetically charged background, it sees its entropy *enhanced* with respect to its flat space one. The supergravity solutions sourced by supertubes are regular and horizonless, and hence can be seen, in the "fuzzball proposal", as black hole microstates. This enhanced entropy could therefore contribute for a large part in a microscopic counting of the black hole entropy. In the second part of the thesis, I present a new class of five-dimensional *non-supersymmetric* solutions, called "floating brane" solutions. The equations giving these new solutions generalize the BPS equations and have the key property to still be partially *first order* and *linear*. The BPS equations, and thus all the known supersymmetric solutions, are recovered as a subcase of the floating brane equations. Some of the new solutions have a horizon and are thus black holes – with different horizon topologies – but some are completely regular and horizonless and should correspond to microstates of *non extremal* black holes.

Résumé

Dans cette thèse je présente les travaux effectués lors de mon doctorat à l'Institut de Physique Théorique (IPhT) du CEA de Saclay, sous la direction de Iosif Bena. Ceux-ci ont pour cadre la théorie des cordes, et plus précisément la supergravité à dix et onze dimensions, comme limite de basse énergie de la théorie des cordes. La première partie concerne l'étude des trous noirs et microétats de trous noirs supersymétriques à trois charges. En utilisant une D-brane supersymétrique appelée supertube, nous avons effectué une approche test et montré que cette approche capture dans tous les cas connus les propriétés physiques de la solution complète de supergravité. Nous avons aussi prouvé que le supertube, quand il est placé dans un fond ayant des charges magnétiques, voit son entropie *augmentée* par rapport à celle qu'il a en espace plat. Les solutions de supergravité sourcées par des supertubes étant régulières et sans horizon, elles peuvent être vues, dans le contexte du "fuzzball proposal", comme des microétats de trous noirs. Cette entropie augmentée pourrait donc contribuer pour une large part dans le cadre d'un comptage microscopique de l'entropie de trou noir, . Dans la deuxième partie de la thèse, je présente une nouvelle classe de solutions *non supersymétriques* de supergravité à onze dimensions, appelées solutions "à branes flottantes". Les équations donnant ces nouvelles solutions généralisent les équations BPS, et ont, comme ces dernières, l'énorme avantage d'être partiellement *du premier ordre* et *linéaires*. Les équations BPS, et donc toutes les solutions supersymétriques, se retrouvent comme une sous-famille des équations à branes flottantes. Certaines de ces nouvelles solutions ont un horizon et sont donc des trous noirs – avec des topologies d'horizon variées – mais certaines sont complètement régulières et sans horizons et correspondraient à des microétats de trous noirs *non extrémaux*.

Publications

Published papers

- **The Nuts and Bolts of Einstein-Maxwell Solutions**,
  N. Bobev, C. Ruef,
  JHEP **01** (2010) 124, arXiv:0912.0010 [hep-th]

- **Supergravity Solutions from Floating Branes**,
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  JHEP **03** (2010) 047, arXiv:0910.1860 [hep-th]

- **A (Running) Bolt for New Reasons**,
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  I. Bena, G. Dall'Agata, S. Giusto, C. Ruef, N. Warner,
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  I. Bena, N. Bobev, C. Ruef, N. Warner,
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Remerciements

Certains disent que le plus important dans une thèse sont les remerciements. Je ne suis pas loin d'être de cet avis, car si vous avez cette thèse entre les mains, c'est en grande partie grâce à tous ceux qui m'ont aidés plus ou moins directement au cours de mes trois années de doctorat.

Mes premiers remerciements vont naturellement à Iosif Bena, mon directeur de thèse. La théorie des cordes est un domaine vaste et compliqué, dans lequel il est facile de se perdre. Grâce à son aide, j'ai pu y entrer progressivement en apprenant les bases tout en commençant mes premiers travaux de recherche. Il m'a aussi beaucoup appris sur le monde plus général de la recherche et m'a toujours poussé à devenir indépendant et à travailler avec d'autres. Cela m'a permis de voir d'autres approches, d'autres méthodes de travail, et je suis conscient que ces interactions m'ont été très bénéfiques. C'est pourquoi je tiens aussi à remercier chaleureusement mes collaborateurs, et particulièrement les plus proches: Stefano Giusto pour m'avoir beaucoup apporté en étant l'exact complémentaire de Iosif, et sans qui je serais sûrement encore bloqué quelque part entre un facteur 2π et la définition d'une charge invariante de jauge, et Nick Warner et Nikolay Bobev dont j'ai pu apprécié non seulement les qualités scientifiques mais aussi le sérieux sens de l'accueil en passant un excellent séjour en Californie. J'espère pouvoir continuer à travailler longtemps avec eux.

Le plaisir que l'on prend à faire de la recherche peut changer du tout au tout en fonction de l'environnement dans lequel on le fait. Je suis conscient que si j'ai apprécié ces trois années de thèse, c'est aussi beaucoup grâce à l'ambiance chaleureuse et dynamique de l'IPhT. Dans ce cadre, je tiens à remercier les chercheurs de l'IPhT avec lesquels j'ai interagi, tous les membres du groupe de théorie des cordes, particulièrement Sheer El Showk et Mariana Graña pour avoir toujours été disponible quand j'arrivais dans leur bureau avec une question à poser. Je pense ensuite tout naturellement aux autres thésards du laboratoire. Non seulement j'ai beaucoup discuté avec eux de physique, de maths, de la vie, de l'univers et du reste, mais leurs coups de main en latex ou en mathematica m'ont aussi servis plus d'une fois. Enfin, je souhaite remercier chaleureusement les secrétaires du laboratoire, Sylvie Zaffanella, Catherine Cathaldi et Laure Sauboy. Tout membre de l'IPhT, je pense, sait à quel point leur aide logistique est précieuse et efficace, leur sourire agréable, et la réserve de petits chocolats dans le bureau de Laure inépuisable.

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Contents

Introduction générale

en français

Vous avez entre les mains ma thèse de doctorat, qui traite des trous noirs, de théorie des cordes et de gravité quantique. Comme tout travail de recherche, le contenu est assez pointu et technique par endroit, ce qui le rend difficile d'accès aux néophytes. Malgré cela, le sujet est à mon avis très intéressant, et pas uniquement pour les spécialistes. J'aimerais donc en présenter les enjeux et motivations à tous ceux qui pourraient être intéressés. C'est le but de cette première introduction : (essayer d') expliquer simplement, quoiqu'un peu rapidement[1], de manière accessible, le contexte et les motivations de mes travaux de thèse. Si vous êtes déjà familier avec la physique des trous noirs et les problèmes liés à la gravité quantique, vous pouvez directement aller à la deuxième introduction, où les motivations sont plus précises et plus ancrées directement en théorie des cordes, ou au corps de la thèse proprement dit.

Pour expliquer mes motivations à travailler sur ce sujet, de nombreuses questions se bousculent : "Qu'est ce qu'un trou noir ?", "Qu'est ce que la gravité quantique ?", "Pourquoi faire de la gravité quantique, puisque les expériences qu'on peut faire n'en ont pas besoin ?", "Quel est le lien entre trou noir et gravité quantique ?", "Et le lien avec la théorie des cordes, dans tout ça ?", ... Si plusieurs angles d'attaque sont possibles pour parler de toutes ces questions, il faut bien en choisir un. Commençons donc par les trous noirs :

Petite histoire des trous noirs

La première idée des trous noirs remonte au XVIIIème siècle. Partant de l'idée qu'à tout astre est associée une vitesse d'échappement (expliquée sur la figure 1), l'anglais John Michell en 1783 et le français Pierre-Simon Laplace en 1796 intuitent de façon indépendante l'idée qu'un astre puisse être tellement lourd que même la lumière ne puisse pas s'en échapper. "*Si le demi-diamètre d'une sphère de la même densité que le soleil et qui excéderait celui du soleil d'une proportion de 500 à 1, un corps tombant depuis une hauteur infinie vers elle aurait acquis à sa surface une vitesse plus grande que celle de la lumière. En conséquence, supposant que la lumière est attirée par la même force en proportion de sa "vis inertiae" [masse d'inertie], comme les autres corps, toute lumière émise depuis ce corps reviendrait sur elle-même par sa propre gravité.*" dit Michell. Ils comprennent que, bien qu'invisibles – car n'émettant pas de lumière – de tels astres auraient de forts effets gravitationnels sur leur entourage. Physiquement, on peut

[1] Cette thèse n'étant pas vouée à un réel travail de vulgarisation, les explications que je donne dans cette introduction sont un peu trop rapides et imprécises. J'espère toutefois donner l'idée de ce que je fais dans cette thèse de manière simple.

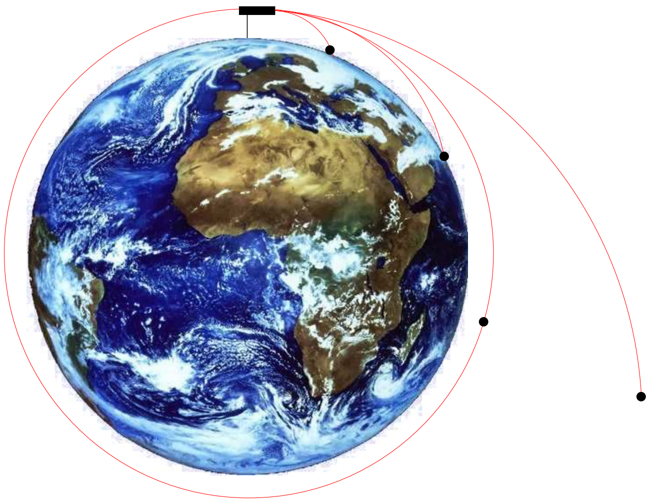

FIG. 1 – La vitesse d'échappement d'un astre est la vitesse minimale qu'il faut donner à un objet pour que celui-ci puisse s'échapper de l'attraction de l'astre (en négligeant les frottements dus à une éventuelle atmosphère). A titre d'exemple, on peut voir sur ce schéma un canon situé au pôle nord de la terre tirer des projectiles à differentes vitesses. Les deux premiers n'ont pas une vitesse suffisante pour s'échapper de l'attraction terrestre. Le troisième est envoyé exactement à la vitesse d'échappement, et effectue donc le tour de la terre. Le dernier projectile est envoyé plus vite que la vitesse d'échappement, et arrive donc à sortir de l'attraction terrestre.

voir cela naïvement de la façon suivante : de la surface de la terre, pour réussir à envoyer un objet assez fort pour qu'il sorte de l'attraction terrestre, il faut lui donner une certaine vitesse initiale. Pour fixer les idées, cette vitesse est d'environ 11 m/s pour la terre. Si on imagine un astre beaucoup plus lourd, cette vitesse de libération pourrait devenir plus grande que la vitesse de la lumière, et donc les rayons lumineux ne pouraient pas s'échapper de l'attraction de cet astre. Celui-ci deviendrait donc noir et ne pourrait pas émettre de lumière.

Les savants de l'époque ne prirent pas cette observation au sérieux, la trouvèrent au plus intéressante et sans sens physique réel ; il fallut attendre 1915 et la célèbre théorie de la relativité générale d'Einstein pour voir les trous noirs revenir. La relativité générale explique, entre autre, que le temps et l'espace, contrairement à notre intuition, ne sont pas statiques et immuables, mais qu'ils sont tout d'abord un seul et même objet, l'espace-temps, et que cet objet est dynamique, c'est à dire qu'il évolue comme toute autre quantité physique. Cette évolution est liée à son contenu en matière et en énergie par les fameuses équations d'Einstein. Malgré la complexité de ces équations, à peine quelques mois après la publication des travaux d'Einstein, Karl Schwarzschild fut le premier à en trouver une solution. Cette solution correspond à une masse ponctuelle immobile au centre de l'espace. Loin de cette masse ponctuelle, cela décrit en très bonne approximation l'espace-temps créé par un astre, le soleil par exemple. Et loin de cet astre, on retrouve la physique Newtonienne. Mais si l'on s'approche de la masse ponctuelle, on arrive dans un régime où la gravitation de Newton n'est plus valide et le comportement de l'espace-temps devient alors bien étrange. À partir d'un rayon particulier, appelé rayon de Schwarzschild et proportionnel à la masse ponctuelle qu'on a placé au centre de l'espace, on peut dire de manière un peu grossière que le temps prend la place de l'espace et l'espace celle du temps. En analysant plus finement la solution, les physiciens ont pu démontrer, dans ce cadre

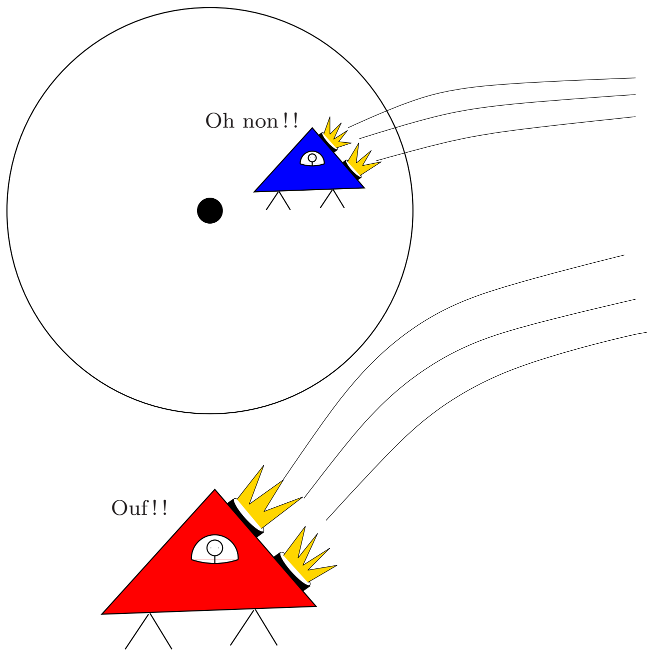

FIG. 2 – Deux vaisseaux spatiaux à proximité d'un trou noir. Le premier, en rouge, arrive à rester à l'extérieur de l'horizon des évènements et donc à échapper à l'attraction du trou noir. Le vaisseau bleu a lui passé l'horizon du trou noir. Malgré tous ses efforts, il ne pourra donc jamais ressortir.

précis, que tout ce qui tombait à l'intérieur de ce rayon ne pouvait jamais ressortir. Jamais au sens strict du terme : ressortir est théoriquement impossible (voir figure 2). Et ceci est vrai tant pour de la matière que pour la lumière. Autrement dit, cette solution a les propriétés de ces fameux astres noirs de Laplace et Michell. On appelle la surface délimitant l'endroit d'où plus rien ne peut s'échapper l'horizon des évènements, ou simplement horizon, du trou noir.

Il est important de s'arrêter un instant pour bien comprendre les différents ordres de grandeur. Pour cela, prenons l'exemple du soleil. La masse de celui-ci est d'environ 10^{30} kg, pour un rayon de 500 000 km. Le rayon de Schwarzschild associé à la masse du soleil est, lui, d'environ 3 km. L'approximation de masse ponctuelle est valide pour le soleil pour des distances plus grande que son rayon. Pour des distances plus petites, en dessous de 500 000 km, la solution de Schwarzschild ne décrit plus du tout le soleil, qui ne peut plus être considéré comme ponctuel. Pour envisager que les comportements étranges dont nous avons parlé puissent avoir un sens physique, il faut imaginer l'existence d'un objet qui aurait toute la masse du soleil dans une boule de moins de trois km de rayon ! Ce doit donc être un objet extrèmement compact.

Pendant longtemps, la grande majorité des physiciens a pensé que des objets aussi compacts ne pouvaient pas exister, et que ce fameux rayon de Schwarzschild n'était en fait qu'un artefact mathématique sans réel sens physique. Faisons alors un nouveau saut pour arriver directement dans les années 60. A cette époque, les améliorations des techniques d'astrophysique observationnelle ont assez avancé pour que les physiciens commencent à penser sérieusement que de tels objets pourraient exister réellement. L'un des plus ardents défenseurs de la possibilité de l'existence des astres noirs fut John Wheeler. Il est d'ailleurs l'inventeur du nom de “trou noir”, “trou” car si on tombe dedans on ne peut pas ressortir, et “noir” car la lumière elle-même ne

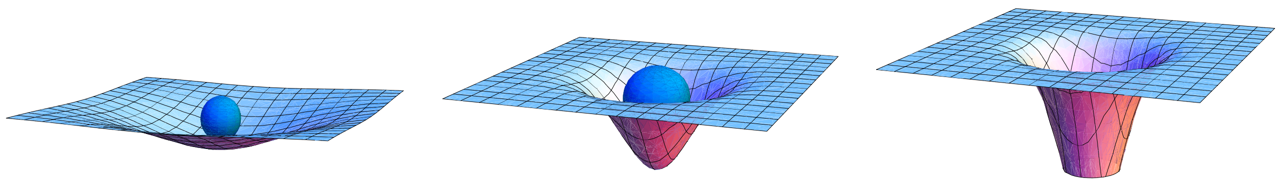

FIG. 3 – La relativité générale explique que la présence d'une masse en un point de l'espace-temps le courbe, et c'est cette courbure qui crée l'attraction gravitationnelle. Sur ce schéma est représenté la courbure de l'espace-temps en présence de différentes masses. Dans le premier cas, la masse est faible, et la courbure n'est donc pas très importante. Dans le deuxième cas, la masse est plus importante, et l'espace-temps est fortement courbé. Dans le dernier cas, la masse est si compacte qu'un trou noir apparaît : la courbure devient infinie en son centre. Sur le schéma n'est représenté l'espace-temps qu'en dehors de l'horizon des évènements.

pouvant pas sortir, cet astre de l'extérieur doit être complètement noir. A partir des années 60, les physiciens comprirent qu'un trou noir pouvait se former à la fin de la vie de certaines étoiles. Je ne m'étendrai pas ici sur les techniques observationnelles permettant d'observer (indirectement) les astres pouvant possiblement être des trous noirs, et dans la suite, je m'intéresserai uniquement à leurs propriétés théoriques.

Après Wheeler, de nombreux physiciens commencèrent à travailler pour comprendre ces objets étranges, à commencer par Stephen Hawking. Leurs travaux ont permis une meilleure compréhension des trous noirs, mais ont aussi montré qu'un certain nombre de problèmes restaient très difficile, voire impossible à comprendre dans le cadre de la relativité générale. Essayons de résumer tout cela :

- **La singularité centrale.** Le premier problème concerne ce qui se passe au centre du trou noir. Un trou noir se forme à la fin de la vie d'une étoile si rien n'arrive à s'opposer à l'effondrement gravitationnel. Tous les autres astres ou objets astrophysiques sont maintenus d'une part par l'attraction gravitationnelle et d'autre part par une autre force permettant de lutter contre cette attraction, et donc de les stabiliser à une taille fixe. Par exemple, pour la terre, cette deuxième force est créée par les interactions électromagnétiques alors que pour le soleil, elle est due aux réactions de fusion du coeur. Dans un trou noir, au contraire, rien ne s'oppose à la force de gravitation, celui-ci s'effondre donc sur lui-même jusqu'à ce que la masse se retrouve entièrement concentrée au milieu, formant une "singularité", une explosion de l'espace-temps où de nombreuses quantitées mesurables deviennent infinies (voir figure 3). Cette singularité indique que nous arrivons aux limites de notre modèle théorique (la relativité générale), et qu'un autre phénomène physique qui n'est pas pris en compte dans notre théorie apparaît. Le premier point à comprendre est donc ce qu'est ce nouvel élément, ce nouveau phénomène physique qui vient corriger la singularité.
- **L'entropie du trou noir.** En 1967, Werner Israel et Brandon Carter démontrèrent le "théorème d'unicité des trous noirs". Celui-ci nous dit qu'un trou noir stationnaire est déterminé de manière unique par un petit nombre de quantités : sa masse, sa charge

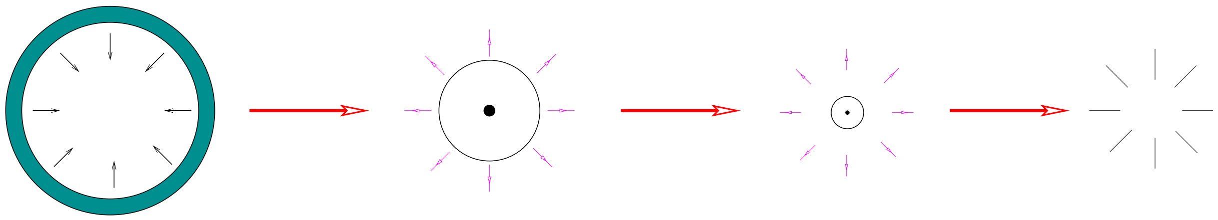

FIG. 4 – Ce schéma représente la “vie” d'un trou noir. Au départ, de la matière s'effondre sur elle-même er forme le trou noir. Celui-ci émet alors un peu de lumière sous forme de radiation de Hawking. Ce faisant, il perd peu à peu son énergie, donc sa masse, et réduit. Si le phénomène ne s'arrête pas, le trou noir finit par disparaître complètement. La radiation émise par le trou noir étant thermale, elle n'a pas pu emporter d'information sur la matière qui constituait le trou noir. On a donc à la fin perdu l'information sur la façon dont il a été formé.

éléctrique et son moment cinétique (qui traduit comment il tourne sur lui-même). Ce théorème est un résultat très important, mais, comme souvent en physique, il pose en fait plus de nouveaux problèmes qu'il ne donne de réponses. En effet, il stipule entre autres que vous pouvez former un trou noir avec n'importe quoi, la seule chose qui importe au final est la valeur de ces quantités, qui le déterminerons de manière unique : prenez un trou noir et rajoutez y un vélo ou votre petite soeur, pourvu qu'ils aient la même masse, le trou noir résultant sera le même. Cela, en plus d'être conceptuellement troublant, pose un problème à cause d'une quantité physique appelée l'entropie. L'entropie est une quantité thermodynamique qui traduit grossièrement le désordre d'un système. Plus elle est grande, plus le système est désordonné. En particulier, l'entropie d'un système isolé ne peut que croître (on ne peut pas ranger chez soi sans aucun apport extérieur d'énergie). D'un point de vue microscopique, l'entropie d'un système est directement relié au nombre de façons qu'on a de réaliser ce système microscopiquement, de “microétats” du système. Pour clarifier, prenons un exemple usuel : l'air dans une pièce est principalement déterminé par sa température et sa pression. Cela suffit à le décrire macroscopiquement, globalement. Mais il est microscopiquement constitué de particules, et il y a de très nombreuses façons d'organiser ces particules qui donnent globalement le même gaz. Ce sont les “microétats”. C'est une notion très importante sur laquelle nous reviendrons plus tard. Si on applique tout cela aux trous noirs, on peut facilement montrer qu'en y jetant deux choses différentes mais de même masse, on a moins d'états à la fin qu'au début. Beckenstein et Hawking introduisirent alors une nouvelle entropie, associée à un trou noir. Celle-ci est proportionnelle à l'aire de son horizon. Si on prend en compte cette entropie supplémentaire, on retrouve bien le fait que, même en présence d'un trou noir, l'entropie d'un système isolé ne peut que grandir. La théorie physique “habituelle” est donc retrouvée, au prix de l'introduction de cette nouvelle entropie. Le problème, encore ouvert à l'heure actuelle, est alors de comprendre cette entropie de manière microscopique, autrement dit, de comprendre ce que sont les “microétats de trou noir”.

– **Le paradoxe de perte d'information.** Qu'en est-il de l'information sur la façon dont on a formé le trou noir ? Quand on forme un trou noir en compactant de la matière, on ne peut plus, de l'extérieur, avoir d'informations sur ce qui constitue le trou noir à cause de l'horizon des évènements. En effet, comme rien ne peut ressortir, ni matière ni radiation ne peuvent sortir pour nous dire ce qu'il y a à l'intérieur. De dehors, à cause du théorème d'unicité, on ne peut avoir accès qu'à la masse, au moment cinétique et à la charge du trou noir, et aucunement aux détails de ce qui le constitue. Autrement dit une partie de l'information est cachée à l'intérieur de l'horizon, mais on peut penser qu'elle n'a pas disparu. Le problème vient alors de la suite de l'histoire : Hawking, poursuivant ces travaux, se demanda comment les propriétés quantiques de la matière pouvaient modifier la compréhension des trous noirs. Il réussit à montrer que si la matière est quantique, le trou noir n'est en fait pas complètement noir mais émet un peu de lumière. Cette émission, appelée radiation de Hawking, est émise à partir de l'horizon, et ne dépend que de la température du trou noir, liée à sa masse, et aucunement de ce qu'on a pu y mettre au départ. En d'autre termes, elle ne donne aucune information sur les détails de son intérieur. Cette émission ayant un peu d'énergie, en l'émettant cette radiation, le trou noir perd peu à peu de énergie, jusqu'à sa disparition complète. Or si le trou noir s'évapore en émettant de la lumière qui ne contient pas d'information sur sa formation, on a au total perdu de l'information à jamais. Ce phénomène est illustré schématiquement sur la figure 4. Dans un tel scénario, on ne peut après évaporation jamais remonter à la façon dont le trou noir avait été formé. Cette perte d'information va contre tous les principes connus de la mécanique quantique. Il est donc important de comprendre s'il y a vraiment perte d'information – et dans ce cas comment allier cela avec le reste de la physique – ou si l'une des étapes du raisonnement est fausse – et dans ce cas laquelle et qu'est ce qui la remplace.

Gravitation quantique et théorie des cordes

Les trois problèmes que sont la singularité, l'entropie et la perte d'information, différents bien que reliés, ne sont actuellement toujours pas clairement compris. Il est clair qu'on ne pourra pas leur donner une réponse satisfaisante dans la cadre de la relativité générale, et ceci pour la raison suivante : comme nous l'avons déjà dit, rien, classiquement, ne s'oppose à l'effondrement gravitationnel dans un trou noir, et cela conduit à une singularité. Or, dans un modèle physique, les singularités et l'apparition de l'infini dans les calculs sont très souvent le signe que quelque chose de nouveau apparaît, de nouveaux phénomènes physiques qui ne sont pas prise en compte dans notre description. Qu'est ce que cette nouvelle physique dans le cas qui nous intéresse ici ? Pour comprendre cela, faisons un petit détour théorique, illustré sur la figure 5 : la relativité générale décrit les grandes masses, les forts champs gravitationnels, et ceci de manière classique (par opposition à la mécanique quantique). La physique quantique s'intéresse quant à elle à la physique à petite échelle. Ces deux théories, bien que parfaitement vérifiées expérimentalement chacune de leur côté, ne décrivent pas la même chose et sont incompatibles dans leur forme actuelle. Si on veut décrire de grandes masses dans de petits volumes – autrement dit des trous noirs –, il nous faut une nouvelle théorie, qui arrive à allier la relativité générale et la mécanique quantique : la gravité quantique.

Depuis maintenant plus de trente ans, plusieurs théories candidates se sont développées. Celle dans laquelle je travaille, la théorie des cordes, est l'une des plus connues et des plus

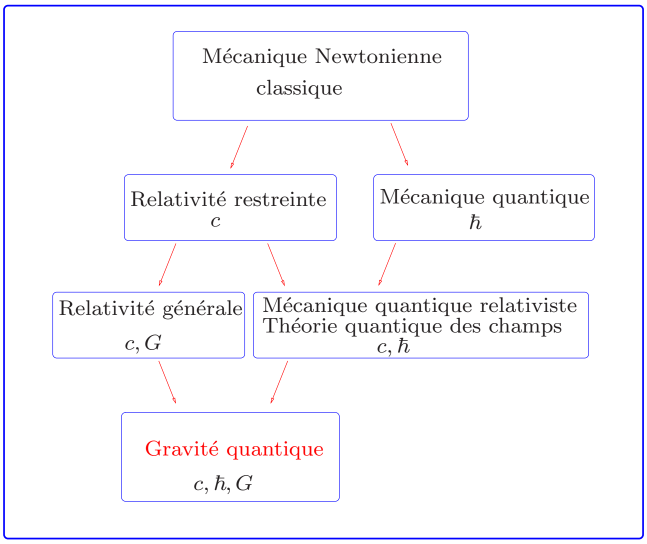

FIG. 5 – Représentation simplifiée des liens entre les différentes théories physiques : la mécanique Newtonienne décrit très bien la physique classique, à notre échelle. Il existe alors deux limites différentes dans lesquelles elle n'est plus pertinente : la première quand il s'agit d'expliquer des phénomènes à très petite échelle, auquel cas elle doit être remplacée par la mécanique quantique (liée à la constante fondamentale $\hbar$, appelée constante de Planck). Le deuxième limite est prise si les vitesses en jeu ne sont plus négligeables devant le vitesse de la lumière c ; la relativité restreinte doit alors être introduite. On peut ensuite réunifier la mécanique quantique et la relativité restreinte, pour prendre en compte à la fois les vitesses importantes et les petites échelles, et cela forme la théorie des champs. D'autre part, pour décrire les interactions d'objets très massifs, générant de forts champs gravitationnels (liés à la constante de gravitation G), la relativité restreinte doit être généralisée, ce qui donne la relativité générale. Enfin, il nous reste encore à trouver la théorie de gravité quantique, qui permettra d'unifier relativité générale et théorie des champs.

développées[1]. Bien qu'étant née dans un autre contexte, elle peut naturellement être vue comme une théorie de gravitation quantique. Dans le cadre de cette théorie, il a été possible de développer des outils précis pour s'attaquer aux problèmes relatifs aux trous noirs :

En utilisant une dualité de la théorie, c.à.d. une façon de décrire le même système de deux façons différentes, Strominger et Vafa ont, en 1997, réussi à compter, dans un certain régime, les configurations microscopiques d'un système dual à un trou noir. Et ce comptage microscopique donne au final exactement l'entropie de Beckenstein-Hawking associée. Autrement dit, ils ont effectué le premier comptage microscopique de l'entropie d'un trou noir. Ce résultat, bien que valable uniquement pour une classe de trous noirs assez particulière est un résultat très important, qui donne des éléments de compréhension nouveaux. Il ne résout toutefois pas tout. En effet, le comptage est fait dans un régime où le trou noir n'existe pas vraiment, et utilise un argument extérieur – cette dualité – pour expliquer que le comptage correspond bien à l'entropie de Beckenstein-Hawking. Il n'explique donc pas vraiment ce qu'est un microétat de trou noir, ni comment le problème de perte d'information peut être résolu.

Pour répondre à ces questions, Samir Mathur, au début des années 2000, fit une proposition

[1] La deuxième autre principale théorie candidate est la gravité quantique à boucles, basée sur ue quantification directe de la théorie d'Einstein, mais exprimée en termes de variables plus appropriées. On peut également citer la géométrie non-commutative ou encore la gravité asymptotiquement régulière.

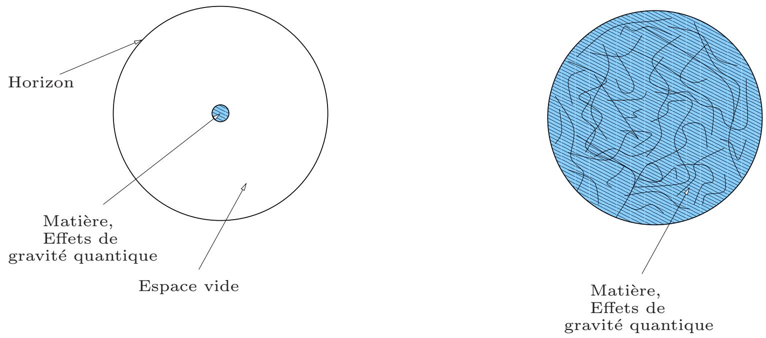

FIG. 6 – Sur ce schéma sont représentées deux possibilités quant à l'intérieur d'un trou noir. Le point de vue classique est représenté à gauche : toute la masse constituant le trou noir est concentrée dans un petit volume de la taille de Planck en son centre, et le reste est complètement vide. En particulier l'espace est vide aux alentours de l'horizon. À droite, on voit l'intérieur du trou noir dans l'hypothèse du "fuzzball proposal". Ici, les effets quantiques ne se limitent pas au centre du trou noir, mais s'étendent jusqu'à son horizon. Dans cette image, l'ensemble du trou noir est donc contitué de matière.

intéressante[1], connue sous le nom de conjecture de Mathur, ou "fuzzball proposal". Les idées physiques soutendant cette conjecture ne sont pas fondamentalement liées à la théorie des cordes, je vais donc essayer de les expliquer simplement :

À quel ordre de grandeur peut-on s'attendre à avoir des effets de gravité quantique ? Un raisonnement dimensionnel nous dit qu'à l'aide des trois constantes fondamentales que sont la vitesse de la lumière c, la constante de gravitation G et la constante de Planck $\hbar$, la seule quantité homogène à une longueur qu'on peut fabriquer est $l_P = \sqrt{\hbar G/c^3} \approx 10^{-35}$m, qu'on appelle la longueur de Planck. Cela nous dit donc qu'on s'attend naturellement à voir apparaître les effets de gravité quantique à cette faramineusement petite échelle. Pour comparer, la taille d'un proton par exemple est d'environ 10^{-15}m. La longueur de Planck est donc 100 000 fois plus petite qu'un objet qui serait aussi petit pour le proton qu'un proton pour nous. Bref, c'est petit, bien plus que le rayon de Schwarzschild d'un trou noir. Naïvement, on s'attend donc à ce qu'à l'intérieur d'un trou noir, les effets de gravité quantique corrigent la singularité centrale sur une cellule de la taille de Planck, bien plus petite que la taille du trou noir. Mais un deuxième cas est possible : comme au trou noir est associé un certain nombre N de microétats, cette taille l_P peut être multipliée par un facteur adimensionnel dépendant de N. Comme N peut être très grand, cela pourrait faire grandir les effets quantiques jusqu'à des tailles macroscopiques, typiquement jusqu'au rayon de Schwarzschild. C'est l'essence de la conjecture de Mathur, représentée sur le schéma 6. Remarquons que ce genre de comportement n'est pas nouveau : dans les naines blanches et les étoiles à neutrons, l'effondrement gravitationnel est contré par un effet purement quantique, la pression de Fermi, qui s'étend grâce au grand nombre de particules jusqu'à la taille de l'étoile, bien plus grande que les échelles habituelles de physique quantique. Si l'on suppose que la proposition de Mathur est vraie, et que les effets quantiques s'étendent jusqu'à la taille de l'horizon, cela veut dire que la description donnée par la relativité générale, qui ne prend pas en compte ces phénomènes quantiques, n'est plus valide dès l'horizon, et non pas seulement au centre du trou noir. Un bon moyen de comprendre

[1]La conjecture de Mathur n'est évidemment pas la seule proposition de réponse. Je ne m'intéresserais ici toutefois qu'à elle.

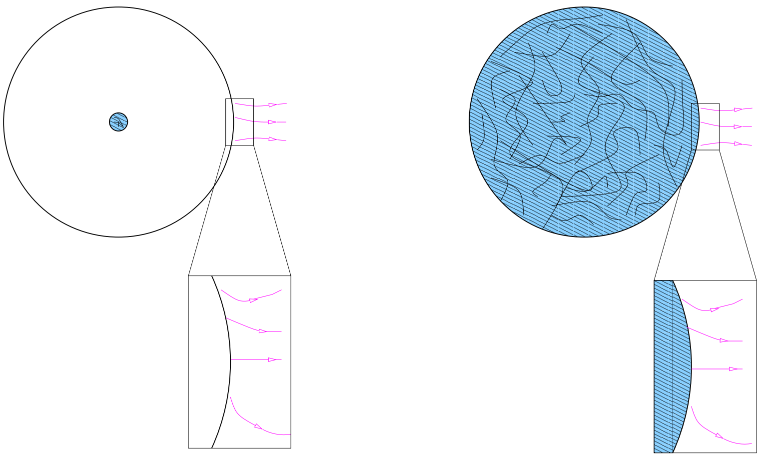

FIG. 7 – Ce schéma reprend les deux images possibles pour l'intérieur des trous noirs, présentés figure 6, et illustre en quoi le paradoxe de perte d'information pourraît être résolu dans le cadre de la conjecture de Mathur. La radiation de Hawking (en violet sur le dessin) émise par le trou noir prend son départ à l'horizon des évènements. Dans la vision classique, à gauche, on voit en zoomant que cette radiation part donc d'un endroit où l'espace est vide ; elle ne peut donc pas emporter d'information sur la matière contenue loin au centre du trou noir. Dans le cadre du fuzzball proposal, à droite, la matière s'étend au contraire jusqu'à l'horizon. La radiation partant de là est donc bien émise par la matière elle-même, et porte donc une certaine information sur celle-ci.

cela est de faire un parallèle avec la thermodynamique usuelle. A grande échelle, on décrit extrèmement bien un gaz en supposant que c'est un fluide continu, déterminé par quelques quantités macroscopiques, comme sa température et sa pression. Mais si l'on zoome, à partir d'une certaine échelle on s'aperçoit que ce n'est en fait pas un fluide continu mais un ensemble de particules, qui ont chacune une vitesse et une position. Si l'on veut décrire des effets à grande échelle, la description en terme de fluide est très bonne, mais elle ne marche plus pour étudier la physique microscopique du gaz. L'idée est la même pour le trou noir : pour des effets plus grands que l'horizon, comme le lentillage gravitationnel ou le mouvement d'un astre loin du trou noir, la description classique de la relativité générale est parfaitement valide. Mais si l'on étudie la physique à l'échelle de l'horizon, comme la radiation de Hawking, alors la relativité générale n'est plus valable et il est nécéssaire d'avoir une description en terme des microétats.

Si l'on suit cette conjecture, on peut comprendre comment ces corrections quantiques répondent aux trois problèmes des trous noirs, sans se poser la question, plus technique, de ce que sont exactement ces microétats :

- **La singularité centrale.** La singularité, premièrement, est corrigée par la gravité quantique. Ceci marche que les effets soit de la taille de l_P ou de celle de l'horizon, il est certain que les effets de gravité quantique vont à un moment contrer l'attraction gravitationnelle pour empêcher toute la masse de se concentrer en un point.
- **L'entropie du trou noir.** Si la conjecture de Mathur est vraie, cela veut qu'il existe des microétats de trou noir, des configurations quantiques, qui par construction – ou par hypothèse – fournissent une description microscopique des trous noirs. Si on a assez de microétats, on retrouvera l'entropie du trou noir macroscopique. De même qu'un gaz en thermodynamique doit être compris comme une approximation, une "moyenne" sur les

configurations microscopiques, le trou noir relativiste est une “moyenne” de ces microétats, une approximation, qui est valable à l’extérieur du rayon de Schwarzschild.
– **Le paradoxe de perte d’information.** La dernière question est celle de l’information. Comme on l’a vu, le problème est que, dans une description classique, la radiation de Hawking émise par le trou noir est uniquement thermale, et ne porte donc aucune infomation sur la façon dont il a été formé. Ce résultat est basé sur le fait que l’émission se fait au niveau de l’horizon, où classiquement il n’y a pas de matière, puisque celle-ci est uniquement au centre du trou noir. Dans le cadre la conjecture de Mathur, la matière s’étend jusqu’à la taille de l’horizon, la radiation partant de l’horizon est donc bien émise par la matière et non par l’espace vide (voir figure 7). Elle va donc en s’en allant emporter un peu d’information sur le contenu. De cette manière, contrairement à la vision classique de la radiation de Hawking, si le trou noir s’évapore on peut savoir de quoi il était constitué en regardant précisément la radiation qui s’en échappe. L’information sur le contenu du trou noir est donc encodée dans ce qui est émis, et non pas perdue.

Toutes les idées présentées ainsi sont élégantes, mais ne resteraient que des mots en l’air si l’on ne pouvait pas les tester explicitement, les comprendre dans un cadre précis. En théorie des cordes, il est possible de les implémenter réellement dans un contexte technique bien compris, et ultimement de démontrer ou d’infirmer la conjecture de Mathur. Depuis bientôt dix ans, de nombreux physiciens ont travaillé sur cette question, et les travaux que de ma thèse s’inscrivent principalement dans ce contexte. Je réexplique ce contexte plus précisément dans la deuxième introduction. Je tiens tout de même à dire avant de finir cette présentation générale que la conjecture a été prouvée dans le cas d’une certaine classe de trous noirs, mais cette classe est d’une certaine manière “dégénérée”, et il n’est pas encore certain qu’elle soit valide dans le cas général. Toutefois, ce réultat est déjà très important, car il a tout d’abord permis de comprendre quels étaient, dans cette classe, réellement les microétats, et il donne deuxièmement une motivation très nette en faveur de la conjecture, qui montre que même si elle se trouve être au final infirmée, elle n’est absolument pas dénuée de sens et nous apprendra beaucoup sur la physique microscopique des trous noirs.

Introduction

With general relativity on one side and quantum mechanics on the other side, physics provides today an extremely good description of our world, that has been perfectly verified experimentally. But from a theoretical point of view, we know that these two theories are incompatible, and finding a complete theory of quantum gravity is one of the major challenges of the physics of the twenty first century. Very generally, one expects to have gravity effects for large masses, and quantum effects for small sizes. Quantum gravity therefore should appear when one has a very large mass in a small regions, in other words for compact objects. And what is more compact than a black hole? One thus naturally expects black holes to be the first place for quantum gravity to reveal itself, and understanding quantum gravity begins with understanding black hole physics. Reversing the point of view, in an astrophysical picture a black hole is what one obtains when nothing (classically) forbids gravitational collapse to go on: matter shrinks to zero size and creates a singularity. But we know that generically singularities reflect the limits of a given theory: a new physics is appearing. Namely here, quantum gravity. But singularities are not the only problem of black holes. The three, different albeit related, main problems in black hole physics are

- resolving the central singularity,
- having a microscopic understanding of the Beckenstein-Hawking entropy, and
- solving the information paradox.

One strongly expects that these three issues will be correctly understood only in a proper theory of quantum gravity.The other way around, understanding these problems will be an enormous step towards understanding quantum gravity.

In order to address this issues, one needs a precise framework. String theory, being naturally a theory of quantum gravity, provides such a framework. More than that, within string theory, a lot of results have already been found: first of all, using the duality between the weakly coupled, open string picture and the strongly coupled, closed string one, Sen, in 1995, has first been able to perform a microscopic counting of the states of supersymmetric two-charge systems [1]: the idea is that at zero string coupling constant, the F1-P two-charge system is in an open string regime and thus described by a conformal field theory (CFT). In this CFT, one can use Cardy's formula to show that the number of configurations is $e^{2\pi\sqrt{N_1 N_P}}$. Turning on g_s, the two-charge system backreacts to form a black hole, and since supersymmetry protects the number of states as g_s is changing, this number should match the exponential of the black hole entropy. However, the story is more complicated than that: indeed, two-charge black holes have vanishing horizon area in supergravity, their entropy is therefore given by the first stringy corrections. But these corrections were not known in 1995, so even if Sen could already argue that the two-charge black hole entropy scales like $\sqrt{N_1 N_P}$, he could not finish the proof. It is only in 2004 that Dabholkar, using results on stringy corrections to supergravity [2], finally showed that the black hole entropy, in the two-charge case, was exactly $4\pi\sqrt{N_1 N_P}$ and thus matched Sen's microscopic counting [3].

Because the two-charge system is degenerate, in the sense that it does not correspond to a large black hole, one cannot completely claim victory. Indeed, it was not clear that the microscopic counting will still hold in the case of black holes with macroscopic horizons. But in a now famous paper [4], Strominger and Vafa extended Sen's result to the case of the supersymmetric three-charge D1-D5-P black hole, that has a non-vanishing entropy $2\pi\sqrt{Q_{\rm D1}Q_{\rm D5}Q_P}$. Strominger and Vafa proved that in this case, the open string CFT counting exactly matches the Beckenstein-Hawking one. One can note that this major result has now a natural understanding in the AdS/CFT correspondence.

Despite its importance, this result is not the end of the game because it does not, for example, address the information paradox question. The problematic point is that the microscopic counting is done in the open string regime, where the black hole does not exist. One can therefore go one step further and ask "How does a black hole microscopic degree of freedom look like in the gravity regime?", or in other words "How does a black hole microstate look like?". There has been different proposal to answer this question. In a first one [5], Horowitz and Polchinski conjectured that every black hole microstates should be corrected from the black hole geometry only on a Planck-sized scale, and would still have a horizon. A different approach is the one of Samir Mathur, now known as "Mathur's conjecture" or the "fuzzball proposal" (see [6] for reviews). There are now different variants, different points of view, since the original proposal was done, but the common physical idea is the following: by dimensionality, the only length one can construct out of the fundamental constants c; G and $\hbar$ is the Planck length $l_P = \sqrt{\hbar G/c^3} \approx 10^{-35}$m. One thus naturally expects quantum gravity effects to become important at this very small scale. But to a black hole entropy S should correspond $N = \mathrm{e}^S$ degrees of freedom, N microstates, it could thus be possible for quantum gravity effects to occur not at an l_P scale but at $N^\alpha l_P$, α being some positive constant to determine. Since N is very large, $N^\alpha l_P$ could be much larger than l_P itself, and eventually extend to the second natural scale associated to the black hole, namely its Schwarzschild radius r_S. Thinking for a second, one can remark that this phenomenon already occurs in other contexts: in white dwarfs or neutron stars, the gravitational collapse is forbidden by quantum effects that extend, because of the very large number of particles, not only to macroscopic scales but to astrophysical ones. In string theory, it is also well-known that the timelike singularities are in some systems resolved by modifying the solution on a large scale, like in Polchinski-Strassler [7], Klebanov-Strassler [8], giant gravitons and the LLM geometries [9, 10].

If quantum gravity effects extend to the horizon size, one should in this picture understand the classical black hole as a thermodynamical, statistical description, valid on scales larger than r_S. Inside the horizon, this statistical description should not be relevant anymore, one would see the details of the "black hole microstate", exactly as usual thermodynamics breaks down at the scale of particle's mean free path. Replacing the black hole with one of its microstates, it is important to understand what generic features such states should have. What do we expect of such a microstate? Generically one knows that, by construction, it is a pure state microscopic configuration, and should thus not carry any entropy and consequently not have any horizon. In addition, it should have the same asymptotic behavior as the classical black hole, in particular the same mass and charges. This leaves a large freedom to the details of the states, and it is still an open question to know if a generic, typical microstates is a complicated non-geometrical stringy configuration, a "fuzzball", or not. Even if the generic states are very stringy, some of the microstates could also be well-described in supergravity, very much like quantum mechanics coherent states. Because this states are, technically speaking, easier to find, the natural question that rises is "Is there enough of such coherent states to accurately sample the phase space?". If the answer is yes, then counting the entropy corresponding to this supergravity microstates would be enough to have the leading order entropy, and could therefore give a positive answer to Mathur's conjecture, at least for supersymmetric back holes.

An important effort to prove this proposal, initiated by Lunin and Mathur, has been done in the last ten years, and in the two-charge case a large family of smooth, horizonless supergravity states has been constructed [11, 12, 13, 14, 15]. Counting these solutions has been done in [12, 16, 17, 18], where it has been shown that they were enough to account for the black hole entropy. It is interesting to mention that the key ingredient for this construction has been particular D-brane configurations called supertubes [19, 20, 21], carrying two electric and one magnetic dipole charge. This objects can have classically an arbitrary shape while still being BPS, and their entropy comes from the quantization of the corresponding infinite dimensional moduli space.

We could here claim that the fuzzball proposal has been proved for supersymmetric two-charge black holes, but, despite the very strong hints it gives in favor of the Mathur's proposal, things are not that clear. Indeed, in a recent paper [22] argue that one cannot find in the same duality frame both smooth configurations and small black holes. Thus, the two pictures – one single black hole or many horizonless configurations – may not be seen as two alternative descriptions of the same physics, but should be added in order to match the microscopic CFT counting. While this remark is very pertinent, it is still unclear which of the two descriptions is the correct one.

In the three-charge case, an impressive body of work has already been done (see [6] for recent reviews, and references therein). Although a very large class of smooth, horizonless solutions, of microstates, have already been constructed, they are still too few to account for the entropy of the three-charge black hole. One of the reason for this is that most of the known solutions assume, for tractability, a $U(1)$ isometry. Relaxing this condition has been, in the two-charge case, crucial to obtain an infinite dimensional moduli space and get the black hole entropy. It is therefore very important to find and understand more general three-charge microstates solutions. The work presented in the first part of this thesis is a step towards more general supersymmetric three-charge solutions. The precise results and motivations are presented at the beginning of the first part.

It is important to note that all of this results have been obtained for supersymmetric (BPS) black holes. Because our world is not supersymmetric, one should ultimately address the same questions for non-BPS ones. In the latter case, we also expect to see new features appearing. In particular, supersymmetric black holes being extremal, they do not emit any radiation, and do not evaporate. Therefore, in order to study advanced questions like "How does one understand Hawking radiation in the fuzzball proposal context?", or "What does Mathur's conjecture tell us about the information paradox?", one has to look at non-BPS black holes and black hole microstates. Because of technical difficulties, very few non-BPS black hole microstates are known, and even fewer non-extremal ones. The first one has been found in [23] (see also [24, 25]). This solution is known to be non-generic, but is still very interesting to understand. In [26] it was proved that this solutions were unstable. After that, Mathur and collaborators, in [27], made a careful analysis of the solution and have been able to study the emission created by the this instability. They could showed that this emission agreed perfectly with what one can compute on the CFT side. Because CFT states are the one that have been counted to find the microscopic entropy of the black hole, it is a strong hint that this smooth, horizonless geometries have really to be understood as "black hole microstates", and consequently that the fuzzball proposal does make sense.

Because very few non-BPS solutions are known, it is very important to find more of them, both black holes and smooth solutions, and then to understand the structure of the solution space. This will give us some insight and will allow to study them in details. The work presented in the second part of the thesis takes place into this effort to find and understand new non-BPS class of supergravity solutions. The precise results and motivations are presented at the beginning of the second part.

Presentation of the results

The work I did over the three years of my PhD is part of the common global context explained before, but are organized in two different parts. The first one deals with the study and detailed analysis of supersymmetric solutions of supergravity, corresponding either to black rings – five-dimensional black holes whose horizon topology is $S^1 \times S^2$ – or to regular horizonless solutions, in other words black hole microstates. This analysis is done via a probe D-brane approach and uses supertubes [19, 20, 21]. These supertubes are very interesting because they preserve the same supersymmetries as the background they are testing. This allows us to test many physical background properties, such as the fact that the entropy of a black ring does increase when merged with a supertube. In parallel, when the complete supergravity solution corresponding to the probe experiment is known, this allowed us to check that in all known cases, the physics is essentially completely captured by the probe analysis. Finally, this probe computation leads us to the "entropy enhancement mechanism", which reflects the fact that supertubes in magnetically charged backgrounds see their entropy enhanced by the dipole dipole interactions with the background. In certain configurations, this entropy can be much larger than what one would naively expect. Since supertubes source smooth supergravity solutions, this results is particularly interesting in the context of the fuzzball proposal, where this entropy can possibly account for a significant part of the corresponding black hole entropy.

The second part of my work concerns the search of new non-supersymmetric supergravity solutions. Supersymmetry simplifying greatly the Einstein equations, finding and analyzing BPS solutions has been very fruitful and a lot of efforts have been done in this direction in the last decade. Without supersymmetry, solving Einstein equations becomes in general much more complicated. However, different groups found new ways to try to factorize Einstein equations into an easier first order system (see [28, 29, 30, 31, 32, 33, 34] for a small part of the references), and consequently discovered more and more non-BPS solutions. In this context, I have tried, with my collaborators, not only to go on in the search of new solutions, but also the structure classifying them. This allowed us to find a new system of equation generalizing the BPS one, and having the crucial property to still be of first order and also solvable in a linear way. We then solved this system in numerous different physically interesting cases.

In order to avoid having a too long introduction, I postpone the detailed presentation of the motivations and results of my works to the beginning of each of the two parts of the thesis.

Organization of the thesis

This thesis is organized in eight chapters and five additional appendices.

In the first chapter, I introduce all the tools that will be necessary for the following work. I first introduce the type IIA, type IIB and eleven-dimensional supergravities and then present the different dualities relating them. The rest of the chapter contains an in-depth review of supersymmetric 3-charge solutions of these theories, given in all the contexts that will be needed in the following.

The rest of the thesis is organized into two parts. The following chapters use the results of the first one, but are presented, for the reader's convenience, as two independent parts. The first one, constituted by the chapters 2, three and 4, deals with the detailed analysis of known supersymmetric black holes and black hole microstates. The second one contains the chapters 5 to 8 and concerns new non-BPS supergravity solutions.

In chapters 2 and 3, I present the result of a probe analysis, using a supertube, for different supergravity backgrounds. After having precisely introduced the tools and formalism in chapter 2, I first show how the probe approach essentially captures all the physics of the backreacted solution. I study more precisely in chapter 3 magnetically charged backgrounds, and particularly the dipole-dipole interaction between the probe and the background. In chapter 4, I show how this new interactions effectively "enhanced" the electric charges of the tube and that this one consequently sees his entropy being "enhanced" way more than what one would naively think. Supertubes backreacting into smooth, regular supergravity solutions, this enhanced entropy could account for a large part in a microscopic counting of the corresponding black hole entropy.

The fifth chapter contains the first class of non-supersymmetric black holes. These solutions, the "almost BPS black holes" are solutions of five-dimensional supergravity with a Taub-NUT Euclidean space. Although being composed of BPS objects, supersymmetry is broken by the relative orientation of the D-branes. The "almost BPS equations" giving these solutions are very similar to the BPS ones, given in chapter 1, and the new class of solutions is very large, describing rotating black holes as well as black rings or multicentered solutions.

Chapter 6 presents a generalization of the approach that lead to the almost BPS equations of chapter 5. It consists in a complete derivation of the equations of motion assuming a "floating brane Ansatz". This hypothesis translates the fact that a probe M2-brane does not feel any force in such a "floating brane" supergravity background. Roughly speaking, it corresponds to an extremality requirement – even if , as we will see in chapter 8, one can find non extremal solutions within this Ansatz. With an extra simple hypothesis, the final equations are finally partially first order and linear if solved in the correct way. They generalize the known BPS and almost BPS equations of chapter 1 and 5.

The seventh chapter is about new solutions based on an Israel-Wilson four-dimensional space. These solutions, in addition to their intrinsic value, are particularly interesting because they interpolate between the BPS and almost BPS solutions. They thus allow us to have a better understanding of the global structure of the new solution space.

Finally, in chapter 8, I present the "Bolt solutions". These solutions to the system of equation of chapter 6, are built using an Euclideanized four-dimensional black hole as a base. They are horizonless and completely regular and consequently correspond, in the context of the "fuzzball proposal" to possible black hole microstates. Furthermore, the black holes corresponding to these solutions are non-extremal.

Appendix A contains details of the dualizations of the BPS solutions of chapter 1. Appendix B gives all the conventions we used in the first part of the thesis. Appendix C presents the details of the angular momentum computation for the probe supertubes. Finally, one can find in Appendix D and E some details and limits of the Bolt solutions presented in chapter 8.

Chapter 1

Framework and review of three charge solutions

The aim of this first chapter is to introduce, in a selfconsistent way, all the necessary tools and material needed for the work presented in this thesis. Except for a few additions, it is a review of known results. We first present, in section 1.1 the supergravity theories that we will use and the different dualities relating them. In section 1.2, we show how to obtain supersymmetric (BPS) solutions of eleven-dimensional supergravity, and present a simple example corresponding to the background sourced by a single family of M2-branes. We then present in section 1.3 and 1.4 a detailed review of three-charge BPS solutions, corresponding to eleven-dimensional solutions reduced on a six-torus. The notations and conventions introduced in this chapter will be followed throughout the thesis. We want to remind the reader that the two parts of the thesis are independent, but both are intensively using the material of this chapter. Indeed, in the first part we will probe the introduced supergravity backgrounds while in the second part, we will present new non-supersymmetric solutions, howevwe that have a structure comparable to the supersymmetric ones presented here, and are best understood when compared to them.

1.1 Supergravity actions and dualities

1.1.1 Supergravity actions

Supersymmetric string theory can present itself under different aspects, leading to different low energy supergravities, which are related by dualities. The two framework that we will need here are type IIA and type IIB supergravities, in ten dimensions [35]. Their bosonic content is

- a gravitational field $G_{\mu\nu}$,
- an antisymmetric field[1] $B^{(2)}$. We will denote $H^{(3)} = \mathrm{d}B^{(2)}$,
- The dilaton Φ, which is a scalar field. These three fields are called Neveu-Schwarz Neveu-Schwarz (NS-NS) fields,

[1]More generally, we will denote $M^{(n)}$ an n-form

- Ramond-Ramond (R-R) fields $C^{(p)}$, with $p = 1, 3, 5, 7$ for type IIA and $p = 0, 2, 4, 6, 8$ for type IIB. We will denote $F^{(p+1)} = \mathrm{d}C^{(p)}$.

The gravitational field encodes for the dynamics of the spacetime, while $B^{(2)}$ is the Maxwell gauge field coupling electrically to F1 fundamental strings and magnetically to NS5-branes. The dilaton is related to the string coupling g_s giving the interaction strength between stringy objects through $e^{\Phi_\infty} = g_s$, Φ_∞ being the value of the dilaton at spatial infinity. Ramond-Ramond fields $C^{(p+1)}$ are gauge fields coupling electrically to p-dimensional D-branes, or Dp-branes, and magnetically to D$(6-p)$-branes.

The bosonic part of the type IIA action can be written as [35]:

$$S_{\mathrm{IIA}} = S_{\mathrm{NS}} + S_{\mathrm{R}} + S_{\mathrm{CS}}, \tag{1.1.1}$$

with

$$S_{\mathrm{NS}} = \frac{1}{2\kappa_{10}^2} \int \mathrm{d}^{10}x\sqrt{-G}\, \mathrm{e}^{-2\Phi} \left(R + 4\partial_\mu\Phi\partial^\mu\Phi\right) - \frac{1}{4\kappa_{10}^2} \int \mathrm{e}^{-2\Phi} H^{(3)} \wedge \star H^{(3)} \,, \tag{1.1.2}$$

$$S_{\mathrm{R}} = -\frac{1}{4\kappa_{10}^2} \int F^{(2)} \wedge \star F^{(2)} + \widetilde{F}^{(4)} \wedge \star \widetilde{F}^{(4)} \tag{1.1.3}$$

and

$$S_{\mathrm{CS}} = -\frac{1}{4\kappa_{10}^2} \int B^{(2)} \wedge F^{(4)} \wedge F^{(4)} \,, \tag{1.1.4}$$

where

$$\widetilde{F}^{(p)} = F^{(p)} + H^{(3)} \wedge C^{(p-3)} \tag{1.1.5}$$

. The constant in front of the action is

$$2\kappa_{10}^2 = 16\pi G_{10} = (2\pi)^7 g_s^2 l_s^8 \,, \tag{1.1.6}$$

with G_{10} the Newton constant in ten dimensions, g_s the string coupling and l_s the string length.

In S_{NS}, the first term is a generalization of the purely gravitational action $\int \sqrt{-g}R$, then we have the kinetic terms for the dilaton and $B^{(2)}$. One can remark that the metric couples to the dilaton through the $\sqrt{-G}\mathrm{e}^{-2\Phi}R$. This metric is called the *string frame* metric. It is possible to redefine the gravitational field in order to recover a more conventional kinetic term for the metric $\int \sqrt{-G}R$, and this redefinition is

$$G_{E\mu\nu} = \mathrm{e}^{\Phi/2}\, G_{\mu\nu} \,. \tag{1.1.7}$$

$G_{E\mu\nu}$ is called the *Einstein frame* metric. The $S_{\rm R}$ action gives us the kinetic terms for the Ramond Ramond fields and finally the Chern-Simons term $S_{\rm CS}$ gives us an interaction term required by supersymmetry.

The action for type IIB supergravity has the same form (1.1.1), only the Ramond-Ramond field contents are different of type IIA, and thus $S_{\rm NS}$ is identical to (1.1.2). $S_{\rm R}$ and $S_{\rm CS}$ are given by:

$$S_{\rm R} = -\frac{1}{4\kappa_{10}^2}\int F^{(1)}\wedge\star F^{(1)} + \widetilde{F}^{(3)}\wedge\star\widetilde{F}^{(3)} + \frac{1}{2}\widetilde{F}^{(5)}\wedge\star\widetilde{F}^{(5)} \tag{1.1.8}$$

and

$$S_{\rm CS} = -\frac{1}{4\kappa_{10}^2}\int C^{(4)}\wedge H^{(3)}\wedge F^{(3)}\,, \tag{1.1.9}$$

where $\widetilde{F}^{(3)}$ and $\widetilde{F}^{(5)}$ are given by (1.1.5). It is important to note that in addition to the equation of motion following from this action, one has to impose, as an additional constraint on the solutions, that the five-form field strength $\widetilde{F}^{(5)}$ has to be self-dual

$$\widetilde{F}^{(5)} = \star\widetilde{F}^{(5)} \tag{1.1.10}$$

Type IIA supergravity action can be naturally seen as the dimensional reduction, or Kaluza-Klein (KK) reduction of an eleven-dimensional action describing 2- and 5-dimensional membranes, called M2 and M5-branes. The fields appearing in this action are the metric $G_{\mu\nu}$ and a three-form $A^{(3)}$ (we still note $F^{(4)}=\mathrm{d}A^{(3)}$):

$$S_{11} = \frac{1}{2\kappa_{11}^2}\left(\int \mathrm{d}^{11}x\sqrt{-G}\,R - \int\frac{1}{2}F^{(4)}\wedge\star F^{(4)} + \frac{1}{6}A^{(3)}\wedge F^{(4)}\wedge F^{(4)}\right)\,, \tag{1.1.11}$$

with

$$2\kappa_{11}^2 = 16\pi G_{11} = (2\pi)^8 l_P^9\,, \tag{1.1.12}$$

G_{11} being the Newton constant and l_P the Planck length in eleven dimension. l_P (G_{11}) is related to the string length (G_{10}) by[1]

$$l_P = g_s^{1/3} l_s\,,\quad G_{11} = 2\pi R_{11} G_{10} = 2\pi g_s l_s G_{10}\,. \tag{1.1.13}$$

This action is easier to work with than the precedent ones, because it only involves two different fields and thus permits, as we will see in the following, to treat the black hole charges in a more symmetric way. We will therefore in this thesis mostly work with this 11D-action, and then, if needed, reduce the solutions to IIA solutions by a KK reduction, as explained in the following section.

[1] All the conventions we are using here are summarized in Appendix B

1.1.2 Dualities

Finding solutions of supergravity is in general a very difficult task. We will present in the following section a way to find supersymmetric solutions, and in the second part of this thesis one approach to find non-supersymmetric solutions. But once one has an explicit solution, it is possible to “dualize” it to find new solutions, or in other word new descriptions of the same solution, description that can be explicitly very different, but will describe the same physics. Depending on what one wants to do with the solutions, as we will see, it will be easier to work in one framework or another, and it is therefore very important to understand this different dualities. We present here first of all how to perform a Kaluza-Klein (KK) reduction of an eleven-dimensional solution with a compact direction to obtain a type IIA solution. Being a weak coupling - weak coupling duality, this can also be done at the level of the action. The second type of duality that we present here is the T-duality. It maps a type IIA solution to a type IIB one, and vice-versa. Finally, S-duality is a weak coupling - strong coupling duality, mapping a type IIB solution to another type IIB solution.

Dimensional reduction

As we mentioned quickly in section 1.1.1, the type IIA action can be seen as the reduction of an eleven-dimensional theory along a compact direction. The principal interest of this reduction is to first find a solution in 11 dimensions, and then reduce it in 10, where the physical problem is usually easier to understand. Explicitly, one notes $\mathrm{d}s_p^2$ the metric in p dimensions, and one reduces along x_{10}, with radius $R_{10} = g_s l_s$, one has [36]:

$$\mathrm{d}s_{11}^2 = \mathrm{e}^{4\Phi/3}\left((\mathrm{d}x^{10} + C_\mu^{(1)}\mathrm{d}x^\mu)^2 + \mathrm{e}^{-2\Phi}\mathrm{d}s_{10}^2\right) . \tag{1.1.14}$$

This defines the dilaton, the ten-dimensional metric, and the $C^{(1)}$ Ramond Ramond field. The reduction of $A^{(3)}$ defines $C^{(3)}$ and $B^{(2)}$, depending on the direction of the compactification:

$$\begin{aligned} \widetilde{A}_{ijk}^{(3)} &\overset{KK}{\to} C_{ijk}^{(3)} \quad \forall i,j,k \neq 10\,, \\ \widetilde{A}_{ij10}^{(3)} &\overset{KK}{\to} B_{ij}^{(2)}\,. \end{aligned} \tag{1.1.15}$$

One can remark that this transformation can be easily inverted, and therefore that this is possible to “oxydize” a IIA solution to eleven dimension.

T-duality

T-duality relates a type IIA solution to a type IIB one and vice-versa, if we have an isometry along a compact direction. If one want to T-dualizes along y, it is useful to write the fields in the following way

$$\begin{aligned} ds^2 &= G_{yy}(dy + A_\mu dx^\mu)^2 + \hat{g}_{\mu\nu}dx^\mu dx^\nu\,, \\ B^{(2)} &= B_{\mu y}dx^\mu \wedge (dy + A_\mu dx^\mu) + \hat{B}^{(2)}\,, \\ C^{(p)} &= C_y^{(p-1)} \wedge (dy + A_\mu dx^\mu) + \hat{C}^{(p)}\,, \end{aligned} \tag{1.1.16}$$

where the $\hat{B}^{(2)}$ and $\hat{C}^{(p)}$ fields do not have any leg along dy. The transformed fields are then given by the following transformation rules [37][1]:

$$\begin{aligned} d\tilde{s}^2 &= G_{yy}^{-1}(dy + B_{\mu y}dx^\mu)^2 + \hat{g}_{\mu\nu}dx^\mu dx^\nu \,, \\ \mathrm{e}^{2\widetilde{\phi}} &= G_{yy}^{-1}\mathrm{e}^{2\phi} \,, \\ \widetilde{B}^{(2)} &= A_\mu dx^\mu \wedge dy + \hat{B}^{(2)} \,, \\ \widetilde{C}^{(p)} &= \hat{C}^{(p-1)} \wedge (dy + B_{\mu y}dx^\mu) + C_y^{(p)} \,. \end{aligned} \tag{1.1.17}$$

Alternatively one can transform the RR field strengths as follows[2]: one writes the field strengths as

$$F^{(p)} = F_y^{(p-1)} \wedge (dy + A_\mu dx^\mu) + \hat{F}^{(p)} \,, \tag{1.1.18}$$

and the transformed ones are

$$\widetilde{F}^{(p)} = \hat{F}^{(p-1)} \wedge (dy + B_{\mu y}dx^\mu) + F_y^{(p)} \,. \tag{1.1.19}$$

This rules have a clear physical interpretation, and it is worth mentioning it: the exchange of $B^{(2)}_{\mu y}$ and $G_{\mu y}$ corresponds, in the T-duality direction, to the transformation of string modes into momentum modes and vice-versa. What happens to a Dp-brane? Under T-duality, it transforms into a D$(p-1)$-brane if it is initially wrapping the T-duality compact direction, or into a D$(p+1)$-brane if not.

As we already said, dualities show us different faces of the same physics. By applying T-duality many times, one thus understands that the dimension of a brane is not really physically important, Every brane is equivalent to the other ones. On the other hand, if we have a system of different branes in different directions, their relative orientation and their dimension difference will translate their interactions, and be preserved by T-duality.

S-duality

In type IIB theory, there are two different sorts of one-dimensional objects: the fundamental strings F1 and the D1-branes, or D1-strings. It is thus rather natural to look for their differences, and if the fundamental strings are really "more fundamental" than D1-branes. It is in fact possible to interchange them, by what is called S-duality. S-duality (see for example [38]) is a strong-weak coupling duality that acts on supergravity fields with rather simple rules. It clearly shows this exchange between D1 and F1 ($B^{(2)} \leftrightarrow C^{(2)}$):

$$\begin{aligned} &\widetilde{\Phi} = -\Phi \,, \\ &\widetilde{G}_{\mu\nu} = \mathrm{e}^{-\Phi}G_{\mu\nu} \,, \\ &\widetilde{B}^{(2)} = C^{(2)} \,, \\ &\widetilde{C}^{(2)} = -B^{(2)} \,, \end{aligned} \tag{1.1.20}$$

[1] In the T-duality expressions, we adopt the conventions of [37]

[2] for a detailed derivation of these rules see Appendix A of [14]

the other fields being invariant. One word on the transformation of the dilaton: the main difference between D1 and F1 strings is that their tension is different, one has $T_{F1}/T_{D1} = g_s = e^{\Phi}$. By inverting D1 and F1 strings, one also has to redefine the dilaton into its opposite, and this gives a better understanding of what is S-duality: it just changes g_s in $1/g_s$. This is therefore a weak coupling - strong coupling duality. The present knowledge of the theory being mostly perturbative, one can only see this duality at the level of solutions, and not in the action. We recall that this is not the case for KK reduction or T-duality, that can be seen at the level of the solutions or of the actions.

To summarize, S-duality relates F1 and D1 strings and T-duality relates all the D-branes together. It means that with both, one sees that every Dp-brane is "as fundamental as" F1 strings, they play the same role and are dynamical objects in the same sense. Another way to understand this is to start from eleven dimensions: the M2-branes can be reduced into D2-branes or fundamental strings, depending on their directions, and therefore D2 and F1 are coming from the same object, and have to be considered in the same way[1]. Note that the S-duality is a $\mathbb{Z}_2$ symmetry. It can in fact be generalized to a more general SL(2,$\mathbb{Z}$) symmetry, that mixes F1- and D1-strings. We will not need it here, and therefore don't write it explicitly. It can be found in [35].

1.2 How to find a supersymmetric solution?

1.2.1 Supersymmetry equations

We now work in the context of eleven-dimensional supergravity, with the action (1.1.11). This action leads to second order non-linear differential equations of motions, which are in general very hard to solve. This is why it is interesting to find some easier way to find solutions. In this context, supersymmetry will be of great help. If we want to find a supersym

$$\delta e^A{}_\mu = \bar{\varepsilon}\,\Gamma^A \Psi_\mu\,, \tag{1.2.1}$$

$$\delta A^{(3)}_{\mu\nu\rho} = -3\,\bar{\varepsilon}\,\Gamma_{[\mu\nu}\Psi_{\rho]}\,, \tag{1.2.2}$$

$$\delta\Psi_\mu = D_\mu\varepsilon + \frac{1}{288}\left(\Gamma_\mu{}^{\nu\rho\sigma\tau}F^{(4)}_{\nu\rho\sigma\tau} - 8\Gamma^{\nu\rho\sigma}F^{(4)}_{\mu\nu\rho\sigma}\right)\varepsilon\,. \tag{1.2.3}$$

To write this equations, we introduce some new ingredients: we work with the eleven-dimensional vielbeins e^A, defined by

$$ds^2 = G_{\mu\nu}\,dx^\mu\,dx^\nu = \eta_{AB}\,e^A\,e^B\,. \tag{1.2.4}$$

Each e^A is a one-form, $e^A = e^A{}_\mu dx^\mu$. We denote curved indices with greek letters $\mu, \nu, ...$ and flat indices with capital roman letters $A, B, ...$. We recall that the corresponding spin-connection two-form ω^{AB} is related to the vielbein by

$$de^A + \omega^A{}_B \wedge e^B = 0\,. \tag{1.2.5}$$

[1]Unfortunately, while the quantization of the fundamental string is at the very beginning of string theory, we still don't know how to quantize the action of D-branes.

It defines the covariant derivative appearing in (1.2.3)

$$D_\mu \varepsilon = \partial_\mu \varepsilon + \frac{1}{4}\omega^{AB}{}_\mu \Gamma_{AB}\,. \tag{1.2.6}$$

We also introduced the 32×32 gamma matrices Γ^μ in eleven dimensions, and defined

$$\Gamma^{\mu\nu\ldots\rho} = \Gamma^{[\mu}\Gamma^{\nu}\ldots\Gamma^{\rho]}\,. \tag{1.2.7}$$

As usual, greek indices are uppered or lowered with $G_{\mu\nu}$ or its inverse, and flat indices are uppered or lowered with η_{AB}. Finally, the ε spinor appearing in (1.2.1), (1.2.2) and (1.2.3) is called the Killing spinor, has 32 components, and translate the number of supersymmetries of the solution.

We will only be interested in purely bosonic solutions, this means solutions where the gravitino Ψ_μ is turned off. By construction, it automatically implies that the supersymmetric variations of the bosonic fields, (1.2.1) and (1.2.2), are zero. In order to have a supersymmetric solution, we will therefore only have to make sure that the variation of the gravitino, (1.2.3), is zero. We will call this equation, $\delta\Psi_\mu = 0$, the supersymmetry equation. The interest of looking at supersymmetric solution is that in order to make sure that Einstein equations are verified, one only have to satisfy the 00 component of Einstein equations and the supersymmetry equation [39]. But the supersymmetry equation is a *first order* equation, and therefore easier to solve that the full, second order, Einstein equations of motion. This is a crucial point, and one of the reasons why it has been possible to find large classes of supersymmetric solutions. Note that one still has to solve the second order Maxwell equation. To summarize, to have a supersymmetric solution with only bosonic fields turned on, one has to solve

$$D_\mu \varepsilon + \frac{1}{288}\left(\Gamma_\mu{}^{\nu\rho\sigma\tau}F^{(4)}_{\nu\rho\sigma\tau} - 8\Gamma^{\nu\rho\sigma}F^{(4)}_{\mu\nu\rho\sigma}\right)\varepsilon = 0\,, \tag{1.2.8}$$

$$d \star F^{(4)} = \frac{1}{2}F^{(4)} \wedge F^{(4)}\,, \tag{1.2.9}$$

and the 00 component of Einstein equation. In practice, for the solutions we are interested in in this thesis, this 00 component will always be implied by (1.2.8)-(1.2.9), and therefore we will forget this last equation.

1.2.2 An example: one single family of M2-branes

In order to illustrate the previous equations and how one can solve them, the best is to present a simple example, corresponding to the background created by a single family of M2-branes. Despite the fact that this solution will be much easier to solve than the ones that we will see in the next section, and in the rest of the thesis, it is very important to understand, on this simplest example, the physics of supersymmetric objects. Indeed, this can really be seen as the building block of every supersymmetric solution, and its physical features will be very generic.

One first has to make an Ansatz for the fields. We want the brane to be along the direction $x^0 = t$, x^1 and x^2, so we expect to have an $SO(1,2)$ symmetry in the direction of the branes, and an $SO(8)$ symmetry in the transverse space. The three-form gauge field sourced by the branes is along $dt \wedge dx_1 \wedge dx_2$. The Ansatz that we take is thus

$$\begin{aligned} ds^2 &= Z^{-2/3}(-dt^2 + dx_1^2 + dx_2^2) + Z^{1/3} ds_8^2 \,, & (1.2.10)\\ A^{(3)} &= (1 - Z^{-1}) dt \wedge dx_1 \wedge dx_2 & (1.2.11) \end{aligned}$$

with ds_8^2 the flat metric on the eight-dimensional space transverse to the branes, and Z an a priori arbitrary function. In general, we should have three different functions for $-G_{tt} = G_{11} = G_{22}$, $G_{33} = \ldots = G_{1010}$ and $A^{(3)}_{012}$ but we will see that our Ansatz is consistent with the equations of motion. The brane being extended along $t = x^0$, x^1 and x^2, Z will only depend on the transverse directions, $Z = Z(x^3, \ldots, x^{10})$.

The vielbein corresponding to our Ansatz is

$$\begin{aligned} e^A &= Z^{-1/3}\, \delta^A_\mu \, dx^\mu \,, \quad \text{for } A = 0,1,2 \,, \\ e^B &= Z^{1/6}\, \delta^B_\nu \, dx^\nu \,, \quad \text{for } B = 3, \ldots, 10 \,, \end{aligned} \tag{1.2.12}$$

and the corresponding spin connection is given by

$$\begin{aligned} \omega^A{}_B &= 0 \quad \text{for } A, B = 0,1,2 \,, \\ \omega^A{}_B &= -\frac{1}{3} Z^{-3/2}\, \delta^A_\mu\, \delta^\nu_B\, \partial_\nu Z\, dx^\mu \quad \text{for } A = 0,1,2 \text{ and } B = 3, \ldots, 10 \,, \\ \omega^B{}_C &= -\frac{1}{6} Z^{-1}\, \delta^B_\mu\, \delta^\nu_C \left(\partial_\nu Z\, dx^\mu - \partial_\mu Z\, dx^\nu\right) \quad \text{for } B, C = 3, \ldots, 10 \,. \end{aligned} \tag{1.2.13}$$

We can now write explicitly the supersymmetry equations $\delta\Psi_\mu = 0$ for this particular ansatz, and this gives

$$\begin{gathered} -\frac{1}{6} Z^{-3/2}\, \delta^A_\mu\, \delta^\nu_B\, \partial_\nu Z\, \Gamma_A{}^B \left(1 + \Gamma^{012}\right) \varepsilon = 0 \quad \text{for } \mu = 0,1,2 \,, \\ \left(\partial_\mu + \frac{1}{6} Z^{-1}\, \partial_\mu Z\right) \varepsilon = 0 \quad \text{for } \mu = 3, \ldots 10 \,, \end{gathered} \tag{1.2.14}$$

where the indices of Γ^{012} in the first equation are flat indices. The second equation is solved by $\varepsilon = Z^{1/6}\varepsilon_0$, for ε_0 a constant spinor. To solve the first equation, one has to impose the projection

$$(1 + \Gamma^{012})\, \varepsilon = (1 + \Gamma^{012})\, \varepsilon_0 = 0 \,. \tag{1.2.15}$$

This projection has a physical sense: it fixes half of the components of ε_0 to be zero, leaving the other half free. Physically speaking, it means that it breaks half of the supersymmetries of the action. This is why we will call such a solution a 1/2-supersymmetric, or 1/2-BPS solution. In the following, we will see more generally $1/N$-BPS solutions, solutions preserving $1/N$ of the 32 supersymmetries of the action. For example, the three charge solutions of the next section will be 1/8-BPS solutions.

Let's now move to Maxwell's equations. For our particular ansatz, it just gives

$$d \star_8 d Z = 0 \,, \tag{1.2.16}$$

where $\star_8$ is the hodge dual with respect to the eight-dimensional flat metric ds_8^2. This means that Z must be a harmonic function in the transverse space. If we now assume in addition that the M2-branes are in the center of the transverse space, we have a spherical symmetry in this space: if $ds_8^2 = dr^2 + r^2 d\Omega_7^2$, $Z = Z(r)$. In this case, (1.2.16) is solved by

$$Z = 1 + \frac{\alpha}{r^6} \,. \tag{1.2.17}$$

The constant in Z has been fixed by the requirement that the metric be flat at infinity. The α constant is related, by Gauss theorem, to the number of M2-branes of the solution

$$N_{M2} = \frac{1}{(2\pi l_P)^3} \int_{S^7} \star F^{(4)} = \frac{6\alpha \mathrm{Vol}(S^7)}{(2\pi l_P)^3} \,, \tag{1.2.18}$$

where the integral is performed on the seven-sphere of any radius of $\mathbb{R}^8$. This completely solves the equations, and we now have the full solution corresponding to a single family of M2-branes.

Before moving to more complicated solutions, it is interesting to discuss a little bit more the physics of the solution we found, in particular the fact that the metric warp factors and the 3-form gauge field contain the same function Z. The metric is related to the mass of the M2-branes, and the gauge field to its charge. Therefore, the fact that they are related states that the mass and the charge of the brane are equal, as one can also check explicitly. This property is means that they are saturating the Bogomolny-Prasad-Sommerfeld (BPS). We will from now on speak indifferently about supersymmetric objects or BPS objects[1]. Using this equality of mass and charge, if one puts a probe M2-brane in this background, it will feel a attractive force because of its mass and a repulsive force because of its charge. If the mass is equal to the charge, this two forces will compensate and the probe brane will not feel any force: the objects will be mutually BPS. This will be deeply investigate in the next chapter of the thesis.

1.3 Three charge BPS solutions

In this section, we present the most general class of supersymmetric three-charge solutions known up to now. This is mostly a review of previous work, but some of the material is new. For the rest of this thesis, we will be mostly concern by this three-charge solutions, which corresponds physically after reduction to five-dimensional black holes, black rings, or smooth solutions. Each of the charge – each of the family of branes – breaking one half of the supersymmetries, this solutions are 1/8-BPS, preserve four supersymmetries. We first present the solutions in a general way, in different duality frames, and in the next section we will detail a bit the possible particular cases.

[1] In the second part of the thesis, we will see that one can have non-supersymmetric objects that still have their mass being equal to their charge, and thu in a sense are also saturating the BPS bound. Despite this remark, we will call BPS only supersymmetric solutions, in order not to confuse the reader.

	0	1	2	3	4	5	6	7	8	9	10
M2	$\updownarrow$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\updownarrow$	$\updownarrow$	$\leftrightarrow$	$\leftrightarrow$	$\leftrightarrow$	$\leftrightarrow$
M2	$\updownarrow$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\leftrightarrow$	$\leftrightarrow$	$\updownarrow$	$\updownarrow$	$\leftrightarrow$	$\leftrightarrow$
M2	$\updownarrow$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\leftrightarrow$	$\leftrightarrow$	$\leftrightarrow$	$\leftrightarrow$	$\updownarrow$	$\updownarrow$
M5	$\updownarrow$	$y^\mu(\phi)$	$y^\mu(\phi)$	$y^\mu(\phi)$	$y^\mu(\phi)$	$\leftrightarrow$	$\leftrightarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
M5	$\updownarrow$	$y^\mu(\phi)$	$y^\mu(\phi)$	$y^\mu(\phi)$	$y^\mu(\phi)$	$\updownarrow$	$\updownarrow$	$\leftrightarrow$	$\leftrightarrow$	$\updownarrow$	$\updownarrow$
M5	$\updownarrow$	$y^\mu(\phi)$	$y^\mu(\phi)$	$y^\mu(\phi)$	$y^\mu(\phi)$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\leftrightarrow$	$\leftrightarrow$

Table 1.1: *The configuration of branes in M-theory that preserves the four supersymmetries of the M2-M2-M2 three-charge black hole [40]. The vertical arrows represent the directions along which the branes are extended and the horizontal arrows represent smearing directions. The functions $y^\mu(\phi)$ describe a closed curve which is wrapped by the M5 branes.*

1.3.1 Three-charge solutions in the M2-M2-M2 (M-theory) frame

Three-charge solutions with four supercharges are most simply written in the M-theory duality frame in which the three charges are treated most symmetrically and correspond to three types of M2 branes wrapping three T^2's inside T^6 [40], as presented in Table 1.3.1. The metric is:

$$\begin{aligned} ds_{11}^2 =& -(Z_1Z_2Z_3)^{-\frac{2}{3}}(dt+k)^2 + (Z_1Z_2Z_3)^{\frac{1}{3}}\, ds_4^2 \\ &+ \left(Z_2Z_3Z_1^{-2}\right)^{\frac{1}{3}}(dx_5^2+dx_6^2) + \left(Z_1Z_3Z_2^{-2}\right)^{\frac{1}{3}}(dx_7^2+dx_8^2) + \left(Z_1Z_2Z_3^{-2}\right)^{\frac{1}{3}}(dx_9^2+dx_{10}^2)\,, \end{aligned} \quad (1.3.1)$$

where ds_4^2 is a four-dimensional hyper-Kähler metric [40, 41, 42][1]. The solution has a non-trivial three-form potential, sourced both by the M2 branes (electrically) and by the M5 dipole branes (magnetically):

$$\mathcal{A} = A^{(1)} \wedge dx_5 \wedge dx_6 + A^{(2)} \wedge dx_7 \wedge dx_8 + A^{(3)} \wedge dx_9 \wedge dx_{10}. \quad (1.3.2)$$

The magnetic contributions can be separated from the electric ones by defining the "magnetic field strengths:"

$$\Theta^{(I)} \equiv dA^{(I)} \; + \; d\left(\frac{(dt+k)}{Z_I}\right), \qquad I = 1,2,3. \quad (1.3.3)$$

Finding supergravity solutions for this system, solutions to (1.2.8)-(1.2.9), then boils down to solving the following system of BPS equations[2]:

$$\begin{aligned} \Theta^{(I)} &= \star_4 \Theta^{(I)}\,, \\ \nabla^2 Z_I &= \frac{1}{2} C_{IJK} \star_4 (\Theta^{(J)} \wedge \Theta^{(K)})\,, \\ dk + \star_4 dk &= Z_I \Theta^{(I)}\,. \end{aligned} \quad (1.3.4)$$

[1] This metric can have regions of signature +4 and signature −4 [43, 44, 45, 46, 47], and for this reason we usually refer to it as ambipolar. We will come back to this point in section 1.4.3

[2] These equations also give supersymmetric solutions when the T^6 is replaced by a Calabi-Yau three-fold, and C_{IJK} is replaced by the triple intersection numbers of this three-fold.

In these equations, $\star_4$ is the Hodge dual in the four-dimensional hyper-Kähler base space, ds_4^2, and $C_{IJK} = |\epsilon_{IJK}|$. The associated BPS projections, generalizing (1.2.15) are

$$(1 + \Gamma^{056})\,\varepsilon = (1 + \Gamma^{078})\,\varepsilon = (1 + \Gamma^{0910})\,\varepsilon = 0\,, \tag{1.3.5}$$

so the solutions are 1/8-BPS. Remembering that the Γ matrices have to verify $\Gamma^{012345678910}\varepsilon = \varepsilon$, this also automatically implies

$$(1 - \Gamma^{1234})\,\varepsilon = 0\,, \tag{1.3.6}$$

which is the requirement for the base to be hyper-Kähler [42]. As we will see below, this means that we will be able to have an extra KK-monopole charge in the solution, without breaking more supersymmetries.

In the equations (1.3.4), the $\Theta^{(I)}$ are the magnetic field strength, Z_I corresponds to the electric charge and k is the angular momentum of the solution. The first equation tells us that the magnetic field strengths have to be self-dual. The magnetic-magnetic interaction then acts as a electric source for Z_I, and in the last equation, the electric-magnetic interaction sources angular momentum, very much like a Pointing vector in electromagnetism. The very important feature if this equations is that the are *linear* if solved in the correct order. The nonlinear terms can always be seen as known sources in a linear equation for an other field. This is a very important simplification, and this is what allows us to find explicit solutions. We will see in the second part how it is possible to extend this property to non-supersymmetric solutions.

If the four-dimensional base manifold has a triholomorphic $U(1)$ isometry[1] then the metric on the base can be put in a Gibbons-Hawking (GH) form [49, 50]:

$$ds_4^2 \;=\; V^{-1}\left(d\psi + A\right)^2 \;+\; V\, d\vec{y}\cdot d\vec{y}\,, \tag{1.3.7}$$

where V is a harmonic function on the $\mathbb{R}^3$ spanned by (y_1, y_2, y_3) and $\vec{\nabla}\times\vec{A} = \vec{\nabla}V$. For such metrics, the BPS equations (1.3.4) can be solved explicitly [51, 52]. The most general solution can be written in terms of eight harmonic functions (V, K^I, L_I, M) on the $\mathbb{R}^3$ base of the GH space[2]. It is convenient to introduce the vielbeins on ds_4^2:

$$\hat{e}^1 \;=\; V^{-\frac{1}{2}}\,(d\psi \;+\; A)\,, \qquad \hat{e}^{a+1} \;=\; V^{\frac{1}{2}}\,dy^a\,, \quad a = 1,2,3\,, \tag{1.3.8}$$

then one has

$$\Theta^{(I)} \;=\; -\sum_{a=1}^{3}\left(\partial_a\left(V^{-1}K^I\right)\right)\left(\hat{e}^1\wedge\hat{e}^{a+1} \;+\; \frac{1}{2}\,\epsilon_{abc}\,\hat{e}^{b+1}\wedge\hat{e}^{c+1}\right)\,. \tag{1.3.9}$$

The three gauge fields, $A^{(I)}$, can be written as

$$A^{(I)} = -\frac{1}{Z_I}(dt + k) + B^{(I)}\,, \tag{1.3.10}$$

[1]For a discussion on BPS solution with more general $U(1)$ isometries see [48].

[2]For M-theory compactifications on a generic Calabi-Yau three-fold the number of harmonic functions will be $2h^{1,1} + 2$. See [53] for a discussion of such solutions.

where

$$B^{(I)} = V^{-1}K^I\,(d\psi + A)\; +\; \vec{\xi}^{(I)}\cdot d\vec{y}\,, \qquad \vec{\nabla}\times\vec{\xi}^{(I)} \;\equiv\; -\vec{\nabla}K^I\,. \tag{1.3.11}$$

The functions Z_I and the angular momentum one-form k are given by

$$Z_I = L_I + \frac{C_{IJK}}{2}\frac{K^JK^K}{V}\,, \qquad k = \mu(d\psi + A) + \vec{\omega}\cdot d\vec{y}\,, \tag{1.3.12}$$

where

$$\mu \;=\; M \;+\; \frac{K^IL_I}{2V} \;+\; \frac{C_{IJK}}{6}\frac{K^IK^JK^K}{V^2} \tag{1.3.13}$$

and $\vec{\omega}$ satisfies the equation

$$\vec{\nabla}\times\vec{\omega} = V\vec{\nabla}M - M\vec{\nabla}V + \frac{1}{2}\left(K^I\vec{\nabla}L_I - L_I\vec{\nabla}K^I\right). \tag{1.3.14}$$

This solution can describe five-dimensional black holes, circular black rings and supertubes, as well as smooth "bubbling solutions" and an arbitrary superposition of these objects. We will detail this a bit more in the next section. Upon compactifying to four dimensions, by reducing along the S^1 fiber of the transverse space and the T^6 torus, all these reduce to BPS multi-center black-hole configurations [54, 55] of the type first considered in [56, 57].

The harmonic functions are usually chosen to be sourced by simple poles:

$$\begin{aligned} V \;&=\; \epsilon_0 \;+\; \sum_{j=1}^{N}\frac{q_j}{r_j}\,, \qquad & K^I \;&=\; \kappa_0^I \;+\; \sum_{j=1}^{N}\frac{k_j^I}{r_j}\,,\\ L_I \;&=\; l_0^I \;+\; \sum_{j=1}^{N}\frac{l_j^I}{r_j}\,, \qquad & M \;&=\; m_0 \;+\; \sum_{j=1}^{N}\frac{m_j}{r_j}\,, \end{aligned} \tag{1.3.15}$$

where $r_j = |\vec{y}-\vec{y}_j|$, $\vec{y}_j$ the location of each center and N is the number of centers. We think of the residues of the poles of these functions as defining the Gibbons-Hawking (GH) charges of the corresponding solution. Explicitly, in the M-theory frame, q_j corresponds to the KK-monopole charge, l_j^I to the M2-brane charge, k_j^I to the M5-charge and m_j to the angular momentum charge, each at the center $\vec{y}_j$. The solutions are invariant under the following gauge transformation:

$$\begin{aligned} V &\;\to\; V\,,\\ K^I &\;\to\; K^I + c^IV\,,\\ L_I &\;\to\; L_I - C_{IJK}c^JK^K - \frac{1}{2}C_{IJK}c^Jc^KV\,,\\ M &\;\to\; M - \frac{1}{2}c^IL_I + \frac{1}{4}C_{IJK}c^Ic^JK^K + \frac{1}{12}C_{IJK}c^Ic^Jc^KV\,, \end{aligned} \tag{1.3.16}$$

with c^I being three arbitrary constants, and $C_{IJK} = |\epsilon_{IJK}|$. As was discussed in [58], such gauge transformations and spectral flow[1] can reshuffle these charges, but this produces physically equivalent solutions.

[1] We will explain what is a spectral flow at the end of this section.

A necessary (but not sufficient) condition for the solutions to be free of closed timelike curves (CTC's) is to satisfy the "integrability equations," or "bubble equations," [44, 45, 56, 57]:

$$\sum_{j=1,j\neq i}^{N} \frac{\langle \hat{Q}_i|\hat{Q}_j\rangle}{r_{ij}} \;=\; 2(\varepsilon_0 m_i - m_0 q_i) + \sum_{I=1}^{3}(k_0^I l_i^I - l_0^I k_i^I) \tag{1.3.17}$$

where $\langle \hat{Q}_i|\hat{Q}_j\rangle$ is the symplectic product[1] between the eight-vectors of charges at the points i and j

$$\langle \hat{Q}_i|\hat{Q}_j\rangle \equiv 2(m_j q_i - q_j m_i) + \sum_{I=1}^{3}(l_j^I k_i^I - k_j^I l_i^I)\ . \tag{1.3.18}$$

One can arrange for the global absence of CTC's by requiring that there is a well-defined, global time function [45]. This is much more stringent than the bubble equations (which only eliminate CTC's in the neighborhood of the GH points) and means that the following inequality should be satisfied globally [44, 45]:

$$Z_1 Z_2 Z_3 V \;-\; \mu^2 V^2 \;-\; |\omega|^2 \;\geq\; 0\,. \tag{1.3.19}$$

This condition is very hard to check in general and usually has to be checked numerically for particular solutions.

In the next chapters, we will study two-charge supertubes (presented in the next section) in backgrounds like those presented here. In order to do this, it is useful to dualize to a frame in which the two-charge supertube action is simple. One such frame is where the three electric charges correspond to D0 branes, D4 branes and F1 strings and the supertube carries D0 and F1 electric charges and D2 dipole charge [19]. On the other hand, in order to study the supergravity solutions describing supertubes in black-ring or bubbling backgrounds, it is useful to work in a duality frame in which the supergravity solution for the supertubes is smooth. In this frame the electric charges of the background correspond to D1 branes, D5 branes, and momentum P, and the supertube carries D1 and D5 charges, with KKM dipole charge. We therefore dualize the foregoing M-theory solution to these frames and give all the details of the solutions explicitly.

1.3.2 Three-charge solutions in the D0-D4-F1 duality frame

Here we will present the three-charge solutions in the duality frame in which they have electric charges corresponding to D0 branes, D4 branes, and F1 strings, and dipole charges corresponding to D6, D2 and NS5 branes. We use the T-duality rules (given in Section 1.1.2) to transform field-strengths. It should be emphasized that our results are correct for any three-charge solution (including those without a tri-holomorphic $U(1)$ [48]), however, finding the explicit form of the RR and NS-NS *potentials* (which is crucial if we want to investigate this solution using probe supertubes) is straightforward only when the solution can be written in Gibbons-Hawking form.

[1] This product is sometimes called the Dirac-Schwinger-Zwanziger product.

Label the coordinates by $(x^0, \ldots, x^8, z)$[1]. The electric charges N_1, N_2 and N_3 of the solution then correspond to:

$$N_1 : \text{D0} \qquad N_2 : \text{D4}\,(5678) \qquad N_3 : \text{F1}\,(z) \tag{1.3.20}$$

where the numbers in the parentheses refer to spatial directions wrapped by the branes and $z \equiv x^{10}$. The magnetic dipole moments of the solutions correspond to:

$$n_1 : \text{D6}\,(y5678z) \qquad n_2 : \text{D2}\,(yz) \qquad n_3 : \text{NS5}\,(y5678)\,, \tag{1.3.21}$$

where y denotes the brane profile in the spatial base, $(x^1, \ldots, x^4)$. The metric of the solution is:

$$ds^2_{IIA} = -\frac{1}{Z_3\sqrt{Z_1Z_2}}(dt+k)^2+\sqrt{Z_1Z_2}ds_4^2+\frac{\sqrt{Z_1Z_2}}{Z_3}dz^2+\sqrt{\frac{Z_1}{Z_2}}(dx_5^2+dx_6^2+dx_7^2+dx_8^2)\,. \tag{1.3.22}$$

The dilaton and the Kalb-Ramond fields are:

$$e^\Phi = \left(\frac{Z_1^3}{Z_2Z_3^2}\right)^{1/4}, \qquad B = -dt\wedge dz - A^{(3)}\wedge dz\,. \tag{1.3.23}$$

The RR field strengths are

$$F^{(2)} = -\mathcal{F}^{(1)}\,, \qquad \widetilde{F}^{(4)} = -\left(\frac{Z_2^5}{Z_1^3Z_3^2}\right)^{1/4} \star_5 (\mathcal{F}^{(2)})\wedge dz\,, \tag{1.3.24}$$

where we define $\mathcal{F}^{(I)} \equiv dA^{(I)}$ and $\star_5$ is the Hodge dual with respect to the five dimensional metric:

$$ds_5^2 = -\frac{1}{Z_3\sqrt{Z_1Z_2}}(dt+k)^2 + \sqrt{Z_1Z_2}ds_4^2\,. \tag{1.3.25}$$

The foregoing results are valid for any three-charge solution with an arbitrary hyper-Kähler base. As we show in Appendix A, when the base has a Gibbons-Hawking metric one can easily find the RR 3-form potential:

$$C^{(3)} \;=\; \big(\zeta_a \;+\; V^{-1}K^3\xi_a^{(1)}\big)\,\Omega_-^{(a)}\wedge dz - \big(Z_3^{-1}(dt+k)\wedge B^{(1)} \;+\; dt\wedge A^{(3)}\big)\wedge dz\,, \tag{1.3.26}$$

where $\xi_a^{(1)}$ and ζ_a are defined by equations (1.3.11) and (A.31). Thus we have the full three-charge supergravity solution in the D0-D4-F1 duality frame. In the next chapter we will perform a probe analysis in this class of backgrounds using the DBI action for supertubes with D0 and F1 electric and D2 dipole charge.

1.3.3 Three-charge solutions in the D1-D5-P duality frame

One can T-dualize the solution above along z to obtain a solution with D1, D5 and momentum charges:

$$N_1 : \text{D1}\,(z) \qquad N_2 : \text{D5}\,(5678z) \qquad N_3 : \text{P}\,(z) \tag{1.3.27}$$

[1] From (1.3.1), we reduce along x_{10} and call $z = x^9$.

and dipole moments corresponding to wrapped D1 branes, D5 branes and Kaluza Klein Monopoles (KKm):

$$n_1 : \text{D5}\,(y5678) \qquad n_2 : \text{D1}\,(y) \qquad n_3 : \text{KKm}\,(y5678z)\,. \tag{1.3.28}$$

The metric is

$$ds^2_{IIB} = -\frac{1}{Z_3\sqrt{Z_1 Z_2}}\,(dt+k)^2 \;\;+\;\; \sqrt{Z_1 Z_2}\,ds_4^2 + \frac{Z_3}{\sqrt{Z_1 Z_2}}\,(dz + A^{(3)})^2 \tag{1.3.29}$$

$$+ \;\; \sqrt{\frac{Z_1}{Z_2}}\,(dx_5^2 + dx_6^2 + dx_7^2 + dx_8^2) \tag{1.3.30}$$

and the dilaton and the Kalb-Ramond field are:

$$e^{\Phi} = \left(\frac{Z_1}{Z_2}\right)^{1/2}, \qquad\qquad B = 0\,. \tag{1.3.31}$$

The only non-zero RR three-form field strength is:

$$F^{(3)} = -\left(\frac{Z_2^5}{Z_1^3 Z_3^2}\right)^{1/4} \star_5 (\mathcal{F}^{(2)}) - \mathcal{F}^{(1)} \wedge (dz - A^{(3)})\,. \tag{1.3.32}$$

If we specialize our general result to the supersymmetric black ring solution in the D1-D5-P frame then it agrees (up to conventions) with [59]. It is also elementary to find the RR two-form potential for a general BPS solution with GH base in D1-D5-P frame. This can be done by T-dualizing the IIA D0-D4-F1 result (1.3.26), to obtain:

$$C^{(2)} \;=\; \left(\zeta_a \;+\; V^{-1}K^3\xi_a^{(1)}\right)\Omega_-^{(a)} \;\;-\;\; \left(Z_3^{-1}(dt+k)\wedge B^{(1)} \;+\; dt \wedge A^{(3)}\right) \tag{1.3.33}$$

$$+ \;\; A^{(1)} \wedge (A^{(3)} - dz - dt) \;+\; dt \wedge (A^3 - dz)\,, \tag{1.3.34}$$

where again $\xi_a^{(1)}$ and ζ_a are defined in equations (1.3.11) and (A.31). This is the full three-charge supergravity solution in the D1-D5-P duality frame. As shown in [13], two-charge supertubes in flat space are regular only in this duality frame, so our general result can be used to analyze the regularity of two charge supertubes in a general three-charge solution. This will be the subject of the section 1.4.4.

1.3.4 Spectral flow transformations

The solutions presented here can be transformed using the *spectral flow transformations* [58]. Spectral flow is a general transformation – or solution generating technique – for solutions having two different $U(1)$ isometries. We will not be interested here in the general transformation, and thus we will not present it here; we refer to [58] for a complete discussion. In this section, we only present how a spectral flow transformation transforms the solutions we just presented, and discuss its physical interpretation. We will in the second part of this thesis be interested in more general spectral flow transformations for non-supersymmetric solutions.

Spectral flow transformations transform one solution of the class presented above in another solution of this class. A single spectral flow transformation is given by its direction 1, 2 or 3, related to the charge 1,2 or 3, and a constant γ with the following rules (for a flow along 3):

$$\begin{aligned} M &\rightarrow M\,, \\ L_1 &\rightarrow L_1\,, \quad L_2 \rightarrow L_2\,, \quad L_3 \rightarrow L_3 - 2\gamma M\,, \\ K_1 &\rightarrow K_1 - \gamma L_2\,, \quad K_2 \rightarrow K_2 - \gamma L_1\,, \quad K_3 \rightarrow K_3\,, \\ V &\rightarrow V + \gamma K^3\,. \end{aligned} \tag{1.3.35}$$

A straightforward generalization is to do a spectral flow along each of the directions, with parameters γ_I:

$$\begin{aligned} M &\rightarrow M\,, \\ L_I &\rightarrow L_I - 2\gamma_I M\,, \\ K_I &\rightarrow K_I - C^{IJK}\gamma_J L_K + C^{IJK}\gamma_J\gamma_K M\,, \\ V &\rightarrow V + \gamma_I K^I - \frac{1}{2}C^{IJK}\gamma_I\gamma_J L_K + \frac{1}{3}C^{IJK}\gamma_I\gamma_J\gamma_K M\,, \end{aligned} \tag{1.3.36}$$

with $C^{IJK} = C_{IJK} = |\epsilon_{IJK}|$. If one remember that M, L_I, K^I and V represents respectively angular momentum, M2-branes, M5-branes and KK-monopoles, the spectral flow has a clear physical interpretation: starting from one solution, one obtains the new one by mixing the charges together, angular momentum becomes M2, M2 becomes M5 and M5 KKm. The fact that the new solution is in the same class of solutions as the starting one is a consequence of supersymmetry: supersymmetry imposes the orientation of the branes to be compatible one with the others, as given by equations (1.3.5)-(1.3.6). Therefore, when one mixes the brane content with a spectral flow, the relative orientation of the branes does not change, and (1.3.5)-(1.3.6) are still verified. We will see in the second part of the thesis that is will not be the case for non-supersymmetric solutions.

1.4 Particular interesting cases

1.4.1 Single-center black hole

First of all, it is important to see that BMPV black hole [60] – a single-center five-dimensional supersymmetric rotating black hole – is in the class of solution that we have. To do so, we first have to remind that flat $\mathbb{R}^4$ is a Hyper-Kähler metric: if we denote $(u, \varphi_1, v, \varphi_2)$ the usual $\mathbb{R}^4$ coordinates, have the Gibbons-Hawking form (1.3.7)

$$ds^2_{\mathbb{R}^4} = du^2 + u^2 d\varphi_1^2 + dv^2 + v^2 d\varphi_2^2\,, \tag{1.4.1}$$

and define new coordinates (ψ, r, χ, ϕ) by

$$r = \frac{1}{4}(u^2+v^2)\,, \qquad \chi = 2\arctan\frac{u}{v}\,, \qquad \psi = 2\,\varphi_1\,, \qquad \phi = -(\varphi_2+\varphi_1)\,, \tag{1.4.2}$$

the four-dimensional metric becomes

$$ds^2_{\mathbb{R}^4} = r(d\psi + (\cos\chi + 1)d\phi)^2 + \frac{1}{r}(dr^2 + r^2 d\chi^2 + r^2\sin^2\chi d\phi^2)\,. \tag{1.4.3}$$

This is exactly of the form (1.3.7) with $V = 1/r$ and $\mathbb{R}^3$ being described by the spherical coordinates (r, χ, ϕ). We will in the following also work in spherical coordinates for $\mathbb{R}^4$

$$ds^2_{\mathbb{R}^4} = d\rho^2 + \rho^2(d\vartheta^2 + \sin^2\vartheta d\varphi_1^2 + \cos^2\vartheta d\varphi_2^2)\,, \tag{1.4.4}$$

with $u = \rho\sin\vartheta$ and $v = \rho\cos\vartheta$.

We can now describe the BMPV black hole. This is a solution without any magnetic charges: $K^I = \Theta^{(I)} = 0$. This means that there are no sources in (1.3.4), and thus the solution simplifies:

$$Z_I = L_I\,, \quad \mu = M \tag{1.4.5}$$

We want a single-center solution and thus take the functions Z_I and the one-form k to be

$$\begin{aligned} Z_I &= 1 + \frac{Q_I}{4r} = 1 + \frac{Q_I}{\rho^2}\,, \\ M &= \frac{J}{8r} = \frac{J}{2\rho^2} \text{ or equivalently } k = k_1 d\varphi_1 + k_2 d\varphi_2 = \frac{J}{\rho^2}(\sin^2\vartheta d\varphi_1 - \cos^2\vartheta d\varphi_2)\,. \quad , \end{aligned} \tag{1.4.6}$$

where J is the angular momentum of the black hole and Q_1, Q_2 and Q_3 its charges. In the type IIA framework presented in section 1.3.2, these charges correspond respectively to the D0-brane, D4-brane and F1-string charges of the black hole. We will study this black hole in the next chapter.

We finally recall that this is a black hole solution, and therefore has an horizon, at $r = 0$. Its entropy is given by

$$S_{BMPV} = 2\pi\sqrt{Q_1Q_2Q_3 - J^2}\,. \tag{1.4.7}$$

For $J^2 > Q_1Q_2Q_3$ the solution has closed time-like curves and is unphysical.

1.4.2 Black rings

The black-ring solution

The three-charge, three-dipole charge black ring solution [40, 59, 61, 62, 52] is in the class of solutions described in the previous section. It is an important solution, that we will study in detail in chapter 3, and thus it is worth writing the solution completely here. We give it in the D0-D4-F1 IIA duality frame, this is the one that we will be using in the probe analysis.

In this frame, the ring has D0, D4 and F1 electric charges and D6, D2 and NS5 dipole charges and its solution is given by

$$\begin{aligned} ds^2 &= -(Z_2Z_1)^{-1/2}Z_3^{-1}(dt+k)^2 + (Z_2Z_1)^{1/2}ds^2_{\mathbb{R}^4} + (Z_2Z_1)^{1/2}Z_3^{-1}dz^2 + Z_2^{-1/2}Z_1^{1/2}ds^2_{T^4}, \\ e^{2\Phi} &= Z_2^{-1/2}Z_1^{3/2}Z_3^{-1}, \\ B &= (Z_3^{-1} - 1)dt\wedge dz + Z_3^{-1}k\wedge dz - B^{(3)}\wedge dz\,\,, \end{aligned} \tag{1.4.8}$$

for the NS-NS fields, and

$$\begin{aligned} C^{(1)} &= (Z_1^{-1}-1)dt + Z_1^{-1}k - B^{(1)}, && (1.4.9)\\ C^{(3)} &= Z_3^{-1}dt\wedge k\wedge dz - Z_3^{-1}(dt+k)\wedge B^{(1)}\wedge dz + B^{(3)}\wedge dt\wedge dz - \gamma_1\wedge dz\ , && (1.4.10)\end{aligned}$$

for the R-R fields. The one-forms, $B^{(I)}$, are the potentials defined in section 1.3.1 with $dB^{(I)} = \Theta^{(I)}$. These fields are sourced by the magnetic charges of the ring. The two-form, γ_1, must satisfy

$$d\gamma_1 = \star_4 dZ_2 - B^{(1)}\wedge\Theta^{(3)}\,. \tag{1.4.11}$$

We use the canonical coordinates that are adapted to the symmetries of the black ring in the flat metric of the $\mathbb{R}^4$ base [61]:

$$ds^2_{\mathbb{R}^4} = g_{\mu\nu}dy^\mu dy^\nu = \frac{R^2}{(x-y)^2}\left(\frac{dy^2}{y^2-1} + (y^2-1)d\varphi_1^2 + \frac{dx^2}{1-x^2} + (1-x^2)d\varphi_2^2\right). \tag{1.4.12}$$

We will also use the orientation: $\epsilon_{yx\varphi_1\varphi_2} = 1$. In these coordinates, the black ring horizon is located at $y\to-\infty$. It is useful to recall that the change of coordinates:

$$x = -\frac{u^2+v^2-R^2}{\sqrt{((u-R)^2+v^2)((u+R)^2+v^2)}}\,,\qquad y = -\frac{u^2+v^2+R^2}{\sqrt{((u-R)^2+v^2)((u+R)^2+v^2)}} \tag{1.4.13}$$

takes one back to the standard flat metric on $\mathbb{R}^2\times\mathbb{R}^2$ (1.4.1). In this coordinates, the ring horizon is at $u=R, v=0$.

The warp factors Z_I are

$$Z_I = 1 + \frac{\overline{Q}_I}{2R^2}(x-y) - \frac{C_{IJK}}{2}\frac{q^J q^K}{4R^2}(x^2-y^2), \tag{1.4.14}$$

where $\overline{Q}_I$ are what we refer to as "constituent charges" of the black ring, and differ from the charges measured at infinity. The angular momentum vector is given by

$$\begin{aligned} k &= k_1 d\varphi_1 + k_2 d\varphi_2\\ &= -\big((y^2-1)\left(C(x+y)+D\right) - A(y+1)\big)d\varphi_1 - \big((x^2-1)\left(C(x+y)+D\right)\big)d\varphi_2 && (1.4.15)\end{aligned}$$

with $A = (q^1+q^2+q^3)/2$, $D = (q^1\overline{Q}_1 + q^2\overline{Q}_2 + q^3\overline{Q}_3)/8R^2$ and $C = -q^1q^2q^3/8R^2$. The vector fields, $B^{(I)}$, are given by

$$B^{(I)} = \frac{q^I}{2}\left((y+d)d\varphi_1 - (x+c)d\varphi_2\right)\,. \tag{1.4.16}$$

The constants c and d are locally pure gauge and are not fixed by the equations of motion. Indeed, because the ring carries a magnetic current there will Dirac strings in any attempt at a global definition of $B^{(I)}$. In the $(u, v, \varphi_1, \varphi_2)$ coordinate patch, defined by (1.4.13), the vector fields, $B^{(I)}$, are potentially singular at either $u=0$, or $v=0$. To remove these singularities we must have $(y+d)=0$ at $u=0$ and $(x+c)=0$ at $v=0$. From (1.4.13) we see that this unambiguously requires $d = +1$ but that one has $x = +1$ for $v = 0, u < R$ and $x = -1$ for $v = 0, u > R$ and so to remove the Dirac strings we must take:

$$d = +1\,,\ c = -1 \quad \textit{inside the ring}\,; \qquad d = +1\,,\ c = +1 \quad \textit{outside the ring}\,. \tag{1.4.17}$$

The coordinates (x, φ_2) in fact define a Gaussian two-sphere around the ring and the choices (1.4.17) represent the familiar gauge field patches surrounding a magnetic monopole. In the following we will set $d = 1$ and retain c with the understanding that it is to be chosen as in (1.4.17).

The two-form γ_1 in $C^{(3)}$ has the form $\gamma_1 = f(x,y)d\varphi_1 \wedge d\varphi_2$ where

$$f(x,y) = -\frac{\overline{Q}_2}{2}\frac{1-xy}{x-y} + \frac{q_1 q_3}{4}\left[\frac{(1-xy)(x+y)}{x-y} + cy - dx\right] + f_0. \tag{1.4.18}$$

where f_0 is another integration constant. It is shown in Appendix A that γ_1 satisfies (1.4.11).

We want to stress that our conventions, given in Appendix B, are such that

$$\overline{Q}_I \;=\; \overline{N}_I \qquad \text{and} \qquad q_I = n_I \tag{1.4.19}$$

where $\overline{N}_I$ and n_I are integers and specify the number of "electric" and "dipole" D-branes comprising the black ring. It is also useful to note that the angular momentum of the black ring is related to its dipole charges by

$$J = 4(q_1 + q_2 + q_3)R\,. \tag{1.4.20}$$

The entropy of the ring is given by the Beckenstein-Hawking formula

$$S = \frac{A}{4G_{11}} = \pi\sqrt{\mathcal{M}} \tag{1.4.21}$$

with

$$\mathcal{M} = 2n_1n_2\overline{N}_1\overline{N}_2 + 2n_1n_3\overline{N}_1\overline{N}_3 + 2n_2n_3\overline{N}_2\overline{N}_3 - n_1^2\overline{N}_1^2 - n_2^2\overline{N}_2^2 - n_3^2\overline{N}_3^2 - 4n_1n_2n_3J \tag{1.4.22}$$

The black ring as a solution with a Gibbons-Hawking base.

As for the black hole, it is useful to rewrite the solution in the Gibbons-Hawking form (1.3.7). The four-dimensional base metric is still only $\mathbb{R}^4$, and the change of coordinates is (1.4.2) (with (1.4.13)).

The black ring solution is written in terms of eight harmonic functions V, L_I, K^I and M [51, 65, 52, 63, 64]. However, as we noted in the last subsection, the black ring has a monopolar magnetic field and so we need two patches that are related by a gauge transformation. Remembering that the vector potentials in solutions with a GH base are given by

$$B^{(I)} = V^{-1}K^I(d\psi + A) + \xi^I\,, \tag{1.4.23}$$

one can easily identify the K^I that give these fields, and observe that changing the patch from $c{=}-1$ to $c{=}+1$ corresponds, in the GH solution, to the gauge transformation:

$$\begin{aligned} K^I &\to K^I + c^I V\,, \qquad\qquad L_I \to L_I - C_{IJK} c^J K^K - \frac{1}{2} C_{IJK} c^J c^K V, \\ M &\to M - \frac{1}{2}\, c^I\, L_I + \frac{1}{12}\, C_{IJK} \left(V\, c^I\, c^J\, c^K + 3\, c^I\, c^J\, K^K \right)\,, \end{aligned} \tag{1.4.24}$$

with $c^I = q^I/2$. Thus, we can now completely specify the eight harmonic functions, once we choose a patch. For $c = -1$, we have

$$\begin{aligned} V &= \frac{1}{r}\,, \quad K^I = -\frac{q_I}{2|\vec{r} - \vec{r}_{BR}|}\,, \\ L_I &= 1 + \frac{\overline{Q}_I}{4|\vec{r} - \vec{r}_{BR}|}\,, \quad M = -\frac{J}{16|\vec{r} - \vec{r}_{BR}|} + \frac{J}{16R}\,, \end{aligned} \tag{1.4.25}$$

and for $c = +1$ they become

$$\begin{aligned} V &= \frac{1}{r}\,, \qquad K^I = -\frac{q_I}{2|\vec{r} - \vec{r}_{BR}|} + \frac{q_I}{2r}\,, \\ L_I &= 1 + \frac{\overline{Q}_I + C_{IJK} q^J q^K}{4|\vec{r} - \vec{r}_{BR}|} - \frac{C_{IJK} q^J q^K}{8r}\,, \qquad M = -\frac{J + q^I \overline{Q}_I + 3 q^1 q^2 q^3}{16|\vec{r} - \vec{r}_{BR}|} - \frac{q^1 q^2 q^3}{16r}\,. \end{aligned} \tag{1.4.26}$$

As noted earlier, these formulae *define* the GH charges of the black ring and these, in turn, define the electric charges of the four-dimensional black hole corresponding to the ring. The electric GH charges Q_I^{GH} are four times the coefficients of the pole at the location of the ring in the L_I functions, the GH dipole charges q_I^{GH} are minus two times the coefficients of the pole in the K^I functions, and the GH angular momentum J^{GH} is minus sixteen times the coefficient of the pole in M (we use the conventions of [51]). Thus, we have:

$$Q_I^{GH} = \overline{Q}_I\,, \qquad q_I^{GH} = q_I\,, \qquad J^{GH} = J \tag{1.4.27}$$

for $c = -1$ and

$$Q_I^{GH} = \overline{Q}_I + C_{IJK} q^J q^K\,, \qquad q_I^{GH} = q_I \qquad J^{GH} = J + q^I \overline{Q}_I + 3 q^1 q^2 q^3 \tag{1.4.28}$$

for $c = +1$.

The dipole charges are patch-independent, but the GH electric charges and the GH angular momentum are *gauge dependent* notions, and are different in different patches. This will be important in the following discussion.

1.4.3 Smooth solutions

The solutions of (1.3.4)-(1.3.7), given by the harmonic functions (1.3.15) are generically a multicenter configuration of four-charge black objects. In the harmonic functions, one has a pole at each center and therefore a lot of singularities. However, if one chooses the residues of the poles at each center in a very particular way, one can manage to cancel all the singularities and end up with a completely regular solution [43, 44, 45, 66]. This solution will be smooth and horizonless everywhere, without any region with a high curvature that would break the supergravity approximation. This is a very non-trivial behavior. Indeed, in order to do so, remembering that the harmonic functions are

$$\begin{aligned} V &= \epsilon_0 + \sum_{j=1}^{N} \frac{q_j}{r_j}\,, \qquad & K^I &= \kappa_0^I + \sum_{j=1}^{N} \frac{k_j^I}{r_j}\,, \\ L_I &= l_0^I + \sum_{j=1}^{N} \frac{l_j^I}{r_j}\,, \qquad & M &= m_0 + \sum_{j=1}^{N} \frac{m_j}{r_j}\,, \end{aligned} \tag{1.4.29}$$

one has to have, to begin with, the q_j to have positive or negative sign. It implies that the four-dimensional base will switch from regions of (+,+,+,+) to regions of (−,−,−,−) signature. We call this an ambipolar base, and the corresponding solution an ambipolar solution. This gives a very singular behavior to the metric (1.3.7) but the important point to understand is that the base is not directly a physical quantity; the only metric that has to be regular is the eleven-dimensional one (1.3.1). For it to be regular, one first need the Z_I functions to remain finite everywhere. Recalling that

$$Z_I = L_I + \frac{C_{IJK}}{2}\frac{K^J K^K}{V}\,, \tag{1.4.30}$$

one can cancel the singularities at each pole by assuming

$$l_j^I = -\frac{C_{IJK}}{2}\frac{k_j^J k_j^K}{q_j}\,, \tag{1.4.31}$$

for each center $\vec{y}_j$. One then also have to assume that the μ function does not have any pole,

$$\mu = M + \frac{1}{2}\frac{K^I L_I}{V} + \frac{C_{IJK}}{6}\frac{K^I K^J K^K}{V^2}\,, \tag{1.4.32}$$

and this happens if

$$m_j = \frac{k_j^1 k_j^2 k_j^3}{2 q_j^2}\,. \tag{1.4.33}$$

With these choices the integrability equations (1.3.17) reduce to the bubble equations considered in [44, 45]. We refer to [44] for a complete analysis of this case and the discussion of closed timelike curves (CTCs). The very important point to understand is that with the constraints (1.4.31)-(1.4.33) together with the bubble equations (1.3.17), the solution ends up being *completely regular*, without any singularities nor horizons. And we want to emphasize that the solution is never Planck-sized and its curvature remains always small enough such that supergravity is a valid approximation of string theory. In the context of the fuzzball proposal, these solutions can be seen as black hole microstates: they carry the same mass and charges that the

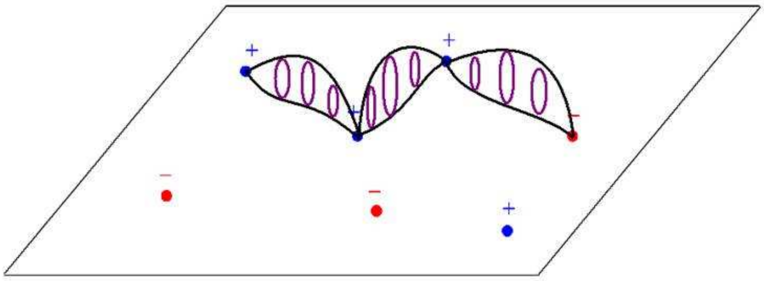

Figure 1.1: *On this graph, we present a schematic picture of the multi center Taub-NUT space. The space has Gibbons-Hawking centers with positive (in blue) or negative (in red) topological charges. The fiber of the space reduces to zero size at the location of each center, and has a finite size in between. It therefore forms two-spheres between every pair of centers, and this two-spheres are the one supporting the magnetic fluxes of the solution.*

ones of an extremal black hole, but when one gets close to what would be the horizon, one sees that there is no horizon, or singular charge distribution. The solutions does not contain any localized sources. So what sources this charges? As it is clear from the BPS equations (1.3.4), the charges comes from magnetic-magnetic interactions. Physically, we replace the singular sources by a non-trivial topology, a multi-center V function, and magnetic fluxes; if one looks at the Gibbons-Hawking metric (1.3.7), it is clear that the size of the fiber shrinks when V blows up, at each center, as pictured on Figure 1.1.

Going from one center to another, the fiber grows up, and reduces size to zero again, and it forms S^2 spheres. These 2-spheres are supporting the magnetic fluxes $\Theta^{(I)}$, and the interaction between the fluxes creates the electric charges, without any need of singular sources. It is interesting to note that this behavior works with an ambipolar base, a base that changes signature from $(+,+,+,+)$ to $(-,-,-,-)$ from one region to another. This *a priori* very singular, very odd, behavior is the key for a mechanism that allows to construct black hole microstates. This mechanism – replacing singular sources by fluxes on 2-spheres – is in principle quite general, and we will see in chapter 8 that it also applies to non-supersymmetric, non-extremal solutions.

1.4.4 Supertubes

The last particular subcase we want to look at is that of supertubes. First introduced in a open string picture [19, 20, 21] (that we will review in the next chapter), one now knows the corresponding supergravity, backreacted, solution. Supertubes are brane bound states carrying two electric charges Q_1 and Q_2 and one dipole magnetic charge d. They are 1/4-BPS objects, and have an angular momentum J given by

$$J = \frac{Q_1 Q_2}{d} \tag{1.4.34}$$

for it to be regular. In the first place, they were introduced as D2-branes with a compact direction, carrying F1 and D0 charges, but they can be dualized to other duality frames. Upon dualization, this is also possible to add charges to obtain three-charge, two dipole-charge supertubes. Being supersymmetric, this tubes are very interesting in the context of probing supergravity supersymmetric backgrounds, as we will see in the next two chapters.

As supergravity solutions, supertubes with a $U(1)$ isometry are in the class of solutions we described, with only some of the charges. Explicitly, a center is a supertube if at this center, only L_1 L_2 K^3 and M have poles, while V, K^1, K^2 and L_3 do not (or permutations of 1,2 and 3). If one writes the supergravity solutions corresponding to a supertube in any duality frame, they will not in general be regular. However, if one dualizes in the duality frame where the electric charges are D1 and D5 charges (the one presented in 1.3.3), they become completely regular. We will therefore consider them to be regular solutions, keeping in mind that this is true only in this particular duality frame. There is also another argument for which supertubes are regular objects: if one starts from a regular solution of the type considered in section 1.4.3, verifying (1.4.31)-(1.4.33), and performs a spectral flow transformation (1.3.35), it is possible to transform one of the center of the solution into a supertube. Conversely, a spectral flow transforms generically a tube into a smooth center. Spectral flow transforming solutions into physically equivalent solutions, it is natural to think about the supertubes on the same footing as the smooth centers, verifying (1.4.31)-(1.4.33). In this section, we study, from a supergravity point of view, the regularity constraints for a supertube configuration. In addition to its implicit interest, it will be important to have this constraints to be able to compare them to what we will obtain in the two following chapters, where we will probe a supergravity background with a supertube.

Constraints from supertube regularity

Consider the D1-D5-P solutions in which one of the centers has vanishing GH charge, and non-trivial D1 and D5 electric charges. Generally such a solution is not regular and can have a horizon or a naked singularity. However, the solution will be regular if one arranges the charges at this point to be those of a two-charge supertube.

Suppose that at $r_1 = 0$ we have a round two-charge supertube with one dipole charge. We take the latter to be k_1^3 and so we have $k_1^1 = k_1^2 = 0$ and $l_1^3 = 0$. This means that in the neighborhood of a two-charge supertube at $r_1 = 0$, we must have:

$$Z_I \;\sim\; \mathcal{O}(r_1^{-1})\,, \quad I = 1, 2\,; \qquad\qquad V, Z_3 \;\sim\; finite\,. \tag{1.4.35}$$

The six-dimensional metric in IIB frame can be re-written as:

$$ds_6^2 \;=\; -\frac{1}{Z_3\sqrt{Z_1 Z_2}}\,(dt+k)^2 + \sqrt{Z_1 Z_2}\, ds_4^2 + \frac{Z_3}{\sqrt{Z_1 Z_2}}\,(dz + A^{(3)})^2\,. \tag{1.4.36}$$

To check regularity along the supertube one must examine potential singularities along the ψ-fiber by collecting all the $(d\psi + A)^2$ terms in (1.4.36):

$$(Z_1 Z_2)^{-\frac{1}{2}} V^{-2} \big[Z_3 \, (K^3)^2 \; - \; 2\mu V K^3 \; + \; Z_1 Z_2 V \big] \; (d\psi + A)^2 \, . \tag{1.4.37}$$

For regularity as $r_1 \to 0$, one must have:

$$\lim_{r_1 \to 0} \, r_1^2 \, \big[Z_3 \, (K^3)^2 \; - \; 2\mu V K^3 \; + \; Z_1 Z_2 V \big] \; = \; 0 \, . \tag{1.4.38}$$

Next there is a potential problem with CTC's coming from Dirac strings in ω. For ω to have a Dirac string originating at $r_1 = 0$, the source terms in the equation for $\vec{\omega}$ must have a piece that behaves as a constant multiple of $\vec{\nabla} \frac{1}{r_1}$. To examine this, it is easier to use (1.3.14) and recall that Z_3, K^1, K^2 and V are finite as $r_1 \to 0$. Thus the only sources of "dangerous terms" are $V \vec{\nabla} \mu$ and $Z_3 \vec{\nabla} K^3$. Since V and Z_3 are finite at $r_1 = 0$, there will be no Dirac strings starting at $r_1 = 0$ if and only if:

$$\lim_{r_1 \to 0} \, r_1 \, \big[V \mu \; - \; Z_3 \, K^3 \big] \; = \; 0. \tag{1.4.39}$$

The two conditions, (1.4.38) and (1.4.39), guarantee that the supertube smoothly caps off the spatial geometry and are the generalization to three-charge three-dipole backgrounds of the conditions for smooth cap-off in [13].

One can massage these conditions using (1.4.39) to eliminate all the explicit K^3 terms in (1.4.38). The condition (1.4.38) may then be written as

$$\lim_{r_1 \to 0} \, r_1^2 \, \mathcal{Q} \; = \; 0 \, . \tag{1.4.40}$$

where $\mathcal{Q}$ is the E_7 invariant that determines the four-dimensional horizon area [67, 51]:

$$\begin{aligned} \mathcal{Q} \; &\equiv \; Z_1 Z_2 Z_3 V \; - \; \mu^2 \, V^2 & (1.4.41) \\ &= \; -M^2 \, V^2 - \frac{1}{3} \, M \, C_{IJK} K^I \, K^J \, K^k - M \, V \, K^I \, L_I - \frac{1}{4} (K^I L_I)^2 & \\ &\quad + \frac{1}{6} V C^{IJK} L_I L_J L_K + \frac{1}{4} C^{IJK} C_{IMN} L_J L_K K^M K^N \, . & (1.4.42) \end{aligned}$$

We will therefore refer to (1.4.40) as the quartic constraint. Note that the right-hand side of (1.3.14) is the quadratic E_7 invariant, and so we may view (1.4.39) as the "quadratic constraint." It is, however, convenient to rewrite this constraint by eliminating μ from (1.4.38) using (1.4.39). One then obtains:

$$\lim_{r_1 \to 0} \, r_1^2 \, \big[V Z_1 Z_2 \; - \; Z_3 \, (K^3)^2 \big] \; = \; 0 \, . \tag{1.4.43}$$

We will use (1.4.39) and (1.4.43) as the independent constraints because they are simplest to apply.

In flat space the supertube solution has $V = \dfrac{1}{r}$, $K^1 = K^2 = 0$ and $Z_3 = 1$, and equation (1.4.43) determines the radius of the supertube in terms of its charges, and (1.4.39) fixes the parameter m_1 of (1.3.15), and thus determines the angular momentum of the supertube in terms of its radius and charges.

Supertube regularity and spectral flow

As explained in [58], one can obtain a solution with a supertube inside a general three-charge solution by spectrally flowing a smooth horizonless bubbling solution. Since spectral flow is implemented by a coordinate change in six dimensions, it cannot affect the smoothness or the regularity of the solution. Equivalently, regularity is determined by placing conditions on quadratic and quartic E_7 invariants, and as shown in [58], these are invariant under spectral flow transformations.

We therefore expect that the equations that determine the smoothness of supertubes, (1.4.38), (1.4.39) and (1.4.43), should be related by spectral flow to the equations that determine the smoothness of a usual bubbling solution. Indeed, consider the spectral flow transformation (see [58] for more detail):

$$\widetilde{V} = V + \gamma\, K^3\,, \qquad \widetilde{K}^1 = K^1 - \gamma\, L_2\,, \qquad \widetilde{K}^2 = K^2 - \gamma\, L_1\,, \qquad \widetilde{K}^3 = K^3 \tag{1.4.44}$$

$$\widetilde{L}_1 = L_1\,, \qquad \widetilde{L}_2 = L_2\,, \qquad \widetilde{L}_3 = L_3 - 2\,\gamma\, M\,, \qquad \widetilde{M} = M\,, \tag{1.4.45}$$

with

$$\gamma = -\frac{q_1}{k_1^3}\,. \tag{1.4.46}$$

This transformation maps a GH bubbled solution to a GH bubbled solution with a supertube at $r_1 = 0$. Under this spectral flow one also has:

$$\widetilde{Z}_1 = \left(\frac{V}{\widetilde{V}}\right) Z_1\,, \qquad \widetilde{Z}_2 = \left(\frac{V}{\widetilde{V}}\right) Z_2\,, \qquad \tilde{\mu} = \left(\frac{V}{\widetilde{V}}\right)\left(\mu - \gamma\frac{Z_1 Z_2}{\widetilde{V}}\right)\,, \tag{1.4.47}$$

$$\widetilde{Z}_3 = \left(\frac{\widetilde{V}}{V}\right) Z_3 + \gamma^2\left(\frac{Z_1 Z_2}{\widetilde{V}}\right) - 2\,\gamma\mu\,. \tag{1.4.48}$$

In the usual bubbling solution, regularity requires that the Z_I are finite and $\mu \to 0$ as $r_1 \to 0$. In the solution with the supertube one can use this and (1.4.48) to verify that:

$$\lim_{r_1\to 0} r_1 \left[\widetilde{V}\tilde{\mu} - \widetilde{Z}_3\,\widetilde{K}^3\right] = -\gamma \lim_{r_1\to 0} r_1 \left(\frac{V Z_1 Z_2}{\widetilde{V}}\right)\left(1 + \gamma\,\frac{K^3}{V}\right)\,, \tag{1.4.49}$$

$$\lim_{r_1\to 0} r_1^2 \left[\widetilde{V}\widetilde{Z}_1\widetilde{Z}_2 - \widetilde{Z}_3\,(\widetilde{K}^3)^2\right] = \lim_{r_1\to 0} r_1^2 \left(\frac{Z_1 Z_2}{\widetilde{V}}\right)\left(V^2 - \gamma^2 (K^3)^2\right)\,. \tag{1.4.50}$$

Both of these vanish by virtue of (1.4.46) and the finiteness of the Z_I and $\widetilde{V}$ as $r_1 \to 0$. Hence, the equations determining the smoothness and regularity of two-charge supertubes are related by spectral flow to those determining the smoothness and regularity of usual three charge bubbling solution.

Part I

Probing supergravity solutions

Motivations and results

The physics of two-charge supertubes is an essential ingredient in understanding the microstates of the two-charge D1-D5 system. Indeed, as we already explained in the first chapter, supergravity solutions for two charge supertubes with D1 and D5 charges and KKM dipole charge are smooth in six dimensions. In addition, they can have arbitrary shape. Hence, they have an infinite dimensional classical moduli space, which, upon quantization, gives the entropy one expects from counting at weak-coupling: $S = 2\pi\sqrt{2N_1N_5}$ [11, 12, 13, 14, 16, 18, 19].

While this entropy is considerable, it is nowhere near the entropy of a black hole with three charges: $S = 2\pi\sqrt{N_1N_5N_P}$ [4]. Hence, if one's goal is to prove that in the regime of parameters where the classical black hole exists one can find a very large number of string/supergravity configurations that realize enough microstates to account for the Beckenstein-Hawking entropy of this black hole [6], the entropy coming from two-charge supertubes does not appear to be large enough.

However, in a scaling supergravity background with large magnetic dipole fluxes, the humble two-charge supertube has, as we will see in the following, more to it than meets the eye, and can undergo *entropy enhancement*. That is, if one uses the Born-Infeld action to compute the entropy of a probe two-charge supertube placed in a background with three charges and three dipole charges, one finds that such a supertube can have an entropy that is *much* larger than that of the same supertube in empty space. The magnetic dipole-dipole interactions between the supertube and the background can greatly increase the capacity of the supertube to store entropy. Hence, the interaction with the supergravity background can enhance (or decrease) the entropy coming from the fluctuating shape of a supertube.

As yet, the fully back-reacted solution corresponding to a supertube in a non trivial background has only been constructed in the case of round supertubes with constant charge densities (see chapter 1), and so the entropy enhancement calculation has only been done in a probe approximation. Nevertheless, in the absence of the fully back-reacted solutions, one can still pose a very sharp question, whose answer can tilt the balance one way or another in the quest to understand whether the black hole is a thermodynamic description of a very large number of horizonless microstates: "*Do two-charge supertubes that are solutions of the Born-Infeld equations of motion correspond to smooth solutions of supergravity once the back-reaction is included?*"[1]

If the answer to this question is *yes*, then all the supertube microstates that were counted in

[1]In [68], we recently compute the backreaction of a supertube with density modes in a two center background. We do not include this recent results in the discussion, but just mention that in this case, we were able to show that the solution is indeed completely smooth.

this scaling regime give smooth microstate solutions of supergravity, valid in the same regime of parameters where the classical black hole exists. Since the Born-Infeld counting might give a macroscopic (black-hole-like) entropy, this would imply that the same entropy could come from smooth supergravity solutions. Our goal in the first part of this thesis is to show that the Born-Infeld description of a supertube does indeed capture all the essential physics of the complete supergravity solution and argue that the corresponding supergravity solution will be smooth in the D1-D5 duality frame, and to show that the entropy of supertubes in magnetically charged backgrounds is enhanced.

First, we establish that when one has *both* a Born-Infeld and a supergravity description of supertubes in a three-charge, three-dipole-charge background, the two descriptions agree to the last detail. As we will see, this agreement can be rather subtle. For example, a supertube that is merging with a black ring appears to merge at an angle that depends on its charges but when this merger is described in supergravity, the merger appears to be angle-independent. The resolution of this rests upon the correct identification of constituent charges and the fact that such charges can depend upon "large" gauge transformations.

Another important fact we establish is that the solutions of the Born-Infeld action are always such that the corresponding solutions of supergravity are smooth in the duality frame where the supertube has D1 and D5 charges. Indeed, upon carefully relating the Born-Infeld and the supergravity charges, we will find that the equations that insure that a supertube is a solution of the Born-Infeld action are identical to the equations that insure that the corresponding supergravity solution is smooth.

One could take the position that our analysis here only implies the smoothness of round supertubes, which have both Born-Infeld and supergravity descriptions. It is possible that the wiggly supertubes (which, upon entropy enhancement, might give a black-hole-like entropy) could give rise to singular solutions when brought to the supergravity regime. While such a possibility cannot be fully excluded before the construction of the fully back-reacted wiggly supertubes, we have some rather strong reasons to believe it is highly unlikely. Indeed, if one investigates the conditions for smoothness of the supergravity solution and compares them to the Born-Infeld conditions, one finds that both the supergravity conditions and the Born-Infeld conditions are *local*. Hence, since any curve can be locally approximated as flat, our analysis indicates that no local properties of wiggly supertubes (like the absence of regions of high curvature) will differ from the local properties of round or flat supertubes. Thus one has a very reasonable expectation that supertubes of arbitrary shape will source smooth supergravity solutions.

In particular, if one considers supertubes of arbitrary shape in flat space, the solutions of the Born-Infeld action always give smooth supergravity solutions [12, 13]. If one now considers a three-charge, three-dipole charge solution containing supertubes whose wiggling scale is much smaller than the variation scale of the gauge fields of the background, one can perform a gauge transformation that locally removes the gauge fields and transforms a portion of this supertube into a portion with many wiggles of a supertube in flat space. Since the latter supertube is smooth, and since gauge transformations do not affect the smoothness of solutions, this implies that the original wiggly supertube is also giving a smooth solution.

Obviously the foregoing conclusion is restricted to the domain of validity of supergravity. If a supertube of arbitrary shape is very choppy, the local curvature will be roughly proportional to the inverse of the scale of the choppiness, and hence if the choppiness is Planck-sized then the curvature of the solution will also be Planck-sized. Such solutions are thus outside the domain of validity of supergravity. The main conclusion of our analysis is that supertubes whose wiggles are *not* Planck-sized will give smooth, low-curvature supergravity solutions.

Our analysis does not establish whether the typical microstates of a certain black hole will have high curvature or will be well described in supergravity. However, it does establish that if the wiggles of the Born-Infeld supertubes that gave the typical microstates are not Planck-sized, the corresponding supergravity solutions will not be either.

The second aim of this part is to probe in a very detailed way black ring backgrounds. It will first clarify several issues related to the embedding of black rings in Taub-NUT, and to the relation between the electric charges of the ring and those of the corresponding four-dimensional black hole. We show that when embedding a black ring solution in Taub-NUT one needs to use at least two coordinate patches. From the perspective of one patch, the electric charges are the ones found in [51], and the ring "angular" momentum along the Taub-NUT fiber (corresponding to the D0 charge in four dimensions) is given by the difference of the two five-dimensional angular momenta. The entropy is given by the $E_{7(7)}$ quartic invariant of these charges [67], as common for four-dimensional BPS black holes [69].

From the perspective of the other patch, the charges and the Kaluza-Klein angular momentum of the corresponding four-dimensional black hole are shifted, to certain values that have no obvious five-dimensional interpretation[1]. The entropy of the black ring is again given by the $E_{7(7)}$ quartic invariant, but now as a function of the shifted charges. The two four-dimensional black holes corresponding to the black ring are related by a gauge transformation, which shifts the Dirac string in the gauge potentials from one side of the ring to another[2].

Probing black rings, we will also verify chronology protection when supertubes and black rings are merged. While chronology protection is expected to be valid for this merger, the way it works is subtle. We compute the merger condition between a supertube and a black ring, and find that this condition depends on the position on the S^2 of the black ring where the supertube merges. We also find that neither very large nor very small supertubes can merge with the ring, for obvious reasons. If one varies the charge of the supertubes we find that mergers happen when the charge lies in a certain interval: At one extreme the supertube barely merges on the exterior of the ring while at the other it barely merges on the interior of the ring.

We also discuss a subtlety in identifying the constituent charges carried into the black ring by a merging supertube. We find that when the S^1 of the supertube curves around the S^2 of the black ring horizon, the charge brought in by a given supertube *must* depend on the S^2 azimuthal angle at which the supertube merges with the ring. Otherwise chronology is not protected. It would be most interesting to see how this comes about in the full supergravity merger solution.

The last, and one of the most important, aim of the first part of this thesis, presented

[1] The asymptotic five-dimensional electric charge is the average between the four-dimensional electric charges in the two patches.

[2] Note that we can also perform a gauge transformation that shifts the four-dimensional electric charges to the asymptotic five-dimensional charges of the black ring [71]. The corresponding four-dimensional solution has two Dirac strings in the gauge potentials

in chapter 4, is to compute in a detailed way supertube entropy, and to present the entropy enhancement mechanism: in magnetically charged backgrounds, the entropy of a supertube does not scale like $\sqrt{Q_1 Q_2}$, but like $\sqrt{Q_{1\text{eff}} Q_{2\text{eff}}}$, the charges of the supertube are replaced by effective charges that depend on the magnetic-magnetic interaction between the tube and the background. If there are strong magnetic fluxes in the background, as there are in deep, bubbled microstate geometries, these effectives charges can be *much larger* than the asymptotic charges of the configuration, and can thus lead to a very large entropy enhancement! Indeed, one finds that if the supertube is put in certain deep scaling solutions, the effective charges can diverge if the supertube is suitably localized or if the length of the throat goes to infinity. Of course, this divergence is merely the result of not considering the back-reaction of the wiggly supertube on its background: Once this back-reaction is taken into account, the supertube will delocalize and the fine balance needed to create extremely deep scaling solutions might be destroyed if the tube wiggles too much.

Hence, we expect a huge range of possibilities in the the semi-classical configuration space, from very shallow solutions to very deep solutions. In very shallow solutions, the supertubes can oscillate a lot, but they will not have their entropy enhanced and for very deep solutions the supertube will have vastly enhanced charges but, if the solution is to remain deep, the supertube will be very limited in its oscillations. One can thus imagine that the solutions with most of the entropy will be intermediate, neither too shallow (so as to obtain effective charge enhancement), nor too deep (to allow the supertube to fluctuate significantly).

It is interesting to note that entropy enhancement is not just a red-shift effect: There is no entropy enhancement unless there are strong background magnetic fluxes. A three-charge BPS black hole will not enhance the entropy of supertubes: it is only solutions that have dipole charges, like bubbled black holes or black rings that can generate supertube entropy enhancement.

A very interesting ingredient that we use is the generalized spectral flow transformation [58][1], presented in the first chapter. This enables us to start from a simple, bubbled black hole microstate geometry [44, 45] and generate a bubbled geometry in which one or several of the Gibbons-Hawking centers are transformed into smooth two-charge supertubes. Indeed, from a six-dimensional perspective (in a IIB duality frame in which the solution has D1-D5-P charges) this mapping is simply a coordinate transformation. One can then study the particular class of fluctuating microstate geometries that result from allowing the supertube component to oscillate in the deep bubbled geometries. The naive expectation is that one would recover an entropy of the form $\sqrt{Q_1 Q_2}$ but, as we indicated, the Q_I are replaced by the enhanced $Q_{I\text{eff}}$, and the entropy of these supertubes can become "macroscopic" in that it corresponds to the entropy of a black hole with a macroscopic horizon. One can then undo the spectral flow to argue that this entropy is present in the BPS fluctuations of three-charge bubbling solutions in *any* duality frame. In fact, spectrally flowing configurations with oscillating supertubes into other duality frames is not strictly speaking necessary for the purpose of illustrating entropy enhancement and arguing that smooth solutions can give macroscopically large entropy. After all, one could do the full analysis in the D1-D5-P duality frame and consider smooth black hole microstates containing both GH centers and supertubes. Nevertheless, since such solutions have

[1] See [43, 72] for relevant earlier work.

not been studied in the past in great detail, it is easiest to construct them by spectrally flowing multi-center GH solutions, which have been studied much more and are better understood.

Our analysis establishes that supertube entropy enhancement can come from supertube oscillation modes in *both* the internal space of the solution (T^4 in our calculations) and from oscillations of supertubes in the transverse spacetime directions. We analyze entropy enhancement in black-ring backgrounds, in which the detailed computation is more straightforward than in generic solutions with a Gibbons-Hawking base. We find that, despite the presence of different (large) factors in the mode expansions, the fluctuations in the plane transverse to the ring give a contribution to the entropy that is identical to that coming from the fluctuations along the compactification torus.

If, as we expect, the enhanced entropy coming from these fluctuations will be black-hole-like, and therefore the fluctuating supertubes will give the typical microstates of the corresponding black hole, our analysis establishes that these microstates will have a non-trivial transverse size. We believe it important to calculate the amount of entropy enhancement coming from *all* the oscillations of the supertube. If the other transverse oscillations are more entropic than the torus ones, this would suggest that five-dimensional supergravity may be enough to capture the typical states of the black hole. On the other hand, if the torus and the transverse fluctuations are equally entropic (as hinted by our partial analysis), the typical states will probably have a curvature set by the compactification scale. Even if this scale is at the Planck scale, the microstate geometries constructed in supergravity will give a pretty good approximation of the rough features of the typical states (like the size, the density profile, the multipole moments). Hence the smooth microstate geometries will act as *representatives* of the typical black hole microstates [70].

It is also interesting to note that a similar conclusion – that deep, scaling, horizonless configurations can give a macroscopic (black-hole-like) entropy – was also reached in [73] and [74]. In [73] this was done by considering D0 branes in a background of D6 branes with world-volume fluxes, in the regime of parameters where the D0 branes do not back-react. In [74], a similar result was obtained by studying the quiver quantum mechanics of multiple D6 branes, in the regime where the branes do not back-react, but form a finite-sized configuration. Since these computations were performed in a regime in which the gravitational back-reaction of all or some of the branes is neglected, it is not clear how the configurations that give the black hole entropy will develop in the regime of parameters in which the classical black hole exists, and *all* the branes back-react on the geometry. Their size will continue increasing at the same rate as the would-be black hole horizon, and since they are made from primitive branes, it is very unlikely they will develop a horizon. Hence these two calculations do suggest that the black hole entropy comes from horizonless configurations. However, since the D0 branes give rise to naked singularities, the naive strong-coupling extrapolation of these microstate configurations will not be reliable when the classical black hole exists.

We begin in chapter 2 by presenting supertubes in the open string picture, and the Dirac-Born-Infeld (DBI) action for a brane. This ingredients are crucial for the analysis we want to perform, and we therefore spend some time presenting them in simpler backgrounds corresponding to flat space and a BMPV black hole [60]. It is important to understand that one can in principle use any type of branes to probe a supergravity backgrounds, but using supertubes is a very interesting choice. Indeed, supertubes are 1/4-BPS objects and can be mutually BPS with supersymmetric backgrounds, will not feel any force. The flat space analysis will present supertubes and establishes that they are, as expected, stable by themselves. Probing the BMPV black hole will allows us to present all the physical features that one can extract from the probe analysis.

In chapter 3 we study probe two-charge supertubes in general three-charge magnetically charged solutions: black rings and bubbling solutions with a Gibbons-Hawking base. We present a detailed analysis of two-charge and three-charge supertube probes in the background of a supersymmetric three-charge black ring. We also relate the supergravity and Born-Infeld charges of supertubes, and show that the supergravity smoothness conditions derived in the first chapter, section 1.4.4, agree with the ones derived from the Born-Infeld action. In the end of the chapter we study mergers of the supertube with the black ring and discuss chronology protection and black hole thermodynamics during these mergers.

Chapter 4 contains an in-depth derivation of the entropy coming from oscillations of supertubes and present the entropy enhancement mechanism for black rings and general solutions with a Gibbons-Hawking base.

In Appendix A we give the details of the three-charge three-dipole charge solutions in various duality frames. We also show how to compute the RR potentials corresponding to these solutions in various duality frames. In Appendix B we give the units and conventions used throughout our calculations. In Appendix C we compute the angular momentum of a supertube in several three-charge backgrounds.

Chapter 2

Probe supertubes

In this chapter, we begin to probe supergravity backgrounds. We first present the action for a probe brane, and the key ingredients that enters in our supersymmetric probes, namely supertubes [19, 20, 21], already seen in the first chapter in a backreacted, closed string, picture. Then, as a warm up for the next chapter, we review how to probe a three-charge BMPV black hole [75, 76]. We will see through this example what is the physics that one can expect to extract from such a probe approach.

2.1 Probe brane and DBI action

The action of a Dp-brane can be written as

$$S = S_{DBI} + S_{WZ} \tag{2.1.1}$$

with

$$S_{DBI} = -T_{Dp} \int \mathrm{d}^{p+1}\xi \; \mathrm{e}^{-\Phi} \sqrt{-\det(\widetilde{G} + \widetilde{B} + 2\pi\alpha' F)} \tag{2.1.2}$$

and

$$S_{WZ} = -T_{Dp} \int \mathrm{e}^{\widetilde{B}^{(2)} + \mathcal{F}^{(2)}} \wedge \oplus_n \widetilde{C}^{(n)}. \tag{2.1.3}$$

The worldvolume of the brane is parametrized with the ξ^a coordinates. The 2-form $F^{(2)}$ lives only on the brane worldvolume, and not in the full background. The $\widetilde{G}_{\mu\nu}$, $\widetilde{B}^{(2)}$ and $\widetilde{C}^{(p)}$ fields, that appear in the action, have also to be understood as the pullbacks on the worldvolume of the brane of the corresponding background fields:

$$\widetilde{G}_{ab} = G_{\mu\nu} \frac{\partial x^\mu}{\partial \xi^a} \frac{\partial x^\nu}{\partial \xi^b}, \tag{2.1.4}$$

with corresponding expressions for $\widetilde{B}^{(2)}$ and $\widetilde{C}^{(p)}$. $\det(\widetilde{G}+\widetilde{B}+\mathcal{F})$ ($\mathcal{F}=2\pi\alpha' F$) is the determinant of the $(p+1)\times(p+1)$-sized matrix $\widetilde{G}_{ab}+\widetilde{B}_{ab}+\mathcal{F}_{ab}$. Finally, $\mathrm{e}^{\widetilde{B}^{(2)}+\mathcal{F}^{(2)}}\wedge\oplus_n\widetilde{C}^{(n)}$ is a compact way to denote $\widetilde{C}^{(p+1)}+(\widetilde{B}^{(2)}+\mathcal{F}^{(2)})\wedge\widetilde{C}^{(p-1)}+\ldots$, the selection being made by the fact that the integrated form must be a (p+1)-form.

Physically, S_{DBI}, or the Dirac-Born-Infeld action, is the intrinsic action of the brane while S_{WZ}, or the Wess-Zumino action, encodes the coupling to the Ramond-Ramond fields, the potential energy of the brane. One can remark that in this action, the $\widetilde{C}^{(n)}$ and $\widetilde{B}^{(2)}$ fields play completely different roles, but $\widetilde{B}^{(2)}$ and $\mathcal{F}^{(2)}$ only appear through their sum, and thus play the same role despite their different origins.

The $\mathcal{F}^{(2)}$ field deserves a small explanation, and the simplest way to obtain it is to look at S_{WZ}. A Dp-brane naturally couples to $C^{(p+1)}$ through the $T_{Dp}\int\widetilde{C}^{(p+1)}$ term. When one has a non-zero $\mathcal{F}^{(2)}$ field turned on, it allows for a coupling to smaller forms $C^{(p-1)}, C^{(p-3)},\ldots$, that couple naturally to respectively D(p-2) branes, D(p-4)-branes, etc. One can therefore interpret the $F^{(2)}$ as the presence of smaller branes, "dissolved" in the big one. We consequently say that the brane is charged: for example, if one has a D3-brane in the (t,x_1,x_2,x_3) directions, with $\mathcal{F}^{(2)}_{23}$ non-zero, this translate into a D1-brane charge along (t,x_1). The number N_{D1} of dissolved D1-branes is then given by the equality

$$T_{D3}\int\mathcal{F}^{(2)}\wedge\widetilde{C}^{(1)}=N_{D1}T_{D1}\int\widetilde{C}^{(1)}.$$

We will also see in the next subsection that having $\mathcal{F}^{(2)}_{t1}$ non-zero corresponds to having an F1-string charge. Finally, the $\widetilde{B}^{(2)}$ field present in the action cannot have such a clear physical explanation, but one can still justify its existence. Indeed, there exists a gauge transformation changing $\widetilde{B}^{(2)}$ into $\mathcal{F}^{(2)}$ and vice versa, and the consequence is that the physical quantity is the combination $\widetilde{B}^{(2)}+\mathcal{F}^{(2)}$ [35]. This is why $\widetilde{B}^{(2)}$ and $\mathcal{F}^{(2)}$ can only appear through this particular combination. Because of the physical explanation that we just gave for $\mathcal{F}^{(2)}$, and the symmetric roles played by $\widetilde{B}^{(2)}$ and $\mathcal{F}^{(2)}$, it would be natural to also relate $\widetilde{B}^{(2)}$ to a dissolved-brane charge. This is not the case. A more complete study [77] shows that such charges terms coming from $\widetilde{B}^{(2)}$ are always compensate by a counterterm in the full supergravity action, cancelling its contribution.

2.2 Supertubes in flat space

We want to test supergravity backgrounds with a probe brane. If we probe a supersymmetric background, it is interesting to have a probe that is also supersymmetric and respects the same supersymmetries. We will therefore always take this probe to be a supertube [19]. Supertubes, already presented in section 1.4.4, are best understood in an open string picture: in a IIA duality frame, they can be seen as D2-branes with one compact direction. They carry F1 and D0 charges, and are 1/4-BPS objects. The fact that the D2-brane wraps a contractible cycle makes the D2 charge being dipolar. Such a supertube is thus a 2 charge, one dipole charge object. As we will see in the following, the shape of the compact direction can be arbitrary.

To present the idea of the probe computations, and how one can extract interesting properties from it, we start with the easiest possible example: a supertube in flat space. It will be described by $G_{\mu\nu} = \eta_{\mu\nu}$, $B^{(2)} = 0$, $\Phi = 0$, $C^{(p)} = 0$. We take for now the supertube to be a circular D2-brane with radius r in the (u, φ_1) plane of $\mathbb{R}^4$, in the coordinates given by (1.4.1). It is parametrized by (z, θ), and carries two charges. It corresponds to $\mathcal{F}_{z\theta} \neq 0$ for the D0 charge and $\mathcal{F}_{tz} \neq 0$ for the F1 charge. As we will see, $\mathcal{F}_{tz}$ is not exactly the F1 charge, but will be related to that one. We wrap the supertube n_{D2} times along φ_1, which corresponds to having a D2 dipole charge n_{D2}. Therefore θ runs from 0 to $2\pi n_{D2}$. The induced metric on the tube is

$$\widetilde{G} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r^2 \end{bmatrix} . \tag{2.2.1}$$

In flat, empty, space, $S_{WZ} = 0$, and therefore $S = S_{DBI}$,

$$\begin{aligned} S &= -T_{D2} \int \mathrm{d}t\mathrm{d}z\mathrm{d}\theta \sqrt{-\det(\widetilde{G} + \mathcal{F})} \\ &= -T_{D2} \int \mathrm{d}t\mathrm{d}z\mathrm{d}\theta \sqrt{r^2(1 - \mathcal{F}_{tz}^2) + \mathcal{F}_{z\theta}^2}. \end{aligned} \tag{2.2.2}$$

For the particular value $\mathcal{F}_{tz} = 1$, the expression simplifies, and we obtain

$$S = -T_{D2} \int \mathrm{d}t\mathrm{d}z\mathrm{d}\theta\, \mathcal{F}_{z\theta} \, . \tag{2.2.3}$$

One now wants to compute the energy density, which is given by

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \mathcal{F}_{tz}} \mathcal{F}_{tz} - \mathcal{L} \, . \tag{2.2.4}$$

Before, this is useful to compute the conjugate momentum of $\mathcal{F}_{tz}$

$$\Pi = \frac{1}{T_{D2}} \frac{\partial \mathcal{L}}{\partial \mathcal{F}_{tz}} = \frac{r^2 \mathcal{F}_{tz}}{\sqrt{r^2(1 - \mathcal{F}_{tz}^2) + \mathcal{F}_{z\theta}^2}} \, . \tag{2.2.5}$$

This is inverted by

$$\mathcal{F}_{tz} = \frac{\Pi}{r} \sqrt{\frac{r^2 + \mathcal{F}_{z\theta}^2}{r^2 + \Pi^2}} \, . \tag{2.2.6}$$

The energy density, in terms of Π and $\mathcal{F}_{z\theta}$ is followingly given by

$$\mathcal{H} = \frac{1}{r}\sqrt{(r^2 + \mathcal{F}_{z\theta}^2)(r^2 + \Pi^2)}\,. \tag{2.2.7}$$

This energy is minimal when $r = \sqrt{|\Pi \mathcal{F}_{z\theta}|}$, in which case one can see that $\mathcal{F}_{tz} = 1$, and

$$\int_0^{2\pi n_{D2}} \mathrm{d}\theta\, dz\ \mathcal{H}\rfloor_{\mathcal{F}_{tz}=1} = N_{D0} + N_{F1}\,. \tag{2.2.8}$$

where we have

$$N_{D0} = \frac{T_{D2}}{T_{D0}} \int_0^{2\pi n_{D2}} \mathrm{d}\theta\, dz \mathcal{F}_{z\theta} \quad \text{and} \quad N_{F1} = \frac{1}{T_{F1}} \int_0^{2\pi n_{D2}} \mathrm{d}\theta\, \Pi \tag{2.2.9}$$

respectively the number of D0-branes and fundamental strings. The units and conventions that we are using are specified in appendix B. This particular expression of $\mathcal{H}$ ensures that we have a supersymmetric configuration, the energy of the system being equal to the sum of the supertube charges. This charges being constant, the Hamiltonian as well as the Lagrangian densities are constant, and the tube does not feel any force. This is expected, because we are in flat space, but this will be a generic property of a supersymmetric configuration, that will take place in non trivial background. We will in the following always find the supersymmetric configuration by choosing $\mathcal{F}_{tz} = 1$ without explicitly minimizing the Hamiltonian density.

The explicit expression for N_{F1} gives us an equality:

$$N_{D0} N_{F1} = n_{D2}^2 r^2 \tag{2.2.10}$$

This has to be seen as an equation relating the radius of the supertube with its charges, and can be understood, from a physical point of view, as the fact that the charges are what prevent the tube to collapse on itself. In the following, more complicated cases, this is also from the expression of N_{F1} that we will be able to extract a lot of the physical properties of the probe analysis.

The last data that we have to compute is the angular momentum of the tube. The angular momentum along the φ circle is very generally given by

$$J_\varphi \;=\; \int dz d\theta\, \frac{\partial \mathcal{L}_{tot}}{\partial \dot{\varphi}}\;. \tag{2.2.11}$$

The details of the angular momentum derivation are given in Appendix C. For our case, finally gives

$$J = n_{D2} r^2\,. \tag{2.2.12}$$

Together with (2.2.10), this implies

$$J = \frac{N_{D0}N_{F1}}{n_{D2}} \,. \tag{2.2.13}$$

We will see that in more complicated backgrounds, (2.2.10) and (2.2.12) will be changed into more complicated expressions, but the relation $J = N_{D0}N_{F1}/n_{D2}$ will always hold. As we will see in chapter 4, the entropy of a supertube is

$$S^{ST} \propto \sqrt{N_{D0}N_{F1} - n_{D2}J} \,. \tag{2.2.14}$$

The relation (2.2.13) therefore implies that the tube is maximally spinning and has no entropy.

2.3 Second example: probing a black hole background

To continue, we now consider a probe supertube with two charges and one dipole charge in the background of a three-charge (BMPV) black hole. This example was considered in [75, 76] and was generalized to a probe supertube with three charges and two dipole charges in [78]. The full supergravity solution describing a BMPV black hole on the symmetry axis of a black ring with three charges and three dipole charges was found in [40, 52], and a more general solution in which the black hole is not at the center of the ring was found in [79].

We probe the black hole background presented in section 1.4.1. As we explained before, in order to probe it with a supertube that has a D2 dipole charge and D0 and F1 electric charges, we need this black hole background to be in the D0-D4-F1 frame (see section 1.3.2), so the fields are given by (1.3.22), (1.3.23), (1.3.24) with the explicit values (1.4.6). The RR potentials in this case can be rewritten as

$$C^{(1)} = (Z_1^{-1}-1)dt+Z_1^{-1}k \,, \qquad C^{(3)} = -(Z_2-1)\rho^2\cos^2\vartheta d\varphi_1\wedge d\varphi_2\wedge dz+Z_3^{-1}dt\wedge k\wedge dz \,. \tag{2.3.1}$$

We will denote the world-volume coordinates on the supertube by ξ^0, ξ^1 and $\xi^2 \equiv \theta$. To make the supertube wrap z we take $\xi^1 = z$ and we will fix a gauge in which $\xi^0 = t$. Note that $z \in (0, 2\pi L_z)$. The profile of the tube, parameterized by θ, lies in the four-dimensional non-compact $\mathbb{R}^4$ parameterized by $(\rho, \vartheta, \varphi_1, \varphi_2)$ and for a generic profile all four of these coordinates will depend on θ.

It is convenient to use polar coordinate (u, φ_1) and (v, φ_2) in $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$, where the $\mathbb{R}^4$ metric takes the form (1.4.1). There is also a gauge field, $\mathcal{F}$, on the world-volume of the supertube. Supersymmetry requires that $\mathcal{F}$ essentially has constant components and we can then boost the frames so that $\mathcal{F}_{t\theta} = 0$.

In this frame supersymmetry also requires $\mathcal{F}_{tz} = 1$ [19]. For the present we take

$$2\pi\alpha' F \equiv \mathcal{F} = \mathcal{F}_{tz}dt\wedge dz + \mathcal{F}_{z\theta}dz\wedge d\theta \,, \tag{2.3.2}$$

where the components are constant. Keeping $\mathcal{F}_{tz}$ as a variable will enable us to extract the charges below.

The supertube action is a sum of the DBI and Wess-Zumino (WZ)actions:

$$S = -T_{D2}\int d^3\xi e^{-\Phi}\sqrt{-\det\left(\widetilde{G}_{ab}+\widetilde{B}_{ab}+\mathcal{F}_{ab}\right)} + T_{D2}\int d^3\xi[\widetilde{C}^{(3)}+\widetilde{C}^{(1)}\wedge(\mathcal{F}+\widetilde{B})] \,, \tag{2.3.3}$$

where, as before, $\widetilde{G}_{ab}$ and $\widetilde{B}_{ab}$ are the induced metric and Kalb-Ramond field. We have also chosen the orientation such that $\epsilon_{tz\theta} = 1$. It is also convenient to define the following induced quantities on the world-volume:

$$\Delta_{\mu\nu} = \partial_\mu u \partial_\nu u + u^2 \partial_\mu \varphi_1 \partial_\nu \varphi_1 + \partial_\mu v \partial_\nu v + v^2 \partial_\mu \varphi_2 \partial_\nu \varphi_2 \,, \qquad \gamma_\mu = k_1 \partial_\mu \varphi_1 + k_2 \partial_\mu \varphi_2 \,, \tag{2.3.4}$$

where $\partial_\mu \equiv \frac{\partial}{\partial \xi^\mu}$.

After some algebra, the DBI part of the action simplifies to:

$$\begin{aligned} S_{DBI} &= -T_{D2} \int dt dz d\theta \Big(\frac{1}{Z_1^2} \left(\mathcal{F}_{z\theta} - \gamma_\theta (\mathcal{F}_{tz} - 1) \right)^2 \\ &\quad + \frac{Z_2}{Z_1} \Delta_{\theta\theta} [2(1 - \mathcal{F}_{tz}) - Z_3 (\mathcal{F}_{tz} - 1)^2] \Big)^{1/2} \,, \end{aligned} \tag{2.3.5}$$

while the WZ piece of the action takes the form

$$S_{WZ} = T_{D2} \int dt dz d\theta \left[(1 - \mathcal{F}_{tz}) \frac{\gamma_\theta}{Z_1} + \mathcal{F}_{z\theta} \left(\frac{1}{Z_1} - 1 \right) \right] . \tag{2.3.6}$$

For a supersymmetric configuration ($\mathcal{F}_{tz} = 1$) we have

$$S_{\mathcal{F}_{tz}=1} = S_{DBI} + S_{WZ} = -T_{D2} \int dt dz d\theta \, \mathcal{F}_{z\theta} \tag{2.3.7}$$

The foregoing supertube carries D0 and F1 "electric" charges, given by

$$N_1^{ST} = \frac{T_{D2}}{T_{D0}} \int dz d\theta \; \mathcal{F}_{z\theta}, \qquad N_3^{ST} = \frac{1}{T_{F1}} \int d\theta \, \frac{\partial \mathcal{L}}{\partial \mathcal{F}_{tz}} \bigg|_{\mathcal{F}_{tz}=1} . \tag{2.3.8}$$

The Hamiltonian density is:

$$\mathcal{H}|_{\mathcal{F}_{tz}=1} = \left[\frac{\partial \mathcal{L}}{\partial \mathcal{F}_{tz}} \mathcal{F}_{tz} - \mathcal{L} \right]_{\mathcal{F}_{tz}=1} = T_{D2} \mathcal{F}_{z\theta} + \frac{\partial \mathcal{L}}{\partial \mathcal{F}_{tz}} \bigg|_{\mathcal{F}_{tz}=1} . \tag{2.3.9}$$

One can easily integrate this to get the total Hamiltonian of the supertube[1] (we assume constant charge density $\mathcal{F}_{z\theta}$)

$$\int dz d\theta \; \mathcal{H}|_{\mathcal{F}_{tz}=1} = N_1^{ST} + N_3^{ST} \,. \tag{2.3.10}$$

Thus the energy of the supertube is the sum of its conserved charges which shows that the supertube is indeed a BPS object.

Now choose a static *round supertube* profile $u' = v' = \varphi_2' = 0$, $\varphi_1 = \theta$. One then has:

$$\gamma_\theta = k_1 = J \, \frac{u^2}{(u^2 + v^2)^2} \,, \qquad \Delta_{\theta\theta} = u^2 \tag{2.3.11}$$

[1] See Appendix B for details about our units and conventions.

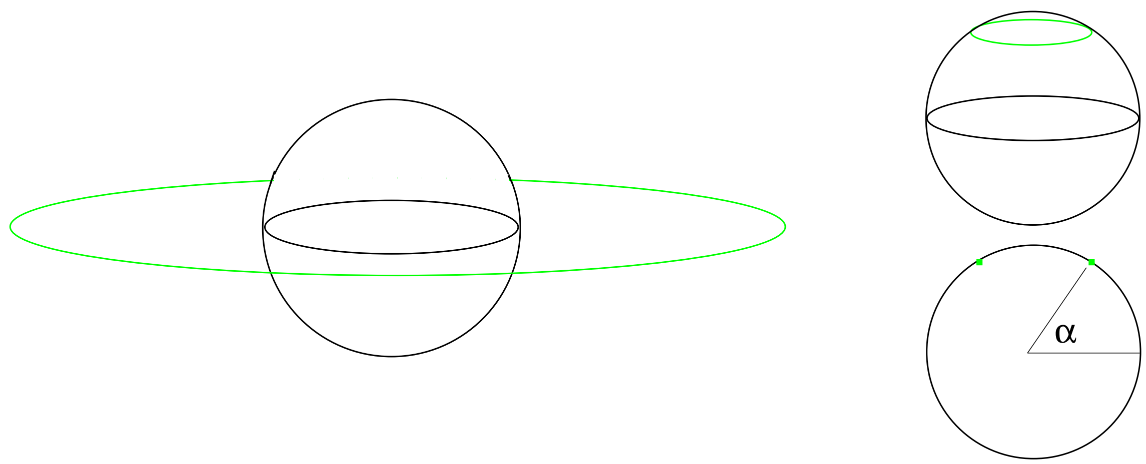

Figure 2.1: *Two different black hole and supertube configurations, for different values of the supertube charges. On the first one, the supertube charges are large, and the tube cannot merge with the black hole. On the second one, the size of the tube is small enough for the merger to be possible. The angle α of the merger depends on the tube charges.*

and the supertube "electric" charges are:

$$N_1^{ST} = n_2^{ST}\mathcal{F}_{z\theta}\ , \qquad\qquad N_3^{ST} = n_2^{ST}\frac{Z_2 u^2}{\mathcal{F}_{z\theta}}\ . \tag{2.3.12}$$

So we find

$$N_1^{ST}N_3^{ST} = (n_2^{ST})^2 u^2\, Z_2\, . \tag{2.3.13}$$

This is an important relation, generalizing (2.2.10), in that it fixes the location of the supertube in terms of its intrinsic charges.

As for the flat space case, one can compute the angular momentum od the tube, given in Appendix C. This gives in this case

$$J^{ST} = n_2^{ST} Z_2 u^2 \tag{2.3.14}$$

and, using (2.3.13), one gets again

$$J^{ST} = \frac{N_1^{ST}N_3^{ST}}{n_2^{ST}}\ . \tag{2.3.15}$$

As we explain for the flat space case, the relation giving the charges of the supertube and its angular momentum have changed because of the non-trivial background, but the relation between them has not.

This computation was used in [75] to study the merger of a supertube and a black hole. Using the explicit expression (1.4.6) for Z_2, (2.3.13) becomes

$$N_1^{ST}N_3^{ST} = (n_2^{ST})^2\rho^2\sin^2\vartheta + (n_2^{ST})^2 N_2 \sin^2\ , \tag{2.3.16}$$

where N_2 is the number of D4 branes in the black hole. In particular, as presented in figure 2.1, a supertube can merge with a black hole if and only if $N_1^{ST} N_3^{ST} \leq (n_2^{ST})^2 N_2$. Moreover, the supertube will "crown" the black hole at "latitude", $\vartheta = \alpha$, given by:

$$\sin\alpha = \sqrt{\frac{N_1^{ST} N_3^{ST}}{(n_2^{ST})^2 N_2}}\,. \tag{2.3.17}$$

The conditions that we obtained here allow us to verify that the chronology protection condition is preserved during the merger, and that the entropy of the black hole increases. As we saw in section 1.4.1, the entropy of the black hole is given by

$$S \propto \sqrt{\mathcal{M}} \equiv \sqrt{N_1 N_2 N_3 - J^2}\,, \tag{2.3.18}$$

Note that the absence of CTCs being given by the inequality $J^2 \leq N_0 N_1 N_4$, verifying that the black hole entropy increases is enough to ensure chronology protection. And this is equivalent to $\Delta\mathcal{M} \geq 0$.

The supertube verifies $J^{ST} = N_1^{ST} N_3^{ST} / n_2^{ST}$ and $J^{ST} \leq N_1^{ST} N_3^{ST}$. Thus one can write ($N_I^{ST} = \Delta N_I$)

$$\begin{aligned}
& (N_1 + \Delta N_1) N_2 (N_3 + \Delta N_3) - (J + \Delta J)^2 \\
= \; & \mathcal{M} + (N_1 \Delta N_3 + N_3 \Delta N_1) N_2 + \Delta N_1 \Delta N_3 N_2 - 2J\Delta J - \Delta J^2 \\
\geq \; & \mathcal{M} + 2N_2 \sqrt{N_1 \Delta N_3 N_3 \Delta N_1} + \Delta J (N_2 - \Delta J) - 2J\Delta J \\
\geq \; & \mathcal{M} + \Delta J (N_2 - \Delta J) + 2\sqrt{N_1 N_2 N_3}\sqrt{N_2 \Delta N_1 \Delta N_3} - 2J\Delta J \\
\geq \; & \mathcal{M} + \Delta J (N_2 - \Delta J) + 2J(\sqrt{N_2 \Delta J} - \Delta J),
\end{aligned} \tag{2.3.19}$$

where we used the arithmetico-geometric inequality for the first inequality and the initial chronology protection for the third. The entry condition of the tube in the black hole being $\Delta J \leq N_4$, and thus $\sqrt{N_4 \Delta J} \geq \Delta J$, this is enough to verify that $S + \Delta S \geq S$.

Chapter 3

Probing a magnetically charged background

We are now ready to get in the real core of this part: in this chapter, we will extend the black hole probe analysis of the previous chapter to black rings and general Gibbons-Hawking smooth backgrounds, presented in the first chapter. The presence of magnetic dipole charges in the background will change the physical analysis in a very interesting way. This will also lead to the "entropy enhancement mechanism", presented in the following chapter.

3.1 Probing a black ring background, a full analysis

3.1.1 Probing the black ring with a two-charge supertube

We now probe the black ring background with a two-charge supertube [19, 21]. The calculation proceeds in much the same way as for the supertube in a black hole background. As before, we parameterize the tube by (t, z, θ), and define an *a priori* arbitrary supertube profile in $\mathbb{R}^4$ by $\vec{y}(\theta)$. Since we are ultimately going to consider a supertube that winds multiple times around the ring direction it will be convenient to take $\theta \in (0, 2\pi n_2^{ST})$ where n_2^{ST} will become this winding number. Thus the supertube will have a dipole charge proportional to n_2^{ST}, and two net charges proportional to N_1^{ST} and N_3^{ST}. Its action is a sum of a DBI and a WZ term

$$S = S_{DBI} + S_{WZ} = -T_{D2} \int dtdzd\theta e^{-\Phi} \sqrt{-\det(\widetilde{G}_{ab} + \widetilde{B}_{ab} + \mathcal{F}_{ab})} + T_{D2} \int dtdzd\theta \left(\widetilde{C}^{(3)}_{tz\theta} + \widetilde{C}^{(1)}_{t} (\widetilde{B}_{z\theta} + \mathcal{F}_{z\theta}) + \widetilde{C}^{(1)}_{\theta} (\widetilde{B}_{tz} + \mathcal{F}_{tz}) \right) \quad (3.1.1)$$

For the supersymmetric configuration one once again finds that $\mathcal{F}_{tz} = 1$ and if one imposes this *ab initio* then one again obtains (2.3.7), (2.3.8) and (2.3.9) and hence the BPS relation for the supertube. The expression for the derivative of the action with respect to $\mathcal{F}_{tz}$ evaluated at $\mathcal{F}_{tz} = 1$ can be most convenient expressed as:

$$\left(\frac{\partial \mathcal{L}}{\partial \mathcal{F}_{tz}} \bigg|_{\mathcal{F}_{tz}=1} + T_{D2} (B^{(1)}_{\varphi_1} \varphi'_1 + B^{(1)}_{\varphi_2} \varphi'_2) \right) \left(\mathcal{F}_{z\theta} + (B^{(3)}_{\varphi_1} \varphi'_1 + B^{(3)}_{\varphi_2} \varphi'_2) \right) = T_{D2} Z_2 g_{\mu\nu} y'^{\mu} y'^{\nu} , \quad (3.1.2)$$

where $'$ denotes the derivative with respect to θ. As for the black hole [75, 76], one can reinterpret this in terms of charge densities and arrive at a generalization of the constraint (2.3.13) that relates the charges to the radius of the supertube. Note that the condition (3.1.2) is local and to get a relation similar to (2.3.13) on has to integrate over the profile of the supertube. There is an important new feature here in that there is a contribution from the interactions of the dipole charges of the supertube and background. This appears through the pull-back of the $B^{(I)}$ to the world-volume of the supertube and it gives an added contribution to the basic supertube charges to yield what we will refer to as the local *effective charges* of the supertube. We will show in section 3.1.3 that this also happens when supertubes are placed in three-charge solutions with a GH base.

It is also important to remember that the Wess-Zumino action of the supertube is only invariant under local small gauge transformations, but is not necessarily invariant under *large gauge transformations.* Indeed, the black ring is a magnetic object, and as such the gauge fields, $B^{(I)}$ are not defined globally but on patches. Their values, and the value of the supertube action, differ from patch to patch by what can be thought of as the effect of a large gauge transformation.

More explicitly, the action depends on the Wilson lines of these gauge fields taken around latitudes of the two-sphere that surrounds the black ring (which is the equivalent of the sphere that contains a monopole charge). The value of these Wilson loops may then be defined using Stokes theorem as the integral of the magnetic flux coming from the black ring through the section of the sphere surrounded by the Wilson line. There is, however, an obvious ambiguity: does one integrate the flux over the upper or the lower cap of the sphere? The difference is, of course, the monopole charge inside the sphere multiplied by the number of times the Wilson loop winds around the latitude circle. These ambiguities will manifest themselves in the definitions of the constituent charges of the supertube.

To analyze the physics of the merger, we consider a supertube embedded in spacetime along the curve $\vec{y}(\theta)$ given by:

$$\varphi_1 = -\theta \,, \qquad\qquad \varphi_2 = -\nu\,\theta \tag{3.1.3}$$

x and y being at fixed values. The projections of the supertube in the (y,φ_1) and (x,φ_2) planes are circular, with winding numbers n_2^{ST} and νn_2^{ST} respectively. For $\nu = 0$, the supertube is circular and simply winds around the plane of the ring n_2^{ST} times. For $\nu \neq 0$, the details of the winding depend upon the equilibrium position of the supertube. We also assume, for simplicity, that the charge densities of the tube are independent of θ. Under these assumptions the condition (3.1.2) becomes:

$$\left[N_1^{ST} - \frac{1}{2}n_2^{ST}n_3(y+1-\nu(x+c))\right]\left[N_3^{ST} - \frac{1}{2}n_2^{ST}n_1(y+1-\nu(x+c))\right] = \\ (n_2^{ST})^2 Z_2 \frac{R^2}{(x-y)^2}((y^2-1)+\nu^2(1-x^2))\,. \tag{3.1.4}$$

We will call this equation the radius relation. Note that this equation is invariant under the exchange of N_1, n_1 with N_3, n_3, as expected by U-duality. Comparing this constraint to the one for a black hole background (2.3.13), we see that the charges of the supertube are enhanced to

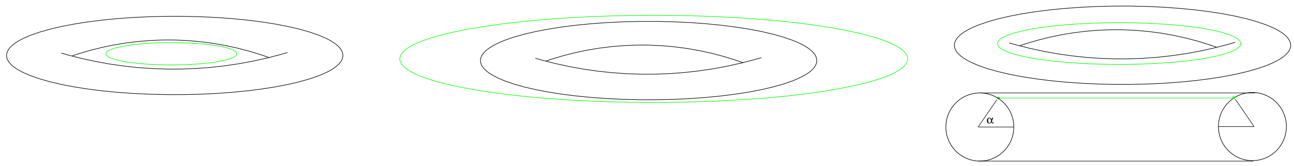

Figure 3.1: *Different black ring and supertube configurations for different values of the supertube charges. In the first picture, the charges of the tube are too small, and hence the tube it is too small, and passes inside the ring. In the second one, the tube is too large and passes on the outside of the ring. In the third picture, the size of the tube is in the correct range for the merger to be possible. The angle α of the merger depends on the tube charges according to (3.1.10).*

their *effective charges* via the interactions of the dipole charges. This is an important result that we will discuss further in the subsequent sections.

To get a better idea of the supertube configuration in the black-ring geometry it is instructive to examine the supertube as it approaches the horizon ($y \to -\infty$). In this limit, the physical metric along the horizon becomes:

$$ds_3^2 \;=\; \big(C^2 R^4\big)^{1/3} \bigg[\big(64\, C^2 R^4\big)^{-1} \mathcal{M}\, d\varphi_1^2 \;+\; \big(d\alpha^2 \;+\; \sin^2\alpha\, (d\varphi_1 + d\varphi_2)^2\big) \bigg] \,, \tag{3.1.5}$$

where we have set $x = \cos\alpha$, and the parameter, $\mathcal{M}$, is proportional to the square of the black-ring entropy

$$S = \pi\sqrt{\mathcal{M}}, \tag{3.1.6}$$

and is given by

$$\mathcal{M} = 2n_1 n_2 \overline{N}_1 \overline{N}_2 + 2n_1 n_3 \overline{N}_1 \overline{N}_3 + 2n_2 n_3 \overline{N}_2 \overline{N}_3 - (n_1 \overline{N}_1)^2 - (n_2 \overline{N}_2)^2 - (n_3 \overline{N}_3)^2 - 4n_1 n_2 n_3 J \,, \tag{3.1.7}$$

where J is the "intrinsic" angular momentum of the ring, and is given by the difference between the two angular momenta of the five-dimensional solution:

$$J = J_1 - J_2 = 4(n_1 + n_2 + n_3)\, R \,. \tag{3.1.8}$$

The topology of the horizon is $S^2 \times S^1$, but observe that for a supertube that winds according to (3.1.3), the winding around the horizon is determined by

$$\varphi_1 \;=\; -\theta \;, \qquad\qquad \varphi_1 + \varphi_2 \;=\; -(\nu + 1)\,\theta \,. \tag{3.1.9}$$

The supertube thus enters the horizon by winding around the S^1 but enters at a point on the S^2 if and only if $\nu = -1$. Otherwise it winds around the S^1 and "crowns" the S^2 by winding $(\nu + 1)$ times around a latitude determined by x.

If we now examine the constraint (3.1.4) and send $y \to -\infty$ the supertube will merge with the black ring and the constraint (3.1.4) will become the merger condition:

$$N_1^{ST} n_1 + N_3^{ST} n_3 - \overline{N}_2 n_2^{ST} \;=\; n_2^{ST} n_1 n_3 ((1 + c) - (\nu + 1)(x + c)) \,. \tag{3.1.10}$$

More explicitly, this condition be written as:

$$N_1^{ST} n_1 + N_3^{ST} n_3 - \overline{N}_2 n_2^{ST} = n_2^{ST} n_1 n_3 \, (\nu+1)(1-x) \quad for \quad c = -1 \,. \tag{3.1.11}$$
$$N_1^{ST} n_1 + N_3^{ST} n_3 - \overline{N}_2 n_2^{ST} = n_2^{ST} n_1 n_3 \, (2-(\nu+1)(1+x)) \quad for \quad c = +1 \,. \tag{3.1.12}$$

The relation (3.1.10) is simply the analogue of the equation giving the merging angle for the supertube in a black-hole background (2.3.17). In particular, as depicted in Figure 3.1, it determines the value of x (which corresponds to an angular variable on the horizon) at which a supertube with a given set of charges enters the black ring horizon. Since $-1 \leq x \leq +1$, this restricts the permissible charges of supertubes that merge with a given black ring.

We can see that the radius relation (3.1.4) and the merger condition (3.1.10) depend both on the gauge choice (by an x-independent factor) and also on $\nu+1$. We can understand this gauge dependance in a physical way: the gauge choice corresponds to a choice for the location of the Dirac string. In other words, the gauge dependance comes from the fact that the tube feels the presence of the Dirac string of the background. Increasing x then corresponds to the supertube wrapping, for $c=-1$, or not wrapping, for $c=+1$ the Dirac string, as can be seen in figure 3.2.

More precisely, if we choose $c=-1$, that is if we choose the Dirac string to extend from the ring location to infinity, then we can put the tube everywhere except on the Dirac string. If we put it at $x=1$, the ϕ circle becomes degenerate and indeed in (3.1.11) the ν dependance disappears. This is expected, because $\nu+1$ is the winding number of the tube around a contractible circle. When the size of this circle is zero, the winding should be irrelevant, which is indeed what happens.

If we now change the location of the ring to approach $x=-1$ without changing the gauge, the tube winds $\nu+1$ times around the Dirac string; this winding is physically-relevant, and hence, as expected, equation (3.1.11) depends on ν when $x \to 1$. However, if we change the gauge to move the Dirac string to the inside of the ring, we can see that when the tube is at $x=1$, where the ϕ circle is degenerate, the winding number is again irrelevant; as expected the merger formula is again independent of ν. We should also note that for the particular value $\nu=-1$, the supertube never wraps the Dirac string, and hence the merger condition does not depend upon x.

In Section 3.1.4 we will examine the details of such a merger and discuss chronology protection and black hole thermodynamics during mergers.

3.1.2 The black ring background: comparing the DBI analysis with supergravity.

We now turn to the main purpose in this section: the relation between the merger conditions obtained from supergravity and from the DBI analysis, and the relation between the GH and the DBI charges of the supertube.

Let begin with the supergravity side. The supergravity solution corresponding to one black ring and one supertube is given as usual the eight harmonic functions V, L_I, K^I and M. The poles of this functions at the location of the ring and of the tube are

Figure 3.2: *The black ring (in blue) with supertubes (in green) at various positions in the* $\mathbb{R}^3$ *base of the Gibbons-Hawking space. The black ring is point-like but the tube is point-like only if it lies on the axis* $x = \pm 1$*. Otherwise, it winds* $\nu + 1$ *times the* ϕ *circle. On the left, the Dirac string starts from the ring and extends to infinity. On the right, the Dirac string extends between the center of the space and the ring location.*

$$
\begin{aligned}
K_1 &= -\frac{q_1}{2|\vec{r}-\vec{r}_{BR}|}\,, \quad K_2 = -\frac{q_2}{2|\vec{r}-\vec{r}_{BR}|} - \frac{q_2^{ST}}{2|\vec{r}-\vec{r}_{ST}|}\,, \quad K_3 = -\frac{q_3}{2|\vec{r}-\vec{r}_{BR}|}\,,\\
L_1 &= \frac{Q_1^{GH}}{4|\vec{r}-\vec{r}_{BR}|} + \frac{Q_1^{GH,ST}}{4|\vec{r}-\vec{r}_{ST}|}\,, \quad L_2 = \frac{Q_2^{GH}}{4|\vec{r}-\vec{r}_{BR}|}\,, \quad L_3 = \frac{Q_3^{GH}}{4|\vec{r}-\vec{r}_{BR}|} + \frac{Q_3^{GH,ST}}{4|\vec{r}-\vec{r}_{ST}|}\,,\\
2M &= -\frac{J^{GH}}{8|\vec{r}-\vec{r}_{BR}|} - \frac{J^{GH,ST}}{8|\vec{r}-\vec{r}_{ST}|}
\end{aligned} \tag{3.1.13}
$$

where Q^{GH} are the GH charges of the black ring defined in Section 1.4.2, and $Q^{GH,ST}$ are the GH charges of the supertube defined in the same way. Recall once again that the GH charges depend upon the choice of patch, as in (1.4.27) and (1.4.28), and the GH charges of both the ring and the tube transform consistently between the patches.

To obtain the merger condition from supergravity observe that the bubble (or integrability) equations (1.3.17) contain a term in which the $E_{7(7)}$ symplectic product of the supertube and black ring GH charge vectors is divided by their separation. Hence, these objects only merge if this symplectic product is zero[1]. Explicitly, this gives[2]

$$
N_1^{GH,ST} n_1 + N_3^{GH,ST} n_3 - N_2^{GH} n_2^{ST} = 0. \tag{3.1.14}
$$

Note that the GH charges of the ring and of the tube are gauge dependent, but the symplectic product is invariant.

To compare the GH merger conditions (3.1.14) to the merger conditions obtained in the previous section using the DBI action, one should recall that this condition describes only those supertubes that correspond to point sources on the $\mathbb{R}^3$ of the GH base. That is, the

[1] One could also imagine in principle the existence of a scaling solution, where the distances in $\mathbb{R}^3$ between the ring, supertube and the center of Taub-NUT go together to zero. In such a solution the ring and the supertube would be spinning very rapidly in opposite directions, which is likely to introduce closed timelike curves. We leave its exploration for future work.

[2] As noted in (1.4.19), we have adopted a set of conventions in which the supergravity charges, Q^{ST}, are the same as the integer charges.

supertubes are embedded into $\mathbb{R}^4$ so as to wind around the GH fiber, and thus preserve the same triholomorphic $U(1)$ isometry as the black ring. From (3.1.3) and (1.4.2) we see that the winding numbers of the supertube in the GH patch are given by $(1, \nu+1)$. (Remember that ψ has period 4π.) Thus a supertube is point-like in the $\mathbb{R}^3$ if and only if it has either $\nu = -1$ or it lies on the polar axis with $x = \pm 1$. We therefore restrict ourselves to mergers with $x = \pm 1$ for any value of ν, or mergers with $\nu = -1$.

For $x = 1$, we need to be on the patch $c = -1$, and (3.1.10) gives:

$$N_1^{ST} n_1 + N_3^{ST} n_3 - \overline{N}_2 n_2^{ST} \;=\; 0\,. \tag{3.1.15}$$

For $x = -1$, we need to be on the patch $c = +1$, and thus have:

$$N_1^{ST} n_1 + N_3^{ST} n_3 - \overline{N}_2 n_2^{ST} \;=\; 2 n_2^{ST} n_1 n_3\,. \tag{3.1.16}$$

But using the relation (1.4.28), we can rewrite it as

$$N_1^{ST} n_1 + N_3^{ST} n_3 - N_2^{GH} n_2^{ST} \;=\; 0 \tag{3.1.17}$$

on both patches. The extra term in (3.1.16) is simply the shift in N^{GH} induced by changing patches. Thus, if we identify the DBI charge of the supertube with the GH charge of the corresponding supergravity solution,

$$N_I^{ST} = N_I^{GH,ST}, \tag{3.1.18}$$

we have a perfect agreement between the supergravity approach (3.1.14) and the DBI approach (3.1.17).

The supertubes with $\nu = -1$ do not wrap the ϕ circle of the $\mathbb{R}^3$ base of the GH space, and thus are point-like in this base for any value of x, and they source a supergravity solution with a GH base for any location. Moreover, since these tubes do not wrap the Dirac string, the merger relations become x independent. Equations (3.1.11) and (3.1.12) then become

$$N_1^{ST} n_1 + N_3^{ST} n_3 - \overline{N}_2 n_2^{ST} \;=\; 0 \quad \text{for} \quad c = -1\,, \tag{3.1.19}$$
$$N_1^{ST} n_1 + N_3^{ST} n_3 - \overline{N}_2 n_2^{ST} \;=\; 2 n_2^{ST} n_1 n_3 \quad \text{for} \quad c = +1\,, \tag{3.1.20}$$

which once again can be re-written as

$$N_1^{ST} n_1 + N_3^{ST} n_3 - N_2^{GH} n_2^{ST} \;=\; 0\,. \tag{3.1.21}$$

Hence we arrive at the same conclusion as for supertubes at $x = \pm 1$: the DBI charges of the supertube give the GH charges of the corresponding supergravity solution:

$$N_I^{ST} \;\equiv\; N_I^{GH,ST}\,. \tag{3.1.22}$$

3.1.3 Black rings and three-charge two-dipole-charge supertubes

One can generalize the foregoing discussion of mergers to examine a three-charge, two dipole charge supertube [75] merging with a generic black ring. This can be done both in the probe approximation, using the DBI action, and in the exact supergravity solution. This supertube is more general than the two-charge supertube, and although it does not source a smooth supergravity solution in any duality frame, it can be used to study rather more general classes of mergers.

The best duality frame to study this merger is that in which the three-charge supertube is a dipolar D6-brane carrying electric D4, D0 and F1 charges. We take our tube to be along the $(x_1, x_2, x_3, x_4, z, \vec{y}(\theta)))$, where $\vec{y}(\theta)$ describes a closed curve in the non-compact space. As before, we take $\theta \in (0{,}2\pi n_1^{ST})$ with n_1^{ST} being the winding number of the supertube which is also its D6 dipole charge. We introduce world-volume electric fields: $\mathcal{F}_{z\theta}$, $\mathcal{F}_{tz}$, $\mathcal{F}_{12}$ and $\mathcal{F}_{34}$. where $\mathcal{F}_{tz}$ and $\mathcal{F}_{z\theta}$ generate the F1 and D4 charges respectively and $\mathcal{F}_{12}$ and $\mathcal{F}_{34}$ are needed for the D0 charge. The integer charges are given by

$$
\begin{aligned}
N_1^{ST} &= N_{D0} = \frac{1}{2\pi}\int d\theta\, \mathcal{F}_{z\theta}\mathcal{F}_{12}\mathcal{F}_{34}\,, && (3.1.23)\\
N_2^{ST} &= N_{D4} = \frac{1}{2\pi}\int d\theta\, \mathcal{F}_{z\theta}\,, && (3.1.24)\\
N_3^{ST} &= N_{F1} = \frac{1}{2\pi}\int d\theta\, \frac{\partial\mathcal{L}}{\partial\mathcal{F}_{tz}}\bigg|_{\mathcal{F}_{tz}=1}\,, && (3.1.25)\\
n_2^{ST} &= n_{D2} = n_1^{ST}\mathcal{F}_{12}\mathcal{F}_{34}\,. && (3.1.26)
\end{aligned}
$$

Note that we can take the D4 dipole moments and D2 charges of the tube to be zero by taking $\mathcal{F}_{12}$ and $\mathcal{F}_{34}$ to be traceless. Supersymmetry requires that $\mathcal{F}_{tz} = 1$ and $\mathcal{F}_{12} = \mathcal{F}_{34}$ [75], and then one can show that

$$
\mathcal{H}|_{\mathcal{F}_{tz}=1,\mathcal{F}_{12}=\mathcal{F}_{34}} = T_{D6}\mathcal{F}_{z\theta}\mathcal{F}_{12}\mathcal{F}_{34} + T_{D6}\mathcal{F}_{z\theta} + \frac{\partial\mathcal{L}}{\partial\mathcal{F}_{tz}}\bigg|_{\mathcal{F}_{tz}=1,\mathcal{F}_{12}=\mathcal{F}_{34}}\,, \tag{3.1.27}
$$

or equivalently

$$
\int d^4x dz d\theta\, \mathcal{H}|_{\mathcal{F}_{tz}=1,\mathcal{F}_{12}=\mathcal{F}_{34}} = N_1^{ST} + N_2^{ST} + N_3^{ST}\,, \tag{3.1.28}
$$

where $\mathcal{H}$ is the energy per unit five-dimensional volume.

As before, we will assume constant charge densities on the supertube worldvolume and the interesting physical condition that generalizes (3.1.4) comes from the variation that define the F1-charge, N_3^{ST}:

$$
\begin{aligned}
&\left[N_3^{ST} - \frac{1}{2}(n_1^{ST}n_2 + n_2^{ST}n_1)(y+1-\nu(x+c))\right]\left[N_2^{ST} - \frac{1}{2}n_1^{ST}n_3(y+1-\nu(x+c))\right] = \\
&\qquad n_1^{ST}\left(n_1^{ST}Z_1 + n_2^{ST}Z_2\right)\frac{R^2}{(x-y)^2}((y^2-1)+\nu^2(1-x^2))\,. \qquad (3.1.29)
\end{aligned}
$$

Note that, using $n_1^{ST}N_1^{ST} = n_2^{ST}N_2^{ST}$, there is a symmetry between (D0,D6) and (D4,D2) charges and dipole moments, as expected from U-duality. However since the tube has no NS5 dipole moment, there is no exchange symmetry between the F1 charge and other charges.

One can extract the merger condition from this as before and one finds that, for a merger with a black ring, (3.1.10) generalizes to:

$$n_1N_1^{ST} + n_2N_2^{ST} + n_3N_3^{ST} - n_1^{ST}\overline{N}_1 - n_2^{ST}\overline{N}_2 = n_3(n_1n_2^{ST} + n_2n_1^{ST})((1+c) - (\nu+1)(x+c)). \tag{3.1.30}$$

When the three-charge supertube respects the GH isometry ($x = \pm 1$ for any ν or $\nu = -1$ for any x), one can also describe this merger in supergravity. The solution is given by the same harmonic functions as in (3.1.13), except that now K_1 and L_2 also have poles at the supertube location:

$$K_1 \to -\frac{q_1^{GH}}{2|\vec{r}-\vec{r}_{BR}|} - \frac{q_1^{GH,ST}}{2|\vec{r}-\vec{r}_{ST}|}\,, \quad L_2 \to \frac{Q_2^{GH}}{4|\vec{r}-\vec{r}_{BR}|} + \frac{Q_2^{GH,ST}}{4|\vec{r}-\vec{r}_{ST}|}\,. \tag{3.1.31}$$

One can see that equation (3.1.30) is equivalent to the vanishing of the $E_{7(7)}$ symplectic product of the GH charges of the black ring and those of the three-charge supertube, and hence the merger conditions obtained from supergravity and from the Born-Infeld analysis of the three-charge supertube are the same. The subtleties associated to the dependence of the charges upon the patch are identical to those for the two-charge supertube, and we will not discuss them again.

3.1.4 Chronology protection

Having obtained the condition under which a supertube and a black ring can merge, both using the Born-Infeld description of supertubes, and (where appropriate) also using the supergravity solution corresponding to the merger, we now turn to verifying that supertube mergers preserve the physical properties of the black ring. For simplicity, and because it is sufficient for capturing all the relevant physics of the merger, we will primarily focus on circular embeddings for the tube (3.1.3).

Mergers of black rings with two-charge supertubes

We begin by considering the merger of a black ring with a two-charge supertube of arbitrary shape. To do this one must first establish what shape can the supertube have when it crosses the black ring horizon. Based on our intuition from supertubes merging with black holes [75] we expect that the supertube will be parallel to the horizon, and that it should not be possible to have a part of the supertube inside the black ring horizon and a part of it is outside.

To see this we can analyze equation (3.1.2) and change variables to $w = \frac{1}{y}$; the merger then happens at $w \to 0$. After some algebra one can see that for $w \to 0$ the leading divergent term in (3.1.2) imposes the constraint $\frac{\partial w}{\partial \theta} = 0$, which implies that the supertube is always tangent to the horizon when it merges to a black ring.

It is particularly important to examine the thermodynamics of mergers and see whether by "throwing in" supertubes one could decrease the entropy of a black ring, or overspin it and introduce closed timelike curves (violating chronology protection). To do this one must determine what are the charges that a supertube brings into a ring. As we saw in the section 3.1.2, there are some subtleties in this determination and we cannot always add the DBI charges of the supertube to the constituent charges, the $\overline{N}$'s, of the ring. We have learned that the DBI charges have to be identified with the GH charges of the supertube, which are patch-dependent, and are not the same as the constituent ones. We have seen this explicitly from the supergravity solution for concentric mergers (when $x = \pm 1$) or alternatively when we take $\nu = -1$ so that the supertube does not wind around latitude circles and crosses the ring horizon at a point on the S^2 of the horizon. We will first focus on mergers where the supertube merges at a point on the S^2, and discuss the other ones at the end of this subsection.

The entropy of the black ring is given by $S = \pi\sqrt{\mathcal{M}}$ where $\mathcal{M}$ is defined in (3.1.7)

$$\mathcal{M} = 2n_1n_2\overline{N}_1\overline{N}_2 + 2n_1n_3\overline{N}_1\overline{N}_3 + 2n_2n_3\overline{N}_2\overline{N}_3 - (n_1\overline{N}_1)^2 - (n_2\overline{N}_2)^2 - (n_3\overline{N}_3)^2 - 4n_1n_2n_3J\,. \tag{3.1.32}$$

Note that $\mathcal{M}$ is in fact the $E_{7(7)}$ quartic invariant and is therefore invariant under a gauge transformation (1.4.24). In terms of GH charges of the ring, we have

$$\begin{aligned}\mathcal{M} \;=\;& 2n_1n_2N_1^{GH}N_2^{GH} + 2n_1n_3N_1^{GH}N_3^{GH} + 2n_2n_3N_2^{GH}N_3^{GH} \\ & -(n_1N_1^{GH})^2 - (n_2N_2^{GH})^2 - (n_3N_3^{GH})^2 - 4n_1n_2n_3J^{GH}\,.\end{aligned} \tag{3.1.33}$$

From the analysis in the previous subsections, we know that the supertube DBI charges correspond to GH charges, and thus should be directly added to the GH charges of the ring.

To keep the expressions simple we will take the three electric and the three dipole charges of the black ring charges to be equal, we will also assume that the two electric charges of the supertube are equal, namely:

$$N_1^{GH} = N_2^{GH} = N_3^{GH} \;\equiv\; N\,, \qquad n_1 = n_2 = n_3 \;\equiv\; n\,, \qquad N_1^{ST} = N_3^{ST} \;\equiv\; \Delta N\,. \tag{3.1.34}$$

Then we have

$$\mathcal{M} \;=\; n^2(3N^2 - 4nJ) \tag{3.1.35}$$

and the charges of physical black rings satisfy: $3N^2 \geq 4nJ$.

Let Δn denote the dipole charge of the tube and ΔJ its angular momentum. The new horizon area parameter, $\widetilde{\mathcal{M}}$, after the merger is then

$$\begin{aligned}\widetilde{\mathcal{M}} \;&=\; 4\,n\,N\,(n+\Delta n)(N+\Delta N) + 2\,n^2(N+\Delta N)^2 - (n+\Delta n)^2N^2 \\ &\qquad\qquad -2\,n^2(N+\Delta N)^2 - 4\,n^2(n+\Delta n)(J+\Delta J) \\ &=\; \mathcal{M} \;+\; n\,\Delta n(3N^2 - 4nJ) \\ &\qquad - \frac{(n+\Delta n)}{\Delta n}\Big[(2\,n\,\Delta N - N\Delta n)^2 \;+\; 4\,n^2\Delta n\Big(\Delta J - \frac{(\Delta N)^2}{\Delta n}\Big)\Big]\,.\end{aligned} \tag{3.1.36}$$

We now need to remember that the angular momentum of the tube is given by (C.38)

$$\Delta J = \frac{(\Delta N)^2}{\Delta n}\,, \tag{3.1.37}$$

and also that that for the charges we consider the merger condition (3.1.17) becomes

$$2\,n\,\Delta N = \Delta n\,N\,. \tag{3.1.38}$$

Using these two equations, we finally have

$$\Delta\mathcal{M} \;\equiv\; \widetilde{\mathcal{M}} - \mathcal{M} \;=\; n\,\Delta n(3N^2 - 4nJ) \;\geq\; 0\,, \tag{3.1.39}$$

with equality if and only if the original black ring has vanishing horizon area. Hence, for mergers with $\nu = -1$ or $x = \pm 1$, we have proved that chronology is protected, and that the second law of black hole thermodynamics holds. This conclusion is similar to that of [75, 76, 79] for supertube-black hole mergers.

However for $\nu \neq -1$ the situation is rather more subtle. First, the complete supergravity solution is not known for mergers in which the supertube winds around an S^1 in the S^2 of the horizon. As a result we cannot identify the supertube DBI charges with simple supergravity charges. In addition it is not clear how to identify directly the charges carried across the horizon during the merger. If one simply chooses one of the patches discussed above and assumes that the supertube carries its constituent DBI or GH charges across the horizon then the x-dependence in the merger condition (3.1.11) can lead to mergers in which the horizon area of the black ring decreases, thus contradicting black hole thermodynamics.

The most likely solution to this conundrum is that the charges carried by the supertube across the horizon are not the same as the constituent supertube charges $\overline{N}^{ST}, \overline{J}^{ST}$, but are modified *in an x-dependent way*, so as not to decrease the horizon area. This would imply that in $\nu \neq -1$, $x \neq \pm 1$ mergers the supertube brings in not only its intrinsic charges, but also some of the charge and angular momentum dissolved in supergravity fluxes. Since it is unclear how the dynamics of this charge can be captured via a Born-Infeld analysis, we believe that the understanding of this phenomenon and a resolution of this puzzle will probably come from finding the fully back-reacted supergravity solution corresponding to the $\nu \neq -1$ mergers[1].

Mergers of black rings with three-charge two-dipole-charge supertubes

Another interesting example for illustrating chronology protection is the merger of a three-charge two-dipole charge supertube with another supertube of the same kind, that can also be thought of as a singular black ring that has one zero dipole charge $n_3^{BR} = 0$. Such a singular black ring must have vanishing horizon area, and to avoid closed timelike curves it must satisfy the charge condition [62]:

$$n_1^{BR} N_1^{BR} \;=\; n_2^{BR} N_2^{BR}\,. \tag{3.1.40}$$

Similarly, the three-charge supertube considered above has no NS5 dipole charge ($n_3 = 0$) and also satisfies

$$n_1^{ST} N_1^{ST} \;=\; n_2^{ST} N_2^{ST}\,. \tag{3.1.41}$$

[1]Such mergers do not have a tri-holomorphic $U(1)$ invariance and hence the supergravity solution will be more complicated than the solutions with a Gibbons-Hawking base presented here.

Since the merger produces another two-dipole three-charge tube, it must also satisfy the regularity condition:

$$(n_1^{BR} + n_1^{ST})(N_1^{BR} + N_1^{ST}) \; - \; (n_2^{BR} + n_2^{ST})(N_2^{BR} + N_2^{ST}) \; = \; 0\,, \tag{3.1.42}$$

which is equivalent to

$$n_1^{BR} N_1^{ST} + n_1^{ST} N_1^{BR} \; - \; (n_2^{BR} N_2^{ST} + n_2^{ST} N_2^{BR}) \; = \; 0\,. \tag{3.1.43}$$

On the other hand, the merger condition (3.1.30) for $n_3^{BR} = 0$ yields:

$$(n_1^{BR} N_1^{ST} + n_2^{BR} N_2^{ST}) \; - \; (n_1^{ST} N_1^{BR} + n_2^{ST} N_2^{BR}) \; = \; 0\,. \tag{3.1.44}$$

To establish chronology protection one must show that (3.1.44) implies (3.1.43).

However, one also knows that the two merging objects obey (3.1.40) and (3.1.41). Multiplying (3.1.44) by $n_2^{BR} n_2^{ST}$ and using (3.1.40) and (3.1.41) one obtains:

$$(n_2^{BR} N_1^{ST} - n_2^{ST} N_1^{BR})\,(n_1^{ST}\, n_2^{BR} + n_2^{ST}\, n_1^{BR}) \; = \; 0\,. \tag{3.1.45}$$

Similarly, one finds that (3.1.43) is equivalent to

$$(n_2^{BR} N_1^{ST} - n_2^{ST} N_1^{BR})\,(n_1^{ST}\, n_2^{BR} - n_2^{ST}\, n_1^{BR}) \; = \; 0\,. \tag{3.1.46}$$

Since all the n's are positive, we see that (3.1.45) implies (3.1.46) and so the merger condition (3.1.44) implies that the regularity condition (3.1.43) is satisfied. Hence, the merger of two three-charge two-dipole charge supertubes always respects chronology protection.

We can also consider a merger of a three-charge two-dipole charge supertube with a fully fledged black ring, we take for simplicity equal charges and dipoles: $n_1^{BR} = n_2^{BR} = n_3^{BR} = n$, $N_1^{BR} = N_2^{BR} = N_3^{BR} = N$, $N_1^{ST} = N_2^{ST} = N_3^{ST} = \Delta N$ and $n_1^{ST} = n_2^{ST} = \Delta n$. The non-negativity of the initial black ring entropy implies that $3N^2 \geq 4nJ$ and the merger condition[1] becomes $3n\Delta N = 2\Delta n N$. Also remembering that angular momentum of the three-charge supertube is given by

$$J^{ST} \; = \; \frac{N_1^{ST} N_3^{ST}}{n_2^{ST}} \; = \; \frac{N_2^{ST} N_3^{ST}}{n_1^{ST}} \tag{3.1.47}$$

and hence $\Delta J = \Delta N^2 / \Delta n$, we obtain

$$\Delta\mathcal{M} \; \equiv \; \widetilde{\mathcal{M}} - \mathcal{M} \; = \; \frac{4}{9}\,(7N^2 - 9nJ)\,(2n\Delta n + \Delta n^2)\;. \tag{3.1.48}$$

Since $N^2 \geq \frac{4}{3} nJ$ this merger is always irreversible, and does not violate chronology protection.

[1]We consider $\nu = -1$ tubes in the $c = -1$ patch; all the subtleties having to do with changing patches are the same as for two-charge supertubes.

3.2 DBI action for a bubbling background

3.2.1 Probing a general Gibbons-Hawking solution

We finally consider two-charge supertubes probing a general three-charge BPS solution with a Gibbons-Hawking base and we will again work in the D0-D4-F1 duality frame. The general BPS solution with three charges and three dipole charges and a GH base is given in Sections 1.3.1 and 1.3.2 and we proceed as we did for the black-hole and black-ring backgrounds. We denote the supertube coordinates as ξ^0, ξ^1 and $\xi^2 \equiv \theta$ and consider the simplified case of a circular supertube along the $U(1)$ fiber of the GH base:

$$\xi^0 \; = \; t \,, \qquad \xi^1 \; = \; z \,, \qquad \theta = \psi \,. \tag{3.2.1}$$

The supertube action (3.1.1) takes the explicit form

$$\begin{aligned} S = T_{D2} \int d^3\xi \Bigg\{ &\left[\left(\frac{1}{Z_1} - 1 \right) \mathcal{F}_{z\theta} + \frac{K^3}{Z_1 V} + \left(\frac{\mu}{Z_1} - \frac{K^1}{V} \right) (\mathcal{F}_{tz} - 1) \right] \\ - &\left[\frac{1}{V^2 Z_1^2} \left[(K^3 - V(\mu(1-\mathcal{F}_{tz}) - \mathcal{F}_{z\theta}))^2 + V Z_1 Z_2 (1 - \mathcal{F}_{tz})(2 - Z_3(1-\mathcal{F}_{tz})) \right] \right]^{1/2} \Bigg\} . \end{aligned} \tag{3.2.2}$$

For $\mathcal{F}_{tz} = 1$ the tube is supersymmetric and, as before, the Hamiltonian density is the sum of the charge densities (2.3.9). Due to the supersymmetry there is a constraint similar to (3.1.4), which determines the location of the supertube in terms of its charges

$$\left[N_1^{ST} + n_2^{ST} \frac{K^3}{V} \right] \left[N_3^{ST} + \frac{K^1}{V} \right] = (n_2^{ST})^2 \frac{Z_2}{V} \,, \tag{3.2.3}$$

where the charges are still defined by (2.3.8).

3.2.2 Gibbons-Hawking backgrounds: comparing the DBI analysis with supergravity.

Equation (3.2.3) determines the position of a supertube in an arbitrary three-charge background with a triholomorphic $U(1)$ isometry. Since both the supertube and the background preserve this isometry, their fully back-reacted supergravity solution will have a Gibbons-Hawking base, and its form is well-known. Hence, one can compare (3.2.3) to the corresponding condition coming from the supergravity analysis of the supertube, and confirm that supertubes that are solutions of the Born-Infeld action always give rise to smooth supergravity solutions.

To do this, it is useful to remember that in any Gibbons-Hawking solution the singularities in the harmonic functions K_2, L_1, L_3 and M at the supertube location are given by (3.1.13). If one now takes equation (1.4.43) for a supertube with charges $Q_1^{GH,ST}, Q_3^{GH,ST}$ and q_2^{ST} and uses the asymptotic behavior of these harmonic functions near the supertube one obtains:

$$\left[Q_1^{GH,ST} - 2 q_2^{ST} \frac{K^3}{V} \right] \left[Q_3^{GH,ST} - 2 q_2^{ST} \frac{K^1}{V} \right] = (q_2^{ST})^2 \frac{Z_2}{V} \,. \tag{3.2.4}$$

Since the supergravity GH charges, $Q_1^{GH,ST}, Q_3^{GH,ST}, q_2^{ST}$, are the same as the integer charges $N_1^{GH,ST}, N_3^{GH,ST}, n_2^{ST}$, one sees that this agrees exactly with the DBI calculation.

It is interesting to observe that the DBI action only gives one equation of motion for the supertube, (3.2.3), while the supergravity analysis of the supertube gives two independent equations, that can be chosen to be any two of (1.4.38), (1.4.39) and (1.4.43). This is because in the Born-Infeld analysis the inputs are the supertube charges and dipole charge, which one first uses to find the embedding, and then one derives the angular momentum of the supertube, J^{ST}, from that solution.

By contrast, in the supergravity analysis the angular momentum of the supertube along the Gibbons-Hawking fiber appears as the coefficient of the singular part in the harmonic function M, and is one of the *inputs* of the calculation. Indeed, in supergravity one can build "supertube" solutions for any value of J_T. However most of these solutions will be singular: if J_T is too large the solutions will have closed timelike curves, and if J_T is too small the solutions will have a naked singularity[1]. Only one specific value of J_T gives a supergravity solution that is smooth and horizonless in the duality frame in which the supertube charges correspond to D1 and D5 branes.

To find this value it is most convenient to use equation (1.4.38), and the expansion of the harmonic functions (3.1.13) near the supertube location to find the supertube angular momentum as a function of the supertube charges $Q_1^{GH,ST}, Q_3^{GH,ST}$ and dipole charge q_2^{ST}:

$$J^{GH,ST} = \frac{N_1^{GH,ST} N_3^{GH,ST}}{n_2^{ST}} \tag{3.2.5}$$

To obtain this equation from the DBI analysis one needs to calculate the angular momentum of the supertube along the Gibbons-Hawking fiber. This calculation is partially shown in Appendix C and gives

$$J^{ST} = \frac{N_1^{ST} N_3^{ST}}{n_2^{ST}}. \tag{3.2.6}$$

This indicates that when supertubes are embedded in a solution with a Gibbons-Hawking base, respecting the triholomorphic $U(1)$ isometry of this solution, their Born-Infeld analysis gives the equations needed for the fully back-reacted supergravity solution of these supertubes to be smooth and free of closed timelike curves.

[1]Such a singularity might be cloaked by a Planck-sized horizon [80].

Chapter 4

Entropy counting, and the entropy enhancement mechanism

This chapter is devoted to an in-depth review of the Born-Infeld calculation of the entropy coming from the shape modes of supertubes, as well as to an extension of this calculation to a supertube in a black-ring background. As we have shown in the previous chapters of this thesis, we expect the latter supertube fluctuations to give rise to smooth horizonless solutions. Hence, our analysis strongly supports the existence of smooth horizonless three-charge solutions that depend on arbitrary continuous functions, and whose entropy is much larger than their typical charge, and might even be as large as the square root of the cube of their charge. That is, it might be black-hole-like.

Our goal is to quantize the small oscillations about round two-charge supertubes in flat space, black-hole, black-ring, and generic three-charge backgrounds, and to examine the entropy coming from these fluctuations. We find it convenient to work in the D0-D4-F1 duality frame, and our approach follows that of [16] (see also [17]).

We begin by reviewing the Marolf-Palmer entropy calculation for a supertube in flat space, and in the following subsections extend this calculation for a supertube in a 3-charge black hole background, in a black ring background and in the background of a general solution with a Gibbons-Hawking base space. In the latter two backgrounds we find a non-trivial enhancement of the entropy of a supertube when the dipole magnetic fields are large. This is what we present in the second section of this chapter. This enhancement arises because the entropy that can be stored in a supertube is governed not by the electric charges of the supertube (as in flat space or in a black hole background) but by its locally-defined *effective charges*, that can get large contributions from the interactions of the dipole moment of the supertube with the magnetic fluxes of the background.

4.1 How to count the entropy of a supertube?

4.1.1 Flat space

In the absence of background fluxes, the WZ action of the supertube is zero, and the DBI action (2.3.3) reduces to

$$S = -T_{D2} \int dt dz d\theta \sqrt{r^2(1-\mathcal{F}_{tz}^2) + \mathcal{F}_{z\theta}^2}\,, \tag{4.1.1}$$

where r is the radius of the supertube and its embedding is

$$t = \xi^0\ , \qquad z = \xi^1\ , \qquad \varphi_1 = \theta\ . \tag{4.1.2}$$

We recall that the charges of the tube are given by (2.2.9):

$$N_1^{ST} \ = \ n_2^{ST} \mathcal{F}_{z\theta}\,, \qquad N_3^{ST} \ = \ n_2^{ST} \frac{r^2}{\mathcal{F}_{z\theta}}\,, \tag{4.1.3}$$

where the factors of n_2^{ST} come from multiple windings in θ. Similarly the radius relation (2.2.10) is

$$N_1^{ST} N_3^{ST} \ = \ \left(n_2^{ST}\right)^2 R^2\,. \tag{4.1.4}$$

The angular momentum of the supertube is (2.2.12)-(2.2.13):

$$J \ = \ \frac{N_1^{ST} N_3^{ST}}{n_2^{ST}} \ = \ n_2^{ST} R^2\,. \tag{4.1.5}$$

The foregoing results apply to round (maximally spinning) supertubes. Supertubes of arbitrary shape will have more complicated expressions for their conserved quantities and will generically have smaller angular momentum.

In this subsection we will perform a simplified version of the analysis in [16], which will be enough to give us the correct dependence of the entropy on the supertube charges. We consider small fluctuations of the supertube in the six directions transverse to its world-volume:

$$x_i \to x_i + \eta_i(t,\theta)\ , \qquad\qquad i = 1, \ldots, 6\,, \tag{4.1.6}$$

where four of these fluctuations take place on the compact T^4 and the other two are radial coordinates in the non-compact space. In general there are eight independent fluctuation modes for the supertube, consisting of seven transverse coordinate motions and a charge density fluctuation (which also affects the shape). To keep the computations simple here, we have restricted to a representative sample of oscillations in both the compactification space and in the space-time. Since we are only interested in BPS fluctuations we will also restrict η_i to depend only upon t and θ [16][1].

The effective Lagrangian for the fluctuations is obtained by expanding the DBI Lagrangian of the supertube

$$L_\eta = -T_{D2}\left[(1-\mathcal{F}_{tz}^2 - \dot\eta_i\dot\eta_i)(R^2+\eta_i'\eta_i') - 2\mathcal{F}_{tz}\mathcal{F}_{z\theta}\dot\eta_i\eta_i' + \mathcal{F}_{z\theta}^2(1-\dot\eta_i\dot\eta_i) + (\dot\eta_i\eta_i')^2\right]^{1/2}\ , \tag{4.1.7}$$

[1]The time dependent modes will break supersymmetry. Hence, we will retain the time dependence of η_i to compute momenta and quantize the system but then we will set $\partial_t\eta_i \equiv \dot\eta_i = 0$.

where the repeated index i is summed over. The canonical momenta conjugate to η_i are:

$$\Pi_i = \int_0^{2\pi L_z} dz \left.\frac{\partial L_\eta}{\partial \dot{\eta}_i}\right|_{\dot{\eta}_i=0\,,\,\mathcal{F}_{tz}=1} = \frac{1}{2\pi}\eta_i'\,, \tag{4.1.8}$$

and the canonical commutation relations are:

$$[\eta_j(t,\theta), \Pi_k(t,\theta')] = i\delta_{jk}\delta(\theta-\theta')\,. \tag{4.1.9}$$

The BPS modes η_i then can be expanded as:

$$\eta_i = \frac{1}{\sqrt{2}}\left[\sum_{k>0} e^{ik\theta/n_2^{ST}} \frac{(a_k^i)^\dagger}{\sqrt{|k|}} \;+\; \text{h.c.}\right] \tag{4.1.10}$$

where $(a_k^i)^\dagger$ and a_k^i are creation and annihilation operators for the k^{th} harmonic. The normalization has been chosen such that[1]:

$$[(a_k^i)^\dagger, a_{k'}^j] = \delta^{ij}\delta_{k,k'} \tag{4.1.11}$$

It is not hard to see that the fluctuations do not change N_1^{ST} and the angular momentum J. The charge N_3^{ST} becomes:

$$N_3^{ST} = \frac{1}{T_{F1}}\int_0^{2\pi n_2^{ST}} d\theta \left.\frac{\partial\mathcal{L}}{\partial\mathcal{F}_{tz}}\right|_{\mathcal{F}_{tz}=1} = \frac{T_{D2}}{T_{F1}}\int_0^{2\pi n_2^{ST}} d\theta\, \frac{(R^2+\eta_i'\eta_i')}{\mathcal{F}_{z\theta}}\,, \tag{4.1.12}$$

from which one finds

$$\begin{aligned}\sum_{i=1}^{6}\sum_{k>0} k(a_k^i)^\dagger a_k^i &= L_z T_{D2}\int_0^{2\pi n_2^{ST}} d\theta \int_0^{2\pi n_2^{ST}} d\theta' \sum_{i=1}^{6}\eta_i'\eta_i' &(4.1.13)\\ &= N_1^{ST}N_3^{ST} - (n_2^{ST})^2R^2 = N_1^{ST}N_3^{ST} - n_2^{ST}J\,. &(4.1.14)\end{aligned}$$

The left hand side of this expression can be thought of as the energy of a system of six massless bosons in (1+1) dimensions. Due to supersymmetry there will also be six corresponding fermionic degrees of freedom. The total central charge of the system is thus $c = 9$, and so the entropy of this system is given by the Cardy formula:

$$S = 2\pi\sqrt{\frac{c}{6}}\sqrt{N_1^{ST}N_3^{ST} - n_2^{ST}J} = 2\pi\sqrt{\frac{3}{2}}\sqrt{N_1^{ST}N_3^{ST} - n_2^{ST}J}\,. \tag{4.1.15}$$

If we had included all eight bosonic fluctuation modes then we would have had eight bosons and eight fermions and hence a theory with $c = 12$ and with the entropy:

$$S_{ST} = 2\pi\sqrt{2}\sqrt{N_1^{ST}N_3^{ST} - n_2^{ST}J}\,. \tag{4.1.16}$$

[1]Technically, to get this normalization correct we need to include the mode expansion of the non-BPS modes in (4.1.10). Ignoring the non-BPS modes gives an incorrect factor of $\sqrt{2}$ in the normalization of the η_i. Here we have given the correctly normalized expressions that one would obtain if one included the non-BPS modes.

This is the correct central charge and it yields the correct supertube entropy [16]. By restricting our analysis to six of the shape modes and ignoring the other supersymmetric modes we have obtained a finite, but well understood, fraction of the supertube entropy. Since our purpose here is to analyze when entropy enhancement happens, and when it does not, we will only be interested on the dependence of the supertube entropy on the macroscopic charges, and not pay particular attention to numerical coefficients. Restricting our analysis in more general backgrounds to transverse BPS fluctuations and counting the entropy coming from these modes will therefore be enough to illustrate the physics of entropy enhancement.

4.1.2 The three-charge black hole

A two-charge round supertube in the background of a three-charge BPS rotating (BMPV) black hole was discussed in section 2.3. Here we will use the metric and background fields presented in sections 1.3.2 and 1.4.1 and consider small shape fluctuations in the directions transverse to the world-volume of the supertube. We are again interested only in BPS excitations, which have the following form

$$x_i \to x_i + \eta_i(t,\theta)\,, \quad i = 1,2,3,4\,, \qquad u \to u + \eta_5(t,\theta)\,, \qquad v \to v + \eta_6(t,\theta)\,, \tag{4.1.17}$$

where we have defined the metric on the four-torus to be

$$ds^2_{T^4} = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2\,. \tag{4.1.18}$$

and the supertube embedding is the same as (4.1.2). One can use the sum of the DBI and WZ actions, find an effective action for the supertube fluctuations and compute the momenta conjugate to η_5, η_6 and η_i:

$$\Pi_{\eta_5} = \int dz \left(\frac{\partial \mathcal{L}}{\partial \dot\eta_5}\right)\Bigg|_{BPS} = \frac{Z_2}{2\pi}\eta_5'\,, \tag{4.1.19}$$

$$\Pi_{\eta_6} = \int dz \left(\frac{\partial \mathcal{L}}{\partial \dot\eta_6}\right)\Bigg|_{BPS} = \frac{Z_2}{2\pi}\eta_6'\,, \tag{4.1.20}$$

$$\Pi_{\eta_i} = \int dz \left(\frac{\partial \mathcal{L}}{\partial \dot\eta_i}\right)\Bigg|_{BPS} = \frac{1}{2\pi}\eta_i'\,, \tag{4.1.21}$$

where the subscript "BPS" means that we have evaluated everything "on shell," which means we have imposed the BPS conditions of no time dependence and $\mathcal{F}_{tz} = 1$.

The BPS modes η_i, η_5 and η_6 then can be expanded as

$$\begin{aligned}
\eta_i &= \frac{1}{\sqrt{2}}\left[\,\sum_{k>0} e^{ik\theta/n_2^{ST}}\frac{(a_k^i)^\dagger}{\sqrt{|k|}} \;+\; \text{h.c.}\,\right], \\
\eta_5 &= \frac{1}{\sqrt{2}}\left[\,\sum_{k>0} e^{ik\theta/n_2^{ST}}\frac{(a_k^5)^\dagger}{\sqrt{|k|}} \;+\; \text{h.c.}\,\right], \\
\eta_6 &= \frac{1}{\sqrt{2}}\left[\,\sum_{k>0} e^{ik\theta/n_2^{ST}}\frac{(a_k^6)^\dagger}{\sqrt{|k|}} \;+\; \text{h.c.}\,\right].
\end{aligned} \tag{4.1.22}$$

At first glance, the physics of the η_i fluctuations along the torus appears very different from that of the fluctuations in the spacetime direction, η_5 and η_6; indeed the latter have a factor of Z_2 in the denominator, and this factor becomes arbitrarily large when the supertube is near the horizon of a black hole.

The charge N_1^{ST} is the same as that of the round supertube, but the charge N_3^{ST} is modified to:

$$N_3^{ST} = \frac{1}{T_{F1}} \int d\theta \left. \frac{\partial \mathcal{L}}{\partial \mathcal{F}_{tz}} \right|_{BPS} = \frac{T_{D2}}{T_{F1}\mathcal{F}_{z\theta}} \int d\theta \left(Z_2 u^2 + Z_2[(\eta_5')^2 + (\eta_6')^2] + \sum_{i=1}^{4} (\eta_i')^2 \right) . \quad (4.1.23)$$

Using similar arguments to those given for the flat space background one finds the entropy of the BPS shape modes to be:

$$S = 2\pi \sqrt{\frac{3}{2}} \sqrt{N_1^{ST} N_3^{ST} - (n_2^{ST})^2 Z_2 u^2} \, . \quad (4.1.24)$$

Hence, despite the presence of the warp factor Z_2 in the radius relation and in the mode expansions (4.1.22), the entropy of the supertube depends on its charges in exactly the same way as in flat space, and hence there is no entropy enhancement. Recalling that the angular momentum of the tube is given by (2.3.14) $J = n_2^{ST} Z_2 u^2$, we can rewrite the entropy as

$$S = 2\pi \sqrt{\frac{3}{2}} \sqrt{N_1^{ST} N_3^{ST} - n_2^{ST} J} \, . \quad (4.1.25)$$

4.2 Fluctuating supertubes and entropy enhancement

4.2.1 The three-charge black ring background

We now consider small shape fluctuations around the round supertube in a black ring background presented in section 1.4.2, that we already probed in section 3.1. The important new element, in comparison to the black hole background, is that this background has non-zero magnetic dipole charges and these will enter the calculation in some very non-trivial ways.

Again we consider the fluctuations (4.1.17) and use the DBI and WZ actions to find an effective action for the fluctuations. After straightforward calculations on can compute the momenta conjugate to η_5, η_6 and η_i:

$$\Pi_{\eta_5} = \int dz \left(\frac{\partial \mathcal{L}}{\partial \dot\eta_5} \right) \Bigg|_{BPS} = \frac{Z_2}{2\pi} \frac{R^2}{(y^2-1)(x-y)^2} \eta_5' \, , \quad (4.2.1)$$

$$\Pi_{\eta_6} = \int dz \left(\frac{\partial \mathcal{L}}{\partial \dot\eta_6} \right) \Bigg|_{BPS} = \frac{Z_2}{2\pi} \frac{R^2}{(1-x^2)(x-y)^2} \eta_6' \, , \quad (4.2.2)$$

$$\Pi_{\eta_i} = \int dz \left(\frac{\partial \mathcal{L}}{\partial \dot\eta_i} \right) \Bigg|_{BPS} = \frac{1}{2\pi} \eta_i' \, , \quad (4.2.3)$$

The BPS modes η_i, η_5 and η_6 can be expanded as:

$$\begin{aligned}
\eta_i &= \frac{1}{\sqrt{2}}\left[\sum_{k>0} e^{ik\theta/n_2^{ST}}\frac{(a_k^i)^\dagger}{\sqrt{|k|}} \;+\; \text{h.c.}\right], \\
\eta_5 &= \sqrt{\frac{(y^2-1)(x-y)^2}{2Z_2R^2}}\left[\sum_{k>0} e^{ik\theta/n_2^{ST}}\frac{(a_k^5)^\dagger}{\sqrt{|k|}} \;+\; \text{h.c.}\right], \qquad (4.2.4)\\
\eta_6 &= \sqrt{\frac{(1-x^2)(x-y)^2}{2Z_2R^2}}\left[\sum_{k>0} e^{ik\theta/n_2^{ST}}\frac{(a_k^6)^\dagger}{\sqrt{|k|}} \;+\; \text{h.c.}\right].
\end{aligned}$$

Suppose that we have a round supertube parallel to the ring ($t = \xi^0,\ z = \xi^1,\ \varphi_1 = -\theta$), then for the F1 charge of the supertube one finds

$$N_3^{ST} = \frac{1}{T_{F1}}\int_0^{2\pi n_2^{ST}}\left(\frac{\partial\mathcal{L}}{\partial\mathcal{F}_{tz}}\right)\Big|_{BPS} \qquad (4.2.5)$$

$$= \frac{T_{D2}}{T_{F1}}n_2^{ST}n_1(1+y) \;+\; \frac{T_{D2}}{T_{F1}(\mathcal{F}_{z\theta}-\frac{n_3}{2}(1+y))}\left[\frac{Z_2R^2(y^2-1)}{(x-y)^2}\right. \qquad (4.2.6)$$

$$\left. + Z_2\frac{R^2}{(y^2-1)(x-y)^2}(\eta_5')^2 + Z_4\frac{R^2}{(1-x^2)(x-y)^2}(\eta_6')^2 + (\eta_i'\eta_i')\right]. \qquad (4.2.7)$$

The expression for the entropy coming from the shape oscillations now becomes:

$$S = 2\pi\sqrt{\frac{3}{2}}\left\{\left[N_1^{ST} - \frac{1}{2}n_2^{ST}n_3(1+y)\right]\left[N_3^{ST} - \frac{1}{2}n_2^{ST}n_1(1+y)\right] \;-\; (n_2^{ST})^2\frac{Z_2R^2(y^2-1)}{(x-y)^2}\right\}^{\frac{1}{2}} \qquad (4.2.8)$$

Note that for a supertube located near the black ring ($y \to -\infty$) one has a huge entropy enhancement due to the dipole-dipole interaction.

For completeness, it is equally easy to consider a round supertube orthogonal to the black ring ($t = \xi^0,\ z = \xi^1,\ \varphi_2 = -\theta$). One then finds that the entropy of the shape modes is:

$$S = 2\pi\sqrt{\frac{3}{2}}\sqrt{\left[N_1^{ST} + \frac{1}{2}n_2^{ST}n_3(x+c)\right]\left[N_3^{ST} + \frac{1}{2}n_2^{ST}n_1(x+c)\right] - (n_2^{ST})^2\frac{Z_2R^2(1-x^2)}{(x-y)^2}} \qquad (4.2.9)$$

While there is still a dipole-dipole interaction, the entropy enhancement does not grow arbitrarily large because the coordinate x has a finite range ($x \in [-1,1]$).

4.2.2 Solution with a general Gibbons-Hawking base

We now make a final generalisation, and study in details the entropy enhancement for a supertube in a three-charge background with a Gibbons-Hawking base. For this background, one can only calculate easily the entropy coming from the internal fluctuations of the supertube. The entropy coming from fluctuations of the supertube in the spacetime directions is more complicated than for the black ring background.

For this background the supertube action becomes:

$$S = T_{D2}\int d^3\xi \left\{ \left[\left(\frac{1}{Z_1}-1\right)\mathcal{F}_{z\theta} + \frac{K^3}{Z_1 V} + \left(\frac{\mu}{Z_1} - \frac{K^1}{V}\right)(\mathcal{F}_{tz}-1)\right] \right.$$

$$\left. - \left[\frac{1}{V^2 Z_1^2}\left[(K^3 - V(\mu(1-\mathcal{F}_{tz}) - \mathcal{F}_{z\theta}))^2 + V Z_1 Z_2 (1-\mathcal{F}_{tz})(2 - Z_3(1-\mathcal{F}_{tz}))\right]\right]^{1/2}\right\}. \quad (4.2.10)$$

Because of the complexity of this background, we consider small shape oscillations in the compactification manifold, T^4, around a round supertube along the GH fiber :

$$t = \xi^0\,, \qquad z = \xi^1\,, \qquad \psi = \theta\,, \qquad x_i \to x_i + \eta_i(t,\theta) \qquad i = 1,2,3,4\,. \quad (4.2.11)$$

The quantization proceeds exactly as before and the conserved electric charges are now:

$$N_1^{ST} = \frac{T_{D2}}{T_{D0}}\int_0^{2\pi L_z} dz \int_0^{2\pi n_2^{ST}} d\theta\, \mathcal{F}_{z\theta} = n_2^{ST}\mathcal{F}_{z\theta}\,, \quad (4.2.12)$$

$$N_3^{ST} = \frac{T_{D2}}{T_{F1}}\int_0^{2\pi n_2^{ST}} d\theta \left[-\frac{K^1}{V} + \frac{1}{\mathcal{F}_{z\theta} + V^{-1}K^3}\left(\frac{Z_2}{V} + \sum_i^4 (\eta_i')^2\right)\right]. \quad (4.2.13)$$

Substituting (4.2.4) into (4.2.13) and rearranging using (4.2.12) leads to:

$$\sum_{i=1}^4 \sum_{k>0} k (a_k^i)^\dagger a_k^i = L_z T_{D2}\int_0^{2\pi n_2^{ST}} d\theta \int_0^{2\pi n_2^{ST}} d\theta \sum_{i=1}^4 \eta_i'\eta_i'$$

$$= \left[N_1^{ST} + n_2^{ST}\frac{K^3}{V}\right]\left[N_3^{ST} + n_2^{ST}\frac{K^1}{V}\right] - (n_2^{ST})^2\frac{Z_2}{V}. \quad (4.2.14)$$

This is the result we have been seeking. As we already explained, the left hand side of (4.2.14) can be thought of as the total energy L_0 of a set of four harmonic oscillators in $1+1$ dimensions. For large L_0, the entropy coming from the different ways of distributing this energy to various modes of these oscillators is given by the Cardy formula:

$$S = 2\pi\sqrt{\frac{cL_0}{6}}\,. \quad (4.2.15)$$

Since we count BPS excitations, there will be also 4 fermionic degrees of freedom, and the central charge associated to the torus oscillations will be $c = 4+2 = 6$, giving the entropy:

$$S = 2\pi\sqrt{\left[N_1^{ST} + n_2^{ST}\frac{K^3}{V}\right]\left[N_3^{ST} + n_2^{ST}\frac{K^1}{V}\right] - (n_2^{ST})^2\frac{Z_2}{V}} = 2\pi\sqrt{N_1^{ST\text{eff}}\,N_3^{ST\text{eff}} \; - \; n_2^{ST\,2}\frac{Z_2}{V}}\,. \quad (4.2.16)$$

4.2.3 The entropy enhancement mechanism

Equation (4.2.14) has two important consequences. The first one has been explained extensively in the previous chapter: for a supertube with a given set of BPS modes, this equation is nothing but a "radius formula" that determines its size by fixing, in the spatial base, the location of the $U(1)$ fiber that it wraps. When the supertube is maximally spinning, and has no BPS modes, this equation simply becomes the radius formula of the maximally spinning supertube. The second consequence is the following: this formula determines the capacity of the supertube to store entropy: In flat space, this capacity is determined by the asymptotic charges[1], N_1 and N_3, whereas, in a more general background, the capacity to store entropy is determined by $N_1^{\rm eff}$ and $N_3^{\rm eff}$. In certain backgrounds, the latter can be made much larger than the former and so a supertube of given asymptotic charges can have a lot more modes and thus store a lot more entropy by the simple expedient of migrating to a location where the effective charges are very large. We will discuss this further below.

For bubbling backgrounds, presented in 1.4.3, and even for black ring backgrounds, the right hand side of (4.2.14) can diverge, and one naively gets an infinite value for the entropy. Nevertheless, as we mentioned in the introduction, this calculation is done in the approximation that the supertube does not back-react on the background, and taking this back-reaction into account will modify this naive conclusion. We will not study this backreaction here, but this has been done in [68].

It is also useful to consider this from the perspective of the intrinsic charges of the supertube. If one rewrites (4.2.16) in terms of the angular momentum of the supertube, J (from the results of Appendix C), one finds the usual formula: $S = 2\pi\sqrt{N_1 N_3 - n_2 J}$ and it appears that the entropy enhancement has disappeared. It has not: The important point is that in a deep scaling solution it is possible for J to become extremely large and negative as the number of BPS modes on the tube increases. If $|J|$ exceeds $|N_1 N_3|$ in flat space one has CTC's *near the supertube*, but the dipole-dipole interaction of the deep scaling solution allows the supertube to exceed this bound on the asymptotic charges so long as one respects the bound on the effective charges.

It is interesting to ask how much entropy can equation (4.2.14) accommodate. The answer is not so simple. At first glance one might say that the both terms in the right hand side of (4.2.14) can be divergent, and hence the entropy of the fluctuating tube is infinite. Nevertheless, one can see that the leading order divergent terms in $N_1^{\rm eff}\, N_3^{\rm eff}$ and in ${n_2}^2 Z_2/V$ come entirely from bulk supergravity fields, and exactly cancel, both for a supertube in a GH background and for a supertube near a black ring.

It is likely that this partial cancelation is an artefact of the extremely symmetric form of the solution, and that in a more general solution such cancellation may not take place. In particular, both $N_1^{\rm eff}$ and N_3^{eff} are integrals of "effective charge" densities on the supertube world-volume, and the right hand side of equation (4.2.14) should be written as

$$N_1^{\rm eff}\, N_3^{\rm eff} - {n_2}^2 \frac{Z_2}{V} = \int \rho_1^{\rm eff} d\theta \int \rho_3^{\rm eff} d\theta - \int \rho_1^{\rm eff}\, \rho_3^{\rm eff} d\theta \tag{4.2.17}$$

[1]For simplicity, and because we will here only speak about the supertube, we will from now on drop the ST superscript for the supertube charges

If this generalized formula is correct, certain density and shape modes will disturb the balance between the product of integrals and the integral of the product, and the leading behavior of the entropy will still be of the order

$$S \sim \sqrt{N_1^{\text{eff}}\, N_2^{\text{eff}}}\,. \tag{4.2.18}$$

Regardless of this, the next-to-leading divergent terms in (4.2.16) are a combination of supertube world-volume terms and bulk supergravity fields. In a scaling solution, or when the tube is close to the black ring, these terms can diverge, giving naively an infinite entropy. As we discussed above, we expect the back-reaction of the supertubes to render this entropy finite.

The idea of *entropy enhancement* is that one can find backgrounds in which the effective charges of a two-charge supertube can be made far larger than the asymptotic charges of the solution, and that, in the right circumstances, the oscillations of this humble supertube could give rise to an entropy that grows with the asymptotic charges much faster than $\sqrt{N^2}$ (as typical for supertubes), and might even grow as fast as $\sqrt{N^3}$, as typical for black holes in five dimensions.

To achieve such a vast enhancement requires a very strong magnetic dipole-dipole interaction and this means that multiple magnetic fluxes must be present in the solution. It is *not* sufficient to have a large red-shift: BMPV black holes have infinitely long throats and arbitrarily large red-shifts but have no magnetic dipole moments to enhance the effective charges and thus increase the entropy that may be stored on a given supertube.

Hence, the obvious places to obtain entropy enhancement are solutions with large dipole magnetic fields, such as black ring or bubbling microstate solutions. Since we are focussing on trying to obtain the entropy of black holes from horizonless configurations, we will focus on the latter. These bubbling solutions are constructed using an ambi-polar base GH metric, and near the "critical surfaces," where V vanishes, the term $\frac{K^I}{V}$ in the effective charge diverges. It is therefore natural to expect entropy enhancement for supertubes that localize near the critical ($V=0$) surfaces.

We also believe that placing supertubes in deep scaling solutions [57, 82] will prove to be an equally crucial ingredient. Indeed, as we will see below, in a deep microstate geometry the K^I at the location of the tube can also become large, and hence there will be a double enhancement of the effective charge, both because of the vanishing V in the denominator and because of the very large K^I in the numerator. There is another obvious reason for this: It is only the scaling microstate geometries that have the same quantum numbers as black holes with macroscopic horizons.

This must mean that the simple entropy enhancement one gets from the presence of critical surfaces is not sufficient for matching the black hole entropy. The fundamental reason for this may well be the following: Even if the round supertube can be brought very close to the $V=0$ surface, once the supertube starts oscillating it will necessarily sample the region around this surface, and the charge enhancement will correspond to the average N_I^{eff} in that region. For this to be very large the entire region where the supertube oscillates must have a very significant charge enhancement. The only such region in a horizonless solution is the bottom of a deep or scaling throat, where the average of the K^I is indeed very large.

All the issues we have raised here have to do with the details of the entropy enhancement mechanism, and involve some very long and complex calculations that we have pursued in more recent work [68][1]. Our goals here are not to have the full backreacted solution, but rather to argue that the supertube, through this enhancement, can store much more entropy that one would naively expect, and possibly enough to account for a large part of the black hole entropy. In the probe computation we just performed, this entropy can even become infinite. We surely expect that the backreaction will regularize this and put a bound on the entropy enhancement. But the important point is that, the effective charges of the supertube depending on the complete background, especially on the total magnetic charges, we expect the bound to come from a condition involving the complete background and not only the local charges of the tube, as for the flat space case. As we already explained, a complementary point of view on the entropy enhancement is to understand it as the possibility for the supertube to spin backward very fast, much faster than the $|N_1N_3|$ bound that exists without the enhancement. If the bound comes from a global requirement, the angular momeentum of the tube will not be bounded by the charges of the supertubes but from some combination of the charges of the all background. Therefore, even after taking into account the backreaction of the supertube on the background, it seems pretty reasonable to think that the enhancement will still be there and that the supertube will be able to store much more entropy than the one it has in a background without any magnetic charges.

Before finishing this discussion, one can try to understand physically how this bound on the enhancement will come from: as we have seen, the supertube sees its entropy enhanced if it is at the bottom of a very long throat, for example near a black ring horizon or in a scaling region. Without any backreaction, the supertube in this throat has very large effective charges, and thus can have a very large angular momentum. Taking into account the backreaction, one will have a frame dragging phenomenon at the bottom of the throat. But if the momentum, and followingly the frame dragging is too large, it will create CTCs upper in the throat, where the effective charges are not very large anymore and do not allow for a large angular momentum. The bound may thus come, after backreaction, from a global causality condition, and it will be related to the global charges of the background, and not on the local charges of the supertube.

This argument finally shows that, even after taking the backreaction of the supertube on the background into account, one can still expect the entripy enhancement to occur, and the suepertube to store a very large entropy. Since supertubes source smooth supergravity solutions, the entropy enhancement mechanism we have discovered in this letter may well provide the key to understanding how fluctuating microstate geometries can provide a semi-classical description of black-hole entropy in the regime of parameters where the classical black hole exists.

[1]In [68], we have been able to show that, taking into account the backreaction of the supertube on the background puts a bound on the enhancement, and that this bound comes from a global requirement of cancelling CTCs. In particular, we confirmed the validity of the following discussion. This has been done only in the particular case of a two-center solution, and a study of the general case still remains to be done. However, it seems pretty unlikely for the results to present some new unwanted behaviours completely absent from the two-center case.

Part II

Non-supersymmetric solutions

Motivations and results

The understanding and classification of supergravity solutions is an important program that has yielded an amazing amount of new physics and, in particular, has greatly enhanced our understanding of the AdS-CFT correspondence, the non-perturbative dynamics of string theory, and the physics of black holes. In pure gravity, there has been extensive work on classifying four-dimensional solutions with horizons and, in higher dimensions, there has been extensive work on systems with enough Killing symmetries to reduce to a two-dimensional problem that can be solved using integrability (see, for example, [83]).

In supergravity and string theory most of the classification work has focused on supersymmetric solutions, and is done essentially by using Killing spinors or G-structures to reduce the second-order supergravity equations of motion to first-order equations (see, for example, [84]). It is clearly important to extend this work to non-supersymmetric solutions, not only because we would like to better understand non-supersymmetric physics, but also because we expect (from the dynamics of string theory probes) to find rather large classes of non-supersymmetric solutions, with very interesting properties.

The aim of the second part of this thesis is to present new non-supersymmetric solutions of eleven-dimensional supergravity on a six-torus[1], and embed them into a general, guiding framework to help us understand not only their structure, but also how these new solutions relate to the BPS ones. Our point of view all along this part is to understand which BPS ingredients one can recycle in the non-supersymmetric case to find a tractable way to attack the resolution while trying to go further and further away from the BPS class of solution to find new interesting non-supersymmetric physics.

Almost BPS solutions

The motivation for the first class of non-BPS solution we present, in chapter 5, is related to the following remark: The supersymmetric solutions presented in the first chapter are well understood in terms of three self-dual two-forms describing magnetic fluxes on a hyper-Kähler four-dimensional base. Implicit in the construction of the supersymmetric solutions is the choice of an orientation for the hyper-Kähler four-dimensional base: The curvature tensor can be arranged to be either self-dual or anti-self dual. For supersymmetry it is crucial that the Riemann curvature of this base has the same duality as the three magnetic two-forms: They must all be self-dual or anti-self-dual. The difference in choice merely amounts to an overall reversal of orientation and is usually neglected. However, there has been a very nice recent

[1] They can equally be seen as solutions of five-dimensional $U(1)^3$ ungauged supergravity.

observation [85] that one can obtain extremal non-supersymmetric solutions of the supergravity equations of motion by flipping the relative dualities of the hyper-Kähler base and the magnetic two-forms. This means that supersymmetries are "locally preserved" by the sources but globally broken.

The supersymmetry breaking is also easily understood in terms of the underlying brane construction. For example, an asymptotically five-dimensional black ring solution (with a flat $\mathbb{R}^4$ base) preserves the four supersymmetries respected by its three constituent electric M2 branes. When one replaces the $\mathbb{R}^4$ base by a Taub-NUT space and considers the solution from the IIA perspective, the M2 branes descend to D2 branes while the tip of Taub-NUT descends to a D6 brane. In the BPS embedding, the four Killing spinors preserved by the three sets of D2 branes are the same as those of the D6 brane, and thus the solution is supersymmetric. In the non-BPS embedding the D6 brane has opposite orientation, and hence it does not preserve any of the four Killing spinors of the D2 branes.

An interesting corollary of this D-brane picture is that five-dimensional objects that preserve the same eight Killing spinors as two sets of D2 branes, will still be supersymmetric when embedded in self-dual or anti-self-dual Taub-NUT. Indeed, if only two sets of D2 branes are present, the D6 brane will be mutually BPS with them irrespective of its orientation. Hence, a two-charge supertube embedded in Taub-NUT in the "duality-matched" embedding [54] or in the "duality-flipped" embedding [85] will still be supersymmetric. We will see the rather unexpected fashion in which this is realized.

In this chapter 5, our purpose is to give a general algorithm for constructing the most general solution to the "almost BPS equations" presented in [85]. If the location for the D2 and D6-branes are different, we know that the solutions "locally preserve" supersymmetry but break it globally. We consequently expect that local properties should be the same as those of the BPS counterparts. Indeed, we find that the near-horizon geometry of the non-BPS extremal ring is identical to that of its BPS cousin, and its entropy is given by the $E_{7(7)}$ quartic invariant as a function of its charges [67]. On the other hand, the global properties, such as the location, or "radius" of the non-BPS ring in Taub-NUT is a more global property and is generically different for BPS and non-BPS solutions.

As observed in [85], the almost BPS equations can be used to re-derive the non-rotating extremal non-BPS four-dimensional single-center black hole obtained in [29, 86]. However their power is much greater, even for single-center solutions: by adding to the angular momentum harmonic function a "dipole" piece of the form $\cos\theta/r^2$ centered at the black hole location, we can give this black hole rotation. The resulting solution is a new rotating extremal non-BPS solution in four dimensions. This solution has five (four-dimensional) quantized charges (corresponding to D6, D0 and three sets of D2 branes) as well as angular momentum[1].

For particular values of the charges and moduli one can show that this black hole can be related by dualities to the "slowly-rotating" or "ergo-free" extremal limit[2] of the D6-D0 (Rasheed-Larsen) black hole [87] or its D6-D2-D2-D0 dual [24]. However, our solution is much more general, as it can have arbitrary D6-D2-D2-D2-D0 charges. Hence this solution is the

[1]It is also trivial to introduce Wilson lines for the magnetic gauge fields, because they do not affect the rest of the solution in any way (unlike for BPS solutions).

[2]See, for example, [88] or [89] for a discussion of the two extremal limits of this black hole.

seed solution for the most generic extremal under-rotating black hole of the STU model and of $\mathcal{N} = 8$ supergravity in four dimensions.

In this chapter, a technical but important part is the case of multi-center solutions. BPS multi-center solutions exhibit a very interesting structure, and have played a crucial role in several ares of research aimed at understanding the quantum structure of black holes in string theory [56, 57, 42, 52, 44, 45, 66]. In the almost-BPS case, one can also find a very rich structure of such multi-center solutions. Just as for BPS solutions, the locations of the centers are not arbitrary, but the absence of closed time-like curves and of Dirac strings imposes certain "bubble" or "integrability" equations that these locations must satisfy. The multi-center solutions also admit scaling solutions. It is believed that these BPS scaling solutions play an important role in the microscopic black hole entropy counting, and it is therefore very important to try to understand how the non-BPS scaling solutions differ/parallel the BPS ones.

Before going on, it is important to mention that there exists a rather large body of work on constructing extremal black holes in four-dimensional supergravity, that started from the observation of [28] that the second-order equations underlying these solutions can be factorized as products of easier-to-solve first-order equations[1]. So far, the single-center solutions obtained in this way appear to be captured in the Ansatz in [85], so one can really understand these different methods as different point of views of the same physics of non-BPS extremal black holes.

Equations of motion in the "floating brane" Ansatz

The almost BPS Ansatz tells us that in some cases, even without supersymmetry, the equations of motions are still as simple as the BPS equations, in particular partially *first order* and *linear* if solved in the correct order. The natural question that arise is then: when is it possible to have such a factorization of Einstein's equations? What can more generally lead us to a rich but tractable structure? In full generality, finding non-BPS solutions implies solving a set of non-linear partial differential equations. If the system has enough symmetry to reduce it to an effective two-dimensional system then the equations of motion often become those of an integrable system and thus substantial progress can be, and has been, made (see, for example, [91, 92, 93, 94, 95]). But finding a general guiding structure for non-supersymmetric solutions is *a priori* very complicated. To make progress more generally one must incorporate some of the physics that one wishes to solve so as to provide a guiding structure that enables one to solve the equations. Hopefully, in our case, one can gain some insight by looking trying to analyze the almost BPS class of solutions: we will use an Ansatz where the warp factors and the electric potentials are equal and hence any probe M2-brane that have the same charge vector as the solution will not feel any force. Therefore, we will call this the "floating brane" Ansatz.

This Ansatz naturally incorporates the known BPS [42, 41, 40] and almost BPS solutions of five-dimensional ungauged supergravity, but, as we will see, the equations governing the general floating-brane solutions are much more general. The mass of these solutions depends linearly on their M2 charges (this comes from the equality of the warp factors and electric potential) and thus many of the floating-brane solutions will be extremal, but there are also some interesting

[1]See, for example, [29, 30, 86, 31, 32, 33, 34].

(but rather restrictive) classes of non-extremal floating-brane solutions.

The purpose of chapter 6 is to examine the full supergravity equations of motion using the floating-brane Ansatz. We first show how to obtain the usual BPS and almost-BPS solutions and find in addition that the linear equations governing these solutions lead to solutions to the supergravity equations of motion not only when the base space is hyper-Kähler, but also when the base space is merely an arbitrary four-dimensional Ricci-flat manifold. We secondly find, within the floating-brane Ansatz and after a few simplifying assumptions, a new class of solutions that solve the five-dimensional supergravity equations of motion. The equations governing these simplified-floating-brane solutions can still be solved in a *linear* fashion, but they are more general than both the BPS and the almost-BPS equations, and reduce to these in certain limits. In particular, the four-dimensional base space of these solutions does not need to be Ricci-flat but rather an Euclidean "electrovac" solution of the Einstein-Maxwell equations.

Solutions on an Israel-Wilson base space

The purpose of chapter 7 is to illustrate the equations found above to construct simplified-floating-brane solutions using Israel-Wilson geometries as base spaces. These geometries are a special class of non-Ricci-flat electrovac solutions that have a $U(1)$ isometry. The new equations imply that the functions determining the magnetic field strengths are no longer harmonic in the $\mathbb{R}^3$ base of the Israel-Wilson space, but satisfy a linear system of coupled differential equations that relate them to some of the warp factors. We solve this system for the particular example of a double-center Israel-Wilson base whose fiber degenerates at two locations, and obtain a solution that, in a certain limit, reduces to a BPS black hole in Taub-NUT, and in a different limit reduces to a non-BPS black ring in Taub-NUT. This is an important remark, because it tells us that the Israel-Wilson metrics, in addition to their intrinsic interest, give us a way to interpolate between BPS and almost BPS solutions. Naively, BPS and almost BPS are related by the change of orientation of the constituent D6-branes and thus seem to be disconnected (it corresponds to a $\mathbb{Z}_2$ transformation). With the Israel-Wilson metrics, one is able to relate them in continuous way, but the price to pay is to have the base space not to be Hyper-Kähler, except at the two ends, but electrovac.

The second aim of this chapter is also to relate the solutions in our new class that are constructed using an asymptotically $\mathbb{R}^3 \times S^1$ Israel-Wilson base space to the known multi-center almost-BPS solutions. We find that the two classes of solutions can be transformed into each other upon applying the "spectral flow" transformation of supergravity solutions with a $U(1)$ isometry discussed in [58]. From the perspective of six-dimensional supergravity (or of the full solution written in a IIB duality frame where the M2 charges correspond to D1, D5, and P charges) this transformation mixes the Kaluza-Klein[1] $U(1)$ and the $U(1)$ of the base. For BPS solutions, this spectral flow transformation re-shuffles the D6, D4, D2 and D0 charges and moduli, but the resulting solution is still BPS and hence remains in the class of solutions of [42, 41, 40]. However, when applying this spectral flow transformation to an almost-BPS solution, the resulting solution is no longer an almost-BPS solution, but is a simplified-floating-brane solution with an Israel-Wilson base space.

[1] That is, the $U(1)$ common to the D1 and D5 branes.

An immediate corollary of this observation is that among the floating-brane solutions there exist not only multiple black holes, but also new smooth horizonless bubbling solutions, that have non-trivial magnetic fluxes on the two-cycles of the Israel-Wilson base. Recall that this was not possible for the almost-BPS solutions: The anti-self-dual flux on the two-cycles of a multi-center Taub-NUT space is non-normalizable, and does not lead to asymptotically-flat solutions. Given that solutions in our new class can be obtained by spectral flow from almost-BPS solutions, it is straightforward to obtain smooth solutions with non-trivial fluxes by spectrally-flowing multiple supertubes.

Bolt solutions

It is now time to go back to the "fuzzball proposal". As we have seen in the introduction and first part of the thesis, there is now growing evidence (see [6] for reviews) that this proposal might well be realized for BPS black holes. In hind-sight, this may not appear so strange: extremal BPS black holes have a timelike singularity, and, as is fairly well known, string theory oftentimes resolves such singularities in terms of configurations that contain extra brane dipole moments, and that have a size that is parametrically much larger than the "size" of the original region of high curvature. The BPS microstate geometries constructed thus far [82, 96] indicate that the timelike singularity of extremal BPS black holes is resolved in a similar manner and that the size of the configurations that resolve the singularity is of the same order as the size of the black-hole horizon. This means that one can no longer trust the "classical" space-time description of the region between the timelike singularity and the horizon of the extremal black hole. If the fuzzball proposal also applies for *non-extremal* black holes, the would-be singularity resolution mechanism would be even more remarkable: The singularity of a non-extremal black hole is in the future of the horizon and if the classical black hole is to be replaced by a superposition of horizon-sized horizonless configurations this would imply that the resolution of non-extremal black hole singularities will affect the spacetime for a macroscopically-large distance in the past of the singularity! It is clearly important to understand this singularity-resolution mechanism, not only because there is, as yet, no rigorous example in string theory of how one might expect a space-like singularity to be resolved, but also because we live in a universe in which such singularities appear to be ubiquitous.

To establish that the singularity of non-extremal black holes is resolved by horizon-sized horizonless geometries that have the same mass and charges as the black hole, one first needs to construct such geometries, which is no easy task – only three such geometries are known at present [23, 24, 25] and some of their properties are studied in [97]. One then needs to see whether the physical properties of these geometries support thinking about them as microstates of the non-extremal black hole (and thus as examples of resolution of the black hole singularity). For example, the geometry constructed in [23], which has an ergosphere but no horizon was found to be unstable in [26] but the decay time was then computed in the dual CFT [27], and found to match exactly the decay time computed in gravity. This remarkable agreement strongly supports thinking about the geometries of [23] as microstates of non-extremal black holes, and brings hope that the fuzzball proposal will equally apply to such black holes.

Our purpose in this last chapter is to show that, within the class of solutions solving the

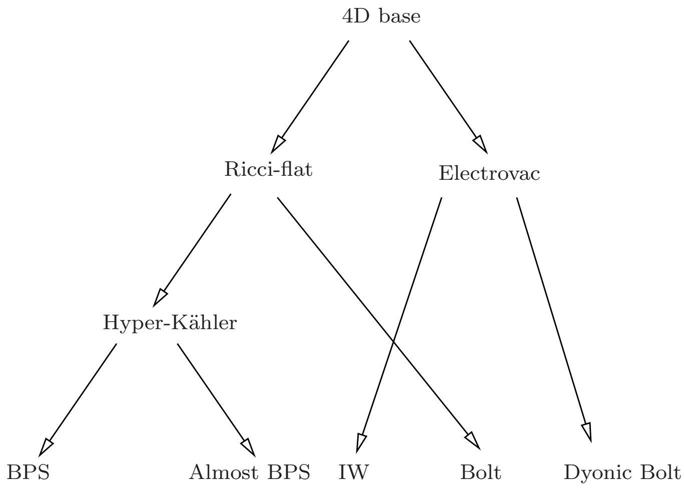

Figure 4.1: In this Figure, we schematically summarize the different results. On it, one sees how all the the new "floating brane" solutions organize in a general framework. In term of the four-dimensional base space, the well-known BPS solutions are only a small part of the solution space, having their base restricted to be Hyper-Kähler. The almost BPS solutions have also a Hyper-Kähler base, but whose orientation breaks supersymmetry. In the middle, the Israel-Wilson class of solutions, that can still be understood in terms of underlying branes, interpolates between the first two classes. Finally, one obtains smooth solutions starting from Euclideanized four-dimensional black holes with either a Ricci-flat or an electrovac base.

simplified-floating-brane system of equations of chapter 6, one can construct a family of such smooth, horizonless solutions that have the same charges and mass as *non-extremal black holes*, and that exist in the same regime of parameters where the black hole exists. To do so, one only has to remark that the simplest, most interesting Ricci-flat – or solutions to electrovac Einstein-Maxwell equations – metrics arise from the Euclideanization of various four-dimensional black-hole metrics. Such solutions have a periodic "imaginary time" coordinate and are thus asymptotic to $\mathbb{R}^3 \times S^1$. The very interesting point of these solutions is that they come with a "bolt". That is, the center of these solutions is topologically $\mathbb{R}^2 \times S^2$ where the S^2 remains of finite size. This is because the Euclideanized geometry closes off smoothly where the (outer) horizon of the original Lorentzian black hole used to be and the S^2 at the center is the same size as the original black-hole horizon. Using these Euclideanized black holes as base space, one can built completely regular five-dimensional solutions by putting some fluxes on the bolt. In chapter 8, we build such regular solutions starting in the easiest case from the Euclidean Schwarzschild black hole, with only a mass m, and then generalizing to the Euclidean dyonic Kerr-Newman-Bolt metric that has a mass m, electric and magnetic charges q and p, a NUT charge N and angular momentum α.

All the results are schematically summarized on the Figure 4.1.

Chapter 5

The "almost-BPS" approach

In this chapter we study our first class of non-BPS solutions by working with an Hyper-Kähler base whose orientation does not preserve supersymmetry. In terms of type IIA D-brane construction, this corresponds to choosing the orientation of the D6-brane to be incompatible with the ones of the three families of D2s. We built and analyze different solutions corresponding to rotating black holes (single-center solutions), black rings in Taub-NUT (double-center solution) and multi-center configurations.

5.1 Idea and equations of motions

As explain in the first chapter of this thesis, section 1.3.1, BPS solutions of eleven-dimensional supergravity carrying M2 and M5 charges are of the form

$$\begin{aligned} ds^2 &= -(Z_1Z_2Z_3)^{-2/3}(dt+k)^2 + (Z_1Z_2Z_3)^{1/3}ds_4^2 \\ &\quad + \Big(\frac{Z_2Z_3}{Z_1^2}\Big)^{1/3}(dx_1^2+dx_2^2) + \Big(\frac{Z_1Z_3}{Z_2^2}\Big)^{1/3}(dx_3^2+dx_4^2) + \Big(\frac{Z_1Z_2}{Z_3^2}\Big)^{1/3}(dx_5^2+dx_6^2) \quad (5.1.1) \\ C^{(3)} &= \Big(-\frac{dt+k}{Z_1}+B^{(1)}\Big)\wedge dx_1\wedge dx_2 + \Big(-\frac{dt+k}{Z_2}+B^{(2)}\Big)\wedge dx_3\wedge dx_4 \\ &\quad + \Big(-\frac{dt+k}{Z_3}+B^{(3)}\Big)\wedge dx_5\wedge dx_6\,, \quad (5.1.2) \end{aligned}$$

where ds_4^2 is a hyper-Kähler four-dimensional metric. Defining the "dipole" field strengths as

$$\Theta^{(I)} = dB^{(I)}\ ,\ I=1,2,3\,, \quad (5.1.3)$$

the equations following from supersymmetry for a self-dual hyper-Kähler base metric are[1] (1.3.4):

$$\Theta^{(I)} = \star_4\Theta^{(I)}\,, \quad (5.1.4)$$

$$d\star_4 d\, Z_I = \frac{C_{IJK}}{2}\,\Theta^{(J)}\wedge\Theta^{(K)}\,, \quad (5.1.5)$$

$$dk + \star_4 dk = Z_I\Theta^{(I)}\,, \quad (5.1.6)$$

[1] If one uses a hyper-Kähler base with an anti-self-dual curvature then the dualities in (5.1.4)–(5.1.6) are flipped to the form (5.1.7)–(5.1.9).

where $\star_4$ is the Hodge duality operation performed with the metric ds_4^2, and $C_{IJK} = |\epsilon_{IJK}|$. The foregoing equations also govern the solutions of arbitrary $U(1)^N$ ungauged supergravities in five dimensions [41], with C_{IJK} the corresponding triple intersection number .

It was observed in [85] that a class of extremal solutions of the equations of motion is obtained by reversing the duality of the $\Theta^{(I)}$ and of k relative to the duality of the curvature of the four-dimensional base. That is, one preserves the metric, ds_4^2, and the duality of its Riemann tensor but flips $\star_4 \to -\star_4$ in (5.1.4)–(5.1.6):

$$\Theta^{(I)} = -\star_4 \Theta^{(I)} \tag{5.1.7}$$

$$d \star_4 d Z_I = -\frac{C_{IJK}}{2} \Theta^{(J)} \wedge \Theta^{(K)} \tag{5.1.8}$$

$$dk - \star_4 dk = Z_I \Theta^{(I)} . \tag{5.1.9}$$

When the base metric ds_4^2 is flat $\mathbb{R}^4$, the flip of orientation can be re-written as a change of coordinates, and solutions to equations (5.1.7)–(5.1.9) are still BPS. When ds_4^2 is not flat, as in Taub-NUT space, equations (5.1.7)–(5.1.9) define, in general, non-BPS solutions, which were named "almost BPS" in [85].

Before going further on with explicit equations, it is worth trying to understand the physics of this "almost BPS" Ansatz. How can we physically interpret the flip of signs from (5.1.4)–(5.1.6) to (5.1.7)–(5.1.9) ? In flat space, we already said that both orientations give supersymmetric solutions. But when the space is not flat anymore, but hyper-Kähler, then the orientation of the four-dimensional space has to be compatible with the one of the system of equation, (5.1.4)–(5.1.6) or (5.1.7)–(5.1.9). This is directly related to the orientation of the branes. As we saw in the first chapter of this thesis, a family of M2-branes has an orientation, encoded mathematically in the projection condition for the Killing spinor. A family of M2-branes along $0, 5, 6$ with positive orientation implies

$$(1 + \Gamma^{056})\, \varepsilon = 0 , \tag{5.1.10}$$

while a negative orientation implies

$$(1 - \Gamma^{056})\, \varepsilon = 0 , \tag{5.1.11}$$

For a three-charge solution, we therefore have

$$(1 + \Gamma^{056})\, \varepsilon = (1 + \Gamma^{078})\, \varepsilon = (1 + \Gamma^{0910})\, \varepsilon = 0 \tag{5.1.12}$$

for so the (5.1.4)–(5.1.6) system, and

$$(1 - \Gamma^{056})\, \varepsilon = (1 - \Gamma^{078})\, \varepsilon = (1 - \Gamma^{0910})\, \varepsilon = 0 \tag{5.1.13}$$

for the (5.1.7)–(5.1.9) system of equations. If the space is flat, the choice of orientation is a simple convention. But remembering that the Γ matrices have to verify $\Gamma^{012345678910}\varepsilon = \varepsilon$, this also automatically implies

$$(1 \mp \Gamma^{1234})\, \varepsilon = 0 , \tag{5.1.14}$$

depending on the convention. In other words, taking the space not to be flat corresponds to add a D6-brane in the system. If the orientation of the D6 is compatible with the ones of the three D2s, this doesn't brake more supersymmetries and gives a 1/8 BPS solution. But if the orientation of the D6-brane is not compatible with the ones of the D2s, it will brake supersymmetry. This is exactly what happens here for the “almost BPS” Ansatz: we choose the orientation of the D2s and of the D6 to be non-compatible. One can remark that this is a very strange way to break supersymmetry. Indeed, the solution built in the Ansatz are all made out of supersymmetric objects, but put in a non-BPS configuration. While this seems to be rather restrictive, we will see in the rest of this chapter that this will lead to a very large class of interesting solutions.

5.1.1 Gibbons-Hawking base

As with the BPS solutions, equations (5.1.7)–(5.1.9) are easier to solve if one specializes to Gibbons-Hawking base metrics:

$$ds_4^2 = V^{-1}(d\psi + \vec{A})^2 + V ds_3^2 \,, \qquad \star_3 d\vec{A} = dV \,. \tag{5.1.15}$$

We will also only look for solutions that are invariant under ψ-translations.

The four-dimensional geometry is encoded in the function V, which is harmonic with respect to the flat three-dimensional euclidean metric ds_3^2. The Hodge star operation in $\mathbb{R}^3$ is denoted by $\star_3$ and one-forms on $\mathbb{R}^3$ are denoted by a vector superscript. In general, for a GH base one can take $\star_3 d\vec{A} = \pm dV$ and this leads to self-dual or anti-self-dual Riemann tensors. The choice in (5.1.15) means we are choosing a self-dual curvature.

The one-form potentials for the anti-self dual field strengths have the form:

$$B^{(I)} = K^I(d\psi + \vec{A}) + \vec{a}^I \,, \qquad \star_3 d\vec{a}^I = V dK^I - K^I dV \,, \tag{5.1.16}$$

where K^I is a harmonic function on $\mathbb{R}^3$. Such $B^{(I)}$'s thus provide the general solution to eq. (5.1.7).

Using this result in eq. (5.1.8), one finds that the warp factors Z_I must satisfy

$$d \star_3 dZ_I = \frac{1}{2} C_{IJK} V d \star_3 d\,(K^J K^K) \,. \tag{5.1.17}$$

Unlike the BPS solution, this equation does not, in general, admit a closed form solution written solely in terms of the functions V and K^I. However, in practice, it is still relatively straightforward to obtain exact solutions for Z_I.

Expanding k along the fiber and base of the Gibbons-Hawking space:

$$k \;=\; \mu\,(d\psi + \vec{A}) + \vec{\omega} \,, \tag{5.1.18}$$

one can reduce (5.1.9) to:

$$d(V\mu) \;+\; \star_3 d\vec{\omega} \;=\; V Z_I dK^I \,. \tag{5.1.19}$$

Acting with $d\star_3$ one obtains the following equation for μ:

$$d \star_3 d(V\mu) \;=\; d(VZ_I) \wedge \star_3 dK^I \,. \tag{5.1.20}$$

This equation is the integrability condition for (5.1.19). Again, one does not seem to be able to find a simple, general solution to this equation, but we will obtain particular solutions in later sections.

5.2 Single-center solution: non-BPS rotating black hole

In this section we present a solution corresponding to a rotating five-charge extremal non-BPS black hole in four dimensions. Because this is a single-center solution, we will see that it will be relatively easy to find. In the following sections, we will also present more complicated, multi-center solutions.

This black hole can serve as the seed solution for the most generic under-rotating non-BPS extremal black hole in the STU model and in $\mathcal{N} = 8$ supergravity in four dimensions, and can be thought of as coming from the non-BPS extension of the five-dimensional BPS rotating (BMPV) black hole to an asymptotically Taub-NUT solution[1].

We first construct and analyze this black hole, and then show that for special values of the charges it can be dualized to the under-rotating D0-D6 extremal black hole [87].

5.2.1 The solution

The harmonic functions associated with the KK-monopole and electric (M2) charges have the usual form

$$V = h + \frac{Q_6}{r}\,, \quad L_I = 1 + \frac{Q_I}{r}\,, \tag{5.2.1}$$

where for simplicity we have set to one the constants l_I in the L_I harmonic functions. The solution with arbitrary moduli is presented in Section 5.3.5.

The dipole charges vanish, and hence $K^I = 0$. The harmonic function, M, which encodes the angular momentum of the solution is taken to have the form:

$$M = m_0 + \frac{m}{r} + \alpha\frac{\cos\theta}{r^2}\,. \tag{5.2.2}$$

The term proportional to α is the harmonic potential is sourced by a dipole at the origin of Taub-NUT space and, as we will see, is needed to generate the angular momentum of the black hole.

With this choice of harmonic functions, the "almost BPS" equations (5.1.7-5.1.9) are solved by[2]

$$\Theta^{(I)} = 0\,, \quad Z_I = L_I\,, \quad \mu = \frac{M}{V} = \frac{m_0}{V} + \frac{m}{Vr} + \alpha\frac{\cos\theta}{Vr^2}\,, \quad \vec{\omega} = -m\cos\theta d\phi + \alpha\frac{\sin^2\theta}{r}d\phi \tag{5.2.3}$$

[1]For a recent discussion of the BPS extension of this black hole to Taub-NUT see [90].

[2]The vector potential $\vec{\omega}$ dual to the dipole field $\frac{\cos\theta}{r^2}$ follows from the identity

$$\star_3\, d\Big(\frac{\sin^2\theta}{r}d\phi\Big) = -d\Big(\frac{\cos\theta}{r^2}\Big)\,.$$

Absence of Dirac-Misner strings requires that $\vec{\omega}$ vanish both at $\theta = 0$ and $\theta = \pi$, and hence we must take

$$m = 0\,. \tag{5.2.4}$$

Nevertheless, α remains as a free parameter of the solution and it encodes the angular momentum. To see this more explicitly we compute the conserved charges. As shown in section 5.3.4, the four-dimensional Lorentzian metric, after reduction along the ψ fiber, is:

$$ds_E^2 = -I_4^{-1/2}(dt + \vec{\omega})^2 + I_4^{1/2} ds_3^2\,, \quad I_4 = Z_1 Z_2 Z_3 V - \mu^2 V^2\,, \tag{5.2.5}$$

and the electric component of the KK gauge field coming from the reduction along the Taub-NUT fiber is

$$A_{KK} = -\frac{\mu V^2}{I_4}\,. \tag{5.2.6}$$

We want the four-dimensional metric to be flat at infinity, this requires the normalization condition $I_4 \to 1$ for large r. It imposes

$$h - m_0^2 = 1\,. \tag{5.2.7}$$

The KK momentum along ψ, found from the asymptotic expansion of A_{KK}, is

$$P = m_0(h^2(Q_1 + Q_2 + Q_3) + m_0^2 Q_6)\,, \tag{5.2.8}$$

and the $\mathbb{R}^3$ angular momentum, encoded in $\vec{\omega}$, is

$$J = \alpha\,. \tag{5.2.9}$$

One can also show that this solution has a regular horizon of finite area. In the near-horizon ($r \to 0$) limit, one has

$$I_4 \to \frac{Q_1 Q_2 Q_3 Q_6 - \alpha^2 \cos^2\theta}{r^4}\,, \qquad \omega_\phi \to \alpha \frac{\sin^2\theta}{r}\,, \tag{5.2.10}$$

and thus the volume element of the metric induced on the horizon is

$$\sqrt{g_H} = r(I_4 r^2 \sin^2\theta - \omega_\phi^2)^{1/2} \approx \sin\theta (Q_1 Q_2 Q_3 Q_6 - \alpha^2)^{1/2}\,. \tag{5.2.11}$$

The horizon area is

$$A_H = (4\pi Q_6)(4\pi)\sqrt{Q_1 Q_2 Q_3 Q_6 - \alpha^2}\,, \tag{5.2.12}$$

which coincides with the area of the corresponding BMPV black hole.

5.2.2 The extremal rotating D0-D6 black hole

We now discuss the relationship between the solution presented above to the one of Rasheed and Larsen [87]. First of all, the solution of Rasheed and Larsen can be compared to ours only in the “slowly rotating” or “ergo-free” extremal limit: $a \to 0$, $m \to 0$, keeping $a/m = J$ fixed. In this limit the metric of [87] can be recast in a form similar to the one of (5.2.5):

$$ds^2 = -\frac{r^2}{\sqrt{H_1 H_2}}\left(dt + \mathbf{B}\right)^2 + \frac{\sqrt{H_1 H_2}}{r^2} ds_3^2, \tag{5.2.13}$$

where

$$\mathbf{B} = \frac{(pq)^{3/2}}{2(p+q)}\, J\, \frac{\sin^2\theta}{r}\, d\phi, \tag{5.2.14}$$

$$H_1 = r^2 + rp + \frac{p^2 q}{2(p+q)} - \frac{p^2 q}{2(p+q)} J\cos\theta, \tag{5.2.15}$$

$$H_2 = r^2 + rq + \frac{q^2 p}{2(p+q)} + \frac{q^2 p}{2(p+q)} J\cos\theta. \tag{5.2.16}$$

This solution has a single scalar field running

$$z = i\sqrt{\frac{H_2}{H_1}} \tag{5.2.17}$$

and a vanishing axion. The physical D0 and D6 charges Q and P are related to p and q by

$$Q^2 = \frac{q^3}{4(p+q)}, \qquad P^2 = \frac{p^3}{4(p+q)}. \tag{5.2.18}$$

This solution is related, by a U-duality transformation, to the solution presented above. We will establish this by applying an appropriate transformation to the scalar field (5.2.17) and showing that the resulting fields and charges fall in a special subset of those presented above. Since we are starting from a special configuration with only two charges turned on and no axion, we do not expect to be able to generate the most general solution, but we will obviously obtain some constraints on the allowed values for the moduli at infinity.

In order to simplify computations, we consider the $\mathcal{N} = 2$ truncation of the M-theory description used earlier. Hence we will look at compactifications on $T^6/(Z_2 \times Z_2) \times S^1$, where the last S^1 is parametrized by ψ and the orbifold action is the trivial one preserving the 2-forms $dx_1 \wedge dx_2$, $dx_3 \wedge dx_4$ and $dx_5 \wedge dx_6$. The resulting $\mathcal{N} = 2$ effective theory is described by an STU model, with scalar fields in the vector multiplets parametrizing:

$$\left[\frac{SU(1,1)}{U(1)}\right]^3 \simeq \frac{SU(1,1)}{U(1)} \times \frac{SO(2,2)}{SO(2)\times SO(2)}. \tag{5.2.19}$$

The three complex moduli for our solution are given by

$$t_I = \frac{4M}{V Z_I} + 4i\, \mathrm{e}^{-\phi} B_I, \tag{5.2.20}$$

where

$$B_I = \frac{(\frac{1}{2} C_{IJK} Z_J Z_K)^{1/3}}{Z_I^{2/3}}, \tag{5.2.21}$$

and the dilaton is

$$\mathrm{e}^{-2\phi} = \frac{I_4}{(Z_1 Z_2 Z_3)^{2/3} V^2}. \tag{5.2.22}$$

The duality action on the three scalar fields then acts as follows:

$$t_I \to \frac{a_I t_I + b_I}{c_I t_I + d_I} \qquad (\textit{no sum}) \tag{5.2.23}$$

where

$$M_I = \begin{pmatrix} a_I & b_I \\ c_I & d_I \end{pmatrix}, \tag{5.2.24}$$

are $\mathrm{SL}(2,\mathbb{R})$ matrices.

Without rotation one can immediately check that our solution reduces to

$$t_I = \frac{4}{V Z_I}(m_0 + i\,\mathrm{e}^{-2U}), \tag{5.2.25}$$

with $\mathrm{e}^{-2U} = \sqrt{I_4}$, which is the one presented in Equation (4.34) of [29]. This is easily dualized to the generating solution by [86] by taking

$$M_I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{5.2.26}$$

which yields

$$t_I = \frac{1}{2\,C_{IJK} Z_J Z_K}(m_0 - i\,\mathrm{e}^{-2U}). \tag{5.2.27}$$

At this point one can further dualize to D0-D6 charges by following the duality rotations described in [86]. The complete duality transformation mapping the D6-D2-D2-D2 system into the D0-D6 is then given by

$$M_I = -\frac{1}{\sqrt{2\lambda\rho_I}} \begin{pmatrix} -\rho_I & 1 \\ -\rho_I\lambda & -\lambda \end{pmatrix}, \tag{5.2.28}$$

where

$$\lambda = \left(\frac{P}{Q}\right)^{1/3}, \quad \rho_I = \sqrt{\frac{p^0 q_I}{\frac{1}{2} C_{IJK} q_J q_K}}, \tag{5.2.29}$$

with $16 p^0 = Q_6$, $q_I = Q_I$ and $(PQ)^2 = 4 p^0 q_1 q_2 q_3$.

Following the inverse route, we can start from (5.2.13)–(5.2.17) and apply the inverse transformation:

$$M_I = -\frac{1}{\sqrt{2\lambda\rho_I}} \begin{pmatrix} -\lambda & -1 \\ \rho_I\lambda & -\rho_I \end{pmatrix}. \tag{5.2.30}$$

The four-dimensional dilaton can be identified to the diagonal scalar $t_1 = t_2 = t_3 = z$. After applying the duality transformation we obtain

$$t_I = -\frac{1}{\rho_I}\frac{\lambda z + 1}{\lambda z - 1} \tag{5.2.31}$$

which we expect to match the moduli of our metric (5.2.20), which become[1]

$$t_I = \frac{4}{VZ_I}\left(\mu V + i\sqrt{I_4}\right). \tag{5.2.32}$$

Using the explicit expression for z given in (5.2.17) we can see that one needs to identify

$$VZ_I = \frac{2\rho_I}{\lambda}\frac{H_1 + \lambda^2 H_2}{r^2} \tag{5.2.33}$$

and

$$V\mu = \frac{1}{2\lambda}\frac{H_1 - \lambda^2 H_2}{r^2}. \tag{5.2.34}$$

This can be achieved for

$$\lambda = \sqrt{\frac{p}{q}}, \quad \rho_I = \frac{p+q}{2(pq)^{3/2}}\, Q_6 q_I, \tag{5.2.35}$$

which is equivalent to (5.2.29) and

$$h = \frac{p+q}{pq}Q_6, \quad l_I = \frac{p+q}{pq}q_I, \quad m_0 = \frac{q-p}{2\sqrt{pq}}, \quad \alpha = -\frac{(pq)^{3/2}}{2(p+q)}J, \tag{5.2.36}$$

where the l_I are the constants in the harmonic functions L_I, which, for simplicity, we have set to one in equation (5.2.1), but which we will explicitly include in the next section (see equation (5.3.64)) when discussing the general black-hole-black-ring solution.

Hence for special values of the charges and of the moduli, our solution can be dualized to the under-rotating extremal limit of the D0-D6 Rasheed-Larsen black hole. However, our solution has generic charges and moduli and hence it is more general; its duality orbit includes all the under-rotating extremal black hole solutions of the STU model or of $\mathcal{N} = 8$ supergravity in four dimensions.

5.3 Two-center solutions: non-BPS black ring in Taub-NUT

In this section we derive two-center solution representing a non-BPS extremal regular black ring in Taub-NUT space. This space is described by the Gibbons-Hawking potential

$$V = h + \frac{Q_6}{r} \quad \Rightarrow \quad \vec{A} = Q_6 \cos\theta d\phi\,. \tag{5.3.1}$$

We have introduced a generic constant h in V to facilitate comparison with the flat space ($\mathbb{R}^4$) limit, which corresponds to taking $h = 0$. Taking $Q_6 = 0$ corresponds to the infinite radius limit of the black ring, in which the base reduces to $\mathbb{R}^3 \times S^1$. In both of these limits the non-BPS solution must reduce to the known BPS black ring solution.

[1] As in [86], we use conventions in which $|t_I| = \frac{1}{\rho_I}$ at infinity.

5.3.1 Solving the equations

We take the position of the black ring in $\mathbb{R}^3$ to be along the positive z axis at a distance R from the origin of Taub-NUT. We denote polar coordinates centered at the black ring position by (Σ, θ_Σ). Their relation to the polar coordinates (r, θ) centered at the origin is:

$$\Sigma = \sqrt{r^2 + R^2 - 2rR\cos\theta}\,, \qquad \cos\theta_\Sigma = \frac{r\cos\theta - R}{\Sigma}\,. \tag{5.3.2}$$

The black ring carries dipole charges associated with the harmonic functions[1]

$$K^I \;=\; \frac{d^I}{\Sigma}\,, \qquad I = 1,2,3\,. \tag{5.3.3}$$

According to eq. (5.1.16), the corresponding dipole gauge fields are given by:

$$B^{(I)} = \frac{d^I}{\Sigma}(d\psi + \vec{A}) + \vec{a}^I\,, \quad \vec{a}^I = h\, d^I \frac{r\cos\theta - R}{\Sigma} d\phi + Q_6 d^I \frac{r - R\cos\theta}{R\,\Sigma} d\phi\,. \tag{5.3.4}$$

The warp factors Z_I are determined by the equation:

$$d\star_3 d\, Z_I = \frac{C_{IJK}}{2} V d\star_3 d(K^J K^K) = \frac{C_{IJK}}{2}\Big(h + \frac{Q_6}{r}\Big) d\star_3 d\Big(\frac{d^J d^K}{\Sigma^2}\Big)\,. \tag{5.3.5}$$

The solution Z_I can be written as the linear combination of two terms. The first term satisfies the equation:

$$d\star_3 d\, Z_I^{(1)} = \frac{C_{IJK}}{2}\, h\, d\star_3 d\Big(\frac{d^J d^K}{\Sigma^2}\Big)\,, \tag{5.3.6}$$

which is trivially solved by:

$$Z_I^{(1)} = \frac{C_{IJK}}{2}\, h\, \frac{d^J d^K}{\Sigma^2}\,. \tag{5.3.7}$$

The second term is found by solving

$$d\star_3 d\, Z_I^{(2)} = \frac{C_{IJK}}{2} \frac{Q_6}{r} d\star_3 d\Big(\frac{d^J d^K}{\Sigma^2}\Big)\,. \tag{5.3.8}$$

This is the same equation as the one in a flat $\mathbb{R}^4$ base and BPS and “almost BPS” solutions are related by simple change of coordinates (essentially, the exchange of the coordinates ψ and ϕ). One can therefore borrow the known BPS solution and see that the equation above is solved by:

$$Z_I^{(2)} = \frac{C_{IJK}}{2} \frac{Q_6\, d^J d^K}{R^2} \frac{r}{\Sigma^2}\,. \tag{5.3.9}$$

Moreover we can add to Z_I a harmonic function L_I, which has a pole at the location of the ring:

$$L_I = l_I + \frac{Q_I}{\Sigma}\,. \tag{5.3.10}$$

[1] As one can see from (5.1.16), adding a constant κ_I to K^I has the only effect of shifting the dipole potential $B^{(I)}$ by the constant one-form $k^I d\psi$. Hence a constant in K^I is physically irrelevant.

It is not much more difficult to add a pole in L_I at the center of the TN space, which corresponds to placing a black hole inside the black ring. We will construct this more general solution in section 5.3.5. The total solution for Z_I is then

$$Z_I = l_I + \frac{Q_I}{\Sigma} + \frac{C_{IJK}}{2}\frac{d^J d^K}{\Sigma^2}\Big(h + \frac{Q_6 r}{R^2}\Big)\,. \tag{5.3.11}$$

The equation for $k = \mu\,(d\psi + \vec{A}) + \vec{\omega}$ is now:

$$\begin{aligned} &d(V\mu) + \star_3 d\vec{\omega} = V Z_I dK^I \\ &= \Big[\Big(h + \frac{Q_6}{r}\Big)\Big(l_I + \frac{Q_I}{\Sigma}\Big) + \Big(h^2 + \frac{Q_6^2}{R^2} + Q_6 h\Big(\frac{1}{r} + \frac{r}{R^2}\Big)\Big)\frac{C_{IJK}}{2}\frac{d^J d^K}{\Sigma^2}\Big] d\Big(\frac{d^I}{\Sigma}\Big)\,, \end{aligned} \tag{5.3.12}$$

and we then expand the source term on the right-hand side into simpler component pieces. It is then straightforward to find a solution for each piece. We list in the following the solutions for the various terms:

$$\begin{aligned} &d(V\mu_1) + \star_3 d\vec{\omega}_1 = \Big(h + \frac{Q_6}{r}\Big) l_I d\Big(\frac{d^I}{\Sigma}\Big) \\ &\Rightarrow\ \mu_1 = \frac{l_I d^I}{2\Sigma}\,,\quad \vec{\omega}_1 = \frac{h\, l_I d^I}{2}\frac{r\cos\theta - R}{\Sigma} d\phi + \frac{Q_6 l_I d^I}{2}\frac{r - R\cos\theta}{R\Sigma} d\phi\,. \end{aligned} \tag{5.3.13}$$

$$d(V\mu_2) + \star_3 d\vec{\omega}_2 = h\,\frac{Q_I}{\Sigma} d\Big(\frac{d^I}{\Sigma}\Big) \quad\Rightarrow\quad \mu_2 = h\,\frac{Q_I d^I}{2V\,\Sigma^2}\,,\quad \vec{\omega}_2 = 0\,. \tag{5.3.14}$$

$$d(V\mu_3) + \star_3 d\vec{\omega}_3 = \frac{Q_6}{r}\frac{Q_I}{\Sigma} d\Big(\frac{d^I}{\Sigma}\Big)\,. \tag{5.3.15}$$

$$d(V\mu_4) + \star_3 d\vec{\omega}_4 = \Big(h^2 + \frac{Q_6^2}{R^2}\Big)\frac{C_{IJK}}{2}\frac{d^J d^K}{\Sigma^2}\, d\Big(\frac{d^I}{\Sigma}\Big)\,. \tag{5.3.16}$$

$$d(V\mu_5) + \star_3 d\vec{\omega}_5 = Q_6 h\Big(\frac{1}{r} + \frac{r}{R^2}\Big)\frac{C_{IJK}}{2}\frac{d^J d^K}{\Sigma^2} d\Big(\frac{d^I}{\Sigma}\Big)\,. \tag{5.3.17}$$

To find a solution to the third equation it is useful to reinterpret it as the equation for a one-form $\tilde{k} \equiv rV\mu_3(d\psi + \vec{A}) + \vec{\omega}_3$ in a flat $\mathbb{R}^4$ base, and use the fact that BPS and almost BPS solutions are related by a $\psi \leftrightarrow \phi$ exchange, in flat space. In this way one arrives at the following solutions

$$\mu_3 = Q_6 Q_I d^I \frac{\cos\theta}{2R\,V\Sigma^2}\,,\qquad \vec{\omega}_3 = Q_6 Q_I d^I \frac{r\sin^2\theta}{2R\,\Sigma^2} d\phi\,. \tag{5.3.18}$$

For the fourth equation one can easily verify that the following expressions

$$\mu_4^{(1)} = \Big(h^2 + \frac{Q_6^2}{R^2}\Big)\frac{C_{IJK}}{6}\frac{d^I d^J d^K}{V\,\Sigma^3}\,,\qquad \vec{\omega}_4^{\,(1)} = 0\,, \tag{5.3.19}$$

and

$$\mu_4^{(2)} = \Big(h^2 + \frac{Q_6^2}{R^2}\Big)\frac{C_{IJK}}{6}\, d^I d^J d^K \frac{r\cos\theta}{RV\,\Sigma^3}\,,\quad \vec{\omega}_4^{\,(2)} = \Big(h^2 + \frac{Q_6^2}{R^2}\Big)\frac{C_{IJK}}{6}\, d^I d^J d^K \frac{r^2\sin^2\theta}{R\,\Sigma^3} d\phi. \tag{5.3.20}$$

both solve the equation. Hence we will take

$$\mu_4 = \mu_4^{(2)} + \alpha(\mu_4^{(2)} - \mu_4^{(1)})\,, \quad \vec{\omega}_4 = (1+\alpha)\vec{\omega}_4^{(2)}\,, \tag{5.3.21}$$

and, for the moment, we will keep the parameter, α, arbitrary.

The fifth equation is the only one whose solution cannot be found by simply recycling pieces of the black ring solutions in $\mathbb{R}^4$ or $\mathbb{R}^3 \times S^1$, because the right hand side vanishes in both limits ($Q_6 \to 0$ or $h \to 0$). However, it is possible to think about the right hand side as coming from a fake solution in $\mathbb{R}^4$ whose warp factor is

$$Z_{\mathrm{fake}} \sim \frac{r^2 + R^2}{\Sigma^2}\,. \tag{5.3.22}$$

One can then express Z_{fake} in the x-y coordinate system used to find the black ring in $\mathbb{R}^4$ [61], solve the corresponding equations[1] for k_1 and k_2, and express the $\mathbb{R}^4$ solution as a solution of the almost BPS equations to read off $\mu_5 V$ and $\vec{\omega}_5$. This gives

$$\begin{aligned}
\mu_5 &= Q_6 h \frac{C_{IJK}}{6}\, d^I d^J d^K \frac{3r^2 + R^2}{2R^2 V\, r\, \Sigma^3}\,, && (5.3.23)\\
\vec{\omega}_5 &= Q_6 h \frac{C_{IJK}}{6}\, d^I d^J d^K \frac{r(3R^2 + r^2) - R(3r^2 + R^2)\cos\theta}{2R^3\,\Sigma^3} d\phi\,, && (5.3.24)
\end{aligned}$$

which one can also verify directly to be a solution of (5.3.17).

Finally one has the freedom to add a solution of the homogeneous equation, that is, a one-form in TN space with self-dual field strength. Such a one-form has the general form

$$k = \frac{M}{V}(d\psi + \vec{A}) + \vec{\omega}\,, \quad \star_3 d\vec{\omega} = -dM\,, \tag{5.3.25}$$

with M any harmonic form on $\mathbb{R}^3$. We take M of the form

$$M = m_0 + \frac{m}{\Sigma} + \frac{\tilde{m}}{r}\,. \tag{5.3.26}$$

We will see that, unlike the BPS solution, a pole in M at $r = 0$ is necessary to produce a regular solution. Hence the final possible contributions to μ and $\vec{\omega}$ are

$$\mu_6 = \frac{m_0}{V} + \frac{m}{V\,\Sigma} + \frac{\tilde{m}}{V\,r}\,, \qquad \vec{\omega}_6 = -m\frac{r\cos\theta - R}{\Sigma} d\phi - \tilde{m}\cos\theta d\phi\,. \tag{5.3.27}$$

We should also note that one should think of the term proportional to α in μ_4 and $\vec{\omega}_4$ as coming from an extra harmonic term in M. Thus, the harmonic function M that determines the black ring solution is really

$$M = m_0 + \frac{m}{\Sigma} + \frac{\tilde{m}}{r} + \alpha\frac{C_{IJK}}{6R}\, d^I d^J d^K \Big(h^2 + \frac{Q_6^2}{R^2}\Big)\frac{\cos\theta_\Sigma}{\Sigma^2}\,, \tag{5.3.28}$$

[1] Equations (46) and (47) in [40].

where θ_Σ was defined in (5.3.2). In the next section we will show that the coefficient of the dipole term, $\frac{\cos\theta_\Sigma}{\Sigma^2}$, is fixed by requiring regularity at the black ring horizon. We will see in Section (5.2) that such a term is not fixed by regularity at black-hole horizons, and in fact is required for allowing the black hole to rotate.

Adding all the terms together, we arrive at the final answer

$$\begin{aligned}
\mu &= \frac{m_0}{V} + \frac{m}{V\,\Sigma} + \frac{\tilde m}{V\,r} + \frac{l_I d^I}{2\Sigma} + \frac{h Q_I d^I}{2V\,\Sigma^2} + Q_6 Q_I d^I \frac{\cos\theta}{2R\,V\Sigma^2} \\
&\quad + \frac{C_{IJK}}{6}\, d^I d^J d^K \Big[\Big(h^2 + \frac{Q_6^2}{R^2}\Big)\Big(\frac{r\cos\theta}{R\,V\,\Sigma^3} + \alpha \frac{r\cos\theta - R}{R\,V\,\Sigma^3}\Big) + Q_6 h \frac{3r^2+R^2}{2R^2 V\,r\,\Sigma^3}\Big]\,, \\
\vec\omega &= \Big[\kappa - m\frac{r\cos\theta - R}{\Sigma} - \tilde m\cos\theta + \frac{h l_I d^I}{2}\frac{r\cos\theta - R}{\Sigma} + \frac{Q_6 l_I d^I}{2}\frac{r - R\cos\theta}{R\Sigma} \\
&\quad + Q_6 Q_I d^I \frac{r\sin^2\theta}{2R\,\Sigma^2} + \Big(h^2 + \frac{Q_6^2}{R^2}\Big)\frac{C_{IJK}}{6}\, d^I d^J d^K (1+\alpha)\frac{r^2\sin^2\theta}{R\,\Sigma^3} \\
&\quad + Q_6 h \frac{C_{IJK}}{6}\, d^I d^J d^K \frac{r(3R^2+r^2) - R(3r^2+R^2)\cos\theta}{2R^3\,\Sigma^3}\Big] d\phi\,.
\end{aligned} \tag{5.3.29}$$

We have included a constant term $\kappa d\phi$ in $\vec\omega$ and this will be needed to cancel Dirac-Misner strings.

5.3.2 Regularity

The angular coordinates ψ and ϕ both shrink to zero size at the center of Taub-NUT space, $r = 0$. Hence regularity of the one-form k requires that μ and $\vec\omega$ vanish at $r = 0$ and imposes the following constraints on the parameters of the solution:

$$\mu_{r=0} = 0 \;\Rightarrow\; \frac{\tilde m}{Q_6} + \frac{l_I d^I}{2R} + \frac{C_{IJK}}{6}\frac{h\, d^I d^J d^K}{2R^3} = 0\,, \tag{5.3.30}$$

$$\vec\omega_{r=0} = 0 \;\Rightarrow\; \kappa + m - \frac{h l_I d^I}{2} - \Big(\tilde m + \frac{Q_6 l_I d^I}{2R} + \frac{C_{IJK}}{6}\frac{Q_6 h\, d^I d^J d^K}{2R^3}\Big)\cos\theta = 0\,. \tag{5.3.31}$$

Moreover the coordinate ϕ degenerates on the z axis (*i.e.* for $\theta = 0$ or π): one should thus require that $\vec\omega$ vanishes on this axis. The constraint one obtains for $\theta = \pi$ is

$$\vec\omega_{\theta=\pi} = 0 \;\Rightarrow\; \kappa + m - \frac{h l_I d^I}{2} + \Big(\tilde m + \frac{Q_6 l_I d^I}{2R} + \frac{C_{IJK}}{6}\frac{Q_6 h\, d^I d^J d^K}{2R^3}\Big) = 0\,, \tag{5.3.32}$$

and is thus already implied by the two previous constraints (5.3.30) and (5.3.31). Vanishing of $\vec\omega$ at $\theta = 0$ imposes the further condition

$$\vec\omega_{\theta=0} = 0 \quad\Rightarrow\quad \kappa - \tilde m + \mathrm{sign}(r-R)\Big(-m + \frac{h l_I d^I}{2} + \frac{Q_6 l_I d^I}{2R} + \frac{C_{IJK}}{6}\frac{Q_6 h\, d^I d^J d^K}{2R^3}\Big) = 0\,. \tag{5.3.33}$$

All the regularity conditions are solved by taking

$$\begin{aligned}
m &= \Big(h + \frac{Q_6}{R}\Big)\frac{l_I d^I}{2} + \frac{C_{IJK}}{6}\frac{Q_6 h\, d^I d^J d^K}{2R^3}\,, \\
\tilde m &= \kappa = -Q_6\Big(\frac{l_I d^I}{2R} + \frac{C_{IJK}}{6}\frac{h\, d^I d^J d^K}{2R^3}\Big)\,.
\end{aligned} \tag{5.3.34}$$

The parameter $\tilde{m}$ determines the value of μ at the center of Taub-NUT, and the second equation determines the value of this parameter that gives regular geometries (much like for BPS solutions). As we will see later, the parameter m gives the D0 charge of the ring, and hence the first equation determines the distance between the two centers, R, as a function of the charges. This equation is the generalization of the bubble equations (1.3.17) [56, 57, 44, 45, 66] to non-BPS black holes, and reduces to these equations in the BPS limits ($h \to 0$ or $Q_6 \to 0$). For BPS solutions this equation is a simple, linear equation for R, but for the non-BPS solutions this equation is cubic in R, and its structure is much richer. Since the charges of the black ring are quantized, for given values of the moduli this equation quantizes the possible values of R.

Note that the foregoing conditions do not depend upon the parameter α that governs the "dipole" piece, proportional to $\frac{\cos\theta_\Sigma}{\Sigma^2}$, in μ. We will see in the next subsection that a careful analysis of regularity near the horizon fixes α to a non-zero value.

We should note that the authors of [85] conjectured some expressions for the harmonic functions that underlie the non-BPS black ring solution. The proposed solutions for K^I, L_I and M had poles at the black ring location (much like for BPS black rings) but our analysis here shows that such a solution will always be pathological. Regular solutions must have a source in M at the center of Taub-NUT, with coefficient $\tilde{m}$ given by (5.3.34). Similarly, there must also be very specific, non-zero "dipole" pieces, proportional to α, in μ and $\vec{\omega}$.

5.3.3 Near-horizon geometry

We now examine the metric in the vicinity of the horizon, which is located at $\Sigma = 0$. We will work in the coordinates (Σ, θ_Σ) defined in (5.3.2). Neglecting the torus directions x_i, the horizon is spanned by the coordinates ψ, ϕ and θ_Σ, and its induced metric (in the eleven-dimensional Einstein frame) is

$$\begin{aligned} ds_H^2 &= \frac{I_4}{(Z_1Z_2Z_3)^{2/3}V^2}(d\psi+\vec{A})^2 - 2\frac{\mu\,\omega_\phi}{(Z_1Z_2Z_2)^{2/3}}(d\psi+\vec{A})d\phi \\ &\quad +(Z_1Z_2Z_3)^{1/3}\Big(V\Sigma^2\sin^2\theta_\Sigma - \frac{\omega_\phi^2}{Z_1Z_2Z_3}\Big)d\phi^2 + (Z_1Z_2Z_3)^{1/3}V\Sigma^2 d\theta_\Sigma^2\,, \end{aligned} \quad (5.3.35)$$

where

$$I_4 = Z_1Z_2Z_3V - \mu^2V^2\,. \quad (5.3.36)$$

The volume element of this metric is

$$\sqrt{g_H} = \Sigma(I_4\Sigma^2\sin^2\theta_\Sigma - \omega_\phi^2)^{1/2}\,. \quad (5.3.37)$$

For generic values of the parameter α one has

$$I_4 \sim \Sigma^{-5}\,, \quad \omega_\phi \sim \Sigma^{-1}\,, \quad (5.3.38)$$

and thus $\sqrt{g_H} \sim \Sigma^{-1/2}$. So for generic α the geometry does not have a regular horizon of finite area. However the term of order Σ^{-5} in I_4 can be canceled by taking

$$\alpha = -\frac{h^2R^2}{h^2R^2+Q_6^2}\,. \quad (5.3.39)$$

One can think about α as the coefficient of a harmonic function that determines a momentum one-form whose field strength is self-dual, and hence lies in the kernel of the $(1-\star)d$ operator in equation (5.1.9). Adding this self-dual piece with the right coefficient is crucial for the regularity of the solution.

For this value of α, the metric coefficients have the following near-horizon expansions:

$$\begin{aligned} I_4 &= \frac{J_4}{\Sigma^4} + \Big(\frac{C_{IJK}}{6}\hat{d}^I\hat{d}^J\hat{d}^K\Big)^2 \frac{Q_6^2}{R^4\,V_R^4\,\Sigma^4}\sin^2\theta_\Sigma + O\Big(\frac{1}{\Sigma^3}\Big) &(5.3.40)\\ Z_I &= \frac{C_{IJK}}{2}\frac{\hat{d}^J\hat{d}^K}{V_R\,\Sigma^2} + O\Big(\frac{1}{\Sigma}\Big) &(5.3.41)\\ \mu &= \frac{C_{IJK}}{6}\frac{\hat{d}^I\hat{d}^J\hat{d}^K}{V_R^2\,\Sigma^3} + O\Big(\frac{1}{\Sigma^2}\Big) &(5.3.42)\\ \omega_\phi &= \frac{C_{IJK}}{6}\frac{Q_6^2\,\hat{d}^I\hat{d}^J\hat{d}^K}{R^2\,V_R^2\,\Sigma}\sin^2\theta_\Sigma + O(\Sigma^0)\,, &(5.3.43)\end{aligned}$$

where J_4 is the usual quartic invariant:

$$J_4(Q_I,\hat{d}^I,\hat{m}) = \frac{1}{2}\sum_{I<J}\hat{d}^I\hat{d}^J\,Q_IQ_J - \frac{1}{4}\sum_I(\hat{d}^I\,Q_I)^2 - \frac{C_{IJK}}{3}\hat{m}\,\hat{d}^I\hat{d}^J\hat{d}^K\,. \tag{5.3.44}$$

We have also defined the "effective" dipole and angular momentum parameters of the ring, $\hat{d}^I$, $\hat{m}$, via:

$$\hat{d}_I = V_R\,d^I\,,\quad \hat{m} = V_R^{-1}\,m\,,\quad V_R = \Big(h + \frac{Q_6}{R}\Big)\,. \tag{5.3.45}$$

One can see from these expressions that the horizon volume element has a finite limit for $\Sigma \to 0$:

$$\sqrt{g_H} \to J_4^{1/2}\sin\theta_\Sigma\,, \tag{5.3.46}$$

and that the five-dimensional horizon area is given by

$$A_H = (4\pi Q_6)(4\pi)J_4^{1/2}\,. \tag{5.3.47}$$

To compare this area to that of the BPS black ring in Taub-NUT, it is easier to choose moduli so that the five-dimensional Newton's constant is given by $G_5 = \frac{\pi}{4}$ and the three tori have equal volume. When $Q_6 = 1$ one can compare the singular parts of the harmonic functions to those of [51], and observe that the integer M2, M5 and KK momentum charges are:

$$n_I = -\frac{d^IV_R}{2} = -\frac{\hat{d}^I}{2}\,,\quad N_I = \frac{Q_I}{4}\,,\quad J_{KK} = -\frac{m}{8V_R} = -\frac{\hat{m}}{8}\,. \tag{5.3.48}$$

The entropy of the ring is then

$$S_{BR} = 2\pi\sqrt{J_4(N_I,n_I,J_{KK})}\,, \tag{5.3.49}$$

which is exactly the same as for BPS black rings of identical integer charges [67].

Furthermore, one can use (5.3.35) and the limiting values (5.3.40)–(5.3.43) to obtain the metric induced on the horizon:

$$ds_H^2 = \ell^{-4/3} J_4 (d\psi + Q_6 d\phi)^2 + \ell^{2/3} \Big[d\theta_\Sigma^2 + \sin^2\theta_\Sigma \Big(d\phi - \frac{Q_6}{R^2 V_R^2} (d\psi + Q_6 d\phi) \Big)^2 \Big] , \quad (5.3.50)$$

where

$$\ell = \frac{C_{IJK}}{6} \, \hat{d}^I \hat{d}^J \hat{d}^K \, . \quad (5.3.51)$$

The factor of $\frac{Q_6}{R^2 V_R^2}$ in (5.3.50) appears naively to imply that the metric induced on the horizon has conical singularities at $\theta_\Sigma = 0$ and $\theta_\Sigma = \pi$. Nevertheless, by carefully investigating the periodicity of ψ and ϕ one can show that the angle that becomes degenerate[1] has periodicity 2π and hence no such singularities exist.

5.3.4 Asymptotic charges

To obtain the reduction to four dimensions of the eleven-dimensional metric (5.1.1) one must recast the Gibbons-Hawking $U(1)$ fibration according to:

$$\begin{aligned} ds^2 \; = \; & \frac{I_4}{(Z_1 Z_2 Z_3)^{2/3} V^2} \Big[d\psi + \vec{A} - \frac{\mu V^2}{I_4} (dt + \vec{\omega}) \Big]^2 + \frac{V (Z_1 Z_2 Z_3)^{1/3}}{I_4^{1/2}} ds_E^2 \quad (5.3.52) \\ & + \Big(\frac{Z_2 Z_3}{Z_1^2} \Big)^{1/3} (dx_1^2 + dx_2^2) + \Big(\frac{Z_1 Z_3}{Z_2^2} \Big)^{1/3} (dx_3^2 + dx_4^2) + \Big(\frac{Z_1 Z_2}{Z_3^2} \Big)^{1/3} (dx_5^2 + dx_6^2) \, , \end{aligned}$$

where

$$ds_E^2 = -I_4^{-1/2} (dt + \vec{\omega})^2 + I_4^{1/2} ds_3^2 \quad (5.3.53)$$

is the four-dimensional Lorentzian metric. In order for this metric to have the canonical normalization at infinity one needs that $I_4 \to 1$ at large r. This is achieved if one takes

$$\frac{C_{IJK}}{6} h \, l_I l_J l_K - m_0^2 = 1 \, . \quad (5.3.54)$$

One could also impose that the ψ coordinate be canonically normalized (*i.e.* that $g_{\psi\psi} \to 1$ asymptotically) and this requires that

$$\frac{C_{IJK}}{6} h^3 \, l_I l_J l_K = 1 \, . \quad (5.3.55)$$

One can also see that, if $m_0 \neq 0$, μ does not vanish at infinity, producing a non-vanishing $g_{t\psi}$. This means that one is in a rotating frame at infinity, which can be undone by a re-definition of the coordinate ψ, as

$$\tilde{\psi} = \psi + h m_0 \, t \, . \quad (5.3.56)$$

[1] More explicitly, this angle is $\frac{\psi}{2Q_6} - \frac{\phi}{2}$.

In terms of $\tilde{\psi}$ the metric is explicitly asymptotically flat and it is straightforward to compute the associated asymptotic charges. The M2 charges are:

$$\hat{Q}_I = Q_I + \frac{Q_6}{R^2}\frac{C_{IJK}}{2}d^J d^K\,, \tag{5.3.57}$$

while the KK-monopole charge is simply given by Q_6 and the M5 charges by d^I. The mass is given by the BPS-like formula:

$$M = \frac{C_{IJK}}{6}\frac{l_I l_J l_K}{4}Q_6 + \frac{h}{4}\frac{C_{IJK}}{2}\hat{Q}_I l_J l_K - \frac{m_0\,h}{2}l_I d^I\,. \tag{5.3.58}$$

Note that here Q_6 and $\hat{Q}_I$ denote the absolute values of the charges.

The momentum along the KK direction $\tilde{\psi}$ is:

$$P = h^2\Big(\frac{C_{IJK}}{6}h\,l_I l_J l_K + m_0^2\Big)\,l_I d^I - m_0\,h^2\,\frac{C_{IJK}}{2}\hat{Q}_I l_J l_K - m_0^3\,Q_6\,, \tag{5.3.59}$$

and the angular momentum in the non-compact $\mathbb{R}^3$ is:

$$\begin{aligned} J &= R\Big(m - h\frac{l_I d^I}{2}\Big) + \frac{Q_6}{2R}d^I Q_I + \frac{Q_6^2}{R^3}\frac{C_{IJK}}{6}d^I d^J d^K \\ &= \frac{Q_6}{2}l_I d^I + \frac{Q_6}{2R}d^I Q_I + \frac{Q_6}{2R^2}\Big(h + \frac{2Q_6}{R}\Big)\frac{C_{IJK}}{6}d^I d^J d^K\,. \end{aligned} \tag{5.3.60}$$

If $m_0 = 0$ and the l_I and h are equal to 1, the mass formula takes a more familiar form, as a sum of absolute values of charges:

$$M = \frac{Q_6}{4} + \frac{1}{4}\sum_I \hat{Q}_I\,, \tag{5.3.61}$$

and the KK momentum along the GH fiber is just the sum of the dipole charges (much like for BPS black rings):

$$P = \sum_I d^I = \sum_I \frac{\hat{d}^I}{1 + Q_6/R}\,. \tag{5.3.62}$$

Moreover, the four-dimensional angular momentum becomes

$$J = \frac{Q_6 P}{2} + \frac{Q_6}{2R}d^I Q_I + \frac{Q_6}{2R^2}\Big(1 + \frac{2Q_6}{R}\Big)\frac{C_{IJK}}{6}d^I d^J d^K\,, \tag{5.3.63}$$

where now we can identify the first piece as coming from the Poynting vector caused by the KK electric and magnetic charges and the other pieces as coming from the interactions between the electric M2 charges and the magnetic M5 charges. When the black ring becomes a supertube ($d^1 = d^2 = Q_3 = 0$), the latter interactions are zero, and the KK Poynting term $\frac{Q_6 P}{2}$ is the only one that survives.

5.3.5 Non-BPS black ring in a black-hole background

Making use of the linear structure underlying the equations (5.1.7)–(5.1.9), it is possible to superimpose the solutions constructed in the previous sections to generate the metric describing a non-BPS black ring with a rotating black hole at the origin of Taub-NUT space. Starting from the black ring solution of section 5.3, adding the rotating black hole corresponds to adding a $1/r$ term to the harmonic functions L_I, which therefore becomes

$$L_I = l_I + \frac{Q_I}{\Sigma} + \frac{\tilde{Q}_I}{r}\,, \tag{5.3.64}$$

and a "dipole" source centered at $r = 0$ to the harmonic function M:

$$M = m_0 + \frac{m}{\Sigma} + \frac{\tilde{m}}{r} + \tilde{\alpha}\frac{\cos\theta_\Sigma}{\Sigma^2} + \beta\frac{\cos\theta}{r^2}\,. \tag{5.3.65}$$

The dipole potentials $B^{(I)}$ are left untouched, and are still given by the expressions in (5.3.4). The warp factors Z_I are obtained by replacing the old functions L_I with the new ones given in (5.3.64):

$$Z_I = l_I + \frac{Q_I}{\Sigma} + \frac{\tilde{Q}_I}{r} + \frac{C_{IJK}}{2}\frac{d^J d^K}{\Sigma^2}\Big(h + \frac{Q_6 r}{R^2}\Big)\,. \tag{5.3.66}$$

The new $1/r$ term in Z_I adds the contribution

$$\Big(h + \frac{\tilde{Q}_6}{r}\Big)\frac{\tilde{Q}_I}{r}d\Big(\frac{d^I}{\Sigma}\Big) \tag{5.3.67}$$

to the r.h.s. of the equation for k (5.3.12). Hence k receives two new contributions. The first one is given by the solution of

$$d(V\,\mu_7) + \star_3 d\vec{\omega}_7 = h\frac{\tilde{Q}_I}{r}d\Big(\frac{d^I}{\Sigma}\Big)\,. \tag{5.3.68}$$

This equation is easily solved by

$$\mu_7 = \frac{h\,\tilde{Q}_I d^I}{2V\,r\Sigma}\,, \quad \vec{\omega}_7 = \frac{h\,\tilde{Q}_I d^I}{2}\frac{r - R\cos\theta}{R\Sigma}d\phi\,. \tag{5.3.69}$$

The other new term in k is found by solving

$$d(V\,\mu_8) + \star_3 d\vec{\omega}_8 = \frac{Q_6\,\tilde{Q}_I}{r^2}d\Big(\frac{d^I}{\Sigma}\Big)\,. \tag{5.3.70}$$

Again one can find the solution by using the corresponding solution for a flat base. The result is

$$\mu_8 = \frac{Q_6\,\tilde{Q}_I d^I}{R\,V\,r\,\Sigma}\cos\theta\,, \quad \vec{\omega}_8 = \frac{Q_6\,\tilde{Q}_I d^I}{R\,\Sigma}\sin^2\theta\,d\phi\,. \tag{5.3.71}$$

Furthermore the term proportional to β in M generates an extra contribution given by

$$\mu_9 = \beta\frac{\cos\theta}{V r^2}\,, \quad \vec{\omega}_9 = \beta\frac{\sin^2\theta}{r}d\phi\,. \tag{5.3.72}$$

Adding the new terms to the previous black ring result, one finds the full solution for k:

$$\begin{aligned}
\mu &= \frac{m_0}{V} + \frac{m}{V\,\Sigma} + \frac{\tilde m}{V\,r} + \beta\frac{\cos\theta}{V r^2} + \frac{l_I d^I}{2\Sigma} + \frac{hQ_I d^I}{2V\,\Sigma^2} + Q_6 Q_I d^I\,\frac{\cos\theta}{2R\,V\Sigma^2} + \frac{h\,\tilde Q_I d^I}{2V\,r\Sigma} + \frac{Q_6\,\tilde Q_I d^I}{R\,V\,r\,\Sigma}\\
&\quad + \frac{C_{IJK}}{6} d^I d^J d^K\Big[\Big(h^2 + \frac{Q_6^2}{R^2}\Big)\Big(\frac{r\cos\theta}{R\,V\,\Sigma^3} + \alpha\frac{r\cos\theta - R}{R\,V\,\Sigma^3}\Big) + Q_6 h\frac{3r^2+R^2}{2R^2 V\,r\,\Sigma^3}\Big]\,, \qquad (5.3.73)\\
\vec\omega &= \Big\{\kappa - m\frac{r\cos\theta - R}{\Sigma} - \tilde m\cos\theta + \beta\frac{\sin^2\theta}{r}d\phi + \frac{hl_I d^I}{2}\frac{r\cos\theta - R}{\Sigma} + \frac{Q_6 l_I d^I}{2}\frac{r - R\cos\theta}{R\Sigma}\\
&\quad + Q_6 Q_I d^I\,\frac{r\sin^2\theta}{2R\,\Sigma^2} + \frac{h\,\tilde Q_I d^I}{2}\frac{r - R\cos\theta}{R\Sigma} + \frac{Q_6\,\tilde Q_I d^I}{R\,\Sigma}\,\sin^2\theta\\
&\quad + \frac{C_{IJK}}{6} d^I d^J d^K\Big[\Big(h^2 + \frac{Q_6^2}{R^2}\Big)(1+\alpha)\frac{r^2\sin^2\theta}{R\,\Sigma^3} + Q_6 h\frac{r(3R^2+r^2) - R(3r^2+R^2)\cos\theta}{2R^3\,\Sigma^3}\Big]\Big\}d\phi\,.
\end{aligned}$$

The absence of Dirac-Misner strings requires that $\vec\omega$ vanishes on the z axis. This imposes the following constraints, which are the generalization of (5.3.34)

$$\begin{aligned}
m &= \Big(h + \frac{Q_6}{R}\Big)\frac{l_I d^I}{2} + \frac{C_{IJK}}{6}\frac{Q_6 h d^I d^J d^K}{2R^3} + \frac{h}{2R}\tilde Q_I d^I\\
\tilde m &= \kappa = -Q_6\Big(\frac{l_I d^I}{2R} + \frac{C_{IJK}}{6}\frac{h d^I d^J d^K}{2R^3}\Big) - \frac{h}{2R}\tilde Q_I d^I\,. \qquad (5.3.74)
\end{aligned}$$

The first equation can again be thought of as the generalization of the bubble equations (1.3.17) [56, 57, 44, 45, 66] to the most generic two-center non-BPS extremal solution.

The topology of the black ring horizon at $\Sigma = 0$ is not affected by the black hole. As above, if α is chosen as in (5.3.39), this solution has horizon of finite area at $\Sigma = 0$ with an $S^2 \times S^1$ geometry. The area of this horizon is:

$$A_H = 16\pi^2 Q_6\,\tilde J_4^{1/2}\,, \qquad (5.3.75)$$

where

$$\tilde J_4^{1/2} = \frac12\sum_{I<J}\hat d^I\hat d^J\,Q_I Q_J - \frac14\sum_I(\hat d^I\,Q_I)^2 - \frac{C_{IJK}}{6}\hat d^I\hat d^J\hat d^K\Big(2\hat m + \frac{Q_6}{R^2\,V_R^2}\tilde Q_I\hat d^I\Big)\,. \qquad (5.3.76)$$

As for BPS black rings in black-hole backgrounds [79], the integer D0 charge of the ring is no longer proportional to $\hat m$ but rather to the combination that appears in equation (5.3.76):

$$\hat m + \frac{Q_6}{2R^2\,V_R^2}\tilde Q_I\hat d^I\,. \qquad (5.3.77)$$

The black hole at the center of the Taub-NUT space has five-dimensional horizon area equal to:

$$A_{BH} = (4\pi Q_6)(4\pi)\sqrt{Q_6\frac{C_{IJK}}{6}\tilde Q_I\tilde Q_J\tilde Q_K - \beta^2}\,. \qquad (5.3.78)$$

This black hole carries electric D6 and D2 charges (Q_6 and $\tilde Q_I$), and angular momentum β.

5.4 Almost BPS supertubes

5.4.1 The supertube solution

We already presented supertubes in the first part of this thesis, and used them intensively to probe BPS background. In section 1.4.4, we also presented their BPS backreacted version. We recall it here quickly to compare the known BPS supertubes with "almost BPS" ones, and will show that "almost BPS" supertubes are in fact BPS.

From a supergravity perspective, a supertube can be thought of as a particular black ring with only two charges and one dipole charge. One can thus trivially obtain an "almost BPS" supertube from the non-BPS solution above taking the following harmonic functions

$$K^1 = K^2 = 0\,, \quad K^3 = \frac{d^3}{\Sigma} \quad V = 1 + \frac{Q_6}{r} \tag{5.4.1}$$

$$L_1 = 1 + \frac{Q_1}{\Sigma}\,, \quad L_2 = 1 + \frac{Q_2}{\Sigma}\,, \quad L_3 = 1\,, \tag{5.4.2}$$

$$M = m_0 + \frac{m}{\Sigma} + \frac{\tilde{m}}{r}\,. \tag{5.4.3}$$

The solution simplifies considerably, and one finds

$$\begin{aligned}
&B^{(1)} = B^{(2)} = 0\,, \quad B^{(3)} = K^3(d\psi + \vec{A}) + \vec{a}^3\,, \quad \star_3 d\vec{a}^3 = V dK^3 - K^3 dV \\
&\Rightarrow\ \vec{a}^3 = d^3\frac{r\cos\theta - R}{\Sigma}d\phi + Q_6 d^3\frac{r - R\cos\theta}{R\Sigma}d\phi\,, \\
&Z_I = L_I \\
&\mu = \frac{M}{V} + \frac{1}{2}K^3\,, \quad \star_3 d\vec{\omega} = -dM + \frac{1}{2}(V dK^3 - K^3 dV) \\
&\Rightarrow\ \vec{\omega} = \Big(-m + \frac{d^3}{2}\Big)\frac{r\cos\theta - R}{\Sigma}\,d\phi - \tilde{m}\cos\theta\,d\phi + \frac{Q_6 d^3}{2}\frac{r - R\cos\theta}{R\,\Sigma}d\phi\,.
\end{aligned} \tag{5.4.4}$$

The supertube is smooth in a duality frame in which the electric (M2) charges correspond to D1 and D5 branes and the magnetic (M5) dipole moment corresponds to a KK-monopole wrapped around the Taub-NUT direction (see section 1.4.4). In this frame, we recall that the ten-dimensional string metric is:

$$ds^2 = -\frac{1}{\sqrt{Z_1 Z_2}Z_3}(dt+k)^2 + \frac{Z_3}{\sqrt{Z_1 Z_2}}\Big(dz + B^{(3)} - \frac{dt+k}{Z_3}\Big)^2 + \sqrt{Z_1 Z_2}ds_4^2 + \sqrt{\frac{Z_2}{Z_1}}\sum_{a=1}^{4} dx_a^2\,, \tag{5.4.5}$$

where z the common D1-D5 direction. Standard BPS supertubes are regular in this frame and so we now consider the regularity of the metric of the "almost BPS" supertubes. The coefficient of $d\psi^2$ in the metric is:

$$\begin{aligned}
g_{\psi\psi} &= \frac{1}{\sqrt{Z_1 Z_2}}(Z_3 {B_\psi^{(3)}}^2 - 2\mu B_\psi^{(3)} + Z_1 Z_2 V^{-1}) \\
&= \frac{1}{V\sqrt{L_1 L_2}}(L_1 L_2 - 2MK^3)\,,
\end{aligned} \tag{5.4.6}$$

where in the second line we have used the expressions for Z_I, $B^{(3)}a_3$ and μ given in (5.4.4). We have already seen in section 1.4.4 that the requirement for $g_{\psi\psi}$ to be finite for $\Sigma \to 0$ implies

$$m = \frac{Q_1 Q_2}{2d^3}\,. \tag{5.4.7}$$

In order for $\vec{\omega}$ not to have any Dirac-Misner string pathologies around the point $\Sigma = 0$ it is necessary that $\vec{\omega}$ vanish for $\theta = 0$ and r greater or smaller than R. These conditions imply:

$$m = \frac{V_R d^3}{2} \quad \text{with} \quad V_R = 1 + \frac{Q_6}{R}\,. \tag{5.4.8}$$

Combining these two relations for m one obtains an equation that determines the supertube location R:

$$V_R = \frac{Q_1 Q_2}{d^{3\,2}}\,. \tag{5.4.9}$$

Finally one should look at regularity at the Taub-NUT center $r = 0$. As the coordinate ψ degenerates at $r = 0$, μ must vanish to prevent CTC's, which implies

$$\tilde{m} = -\frac{d^3 Q_6}{2R}\,. \tag{5.4.10}$$

5.4.2 Comparing BPS and "almost BPS" supertubes

Having found a smooth supertube metric that solves the "almost BPS" equations (5.1.7)–(5.1.9), we can compare it to that of a BPS supertube, and show that despite their rather different appearance, the two solutions are identical.

Denoting with a "hat" the quantities associated with the BPS solution, we recall that the BPS supertube solution is given by:

$$\begin{aligned}
\hat{B}^{(3)} &= \frac{\hat{K}^3}{V}(d\psi + \vec{A}) + \hat{\vec{a}}^3\,, \quad \star_3 d\hat{\vec{a}}^3 = -d\hat{K}^3\,, \quad \hat{Z}_I = \hat{L}_I\,, \\
\hat{k} &= \Big(\hat{M} + \frac{\hat{K}^3}{2V}\Big)(d\psi + \vec{A}d\phi) + \hat{\vec{\omega}}\,, \quad \star_3 d\hat{\vec{\omega}} = V d\hat{M} - \hat{M} dV - \frac{1}{2} d\hat{K}^3\,.
\end{aligned} \tag{5.4.11}$$

Since the supertube solution has $Z_3 = 1$, one can absorb the term $-dt/Z_3$ in equation (5.4.5) by the coordinate shift $z \to z + t$. Thus the dipole potential $B^{(3)}$ only enters in the metric via the combination $(dz + B^{(3)} - k)^2$. Comparing the BPS expressions (5.4.11) to the "almost BPS" ones (5.4.4), one sees that, under the identifications

$$\hat{K}^3 = 2M\,, \quad \hat{M} = \frac{K^3}{2}\,, \quad \hat{L}_I = L_I \tag{5.4.12}$$

one has

$$\hat{B}^{(3)} - \hat{k} = -(B^{(3)} - k)\,, \quad \hat{Z}_I = Z_I\,, \quad \hat{k} = k\,. \tag{5.4.13}$$

Hence, the BPS and "almost BPS" supertube solutions can be related to each other by flipping the sign of z and interchanging harmonic functions. In other words, the "almost BPS" supertube is in some hidden way still supersymmetric. As we discussed at the beginning of this chapter, this was expected. Let us quickly reexplain why: in the "almost BPS" class of solutions, all the objects that we put to build the solution are BPS, but sypersymmetry is broken because they are put such that their orientations are not compatible. If one puts three families of M2-branes, like explained in section 1.3.1, the projection condition on the Killing spinor ε are

$$(1+\Gamma^{056})\,\varepsilon = (1+\Gamma^{078})\,\varepsilon = (1+\Gamma^{0910})\,\varepsilon = 0\,. \tag{5.4.14}$$

This is only compatible with an hyper-Kähler base with the orientation verifying

$$(1-\Gamma^{1234})\,\varepsilon = 0\,. \tag{5.4.15}$$

This is what is done in the BPS case. In the "almost BPS" case, we choose the same base, but we explicitly reverse the orientations of the M2-branes

$$(1-\Gamma^{056})\,\varepsilon = (1-\Gamma^{078})\,\varepsilon = (1-\Gamma^{0910})\,\varepsilon = 0\,. \tag{5.4.16}$$

and therefore break supersymmetry. But this breaking comes from a *four-object interaction*; if one take out either one of the M2-charges, or the KKm charge, supersymmetry is restored. Killing the KKm charge is just assuming the base space to be flat, and it is well-known that the two orientations are supersymmetric. This was the original motivation for introducing the almost BPS class [85]. But one can also have the KKm charge to be non-zero and one of the M2-charges to vanish, this has the same effect of restoring supersymmetry. This is exactly what happens for supertubes: one of the charge being zero, both orientations for the supertube preserve supersymmetry.

5.5 Multicenter solutions

5.5.1 The Solution with a Taub-NUT base

We finally construct in this section general multi-center solutions with a Taub-NUT base:

$$ds_4^2 = V^{-1}(d\psi + A) + V ds_3^2 \tag{5.5.1}$$

with

$$V = h + \frac{q}{r}\,, \quad A = q\cos\theta d\phi\,, \quad ds_3^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\,. \tag{5.5.2}$$

Let a_i, $i = 1, \ldots, N$ denote a succession of points along the z axis in $\mathbb{R}^3$, distinct from the Taub-NUT origin ($a_i \neq 0$). In the $\mathbb{R}^3$ base of the Taub-NUT space, the distance from a given point (r, θ, ϕ) to any one of these points is

$$\Sigma_i = \sqrt{r^2 + a_i^2 - 2ra_i\cos\theta}\,, \tag{5.5.3}$$

and the polar angle of that point with respect to the point i is

$$\cos\theta_i = \frac{r\cos\theta - a_i}{\Sigma_i}\,. \tag{5.5.4}$$

The M5 (magnetic) charges are determined by harmonic functions K^I, and we assume that they have generic poles at the points a_i[1]

$$K^I = \sum_{i=1}^{N} \frac{d_i^{(I)}}{\Sigma_i} \,. \tag{5.5.5}$$

The harmonic functions L_I associated with the M2 (electric) charges can have poles both at the points, a_i, and at the Taub-NUT center:

$$L_I \;=\; \ell_I + \frac{Q_0^{(I)}}{r} + \sum_i \frac{Q_i^{(I)}}{\Sigma_i} \;=\; \ell_I + \sum_{i=0}^{N} \frac{Q_i^{(I)}}{\Sigma_i} \,, \tag{5.5.6}$$

where $\Sigma_0 \equiv r$. We slightly changed notation in this section for the $Q_i^{(I)}$ and $d_i^{I)}$, in order nit to mix center indices with charges ($I = 1, 2, 3$) indices. A solution of the almost-BPS equations (5.1.7), (5.1.8) and (5.1.9) can now be constructed from these harmonic functions.

Dipole field strengths

The two-form field strengths, $\Theta^{(I)}$, are closed and anti-self dual in the Taub-NUT space and have the form:

$$\Theta^{(I)} = d[K^I(d\psi + A) + \vec{a}^I] \,, \tag{5.5.7}$$

where K^I is given in (5.5.5) and $\vec{a}^I$ is given by

$$\star_3\, d\vec{a}^I = V dK^I - K^I dV \;\Rightarrow\; \vec{a}^I = \sum_i \frac{d_i^{(I)}}{\Sigma_i} \Big(h(r\cos\theta - a_i) + q\,\frac{r - a_i \cos\theta}{a_i} \Big) d\phi \,. \tag{5.5.8}$$

Warp factors

The warp factors, Z_I, which encode the M2 charges, are determined by (5.1.8), and for the dipole field strengths in (5.5.7) this equation becomes:

$$d \star_3 d\, Z_I = V \frac{C_{IJK}}{2} d \star_3 d\, (K^J K^K) = \Big(h + \frac{q}{r} \Big) \sum_{j,k} \frac{C_{IJK}}{2} d \star_3 d \left(\frac{d_j^{(J)} d_k^{(K)}}{\Sigma_j \Sigma_k} \right) , \tag{5.5.9}$$

where sums over repeated J, K indices are implicit (as they will be throughout this paper). It is completely trivial to solve this equation for the terms proportional to h and for the term proportional to q we use the identity:

$$d \star_3 d \left[\frac{r}{a_i a_j} \frac{1}{\Sigma_i \Sigma_j} \right] = \frac{1}{r} d \star_3 d \left[\frac{1}{\Sigma_i \Sigma_j} \right] . \tag{5.5.10}$$

If one also includes the freedom to add to Z_I a generic harmonic function, L_I, given in (5.5.6), the complete solution for Z_I is

$$Z_I \;=\; L_I + \frac{C_{IJK}}{2} \sum_{j,k} \Big(h + \frac{qr}{a_j a_k} \Big) \frac{d_j^{(J)} d_k^{(K)}}{\Sigma_j \Sigma_k} \,. \tag{5.5.11}$$

[1]Allowing K^I to have poles at $r = 0$ appears to lead to singular solutions.

The angular momentum one-form

We again decompose the angular momentum one-form, k as

$$k = \mu(d\psi + A) + \omega\,, \tag{5.5.12}$$

where ω is a one-form on $\mathbb{R}^3$. Equation (5.1.9) then becomes[1]:

$$\begin{aligned} &d(V\mu) + \star_3 d\omega = V Z_I dK^I \\ &= V \sum_i \ell_I d_i^{(I)} d\frac{1}{\Sigma_i} + \Big(h + \frac{q}{r}\Big) \sum_{i,i'} Q_i^{(I)} d_{i'}^{(I)} \frac{1}{\Sigma_i} d\frac{1}{\Sigma_{i'}} \\ &+ \frac{C_{IJK}}{2} \sum_{i,j,k} d_i^{(I)} d_j^{(J)} d_k^{(K)} \Big[h^2 + \frac{q^2}{a_j a_k} + hq\Big(\frac{1}{r} + \frac{r}{a_j a_k}\Big)\Big] \frac{1}{\Sigma_j \Sigma_k} d\frac{1}{\Sigma_i}\,. \end{aligned} \tag{5.5.13}$$

It is convenient to rewrite the term cubic in $d_i^{(I)}$ as

$$\begin{aligned} &\frac{C_{IJK}}{2} \sum_{i,j,k} d_i^{(I)} d_j^{(J)} d_k^{(K)} \Big[h^2 + \frac{q^2}{a_j a_k} + hq\Big(\frac{1}{r} + \frac{r}{a_j a_k}\Big)\Big] \frac{1}{\Sigma_j \Sigma_k} d\frac{1}{\Sigma_i} \\ &\quad = \sum_{i,j,k} d_i^{(1)} d_j^{(2)} d_k^{(3)} (h^2\, T_{ijk}^{(1)} + q^2\, T_{ijk}^{(2)} + hq\, T_{ijk}^{(3)})\,, \end{aligned} \tag{5.5.14}$$

where

$$\begin{aligned} T_{ijk}^{(1)} &\equiv \frac{1}{\Sigma_j \Sigma_k} d\frac{1}{\Sigma_i} + \frac{1}{\Sigma_i \Sigma_k} d\frac{1}{\Sigma_j} + \frac{1}{\Sigma_i \Sigma_j} d\frac{1}{\Sigma_k} \\ T_{ijk}^{(2)} &\equiv \frac{1}{a_j a_k} \frac{1}{\Sigma_j \Sigma_k} d\frac{1}{\Sigma_i} + \frac{1}{a_i a_k} \frac{1}{\Sigma_i \Sigma_k} d\frac{1}{\Sigma_j} + \frac{1}{a_i a_j} \frac{1}{\Sigma_i \Sigma_j} d\frac{1}{\Sigma_k} \\ T_{ijk}^{(3)} &\equiv \Big(\frac{1}{r} + \frac{r}{a_j a_k}\Big) \frac{1}{\Sigma_j \Sigma_k} d\frac{1}{\Sigma_i} + \Big(\frac{1}{r} + \frac{r}{a_i a_k}\Big) \frac{1}{\Sigma_i \Sigma_k} d\frac{1}{\Sigma_j} + \Big(\frac{1}{r} + \frac{r}{a_i a_j}\Big) \frac{1}{\Sigma_i \Sigma_j} d\frac{1}{\Sigma_k}\,, \end{aligned} \tag{5.5.15}$$

with a_i, a_j, a_k any three, possibly coincident, non-vanishing points. Note that in (5.5.15) we have explicitly symmetrized over the three source points and so there is an associated factor of 1/3 but this is canceled in (5.5.14) by the explicit replacement of $\frac{1}{2}C_{IJK}$.

One can thus reduce the complete solution for μ and ω to the solution of the following equations:

$$\begin{aligned} d(V\mu_i^{(1)}) + \star_3 d\omega_i^{(1)} &= V d\frac{1}{\Sigma_i} \\ d(V\mu_i^{(2)}) + \star_3 d\omega_i^{(2)} &= \frac{1}{\Sigma_i} d\frac{1}{\Sigma_i} \quad (i \neq 0) \\ d(V\mu_{ij}^{(3)}) + \star_3 d\omega_{ij}^{(3)} &= \frac{1}{\Sigma_i} d\frac{1}{\Sigma_j} \quad (i \neq j) \\ d(V\mu_i^{(4)}) + \star_3 d\omega_i^{(4)} &= \frac{1}{r\,\Sigma_i} d\frac{1}{\Sigma_i} \quad (i \neq 0) \end{aligned} \tag{5.5.16}$$

[1]All sums over i, i', j, k are from 0 to N, with the convention that $d_0^{(I)} = 0$.

$$
\begin{aligned}
d(V\mu_{ij}^{(5)}) + \star_3 d\omega_{ij}^{(5)} &= \frac{1}{r\,\Sigma_i} d\frac{1}{\Sigma_j} \quad (i \neq j, j \neq 0) \\
d(V\mu_{ijk}^{(6)}) + \star_3 d\omega_{ijk}^{(6)} &= T_{ijk}^{(1)} \quad (i, j, k \neq 0) \\
d(V\mu_{ijk}^{(7)}) + \star_3 d\omega_{ijk}^{(7)} &= T_{ijk}^{(2)} \quad (i, j, k \neq 0) \\
d(V\mu_{ijk}^{(8)}) + \star_3 d\omega_{ijk}^{(8)} &= T_{ijk}^{(3)} \quad (i, j, k \neq 0)\,.
\end{aligned} \tag{5.5.17}
$$

A solution to this is:

$$
\begin{aligned}
V\mu_i^{(1)} &= \frac{V}{2\Sigma_i}\,, \quad \omega_i^{(1)} = \frac{h}{2}\frac{r\cos\theta - a_i}{\Sigma_i}d\phi + \frac{q}{2}\frac{r - a_i\cos\theta}{a_i\Sigma_i}d\phi \\
V\mu_i^{(2)} &= \frac{1}{2\Sigma_i^2}\,, \quad \omega_i^{(2)} = 0 \\
V\mu_{ij}^{(3)} &= \frac{1}{2}\frac{1}{\Sigma_i\Sigma_j} \quad \omega_{ij}^{(3)} = \frac{r^2 + a_ia_j - (a_i + a_j)r\cos\theta}{2(a_j - a_i)\Sigma_i\Sigma_j}d\phi \\
V\mu_i^{(4)} &= \frac{\cos\theta}{2a_i\Sigma_i^2}\,, \quad \omega_i^{(4)} = \frac{r\sin^2\theta}{2a_i\Sigma_i^2}d\phi \\
V\mu_{ij}^{(5)} &= \frac{r^2 + a_ia_j - 2a_jr\cos\theta}{2a_j(a_i - a_j)r\Sigma_i\Sigma_j}\,, \quad \omega_{ij}^{(5)} = \frac{r(a_i + a_j\cos 2\theta) - (r^2 + a_ia_j)\cos\theta}{2a_j(a_i - a_j)\Sigma_i\Sigma_j}d\phi \\
V\mu_{ijk}^{(6)} &= \frac{1}{\Sigma_i\Sigma_j\Sigma_k}\,, \quad \omega_{ijk}^{(6)} = 0 \\
V\mu_{ijk}^{(7)} &= \frac{r\cos\theta}{a_ia_ja_k\Sigma_i\Sigma_j\Sigma_k}\,, \quad \omega_{ijk}^{(7)} = \frac{r^2\sin^2\theta}{a_ia_ja_k\Sigma_i\Sigma_j\Sigma_k}d\phi \\
V\mu_{ijk}^{(8)} &= \frac{r^2(a_i + a_j + a_k) + a_ia_ja_k}{2a_ia_ja_kr\Sigma_i\Sigma_j\Sigma_k} \\
\omega_{ijk}^{(8)} &= \frac{r^3 + r(a_ia_j + a_ia_k + a_ja_k) - (r^2(a_i + a_j + a_k) + a_ia_ja_k)\cos\theta}{2a_ia_ja_k\Sigma_i\Sigma_j\Sigma_k}d\phi\,.
\end{aligned} \tag{5.5.18}
$$

One can also add to k a solution of the homogeneous equation $dk - \star_4 dk = 0$, and we consider a such solution with components:

$$
V\mu^{(9)} = M\,, \quad \star_3 d\omega^{(9)} = -dM\,, \tag{5.5.19}
$$

where M is a harmonic function that generically can be of the form:

$$
M = m + \sum_{i=0}^{N}\frac{m_i}{\Sigma_i} + \sum_{i=0}^{N}\alpha_i\frac{\cos\theta_i}{\Sigma_i^2}\,. \tag{5.5.20}
$$

Note that we have allowed for the possibility of dipole harmonic functions in M because we know, from the two-center solution (from section 5.3, that these are necessary to obtain a rotating black hole at the Taub-NUT center. The corresponding $\omega^{(9)}$ is:

$$
\omega^{(9)} = \kappa d\phi - \sum_{i=0}^{N} m_i\cos\theta_i d\phi + \sum_{i=0}^{N}\alpha_i\frac{r^2\sin^2\theta}{\Sigma_i^3}d\phi\,. \tag{5.5.21}
$$

The complete expression for μ and ω is then

$$\mu = \sum_i \ell_I d_i^{(I)} \mu_i^{(1)} + \sum_i Q_i^{(I)} d_i^{(I)} (h\mu_i^{(2)} + q\mu_i^{(4)}) + \sum_{i \neq i'} Q_i^{(I)} d_{i'}^{(I)} (h\mu_{ii'}^{(3)} + q\mu_{ii'}^{(5)})$$
$$+ \sum_{i,j,k} d_i^{(1)} d_j^{(2)} d_k^{(3)} (h^2 \mu_{ijk}^{(6)} + q^2 \mu_{ijk}^{(7)} + hq\mu_{ijk}^{(8)}) + \mu^{(9)} \tag{5.5.22}$$
$$\omega = \sum_i \ell_I d_i^{(I)} \omega_i^{(1)} + \sum_i Q_i^{(I)} d_i^{(I)} (h\omega_i^{(2)} + q\omega_i^{(4)}) + \sum_{i \neq i'} Q_i^{(I)} d_{i'}^{(I)} (h\omega_{ii'}^{(3)} + q\omega_{ii'}^{(5)})$$
$$+ \sum_{i,j,k} d_i^{(1)} d_j^{(2)} d_k^{(3)} (h^2 \omega_{ijk}^{(6)} + q^2 \omega_{ijk}^{(7)} + hq\omega_{ijk}^{(8)}) + \omega^{(9)} \,, \tag{5.5.23}$$

or, more explicitly,

$$\mu = \sum_i \frac{\ell_I d_i^{(I)}}{2\Sigma_i} + \sum_i \frac{Q_i^{(I)} d_i^{(I)}}{2V\Sigma_i^2} \Big(h + \frac{q\cos\theta}{a_i} \Big) + \sum_{i \neq i'} \frac{Q_i^{(I)} d_{i'}^{(I)}}{2V\Sigma_i \Sigma_{i'}} \Big(h + q\frac{r^2 + a_i a_{i'} - 2a_{i'} r\cos\theta}{a_{i'}(a_i - a_{i'})} \Big)$$
$$+ \sum_{i,j,k} \frac{d_i^{(1)} d_j^{(2)} d_k^{(3)}}{V\Sigma_i \Sigma_J \Sigma_k} \Big(h^2 + q^2 \frac{r\cos\theta}{a_i a_j a_k} + hq\frac{r^2(a_i + a_j + a_k) + a_i a_j a_k}{2a_i a_j a_k r} \Big) + \frac{M}{V} \,, \tag{5.5.24}$$
$$\omega = \sum_i \frac{\ell_I d_i^{(I)}}{2\Sigma_i} \Big(h(r\cos\theta - a_i) + q\frac{r - a_i\cos\theta}{a_i} \Big) d\phi + \sum_i Q_i^{(I)} d_i^{(I)} \frac{qr\sin^2\theta}{2a_i\Sigma_i^2} d\phi$$
$$+ \sum_{i \neq i'} \frac{Q_i^{(I)} d_{i'}^{(I)}}{2(a_{i'} - a_i)\Sigma_i \Sigma_{i'}} \Big(h(r^2 + a_i a_{i'} - (a_i + a_{i'}) r\cos\theta)$$
$$- q\frac{r(a_i + a_{i'}\cos 2\theta) - (r^2 + a_i a_{i'})\cos\theta}{a_{i'}} \Big) d\phi$$
$$+ \sum_{i,j,k} \frac{d_i^{(1)} d_j^{(2)} d_k^{(3)}}{a_i a_j a_k \Sigma_i \Sigma_J \Sigma_k} \Big(q^2 r^2 \sin^2\theta$$
$$+ hq\frac{r^3 + r(a_i a_j + a_i a_k + a_j a_k) - (r^2(a_i + a_j + a_k) + a_i a_j a_k)\cos\theta}{2} \Big) d\phi$$
$$+ \kappa d\phi - \sum_{i=0}^{N} m_i \cos\theta_i d\phi + \sum_{i=0}^{N} \alpha_i \frac{r^2 \sin^2\theta}{\Sigma_i^3} d\phi \,. \tag{5.5.25}$$

5.5.2 Regularity of the solutions

As for the two center case, done in section 5.3, the solutions constructed above satisfy the equations of motion, but are not necessarily regular. Indeed, the angular momentum one-form ω is proportional to $d\phi$, and can have Dirac-Misner string singularities, and these would lead to closed time-like curves (CTC's). One must therefore require ω to vanish on the z-axis (where the ϕ coordinate degenerates). Furthermore, near the poles of the harmonic functions the warp factor and rotation one-form blow up, and this can also lead to CTC's. We now find the conditions that guarantee the absence of CTC's in these two obvious places.

The conditions we will obtain are necessary but not sufficient; to be absolutely sure of regularity and absence of CTC's one must usually check each solution globally and in practice this is usually done individually and numerically. Nevertheless, in our experience (and that of others [63]), when the charges and dipole charges of the rings have the same signs, and there are no Dirac-Misner strings or CTC's at the horizons, the multi-center black ring solution is regular.

Removing closed time-like curves

We require ω_ϕ to vanish for $\theta = 0$ or π. Looking at the various terms contributing to ω we see that only $\omega^{(1)}$, $\omega^{(3)}$, $\omega^{(5)}$, $\omega^{(8)}$ and $\omega^{(9)}$ are non-vanishing on the z-axis. Their values are:

$$\begin{aligned}\omega_i^{(1)} &= \frac{s_i^{(-)}}{2}\Big(h+\frac{q}{a_i}\Big)d\phi\,, \quad \omega_{ij}^{(3)} = \frac{s_i^{(-)}s_j^{(-)}}{2(a_j-a_i)}d\phi\,, \quad \omega_{ij}^{(5)} = \frac{s_i^{(-)}s_j^{(-)}}{2a_j(a_j-a_i)}d\phi\,,\\ \omega_{ijk}^{(8)} &= \frac{s_i^{(-)}s_j^{(-)}s_k^{(-)}}{2a_ia_ja_k}d\phi\,, \quad \omega^{(9)} = (\kappa - m_0 - \sum_{i\neq 0} s_i^{(-)})d\phi\,, \end{aligned} \tag{5.5.26}$$

at $\theta = 0$, while for $\theta = \pi$ one has:

$$\begin{aligned}\omega_i^{(1)} &= \frac{s_i^{(+)}}{2}\Big(-h+\frac{q}{a_i}\Big)d\phi\,, \quad \omega_{ij}^{(3)} = \frac{s_i^{(+)}s_j^{(+)}}{2(a_j-a_i)}d\phi\,, \quad \omega_{ij}^{(5)} = -\frac{s_i^{(+)}s_j^{(+)}}{2a_j(a_j-a_i)}d\phi\,,\\ \omega_{ijk}^{(8)} &= \frac{s_i^{(+)}s_j^{(+)}s_k^{(+)}}{2a_ia_ja_k}d\phi\,, \quad \omega^{(9)} = (\kappa + m_0 + \sum_{i\neq 0} s_i^{(+)})d\phi\,, \end{aligned} \tag{5.5.27}$$

where we have defined

$$s_i^{\pm} \equiv \mathrm{sign}(r \pm a_i)\,. \tag{5.5.28}$$

Hence the absence of Dirac-Misner strings imposes the constraints

$$\begin{aligned}&\sum_i \ell_I d_i^{(I)} \frac{s_i^{(-)}}{2}\Big(h+\frac{q}{a_i}\Big) + \sum_{i\neq i'} Q_i^{(I)} d_{i'}^{(I)} \frac{s_i^{(-)}s_{i'}^{(-)}}{2(a_{i'}-a_i)}\Big(h+\frac{q}{a_{i'}}\Big)\\ &\qquad + hq\sum_{ijk} d_i^{(1)}d_j^{(2)}d_k^{(3)} \frac{s_i^{(-)}s_j^{(-)}s_k^{(-)}}{2a_ia_ja_k} + \kappa - m_0 - \sum_{i\neq 0} s_i^{(-)} m_i = 0\,, \end{aligned} \tag{5.5.29}$$

$$\begin{aligned}&-\sum_i \ell_I d_i^{(I)} \frac{s_i^{(+)}}{2}\Big(h-\frac{q}{a_i}\Big) + \sum_{i\neq i'} Q_i^{(I)} d_{i'}^{(I)} \frac{s_i^{(+)}s_{i'}^{(+)}}{2(a_{i'}-a_i)}\Big(h-\frac{q}{a_{i'}}\Big)\\ &\qquad + hq\sum_{ijk} d_i^{(1)}d_j^{(2)}d_k^{(3)} \frac{s_i^{(+)}s_j^{(+)}s_k^{(+)}}{2a_ia_ja_k} + \kappa + m_0 + \sum_{i\neq 0} s_i^{(+)} m_i = 0\,. \end{aligned} \tag{5.5.30}$$

Note that, taking into account the possible values of the signs $s_i^{(\pm)}$, the conditions above imply $N+2$ independent constraints. One can make these constraints explicit, for example, by solving them with respect to the $N+2$ variables κ, m_0 and m_i for $i = 1, \ldots, N$. If one considers, for definiteness, a configuration in which all the poles a_i lie to the right of the Taub-NUT center ($0 < a_1 < \ldots < a_N$), then the regularity constraints are:

$$
\begin{aligned}
\kappa &= -q \sum_i \frac{\ell_I d_i^{(I)}}{2a_i} - h \sum_{i \neq i'} \frac{Q_i^{(I)} d_{i'}^{(I)}}{2(a_{i'} - a_i)} - hq \sum_{i,j,k} \frac{d_i^{(1)} d_j^{(2)} d_k^{(3)}}{2a_i a_j a_k}\,, && (5.5.31)\\
m_0 &= -q \sum_i \frac{\ell_I d_i^{(I)}}{2a_i} - h \sum_i \frac{Q_0^{(I)} d_i^{(I)}}{2a_i} + q \sum_{i \neq i', i \neq 0} \frac{Q_i^{(I)} d_{i'}^{(I)}}{2a_{i'}(a_{i'} - a_i)} - hq \sum_{i,j,k} \frac{d_i^{(1)} d_j^{(2)} d_k^{(3)}}{2a_i a_j a_k}\,, && (5.5.32)\\
m_i &= \frac{\ell_I d_i^{(I)}}{2}\Big(h + \frac{q}{a_i}\Big) + \sum_j \frac{1}{2|a_i - a_j|}\Big[Q_j^{(I)} d_i^{(I)}\Big(h + \frac{q}{a_i}\Big) - Q_i^{(I)} d_j^{(I)}\Big(h + \frac{q}{a_j}\Big)\Big] \\
&\quad + \frac{hq}{2}\Big[\frac{d_i^{(1)} d_i^{(2)} d_i^{(3)}}{a_i^3} + \frac{C_{IJK}}{2}\frac{d_i^{(I)}}{a_i} \sum_{j,k} \mathrm{sign}(a_j - a_i)\mathrm{sign}(a_k - a_i) \frac{d_j^{(J)} d_k^{(K)}}{a_j a_k}\Big] \quad (i \geq 1)\,, && (5.5.33)
\end{aligned}
$$

where we have used the convention $\mathrm{sign}(0) = 0$.

When there is no black hole and no rotation at the center of Taub-NUT ($Q_0^{(I)} = 0$ and $\alpha_0 = 0$), the metric around $r = 0$ is expected to describe empty space, and hence be completely regular. As both coordinates ψ and ϕ degenerate at $r = 0$, regularity requires that μ and ω vanish. From (5.5.22) and (5.5.23) and the regularity relations (5.5.31), (5.5.32) and (5.5.33), one indeed finds that μ and ω must satisfy:

$$
\begin{aligned}
\mu|_{r=0} &= \sum_i \frac{\ell_I d_i^{(I)}}{2a_i} - \sum_{i \neq i', i \neq 0} \frac{Q_i^{(I)} d_{i'}^{(I)}}{2a_{i'}(a_{i'} - a_i)} + h \sum_{i,j,k} \frac{d_i^{(1)} d_j^{(2)} d_k^{(3)}}{2a_i a_j a_k} + \frac{m_0}{q} = 0\,, \\
\omega|_{r=0} &= \Big[-\sum_i \frac{\ell_I d_i^{(I)}}{2}\Big(h + \frac{q\cos\theta}{a_i}\Big) + \sum_{i \neq i', i \neq 0} \frac{Q_i^{(I)} d_{i'}^{(I)}}{2(a_{i'} - a_i)}\Big(h + \frac{q\cos\theta}{a_{i'}}\Big) \\
&\quad - hq \sum_{i,j,k} \frac{d_i^{(1)} d_j^{(2)} d_k^{(3)} \cos\theta}{2a_i a_j a_k} + \kappa - m_0 \cos\theta + \sum_{i \neq 0} m_i\Big] d\phi = 0\,, && (5.5.34)
\end{aligned}
$$

which are automatically implied by (5.5.31), (5.5.32) and (5.5.33). Hence, these relations are enough to guarantee the regularity of the solution at the center of Taub-NUT space.

Regularity at the horizons

It is also important to study the geometry in the vicinity of the poles $\Sigma_i = 0$, where, for generic charges and not-too-large angular momenta, we expect to find regular horizons. For this purpose it is convenient to define

$$
I_4 = Z_1 Z_2 Z_3 V - \mu^2 V^2\,. \qquad (5.5.35)
$$

The volume element of the horizon around $\Sigma_i = 0$ is

$$\sqrt{g_{H,i}} = \Sigma_i (I_4 \Sigma_i^2 \sin^2\theta_i - \omega_\phi^2)^{1/2} \,. \tag{5.5.36}$$

Consider first the black hole horizon at $\Sigma_0 \equiv r = 0$. The near-horizon expansion gives

$$I_4 \approx \frac{Q_0^{(1)} Q_0^{(2)} Q_0^{(3)} q - \alpha_0^2 \cos^2\theta}{r^4} \,, \qquad \omega_\phi \approx \alpha_0 \frac{\sin^2\theta}{r} \,, \tag{5.5.37}$$

and hence

$$\sqrt{g_{H,0}} \approx (Q_0^{(1)} Q_0^{(2)} Q_0^{(3)} q - \alpha_0^2)^{1/2} \sin\theta \,. \tag{5.5.38}$$

Thus we find a horizon of finite area[1] given by:

$$A_{H,0} = (4\pi q)(4\pi)(Q_0^{(1)} Q_0^{(2)} Q_0^{(3)} q - \alpha_0^2)^{1/2} \,. \tag{5.5.39}$$

As expected, the black hole at the center is the four-charge rotating black hole constructed in the previous sections, and the parameter α_0 encodes its four-dimensional angular momentum.

Consider now the limiting form of the metric near the i^{th} point (around $\Sigma_i = 0$). After several highly non-trivial cancelations one obtains:

$$I_4 = -2\alpha_i d_i^{(1)} d_i^{(2)} d_i^{(3)} \left(h + \frac{q}{a_i} \right)^2 \frac{\cos\theta_i}{\Sigma_i^5} + O(\Sigma_i^{-4}) \tag{5.5.40}$$

and

$$\omega_\phi \sim \Sigma_i^{-1} \,. \tag{5.5.41}$$

This would lead to closed timelike curves outside the horizon unless the term of order Σ_i^{-5} in I_4 vanishes, which requires[2]:

$$\alpha_i = 0 \quad (i \geq 1). \tag{5.5.42}$$

When this condition is imposed, each point $\Sigma_i = 0$ is a black ring horizon of area

$$A_H = 16\pi^2 q J_4^{1/2} \,, \tag{5.5.43}$$

where J_4 is the usual $E_{7(7)}$ quartic invariant that appears in the black ring horizon area [67]:

$$J_4 = \frac{1}{2} \sum_{I<J} \hat{d}_i^{(I)} \hat{d}_i^{(J)} Q_i^{(I)} Q_i^{(J)} - \frac{1}{4} \sum_I (\hat{d}_i^{(I)})^2 (Q_i^{(I)})^2 - 2 \hat{d}_i^{(1)} \hat{d}_i^{(2)} \hat{d}_i^{(3)} \hat{m}_i \,. \tag{5.5.44}$$

[1] As usual, area means the spatial measure of the three-dimensional horizon of the five-dimensional black hole.

[2] In the two-center solution of section 5.3 a non-zero value for α_i was required for regularity at the black ring horizon. However, the parameter α_i in section 5.3 differs from the one used here by a constant coming from the gauge choice for $\mu^{(6)}$, and the two results are consistent.

In order to bring J_4 to its canonical form, we have defined the "effective" dipole and angular momentum parameters;[1]

$$\hat{d}_i^{(I)} = \Big(h + \frac{q}{a_i}\Big) d_i^{(I)}\,, \quad \hat{m}_i = \Big(h + \frac{q}{a_i}\Big)^{-1} m_i\,. \tag{5.5.46}$$

Note that the result (5.5.43) and (5.5.44) coincides with the one for an isolated BPS black ring carrying charges $\hat{d}_i^{(I)}$, Q_i^I and $\hat{m}_i$: the area of the i^{th} horizon is not affected by the presence of the other horizons nor by the switch of orientation of the base space that is characteristic of our non-BPS solutions.

If one chooses units such that the five-dimensional Newton's constant is $G_5 = \frac{\pi}{4}$ and the three tori have equal sizes, the integer M2, M5 and KK momentum charges carried by the i^{th} center are:

$$n_i^{(I)} = -\frac{\hat{d}_i^{(I)}}{2}\,, \quad N_i^{(I)} = \frac{Q_i^{(I)}}{4}\,, \quad J_i^{(KK)} = -\frac{\hat{m}_i}{8}\,. \tag{5.5.47}$$

One can also construct solutions in which some of the centers do not have three M2 charges and three M5 charges, but only two M2 charges and one M5 charge. These solutions describe now two-charge round supertubes, and the geometry near an individual supertube is expected to be smooth in the duality frame in which the dipole charge of the tube corresponds to KK-monopoles, and the electric charges correspond to D1 and D5 branes (see section 1.4.4 or [11, 13]).

For the supertube with dipole charge corresponding to, say, K^3, this regularity condition is (1.4.38):

$$\lim_{\Sigma_i\to 0} \Sigma_i^2 (Z_3 V (K^3)^2 - 2\mu V K^{(3)} + Z_1 Z_2) = 0\,. \tag{5.5.48}$$

Just as for black rings, this requires that the "dipole" harmonic term in M vanish (otherwise $\mu V K_i^3 \sim \Sigma_i^{-3}$):

$$\alpha_i = 0\,. \tag{5.5.49}$$

Furthermore, equation (5.5.48) implies the usual supertube regularity condition:

$$2 d_i^{(3)} m_i = Q_i^{(2)} Q_i^{(1)}\,. \tag{5.5.50}$$

Scaling solutions

Consider the limit in which the positions of the centers are scaled to zero ($a_i \ll \frac{q}{h}$). In this limit the regularity conditions (5.5.32) and (5.5.33), when written in terms of the quantized

[1]The "effective angular momentum" that appears in the J_4 parameter of the non-BPS black ring in Taub-NUT constructed in section 5.3 is not $\hat{m}_i$ but

$$\hat{m}_i^{\text{BDGRW}} \equiv \hat{m}_i - \frac{q}{2a_i^2}\Big(h + \frac{q}{a_i}\Big)^{-2} Q_0^{(I)} \hat{d}_i^{(I)} \qquad \Leftrightarrow \qquad m_i^{\text{here}} = m_i^{\text{BDGRW}} + \frac{q}{2a_i^2} Q_0^{(I)} d_i^{(I)}\,. \tag{5.5.45}$$

We find here, instead, that J_4 simply depends on $\hat{m}_i$. The two results are consistent because here we are using a different (and more natural) gauge choice originating from a different definition of $\mu_{0i}^{(5)}$ and reflected in the equation for m_i^{here}.

charge parameters $\hat{d}_i^{(I)}$, Q_i^I and $\hat{m}_i$, reduce to:

$$
\begin{aligned}
m_0 &= -\sum_i \frac{\ell_I \hat{d}_i^{(I)}}{2} - \frac{h}{q}\sum_i \frac{Q_0^{(I)}\hat{d}_i^{(I)}}{2} + \sum_{i\neq i', i\neq 0} \frac{Q_i^{(I)}\hat{d}_{i'}^{(I)}}{2(a_{i'}-a_i)} - \frac{h}{q^2}\sum_{i,j,k}\frac{\hat{d}_i^{(1)}\hat{d}_j^{(2)}\hat{d}_k^{(3)}}{2}\,, \qquad (5.5.51)\\
q\frac{\hat{m}_i}{a_i} &= \frac{\ell_I \hat{d}_i^{(I)}}{2} + \sum_j \frac{1}{2|a_i-a_j|}\Big[Q_j^{(I)}\hat{d}_i^{(I)} - Q_i^{(I)}\hat{d}_j^{(I)}\Big]\\
&\quad + \frac{h}{2q^2}\Big[\hat{d}_i^{(1)}\hat{d}_i^{(2)}\hat{d}_i^{(3)} + \frac{C_{IJK}}{2}\hat{d}_i^{(I)}\sum_{j,k}\mathrm{sign}(a_j-a_i)\mathrm{sign}(a_k-a_i)\hat{d}_j^{(J)}\hat{d}_k^{(K)}\Big] \quad (i\geq 1) \qquad (5.5.52)
\end{aligned}
$$

These equations are now linear in the inverse of the inter-center distance, much as they are for BPS solutions.

As the parameters $\hat{d}_i^{(I)}$, Q_i^I and $\hat{m}_i$ with $i > 0$ are associated to quantized charges, their value is to be kept finite while the a_i's are scaled to zero. Note however that m_0 does not correspond to any quantized charge, but is a parameter needed for regularity, as indicated by (5.5.31). Hence, one should think about equation (5.5.51) (or (5.5.32) in the full solution) as determining the value of a parameter of the solution as a function of the charges and the positions of the centers, and about equations (5.5.52) (or (5.5.52) in the full solution) as the "bubble equations" of the system, that determine the inter-center distances as a function of the charges and the moduli.

In the small a_i limit, the non-BPS bubble equations become:

$$
\sum_j \frac{1}{2|a_i-a_j|}\Big[Q_j^{(I)}\hat{d}_i^{(I)} - Q_i^{(I)}\hat{d}_j^{(I)}\Big] = q\frac{\hat{m}_i}{a_i}\,, \qquad (5.5.53)
$$

which coincides with the scaling limit of the BPS bubble equations.

Chapter 6

The floating brane equations

This chapter is devoted to a complete analysis of the equations of motions, inside the “floating brane” Ansatz. This Ansatz relates warp factors of the metric with the electric part of the Ramond-Ramond potentials. It physically implies that a probe M2-brane with the same charge vector as the background will not feel any force, and thus will be “floating”. With some simple additionnal assumptions, we are able to find a system of equation where Einstein’s equations factorize into first order equations, and where the system can ultimately be solved in a linear way. We found as sub-cases of our system the BPS and almost BPS equations.

6.1 Equations of motion

6.1.1 Conventions and the floating-brane Ansatz

We consider eleven-dimensional supergravity presented in the first chapter of this thesis, (1.1.11), with the ansatz (1.3.1)-(1.3.2). Upon redusing along the T^6 directions, it is interesting to note that this is equivalent to $\mathcal{N}=2$, five-dimensional supergravity with three $U(1)$ gauged fields[1], whose bosonic action is[2]

$$S = \frac{1}{2\kappa_5}\int \sqrt{-g}\, d^5x\Big(R - \tfrac{1}{2}Q_{IJ}F^I_{\mu\nu}F^{J\mu\nu} - Q_{IJ}\partial_\mu X^I\partial^\mu X^J - \tfrac{1}{24}C_{IJK}F^I_{\mu\nu}F^J_{\rho\sigma}A^K_\lambda\,\bar\epsilon^{\mu\nu\rho\sigma\lambda}\Big)\,, \tag{6.1.1}$$

with $I, J = 1, 2, 3$, and the scalar kinetic term being

$$Q_{IJ} \;=\; \frac{1}{2}\,\mathrm{diag}\left((X^1)^{-2}, (X^2)^{-2}, (X^3)^{-2}\right). \tag{6.1.2}$$

The X^I’s are related to the Z_I functions introduced in (1.3.1) by

$$X^1 = \left(\frac{Z_2\, Z_3}{Z_1^2}\right)^{1/3}, \quad X^2 = \left(\frac{Z_1\, Z_3}{Z_2^2}\right)^{1/3}, \quad X^3 = \left(\frac{Z_1\, Z_2}{Z_3^2}\right)^{1/3}. \tag{6.1.3}$$

[1] We use the conventions of [85].

[2] We take $\sqrt{-g}\,\bar\epsilon^{01234} = -1$.

They correspond to the volumes of the three T^2 tori. The ansatz that we choose imposes that the volume of the T^6 is constant, and set to 1,

$$X^1 X^2 X^3 = 1\,. \tag{6.1.4}$$

From a five-dimensional point of view, this constraint comes from the fact that one of the photons lies in the gravity multiplet and so there are only two vector multiplets and hence only two independent scalars.

We denote Z the third independant scalar

$$Z \;\equiv\; \big(Z_1\, Z_2\, Z_3\big)^{1/3}\,. \tag{6.1.5}$$

Eq (1.3.1) reduces in five dimension to the ansatz

$$ds_5^2 \;=\; -Z^{-2}\,(dt+k)^2 \;+\; Z\,ds_4^2\,. \tag{6.1.6}$$

The powers guarantee that Z becomes an independent scalar from the four-dimensional perspective. We will denote the frames for (6.1.6) by e^A, $A = 0, 1, \ldots, 4$ and let $\hat{e}^a$, $a = 1, \ldots, 4$ denote frames for ds_4^2. That is, we take:

$$e^0 \;\equiv\; -Z^{-1}\,(dt+k)\,, \qquad e^a \;\equiv\; Z^{1/2}\,\hat{e}^a\,. \tag{6.1.7}$$

The heart of the "floating brane" Ansatz is to relate the metric coefficients and the scalars to the electrostatic potentials. The Maxwell Ansatz is thus:

$$A^{(I)} \;=\; -\varepsilon\, Z_I^{-1}\,(dt+k) + B^{(I)}\,, \tag{6.1.8}$$

where $B^{(I)}$ is a one-form on the base (with metric ds_4^2). The parameter, ε, will be related to the self-duality or anti-self-duality of the fields in the solution and is fixed to have $\varepsilon^2 = 1$. Except for the ε orientation, this is the exactly the reduction of (1.3.1)-(1.3.2) to five dimensions. In eleven-dimensional supergravity, or M-theory, this Ansatz implies that M2 brane probes that have the same charge vector as the M2 charge vector of the solution will have equal and opposite Wess-Zumino and Born-Infeld terms in the Lagrangian and hence will feel no force. Such brane probes may be placed anywhere in the base and may thus be viewed as "floating". We will see that the ε factor is mostly a question of conventions, but it will be interesting to keep it. In particular, it will be useful to make the link between the BPS and almost-BPS classes of solutions.

As before, we define the field strengths:

$$\Theta^{(I)} \;\equiv\; dB^{(I)} \;=\; \tfrac{1}{2}\,Z^{-1}\,\Theta^{(I)}_{ab}\,e^a\wedge e^b \;=\; \tfrac{1}{2}\,\Theta^{(I)}_{ab}\,\hat{e}^a\wedge\hat{e}^b\,. \tag{6.1.9}$$

We also introduce

$$K \;\equiv\; dk \;=\; \tfrac{1}{2}\,(\partial_\mu\, k_\nu - \partial_\nu\, k_\mu)\,dx^\mu\wedge dx^\mu \;=\; \tfrac{1}{2}\,K_{ab}\,\hat{e}^a\wedge\hat{e}^b\,. \tag{6.1.10}$$

Note that the frame components are defined relative to the frames on ds_4^2.

Another consequence of the fact that we have used the same function, Z, in both the metric and the electric potential in (6.1.8) is that the mass of our solutions will always be linear in the electric (M2) charges, much like the mass of extremal solutions (although for some orientations the mass may also *decrease* linearly with the charges, as we will see in chapter 8). This also suggests that our solutions should be essentially extremal, however we have made no assumptions about the base metric, ds_4^2, and the choices for this will lead to a very large class of non-BPS solutions that include non-extremal solutions, as we will see in the following.

6.1.2 Einstein's equations

The time (00) components of Einstein's equations give:

$$\sum_I Z_I^{-1}\hat{\nabla}^2 Z_I \;=\; -\tfrac{1}{4}\, Z^{-3} \sum_I Z_I\, \Theta^{(I)}_{ab} \big(Z_I\, \Theta^{(I)}_{ab} \;-\; 2\,\varepsilon\, K_{ab}\big)\,, \tag{6.1.11}$$

where $\hat{\nabla}$ is the covariant derivative in the base metric, ds_4^2.

The off-diagonal ($0a$ components) of Einstein's equations give:

$$\hat{\nabla}^b K_{ba} \;=\; \varepsilon \sum_I \big(\hat{\nabla}^b Z_I\big)\, \Theta^{(I)}_{ba}\,, \tag{6.1.12}$$

or, equivalently,

$$d \star_4 K \;=\; \varepsilon \sum_I dZ_I \,\wedge\, \star_4 \Theta^{(I)}\,. \tag{6.1.13}$$

To give the remaining Einstein's equations it is convenient to define the two-form:

$$\mathcal{P} \;\equiv\; K \;-\; \tfrac{1}{2}\,\varepsilon \sum_{I=1}^{3} Z_I\, \Theta^{(I)}\,. \tag{6.1.14}$$

The components of Einstein's equations on the four-dimensional base are:

$$\begin{aligned}\hat{R}_{ab} \;-\; \tfrac{1}{2}\hat{R}\,\delta_{ab} \;=\; Z^{-3}\Big[&\mathcal{P}_{ac}\,\mathcal{P}_{bc} \;-\; \tfrac{1}{4}\,\delta_{ab}\,\mathcal{P}_{cd}\mathcal{P}_{cd} \\ &+\tfrac{1}{4}\Big(2\sum_I Z_I^2\, \Theta^{(I)}_{ac}\, \Theta^{(I)}_{bc} \;-\; \sum_{I,J} Z_I Z_J\, \Theta^{(I)}_{ac}\, \Theta^{(J)}_{bc}\Big) \\ &-\tfrac{1}{16}\,\delta_{ab}\Big(2\sum_I Z_I^2\, \Theta^{(I)}_{cd}\, \Theta^{(I)}_{cd} \;-\; \sum_{I,J} Z_I Z_J\, \Theta^{(I)}_{cd}\, \Theta^{(J)}_{cd}\Big)\Big]\,,\end{aligned} \tag{6.1.15}$$

where $\hat{R}_{ab}$ and $\hat{R}$ are the Ricci tensor and Ricci scalar of the base metric, ds_4^2. Note that these equations imply that the Ricci scalar of the base must vanish:

$$\hat{R} = 0\,. \tag{6.1.16}$$

6.1.3 Scalar equations

The scalar equations of motion yield equations for the ratios of the Z_I. For example:

$$Z_1^{-1}\hat\nabla^2 Z_1 \;-\; Z_3^{-1}\hat\nabla^2 Z_3 \;=\; \tfrac{1}{2}\,Z^{-3}\Big[Z_1\,\Theta^{(1)}_{ab}\,\big(Z_1\,\Theta^{(1)}_{ab} \;-\; 2\,\varepsilon\,K_{ab}\big) \tag{6.1.17}$$

$$- \; Z_3\,\Theta^{(3)}_{ab}\,\big(Z_3\,\Theta^{(3)}_{ab} \;-\; 2\,\varepsilon\,K_{ab}\big)\Big]\,. \tag{6.1.18}$$

When combined with (6.1.11) one gets:

$$\hat\nabla^2 Z_I \;=\; -\tfrac{1}{4}\,Z_J^{-1}Z_K^{-1}\Big[Z_J\,\Theta^{(J)}_{ab}\,\big(Z_J\,\Theta^{(J)}_{ab} \;-\; 2\,\varepsilon\,K_{ab}\big) \tag{6.1.19}$$

$$+\; Z_K\,\Theta^{(K)}_{ab}\,\big(Z_K\,\Theta^{(K)}_{ab} \;-\; 2\,\varepsilon\,K_{ab}\big) \;-\; Z_I\,\Theta^{(I)}_{ab}\,\big(Z_I\,\Theta^{(I)}_{ab} \;-\; 2\,\varepsilon\,K_{ab}\big)\Big]\,, \tag{6.1.20}$$

where $\{I,J,K\} = \{1,2,3\}$ are all distinct and these indices are not summed.

6.1.4 Maxwell equations

To give the Maxwell equations it is convenient to define:

$$\mathcal{R}^{(I)}_{\pm} \;\equiv\; \tfrac{1}{2}\,\varepsilon\,Z_I\,\big(\Theta^{(I)} \;\pm\; \varepsilon\,\star_4\,\Theta^{(I)}\big)\,, \qquad \mathcal{P}_{\pm} \;\equiv\; \tfrac{1}{2}\,\big(K \;\pm\; \varepsilon\,\star_4\,K\big) \;-\; \tfrac{1}{2}\,\sum_{M=1}^{3}\,\mathcal{R}^{(M)}_{\pm} \tag{6.1.21}$$

with no sum on I. Note that $\mathcal{P} = \mathcal{P}_+ + \mathcal{P}_-$. The parameter, ε, satisfies $\varepsilon^2 = 1$ and so determines whether these combinations are self-dual or anti-self-dual.

The Maxwell equations are:

$$d\star_5\big(Q_{IJ}\,F^J\big) \;=\; \tfrac{1}{4}\,C_{IJK}\,F^J\wedge F^K\,, \tag{6.1.22}$$

with $F^I = dA^I$. Using the Ansatz (6.1.8) one obtains two types of terms: (i) a four form on the four-dimensional base and (ii) e^0 wedged into a three form on the four-dimensional base. The former generates the following equations for $\hat\nabla^2 Z_I$:

$$\hat\nabla^2 Z_I \;=\; \varepsilon\star_4\bigg[\Theta^{(J)}\wedge\Theta^{(K)} \;+\; Z^{-3}Z_I\,K\wedge\Big(K + \varepsilon\star_4 K + 2\mathcal{R}^{(I)}_{-} - \varepsilon\sum_{M=1}^{3} Z_M\Theta^{(M)}\Big)\bigg]\,. \tag{6.1.23}$$

Combining this with (6.1.20) one obtains three algebraic constraints on the forms $\mathcal{P}_+$ and $\mathcal{R}^{(M)}_{\pm}$:

$$\mathcal{P}_+\wedge\mathcal{P}_+ \;+\; \mathcal{P}_+\wedge\mathcal{R}^{(I)}_{+} \;+\; \tfrac{1}{4}\,\big(\mathcal{R}^{(I)}_{-} - \mathcal{R}^{(J)}_{-} + \mathcal{R}^{(K)}_{-}\big)\wedge\big(\mathcal{R}^{(I)}_{-} + \mathcal{R}^{(J)}_{-} - \mathcal{R}^{(K)}_{-}\big) \;=\; 0\,, \tag{6.1.24}$$

where, once again, $\{I,J,K\} = \{1,2,3\}$ are all distinct and these indices are not summed.

The second set of Maxwell equations can be written as:

$$d\Big(Z^{-3}Z_I\,\Big(K + \varepsilon\star_4 K + 2\mathcal{R}^{(I)}_{-} - \varepsilon\sum_{M=1}^{3} Z_M\Theta^{(M)}\Big)\Big) \;=\; 0\,, \tag{6.1.25}$$

where the index I isn't summed.

6.2 Analysis and subcases

6.2.1 General results on the equations

Using the equations of motion one can easily show that:

$$d\big((Z_1 Z_2 Z_3)^{-1}\mathcal{P}_+\big) \;=\; 0\,, \tag{6.2.1}$$

and hence one may write

$$\mathcal{P}_+ \;=\; (Z_1 Z_2 Z_3)\,\omega_+^{(0)}\,, \tag{6.2.2}$$

where $\omega_+^{(0)}$ is harmonic.

One can simplify some of the Maxwell equations by introducing some additional forms, $\omega_-^{(I)}$ defined by:

$$\tfrac{1}{2}\,\varepsilon\left(\Theta^{(I)} - \varepsilon\star_4\Theta^{(I)}\right) \;\equiv\; C_{IJK}\,Z_J\,\omega_-^{(K)}\,. \tag{6.2.3}$$

Since $Z_1 Z_2 Z_3 \neq 0$, this transformation is invertible and so we have made no additional assumptions. In terms of these new ε-anti-self-dual forms, the Maxwell equations (6.1.25) become:

$$d\star_4\omega_-^{(I)} \;=\; (Z_1 Z_2 Z_3)^{-1} dZ_I \wedge \mathcal{P}_+\,. \tag{6.2.4}$$

and (6.1.24) becomes:

$$\mathcal{P}_+\wedge\mathcal{P}_+ \;+\; \mathcal{P}_+\wedge\mathcal{R}_+^{(I)} \;+\; (Z_1 Z_2 Z_3)\,Z_I\,\omega_-^{(J)}\wedge\omega_-^{(K)} \;=\; 0\,, \tag{6.2.5}$$

where $\{I,J,K\} = \{1,2,3\}$ are all distinct and these indices are not summed.

To simplify Einstein's equations, introduce the function, $\mathcal{T}_{ab}$, of a pair of two forms that is defined by:

$$\mathcal{T}_{ab}(X,Y) \;\equiv\; \tfrac{1}{2}\left(X_{ac}\,Y_{bc} \;+\; X_{bc}\,Y_{ac}\right) \;-\; \tfrac{1}{4}\,\delta_{ab}\,X_{cd}\,Y_{cd}\,. \tag{6.2.6}$$

In particular, $\mathcal{T}_{ab}(F,F)$ is the energy momentum tensor associated with the Maxwell field, F. Note that if $X_\pm$ and $Y_\pm$ are the self-dual and anti-self dual parts of X and Y, then

$$\mathcal{T}_{ab}(X_\pm, Y_\pm) \;=\; 0\,, \qquad \mathcal{T}_{ab}(X,Y) \;=\; \mathcal{T}_{ab}(X_+,Y_-) \;+\; \mathcal{T}_{ab}(X_-,Y_+)\,. \tag{6.2.7}$$

Using this in the Einstein equations (6.1.15), one obtains:

$$\hat{R}_{ab} \;=\; 2\,Z^{-3}\,\mathcal{T}_{ab}(\mathcal{P}_+,\mathcal{P}_-) \;-\; \sum_{I=1}^{3}\mathcal{T}_{ab}\Big(\tfrac{1}{2}\,\varepsilon\left(\Theta^{(I)} + \varepsilon\star_4\Theta^{(I)}\right), \omega_-^{(I)}\Big)\,. \tag{6.2.8}$$

Thus far we have made no assumptions other than our floating brane Ansatz.

6.2.2 A simple assumption

The equations of motion dramatically simplify if one takes:

$$\mathcal{P}_+ \;\equiv\; 0\,, \tag{6.2.9}$$

which is, of course, consistent with (6.2.1) and thus with the equations of motion. We will henceforth assume that (6.2.9) is true.

One then finds from (6.2.4) and (6.2.5) that the forms $\omega_-^{(I)}$ must be harmonic and satisfy

$$\omega_-^{(I)} \wedge \omega_-^{(J)} \;=\; 0\,, \quad I \neq J\,. \tag{6.2.10}$$

There are two obvious ways to satisfy this condition:
• (i) Take $\omega_-^{(1)} = \omega_-^{(2)} = 0$ and $\omega_-^{(3)}$ to be an arbitrary ε-anti-self-dual harmonic form.
• (ii) Take the manifold to be hyper-Kähler and let each of the $\omega_-^{(I)}$ be a constant multiple of one the three harmonic two forms associated with the three complex structures[1].

Continuing with the implications of (6.2.9), one finds that the equations for the scalars (6.1.23) reduce to:

$$\hat{\nabla}^2 Z_I \;=\; \varepsilon \star_4 \Big[\Theta^{(J)} \wedge \Theta^{(K)} \;-\; \omega_-^{(I)} \wedge (K - \varepsilon \star_4 K)\Big]\,, \tag{6.2.11}$$

and Einstein's equations collapse to

$$\hat{R}_{ab} \;=\; -\tfrac{1}{2}\,\varepsilon \sum_{I=1}^{3} \mathcal{T}_{ab}\big((\Theta^{(I)} + \varepsilon \star_4 \Theta^{(I)}), \omega_-^{(I)}\big)\,. \tag{6.2.12}$$

Note that the Ricci tensor depends only upon the four-dimensional electromagnetic fluxes.

6.2.3 A first linear system

If one assumes that all the $\omega^{(I)}$'s are set to zero,

$$\omega^{(I)} = 0\,,\; I = 1, 2, 3\,, \tag{6.2.13}$$

the system of equations reduces to

$$\begin{aligned}
\hat{R}_{ab} &= 0\,, && (6.2.14)\\
\Theta^{(I)} &= \varepsilon \star_4 \Theta^{(I)}\,, && (6.2.15)\\
\hat{\nabla}^2 Z_I &= \varepsilon \frac{C_{IJK}}{2} \star_4 \Theta^{(J)} \wedge \Theta^{(K)}\,, && (6.2.16)\\
\mathcal{P}_+ &= 0\,. && (6.2.17)
\end{aligned}$$

For $\varepsilon = 1$, (6.2.15)-(6.2.17) are just the BPS equations (1.3.4) [40, 41]. For $\varepsilon = -1$ they become the almost-BPS equations of the previous chapter[2] [85]. Nevertheless, the known BPS and almost BPS classes of solutions have an hyper-Kähler base space, while here the equations of motion impose only the base to be Ricci-flat (6.2.14). So this system of equations not only encodes all the BPS solutions presented in the first chapter of this thesis, and the new almost

[1] There might be an interesting generalization of (ii) to quaternionic-Kähler spaces.

[2] The global sign of he last equation (6.2.17) is different from the one of the previous chapter (5.1.9), but this sign is not physical but convention dependent.

BPS solutions of the previous chapter, but also allows in general many possible new solutions. We will present some new solutions to this system in chapter 8. The fact that the base only needs to be Ricci-flat is not so strange: The hyper-Kähler condition originally arose because one wanted to preserve supersymmetry, however, Einstein's equations, (6.2.8), only care about the Ricci tensor of the base. As we already discussed in detail in the first chapter of this thesis, one of the key properties of this system of equations is that it is *linear* if solved in the correct order; the non-linear terms only appear as known sources in the equations. Linearity is what allows one to perform explicit computations, and to find interesting solutions.

6.2.4 A second linear system

We now generalize the case of the previous subsection and make a less restrictive assumption: we impose that condition (i) above is satisfied: $\omega_-^{(1)} = \omega_-^{(2)} = 0$ and take $\omega_-^{(3)}$ to be an arbitrary ε-anti-self-dual harmonic form. Then the equations become

$$\begin{aligned} \big(\Theta^{(1)} - \varepsilon \star_4 \Theta^{(1)}\big) &= 2\,\varepsilon\, Z_2\, \omega_-^{(3)} \,, \qquad \big(\Theta^{(2)} - \varepsilon \star_4 \Theta^{(2)}\big) = 2\,\varepsilon\, Z_1\, \omega_-^{(3)} \,, \\ \big(\Theta^{(3)} - \varepsilon \star_4 \Theta^{(3)}\big) &= 0 \,. \end{aligned} \tag{6.2.18}$$

Thus $\Theta^{(3)}$ is a harmonic, ε-self-dual two form.

The background geometry must be chosen so that

$$\hat{R}_{ab} = -\varepsilon\, \mathcal{T}_{ab}\big(\Theta^{(3)}, \omega_-^{(3)}\big) \equiv \tfrac{1}{2}\left(\mathcal{F}_{ac}\,\mathcal{F}_{bc} - \tfrac{1}{4}\,\delta_{ab}\,\mathcal{F}_{cd}\mathcal{F}_{cd}\right), \tag{6.2.19}$$

where $\mathcal{F}$ is defined by

$$\mathcal{F} \equiv \Theta^{(3)} - \varepsilon\, \omega_-^{(3)} \,. \tag{6.2.20}$$

Note that this Maxwell field must be harmonic.

To find a full solution of the supergravity equations of motion one must start from a four-dimensional Euclidean "electrovac" solution to $U(1)$ Einstein-Maxwell theory. The metric of this solution will be the base metric of the full five-dimensional geometry, and the self- and anti-self-dual parts of the electrovac Maxwell field determine $\Theta^{(3)}$ and $\omega_-^{(3)}$. Note that both these forms must be closed, as a consequence of the Maxwell equations and Bianchi identities for $\mathcal{F}$. They will therefore automatically satisfy equations (6.2.18) and (6.2.4) under assumption (i). Conversely, given any solution to our equations, one can always repackage $\Theta^{(3)}$ and $\omega_-^{(3)}$ into a Maxwell field that satisfies (6.2.19), and obtain an electrovac solution.

Given $\Theta^{(3)}$ and $\omega_-^{(3)}$, we then need to solve the following pairs of equations:

$$\hat{\nabla}^2 Z_1 = \varepsilon \star_4 \left[\Theta^{(2)} \wedge \Theta^{(3)}\right], \qquad \big(\Theta^{(2)} - \varepsilon \star_4 \Theta^{(2)}\big) = 2\,\varepsilon\, Z_1\, \omega_-^{(3)} \,; \tag{6.2.21}$$

$$\hat{\nabla}^2 Z_2 = \varepsilon \star_4 \left[\Theta^{(1)} \wedge \Theta^{(3)}\right], \qquad \big(\Theta^{(1)} - \varepsilon \star_4 \Theta^{(1)}\big) = 2\,\varepsilon\, Z_2\, \omega_-^{(3)} \,. \tag{6.2.22}$$

Since $\Theta^{(3)}$ and $\omega_-^{(3)}$ are already known, (6.2.21) represents a coupled *linear* system for $\Theta^{(2)}$ and Z_1 and (6.2.22) represents a coupled *linear* system for $\Theta^{(1)}$ and Z_2. In solving these systems one should, of course, remember that the $\Theta^{(I)}$ should also satisfy the (linear) Bianchi identities $d\Theta^{(1)} = 0$.

Once one knows the solutions of the equations above, one must solve the equations for Z_3 and $K = dk$, which, amazingly enough, are also linear:

$$K \; + \; \varepsilon \star_4 K \;\; = \;\; \tfrac{1}{2}\,\varepsilon \sum_I \, Z_I \left(\Theta^{(I)} + \varepsilon \star_4 \Theta^{(I)}\right) , \tag{6.2.23}$$

$$\hat{\nabla}^2 Z_3 \;\; = \;\; \varepsilon \star_4 \left[\Theta^{(1)} \wedge \Theta^{(2)} \; - \; \omega_-^{(3)} \wedge (K - \varepsilon \star_4 K)\right] . \tag{6.2.24}$$

Hence, starting from an Euclidean electrovac solution one can build a full solution of five-dimensional $U(1)^3$ ungauged supergravity by following a linear procedure. Note that this class of solutions is much larger, and, as we just explained, includes the BPS and almost-BPS solutions.

Chapter 7

Solutions with an Israel-Wilson base, or how to interpolate between BPS and almost BPS solutions

In this chapter, we examine in more details the linear system of equations found in section 6.2.4 of the previous chapter, and solve it for a special class of electrovac base spaces that have a translational $U(1)$ isometry: the Israel-Wilson spaces. From a four-dimensional perspective, the particular solution we find describes a non-BPS two-centered solution where one of the centers is a locally-BPS D6-D4-D2-D0 black hole and the other center is a $\overline{\text{D6}}$ brane. We show that this new Israel-Wilson solutions interpolates between the BPS and almost BPS class of solutions, and therefore help us understanding the global structure of the solution space.

7.1 Israel-Wilson solutions

7.1.1 The Israel-Wilson background

The starting ingredient for constructing non-trivial solutions using the procedure outlined above is an Euclidean electrovac solution that satisfies (6.2.19) and that has a non-trivial harmonic form, $\omega_-^{(3)}$. An interesting choice for such a background is an Israel-Wilson (IW) metric [98, 99, 100, 101]:

$$ds_4^2 = (V_+ V_-)^{-1}(d\psi + \vec{A} \cdot d\vec{y})^2 + (V_+ V_-)(dy_1^2 + dy_2^2 + dy_3^2)\,, \tag{7.1.1}$$

where

$$\vec{\nabla} \times \vec{A} = V_- \vec{\nabla} V_+ - V_+ \vec{\nabla} V_-\,, \tag{7.1.2}$$

and the functions $V_\pm$ are required to be harmonic on the $\mathbb{R}^3$ base. Introducing the frames:

$$\hat{e}^1 \;=\; (V_+ V_-)^{-\frac{1}{2}}\,(d\psi \;+\; \vec{A} \cdot d\vec{y})\,, \qquad \hat{e}^{a+1} \;=\; (V_+ V_-)^{\frac{1}{2}}\,dy^a\,, \quad a = 1,2,3\,, \tag{7.1.3}$$

the associated background Maxwell field is given by [101]:

$$\begin{aligned} \mathcal{F} \;&\equiv\; \tfrac{1}{2}\,\mathcal{F}_{ab}\hat{e}^a \wedge \hat{e}^b \\ &=\; \big[\partial_a\big(V_+^{-1} - V_-^{-1}\big)\big]\, e^1 \wedge \hat{e}^{a+1} \;+\; \tfrac{1}{2}\,\epsilon_{abc}\big[\partial_a\big(V_+^{-1} + V_-^{-1}\big)\big] e^{b+1} \wedge \hat{e}^{c+1}\,. \end{aligned} \tag{7.1.4}$$

This background then satisfies equation (6.2.19), where we choose for the rest of this section the convention $\varepsilon = +1$.

7.1.2 Harmonic forms

Define the sets of two-forms:

$$\Omega_\pm^{(a)} \equiv \hat{e}^1 \wedge \hat{e}^{a+1} \pm \tfrac{1}{2}\epsilon_{abc}\,\hat{e}^{b+1} \wedge \hat{e}^{c+1}\,, \qquad a = 1,2,3\,. \tag{7.1.5}$$

The Maxwell field of the Israel-Wilson solution is then:

$$\mathcal{F} = \big(\partial_a\big(V_+^{-1}\big)\big)\,\Omega_+^{(a)} - \big(\partial_a\big(V_-^{-1}\big)\big)\,\Omega_-^{(a)}\,, \tag{7.1.6}$$

from which one can read off (up to an irrelevant, overall sign) the harmonic forms, $\Theta^{(3)}$ and $\omega_-^{(3)}$, using (6.2.20). However, it is easy to see that (7.1.4) is not the most general Maxwell field one can have for this base. Introducing two arbitrary harmonic functions $K_\pm$ on $\mathbb{R}^3$, the two-forms:

$$\Theta_\pm \equiv -\sum_{a=1}^{3} \big(\partial_a\big(V_\pm^{-1}\,K_\pm\big)\big)\,\Omega_\pm^{(a)} \tag{7.1.7}$$

are also harmonic and self-dual, or anti-self-dual respectively. These forms have (local) potentials:

$$B_\pm = \frac{K_\pm}{V_\pm}(d\psi + \vec{A}\cdot d\vec{y}) + \vec{a}_\pm \cdot d\vec{y}\,, \qquad \vec{\nabla}\times\vec{a}_\pm = \pm\Big(K_\pm\vec{\nabla}V_\mp - V_\mp\vec{\nabla}K_\pm\Big)\,. \tag{7.1.8}$$

From now on we choose $\varepsilon = 1$. The equations for ($\varepsilon = -1$) can be simply obtained by exchanging V_+ and V_-. The two-form, $\Theta^{(3)}$, is then self-dual while $\omega_-^{(3)}$ is anti-self-dual.

One can try to obtain a more general solution for $\Theta^{(3)}$ and $\omega_-^{(3)}$ by taking:

$$\Theta^{(3)} = d\Big(\frac{K_+}{V_+}(d\psi + \vec{A}) + \vec{a}_+\Big)\,, \qquad \omega_-^{(3)} = d\Big(\frac{K_-}{V_-}(d\psi + \vec{A}) + \vec{a}_-\Big)\,. \tag{7.1.9}$$

The Einstein-Maxwell electrovac equations (6.2.19) are then solved if, and only if

$$\partial_i\Big(\frac{K_+}{V_+}\Big)\partial_j\Big(\frac{K_-}{V_-}\Big) = \big(\partial_i V_+^{-1}\big)\,\big(\partial_j V_-^{-1}\big) \tag{7.1.10}$$

for $i, j = 1,2,3$. Hence, one can apparently obtain a more general electrovac Israel-Wilson base by using the solutions to this equation:

$$K_- = \beta\,V_- - \alpha\,, \quad K_+ = \delta\,V_+ - \gamma\,, \tag{7.1.11}$$

with $\alpha,\beta,\gamma,\delta$ constants satisfying the constraint $\alpha\gamma = 1$. However, one can easily see that β and δ are "pure gauge" constants, since they make no contribution to the Maxwell fields (7.1.7). We therefore set $\beta = 0$, which implies that K_- is constant. We could, of course, do the same with K_+, however, we will find it useful in the next sub-section to keep $\delta \neq 0$.

We should also note that the foregoing discussion no longer applies if either V_- or V_+ are constant, because the solutions to (7.1.10) are then different from those in (7.1.11). We will partially address this situation later in the paper, and we leave a more general analysis for further investigation. Given that $K_- = -\alpha$, the two-form $\omega_-^{(3)}$ is a constant multiple of the natural anti-self-dual two-form on the Israel-Wilson base space, $\big(\partial_a\big(V_-^{-1}\big)\big)\,\Omega_-^{(a)}$.

7.1.3 The linear system

We now solve the linear system for the other fields. We write $\Theta^{(1)}$ and $\Theta^{(2)}$ in the form:

$$\Theta^{(1)} \;=\; d\left(\frac{K_1}{V_+}(d\psi + A) + a_1\right)\,, \qquad \Theta^{(2)} \;=\; d\left(\frac{K_2}{V_+}(d\psi + A) + a_2\right) \tag{7.1.12}$$

where K_1, K_2, a_1 and a_2 are unknown functions and one-forms on the $\mathbb{R}^3$ base and determine the dipole charges of the solution[1]. Writing the equations (6.2.21) in the IW base, we obtain

$$\nabla^2 K_2 \;=\; \frac{2\alpha}{V_-}\,\vec\nabla\cdot\left(\frac{V_+}{V_-}Z_1\vec\nabla V_-\right)\,, \tag{7.1.13}$$

$$\nabla^2 Z_1 \;=\; V_-\nabla^2\Big(\frac{K_2K_+}{V_+}\Big) \;-\; 2\alpha\,\vec\nabla\cdot\Big(\frac{Z_1K_+}{V_-}\vec\nabla V_-\Big)\,, \tag{7.1.14}$$

with a_2 given by

$$\vec\nabla\times\vec a_2 \;=\; -V_-\vec\nabla K_2 + K_2\vec\nabla V_- + 2\alpha\,\frac{V_+}{V_-}Z_1\vec\nabla V_-\,. \tag{7.1.15}$$

The corresponding system for Z_2 and $\Theta^{(1)}$ is:

$$\nabla^2 K_1 \;=\; \frac{2\alpha}{V_-}\,\vec\nabla\cdot\left(\frac{V_+}{V_-}Z_2\vec\nabla V_-\right)\,, \tag{7.1.16}$$

$$\nabla^2 Z_2 \;=\; V_-\nabla^2\Big(\frac{K_1K_+}{V_+}\Big) \;-\; 2\alpha\,\vec\nabla\cdot\Big(\frac{Z_2K_+}{V_-}\vec\nabla V_-\Big)\,, \tag{7.1.17}$$

$$\vec\nabla\times\vec a_1 \;=\; -V_-\vec\nabla K_1 + K_1\vec\nabla V_- + 2\alpha\,\frac{V_+}{V_-}Z_2\vec\nabla V_-\,. \tag{7.1.18}$$

We also need the equation for the last warp factor Z_3 and the angular momentum k. We decompose k as usual:

$$k = \mu(d\psi + A) + \omega\,. \tag{7.1.19}$$

Equations (6.2.23) and (6.2.24) for μ and Z_3 then give

$$\begin{aligned}\nabla^2 Z_3 \;=\;& V_-\nabla^2\Big(\frac{K_1K_2}{V_+}\Big) \;-\; 2\alpha\,\vec\nabla\cdot\Big(\frac{Z_1K_1+Z_2K_2}{V_-}\vec\nabla V_-\Big)\\ &+4\alpha\frac{V_+}{V_-}\,\vec\nabla\cdot\Big(\mu\vec\nabla V_-\Big) - 2\alpha\frac{V_+Z_I}{V_-}\,\vec\nabla\cdot\Big(\frac{K_I}{V_+}\vec\nabla V_-\Big)\\ &+2\alpha^2V_+Z_1Z_2\,\nabla^2\Big(\frac{1}{V_-}\Big)\,,\end{aligned} \tag{7.1.20}$$

$$\nabla^2(V_-\mu) \;=\; \frac{1}{V_+}\vec\nabla\cdot\Big(V_-V_+Z_I\vec\nabla\Big(\frac{K_I}{V_+}\Big)\Big) - \frac{2\alpha}{V_+}\vec\nabla\cdot\Big(\frac{V_+Z_1Z_2}{V_-}\vec\nabla V_-\Big) \tag{7.1.21}$$

where ω is given by:

$$\vec\nabla\times\vec\omega = V_+^2\vec\nabla\Big(\frac{V_-}{V_+}\mu\Big) - V_+V_-Z_I\vec\nabla\Big(\frac{K_I}{V_+}\Big) + 2\alpha\frac{V_+Z_1Z_2}{V_-}\vec\nabla V_-\,. \tag{7.1.22}$$

As usual, (7.1.21) is the integrability equation for (7.1.22).

[1] We slightly change notation here from K^I to K_I because, having specified Θ_1, Θ_2 and Θ_3 differently, we will often write K^2 and want to avoid possible confusions with $(K)^2$.

7.1.4 An explicit example: a non-BPS black hole in an Israel-Wilson metric

We now have all the tools to find explicit solutions with an Israel-Wilson base space. Here we will present an M-theory solution that corresponds, in type IIA string theory, to a $\mathrm{D6D4^3D2^3D0}$ black hole in a $\overline{\mathrm{D6}}$ background. We parameterize the flat, three-dimensional $\mathbb{R}^3$ base space using spherical coordinates (r, θ, ϕ) and put the $\overline{\mathrm{D6}}$ brane at the origin of the space and the black hole at a distance R from the origin. We denote polar coordinates centered at the black hole position by (Σ, θ_Σ). Their relation to the polar coordinates (r, θ) centered at the origin is:

$$\Sigma = \sqrt{r^2 + R^2 - 2rR\cos\theta}\,, \qquad \cos\theta_\Sigma = \frac{r\cos\theta - R}{\Sigma}\,. \tag{7.1.23}$$

For $V_+ = 1$, we want the space to be Taub-NUT, and thus we take V_- to be

$$V_- = 1 + \frac{Q_{\bar{6}}}{r}\,. \tag{7.1.24}$$

The parameter, $Q_{\bar{6}}$, is $\overline{\mathrm{D6}}$ or the $\overline{\mathrm{KK}}$-monopole charge of the space[1]. The function, K_+, is harmonic and corresponds to one of the M5 charges of the solution:

$$K_+ = K_3 = \frac{d_3}{\Sigma}\,. \tag{7.1.25}$$

For convenience, we will change notation throughout the rest of the chapter, and refer to K_- as K_3. The relation (7.1.11) then forces V_+ to have a pole at the black hole location. Assuming the space to be asymptotically flat (asymptotic to $\mathbb{R}^3 \times S^1$) means that the constant in V_+ to be finite, and we set it to 1 for convenience. Hence,

$$V_+ = 1 + \alpha K_+ = 1 + \frac{\alpha\, d_3}{\Sigma} \equiv 1 + \frac{Q_6}{\Sigma}\,, \tag{7.1.26}$$

where α was introduced in (7.1.11), and we have defined $Q_6 \equiv \alpha d_3$. Thus, the black hole has a finite D6 (or KKm) charge. The associated vector fields are

$$\begin{aligned} A &= Q_6 \frac{r\cos\theta - R}{\Sigma} d\phi + Q_6\, Q_{\bar{6}}\, \frac{r - R\cos\theta}{\Sigma} d\phi - Q_{\bar{6}}\cos\theta d\phi\,, && (7.1.27)\\ a_3 &= -d_3 \frac{r\cos\theta - R}{\Sigma} d\phi - Q_{\bar{6}}\, d_3\, \frac{r - R\cos\theta}{\Sigma} d\phi && (7.1.28) \end{aligned}$$

The system (7.1.13) and (7.1.14) is not completely straightforward to solve, but, as explained in the previous section, it is linear in the unknowns K_2 and Z_1. We find the following solution:

$$\begin{aligned} Z_1 &= \frac{1}{V_+}\left(1 + \frac{Q_1}{\Sigma} + \frac{d_2 d_3}{\Sigma^2}\left(1 + \frac{Q_{\bar{6}} r}{R^2}\right)\right)\,, && (7.1.29)\\ K_2 &= V_+\left(\frac{d_2}{\Sigma} - \alpha\, \frac{Z_1}{V_-}\right)\,, && (7.1.30) \end{aligned}$$

[1] We will explain below why we refer to this as $\overline{\mathrm{D6}}$ and not as D6 charge.

and similarly for Z_2 and K_1. Here we have also introduced the dipole charge d_2 associated to K_2, and the electric charge Q_1 of the hole, associated to Z_1. The vector field a_2 is then given by

$$a_2 = -(d_2 - \alpha Q_1)\frac{r\cos\theta - R}{\Sigma}d\phi - Q_{\bar{6}}\, d_2\, \frac{r - R\cos\theta}{\Sigma}d\phi + d_2 Q_6 Q_{\bar{6}}\frac{\cos\theta}{\Sigma^2}d\phi\,, \tag{7.1.31}$$

with a similar expression for a_1.

The solution to the last system of equations, (7.1.20) and (7.1.21), is:

$$\begin{aligned}
\mu &= \frac{1}{V_+V_-}\Big(\frac{m}{\Sigma} + \frac{\tilde{m}}{r} + \frac{V_-(d_1+d_2+d_3)}{2\Sigma} + \frac{Q_I d_I}{2\Sigma^2} + Q_{\bar{6}}Q_I d_I\frac{\cos\theta}{2R\Sigma^2} \\
&\quad + \frac{C_{IJK}}{6}\, d_I d_J d_K\Big[\Big(1+\frac{Q_{\bar{6}}^2}{R^2}\Big)\Big(\frac{r\cos\theta}{R\,\Sigma^3} + \lambda\frac{r\cos\theta - R}{R\,\Sigma^3}\Big) + Q_{\bar{6}}\frac{3r^2+R^2}{2R^2 r\,\Sigma^3}\Big]\Big) \\
&\quad -\alpha\frac{Z_1Z_2}{V_-}\,,
\end{aligned}\tag{7.1.32}$$

$$Z_3 = V_+\left(1 + \frac{Q_3}{\Sigma} + \frac{d_1d_2}{\Sigma^2}\Big(1+\frac{Q_{\bar{6}}r}{R^2}\Big)\right) - 2\alpha V_+\mu - \alpha^2\frac{V_+Z_1Z_2}{V_-}\,, \tag{7.1.33}$$

and

$$\begin{aligned}
\omega &= -\Big[\kappa - m\frac{r\cos\theta - R}{\Sigma} - \tilde{m}\cos\theta + \frac{d_1+d_2+d_3}{2}\frac{r\cos\theta - R}{\Sigma} + Q_{\bar{6}}\frac{d_1+d_2+d_3}{2}\frac{r - R\cos\theta}{R\Sigma} \\
&\quad + Q_{\bar{6}}Q_I d_I\frac{r\sin^2\theta}{2R\,\Sigma^2} + \Big(1+\frac{Q_{\bar{6}}^2}{R^2}\Big)\frac{C_{IJK}}{6}\, d_I d_J d_K(1+\lambda)\frac{r^2\sin^2\theta}{R\,\Sigma^3} \\
&\quad + Q_{\bar{6}}\frac{C_{IJK}}{6}\, d_I d_J d_K\frac{r(3R^2+r^2) - R(3r^2+R^2)\cos\theta}{2R^3\,\Sigma^3}\Big]d\phi\,.
\end{aligned}\tag{7.1.34}$$

The constants m, $\tilde{m}$, κ and λ represent homogeneous solutions that are fixed by regularity:

$$\begin{aligned}
m &= \Big(1+\frac{Q_{\bar{6}}}{R}\Big)\frac{d_1+d_2+d_3}{2} + \frac{C_{IJK}}{6}\frac{Q_{\bar{6}}\, d_I d_J d_K}{2R^3}\,, \\
\tilde{m} &= \kappa = -Q_{\bar{6}}\Big(\frac{d_1+d_2+d_3}{2R} + \frac{C_{IJK}}{6}\frac{d_I d_J d_K}{2R^3}\Big)\,. \\
\lambda &= -\frac{R^2}{R^2+Q_{\bar{6}}^2}\,.
\end{aligned}\tag{7.1.35}$$

The reason for this regularity conditions will become clear in the section 7.2.2.

7.1.5 The BPS and almost-BPS limits of solutions with an Israel-Wilson base

The solution presented here seems to be somewhat complicated, but its physical interpretation is rather straightforward. Take first V_+ to be 1 (by setting the parameter α to zero). As we already remarked, the metric then becomes the usual Taub-NUT metric, with negative orientation. Looking at the complete solution, we see that it becomes the non-BPS black ring

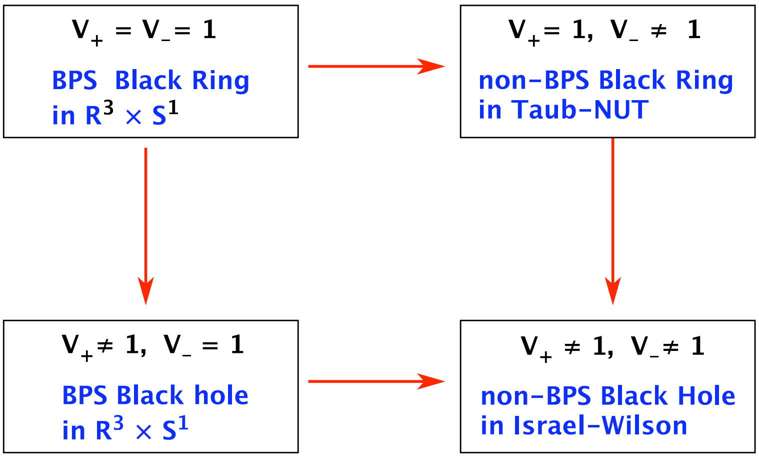

Figure 7.1: *This diagram represents four classes of solutions that can be obtained from our solution for various values of the Israel-Wilson harmonic functions. When both V_+ and V_- are constant, the solution describes a BPS black string in $\mathbb{R}^3 \times S^1$. Turning on a $\overline{KKm}$ charge at the center of the space ($V_- \neq 1$) the space becomes Taub-NUT, the black ring is non-BPS and the solution belongs to the almost-BPS Ansatz. Turning on a KKm charge at the location of the ring, we obtain a BPS D6-D4-D2-D0 black hole. Turning on both types of KKm charges ($V_+ \neq 1$,$V_- \neq 1$), we obtain the more general non-BPS solution constructed here: a D6-D4-D2-D0 four-charge black hole in a* $\overline{\text{D6}}$ *background.*

in Taub-NUT solution presented in chapter 5[1]. From a four-dimensional perspective this is a two-center solution where one center is a D4-D2-D0 black hole located at $z = R$ and the other is a pure $\overline{\text{D6}}$ brane located at $r = 0$. Despite the fact that both objects are locally-BPS, the relative orientation of the $\overline{\text{D6}}$ and the three D2 branes (which determine, locally, the Killing spinors of the D4-D2-D0 black hole) makes the full configuration non-BPS.

On the other hand, setting $Q_{\bar{6}}$ to zero, and hence $V_- = 1$, one can see that the solution becomes one of the BPS solution presented in the first chapter of the thesis, section 1.3, and describes a four-dimensional black hole with D6, D4, D2 and D0 charges. The singular part in the D4 harmonic function can be traded, via the gauge transformation (1.3.16) [44, 58] for a non-trivial Wilson line at infinity, and thus this black hole is in fact a BMPV black hole located at the tip of Taub-NUT (which is now located at at $z = R$ because we set $Q_{\bar{6}} = 0$), or a single-center D6-D2-D0 black hole from a four-dimensional perspective. For this solution the relative orientation of the D6 and D2 branes match, and the solution preserves four supercharges. These two limits are summarized in the Figure 7.1. If one now takes both the D6 and the $\overline{\text{D6}}$ charges to zero, the solution becomes a BPS D4-D2-D0 four-dimensional black hole, which lifts to a BPS three-charge three-dipole-charge black string in $\mathbb{R}^{3,1} \times S^1$ [62].

Having taken these limits, it is now clear that the general solution with an Israel-Wilson base describes a two-center configuration, where one of the centers has D6,D4,D2, and D0 charges and is locally-BPS, and the other has $\overline{\text{D6}}$ charge. Of course, an Israel-Wilson solution with

[1] In chapter 5, the solution is written in the $\varepsilon = -1$ convention, ie $V = V_+$ and not the $\varepsilon = +1$ one we use here. One can rewrite the solution in the new convention by taking $V = V_-$, and changing the signs of the base-space vectors.

multiple D6 branes of opposite orientations is only possible when other charges and fluxes are turned on (6.2.19). Indeed, the D6 and $\overline{\text{D6}}$ charges attract each other and in the absence of other branes, there is nothing to balance this attraction. Introducing D4, D2 and D0 branes creates new interactions: the D4 branes are also attracted, the D2's feel no force, and the D0's are repelled, and thus balance becomes possible.

Note that upon flipping the sign of ε one could also obtain a solution where the D6 charge becomes anti-D6 charge and vice versa; this solution should describe an intrinsically non-BPS $\overline{\text{D6}}$-D2-D0 black hole in a background of a D6 brane that is mutually-BPS with respect to the three sets of D2 branes. When the $\overline{\text{D6}}$ charge is zero the solution should become a BPS black ring in Taub-NUT [64, 65, 51] and when the D6 charge is zero it should becomes the almost-BPS non-rotating $\overline{\text{D6}}$-D2-D0 black hole [29, 86, 106]. When both the $\overline{\text{D6}}$ and the D6 charges are zero, this solution should reduce again to the D4-D2-D0 BPS black hole whose five-dimensional lift is the M5-M2-P black string (or the infinite black ring) of [62].

7.2 Spectral Flow and the Israel-Wilson metric

In this section we study the action of a spectral flow transformation [58] on the solution describing an almost-BPS black ring in Taub-NUT given in chapter 5, and show that it yields the $\varepsilon = 1$ solution with the Israel-Wilson base found in the previous section. We also argue that all the solutions that are constructed starting from a Euclidean electrovac solution given by the Israel-Wilson metric can be generated by the spectral flow of a more-standard "almost-BPS" solution.

7.2.1 The D1-D5-P duality frame

We already presented the spectral flow transformation in the particular case of the BPS class of solutions of section 1.3. But this transformtion is much more general [58]. It is a very useful tool for generating new asymptotically $\mathbb{R}^{3,1} \times S^1$ solutions of five-dimensional $U(1)^3$ ungauged supergravity (or of the STU model in four dimensions) by starting from other such solutions. In asymptotically $AdS_3 \times S^3$ spaces this transformation is the gravity counterpart of a symmetry of the dual CFT, and it is most naturally performed upon dualizing the solution to the D1-D5-P duality frame [102, 103], presented in section 1.3.3. In this frame the solution, which is invariant along the four internal directions wrapped by the D5 branes, corresponds to a solution of six-dimensional ungauged supergravity [104, 105]. The spatial section of the metric can be written as a T^2 fibration over an $\mathbb{R}^3$ base, where the T^2 is made up by the fiber of the Taub-NUT base space and by the (internal) direction common to the D1 and the D5 branes. Spectral flows can then be recast as simply a subgroup of the group, $SL(2, \mathbb{Z})$, of global diffeomorphisms on this T^2. Thus, from a six-dimensional point of view, spectral flow is just a change of coordinates, mixing two different $U(1)$'s. However, upon dualizing back to the duality frame where the charges correspond to three sets of M2 branes, the resulting solution, which is again a solution of $U(1)^3$ supergravity, differs rather non-trivially from the original one[1].

[1]Note that to go from a six-dimensional supergravity solution to the final solution of five-dimensional supergravity one does not KK reduce the six-dimensional solution; rather one trivially uplifts it to a solution

To perform a spectral flow we need to find the metric and RR gauge-field of the solution dualized to the D1-D5-P duality frame. This was presented in section 1.3.3. For completeness, we remaind that this yields

$$\begin{aligned} ds^2 &= -\frac{1}{\sqrt{Z_1 Z_2} Z_3}(dt+k)^2 + \frac{Z_3}{\sqrt{Z_1 Z_2}}(dy + A^3)^2 + \sqrt{Z_1 Z_2} ds_4^2 + \sqrt{\frac{Z_2}{Z_1}} \sum_{a=1}^{4} dx_a^2 \,, && (7.2.1) \\ C^{(2)} &= A^1 \wedge (dy + A^3) + B^{(1)} \wedge \frac{dt+k}{Z_3} + \gamma_2 \,, && (7.2.2) \end{aligned}$$

where

$$A^I = -\frac{dt+k}{Z_I} + B^{(I)} \,, \tag{7.2.3}$$

and

$$d\gamma_2 = \star_4 dZ_2 - B^{(1)} \wedge \Theta^{(3)} \,. \tag{7.2.4}$$

For convenience we take $\varepsilon = 1$; the result for the other sign is equally straightforward to obtain.

7.2.2 The action of spectral flow

We start from the solution of the "almost BPS" equation presented in sectionringsec, corresponding to a non-BPS black ring and thus we assume that the base metric ds_4^2 has GH form:

$$ds_4^2 = V_-^{-1}(d\psi + \vec{A}.d\vec{y})^2 + V_- ds_3^2 \,, \quad \vec{\nabla} \times \vec{A} = -\vec{\nabla} V_- \,. \tag{7.2.5}$$

The one-form potentials are:

$$B^{(I)} = K_I(d\psi + A) + a_I \,, \quad k = \mu(d\psi + A) + \omega \,, \tag{7.2.6}$$

where $I = 1, 2, 3$, the K_I are harmonic and the a_I satisfy the equation:

$$\vec{\nabla} \times \vec{a}_I = -V_- \vec{\nabla} K_I + K_I \vec{\nabla} V_- \,. \tag{7.2.7}$$

In order to perform the spectral flow, we also need to decompose the two-form, γ_2, as

$$\gamma_2 = (d\psi + A) \wedge \gamma_2^{(\psi)} + \gamma_2^{(b)} \,, \tag{7.2.8}$$

where $\gamma_2^{(b)}$ is a two form on the three-dimensional space defined by ds_3^2.

Note that the equation for Z_2,

$$d \star_4 dZ_2 = \Theta^{(1)} \wedge \Theta^{(3)} \,, \tag{7.2.9}$$

implies

$$\begin{aligned} \nabla^2 Z_2 &= V_- \nabla^2 (K_1 K_3) = \vec{\nabla}.(V_- \vec{\nabla}(K_1 K_3)) - \vec{\nabla} V_- . \vec{\nabla}(K_1 K_3) \\ &= \vec{\nabla}.(V_- \vec{\nabla}(K_1 K_3) + \vec{A} \times \vec{\nabla}(K_1 K_3)) && (7.2.10) \end{aligned}$$

of type IIB supergravity, performs three T-dualities, then uplifts the resulting solution to M-theory, and then reads off the new solution of five-dimensional ungauged supergravity

and hence
$$\vec{\nabla} Z_2 = V_- \vec{\nabla}(K_1 K_3) + \vec{A} \times \vec{\nabla}(K_1 K_3) + \vec{\nabla} L_2 \,. \tag{7.2.11}$$
The equation satisfied by γ_2
$$\vec{\nabla} \times \vec{\gamma}_2^{(\psi)} = \vec{\nabla} Z_2 - V_- K_1 \vec{\nabla} K_3 + \vec{a}_1 \times \vec{\nabla} K_3 = \vec{\nabla} L_2 - \vec{\nabla} \times (K_3 \vec{a}_1 + K_1 K_3 \vec{A}) \,, \tag{7.2.12}$$
implies
$$\gamma_2^{(\psi)} = -K_3 a_1 - K_1 K_3 A + \hat{\gamma}_2^{(\psi)} \text{ with } \vec{\nabla} \times \vec{\hat{\gamma}}_2^{(\psi)} = \vec{\nabla} L_2 \,. \tag{7.2.13}$$
Similarly one can define a two-form
$$\gamma_1 = (d\psi + A) \wedge \gamma_1^{(\psi)} + \gamma_1^{(b)} \,, \tag{7.2.14}$$
that satisfies
$$d\gamma_1 = \star_4 dZ_1 - B^{(2)} \wedge \Theta^{(3)} \,. \tag{7.2.15}$$
One has
$$\vec{\nabla} \times \vec{\gamma}_1^{(\psi)} = \vec{\nabla} Z_1 - V_- K_2 \vec{\nabla} K_3 + a_2 \times \vec{\nabla} K_3 = \vec{\nabla} L_1 - \vec{\nabla} \times (K_3 \vec{a}_2 + K_2 K_3 \vec{A}) \tag{7.2.16}$$
which implies
$$\gamma_1^{(\psi)} = -K_3 a_2 - K_2 K_3 A + \hat{\gamma}_1^{(\psi)} \text{ with } \vec{\nabla} \times \vec{\hat{\gamma}}_1^{(\psi)} = \vec{\nabla} L_1 \,. \tag{7.2.17}$$

Spectral flow mixes the internal $U(1)$ coordinate y, associated with the momentum charge, with the GH fiber, ψ. Explicitly, this is just the change of coordinates
$$\psi \to \psi + \alpha\, y \,. \tag{7.2.18}$$
To find the transformation of the metric coefficients, one performs the change of coordinates (7.2.18) and rewrites the metric and gauge field in the exact same form as (7.2.2). Defining the harmonic function V_+ by
$$V_+ = 1 + \alpha K_3, \tag{7.2.19}$$
the transformed metric is
$$ds_4^2 \;=\; (V_+ V_-)^{-1} (d\psi + \vec{\widetilde{A}}.d\vec{y})^2 + V_+ V_- ds_3^2 \,, \qquad \widetilde{A} \;=\; A - \alpha a_3. \tag{7.2.20}$$
Note that $\widetilde{A}$ now satisfies:
$$\vec{\nabla} \times \vec{\widetilde{A}} = V_- \vec{\nabla} V_+ - V_+ \vec{\nabla} V_- . \tag{7.2.21}$$
The rest of the fields can be recast in the exact same form as before, with the new coefficients (obtained after a fair amount of of algebra) given by:
$$\widetilde{K}_1 = K_1 - \alpha\, \frac{Z_2}{V_+ V_-} \,, \quad \widetilde{K}_2 = K_2 - \alpha\, \frac{Z_1}{V_+ V_-} \,, \quad \widetilde{K}_3 = \frac{K_3}{V_+} \,, \tag{7.2.22}$$
$$\widetilde{a}_1 = V_+\, a_1 + \alpha\, \gamma_2^{(\psi)} \,, \quad \widetilde{a}_2 = V_+\, a_2 + \alpha\, \gamma_1^{(\psi)} \,, \quad \widetilde{a}_3 = a_3 \tag{7.2.23}$$
$$\widetilde{Z}_1 = \frac{Z_1}{V_+} \,, \quad \widetilde{Z}_2 = \frac{Z_2}{V_+} \,, \quad \widetilde{Z}_3 = V_+ Z_3 - 2\,\alpha\,\mu + \alpha^2 \frac{Z_1 Z_2}{V_+ V_-} \tag{7.2.24}$$
$$\widetilde{\mu} = \frac{1}{V_+} \Big(\mu - \alpha\, \frac{Z_1 Z_2}{V_+ V_-} \Big) \,, \quad \widetilde{\omega} = \omega \,. \tag{7.2.25}$$

This is exactly the solution with an Israel-Wilson base constructed in Section 7.1. In particular, the relation (7.2.19), which is the same as (7.1.26), between the harmonic function V_+ corresponding to the D6 charge and one of the harmonic functions corresponding to D4 charge emerges directly from the spectral flow transformation.

While this approach to obtaining solutions is rather different from the one outlined in Section 7.1 in that it does not involve starting from a non-trivial Einstein-Maxwell electrovac solution but from a Ricci-flat metric, the resulting solution is the same. This greatly simplifies the regularity analysis, as we know that spectral flow always transforms regular solutions into regular solutions. Hence the regularity of the D6-D4-D2-D0 black hole is ensured by the regularity of the non-BPS black ring in Taub-NUT, which yields the regularity conditions outlined in the previous section (7.1.35).

7.2.3 Spectral flow and smooth horizonless multi-center solutions.

One of the driving forces in our effort to construct large classes of multi-center non-BPS solutions is to obtain smooth horizonless solutions that have the same charges and mass as non-BPS black holes with a macroscopically-large horizon area. For BPS black holes, the existence of large classes of such solutions brings considerable support to the fact that these black holes should be thought of as statistical ensembles of horizonless configurations, thus realizing the fuzzball proposal (see [6] for reviews) for this class of black holes. We would like to extend this to non-BPS black holes.

The most obvious way to look for such non-BPS multi-center horizonless solutions is to use the almost-BPS Ansatz. However, in this Ansatz the anti-self-dual two-forms that one can turn on (for example the harmonic forms dual to the two-cycles of a multi-center Taub-NUT space) source strongly singular solutions to the equations of motion. Hence, at least at first glance, no smooth horizonless solutions exist.

Another way to obtain smooth horizonless solutions is to use spectral flow. It is well known that in the appropriate IIB frame a two-charge supertube with D1 and D5 charges corresponds to a completely regular geometry. Furthermore, using spectral flow, we can change coordinates and then dualize a BPS solution containing such a supertube in a multi-Taub-NUT space into a completely regular multi-Taub-NUT five-dimensional solution with fluxes supported on bubbles [58]. On the other hand, a solution with multiple supertubes of different types (with different dipole charges) cannot simultaneously be dualized via one spectral flow to a smooth geometry. This needs to be done by three subsequent spectral flows, which transform every type of supertube into a Taub-NUT center. Since the near-tube geometry is the same in a BPS and in an almost-BPS solution, we expect the spectral flow to transform multiple supertubes in an almost-BPS solution into a smooth non-BPS horizonless solution with multiple distinct fluxes supported on bubbles.

To illustrate this, consider a single supertube in a Taub-NUT geometry of "opposite orientation." That is, the base space is of the form (7.2.5) while the supertube magnetic dipoles are given by (7.2.6). If $K_1 = K_2 = 0$ this supertube has only one dipole charge, and it can be arranged to give a completely regular geometry in six dimensions. However, as explained in the previous chapter, section 5.4, even if this solution is written as an almost-BPS solution, it still preserves four supersymmetries[1]. One can now perform a spectral flow on this solution exactly as in Section 7.2.2 and obtain a floating-brane solution with an Israel-Wilson base that has V_- unchanged and V_+ given by (7.2.19). The spectral flow transformation preserves the regularity of the solution and replaces the supertube by a fluxed D6 brane, which is also perfectly regular considered either as either a six-dimensional *or as a five-dimensional bubbled geometry.* Hence, one obtains the smooth D6-$\overline{\text{D6}}$ solution with non-trivial flux described above[2].

One can take this procedure further, and consider two or three different types of supertube in a GH geometry of the opposite orientation. Unlike the single supertube, this solution is no longer BPS, as the holonomy of the base metric is inconsistent with the supersymmetry projections associated with all the supertubes (the solution has three D2 and one $\overline{\text{D6}}$ charge).

If one now makes several spectral flows to convert each species of supertube to fluxes supported by geometry, the result must be regular for exactly the same reason that the BPS supertubes produce regular five-dimensional geometries after spectral flow: The almost-BPS supertubes are locally identical to BPS supertubes and so the spectral flow cannot generate singularities. The result of such a multiple spectral flow must therefore be a completely regular, non-BPS geometry with fluxes in five dimensions. We expect that these solutions will go well beyond the Israel-Wilson class: Indeed, the metric coefficients of the base will generically involve products of more than two functions. We also expect this method to yield large classes of smooth horizonless non-BPS scaling solutions, which will be instrumental in extending the fuzball proposal to non-BPS extremal black holes.

[1]Essentially because the supertube only has two D2 charges, that are mutually-BPS with respect to a D6 brane irrespective of its orientation.

[2]It is worth commenting on how much supersymmetry this solution preserves. On one hand, we have obtained this solution by spectral flow from a supersymmetric solution. Since spectral flow is a combination of coordinate transformations and dualities, one would expect the resulting solution to still be supersymmetric. On the other hand, the resulting solution has an Israel-Wilson base, and, as proved in [42, 41], such solutions should not be supersymmetric, since all supersymmetric solutions must have a Hyper-Kähler base. The resolution of the puzzle is a generalization of that described in [101]: For these very special solutions the warp factors and angular momentum vector are such that if one makes a coordinate transformation of the type $\psi \to \psi + \alpha t$, and rewrites the metric as a time fibration over a four-dimensional base, this base space can be made hyper-Kähler. Hence this particular floating-brane solution is secretly BPS.

Chapter 8

Bolt solutions

In this last chapter, we present new smooth, horizonless solutions of the floating brane equations of chapter 6. The starting point of these solutions is a four-dimensional base space that, instead of being hyper-Kähler, is a Euclidean four-dimensional black hole. Euclideanizing a black hole transforms the event horizon into a regular bolt, an S^2 at the center of the space. One can then put some magnetic fluxes on this bolt and use it as a base to construct our five-dimensional solution. We use this method to produce five-dimensional smooth solutions starting from the Euclidean Schwarzschild, Kerr-Taub-Bolt, dyonic Reissner-Nordström and Dyonic Kerr-Newman black hole. While the first two are Ricci-flat, the others are solutions to the electrovac Einstein-Maxwell equations. We finally show that if the Euclidean black hole is rotating, one can assume the base space to be ambipolar and still get a completely regular solution.

8.1 Solution with a Ricci-flat base: Euclidean Schwarzschild and Kerr-Taub-Bolt

Both the BPS and the almost-BPS solutions are given by taking the base to be hyper-Kähler with a self-dual curvature and then solving the linear system presented in chapter 6 [40, 85]:

$$\Theta^{(I)} \;=\; \varepsilon \star_4 \Theta^{(I)}\,, \tag{8.1.1}$$

$$\hat{\nabla}^2 Z_I \;=\; \tfrac{1}{2}\,\varepsilon\, C_{IJK} \star_4 [\Theta^{(J)} \wedge \Theta^{(K)}]\,, \tag{8.1.2}$$

$$dk \;+\; \varepsilon \star_4 dk \;=\; \varepsilon\, Z_I \Theta^{(I)}\,. \tag{8.1.3}$$

As we explained in chapter 6, one can obtain solutions to the equations of motion simply by solving the BPS system (8.1.1)–(8.1.2) with any Ricci-flat base metric on ds_4^2:

$$\hat{R}_{ab} \;=\; 0\,. \tag{8.1.4}$$

The most obvious such base is the Euclidean Schwarzschild metric, which we use in the next subsection to generate new solutions. In subsection 8.1.2, we will extend this to the more general Kerr-Taub-bolt solution.

8.1.1 Adding fluxes to Euclidean Schwarzschild

The Euclidean Schwarzschild metric is given by:

$$ds_4^2 \;=\; \Big(1-\frac{2m}{r}\Big)\,d\tau^2 \;+\; \Big(1-\frac{2m}{r}\Big)^{-1}dr^2 \;+\; r^2 d\theta^2 \;+\; r^2\sin^2\theta\,d\phi^2\,. \tag{8.1.5}$$

It is, of course, Ricci flat, and if one restricts to the region $r \geq 2m$ then the metric is globally regular provided one periodically identifies the Euclidean time by:

$$\tau \;\equiv\; \tau \;+\; 8\pi m\,. \tag{8.1.6}$$

Near $r = 2m$ the manifold is then locally $\mathbb{R}^2 \times S^2$ and at infinity it is asymptotic to $\mathbb{R}^3 \times S^1$. The "bolt" at the origin can be given a magnetic flux and we can take the ε-self-dual harmonic two-forms to be:

$$\Theta^{(I)} \;=\; q_I\,\Big(\frac{1}{r^2}\,d\tau\wedge dr \;+\; \varepsilon\,\sin\theta\,d\theta\wedge d\phi\Big)\,, \tag{8.1.7}$$

for some magnetic charges, q_I. The $B^{(I)}$'s associated potentials, verifying $\Theta^{(I)} = dB^{(I)}$, are

$$B^{(I)} = q_I\,\Big(\frac{1}{r}\,d\tau \;-\; \varepsilon\,\cos\theta\,d\phi\Big)\,. \tag{8.1.8}$$

With this flux it is trivial to solve the second equation (8.1.2) and one finds

$$Z_I \;=\; 1 \;-\; \tfrac{1}{2}\,C_{IJK}\,\frac{q_J q_K}{m}\,\frac{1}{r}\,. \tag{8.1.9}$$

We have chosen the homogeneous solution so as to exclude all other electric sources for Z_I and to arrange that $Z_I \to 1$ as $r \to \infty$.

The last equation (8.1.3) is equally elementary, and setting

$$k \;=\; \mu\,d\tau \;+\; \nu\,d\phi \tag{8.1.10}$$

we find

$$\mu \;=\; (\varepsilon+\alpha)\,(q_1+q_2+q_3)\,\frac{1}{r} \;-\; \frac{3\varepsilon}{2m}\,q_1q_2q_3\,\frac{1}{r^2} \;+\; \gamma\,, \tag{8.1.11}$$

$$\nu \;=\; \alpha(q_1+q_2+q_3)(\beta+\cos\theta)\,, \tag{8.1.12}$$

where α, β and γ parameterize homogeneous solutions. These parameters must be chosen to remove closed time-like curves (CTC's) in (6.1.6). First, to avoid CTC's on ϕ-circles one must make sure that there are no Dirac strings in k, and hence $\alpha = \beta = 0$. Similarly, there are potential CTC's around the small τ circles near $r = 2m$ unless we choose γ so that $\mu = 0$ at $r = 2m$. Thus we must take $\nu = 0$ and

$$\mu \;=\; \varepsilon(q_1+q_2+q_3)\,\Big(\frac{1}{r}-\frac{1}{2m}\Big) \;-\; \frac{3\varepsilon}{2m}\,q_1q_2q_3\,\Big(\frac{1}{r^2}-\frac{1}{4m^2}\Big)\,. \tag{8.1.13}$$

The solution is now completely determined but it still remains to verify the absence of CTC's elsewhere. On the constant time slices the metric in the τ direction is $\mathcal{M}\,d\tau^2$ where $\mathcal{M} \equiv Z^{-2}r^{-4}\mathcal{Q}$ and

$$\mathcal{Q} \;\equiv\; r^4 Z_1Z_2Z_3\Big(1-\frac{2m}{r}\Big) \;-\; \mu^2 r^4\,. \tag{8.1.14}$$

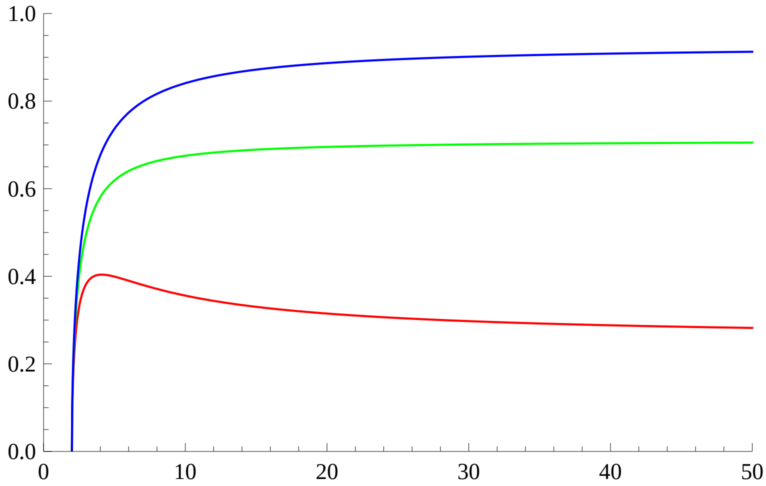

Figure 8.1: *Plot of the scale,* $\sqrt{\mathcal{M}}$*, of the compactification circle as a function of* r/m*. The three plots, from top to bottom, correspond to* q/m *of* $1/4$*,* $1/2$ *and* $3/4$*. Note that as one approaches the upper bound (8.1.17) the circle does not grow uniformly but attains a maximum scale before decreasing asymptotically.*

This is a quartic function of r and must remain non-negative for $2m \le r < \infty$ and this places constraints on m and the q_I. In addition, the Z_I should remain positive definite for $r > 2m$.

To examine these conditions in more detail we simplify the analysis by taking $q_I = q > 0$, $I = 1, 2, 3$. The positivity of the Z_I for $r > 2m$ means that one must have:

$$q \; < \; \sqrt{2}\, m \,. \tag{8.1.15}$$

To analyze the positivity of $\mathcal{M}$, we first look at the behavior at infinity, where one has

$$\mathcal{M} \;\sim\; r^{-4}\mathcal{Q} \;\sim\; \Big(1 - \frac{3q}{8m^3}(q^2 - 4m^2)\Big)\Big(1 + \frac{3q}{8m^3}(q^2 - 4m^2)\Big)\,. \tag{8.1.16}$$

For this to be positive, the two cubics in q must be positive and this implies the stronger condition:

$$0 \;<\; \frac{q}{m} \;<\; \frac{4}{\sqrt{3}}\sin\frac{\pi}{9} \;\approx\; 0.78986\,. \tag{8.1.17}$$

Note that the function $\mathcal{M}$, and hence the condition above, do not depend on ε. More generally, the quartic that sets the scale of the τ-circle is:

$$\mathcal{Q} \;\equiv\; (r-2m)\Big[\Big(r - \frac{q^2}{m}\Big)^3 \;-\; \frac{9q^2}{64m^6}(r-2m)\big((q^2-4m^2)r + 2mq^2\big)^2\Big]\,. \tag{8.1.18}$$

One can verify that this is indeed positive definite for $2m < r < \infty$ for q in the range (8.1.17).

It is interesting to note that (8.1.13) shows that μ asymptotes to a finite value as $r \to \infty$. One can undo the rotation of this frame by shifting $\tau \to \tau + at$ and the condition (8.1.17) simply reflects the fact that this rotation is sub-luminal.

We have thus created a "magnetized bolt" solution in which fluxes have been added to a pre-existing two-cycle. It is interesting to note that in the BPS "bubbled" solutions of

[44, 45, 66, 55], presented in 1.4.3, the fluxes were an essential part of blowing up the two-cycles and these bubbles would collapse without the fluxes. Another element of the BPS bubbled solutions was the presence of an ambi-polar base metric where the metric on the four-dimensional base changes sign but this sign change is canceled in the five-dimensional metric by a simultaneous sign change in warp factor, Z. In more physical terms, there is also a direct D-brane interpretation of the bubbling transition [44, 55].

The solution constructed here does not appear to have such a D-brane interpretation and does not involve an ambi-polar base. One could try to see whether the ranges of parameters or the range of r might be extended to give an ambi-polar four-dimensional base that still yields a smooth Lorentzian five-dimensional solution. There are obvious possibilities, like taking $m < 0$ and trying to extend to $r < 0$ but such extensions do not lead to an *overall* sign change in (8.1.5) and so cannot be canceled by the warp-factor Z. Thus, at least for this solution, the standard Euclidean Bolt is simply decorated by fluxes to give a running Bolt solution with electric and magnetic charges. We shall see later that there are richer possibilities once angular momentum and a NUT charge are included.

Some Remarks on the Mass of the Running Bolt

The computation of the asymptotic charges, mass and angular momentum is resented in Appendix E. In this Appendix, we show that the mass M_0 and the charges Q^I of this bolt solution are related through

$$M_0 \equiv (1-\gamma^2)^{-1/2}(M - \gamma Q_e) = \frac{\pi}{4G_5}\left(16m^2 + \frac{\varepsilon}{4\pi^2}(Q^1 + Q^2 + Q^3)\right) . \tag{8.1.19}$$

As we argued in chapter 6, the solutions in the floating brane ansatz have their mass being linear with the charges. Nevertheless, the bolt in the middle of the space gives a new contribution to the mass, which becomes an affine function of the charges. This extra contribution has really to be seen as a solitonic contribution, and is important for the following reason: if one sees the running bolt solution as a microstate geometry corresponding to a black hole, the mass of this black hole is not equal to the sum of the charges, and therefore will not be extremal. In this sense, this new bolt solution is a non extremal solution.

More precisely, for $\varepsilon = 1$, equation (8.1.19) indicates that the total rest mass is simply the sum of the mass of the uncharged bolt and the masses corresponding to the M2 branes. Hence, if one could ascribe a putative solitonic charge to the uncharged bolt, this formula would look very much like the mass of a BPS object. Furthermore, the fact that the M2 brane charge enters linearly in the total mass is also consistent with the fact that a probe M2 brane feels no force in this background.

For $\varepsilon = -1$ the situation is even more interesting. The mass now decreases linearly with increasing the M2 charge. Hence, the mass formula is still linear, but the sign in front of the M2 charges is negative! We are not aware of any other such mass formula in the literature. One might object to this by noting that one can always flip the sign of M2 charges by reversing some orientations; however, by flipping the signs of some of the q_I one can change the sign of some of the M2 charges. Hence the total mass can either decrease or increase with increasing the mass corresponding to the M2 charges. Alternatively, if one absorbes ε by an orientation change, then the mass formula (3.29) is linear in the charges, and not in their absolute values as for BPS systems.

The fact that the mass of the solutions can decrease with increasing charge and the fact that M2 brane probes feel no force may lead one to believe naively that one could violate energy conservation: one can bring an M2 brane adiabatically from the infinity to the core of a solution, and the resulting solution will have a lower mass than the sum of the masses of the two pre-merger components. However, this does not happen. The charge of the M2 brane probe that feels no force is oriented oppositely to the M2 brane charge of this solution! Bringing in this probe brane actually decreases the total M2 charge and therefore increases the mass of the solution to the mass of the soliton plus the mass of the probe M2 one brought in adiabatically, as expected.

Our analysis thus indicates that the uncharged bolt is the middle point of a family of magnetized solutions that can have both larger and smaller rest masses, and that moreover these masses can grow or decrease linearly with the M2 charges of the solution.

8.1.2 Adding fluxes to the Kerr-Taub-Bolt solution

The Euclidean Schwarzschild solution admits a very interesting generalization with additional quantum numbers: The Euclidean Kerr-Taub-Bolt solution [107]. In this section we add fluxes to the bolt and get a more general regular five-dimensional running Bolt solution. We will for convenience fix the convention $\varepsilon = +1$ in the rest of the chapter.

8.1.3 The Euclidean Kerr-Taub-Bolt Solution

The four-dimensional metric is

$$ds_4^2 \;=\; \Xi\Big(\frac{dr^2}{\Delta} \;+\; d\theta^2\Big) \;+\; \frac{\sin^2\theta}{\Xi}\,(\alpha d\tau + P_r d\phi)^2 \;+\; \frac{\Delta}{\Xi}\,(d\tau + P_\theta d\phi)^2\,, \tag{8.1.20}$$

where

$$\begin{aligned} \Delta \;&\equiv\; r^2 - 2mr - \alpha^2 + N^2\,, \qquad & \Xi \;&\equiv\; P_r - \alpha P_\theta \;=\; r^2 - (N + \alpha\cos\theta)^2\,, \\ P_r \;&\equiv\; r^2 - \alpha^2 - \frac{N^4}{N^2-\alpha^2}\,, \qquad & P_\theta \;&\equiv\; -\alpha\sin^2\theta + 2N\cos\theta - \frac{\alpha N^2}{N^2-\alpha^2}\,. \end{aligned} \tag{8.1.21}$$

This is a Ricci-flat metric where m is the mass, α is the angular momentum and N is the NUT charge. If the metric is to be regular then these parameters are not independent, as we will see in the following. At infinity the metric behaves as

$$ds_4^2 \;\sim\; dr^2 \;+\; r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \;+\; \big(d\tilde\tau - (\alpha\sin^2\theta + 2N(1-\cos\theta))\, d\phi\big)^2\,, \tag{8.1.22}$$

with

$$\tilde\tau \;\equiv\; \tau \;+\; 2N\phi \;-\; \frac{\alpha N^2}{N^2-\alpha^2}\,\phi\,. \tag{8.1.23}$$

Thus the metric is asymptotic to $\mathbb{R}^3 \times S^1$ provided that ϕ has period 2π and the fibration of τ over the two-sphere in $\mathbb{R}^3$ is regular if τ is identified under shifts:

$$\tau \;\equiv\; \tau \;+\; 8N\pi\,. \tag{8.1.24}$$

We now need to examine regularity at the points where some of the metric coefficients vanish. First, at $\theta = 0, \pi$, the circle with $d\tau = -P_\theta d\phi$ pinches off. Substituting $d\tau = -P_\theta d\phi$ into the metric and ignoring the radial terms gives a metric:

$$\Xi\Big[d\theta^2 \;+\; \frac{1}{\Xi^2}\sin^2\theta\,\big((P_r - \alpha P_\theta)\,d\phi\big)^2\Big] \;=\; \Xi\big[d\theta^2 \;+\; \sin^2\theta\,d\phi^2\big]\,, \tag{8.1.25}$$

which is perfectly regular.

The second degeneracy appears at $\Delta = 0$. Define $r_\pm$ by $\Delta = (r - r_+)(r - r_-)$ with $r_+ > r_-$:

$$r_\pm \;=\; m \;\pm\; \sqrt{m^2 - N^2 + \alpha^2}\,. \tag{8.1.26}$$

Since we are interested in Euclidean black-hole solutions with non-trivial bolts, we will consider the situation where these roots are real:

$$m^2 + \alpha^2 \geq N^2\,. \tag{8.1.27}$$

We have to arrange that the metric is regular at $r \to r_+$ and then restrict to $r \geq r_+$. As usual, this will lead to a periodic identification in the τ coordinate. Before proceeding with the analysis here, it is worth noting that in the usual analysis of the Kerr-Taub-Bolt metric [107], one requires the metric to be positive definite and hence one requires $\Xi > 0$ and hence $m > |N|$.

However, if one's purpose is to construct five-dimensional solutions, the four-dimensional base can be ambi-polar and hence Ξ *can* be allowed to change sign. We will indeed find that the warp factors also change sign to compensate for this and give a regular five-dimensional solution. Thus we will not impose that $m > |N|$, but only (8.1.27).

To explore regularity at $r = r_+$ it is useful to define:

$$P_{r+} \;\equiv\; P_r|_{r=r_+} \;=\; r_+^2 - \alpha^2 - \frac{N^4}{N^2 - \alpha^2}\,, \qquad \kappa \;\equiv\; \left|\frac{r_+ - r_-}{2P_{r+}}\right|. \tag{8.1.28}$$

The circle that pinches off at $r = r_+$ has $d\phi = -\alpha d\tau / P_{r+}$. Substituting this into the metric and expanding in $x = (r - r_+)^{1/2}$ one obtains:

$$\frac{4\,\Xi}{r_+ - r_-}\Big[dx^2 \;+\; \kappa^2 x^2 d\tau^2 \;+\; \tfrac{1}{4}(r_+ - r_-)\,d\theta^2\Big]\,. \tag{8.1.29}$$

This is regular as $x \to 0$ provided that τ is periodically identified according to:

$$\tau \;\equiv\; \tau \;+\; \frac{2\pi}{\kappa}\,, \tag{8.1.30}$$

and hence the base space is smooth if

$$\kappa \;=\; \frac{1}{4|N|}\,. \tag{8.1.31}$$

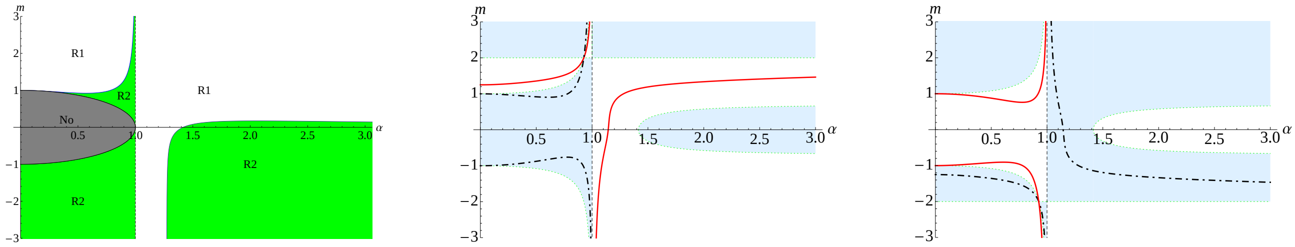

Figure 8.2: *The three graphs are plots of m versus α, in units in which $N = 1$ (this choice can always be made because the equations are homogeneous). The first shows the regions where P_{r+} is either positive (R1, in white) or negative (R2, in green). The grey area, No, is forbidden by the reality condition, (8.1.27). The second and the third graph show the solutions of (8.1.32) and (8.1.33). In the shaded (blue) areas the square root in equation (8.1.31) is equal to a negative expression. Hence, the solutions that belong to these regions, or to the regions of the first graph where P_{r+} has the wrong sign, do not obey (8.1.31) and are "wrong branch" solutions. These solutions are represented using dotted lines, while the physical solutions, that obey (8.1.31), are represented using continuous lines and belong to the white areas.*

This condition, together with (8.1.27) are the two necessary condition for absence of conical singularities in the base. Hence, the ambi-polar Kerr-Taub-Bolt metrics that we will use to generate running Bolt solutions depend on only two independent parameters.

We now explore the implications of equation (8.1.31) for the allowed range of N, m and α. Since this equation only involves $|N|$, one can use the definition of κ to see that the sign of N is irrelevant; hence we will assume in the following that N is positive.

One can now square the square roots in (8.1.31), and obtain a constraint that is cubic in m. This constraint depends on the sign of P_{r+}: if P_{r+} is positive, we have

$$\begin{aligned} &16N(N^2-\alpha^2)^2m^3 - 4(5N^6 - 8\alpha^2N^4 + 2\alpha^4N^2 + \alpha^6)m^2 \\ &\quad -16N(N^2-\alpha^2)^3m + 20N^8 - 52\alpha^2N^6 + 49\alpha^4N^4 - 16\alpha^6N^2 \;=\; 0\,, \end{aligned} \tag{8.1.32}$$

while if P_{r+} is negative equation (8.1.31) implies:

$$\begin{aligned} &-16N(N^2-\alpha^2)^2m^3 - 4(5N^6 - 8\alpha^2N^4 + 2\alpha^4N^2 + \alpha^6)m^2 \\ &\quad +16N(N^2-\alpha^2)^3m + 20N^8 - 52\alpha^2N^6 + 49\alpha^4N^4 - 16\alpha^6N^2 \;=\; 0\,, \end{aligned} \tag{8.1.33}$$

which is the same as (8.1.32) but with $m \to -m$. Note that a solution to (8.1.32) or (8.1.33) is not automatically a solution to (8.1.31). This only happens when two conditions are satisfied: first, P_{r+} must be respectively positive or negative; second, before squaring the square root in (8.1.31) one has to insure that the expression to which this square root is equal is positive. Hence, the cubic equations (8.1.32) and (8.1.33) contain "wrong branch" solutions, that do not solve (8.1.31).

The details of the parameter ranges are shown in Figure 8.2. The first graph depicts the regions in which P_{r+} is positive or negative, and thus where we have to solve, respectively, (8.1.32) or (8.1.33). For $\alpha < 1$, each cubic has three real roots; for (8.1.32) two of them are solutions to (8.1.31), and one lies on the "wrong branch;" for (8.1.33) two of them lie on the

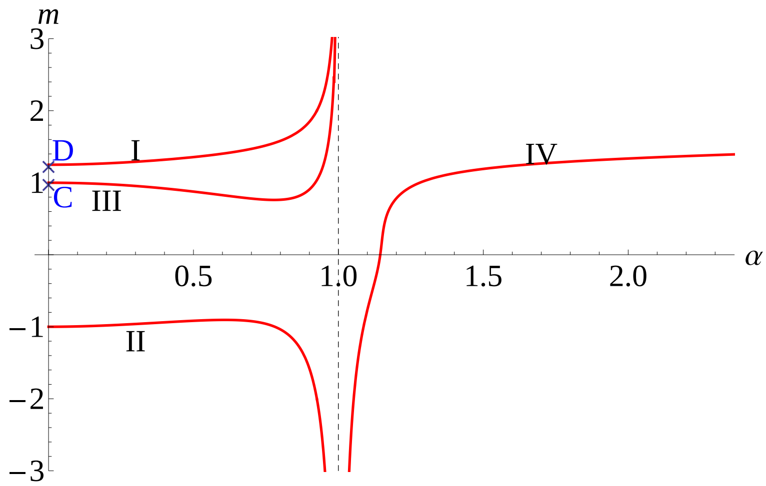

Figure 8.3: *Plot of the values of m that give physical solutions of (8.1.31) for a given α, in units in which $N = 1$. The solution has four disconnected branches: Branches I ,II and III go from $\alpha = 0$ to $\alpha = 1$, diverging as α approaches 1. Branch IV starts from $-\infty$ as $\alpha \to 1_+$ and approaches $m = 2$ as $\alpha \to \infty$. The intercepts, C and D, correspond, respectively, to the Taub-NUT and Taub-Bolt metrics.*

"wrong branch." For $\alpha > 1$, there is one real root to the cubics, and the only physical solution is the one with $P_{r+} > 0$.

The complete solution to (8.1.31) is shown on Figure 8.3.

We would like to note that our analysis does not completely agree with the discussion in [107]. Indeed [107] only analyzes solutions with positive P_{r+}, and thus misses some of the ambi-polar branches of the five-dimensional solution. Furthermore, for $\alpha < 1$ the solutions found and plotted in [107] do not have quite the same shape as the ones in Figure 8.3[1].

At $\alpha = 0$ one obtains two interesting particular solutions: the Taub-NUT solution, for $m = |N|$, and the Taub-Bolt solution of [108] for $m = 5/4N$. It is worth noting that allowing the metric to be ambi-polar (see section 8.3) extends the range of parameters significantly. Indeed, forcing the four-dimensional metric to have a signature $(+,+,+,+)$ imposes $m > |N|$, and thus would forbid the complete branch II, and part of branches III and IV (see Figure 8.3).

Maxwell Fields on the Kerr-Taub-Bolt space

Introduce frames:

$$\begin{aligned} \hat{e}^1 &= \left(\frac{\Xi}{\Delta}\right)^{\frac{1}{2}} dr\,, & \hat{e}^2 &= \Xi^{\frac{1}{2}} d\theta\,, \\ \hat{e}^3 &= \frac{\sin\theta}{\Xi^{\frac{1}{2}}}(\alpha d\tau + P_r d\phi)\,, & \hat{e}^4 &= \left(\frac{\Delta}{\Xi}\right)^{\frac{1}{2}}(d\tau + P_\theta d\phi)\,, \end{aligned} \tag{8.1.34}$$

and define the self-dual and anti-self-dual two-forms by:

$$\Omega_\pm = \frac{1}{(r \mp (N + \alpha\cos\theta))^2}\left[\hat{e}^1 \wedge \hat{e}^4 \pm \hat{e}^2 \wedge \hat{e}^3\right]. \tag{8.1.35}$$

[1]We believe this discrepancy can be most easily explained by evolving Moore's law backwards in time.

These forms are harmonic and have potentials satisfying $dA_\pm = \Omega_\pm$, of the form [109]:

$$A_\pm \;=\; \mp\cos\theta\, d\phi \;-\; \frac{1}{(r\mp(N+\alpha\cos\theta))}\,(d\tau + P_\theta d\phi)\,. \qquad (8.1.36)$$

In the rest of this section we we will focus on the self-dual Maxwell fields and take:

$$\Theta^{(I)} \;=\; q_I\,\Omega_+\,. \qquad (8.1.37)$$

The extension to anti-self-dual Maxwell fields is rather straightforward.

Solving the second equation (8.1.2) yields:

$$Z_I \;=\; 1 \;-\; \tfrac{1}{2}\,C_{IJK}\,\frac{q_J q_K}{(m-N)}\,\frac{1}{(r-(N+\alpha\cos\theta))}\,. \qquad (8.1.38)$$

We have, once again, chosen the homogeneous solution so as to exclude all singular electric sources for Z_I and to arrange that $Z_I \to 1$ as $r \to \infty$. Notice that the denominator of Z_I is one of the factors of Ξ and, if $N \geq 0$, both this denominator and Ξ will change sign when r is small[1]. This suggests that the five-dimensional metric could be regular when the base space is ambi-polar.

The angular momentum vector

Solving the last equation (8.1.3) is a little non-trivial and we find it convenient to make the Ansatz:

$$k \;=\; \mu\,(d\tau + P_\theta\, d\phi) \;+\; \nu\, d\phi\,, \qquad (8.1.39)$$

and solve the system for μ and ν. We find that this system may be recast in terms of a single function, F, for which:

$$\Xi\,\mu \;-\; \alpha\,\nu \;=\; \Delta\partial_r F\,, \qquad \nu \;=\; \sin\theta\,\partial_\theta F\,. \qquad (8.1.40)$$

The equation satisfied by F is:

$$\begin{aligned}\partial_r(\Delta\partial_r F) \;+\; \frac{1}{\sin\theta}\,\partial_\theta(\sin\theta\,\partial_\theta F) - \frac{2}{(r-(N+\alpha\cos\theta))}\,(\Delta\partial_r F + \alpha\sin\theta\,\partial_\theta F) \qquad (8.1.41)\\ =\; (q_1+q_2+q_3)\frac{(r+N+\alpha\cos\theta)}{(r-(N+\alpha\cos\theta))} \;-\; \frac{3\,q_1q_2q_3}{(m-N)}\,\frac{(r+N+\alpha\cos\theta)}{(r-(N+\alpha\cos\theta))^2}\,.\end{aligned}$$

Upon solving this equation we find the following solution for the angular momentum vector:

$$\mu \;=\; \gamma\left[1 \;-\; \frac{2\,N}{(r+N+\alpha\cos\theta)}\right] \;-\; (q_1+q_2+q_3)\,\frac{r}{\Xi} \qquad (8.1.42)$$

$$\qquad +\; \frac{q_1q_2q_3}{2(m-N)^2}\left[\frac{m-N-2\,\alpha\cos\theta}{\Xi} \;+\; \frac{2(m-N)}{(r-(N+\alpha\cos\theta))^2}\right], \qquad (8.1.43)$$

$$\nu \;=\; \gamma\,\alpha\,\sin^2\theta \;-\; \frac{\alpha\,q_1q_2q_3}{(m-N)^2}\,\frac{\sin^2\theta}{(r-(N+\alpha\cos\theta))}\,, \qquad (8.1.44)$$

[1] We will see later that the other factor in Ξ never vanishes for $N > 0$. If $N < 0$ one can produce a similar result by starting with an anti-self-dual flux.

where γ is an arbitrary constant that multiplies terms coming from the homogeneous solution of (8.1.42). As with the Schwarzschild solution, we suppress homogeneous solutions that lead to Dirac strings (and hence CTC's) in the ϕ direction.

The parameter, γ, is now fixed by making sure that there are no CTC's near $r = r_+$. Since Δ vanishes at $r = r_+$, one can make the spatial part of the metric vanish by moving on the circle with $d\phi = -\alpha P_r^{-1} d\tau$. It follows that, to avoid CTC's, the angular momentum vector, k, must vanish on this circle at $r = r_+$ This means that we must impose $\Xi\mu - \alpha\nu = 0$ at $r = r_+$, for any value of θ. This would follow from the first equation in (8.1.40) provided that F has no singularity at $\Delta = 0$. However, F generically has terms proportional to $\log \Delta$. On the other hand, the homogeneous solution to (8.1.42) (that yields the terms proportional to γ in (8.1.44)) also contains such terms:

$$F_{hom} \;=\; \gamma(r - \alpha\cos\theta) + \frac{(m-N)}{\sqrt{m^2 - N^2 + \alpha^2}}(r_+ \log(r - r_+) - \, r_- \log(r - r_-))\,. \qquad (8.1.45)$$

Hence, we can cancel the singular behavior at $r = r_+$ by choosing the coefficient of the homogeneous solution:

$$\gamma \;=\; \frac{(q_1 + q_2 + q_3)}{2\,(m-N)} \;+\; \frac{q_1 q_2 q_3}{4\,(m-N)^3}\left[\frac{m+N}{r_+} \;-\; 2\right]. \qquad (8.1.46)$$

The full non-singular solution than has:

$$\begin{aligned}
\mu \;&=\; (\Delta + \alpha^2 \sin^2\theta)\left[\frac{(q_1 + q_2 + q_3)}{2\,(m-N)\,\Xi} \;+\; \frac{3\,q_1 q_2 q_3}{2\,(m-N)^3\,\Xi}\left(\frac{N}{r_+} - \frac{m-N}{2\,r} - 1\right)\right. \\
&\qquad\qquad \left. -\; \frac{q_1 q_2 q_3}{2\,(m-N)^2}\,\frac{1}{r\,(r - (N + \alpha\cos\theta))^2}\right] \\
&\qquad -\; \frac{3\,q_1 q_2 q_3}{4\,(m-N)^2}\left(1 \;+\; \frac{2\,N}{(r - (N + \alpha\cos\theta))}\right)\left(\frac{1}{r} \;-\; \frac{1}{r_+}\right) \qquad (8.1.47) \\
\nu \;&=\; \alpha\left[\frac{(q_1 + q_2 + q_3)}{2\,(m-N)} \;-\; \frac{q_1 q_2 q_3}{(m-N)}\left(\frac{1}{(r - (N + \alpha\cos\theta))}\right.\right. \\
&\qquad\qquad \left.\left. +\; \frac{1}{4\,(m-N)^2}\left(\frac{m+N}{r_+} \;-\; 2\right)\right)\right]\sin^2\theta\,. \qquad (8.1.48)
\end{aligned}$$

Note that at $r = r_+$, this solution is proportional to $\sin^2\theta$.

The asymptotic charges of the solution are computed in appendix E. As for the non-rotating case, one can see that the mass is an affine function of the four charges (with the NUT charge), the extra part being a solitonic contribution from the bolt.

$$M_0 \equiv (1 - \gamma^2)^{-1/2}(M - \gamma Q_e) = \frac{4\pi N}{G_5}(m + N) + Q_m + \frac{1}{16\pi G_5}\,(Q_1 + Q_2 + Q_3)\,. \qquad (8.1.49)$$

8.2 Solution with an Einstein-Maxwell base: Euclidean Reissner-Nordström and Kerr-Newman

In the previous section we constructed five-dimensional solutions using Ricci-flat four-dimensional Euclidean black holes: the Schwarzschild and Kerr-Taub-Bolt black holes. It is rather natural to try to extend this construction to black holes with electric charges, such as the Euclidean Reissner-Nordström, and ultimately dyonic Kerr-Newman black holes. However in this case, the base is not Ricci-flat anymore, but a solution of the Einstein-Maxwell "electrovac" equations. Fortunately, the analysis of the equations of motion that we performed in chapter 6 gave us a the generalisation of the equations (6.2.14)-(6.2.17) in the case of an electrovac bases, (6.2.19)-(6.2.24). We already solved this system in the case of an Israel-Wilson metric, in chapter 7, but we can use it here in the case of charges Euclidean black holes.

We first quickly recall the useful definitions and relevant equations derived in chapter 6: the two-forms $\omega_-^{(I)}$ are defined from the magnetic field strength by

$$\frac{1}{2}\left(\Theta^{(I)} - \star_4\Theta^{(I)}\right) \equiv C_{IJK} Z_J \omega_-^{(K)} , \tag{8.2.1}$$

and we assume

$$dk + \star_4 dk = \frac{1}{2}\sum_I Z_I \left(\Theta^{(I)} + \star_4\Theta^{(I)}\right) , \qquad \text{and} \qquad \omega_-^{(1)} = \omega_-^{(2)} = 0 . \tag{8.2.2}$$

The four-dimensional base space has to be a solution of Euclidean Einstein-Maxwell theory[1] with (symbols with a ˆ live on the four-dimensional base)

$$\hat{R}_{\mu\nu} = \frac{1}{2}\left(F_{\mu\rho}F_\nu^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right) , \tag{8.2.3}$$

and

$$F = \Theta^{(3)} - \omega_-^{(3)} . \tag{8.2.4}$$

The rest of the equations of motion reduce to

$$\hat{\nabla}^2 Z_1 = \star_4(\Theta^{(2)} \wedge \Theta^{(3)}) , \qquad (\Theta^{(2)} - \star_4\Theta^{(2)}) = 2Z_1\,\omega_-^{(3)} , \tag{8.2.5}$$

$$\hat{\nabla}^2 Z_2 = \star_4(\Theta^{(1)} \wedge \Theta^{(3)}) , \qquad (\Theta^{(1)} - \star_4\Theta^{(1)}) = 2Z_2\,\omega_-^{(3)} , \tag{8.2.6}$$

$$\hat{\nabla}^2 Z_3 = \star_4[\Theta^{(1)} \wedge \Theta^{(2)} - \omega_-^{(3)} \wedge (dk - \star_4 dk)] , \tag{8.2.7}$$

$$dk + \star_4 dk = \frac{1}{2}\sum_{I=1}^{3} Z_I(\Theta^{(I)} + \star_4\Theta^I) . \tag{8.2.8}$$

We remind the reader that we fixed the convention to $\varepsilon = +1$. An important point about this system of equations is that it can be solved in a linear fashion. In order to do that, one has to solve the equations in the right order. The starting point is to choose a four-dimensional metric and its associated two-form field strength that solve (8.2.3). Then using (8.2.4) one can read off $\Theta^{(3)}$ and $\omega_-^{(3)}$ from the field strength. Knowing these fields, (8.2.5) and (8.2.6) become systems of two linear coupled equations for Z_1 and $\Theta^{(2)}$ and Z_2 and $\Theta^{(1)}$ respectively. Finally, k and Z_3 are solutions to the system of linear equations (8.2.7) and (8.2.8). We will show in the next subsections how to solve these equations starting from the Euclidean Reisner-Nordström and Euclidean Kerr-Newman-NUT backgrounds.

[1] The normalization of the flux in this equation is the one of chapter 6, but note that it differs from the standard convention of the general relativity community.

8.2.1 Solutions with Euclidean Reissner-Nordström base

The four-dimensional background

Our starting point in this section will be the Euclidean dyonic Reissner-Nordström background [110]

$$ds_4^2 = \left(1 - \frac{2m}{r} + \frac{p^2 - q^2}{r^2}\right) d\tau^2 + \left(1 - \frac{2m}{r} + \frac{p^2 - q^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ , \quad (8.2.9)$$

$$F = \frac{2q}{r^2}\, d\tau \wedge dr + 2p \sin\theta \, d\theta \wedge d\phi \,, \quad (8.2.10)$$

where m corresponds to the mass, q to the electric charge and p to the magnetic charge of the solution. This background solves the four-dimensional Einstein equations (8.2.3). It is useful to rewrite the metric as

$$ds_4^2 = \frac{(r - r_+)(r - r_-)}{r^2} d\tau^2 + \frac{r^2}{(r - r_+)(r - r_-)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \,. \quad (8.2.11)$$

The constants $r_\pm$ are the Euclidean analogs of the inner and outer horizon of the Reissner-Nordström black hole

$$r_\pm = m \pm \sqrt{m^2 - p^2 + q^2} \,. \quad (8.2.12)$$

To render $r_\pm$ real we restrict to the range of parameters[1] $m^2 > p^2 - q^2$. Near the outer horizon one can set

$$r = r_+ + \frac{r_+ - r_-}{4r_+^2}\, \rho^2 \,, \qquad \chi = \frac{r_+ - r_-}{2r_+^2}\, \tau \,, \quad (8.2.13)$$

and rewrite the metric as

$$ds_{NH}^2 = d\rho^2 + \rho^2 d\chi^2 + r_+^2(d\theta^2 + \sin^2\theta d\phi^2) \,, \quad (8.2.14)$$

which means that for a regular solution we should restrict to $r \geq r_+$ and the coordinate τ should be made periodic

$$\tau \sim \tau + \frac{4\pi r_+^2}{r_+ - r_-} \,. \quad (8.2.15)$$

With this identification the metric is asymptotic to $\mathbb{R}^2 \times S^2$ for $r \to r_+$ (i.e. we have a bolt of radius r_+ [111]) and to $\mathbb{R}^3 \times S^1$ for $r \to \infty$. The angles θ and ϕ are the coordinates on S^2. In the next section we will solve the equations of motion of $\mathcal{N} = 2$ five-dimensional supergravity with this Euclidean metric as a base space.

[1] The case $m^2 = p^2 - q^2$ corresponds to the extremal Euclidean Reissner-Nordström black hole. We discuss this case in Appendix D.

The five-dimensional supergravity solution

A convenient set of frames on the four-dimensional base is given by

$$\hat{e}^1 = \left(1 - \frac{2m}{r} + \frac{p^2 - q^2}{r^2}\right)^{1/2} d\tau\,. \qquad \hat{e}^2 = \left(1 - \frac{2m}{r} + \frac{p^2 - q^2}{r^2}\right)^{-1/2} dr\,, \tag{8.2.16}$$

$$\hat{e}^3 = r\, d\theta\,, \qquad \hat{e}^4 = r \sin\theta\, d\phi\,, \tag{8.2.17}$$

and the usual self-dual and anti-self-dual two-forms are

$$\Omega_\pm = \hat{e}^1 \wedge \hat{e}^2 \pm \hat{e}^3 \wedge \hat{e}^4\,. \tag{8.2.18}$$

With this in hand it is easy to show that

$$\Theta^{(3)} = \frac{p+q}{r^2}\,\Omega_+\,, \qquad \omega_-^{(3)} = \frac{p-q}{r^2}\,\Omega_-\,. \tag{8.2.19}$$

It will be useful to have the explicit expression for the potential $B^{(3)}$ satisfying $\Theta^{(3)} = dB^{(3)}$

$$B^{(3)} = \frac{(p+q)}{r}\, d\tau - (p+q)\cos\theta\, d\phi\,. \tag{8.2.20}$$

The solution to equations (8.2.5) and (8.2.6) is

$$Z_1 = 1 - \frac{2q_2(p+q)}{m}\,\frac{1}{r}\,, \qquad Z_2 = 1 - \frac{2q_1(p+q)}{m}\,\frac{1}{r}\,, \tag{8.2.21}$$

$$\Theta^{(1)} = f_1(r)\Omega_+ + g_1(r)\Omega_-\,, \qquad \Theta^{(2)} = f_2(r)\Omega_+ + g_2(r)\Omega_-\,, \tag{8.2.22}$$

where

$$f_1 = \frac{2q_1}{r^2} - \frac{2q_1(p^2-q^2)}{m\,r^3}\,, \qquad f_2 = \frac{2q_2}{r^2} - \frac{2q_2(p^2-q^2)}{m\,r^3}\,, \tag{8.2.23}$$

$$g_1 = \frac{(p-q)}{r^2} - \frac{2q_1(p^2-q^2)}{m\,r^3}\,, \qquad g_2 = \frac{(p-q)}{r^2} - \frac{2q_2(p^2-q^2)}{m\,r^3}\,. \tag{8.2.24}$$

Note that with these functions $f_I(r)$ and $g_I(r)$ one can show that $d\Theta^{(I)} = 0$, which means that locally one can express $\Theta^{(1)}$ and $\Theta^{(2)}$ in terms of potential one-forms, $\Theta^{(I)} = dB^{(I)}$. Explicitly, these one-forms are

$$B^{(I)} = K_I\, d\tau + a_I\,, \tag{8.2.25}$$

with

$$K_1 = \frac{2q_1 + p - q}{r} - \frac{2q_1(p^2-q^2)}{m\,r^2}\,, \qquad a_1 = (-2q_1 + p - q)\cos\theta\, d\phi\,, \tag{8.2.26}$$

$$K_2 = \frac{2q_2 + p - q}{r} - \frac{2q_2(p^2-q^2)}{m\,r^2}\,, \qquad a_2 = (-2q_2 + p - q)\cos\theta\, d\phi\,. \tag{8.2.27}$$

To solve (8.2.7) and (8.2.8), we will use the Ansatz

$$k = \mu(r)d\tau + \nu(\theta)d\phi\,. \tag{8.2.28}$$

One can then show that

$$\nu(\theta) = \nu_0 + \xi\cos\theta\,, \tag{8.2.29}$$

with ν_0 and ξ constants. Then the problem reduces to a system of two coupled *linear* ordinary differential equatons for $\mu(r)$ and $Z_3(r)$

$$\frac{d\mu}{dr} = -\left(\frac{\xi}{r^2} + Z_1f_1 + Z_2f_2 + \frac{p+q}{r^2}Z_3\right)\,, \tag{8.2.30}$$

$$\hat{\nabla}^2 Z_3 = 2\left(f_1f_2 - g_1g_2 + \frac{\xi(p-q)}{r^4} - \frac{(p-q)}{r^2}\frac{d\mu}{dr}\right)\,. \tag{8.2.31}$$

A solution to these equations is given by

$$Z_3 = 1 - \left(\frac{4q_1q_2(m^2-p^2+q^2)}{m^3} + \frac{2(p-q)(q+q_1+q_2)}{m}\right)\frac{1}{r} + \frac{4q_1q_2(p^2-q^2)}{m^2}\frac{1}{r^2}\,, \tag{8.2.32}$$

$$\begin{aligned}\mu = {} & (p+q+2(q_1+q_2))\left(\frac{1}{r} - \frac{1}{r_+}\right)\\ & -\left(\frac{2q_1q_2(p+q)(3m^2-p^2+q^2)}{m^3} + \frac{(p^2-q^2)(q+2q_1+2q_2)}{m}\right)\left(\frac{1}{r^2} - \frac{1}{r_+^2}\right)\\ & +\frac{4q_1q_2(p^2-q^2)(p+q)}{m^2}\left(\frac{1}{r^3} - \frac{1}{r_+^3}\right)\,.\end{aligned} \tag{8.2.33}$$

To arrive at this particular solution we have chosen

$$\nu_0 = \xi = 0\,, \qquad \rightarrow \qquad \nu = 0\,, \tag{8.2.34}$$

which ensures that there are no closed time-like curves (CTCs) coming from the $d\phi^2$ term in the five-dimensional metric, at $\theta = 0, \pi$. We have also chosen the additive constant in the solution for μ such that $\mu(r_+) = 0$, which ensures the absence of CTCs near the bolt. This implies that μ has a non vanishing value γ at infinity,

$$\begin{aligned}\lim_{r\to\infty}\mu = \gamma \equiv {} & -\frac{1}{r_+}(p+q+2(q_1+q_2))\\ & +\frac{1}{r_+^2}\left(\frac{2q_1q_2(p+q)(3m^2-p^2+q^2)}{m^3} + \frac{(p^2-q^2)(q+2q_1+2q_2)}{m}\right)\\ & -\frac{1}{r_+^3}\frac{4q_1q_2(p^2-q^2)(p+q)}{m^2}\,,\end{aligned} \tag{8.2.35}$$

this will be important in the calculation of the asymptotic charges of the five-dimensional solution. Note also that we have set the constants terms in Z_I to 1 by which we fix the asymptotic values of the scalar fields[1].

[1] In an eleven-dimensional uplift of our solution this choice will fix the asymptotic volumes of the two-cycles of T^6.

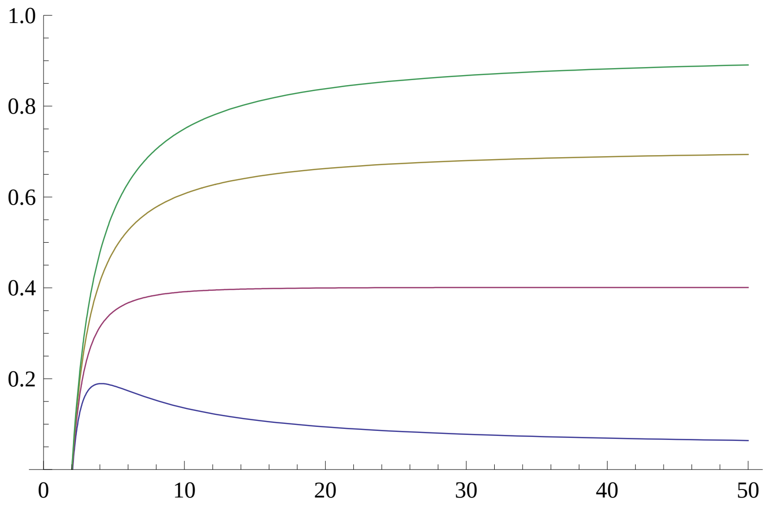

Figure 8.4: *$\mathcal{M}$ as a function of $\rho = r/r_+$ for four different values of Q/m. The curves correspond to $Q/m = (0.1, 0.2, 0.3, 0.4)$ from top to bottom.*

An important difference between this solution and the magnetized Euclidean Schwarzschild solution of the previous section is that the fluxes here are not self-dual. It is clear that if we set

$$q = p = \frac{\tilde{q}_3}{2}\,, \qquad q_1 = \frac{\tilde{q}_1}{2}\,, \qquad q_2 = \frac{\tilde{q}_2}{2}\,, \tag{8.2.36}$$

we will recover the five-dimensional solution based on the Euclidean Schwarzschild black hole[1].

An important step in the analysis of the five-dimensional solution constructed above is to ensure the global absence of CTCs. This means that for constant time slices one should make sure that the coefficient of $d\tau^2$ in the five-dimensional metric is non-negative and all Z_I are positive definite. To analyze this condition in an explicit example we will take

$$q = q_1 = q_2 = Q > 0\,, \qquad p = \frac{Q}{2}\,. \tag{8.2.37}$$

Then we have

$$r_\pm = m \pm \sqrt{m^2 + \frac{3Q^2}{4}}\,, \tag{8.2.38}$$

and the condition that Z_1 and Z_2 are positive for $r \geq r_+$ imposes

$$0 < \frac{Q}{m} < \frac{\sqrt{3}}{2} \approx 0.8660\,. \tag{8.2.39}$$

Requiring that Z_3 is positive for $r > r_+$ leads to

$$0 < \frac{Q}{m} \lessapprox 0.7783\,, \tag{8.2.40}$$

[1]Note that all q_I of the previous section should be identified with $\tilde{q}_I$, this is due to the different conventions in the normalization of the fluxes.

which is clearly a stronger constraint. Finally we have to make sure that the coefficient of $d\tau^2$ is non-negative

$$\mathcal{M} \equiv \frac{1}{r^2(Z_1Z_2Z_3)^{2/3}}[Z_1Z_2Z_3(r-r_+)(r-r_-)-\mu^2r^2] \geq 0\,. \tag{8.2.41}$$

Expanding this expression for $r \to \infty$ we find a sextic algebraic inequality in Q/m, which can be solved numerically. The allowed range of parameters coming from this constraint is

$$0 < \frac{Q}{m} \lessapprox 0.4118\,, \qquad\qquad 0.8811 \lessapprox \frac{Q}{m} \lessapprox 1.2587\,. \tag{8.2.42}$$

The bottom line is that for the choice of parameters (8.2.37) the five-dimensional solution is completely regular and there are no CTCs (globally) if

$$0 < \frac{Q}{m} \lessapprox 0.4118\,. \tag{8.2.43}$$

Some plots of $\mathcal{M}$ for different values of Q/m are presented in figure 8.4. We have performed a detailed numerical analysis for a number of other choices for the parameters (p, q, q_1, q_2) and the conclusions are qualitatively the same. Namely, there is a region in parameter space in which the five-dimensional solution is regular and has no global CTCs.

The relation between the mass and the charges of the solution, as shown in appendix E, is

$$M_0 = \frac{1}{16\pi G_5}\left(\frac{32\pi^2 r_+^2 m}{r_+ - r_-} + Q_1 + Q_2 + Q_3\right)\,. \tag{8.2.44}$$

It is again linear in the charges, with an extra solitonic contribution.

8.2.2 Adding rotation and NUT charge

The four-dimensional background

We finally make a last generalization to include an angular momentum parameter α and a NUT charge N. The metric and the two-form flux are

$$\begin{aligned} ds_4^2 &= \frac{\Sigma}{\Delta}\,dr^2 + \Sigma\,d\theta^2 + \frac{\sin^2\theta}{\Sigma}\,(\alpha d\tau + P_r d\phi)^2 + \frac{\Delta}{\Sigma}\,(d\tau + P_\theta d\phi)^2\,, && (8.2.45)\\ F &= \frac{p+q}{[r-(N+\alpha\cos\theta)]^2}\,\Omega_+ - \frac{p-q}{[r+(N+\alpha\cos\theta)]^2}\,\Omega_-\,, && (8.2.46)\end{aligned}$$

where we defined the functions

$$\begin{aligned} P_r &= r^2 - \alpha^2 - \frac{N^4}{N^2-\alpha^2}\,, \qquad P_\theta = 2N\cos\theta - \alpha\sin^2\theta - \frac{\alpha N^2}{N^2-\alpha^2}\,, && (8.2.47)\\ \Delta &= r^2 - 2mr + N^2 - \alpha^2 + p^2 - q^2 \qquad \Sigma = P_r - \alpha P_\theta = r^2 - (N+\alpha\cos\theta)^2\,. && (8.2.48)\end{aligned}$$

The (anti-)self-dual two-forms $\Omega_\pm$ are

$$\Omega_\pm = \hat{e}^1 \wedge \hat{e}^4 \pm \hat{e}^2 \wedge \hat{e}^3 \,, \tag{8.2.49}$$

with the four-dimensional vielbeins

$$\hat{e}^1 \;=\; \left(\frac{\Sigma}{\Delta}\right)^{1/2} dr \,, \qquad\qquad \hat{e}^2 = (\Sigma)^{1/2}\, d\theta \,, \tag{8.2.50}$$

$$\hat{e}^3 \;=\; \frac{\sin\theta}{(\Sigma)^{1/2}}(\alpha d\tau + P_r d\phi) \,, \qquad\qquad \hat{e}^4 = \left(\frac{\Delta}{\Sigma}\right)^{1/2} (d\tau + P_\theta d\phi) \,. \tag{8.2.51}$$

The four-dimensional metric (8.2.45) and gauge field (8.2.46) are solutions to the Einstein-Maxwell equations (8.2.3). The parameters m, q and p still correspond respectively to the mass, electric charge and magnetic charge of the four-dimensional Euclidean solution. The new parameters are the NUT charge, N, and the angular momentum parameter, α. This background is a generalization of the familiar Kerr-Newman solution [112] to which we have added magnetic and NUT charges. Note also that the Kerr-Newman-NUT metric (8.2.45) has exactly the same form as the uncharged Kerr-Taub-Bolt metric [107], the only difference is in the function Δ. One can recover the Kerr-Taub-Bolt metric of the previous section by taking $p = q$. The Euclidean analogs of the inner and outer horizon of the black hole are given by the zeroes of Δ

$$\Delta = (r - r_+)(r - r_-) \,, \qquad\qquad r_\pm = m \pm \sqrt{m^2 - N^2 + \alpha^2 - p^2 + q^2} \,. \tag{8.2.52}$$

The analysis of the regularity of this four-dimensional background is exactly the same as the one performed before for the Kerr-Taub-Bolt solution. We will not reproduce it here and will present only the conclusions. We are interested in the case where the roots $r_\pm$ of Δ are real, in order to have a non-trivial bolt. This imposes

$$m^2 \geq N^2 - \alpha^2 + (p^2 - q^2) \,. \tag{8.2.53}$$

Then, the metric is regular provided that

$$r \geq r_+ \,, \qquad \phi \sim \phi + 2\pi \,, \qquad \tau \sim \tau + 8\pi N \sim \tau + \frac{2\pi}{\kappa} \,, \tag{8.2.54}$$

where we defined

$$P_{r+} \equiv P_r(r = r_+) = r_+^2 - \alpha^2 - \frac{N^4}{N^2 - \alpha^2} \,, \qquad \kappa \;\equiv\; \left|\frac{r_+ - r_-}{2P_{r+}}\right| . \tag{8.2.55}$$

Regularity imposes two *a priori* independent periodicities for the coordinate τ: $\tau \sim \tau + 8\pi N$ comes from imposing regularity for $r \to \infty$ and $\tau \sim \tau + \frac{2\pi}{\kappa}$ is a regularity condition at $r = r_+$. To have a globally regular four-dimensional base with no conical singularities we have to impose the following constraint

$$\kappa = \frac{1}{4|N|} \,. \tag{8.2.56}$$

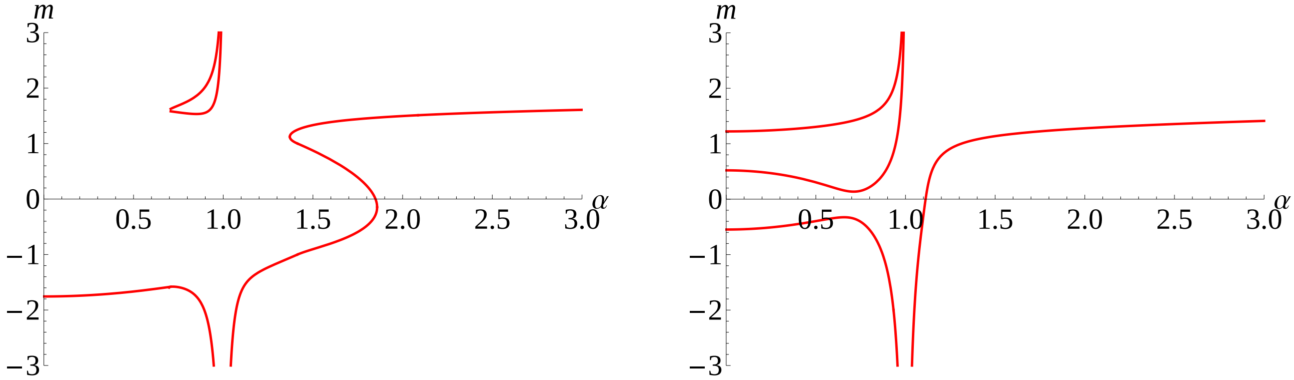

Figure 8.5: *The two graphs represented here are plots of m as a function of α, in units in which $N = 1$ (this choice can always be made because the equations are homogeneous). They show the solutions to* (8.2.56) *for $p^2 - q^2 = 2$ (left) and $p^2 - q^2 = -3/4$ (right). As the value of $p^2 - q^2$ changes, the different branches of the solution evolve and some non trivial differences can be seen. For example, for $p^2 - q^2 = 2$, one can see that there is only one possible value of m for $\alpha = 0$, in contrast with the three different possibilities for $p^2 - q^2 = -3/4$. The important feature is that for any given value of $p^2 - q^2$, there will always be a solution to* (8.2.56)*.*

It is imporant to mention that if we want this metric to have signature (+,+,+,+), in order for it to be a regular Euclidean four-dimensional metric, one has to impose that Σ remains positive, and this will restrict the allowed range of parameters. However, since we are interested here in constructing a regular five-dimensional solution starting from a four-dimensional base, we do not have to impose that the four-dimensional signature stays positive. The only requirement is that we end up with a regular Lorentzian five-dimensional solution, we will discuss this point in section 8.3. Therefore, the physical constraints on the parameters of the solutions are (8.2.53) and (8.2.56). Before constructing the five-dimensional solution, it is worth analyzing what (8.2.56) imposes on the parameters m, N, α, p and q.

One can easily see, using the definition of κ, that (8.2.56) only involves $|N|$ and $|\alpha|$. We will therefore assume N and α to be positive to study this constraint. Note also that p and q only appear in the combination $p^2 - q^2$. In order to solve (8.2.56), the simplest approach is to get rid of the square roots in (8.2.56), and this gives a constraint that is cubic in m, and quadratic in $p^2 - q^2$. This constraint depends on the sign of P_{r+}: if P_{r+} is positive, we have

$$\begin{aligned}
&16N(N^2-\alpha^2)^2m^3 - 4(N^2-\alpha^2)(5N^4-3N^2\alpha^2-\alpha^4)m^2\\
&-16N(N^2-\alpha^2)^2(N^2-\alpha^2+p^2-q^2)m + 20N^8 - 52N^6\alpha^2 + 49N^4\alpha^4 - 16N^2\alpha^6\\
&+2N^2(p^2-q^2)(N^2-\alpha^2)(10N^2-9\alpha^2) + (p^2-q^2)^2(N^2-\alpha^2)^2 \;=\; 0\,,
\end{aligned} \tag{8.2.57}$$

if P_{r+} is negative equation (8.2.56) implies

$$\begin{aligned}
&-16N(N^2-\alpha^2)^2m^3 - 4(N^2-\alpha^2)(5N^4-3N^2\alpha^2-\alpha^4)m^2\\
&+16N(N^2-\alpha^2)^2(N^2-\alpha^2+p^2-q^2)m + 20N^8 - 52N^6\alpha^2 + 49N^4\alpha^4 - 16N^2\alpha^6\\
&+2N^2(p^2-q^2)(N^2-\alpha^2)(10N^2-9\alpha^2) + (p^2-q^2)^2(N^2-\alpha^2)^2 \;=\; 0\,,
\end{aligned} \tag{8.2.58}$$

which is the same as (8.2.57) but with $m \to -m$. Note that a solution to (8.2.57) or (8.2.58) is not automatically a solution to (8.2.56). Indeed, one has first to make sure to solve either (8.2.57) or (8.2.58) in the domains where P_{r+} is respectively positive or negative; secondly, by squaring the square roots, one has to insure that the expression to which this square root is equal is positive. We performed a detailed analysis of these relations for many different values of the parameters, including $p^2 - q^2$. Our analysis shows that, even if the explicit form of the branches of the solutions can differ quite a lot, there are solutions to (8.2.56) for any value of $p^2 - q^2$. For illustration, we present in Figure 8.5 the solution to (8.2.56) for two different values of $p^2 - q^2$.

The five-dimensional supergravity solution

We can use the regular four-dimensional electrovac solution from the previous section to construct a five-dimensional supergravity solution by solving the equations from Section 2.2. From the four-dimensional solution one can read off

$$\Theta^{(3)} = \frac{p+q}{[r-(N+\alpha\cos\theta)]^2}\Omega_+\,, \qquad \omega_-^{(3)} = \frac{p-q}{[r+(N+\alpha\cos\theta)]^2}\Omega_-\,. \tag{8.2.59}$$

These two-forms are d-closed, and thus (at least locally) have corresponding one-form potentials

$$\Theta^{(3)} = (p+q)\,dA_+\,, \qquad \omega_-^{(3)} = (p-q)\,dA_-\,, \tag{8.2.60}$$

which are given by

$$A_\pm = -\frac{1}{r \mp (N+\alpha\cos\theta)}\,(d\tau + P_\theta d\phi) \mp \cos\theta\, d\phi\,. \tag{8.2.61}$$

We now want to solve (8.2.5) and (8.2.6). As noted above, once we know the four-dimensional base space, $\Theta^{(3)}$ and $\omega_-^{(3)}$, (8.2.5) is a coupled system of two linear equations for Z_1 and $\Theta^{(2)}$. Defining

$$\Theta^{(2)} = f_2(r,\theta)\,\Omega_+ + g_2(r,\theta)\,\Omega_-\,, \tag{8.2.62}$$

(8.2.5) can be rewritten as

$$\begin{aligned} \hat\nabla^2 Z_1 &= \frac{2f_2(p+q)}{[r-(N+\alpha\cos\theta)]^2}\,, \\ g_2 &= \frac{p-q}{[r+(N+\alpha\cos\theta)]^2} Z_1\,. \end{aligned} \tag{8.2.63}$$

The solution to this system is given by

$$Z_1 = 1 - \frac{2q_2(p+q)}{m-N}\frac{1}{r-(N+\alpha\cos\theta)}\,, \tag{8.2.64}$$

$$f_2 = \frac{2q_2}{[r-(N+\alpha\cos\theta)]^2} - \frac{2q_2(p^2-q^2)}{m-N}\frac{1}{[r-(N+\alpha\cos\theta)]^2[r+(N+\alpha\cos\theta)]}\,, \tag{8.2.65}$$

$$g_2 = \frac{p-q}{[r+(N+\alpha\cos\theta)]^2} - \frac{2q_2(p^2-q^2)}{m-N}\frac{1}{[r-(N+\alpha\cos\theta)][r+(N+\alpha\cos\theta)]^2}\,. \tag{8.2.66}$$

Similarly, (8.2.6) is solved by

$$Z_2 = 1 - \frac{2q_1(p+q)}{m-N}\frac{1}{r-(N+\alpha\cos\theta)}\,, \tag{8.2.67}$$

$$f_1 = \frac{2q_1}{[r-(N+\alpha\cos\theta)]^2} - \frac{2q_1(p^2-q^2)}{m-N}\frac{1}{[r-(N+\alpha\cos\theta)]^2[r+(N+\alpha\cos\theta)]}\,, \tag{8.2.68}$$

$$g_1 = \frac{p-q}{[r+(N+\alpha\cos\theta)]^2} - \frac{2q_1(p^2-q^2)}{m-N}\frac{1}{[r-(N+\alpha\cos\theta)][r+(N+\alpha\cos\theta)]^2}\,, \tag{8.2.69}$$

and q_1 and q_2 are constants related to the electric charges of the solution[1].

One can show that the 2-forms $\Theta^{(I)}$, $I=1,2$, are d-closed, and the corresponding one form potentials, $B^{(I)}$, are given by

$$B^{(I)} = 2q_I\, A_+ + (p-q)\, A_- + \frac{2q_I(p^2-q^2)}{m-N}\frac{1}{\Sigma}\,(d\tau + P_\theta\, d\phi)\,. \tag{8.2.70}$$

We now have to solve the last system of equations (8.2.7), (8.2.8), to find Z_3 and the angular momentum, k, of the solution. We choose the following Ansatz for k

$$k = \mu(r,\theta)\,(d\tau + P_\theta d\phi) + \nu(r,\theta)\, d\phi\,. \tag{8.2.71}$$

After some work one finds

$$\begin{aligned} dk = &\left(\partial_r\mu - \frac{\alpha}{\Sigma}\partial_r\nu\right)\hat{e}^1\wedge\hat{e}^4 + \frac{\Delta^{1/2}}{\Sigma\sin\theta}\partial_r\nu\ \hat{e}^1\wedge\hat{e}^3 \\ &+ \frac{1}{\Sigma\sin\theta}(\mu\,\partial_\theta P_\theta + \partial_\theta\nu)\ \hat{e}^2\wedge\hat{e}^3 + \frac{\partial_\theta(\Sigma\mu-\alpha\nu)}{\Sigma\,\Delta^{1/2}}\ \hat{e}^2\wedge\hat{e}^4\,. \end{aligned} \tag{8.2.72}$$

Equation (8.2.8) imposes a relation between the functions μ and ν

$$\Delta\partial_r\nu = \sin\theta\,\partial_\theta(\Sigma\mu - \alpha\nu)\,. \tag{8.2.73}$$

Using this constraint one can express μ and ν in terms of a single function $F(r,\theta)$ as

$$\mu = \frac{\Delta\ \partial_r F + \alpha\sin\theta\,\partial_\theta F}{\Sigma}\,, \qquad \nu = \sin\theta\,\partial_\theta F\,. \tag{8.2.74}$$

With this in mind one can rewrite (8.2.7) and (8.2.8) as

$$\mathcal{D}_+ F = Z_1 f_1 + Z_2 f_2 + \frac{(q+p)Z_3}{[r-(N+\alpha\cos\theta)]^2}\,, \tag{8.2.75}$$

$$\hat{\nabla}^2 Z_3 = 2(f_1 f_2 - g_1 g_2) + \frac{2(p-q)}{[r+(N+\alpha\cos\theta)]^2}\mathcal{D}_- F\,, \tag{8.2.76}$$

[1] Note that, as in the Reissner-Nordström solution, our q_I differ from the ones of the previous section by a factor of 2.

where we have defined

$$\mathcal{D}_{\pm}F = \frac{1}{\Sigma}\left[\partial_r(\Delta\partial_r F) \pm \frac{\partial_\theta(\sin\theta\,\partial_\theta F)}{\sin\theta} - \frac{2}{r \mp (N+\alpha\cos\theta)}(\Delta\partial_r F + \alpha\sin\theta\,\partial_\theta F)\right]. \quad (8.2.77)$$

These equations may look complicated, but one can still find an analytic solution. The following is a solution to (8.2.75)

$$\begin{aligned} Z_3 = 1 &- \frac{4q_1q_2}{(m-N)}\frac{1}{r-(N+\alpha\cos\theta)} + \frac{4q_1q_2(p^2-q^2)}{(m-N)^2}\frac{1}{\Sigma} \\ &+ \left(\frac{4q_1q_2(p^2-q^2)}{(m-N)^3} - \frac{2(q+q_1+q_2)(p-q)}{m-N} + \lambda(m-N)\right)\frac{1}{r+(N+\alpha\cos\theta)}\,, \end{aligned} \quad (8.2.78)$$

$$F = F_{nonhom} + F_{hom}\,, \quad (8.2.79)$$

where

$$\begin{aligned} F_{nonhom} = {}& \frac{2q_1q_2(p+q)}{(m-N)^2}\log\left[\frac{\Delta^{1/2}\sin\theta}{[r-(N+\alpha\cos\theta)]^2}\right] \\ &- \frac{2q_1+2q_2+p+q}{r_+-r_-}[r_+\log(r-r_+) - r_-\log(r-r_-)] \\ &- \frac{p^2-q^2}{r_+-r_-}\left(\frac{2q_1q_2(p+q)}{(m-N)^3} - \frac{q+2q_1+2q_2}{m-N}\right)\log\left[\frac{r-r_+}{r-r_-}\right] - \lambda\,\frac{p+q}{2}\log\left(\frac{\sin\theta}{\Delta^{1/2}}\right), \end{aligned} \quad (8.2.80)$$

and

$$\begin{aligned} F_{hom} = \gamma\Bigg(&[r-(N+\alpha\cos\theta)] + \frac{2(m-N)}{r_+-r_-}(r_+\log(r-r_+) - r_-\log(r-r_-)) \\ &- \frac{p^2-q^2}{r_+-r_-}\log\frac{r-r_+}{r-r_-}\Bigg) + \kappa\left(\frac{1}{2}\log(\Delta) - \log(\sin\theta) + \frac{m-N}{r_+-r_-}\log\frac{r-r_+}{r-r_-}\right). \end{aligned} \quad (8.2.81)$$

The function F_{hom} satisfies the equation

$$\mathcal{D}_+F_{hom} = 0\,. \quad (8.2.82)$$

In the expressions above λ, γ and κ are three constants. The functions Z_3 and F presented above are also solutions to the inhomogeneous Laplace equation for Z_3, (8.2.76), if one imposes the following relation between the constants

$$2N\gamma - \kappa = -\frac{\lambda}{2}\left(\frac{m^2-N^2}{p-q} - (p+q)\right) - \frac{2q_1q_2(p+q)(m+N)}{(m-N)^3} + \frac{2N(q+q_1+q_2)}{m-N}\,. \quad (8.2.83)$$

We now have to make sure that there are no CTCs in the solution. First we rewrite k as

$$k = \frac{1}{\Sigma}\left((\Sigma\mu - \alpha\nu)(d\tau + P_\theta\,d\phi) + \nu(\alpha d\tau + P_r\,d\phi)\right). \quad (8.2.84)$$

To avoid CTCs, one has to make sure that ν vanishes for $\theta \to 0, \pi$ and that $\Sigma\mu - \alpha\nu$ vanishes for $r \to r_+$. Using (8.2.74), these conditions lead to the following constraints

$$\kappa = -\lambda\,\frac{p+q}{2} + \frac{2q_1q_2(p+q)}{(m-N)^2}\,, \tag{8.2.85}$$

$$\lambda = \frac{4N(p-q)}{p^2-q^2-2(m+N)\,r_+}\left(\frac{2q_1q_2(p+q)(m+N)}{(m-N)^4} - \frac{(q_1+q_2)(p^2-q^2)}{(m-N)^3} + \frac{(p-q)\,r_+}{(m-N)^2}\right) \tag{8.2.86}$$

These relations, together with (8.2.83), allow to solve for the constants $(\lambda, \kappa, \gamma)$ in terms of the parameters of the four-dimensional base. The explicit form of μ and ν is

$$\begin{aligned}
\mu &= \gamma - \frac{2N\gamma}{r+(N+\alpha\cos\theta)} - \frac{4q_1q_2(p+q)}{(m-N)^2}\,\frac{\Delta+\alpha^2\sin^2\theta}{[r-(N+\alpha\cos\theta)]^2[r-(N+\alpha\cos\theta)]}\\
&\quad - \left(\frac{2q_1q_2(p+q)}{(m-N)^3}(m^2-N^2+p^2-q^2) - (p^2-q^2)\frac{q+2q_1+2q_2}{m-N} + \lambda\,\frac{p+q}{2}(m-N)\right)\frac{1}{\Sigma}\\
&\quad + \left(\frac{4q_1q_2(p+q)}{(m-N)^2} - (2q_1+2q_2+p+q)\right)\frac{r}{\Sigma}\,, \qquad (8.2.87)\\
\nu &= \left(\gamma - \frac{4q_1q_2(p+q)}{(m-N)^2}\,\frac{1}{r-(N+\alpha\cos\theta)}\right)\alpha\sin^2\theta\,, \qquad (8.2.88)
\end{aligned}$$

with λ given by (8.2.86) and γ by

$$\gamma = \frac{q+q_1+q_2}{m-N} - \frac{2q_1q_2(p+q)}{(m-N)^3} - \lambda\frac{m^2-N^2}{4N(p-q)}\,. \tag{8.2.89}$$

Note that the sign of γ and μ in the Kerr-Newman-NUT solution is different from the one for the Reissner-Nordström solution due to the different choice of orientation of the four-dimensional base.

The parameters of the five-dimensional solution should be chosen such that there are no global CTCs. This analysis is rather lengthy and unilluminating, but it suffices to say that one can always find a choice (or range) of parameters for which the solution is regular and free of global CTCs. As we will see in the next subsection, this range of parameters is even bigger than one could naively expect, because the four-dimensional metric can change signature while the complete five-dimensional solution remains regular and free of CTCs.

The relation between the mass and the charges of the solution, as shown in appendix E, is

$$M_0 \equiv (1-\gamma^2)^{-1/2}(M-\gamma Q_e) = \frac{\pi}{G_5\kappa}(m+N) + Q_m + \frac{1}{16\pi G_5}\,(Q_1+Q_2+Q_3)\,. \tag{8.2.90}$$

One can verify that it is still linear in the four charges, with an solitonic contribution.

8.3 Bolt solutions and ambipolar bases

In this final section, we show how the Kerr-Taub-Bolt can be made ambipolar while keeping the five-dimensional solution completely regular. This also happens for the Kerr-Newman metric, that we just presented, but the story for the Kerr-Newman metric almost exactly parallels the one for the Kerr-Taub-Bolt metric, and we therefore choose not to present it here, but will only explain how it differs from the Kerr-Taub-Bolt case.

The Kerr-Taub-Bolt metric (8.1.20) can be recast in the form

$$ds_4^2 = V^{-1}(d\tau + P'_\theta d\phi)^2 + V\Big(\frac{\Delta_\theta}{\Delta}dr^2 + \Delta_\theta d\theta^2 + \Delta\sin^2\theta d\phi^2\Big)\,, \tag{8.3.1}$$

with

$$\Delta_\theta = \Delta + \alpha^2\sin^2\theta\,,\quad V = \frac{\Xi}{\Delta_\theta}\,,\quad P'_\theta = P_\theta + \alpha\frac{\Xi}{\Delta_\theta}\sin^2\theta\,. \tag{8.3.2}$$

In this form, we see that if Ξ becomes negative, the signature changes from (+,+,+,+) to (−,−,−,−). If one is interested in four-dimensional Euclidean metrics, one must require $\Xi > 0$ and thus, keeping in mind that one always has $r \geq r_+$, imposes $m > |N|$. However, we are interested in regular solutions of five-dimensional supergravity, and as it is well-established, such solutions can be obtained from four-dimensional base spaces that have such signature changes [43, 44, 45, 66]. We already discussed such ambipolar BPS solutions in the first chapter, section 1.4.3, of this thesis. We now investigate this possibility in more detail for our non-BPS solutions, and for simplicity we will assume $N > 0$.

The five-dimensional metric is

$$ds^2 = -Z^{-2}(dt+k)^2 + Z\,V^{-1}(d\tau + P'_\theta d\phi)^2 + Z\,V ds_3^2\,, \tag{8.3.3}$$

with $ds_3^2 = \frac{\Delta_\theta}{\Delta}dr^2 + \Delta_\theta d\theta^2 + \Delta\sin^2\theta d\phi^2$. The only factor in V that can change sign is Ξ:

$$\Xi \;\equiv\; r^2 - (N+\alpha\cos\theta)^2 = (r-(N+\alpha\cos\theta))(r+(N+\alpha\cos\theta))\,. \tag{8.3.4}$$

It is easy to see that because $N > 0$, the inequality (8.1.27) implies that the second factor, $(r + (N + \alpha\cos\theta))$, is always positive[1] and so Ξ changes sign when $(r - (N + \alpha\cos\theta))$ changes sign. We therefore define

$$\eta \;\equiv\; (r - (N+\alpha\cos\theta))\,. \tag{8.3.5}$$

As $\eta \to 0$, we have

$$\begin{aligned}
\Xi &= 2(N+\alpha\cos\theta)\,\eta\,(1 + \frac{1}{2(N+\alpha\cos\theta}\,\eta) + O(\eta^3)\,, \qquad (8.3.6)\\
\Delta_\theta &= 2(N-m)(N+\alpha\cos\theta)(1 + \frac{N+\alpha\cos\theta - m}{(N-m)(N+\alpha\cos\theta)}\,\eta) + O(\eta^2)\,,\\
Z_I &= \frac{C_{IJK}}{2}\,\frac{q^Jq^k}{N-m}\,\frac{1}{\eta}\,(1 - \frac{C_{IJK}}{2}\,\frac{N-m}{q^Jq^K}\,\eta) + O(\eta)\,,\\
\mu &= -\frac{q^1q^2q^3}{N-m}\,\frac{1}{\eta^2}\left(1 + \left(-\frac{m-N-2\alpha\cos\theta}{4(N-m)(N+\alpha\cos\theta)} + \frac{(q^1+q^2+q^3)(N-m)}{2q^1q^2q^3}\right)\eta\right) + O(1)\,.
\end{aligned}$$

[1] It might appear that when $\alpha > N$ this term can also become negative. However, the range of r is $r \geq r_+$, and one can check straightforwardly using the regularity constraints that this implies $(r + (N + \alpha\cos\theta)) > 0$. For the Kerr-Newman black hole, this will however not be the case anymore, because of the new $p^2 - q^2$ contribution in Δ. In this generalised case, one will therefore have to restrict the regime of parameters to forbid $(r + (N + \alpha\cos\theta))$ to vanish.

The first possible divergences can come from the coefficient in front of the three-dimensional metric, ZV. But as $\eta \to 0$,

$$ZV = \frac{(q^1 q^2 q^3)^{2/3}}{(N-m)^2} + O(\eta) \tag{8.3.7}$$

which is perfectly regular. The factor of $Z/V \sim \eta^{-2}$ in front of the fiber metric is potentially more troublesome:

$$\begin{aligned} \frac{Z}{V} &= \frac{(q^1 q^2 q^3)^{2/3}}{\eta^2} \\ &+ \left(\frac{(N-m)(q^1+q^2+q^3)}{3(q^1 q^2 q^3)^{1/3}} - \frac{(q^1 q^2 q^3)^{2/3}}{2(N+\alpha\cos\theta)} \left(1 + \frac{2(N-m+\alpha\cos\theta)}{(N-m)}\right)\right) \frac{1}{\eta} + O(1)\,, \end{aligned} \tag{8.3.8}$$

and thus $g_{\tau\tau}$ appears to blow up at $\eta = 0$. However, there is a similar set of terms coming from $-Z^{-2}(dt+k)^2$ and we find that these cancel both the leading and the subleading divergences, and $g_{\tau\tau}$ has a finite value as $\eta = 0$. Finally, one can also verify that the off-diagonal terms $g_{t\tau}$ and $g_{t\phi}$ are finite at $\eta = 0$.

This cancellation of singular terms and the ultimate regularity of the metric exactly parallels the story for the bubbled BPS solutions [43, 44, 45, 66]. Thus, when Ξ changes sign, the ambi-polar base metric leads to a regular five-dimensional metric and therefore, as described earlier, one can allow a wider range of parameters than merely $m > |N|$ and still get a regular, Lorentzian metric in five dimensions.

The analysis that we performed so far ensures that the five-dimensional solution is regular near the $\eta = 0$ surface despite the fact that the four-dimensional base changes signature and seems to be very pathological. We have not presented a detailed analysis of the conditions imposed by global absence of CTCs. Once we have checked that all the coefficients remain regular as one crossed the $\eta = 0$ surface, one still has to verify the positivity of the spacelike metric coeficients, to ensure the absence of CTCs. We do not perform this complete analysis here, but only verify that the Z_I's remain always positive. They are given by

$$Z_I = 1 - \frac{C_{IJK}}{2} \frac{q_J q_K}{m-N} \frac{1}{\eta}\,. \tag{8.3.9}$$

We assume for simplicity the all the q_I's are positive. If $m > N$, then the second term is negative, but η cannot go to zero. One can therefore find parameters such that Z_I always stays positive (and finite). If however $m < N$, then η can diverge. But in this case $-C_{IJK} q_J q_K / 2(m-N)$ is a positive quantity, and thus Z_I diverges but has the sign of η. From the previous singularity analysis, it follows that the physical quantities like the volume of the T^2 tori or ZV will stay finite and positive, and the solution will be causal.

Appendices

Appendix A. Three charge solutions and T-duality

In this Appendix we give the details of the T-duality transformations leading to the solutions presented in section 1.3. The T-duality rules, presented in 1.1.2, are derived in [113] and can be considered a generalization of the Buscher rules [114].

Compactification along x_9

The first step is to compactify the eleven-dimensional solution, presented in Section 2, along x_9, in this way we obtain the following combination of "electric"[1]

$$N_1 : \text{ D2}\,(56) \qquad N_2 : \text{ D2}\,(78) \qquad N_3 : \text{ F1}\,(z) \tag{A.1}$$

and "dipole" branes

$$n_1 : \text{ D4}\,(y78z) \qquad n_2 : \text{ D4}\,(y56z) \qquad n_3 : \text{ NS5}\,(y5678) \tag{A.2}$$

in Type IIA. From now on we will denote $x_{10} = z$. The ten-dimensional *string frame* metric is

$$ds_{10}^2 = -\frac{1}{Z_3\sqrt{Z_1 Z_2}}(dt+k)^2+\sqrt{Z_1 Z_2}ds_4^2+\frac{\sqrt{Z_1 Z_2}}{Z_3}dz^2+\sqrt{\frac{Z_2}{Z_1}}(dx_5^2+dx_6^2)+\sqrt{\frac{Z_1}{Z_2}}(dx_7^2+dx_8^2) \tag{A.3}$$

The dilaton and the Kalb-Ramond field are

$$e^{\Phi} = \left(\frac{Z_1 Z_2}{Z_3^2}\right)^{1/4}, \qquad B = -A^{(3)} \wedge dz\,. \tag{A.4}$$

The RR ("electric") forms are

$$C^{(1)} = 0\,, \qquad C^{(3)} = A^{(1)} \wedge dx_5 \wedge dx_6 + A^{(2)} \wedge dx_7 \wedge dx_8\,, \tag{A.5}$$

and the four-form field strength is[2]

$$\widetilde{F}^{(4)} = dC^{(3)} + dB \wedge C^{(1)} = A^{(1)} \wedge dx_5 \wedge dx_6 + dA^{(2)} \wedge dx_7 \wedge dx_8 \tag{A.6}$$

$$= d\mathcal{F}^{(1)} \wedge dx_5 \wedge dx_6 + \mathcal{F}^{(2)} \wedge dx_7 \wedge dx_8\,, \tag{A.7}$$

[1] We are choosing x_9 to be the M-theory circle in order to match the conventions in the literature for the global signs of the B-field and the RR potentials for the BMPV black hole [60] and the supersymmetric black ring solutions [59].

[2] Note that we are using the notation of [35] $\widetilde{F}^{(4)} = dC^{(3)} + dB \wedge C^{(1)}$.

where we have used the notation $\mathcal{F}^{(I)} = dA^{(I)}$. Now we will perform a chain of T-dualities in order to arrive at the desired frame.

T-duality along x_5

A T-duality along the x_5 direction brings us to Type IIB with the following sets of "electric"

$$N_1 : \text{ D1}\,(6) \qquad N_2 : \text{ D}\,(578) \qquad N_3 : \text{ F1}\,(z) \tag{A.8}$$

and "dipole" branes

$$n_1 : \text{ D5}\,(y578z) \qquad n_2 : \text{ D3}\,(y6z) \qquad n_3 : \text{ NS5}\,(y5678)\,. \tag{A.9}$$

The metric is

$$ds_{10}^2 = -\frac{1}{Z_3\sqrt{Z_1Z_2}}(dt+k)^2+\sqrt{Z_1Z_2}ds_4^2+\frac{\sqrt{Z_1Z_2}}{Z_3}dz^2+\sqrt{\frac{Z_2}{Z_1}}dx_6^2+\sqrt{\frac{Z_1}{Z_2}}(dx_5^2+dx_7^2+dx_8^2)\,. \tag{A.10}$$

The other NS-NS fields are

$$e^{\Phi} = \left(\frac{Z_1^2}{Z_3^2}\right)^{1/4}, \qquad\qquad B = -A^{(3)} \wedge dz\,. \tag{A.11}$$

The RR field strengths are

$$\begin{aligned} F^{(3)} &= -\mathcal{F}^{(1)} \wedge dx_6\,, \\ \widetilde{F}^{(5)} &= \mathcal{F}^{(2)} \wedge dx_5 \wedge dx_7 \wedge dx_8 + \star_{10}(\mathcal{F}^{(2)} \wedge dx_5 \wedge dx_7 \wedge dx_8)\,, \end{aligned} \tag{A.12}$$

where in the expression for $\widetilde{F}^{(5)}$ we have added the Hodge dual piece by hand to ensure self-duality [115]. Note that if one is working in the "democratic formalism" (*i.e.* with both electric and magnetic field strengths) $\widetilde{F}^{(5)}$ will be automatically self-dual, however since we have chosen to T-dualize explicitly only the electric field strengths we have to add the self-dual piece by hand whenever we encounter a five-form field strength after T-dualizing a four-form field strength.

Using the form of the ten-dimensional metric (A.10) one can show that

$$\star_{10}\left(dA^{(2)} \wedge dx_5 \wedge dx_7 \wedge dx_8\right) = -\left(\frac{Z_2^5}{Z_1^3Z_3^2}\right)^{1/4} \star_5\left(dA^{(2)} \wedge dz \wedge dx_6\right), \tag{A.13}$$

where $\star_5$ is the Hodge dual on the five-dimensional subspace given by the metric

$$ds_5^2 = -\frac{1}{Z_3\sqrt{Z_1Z_2}}(dt+k)^2 + \sqrt{Z_1Z_2}ds_4^2\,. \tag{A.14}$$

T-duality along x_6

Now perform T-duality along x_6 to get

$$N_1 : \text{ D0} \qquad N_2 : \text{ D4}\,(5678) \qquad N_3 : \text{ F1}\,(z) \tag{A.15}$$

"electric"

$$n_1 : \text{ D6}\,(y5678z) \qquad n_2 : \text{ D2}\,(yz) \qquad n_3 : \text{ NS5}\,(y5678) \tag{A.16}$$

and "dipole" branes in Type IIA. The metric is

$$ds_{10}^2 = -\frac{1}{Z_3\sqrt{Z_1 Z_2}}(dt+k)^2 + \sqrt{Z_1 Z_2}\,ds_4^2 + \frac{\sqrt{Z_1 Z_2}}{Z_3}dz^2 + \sqrt{\frac{Z_1}{Z_2}}(dx_5^2+dx_6^2+dx_7^2+dx_8^2)\,. \tag{A.17}$$

The dilaton and the Kalb-Ramond fields are

$$e^{\Phi} = \left(\frac{Z_1^3}{Z_2 Z_3^2}\right)^{1/4}, \qquad B = -A^{(3)} \wedge dz\,. \tag{A.18}$$

The RR field strengths are

$$F^{(2)} = -\mathcal{F}^{(1)}\,, \qquad \widetilde{F}^{(4)} = -\left(\frac{Z_2^5}{Z_1^3 Z_3^2}\right)^{1/4} \star_5 (\mathcal{F}^{(2)}) \wedge dz\,. \tag{A.19}$$

Since we are interested in studying probe two charge supertubes in this background, we will also need the RR potentials since they enter the Wess-Zumino action of the supertube.

Finding the RR and NS-NS potentials in the D0-D4-F1 frame

If everything is consistent, then the Bianchi identities for the field strengths should be satisfied. For the solution given by (1.3.22)–(1.3.24), the non-trivial Bianchi identity is:[1]

$$d\widetilde{F}^{(4)} = -F^{(2)} \wedge dB\,. \tag{A.20}$$

Indeed we can use the BPS equations to show that

$$d\widetilde{F}^{(4)} = -d\left(\left(\frac{Z_2^5}{Z_1^3 Z_3^2}\right)^{1/4} \star_5 (\mathcal{F}^{(2)})\right) \wedge dz \tag{A.21}$$

$$= -\left[d\left(\frac{1}{Z_1 Z_3}\right) \wedge dk \wedge (dt+k) - d\left(\frac{(dt+k)}{Z_1}\right) \wedge \Theta^{(3)} - d\left(\frac{(dt+k)}{Z_3}\right) \wedge \Theta^{(1)} + \Theta^{(3)} \wedge \Theta^{(1)}\right] \wedge dz\,. \tag{A.22}$$

On the other hand

$$F^{(2)} \wedge dB = dA^{(1)} \wedge dA^{(3)} \wedge dz \tag{A.23}$$

$$= \left[d\left(\frac{1}{Z_1 Z_3}\right) \wedge dk \wedge (dt+k) - d\left(\frac{(dt+k)}{Z_1}\right) \wedge \Theta^{(3)} - d\left(\frac{(dt+k)}{Z_3}\right) \wedge \Theta^{(1)} + \Theta^{(3)} \wedge \Theta^{(1)}\right] \wedge dz\,. \tag{A.24}$$

[1]See [35] p. 86.

So the Bianchi identity is obeyed and it can be checked in a similar manner that the equations of motion of type IIA supergravity are obeyed. Thus confirms the consistency of our calculations.

We will now find the RR three-form potential $C^{(3)}$ in the same duality frame. It satisfies the following differential equation

$$dC^{(3)} \equiv \tilde{F}^{(4)} + C^{(1)} \wedge H^{(3)} . \tag{A.25}$$

Note that this depends upon a gauge choice for $C^{(1)}$, we choose a gauge in which $C^{(1)}$ is vanishing at asymptotic infinity, namely[1]

$$C^{(1)} = -A^{(1)} - dt . \tag{A.26}$$

Computing explicitly one finds

$$dC^{(3)} = \Big[\big(- \star_4 dZ_2 + B^{(1)} \wedge \Theta^{(3)}\big) - d\big(Z_3^{-1}(dt+k) \wedge B^{(1)} + dt \wedge A^{(3)}\big)\Big] \wedge dx_5 , \tag{A.27}$$

and hence

$$C^{(3)} = -\big(\gamma + Z_3^{-1}(dt+k) \wedge B^{(1)} + dt \wedge A^{(3)}\big) \wedge dx_5 , \tag{A.28}$$

where

$$d\gamma = \big(\star_4 dZ_2 - B^{(1)} \wedge \Theta^{(3)}\big) . \tag{A.29}$$

So the calculation boils down to integrating for the 2-form γ. Up to this stage we have not assumed any particular form of the four-dimensional base space. If this space is Gibbons-Hawking then the equation for γ can be integrated explicitly. Using the BPS supergravity solutions presented in section 1.4.3 it is not hard to show that

$$\begin{aligned}\star_4 dZ_2 - B^{(1)} \wedge \Theta^{(3)} = {} & \big(-\partial_a Z_2 + K^1 \partial_a (V^{-1} K^3)\big) \frac{1}{2} \epsilon_{abc} (d\psi + A) \wedge dy^b \wedge dy^c \\ & - \xi^{(1)}_a \big(\partial_b (V^{-1} K^3)\big)(d\psi + A) \wedge dy^a \wedge dy^b \\ & + V \big(\vec{\xi}^{(1)} \cdot \vec{\nabla}(V^{-1}K^3)\big)\, dy^1 \wedge dy^2 \wedge dy^3 .\end{aligned} \tag{A.30}$$

Recall that $Z_2 = L_2 + V^{-1}K^1K^3$ and define $\vec{\zeta}$ by:

$$\vec{\nabla} \times \vec{\zeta} \equiv -\vec{\nabla} L_2 , \tag{A.31}$$

then using

$$\Omega^{(a)}_{\pm} = \hat{e}^1 \wedge \hat{e}^{a+1} \pm \frac{1}{2} \epsilon_{abc} \hat{e}^{b+1} \wedge \hat{e}^{c+1} , \tag{A.32}$$

one can show that:

$$\begin{aligned}\star_4 dZ_2 - B^{(1)} \wedge \Theta^{(3)} = {} & d\big[\big(- \zeta_a - V^{-1} K^3 \xi^{(1)}_a\big)\Omega^{(a)}_-\big] \\ & - \big(V\, \vec{\nabla} \cdot \vec{\zeta} + K^3\, \vec{\nabla} \cdot \vec{\xi}^{(1)}\big)\, dy^1 \wedge dy^2 \wedge dy^3 .\end{aligned} \tag{A.33}$$

The last term is a multiple of the volume form on $\mathbb{R}^3$ and so is necessarily exact, however, it can be simplified if we chose a gauge for $\vec{\xi}^{(1)}$ and $\vec{\zeta}$:

$$\vec{\nabla} \cdot \vec{\zeta} = \vec{\nabla} \cdot \vec{\xi}^{(1)} = 0 . \tag{A.34}$$

[1] We have fixed $Z_I \sim 1 + \mathcal{O}(r^{-1})$.

Then one has:

$$\gamma \;=\; -\big[\big(\zeta_a \;+\; V^{-1}K^3\xi_a^{(1)}\big)\,\Omega_-^{(a)}\big]\,. \tag{A.35}$$

Finally, let $\vec{r}_i = (y_1 - a_i, y_2 - b_i, y_3 - c_i)$ and let $F \equiv \frac{1}{r_i}$ and then define $\vec{w}$ by $\vec{\nabla}\times\vec{w} \;\equiv\; -\vec{\nabla}F$, then the standard solution for $\vec{w}$ is:

$$w \;=\; -\frac{y_3 - c_i}{r_i}\,\frac{(y_1 - a_i)\,dy_2 - (y_2 - b_i)\,dy_1}{((y_1 - a_i)^2 + (y_2 - b_i)^2)}\,. \tag{A.36}$$

It is elementary to verify that $\vec{\nabla}\cdot\vec{w} = 0$ and so this is the requisite gauge. Finally the explicit form of the RR three-form potential for a solution with GH base in the D0-D4-F1 frame is

$$C^{(3)} \;=\; \big(\zeta_a \;+\; V^{-1}K^3\xi_a^{(1)}\big)\,\Omega_-^{(a)}\wedge dz - \big(Z_3^{-1}(dt+k)\wedge B^{(1)} \;+\; dt\wedge A^{(3)}\big)\wedge dz\,. \tag{A.37}$$

T-duality along z

Another T-duality along z transforms the system into D1-D5-P frame with

$$N_1:\ \mathrm{D1}\,(z) \qquad N_2:\ \mathrm{D5}\,(5678z) \qquad N_3:\ \mathrm{P}\,(z) \tag{A.38}$$

"electric"

$$n_1:\ \mathrm{D5}\,(y5678) \qquad n_2:\ \mathrm{D1}\,(y) \qquad n_3:\ \mathrm{kkm}\,(y5678z) \tag{A.39}$$

and "dipole" branes. The metric is

$$ds^2_{IIB} = -\frac{1}{Z_3\sqrt{Z_1Z_2}}(dt+k)^2 + \sqrt{Z_1Z_2}ds_4^2 + \frac{Z_3}{\sqrt{Z_1Z_2}}(dz+A^3)^2 + \sqrt{\frac{Z_1}{Z_2}}(dx_5^2+dx_6^2+dx_7^2+dx_8^2)\,. \tag{A.40}$$

The dilaton and the Kalb-Ramond field are:

$$e^{\Phi} = \left(\frac{Z_1}{Z_2}\right)^{1/2}, \qquad\qquad B = 0\,. \tag{A.41}$$

The RR three-form field strength (it is the only non-zero field strength) is:

$$F^{(3)} = -\left(\frac{Z_2^5}{Z_1^3Z_3^2}\right)^{1/4}\star_5\left(\mathcal{F}^{(2)}\right) - \mathcal{F}^{(1)}\wedge\left(dz - A^{(3)}\right). \tag{A.42}$$

For the supersymmetric black ring solution in D1-D5-P frame then our general result agrees (up to conventions) with [59]. We can also easily find the RR 2-form potential by T-dualizing (A.37)

$$\begin{aligned} C^{(2)} \;=\; &\big(\zeta_a \;+\; V^{-1}K^3\xi_a^{(1)}\big)\,\Omega_-^{(a)} - \big(Z_3^{-1}(dt+k)\wedge B^{(1)} \;+\; dt\wedge A^{(3)}\big) \\ &+ A^{(1)}\wedge\left(A^{(3)} - dz - dt\right) \;+\; dt\wedge\left(A^{(3)} - dz\right). \end{aligned} \tag{A.43}$$

Appendix B. Units and conventions

Here we summarize some of the conventions we use in this paper (see [35, 36] for more details).

The tensions of the extended objects in string and M-theory are:

$$T_{F1} = \frac{1}{2\pi\alpha'} \,, \qquad T_{Dp} = \frac{1}{g_s (2\pi)^p (l_s)^{p+1}} \,, \qquad T_{NS5} = \frac{1}{g_s^2 (2\pi)^5 (l_s)^6} \,, \tag{B.1}$$

$$T_{M2} = \frac{1}{(2\pi)^2 (l_{11})^3} \,, \qquad T_{M5} = \frac{1}{(2\pi)^5 (l_{11})^6} \tag{B.2}$$

where $\alpha' = l_s^2$, l_s is the string length, g_s is the string coupling constant (in the particular duality frame in which one works) and l_D is the D-dimensional Planck length. The Newton's constant in different dimensions is

$$16\pi G_{11} = (2\pi)^8 (l_{11})^9 \,, \qquad 16\pi G_{10} = (2\pi)^7 (g_s)^2 (l_s)^8 \,, \qquad 16\pi G_D = (2\pi)^{D-3} (l_D)^{D-2} \,. \tag{B.3}$$

One can show that

$$l_{11} = g_s^{1/3} l_s = g_s^{1/3} (\alpha')^{1/2} \,. \tag{B.4}$$

T-duality along a circle of radius R changes the coupling constants to:

$$\widetilde{R} = \frac{\alpha'}{R} \,, \qquad \tilde{g}_s = \frac{l_s}{R} g_s \,, \qquad \tilde{l}_s = l_s \,. \tag{B.5}$$

where $\widetilde{R}$ is the radius after T-duality:

When one compactifies M-theory on a circle of radius L_9, the coupling constants of the resulting type IIA string theory satisfy:

$$L_9 = g_s l_s \,. \tag{B.6}$$

If one compactifies M-theory on a T^6 (along the directions $5, 6, 7, 8, 9, 10$) and the radius of each circle is L_i ($i = \{5, 6, 7, 8, 9, 10\}$), the five-dimensional Newton's constant is

$$G_5 = \frac{G_{11}}{vol(T^6)} = \frac{G_{11}}{(2\pi)^6 L_5 L_6 L_7 L_8 L_9 L_{10}} = \frac{\pi}{4} \frac{(l_{11})^9}{L_5 L_6 L_7 L_8 L_9 L_{10}} \,. \tag{B.7}$$

The relations between the number of M2 and M5 branes, N_I and n_I, and the physical charges of the five-dimensional solution obtained by compactifying M-theory on a T^6, Q_I and q_I, are

$$Q_1 = \frac{(l_{11})^6}{L_7 L_8 L_9 L_{10}} \, N_1 \,, \qquad Q_2 = \frac{(l_{11})^6}{L_5 L_6 L_9 L_{10}} \, N_2 \,, \qquad Q_3 = \frac{(l_{11})^6}{L_5 L_6 L_7 L_8} \, N_3 \,, \tag{B.8}$$

$$q_1 = \frac{(l_{11})^3}{L_5 L_6} \, n_1 \,, \qquad q_2 = \frac{(l_{11})^3}{L_7 L_8} \, n_2 \,, \qquad q_3 = \frac{(l_{11})^3}{L_9 L_{10}} \, n_3 \,. \tag{B.9}$$

We will choose a system of units in which all three T^2 are of equal volume

$$L_5 L_6 = L_7 L_8 = L_9 L_{10} = (l_{11})^3 \equiv g_s l_s^3 \,, \tag{B.10}$$

note that this is a numerical identity and is not dimensionally correct since g_s is dimensionless. With this choice we will have

$$G_5 = \frac{\pi}{4} \,, \qquad Q_I = N_I \,, \qquad q_I = n_I \,. \tag{B.11}$$

and these identities hold in every duality frame we use in the paper. Furthermore we will choose

$$g_s l_s = 1 \,. \tag{B.12}$$

Since we are compactifying M-theory on L_9 we will have $L_9 = g_s l_s = 1$ and $L_{10} = l_s^2$, this implies (note that throughout the paper we put $L_{10} \equiv L_z$)

$$T_{D0} = 1 \,, \qquad 2\pi T_{F1} L_{10} = 1 \,, \qquad \text{and} \qquad \frac{2\pi T_{D2}}{T_{F1}} = 1 \,. \tag{B.13}$$

We have fixed $l_s = g_s^{-1}$ so that a lot of the various brane tension factors, appearing in the probe supertube calculations throughout the paper, cancel. Note that with our choices g_s is still a free parameter but we have fixed the volume of the compactification torii.

Appendix C. The angular momentum of the supertube

Generalities

Our goal in this Appendix is to compute the angular momentum of a supertube in the background of three-charge black holes and black rings. Because that's the framework we are using in chapter 2 and 3, we work in the D0-D4-F1 duality frame, presented in section 1.3.2:

$$ds^2_{IIA} = -\frac{1}{Z_3\sqrt{Z_1 Z_2}}(dt+k)^2 + \sqrt{Z_1 Z_2}ds_4^2 + \frac{\sqrt{Z_1 Z_2}}{Z_3}dz^2 + \sqrt{\frac{Z_1}{Z_2}}(dx_5^2 + dx_6^2 + dx_7^2 + dx_8^2) \,. \tag{C.1}$$

For the purpose of our calculations we can restrict without loss of generality to a (non-generic) $U(1) \times U(1)$ invariant base metric of the form:

$$ds_4^2 = g_1(u,v)du^2 + g_2(u,v)d\varphi_1^2 + h_1(u,v)dv^2 + h_2(u,v)d\varphi_2^2 \,, \tag{C.2}$$

in which the angular momentum vector has the form

$$k = k_1(u,v)d\varphi_1 + k_2(u,v)d\varphi_2 \,. \tag{C.3}$$

This is general enough to describe all the matric that we will study, in particular the Gibbons-Hawking metrics with a flat $\mathbb{R}^3$ base. The solutions we consider also have RR and NS-NS fields, which have the general form

$$B = (Z_3^{-1} - 1)dt \wedge dz + Z_3^{-1} k \wedge dz - B^{(3)} \wedge dz \tag{C.4}$$

$$C^{(1)} = (Z_1^{-1} - 1)dt + Z_1^{-1} k - B^{(1)} \tag{C.5}$$

$$C^{(3)} = Z_3^{-1} dt \wedge k \wedge dz - Z_3^{-1}(dt + k) \wedge B^{(1)} \wedge dz + B^{(3)} \wedge dt \wedge dz - f(u,v) d\varphi_1 \wedge d\varphi_2 \wedge dz \quad \text{(C.6)}$$

where the self-dual harmonic two-forms are $\Theta^{(I)} = dB^{(I)}$, $I = 1,2,3$ and

$$B^{(I)} = B^{(I)}_{\varphi_1} d\varphi_1 + B^{(I)}_{\varphi_2} d\varphi_2 \,. \quad \text{(C.7)}$$

Consider a probe supertube with world-volume coordinates $\xi = \{\xi^0, \xi^1, \xi^2 \equiv \theta\}$ in the above background and suppose that the supertube is embedded as follows:

$$t = \xi^0 \,, \qquad z = \xi^1 \,, \qquad \varphi_1 = \nu_1 \theta \,, \qquad \varphi_2 = \nu_2 \theta \quad \text{(C.8)}$$

where $0 \leq \theta \leq 2\pi n_2^{ST}$ and $0 \leq z \leq 2\pi L_z$. We remaind the reader of the importance of the different embeddings in the case of the black ring solutnio, see section 3.1. The supertube "electric" charges are:

$$N_1^{ST} = \frac{T_{D2}}{T_{D0}} \int dz d\theta \mathcal{F}_{z\theta} = n_2^{ST} \mathcal{F}_{z\theta} \quad \text{(C.9)}$$

$$N_3^{ST} = \frac{1}{T_{F1}} \int d\theta \left(\frac{\partial \mathcal{L}_{tot}}{\partial \mathcal{F}_{tz}} \right) \Bigg|_{BPS} = n_2^{ST} \left[Z_2 \left(\frac{\nu_1^2 g_2(u,v) + \nu_2^2 h_2(u,v)}{\mathcal{F}_{z\theta} + \nu_1 B^{(3)}_{\varphi_1} + \nu_2 B^{(3)}_{\varphi_2}} \right) - (\nu_1 B^{(1)}_{\varphi_1} + \nu_2 B^{(1)}_{\varphi_2}) \right] \quad \text{(C.10)}$$

Since the background is independent of φ_1 and φ_2, the supertube has two conserved angular momenta:

$$J^{ST}_{\varphi_1} = \int dz d\theta \, \frac{\partial \mathcal{L}_{tot}}{\partial \dot{\varphi}_1} \,, \qquad J^{ST}_{\varphi_2} = \int dz d\theta \, \frac{\partial \mathcal{L}_{tot}}{\partial \dot{\varphi}_2} \,. \quad \text{(C.11)}$$

One can compute them explicitly and find

$$\begin{aligned} J^{ST}_{\varphi_1} = n_2^{ST} \Bigg[& \nu_1 Z_2 g_2 - \mathcal{F}_{z\theta} B^{(1)}_{\varphi_1} - Z_2 B^{(3)}_{\varphi_1} \left(\frac{\nu_1^2 g_2 + \nu_2^2 h_2}{\mathcal{F}_{z\theta} + \nu_1 B^{(3)}_{\varphi_1} + \nu_2 B^{(3)}_{\varphi_2}} \right) \\ & + \nu_2 (B^{(1)}_{\varphi_2} B^{(3)}_{\varphi_1} - B^{(1)}_{\varphi_1} B^{(3)}_{\varphi_2}) + \nu_2 f(u,v) \Bigg] \,, \end{aligned} \quad \text{(C.12)}$$

$$\begin{aligned} J^{ST}_{\varphi_2} = n_2^{ST} \Bigg[& \nu_2 Z_2 h_2 - \mathcal{F}_{z\theta} B^{(1)}_{\varphi_2} - Z_2 B^{(3)}_{\varphi_2} \left(\frac{\nu_1^2 g_2 + \nu_2^2 h_2}{\mathcal{F}_{z\theta} + \nu_1 B^{(3)}_{\varphi_1} + \nu_2 B^{(3)}_{\varphi_2}} \right) \\ & + \nu_1 (B^{(1)}_{\varphi_1} B^{(3)}_{\varphi_2} - B^{(1)}_{\varphi_2} B^{(3)}_{\varphi_1}) - \nu_1 f(u,v) \Bigg] \,. \end{aligned} \quad \text{(C.13)}$$

One can also define a "total" angular momentum of the supertube, as the angular momentum along the direction of the supertube

$$J^{ST}_{TOT} = \nu_1 J^{ST}_{\varphi_1} + \nu_2 J^{ST}_{\varphi_2} \quad \text{(C.14)}$$

and one can show that

$$J^{ST}_{TOT} = \nu_1 J^{ST}_{\varphi_1} + \nu_2 J^{ST}_{\varphi_2} = \frac{N_1^{ST} N_3^{ST}}{n_2^{ST}} \,. \quad \text{(C.15)}$$

Flat Space

For flat space we have

$$Z_I = 1 \ , \qquad B^{(I)}_{\varphi_1} = B^{(I)}_{\varphi_2} = 0 \ , \qquad k_1(u,v) = k_2(u,v) = 0 \ , \qquad f(u,v) = 0 \ , \tag{C.16}$$

and using the change of variables $u = \rho \sin\vartheta$, $v = \rho\cos\vartheta$ one has:

$$g_1(u,v) = h_1(u,v) = 1 \ , \qquad g_2 = \rho^2 \sin^2\vartheta \ , \qquad h_2 = \rho^2\cos^2\vartheta \ . \tag{C.17}$$

The conserved "electric" charges of the supertube are

$$N_1^{ST} = n_2^{ST}\mathcal{F}_{z\theta} \tag{C.18}$$

$$N_3^{ST} = n_2^{ST}\left(\frac{\nu_1^2\rho^2\sin^2\vartheta + \nu_2^2\rho^2\cos^2\vartheta}{\mathcal{F}_{z\theta}}\right) \tag{C.19}$$

From these expressions one recovers the familiar radius relation of the supertube

$$\frac{N_1^{ST}N_3^{ST}}{(n_2^{ST})^2} = \rho^2(\nu_1^2\sin^2\vartheta + \nu_2^2\cos^2\vartheta) \ . \tag{C.20}$$

The components of the supertube angular momentum are

$$J^{ST}_{\varphi_1} = \nu_1 n_2^{ST}\rho^2\sin^2\vartheta \ , \tag{C.21}$$

$$J^{ST}_{\varphi_2} = \nu_2 n_2^{ST}\rho^2\cos^2\vartheta \ . \tag{C.22}$$

Of course we again have

$$J^{ST}_{TOT} = \nu_1 J^{ST}_{\varphi_1} + \nu_2 J^{ST}_{\varphi_2} = \frac{N_1^{ST}N_3^{ST}}{n_2^{ST}} \ . \tag{C.23}$$

BMPV Black Hole

For a BMPV black hole we have

$$Z_I = 1 + \frac{Q_I}{\rho^2} \ , \qquad B^{(I)}_{\varphi_1} = B^{(I)}_{\varphi_2} = 0 \ , \qquad k_1 = \frac{J\sin^2\vartheta}{\rho^2} \ , \qquad k_2 = -\frac{J\cos^2\vartheta}{\rho^2} \ , \tag{C.24}$$

$$f = (Z_2 - 1)\rho^2\cos^2\vartheta \ , \qquad g_1(u,v) = h_1(u,v) = 1 \ , \tag{C.25}$$

$$g_2 = \rho^2\sin^2\vartheta \ , \qquad h_2 = \rho^2\cos^2\vartheta \, . \tag{C.26}$$

The conserved "electric" charges of the supertube are

$$N_1^{ST} = n_2^{ST}\mathcal{F}_{z\theta} \ , \tag{C.27}$$

$$N_3^{ST} = n_2^{ST}\left(1 + \frac{Q_2}{\rho^2}\right)\left(\frac{\nu_1^2\rho^2\sin^2\vartheta + \nu_2^2\rho^2\cos^2\vartheta}{\mathcal{F}_{z\theta}}\right) \ . \tag{C.28}$$

These again lead to a radius relation for the supertube in the background of the BMPV black hole

$$\frac{N_1^{ST} N_3^{ST}}{(n_2^{ST})^2} = \left(1 + \frac{Q_2}{\rho^2}\right) \rho^2 (\nu_1^2 \sin^2\vartheta + \nu_2^2 \cos^2\vartheta) \ . \tag{C.29}$$

The components of the supertube angular momentum are

$$J_{\varphi_1}^{ST} = n_2^{ST} \left[\nu_1 \left(1 + \frac{Q_2}{\rho^2}\right) \rho^2 \sin^2\vartheta + \nu_2 Q_2 \cos^2\vartheta\right] \ , \tag{C.30}$$

$$J_{\varphi_2}^{ST} = n_2^{ST} \left[\nu_2 \left(1 + \frac{Q_2}{\rho^2}\right) \rho^2 \cos^2\vartheta - \nu_1 Q_2 \cos^2\vartheta\right] \ . \tag{C.31}$$

One can compare this result to the one obtained in [76] where the special case $\nu_1 = n_2^{ST} = 1$, $\nu_2 = 0$ was considered. For these special values (C.30) and (C.31) are identical to (4.4) and (4.5) in [76].

Three-charge BPS Black Ring

For a three-charge BPS black ring we have :

$$g_1 = \frac{R^2}{(x-y)^2(y^2-1)} \ , \quad g_2 = \frac{R^2(y^2-1)}{(x-y)^2} \ , \quad h_1 = \frac{R^2}{(x-y)^2(1-x^2)} \ , \quad h_2 = \frac{R^2(1-x^2)}{(x-y)^2} \ . \tag{C.32}$$

The functions, Z_I, appearing in the ten-dimensional metric, the one-forms $B^{(I)}$ and the function $f(x,y)$ are given by (1.4.14), (1.4.16) and (1.4.18) respectively. The explicit form of the angular momentum components of the black ring, $k_1(x,y)$ and $k_2(x,y)$, is not needed here.

The conserved "electric" charges of the supertube are

$$N_1^{ST} = n_2^{ST} \mathcal{F}_{z\theta} \ , \tag{C.33}$$

$$\begin{aligned} N_3^{ST} \ &= \ n_2^{ST} \left[\frac{n_1}{2}(-\nu_1(d+y) + \nu_2(c+x)) \right. \\ &+ \left. \frac{Z_2}{\mathcal{F}_{z\theta} + \frac{n_3}{2}(-\nu_2(c+x) + \nu_1(d+y))} \left(\nu_1^2 R^2 \frac{(y^2-1)}{(x-y)^2} + \nu_2^2 R^2 \frac{(1-x^2)}{(x-y)^2}\right)\right] \ , \end{aligned} \tag{C.34}$$

which leads to the radius relation

$$\begin{aligned} &\left[N_1^{ST} + \frac{1}{2} n_2^{ST} n_3 (\nu_1(y+d) - \nu_2(x+c))\right] \left[N_3^{ST} + \frac{1}{2} n_2^{ST} n_1 (\nu_1(y+d) - \nu_2(x+c))\right] \ = \\ &\qquad (n_2^{ST})^2 Z_2 \frac{R^2}{(x-y)^2} (\nu_1^2(y^2-1) + \nu_2^2(1-x^2)) \end{aligned} \tag{C.35}$$

The components of the supertube angular momentum are

$$J_{\varphi_1}^{ST} = n_2^{ST}\Bigg[-\mathcal{F}_{z\theta}\frac{n_1}{2}(d+y)+\nu_1 Z_2 R^2\frac{(y^2-1)}{(x-y)^2}+\nu_2 f(x,y) - Z_2\frac{n_3(d+y)}{2}\left(\frac{\nu_1^2R^2\frac{(y^2-1)}{(x-y)^2}+\nu_2^2R^2\frac{(1-x^2)}{(x-y)^2}}{\mathcal{F}_{z\theta}+\frac{n_3}{2}(-\nu_2(c+x)+\nu_1(d+y))}\right)\Bigg] \quad (C.36)$$

$$J_{\varphi_2}^{ST} = n_2^{ST}\Bigg[\mathcal{F}_{z\theta}\frac{n_1}{2}(c+x)+\nu_2 Z_2 R^2\frac{(1-x^2)}{(x-y)^2}-\nu_1 f(x,y) + Z_2\frac{n_3(c+x)}{2}\left(\frac{\nu_1^2R^2\frac{(y^2-1)}{(x-y)^2}+\nu_2^2R^2\frac{(1-x^2)}{(x-y)^2}}{\mathcal{F}_{z\theta}+\frac{n_3}{2}(-\nu_2(c+x)+\nu_1(d+y))}\right)\Bigg] \quad (C.37)$$

And we again have

$$J_{TOT}^{ST} = \nu_1 J_{\varphi_1}^{ST} + \nu_2 J_{\varphi_2}^{ST} = \frac{N_1^{ST}N_3^{ST}}{n_2^{ST}}\ . \quad (C.38)$$

Appendix D. Extremal Reissner-Nordström

An interesting limiting case of the solution presented in Section 8.2.1 is when the two horizons of the four-dimensional base coincide. This is the extremal Euclidean dyonic Reissner-Nordström background

$$ds_4^2 = \left(1-\frac{m}{r}\right)^2 d\tau^2 + \left(1-\frac{m}{r}\right)^{-2} dr^2 + r^2(d\theta^2+\sin^2\theta d\phi^2)\ , \quad (D.1)$$

$$F = \frac{2q}{r^2}\,d\tau\wedge dr + 2p\sin\theta\,d\theta\wedge d\phi\,. \quad (D.2)$$

This background is a limit of the dyonic Reissner-Nordström black hole which is obtained by taking $m^2 = p^2 - q^2$. The two horizons degenerate and we have

$$r_+ = r_- = m\,. \quad (D.3)$$

The near horizon limit of the Lorentzian extremal Reissner-Nordström black hole is the Bertoti-Robinson solution which is $AdS_2 \times S^2$ with electric and magnetic flux [116]. In the Euclidean solution of interest the horizon has become a bolt of radius m and near the bolt we can set

$$r = m + \frac{m^2}{\rho^2}\ , \quad (D.4)$$

and rewrite the metric as

$$ds_{NH}^2 = m^2\left(\frac{d\rho^2+d\tau^2}{\rho^2}+d\theta^2+\sin^2\theta d\phi^2\right)\ . \quad (D.5)$$

This is the metric on $H_2^+\times S^2$, where H_2^+ is the Poincaré half plane and we have the following range of coordinates $\tau\in(-\infty,\infty)$ and $\rho\in(0,\infty)$. Note that we still have a finite size bolt

(S^2) at $r = m$ on which we can put flux. At asymptotic infinity the metric approaches the flat metric on $\mathbb{R}^4$. This should be contrasted with the case of the non-extremal Euclidean Reissner-Nordström black hole of Section 8.2.1, where we had to periodically identify the coordinate τ to get a regular metric near the outer horizon. The five-dimensional supegravity solution based on this four-dimensional base has the same warp factors and fluxes as the solution in Section 8.2.1, however one should remember to set $m^2 = p^2 - q^2$. The coordinate τ is non-compact but it is still an isometry of the five-dimensional solution. This means that we have the electric charges corresponding to the three $U(1)$ gauge fields smeared along τ. What happens effectively is that in the extremal limit the coordinate τ decompactifies and the five-dimensional solution is asymptotic to $\mathbb{R}^{1,4}$ and corresponds to a smeared distribution of charges along τ. With this in mind one can proceed in the same way as in Section 8.2.1 and compute the asymptotic charges and mass densities of the five-dimensional solution[1]

$$
\begin{aligned}
Q_1 &= -4\pi \left(\frac{2(p+q)q_2}{m} + \gamma(q+q_2) \right) , \\
Q_2 &= -4\pi \left(\frac{2(p+q)q_1}{m} + \gamma(q+q_1) \right) , \\
Q_3 &= -4\pi \left(\frac{4q_1q_2}{m} + \gamma(q_1+q_2+p-q) + \frac{2(p-q)(q+q_1+q_2)}{m} - \frac{4q_1q_2(p^2-q^2)}{m^3} \right) , \\
M_0 &= \frac{1}{16\pi G_5} \left(8\pi m + Q_1 + Q_2 + Q_3 \right) .
\end{aligned}
\tag{D.6}
$$

It is clear from the dependence of the mass on the charges that we again have a non-BPS five-dimensional solution that has the same asymptotic charges as a non-extremal black hole. This may seem somewhat strange because we have started with an extremal four-dimensional solution, which is also known to be BPS[2]. There is nothing puzzling going on here, to get the five-dimensional solution we have added fluxes to the four-dimensional base which break the supersymmetry completely. In addition the difference between the mass and the sum of the electric charges corresponds to the “solitonic” contribution of the bolt, and therefore one should not expect to have a solution with the same charges as an extremal black hole.

Finally we will provide some comments on the extremal limit of the Kerr-Newmann-NUT solution of Section 8.2.2. This limit arises when we have $m^2 = N^2 - \alpha^2 + p^2 - q^2$. There is no need to compactify the coordinate τ near $r = r_+ = m$, however we still have to compactify τ to ensure regularity at spatial infinity. Since we will have an unique identification of τ, $\tau \sim \tau + 8\pi N$, we will not have to impose the constraint (8.2.56). The five-dimensional solution will be the same as in Section 8.2.2 and will still be asymptotic to $\mathbb{R}^{1,3} \times S^1$.

---

[1]Note that, since the τ coordinate is not compact anymore, we are now computing charge and mass densities.

[2]The Lorentzian extremal Reissner-Nordström solution is a BPS background interpolating between $AdS_2 \times S^2$ and $\mathbb{R}^{1,3}$. Going to the Euclidean regime does not spoil the supersymmetry of the solution which now interpolates between $H_2^+ \times S^2$ and $\mathbb{R}^4$, see for example [117].

Appendix E. Asymptotic charges of Bolt Solutions

In this appendix we compute the asymptotic mass and charges of the different bolt solutions. Even if each of the solution can be seen as the general dyonic Kerr-Newman solutnio, with some of the parameters being null, we think this is worse giving the mass and charges for each of the solution.

In general, in the M-theory frame, the solutions carries M5 charges, which are encoded in the magnetic part of the gauge field, $B^{(I)}$, and are essentially equal to q_I, up to some conventional factor. The solution also carries M2 charges. Note that the gauge field equations involve Chern-Simons terms:

$$d((X^I)^{-2} \star_5 dA^I) = \tfrac{1}{2} C_{IJK} dA^J \wedge dA^K \,. \tag{E.1}$$

In the presence of such terms, the proper definition of the conserved electric charge associated with A^I is

$$Q_I = \int_{S^1\times S^2} \Big[(X^I)^{-2} \star_5 dA^I - \tfrac{1}{2} C_{IJK} A^J \wedge dA^K \Big] \,, \tag{E.2}$$

where the integral is computed over the S^1 circle parameterized by τ and the S^2 sphere at spatial infinity. The Chern-Simons term gives a non-vanishing contribution to the charge, due to the fact that the one-form k goes to a constant non-zero value at infinity:

$$k \to \gamma d\tau \,, \tag{E.3}$$

For all the solutions, one has the identity

$$\begin{aligned} &(X^I)^{-2} \star_5 dA^I - \tfrac{1}{2} C_{IJK} A^J \wedge dA^K \\ &\quad = \; \varepsilon \star_4 dZ_I - \tfrac{1}{2} C_{IJK} B^{(J)} \wedge \Theta^{(K)} + \frac{\varepsilon}{2} C_{IJK} d\Big[(dt + k) \wedge \frac{B^{(J)}}{Z_K} \Big] \,. \end{aligned} \tag{E.4}$$

To compute the mass and the KK electric charge of the solution one has to analyze the asymptotic form of the metric. The fact that the one-form, k, does not vanish at infinity implies that the coordinates (τ, t) define a frame which is not asymptotically at rest, much like for the black ring in Taub-NUT constructed in [64]. One can go to an asymptotically static frame by re-writing the large r limit of the metric in the form

$$ds^2 \approx (1-\gamma^2)\Big(d\tau - \frac{\gamma}{1-\gamma^2} dt \Big)^2 - dt^2 (1-\gamma^2)^{-1} + dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \,, \tag{E.5}$$

and redefining the coordinates as

$$\hat{\tau} = (1-\gamma^2)^{1/2} \Big(\tau - \frac{\gamma}{1-\gamma^2} t \Big) \,, \quad \hat{t} = (1-\gamma^2)^{-1/2} t \,. \tag{E.6}$$

The condition for the absence of CTCs reflects the fact that the rotation is sub-luminal ($\gamma < 1$) and hence this change of coordinates is well-defined.

In order to read off the mass and electric charge of the solution, on have to reduce to four dimension, along the $\hat{\tau}$:

$$ds_5^2 = g_{\hat{\tau}\hat{\tau}} \Big[d\hat{\tau} + A_{KK} \Big]^2 + g_{\hat{\tau}\hat{\tau}}^{-1/2} ds_E^2 \,. \tag{E.7}$$

ds^2_E is the four-dimensional Einstein metric, and A_{KK} the Kalza-Klein gauge field. From the coefficients of $d\hat{t}^2$ and $d\hat{t}d\phi$ in the Einstein metric ds^2_E one can read off the mass of the solution:

$$- g_{\hat{t}\hat{t}} = 1 - \frac{2\,G_4 M}{r} + o\left(\frac{1}{r}\right), \qquad \frac{g_{\hat{t}\phi}}{g\hat{t}\hat{t}} = \frac{2\,G_4 J\,\sin^2\theta}{r} + o\left(\frac{1}{r}\right). \tag{E.8}$$

Here G_4 is the four-dimensional Newton's constant, whose relation with the five-dimensional Newton's constant G_5 is

$$G_4 = \frac{G_5}{\text{Vol}(\hat{\tau})}, \tag{E.9}$$

vol(τ) being the length of the S^1 parametrized by $\hat{\tau}$. The KK electric Q_e and magnetic Q_m charges are encoded in the KK gauge field[1]

$$A_{KKt} = \frac{4\,G_4 Q_e}{r} + o\left(\frac{1}{r}\right), \qquad A_{KK\phi} = -4\,G_4 Q_m\,\cos\theta + o(1)\,. \tag{E.10}$$

Asymptotic charges for the Schwarzschild solution

The M5 charges are here exactly equal to q_I. The angular momentum one-form k goes to a constant non-zero value at infinity

$$k \to \gamma d\tau\,, \quad \gamma = -\varepsilon\frac{q_1+q_2+q_3}{2m} + 3\varepsilon\,\frac{q_1 q_2 q_3}{8m^3}\,. \tag{E.11}$$

From (E.4), one finds

$$Q_I = -(8\pi m)(4\pi)\tfrac{1}{2}\,C_{IJK}\Big[\varepsilon\frac{q_J q_K}{m} + \frac{\gamma}{2}(q_J+q_K)\Big]\,. \tag{E.12}$$

One computes the mass and electric charge of the solutions by reducing it to four dimension, in the $(\hat{t},\hat{\tau})$ variables (E.6). It yields

$$ds^2_5 = \Big(1-\frac{2m}{r}\Big)^2 Z^{-2}\hat{I}_4\Big[d\hat{\tau} - \Big(1-\frac{2m}{r}\Big)^{-2}\mu\hat{I}_4^{-1}d\hat{t} + \gamma d\hat{t}\Big]^2 + \Big(1-\frac{2m}{r}\Big)^{-1} Z\hat{I}_4^{-1/2}ds^2_E\,, \tag{E.13}$$

where

$$\hat{I}_4 = (1-\gamma^2)^{-1}\Big[Z^3\Big(1-\frac{2m}{r}\Big)^{-1} - \mu^2\Big(1-\frac{2m}{r}\Big)^{-2}\Big]\,, \tag{E.14}$$

and

$$ds^2_E = -\hat{I}_4^{-1/2}d\hat{t}^2 + \hat{I}_4^{1/2}\Big[dr^2 + \Big(1-\frac{2m}{r}\Big)(r^2 d\theta^2 + r^2\sin^2\theta d\phi^2)\Big]\,. \tag{E.15}$$

From (E.8) and (E.10), and obtains

$$G_4 M = \frac{1}{(1-\gamma^2)}\Big(\frac{m}{2} - \frac{q_1q_2+q_1q_3+q_2q_3}{4m} - \frac{3}{8}\frac{\gamma}{\varepsilon}\frac{q_1q_2q_3}{m^2}\Big)\,, \tag{E.16}$$

$$G_4 Q_e = -\frac{1}{4(1-\gamma^2)}\Big(\frac{3\varepsilon}{4}\frac{q_1q_2q_3}{m^2} + \gamma\frac{q_1q_2+q_1q_3+q_2q_3}{m} + \varepsilon\gamma^2(q_1+q_2+q_3)\Big)\,. \tag{E.17}$$

[1] We use the conventions of [64].

Here the four-dimensional Newton's constant G_4 is related to the five-dimensional Newton's constant G_5 by

$$G_4 = \frac{G_5}{(1-\gamma^2)^{1/2}(8\pi m)} \,. \tag{E.18}$$

If one now computes the rest-mass, M_0, of the solution (i.e. the mass with respect to the (t,τ) frame) one obtains:

$$M_0 \equiv (1-\gamma^2)^{-1/2}(M-\gamma Q_e) = \frac{\pi}{4G_5}\Big(16m^2 + \frac{\varepsilon}{4\pi^2}(Q_1+Q_2+Q_3)\Big) \,. \tag{E.19}$$

It is clear from this expression that if we set the mass of the four-dimensional black hole to zero we will recover the usual relation between the mass and the charges of a BPS black hole solution.

Asymptotic charges for the Kerr-Taub-Bolt solution

When one add rotation and a NUT charge, the magnetic charges do not change. The M2 charges, given by (E.4), are

$$Q_I = -(8\pi N)(4\pi)\tfrac{1}{2}C_{IJK}\Big[\frac{q_J q_K}{m-N} + \frac{\gamma}{2}(q_J+q_K)\Big] \,, \tag{E.20}$$

with the parameter γ given in (8.1.46).

As for the previous case, μ goes to a finite non-zero value, γ, at infinity. To find the mass of the solution one must then introduce coordinates $\hat\tau$ and $\hat t$ as in (E.6). It is also convenient to use the form (8.3.1) for the Kerr-Taub-Bolt metric and to rewrite the one-form k as

$$k = \mu(d\tau + P'_\theta d\phi) + \nu' d\phi \,, \tag{E.21}$$

with

$$\nu' = \nu - \alpha\frac{\Xi}{\Delta_\theta}\sin^2\theta\mu = \alpha\frac{q_1q_2q_3(m+N)}{2(m-N)^2\Delta_\theta}\Big(1-\frac{r}{r_+}\Big)\sin^2\theta \,. \tag{E.22}$$

One can then rewrite the five-dimensional metric in a form ready for Kaluza-Klein reduction along $\hat\tau$:

$$ds^2 = \frac{\hat I_4}{(ZV)^2}\Big(d\hat\tau + \hat P_\theta d\phi - \frac{\mu V^2}{\hat I_4}(d\hat t + \hat\nu d\phi)\Big)^2 + \frac{VZ}{\hat I_4^{1/2}}ds_E^2 \,, \tag{E.23}$$

where

$$ds_E^2 = -\hat I_4^{-1/2}(d\hat t + \hat\nu d\phi)^2 + \hat I_4^{1/2}\Big(\frac{\Delta_\theta}{\Delta}dr^2 + \Delta_\theta d\theta^2 + \Delta\sin^2\theta d\phi^2\Big) \tag{E.24}$$

and

$$\hat I_4 = (1-\gamma^2)^{-1}(Z_1Z_2Z_3V - \mu^2V^2) \,, \quad \hat P_\theta = (1-\gamma^2)^{1/2}P'_\theta \,, \quad \hat\nu = (1-\gamma^2)^{-1/2}\nu' \,. \tag{E.25}$$

From this and (E.8), one can read off the mass, M and the four-dimensional angular momentum, J:

$$\begin{aligned} G_4 M &= \frac{1}{4(1-\gamma^2)}\Big[2m - \frac{q_1q_2+q_1q_3+q_2q_3}{m-N} \\ &\quad + \frac{q_1q_2q_3(q_1+q_2+q_3)}{2(m-N)^3}\Big(2-\frac{m+N}{r_+}\Big) - \frac{(q_1q_2q_3)^2}{4(m-N)^5}\Big(2-\frac{m+N}{r_+}\Big)^2\Big], \quad \text{(E.26)} \\ G_4 J &= -\frac{1}{(1-\gamma^2)^{1/2}}\frac{\alpha q_1q_2q_3(m+N)}{4(m-N)^2 r_+}. \quad \text{(E.27)} \end{aligned}$$

The geometry also carries Kaluza-Klein electric charge, Q_e, and the Kaluza-Klein magnetic charge, Q_m, given by (E.10)

$$\begin{aligned} G_4 Q_e &= -\frac{1}{8(1-\gamma^2)(m-N)^3}\Big[q_1q_2q_3\Big(\frac{m}{2}-2N+\frac{N^2-m^2}{r_+}\Big) \\ &\quad +\frac{1}{2}(q_1q_2+q_1q_3+q_2q_3)(q_1+q_2+q_3)(m+2N)+\frac{1}{2}(q_1^3+q_2^3+q_3^3)m \\ &\quad -\frac{1}{2}q_1q_2q_3(q_1q_2+q_1q_3+q_2q_3)\frac{m+2N}{(m-N)^2}\Big(2-\frac{m+N}{r_+}\Big) \\ &\quad -\frac{1}{4}q_1q_2q_3(q_1^2+q_2^2+q_3^2)\frac{2m+N}{(m-N)^2}\Big(2-\frac{m+N}{r_+}\Big) \\ &\quad +\frac{1}{8}q_1^2q_2^2q_3^2(q_1+q_2+q_3)\frac{m+2N}{(m-N)^4}\Big(2-\frac{m+N}{r_+}\Big)^2 \\ &\quad -\frac{1}{16}q_1^3q_2^3q_3^3\frac{N}{(m-N)^6}\Big(2-\frac{m+N}{r_+}\Big)^3\Big], \quad \text{(E.28)} \\ G_4 Q_m &= (1-\gamma^2)^{1/2}\frac{N}{2}, \quad \text{(E.29)} \end{aligned}$$

where G_4 is the four-dimensional Newton's constant

$$G_4 = \frac{G_5}{(1-\gamma^2)^{1/2}(8\pi N)}. \quad \text{(E.30)}$$

Finally, the rest mass of the Kerr-Taub-Bolt solution is

$$M_0 \equiv (1-\gamma^2)^{-1/2}(M-\gamma Q_e) = \frac{4\pi N}{G_5}(m+N) + Q_m + \frac{1}{16\pi G_5}(Q_1+Q_2+Q_3). \quad \text{(E.31)}$$

The asymptotic charges for the Reissner-Nordström solution

For the solution based on the Reissner-Nordström base, we changed a bit the conventions, and thus the magnetic charges are given, from (8.2.20), (8.2.26) and (8.2.27), by

$$\begin{aligned} d_1 &= 2q_1 - p + q, \\ d_2 &= 2q_2 - p + q, \\ d_3 &= p + q. \end{aligned} \quad \text{(E.32)}$$

From (E.4), one has

$$
\begin{aligned}
Q_1 &= -\frac{16\pi^2 r_+^2}{r_+ - r_-}\left(\frac{2(p+q)q_2}{m} + \gamma(q+q_2)\right), \\
Q_2 &= -\frac{16\pi^2 r_+^2}{r_+ - r_-}\left(\frac{2(p+q)q_1}{m} + \gamma(q+q_1)\right), \\
Q_3 &= -\frac{16\pi^2 r_+^2}{r_+ - r_-}\left(\frac{4q_1q_2}{m} + \gamma(q_1+q_2+p-q) + \frac{2(p-q)(q+q_1+q_2)}{m} - \frac{4q_1q_2(p^2-q^2)}{m^3}\right).
\end{aligned} \tag{E.33}
$$

with γ given by (8.2.35). To compute the mass and the Kaluza-Klein (KK) electric charge of the solution one reduces the solution to four dimensions

$$ds_5^2 = \frac{g^2}{Z^2}\hat{I}_4\Big[d\hat{\tau} + \Big(\gamma - \frac{\mu}{g^2\hat{I}_4}\Big)d\hat{t}\Big]^2 + \frac{Z}{g\hat{I}_4^{1/2}}ds_E^2\,, \tag{E.34}$$

where we have defined,

$$g = 1 - \frac{2m}{r} + \frac{p^2-q^2}{r^2}\,, \qquad \hat{I}_4 = \frac{1}{1-\gamma^2}\left(g^{-1}Z^3 - g^{-2}\mu^2\right), \tag{E.35}$$

and

$$ds_E^2 = -\hat{I}_4^{-1/2}d\hat{t}^2 + \hat{I}_4^{1/2}\Big[dr^2 + gr^2(d\theta^2 + \sin^2\theta d\phi^2)\Big] \tag{E.36}$$

is the four-dimensional Einstein metric. From (E.8) one can read off the mass of the solution

$$
\begin{aligned}
M \;=\; & \frac{1}{G_4(1-\gamma^2)}\Big[\frac{m}{2}(1-2\gamma^2) - \frac{q_1q_2 + pq_1 + pq_2 + \frac{q(p-q)}{2}}{m} \\
& -\gamma(q_1+q_2+\frac{p+q}{2}) + \frac{q_1q_2(p^2-q^2)}{m^3}\Big],
\end{aligned} \tag{E.37}
$$

with

$$G_4 = \frac{G_5}{(1-\gamma^2)^{1/2}}\frac{(r_+ - r_-)}{4\pi r_+^2}\,, \tag{E.38}$$

The KK electric charge, Q_e, given by (E.10) is

$$
\begin{aligned}
Q_e \;=\; & -\frac{1}{G_4(1-\gamma^2)}\Big[\gamma\frac{m}{2} + \gamma\frac{q_1q_2 + pq_1 + pq_2 + \frac{q(p-q)}{2}}{m} + \frac{1+\gamma^2}{2}(q_1+q_2+\frac{p+q}{2}) \\
& -\gamma\frac{q_1q_2(p^2-q^2)}{m^3}\Big].
\end{aligned} \tag{E.39}
$$

Finally it is instructive to compute the rest-mass, M_0, of the solution, i.e. the mass with respect to the (t,τ) frame

$$M_0 \equiv (1-\gamma^2)^{-1/2}(M - \gamma Q_e) = \frac{1}{16\pi G_5}\left(\frac{32\pi^2 r_+^2 m}{r_+ - r_-} + Q_1 + Q_2 + Q_3\right). \tag{E.40}$$

Note also that despite the fact that we start our construction from a four-dimensionnal black hole with a magnetic charge p, A_{KK} has a component only along $d\hat{t}$, which implies that the final solution does not carry any global magnetic charge.

The asymptotic charges for the Kerr-Newman-NUT solution

We finally give the asymptotics of the Kerr-Newman solution: first of all, the magnetic charges are given by the same formulae as for the Reissner-Nordtröm case

$$\begin{aligned} d_1 &= 2q_1 - p + q\,, \\ d_2 &= 2q_2 - p + q\,, \\ d_3 &= p + q\,. \end{aligned} \tag{E.41}$$

We now have to compute the electric charges Q_I. They are still given by the general formula (E.4), which yields

$$\begin{aligned} Q_1 &= -\frac{8\pi^2}{\kappa}\Big(\frac{2(p+q)q_2}{m-N} - \gamma(q+q_2)\Big)\,, \\ Q_2 &= -\frac{8\pi^2}{\kappa}\Big(\frac{2(p+q)q_1}{m-N} - \gamma(q+q_1)\Big)\,, \\ Q_3 &= -\frac{8\pi^2}{\kappa}\Big(\frac{4q_1q_2}{m-N} - \gamma(q_1+q_2+p-q) \\ &\quad +2(q+q_1+q_2)\frac{p-q}{m-N} - \frac{4q_1q_2(p^2-q^2)}{(m-N)^3} - \lambda(m-N)\Big)\,. \end{aligned} \tag{E.42}$$

As for the non-rotating case, μ goes to a finite non-zero value, γ, at infinity. One therefore has to introduce the coordinates $(\hat{t},\hat{\tau})$, given by (E.6) in order to compute the mass, angular momentum and KK charge of the solution. It is also convenient to use the form (8.3.1) for the Kerr-Newman-Taub-Bolt metric, with Δ given by (8.2.47) and to rewrite the one-form k as

$$k = \mu(d\tau + P'_\theta d\phi) + \nu' d\phi\,, \tag{E.43}$$

with

$$\nu' = \nu - \alpha\frac{\Sigma}{\Delta_\theta}\sin^2\theta\,\mu\,. \tag{E.44}$$

One can now rewrite the five-dimensional metric in a form suitable for Kaluza-Klein reduction along $\hat{\tau}$

$$ds_5^2 = \frac{\hat{I}_4}{(ZV)^2}\Big(d\hat{\tau} + \Big(\gamma - \frac{\mu V^2}{\hat{I}_4}\Big)d\hat{t} + \Big(\hat{P}_\theta - \frac{\hat{\nu}\mu V^2}{\hat{I}_4}\Big)d\phi\Big)^2 + \frac{ZV}{\hat{I}_4^{1/2}}ds_E^2\,, \tag{E.45}$$

where

$$ds_E^2 = -\hat{I}_4^{-1/2}(d\hat{t} + \hat{\nu}d\phi)^2 + \hat{I}_4^{1/2}\Big(\frac{\Delta_\theta}{\Delta}dr^2 + \Delta_\theta d\theta^2 + \Delta\sin^2\theta d\phi^2\Big) \tag{E.46}$$

is the four-dimensional Einstein metric and

$$\hat{I}_4 = (1-\gamma^2)^{-1}(Z_1Z_2Z_3V - \mu^2V^2)\,, \quad \hat{P}_\theta = (1-\gamma^2)^{1/2}P'_\theta\,, \quad \hat{\nu} = (1-\gamma^2)^{-1/2}\nu'\,. \tag{E.47}$$

From this metric, it is easy to read off the mass, M

$$\begin{aligned} M &= \frac{1}{G_4(1-\gamma^2)}\Bigg[\frac{m}{2} - (m-N)\gamma^2 - \frac{q_1q_2 + pq_1 + pq_2 + \frac{q(p-q)}{2}}{m-N} \\ &\quad +\gamma\left(q_1 + q_2 + \frac{p+q}{2}\right) + q_1q_2\frac{p^2-q^2}{(m-N)^3} + \frac{\lambda}{4}(m-N)\Bigg]\,, \end{aligned} \tag{E.48}$$

with

$$G_4 = \frac{G_5}{(1-\gamma^2)^{1/2}} \frac{\kappa}{2\pi} \,. \tag{E.49}$$

From (E.46), one can also read off the angular momentum of the solution

$$J = \frac{\alpha}{G_4(1-\gamma^2)^{1/2}} \left(-\frac{2q_1q_2(p+q)}{(m-N)^2} + \left(q_1 + q_2 + \frac{p+q}{2} \right) - \gamma(m-N) \right) \,. \tag{E.50}$$

We finally need the Kaluza-Klein electric and magnetic charges Q_e and Q_m, encoded in the one-form

$$A_{KK} = \left(\gamma - \frac{\mu V^2}{\hat{I}_4} \right) d\hat{t} + \left(\hat{P}_\theta - \frac{\hat{\nu}\mu V^2}{\hat{I}_4} \right) d\phi \,. \tag{E.51}$$

This gives

$$\begin{aligned} Q_e \;=\; & \frac{1}{G_4(1-\gamma^2)} \Bigg[-\gamma\frac{m}{2} + \gamma(1+\gamma^2)\frac{N}{2} - \gamma \frac{q_1q_2 + pq_1 + pq_2 + \frac{q(p-q)}{2}}{m-N} \\ & + \frac{1+\gamma^2}{2} \left(q_1 + q_2 + \frac{p+q}{2} \right) + \gamma q_1 q_2 \frac{p^2-q^2}{(m-N)^3} + \gamma\frac{\lambda}{4}(m-N) \Bigg] \,, \end{aligned} \tag{E.52}$$

and

$$Q_m = -(1-\gamma^2)^{1/2} \frac{N}{2G_4} \,. \tag{E.53}$$

Finally, one can compute the rest mass of the solution

$$M_0 \equiv (1-\gamma^2)^{-1/2}(M - \gamma Q_e) = \frac{\pi}{G_5\kappa}(m+N) + Q_m + \frac{1}{16\pi G_5} \left(Q_1 + Q_2 + Q_3 \right) \,. \tag{E.54}$$

It is clear from this expression that the solution has the same mass and charges as a non-extremal black hole. The mass of the five-dimensional solution is a sum of the electric charges and the solitonic charges of the four-dimensional base. The dependence of the mass on the charges is still linear due to the “floating brane” Ansatz.

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