Stochastic geometric networks : connectivity and comparison.

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Stochastic geometric networks : connectivity and comparison

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To my father

I do not know what I may appear to the world, but to myself I seem to have been only like a boy playing on the sea-shore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.

Sir Isaac Newton.

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It is ten years since i left school and of the many things i wanted to do then, a Ph.D.

in mathematics did not even feature in my far-fetched dreams. For the unexpected tryst with mathematics in the ensuing decade, the catalysts were the inspirational mathematics teachers at my alma-mater - Sri Sathya Sai Institute of Higher Learning, Puttaparthi. Many thanks to Prof. Venkateswarulu, Prof. Dev Sen, Prof. Pallabh Baruah, Dr. D. J. Das and Dr. J. Balasubramaniam for their courses and guidance. As concerns research in mathematics, the two special courses that clinched the deal for me were Topology by Prof. C.J.M. Rao and Analysis by Prof. G.V. Prabhakar Rao. Prof. Krishna Tiwari has been of constant support to me since my undergraduate days. The lenses of time have only amplified to me the care and affection shown by all my teachers at Puttaparthi. I would not want to forget the efforts of my school teachers either and especially my mathematics teachers, of whom i have always had the least complaints.

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I resist the temptation to wax eloquent on my mother and brother for their role in my development as mere words hardly qualify for such an exercise. And it would be gross injustice not to acknowledge the soothing effect of indian classical music and bhajans on my frayed nerves.

Despite my best efforts, if anyone thinks that i have ungratefully chosen not to acknowledge them or have not acknowledged them enough, please accept my sincere apologies and i hope to have another occassion in future to express my thankfulness to you.

Contents

Résumé

Le cadre de la géométrie aléatoire pour les réseaux comporte deux objets : un processus ponctuel Φ dans $\mathbb{R}^d$, l'espace euclidien de dimension d, et une fonctionelle $F(X, Y, \Phi) \in \{0, 1\}$, definie pour toutes les paires de points $X, Y \in \Phi$. La valeur $F = 1$ se traduit par l'existence d'une arête entre les points X et Y. F pourra également dépendre des marques éventuelles du processus Φ. Dans cette thèse nous nous intéressons aux problèmes de connexité et particulièrement à l'existence d'une composante infinie du graph engendré par Φ et F. On appelle ce dernier phénomène la percolation du graph. Dans le langage des communications, les points de Φ représentent les nœuds du réseau et les arêtes les communications possibles entre ces nœuds. Pour modéliser le réseau géométrique, le processus Φ le plus fréquemment utilisé est le processus ponctuel de Poisson. Quant à la fonctionnelle F, le choix le plus naturel est celui du modèle booléen, où $F(x, y, \Phi) = \mathbf{1}[|x - y| \leq 2r]$: seuls les nœuds situés à une distance au plus $2r$ peuvent communiquer. Nous nous intéressons dans cette thèse à des extensions par rapport au choix de ces deux eléments fondamentaux Φ et F. Dans un premier temps, nous considérons des fonctionelles F plus complexe, définies sur deux processus ponctuels de Poisson indépendants, ce qui nous conduit à un nouveau modèle booléen, dit "de type AB". En second lieu, nous nous intéressons à des modèles non poissoniens. Nous faisons appel à la theorie des ordres stochastiques pour élaborer des outils permettant la comparaison systématique des processus ponctuels en fonction du comportement d'une certaine classe de fonctionnelles. En particulier, cela nous permet de comparer qualitativement les performances des réseaux non-poissoniens à celles des réseaux poissoniens. Voici un bref aperçu des résultats décrits dans les quatre derniers chapitres.

Le modèle Booléen dit "de type AB" : Étant donnés deux processus ponctuels de Poisson indépendants $\Phi^{(1)}$ et $\Phi^{(2)}$ sur $\mathbb{R}^d$, le modèle booléen de type AB est le graphe

biparti dans lequel $X \in \Phi^{(1)}$ est relié à $Y \in \Phi^{(2)}$ si et seulement si $|X - Y| \leq 2r$. Il s'agit d'une généralisation du modèle de percolation AB sur les grilles. Dans le chapitre 3, nous montrons l'existence d'une percolation pour toute dimension $d \geq 2$, et nous établissons des bornes pour l'intensité critique. Dans le cas $d = 2$, nous caractérisons exactement l'intensité critique. Afin d'étudier les propriétés de connexité de ce réseau infini, nous considérons le modèle booléen fini de type AB avec rayon r (au lieu de $2r$) sur deux processus ponctuels de Poisson indépendants dans le cube unité $[0,1]^d$, avec intensités respectives n et cn. Lorsque $n \to \infty$, nous établissons une loi faible pour la plus grande distance au plus proche voisin, ainsi que des bornes asymptotiques presque sûres pour le seuil de connexité. La section 1.2.1 contient une description des réseaux de communications réels ayant motivé cette étude.

L'ordre directionnellement convexe des processus ponctuels : L'ordre stochastique que l'on utilisera pour comparer les processus ponctuels est l'ordre directionnellement convexe (dcx) : Etant donnés une classe $\mathfrak{F}$ de fonctions réelles et deux vecteurs aléatoires X, Y, nous écrirons $X \leq_{\mathfrak{F}} Y$ si pour tout $f \in \mathfrak{F}$, $\mathsf{E}(f(X))$ et $\mathsf{E}(f(Y))$ existent et satisfont $\mathsf{E}(f(X)) \leq \mathsf{E}(f(Y))$. Nous dirons que X est inférieur à Y pour l'ordre $\mathfrak{F}$. L'ordre directionnellement convexe sur les vecteurs alatoires est l'ordre $\leq_{\mathfrak{F}}$ obtenu en prenant pour $\mathfrak{F}$ la classe des fonctions *dcx*. Cet ordre est utile pour comparer la structure de dépendence des vecteurs aléatoires en prenant en compte la variabilité de leurs marginales. Il s'étend naturellement aux champs aléatoires par comparaison de toutes leurs marginales finidimensionnelles. Pour definir l'ordre *dcx* sur les mesures aléatoires localement finies, nous verrons celles-ci comme des champs aléatoires positifs indexés par les ensembles boréliens bornés. C'est précisément cet ordre *dcx* sur les mesures aléatoires dans les espaces Polonais qui fait l'objet de notre étude dans le chapitre 4.

Nous montrons que l'ordre *dcx* est preservé par certaines transformations naturelles comme la translation deterministe des points, la superposition indépendante, le marquage ou la suppression indépendante de points. Nous prouvons également que certaines transformations plus sophistiquées, comme un marquage ou une translation qui dépend de la position du point, sont encore preservées par l'ordre *dcx* pourvu que l'on se restreigne aux processus ponctuels de Cox. Nous traitons également l'effet de l'ordre sur les propriétés de second moment, notamment la tendance au regroupement des points et les mesures de Palm. Tout semble indiquer que la tendance au regroupement est d'autant plus importante

que le processus est élevé pour l'ordre *dcx*.

Les champs intégraux dits de bruit grenaille (shot-noise) de mesures aléatoires sont des champs aléatoires qui peuvent s'écrire comme l'intégrale d'un noyau mesurable par rapport à une mesure aléatoire. Le résultat principal du chapitre 4 est que des champs intégraux de bruit grenaille engendré par des mesures aléatoires ordonnées selon l'ordre *dcx* sont aussi ordonnés selon l'ordre *dcx*. Les champs dit extrémaux de bruit grenaille de processus ponctuels sont les champs aléatoires qui s'écrivent comme le supremum d'un noyau sur tous les points d'un processus ponctuel. Lorsque ces processus ponctuels sont ordonnés selon l'ordre *dcx*, nous montrons que les champs extrémaux résultants sont toujours ordonnés selon l'ordre dit orthant inférieur. Pour conclure de la chapitre 4, nous expliquons pourquoi d'autres ordres – supermodulaire et convexe par rapport à chaque coordonées – ne sont pas appropriés à la comparaison de processus ponctuels.

Exemples et Applications : Le but du chapitre 5 est de fournir des exemples de mesures aléatoires et plus particulièrement de processus ponctuels ordonnés en *dcx*. Nous exhibons des processus ponctuels supérieurs en *dcx* aux processus de Poisson (i.e. ayant une plus forte tendance au regroupement des points) et des processus ponctuels inférieurs en *dcx* aux processus de Poisson (i.e. ayant une plus forte tendance à l'éparpillement des points). Parmi les premiers se trouvent notamment les processus ponctuels d'Ising-Poisson et les processus ponctuels de Poisson regroupés. Quant-aux seconds, ils incluent en particulier les perturbations de grilles régulières. Nous décrivons également certaines fonctionnelles *dcx* de champs intégraux de bruit grenaille utilisées dans les réseaux sans fil : couverture des reseaux de capteurs, degré d'un nœud typique dans un réseau géométrique aléatoire, probabilité de connectivité dans les réseaux SINR.

Percolation et ordre *dcx* : Dans le chapitre 6 nous revenons aux problèmes de percolation et nous montrons comment exploiter l'ordre *dcx* sur les processus ponctuels pour comparer des propriétés de percolation dans des modèles de percolation qu'ils engendrent. Après avoir étudié l'effet de l'ordre *dcx* sur des fonctionnelles locales comme la couverture, une question naturelle consiste à étudier l'effet de l'ordre sur des fonctionnelles globales, au premier rang desquelles figure évidemment la percolation. Nous commençons par comparer les fonctions de capacité des modèles booléens. Ce résultat nous conduit à définir deux nouveaux rayons critiques, qui sont des bornes respectivement inférieure et supérieure du rayon critique de percolation. Nous montrons que la borne supérieure croît avec l'ordre

dcx, tandis que la borne inférieure décroît. Ces deux résultats restent vrais si nous comparons des processus ponctuels de Cox aux processus ponctuels de Poisson. Le résultat pour la borne inférieure reste vrai si nous comparons des processus ponctuels determinantaux aux processus ponctuels de Poisson.

Plusiers modèles de percolation comme la k-percolation, la percolation de mots et la percolation SINR sont définis par des champs intégraux de bruit grenaille de processus ponctuels. Si l'on souhaite utiliser l'argument de Peierls pour établir l'existence d'une transition de phase non-triviale dans ces modèles, il faut d'abord obtenir des bornes sur les probabilités de depassement de niveau pour les champs intégraux de bruit grenaille. Nous obtenons ces bornes en comparant la transformée de Laplace du champ intégral de bruit grenaille en question 'a celle du champ intégral de bruit grenaille induit par un processus ponctuel de Poisson. Ces bornes sont valides dès que le processus ponctuel considéré est inférieur au processus ponctuel de Poisson pour l'ordre dcx. À l'aide d'une approximation discrète appropriée, nous déduisons finalement l'existence d'une transition de phase pour cette classe de modèles.

Mots-clés : processus ponctuels, percolation, modèle booléen, graphe géométrique aléatoire, measures aléatoires, champs aléatoires, l'ordre directionellement convexe.

Abstract

The stochastic geometric setting for a network consists of two components : a point process Φ in $\mathbb{R}^d$, the d-dimensional Euclidean space and then a functional $F(X, Y, \Phi) \in \{0, 1\}$ for any pair of points $X, Y \in \Phi$. If $F = 1$, we say that an edge exists between the two points. In many cases, the point process could have marks associated with the points and in such cases the functional F shall depend on the marks. This thesis focuses on connectivity problems in a network and in particular, the existence of an infinite component in the graph generated by Φ and F. This latter phenomenon is called as percolation of the graph. In network context, the points of the point process are nodes of the network and an edge between a pair of points indicates that the nodes can communicate with each other. The simplest continuum model is the Boolean model in which Φ is a point process in $\mathbb{R}^d$ and $F(x, y, \Phi) = \mathbf{1}[|x-y| \leq 2r]$ i.e, edges exist between points that are at a distance $2r$ of each other. A ubiquitous assumption in stochastic geometric networks is that Φ is a Poisson point process. In this thesis, we study extensions in the above set-up with respect to the choice of the two components – point process Φ and functional F. Firstly, we consider a functional F acting on two independent Poisson point processes and define what we call the AB Boolean model. Secondly, we construct tools for comparison of certain functionals of point processes using the theory of stochastic ordering for random vectors and thereby compare behaviour of networks with non-Poissonian node distributions to those with Poissonian node distributions. In what follows, we shall provide a brief summary of the results obtained in the thesis.

AB Boolean model : Given two independent Poisson point processes $\Phi^{(1)}, \Phi^{(2)}$ in $\mathbb{R}^d$, the AB Poisson Boolean model is the graph with edges only between $X \in \Phi^{(1)}$ and $Y \in \Phi^{(2)}$ iff $|X-Y| \leq 2r$. This is a generalization of the AB percolation model on discrete lattices. In chapter 3, we show the existence of percolation for all $d \geq 2$ and derive bounds

for a critical intensity. We also provide a characterization for this critical intensity when $d = 2$. To study the connectivity problem, we consider independent Poisson point processes of intensities n and cn in the unit cube. The AB random geometric graph is defined as above but with balls of radius r. We derive a weak law result for the largest nearest neighbour distance and almost sure asymptotic bounds for the connectivity threshold. For network models that resemble this model, see Section 1.2.1.

Directionally convex ordering of point processes : The stochastic order that we shall use to compare point processes is the directionally convex (*dcx*) order. For a class of real-valued functions, $\mathfrak{F}$ and two random vectors X, Y, we say that $X \leq_{\mathfrak{F}} Y$ (i.e, X is less than Y in $\mathfrak{F}$ order) if $\mathsf{E}(f(X)) \leq \mathsf{E}(f(Y))$ for all $f \in \mathfrak{F}$ and the expectations exist. Directionally convex order is the order defined when $\mathfrak{F}$ is taken to be the class of *dcx* functions. Directionally convex ordering is a tool for comparison of dependence structure of random vectors that also takes into account the variability of the marginal distributions. When extended to random fields it concerns comparison of all finite dimensional distributions. Viewing locally finite measures as non-negative fields of measure-values indexed by the bounded Borel subsets of the space, in the chapter 4 we formulate and study the *dcx* ordering of random measures on Polish spaces. We show that the *dcx* order is preserved under some of the natural operations considered on random measures and point processes, such as deterministic displacement of points, independent superposition and thinning as well as independent, identically distributed marking. Further operations such as the position dependent marking and displacement of points are shown to preserve the order on Cox point processes. We also examine the impact of *dcx* order on the second moment properties, in particular on clustering and on Palm distributions. Comparisons of Ripley's functions, pair correlation functions as well as examples seem to indicate that point processes higher in *dcx* order cluster more.

Integral shot-noise fields of random measures are integrals of a measurable kernel with respect to the random measure. As the main result of the chapter 4, we show that integral shot-noise fields of non-negative kernels with respect to *dcx* ordered random measures inherit this ordering from the measures. Extremal shot-noise fields of point processes are supremum of a kernel over a point process. We show that extremal shot noise fields of *dcx* ordered point processes are lower orthant ordered. Further, it is shown that two other dependence orders – supermodular and componentwise convex – for random vectors, when

extended to point processes are stronger than coupling or called as strong ordering in the terminology of stochastic ordering.

Examples and Applications : In the chapter 5, we give examples of random measures and especially point processes that are *dcx* ordered. We give examples of point processes that are greater as well as lesser in *dcx* order than the Poisson point process. The clustered point processes are Ising-Poisson cluster point process, Poisson-Poisson cluster point process, Generalized shot-noise cox point process, Lévy-based Cox point process and Log Cox point process while the sparse point processes are perturbed lattices and generalizations. We also give examples of *dcx* functionals of shot-noise fields generated by point processes that are used in wireless communication networks. These functionals pertain to coverage in sensor networks, typical vertex degree in geometric networks and connectivity probability in SINR networks.

Percolation and *dcx* order : Finally in chapter 6, we turn our attention to the relation between percolation and *dcx* order. After studying impact of *dcx* order on local functionals such as coverage, a natural question is to study the effect on global functionals and percolation is the obvious candidate of interest for network theorists. The starting point is the ordering of capacity functionals of Boolean models. With this, we define two newer critical radii which are lower and upper bounds respectively to the critical radius for percolation. We show that the lower bound is preserved and the upper bound is reversed by the *dcx* order. As a corollary we get that when the upper and lower bounds are on-trivial for Poisson point process, so are they for sub-Poisson point processes (point processes lesser than Poisson in *dcx* order). Both the results are also proved for any Cox point process in comparison with the Poisson point process. The ordering of the lower bound also holds true for determinantal point processes in comparison with Poisson point process.

Many percolation models such as k-percolation, word percolation and SINR percolation are defined in terms of integral shot-noise fields of point processes. To show non-trivial phase transitions in such models, one needs bounds on level crossing probabilities of the integral shot-noise field and then it is possible to use Peierls argument. These bounds are obtained by using their Laplace transforms and bounding them by the Laplace transforms of the Poisson shot-noise fields. For sub-Poisson point processes, using these bounds and suitable discrete approximations, we show the existence of phase

transition in the models described above. We prove these phase transitions by obtaining uniform (over all sparse point processes with the same mean measure) upper and lower bounds for the critical parameters.

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KEYWORDS : point processes, percolation, Boolean model, random geometric graphs, random measures, random fields, directionally convex order.

Chapter 1

Introduction

This thesis aims to answer some questions arising in the percolation theoretic studies of communication networks. Percolation is the mathematical term for existence of long-range multi-hop communication in networks. Before describing the mathematical framework, in this introductory chapter consisting of two parts, we will informally introduce percolation and then provide a brief overview of the thesis. Let us outline the contents of each of the two parts.

The first part (Section 1.1) will elucidate on the appearance and usage of stochastic geometric tools in communication networks. Without being unduly reliant on specialist terminology, we shall describe the basic stochastic models (both discrete and continuum) used to study connectivity and long-range communication in wireless networks. We shall also provide a flavour of the percolation theoretic results in these basic models.

In the second part (Section 1.2), the focus is on a few problems of interest to us and some of their answers that can be found within the pages of this thesis. Section 1.2.1 motivates these problems of interest and in parallel, surveys the efforts of other people in similar directions that have greatly aided our approach to the problems. A more extended version of the abstract with a detailed chapter-wise presentation of results can be found in Section 1.2.2. In that section, we highlight the important results in the thesis by stating them rigorously but leaving the most general formulation to the respective chapters.

1.1 Stochastic models of communication networks

Many models we consider have been initially motivated by various other applications but we do not strive for an encyclopaedic presentation of the applications. We shall restrict ourselves to a few applications in wireless communications. Further, we warn the reader that the choice of models is greatly influenced by the contents of the thesis and hence we absolve ourselves of any perceived injustice to the varied stochastic models in vogue for wireless network applications. However, where we might have been frugal with our descriptions of models or applications, we have tried to be generous in citing the necessary literature.

A *network*, in the simplest terms, is a collection of points in space called *nodes or vertices* and a collection of node pairs called *edges*. The presence of an edge between two nodes indicates that they can communicate with each other. Communication between nodes is not necessarily bi-directional. The mathematical name for a network is *graph* and henceforth we shall alternate between the two terms – graph and network. The choice of nodes or edges can be random and it is here that probability makes its way into these models. Many models of communication networks also have a natural interpretation in terms of spread of epidemics. It suffices to note that in this context, edges between two nodes just mean that one node can infect the other node.

In the context of communications, the two basic questions one would like to ask about a graph are – (1) When is the graph connected ? and (2) What is the maximum amount of information that can be reliably communicated in the network per node ? The latter quantity is called the *capacity* and it is measured in units of information per unit of time. As we will hint later, recent methods do use graph connectivity to compute the capacity of the network. When we say a subset of vertices is connected, we mean that there exists a path (collection of edges) linking every vertex in the subset to every other vertex in the subset. By graph *connectivity*, we imply that the set of all vertices forms a connected set. Since an edge represents the presence of communication between nodes, connectivity of the graph implies that any two nodes in the network can communicate via intermediate nodes. Thus, we have made it clear as to why connectivity of a graph is important from a communication perspective. Once communication is estabilished between two nodes, it is important to know the amount of information they can communicate or transmit to each other. Since there are costs associated in building and maintaining connections between nodes in a network, it does not greatly benefit a network to have nodes connected but

transmitting very little information.

Apart from the randomness in the choice of nodes or edges, there could be further randomness in the graph arising from the medium or channel in which the nodes communicate. When nodes transmit information across the same channel, their transmissions interfere with each other and hence the presence of an edge worsens the chances of neighbouring edges. This in particular restrains too many nodes from communicating at the same time. The perfect analogy is the following scenario : Consider a room with people where each person is talking to his friend. If many people talk at the same time, it disadvantages everyone from listening clearly to their friends. In particular, a simple question would be how many people can talk to their friends at the same time despite the noise from others ? Such a question in a network is related to its capacity. We shall look at specific models of graphs which can satisfiably capture some of these phenomena and also be rigorously analysed to answer the questions of connectivity and capacity. Our goal is mostly to understand these phenomena in infinite graphs. The reason being that in practice one is interested in understanding the behaviour of large finite networks but in many cases these do not seem to be tractable unlike their limiting infinite network i.e, the network with infinite nodes.

1.1.1 Discrete networks

We shall start with discrete models not merely as a matter of customary lip service but also because as we will see in the thesis, results for spatial models rely heavily on discrete approximations. We shall be economical with our descriptions on the problem origin but the interested reader can read the excellent introduction to the recent book of [Franceschetti and Meester 2007] or the cited articles.

We mentioned a few lines earlier that one of the first questions of interest is connectivity. The question of connectivity in networks tends to veer between the trivial and the very hard. So first, we try to address the relatively easier question of percolation. By *percolation* one refers to existence of a giant component i.e, a connected subset of the graph with infinitely many nodes. Of course, the question makes sense only when the graph has infinitely many nodes and as already said, this is true for all of our graphs. Once a network percolates, it means that there can be long-range communication in the network i.e, nodes located very far away in the network can relay information to each other via intermediate nodes. Percolation has turned out to be a more reasonable question to answer in many

communication networks and starting with the next section, we shall see many models where study of percolation has turned out to be interesting and illuminating. Without further ado, let us see some percolation models.

Bernoulli percolation

This is the simplest of the percolation models. Let us consider an arbitrary connected graph G with V as the (infinite) set of vertices and E as the set of edges between the vertices. Now we delete each edge independently with a probability $1-p$. The simpler and a probabilist's favourite way of describing the procedure is as follows : At each edge one flips a coin and if it is heads (with probability p), the edge stays else it is removed. We call the resulting random sub-graph (i.e, the graph with the remaining edges after the edge-removal procedure) as $G(p)$. This is called the *Bernoulli bond percolation* model. The word 'bond' implies that the removal procedure is applied to the bonds or edges. If we were to apply the removal procedure to the sites or vertices, the resulting model is called the *Bernoulli site percolation* model. In the site percolation model, edges are kept only if both the end vertices are not removed. Both the bond and site percolation models shall be used in the thesis.

Consider the Bernoulli bond percolation model. Fix a vertex O in V. To introduce percolation into the picture, it helps to think of the edges as pipes that allow the flow of some fluid. Now percolation of the graph means that if one pours liquid into the pipe at O, will it flow to infinity with a positive probability ? This means that with a positive probability, there exists an infinite sequence of adjacent edges $e_1, e_2, \ldots$ with the edge e_1 originating from O such that none of them are deleted by the removal procedure. A little more formally, let us denote by $\theta_O(p)$ the probabilty that O is contained in an infinite component of $G(p)$. Then, by percolation we mean that $\theta_O(p) > 0$ i.e, with a positive probability, nodes located very far away are connected to O even after the edge-removal procedure. Before introducing other notions such as phase transition and critical parameters, we describe two examples of graphs and Bernoulli bond percolation on them. These models will help one to understand percolation better. In many graphs (including the two examples below), it is the case that the graph $G(p)$ is connected only when $p = 1$. Notice that our definition of percolation depends on the choice of vertex O. However, in any connected graph, for two vertices v_1 and v_2, $\theta_{v_1}(p) > 0$ iff $\theta_{v_2}(p) > 0$. Thus, one need not fret over the choice of O when studying percolation.

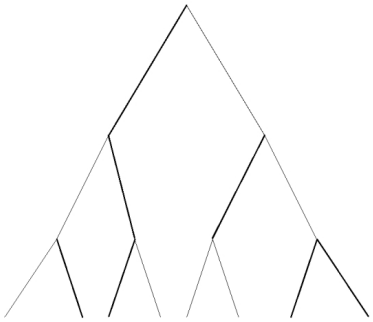

Figure 1.1: The binary tree i.e, Galton-Watson tree with exactly two children at every node. The edges that remain after the deletion procedure are shaded in black.

We also note here that this thesis will be oblivious to the *Erdös-Rényi random graphs* and its many flourishing variants that started with the works of [Ford and Uhlenbeck 1957, Gilbert 1959, Austin et al. 1959, Erdős and Rényi 1959]. The interested reader can refer to the recent comprehensive course notes [Hofstad 2010] or the books [Bollobás 2001, Durrett 2007]. For a one stop overview of percolation theory for the models in this chapter, we recommend [Bollobás and Riordan 2006].

Galton-Watson trees :

This simplest of random graphs is obtained by imposing a graph structure on the *Bienaymé-Galton-Watson processes*. The model can be described more easily in terms of a population growth process. Indeed, the widely known origin of the model is the study by Galton and Watson in the context of survival of family names (see [Watson and Galton 1874]). However, an earlier work on the same model by I.J.Bienaymé (1845) was discovered later. See [Heyde and Seneta 1977, Kendall 1975] for more on the history of branching processes.

We ask the reader to refer to Figure 1.1.1 for a respresentation of the following graph. Let us start with a node O called the root. It gives birth to X number of nodes where X is a random number. These X nodes are called as the children of O. Each of the children has further children and so on the procedure is repeated indefinitely. The number of children of a node is independent of other nodes and random with the same distribution as X. Give an edge between a child and its parent. This now gives a graph structure called the *Galton-Watson tree*. Call it GW. In all the following discussion, we omit the

trivial case when $X \equiv 1$ i.e, every parent has exactly one child. Now consider $GW(p)$, the p-thinned subgraph obtained after the edge-removal procedure as described above. The component of $GW(p)$ containing the root O will still be a Galton-Watson tree with the number of children being distributed as the random number $X_* := \sum_{i=1}^{X} Y_i$ where $Y_i = 1$ with probability p and else $Y_i = 0$. Let $\mathsf{E}(X)$ denote the expected (or average) number of children in the original Galton-Watson tree. Then $\mathsf{E}(X_*) = p\mathsf{E}(X)$.

It is clear that the graph corresponds to a population growth process. Such growth processes called the branching processes have a tendency to show up in various stochastic models. For a detailed study of branching processes, refer to [Athreya and Ney 1972, Harris 1963].

Percolation of the root in the graph is tantamount to non-extinction of the population in the corresponding growth process. The famous result for Galton-Watson trees is that the process survives with a positive probability only when the average number of children is strictly greater than one or every parent has exactly one child i.e, $\mathsf{E}(X) > 1$. Hence for $GW(p)$, percolation occurs (i.e, $\theta_O(p) > 0$) iff $p\mathsf{E}(X) > 1$. We must remark here that even if the graph looks very simple, it has turned to be a very useful tool. For example, the standard technique to show non-percolation in most graphs is to bound the growth of its connected component by a sub-critical branching process.

Grid percolation model :

We shall describe the following model on a planar grid and it does not require a great strech of imagination to construct the model in higher dimensions. The figure 1.1.1 conveys more easily the grid model delineated below in words. Let $\mathbb{L}^2 := (\mathbb{Z}^2, \mathbb{E}^2)$ be the graph with vertex set $\mathbb{Z}^2 := \{(i,j) : i,j \in \mathbb{Z}, \text{the set of integers}\}$ and $\mathbb{E}^2 := \{\langle (i,j),(i_*,j_*)\rangle : i = i_*, j = j_* \overset{+}{-} 1 \text{ or vice-versa}\}$. This the graph representing the grid.

The first percolation problem of this type appeared in [De Volson Wood 1894] where an incorrect solution was given and almost after half a century of dormancy, the subject had its rebirth in ([Broadbent and Hammersley 1957]). See [Kesten 2006, Kesten 2010] for a very accessible, up-to-date introduction to percolation theory. This classical Bernoulli bond percolation model on the grid $\mathbb{L}^2$ has been of interest to both mathematicians and physicists. For further reading, we direct the reader to [Grimmett 1999, Kesten 1982]. Let us at least describe one of the deeper and beautiful results in percolation theory : The random sub-graph of the grid, $\mathbb{L}^2(p)$ percolates if and only if $p > \frac{1}{2}$.

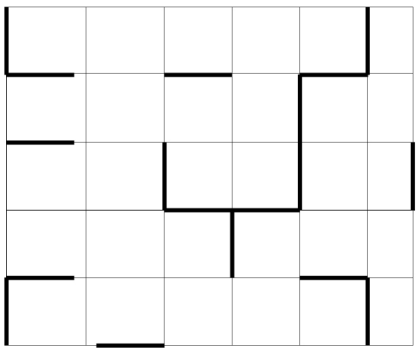

Figure 1.2: The network of pipes described below as $\mathbb{L}^2$. The pipes that remain after the removal procedure have been coloured black.

Phase transition and Critical values : Though phase transition and critical values are more generic terms in statistical physics literature used in a great many contexts, we shall describe them only in the context of percolation. However, this will give the reader an idea of these terms in their full generality too. These terms will be used often in the thesis and so we deem it worthwhile to explain them in this simple context.

In the Bernoulli bond percolation model on a graph, as we increase the parameter p, the percolation probability $\theta_O(p)$ increases. From this increasing property and the fact that $\theta_O(0) = 0$, $\theta_O(1) = 1$, we can say that there exists a real number $p_c, 0 \leq p_c \leq 1$ such that the graph $G(p)$ percolates ($\theta_0(p) > 0$) for $p > p_c$ and does not percolate ($\theta_0(p) = 0$) if $p < p_c$. We call such a p_c the *critical value*. It could so happen that $p_c = 0$ or $p_c = 1$. But such cases are generally less interesting to study. Thus one of the first aims in percolation studies of a graph is to show that $0 < p_c < 1$. The graph $G(p)$ is called sub-critical,super-critical or critical depending on whether $p < p_c$, $p > p_c$ or $p = p_c$ respectively. For the above two examples, $p_c = \frac{1}{\mathsf{E}(X)}$ for a Galton-Watson tree with X as the distribution of the children and $p_c = \frac{1}{2}$ for the grid. Further, observe that the Galton-Watson tree does not percolate at criticality i.e, $GW(\frac{1}{\mathsf{E}(X)})$ does not percolate. The same is also known to be true for the Grid model too. Though it is believed that percolation is absent at criticality in many models, it has been proven only for a few. Further, there is no reason for this statement to be true in general.

Percolation also provides an example of the phenomenon of *phase transition*. The latter term refers to a drastic change in a global property of the system (such as a graph in our

case) due to a small variation in some of the local parameters governing the system. Global property of a system, by its very name, is a property of the system that is not affected by change in its local structure. In the Bernoulli bond percolation model, the percolation probability $\theta_O(p)$ is a global property whereas the parameter p is a local parameter. From our above discussions, we see that the percolative property of the graph changes suddenly at $p = p_c$ i.e, a small increase in p beyond p_c leads to percolation of the graph. This is precisely what we earlier referred to as drastic change in a global property due to a small variation in a local parameter. It must be remarked here that in the case $\theta_O(p)$ is not monotonic, there is nothing to prevent multiple phase transitions. In such cases, we also have multiple critical values and multiple singularity points. When $0 < p_c < 1$, the model is said to have a non-trivial phase transition. Showing such non-trivial phase transitions in various continuum percolation models is one of the contributions of this thesis.

1.1.2 Stochastic geometric networks

As briefly previewed above, discrete models are interesting in their own right and have been crucial to various applications. In Galton-Watson trees as well as in Erdös-Rényi random graphs, there is no spatial location associated with the nodes. In the grid model, the node locations are deterministic but the nodes are regularly spaced. Though these assumptions are helpful to study the models but from the perspective of applications, it is also useful to construct models with more randomly spaced nodes and communication links depending on inter-node distances. This necessity leads one to continuum models i.e, models of network where the vertices are points randomly deployed in space and edges depend on the distance between vertices and locations of other vertices.

The one stark omission in this subsection is of tessellations. These models have influenced both the theory and applications of stochastic geometry. We shall refer to [Møller 1994] for a reading of this fascinating topic. For a more broader overview of applications of stochastic geometry in wireless networks, we refer the reader to [Baccelli and Błaszczyszyn 2009b]. The compiled volume [Eds. Kendall and Molchanov 2009] is a good starting point to learn about some recent developments in stochastic geometry. For more information on the percolation, connectivity, coverage and colouring of random geometric graphs, we point the accessible recent survey [Balister et al. 2009].

Boolean model :

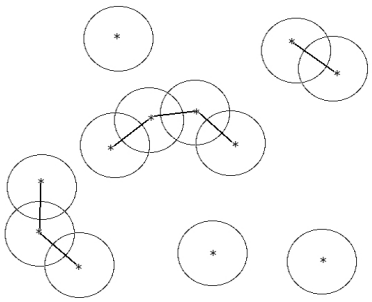

Figure 1.3: The Boolean model. Balls are centred at points (stars in the figure) and edges (lines in the figure) between points with intersecting balls.

The first simple stochastic geometric model of percolation was introduced and studied in [Gilbert 1961] as a model of radio communication networks. This model is now called the Boolean model or the Gilbert disk graph. Let $\Phi = \{X_i, i = 1, 2...\}$, a collection of random points in space (called the point process) be the set of vertices and edges are between any pair X_i, X_j within a distance 2 of each other i.e, $|X_i - X_j| \leq 2$. Alternatively, one places unit balls (balls of radius 1) at each point of Φ and connects two points if their balls intersect. Refer to Figure 1.1.2. The interpretation in terms of radio communication networks is that the points of the point process represent the locations of radio stations and stations can communicate to one another only if they are within a distance 2 of each other. In this context, percolation guarantees long-distance relaying of messages in the network. In most studies, Φ is assumed to be a Poisson point process of intensity $\lambda > 0$ and in such cases we shall call the model as Poisson-Boolean model. Informally, a Poisson point process can be constructed as follows : Divide the space into disjoint unit cubes. For each of them, independently choose a random number N according to Poisson(λ) distribution. Now throw these N points independently and uniformly over the unit cube. The collection of all these points from all the unit cubes form a Poisson point process of intensity λ. It has two significant properties - the number of points in different regions are independent and have a Poisson distribution - that make it amenable to computations. It was shown in [Gilbert 1961] that, there exists a small λ for which the Poisson-Boolean model does not percolate and also there exists a large λ for which it percolates. In a formal language,

one can define a critical intensity $\lambda_c(1)$ such that the Poisson-Boolean model percolates for $\lambda > \lambda_c(1)$ and does not percolate for $\lambda < \lambda_c(r)$. The above result of [Gilbert 1961] guarantees that in a Poisson-Boolean model $0 < \lambda_c(r) < \infty$ i.e, there is a non-trivial phase transition. Equivalently, one can fix λ and look for the least radius at which the Gilbert disk graph percolates. This critical radius is denoted by $r_c(\lambda)$. Thus, it is true that for $\lambda > 0$, we have $0 < r_c(\lambda) < \infty$. Further, due to scaling laws $(r_c(\lambda_1)^d\lambda_1 = r_c(\lambda_2)^d\lambda_2)$, it is good enough to study $r_c(1)$. Unfortunately though, even in this case exact values have remained elusive despite tighter bounds. Currently, the best known bounds are from [Hall 1985]. These are

$$0.834 \leq r_c(1) \leq 1.836.$$

Simulations ([Quintanilla et al. 2000]) have given the improved bounds

$$1.19845 \leq r_c(1) \leq 1.19846,$$

while in recent times, it has been shown in [Balister et al. 2005] that with confidence 99.99%,

$$1.19789 \leq r_c(1) \leq 1.19882.$$

For a very detailed presentation of percolation in Boolean model, refer to [Meester and Roy 1996]. Many variants of this basic Boolean model have been studied (see [Balister et al. 2009]). We shall sketch a variant called the SINR graph in this chapter (see Section 1.1.2) while another variant called the AB Boolean model is one of the contributions of this thesis (see Chapter 3). We shall briefly mention a third variant called the random connection model in the Section 1.2.1.

Random geometric graphs : For a detailed study of random geometric graphs, refer [Penrose 2003]. The Poisson-Boolean model (and Boolean models driven by any ergodic point process) will be fully connected if and only if the radius is infinite. Sometimes, the more pertinent question is connectivity in a large finite region rather than connectivity in the infinite space. Thus, one would like to know the radius required for connectivity of nodes in a large region (sparse network). The other equivalent formulation is to keep the region fixed and study the radius required for connectivity of a large number of nodes (dense network). The radii obtained from these two formulations differ only by a scaling factor.

The mathematical formulation of the two approaches are as follows : We either consider a Poisson point process of intensity 1 in a box of volume n or a Poisson point process of intensity n in a box of volume 1. Call the Boolean model formed on these two point processes with radius r as $G_n^1(r)$ and $G_n^2(r)$. Let the critical radii for connectivity in the two cases be $r_{ni}, i = 1, 2$ respectively. By r_{ni} being the critical radius, we mean that it is the least radius at which the corresponding graph is connected. It is known that for large n, r_{n1}^d is approximately $\frac{\log n}{\theta_d}$ and r_{n2}^d is approximately $\frac{\log n}{n\theta_d}$ where θ_d is the volume of a unit-radius ball in d-dimensions. In both the cases, the expected number of neighbours (average degree of a node in the random geometric graph) is $logn$. So, the connectivity result can be rephrased as follows : If the expected number of neighbours is more than $\log n$, the random geometric graph will be asymptotically connected with high probability and else asymptotically disconnected with high probability. An analogous result exists for Erdös-Rényi random graphs ([Bollobás 2001]).

Coverage processes : In a sensor network, the goal is to effectively monitor a given region. Thus, in such networks the interest is the region covered by the nodes in the network. In the simplest model, one can assume that each node can monitor its surrounding area at a distance r from itself. Thus, the total region covered can be expressed as $C = \bigcup_{X\in\Phi} B_X(r)$ where $B_X(r)$ denotes the ball of radius r centred at X and Φ the underlying point process. Such models are known as coverage processe and are the central objects of study in [Hall 1988]. For a wide class of point process (ergodic point processes to be more precise), both the volume of space covered as well as not covered will be infinite. Hence, we will use two approaches as in random geometric graphs for a better understanding i.e, fixing the point process intensity and increasing the volume or fixing the volume and increasing the point process intensity. As with random geometric graphs, let n be the intensity of the Poisson point process and r_n be the radius. For this Poisson driven coverage process, it is known that when $n\theta_d r_n^d \to c$(a constant), the volume of the covered region C in a region of unit volume is close to $(1 - \exp\{-c\})$ for a large n.

A performance guarantee one might demand a given set of sensors is the sensor power required for complete coverage of a given large region. As sensor power directly relates to the radius of the sensor in the Boolean model, we ask for the minimum radius at which one can guarantee complete coverage. One might be tempted to ask the same for complete k-coverage i.e, every point is to be covered by at least k sensors. More generally, it has

been shown that (see [Janson86] for very general results and see [Balister et al. 2010] for a shorter proof) in 2-dimensions, when $\pi r^2 = \log n + k \log\log n + x$, the probability that the region $[0, \sqrt{n}]$ is k-covered converges to $e^{-e^{-x}/(k-1)}$. This result gives a choice of radius r (by chosing x) such that we can achieve complete k-coverage with as high a probability as one wants to. In the above result, πr^2 is actually the expected number of neighbours in the corresponding Gilbert disk graph. Thus though connectivity is necessary for coverage, we see that a significantly higher radius than critical connectivity radius is needed for coverage.

SINR graphs :

The Boolean model is not sufficient for analyzing wireless networks as it ignores the fact that in radio communications signal received from one particular transmitter is jammed by the signals received from the other transmitters and the environmental noise. According to information theory as well as existing technology, the quality of a given radio communication link is determined by the so called *signal to interference and noise ratio* (SINR) at the receiver of this link. First we shall consider the scenario of only environmental noise and no interfering transmissions from other transmitters.

Suppose that transmitters send signals at a power P. This signal decays over distances, say as per a decreasing function $l(r)$. Thus, the signal received at a location x_j from the transmitter at x_i (and vice-versa) is $Pl(|x_i - x_j|)$ where $|x_i - x_j|$ denotes the distance between the two locations. The function $l(.)$ denoting the transmission gain factor decreases with distance i.e, the received power decreases as the distance increases. If the environmental noise is N, then the receiver at x_j can decode the signal (and vice-versa) from x_i if only $\frac{Pl(|x_i - x_j|)}{N} > T$. The ratio is called the *signal to noise ratio* (SNR) and the condition means that the the nodes at x_i and x_j can communicate with each other only when the SNR ratio is greater than a certain threshold T. One can construct the SNR graph using this condition. The nodes are points of a point process Φ and edges are placed between points if there is communication between the points as defined by the SNR ratio. This graph is called the SNR graph. It can be observed that this graph is equivalent to the Boolean model as there exists an edge between x_i and x_j only when $|x_i - x_j| \leq r_l = l^{-1}(\frac{TN}{P})$.

SNR graph is an accurate model when the nodes are placed far apart from each other or each pair of nodes have a seperate channel for communication. In such a case, the

interference from other nodes is minimal and hence can be neglected. Let us now also consider a model in which concurrent transmissions interfere with each other. We treat all other signals as noise and hence the total interference is the sum of all the transmissions received at a node plus the environmental noise. Thus, the SINR ratio at a receiver for a signal from a transmitter is the ratio of the received power from the transmitter to the interference and noise (environmental or background). A receiver can decode the signal from the transmitter (or a transmitter can communicate to a receiver) only if this SINR ratio is high enough. The SINR graph is the graph with vertices as a point process Φ and edges between points such that both the points can communicate with each other. Unlike the SNR graph or the Boolean model, the edges in this graph depend on other points and sometimes on all the points.

The foundations of the theory of SINR models are quite recent (see [Gupta and Kumar 2000, Baccelli and Błaszczyszyn 2001, Dousse et al. 2006, Baccelli et al. 2006]). From a mathematical point of view, the interference in the above considerations is called a shot-noise field. Shot-noise fields play a key role in determining the connectivity and the capacity of the network in a broad sense. Let us now then make a more mathematical formulation of the model. As above, let P be the signal power, $l(.)$ be the transmission loss factor, T the threshold, N be the environmental noise and γ be the interference factor. Thus the SINR ratio at node Y for a transmission from X is

$$SINR(X,Y) := \frac{Pl(|X-Y|)}{N + \gamma \sum_{Z\in\Phi, z\neq X,Y} Pl(|Z-Y|)}.$$

So two nodes X_i, X_j in Φ can communicate if only $SINR(X_i, X_j) > T$ and $SINR(X_j, X_i) > T$. The SINR graph without the interference term (i.e, $\gamma = 0$) is the SNR graph. In the above definition, we have considered only self-interference i.e, interference from other communicating nodes alone. But, in reality, there could be interference from other miscallaneous nodes as well. Suppose Φ denotes the set of communicating nodes, Φ_*, the additional set of interferers, then we denote the corresponding SINR network as $G(\Phi, \Phi_*, \gamma)$. Observe that SNR network is now nothing but $G(\Phi, \emptyset, 0)$.

Let us describe one of the interesting results as regards percolation of SINR graphs. Assuming that Φ is a Poisson point process, we have that whenver the SNR graph percolates then there exists a $\gamma > 0$ such that the SINR graph percolates as well (see [Dousse et al. 2006]). We will extend this result in this thesis to the case when apart from interference due to the transmitters one has interference from some other miscalla-

neous nodes.

Fading : One key phenomenon, we have omitted in the above discussion but that shall feature in Section 5.2 is *fading*. So far, we have only mentioned about loss in transmission due to the distance between the emitter and receiver. However, there could be many obstacles or reflectors in the transmission medium that divert the signal along multiple paths. Thus the signal at the receiver is a superposition of multiple copies of these signals that have traversed different paths. This deviation of the signal power from the attenuation of path-loss is what is termed as fading. The various factors at play in fading explain why it could be constructive as well as destructive. Thus, fading could either amplify or attenuate the signal. In reality, fading is a random process that varies over time and space. Since this thesis considers models that do not change over time, we study the effect of fading across space alone.

The way to incorporate fading into our stochastic geometric models is to replace the power P by $PS_{X,Y}$ where $S_{X,Y}$ is a random variable depending on the emitter location X and receiver location Y. In theory, since all the signals from an emitter X traverse via the same medium, the fading random variables need not be independent. However, if we asssume that the locations of the emitters and receivers are located far away from each other, it is not too far-fetched to assume that the paths traversed are disjoint and hence the fading random variables are independent. This assumption of independence is crucial to many computations.

The next issue is of choice of distribution for these fading random variables. As expected and as it turns out, different scenarios necessitate different choice of fading distributions. Based on the choice of fading distribution, various fading models such as Nakagami fading, Rayleigh fading, Weibull fading, Rician fading, Log-Normal shadow fading etc... have been proposed. We shall assume Rayleigh fading in one of our models (see Section 5.2).

Capacity and percolation :

We shall now turn our attention to the information-theoretic question of capacity. This notion pertains to the rate at which the information flows in a network. In the earlier models, by an edge we assumed the existence of a communication link between the two nodes. However, nodes closer to each other can transmit information faster than far away

nodes. To this end, one associates with each link its capacity denoting the number of bits that can be transmitted across the link per second. The capacity clearly decreases with the distance between the nodes but it is also known to be non-linear. Following Shannon's result on Gaussian channels, one defines the capacity of link in a SINR model as follows (see [Franceschetti and Meester 2007, Chapter 5]) :

$$C(X_i, X_j) := \log(1 + SINR(X_i, X_j)) = \log(1 + \frac{Pl(|X - Y|)}{N + \gamma \sum_{Z \in \Phi, z \neq X,Y} Pl(|Z - Y|)}).$$

Under this definition of capacity, any two nodes that are connected by an edge in the SINR graph as described in the subsection above can transmit atleast $\log(1 + T)$ bits per second i.e, the link sustains a constant information flow. One can consider the per-node capacity which is the capacity averaged over all nodes in the network.

We shall now discuss some results regarding the capacity of the wireless networks that utilise results from percolation theory. In [Gupta and Kumar 2000], it was shown that in a network with randomly deployed nodes, the capacity between each source-destination pair must decay atleast at the rate $\frac{1}{\sqrt{n}}$ as the number of nodes n increases to infinity. But the authors could only construct a transmission strategy to achieve a capacity of the order $\frac{1}{\sqrt{n \log n}}$. This seminal work triggered efforts to answer similar questions for more general models of transmission and interference but a tranmission strategy to eliminate the logarithmic factor remained elusive. This gap was closed in [Franceschetti et al. 2007] by using a specific multihop transmission strategy that involved percolation theory i.e., it was shown that this strategy achieves a capacity rate of the order of $\frac{1}{\sqrt{n}}$. The specific strategy was to divide the nodes into disjoint sets that cross the network area forming what the authors term as a "highway system". Along this highway, nodes transmit information in short hops at a constant capacity. The source and destination nodes access this highway in a single hop of longer length. However, after accounting for the fact that various source-destination paris would access this highway, it turns out that the capacity on the highway path is actually of order $\frac{1}{\sqrt{n}}$, the correct order. The construction of this highway, which as expected relies heavily on percolation theory, is at the heart of the above transmission strategy. Such connections between percolation theory and information theory is envisioned to play key role in solving various other information-thoeretic questions.

1.2 Overview of the thesis

As explained in the above sections, there are fairly rich models of stochastic geometric networks. The Boolean model is generally considered a good starting point for stochastic geometric networks. The two important aspects of a network are its nodes (Φ) and the notion of communication between nodes. The existence of an edge between two nodes (i.e, their ability to communicate) can be seen as a functional $F(x, y, \Phi)$ dependent on the positions of the two nodes x, y and the positions of other nodes Φ. In the case of a Boolean model, this functional is simple in the sense that it depends only on the relative positions of the two nodes i.e, $F(x, y, \Phi) = \mathbf{1}[\|x - y\| \leq r]$.

We shall endeavour to study Boolean models where the point process (nodes) is not necessarily a Poisson point process as well as a model where the functional F is not solely dependent on inter-nodal distances. We have already seen the SINR graph which is a variant of the Boolean model in the sense that the functional F depends on entire point process. As avowed in the outlining of the introduction, the background and context of such extensions shall be elaborated upon in the following section (Section 1.2.1). After having broadly laid out the goals of the thesis in that section, we shall then preview the remaining chapters of the thesis in Section 1.2.2.

1.2.1 Motivation and related works

AB Boolean model :

We shall describe wireless networks that motivate the study of a Boolean model with two types of nodes which we shall call as the *AB* Boolean model. In the standard Boolean model, points (nodes) communicate only if they lie within a distance r of each other. In practice, this implies that all nodes transmit and receive at the same frequencies. However certain transmission strategies entail transmission at a particular frequency and reception at a different frequency. For example, consider the transmission scheme called the frequency division half duplex, where each node transmits at a frequency f_1 and receives at a frequency f_2 or vice-versa ([Tse and Vishwanath 2005]). Thus nodes with transmission-reception frequency pair (f_1, f_2) can communicate only with nodes that have transmission-reception frequency pair (f_2, f_1) that are located within the cutoff distance r. Thus, in this model, one would like to model the node locations as two point processes $\Phi^{(1)}, \Phi^{(2)}$; $\Phi^{(1)}$ consisting of nodes with transmission-reception frequency pair (f_1, f_2),

$\Phi^{(2)}$ consisting of nodes with transmission-reception frequency pair (f_2, f_1) and edges only between points of the opposite point processes provided they are within a distance r of each other. Observe that the requirements for percolation in this graph are different from those of the Boolean model.

Another example where such a model would be applicable is in communication between nodes deployed at two different levels, for example surface (or underwater) and in air. Nodes in a level can communicate only with those at the other level that are within a certain range. A third example is in secure communication in wireless sensor networks with two types of nodes, tagged and normal. Upon deployment, each tagged node broadcasts a key over a predetermined secure channel, which is received by all normal nodes that are within transmission range. Two normal nodes can then communicate provided there is a tagged node from which both these normal nodes have received a key, that is, the tagged node is within the transmission range of both the normal nodes.

Our model is named so because such a model of communication between nodes of opposite types has been already considered in the discrete set-up and is known as the AB percolation model ([Grimmett 1999, Wu and Popov 2003]). Given a graph $G := (V, E)$, each vertex is given a mark A or B independent of other vertices. Edges between vertices with similar marks (A or B) are removed. The resulting random sub-graph is the AB graph model. Percolation is said to happen in this model if there exists, with positive probability, an infinite path of vertices with marks alternating between A and B. This model has been studied on lattices and some related graphs. The AB percolation model behaves quite differently as compared to the usual percolation model. For example, it is known that AB percolation does not occur in $\mathbb{Z}^2$ ([Appel and Wierman 1987]), but occurs on the planar triangular lattice ([Wierman and Appel 1987]), some periodic two-dimensional graphs ([Scheinerman and Wierman 1987]) and the half close-packed graph of $\mathbb{Z}^2$ ([Wu and Popov 2003]).

Beyond Poisson point process :

Throughout this introduction so far, point processes have only played the role of nodes in a communication network. However, we must emphasise that point processes lie at the heart of various studies in stochastic geometry, both theoretical and applied. Their applications run the gamut from medical imaging to military applications.

Most of the work involving quantitative analysis of point processes have dealt with

Poisson point process. One of the main reasons being that characteristics of Poisson point process are amenable to computations and yield nice closed form expressions in many cases. Even for Cox (doubly stochastic Poisson) point processes, the computations have been cumbersome and often leading to complicated expressions.

To improve upon this situation, qualitative, comparative studies of point process have emerged as useful tools. The first method of comparison of point processes has been coupling or stochastic domination (see [Kamae et al. 1977, Lindvall 1992]). In our terminology, these are known as strong ordering of point processes. When two point processes can be coupled, one turns out to be a subset of the other (see [Rolski and Szekli 1991]). This ordering is very useful for obtaining various bounds and proving limit theorems. However, using it one cannot compare two different point processes with same mean measures. An obvious example is an homogeneous Poisson point process and a stationary Cox point process with the same intensity. The question arises of how to compare such point processes? This is an important question since it is expected that by comparing point processes of the same intensity one should achieve a tighter bound than by coupling and also one can analyse more point processes. For some more details on strong ordering of point processes and need for other orders, see remarks in [Müller and Stoyan 2002, Sections 5.4 and 7.4.2].

Our interest in point processes, and in particular in the shot-noise fields they generate, as seen earlier has roots in the analysis of wireless communications. According to a new emerging methodology, stochastic geometric modeling of wireless communications provides a way of defining and computing macroscopic properties of large wireless networks by averaging over all potential random patterns for node locations in an infinite plane and radio channel characteristics. This averaging procedure is in the same way as queuing theory provides averaged response times or congestion over all potential arrival patterns within a given parametric class. These macroscopic properties will allow one to characterize the key dependencies of the network performance characteristics as a function of a relatively small number of parameters. Shot-noise fields being one of the simplest functionals of a point process occur in various contexts. For example, shot-noise fields are used to model spike trains in neuronal networks. (refer to [Huffer 1984a]).

In the above context, Poisson distribution of emitters/receiver/users is often too simplistic. Statistics show that the real patterns of users exhibits more clustering effects ("hot spots") than observed in an homogeneous Poisson point processes. On the other

hand, good packet-collision-avoidance mechanisms scheme should create some "repulsion" in the pattern of nodes allowed to access concurrently to the channel. Social networks also exhibit such repulsive or hot spots behaviour. During major festive events in a town, one observes people crowding certain spots in a town. Perhaps, in beaches or parks, there would not be too many people close to each other. If one considers people as mobile users, then they act as receivers or transmitters. Thus in the situations described above, one would observe the nodes are either clustered or spread-out. This rises questions about the analysis of non-Poisson models, which we wish to tackle using the theory of stochastic ordering that will be introduced in the next subsection.

Stochastic orders :

Let us briefly dwell on stochastic ordering before proceeding to relating it to the comparison of point processes. Two random variables X and Y with the same mean $\mathsf{E}(X) = \mathsf{E}(Y)$ can be compared by how "spread out" their distributions are. This *statistical variability* (in a statistical ensemble) is captured to a limited extent by the variance, but more fully by convex ordering, under which X is less than Y if and only if for all convex f, $\mathsf{E}(f(X)) \leq \mathsf{E}(f(Y))$. In multi-dimensions, besides different statistical variability of marginal distributions, two random vectors can exhibit different *dependence properties* on their coordinates. The most evident example here is comparison of the vector composed of several copies of one random variable to a vector composed of independent copies sampled from the same distribution. A useful tool for comparison of the dependence structure of random vectors with fixed marginals is the *supermodular order*. The *dcx* order is another integral order (generated by a class of *dcx* functions in the same manner as convex functions generate the convex order) that can be seen as a generalization of the supermodular one, which in addition takes into account the variability of the marginals (cf [Müller and Stoyan 2002, Section 3.12]). More generally, one can compare two random vectors with respect to a class of functions $\mathfrak{F}$ i.e, we say $X \leq Y$ in $\mathfrak{F}$ order if $\mathsf{E}(f(X)) \leq \mathsf{E}(f(Y))$ for all $f \in \mathfrak{F}$ and such that the expectations exist. Such orders were named *integral stochastic orders* in [Whitt 1986]. The goals of the theory are two-fold - (1) To construct tools that shall simplify proving $X \leq_{\mathfrak{F}} Y$ and (2) Given $\mathfrak{F}$, what is the maximal class $\mathfrak{G} \supset \mathfrak{F}$ such that $X \leq_{\mathfrak{F}} Y$ implies $X \leq_{\mathfrak{G}} Y$? Finding a minimal class $\mathfrak{G} \subset \mathfrak{F}$ is also of interest and is one of the tools noted in the first statement above.

This notion of ordering of random vectors appears natural from the viewpoint of sub-

jective expected utility theory axiomatically derived in the seminal work of Von Neumann and Morgenstern (see [Neumann and Morgenstern 1947]). The concept was known to have been already used by Bernoulli in the 18th century. Every agent has a utility function which determines his actions. This utility function can be thought of as profit from an action to agent. An agent chooses an action Y instead of X if on an average the action Y is more beneficial to him i.e, $\mathsf{E}(u(X)) \leq \mathsf{E}(u(Y))$ where $u(.)$ is the utility function of the agent. In many cases, one does not have complete knowledge of the utility function of an agent but only the partial knowledge that the utility function belongs to a certain class. Also, sometimes one considers a group of agents instead of a single agent and in this case the utility functions belong to the class generated by all the individual utility functions. These two possible scenarios lead to a definition of stochastic ordering as in the last paragraph.

The theory of stochastic ordering provides elegant and efficient tools for comparison of random objects and is now being used in many fields. The applicability of these results has generated sufficient interest in the theory of stochastic ordering as can be seen from the diverse results in the book of Müller and Stoyan ([Müller and Stoyan 2002]). We shall skim through the applications to queueing theory and the major part of the thesis is application to point processes.

Ross conjecture and dependence orders in queueing theory :

Generalizing the viewpoint of comparison expounded in the above section, to compare random elements in an arbitrary space, we need to define suitable "utlity functions" for elements in that space. Using these utility functions, one can define a ordering of random elements. If the space is an Euclidean space, there are a host of utility functions such as increasing functions, convex functions and many more readily available. However, it is less clear what kind of interesting utility functions exist on the space of locally finite measures, space of locally finite point processes or the space of closed sets. In the hope of enticing the reader to continue reading the thesis, we mention that this thesis does study a class of utility functions on the former two spaces. Due to the relation between queues and point process, the comparison of queues was studied first and this later paved the way for comparitive studies of point processes. Let us then make a brief foray into queues before relating them to point processes.

The genesis of mathematical study of queues is unanimously attributed to

[Erlang 1909]. A queuing network is a network of servers and customers. Customers arrive in the network and are allocated to or choose a server. The server provides service to the customer for a certain duration if it is free. Depending on the modalities of the network, when the server is busy the protocol for re-access varies. In telephone networks, the customer is routed to any free server and if there is no free server, the customer retries after some time. In other networks, customers are forced to wait in a queue and then based on some priority value assigned to the customer, he is served. In reservation counters, one would follow the simple FIFO (First-In-First-Out) rule but whereas in hospitals the service rule would be based on the patient's profile. The link between queues and point processes is conspicuous once we observe that the arrival times of customers in a network form a sequence of points on the positive real line. In other words, the arrival times of customers are a one-dimensional point process and so are the exit times. As with point processes, in most queueing networks also one assumes the arrival process to be a Poisson point process. For more on queueing theory, refer [Baccelli and Brémaud 2003].

In the queueing theory context, it was conjectured [Ross 1978] that replacing a stationary Poisson arrival process (the sequence of arrival times) in a single server queue (with FIFO rule) by a stationary Cox point process with the same intensity would increase the average customer delay. There have been many variations of these conjectures which are now known as Ross-type conjectures. They triggered the interest in comparison of queues with similar inputs ([Chang et al. 1991, Miyoshi and Rolski 2004, Rolski 1989]). These comparisons relied heavily on the theory of stochastic ordering. The notion of a *dcx* function was partially developed and used in conjunction with the proving of Ross-type conjectures ([Meester and Shanthikumar 1993, Meester and Shanthikumar 1999, Shaked and Shanthikumar 1990]). Directionally convex (*dcx*) order of the point process of arrival times in queues was used to settle many of the Ross-type conjectures. Much earlier than these works, a comparative study of queues using supermodular order and motivated by neuron-firing models can be found in [Huffer 1984a].

Comparison of point processes and clustering effects :

As most works on ordering of point processes were motivated by applications to queueing theory, results were primarily focused on one-dimensional point processes. However, from the relation between queues and point processes, it is to be expected that comparisons of queues with similar inputs shall play a crucial role in our work of extending them to

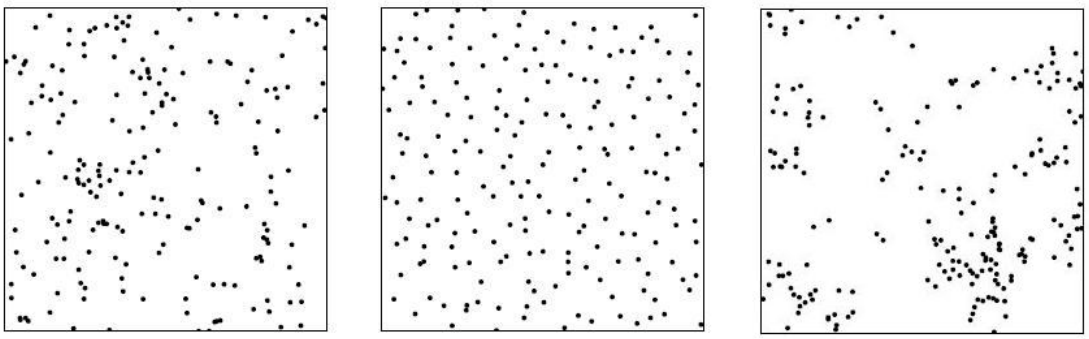

Figure 1.4: Samples of translation invariant point processes in the plane: Poisson (left), determinantal (center) and permanental (right) for kernel $K(z,w) = \frac{1}{\pi}e^{z\overline{w}-\frac{1}{2}(|z|^2+|w|^2)}$. These are examples of independent, sparse and clustered point processes. This figure is reproduced from [Ben Hough et al. 2006].

point processes. An attempt to rectify the lack of work in higher dimensions was made in [Miyoshi 2004], where comparison results for shot-noise fields of spatial stationary Cox point process were given. In particular, it was shown that one-dimensional marginals of some spatial stationary Cox point processes were greater in *dcx* order than those corresponding to Poisson point process. The results of [Miyoshi 2004] are the starting point of our investigation. Also comparison of variances of point processes and fibre processes was studied in [Stoyan 1983] and hence it can be considered as a forerunner to our work.

A natural phenomenon that could be the basis for comparison among point processes of similar intensity is clustering. This refers to the spread of the points of a point process across space. In some point processes, the points tend to occur in clusters whereas in others they are more evenly spread out. For example, in Figure 1.2.1 we can see three different point processes - Poisson, Determinantal and Permanental. For proper definitions, see Definitions 2.1.1 and 6.2.13. In a Poisson point process, the points are spread independent of each other. In determinantal point processes, the points repel each other and hence are spread out more sparsely. Permanental point processes lie at the other end of spectrum. The points occur in clusters. We will see some more figures of sparse and clustered point processes in Chapter 5.

There have been various statistical measures of clustering such as Ripley's K-function and pair correlation function (see Section 4.2.2). While these measures are sufficient for the purposes of comparison of clustering properties of point processes, it is far from obvious

whether they could be used to ascertain better coverage or better connectivity for less clustered point processes. In our language of stochastic ordering, we are looking for a larger class of utility functions than the afore-mentioned statistical measures of clustering. While it should be rich enough to compare functionals of point processes such as coverage, connecvitity or percolation, it should also be amenable so that we can provide examples of point processes that can be compared via this class of utility functions. We shall propose our contribution towards such a goal in Chapter 4 and demonstrate its amenability and applications in the following chapters (chapters 5 and 6). These are inspired by the success of *dcx* functions in the proving of Ross-type conjectures and by their spatial extensions in [Miyoshi 2004].

Comparison studies in percolation theory :

We shall switch back to percolation from stochastic ordering and survey literature on comparison studies in continuum percolation. Optimization of percolation over shapes was first considered in [Jonasson 2001]. In particular, one can consider a general Boolean model on Poisson point process with compact convex sets C of unit area instead of balls in Figure 1.1.2. In [Jonasson 2001], it was shown that the critical intensity for percolation required to percolate is minimized when C is a triangle and maximized for some centrally symmetric set C. Critical intensity for percolation is the least intensity λ of the Poisson point process above which the Boolean model percolates. The proof argument using difference-body inequalities is applicable for a more general class of point processes. Similar result was proved using more probabilistic arguments (namely enhancement technique) for the case when C is a polygon in [Roy and Tanemura 2002]. Further, the usage of enhancement arguments allowed [Roy and Tanemura 2002] to show strict inequalities as well. This idea was also used for comparison of percolation models with different shapes in three dimensions.

Another percolation model where comparison has been used is the random connection model introduced in [Penrose 1991]. In this model, one connects points of a Poisson point process in the plane at a distance r with probability $f(r)$. Under reasonable assumptions on f, it is shown that the critical intensity $\lambda_c(f)$ is non-degenerate i.e, $0 < \lambda_c(f) < \infty$. In [Franceschetti et al. 2005] two transformations to f were considered : $f_p(r) = pf(\sqrt{p}r)$ and $f_s(r) = f(c^{-1}(x-s))$ for $p, s > 1$. Using a nice coupling argument and the fact that bond percolation percolates better than site percolation on any graph, it

is shown that these transformations help percolation i.e, $\lambda_c(f_p) \leq \lambda_c(f)$ and $\lambda_c(f_s) \leq \lambda_c(f)$. In other words, this says that spreading out the connections help percolation. The strict inequalities for the first of these inequalities can be found in the recent pre-print [Franceschetti et al. 2010]. Though the mean degree of nodes remain same under the two transformations, it is conspicuous that nodes have more longer range edges in the random connection models with the transformed connectivity function. For applications, this means that it is beneficial to replace a few short-range communications by some long-range ones.

Clustering and percolation :

In the above comparison studies, the underlying point process has been a Poisson point process. It is in this regard that the thesis differs from previous efforts. We will try to study the change in percolative properties when the underlying point process is changed. One of the common heuristics is that more clustering in a point process impacts percolation negatively. A more precise mathematical statement of the same would be that the critical radius for percolation increases with increase in clustering of a point process. The focus of the final part of the thesis is to study this heuristic.

For example, let us denote the Poisson, determinantal and permanental point processes by $\Phi_i, i = 1, 2, 3$ and their critical radii for percolation as $r_c(\Phi_i), i = 1, 2, 3$ respectively. Then we expect that $r_c(\Phi_1) \leq r_c(\Phi_2) \leq r_c(\Phi_3)$ i.e, it should be easier for a determinantal point process to percolate than the Poisson point process and which in turn should percolate better than a permanental point process. Now let us look at more concrete examples. For the following three point processes known as perturbed lattices (see 5.1.7), we have simulated their corresponding Boolean models or Gilbert disk graph for different radii. These are obtained by independently replicating each vertex of an hexagonal grid random number of times and then shifting each replication independently and uniformly within its hexagon. In the following, the number of replications are distributed as $Bin(n, 1/n)$ for $n = 1, 2, 5$ where $Bin(n, 1/n)$ stands for a Binomial distribution with parameters n and $1/n$. Thus, all the three point processes have on a average same number (1 in this case) of points within a hexagon.

A glance at the figures tell you that the clustering in the above point processes increases from left to right. Heuristically, the radius required for percolation also should increase from left to right. The simulations of their Gilbert disk graphs in Figure 1.2.1 show that

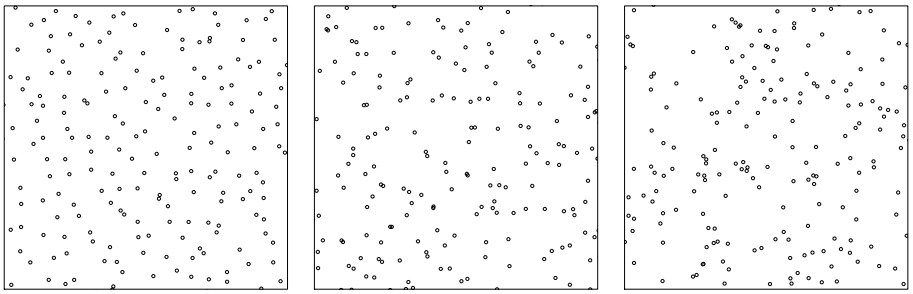

Figure 1.5: Three different perturbed lattices. See Section 5.1.7 for more explanations.

our heuristic is not too far off the mark.

The most general statement should be of the form $\Phi_0 \leq_{\mathfrak{F}} \Phi$ implies that $r_c(\Phi_0) \leq r_c(\Phi)$ where $\mathfrak{F}$ is the class of utility functions defining the ordering of point processes. What would be a suitable class $\mathfrak{F}$ that includes some reasonable point processes as examples ? If we are greedy, we would demand the order to include the above point processes also as examples. Surely, the class of increasing functions (corresponding to coupling) fails as it cannot compare point processes of same mean intensity. In Chapter 6, we shall see how far do *dcx* functions take us in this direction. As we have explained in the above subsection, *dcx* ordering of point processes is a strong measure of clustering. Hence, it is natural to ask whether the order can be used to compare the percolative properties as well. We will see that though *dcx* order does not include the determinantal and permanental point processes as examples, our proofs can be modified easily enough to prove comparison of certain critical radii for these class of point processes also (see first Theorem in Section 1.2.2 below). In Chapter 6, we will also present some simulations of the Boolean model on different point processes that confirm the above heuristic.

1.2.2 Presentation of the results

In the chapter on preliminaries (Chapter 2), we lay out the stochastic geometric framework encompassing the reccurrent themes of the thesis – point processes (pp), percolation and directionally convex ordering. The main goal of the chapter is to find the middle between over-crowding of the main chapters with standard results and self-containement of the thesis. Firstly, we shall make a quick run through the definitions of random measures,

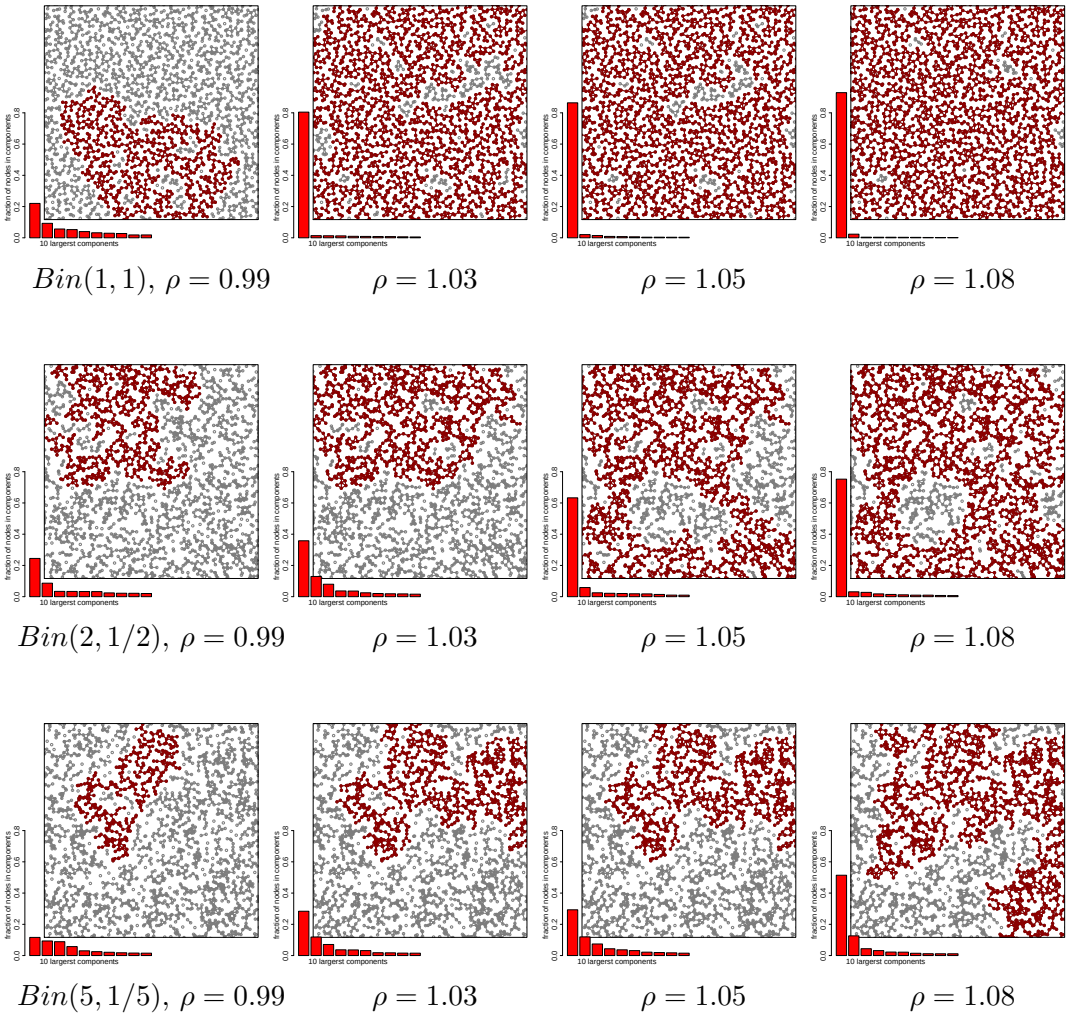

Figure 1.6: Gilbert graph with communication range ρ and nodes in perturbed lattice pp with Binomial $Bin(n, 1/n)$ number of replicas uniformly distributed in hexagonal cells. The largest component in the simulation window is highlighted. Bar-plots show the fraction of nodes in ten largest components.

Palm measures, random closed sets, stationarity, ergodicity, Campbell-Mecke formula and Choquet's theorem. Secondly, we shall sketch the Peierls argument for proving percolation on lattices as well as give a sufficient condition for non-percolation on lattices. Both these conditions form the backbone of proofs of percolation and non-percolation in Boolean

models respectively. Thirdly and finally, we shall state some results on directionally convex ordering of random vectors that shall be used in the subsequent chapters.

For the ease of referencing, we would like to mention that the content of the four remaining chapters are sourced from the following articles :

- The results of the chapter 3 can be found in the submitted pre-print [Iyer and Yogeshwaran 2010].
- Most parts of the chapters 4 and 5 have already appeared in the publication [Blaszczyszyn and Yogeshwaran 2009].
- Finally, the results of the chapter 6 are from an unsubmitted manuscript [Blaszczyszyn and Yogeshwaran 2010b] and a more concise version in [Blaszczyszyn and Yogeshwaran 2010] explaining the possible applications to communication networks. An extended version of the above article with more figures can be found in the technical report [Blaszczyszyn and Yogeshwaran 2010a].

AB random geometric graphs - Chapter 3 :

In Section 1.2.1, we have described models with communication only between opposite types of nodes. In Chapter 3, we make a more rigorous formulation of this model. To be consistent with the respective literatures, the model formulations are different for the problems of percolation and connectivity. We denote the Boolean model as described in subsection 1.1.2 as $C(\lambda, 2r)$ where λ is the intensity of the Poisson point process and r is the radius of the balls centred at the points.

A natural analog of the discrete AB percolation model or the usual Poisson Boolean model to the AB set-up would be to consider a graph with vertex set $\Phi^{(1)}$ where each vertex is independently marked A or B. However, we will consider a more general model from which the results for the above model will follow as a corollary. Let $\Phi^{(1)}, \Phi^{(2)}$ be two independent Poisson point processes in $\mathbb{R}^d$ with intensities λ, μ respectively. Then, the AB Poisson Boolean model $G(\lambda, \mu, r)$ is the graph with points of $\Phi^{(1)}$ as vertices and with edges between any pair of points for which the intersection of balls of radius $2r$ centred at these points contains at least one point of $\Phi^{(2)}$ i.e, $X_i, X_j \in \Phi^{(1)}$ have an edge if there exists $Y \in \Phi^{(2)} \cap B_{X_i}(2r) \cap B_{X_j}(2r)$ or in other words a $Y \in \Phi^{(2)}$ such that $B_{X_i}(r) \cap B_Y(r) \neq \emptyset, B_{X_j}(r) \cap B_Y(r) \neq \emptyset$. In two dimensions, the ease of geometry allows us to characterize the point of transition of the critical intensity $\mu_c(\lambda, r) := \sup\{\mu :$

$\mathsf{P}\left(G(\lambda,\mu,r) \text{ percolates}\right) = 0\}$. Let $\lambda_c(r) := \sup\{\lambda : \mathsf{P}\left(C(\lambda,r) \text{ percolates}\right) = 0\}$ be the critical intensity for percolation of the Boolean model.

Theorem (Theorem 3.1.5)**.** *For $d = 2$, we have that $\mu_c(\lambda, r) < \infty$ iff $\lambda > \lambda_c(2r)$.*

In higher dimensions, we do not have such precise characterizations but we are still able to derive bounds for the critical intensity μ required for percolation when r is fixed and λ is chosen suitably large. This assures us of the existence of phase transitions in the AB Boolean model in all dimensions starting from two. These results are obtained from more general results on a continuum word percolation model. Word percolation is the occurence of a given infinite sequence from $\{1,\ldots,k\}^\infty$ in a giant component of the Boolean model when the points are independently marked with marks in $1,\ldots,k$. By using these results, we show that for large λ_* , the AB Boolean model $G(\frac{\lambda_*}{2}, \frac{\lambda_*}{2}, r)$ percolates. By continuity of the percolation function, we can deduce that the AB Boolean model $G(\lambda_* p, \lambda_*(1-p), r)$ percolates for all p sufficiently close to $\frac{1}{2}$ and λ^* large. Observe that this is the natural extension of the AB grid percolation model in the continuum as independent thinning of Poisson point process is also a Poisson point process.

To study the full connectivity problem, we consider independent Poisson point processes of intensities n and cn in the unit cube. The AB random geometric graph $G_n(c,r)$ is defined on these two point processes similar to the AB Boolean model but with balls of radius $\frac{r}{2}$. We first describe the radius regime $r_n(c) = \frac{\log(n/\beta)}{cn\theta_d}^{\frac{1}{d}}$ such that the expected number of isolated nodes in $G_n(cn, r_n(c))$ converges to a non-degenerate Poisson random variable. We prove such a total variation convergence result for all $c < 1$ in $d \geq 3$ and for all $c < c_0 < 4$ in $d = 2$. Further, under this radius regime for $c > 2^d$, we have that the expected number of isolated nodes diverges. This indicates a phase transition for the expected number of isolated nodes under this radius regime. Using this convergence result, we derive a weak law result for the largest nearest neighbour distance and almost sure asymptotic lower bounds for the connectivity threshold. The upper bounds are derived via coupling with usual random geometric graphs.

Theorem (Theorem 3.2.6)**.** *Define $\alpha_n^*(c) := \inf\{a : G_n(cn, a^{\frac{1}{d}} r_n(c))$ is connected$\}$. Then there are constants c_0 and $\alpha(c)$ such that*

$$\begin{aligned} \forall c < c_0, a < 1, \qquad & \lim_{n\to\infty} \mathsf{P}\left(G_n(cn, ar_n) \text{ is connected }\right) = 0 \text{ and} \\ \forall c > 0, \qquad & \limsup_{n\to\infty} \alpha_n^*(c) \leq \alpha(c) \quad a.s.. \end{aligned}$$

In less mathematical terms, the result means that the AB random geometric graph $G_n(cn, a^{\frac{1}{d}} r_n(c))$ will remain connected asymptotically for $c > 0, a > \alpha(c)$ and disconnected asymptotically for $c < c_0, a < 1$.

Directionally convex ordering of point processes - Chapters 4 and 5 :

The order $\leq$ on $\mathbb{R}^n$ shall denote the component-wise partial order, i.e., $(x_1, \ldots, x_d) \leq (y_1, \ldots, y_d)$ if $x_i \leq y_i$ for every i. We shall abbreviate $x \leq q, y \leq q$ to $[x, y] \leq q$ (and similarly $p \leq [x, y]$).We say that a function $f : \mathbb{R}^d \to R$ is *directionally convex* (*dcx*) if for every $x, y, p, q \in \mathbb{R}^d$ such that $p \leq [x, y] \leq q$ and $x + y = p + q$,

$$f(x) + f(y) \leq f(p) + f(q).$$

Two random vectors are said to be *dcx* ordered i.e, $X \leq_{dcx} Y$ if $\mathsf{E}(f(X)) \leq \mathsf{E}(f(Y))$ for all f *dcx* such that the expectations exist. We make a further extension that consists in *dcx* ordering of locally finite measures on a Polish space $\mathbb{E}$ (to which belong point processes) viewed as non-negative fields of measure-values on all bounded Borel subsets of the space. More precisely, two random measures are said to be ordered and denoted $\Lambda_1 \leq_{dcx} \Lambda_2$ if $(\Lambda_1(B_1), \ldots, \Lambda_1(B_n)) \leq_{dcx} (\Lambda_2(B_1), \ldots, \Lambda_2(B_n))$ for any finitely many bounded Borel subsets. This is the ordering of marginal vectors when the measures are viewed as random fields indexed by bounded Borel subsets. A simple but powerful result is that random measures are *dcx* ordered provided the above condition on ordering of their finite dimensional marginals holds for all disjoint bounded Borel subsets. We show that the *dcx* order is preserved under some of the natural operations considered on random measures and point processes, such as independent superposition, i.i.d marking, deterministic displacement, projection and thinning. The last three properties can also be viewed as special cases of i.i.d. marking. It is known that that random measures with random densities (i.e, Radon-Nikodyn derivatives) are *dcx* ordered when the densities are *dcx* ordered. We take it one step further by showing that the Cox point process generated by random measures preserve the *dcx* order. This can be exploited to show that intensity measures of independently (not necessarily identical) marked Cox point processes preserve the *dcx* order as well. This gives greater leverage when working with Cox point processes.

We next study the relation between clustering of point processes and *dcx* order. The well-known statistical measures of clustering are the Ripley's $K(.)$-function (the expected

empirical average of neigbhours within a distance r) and the second moment densities. We show that these respect the *dcx* order by proving that moment measures are preserved by *dcx* order. We then turn our attention to Palm measures. Palm measure of a point process is the distribution of the point process conditioned on existence of a point at a given location x. We show that *dcx* ordering of random measures implies *idcx* (increasing and *dcx*) ordering of their Palm measures. We also give counter-examples to show that this result cannot be improved in full generality.

Many interesting characteristics of random measures, both in the theory and in applications have the form of integrals of some non-negative kernels. Suppose that $k(.,.)$ is a measurable (in the first variable alone) non-negative kernel on $\mathbb{E} \times S$ (for an arbitrary index set S), then *integral shot-noise field* $V_\Lambda(.)$ of a random measure Λ and *extremal shot-noise field* $U_\Phi(.)$ of a point process Φ are defined as follows :

$$V_\Lambda(y) := \int_{\mathbb{E}} k(x,y)\Lambda(dx) \ ; \ U_\Phi(y) := \sup_{X \in \Phi} \{k(X,y)\}.$$

For example, many classes of Cox point process, with the most general being Lévy based Cox point process (cf. [Hellmund et al. 2008]), have stochastic intensity fields, which are integrable shot-noise fields. They are also key ingredients of the recently proposed, so-called "physical" models for wireless networks, as was already explained in Section 1.1.2. It is particularly appealing to study the shot-noise fields generated by *dcx* ordered random measures as they appear in a variety of contexts.

Since integrals are linear operators on the space of measures, and knowing that a linear function of a vector is trivially *dcx*, it is naturally to expect that the integral shot-noise fields with respect to *dcx* ordered random measures will inherit this ordering from the measures. However, this property cannot be concluded immediately from the finite dimensional *dcx* ordering of measures. The formal proof of this fact is one of the central results of the thesis (see Chapter 4) involves some arguments from the theory of integration combined with the closure property of *dcx* order under joint weak convergence and convergence in mean. Further for *dcv* (directionally concave) ordered point processes, we show that the extremal shot-noise fields are lower orthant(lo) ordered. Formally, these two results are together stated as follows :

Theorem (Theorem 4.1.3 and Proposition 4.3.3)**.** *Let S be an index set and $\mathbb{E}$ be a Polish*

space. Let $\Lambda_1 \leq_{dcx} \Lambda_2$, $\Phi_1 \leq_{dcx} \Phi_2$ *and* $\mathsf{E}(V_{\Lambda_i}(y)) < \infty$, *for all* $y \in S$, $i = 1, 2$. *Then*

$$\{V_{\Lambda_1}(y)\}_{y\in S} \leq_{dcx} \{V_{\Lambda_2}(y)\}_{y\in S} \ ; \ \{U_{\Phi_1}(y)\}_{y\in S} \geq_{lo} \{U_{\Phi_2}(y)\}_{y\in S},$$

where by $X \geq_{lo} Y$ *for two random vectors, we mean that* $\mathsf{P}(X \leq a) \leq \mathsf{P}(Y \leq a)$ *for all* $a \in \mathbb{R}^d$.

We shall give examples of point processes that are smaller and larger than the Poisson point process in the *dcx* order in Chapter 5. The point processes greater than the Poisson point process in the *dcx* order are a wide class of Cox point processes - Cox point processes with conditionally increasing intensity fields, Poisson-Poisson cluster point processes, Lévy based Cox point processes, Log Lévy based Cox point processes and Generalized shot-noise Cox point processes. Conditionally increasing property of random fields can be seen as a continuum version of the famed FKG inequality in statistical physics. We shall indeed show that when spin models on lattices satisfy the FKG inequality (such as random cluster measures) they can be used to construct a point process with a conditionally increasing intensity field and thereby a clustered point process. Lévy based Cox point processes are Cox point processes with intensity fields being integral shot-noise fields driven by a "nice" kernel and the random measure as a Lévy basis (a completely independent, infinitely divisible random measure). When the Lévy basis is a Poisson point process, we call the corresponding Cox point process as a Poisson-Poisson cluster point process. These are the most commonly studied models of clustered point processes. Further, under some regularity conditions on the intensity field, it is possible to construct a family of *dcx* ordered intensity fields in which the lower limit is the constant intensity field and the upper limit is the homogeneous but random intensity field. These two limits correspond to intensity fields of Poisson and mixed Poisson point process. The point processes smaller than Poisson point process in the *dcx* order (called sub-Poisson point processes now onwards) are perturbed lattices i.e, point processes obtained by making i.i.d. perturbations of vertices in a discrete lattice. These point processes are considered as approximations to the zeros of Gaussian analytic functions.

We will also give examples of interesting functionals of point processes that are a *dcx* function of a shot-noise field generated by the point process. Let $C(\Phi, r)$ denote the Boolean model driven by a point process Φ and balls of radius r. One would guess that the expected region covered in a Boolean model decreases with clustering and we show

that it is indeed true for *dcx* ordered point processes i.e, $\mathsf{E}(\|C(\Phi_1, r)\|) \geq \mathsf{E}(\|C(\Phi_2, r)\|)$ when $\Phi_1 \leq_{dcx} \Phi_2$. In contrast to Ross-type conjectures, these examples show that there are certain performance characteristics in wireless networks that *improve* with more variability in the input process. More accurately, the performance measures that improve with variablity are the probability of connectivity of nodes in a SINR network and the capacity of a SINR network. We consider the set of interferers independent of the communicating nodes Φ. For more clarity, define $G(\Phi, \Phi_I, \gamma)$ the SINR network on Φ (see subsection 1.1.2) but with interference from both the points of Φ as well as points of Φ_I, the set of external interferers. Pick any finitely many source-destination pairs to communicate among the points of Φ. Let $p(\Phi_I)$ denote the probability of success of all the transmissions in the SINR network and C_{Φ_I} the capacity of the network. If Φ_1, Φ_2 are two sets of interferers such that $\Phi_1 \leq_{dcx} \Phi_2$, then $p(\Phi_1) \leq p(\Phi_2)$ and $C_{\Phi_1} \leq C_{\Phi_2}$. Many of these performance characteristics are dependent on the value of the shot-noise fields only at finitely many points. One could conceive an extension to functionals of shot-noise fields dependent on a bounded region but whether one could extend it to functionals that depend on the values of shot-noise fields over unbounded regions is not clear. In the next subsection, we shall throw more light on such a question.

Percolation and directionally convex ordering of point processes - Chapter 6 :

Percolation is the first candidate for a functional that depends on the behaviour of point process over large regions. The hope for a connection between percolation and *dcx* order comes from the fact that point processes smaller in *dcx* order have a larger capacity functional of their Boolean models. This is proved using the lower orthant ordering of the extremal shot-noise fields generating the Boolean model.

We define two new critical radii $\underline{r}_c(\Phi)$ and $\overline{r}_c(\Phi)$ for a point process that are respectively lower and upper bounds to the critical radius $r_c(\Phi)$ for percolation of the Boolean model. $\underline{r}_c(\Phi)$ is the critical radius defined with respect to the expected number of paths in $C(\Phi, r)$ to the boundary of a box $[-m, m]^d$ as m tends to ∞. $\overline{r}_c(\Phi)$ is the critical radius such that the super-critical Boolean model as per this radius satisfies Peierls argument on a sequence of discrete models approximating the Boolean model. We show that $\underline{r}_c(\Phi)$ decreases with the *dcx* order while $\overline{r}_c(\Phi)$ increases. The precise formulation of this result as follows will be proved for Boolean models with arbitrary grains instead of just balls of raidus r as we have seen so far.

Theorem (Theorems 6.2.15 and 6.2.14)**.** *Let $\Phi_1 \leq_{dcx} \Phi_2$ be two point processes. Then,*

$$\underline{r}_c(\Phi_2) \leq \underline{r}_c(\Phi_1) \leq r_c(\Phi_1) \leq \overline{r}_c(\Phi_1) \leq \overline{r}_c(\Phi_2).$$

Further, both the inequalities shall hold in the case when Φ_2 is a Cox process with intensity measure $\Lambda(.)$ and Φ_1 is a Poisson point process with intensity measure $\mathsf{E}(\Lambda(.))$. The inequality for the lower bound alone stays true when Φ_1 is a stationary determinantal point process with kernel $K(x, y)$ and Φ_2 is a Poisson point process with intensity $K(0, 0)$.

Percolation or non-percolation in various continuum models are proved by discretisation and then bounding the level crossing probabilities of integral shot-noise fields of point processes – $\mathsf{P}\left(V_\Phi(z_i) \leq k, 1 \leq i \leq n\right)$ (or $\mathsf{P}\left(V_\Phi(z_i) \geq k, 1 \leq i \leq n\right)$) – on the lattices. We observe that these level crossing probabilities can be bounded by the corresponding Laplace transforms using Chernoff's bound. If these point processes are less in *dcx* order than Poisson point process, this can in turn be bounded by the Laplace transforms of the corresponding Poissonian shot-noise fields. These latter fields have closed form expressions. On suitable discrete approximations for nice functions, one would hope to show that the discrete model induced by the level crossings of the shot-noise fields can be coupled with a sub-critical branching process for small values of the parameter and in the other extreme, it can be proved that the discrete model will percolate by using the Peierls argument. We carry out such a program to show non-trivial phase transition for k-percolation and word percolation in Boolean models. A Boolean model is said to k-percolate if the region is covered by at least k grains of the Boolean model percolates. By non-trivial phase transition, we mean the existence of an $r > 0$ such that the Boolean model does not k-percolate and also the existence of an $r < \infty$ such that the Boolean model k-percolates. If we denote the critical radius for k-percolation as $r_c^k(\Phi)$ and Φ_λ is the Poisson point process, the above discussion amounts to the following result.

Theorem (Theorem 6.3.4)**.** *For $\Phi \leq_{dcx} \Phi_\lambda$, we have that*

$$0 < c(\lambda) \leq r_c^1(\Phi) \leq r_c^k(\Phi) \leq C(\lambda, k) < \infty,$$

where the constants depend only on λ, k and not on the finer structure of Φ.

While finiteness of the critical radius is intuitively a desired property, its positivity might been seen as irrelevant if not a disadvantage from the networking point of view.

However, as [Blaszczyszyn and Yogeshwaran 2010, Example 3.2] (Gilbert's carrier sense network) demonstrates, certain models do require non-percolation as well.

As in the former theorem, we prove that the lower bound also holds true for stationary determinantal point processes with kernel $K(x, y)$ by comparing them with a Poisson point process of intensity $K(0, 0)$. The uniform bounds in the theorem over all Φ such that $\Phi \leq_{dcx} \Phi_\lambda$ will be a feature of all the results in this section on existence of non-trivial phase transitions. In word percolation by super-criticality, we mean the almost sure occurence of all words. As with k-percolation, we give an uniform bound on the radius r above which all words occur almost surely for any sub-Poisson point process.

For Poisson SINR networks, it is known that when the SNR network $G(\Phi_\lambda, \emptyset, 0)$ (i.e, network without any intereference) percolates, the SINR network $G(\Phi_\lambda, \emptyset, \gamma)$ percolates for some $\gamma > 0$ (See Section 1.1.2). We show that this is true even if one adds an additional set of sub-Poisson interferers. More precisely, if $\Phi \leq_{dcx} \Phi_\lambda$ and $G(\Phi_\lambda, \Phi, 0)$ percolates, the $G(\Phi_\lambda, \Phi, \gamma)$ percolates for some $\gamma > 0$. Further, for a SINR network on sub-Poisson point process $G(\Phi, \Phi, \gamma)$, we show that by increasing the signal power P well above the critical power one can find a $\gamma > 0$ such that the SINR network percolates.

Chapter 2

Preliminaries

2.1 Random measures, point processes and random sets.

For a more detailed and classical presentation on the subject matter of this section, refer to one or all of the following books : [Kallenberg 1983, Daley and Vere-Jones 1988, Stoyan et al. 1995, Schneider and Weil 2008]. For a more hands-on introduction to the subject (especially for readers interested in wireless networks), we suggest the recent monograph [Baccelli and Błaszczyszyn 2009a].

Random Measure : As concerns random measures, we shall work in the set-up of [Kallenberg 1983]. Let $\mathbb{D}$ be a locally compact, second countable Hausdorff (LCSC) space. Such spaces are Polish, i.e., complete and separable metric space. Let $\mathtt{B}(\mathbb{D})$ be the Borel σ-algebra and $\mathtt{B}_b(\mathbb{D})$ be the σ-ring of *bounded, Borel subsets* (bBs). Let $\mathbb{M} = \mathbb{M}(\mathbb{D})$ be the space of non-negative Radon measures on $\mathbb{D}$. The Borel σ-algebra $\mathcal{M}$ is generated by the mappings $\mu \mapsto \mu(B)$ for all B bBs. A random measure Λ is a random element in $(\mathbb{M}, \mathcal{M})$ i.e, a measurable map from a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ to $(\mathbb{M}, \mathcal{M})$. We shall call a random measure Φ a point process (pp) if $\Phi \in \bar{\mathbb{N}}$, the subset of counting measures in $\mathbb{M}$. The corresponding σ-algebra is denoted by $\mathcal{N}$. Further, we shall say a pp Φ is simple if a.s. $\Phi(\{x\}) \leq 1$ for all $x \in \mathbb{D}$. Throughout, we shall use Λ for an arbitrary random measure and Φ for a pp.

Though pp are a class of random measures, due to the additional properties it can be represented in various ways. We can represent Φ on a LCSC space $\mathbb{D}$ as a countable sum of Dirac measures : $\Phi = \sum_i \varepsilon_{X_i}$ ($\varepsilon_x(A) = 1$ if $x \in A$ and 0 otherwise) where X_i are

random elements in $\mathbb{D}$. Another representation as a random set shall be given in Section 2.1.1.

Two classes of pp - Poisson pp and the more general Cox pp (also known as doubly stochastic Poisson pp) - are the more tractable ones among various pp. Let us define them.

Definition 2.1.1. *Let $\Lambda(.)$ be a random measure as defined above. A pp $\Phi \subset \mathbb{R}^d$ is said to be a Cox pp with intensity measure $\Lambda(.)$ and denoted by Cox(Λ) if given $\Lambda(.)$, the following properties hold :*

1. *for a bBs A, $\Phi(A)$ is distributed as a Poisson rv with mean $\Lambda(A)$ and*
2. *for any finite collection of disjoint bounded Borel subsets $A_1, \ldots, A_n$, $\Phi(A_i), 1 \leq i \leq n$ conditioned on $\Lambda(A_i), 1 \leq i \leq n$ are independent rvs.*

If a random intensity measure has a (random) density $\lambda(x)$ (where $\{\lambda(x)\}_{x\in\mathbb{D}}$ is a stochastic process) with respect to a locally finite measure $\mu(.)$ on E i.e, $\Lambda(\mathrm{d}x) = \lambda(x)\mu(\mathrm{d}x)$, then $\lambda(.)$ is called the *intensity field* and the corresponding Cox pp will be denoted by Cox($\lambda(.)$). When we speak of intensity fields of random measures on $\mathbb{R}^d$, the measure $\mu(.)$ will be taken to be the Lebesgue measure. If $\Lambda(A) = \int_A \lambda(x)\mathrm{d}x$ for a deterministic function $\{\lambda(x)\}_{x\in\mathbb{R}^d}$ and for all bBs A, then we call Φ to be an inhomogeneous Poisson ($\lambda(x)$) pp. If $\lambda(x) \equiv \lambda$, then Φ is said to be the Poisson pp with intensity λ or Poisson (λ) pp in short. We denote it by Φ_λ.

Moment measures : By the nth power of a random measure Λ, we understand a random measure Λ^k on the product space $\mathbb{D}^k$ given by $\Lambda^k(A_1 \times \ldots \times A_k) = \prod_{j=1}^k \Lambda(A_j)$. Its expectation, $\alpha^k(\cdot) = \mathsf{E}(\Lambda^k(\cdot))$ is called the *kth moment measure*. The first moment measure $\alpha(\cdot) = \alpha^1(\cdot)$ is called the *mean measure*.

The nth power *joint intensity* of a pp $\Phi \subset \mathbb{R}^d$ w.r.t. Lebesgue measure is a function (if it exists) $\rho^{(n)} : \mathbb{D}^n \to [0, \infty)$ such that for any n disjoint subsets $B_1, \ldots, B_n$, we have that

$$\mathsf{E}\left(\prod_{i=1}^n \Phi(B_i)\right) = \int_{\prod_{i=1}^n B_i} \rho^{(n)}(x_1, \ldots, x_n)\mathrm{d}x_1 \ldots \mathrm{d}x_n.$$

Stationarity and Ergodicity : The distribution of Λ is the measure $\mathsf{P}\Lambda^{-1}(.)$ on $(\mathbb{M}, \mathcal{M})$. For a measure $\psi \in \mathbb{M}(\mathbb{R}^d)$, let $\psi_{(x)}$ be the translate measure given by $\psi_{(x)}(B) = \psi(B - x)$ for $x \in \mathbb{R}^d, B \in \mathsf{B}(\mathbb{R}^d)$. A random measure is said to be *stationary* if the distribution of $\Lambda_{(x)}$ remains invariant i.e, $\mathsf{P}\Lambda_{(x)}^{-1}(M) = \mathsf{P}\Lambda^{-1}(M)$ for all

$x \in \mathbb{R}^d, M \in \mathcal{M}$. For a stationary random measure in $\mathbb{R}^d$, $\alpha(B) = \lambda\|B\|$ for all B bBS and this constant λ is called the *intensity* of the random measure. For $\mathbf{Y} \in \mathcal{N}$, let the translate family be $\mathbf{Y}_x := \{\phi_{(x)} : \phi \in \mathbf{Y}\}$. A pp Φ is said to be *ergodic* if $\mathsf{P}(\Phi \in \mathbf{Y}) \in \{0, 1\}$ for any $\mathbf{Y} \in \mathcal{N}$ such that $\mathsf{P}(\Phi \in (\mathbf{Y}/\mathbf{Y}_x) \cup (\mathbf{Y}_x/\mathbf{Y})) = 0$ for all $x \in \mathbb{R}^d$.

Palm measures: In the simplest informal terms, Palm measure is the probability measure of a point process conditioned to have a point at a given location. We shall now define them formally. For more details regarding Palm distributions of random easures see [Kallenberg 1983, Section 10].

Definition 2.1.2. *For a fixed measurable f such that $0 < \mathsf{E}(\int_{\mathbb{D}} f(x)\Lambda(dx)) < \infty$, the f-mixed Palm version of Λ, denoted by $\Lambda_f \in \mathbb{M}$, is defined as having the distribution*

$$\mathsf{P}(\Lambda_f \in M) = \frac{\mathsf{E}(\int_{\mathbb{D}} f(x)\Lambda(dx)\mathbf{1}[\Lambda \in M])}{\mathsf{E}(\int_{\mathbb{D}} f(x)\Lambda(dx))}, \quad M \in \mathcal{M}.$$

In case Λ (say on the Euclidean space $\mathbb{D} = \mathbb{R}^d$) has a density $\{\lambda(x)\}_{x\in\mathbb{R}^d}$, we define for each $x \in \mathbb{R}^d$ the Palm version Λ_x of Λ by the formula

$$\mathsf{P}(\Lambda_x \in M) = \frac{\mathsf{E}(\lambda(x)\mathbf{1}[\Lambda \in M])}{\mathsf{E}(\lambda(x))}, \quad M \in \mathcal{M}.$$

Palm versions Λ_x can be defined for a general random measure via some Radon-Nikodym derivatives. However, we shall use only Λ_x and mixed Palm versions Λ_f as defined above in order to avoid the arbitrariness related to the non-uniqueness of Radon-Nikodym derivatives. From the above definition, it is easy to see that the expecations with respect to Palm measure can be expressed as follows. For a function $g : \mathbb{M} \to \mathbb{R}$,

$$\begin{aligned} \mathsf{E}(g(\Lambda_f)) &= \frac{\mathsf{E}(\int_{\mathbb{D}} f(x)\Lambda(\mathrm{d}x)g(\Lambda))}{\mathsf{E}(\int_{\mathbb{D}} f(x)\Lambda(\mathrm{d}x))}, \\ \mathsf{E}(g(\Lambda_x)) &= \frac{\mathsf{E}(\lambda(x)g(\Lambda))}{\mathsf{E}(\lambda(x))}. \end{aligned} \tag{2.1}$$

The most common application of Palm measures is via the Campbell-Mecke formula (see [Stoyan et al. 1995, Pg 119]). Though the result can be stated for stationary random measures, we restrict ourselves to the case of point processes on Euclidean spaces.

Theorem 2.1.3 (Campbell-Mecke Formula)**.** *Suppose Φ is a stationary point process, with*

finite non-zero intensity λ. *Then for any non-negative measurable function* $h : \mathbb{R}^d \times \mathbb{N} \to \mathbb{R}_+$,

$$\mathsf{E}\left(\sum_{X\in\Phi} h(X,\Phi)\right) = \lambda \int_{\mathbb{R}^d} \mathsf{E}(h(x,\Phi_x))\,dx.$$

2.1.1 Random closed sets (racs)

Definition of racs : We shall work only with racs in Euclidean spaces and for general racs, refer to [Stoyan et al. 1995]. Let $\mathbb{F}$ and $\mathbb{K}$ be respectively the family of all closed and compact sets in $\mathbb{R}^d$. Let $\mathcal{F}$ be the σ-algebra generated by the "hitting sets" : $\mathbb{F}_K := \{F \in \mathbb{F} : F \cap K \neq \emptyset\}$ for $K \in \mathbb{K}$. Thus in the manner of random measures, we define racs Ξ to be a rv taking values in $(\mathbb{F}, \mathcal{F})$. A simple pp can be viewed naturally as the following racs : $\Xi = \{x \in \mathbb{R}^d : \Phi(\{x\}) > 0\}$. Observe that the x's are random due to the randomness of Φ and also that the racs has countable cardinality. Because of this representation, we shall sometimes use the notation $\Phi = \{X_i\}_{i\geq 1}$. Such dual viewpoint shall be used in the thesis. The *capacity functional* $\mathbf{T}_\Xi(.)$ of a racs is defined as:

$$\mathbf{T}_\Xi(K) := \mathsf{P}\left(\Xi \cap K \neq \emptyset\right), \quad K \in \mathbb{K}. \tag{2.2}$$

The famed Choquet's theorem (see [Matheron 1975]) states that the capacity functional characterizes the distribution of racs. Thus stationarity ($\Xi \stackrel{d}{=} \Xi + x$ for all $x \in \mathbb{R}^d$) is equivalent to $\mathbf{T}_\Xi(K) = \mathbf{T}_\Xi(K + x)$ for all $x \in \mathbb{R}^d$ and isotropy ($\Xi \stackrel{d}{=} \mathbf{m}(\Xi)$ for all *rigid motions* $\mathbf{m}$) is equivalent to $\mathbf{T}_\Xi(K) = \mathbf{T}_\Xi(\mathbf{m}(K))$ for all *rigid motions* $\mathbf{m}$.

Let us remark at this point that even though we have formally defined random sets among the class of closed sets alone, we would still take the liberty to use random compact sets or random open sets when needed. Refer to [Hall 1988, Section 3.1] for formal definition for both these classes of random sets.

Ergodicity: For $\mathbf{F} \in \mathcal{F}$ and $x \in \mathbb{R}^d$, the translate family is defined as $\mathbf{F}_x := \{F + x : F \in \mathbf{F}\}$. A racs Ξ is said to be *ergodic* if $\mathsf{P}(\Xi \in \mathbf{F}) \in \{0, 1\}$ whenever for all $x \in \mathbb{R}^d$, $\mathsf{P}(\Xi \in (\mathbf{F}/\mathbf{F}_x) \cup (\mathbf{F}_x/\mathbf{F})) = 0$.

2.2 Discrete percolation

Many proofs in continuum percolation proceed via discretisation and thus we deem it necessary to collect some sufficient conditions used to prove percolation and non-percolation on integer lattices. Let $G := (V, E)$ be a graph with a countable vertex set V and edge set E. Edges are also called as bonds. Let $\mathcal{X} := \{X_v\}_{v\in V}$ be Bernoulli rvs indexed by V. We say that a vertex is *open* if $X_v = 1$ and *closed* otherwise. The random field $\mathcal{X}$ is said to *percolate* on G if the random subset $\{v : X_v = 1\} \subset V$ has an infinite connected component. If the underlying graph (or the random field) is disambiguous, we shall only say G percolates (or X percolates). This is referred to as *site percolation*, as the random field is indexed by vertices of the graph. Alternatively, one can define *bond percolation* as above with the difference being that the Bernoulli random field will be indexed by the edges and the random subset being vertices when one of their incident edges is open.

Peierls argument: The standard technique to prove percolation of graphs or random fields is via Peierls argument. This was first used by Rudolf Peierls for showing phase transition in the 2-dimensional Ising model ([Peierls 1936]). We give here details about the Peierls argument that will be used in some of the proofs in the thesis. We shall sketch the argument for site percolation in the close-packed lattice of all dimensions and in two dimensions for bond percolation as well.

Definition 2.2.1. *Let $d \geq 2$. The (integer) lattice graph $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$ is the graph with usual d-dimensional integer lattice $\mathbb{Z}^d$ as the vertex set and edge set $\mathbb{E}^d := \{< z, z_1 >: |z - z_1| = 1\}$. The close-packed lattice graph is $\mathbb{L}^{*d} = (\mathbb{Z}^d, \mathbb{E}^{*d})$ with $\mathbb{Z}^d$ as the vertex set and edge set $\mathbb{E}^{*d} := \{< z, z_1 >: \|z - z_1\|_\infty = 1\}$, where for any $x \in \mathbb{R}^d$, $\|x\|_\infty := \max\{|x_1|, \ldots, |x_d|\}$ denotes the L_∞ or* maximum *norm.*

To know more about the notion of contours that shall be introduced in the following proof and bounds on the number of contours surrounding the origin, the reader is referred to [Lebowitz and Mazel 1998, Balister and Bollobás 2007]

Lemma 2.2.2. *Let $\mathcal{X}(h) := \{X_z(h)\}_{z\in\mathbb{Z}^d}$ be a collection of Bernoulli rvs indexed by the sites of the lattice and parametrized by $h \in \mathbb{R}_+$. Suppose that there exists a function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $g(h) \downarrow 0$ as $h \uparrow \infty$ and for any $z_1, \ldots, z_n \in \mathbb{Z}^d$, the following holds*

for some constant $c(h) > 0$ *:*

$$\mathsf{P}\left(X_{z_i}(h) = 0, 1 \le i \le n\right) \le c(h)g(h)^n.$$

Then there exists a $h_c < \infty$ *such that for all* $h > h_c$ $\mathcal{X}(h)$ *percolates in* $\mathbb{L}^d$ *and so in* $\mathbb{L}^{*d}$ *also a.s..*

Proof. The argument we sketch below is along the lines of the proof of [Meester and Roy 1996, Theorem 8.3]. As in standard percolation theory, we shall call a vertex z *h-open* if $X_z(h) = 1$, else *h-closed*. Thus, a set A is said to be h-closed if $X_z(h) = 0, \forall z \in A$. A finite connected set in $\mathbb{L}^{*d}$ is called a *contour surrounding the origin* if it cuts off the origin from infinity and the set is minimal in the sense that no strict subset of it has similar property. Now, note that $\mathcal{X}(h)$ does not percolate in $\mathbb{L}^d$ iff there are infinitely many disjoint closed contours surrounding the origin in $\mathbb{L}^{*d}$. Since a vertex has at most $(3^d - 1)$ neighbours, the number of closed contours surrounding the origin with n vertices is at most $n(3^d - 2)^{n-1}$ and the probability such a contour is closed is at most $c(h)g(h)^n$. Thus the expected number of closed contours surrounding the origin is bounded by $\sum_n n(3^d - 2)^{n-1}c(h)g(h)^n$. One can thus find a $h < \infty$ such that the latter sum is finite and hence by Borel-Cantelli lemma, the number of disjoint closed contours surrounding the origin is finite and hence $\mathcal{X}(h)$ percolates. □

Lemma 2.2.3. *Let* $\mathcal{X}(h) := \{X_e(h)\}_{e \in \mathbb{E}^2}$ *be a collection of Bernoulli rvs indexed by the bonds of the planar lattice and parametrized by* $h \in \mathbb{R}_+$*. Suppose that there exists a function* $g : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $g(h) \downarrow 0$ *as* $h \uparrow \infty$ *and for any* $e_1, \ldots, e_n \in \mathbb{Z}^d$*, the following holds for some constant* $c(h) > 0$ *:*

$$\mathsf{P}\left(X_{e_i}(h) = 0, 1 \le i \le n\right) \le c(h)g(h)^n.$$

Then there exists a $h_c < \infty$ *such that* $\mathcal{X}(h)$ *percolates a.s. in* $\mathbb{L}^2$ *for all* $h > h_c$*.*

Proof. Consider the dual graph $\overline{\mathbb{L}}^2 := (\mathbb{Z}^2, \mathbb{E}^2) + (\frac{1}{2}, \frac{1}{2})$. This is the graph $\mathbb{L}^2$ with vertices and edges shifted by $(\frac{1}{2}, \frac{1}{2})$. Each edge of the original graph intersects a unique edge of the dual graph. Declare a bond in the dual graph open iff the corresponding bond in the original graph is open under $\mathcal{X}(h)$. Note that the original graph does not percolate iff there are infinitely many disjoint closed contours around the origin in the dual graph. Since the number of contours of length n around the origin is atmost $n4^n$, as in the above

proof one can estimate the expected number of closed contours around the origin to be finite and thus the proof follows. □

Condition for non-percolation: We shall now give a sufficient condition for non-percolation of random fields on lattices. The result shall be stated for site percolation in the close-packed lattice and it is easy to see that it extends to bond percolation in the lattice too. Further, we have stated the above lemmas for $h \to \infty$ and the following lemma will be for $h \to 0$ but in usage the limits could be interchanged depending on the parametrization.

Lemma 2.2.4. *Let $\mathcal{X}(h) := \{X_z(h)\}_{z\in\mathbb{Z}^d}$ be a collection Bernoulli rvs indexed by the lattice and parametrized by $h \in \mathbb{R}_+$. Suppose that there exists a function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $g(h) \downarrow 0$ as $h \downarrow 0$ and for any $z_1, \ldots, z_n \in \mathbb{Z}^d$, the following holds for some constant $c(h) > 0$:*

$$\mathsf{P}\left(X_{z_i}(h) = 1, 1 \le i \le n\right) \le c(h)g(h)^n.$$

Then there exists a $h_c > 0$ such that $\mathcal{X}(h)$ does not percolates a.s. in $\mathbb{L}^{*d}$ for all $h < h_c$.

Proof. Let k_v^n and $N_v^n(h)$ denote respectively the number of paths in $\mathbb{L}^{*d}$ and open paths in $\mathbb{L}^{*d}$ (w.r.t $\mathcal{X}(h)$) of length n starting from a vertex v. It is easily seen that for all $v \in \mathbb{Z}^d$, $n \ge 1$, $k_v^n \le (3^d - 1)^n$. Denoting the probability of percolation of a vertex v (i.e, v is in a giant component) in $\mathcal{X}(h)$ by $p_v(h)$, we have that

$$p_v(h) = \liminf_{n\to\infty} \mathsf{P}\left(N_v^n(h) \ge 1\right) \le \liminf_{n\to\infty} \mathsf{E}(N_v^n(h)) \le \liminf_{n\to\infty} k_v^n c(h)g(h)^n.$$

Thus if $g(h) < (3^d - 2)^{-1}$, then $\mathsf{E}(N_v^n(h)) \to 0$ as $n \to \infty$ and so $p_v(h) = 0$. Thus there exists a $h_c > 0$ such that for $h < h_c$, $p_v(h) = 0$ for all $v \in \mathbb{Z}^d$. Now if we denote the probability of percolation of $\mathcal{X}(h)$ as $p(h)$, then the proof is complete by the following observation

$$p(h) = \mathsf{P}\left(\cup_{v\in\mathbb{Z}^d}\{v \text{ percolates}\}\right) \le \sum_{v\in\mathbb{Z}^d} p_v(h).$$

□

2.3 Directionally convex ordering

We review here definitions and some useful results in the theory of directionally convex ordering that are used in the thesis. Some of the other orders used occasionally are defined when needed. To understand the later results, it is enough to read the subsection on definitions alone. The auxiliary results used shall be cited in the thesis and hence can be consulted when necessary.

2.3.1 Definitions

First, we start with a very brief introduction to *dcx* order. The order $\leq$ on $\mathbb{R}^n$ shall denote the component-wise partial order, i.e., $(x_1, \ldots, x_d) \leq (y_1, \ldots, y_d)$ if $x_i \leq y_i$ for every i. We shall abbreviate $x \leq q, y \leq q$ to $[x, y] \leq q$ (and similarly $p \leq [x, y]$).

Definition 2.3.1.

- *We say that a function $f : \mathbb{R}^d \to R$ is* directionally convex *(dcx) if for every $x, y, p, q \in \mathbb{R}^d$ such that $p \leq [x, y] \leq q$ and $x + y = p + q$,*

$$f(x) + f(y) \leq f(p) + f(q).$$

- *Function f is said to be* directionally concave *(dcv) if the inequality in the above equation is reversed.*

- *Function f is said* directionally linear *(dl) if it is dcx and dcv.*

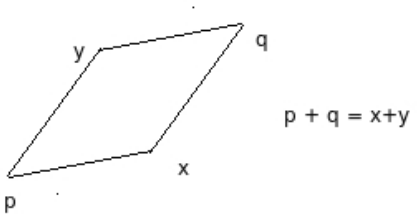

Figure 2.1: Representation of the quadruple in the definition of *dcx* function for $d = 2$.

We shall try to explain geometrically what *dcx* means in two dimensions atleast. Consider a parallelogram $pxyq$ with vertices p, x, y, q in the upper orthant of p with the diagonals $\overrightarrow{pq}$ and $\overrightarrow{xy}$ (see Fig. 2.1). Such a quadruple satisfies $p \leq [x, y] \leq q$ and $p + q = x + y$.

Thus, a function f is *dcx* means that $f(x)+f(y) \leq f(p)+f(q)$ for all such paralleograms $pxyq$.

Function $f = (f_1, \ldots, f_n) : \mathbb{R}^d \to \mathbb{R}^n$ is said to be *dcx*(*dcv*) if each of its component f_i is *dcx*(*dcv*). Also, we shall abbreviate *increasing* (with respect to the component-wise partial order on $\mathbb{R}^d$) and *dcx* by *idcx* and *decreasing* and *dcx* by *ddcx*. Similar abbreviations shall be used for *dcv* functions.The following characterizations from [Müller and Stoyan 2002, Theorem 3.12.2] is very useful to understand as well as verify *dcx* property of functions. Many other characterizations can be found in the cited Theorem.

Lemma 2.3.2. *([Shaked and Shanthikumar 1990])*

1. *A function f is dcx if and only if*

$$f(x+\delta e_j) + f(x+\epsilon e_i) \leq f(x+\epsilon e_i+\delta e_j) + f(x),$$

   for all $\epsilon, \delta > 0, x \in \mathbb{R}^d, 1 \leq i, j \leq d$ and where $e_i = (0, \ldots, 1, \ldots, 0)$ with the 1 in the ith coordinate.

2. *A twice differentiable function f is directionally convex if and only if $\dfrac{\partial^2}{\partial x_i \partial x_j} f(x) \geq 0$, for all $x \in \mathbb{R}^d, 1 \leq i, j \leq d$.*

Proof. We shall skip the proof of (2) as it is a consequence of (1). To prove (1), observe that it is enough to show that the inequality implies the following inequality

$$f(x+z) - f(x) \leq f(y+z) - f(y),$$

for all vectors $x \leq y$ and $z \geq 0$. Suppose that $y = x + \sum_i y_i e_i$ and $z = \sum_i z_i e_i$ for some $y_i, z_i \geq 0$, $\forall i$. By assumption we have that

$$f(x+z_1e_1)-f(x) \leq f(x+z_1e_1+y_1e_1)-f(x+y_1e_1) \leq \ldots \leq f(x+\sum_i y_ie_i+z_1e_1)-f(x+\sum_i y_ie_i).$$

Thus we have that $f(y)-f(x) \leq f(y+z_1e_1)-f(x+z_1e_1)$. Now using the above argument inductively on $z_2, \ldots, z_n$, we get the required inequality for x, y, z.

□

Example 2.3.3. *By Lemma 2.3.2, it is easy to see that the following functions are dcx :*

1. $f(x_1, \ldots, x_n) := \prod f_i(x_i)$ *where* $f_i(.)$ *are positive increasing (or all the* f_i *are positive decreasing) convex functions on the real line ;*

2. $f(x_1, \ldots, x_n) := \sum a_i f_i(x_i)$ *where* $f_i(.)$ *are convex functions on the real line and* a_i *are positive constants ;*

3. $f(x, y) = xy$.

In the following, let $\mathfrak{F}$ denote some class of functions from $\mathbb{R}^d$ to $\mathbb{R}$. The dimension d is assumed to be clear from the context. Unless mentioned, when we state $\mathsf{E}(f(X))$ for $f \in \mathfrak{F}$ and X a random vector, we assume that the expectation exists, i.e., for each random vector X we consider the sub-class of $\mathfrak{F}$ for which the expectations exist with respect to (w.r.t) X.

Definition 2.3.4.

- *Suppose* X *and* Y *are real-valued random vectors of the same dimension. Then* X *is said to be* less than Y in $\mathfrak{F}$ order *if* $\mathsf{E}(f(X)) \leq \mathsf{E}(f(Y))$ *for all* $f \in \mathfrak{F}$ *(for which both expectations are finite). We shall denote it as* $X \leq_{\mathfrak{F}} Y$.

- *Suppose* $\{X(s)\}_{s \in S}$ *and* $\{Y(s)\}_{s \in S}$ *are real-valued random fields, where* S *is an arbitrary index set. We say that* $\{X(s)\} \leq_{\mathfrak{F}} \{Y(s)\}$ *if for every* $n \geq 1$ *and* $s_1, \ldots, s_n \in S$, $(X(s_1), \ldots, X(s_n)) \leq_{\mathfrak{F}} (Y(s_1), \ldots, Y(s_n))$.

In the remaining part of the thesis, we will mainly consider $\mathfrak{F}$ to be the class of *dcx*, *idcx* and *idcv* functions; the negation of these functions give rise to *dcv*, *ddcv* and *ddcx* orders respectively. If $\mathfrak{F}$ is the class of increasing functions, we shall replace $\mathfrak{F}$ by *st* (strong) in the above definitions. These are standard notations used in literature. Here is a simple condition for *dcx* ordering of random vectors with independent marginals.

Lemma 2.3.5. *Let* $X_i, 1 \leq i \leq n$ *and* $Y_i, 1 \leq i \leq n$ *be a collection of independent random variables such that* $X_i \leq_{cx} Y_i, \forall i$ *where cx stand for convex order, the stochastic order generated by convex functions. Then* $(X_1, \ldots, X_n) \leq_{dcx} (Y_1, \ldots, Y_n)$.

The proof just follows by noting that a *dcx* function is convex on each coordinate and by conditioning on coordinates one by one, we get the required ordering.

2.3.2 Auxilliary results

In order to make the thesis more self-contained, we shall recall now some basic results on stochastic orders used in the main stream of the thesis. The proofs in the this section will

be detailed without being complete. The following lemmas which are true for many other orders as well can be found in [Müller and Stoyan 2002, Chapter 3].

Lemma 2.3.6. *If $X \leq_{dcx\,(resp.\,idcx;idcv)} Y$, then for Z_1, Z_2 independent of X, Y and $Z_1 \leq_{dcx\,(resp.\,idcx;idcv)} Z_2$, we have that $X + Z_1 \leq_{dcx\,(resp.\,idcx;idcv)} Y + Z_2$.*

Proof. We shall prove in the case of *dcx* alone and the proof is similar for the other two orders. Also, observe that it suffices to prove assuming $Z_1 = Z_2 = Z$ as by re-using the same arguments one can get the result stated in the lemma. For $z \in \mathbb{R}^d$ and a *dcx* function $f : \mathbb{R}^d \to \mathbb{R}$, define $f_z(x) := f(x + z)$. Clearly, $f_z(.)$ is *dcx* for all $z \in \mathbb{R}^d$, whenever $f(.)$ is. Thus,

$$\mathsf{E}(f(X + Z)) = \mathsf{E}(\mathsf{E}(f_Z(X)|Z)) \leq \mathsf{E}(\mathsf{E}(f_Z(Y)|Z)) = \mathsf{E}(f(Y + Z))\,.$$

□

Lemma 2.3.7. *The stochastic order relation $\leq_{dcx}$ is generated by infinitely differentiable dcx functions i.e, it suffices to check Definition 2.3.4 for infinitely differentiable dcx functions alone.*

Proof. The proof is an adaptation to *dcx* order of a more general result, [Müller and Stoyan 2002, Theorem 2.5.5]. Since every dcx function is continuous also, a *dcx* function f can be approximated by a sequence $\{f_n\}$ of infinitely differentiable functions as follows (see [Rudin 1987, Theorem 9.10]) :

$$f_n(x) = \int_{\mathbb{R}^d} f(x - y) g_n(y) \mathrm{d}y \text{ and } \lim_{n\to\infty} \mathsf{E}(f_n(X)) = \mathsf{E}(f(X))\,, \text{for any random vector } X$$

where $g_n(y) := n^d g(nx)$ for some infinitely differentiable probability density g with compact support. Using the fact that $f(. - y)$ is *dcx*, it is easy to see that f_n is also *dcx* and so every *dcx* function can be approximated by a sequence of infinitely differentiable *dcx* functions such that the expectations converge as well. This proves the result. □

Lemma 2.3.8. *Let $(X^{(k)} : k = 1, \ldots)$ and $(Y^{(k)} : k = 1, \ldots)$ be sequences of random vectors. Suppose $X^{(k)} \leq_{dcx} Y^{(k)}$ for all $k \in \mathbb{N}$. If $X^{(k)} \xrightarrow{d} X$ and $Y^{(k)} \xrightarrow{d} Y$ and moreover $\mathsf{E}(X^{(k)}) \to \mathsf{E}(X)$ and $\mathsf{E}(Y^{(k)}) \to \mathsf{E}(Y)$, then $X \leq_{dcx} Y$.*

Proof. Weak convergence and convergence of expectations together are equivalent to $\mathsf{E}\big(f(X^{(k)})\big) \to \mathsf{E}(f(X))$ for all continuous f such that $f(x) = O(\|x\|)$ at infinity (see

[Bickel and Freedman 1981, Lemma 8.3]). By this result and the continuity of a directionally convex function, we have that $\mathsf{E}(f(X)) \leq \mathsf{E}(f(Y))$ for all directionally convex f such that $f(x) = O(\|x\|)$ at infinity. Now it is enough to show that a *dcx* function can be approximated monotonically by a sequence of *dcx* functions with bounded growth behaviour at infinity as described above. Such an approximation is given in what follows. Define $\psi_m : \mathbb{R} \to \mathbb{R}$ as

$$\psi_m(x) = \begin{cases} m & \text{if } x > m \\ x & \text{if } -m \leq x \leq m \\ -m & \text{if } x < -m, \end{cases}$$

and now define $g_{m,i} : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, n$ as

$$g_{m,i}(x) = \begin{cases} \frac{\partial^-}{\partial x_i} f(\psi_m(x_1), \ldots, \psi_m(x_n)) & \text{if } x_i > m \\ \frac{\partial^+}{\partial x_i} f(\psi_m(x_1), \ldots, \psi_m(x_n)) & \text{if } x_i < -m \\ 0 & \text{if else,} \end{cases}$$

where $\frac{\partial^-}{\partial x_i}, \frac{\partial^+}{\partial x_i}$ are the left and right partial derivatives respectively. Now one can verify that the following sequence of functions $\{f_m\}$ are *dcx* (by taking their second partial derivatives), monotonically increasing and have bounded growth behaviour at infinity.

$$f_m(x) = f(\psi_m(x_1), \ldots, \psi_m(x_n)) + \sum_{i=1}^{n} g_{m,i}(x)(x_i - \psi_m(x_i)).$$

Using the above approximation, one can show that $\mathsf{E}(f_m(X)) \nearrow \mathsf{E}(f(X))$ for any directionally convex f. This proves the lemma. □

Let $dcx, idcx, idcv$ functions be defined on $\mathbb{Z}^d$ similar to Definition 2.3.1 using the natural partial orders on those spaces respectively. Let $sgn(x) := \frac{x}{|x|}$ denote the sign of a number.

Lemma 2.3.9. *([Meester and Shanthikumar 1993, Lemmas 2.17,2.18]) For $(\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$, let $N_i, i = 1, \ldots, k$ denote k mutually independent Poisson rv where the mean of N_i is $|\lambda_i|$. If $f : \mathbb{Z}^k \to \mathbb{R}$ is dcx (resp. idcx; idcv), then $g : \mathbb{R}^k \to \mathbb{R}$ defined by $g(\lambda_1, \ldots, \lambda_k) := \mathsf{E}(f(sgn(\lambda_1)N_1, \ldots, sgn(\lambda_k)N_k))$ is also dcx (resp. idcx; idcv).*

Proof. We provide the proof in the case of f being *dcx* and the other two are similar.

Choose a quadruple in $\mathbb{R}^k$ such that $\lambda^1 \leq [\lambda^2, \lambda^3] \leq \lambda^4, \lambda^1 + \lambda^4 = \lambda^2 + \lambda^3$. Let $N_i(|\lambda|), i = 1, \ldots, k$ and $\tilde{N}_i(|\lambda|), i = 1, \ldots, k$ denote independent families of mutually independent Poisson rv with mean $|\lambda|$. As a consequence of the following obvious equalities,

$$sgn(\lambda_i^3)|\lambda_i^3| = sgn(\lambda_i^1)|\lambda_i^1| + \lambda_i^3 - \lambda_i^1, \quad sgn(\lambda_i^3)|\lambda_i^3| = sgn(\lambda_i^1)|\lambda_i^1| + \lambda_i^3 - \lambda_i^1,$$

we can construct a probability space such that $\forall i = 1, \ldots, k$ we have the following $a.s.$ equalities :

$$\begin{aligned} sgn(\lambda_i^3)N_i(|\lambda_i^3|) &\stackrel{a.s.}{=} sgn(\lambda_i^1)\tilde{N}_i(|\lambda_i^1|) + \tilde{N}_i(\lambda_i^3 - \lambda_i^1), \\ sgn(\lambda_i^4)N_i(|\lambda_i^4|) &\stackrel{a.s.}{=} sgn(\lambda_i^2)N_i(|\lambda_i^2|) + \tilde{N}_i(\lambda_i^3 - \lambda_i^1). \end{aligned}$$

Thus, we get that,

$$\begin{aligned} & g(\lambda^2) + g(\lambda^3) \\ = \; & \mathsf{E}\big(f(sgn(\lambda_1^2)N_1(|\lambda_1^2|), \ldots, sgn(\lambda_1^2)N_k(|\lambda_k^2|))\big) + \\ & \mathsf{E}\Big(f(sgn(\lambda_1^1)N_1(|\lambda_1^1|) + \tilde{N}_1(\lambda_1^3 - \lambda_1^1), \ldots, sgn(\lambda_1^1)N_k(|\lambda_k^1|) + \tilde{N}_k(\lambda_k^3 - \lambda_k^1))\Big) \\ = \; & \mathsf{E}\left(f(sgn(\lambda_1^2)N_1(|\lambda_1^2|), \ldots, sgn(\lambda_1^2)N_k(|\lambda_k^2|)) + f(sgn(\lambda_1^2)N_1(|\lambda_1^1|) + \right. \\ & \left. \tilde{N}_1(\lambda_1^3 - \lambda_1^1), \ldots, sgn(\lambda_k^1)N_k(|\lambda_k^1|) + \tilde{N}_k(\lambda_k^3 - \lambda_k^1))\right) \\ \leq \; & \mathsf{E}\Big(f(sgn(\lambda_1^2)N_1(|\lambda_1^2|) + \tilde{N}_1(\lambda_1^3 - \lambda_1^1), \ldots, sgn(\lambda_k^2)N_k(|\lambda_k^2|) + \tilde{N}_k(\lambda_k^3 - \lambda_k^1)) + \\ & f(sgn(\lambda_1^1)N_1(|\lambda_1^1|), \ldots, sgn(\lambda_k^1)N_k(|\lambda_k^1|))\Big) \\ = \; & \mathsf{E}\big(f(sgn(\lambda_1^4)N_1(|\lambda_1^4|), \ldots, sgn(\lambda_k^4)N_k(|\lambda_k^4|))\big) + \\ & \mathsf{E}\big(f(sgn(\lambda_1^1)N_1(|\lambda_1^1|), \ldots, sgn(\lambda_k^1)N_k(|\lambda_k^1|))\big) = g(\lambda^4) + g(\lambda^1). \end{aligned}$$

□

Chapter 3

AB random geometric graphs

3.1 Percolation in the AB Poisson Boolean model

3.1.1 Model definition

We shall now describe the AB Poisson Boolean model. Let $\Phi^{(1)} = \{X_i\}_{i\geq 1}$ and $\Phi^{(2)} = \{Y_i\}_{i\geq 1}$ be independent Poisson point processes in $\mathbb{R}^d$, $d \geq 2$, with intensities λ and μ respectively. Recall that by $B_x(r)$, we denote the closed ball of radius r centred at $x \in \mathbb{R}^d$. For a definition of Poisson point process, see Section 2.1.

The usual continuum percolation model is defined as follows.

Definition 3.1.1. *Define the Poisson Boolean model $\tilde{G}(\lambda, r) := (\Phi^{(1)}, \tilde{E}(\lambda, r))$ to be the graph with vertex set $\Phi^{(1)}$ and edge set*

$$\tilde{E}(\lambda, r) = \{\langle X_i, X_j\rangle : X_i, X_j \in \Phi^{(1)}, |X_i - X_j| \leq 2r\}.$$

The edges in all the graphs that we consider are undirected, that is, $\langle X_i, X_j\rangle \equiv \langle X_j, X_i\rangle$. We will use the notation $X_i \sim X_j$ to denote existence of an edge between X_i, X_j when the underlying graph is unambiguous. By percolation, we mean the existence of an infinite connected component in the graph. Topologically, percolation means existence of an unbounded connected subset of $\bigcup_{X_i\in\Phi^{(1)}} B_{X_i}(r)$. For fixed $r > 0$, define

$$\lambda_c(r) := \inf\left\{\lambda > 0 : \mathsf{P}\left(\tilde{G}(\lambda, r) \text{ percolates }\right) > 0\right\}. \tag{3.1}$$

For this usual continuum percolation model ([Meester and Roy 1996]), it is known that

$0 < \lambda_c(r) < \infty$.

A natural analog of this model to the AB set-up would be to consider a graph with vertex set $\Phi^{(1)}$ where each vertex is independently marked A or B. We will consider a more general model from which results for the above model will follow as a corollary.

Definition 3.1.2. *The AB Poisson Boolean model* $G(\lambda, \mu, r) := (\Phi^{(1)}, E(\lambda, \mu, r))$ *is the graph with vertex set* $\Phi^{(1)}$ *and edge set*

$$E(\lambda, \mu, r) := \{\langle X_i, X_j \rangle : X_i, X_j \in \Phi^{(1)}, |X_i - Y| \leq 2r, |X_j - Y| \leq 2r, \text{ for some } Y \in \Phi^{(2)}\}.$$

Let $\theta(\lambda, \mu, r) = \mathsf{P}\left(G(\lambda, \mu, r) \text{ percolates}\right)$. Since percolation of the model depends on points outside $B_0(R)$ for any $R > 0$, it is in the tail sigma-field of $(\Phi^{(1)} \cup \Phi^{(2)}) \cap B_0(N)^c, N \in \mathbb{N}$. Thus, it follows from Kolmogorov's $0-1$ law and the complete independence property of the Poisson pp that $\theta(\lambda, \mu, r) \in \{0, 1\}$. We are interested in characterizing the region formed by (λ, μ, r) for which $\theta(\lambda, \mu, r) = 1$.

Definition 3.1.3. *For fixed* $\lambda, r > 0$, *define the critical intensity* $\mu_c(\lambda, r)$ *by*

$$\mu_c(\lambda, r) := \sup\{\mu : \theta(\lambda, \mu, r) = 0\}.$$

3.1.2 Main results

In this section, we shall state our results on percolation in the AB Boolean model and the proofs are postponed to Section 3.3. We start with some simple lower bounds for the critical intensity $\mu_c(\lambda, r)$.

Proposition 3.1.4. *Fix* $\lambda, r > 0$. *Let* $\lambda_c(r)$, $\mu_c(\lambda, r)$ *be the critical intensities as in (3.1) and Definition 3.1.3, respectively. Then the following statements are true.*

1. *If* $\lambda_c(2r) < \lambda < \lambda_c(r)$, *then* $\mu_c(\lambda, r) \geq \lambda_c(r) - \lambda$.
2. *If* $\lambda \leq \lambda_c(2r)$, $\mu_c(\lambda, r) = \infty$.

However, it is not clear that $\mu_c(\lambda, r) < \infty$ for $\lambda > \lambda_c(2r)$. We answer this in affirmative for $d = 2$.

Theorem 3.1.5. *Let* $d = 2$ *and* $r > 0$ *be fixed. Then for any* $\lambda > \lambda_c(2r)$, *we have* $\mu_c(\lambda, r) < \infty$.

Thus the AB Boolean model exhibits a *phase transition* in the plane. However, the above theorem does not tell us how to choose a μ for a given $\lambda > \lambda_c(2r)$ for $d = 2$ such that AB percolation happens, or if indeed there is a phase transition for $d \geq 3$. In other words, we would like to obtain estimates on $\mu_c(\lambda, r)$. We have already obtained a lower bound and we will obtain an upper bound for $\mu_c(\lambda, r)$ as a special case of a more general result which is the continuum analog of *word percolation* on discrete lattices described in the following paragraph.

Word percolation is a generalization of the discrete AB percolation model and has been studied on various graphs by Kesten *et. al.* (see [Benjamini and Kesten 1995, Kesten et al. 1998, Kesten et al. 2001]). The model is as follows : Mark each vertex or site of a graph $G := (V, E)$ independently as 0 or 1 with probability p and $1-p$ respectively. Given any infinite sequence (referred to as a word) $w \in \{0,1\}^\infty$, the question is whether w occurs in the graph G or not. The words $(1,0,1,0...), (0,1,0,1..)$ correspond to AB percolation and the word $(1,1,1...)$ corresponds to usual percolation. More generally Kesten *et. al.* answer whether all (or almost all) infinite sequences (words) are seen in G or not. The graphs for which the answer is known in affirmative are $\mathbb{Z}^d$ for d large, triangular lattice and $\mathbb{L}^{*2}$, the close-packed graph of $\mathbb{Z}^2$ (see Definition 2.2.1). Our results provide partial answers to these questions in the continuum.

For the purpose of stating our results, let us recall the *Bernoulli or independent site percolation model* on a graph G (see 1.1.1) with parameter p is the subgraph $G(p)$ obtained by keeping vertices independently with probability p and else removing them. Edges are retained only if both the corresponding vertices are in the sub-graph. Percolation, as always, denotes the existence of an infinite connected component in this subgraph. The critical probability of an independent site percolation on G is defined as $p_c(G) := \inf\{p : \mathsf{P}(G(p) \text{ percolates}) > 0\}$.

Definition 3.1.6. *For each $d \geq 2$, define the critical probabilities $p_c(d)$, and the functions $a(d,r)$ as follows.*

1. *For $d = 2$, consider the triangular lattice $\mathbb{T}$ (see Figure 3.1) with edge length $r/2$. Let $p_c(2) = p_c(\mathbb{T})$ be the critical probability for independent site percolation on this lattice. Around each vertex of $\mathbb{T}$, place a "flower" formed by six circular arcs (see Figure 3.1). These arcs are formed by circumferences of circles of radius $\frac{r}{2}$ drawn from the mid-points of the six adjacent edges to the vertex. Let $a(2,r)$ be the area of a flower.*

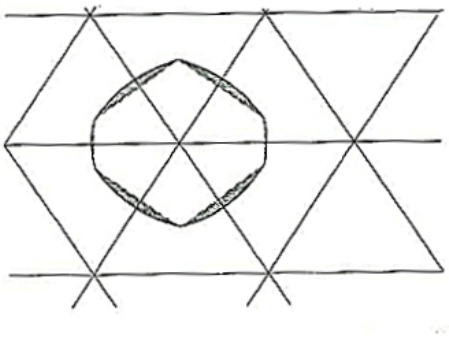

Figure 3.1: The triangular lattice and flower in $\mathbb{R}^2$ with area $a(2,r)$. The figure is reproduced from [Meester and Roy 1996, Fig. 3.2].

2. *For $d \geq 3$, let $p_c(d) = p_c(\mathbb{L}^{*d})$ be the critical probability for independent site percolation on $\mathbb{L}^{*d}$, the close-packed integer lattice and define $a(d,r) = (r/2\sqrt{d})^d$.*

It is known that $p_c(2) = \frac{1}{2}$, and $p_c(d) < 1$ for $d \geq 3$ (see [Grimmett 1999, Theorem 1.33 and pp. 56]).

Definition 3.1.7. *Fix $k \in \mathbb{N}$ and let $(r_1, \ldots, r_k) \in \mathbb{R}_+^k$. For $i = 1, \ldots, k$, let $\Phi^{(i)}$ be independent Poisson point processes of intensity $\lambda_i > 0$. A word $\omega = \{w(i)\}_{i \geq 1} \in \{1, 2, \ldots, k\}^\infty$ is said to occur if there exists a sequence of distinct elements $\{X_i\}_{i \geq 1} \subset \mathbb{R}^d$, such that $X_i \in \Phi^{(w(i))}$, and $|X_i - X_{i+1}| \leq r_{w(i)} + r_{w(i+1)}$, for $i \geq 1$.*

Proposition 3.1.8. *For any $d \geq 2$, let $p_c(d)$, $a(d,r)$ be as in Definition 3.1.6. Let $k \in \mathbb{N}, (r_1, \ldots, r_k) \in \mathbb{R}_+^k$ and $\Phi^{(i)}, i = 1, \ldots, k$ be independent Poisson point processes of intensity $\lambda_i > 0$. Set $r_0 = \inf_{1 \leq i,j \leq k}\{r_i + r_j\}$. If $\prod_{i=1}^k (1 - e^{-\lambda_i a(d,r_0)}) > p_c(d)$, then almost surely, all words occur.*

The following corollary gives an upper bound for $\mu_c(\lambda, r)$ for large λ.

Corollary 3.1.9. *Suppose that $d \geq 2$, $r > 0$, and $\lambda > 0$ satisfies*

$$\lambda > -\frac{\log(1 - p_c(d))}{a(d, 2r)},$$

where $p_c(d)$, $a(d,r)$ are as in Definition 3.1.6. Let $\mu_c(\lambda, r)$ be the critical intensity as in

Definition 3.1.3. Then

$$\mu_c(\lambda, r) \leq -\frac{1}{a(d, 2r)} \log\left[1 - \left(\frac{p_c(d)}{1 - e^{-\lambda a(d,2r)}}\right)\right]. \tag{3.2}$$

Remark 3.1.10. *A simple calculation (see [Meester and Roy 1996], pg.88) gives* $a(2,2) \simeq 0.8227$*, and so*

$$-(a(2,2))^{-1} \log(1 - p_c(2)) \simeq 0.843.$$

Using these we obtain from Corollary 3.1.9 that $\mu_c(0.85, 1) \leq 6.2001$.

Remark 3.1.11. *It can be shown that the number of infinite components in the AB Boolean model is atmost one, almost surely. The proof of this fact follows along the same lines as the proof in Poisson Boolean model (see [Meester and Roy 1996, Proposition 3.3, Proposition 3.6]), since it relies on the ergodic theorem and the topology of infinite components, and not on the specific nature of the infinite components.*

The above proposition can be used to show existence of AB percolation in the natural analog of the discrete AB percolation model (refer to the two sentences above Definition 3.1.2). Recall that $\Phi^{(1)}$ is a Poisson point process in $\mathbb{R}^d$ of intensity $\lambda > 0$. Let $\{m_i\}_{i \geq 1}$ be a sequence of i.i.d. marks distributed as $m \in \{A, B\}$, with $\mathsf{P}(m = A) = p = 1 - \mathsf{P}(m = B)$. Define the point processes Φ^A, Φ^B as

$$\Phi^A := \{X_i \in \Phi^{(1)} : m_i = A\}, \qquad \Phi^B := \Phi^{(1)} \setminus \Phi^A.$$

Definition 3.1.12. *For any* $\lambda, r > 0$*, and* $p \in (0,1)$*, let* Φ^A, Φ^B *be as defined above and* $\widehat{G}(\lambda, p, r) := (\Phi^A, \widehat{E}(\lambda, p, r))$ *be the graph with vertex-set* Φ^A *and edge-set*

$$\widehat{E}(\lambda, p, r) := \{< X_i, X_j >: X_i, X_j \in \Phi^A,\ |X_i - Y| \leq 2r, |X_j - Y| \leq 2r, \ \text{ for some } Y \in \Phi^B\}.$$

Corollary 3.1.13. *Let* $\widehat{\theta}(\lambda, p, r) := P(\widehat{G}(\lambda, p, r)$ *percolates). Then for any* λ *satisfying*

$$\lambda > -\frac{2\log\left(1 - \sqrt{p_c(d)}\right)}{a(d, 2r)},$$

there exists a $p(\lambda) < \frac{1}{2}$*, such that* $\widehat{\theta}(\lambda, p, r) = 1$*, for all* $p \in (p(\lambda), 1 - p(\lambda))$.

3.2 Connectivity in AB random geometric graphs

3.2.1 Model definition

The set up for the study of connectivity in AB random geometric graphs is as follows. For each $n \geq 1$, let $\mathcal{P}_n^{(1)}$ and $\mathcal{P}_n^{(2)}$ be independent homogenous Poisson point processes in $U = [0,1]^d$, $d \geq 2$, of intensity n. We also nullify some of the technical complications arising out of boundary effects by choosing to work with the toroidal metric on the unit cube, defined as

$$d(x,y) := \inf\{|x - y + z| : z \in \mathbb{Z}^d\}, \qquad x, y \in U. \tag{3.3}$$

Definition 3.2.1. *For any $m, n \geq 1$, the AB random geometric graph $G_n(m,r)$ is the graph with vertex set $\mathcal{P}_n^{(1)}$ and edge set*

$$E_n(m,r) := \{\langle X_i, X_j\rangle : X_i, X_j \in \mathcal{P}_n^{(1)}, d(X_i,Y) \leq r, d(X_j,Y) \leq r, \text{ for some } Y \in \mathcal{P}_m^{(2)}\}.$$

Our goal in this section is to study the *connectivity threshold* in the sequence of graphs $G_n(cn,r)$ as $n \to \infty$ for $c > 0$. The constant c can be thought of as a measure of the relative density of $\mathcal{P}_n^{(1)}$ with respect to $\mathcal{P}_{cn}^{(2)}$ (see Remark 3.2.5 below). It is easier to first consider the critical radius required to eleminate isolated nodes.

Definition 3.2.2. *For each $n \geq 1$, let $W_n(r)$ be the number of isolated nodes, that is, vertices with degree zero in $G_n(cn,r)$, and define the largest nearest neighbor radius as*

$$M_n := \sup\{r \geq 0 : W_n(r) > 0\}.$$

3.2.2 Main results

In this section, we shall state our results on connectivity in the AB random geometric graph and the proofs are postponed to Section 3.4. Let $\theta_d := \|B_O(1)\|$ be the volume of the d-dimensional unit closed ball. For any $\beta > 0$, and $n \geq 1$, define the sequence of cut-off functions,

$$r_n(c,\beta) = \left(\frac{\log(n/\beta)}{cn\theta_d}\right)^{\frac{1}{d}}, \tag{3.4}$$

and let

$$r_n(c) = r_n(c,1). \tag{3.5}$$

Let $\phi(a) = \arccos(a)$. For $d = 2$, define

$$A(c) = \pi^{-1}\left[2\phi\left(\frac{c^{\frac{1}{2}}}{2}\right) - \sin\left(2\phi\left(\frac{c^{\frac{1}{2}}}{2}\right)\right)\right]. \tag{3.6}$$

Define the constant c_0 to be

$$c_0 := \begin{cases} \sup\{c : A(c) + \frac{1}{c} > 1\} & \text{if } d = 2 \\ 1 & \text{if } d \geq 3. \end{cases} \tag{3.7}$$

The derivative of the function $A(c)$ is $A'(c) = 2\frac{\phi(c^{\frac{1}{2}}/2)}{c}\left(1 - \cos(2\phi(\frac{c^{\frac{1}{2}}}{2}))\right) \leq 0$ as the first term is negative and the second term is positive in the interval $0 \leq c \leq 4$. Thus the function $A(c)$ is decreasing and hence $1 < c_0 \leq 4$ for $d = 2$. The first part of the following Lemma shows that for $c < c_0$, the above choice of radius stabilizes the expected number of isolated nodes in $G_n(cn, r_n(c, \beta))$ as $n \to \infty$. The second part shows that the assumption $c < c_0$ is not merely technical. The Lemma also suggests a *phase transition* at some $\tilde{c} \in [c_0, 2^d]$, in the sense that, for $c < \tilde{c}$ the expected number of isolated nodes in $G_n(cn, r_n(c, \beta))$ converges to a finite limit and diverges for $c > \tilde{c}$.

Lemma 3.2.3. *For any $\beta, c > 0$, let $r_n(c, \beta)$ be as defined in (3.4), and $W_n(r_n(c, \beta))$ be the number of isolated nodes in $G_n(cn, r_n(c, \beta))$. Let c_0 be as defined in (3.7). Then as $n \to \infty$,*

1. $\mathsf{E}(W_n(r_n(c, \beta))) \to \beta$ *for* $c < c_0$, *and*
2. $\mathsf{E}(W_n(r_n(c, \beta))) \to \infty$ *for* $c > 2^d$.

For $c < c_0$, having found the radius that stabilizes the mean number of isolated nodes, the next theorem shows that the number of isolated nodes and the largest nearest neighbour radius in $G_n(cn, r_n(c, \beta))$ converge in distribution as $n \to \infty$. Let $Po(\beta)$ denote a Poisson rv with mean β and recall that $\xrightarrow{d}$ stands for convergence in distribution.

Theorem 3.2.4. *Let $r_n(c, \beta)$ be as defined in (3.4) with $\beta > 0$ and $0 < c < c_0$. Then as $n \to \infty$,*

$$W_n(r_n(c, \beta)) \xrightarrow{d} Po(\beta), \tag{3.8}$$

$$\mathsf{P}\left(M_n \leq r_n(c, \beta)\right) \to e^{-\beta}. \tag{3.9}$$

Remark 3.2.5. *Let $B_x(r)$ denote the closed ball of radius r centred at $x \in \mathbb{R}^d$. For any locally finite point process $\mathcal{X}$ (for example $\mathcal{P}_n^{(1)}$ or $\mathcal{P}_n^{(2)}$), we denote the number of points of $\mathcal{X}$ in A, $A \subset \mathbb{R}^d$ by $\mathcal{X}(A)$. Define*

$$W_n^0(c,r) = \sum_{Y_i \in \mathcal{P}_{cn}^{(2)}} \mathbf{1}[\mathcal{P}_n^{(1)}(B_{Y_i}(r)) = 0],$$

that is, $W_n^0(c,r)$ is the number of $\mathcal{P}_{cn}^{(2)}$ nodes isolated from $\mathcal{P}_n^{(1)}$ nodes. From Palm calculus for Poisson point processes (Theorem 2.1.3) and the fact that the metric is toroidal, we have

$$\mathsf{E}\big(W_n^0(c, r_n(c,\beta))\big) = cn \int_U \mathsf{P}\left(\mathcal{P}_n^{(1)}(B_x(r)) = 0\right) dx = cn \exp(-n\theta_d r_n(c,\beta)^d).$$

Substituting from (3.4) we get

$$\lim_{n\to\infty} \mathsf{E}\big(W_n^0(c, r_n(c,\beta))\big) \begin{cases} 0 & \text{if } c < 1 \\ \beta & \text{if } c = 1 \\ \infty & \text{if } c > 1. \end{cases} \tag{3.10}$$

Thus there is a trade off between the relative density of the nodes and the radius required to stabilise the expected number of isolated nodes.

The next theorem gives asymptotic bounds for strong connectivity threshold in the *AB* random geometric graph. While the lower bound can be derived using Theorem 3.2.4, for the upper bound, we couple the AB random geometric graph with the Poisson random geometric graph and use the connectivity threshold for the usual random geometric graph (see Theorem 3.4.5). As will become obvious, the bounds are very tight for small c. We will take $\beta = 1$ in (3.4) and work with the cut-off functions $r_n(c)$ as defined in (3.5). Define the function $\eta : \mathbb{R}_+^2 \to \mathbb{R}$ by

$$\eta(a,c) \begin{cases} \frac{1}{\pi}\left[2\phi\left(\frac{1}{2}\left(\frac{c}{a}\right)^{\frac{1}{2}}\right) - \sin\left(2\phi\left(\frac{1}{2}\left(\frac{c}{a}\right)^{\frac{1}{2}}\right)\right)\right] & \text{if } d = 2 \\ \left(1 - \frac{1}{2}\left(\frac{c}{a}\right)^{\frac{1}{d}}\right)^d & \text{if } d \geq 3, \end{cases} \tag{3.11}$$

where $\phi(a) = \arccos(a)$. Define the function $\alpha : \mathbb{R}_+ \to \mathbb{R}$ by

$$\alpha(c) := \inf\{a : a\eta(a, c) > 1\}. \tag{3.12}$$

It is easily seen that $\alpha(c) \leq \left(1 + \frac{c^{\frac{1}{d}}}{2}\right)^d$ for $d \geq 2$ with equality for $d \geq 3$.

Theorem 3.2.6. *Let $\alpha(c)$ be as defined in (3.12), $r_n(c)$ be as defined in (3.5) and c_0 be as in (3.7). Define $\alpha_n^*(c) := \inf\{a : G_n(cn, a^{\frac{1}{d}} r_n(c))$ is connected$\}$. Then*

$$\begin{aligned} \forall c < c_0, a < 1, \qquad & \lim_{n\to\infty} \mathsf{P}\left(G_n(cn, ar_n) \text{ is connected }\right) = 0 \text{ and} \\ \forall c > 0, \qquad & \limsup_{n\to\infty} \alpha_n^*(c) \leq \alpha(c) \quad a.s.. \end{aligned}$$

Strong law of large numbers and central limit theorems for stabilizing functionals of Poisson p.p. have been developed in [Penrose and Yukich 2005, Penrose 2007]. Heuristically, stabilizing functionals are functionals of the point process such that the contribution of each point to the functional depends only on points in a bounded window around the point. Though we do not go into details here, it is expected that, strong laws and central limit theorems can be derived for stabilizing functionals of AB random geometric graph as they are nothing but stabilizing functionals of a marked Poisson process with intensity $(1 + c)n$.

3.3 Proofs for Section 3.1

Proof of Proposition 3.1.4

(1). Recall from Definition 3.1.2 the graph $G(\lambda, \mu, r)$ with vertex set $\Phi^{(1)}$ and edge set $E(\lambda, \mu, r)$. Consider the graph $\tilde{G}(\lambda + \mu, r)$ (see Definition 3.1.1), where the vertex set is taken to be $\Phi^{(1)} \cup \Phi^{(2)}$ and let the edge set of this graph be denoted $\tilde{E}(\lambda + \mu, r)$.

If $< X_i, X_j > \in E(\lambda, \mu, r)$, then there exists a $Y \in \Phi^{(2)}$ such that $< X_i, Y >, < X_j, Y > \in \tilde{E}(\lambda+\mu, r)$. It follows that $G(\lambda, \mu, r)$ has an infinite component only if $\tilde{G}(\lambda+\mu, r)$ has an infinite component. Consequently, for any $\mu > \mu_c(\lambda, r)$ we have $\mu + \lambda > \lambda_c(r)$, and hence $\mu_c(\lambda, r) + \lambda \geq \lambda_c(r)$. Thus for any $\lambda < \lambda_c(r)$, we obtain the (non-trivial) lower bound $\mu_c(\lambda, r) \geq \lambda_c(r) - \lambda$.

(2). Again $< X_i, X_j > \in E(\lambda, \mu, r)$ implies that $|X_i - X_j| \leq 4r$. Hence, $G(\lambda, \mu, r)$ has an infinite component only if $\tilde{G}(\lambda, 2r)$ has an infinite component. Thus $\mu_c(\lambda, r) = \infty$ if

$\lambda \leq \lambda_c(2r)$. □

Proof of Theorem 3.1.5

Fix $\lambda > \lambda_c(2r)$. The proof adapts the idea used in [Dousse et al. 2006] (see also proof of Theorem 6.3.10) of coupling the continuum percolation model to a discrete percolation model. For $l > 0$, let $l\mathbb{L}^2$ be the graph with vertex set $l\mathbb{Z}^2$, the expanded two-dimensional integer lattice, and endowed with the usual graph structure, that is, $x, y \in l\mathbb{Z}^2$ share an edge if $|x - y| = l$. Denote the edge-set by $l\mathbb{E}^2$. For any edge $e \in l\mathbb{E}^2$ denote the mid-point of e by (x_e, y_e). For every horizontal edge e, define three rectangles $R_{ei}, i = 1, 2, 3$ as follows : R_{e1} is the rectangle $[x_e - 3l/4, x_e - l/4] \times [y_e - l/4, y_e + l/4]$; R_{e2} is the rectangle $[x_e - l/4, x_e + l/4] \times [y_e - l/4, y_e + l/4]$ and R_{e3} is the rectangle $[x_e + l/4, x_e + 3l/4] \times [y_e - l/4, y_e + l/4]$. Let $R_e = \cup_i R_{ei}$. The corresponding rectangles for vertical edges are defined similarly. The reader can refer to Figure 3.2.

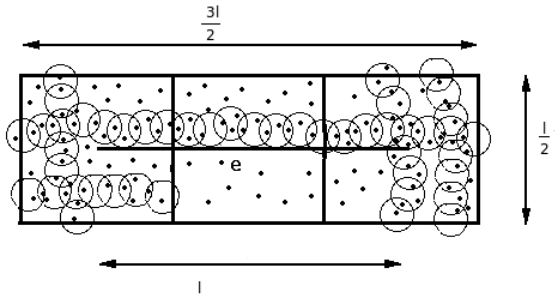

Figure 3.2: An horizontal edge e that satisfies the condition for $B_e = 1$. The balls are of radius $2r$, centered at points of $\Phi^{(1)}$ and the adjacent centers are of at most distance r_1. The dots are the points of $\Phi^{(2)}$.

Due to continuity of $\lambda_c(2r)$ (see [Meester and Roy 1996, Theorem 3.7]), there exists $r_1 < r$ such that $\lambda > \lambda_c(2r_1)$. We shall now define some rvs associated with horizontal edges and the corresponding definitions for vertical edges are similar. Let A_e be the indicator rv for the event that there exists a left-right crossing of R_e by a component of $\tilde{G}(\lambda, 2r_1)$ and top-down crossings of R_{e1} and R_{e3} by a component of $\tilde{G}(\lambda, 2r_1)$. Suppose that $A_e = 1$. Draw balls of radius $2r_1$ around each vertex of any left-right crossing of R_e and every top-down and left-right crossing of R_{e1} and R_{e3}. Let C_e be the indicator rv of the event that, for each pair of balls drawn above that have non-empty intersection, when expanded to balls of radius $2r$ contain atleast one point of $\Phi^{(2)}$. Let B_e be the indicator rv for the event that $\{A_e = 1\} \cap \{C_e = 1\}$.

Declare an edge $e \in l\mathbb{E}^2$ to be open if $B_e = 1$. We first show that for $\lambda > \lambda_c(2r)$ there

exists a μ, l such that $l\mathbb{L}^2$ percolates (Step 1). The next step is to show that this implies percolation in the continuum model $G(\lambda, \mu, r)$. (Step 2).

Step 1: The rvs $\{B_e\}_{e \in l\mathbb{E}^2}$ are 1-dependent, that is, B_e's indexed by two non-adjacent edges are independent. Hence, given edges $e_1, \ldots, e_n \in l\mathbb{E}^2$, there exists $\{k_j\}_{j=1}^m \subset \{1, \ldots, n\}$ with $m \geq n/4$ such that $\{B_{e_{k_j}}\}_{1 \leq j \leq m}$ are i.i.d. Bernoulli rvs. Hence,

$$\mathsf{P}\left(B_{e_i} = 0,\ 1 \leq i \leq n\right) \leq \ \mathsf{P}\left(B_{e_{k_j}} = 0, 1 \leq j \leq m\right) \leq \mathsf{P}\left(B_e = 0\right)^{n/4}. \tag{3.13}$$

We need to show that for a given $\epsilon > 0$ there exists l, μ for which $\mathsf{P}\left(B_e = 0\right) < \epsilon$ for any $e \in l\mathbb{E}^2$. Fix an edge e. Observe that

$$\begin{aligned} \mathsf{P}\left(B_e = 0\right) &= \mathsf{P}\left(A_e = 0\right) + \mathsf{P}\left(B_e = 0 | A_e = 1\right) \mathsf{P}\left(A_e = 1\right) \\ &\leq \mathsf{P}\left(A_e = 0\right) + \mathsf{P}\left(B_e = 0 | A_e = 1\right). \end{aligned} \tag{3.14}$$

Since $\lambda > \lambda_c(2r_1)$, $\tilde{G}(\lambda, 2r_1)$ percolates. Hence by [Meester and Roy 1996, Corollary 4.1], we can and do choose a l large enough so that

$$\mathsf{P}\left(A_e = 0\right) < \frac{\epsilon}{2}. \tag{3.15}$$

Now consider the second term on the right in (3.14). Given $A_e = 1$, there exist crossings as specified in the definition of A_e in $\tilde{G}(\lambda, 2r_1)$. Draw balls of radius $2r (> 2r_1)$ around each vertex. Any two vertices that share an edge in $\tilde{G}(\lambda, 2r_1)$ are centered at a distance of at most $4r_1$. The width of the lens of intersection of two balls of radius $2r$ whose centers are at most $4r_1 (< 4r)$ apart is bounded below by a constant, say $b(r, r_1) > 0$. Hence if we cover R_e with disjoint squares of diagonal-length $b(r, r_1)/3$, then every lens of intersection will contain at least one such square. Let $S_j, j = 1, \ldots, N(b)$, be the disjoint squares of diagonal-length $b(r, r_1)/3$ that cover R_e. Note that

$$\begin{aligned} \mathsf{P}\left(B_e = 1 | A_e = 1\right) &\geq \mathsf{P}\left(\Phi^{(2)} \cap S_j \neq \emptyset, 1 \leq j \leq N(b)\right) \\ &= (1 - \exp(-\frac{\mu b(r, r_1)^2}{18}))^{N(b)} \to 1, \text{ as } \mu \to \infty. \end{aligned}$$

Thus for the choice of l satisfying (3.15), we can choose a μ large enough so that

$$\mathsf{P}\left(B_e = 0 | A_e = 1\right) < \frac{\epsilon}{2}. \tag{3.16}$$

From (3.14) - (3.16), we get $\mathsf{P}(B_e = 0) < \epsilon$. Hence given any $\epsilon > 0$, it follows from (3.13) that there exists l, μ large enough so that $\mathsf{P}(B_{e_i}, 1 \le i \le n) \le \epsilon^{n/4}$. That $l\mathbb{L}^2$ percolates now follows from a standard Peierls argument (see Lemma 2.2.3).

STEP 2: By Step 1, choose l, μ so that $l\mathbb{L}^2$ percolates. Consider any infinite component in $l\mathbb{L}^2$. Let e, f be any two adjacent edges in the infinite component. In particular $B_e = B_f = 1$. This has two implications, the first one being that there exists crossings I_e and I_f of R_e and R_f respectively in $\tilde{G}(\lambda, 2r_1)$. Since e, f are adjacent, $R_{ei} = R_{fj}$ for some $i, j \in \{1, 3\}$. Hence there exists a crossing J of R_{ei} in $\tilde{G}(\lambda, 2r_1)$ that intersects both I_e and I_f. Draw balls of radius $2r$ around each vertex of the crossings J, I_e, I_f. The second implication is that every pairwise intersection of these balls will contain atleast one point of $\Phi^{(2)}$. This implies that I_e and I_f belong to the same AB component in $G(\lambda, \mu, r)$. Therefore $G(\lambda, \mu, r)$ percolates when $l\mathbb{L}^2$ does. □.

Proof of Proposition 3.1.8. Recall Definition 3.1.6. For $d = 2$, let $\mathbb{T}^*$ be the triangular site percolation model with edge length $r_0/2$, and let Q_z be the flower centred at $z \in \mathbb{T}^*$ as shown in Figure 3.1. For $d \ge 3$, let $\mathbb{Z}^{*d}_{r_0} = \frac{r_0}{2\sqrt{d}}\mathbb{Z}^d$, and Q_z be the cube of side-length $\frac{r_0}{2\sqrt{d}}$ centred at $z \in \mathbb{Z}^{*d}_{r_0}$. Note that the flowers or cubes are disjoint. We declare z open, if $Q_z \cap \Phi^{(i)} \ne \emptyset$, $1 \le i \le k$. This is clearly an independent site percolation model on $\mathbb{T}^*$ $(d = 2)$ or $\mathbb{L}^{*d}$ $(d \ge 3)$ with probability $\prod_{i=1}^k (1 - e^{-\lambda_i a(d, r_0)})$ of z being open. By hypothesis, $\prod_{i=1}^k (1 - e^{-\lambda_i a(d, r_0)}) > p_c(d)$, the critical probability for site percolation on $\mathbb{T}^*$ $(d = 2)$ or $\mathbb{L}^{*d}$ $(d \ge 3)$ and hence the corresponding graphs percolate. Let $< z_1, z_2, ... >$ denote the infinite percolating path in $\mathbb{T}^*$ $(d = 2)$ or $\mathbb{L}^{*d}$ $(d \ge 3)$. Since it is a percolating path, *almost surely*, for all $i \ge 1$, and every $j = 1, 2, \ldots, k$, $\Phi^{(i)}(Q_{z_i}) > 0$, that is, each (flower or cube) Q_{z_i} contains a point of $\Phi^{(i)}$. Hence almost surely, for every word $\{w(i)\}_{i \ge 1}$ we can find a sequence $\{X_i\}_{i \ge 1}$ such that for all $i \ge 1$, $X_i \in \Phi^{(w(i))} \cap Q_{z_i}$. Further, $|X_i - X_{i+1}| \le r_0 \le r_{w(i)} + r_{w(i+1)}$. Thus, almost surely, every word occurs. □

Proof of Corollary 3.1.9. Apply Proposition 3.1.8 with $k = 2$, $\lambda_1 = \lambda$, $\lambda_2 = \mu$, $r_1 = r_2 = r$, and so $r_0 = 2r$. It follows that almost surely, every word occurs provided $(1 - e^{-\lambda a(d, 2r)})(1 - e^{-\mu a(d, 2r)}) > p_c(d)$. In particular, under the above condition, almost surely, the word $(1, 2, 1, 2, \ldots)$ occurs. This implies that there is a sequence $\{X_i\}_{i \ge 1}$ such that $X_{2j-1} \in \Phi^{(1)}$, $X_{2j} \in \Phi^{(2)}$, and $|X_{2j} - X_{2j-1}| \le 2r$, for all $j \ge 1$. But this is equivalent to percolation in $G(\lambda, \mu, r)$ as the pp is locally finite. This proves the corollary once we note that there exists a $\mu < \infty$ satisfying the condition above only if $(1 - e^{-\lambda a(d, 2r)}) > p_c(d)$, or equivalently $a(d, 2r)\lambda > \log(\frac{1}{1 - p_c(d)})$ and the least such μ is given in the RHS of (3.2). □

Proof of Corollary 3.1.13. By the given condition $(1 - e^{-\lambda a(d,2r)/2}) > \sqrt{p_c(d)}$, and continuity, there exists an $\epsilon > 0$ such that for all $p \in (1/2 - \epsilon, 1/2 + \epsilon)$, we have $(1-e^{-\lambda pa(d,2r)}) > \sqrt{p_c(d)}$. Thus for all $p \in (1/2-\epsilon, 1/2+\epsilon)$, we get that $(1-e^{-\lambda pa(d,2r)})(1-e^{-\lambda(1-p)a(d,2r)}) > p_c(d)$. Hence by invoking Proposition 3.1.8 as in the proof of Corollary 3.1.9 with $\lambda_1 = \lambda p, \lambda_2 = \lambda(1-p), r_1 = r_2 = r$, we get that $\widehat{\theta}(\lambda, p, r) = 1$. □

3.4 Proofs for Section 3.2

For any locally finite point process $\mathcal{X} \subset U$, the coverage process is defined as

$$\mathcal{C}(\mathcal{X}, r) := \cup_{X_i \in \mathcal{X}} B_{X_i}(r), \tag{3.17}$$

and we abbreviate $\mathcal{C}(\mathcal{P}_n^{(1)}, r)$ by $\mathcal{C}(n, r)$. Recall that for any $A \subset \mathbb{R}^d$, we write $\mathcal{X}(A)$ to be the number of points of $\mathcal{X}$ that lie in the set A. We will need the following vacancy estimate similar to [Hall 1988, Theorem 3.11] for the proof of Lemma 3.2.3. $\|\cdot\|$ denotes the Lebesgue measure on $\mathbb{R}^d$.

Lemma 3.4.1. *For $d = 2$ and $0 < r < \frac{1}{2}$, define $V(n,r) := 1 - \frac{\|B_O(r) \cap \mathcal{C}(n,r)\|}{\pi r^2}$, the normalised vacancy in the r-ball. Then*

$$\mathsf{P}\left(V(n,r) > 0\right) \leq (1 + n\pi r^2 + 4(n\pi r^2)^2)\ \exp(-n\pi r^2).$$

Proof of Lemma 3.4.1. Write $\mathsf{P}\left(V(n,r) > 0\right) \leq p_1 + p_2 + p_3$, where

$$\begin{aligned}
p_1 &= \mathsf{P}\left(\mathcal{P}_n^{(1)}(B_O(r)) = 0, V(n,r) > 0\right) = \mathsf{P}\left(\mathcal{P}_n^{(1)}(B_O(r)) = 0\right) = \exp(-n\pi r^2), \\
p_2 &= \mathsf{P}\left(\mathcal{P}_n^{(1)}(B_O(r)) = 1, V(n,r) > 0\right) \leq \mathsf{P}\left(\mathcal{P}_n^{(1)}(B_O(r)) = 1\right) = n\pi r^2 \exp(-n\pi r^2), \\
p_3 &= \mathsf{P}\left(\mathcal{P}_n^{(1)}(B_O(r)) > 1, V(n,r) > 0\right).
\end{aligned}$$

We shall now upper bound p_3 to complete the proof. A *crossing* is defined as a point of intersection of two r-balls centred at points of $\mathcal{P}_n^{(1)}$. A crossing is said to be *covered* if it lies in the interior of another r-ball centred at a point of $\mathcal{P}_n^{(1)}$, else it is said to be *uncovered*. If there is more than one point of $\mathcal{P}_n^{(1)}$ in $B_O(r)$, then there exists atleast one crossing in $B_O(r)$. If $V(n,r) > 0$ and there exists more than one r-ball centred at a point of $\mathcal{P}_n^{(1)}$ in $B_O(r)$, then there exists atleast one such r-ball with two uncovered crossings on

its boundary. Denoting the number of uncovered crossings by M, we have that

$$p_3 \leq \mathsf{P}\left(M \geq 2\right) \leq \frac{\mathsf{E}(M)}{2}.$$

Given a disk, the number of crossings is twice the number of r-balls centred at a distance within $2r$. This number has expectation $2\int_0^{2r} 2n\pi x \mathrm{d}x = 8n\pi r^2$, where $2n\pi x \mathrm{d}x$ is the expected number of r-balls whose centers lie between x and $x + \mathrm{d}x$ of the center of the given r-ball. Thus,

$$\mathsf{E}(M) = \mathsf{E}\Big(\mathcal{P}_n^{(1)}(B_O(r))\Big)\, 8n\pi r^2 \mathsf{P}\left(\text{a crossing is uncovered}\right) = 8(n\pi r^2)^2 \exp(-n\pi r^2). \quad \square$$

Lemma 3.4.2. *For any $r > 0$ and $x \in \mathbb{R}^d$ with $0 \leq R = \|x\| \leq 2r$, define $L(r, R) := \|B_O(r) \cap B_x(r)\|$. Then*

$$\begin{aligned} L(r,R) &= \left(2\phi\left(\frac{R}{2r}\right) - \sin\left(2\phi\left(\frac{R}{2r}\right)\right)\right) r^2, \qquad \text{if } d = 2, \\ L(r,R) &\geq \theta_d \left(r - \frac{R}{2}\right)^d, \qquad \text{if } d \geq 3, \end{aligned} \tag{3.18}$$

where $\phi(a) = \arccos(a)$.

Proof of Lemma 3.4.2.

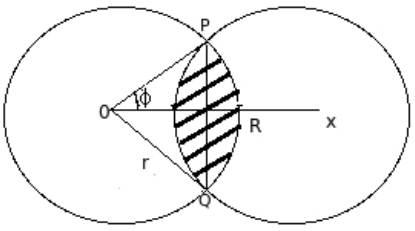

Figure 3.3: $|x| = R, \phi = \phi(r, R)$ and $L(r, R)$ is the area of the lens of intersection, the shaded region.

Let $d = 2$. From Figure 3.3, it is clear that $L(r, R)$ is cut into two equal halves by the line PQ and the area of each of those halves is the area enlosed between the chord PQ in the circle $B_O(r)$ and its circumference. The area of the segment OPQ (with PQ considered as the arc along the circumference of the circle) is $\phi\left(\frac{R}{2r}\right) r^2$. The area of the

triangle OPQ is

$$r \sin\left(\phi\left(\frac{R}{2r}\right)\right) \times r\cos\left(\phi\left(\frac{R}{2r}\right)\right) = \frac{r^2}{2}\sin\left(2\phi\left(\frac{R}{2r}\right)\right).$$

Hence $L(r,R) = \left(2\phi\left(\frac{R}{2r}\right) - \sin\left(2\phi\left(\frac{R}{2r}\right)\right)\right) r^2$. Consider the case $d \geq 3$. The width of the lens of intersection of the balls $B_O(r)$ and $B_x(r)$ is $2r - R$. Thus the lens of intersection contains a ball of diameter $2r - R$. Hence the volume of such a ball, $\theta_d(r - \frac{R}{2})^d$, is a lower bound for $L(r,R)$. □

Proof of Lemma 3.2.3. We first prove the second part of the Lemma which is easier.

(2). Let $\widehat{W}_n(r)$ be the number of $\mathcal{P}_n^{(1)}$ nodes for which there are no other $\mathcal{P}_n^{(1)}$ node within a distance r. Note that $\widehat{W}_n(2r) \leq W_n(r)$. By this inequality and the Palm calculus (Theorem 2.1.3), we get

$$\begin{aligned}
\mathsf{E}(W_n(r_n(c,\beta))) &\geq \mathsf{E}\Big(\widehat{W}_n(2r_n(c,\beta))\Big) \\
&= n\int_U \mathsf{P}\left(\mathcal{P}_n^{(1)}(B_x(2r_n(c,\beta))) = 0\right) \mathrm{d}x \\
&= n\exp(-2^d n\theta_d r_n^d(c,\beta)) \;=\; n\exp\left(-\frac{2^d}{c}\log(\frac{n}{\beta})\right) \;\to\; \infty,
\end{aligned}$$

as $n \to \infty$ since $c > 2^d$.

(1). We prove the cases $d = 2$ and $d \geq 3$ separately. Let $d \geq 3$ and fix $c < 1$. Define $\widetilde{W}_n(c,r)$ to be the number of $\mathcal{P}_n^{(1)}$ nodes for which there are no $\mathcal{P}_{cn}^{(2)}$ nodes within a distance r and $\overline{W}_n(c,r)$ be the number of $\mathcal{P}_{cn}^{(2)}$ nodes with only one $\mathcal{P}_n^{(1)}$ node within a distance r. Note that

$$\widetilde{W}_n(c,r) \leq W_n(r) \leq \widetilde{W}_n(c,r) + \overline{W}_n(c,r). \tag{3.19}$$

By Palm calculus for Poisson point processes, we have

$$\begin{aligned}
\mathsf{E}\Big(\widetilde{W}_n(c, r_n(c,\beta))\Big) &= n\int_U \mathsf{P}\left(\mathcal{P}_{cn}^{(2)}(B_x(r_n(c,\beta))) = 0\right) \mathrm{d}x \\
&= n\exp(-cn\theta_d r_n^d(c,\beta)) \;=\; \beta, \qquad (3.20) \\
\mathsf{E}\big(\overline{W}_n(c, r_n(c,\beta))\big) &= cn\int_U \mathsf{P}\left(\mathcal{P}_n^{(1)}(B_x(r_n(c,\beta))) = 1\right) \mathrm{d}x \\
&= c\,n\exp(-n\theta_d r_n^d(c,\beta))\, n\,\theta_d\, r_n^d(c,\beta) \to 0, \qquad (3.21)
\end{aligned}$$

since $c < 1$. It follows from (3.19), (3.20) and (3.21) that $\mathsf{E}(W_n(r_n(c,\beta))) \to \beta$, as $n \to \infty$,

if $d \geq 3$ and $c < 1$.

Now let $d = 2$, fix $c < c_0$, where c_0 is as defined in (3.7) and choose n large enough such that $r_n(c, \beta) < \frac{1}{2}$. For any $X \in \mathcal{P}_n^{(1)}$, using (3.17), the degree of X in the graph $G_n(cn, r)$ can be written as

$$deg_n(cn, X) := \sum_{X_j \in \mathcal{P}_n^{(1)}} \mathbf{1}\{< X_j, X > \in E_n(cn, r)\} = \mathcal{P}_n^{(1)}(\mathcal{C}((\mathcal{P}_{cn}^{(2)} \cap B_X(r)), r) \setminus \{X\}),$$

Since

$$\{\mathcal{P}_n^{(1)}(\mathcal{C}((\mathcal{P}_{cn}^{(2)} \cap B_X(r)), r) \setminus \{X\}) = 0\} = \{\mathcal{P}_{cn}^{(2)}(B_X(r) \cap \mathcal{C}(\mathcal{P}_n^{(1)} \setminus \{X\}, r)) = 0\}, \quad (3.22)$$

we have

$$W_n(r) = \sum_{X_i \in \mathcal{P}_n^{(1)}} \mathbf{1}\{deg_n(cn, X_i) = 0\} = \sum_{X_i \in \mathcal{P}_n^{(1)}} \mathbf{1}\{\mathcal{P}_{cn}^{(2)}(B_{X_i}(r) \cap \mathcal{C}(\mathcal{P}_n^{(1)} \setminus \{X\}, r)) = 0\}. \quad (3.23)$$

By Palm calculus for Poisson point processes and the metric being toroidal, we have,

$$\mathsf{E}(W_n(r)) = n \int_U \mathsf{E}(\mathbf{1}\{deg_n(cn, x) = 0\})\, \mathrm{d}x = n\mathsf{P}\left(\mathcal{P}_{cn}^{(2)}(B_O(r) \cap \mathcal{C}(n, r)) = 0\right), \quad (3.24)$$

where $\mathcal{C}(n, r) = \mathcal{C}(\mathcal{P}_n^{(1)}, r)$. For any bounded random closed set F, conditioning on F and then taking expectation, we have

$$\mathsf{P}\left(\mathcal{P}_{cn}^{(2)}(F) = 0\right) = \mathsf{E}(\exp(-cn\|F\|)). \quad (3.25)$$

Thus from (3.24), (3.25) we get

$$\mathsf{E}(W_n(r)) = n\ \mathsf{E}(\exp(-cn\|B_O(r) \cap \mathcal{C}(n, r)\|)) = n\ \mathsf{E}\big(\exp(-cn\pi r^2(1 - V(n, r)))\big), \quad (3.26)$$

where $V(n, r)$ is as defined in Lemma 3.4.1. Let $A(c)$ be as defined in (3.6) and $e_1 = (1, 0)$. Since $\frac{r_n(1,\beta)}{2r_n(c,\beta)} = \frac{c^{\frac{1}{2}}}{2}$, by Lemma 3.4.2, we have

$$\begin{aligned} \frac{\|B_O(r_n(c, \beta)) \cap B_{r_n(1,\beta)e_1}(r_n(c, \beta))\|}{\pi r_n(c, \beta)^2} &= \pi^{-1}\left(2\phi\left(\frac{r_n(1, \beta)}{2r_n(c, \beta)}\right) - \sin\left(2\phi\left(\frac{r_n(1, \beta)}{2r_n(c, \beta)}\right)\right)\right) \\ &= A(c). \end{aligned}$$

Given $c < c_0$, by continuity, we can choose an $\epsilon \in (0,1)$, such that

$$A(c,\epsilon) = \frac{\|B_O(r_n(c,\beta)) \cap B_{r_n(1-\epsilon,\beta)e_1}(r_n(c,\beta))\|}{\pi r_n(c,\beta)^2} \quad \text{satisfies} \quad A(c,\epsilon) + \frac{1}{c} > 1. \tag{3.27}$$

From Lemma 3.4.1, we obtain the bound,

$$\mathsf{P}\left(V(n, r_n(c,\beta)) > 0\right) \le D(1 + \log n + 4(\log n)^2) n^{-\frac{1}{c}}, \tag{3.28}$$

for some constant D. Let $N_n = \mathcal{P}_n^{(1)}(B_O(r_n(1-\epsilon,\beta)))$. On the event $\{N_n > 0\}$, we have

$$1 - V(n, r_n(c,\beta)) \ge A(c,\epsilon). \tag{3.29}$$

From (3.26), we get

$$\begin{aligned}
\mathsf{E}(W_n(r_n(c,\beta))) &= n\,\mathsf{E}\Big(e^{-cn\pi r_n^2(c,\beta)(1-V(n,r_n(c,\beta)))}\mathbf{1}\{V(n,r_n(c,\beta)) = 0\}\Big) \\
&+ n\,\mathsf{E}\Big(e^{-cn\pi r_n^2(c,\beta)(1-V(n,r_n(c,\beta)))}\mathbf{1}\{V(n,r_n(c,\beta)) > 0, N_n = 0\}\Big) \\
&+ n\,\mathsf{E}\Big(e^{-cn\pi r_n^2(c,\beta)(1-V(n,r_n(c,\beta)))}\mathbf{1}\{V(n,r_n(c,\beta)) > 0, N_n > 0\}\Big)
\end{aligned} \tag{3.30}$$

Consider the first term in (3.30).

$$\begin{aligned}
n\,\mathsf{E}\Big(e^{-cn\pi r_n^2(c,\beta)(1-V(n,r_n(c,\beta)))}\mathbf{1}\{V(n,r_n(c,\beta)) = 0\}\Big) &= n\exp(-cn\pi r_n(c,\beta)^2)\mathsf{P}\left(V(n,r_n(c,\beta)) = 0\right) \\
&= \beta\,\mathsf{P}\left(V(n,r_n(c,\beta)) = 0\right) \to \beta,
\end{aligned} \tag{3.31}$$

as $n \to \infty$, since $\mathsf{P}\left(V(n,r_n(c,\beta)) = 0\right) \to 1$ by (3.28). The second term in (3.30) is bounded by

$$n\,\mathsf{P}\left(N_n = 0\right) = n\exp(-n\pi r_n(1-\epsilon,\beta)^2) = n^{1-\frac{1}{1-\epsilon}}\beta^{\frac{1}{1-\epsilon}} \to 0, \tag{3.32}$$

as $n \to \infty$. Using (3.29) first and then (3.28), the third term in (3.30) can be bounded by

$$\begin{aligned}
ne^{-cn\pi r_n(c,\beta)^2 A(c,\epsilon)}\mathsf{P}\left(V(n,r_n(c,\beta)) > 0, N_n > 0\right) &\le n^{1-A(c,\epsilon)}\beta^{A(c,\epsilon)}\mathsf{P}\left(V(n,r_n(c,\beta)) > 0\right) \\
&\le D\, n^{1-A(c,\epsilon)-\frac{1}{c}}(1 + \log n + 4(\log n)^2)\beta^{A(c,\epsilon)} \\
&\to 0,
\end{aligned} \tag{3.33}$$

as $n \to \infty$ by (3.27).

It follows from (3.30) - (3.33) that

$$\mathsf{E}(W_n(r_n(c,\beta))) \to \beta, \qquad \text{as } n \to \infty. \quad \square$$

We shall need the notion of dependency graphs and Poisson approximation lemma for the proof of Theorem 3.2.4. Let (V, E) be a graph with countable vertex set V. We shall denote the edges by $< i, j >$ when $(i, j) \in E$. The adjacency neigbourhood of a vertex $i \in V$ is defined as $\mathcal{N}_i := \{i\} \cup \{j :< i, j >\in E\}$. The graph (V, E) is called a *dependency graph* for a collection of rvs $\{\Psi_i\}_{i\geq 1}$ if $\{\Psi_i\}_{i\in V_1}$ and $\{\Psi_i\}_{i\in V_2}$ are independent for two disjoint subsets $V_1, V_2 \subset V$ such that there are no edges between V_1 and V_2. Let $d_{TV}(.,.)$ be the total variation distance between two integer valued rvs ψ, ζ defined as

$$d_{TV}(\psi, \zeta) = \sup_{A\subset\mathbb{Z}} |\mathsf{P}(\psi \in A) - \mathsf{P}(\zeta \in A)|. \tag{3.34}$$

The following theorem originally proved by [Arratia et al. 1989] using the Stein-Chen method can be found in [Penrose 2003, Theorem 2.1].

Theorem 3.4.3. *Let $\{\Psi_i\}_{i\geq 1}$ be a collection of Bernoulli rvs with dependency graph (V, E). Define $p_i := \mathsf{E}(\Psi_i)$, $p_{ij} := \mathsf{E}(\Psi_i\Psi_j)$, $\lambda := \sum_{i\in V} p_i$ and $W := \sum_{i\in V} \Psi_i$. Then*

$$d_{TV}(W, Po(\lambda)) \leq min(3, \frac{1}{\lambda})\{\sum_{i\in V, j\in\mathcal{N}_i/\{i\}} p_{ij} + \sum_{i\in V, j\in\mathcal{N}_i} p_i p_j\}.$$

The following estimate in the spirit of Theorem 6.7([Penrose 2003]) will be our main tool in proving Poisson convergence of $W_n(r_n(c, \beta))$. We denote the Palm version $\mathcal{P}_n^{(1)} \cup \{x\}$ of $\mathcal{P}_n^{(1)}$ by $\mathcal{P}_n^{(1,x)}$.

Lemma 3.4.4. *Let $0 < r < 1$ and let $\mathcal{C}(.,.)$ be the coverage process defined by (3.17). Define the integrals $I_{in}(r)$, $i = 1, 2$, and $n \geq 1$ by*

$$\begin{aligned} I_{1n}(r) &:= n^2 \int_U dx \int_{B_x(5r)\cap U} dy\ \mathsf{P}\left(\mathcal{P}_n^{(1)}(\mathcal{C}(\mathcal{P}_{cn}^{(2)} \cap B_x(r), r)) = 0\right) \mathsf{P}\left(\mathcal{P}_n^{(1)}(\mathcal{C}(\mathcal{P}_{cn}^{(2)} \cap B_y(r), r)) = 0\right), \\ I_{2n}(r) &:= n^2 \int_U dx \int_{B_x(5r)\cap U} dy\ \mathsf{P}\left(\mathcal{P}_n^{(1,x)}(\mathcal{C}(\mathcal{P}_{cn}^{(2)} \cap B_y(r), r)) = 0 = \mathcal{P}_n^{(1,y)}(\mathcal{C}(\mathcal{P}_{cn}^{(2)} \cap B_x(r), r))\right). \end{aligned}$$

Then,

$$d_{TV}\left(W_n(r), Po(\mathsf{E}(W_n(r)))\right) \leq \min\left(3, \frac{1}{\mathsf{E}(W_n(r))}\right)(I_{1n}(r) + I_{2n}(r)). \tag{3.35}$$

Proof of Lemma 3.4.4. The proof follows along the same lines as the proof of Theorem 6.7 ([Penrose 2003]). For every $m \in \mathbb{N}$, partition U into disjoint cubes of side-length m^{-1} and corners at $m^{-1}\mathbb{Z}^d$. Let the cubes and their centres be denoted by $H_{m,1}, H_{m,2}, ...$ and $a_{m,1}, a_{m,2}...$ respectively. Let

$$\xi_{m,i} := 1_{\{\mathcal{P}_n^{(1)}(H_{m,i})=1\}\cap\{\mathcal{P}_n^{(1)}(\mathcal{C}(\mathcal{P}_{cn}^{(2)}\cap B_{a_{m,i}}(r),r)\cap H_{m,i}^c)=0\}}.$$

$\xi_{m,i} = 1$ provided there is exactly one point of $\mathcal{P}_n^{(1)}$ in the cube $H_{m,i}$ which is not connected to any other point of $\mathcal{P}_n^{(1)}$ that falls outside $H_{m,i}$ in the graph $G_n(cn, r)$.

Define $V_m := \{i \in \mathbb{N} : H_{m,i} \subset [0,1]^d\}$ and $E_m := \{< i, j >: i, j \in V_m,\ 0 < \|a_{m,i} - a_{m,j}\| < 5r\}$. The graph $G_m = (V_m, E_m)$ forms a dependency graph (see Theorem 3.4.3) for the rvs $\{\xi_{m,i}\}_{i\in V_m}$. The dependency neighbourhood of a vertex i is $N_{m,i} = i \cup \{j :< i, j >\in E_m\}$. Let $W^m = \sum_{i\in V_m}\xi_{m,i}$, $p_{m,i} = \mathsf{E}(\xi_{m,i})$ and $p_{m,i,j} = \mathsf{E}(\xi_{m,i}\xi_{m,j})$. By Theorem 3.4.3, we have

$$d_{TV}(W^m, Po(\mathsf{E}(W^m))) \leq min(3, \frac{1}{\mathsf{E}(W^m)})(b_1(m) + b_2(m)), \tag{3.36}$$

where $b_1(m) = \sum_{i\in V_m}\sum_{j\in N_{m,i}} p_{m,i}p_{m,j}$ and $b_2(m) = \sum_{i\in V_m}\sum_{j\in N_{m,i}/\{i\}} p_{m,i,j}$. The result follows if we show that the expressions on the left and right in (3.36) converge to the left and right hand expressions respectively in (3.35).

Note that,

$$W_n(r) = \lim_{m\to\infty} W^m, a.s.. \tag{3.37}$$

Let $w_m(x) = m^d p_{m,i}$ for $x \in H_{m,i}$. Then $\sum_{i\in V_m} p_{m,i} = \int_U w_m(x)\ \mathrm{d}x$. Clearly,

$$\lim_{m\to\infty} w_m(x) = n\mathsf{P}\left(\mathcal{P}_n^{(1,x)}(\mathcal{C}((\mathcal{P}_{cn}^{(2)}\cap B_x(r))/\{x\}, r)) = 0\right) = n\mathsf{P}\left(\mathcal{P}_n^{(1)}(\mathcal{C}(\mathcal{P}_{cn}^{(2)}\cap B_x(r), r)) = 0\right).$$

Since $w_m(x) \leq m^d\mathsf{P}\left(\mathcal{P}_n^{(1)}(H_{m,i}) = 1\right) \leq n$,

$$\lim_{m\to\infty}\mathsf{E}(W^m) = n\int_U \mathsf{P}\left(\mathcal{P}_n^{(1)}(\mathcal{C}(\mathcal{P}_{cn}^{(2)}\cap B_x(r), r)) = 0\right)\ \mathrm{d}x = \mathsf{E}(W_n(r)),$$

where the first equality is due to the dominated convergence theorem and the second follows from (3.22) - (3.24). Similarly by letting $u_m(x,y) = m^{2d} p_{m,i} p_{m,j} 1_{[j \in N_{m,i}]}$ and $v_m(x,y) = m^{2d} p_{m,i,j} 1_{[j \in N_{m,i}/\{i\}]}$ for $x \in H_{m,i},\ y \in H_{m,j}$, one can show that

$$\begin{aligned} b_1(m) &= \int_{U \times U} u_m(x,y)\, \mathrm{d}x\, \mathrm{d}y \ \to\ I_{1n}(r), \\ b_2(m) &= \int_{U \times U} v_m(x,y)\, \mathrm{d}x\, \mathrm{d}y \ \to\ I_{2n}(r). \quad \square \end{aligned}$$

Proof of Theorem 3.2.4. (3.9) follows easily from (3.8) by noting that

$$\mathsf{P}\left(M_n \le r\right) = \mathsf{P}\left(W_n(r) = 0\right).$$

Hence, the proof is complete if we show (3.8) for which we will use Lemma 3.4.4. Let $I_{in}(r_n(c,\beta))$, $i = 1, 2$, be the integrals defined in (3.35) with r taken to be $r_n(c,\beta)$ satisfying (3.4). From Lemma 3.2.3, $\mathsf{E}(W_n(r_n(c,\beta))) \to \beta$ as $n \to \infty$. Using (3.24) and Lemma 3.2.3, we get for some finite positive constant C that

$$I_{1n}(r_n(c,\beta)) = \int_U \mathrm{d}x \int_{B_x(5r_n(c,\beta)) \cap U} \mathrm{d}y\, (\mathsf{E}(W_n(r_n(c,\beta))))^2 \le C(5r_n(c,\beta))^d \to 0, \qquad \text{as } n \to \infty.$$

We now compute the integrand in the inner integral in $I_{2n}(r)$. Let $\Gamma(x,r) = \|B_O(r) \cap B_x(r)\|$. For $x, y \in U$, using (3.25) we get

$$\begin{aligned} &\mathsf{P}\left(\{\mathcal{P}_n^{(1,x)}(\mathcal{C}(\mathcal{P}_{cn}^{(2)} \cap B_y(r), r)) = 0\} \cap \{\mathcal{P}_n^{(1,y)}(\mathcal{C}(\mathcal{P}_{cn}^{(2)} \cap B_x(r), r)) = 0\}\right) \\ &= \mathsf{P}\left(\mathcal{P}_{cn}^{(2)}(B_y(r) \cap (\mathcal{C}(n,r) \cup B_x(r))) = 0, \mathcal{P}_{cn}^{(2)}(B_x(r) \cap (\mathcal{C}(n,r) \cup B_y(r))) = 0\right) \\ &\le \mathsf{P}\left(\mathcal{P}_{cn}^{(2)}(B_y(r) \cap \mathcal{C}(n,r)) = 0, \mathcal{P}_{cn}^{(2)}(B_x(r) \cap \mathcal{C}(n,r)) = 0\right) \\ &= \mathsf{P}\left(\mathcal{P}_{cn}^{(2)}((B_y(r) \setminus B_x(r)) \cap \mathcal{C}(n,r)) = 0, \mathcal{P}_{cn}^{(2)}(B_x(r) \cap \mathcal{C}(n,r)) = 0\right) \\ &= \mathsf{E}(\exp(-cn\|(B_y(r) \setminus B_x(r)) \cap \mathcal{C}(n,r)\|) \exp(-cn\|B_x(r) \cap \mathcal{C}(n,r)\|)). \end{aligned} \tag{3.38}$$

We can and do choose an $\eta > 0$ so that for any $r > 0$ and $\|y - x\| \le 5r$ (see [Penrose 2003, Eqn 8.21]), we have

$$\|B_x(r) \setminus B_y(r)\| \ \ge\ \eta\, r^{d-1}\, \|y - x\|.$$

Hence if $\|y-x\| \leq 5r$, the left hand expression in (3.38) will be bounded above by

$$\mathsf{E}\Big(\exp\left(-cn\eta r^{d-1}\|y-x\|\frac{\|(B_y(r)\setminus B_x(r))\cap \mathcal{C}(n,r)\|}{\|B_y(r)\setminus B_x(r)\|}\right)\exp\left(-cn\|B_x(r)\cap\mathcal{C}(n,r)\|\right)\Big).$$

Using the above bound, we get

$$\begin{aligned} I_{2n}(r_n(c,\beta)) &\leq \int_U \int_{B_O(5r_n^d(c,\beta))\cap U} n^2\mathsf{E}\Big(\exp\left(-cn\|B_O(r_n(c,\beta))\cap\mathcal{C}(n,r_n(c,\beta))\|\right) \\ &\quad \exp\left(-cn\eta r_n(c,\beta)^{d-1}\|y\|\frac{\|(B_y(r_n(c,\beta))\setminus B_O(r_n(c,\beta)))\cap\mathcal{C}(n,r_n(c,\beta))\|}{\|B_y(r_n(c,\beta))\setminus B_O(r_n(c,\beta))\|}\right)\Big)\,\mathrm{d}y\,\mathrm{d}x. \end{aligned}$$

Making the change of variable $w = nr_n(c,\beta)^{d-1}y$ and using (3.26), we get

$$\begin{aligned} I_{2n}(r_n(c,\beta)) &\leq \int_{B_O(5nr_n(c,\beta)^d)\cap U} (nr_n(c,\beta)^d)^{1-d}\mathsf{E}\Bigg(n\exp(-cn\|B_O(r_n(c,\beta))\cap\mathcal{C}(n,r_n(c,\beta))\|) \\ &\quad \exp\left(-c\eta\|w\|\frac{\|(B_{w(nr_n(c,\beta)^{d-1})^{-1}}(r_n(c,\beta))\setminus B_O(r_n(c,\beta)))\cap\mathcal{C}(n,r_n(c,\beta))\|}{\|B_{w(nr_n(c,\beta)^{d-1})^{-1}}(r_n(c,\beta))\setminus B_O(r_n(c,\beta))\|}\right)\Bigg)\,dw \\ &\leq (nr_n(c,\beta)^d)^{1-d}\mathsf{E}(W_n(r_n(c,\beta))) \to 0, \end{aligned}$$

as $n\to\infty$, since by Lemma 3.2.3, $\mathsf{E}(W_n(r_n(c,\beta)))\to\beta$ as $n\to\infty$ and $nr_n^d(c,\beta) = \log\frac{\frac{n}{\beta}}{c\theta_d}$. We have shown that for $i=1,2$, $I_{in}(r_n(c,\beta))\to 0$, and hence by Lemma 3.2.3,

$$d_{TV}(W_n(r_n(c,\beta)), Po(\mathsf{E}(W_n(r_n(c,\beta))))) \to 0,$$

as $n\to\infty$. Again, since $\mathsf{E}(W_n(r_n(c,\beta)))\to\beta$, we have $Po(\mathsf{E}(W_n(r_n(c,\beta))))\xrightarrow{d} Po(\beta)$. Consequently, $d_{TV}(W_n(r_n(c,\beta)), Po(\beta)\to 0$ as $n\to\infty$. As convergence in *total variation distance* implies convergence in distribution, we get (3.8). □

We now prove Theorem 3.2.6. In the second part of this proof, we will couple our sequence of AB RGGs with a sequence of usual RGGs. By usual RGG we mean the sequence of graphs $\underline{G}_n(r)$ with vertex set $\mathcal{P}_n^{(1)}$ and edge set $\{\langle X_i, X_j\rangle : X_i, X_j\in\mathcal{P}_n^{(1)}, d(X_i,X_j)\leq r\}$, where d is the toroidal metric defined in (3.3). We will use the following well known result regarding strong connectivity in the graphs $\underline{G}_n(r)$.

Theorem 3.4.5 (Theorem 13.2, [Penrose 2003]). *For $R_n(A_0) = \left(\frac{A_0\log n}{n\theta_d}\right)^{1/d}$, almost surely, the sequence of graphs $\underline{G}_n(R_n(A_0))$ is connected eventually if and only if $A_0>1$.*

We will also need the following estimate ([Penrose 2003, Lemma 1.4]).

Lemma 3.4.6. *Let $\gamma > \frac{1}{2}$. Then there exists a constant $\lambda_1 = \lambda_1(\gamma)$ such that for all $\lambda > \lambda_1$,*

$$\begin{aligned} \mathsf{P}\left(Po(\lambda) > \lambda + \lambda^{\frac{\gamma}{2}}\right) &\leq \exp\{-\frac{\lambda^{2\gamma-1}}{9}\}, \\ \mathsf{P}\left(Po(\lambda) > \lambda - \lambda^{\frac{\gamma}{2}}\right) &\leq \exp\{-\frac{\lambda^{2\gamma-1}}{9}\}. \end{aligned}$$

Proof of Thm 3.2.6. Let $r_n = a^{\frac{1}{d}} r_n(c)$, where $r_n(c) = r_n(c,1)$ is as defined in (3.5). It is enough to show the following :

$$\text{For all } c < c_0 \text{ and } a < 1, \qquad \lim_{n\to\infty} \mathsf{P}\left(G_n(cn, r_n) \text{ is not connected }\right) = 1. \tag{3.39}$$

$$\text{For all } c > 0 \text{ and } a > \alpha(c), \qquad \mathsf{P}\left(G_n(cn, r_n) \text{ is not connected i.o.}\right) = 0, \tag{3.40}$$

where i.o. stands for infinitely often. To show (3.39), note that for $a < 1$

$$r_n^d = \frac{\log(\frac{n}{n^{1-a}})}{cn\theta_d} < \frac{\log(\frac{n}{\beta})}{cn\theta_d},$$

for any $\beta > 0$ and sufficiently large n. From Theorem 3.2.4, if $c < c_0$, then the largest nearest neighbour radius is asymptotically greater than r_n with probability tending to one. This gives (3.39) and thus we have proved the lower limit.

Let $R_n(A_0)$ be as in Theorem 3.4.5. We will show (using a subsequence argument) that if $a > \alpha(c)$, then we can find $A_0 > 1$, such that the probability of the event that not every point of $\mathcal{P}_n^{(1)}$ is connected to all points of $\mathcal{P}_n^{(1)}$ that fall within a distance $R_n(A_0)$ in $G_n(cn, r_n)$, is summable. (3.40) then follows from Theorem 3.4.5 and the Borel-Cantelli Lemma.

Since $a > \alpha(c)$, by definition $a\eta(a,c) > 1$. By continuity, we can choose $A_0 > 1$ such that $a\eta(a, A_0 c) > 1$. Choose $\epsilon \in (0,1)$ so that

$$(1-\epsilon)^2 a\eta(a, A_0 c) > 1. \tag{3.41}$$

For each $X_i \in \mathcal{P}_n^{(1)}$, define the event

$$A_i(n, m, r, R) := \{X_i \text{ connects to all points of } \mathcal{P}_n^{(1)} \cap B_{X_i}(R) \text{ in } G_n(m, r)\},$$

and let

$$B(n,m,r,R) = \cup_{X_i \in \mathcal{P}_n^{(1)}} A_i(n,m,r,R)^c.$$

We shall explain a coupling of $\mathcal{P}_n^{(1)}$'s and a similar coupling for $\mathcal{P}_n^{(2)}$'s also shall be used. Suppose that $\widehat{\mathcal{P}}_i^{(1)}, i \geq 1$ be a sequence of i.i.d. homogeneous Poisson pp of unit intensity in U. Set $\mathcal{P}_0^{(1)} = \emptyset$ and for $n \geq 1$, let $\mathcal{P}_n^{(1)} = \mathcal{P}_{n-1}^{(1)} \cup \widehat{\mathcal{P}}_n^{(1)}$. Clearly we have that for all $n \geq 1$, $\mathcal{P}_n^{(1)}$ is an homogeneous Poisson pp of intensity n and $\mathcal{P}_n^{(1)} \subset \mathcal{P}_{n+1}^{(1)}$. By such a coupling, we have that $B(n,m,r,R) \subset B(n_1,m_1,r_1,R_1)$, provided $n \leq n_1, m \geq m_1, r \geq r_1, R \leq R_1$. Let $n_j = j^b$ for some integer $b > 0$ that will be chosen later. Since $B(n,cn,r_n,R_n) \subset B(n_{j+1},cn_j,r_{n_{j+1}},R_{n_j})$, for $n_j \leq n \leq n_{j+1}$,

$$\cup_{n=n_j}^{n_{j+1}} B(n,cn,r_n,R_n) \subset B(n_{j+1},cn_j,r_{n_{j+1}},R_{n_j}). \tag{3.42}$$

Let $p_j = \mathsf{P}\left(A_i(n_{j+1},cn_j,r_{n_{j+1}},R_{n_j})^c\right)$. Let $N_n = \mathcal{P}_n^{(1)}([0,1]^2)$. From (3.42) and the union bound we get

$$\begin{aligned}
\mathsf{P}\left(\cup_{n=n_j}^{n_{j+1}} B(n_k,cn,r_n,R_n)\right) &\leq \mathsf{P}\left(B(n_{j+1},cn_j,r_{n_{j+1}},R_{n_j})\right) \\
&\leq \mathsf{P}\left(\cup_{i=1}^{N_{n_{j+1}}} A_i(n_{j+1},cn_j,r_{n_{j+1}},R_{n_j})^c\right) \\
&\leq \sum_{i=1}^{n_{j+1}+n_{j+1}^{\frac{3}{4}}} \mathsf{P}\left(A_i(n_{j+1},cn_j,r_{n_{j+1}},R_{n_j})^c\right) + \mathsf{P}\left(|N_{n_{j+1}} - n_{j+1}| > n_{j+1}^{\frac{3}{4}}\right) \\
&\leq 2n_{j+1}p_j + \mathsf{P}\left(|N_{n_{j+1}} - n_{j+1}| > n_{j+1}^{\frac{3}{4}}\right).
\end{aligned} \tag{3.43}$$

We now estimate p_j. Let $e_1 = (1,0,\ldots,0) \in \mathbb{R}^d$. Conditioning on the number of points of $\mathcal{P}_{n_{j+1}}$ in $B_O(R_{n_j})$ and then using the Boole's inequality, we get

$$\begin{aligned}
p_j &\leq \sum_{k=0}^{\infty} \frac{(n_{j+1}\theta_d R_{n_j}^d)^k e^{-n_{j+1}\theta_d R_{n_j}^d}}{k!} \frac{k}{\theta_d R_{n_j}^d} \int_{B_O(R_{n_j})} e^{-cn_j \|B_0(r_{n_{j+1}}) \cap B_x(r_{n_{j+1}})\|} \mathrm{d}x \\
&\leq \sum_{k=0}^{\infty} \frac{(n_{j+1}\theta_d R_{n_j}^d)^k e^{-n_{j+1}\theta_d R_{n_j}^d}}{k!} \frac{k}{\theta_d R_{n_j}^d} \int_{B_O(R_{n_j})} e^{-cn_j \|B_0(r_{n_{j+1}}) \cap B_{R_{n_j}e_1}(r_{n_{j+1}})\|} \mathrm{d}x, \\
&= n_{j+1}\theta_d R_{n_j}^d e^{-cn_j L(r_{n_{j+1}},R_{n_j})},
\end{aligned}$$

where $L(r, R)$ is as defined in Lemma 3.4.2. Since

$$\frac{R_{n_j}}{r_{n_{j+1}}} = \left(\frac{A_0 \log n_j}{\theta_d n_j} \frac{cn_{j+1}\theta_d}{a \log n_{j+1}}\right)^{\frac{1}{d}} \to \left(\frac{A_0 c}{a}\right)^{\frac{1}{d}},$$

by Lemma 3.4.2, we have

$$L(r_{n_{j+1}}, R_{n_j}) \geq (1-\epsilon)\, \eta(a, A_0 c)\, \theta_d r_{n_{j+1}}^d, \tag{3.44}$$

for all sufficiently large j, where η is as defined in (3.11). For all j sufficiently large, we have $(\frac{j}{j+1})^b \geq (1-\epsilon)$. Using (3.44) and simplifying by substituting for R_{n_j} and $r_{n_{j+1}}$, for all sufficiently large j, we have

$$\begin{aligned} p_j &\leq \frac{(j+1)^b\, A_0\, b\, \log j}{j^b} e^{-\frac{j^b}{(j+1)^b}\,(1-\epsilon)\,\eta(a,A_0c)\,a\,b\,\log(j+1)} \\ &\leq \frac{A_0\, b\, \log j}{(1-\epsilon)} e^{-(1-\epsilon)^2\,\eta(a,A_0c)\,a\,b\,\log(j+1)} \\ &= \frac{A_0\, b\, \log j}{(1-\epsilon)(j+1)^{(1-\epsilon)^2\,\eta(a,A_0c)\,a\,b}}. \end{aligned}$$

Hence

$$n_{j+1}\, p_j \leq \frac{A_0\, b\, \log j}{(1-\epsilon)(j+1)^{((1-\epsilon)^2\,\eta(a,A_0c)\,a\,-1)b}}. \tag{3.45}$$

Using (3.41), we can choose b large enough so that $((1-\epsilon)^2\,\eta(a, A_0c)\,a\, -1)b > 1$. It then follows from (3.45) that the first term on the right in (3.43) is summable in j. From Lemma 3.4.6, the second term on the right in (3.43) is also summable.

Hence by the Borel-Cantelli Lemma, almost surely, only finitely many of the events

$$\cup_{n=n_j}^{n_{j+1}} B(n, cn, r_n, R_n)$$

occur, and hence only finitely many of the events $B(n, cn, r_n, R_n)$ occur. This implies that almost surely, every vertex in $G_n(cn, r_n)$ is connected to every other vertex that is within a distance $R_n(A_0)$ from it, for all large n. Since $A_0 > 1$, it follows from Theorem 3.4.5 that almost surely, $G_n(cn, r_n)$ is connected eventually. This proves (3.40). □

Chapter 4

Directionally convex ordering of random measures and shot-noise fields.

4.1 Definitions and the main result

A random measure Λ can be viewed as the following random field $\{\Lambda(B)\}_{B\in\mathtt{B}_b(\mathbb{D})}$. With this viewpoint and the previously introduced notion of ordering for random fields (see Section 2.3), we define ordering on random measures.

Definition 4.1.1. *Suppose $\Lambda_1(\cdot)$ and $\Lambda_2(\cdot)$ are random measures on $\mathbb{D}$. We say that $\Lambda_1(\cdot) \leq_{dcx} \Lambda_2(\cdot)$ if for any $I_1, \ldots, I_n$ bBs in $\mathbb{D}$,*

$$(\Lambda_1(I_1), \ldots, \Lambda_1(I_n)) \leq_{dcx} (\Lambda_2(I_1), \ldots, \Lambda_2(I_n)). \tag{4.1}$$

The definition is similar for other orders, i.e., when $\mathfrak{F}$ is the class of *idcx/idcv/ddcx/ddcv/st* functions.

Definition 4.1.2. *Let S be any set and $\mathbb{D}$ a LCSC space. Given a random measure Λ on $\mathbb{D}$ and a measurable (in the first variable alone)* response function $h(x, y) : \mathbb{D} \times S \to \bar{\mathbb{R}}^+$ *where $\bar{\mathbb{R}}^+$ denotes the completion of positive real-line with infinity, the* (integral) shot-noise field *is defined as*

$$V_\Lambda(y) = \int_{\mathbb{D}} h(x, y)\Lambda(dx). \tag{4.2}$$

With this brief introduction, we are ready to state our key result that will be proved in Section 4.3.1.

Theorem 4.1.3. *1. If* $\Lambda_1 \leq_{idcx\,(resp.\,idcv)} \Lambda_2$, *then* $\{V_{\Lambda_1}(y)\}_{y\in S} \leq_{idcx\,(resp.\,idcv)} \{V_{\Lambda_2}(y)\}_{y\in S}$.

2. Let $\mathsf{E}(V_{\Lambda_i}(y)) < \infty$, *for all* $y \in S$, $i = 1, 2$. *If* $\Lambda_1 \leq_{dcx} \Lambda_2$, *then* $\{V_{\Lambda_1}(y)\}_{y\in S} \leq_{dcx} \{V_{\Lambda_2}(y)\}_{y\in S}$.

The first part of the above theorem for the one-dimensional marginals of bounded shot-noise fields generated by lower semi-continuous response functions is proved in [Miyoshi 2004] for the special case of spatial stationary Cox pp. It is conspicuous that we have generalized the earlier result to a great extent. This simple-looking theorem will be at the heart of all our future analysis of directionally convex ordering of point processes and its applications. In particular we will use it to prove ordering of i.i.d. marked pp (Proposition 4.2.3), moment measures (Proposition 4.2.4), Ripley's K-functions (Proposition 4.2.6), Palm measures (Proposition 4.2.8), independently marked Cox processes (Proposition 4.2.11), extremal shot-noise fields (Proposition 4.3.5). Apart form these results, Sections 5.1 and 5.2 shall provide enough examples and applications that shall need Theorem 4.1.3. We will also use the theorem to comparse percolative properties of point processes in various percolation models in Chapter 6. In short, all the future sections in the thesis would use this theorem.

4.2 Ordering of random measures and point processes

We shall now give a sufficient condition for random measures to be ordered, namely that the condition (4.1) in Definition 4.1.1 needs to be verified only for disjoint bBs. The necessity is trivial. This is a much easier condition and will be used many times in the future.

Proposition 4.2.1. *Suppose* $\Lambda_1(\cdot)$ *and* $\Lambda_2(\cdot)$ *are two random measures on* $\mathbb{D}$. *Then* $\Lambda_1(\cdot) \leq_{dcx} \Lambda_2(\cdot)$ *if and only if condition (4.1) holds for all* mutually disjoint *bBs. The same results holds true for idcx and idcv order.*

Proof. We need to prove the 'if' part alone. We shall prove for *dcx* order and the same argument is valid for f being *idcx* or *idcv*. Let condition (4.1) be satisfied for all mutually

disjoint bBs. Let $f : \mathbb{R}^n \to R$ be *dcx* function and $B_1, \ldots, B_n$ be bBs. We can choose mutually disjoint bBs $A_1, \ldots, A_m$ such that $B_i = \cup_{j \in J_i} A_j$ for all i. Hence $\Lambda(B_i) = \sum_{j \in J_i} \Lambda(A_j)$. Now define $g : \mathbb{R}^m \to \mathbb{R}^n$ as $g(x_1, \ldots, x_m) = (\sum_{j \in J_1} x_j, \ldots, \sum_{j \in J_n} x_j)$. Then g is *idl* and so $f \circ g$ is *dcx*. Moreover, $f(\Lambda(B_1), \ldots, \Lambda(B_n)) = f \circ g(\Lambda(A_1), \ldots, \Lambda(A_m))$ and thus the result for *dcx* follows. □

Proposition 4.2.2. *Let $\mathcal{C} \subset \mathcal{B}(\mathbb{D})$ be a semi-ring of $\mathbb{D}$ and $\Lambda_i, i = 1, 2$ be two random measures such that for any $n \geq 1$, $C_1, \ldots, C_n \in \mathcal{C} \cap \mathcal{B}_b(\mathbb{D})$ and for all f dcx ((resp. idcx; idcv))*

$$\mathsf{E}(f(\Lambda_1(C_1), \ldots, \Lambda_1(C_n))) \leq \mathsf{E}(f(\Lambda_2(C_1), \ldots, \Lambda_2(C_n)))\,.$$

Further, let $\alpha_1(.)$ and $\alpha_2(.)$ be σ-finite measures. Then $\Lambda_1 \leq_{dcx\,(resp.\,idcx;\,idcv)} \Lambda_2$.

Proof. Let $B^1, \ldots, B^m$ be disjoint bBs. Recall that $\alpha_l(.) = \mathsf{E}(\Lambda_l(.))\,, l = 1, 2$ denote the respective mean measures. Since $\alpha_l(.)$ is an outer measure on $\mathcal{C}$, for $1 \leq i \leq m$, we have that there exist sequences $\{B^i_{nj}\}_{n \geq 1, 1 \leq j \leq k_n < \infty} \subset \mathcal{C} \cap \mathsf{B}_b(\mathbb{D})$ such that

$$\forall n \;\&j \neq k \; B^i_{nj} \cap B^i_{nk} = \emptyset, \;\; ; \quad \cup_{j \geq 1} B^i_{(n+1)j} \subset \cup_{j \geq 1} B^i_{nj}; \quad B^i \subset \cup_{j \geq 1} B^i_{nj};$$

$$\alpha_l(\cup_{j \geq 1} B^i_{nj} / B^i) \searrow 0, \;\text{ as } n \to \infty \quad l = 1, 2.$$

Existence of such an approximating sequence is guaranteed by Carathéodory's extension theorem and uniqueness due to the σ-finiteness of the measures.

First note that for f *dcx* ((resp. *idcx*; *idcv*)) as in the proof of Proposition 4.2.1, we have that for all $n \geq 1$,

$$\mathsf{E}\left(f(\sum_{j \geq 1} \Lambda_1(B^1_{nj}), \ldots, \sum_{j \geq 1} \Lambda_1(B^m_{nj})) \right) \leq \mathsf{E}\left(f(\sum_{j \geq 1} \Lambda_2(B^1_{nj}), \ldots, \sum_{j \geq 1} \Lambda_2(B^m_{nj})) \right).$$

Since for all $n \geq 1$ and $1 \leq i \leq k$, $\alpha_l(\cup_{j \geq 1} B^i_{nj} / B^i) \searrow 0$, $l = 1, 2$, we have that $\sum_{j \geq 1} \Lambda_l(B^1_{nj}) \searrow \Lambda_l(B^i)$ a.s.. Now use Lemma 2.3.8 for f *dcx* (with approximating sequences $(\sum_{j \geq 1} \Lambda_1(B^1_{nj}), \ldots, \sum_{j \geq 1} \Lambda_1(B^m_{nj})$ and $(\sum_{j \geq 1} \Lambda_2(B^1_{nj}), \ldots, \sum_{j \geq 1} \Lambda_2(B^m_{nj}))$ and monotone convergence theorem for f *idcx* or *idcv*, to show that $\mathsf{E}\big(f(\Lambda_1(B^1), \ldots, \Lambda_1(B^m)\big) \leq \mathsf{E}\big(f(\Lambda_2(B^1), \ldots, \Lambda_2(B^m)\big)\,.$ □

4.2.1 Simple operations preserving order

Point processes are special cases of random measures and as such will be subject to the considered ordering. Let $\Phi = \sum_i \varepsilon_{X_i}$ ($\varepsilon_x(.)$ is the Dirac measure) for random elements X_i in $\mathbb{D}$. We shall now show that all the three orders $dcx, idcx, idcv$ preserve some simple operations such as deterministic mapping, independent identically distributed (i.i.d.) thinning and independent superposition on random measures and pp.

Let $\phi : \mathbb{D} \to \mathbb{D}'$ be a measurable mapping to some LCSC space $\mathbb{D}'$. By the image of a (random) measure Λ by ϕ we understand $\Lambda'(\cdot) = \Lambda(\phi^{-1}(\cdot))$. Note that the image of a pp Φ by ϕ consists in deterministic displacement of all its points by ϕ.

Let $\Phi = \sum_i \varepsilon_{x_i}$. By *i.i.d. marking* of Φ, with marks in some LCSC space $\mathbb{D}'$, we understand a pp on the product space $\mathbb{D} \times \mathbb{D}'$, with the usual product Borel σ-algebra, defined by $\tilde{\Phi} = \sum_i \varepsilon_{(x_i, Z_i)}$, where $\{Z_i\}$ are i.i.d. rvs, so called marks, on $\mathbb{D}'$. By *i.i.d. thinning* of Φ, we understand $\overline{\Phi} = \sum_i Z_i \varepsilon_{x_i}$, where Z_i are i.i.d. 0-1 Bernoulli rvs. The probability $\mathsf{P}\{Z = 1\}$ is called the *retention probability*. *Superposition* of random measures is understood as addition of measures. Measures on Cartesian products of LCSC spaces are always considered with their corresponding product Borel σ-algebras.

Proposition 4.2.3. *Suppose $\Lambda_i, i = 1, 2$ are random measures and $\Phi_i, i = 1, 2$ are pp. Assume that $\Lambda_1 \leq_{dcx\,(resp.\,idcx;\,idcv)} \Lambda_2$ and $\Phi_1 \leq_{dcx\,(resp.\,idcx;\,idcv)} \Phi_2$.*

1. *Let Λ_i' be the image of Λ_i, $i = 1, 2$, by some mapping $\phi : \mathbb{D} \to \mathbb{D}'$. Then $\Lambda_1' \leq_{dcx\,(resp.\,idcx;\,idcv)} \Lambda_2'$. As a special case, the same holds true for the displacement of points of Φ_i's by ϕ.*

2. *Let $\Phi_i, i = 1, 2$, be simple pp and $\tilde{\Phi}_i, i = 1, 2$, be the corresponding i.i.d. marked pp with the same distribution of marks. Also assume that $\tilde{\alpha}_i(.), i = 1, 2$ are σ-finite measures. Also, assume that $\alpha(.) = \alpha_1(.) = \alpha_2(.)$ is a $\sigma-$ finite measure. Then $\tilde{\Phi}_1 \leq_{dcx} \tilde{\Phi}_2$.*

3. *Then $\overline{\Phi}_i$ be i.i.d. thinning of Φ_i, $i = 1, 2$, with the same retention probability. Also assume that $\alpha_i(.), i = 1, 2$ are σ-finite measures. Then $\overline{\Phi}_1 \leq_{dcx} \overline{\Phi}_2$.*

4. *Let Λ_1' and Λ_2' be two random measures such that $\Lambda_1' \leq_{dcx\,(resp.\,idcx;\,idcv)} \Lambda_2'$. Assume that Λ_i''s are independent of Λ_i's. Then $\Lambda_1 + \Lambda_1' \leq_{dcx\,(resp.\,idcx;\,idcv)} \Lambda_2 + \Lambda_2'$, where $+$ is understood as the addition of measures.*

5. *Suppose the random measures are on the product space* $\mathbb{D} \times \mathbb{D}'$. *Then* $\Lambda_1(\mathbb{D} \times \cdot) \leq_{dcx\,(resp.\,idcx;\,idcv)} \Lambda_2(\mathbb{D} \times \cdot)$, *provided the respective projections are Radon measures.*

Proof. (1): The result follows immediately from the Definition 4.1.1.

(2): We shall prove $\tilde{\Phi}_1 \leq_{dcx} \tilde{\Phi}_2$. Since $\mathbb{D}$ is a LCSC space, for every B bBs, there exists a *null-array of partitions* $\{B_{n,j}\}_{n\geq 1, j\geq 1}$, i.e., $B_{n,j}, n \geq 1, j \geq 1$ are bBs and form a finite disjoint partition of B for every n and $\max_{j\geq 1}\{|B_{n,j}|\} \to 0$ as $n \to \infty$ where $|\cdot|$ denotes the diameter in any fixed metric (see [Kallenberg 1983, page 11]). For every $x \in \mathbb{D}$, let $j(n,x)$ be the unique index such that $x \in B_{n,j(n,x)}$. For $X_k \in \Phi_1$ (and similarly for Φ_2), define $n(X_k) := \inf\{n : j(n, X_k) \neq j(n, X_i)\ \forall\ i \neq k\}$. Let $\overline{Z} = \{Z_{n,j}\}_{n\geq 1, j\geq 1}$ be a family of $\mathbb{D}'$-valued i.i.d. rvs with distribution $F(\cdot)$. Now define marked pp $\tilde{\Phi}_i, \tilde{\Phi}_i^n, i = 1, 2, n \geq 1$ as follows :

$$\tilde{\Phi}_i^n = \sum_{X_k \in \Phi_i} \varepsilon_{(X_k, Z_{n,j(n(X_k)\wedge n, X_k)})}; \quad \tilde{\Phi}_i = \sum_{X_k \in \Phi_i} \varepsilon_{(X_k, Z_{n,j(n(X_k), X_k)})}.$$

We shall now verify that the sequences $\tilde{\Phi}_i^n$'s satisfy the assumption of Lemma 2.3.8 with limits $\tilde{\Phi}_i$'s respectively.

Firstly let $B_1, \ldots, B_m \subset \mathbb{D} \times \mathbb{D}'$ be bBs and $g : \mathbb{R}^m \to \mathbb{R}$ be a continuous bounded function. Let $B_i^{'}, B_i^{''}$ denote the projection of B_i onto $\mathbb{D}$ and $\mathbb{D}'$ respectively for all $1 \leq i \leq m$. Since B_j's are bounded and Φ_i's are simple, given $\Phi_i, i = 1, 2$, there exists a.s. $N(\Phi_i, B_1, \ldots, B_m) \in \mathbb{N}$ such that $n(X_k) \leq N(\Phi_i, B_1, \ldots, B_m)$ for all $X_k \in (B_1^{'} \cup \ldots \cup B_m^{'}) \cap \Phi_i$. For $n \geq N(\Phi_i, B_1, \ldots, B_m)$,$i = 1, 2$

$$(\tilde{\Phi}_i^n(B_1), \ldots, \tilde{\Phi}_i^n(B_m)) = (\tilde{\Phi}_i(B_1), \ldots, \tilde{\Phi}_i(B_m)).$$

As a consequence we have that for $i = 1, 2$,

$$(\tilde{\Phi}_i^n(B_1), \ldots, \tilde{\Phi}_i^n(B_m)) \overset{n\to\infty}{\to} (\tilde{\Phi}_i(B_1), \ldots, \tilde{\Phi}_i(B_m))\ \ a.s.$$

and so

$$(\tilde{\Phi}_i^n(B_1), \ldots, \tilde{\Phi}_i^n(B_m)) \overset{d}{\to} (\tilde{\Phi}_i(B_1), \ldots, \tilde{\Phi}_i(B_m)).$$

Secondly it is easy to check that for $B_1 = B' \times B''$ and $i = 1, 2$ we have $\mathsf{E}(\tilde{\Phi}_i^n(B_1)) = \mathsf{E}(\Phi_i(B'))F(B'') = \mathsf{E}(\tilde{\Phi}_i(B_1))$ and hence by an appropriate approximation as in Proposi-

tion 4.2.2, we have that $\mathsf{E}(\tilde{\Phi}_i^n(B_1)) = \mathsf{E}(\tilde{\Phi}_i(B_1))$ for any bBs B_1.

Finally for any bBs $B \subset \mathbb{D} \times \mathbb{D}'$ and any realization $\overline{Z} = \overline{z} = \{z_{n,j}\}_{n\geq 1, j\geq 1}$, define

$$V_i^{\overline{z}}(B) := \tilde{\Phi}_i^n(B) | \{\overline{Z} = \overline{z}\} = \int_{\mathbb{D}} \mathbf{1}[(x, z_{n,j(n,x)}) \in B] \Phi_i(\mathrm{d}x).$$

Since $z_{n,j(n,\cdot)}$ is a piecewise constant function, $1[(x, z_{n,j(n,x)}) \in B]$ is a measurable function in x and so $V_i^{\overline{z}}$'s are integral shot-noise fields (as per Definition 4.1.2) indexed by bBs of $\mathbb{D} \times \mathbb{D}'$. Thus from Theorem 4.1.3, we have that for any *dcx* function f,

$$\begin{aligned} &\mathsf{E}(f(\tilde{\Phi}_1^n(B_1), \ldots, \tilde{\Phi}_1^n(B_m)) | \overline{Z} = \overline{z}) = \mathsf{E}(f(V_1^{\overline{z}}(B_1), \ldots, V_1^{\overline{z}}(B_m))) \\ &\leq \quad \mathsf{E}(f(V_2^{\overline{z}}(B_1), \ldots, V_2^{\overline{z}}(B_m))) = \mathsf{E}(f(\tilde{\Phi}_2^n(B_1), \ldots, \tilde{\Phi}_2^n(B_m)) | \overline{Z} = \overline{z}) \end{aligned}$$

Now, taking further expectations we get $(\tilde{\Phi}_1^n(B_1), \ldots, \tilde{\Phi}_1^n(B_m)) \leq_{dcx} (\tilde{\Phi}_2^n(B_1), \ldots, \tilde{\Phi}_2^n(B_m))$. Since the approximation satisfies the assumption of Lemma 2.3.8, the proof follows.

(3): Given a pp Φ, consider the marked pp $\tilde{\Phi}$ with marks as i.i.d. 0-1 Bernoulli rvs Z_i. Now note that the thinned pp can be represented in terms of the marked pp as follows : $\overline{\Phi}(B) = \tilde{\Phi}(B \times \{1\})$ for any bBS B. Now the result follows from the second part of the proposition.

(4): This follows from Lemma 2.3.6.

(5): This result follows easily from Lemma 2.3.8 using an increasing approximation of $\mathbb{D}$ by bBs.

□

4.2.2 Impact on higher order properties

We will state now some results involving ordering of moments of random measures (refer Section 2.1) and draw some conclusions concerning the so called second order properties. These latter ones make it possible to characterize the clustering in pp.

Proposition 4.2.4. *Consider random measures $\Lambda_1 \leq_{idcx} \Lambda_2$. Also assume that $\alpha_i^k(.), i = 1, 2$ are σ-finite measures. Then $\Lambda_1^k \leq_{idcx} \Lambda_2^k$ and $\alpha_1^k(\cdot) \leq \alpha_2^k(\cdot)$. Moreover, if $\Lambda_1 \leq_{dcx} \Lambda_2$ then $\alpha_1(\cdot) = \alpha_2(\cdot)$.*

Proof. Since the set of rectangles form a semi-ring in $\mathbb{D}^k$, by Proposition 4.2.2, it is enough to show the inequality (4.1) with *idcx* functions for the random measures $\Lambda_1^k(.)$ and $\Lambda_2^k(.)$

on rectangles alone. For the same, we consider a *idcx* function $f : \mathbb{R}^l \to \mathbb{R}$ taken of the values of the moment measures on l rectangles in $\mathbb{D}^k$. In this context, consider $g : \mathbb{R}^m \to \mathbb{R}$ given by

$$g(y_1, \ldots, y_m) = f\Big(\prod_{j \in J_1} y_j^+, \ldots, \prod_{j \in J_l} y_j^+ \Big),$$

where $J_1, \ldots, J_l$ are k-element subsets of the set $\{1, \ldots, m\}$ and for $y \in \mathbb{R}$, $y^+ := \max\{y, 0\}$. Since $y_j^+, j \in [m]$ are increasing non-negative convex functions, the products are *idcx* functions as well (see Example 2.3.3) and hence g is *idcx* as it is the composition of an *idcx* function f with another *idcx* function.

The second statement follows easily from the first one by the fact that $f(x) = x$ is *idcx*. For the first moment (mean measure) note that both $f(x) = x$ and $f(x) = -x$ are *dcx*. □

A useful characteristic for measuring clustering effect in point processes is the *pair correlation function* defined on $\mathbb{R}^2$ as $g(x, y) = \frac{\rho^{(2)}(x,y)}{\rho^{(1)}(x)\rho^{(1)}(y)}$, where $\rho^{(k)}$ is the *k th joint intensity* (see Section 2.1).

Corollary 4.2.5. *Let $\Phi_1 \leq_{idcx} \Phi_2$ be two pp with joint intensities $\rho_i^{(k)}(x_1, \ldots, x_k), i = 1, 2$ respectively. Then $\rho_1^{(k)}(x_1 \ldots, x_k) \leq \rho_1^{(k)}(x_1 \ldots, x_k)$ for Lebesgue a.e. $(x_1 \ldots, x_k)$. If $\Phi_1 \leq_{dcx} \Phi_2$, then $g_1(x, y) \leq g_2(x, y)$ for Lebesgue a.e. (x, y).*

Proof. The proof follows from the following two observations :For any disjoint bBs $B_1, \ldots, B_n$,

1. By Proposition 4.2.4, we have that $\mathsf{E}(\prod_{i=1}^n \Phi_1(B_i)) \leq \mathsf{E}(\prod_{i=1}^n \Phi_2(B_i))$ and
2. we have from definition of joint intensities, $\mathsf{E}(\prod_{i=1}^n \Phi_j(B_i)) = \int_{\prod_{i=1}^n B_i} \rho_j^{(n)}(x_1, \ldots, x_n) \mathrm{d}x_1 \ldots \mathrm{d}x_n$, $j = 1, 2$.

□

We shall now explore further the relation between *dcx* ordering and clustering of points in a pp. One of the most popular functions for the analysis of this effect is the *Ripley's K function* $K(r)$ (reduced second moment function); see [Stoyan et al. 1995]. Assume that Φ is a stationary pp on $\mathbb{R}^d$ with finite intensity $\lambda = \alpha(B)$, where B is a bBs such that $\|B\| = 1$. Then

$$K(r) = \frac{1}{\lambda \|G\|} \mathsf{E}\Big(\sum_{X_i \in \Phi \cap G} (\Phi(B_{X_i}(r)) - 1) \Big),$$

where $B_x(r)$ is the ball centered at x of radius r and $\|G\|$ denotes the Lebesgue measure of a bBs G; due to stationarity, the definition does not depend on the choice of G.

Proposition 4.2.6. *Consider two stationary pp Φ_i, $i = 1, 2$, with same finite intensity and denote by $K_i(r)$ their Ripley's K functions. If $\Phi_1 \leq_{dcx} \Phi_2$ then $K_1(\cdot) \leq K_2(\cdot)$.*

Proof. Denote $I_i = \mathsf{E}\Big(\sum_{X_j \in \Phi_i \cap G} (\Phi_i(B_{X_j}(r)) - 1)\Big)$, $i = 1, 2$. By the equality of mean measures (Proposition 4.2.4), it is enough to prove that $I_1 \leq I_2$. Note that I_i can be written as the value of some shot noise evaluated with respect to Φ_i^2, the second product of the pp.

$$I_i = \sum_{X_j, X_k \in \Phi_i} \mathbf{1}[X_j \in G] \mathbf{1}[0 < |X_k - X_j| \leq r],$$

where $\mathbf{1}[\cdot]$ denotes the indicator function. Thus, the result follows from Proposition 4.2.4 and Theorem 4.1.3. □

4.2.3 Impact on Palm measures

We shall start with a lemma that guarantees that intensity measures of ordered intensity fields are ordered. The first part of the following lemma is an easy extension of the one-dimensional version in [Meester and Shanthikumar 1993, Lemma 3.3]. The second part, which we prove in what follows, is a further extension of it.

Lemma 4.2.7. *Suppose $\{\lambda_1(s)\}_{s \in \mathbb{R}^d}$ and $\{\lambda_2(s)\}_{s \in \mathbb{R}^d}$ are two non-negative real-valued and a.s. locally Riemann integrable random fields. Define random measures $\Lambda_i(B) := \int_B \lambda_i(s) ds$,$i = 1, 2$ for any B bBs.*

1. *If $\{\lambda_1(s)\} \leq_{idcx\,(resp.\,idcv)} \{\lambda_2(s)\}$, then $\Lambda_1 \leq_{idcx;idcv} \Lambda_2$.*
2. *Suppose further that $\mathsf{E}(\int_A \lambda_1(x) dx) < \infty$ for all bBs A in $\mathbb{R}^d$ and similarly for $\{\lambda_2(x)\}$. If $\{\lambda_1(x)\} \leq_{dcx} \{\lambda_2(x)\}$, then $\Lambda_1 \leq_{dcx} \Lambda_2$.*

Proof. (2) We shall prove for $d = 1$ and as can be seen from the proof, the generalization is fairly straightforward. The proof of the first part is also similar.

Due to Proposition 4.2.1, Proposition 4.2.2 and the fact that intervals form a semi-ring on the real line, we only need to prove that $(\int_{I_1} X(s) ds, \ldots, \int_{I_n} X(s) ds) \leq_{dcx} (\int_{I_1} Y(s) ds, \ldots, \int_{I_n} Y(s) ds)$, for $I_i, i = 1, \ldots, n$ disjoint intervals. We shall give an approximation satisfying the assumptions of Lemma 2.3.8. Let $I_i = [a_i, b_i]; a_i, b_i \in \mathbb{R}, i = 1, \ldots, n$.

Let $\{(t^i_{mj})_{1\le j\le k_m}, i = 1,\dots,n\}$ be the sequences of mth nested partition of each interval. The middle Riemann sum can be given as follows : $X^m(I_i) = \sum_j X(t^i_{mj})(t^i_{m(j+1)} - t^i_{mj}), i = 1,\dots,n, m \in \mathbb{N}$ and similarly for $Y(x)$. These are the variables satisfying the approximation as in Lemma 2.3.8. As $X(s)$ is Riemann integrable,

$$(X^m(I_1),\dots,X^m(I_n)) \to (J^1_X,\dots,J^n_X) \ \ a.s.$$

and hence in distribution. It is also clear the middle Riemann sums of $X(\cdot)$ and $Y(\cdot)$ are ordered. The proof is complete by showing that for all $i = 1,\dots,n$,

$$\mathsf{E}X^m(I_i) = \sum_j \mathsf{E}X(t^i_{mj})(t^i_{m(j+1)} - t^i_{mj}) \to \int_{I_i} \mathsf{E}X(s)ds = \mathsf{E}J^i_X,$$

where the convergence follows from integrability of $\mathsf{E}(X(x))$ and last equality is due to Fubini's theorem. The proof of (1) also uses the same idea of approximation via Riemann sums with the difference being in the usage of upper and lower Riemann sums while convergence is shown via monotone convergence theorem for *idcx* or *idcv* functions.

The generalization to higher dimensions is by approximating the integral over disjoint rectangles by a linear combination of appropriately chosen $X(t_i)$'s as above and then again using Propositions 4.2.1 and 4.2.2. □

Refer to Section 2.1 for definition of Palm measures.

Proposition 4.2.8. *Suppose $\Lambda_i, i = 1,2$ are random measures.*

1. *If $\Lambda_1 \le_{dcx} \Lambda_2$ then $(\Lambda_1)_f \le_{idcx} (\Lambda_2)_f$ for any non-negative measurable function f such that $0 < \int_{\mathbb{D}} f(x)\alpha(dx) < \infty$, where α is the (common) mean measure of Λ_i, $i = 1,2$.*

2. *Suppose that Λ_i has locally finite mean measure and almost surely (a.s.) locally Riemann integrable density λ_i, $i = 1,2$. If $\{\lambda_1(x)\} \le_{dcx} \{\lambda_2(x)\}$, then for every $x \in \mathbb{R}^d$, $(\Lambda_1)_x \le_{idcx} (\Lambda_2)_x$.*

Proof. (1): Denote $I_i = \int_{\mathbb{D}} f(x)\Lambda_i(\mathrm{d}x)$, $i = 1,2$. By Proposition 4.2.4, $\Lambda_1 \le_{dcx} \Lambda_2$ implies that the mean measures are equal and thus $\mathsf{E}(I_1) = \mathsf{E}(I_2)$. From (2.1), it remains to prove

$$\mathsf{E}(g(\Lambda_1(B_1),\dots,\Lambda_1(B_n))I_1) \le \mathsf{E}(g(\Lambda_2(B_1),\dots,\Lambda_2(B_n))I_2)$$

for *idcx* function g. Now, if we show that

$$(I_1, \Lambda_1(B_1), \ldots, \Lambda_1(B_n)) \leq_{dcx} (I_2, \Lambda_2(B_1), \ldots, \Lambda_2(B_n))$$

then the proof follows from the fact that $h(x_0, x) = x_0^+ g(x) : \mathbb{R}^{n+1} \to \mathbb{R}$ is *dcx* (use Lemma 2.3.2). To show the above inequality, use Theorem 4.1.3, with $S = \{0, \ldots, n\}$ and $h(x, 0) = f(x)$ and $h(x, i) = 1[x \in B_i], 1 \leq i \leq n$.

(2): The first part follows immediately from the second statement of Lemma 4.2.7. For the second part, use the same argument about $h(x_0, x) = x_0^+ g(x)$ being *dcx* as above. □

Remark 4.2.9. *Compared to earlier results where dcx ordering led to dcx ordering, one might tend to believe that the loss here (as dcx implies idcx only) is more technical. However the following illustrates that it is natural to expect so: consider a Poisson pp Φ and its (deterministic) intensity measure $\alpha(\cdot)$ (i.e., its mean measure $\alpha(\cdot) = \mathsf{E}(\Phi(\cdot))$. Using the complete independence property of the Poisson pp and the fact that each dcx function is component-wise convex, one can show that for disjoint bBs $A_1, \ldots, A_n$ and any dcx function f, $f(\alpha(A_1), \ldots, \alpha(A_n)) \leq \mathsf{E}(f(\Phi(A_1), \ldots, \Phi(A_n))$. Thus $\alpha \leq_{dcx} \Phi$. It is easy to see that for any "nice" function f, $\alpha_f(\cdot) = \alpha(\cdot)$ (mixed Palm version of a deterministic measure is equal to the original measure). Take $f(x) = 1[x \in A]$ for some bBs A. Then $\mathsf{E}(\Phi_f(A)) = \mathsf{E}((\Phi(A))^2)/\alpha(A) = \alpha(A) + 1$ since $\Phi(A)$ is a Poisson rv. Thus $\alpha_f(A) < \mathsf{E}(\Phi_f(A))$ disproving $\alpha_f(A) \leq_{dcx} \Phi_f(A)$. Another counterexample involving Poisson-Poisson cluster pp will be given in Remark 5.1.9. Further, as will be seen in the discussion after Corollary 5.2.3, it is also not possible to get ddcx ordering of Palm measures assuming dcx ordering of the random measures. Thus, it is clear that the above result is the best possible in full generality.*

4.2.4 Cox point processes

We will consider now Cox pp (see Section 2.1), which constitute a rich class often used to model patterns which exhibit more clustering than in Poisson pp. Note that Cox pp may be seen as a result of an operation transforming some random (intensity) measure into a point (Cox) pp. One can easily show that this operation preserves our orders.

Proposition 4.2.10. *Consider two ordered random measures $\Lambda_1 \leq_{dcx\,(resp.\,idcx;\,idcv)} \Lambda_2$. Then $\Phi_{\Lambda_1} \leq_{dcx\,(resp.\,idcx;\,idcv)} \Phi_{\Lambda_2}$.*

Proof. Taking a *dcx* (resp. *idcx*; *idcv*) function ϕ, assuming (by Proposition 4.2.1) mutually disjoint bBs A_k, $k = 1, \ldots, n$, using the definition of Cox pp and Lemma 2.3.9, one shows for $i = 1, 2$ that given the intensity measure Λ_i, the conditional expectation

$$g(\Lambda_i(A_1), \ldots, \Lambda_i(A_n)) := \mathsf{E}(\phi(\Phi_{\Lambda_i}(A_1), \ldots, \Phi_{\Lambda_i}(A_n))|\Lambda_i)$$

is a *dcx* (resp. *idcx*; *idcv*) function. The result follows thus from the assumption of the measures Λ_i being *dcx* ordered. □

We will show now using Theorem 4.1.3 that *dcx*, *idcx*, *idcv* ordering of Cox intensity measures is preserved by independent (not necessarily identically distributed) marking and thinning, as well as independent displacement of points of the pp.

By *independent marking* of pp Φ on $\mathbb{D}$ with *marks* on some LCSC space $\mathbb{D}'$, we understand a pp $\tilde{\Phi} \sum_i \varepsilon_{(x_i, Z_i)}$ such that given $\Phi = \sum_i \varepsilon_{x_i}$, Z_i are independent random elements in $\mathbb{D}'$, with distribution $\mathsf{P}\{Z_i \in \cdot | \Phi = \sum_i \varepsilon_{x_i}\} = F_{x_i}(\cdot)$ given by some probability (mark) kernel $F_x(\cdot)$ from $\mathbb{D}$ to $\mathbb{D}'$. The fact that $F_x(\cdot)$ may depend on x (in contrast to i.i.d. marking) is sometimes emphasized by calling $\tilde{\Phi}$ a "position dependent" marking. Independent thinning can be seen as the projection on $\mathbb{D}$ of the subset $\tilde{\Phi}(\cdot, \{1\})$ of the independently marked pp $\tilde{\Phi}$ where the marks $Z_i \in \{0, 1\} = \mathbb{D}'$, are independent Bernoulli thinning variables $Z_i = Z_i(x)$, whose distributions may be dependent on x_i. Similarly, the projection of an independently marked pp $\tilde{\Phi} = \sum_i \varepsilon_{(x_i, Z_i)}$ on the space of marks $\mathbb{D}'$; i.e., $\tilde{\Phi}(\mathbb{D} \times \cdot) = \sum_i \varepsilon_{Z_i}$ can be seen as independent displacement of points of Φ to the space $\mathbb{D}'$. Special examples are i.i.d. shifts of points in the Euclidean space, when $Z_i = x_i + Y_i$, where Y_i are i.i.d.

Proposition 4.2.11. *Suppose $\Phi_i, i = 1, 2$, are two Cox(Λ_i) pp be such that their intensity measures are ordered $\Lambda_1 \leq_{dcx\,(resp.\,idcx;\,idcv)} \Lambda_2$. Let $\tilde{\Phi}_i, i = 1, 2$ be the corresponding independently marked pp with the same mark kernel $F_x(\cdot)$. Then $\tilde{\Phi}_1 \leq_{dcx\,(resp.\,idcx;\,idcv)} \tilde{\Phi}_2$.*

From the above Proposition, the following corollary follows immediately by the last statement of Proposition 4.2.3.

Corollary 4.2.12. *Independent thinning and displacement of points preserves dcx (resp. idcx; idcv) order of the intensities of Cox pp.*

Proof. (*Prop. 4.2.11*) Let Φ_i be Cox (Λ_i) $i = 1, 2$ respectively and $\Lambda_1 \leq_{dcx(idcx,idcv)} \Lambda_2$. It is known that independent marking of Cox (Λ_i) pp is a Cox $(\tilde{\Lambda}_i)$ pp with intensity measure

$\tilde{\Lambda}_i$ on $\mathbb{D} \times \mathbb{D}'$ given by $\tilde{\Lambda}_i(\cdot) = \int_{\mathbb{D}} \int_{\mathbb{D}'} \mathbf{1}[(x,y) \in \cdot]\, F_x(dy)\Lambda_i(dx)$; cf. [Stoyan et al. 1995, Secs 4.2 and 5.2]. Let S be the family of bBs in $\mathbb{D} \times \mathbb{D}'$; for $x \in \mathbb{D}$ and bBs $C \subset \mathbb{D} \times \mathbb{D}'$ consider $h(x,C) = \int_{\mathbb{D}'} \mathbf{1}[(x,y) \in C]\, F_x(dy)$. Then the integral shot noise $V_{\Lambda_i}(C) = \int_{\mathbb{D}} h(x,C)\, \Lambda_i(dx)$ satisfies $V_{\Lambda_i}(C) = \tilde{\Lambda}_i(C)$ for all bBs C. Thus, by Theorem 4.1.3 $\tilde{\Lambda}_1 \leq_{dcx\,(\text{resp. } idcx;\, \text{id}xv)} \tilde{\Lambda}_2$ and the result follows from Proposition 4.2.10. □

Recall that if $\Lambda(\cdot) \in \mathbb{M}(\mathbb{R}^d)$ a.s has a density $\{\lambda(x)\}_{x \in \mathbb{R}^d}$ with respect to Lebesgue measure then the density is referred to as the intensity field of the Cox pp, which will be called in this case Cox (λ) pp and denoted by Φ_λ.

It is known that Cox pp is over-dispersed with respect to the Poisson pp, i.e., $\text{Var}(\Phi_1(B)) \leq \text{Var}(\Phi_2(B))$ where Φ_1, Φ_2 are, respectively, Poisson and Cox pp with the same mean measure. Hence, it is clear that a Cox pp can only be greater in *dcx* order than a Poisson pp with the same mean measure. Indeed, in Section 5.1 we will show several examples when this stronger result holds, namely Cox pp that are *dcx* ordered (larger) with respect to the corresponding Poisson pp, as well as Cox pp *dcx* ordered with respect to each other.

4.2.5 Alternative definition of *dcx* order

We viewed a random measure as a random field and have defined ordering from this viewpoint. Alternatively, one can consider a random measure as an element of the space of Radon measures $\mathbb{M}$ and define ordering between two $\mathbb{M}$-valued random elements. This can be done once we define what is a *dcx* function on $\mathbb{M}$. The *dcx* order can be defined on more general spaces; [Meester and Shanthikumar 1999] extends the notion of *dcx* ordering to lattice ordered Abelian semigroups ($LOAS^+$) with some compatibility conditions between the lattice structure and the Abelian structure. The space $\mathbb{M}$ can be equipped with the following lattice and algebraic structure. Consider the following partial order: for $\mu, \nu \in \mathbb{M}$, we say $\mu \leq \nu$ if $\mu(B) \leq \nu(B)$ for all bBs B in $\mathbb{D}$ and addition $(\mu+\nu)(B) = \mu(B)+\nu(B)$. Under this definition, the space $\mathbb{M}$ forms a $LOAS^+$ as required by [Meester and Shanthikumar 1999]. Then one can define a directionally convex function on $\mathbb{M}$ as in Definition 2.3.1. Call it a dcx^1 function. This gives rise to dcx^1 order of random measures analogously to the first part of the Definition 2.3.4.

Now we have two reasonable definitions of ordering of random measures. It is easy to see that dcx^1 ordering implies *dcx* ordering. In light of Example 5.1.7 of

[Müller and Stoyan 2002], existence of a counterexample to the converse looks plausible, though we failed in our attempts to construct one. However, the result of [Bassan and Scarsini 1991] proves that *convex* ordering of real valued stochastic process $\{X_n\}_{n\in\mathbb{N}}$ implies *continuous, convex* ordering of the corresponding elements of the infinite-dimensional Euclidean spaces $\mathbb{R}^{\mathbb{N}}$. This suggests that *dcx* of random measures may imply a dcx^{1*} order induced by some subclass of dcx^1 functionals of random measures, which are regular in some sense. Leaving this general question as an open problem, we remark only that the integral shot-noise fields studied in the next section can be seen as some particular class of functionals of random measures, which are dcx^1 (in fact linear on $\mathbb{M}$) and regular enough for their means to satisfy the required inequality provided the random measures are *dcx* ordered. It is natural thus to have them in the suggested dcx^{1*} class.

Recall also that for *strong* order of pp there is the full equivalence between these two definitions, and both imply the possibility of a coupling of the ordered pp such that the smaller one is a.s. a subset of the greater one; cf [Rolski and Szekli 1991].

4.3 Ordering of shot-noise fields

In this section we will prove Theorem 4.1.3 concerning *dcx* ordering of integral shot-noise fields, which is the main result of this chapter. We will also consider the so called extremal shot-noise fields.

4.3.1 Integral shot-noise fields

Usually shot-noise fields are defined for pp as the following sum (thus sometimes called additive shot-noise fields) $V_\Phi(y) = \sum_{X_n\in\Phi} h(X_n, y)$ where $\Phi = \sum_n \varepsilon_{X_n}$ and h is a non-negative response function. In definition 4.1.2, we have made a significant but natural generalization of this definition. It is pretty clear as to why we call this generalization the integral shot-noise field. The extension to unbounded response functions is not just a mathematical generalization alone. It shall provide us a simple proof of ordering for extremal-shot-noise fields of pp.

Now, we shall prove Theorem 4.1.3. The proof is inspired by [Miyoshi 2004].

Proof. (*Theorem 4.1.3*) We shall prove the second statement first. The necessary modifications for the proof of the first statement shall be indicated later on.

(2): We need to show that $(V^1(y_1),\ldots,V^1(y_m)) \leq_{dcx} (V^2(y_1),\ldots,V^2(y_m))$ for $y_i \in S, 1 \leq i \leq m$ and $V^j(\cdot) = V_{\Lambda_j}(\cdot)$, $j = 1,2$. The proof relies on the construction of two sequences of random vectors $(V_k^j(y_1),\ldots,V_k^j(y_m))$, $k = 1,2\ldots.$, $j = 1,2$ satisfying the assumptions of Lemma 2.3.8.

Choose an increasing sequence of compact sets K_k, $k \geq 1$ in $\mathbb{D}$, such that $K_k \nearrow \mathbb{D}$. Since h is measurable in its first argument, we know that there exists a sequence of simple functions $h_k(\cdot,y_i), k \in \mathbb{N}$ supported by K_k such that as $k \to \infty$, $h_k(\cdot,y_i) \uparrow h(\cdot,y_i)$ for $1 \leq i \leq m$. They can be written down explicitly as follows:

$$h_k(x,y_i) = \gamma_k \mathbf{1}[x \in I^i_{k\infty}] + \sum_{n=1}^{\gamma_k} \frac{n-1}{2^k} \mathbf{1}[x \in I^i_{kn}]$$

for $1 \leq i \leq m$, where $\gamma_k = k2^k$, $I^i_{kn} = \{x \in K_k : \frac{n-1}{2^k} \leq h(x,y_i) < \frac{n}{2^k}\}$ and $I^i_{k\infty} = \{x \in K_k : h(x,y_i) = \infty\}$ for $1 \leq i \leq m$ and $1 \leq n \leq \gamma_k$. Note that all I^i_{kn} $n = 1,\ldots,\infty$ are bBs and the sequence of random vectors we are looking for is

$$V_k^j(y_i) = \int_{\mathbb{D}} h_k(x,y_i)\Lambda_j(\mathrm{d}x) = \gamma_k \Lambda_j(I^i_{k\infty}) + \sum_{n=1}^{\gamma_k} \frac{n-1}{2^k} \Lambda_j(I^i_{kn}),$$

for $j = 1,2$. By the monotone convergence theorem, it is clear that for $j = 1,2$ as $k \to \infty$, $(V_k^j(y_1),\ldots,V_k^j(y_m)) \uparrow (V^j(y_1),\ldots,V^j(y_m))$ a.s. and hence in distribution. By monotone convergence theorem, the expectations, which are finite by the assumption, also converge. What remains to prove is that for each $k \in \mathbb{N}$, the vectors are *dcx* ordered.

Fix $k \in \mathbb{N}$. Now observe that for $j = 1,2$, $i = 1,\ldots,m$, $V_k^j(y_i)$ are *idl* functions of the vectors $(\Lambda_j(I^i_{kn}) : n = 1,\ldots,\gamma_k,\infty)$, $j = 1,2$. The latter are *dcx* ordered by the assumptions. And since composition of *dcx* with *idl* functions is *dcx*, it follows that $(V_k^1(y_1),\ldots,V_k^1(y_m)) \leq_{dcx} (V_k^2(y_1),\ldots,V_k^2(y_m))$.

(1): For vectors $(V_k^j(y_1),\ldots,V_k^j(y_m))$, $k = 1,2\ldots.$, $j = 1,2$ defined as above, $f(V_k^j(y_1),\ldots,V_k^j(y_m)) \uparrow f(V^j(y_1),\ldots,V^j(y_m))$ a.s. for f *idcx* (resp. *idcv*) and hence $\mathsf{E}(f(V_k^j(y_1),\ldots,V_k^j(y_m))) \uparrow \mathsf{E}(f(V^j(y_1),\ldots,V^j(y_m)))$, $j = 1,2$. The proof is complete by noting that $\mathsf{E}f(V_k^1(y_1),\ldots,V_k^1(y_m)) \leq \mathsf{E}(f(V_k^2(y_1),\ldots,V_k^2(y_m))$ for all $k \geq 1$ and f *idcx* (resp. *idcv*). □

4.3.2 Extremal shot-noise fields

We recall now the definition of the extremal shot-noise field, first introduced in [Heinrich and Molchanov 1994].

Definition 4.3.1. *Let S be any set and $\mathbb{D}$ a LCSC space. Given a pp Φ on $\mathbb{D}$ and a measurable (in the first variable alone) response function $h(x, y) : \mathbb{D} \times S \to \mathbb{R}$, the extremal shot-noise field is defined as*

$$U_\Phi(y) = \sup_{X_i \in \Phi} \{h(X_i, y)\}. \tag{4.3}$$

In order to state our result for extremal shot-noise fields, we shall use the *lower orthant* (*lo*) order.

Definition 4.3.2. *Let X and Y be random $\mathbb{R}^d$ vectors. We say $X \leq_{lo} Y$ if $\mathsf{P}(X \leq t) \geq \mathsf{P}(Y \leq t)$ for every $t \in \mathbb{R}^d$.*

On the real line, this is the same as the strong order (i.e., when $\mathfrak{F}$ consists of increasing functions) but in higher dimensions it is different. Obviously *st* order implies *lo* order and examples of random vectors which are ordered in *lo* but not in *st* are known; see ([Müller and Stoyan 2002]). Thus, it is clear that the following proposition is a generalization of the corresponding one-dimensional result in [Miyoshi 2004] where the proof method was similar to the proof of the ordering of integral shot-noise fields. We shall give a much simpler proof using the already proved result.

Proposition 4.3.3. *Let $\Phi_1 \leq_{idcv} \Phi_2$. Then $\{U_{\Phi_1}(y)\}_{y \in S} \leq_{lo} \{U_{\Phi_2}(y)\}_{y \in S}$.*

Proof. The probability distribution function of the extremal shot-noise can be expressed by the Laplace transform of some corresponding (additive) one as follows. Let $\{x_1, \ldots, x_m\} \subset$

S and $(a_1, \ldots, a_m) \in R^m$. Then

$$\begin{aligned}
\mathsf{P}(U(y_i) \leq a_i, 1 \leq i \leq m) &= \mathsf{E}(\prod_i \mathbf{1}[\sup_n h(X_n, y_i) \leq a_i]) \\
&= \mathsf{E}(\prod_i \prod_n \mathbf{1}[h(X_n, y_i) \leq a_i]) \\
&= \mathsf{E}(\prod_i \prod_n e^{\log \mathbf{1}[h(X_n, y_i) \leq a_i]}) \\
&= \mathsf{E}(\prod_i e^{-\sum_n - \log \mathbf{1}[h(X_n, y_i) \leq a_i]}) \\
&= \mathsf{E}(e^{-\sum_i \hat{U}(y_i)})
\end{aligned}$$

where $\hat{U}(y_i) = \sum_n - \log \mathbf{1}[h(X_n, y_i) \leq a_i]$ is an additive shot-noise with response function taking values in $[0, \infty]$. The response function is clearly non-negative and measurable. The function $f(x_1, \ldots, x_m) = e^{-\sum_i x_i}$ is a ddcx function on $(-\infty, \infty]$. The result follows by the first statement of Theorem 4.1.3. □

4.3.3 Ordering of the capacity functional of Boolean models

The extremal shot-noise field can be used to define the Boolean model. We gave the graph-theoretic definition in Definition 3.1.1 but now we shall give a more topological definition.

Definition 4.3.4. *Given a (generic) random compact set (racs; see Section 2.1.1) G, let $h((x, G), y) = \mathbf{1}[y \in x + G]$. By a* Boolean model *with the pp of germs Φ and the typical grain G we call the random set $C(\Phi, G) = \{y : U_{\tilde{\Phi}}(y) > 0\}$ where $\tilde{\Phi} = \sum_i \varepsilon_{(X_i, G_i)}$ is i.i.d. marking of Φ with the mark distribution equal to this of G.*

We shall call G a fixed grain if there exists a closed set B such that $G = B$ a.s.. In the case $B = B_O(r)$, we shall abbreviate by $C(\Phi, r)$. In the case when, $G = B_O(\rho)$ for a random variable ρ, we shall denote the Boolean model by $C(\Phi, \rho)$.

The following result (Proposition 4.3.5) is the starting point for our investigation of the connections between percolation and directionally convex ordering of point processes (see Chapter 6 and Section 5.2.1).

Proposition 4.3.5. *Let Φ_1, Φ_2 be two point processes and $U_{\Phi_i}(x)$ their respective extremal shot-noise fields with response functions $h(.,.)$. Define level sets of the extremal shot-noise*

fields for a level h as $S_{>h}(\Phi_i) := \{x : U_{\Phi_i}(x) > h\}$, $i = 1, 2$. Let B be a Borel set such that $\tilde{h}(., B) := \sup_{y \in B} h(., y)$ is measurable. Then $\Phi_1 \leq_{ddcx} \Phi_2 \Rightarrow \mathsf{P}(S_{>h}(\Phi_1) \cap B = \emptyset) \leq \mathsf{P}(S_{>h}(\Phi_2) \cap B = \emptyset)$.

In the special case of the Boolean model, we have that $\Phi_1 \leq_{ddcx} \Phi_2 \Rightarrow \mathbf{T}_{C(\Phi,G)}(B) \geq \mathbf{T}_{C(\Phi,G)}(B)$ for all Borel sets B.

Proof. The proof for the general case follows from Proposition 4.3.3 and the measurability of $\tilde{h}(., B)$, once we note the following for $i = 1, 2$:

$$\begin{aligned}
\mathsf{P}(S_{>h}(\Phi_i) \cap B = \emptyset) &= \mathsf{P}\left(\sup_{x \in B} U_{\Phi_i}(x) \leq h\right) \\
&= \mathsf{P}\left(\sup_{x \in B} \sup_{X \in \Phi_i} h(X, x) \leq h\right) \\
&= \mathsf{P}\left(\sup_{X \in \Phi_i} \tilde{h}(X, B) \leq h\right).
\end{aligned}$$

We have expressed $\mathsf{P}(S_{>h}(\Phi_i) \cap B = \emptyset)$ in terms of a level non-crossing probability of another extremal shot-noise field and thus we can use Proposition 4.3.3 with response function $\tilde{h}(., B)$.

As noted in the Definition 4.3.4, when the response function is $h((x, S), y) = \mathbf{1}[y \in x + S]$, we get that $S_{>0}(\tilde{\Phi}_i) = C(\Phi, G)$, the Boolean model. In this case the proof follows from the first part because $\tilde{h}((x, S), B) = \mathbf{1}[(x + S) \cap B \neq \emptyset]$ is measurable in (x, S) for all Borel sets B.

□

4.4 Relations to other orders

The entire chapter has focussed on directionally convex ordering of point processes and a natural question arises that are these properties specific to *dcx* order or not ? We will see some partial results that rule out some of the other orders as suitable for comparison of point processes with same mean measures.

The other well studied dependence orders for rvs are supermodular order (sm), convex order (cx) and componentwise convex order (ccx). The latter two are defined in similar fashion as *dcx* order but with convex and componentwise convex functions i.e, a function is *ccx* if it is convex in each variable when the other variables are fixed. A function $f : \mathbb{R}^d \to \mathbb{R}$

is said to be *supermodular* (see [Müller and Stoyan 2002, § 3.9]) if for $x, y \in \mathbb{R}^d$,

$$f(x) + f(y) \leq f(x \wedge y) + f(x \vee y),$$

where $x \wedge y := (x_1 \wedge y_1, \ldots, x_d \wedge y_d), x \vee y := (x_1 \vee y_1, \ldots, x_d \vee y_d), a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$ for $a, b \in \mathbb{R}$. This immediately leads to the definition of the supermodular order (denoted by *sm*) on vectors as in Definition 2.3.4. Similar to Lemma 2.3.2, one can characterize *sm* order and *ccx* order for twice differentiable functions as below. Let f be a twice differentiable function.

- f is *sm* iff $\dfrac{\partial^2}{\partial x_i \partial x_j} f(x) \geq 0$, for all $x, 1 \leq i \neq j \leq n$.
- f is *ccx* iff $\dfrac{\partial^2}{\partial x_i^2} f(x) \geq 0$, for all $x, 1 \leq i \leq n$.

Thus a function is *dcx* iff it is *sm* and *ccx*.

Definition 4.4.1. *Let X, Y be two random vectors. We say that X is less than Y in upper orthant (uo) order and denote it by $X \leq_{uo} Y$ if $\mathsf{P}(X \geq x) \leq \mathsf{P}(Y \geq x)$ for every $x \in \mathbb{R}^d$.*

Figure 4.1 (see [Müller and Stoyan 2002, Sec 3.14]) explains the various implications amidst the orders we have defined on random vectors. Note that the implications of the orders on random vectors are reversed with respect to the respective implications on the classes of the functions. For example, a *dcx* function is *sm* also and hence if two vectors are ordered in *sm* order, they will be ordered in *dcx* order also.

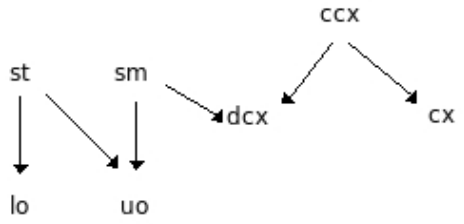

Figure 4.1: Relations between various orders on random vectors

For the rest of the section, we shall restrict ourselves to $\mathbb{R}^d$. Given two racs Ξ_1, Ξ_2, we shall say that $\Xi_1 \leq_{st} \Xi_2$ if there exists a coupling $\widehat{\Xi}_1 \stackrel{d}{=} \Xi_1, \widehat{\Xi}_2 \stackrel{d}{=} \Xi_2$ such that $a.s.$ $\widehat{\Xi}_1 \subset \widehat{\Xi}_2$. More rigorously, *st* order on racs can be defined as the integral order generated by the class of increasing functions where increasing is taken with respect to the partial ordering of set inclusion. Even though the capacity functional characterizes the distribution of racs (see 2.2), ordering of capacity functionals is not enough to guarantee strong ordering of racs. The counter-example for the same as well as a sufficient and necessary condition for strong ordering of racs was provided in [Norberg 1992].

Theorem 4.4.2. *Let Ξ_1, Ξ_2 be two racs in $\mathbb{R}^d$. Then we have that,*

$$\Xi_1 \leq_{st} \Xi_2 \quad \textit{iff} \quad \mathsf{P}\left(\cap_{i=1}^n \{\Xi_1 \cap K_i \neq \emptyset\}\right) \leq \mathsf{P}\left(\cap_{i=1}^n \{\Xi_2 \cap K_i \neq \emptyset\}\right),$$

for all n and all compact subsets $K_1, \ldots, K_n$ of $\mathbb{R}^d$.

Thus when simple pp are viewed as racs, we can restate the result in the following manner :

$$\Phi_1 \leq_{st} \Phi_2 \quad \text{iff} \quad \mathsf{P}\left(\cap_{i=1}^n \{\Phi_1(K_i) \geq 1\}\right) \leq \mathsf{P}\left(\cap_{i=1}^n \{\Phi_2(K_i) \geq 1\}\right), \tag{4.4}$$

for all n and all compact subsets $K_1, \ldots, K_n$ of $\mathbb{R}^d$. Thus, immediately it is clear that *uo* order for simple pp implies *st* order. Instead of considering *ccv* order, let us consider the stronger order of ccv_+ - the order generated by functions that are *ccv* on the non-negative orthant. Define $g : \mathbb{R}_+ \to \mathbb{R}$ as $g(x) = 1, x \geq 1$ and else $g(x) = x$. Then, note that $f(x_1, \ldots, x_n) = \prod_{i=1}^n g(x_i)$ is a *ccv* function on the positive orthant. Thus if $\Phi_1 \leq_{ccv_+} \Phi_2$, then they satisfy the inequality on the rhs of (4.4) and hence $\Phi_1 \leq_{st} \Phi_2$. This indicates the infeasibility of ccx_+ as an order for comparison of point processes with same mean measure. The function g above is also sm_+ i.e, *sm* on the positive orthant.

Given a function $g : \mathbb{R} \to \mathbb{R}$, let $f(x_1, \ldots, x_n) := g(x_1)$. It is easy to see that both f and $-f$ are *sm* and from this one can show that the one-dimensioanl marginals are equal if $X \leq_{sm} Y$. This helps us to rule out supermodular order as a reasonable one in the context of random measures. The reason is that it allows to compare only measures with the same one dimensional marginals, and thus a Poisson pp can only be (trivially) compared to itself in this order on the class of pp. Indeed, Poisson one dimensional marginal imply total independence property and thus uniquely characterize Poisson pp cf [Daley and Vere-Jones 1988, Lemma 2.3.I]. Now let us see a more stronger reasoning to

rule out *sm* order. The function $f(x_1, \ldots, x_n) := \prod_{i=1}^{n} 1[x_i \geq 1]$ is *sm* and so from (4.4), we get that $\Phi_1 \leq_{sm} \Phi_2$ implies that $\Phi_1 \leq_{st} \Phi_2$. The reverse implication is trivial. Hence, among the dependence orders, this leaves only the *cx* order as unexplored and hinting that *cx* order could be the other order of interest apart from *dcx* order when one wishes to compare point processes with same mean measures.

Chapter 5

Examples and applications of *dcx* order

5.1 Examples of *dcx* ordered random measures and point processes

In this section, we shall provide some examples of *dcx* ordered measures and pp on the Euclidean space $\mathbb{E} = \mathbb{R}^d$. The examples are intended to be illustrative and not encyclopaedic. The purpose of the examples is to show that there are *dcx* ordered pp as well as demonstrate some methods to prove *dcx* ordering of two pp. Many of the examples seem to indicate that pp higher in *dcx* order cluster more, at least for Cox pp. Due to Proposition 4.2.8, we shall in many examples prove ordering of the intensity fields alone.

5.1.1 Cox point processes with conditionally increasing intensity fields

Let $\{\lambda(s)\}_{s\in\mathbb{R}^d}$ be a stationary random intensity field. Define a new field, which is random but constant in space $\{\lambda_m(s) = \lambda(O)\}$ and deterministic constant field $\{\lambda_h(s) = \mathsf{E}(\lambda(0))\}$. $\mathrm{Cox}(\lambda_m)$ is known as mixed Poisson pp and $\mathrm{Cox}(\lambda_h)$ is just the well-known homogeneous Poisson pp. Denote the random intensity measures of the Cox, mixed and homogeneous Poisson pp, by Λ, Λ_m and Λ_h respectively (i.e., $\Lambda(dx) = \lambda(x)\,dx$, etc.)

A random field $\{X(s)\}$ is a *conditionally increasing field* if for any k and $s_1, \ldots, s_k \in \mathbb{R}^d$ the expectation $\mathsf{E}(f(X(s_1))|X(s_j) = a_j \ \forall\ 2 \le j \le k)$ is increasing in a_j for all increasing f whenever the conditioning event has a positive probability. The following proposition

is contained in the proof of [Miyoshi 2004, Theorem 1].

Proposition 5.1.1. *Let $\lambda_h(.), \lambda(.), \lambda_m(.)$ be respectively the intensity fields of homogeneous Poisson pp, Cox pp and mixed Poisson pp as defined above. Then we have that*

$$\{\lambda(s)\}_{s\in\mathbb{R}^d} \leq_{dcx} \{\lambda_m(s)\}_{s\in\mathbb{R}^d},$$

and when $\{\lambda(s)\}$ is a conditionally increasing field,

$$\{\lambda_h(s)\}_{s\in\mathbb{R}^d} \leq_{dcx} \{\lambda(s)\}_{s\in\mathbb{R}^d}.$$

Proof. The first part follows easily from *Lorentz's inequality*([Müller and Stoyan 2002, Th. 3.9.8]) : If $X_1, \ldots, X_d$ are identically distributed r.vs as X, then $(X_1, \ldots, X_d) \leq_{sm} (X, \ldots, X)$, where *sm* stands for supermodular (Section 4.4). The above result was proved in [Tchen 1980].

It is proved in [Meester and Shanthikumar 1993, Theorem 3.8] that if $\{Y_i\}_{i=1}^n$ is a conditionally increasing field and $\{X_i\}_{i=1}^n$ is a sequence of independent random variables such that $X_i \stackrel{d}{=} Y_i, \forall i$, then $(X_1, \ldots, X_n) \leq_{sm} (Y_1, \ldots, Y_n)$ and so $(X_1, \ldots, X_n) \leq_{dcx} (Y_1, \ldots, Y_n)$. By Jensen's inequality, we have that $\mathsf{E}(Y_i) = \mathsf{E}(X_i) \leq_{cx} X_i, \forall i$. Now by independence of co-ordinates (see 2.3.5) and transitivity of the order, $(\mathsf{E}(Y_1), \ldots, \mathsf{E}(Y_n)) \leq_{dcx} (Y_1, \ldots, Y_n)$. Thus when $\{\lambda(s)\}$ is a conditionally increasing field, $\{\lambda_h(s)\}_{s\in\mathbb{R}^d} \leq_{dcx} \{\lambda(s)\}_{s\in\mathbb{R}^d}$. □

Conditionally increasing random field : Suppose that a random vector $X = (X_1, \ldots, X_n)$ has a density f with respect to the Lebesgue measure on $\mathbb{R}^n$. We say that X (or f) is said to be *multivariate totally positive of order 2* ($MTP-2$ for short) if $f(x \wedge y)f(x \vee y) \geq f(x)f(y)$ for $x, y \in \mathbb{R}^n$ and $\cdot \wedge \cdot, \cdot \vee \cdot$ are coordinate-wise minimum and maximum respectively ; cf. [Karlin and Renott 1980]. This can be considered as the continnum version of the famed FKG condition on lattices in statistical mechanics (see (5.1)). Equivalently, f is $MTP-2$ iff $\log f$ is *sm*. For a finite $S \subset \mathbb{R}^d$, let f_S denote the marginal probability denisty of $\{\lambda(s)\}_{s\in S}$. From [Müller and Stoyan 2002, Theorem 3.10.14, Theorem 3.10.16], we know that if f_S is $MTP-2$ for all finite $S \subset \mathbb{R}^d$, then the random intensity field $\{\lambda(s)\}_{s\in\mathbb{R}^d}$ is conditionally increasing.

Let us now give some $MTP-2$ functions. The simplest $MTP-2$ function is perhaps $f(x_1, \ldots, x_n) = \prod_{i=1}^n h(x_i)$ for some function h on $\mathbb{R}$. This corresponds to the case of an

intensity with independent 1-dimensional marginals. From [Rüschendorf 1981] (see also [Müller and Stoyan 2002, Theorem 3.10.18]), we have that for a random vector having a multivariate normal distribution with an invertible covariance matrix Σ, both $MTP-2$ and conditionally increasing property are equivalent to the off-diagonal entries in Σ^{-1} being non-positive (i.e, Σ^{-1} is an M-matrix).

1-monotonic measures : We shall make a short digression into 1-monotonic measures as they will lead naturally to construction of a conditionally increasing intensity field for pp. For more details on monotonic measures, refer to [Grimmett 2006, Chapter 2].

Definition 5.1.2. *For a finite set $S \subset \mathbb{Z}^d$, let $\Omega_S := \{0,1\}^S$ and $\mathcal{F}_S$ be the corresponding σ-field. Let ν be a strictly positive measure on Ω_S i.e, $\nu(w) > 0$, $\forall w \in \Omega_S$. For a $x \in S, \xi \in \Omega_{S\setminus\{X\}}$, define*

$$\Omega^{\xi}_{x,S} := \{w \in \Omega_S : w(s) = \xi(s), s \in S \setminus \{x\}\}.$$

A strictly positive measure ν on Ω_S is said to be 1-monotonic if for every $\xi \leq \eta$ on $\Omega_{S\setminus\{x\}}$, ie., $\xi(s) \leq \eta(s), s \in S \setminus \{x\}$, we have that

$$\nu(w(x) = 1 | \Omega^{\xi}_{x,S}) \leq \nu(w(x) = 1 | \Omega^{\eta}_{x,S}).$$

A measure ν on Ω is said to be 1-monotonic if it is strictly positive and 1-monotonic on every finite subset.

From [Grimmett 2006, Theorem 2.27], we get that for strictly positive measures on Ω_S (S finite), 1-monotonicity is equivalent to the FKG lattice condition i.e, for all $w_1, w_2 \in \Omega_S$

$$\nu(w_1 \wedge w_2)\nu(w_1 \vee w_2) \geq \nu(w_1)\nu(w_2). \tag{5.1}$$

This condition on the lattices is named so in honor of the authors of the paper [Fortuin, Kasteleyn and Ginibre 1971], who proved it to be a sufficient condition for positive association of measures. From [Grimmett 2006, Theorem 2.22]), FKG lattice condition is true if it holds for pairs w_1, w_2 such that $w_1 \not\leq w_2, w_1 \not\geq w_2$ and w_1, w_2 differ in two coordinates alone i.e, $\sum_{s\in S} \mathbf{1}[w_1(s) \neq w_2(s)] = 2$. This simplifies greatly the construction of 1-monotonic measures. It is trivial to see that if μ is the product measure (i.e, $X(z)$ are i.i.d), then μ is 1-monotonic. A very general measure μ that is 1-monotonic is the

random cluster measure ([Grimmett 2006, Chapter 1]) and of which the specific cases are the Ising model and Potts model. Now we will construct conditionally increasing intensity fields using 1-monotonic measures on lattices.

Ising-Poisson cluster point process : Consider the d-dimensional lattice $\mathbb{Z}^d$. Let $\{X(z)\}_{z\in\mathbb{Z}^d}$ be a stationary (with respect to $\mathbb{Z}^d$-shifts) random field taking values in $\{0,1\}$. Call $\{X(z)\}$ a (random) configuration of spins and let μ, the induced measure on $\Omega = \{0,1\}^{\mathbb{Z}^d}$ be 1-monotonic. In order to obtain a stationary field consider a random shift of the origin of $\mathbb{Z}^d$ to U with uniform distribution on $[0,1]^d$ (U independent of $\{X(z)\}$). Let the lattice shifted by U be denoted by $\mathbb{Z}^d_*$. Pick two numbers $c_2 \leq c_1$. For $s \in \mathbb{R}^d$, define $\lambda_\mu(s) = c_1\mathbf{1}[X(\dot{s}) = 1] + c_2 1[X(\dot{s}) = 0]$ where $\dot{s}$ represents the unique "lower left" point in $\mathbb{Z}^d_*$ nearest to s. The intensity field is clearly stationary. We shall now show that for μ 1-monotonic, $\{\lambda_\mu(s)\}$ is conditionally increasing. Note that

$$f(\lambda_\mu(s)) = 1[x(\dot{s}) = 1](f(c_1) - f(c_2)) + f(c_2). \tag{5.2}$$

From Theorem 1.2.15 of [Müller and Stoyan 2002], it is sufficient to show the conditional increasing property conditioned on U, the random origin of the lattice $\mathbb{Z}^d_*$. Hence it is equivalent to the spin model possessing the following property:

$$\mathsf{P}\left(X(z_1) = 1 | X(z_j) = a_j, j = 2,\ldots,k\right) \;\leq\; \mathsf{P}\left(X(z_1) = 1 | X(z_j) = b_j, j = 2,\ldots,k\right) \tag{5.3}$$

where $a_i \leq b_i \in \{0,1\}$ and $z_i \in \mathbb{Z}^d, i = 1,\ldots,k$. From the definition 5.1.2, it is clear that any 1-monotonic measure μ will satisfy (5.3).

We call the Cox pp generated by the above conditionally increasing field $\{\lambda_\mu(s)\}$ the *Ising-Poisson cluster* pp. Thus by Proposition 5.1.1, it is *dcx* larger than the homogeneous Poisson pp with the same intensity. Note that intuitively the Ising-Poisson cluster pp "clusters" its points more than a homogeneous Poisson pp. In what follows, we will see more examples of cluster (Cox) pp which are *dcx* larger than the corresponding homogeneous Poisson pp.

5.1.2 Lévy based Cox point processes (LCPs)

This class of pp is being introduced in [Hellmund et al. 2008]. One can find many examples of LCPs in the above mentioned paper. In simple terms, a LCP is a pp whose intensity

field is an integral shot-noise field of a Lévy basis. A collection of random variables $\{L(B) : B\text{bBs}\}$ is said to be a non-negative Lévy basis if

- for any sequence $\{A_n\}$ of disjoint, bBs of $\mathbb{R}^d$, $L(A_n)$ are independent random variables (*complete independence*) and $L(\bigcup A_n) = \sum L(A_n)$ a.s. provided $\cup A_n$ is also a bBs of $\mathbb{R}^d$.
- for every bBs A of $\mathbb{R}^d$, $L(A)$ is infinitely divisible.

We shall consider only non-negative Lévy bases, even though there exist signed Lévy bases too (see [Hellmund et al. 2008]). If we assume non-negativity i.e, $L(B) \geq 0 \forall B$ bBS, then L is equivalent to a random measure (see [Daley and Vere-Jones 1988, Theorem 6.1.VI]). Hence, we shall omit the reference to non-negativity in future and thus for us Lévy bases will be random measures.

A Cox pp Φ is said to be a LCP, if has an intensity field of the form

$$\lambda(y) = \int_{\mathbb{R}^d} k(x, y) L(dx),$$

where L is a Lévy basis and the kernel k is a non-negative function such that $k(., y)$ is a.s. integrable with respect to L and $k(x, .)$ is integrable with respect to Lebesgue measure. As in [Hellmund et al. 2008] the response function k and the Lévy basis L are chosen such that $\int_B \lambda(y)\, dy < \infty$ a.s. for all bBs B, for which a sufficient condition is $\int_B \mathsf{E}(\lambda(y))\, dy < \infty$. In our considerations, in order to be able to use 4.2.7 for showing ordering results, we will require that $\lambda(y)$ is a.s. locally Riemann integrable.

Remark 5.1.3. *Note that a sufficient condition for $\lambda(y)$ being a.s. locally Riemann integrable is that $\lambda(y)$ is a.s. continuous, for which, in turn, it is enough to assume that k is continuous in its second argument and that for all $x \in \mathbb{R}^d$, there exist $B_x(\epsilon_x)$, $\epsilon_x > 0$ such that $\int_{\mathbb{R}^d} \sup_{z \in B_x(\epsilon_x)} k(z, y)\, \alpha(dx) < \infty$ for all y, where $\alpha(B) = \mathsf{E}(L(B))$, the mean measures of the Lévy bases; (cf [Baccelli and Błaszczyszyn 2001]).*

Proposition 5.1.4. *Let L_1 and L_2 be Lévy bases with mean measures α_1 and α_2 respectively. Let $\Phi_i, i = 1, 2$ be LCPs with sam response function $k(., .)$ and Lévy bases $L_i, i = 1, 2$ respectively.*

1. *$L_1 \leq_{dcx\,(resp.\,idcx;\,idcv)} L_2$ if and only if $L_1(A) \leq_{cx\,(resp.\,icx;\,icv)} L_2(A)$ for all bBs A of $\mathbb{R}^d$, where cx, icx, icv stands, respectively for convex, increasing convex and increasing concave.*

2. *If* $L_1 \leq_{dcx\,(resp.\,idcx;\,idcv)} L_2$, *then* $\Phi_1 \leq_{dcx\,(resp.\,idcx;\,idcv)} \Phi_2$ *provided the intensity fields*

   $\lambda_i(y)$ *of LCP* Φ_i *are a.s. locally Riemann integrable with these integrals, in case of dcx, having finite means.*

3. $\alpha_i \leq_{dcx} L_i$.

Proof. The first part is due to Proposition 4.2.1, the complete independence property of Lévy bases and 2.3.5. As for the second part, from Theorem 4.1.3 we get that the respective intensity fields are *dcx* ordered and then from Propositions 4.2.8 and Proposition 4.2.10, we can conclude that the corresponding intensity measures as well as the point processes are also *dcx* ordered. The third part follows from complete independence and Jensen's inequality (see Lemma 2.3.5). □

We shall now give some examples of *dcx* ordered Lévy basis.

Example 5.1.5. *Let* $\{x_i\}$ *be a locally finite deterministic configuration of points in* $\mathbb{R}^d$. *Let* $\{X_i^j\}_{i\geq 1}, j = 1, 2$ *be i.i.d sequence of infinite divisible random variables such that* $X_1^1 \leq_{cx} X_1^2$. *(For example,* X_1^1 *can be sum of two independent exponential r.v. with mean* $1/2$ *and* X_1^2 *be an exponential r.v. with mean* 1.*) Define the Lévy bases as follows:*

$$L_j(A) = \sum_{x_i \in A} X_i^j,$$

where A is a bBs of $\mathbb{R}^d$ *and* $j = 1, 2$. *By 5.1.4 and the fact that* $X_1^1 \leq_{cx} X_1^2$ *it follows that* $L_1 \leq_{dcx} L_2$.

Example 5.1.6. *Let* $\tilde{\Phi} = \sum_i \varepsilon_{(x_i, Z_i)}$ *be an homogeneous Poisson pp on* $\mathbb{R}^d$ *marked by independent non-negative random variables* $\{Z_i\}$ *with mean* λ_0. *Consider two random measures* $\Lambda_1 = \sum_{(x_i,Z_i)\in\tilde{\Phi}} \lambda_0 \varepsilon_{x_i}$ *and* $\Lambda_2 = \sum_{(x_i,Z_i)\in\tilde{\Phi}} Z_i \varepsilon_{x_i}$. *Note that* L_i, $i = 1, 2$ *are Lévy bases. By 5.1.4 and the fact that* $\lambda_0 \leq_{cx} Z_i$, *conditioning on the number of points and using the same arguments as in the proof of the second statement of Proposition 4.2.3, one can prove that* $\Lambda_1 \leq_{dcx} \Lambda_2$.

5.1.3 Poisson cluster point processes

By Poisson cluster pp, we understand a LCP with the Lévy basis being a Poisson pp. This class deserves a separate mention due to the generality of the ordering results that are

possible.

We shall now give an example of a parametric family of *dcx* ordered Poisson cluster pp. Fix $\lambda > 0$. Let $\Phi_c, c > 0$ be a family of homogeneous Poisson pp on $\mathbb{R}^d$ of intensity $c\lambda$. Let a non-negative function $h : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be given and consider a family of shot noise fields $\lambda_c(y) = \int_{\mathbb{R}^d} (h(x,y)/c)\ \Phi_c(dx)$, which are assumed a.s. locally Riemann integrable with $\int_B \mathsf{E}(\lambda_c(y))\, dy < \infty$ for bBs B.

Proposition 5.1.7. *The family of shot-noise fields $\{\lambda_c(y)\}_{y\in\mathbb{R}^d}$ is decreasing in dcx, i.e., for $0 < c_1 \le c_2$ we have $\{\lambda_{c_2}(y)\} \le_{dcx} \{\lambda_{c_1}(y)\}$. Consequently $Cox(\lambda_{c_2}) \le_{dcx} Cox(\lambda_{c_1})$.*

Proof. Note that $\{\lambda_c(x)\}$ can be seen as a shot-noise field generated by the response function h and the Levy basis $L_c = (1/c)\Phi_{\lambda c}$. By 5.1.4 and Theorem 4.1.3, it is enough to prove that $L_{c_2}(A) \le_{cx} L_{c_1}(A)$ for A bBs and $c_2 > c_1 > 0$.

Since, $X \le_{cx} Y$ implies that $aX \le_{cx} aY$ for all scalars $a > 0$, it suffices to prove that $L_{ca}(A) \le_{cx} L_a(A)$ for A bBs and $c > 1, a > 0$. This essentially boils down to proving that $N_{ca} \le_{cx} cN_a,\ c > 1, a > 0$, where N_a stands for a Poisson r.v. with mean a. Let $\{X_i^n\}_{1\le i\le n}$ and $\{Y_i^n\}_{1\le i\le n},\ n \ge 1$ be i.i.d. sequences of Bernoulli r.v's with probability of success ca/n and a/n, respectively, with $n \ge ca$. Let $X^n = \sum_{i=1}^n X_i^n$ and $Y^n = \sum_{i=1}^n cY_i^n$. It is well known that X^n, Y^n converge weakly and in expectations to N_{ca}, N_a respectively, as $n \to \infty$. As convex order preserves weak convergence with L_1-convergence (see [Müller and Stoyan 2002, Theorem 3.4.6] which is similar to 2.3.8), we need to only prove that $X^n \le_{cx} cY^n$. Due to 2.3.6 and the independence of summands, it is enough to prove that $X_i^n \le_{cx} cY_i^n$, which we shall do in what follows. Let f be a convex and differentiable function. Define $g(c) := \mathsf{E}f(X_i^n) - \mathsf{E}f(cY_i^n) = \frac{a}{n}\{c(f(1)-f(0)) - f(c) + f(0)\}$. Note that $g(1) = 0$. Hence, our proof is complete if we show that g is decreasing in $c > 1$. Indeed,

$$\begin{aligned} g'(c) &= \frac{a}{n}\{(f(1) - f(0)) - f'(c)\} \\ &= \frac{a}{n}\{f'(b) - f'(c)\} \le 0, \quad (b < c) \end{aligned}$$

where $b \in (0,1)$ by mean-value theorem and f' is increasing due to convexity. □

Since Poisson pp can be seen as the limit of the parametric pp at infinity, it is expected that Poisson pp can be compared in *dcx* order to Poisson cluster pp. For rest of the section, assume that $h(x)$ is a non-negative measurable function such that $\int_{\mathbb{R}^d} h(x)dx = \lambda_0 < \infty$.

Proposition 5.1.8. *Let Φ and Φ' be homogeneous Poisson pp on $\mathbb{R}^d$ with intensities $\lambda < \infty$ and $\lambda \times \lambda_0$ respectively. Define $\mu(y) = \sum_{X_i \in \Phi} h(X_i - y)$. Let Φ'' be Cox($\mu(x)$). Assume that $\mu(y)$ is a.s. locally Riemann integrable and $\mathsf{E}(\mu(y)) = \mathsf{E}(\mu(0)) < \infty$. Then $\Phi' \leq_{dcx} \Phi''$.*

Proof. By the last statement of 5.1.4 we have $\lambda\, dx \leq_{dcx} \Phi(dx)$. Note that $\lambda \times \lambda_0 = \int_{\mathbb{R}^d} h(x-y)\, \lambda dx$ and thus by the second statement of Theorem 4.1.3 (note the assumption $\mathsf{E}(\mu(y)) < \infty$) $\{\lambda \times \lambda_0\} \leq_{dcx} \{\mu(y)\}$, where the *dcx* smaller field is a deterministic, constant. The result follows now from Proposition 4.2.10 and the second statement of 4.2.7 by assumption that $\mu(y)$ is a.s. Riemann integrable and observing that $\mathsf{E}(\int_A \mu(y)\, dy) = \mathsf{E}(\mu(0)) \int_A dy < \infty$ for all bBs A. □

Remark 5.1.9. *Consider Poisson pp Φ' and Cox(μ) as in Proposition 5.1.8. It is known that the Palm version (given a point at the origin) of Φ' can be constructed taking $\Phi' + \varepsilon_0$. By [Møller 2003, Proposition 2], analogously, Palm version of Cox(μ) can be taken as Cox(μ) $+ \varepsilon_0 + \Phi''$, where Φ'' is an independent of Cox(μ) Poisson pp with intensity $h(y-\xi)$ where ξ is sampled from the distribution $h(dx)/\int h(y)dy$. This shows that one cannot expect dcx ordering of the Palm versions of Φ' and Cox(μ).*

5.1.4 Log Cox point processes

This class of pp are defined by the logarithm of their intensity fields.

An extension of LCP studied in [Hellmund et al. 2008] is the Log-Lévy driven Cox process (LLCPs). Under the notation of the previous subsection, a Cox pp Φ is said to be a LLCP if it has a intensity field of the form

$$\lambda(y) = \exp\left(\int_{\mathbb{R}^d} k(x,y) L(dx)\right).$$

[Hellmund et al. 2008] allows for negative kernels and signed Lévy measures but these do not fit into our framework. Suppose that $L_1 \leq_{idcx} L_2$, then $\Phi_1 \leq_{idcx} \Phi_2$ where $\Phi_i, i = 1, 2$ are the respective LLCPs of $L_i, i = 1, 2$ with kernel $k(.,.)$. These are simple consequences of Theorem 4.1.3 and the exponential function being *icx*.

Another class is the Log-Gaussian Cox process (LGCPs)(see [Møller et al. 1998]). A pp Φ is said to be a LGCP if its intensity field is $\lambda(y) = \exp\{X(y)\}$ where $\{X(y)\}$ is a Gaussian random field. Suppose $\{X_i(y)\}, i = 1, 2$ are two Gaussian random fields, then

$\{X_1(y)\} \leq_{idcx} \{X_2(y)\}$ if and only if $\mathsf{E}(X_1(y_1)) \leq \mathsf{E}(X_2(y_1))$ and $\mathsf{Cov}(X_1(y_1), X_1(y_2)) \leq \mathsf{Cov}(X_2(y_1), X_2(y_2))$ for all $y_1, y_2 \in \mathbb{R}^d$ (see [Müller and Stoyan 2002, Theorem 3.13.6]). From the composition rules of *idcx* order ($f(g(.))$ is *idcx* whenever f and g are ; use partial derivative conditions of 2.3.2 to verify this) is *idcx*, it is clear that *idcx* ordering of Gaussian random fields implies *idcx* ordering of the corresponding LGCPs. An example of a parametric family of *dcx* ordered Gaussian random fields from [Miyoshi 2004, Sec 4] is presented in Section 5.1.6.

5.1.5 Generalized shot-noise Cox processes (GNSCPs)

This class of Cox pp was first introduced and its various statistics were studied in [Møller and Torrisi 2005]. In simple terms, these are Cox pp whose random intensity field is a shot-noise field of a pp We say a Cox pp is GNSCP if the random intensity field $\{\lambda(y)\}_{y\in\mathbb{R}^d}$ driving the Cox pp is of the following form : $\lambda(y) = \sum_j \gamma_j k_{b_j}(c_j, y)$ where $(c_j, b_j, \gamma_j) \in \Phi$, a pp on $\mathbb{R}^d \times (0,\infty) \times (0,\infty)$. Also we impose the following condition on the kernel k : $k_{b_j}(c_j, y)\frac{k_1(c_j/b_j, y/b_j)}{b_j^d}$ where $k_1(c_j, .)$ is a density with respect to the Lebesgue measure on $\mathbb{R}^d$. We shall denote the GNSCP driven by Φ as Φ^G. This class includes various known pp such as Neyman-Scott pp, Thomas pp, Matérn Cluster pp among others. The case when b_j's are constants and $\{(c_j, \gamma_j)\}$ is a Poisson pp is called as Shot Noise Cox process (See [Møller 2003]). Shot Noise Cox process are also LCPs. Suppose two pp $\Phi_1 \leq_{dcx\,(\text{resp. } idcx;\, idcv)} \Phi_2$, then from Theorem 4.1.3, we infer that $\Phi_1^G \leq_{dcx\,(\text{resp. } idcx;\, idcv)} \Phi_2^G$.

5.1.6 Parametric family of *dcx* ordered fields

We shall now construct a parametric family of *dcx* ordered random fields starting with the following regularity condition on a random field (see [Miyoshi 2004, Miyoshi and Rolski 2004]). This section is based on [Miyoshi 2004, Section 4].

Definition 5.1.10. *A stationary and isotropic random field $\{X(s)\}_{s\in\mathbb{R}^d}$ is said to be $dcx(resp.idcx; idcv)-$ regular if for any $k \in \mathbb{N}$ and any $s_1, \ldots, s_k, t_1 \ldots, t_k \in \mathbb{R}^d$ such that $\|s_i - s_j\| \leq \|t_i - t_j\|, 1 \leq i, j \leq k$, the following holds :*

$$(X(t_1), \ldots, X(t_k)) \leq_{dcx(resp.idcx;idcv)} (X(s_1), \ldots, X(s_k)).$$

The following follows easily from the definition and is our main tool in constructing *dcx* ordered random fields.

Lemma 5.1.11 ([Miyoshi 2004])**.** *Let $\{X(s)\}_{s\in\mathbb{R}^d}$ be a stationary and isotropic dcx(resp.idcx; idcv) regular random field. Then the family of random fields $\{X_c(s)\}_{s\in\mathbb{R}^d}$ for $c > 0$ defined by $X_c(s) = X(cs)$ are dcx(resp.idcx; idcv) decreasing in c i.e, $\{X_{c_1}(s)\}_{s\in\mathbb{R}^d} \leq \{X_{c_2}(s)\}_{s\in\mathbb{R}^d}$ for $c_1 > c_2 > 0$.*

Example 5.1.12. *Let $\{X(s)\}_{s\in\mathbb{R}^d}$ be a stationary and isotropic Gaussian random field. Then $\mathsf{E}(X(s))$ is a constant and $\mathsf{Cov}\,(X(s), X(t)) = C(\|s-t\|)$ for a function $C(.)$. From [Müller and Stoyan 2002, Theorem 3.13.6], $\{X(s)\}_{s\in\mathbb{R}^d}$ is dcx regular iff $C(.)$ is decreasing. Thus the random intensity field $Y(s) = \exp\{-X(s)\}$ is ddcx regular if $C(.)$ is decreasing.*

The intensity fields corresponding to Poisson pp and the mixed Poisson pp can be seen as extreme cases of a Cox intensity field. Let $\{\lambda(s)\}_{s\in\mathbb{R}^d}$ be a *dcx* regular random field with a.s. continuity at O and for $c > 0$, let $\Lambda_c(.)$ denote the intensity measure with $\{\lambda_c(s)\}_{s\in\mathbb{R}^d}$ as the intensity field and $\overline{\lambda} = \mathsf{E}(\lambda(s))$. Then for any bBs B of $\mathbb{R}^d$,

$$\lim_{c\to 0} \Lambda_c(B) = \lim_{c\to 0} \int_{s\in B} \lambda(cs)ds = \lambda(O)\|B\|, \ \ a.s.$$

where O denotes the origin and if $\Lambda_1(.)$ is an ergodic random measure,

$$\lim_{c\to\infty} \Lambda_c(B) = \lim_{c\to\infty} \int_{s\in B} \lambda(cs)ds = \lim_{c\to\infty} \frac{\|B\|}{\|cB\|} \int_{s\in cB} \lambda(s)ds = \overline{\lambda}\|B\|. \ a.s. \tag{5.4}$$

Thus Proposition 5.1.1 can be seen as the the above comparison result (5.1.11) for the extremal cases.

5.1.7 Perturbed point processes

Let Φ be a pp. Given Φ, consider the following family of mutually independent random variables and vectors $\{N_X, Y_{iX} : X \in \Phi\,, i = 1, 2, \ldots\}$, where *the replication field* $\mathcal{N} = \{N_X\}_{X\in\Phi}$ is a collection of i.i.d integer-valued random variables with finite expectations and for each $X \in \Phi$, $\{Y_{iX} : i = 1, 2, \ldots\}$ are identically distributed vectors in $\mathbb{R}^d$, with Y_X denoting a generic copy, such that *a.s.*, $\sum_{X\in\Phi} \mathsf{P}\,(X + Y_X \in A) < \infty$ for any bounded Borel subset $A \subset \mathbb{R}^d$. Note that the following assumption suffices for the same :

$\int_{\mathbb{R}^d} \mathsf{P}\left(x + Y_x \in A\right) \alpha(dx) < \infty$, where we recall that $\alpha(.) = \mathsf{E}(\Phi)\,(.)$ is the mean measure of the pp. The condition guarantees that the following pattern of points in $\mathbb{R}^d$ is locally finite and thus can be considered as a pp:

$$\Phi(\mathcal{N}) = \bigcup_{X \in \Phi} \bigcup_{j=1}^{N_X} \{X + Y_{jX}\}\,, \tag{5.5}$$

where the inner union is interpreted as $\emptyset$ when $N_X = 0$. Point process $\Phi(\mathcal{N})$ can be seen as replicating and dispersing points from the lattice Φ, with the number of point replications driven by the field $\mathcal{N}$. We will call $\Phi(\mathcal{N})$ the *perturbed point process* of Φ. The class of perturbed pp encompasses as a special case some of the perturbed lattice pp considered in [Sodin and Tsirelson 2004] and also, surprisingly enough, homogeneous Poisson pp. The former shall be shown *dcx* smaller that the latter.

Proposition 5.1.13. *Consider a pp Φ and two replication fields $\mathcal{N}_i$, $i = 1, 2$ such that $\mathcal{N}_1 \leq_{dcx} \mathcal{N}_2$ (due to the independence this is equivalent to usual convex ordering $N_{1X} \leq_{cx} N_{2X}$ for all $X \in \Phi$, where $\mathcal{N}_i = \{N_{iX}\}$). Then $\Phi(\mathcal{N}_1) \leq_{dcx} \Phi(\mathcal{N}_2)$ where the same perturbation vectors $\{Y_X\}_{X \in \Phi}$ are used for both.*

Proof. Let $\Phi_{iX} = \bigcup_{j=1}^{N_{iX}} \{X + Y_{jX}\}$ for $i = 1, 2$ and every $X \in \Phi$. Thus conditioned on Φ, $\Phi(\mathcal{N}_i)$, $i = 1, 2$ are independent superpositions of Φ_{iX} for $X \in \Phi$ and hence by Propn. 4.2.3(4) and 2.3.8, it is enough to show that conditioned on Φ, $\Phi_{1X} \leq_{dcx} \Phi_{2X}$ for every $X \in \Phi$.

Let $A_1, \ldots, A_j$ be disjoint bounded Borel subsets and $f : \mathbb{R}^j \to \mathbb{R}$, a *dcx* function. Define $g : \mathbb{Z}^+ \to \mathbb{R}$ as $g(k) = \mathsf{E}\Big(f(\sum_{i=1}^{k}(\mathbf{1}[Y_{iX} \in A_1 - X], \ldots, \mathbf{1}[Y_{iX} \in A_j - X]))|\Phi\Big)$. Note that $\mathsf{E}(g(N_{iX})|\Phi) = \mathsf{E}(f(\Phi_{iX}(A_1), \ldots, \Phi_{iX}(A_j))|\Phi)$ for $i = 1, 2$, $X \in \Phi$. Thus by assumption, the proof is complete once we show that $g(k)$ is a *dcx* function.

Take a quadruple $(p, q, n, m) \in \mathbb{N}^4$ with $p \leq n, m \leq q$ such that $p + q = n + m$ and without loss of generality assume $n \leq m$. For $k \leq l$ define

$$G(k, l) := \sum_{i=k+1}^{l} (\mathbf{1}[Y_{iX} \in A_1 - X], \ldots, \mathbf{1}[Y_{iX} \in A_j - X])$$

Note that $G(p, n)$ and $G(m, q)$ have the same distribution and that $G(0, p), G(p, n), G(m, q)$

are mutually independent. Thus,

$$\begin{aligned} g(n)+g(m) &= \mathsf{E}(f(G(0,p)+G(p,n))|\Phi)+\mathsf{E}(f(G(0,p)+G(p,n)+G(n,m))|\Phi) \\ &= \mathsf{E}(f(G(0,p)+G(m,q))|\Phi)+\mathsf{E}(f(G(0,p)+G(p,n)+G(n,m))|\Phi) \\ &\leq \mathsf{E}(f(G(0,p))|\Phi)+\mathsf{E}(f(G(0,p)+G(p,n)+G(n,m)+G(m,q))|\Phi) \\ &= g(p)+g(q)\,, \end{aligned}$$

where the inequality follows from the assumption that f is *dcx*. This completes the proof. □

Now,we shall discuss some specific examples in the above framework that served as a motivation. The first is to recast Poisson cluster pp within this set-up and the second is a minor extension of the 'toy models' in [Sodin and Tsirelson 2004].

Example 5.1.14. *Let Φ be a Poisson (λ) pp, $N_{1X} \equiv 1$, N_{2X}, be i.i.d. Poisson variables with parameter 1 for all $X \in \Phi$ and Y_X be i.i.d. random vectors with density $h(.)$. Then $\Phi(\mathcal{N}_1)$ is also a Poisson (λ) pp and $\Phi(\mathcal{N}_2)$ is a Cox pp with random intensity field $\lambda(x) = \sum_{X\in\Phi} h(x-X)$. The latter pp is nothing but the Poisson cluster pp introduced in Section 5.1.3.*

Example 5.1.15. *Let $\Phi = \mathbb{Z}^d$, $N_{1z} \equiv 1$ for all $z \in \mathbb{Z}^d$ and N_{2z}, be i.i.d. Poisson variables with parameter 1. Moreover, let $Y_X, z \in \mathbb{Z}^d$ be i.i.d. random vectors with some given distribution $G(dx)$ on $\mathbb{R}^d$. With these choice of parameters, $\Phi(\mathcal{N}_1)$ is the perturbed lattice as considered in [Sodin and Tsirelson 2004] and $\Phi(\mathcal{N}_2)$ is Cox(Λ) pp, where $\Lambda(dx)\sum_{z\in\mathbb{Z}^d} G(dx-z)$. By Proposition 5.1.13, the former process is dcx smaller than the latter (indeed, $1 \leq_{dcx} N$, where N is a random variable with mean 1).*

It is also clear that if for $m_i \in \mathbb{Z}, 1 \leq i \leq d$, we set $G(dx) = \mathbf{1}\{x \in \prod_{i=1}^d [0,m_i]\}dx/\left(\prod_{i=1}^d m_i\right)$, then $\Lambda(dx) = dx$ i.e., $\Phi(\mathcal{N}_2)$ is Poisson pp with intensity 1. Thus, perturbed lattice $\Phi(\mathcal{N}_1)$ is dcx smaller than Poisson pp with the same intensity.

Another 'toy' model in [Sodin and Tsirelson 2004] assumes $G(dx)$ being the Gaussian distribution on the plane.

Here is another example of a point process that is lesser in dcx than Poisson : let Φ_1 be the pp on the unit cube $[0,1]^d$ obtained by distributing n points uniformly and independently. Let Φ_2 be the Poisson pp with intensity n. An easy corollary of the above example is $\Phi_1 \leq_{dcx} \Phi_2$.

The interest in perturbed lattices stems from their relations to zeros of Gaussian analytic functions (GAFs) (see [Peres and Virag 2005, Sodin and Tsirelson 2004]). More precisely, [Sodin and Tsirelson 2006] shows that zeros of GAFs have the same distribution as the pp $\bigcup_{z\in\mathbb{Z}^d}\{z + X_z\}$ for a $\mathbb{Z}^2$-shift invariant sequence $\{X_z\}_{z\in\mathbb{Z}^2}$. Simulations and second-moment properties ([Peres and Virag 2005]) indicate that the zero set of GAFs are sparsely distributed than Poisson pp. The above example when seen in the light of the above-mentioned papers, asks the question whether zeros of GAF are less in *dcx* than Poisson pp ? Before giving further motivation to study such a question, we present some simulations of the many perturbed lattices discussed above. In the next chapter, we will also see their corresponding Boolean models.

5.1.8 Ginibre-radii like point process

A class of pp related to the zeros of GAF are determinantal pp and it is natural to ask if they are less in *dcx* than Poisson pp. Here we take the first step towards proof of such a statement for the Ginibre pp, one of the well-known examples of determinantal pp.

Let $\{\Phi_i\}_{i\geq 0}$ be an i.i.d. family of pp on $\mathbb{R}^+$. So, the points of each pp Φ_i can be sequenced based on their distance from the origin. Let Φ be the pp formed *by picking the ith point of* Φ_i for $i \geq 1$. We shall from now on abbreviate $\Phi([0, b])$ by $\Phi(b)$ for $b > 0$ and similarly for other pp used. Note the following representation for $\Phi(b)$ and $\Phi_0(b)$:

$$\Phi(b) = \sum_{k\geq 1} \mathbf{1}[\Phi_k(b) \geq k] \ ; \ \Phi_0(b) = \sum_{k\geq 1} \mathbf{1}[\Phi_0(b) \geq k].$$

Let

$$\Phi^m(b) = \sum_{k=1}^{m} \mathbf{1}[\Phi_k(b) \geq k] \ ; \ \Phi_0^m(b) = \sum_{k=1}^{m} \mathbf{1}[\Phi_0(b) \geq k].$$

By Lorentz's inequality (see the proof of Proposition 5.1.1), it follows that $(\Phi_1(b), \ldots, \Phi_m(b)) \leq_{sm} (\Phi_0(b), \ldots, \Phi_0(b))$ where *sm* stands for supermodular as defined in Section 4.4. Define the $f : \mathbb{R}^m \to \mathbb{R}$ as follows :

$$f(x_1, \ldots, x_m) = \sum_{k=1}^{m} 1 \wedge (x_k - k + 1).$$

It is easy to verify that both f and $-f$ are *sm* since for each $k = 1, \ldots, m$ and for all

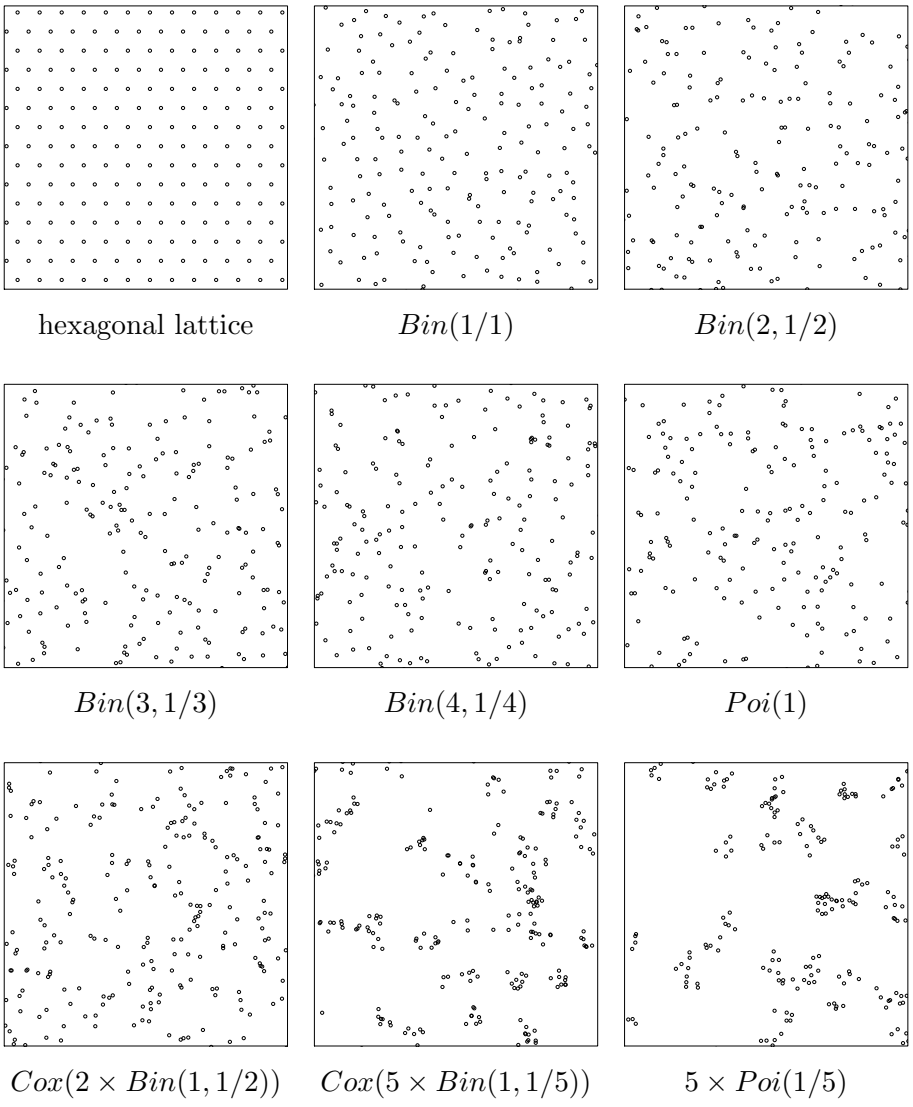

Figure 5.1: "Unperturbed" hexagonal lattice and sub-Poisson perturbed lattices with the number of replicas N having binomial distribution $B(n, 1/n)$. In the last row three examples of super-Poisson perturbed lattices: with N having double stochastic Poisson (Cox) distribution of random mean L having Bernoulli distribution $n \times Bin(1, 1/n)$ (i.e., $\Pr\{ L = n \} = 1 - \Pr\{ L = 0 \} = 1/n$) and "rescaled" Poisson distribution $N \sim n \times Poi(1/n)$.

$x, y \in \mathbb{R}$,

$$1 \wedge (x \wedge y - k + 1) + (1 \wedge (x \vee y - k + 1) = 1 \wedge (x - k + 1) + (1 \wedge (y - k + 1).$$

Thus for all $x, y \in \mathbb{R}^d$, we have that

$$f(x \wedge y) \leq [f(x), f(y)] \leq f(x \vee y), f(x) + f(y) = f(x \wedge y) + f(x \vee y).$$

So for any *cx* $g : \mathbb{R} \to \mathbb{R}$, we have that for all $x, y \in \mathbb{R}^d$,

$$(g \circ f)(x) + (g \circ f)(y) \leq (g \circ f)(x \wedge y) + (g \circ f)(x \vee y)$$

and thus $g \circ f$ is also *sm*. Thus

$$\mathsf{E}(g(\Phi^m(b))) = \mathsf{E}(g \circ f(\Phi_1(b), \ldots, \Phi_m(b))) \leq \mathsf{E}(g \circ f(\Phi_0(b), \ldots, \Phi_0(b))) = \mathsf{E}(g(\Phi_0^m(b))).$$

Hence $\Phi^m(b) \leq_{cx} \Phi_0^m(b)$ and using 2.3.8, we get that $\Phi(b) \leq_{cx} \Phi_0(b)$. To complete the proof $\Phi \leq_{dcx} \Phi_0$, one would require a multi-variate generalization of Lorentz's inequality which we have been unable to prove.

We shall now explain the reasons for considering the above pp Φ.

Example 5.1.16. *If we assume that Φ_i above are Poisson pp, then Φ is known to be a representation of the pp of the squared radii $|\Phi_G|^2 = \{|X_n|^2 \ : \ X_n \in \Phi_G\}$ of the* Ginibre process Φ_G *(see [Kostlan 1992, Ben Hough et al. 2006]). The Ginibre point process ([Ben Hough et al. 2006, Example 15]) is defined as the limiting point process of the following sequence Φ_{G_n} of ensembles. Φ_{G_n} is the ensemble of the n eigenvalues of a $n \times n$ matrix with i.i.d. standard normal entries. This was shown to be a* determinantal *process in [Ginibre 1965]. It has been observed in simulations that this determinental pp (see Figure 1.2.1) exhibits less clustering than the homogeneous Poisson pp.*

5.2 Applications to wireless communication networks

From the point of view of applications of our main result (Theorem 4.1.3), what remains is examples of interesting *dcx* functions. In what follows, we will provide such functions arising in the context of wireless networks. In many of the models we have assumed ordered point processes with i.i.d. marks. However due to Proposition 4.2.11, the results hold for independently marked Cox pp provided the respective intensity measures are ordered. The applications here concern with functionals of the point process that mainly depend on the local configuration of the points. Comparison of some functionals on the global properties

of the pp will be analyzed in Chapter 6. All the point processes in this section will be lie in the Euclidean space $\mathbb{R}^d$ for some $d \geq 1$.

5.2.1 Coverage number in Boolean model

The Boolean model $C(\Phi, G)$ defined earlier (see Definition 4.3.4) is also the main object of analysis in the theory of Coverage processes (see [Hall 1988]). Applications of Boolean model and coverage processes have been already discussed in Section 1.1.2. For $\tilde{\Phi}$ as in the Definition 4.3.4 of the Boolean model, denote by $V(y) = \sum_{(X_i,G_i)\in\tilde{\Phi}} \mathbf{1}[y \in X_i + G_i]$ the number of grains covering $y \in \mathbb{R}^d$. Denote by $\psi(s_1, \ldots, s_n)$ the *joint probability generating functional (p.g.f) of the number of grains* covering locations $y_1, \ldots, y_n \in \mathbb{R}^d$ i.e, $\psi(s_1, \ldots, s_n) = \mathsf{E}\Big(\prod_{j=1}^n s_j^{V(y_j)}\Big)$, $s_j \geq 0$, $j = 1, \ldots, n$.

Thus the following result follows from Theorem 4.1.3, Proposition 4.2.3, Proposition 4.2.11 once we note that the the function $g(v_1, \ldots, v_n) = \prod_j^n s_j^{v_j}$ is *idcx* when $s_j \geq 1$ for all $j = 1, \ldots, n$ while it is *ddcx* when $0 \leq s_j \leq 1$ for all j. Also note that the function $x^\beta \vee 0, x \in \mathbb{R}$ is *idcx* for $\beta \geq 1$ (Use 2.3.2).

Corollary 5.2.1. *Let Φ_i, $i = 1, 2$ be a simple pp (of germs) on $\mathbb{R}^d$. Consider the corresponding Boolean models with the typical grain G and, as above, denote the respective coverage number fields by $\{V_i(y)\}$ and their p.g.f by ψ_i. If $\Phi_1 \leq_{dcx\,(resp.\,idcx;\,idcv)} \Phi_2$ then $\{V_1(y)\} \leq_{dcx\,(resp.\,idcx;\,idcv)} \{V_2(y)\}$, with the result for dcx holding provided $\mathsf{E}(V_i(y)) < \infty$ for all y. In particular, if $\Phi_1 \leq_{idcx} \Phi_2$ then $\mathsf{E}(V_1(y)^\beta) \leq \mathsf{E}(V_2(y)^\beta)$ for all $\beta \geq 1$. If $\Phi_1 \leq_{idcx\,(resp.\,ddcx)} \Phi_2$ then $\psi_1(s_1, \ldots, s_n) \leq \psi_2(s_1, \ldots, s_n)$ for $s_j \geq 1$ (resp. $s_j \leq 1$) $j = 1, \ldots, n$.*

Note that $1 - \psi(0, \ldots, 0)$ represents the expected *coverage measure*, i.e.,the probability whether the locations $y_1, \ldots, y_n$ are covered by at least one grain. In [Hall 1988, Section 3.8] it is shown that expected one-point coverage (or volume fraction in case of stationary pp) for a stationary Cox pp and some clustered pp is lower than that of a stationary, homogeneous Poisson pp.

5.2.2 Typical vertex degrees in geometric intersection graphs

Recall that, a random geometric graph is defined as a graph with $\Phi \subset \mathbb{R}^d, d \geq 1$ as the vertex set and the edge-set $E = \{\{X_i, X_j\} : |X_i - X_j| \leq r\}$. Clearly this is related to the Boolean model defined in the previous subsection. The obvious generalization of

RGG to arbitrary shapes instead of balls of fixed radius is called an *geometric intersection graph*. One of the objects of interest in a geometric intersection graph is the typical degree. Under the notation of the previous subsection, the typical degree ($deg(\Phi, G)$) for a geometric intersection graph formed by a stationary pp Φ and grain distribution G is $deg(\Phi, G) = \frac{1}{\lambda|A|}\sum_{X_i, X_j \in \Phi} \mathbf{1}[X_i \in U]\mathbf{1}[X_i \neq X_j]\mathbf{1}[(X_i + G_i) \cap (X_j + G_j) \neq \emptyset]$, where $U = [0,1]^d$ is the unit cube. If $G = B_0(r), r > 0$, then $\mathsf{E}(deg(\Phi, G)) = K(r)$ is the *Ripley's K function* defined in Section 4.2.2. The following result follows easily from Theorem 4.1.3, Proposition 4.2.4 and Proposition 4.2.11.

Corollary 5.2.2. *Suppose that simple pp $\Phi_1 \leq_{dcx} \Phi_2$, then $deg(\Phi_1, G) \leq_{idcx} deg(\Phi_2, G)$.*

5.2.3 Connectivity and capacity in signal to interference noise ratio (SINR) graph

In Section 1.1.2, we have already introduced SINR graphs informally. Now we shall place them in a more formal set-up and give comparison results on some simple properties of the network. The more complex properties of the network such as percolation will be analyzed in the next chapter. As said earlier, the pp considered in this section are assumed to be lie in the Euclidean space $\mathbb{R}^d, d \geq 1$. Let the set of emitters be a pp $\Phi_e = \{Y_i^e\}_{i=1}^n$ and the set of receivers be a pp $\Phi_r = \{Y_i^r\}_{i=1}^n$. For the ease of stating the results, we have assumed an enumeration of the points of the two pp. Let $l : \mathbb{R}_+ \to \mathbb{R}_+$ denote the omni-directional path-loss function. We assume that $l(.)$ is a non-increasing function. Let $\{S_{ik}\}$ be independent random variables and independent of the emitter and receiver pp too. The signal received by a receiver Y_k^r from Y_i^e is given by $PS_{ik}l(|Y_i^e - Y_k^r|)$ where P is the power of the signal transmitted by an emitter Y_i^e. The presence of a non-increasing function is justified by the decay of signal power over distances and the random variables S_{ik}'s appear due to the phenomenon of fading (see Section 1.1.2).

Let the set of additional interferers be modeled by an i.i.d. marked pp $\tilde{\Phi} = \sum_j \varepsilon_{(X_j, (Z_j^1, \ldots, Z_j^n))}$, independent of $\{S_{ik}\}, \Phi_e, \Phi_r$, where Z_j^i is the power received by the receiver Y_i^r from the interferer located at X_j. Denote the *background noise* random variable by N.

We say that the signal from Y_i^e is successfully received by Y_i^r if $SINR_i(\tilde{\Phi}) > T$ where $SINR_i(\tilde{\Phi}) := S_{ii}/(N + I_i + V_i)$ with $I_i = \sum_{k \neq i} PS_{ki}l(|Y_i^e - Y_k^r|)$ and $V_i = \sum_j Z_j^i$ is the interference received at Y_i^r from the set of other emitters $\{Y_k^e : k \neq i\}$ and interferers in $\tilde{\Phi}$, respectively. $T > 0$ is some (assume constant) required *SINR threshold*. If we denote by

$p(n)$, the probability of successful reception of signals from each Y_i^e to Y_i^r for $1 \leq i \leq n$, then

$$\begin{aligned} p(n) &= \mathsf{P}(S_{ii} > (N + I_i + V_i)\frac{Tl(|Y_i^e - Y_k^r|)}{P} \quad \forall i = 1, \ldots, n) \\ &= \mathsf{E}(\prod_i \overline{F}_{ii}(\frac{Tl(|Y_i^e - Y_k^r|)}{P}(N + I_i + V_i))), \end{aligned} \tag{5.6}$$

where $\overline{F}_{ii}(s) = \mathsf{P}(S_{ii} > s)$ and the second equality is due to independence. These communicating pairs (Y_i^e, Y_i^r) are called the source-destination pairs.

Given $\Phi_e, \Phi_r, \{I_i : i = 1, \ldots, n\}$ and N, the expression under expectation in (5.6) can be viewed as a function of the value of the shot-noise vector $(V_1, \ldots, V_n)$ evaluated with respect to $\tilde{\Phi}$. Theorem 4.1.3 implies the following result concerning the impact of the structure of the set of interferers $\tilde{\Phi}$ on $p(n)$.

Corollary 5.2.3. *Consider emitters $\Phi_e = \{Y_i^e\}$, receivers $\Phi_r = \{Y_i^r\}$ and powers $\{S_{ki}\}$ as above. Let $\tilde{\Phi}_u$, $u = 1, 2$ be two simple marked pp of interferers. Denote by $p_u(n)$, $u = 1, 2$ the probability of successful reception given by (5.6) and the capacity (per-node throughput) of a network given by $C_u(n) = \frac{1}{n}\sum_{i=1}^n \mathsf{E}\Big(\log(1 + SINR_i(\tilde{\Phi}_u))\Big)$ in the model with the set of interferers $\tilde{\Phi}_u$. Assume the product of tail distribution functions of the received powers $\prod_{i=1}^n \overline{F}_{ii}(s_i)$ be dcx. If $\tilde{\Phi}_1 \leq_{ddcx} \tilde{\Phi}_2$ then $p_1(n) \leq p_2(n)$ and $C_1(n) \leq C_2(n)$.*

Proof. $p_1(n) \leq p_2(n)$ follows from Theorem 4.1.3 and the fact that the function in (5.6) is a *ddcx* function conditioned on $N, I_i, 1 \leq i \leq n$. The ordering of capacities $C_u(n)$ follows from the ordering of $p_u(n)$ and the following observation : for $u = 1, 2$ and $1 \leq i \leq n$,

$$\mathsf{E}\Big(\log(1 + SINR_i(\tilde{\Phi_u}))\Big) = \int_{\mathbb{R}_+} \mathsf{P}\left(SINR_i(\tilde{\Phi_u}) > e^x - 1\right) dx.$$

Note that the integrand is the same as $p_u(1)$ excepting that we have $e^x - 1$ instead of a T. □

It is quite natural to assume *ddcx* $\prod_{i=1}^n \overline{F}_{ii}(s_i)$. For example *Rayleigh fading* in the radio channel implies S_{ki} are i.i.d. exponential random variables with mean 1. In this case $\prod_{i=1}^n \overline{F}_{ii}(s_i)$ is ddcx. Recently in [Ganti and Haenggi 2009], under the assumption of Rayleigh fading, direct analytical methods have been used to compare the probability of successful reception in Poisson pp and a class of Poisson-Poisson cluster pp known as

Neyman-Scott pp for both stationary and Palm versions. These results relay on explicit expressions for this probability known in the considered cases. Further, it is shown that for a certain choice of parameters, Palm version of the Poisson-Poisson cluster pp has a worse probability of successful reception than the Poisson pp. In our terminology, it simply means that the corresponding Palm versions aren't *ddcx* ordered as the connectivity probability is a *ddcx* function (Eqn. 5.6) of the integral shot-noise fields of the corresponding Palm versions. This strengthens Remark 5.1.9 by showing that *idcx* ordering of Palm versions is the best one can obtain in full generality.

Chapter 6

Percolation and directionally convex ordering of point processes

6.1 Notations

Though we focus on continuum percolation models, but as is the wont in the subject we shall extensively use discrete percolation models. Here we give the generic construction that shall be used with minor modifications often in the chpater. Many percolation models in $\mathbb{R}$ are degenerate and so we shall work with higher dimensional models, i.e, models in $\mathbb{R}^d, d \geq 2$. For $r > 0, x \in \mathbb{R}^d$, define the following subsets of $\mathbb{R}^d$:

$$\begin{aligned} Q^r &:= (-\frac{1}{2r}, \frac{1}{2r}]^d, \ Q^r(x) := x + Q^r; \\ Q_r &:= (-r, r]^d, \ Q_r(x) := x + Q_r, \ Q_{r,3r,\ldots,3r} := [0, r] \times [0, 3r] \times \ldots \times [0, 3r]. \end{aligned}$$

Let $\mathfrak{A} = \{K_z\}_{z \in \mathbb{Z}^d} \subset (\mathbb{K}_d)^{\mathbb{Z}^d}$ be a sequence of compact sets. The three discrete graphs we consider often are the following : For $n \in \mathbb{N}$,

$$\begin{aligned} \mathbb{L}_n^{*d} &:= (\mathbb{Z}_n^d = \frac{1}{n}\mathbb{Z}^d, \mathbb{E}_n^{*d}), \ \mathbb{E}_n^{*d} := \{\langle z_i, z_j \rangle : Q^{\frac{n}{2}}(z_i) \cap Q^{\frac{n}{2}}(z_j) \neq \emptyset\}; \\ \mathbb{L}_n^{*d}(\mathfrak{A}) &:= (\mathbb{Z}_n^d, \mathbb{E}_n^{*d}(\mathfrak{A})), \ \mathbb{E}_n^{*d}(\mathfrak{A}) := \{\langle z_i, z_j \rangle : \{Q^n(z_i) + K_{z_i}\} \cap \{Q^n(z_j) + K_{z_j}\} \neq \emptyset\}; \\ \mathbb{L}_n^{*d}(r) &:= (r\mathbb{Z}^d, \mathbb{E}^{*d}(r)), \ \mathbb{E}^{*d}(r) := \{\langle z_i, z_j \rangle \in (r\mathbb{Z}^d)^2 : Q_r(z_i) \cap Q_r(z_j) \neq \emptyset\}. \end{aligned}$$

A path in $\mathbb{L}_n^{*d}(\mathfrak{A})$ from 0 to ∂Q_m is $\gamma = \langle z_1, \ldots, z_{k+1} \rangle \subset \mathbb{Z}_n^d$ such that for $1 \leq i \leq k$, $z_i \in Q_m, \langle z_i, z_{i+1} \rangle \in \mathbb{E}_n^d(\mathfrak{A}), \{Q^n(z_i) + K_{z_i}\} \cap \partial Q_m = \emptyset$ and $0 \in \{Q^n(z_1) + K_{z_1}\}$,

$\{Q^n(z_{k+1}) + K_{z_{k+1}}\} \cap \partial Q_m \neq \emptyset$. Let $\Upsilon_m^n(\mathfrak{A})$ be the set of all such paths from origin to ∂Q_m in $\mathbb{L}_n^{*d}(\mathfrak{A})$. Let Γ_n be the set of all contours around the origin in $\mathbb{L}_n^{*d}$. Recall that a contour in $\mathbb{L}_n^{*d}$ is a minimal collection of vertices such that any infinite path from the origin has to contain one of these vertices. The minimality condition implies that removal of any vertex from the collection will lead to existence of an infinite path from the origin without any intersection with the remaining vertices in the collection. For a path $\gamma \in \Upsilon_m^n(\mathfrak{A}), \Gamma_n$, we associate $Q_\gamma = \bigcup_{z\in\gamma} Q^n(z)$.

6.2 Comparison of a critical intensity

6.2.1 Percolation of racs

By *percolation of a racs* Ξ, we refer to the existence of an unbounded connected subset (giant component) of Ξ. We say that O percolates in Ξ if the origin O is contained in a giant component. We shall always use W to denote the component containing O. Typically, the giant component is unique. For a set A, let $A^{(r)} := \bigcup_{x\in A} B_x(r)$, the r-th parallel set of A.

Definition 6.2.1. *For a racs Ξ, we define some critical radii for percolation as follows:*

$$\begin{aligned} r_c(\Xi) &:= \inf\{r : \mathsf{P}\left(\Xi^{(r)} \textit{ percolates}\right) > 0\}. \\ \overline{r}_c(\Xi) &:= \inf\{r : \forall n, \sum_{\gamma\in\Gamma_n} \mathsf{P}\left(\Xi^{(r)} \cap Q_\gamma = \emptyset\right) < \infty\}. \end{aligned}$$

For an ergodic racs, it is the case that $\mathsf{P}\left(\Xi^{(r)} \text{ percolates}\right) \in \{0, 1\}$ for all $r > 0$. Further, assuming stationarity of the racs, we get that $\mathsf{P}\left(\Xi^{(r)} \text{ percolates}\right) = 1$ iff $\mathsf{P}\left(O \text{ percolates in } \Xi^{(r)}\right) > 0$. Thus for ergodic racs, there is a strict phase-transition.

$r_c(\Xi)$ is similar to the critical radius defined for percolation in various continuum percolation models and the second critical radius $\overline{r}_c(\Xi)$ defined above might be seen as a critical radius corresponding to a phase transition when some suitable discrete site percolation models approximating the racs start percolating through the Peierls argument (cf the proof of Lemma 6.2.2). In consequence $\overline{r}_c(\Xi)$, is an upper bound for the actual critical radii as we will see now.

Lemma 6.2.2. *For a racs Ξ, $\overline{r}_c(\Xi) \geq r_c(\Xi)$.*

Proof. The proof as indicated above shall use Peierls argument on a suitable discrete approximation. Consider the graph $\mathbb{L}_n^{*d}$. We shall define a random field $\mathcal{X}_r := \{X_r(z)\}_{z\in\mathbb{Z}_n^d}$ such that $X_r(z) = \mathbf{1}[\Xi^{(r)} \cap Q_n(z) \neq \emptyset]$. By Peierls argument (cf. the proof of Lemma 2.2.2) for $r > \overline{r}_c(\Xi)$, $\mathcal{X}_r$ percolates in $\mathbb{L}_n^{*d}$ a.s. for all n. Thus for all n, $\Xi^{(r+\frac{\sqrt{d}}{n})}$ percolates a.s. and so $r + \frac{\sqrt{d}}{n} > r_c(\Xi)$. Hence, $r \geq r_c(\Xi)$. □

The following lemma follows directly from the definition of $\overline{r}_c(\Xi)$.

Lemma 6.2.3. *Let Ξ_1, Ξ_2 be two racs such that their capacity functionals satisfy $\mathbf{T}_{\Xi_1}(B) \leq \mathbf{T}_{\Xi_2}(B)$ for all bBs B. Then, $\overline{r}_c(\Xi_1) \geq \overline{r}_c(\Xi_2)$.*

6.2.2 Percolation of level sets of random fields.

While little can be said about percolative properties of racs in general, much can be said about percolation of level sets of random fields and also about percolation of Boolean model (see Section 6.2.3). The interest is to characterize the levels (h) for percolation of the level sets.

For a real valued stationary random field $\mathcal{X} = \{X(x)\}_{x\in\mathbb{R}^d}$, we define its lower and upper level sets respectively as follows: $S_{\leq h}(\mathcal{X}) := \{x : X(x) \leq h\}$, $S_{>h}(\mathcal{X}) := \{x : X(x) > h\}$. We denote the closure and the interior of a set A by $\overline{A}$ and $\widehat{A}$ respectively. We will also work with the closures or the interiors of the level sets : $\overline{S}_{>h}(\mathcal{X}) := \overline{S_{>h}(\mathcal{X})}$, $\widehat{S}_{>h}(\mathcal{X}) := \widehat{S_{>h}(\mathcal{X})}$ and similarly for $S_{\leq h}(\mathcal{X})$. If a random field $\{X(x)\}_{x\in\mathbb{R}^d}$ a.s. lower semi-continuous (LSC) i.e., if a.s. its sample paths are lower semi-continuous, then $\widehat{S}_{>h}(\mathcal{X}) = S_{>h}(\mathcal{X})$ and $\overline{S}_{\leq h}(\mathcal{X}) = S_{\leq h}(\mathcal{X})$. We refer the reader to [Hall 1988, Chapter 3] for more details on random open sets. Note that when $\mathcal{X} = U_{\tilde{\Phi}}$, $S_{\leq 0}(\mathcal{X}) = \mathbb{R}^d \setminus C(\Phi, G)$ and $S_{>0}(\mathcal{X}) = C(\Phi, G)$, the Boolean model (see Definition 4.3.4).

For a given constant $0 \leq R < \infty$, denote

$$g^n(\mathcal{X}, h, R) := \left(\mathsf{P}\left(\widehat{S}_{>h}(\mathcal{X}) \cap Q_n(0) \neq \emptyset\right)\right)^{\frac{1}{n(2R+3)^d}}.$$

Definition 6.2.4. *Define*

$$\begin{aligned}
h_c^l(\mathcal{X}) &:= \inf\{h : \mathsf{P}\left(\overline{S}_{\leq h}(\mathcal{X}) \text{ percolates}\right) > 0\}, \\
h_c^u(\mathcal{X}) &:= \sup\{h : \mathsf{P}\left(\overline{S}_{>h}(\mathcal{X}) \text{ percolates}\right) > 0\}, \\
\overline{h}_c^l(\mathcal{X}, R) &:= \inf\{h : \exists n, g^n(\mathcal{X}, h, R) < (3^d - 2)^{-1}\}, \\
\underline{h}_c^u(\mathcal{X}) &:= \sup\{h : \forall n, \sum_{\gamma \in \Gamma_n} \mathsf{P}\left(\widehat{S}_{>h}(\mathcal{X}) \cap Q_\gamma = \emptyset\right) < \infty\}, \\
\overline{h}_c^u(\mathcal{X}) &:= \sup\{h : \forall r > 0, \mathsf{P}\left(\overline{S}_{>h}^{(r)}(\mathcal{X}) \text{ percolates}\right) > 0\}.
\end{aligned}$$

Remark 6.2.5. *$h_c^l(\mathcal{X})$ and $h_c^u(\mathcal{X})$ are the actual critical levels for the percolation of the closure of the respective lower and upper level sets ,i.e., for the appearance of their giant components. Note that the upper set $\overline{S}_{>h}(\mathcal{X})$ percolates* below *its critical level $h_c^u(\mathcal{X})$. $\overline{h}_c^u(\mathcal{X})$ can be considered as a critical level for $\overline{S}_{>h}(\mathcal{X})$ "nearly percolating" and clearly $\overline{h}_c^u(\mathcal{X}) \geq h_c^u(\mathcal{X})$. We shall see in Remark 6.2.11, cases when $\overline{h}_c^u(\mathcal{X}) = h_c^u(\mathcal{X})$. As for the other two defined critical levels, they correspond to the phase transition when some suitable discrete approximating site percolation models start percolating through the Peierls argument. This suggests that the new critical radii $\overline{h}_c^l(\mathcal{X}, R)$ and $\underline{h}_c^u(\mathcal{X})$ are, respectively, upper and lower bounds for the critical levels $h_c^l(\mathcal{X})$ and $\overline{h}_c^u(\mathcal{X})$, which will be proved under some conditions.*

In fact, the announced inequality for $\overline{h}_c^l(\mathcal{X}, R)$ will be proved under the following assumption regarding the constant R involved in the definition of $g^n(\mathcal{X}, h, R)$. Namely, we say a random field $\mathcal{X}$ is R-independent if for any collection of bounded measurable sets $A_1, \ldots, A_n$ with $d(A_i, A_j) > R$, $\forall i, j$, the $\sigma-$algebras generated by $\{X(x)\}_{X \in A_i}; i = 1, \ldots, n$ are independent.

Lemma 6.2.6. *For a stationary random field $\mathcal{X} = \{X(x)\}_{x \in \mathbb{R}^d}$, we have $\underline{h}_c^u(\mathcal{X}) \leq \overline{h}_c^u(\mathcal{X}) \geq h_c^u(\mathcal{X})$ and further assuming R-independence of $\mathcal{X}$, we have that $\overline{h}_c^l(\mathcal{X}, R) \geq h_c^l(\mathcal{X})$.*

Proof. We begin with the second statement. Defining a site $z \in \mathbb{Z}_n^d$ open if $\overline{S}_{\leq h}(\mathcal{X}) \supset Q_n(z)$, we obtain a site percolation model on $\mathbb{L}_n^{*d}$. Note that the percolation of this model for some $n \geq 1$ implies percolation of $\overline{S}_{\leq h}(\mathcal{X})$. Thus in order to conclude the proof, it is enough to show that for any $h > \overline{h}_c^l(\mathcal{X}, R)$ the auxiliary discrete site percolation model percolates for some $n \geq 1$. This can be shown via Peierls argument (Lemma 2.2.2) by observing that the probability that a given site is closed is equal to $\mathsf{P}\left(\overline{S}_{\leq h}(\mathcal{X}) \not\supset Q_n(0)\right) =$

$\mathsf{P}\left(\widehat{S}_{>h}(\mathcal{X}) \cap Q_n(0) \neq \emptyset\right)$ and for R-independent field $\mathcal{X}(\cdot)$ we have

$$\mathsf{P}\left(\widehat{S}_{>h}(\mathcal{X}) \cap Q_n(z_i) \neq \emptyset, 1 \leq i \leq k\right) \leq (g^n(\mathcal{X}, h))^k .$$

Thus the proof of second statement is complete.

Now, to the first part, define a site $z \in \mathbb{Z}_n^d$ open if $\overline{S}_{>h}(\mathcal{X}) \cap Q_n(z) \neq \emptyset$. Note that for $h < \underline{h}_c^u(\mathcal{X})$, this adjacent discrete site model percolates via the Peierls argument for any $n \geq 1$. Indeed, the probability that a given site is closed satisfies $\mathsf{P}\left(\overline{S}_{>h}(\mathcal{X}) \cap Q_n(z) = \emptyset\right) \leq \mathsf{P}\left(\widehat{S}_{>h}(\mathcal{X}) \cap Q_n(z) = \emptyset\right)$ and thus for $h < \underline{h}_c^u(\mathcal{X})$ the expected number of closed contours around the origin is finite for all $n \geq 1$ (see Lemma 2.2.2). It remains to show, that the fact that the auxiliary discrete site model percolates for every $n \geq 1$ implies percolation of $\overline{S}_{>h}^{(r)}(\mathcal{X})$ for every $r > 0$. For a $n \geq 1$, if $\overline{S}_{>h}(\mathcal{X}) \cap Q_n(z) \neq \emptyset$, then $Q_n(z) \subset \overline{S}_{>h}^{(r)}(\mathcal{X})$ for any $r > \frac{\sqrt{d}}{n}$. Thus percolation in the auxiliary discrete site percolation model for a $n \geq 1$ implies percolation of $\overline{S}_{>h}^{(r)}(\mathcal{X})$ for any $r > \frac{\sqrt{d}}{n}$. This proves that $\underline{h}_c^u(\mathcal{X}) \leq \overline{h}_c^u(\mathcal{X})$ and the other inequality is obvious as mentioned in Remark 6.2.5. □

It is easy to see that $\mathcal{X}_1 \leq_{st} \mathcal{X}_2$ implies ordering of all the critical levels defined above. But we shall show here that a weaker condition suffices for comparison of two bounds for the critical levels. This can be used to ascertain non-degeneracy of the critical level in a random field if the other field has a non-degenerate critical level. Examples satisfying the assumptions follow from Propositions 4.3.5, 4.3.3 and Section 5.1. The starting point is an analogous proposition to Proposition 4.3.5.

Proposition 6.2.7. *Define $\mathbb{K}_0$ to be the class of subsets $B \subset \mathbb{R}^d$ such that B is (a) countable set or (b) such that $B \subset \overline{\widehat{B}}$ or (c) a union of countably many sets satisfying (a) or (b). Let $\mathcal{X}_1, \mathcal{X}_2$ be two LSC random fields on $\mathbb{R}^d$ such that $\mathcal{X}_1 \leq_{lo} \mathcal{X}_2$. Then $\mathbf{T}_{\widehat{S}_{>h}(\mathcal{X}_1)}(B) \leq \mathbf{T}_{\widehat{S}_{>h}(\mathcal{X}_2)}(B)$ for all $B \in \mathbb{K}_0$.*

Proof. Assume first that $B = \{x_1, x_2, \dots\}$ is countable. By *lo* ordering of the random fields, we have

$$\begin{aligned} \mathsf{P}\left(\widehat{S}_{>h}(\mathcal{X}_1) \cap \{x_1, \dots, x_n\} = \emptyset\right) &= \mathsf{P}\left(X_1(x_i) \leq h \forall i\right) \\ &\geq \mathsf{P}\left(X_2(x_i) \leq h, \forall i\right) \\ &= \mathsf{P}\left(\widehat{S}_{>h}(\mathcal{X}_2) \cap \{x_1, \dots, x_n\} = \emptyset\right) . \end{aligned}$$

Moreover, $\bigcap_{n=1}^{\infty}\{\,\widehat{S}_{>h}(\mathcal{X}_i)\cap\{x_1,\ldots,x_n\,\}=\emptyset\}=\{\,\widehat{S}_{>h}(\mathcal{X}_i)\cap B=\emptyset\,\}$ for $i=1,2$. Thus, the result (for countable B) follows by the continuity of probability.

Assume now that $B\subset\overline{\widehat{B}}$. The proposition rests on the following simple observation $\{\widehat{S}_{>h}(\mathcal{X}_i)\cap B=\emptyset\}=\{\widehat{S}_{>h}(\mathcal{X}_i)\cap\mathbb{Q}^d\cap B=\emptyset\}$, where $\mathbb{Q}^d$ is the set of all d-dimensional rational vectors, which reduces the proof to the previously consider case of countable B. In order to prove this fact note first that the inclusion $\subset$ is trivial. Suppose now that $x\in\widehat{S}_{>h}(\mathcal{X}_i)\cap B$. By the assumption on B we have $x\in\overline{\widehat{B}}$ and thus there exist an open neighborhood U of x in $\widehat{S}_{>h}(\mathcal{X}_i)$ in which we can further find a non-empty open set U' satisfying $U'\subset\widehat{S}_{>h}(\mathcal{X}_i)\cap B$ and therefore $\widehat{S}_{>h}(\mathcal{X})\cap B\cap\mathbb{Q}^d\neq\emptyset$.

To show the result under assumption (c), observe that whenever countably many subsets satisfy assumption (a) or (b), their union also satisfies assumption (a) or (b) respectively. Thus without loss of generality, to prove (c), we can consider the case of $B=A\cup A_0$ where $A=\{x_1,\ldots,\}$ is a countable subset and $A_0\subset\overline{\widehat{A_0}}$. Now note that

$$\{\widehat{S}_{>h}(\mathcal{X}_i)\cap B=\emptyset\}=\{\widehat{S}_{>h}(\mathcal{X}_i)\cap\cap A_0=\emptyset\}\cap\{\widehat{S}_{>h}(\mathcal{X}_i)\cap A=\emptyset\}.$$

Now we can complete the proof by using arguments as used in the first two parts of the proof. □

We give here a comparison result for LSC random fields and we shall see in the next section, comparison results for non-LSC random fields (namely, the Boolean model) also.

Theorem 6.2.8. *Let $\mathcal{X}_1,\mathcal{X}_2$ be two stationary LSC random fields on $\mathbb{R}^d$ such that $\mathcal{X}_1\leq_{lo}\mathcal{X}_2$. Then $\underline{h}_c^u(\mathcal{X}_1)\leq\underline{h}_c^u(\mathcal{X}_2)$ and we have that $\overline{h}_c^l(\mathcal{X}_1)\leq\overline{h}_c^l(\mathcal{X}_2)$.*

Proof. From Proposition 6.2.7, we have that $g^n(\mathcal{X}_1,h)\leq g^n(\mathcal{X}_2,h)$ as well as

$$\mathsf{P}\left(\widehat{S}_{>h}(\mathcal{X}_1)\cap Q_\gamma=\emptyset\right)\geq\mathsf{P}\left(\widehat{S}_{>h}(\mathcal{X}_2)\cap Q_\gamma=\emptyset\right)$$

and thus the proof follows by noting that $Q_\gamma\in\mathbb{K}_0$. □

However, in view of Lemma 6.2.6, the comparison result for lower level sets makes sense only when $\mathcal{X}_i, i=1,2$ are R-independent.

6.2.3 Percolation of a Boolean model

For a pp Φ and a random compact set S (i.e, $S \in \mathbb{K}_d$), recall that the Boolean model (Definition 4.3.4) denoted by $C(\Phi, S)$ is the union of 'grains' S_X (i.i.d. distributed as S) centred at 'germs' $X \in \Phi$. Further, we assume throughout that S is a.s. path connected.

Let the set of grains associated with the pp Φ be $\mathfrak{S} = \{S_X\}_{X\in\Phi}$, where S_X are i.i.d distributed as S. We say that the *origin percolates* in $C(\Phi, S)$ if there exists an infinite component $W \subset C(\Phi, S)$ such that the origin $O \in W$. We shall always use W to denote the component containing O. An open path γ in $C(\Phi, S)$ from the origin to ∂Q_m is $\langle X_1, \ldots, X_{k+1}\rangle \subset \Phi$ such that $\forall 1 \leq i < k$, $\{X_i\}_{i\geq 1} \subset Q_m$, $\{X_{i+1}+S_{X_{i+1}}\}\cap\{X_i+S_{X_i}\} \neq \emptyset$ and $\{X_{k+1} + S_{X_{k+1}}\} \cap \partial Q_m \neq \emptyset$, $0 \in \{X_1 + S_{X_1}\}$. Let $N_m(\Phi, S)$ be the number of paths from the origin to ∂Q_m. We call a path $\gamma = \langle X_1, \ldots, X_{k+1}\rangle \subset \Phi$ an open crossing of $Q_{m,3m,\ldots,3m}$ if $\forall 1 < i \leq k$, $\{X_{i+1} + S_{X_{i+1}}\} \cap \{X_i + S_{X_i}\} \neq \emptyset$ and $\{X_1 + S_{X_1}\} \cap L_0^m \neq \emptyset, \{X_{k+1} + S_{X_{k+1}}\} \cap L_m^m \neq \emptyset$, where $L_j^m = \{j\} \times [0, 3m] \times \ldots \times [0, 3m], j \in \mathbb{Z}$. Let $\sigma_m(\Phi, S)$ be the number of open crossings of $Q_{m,3m,\ldots,3m}$. Note that the crossings of $Q_{m,3m,\ldots,3m}$ can be formed totally by points outside $Q_{m,3m,\ldots,3m}$.

We shall now show that our definition of crossing is consistent with that already defined in the literature. Crossing in $C(\Phi, S)$ of a box $B = [0, l_1] \times \ldots \times [0, l_d]$ in the i-th direction was defined in [Meester and Roy 1996, Example 2.2] as a continuous curve $\gamma \in C(\Phi, S)\cap B$ and $\gamma \cap [0, l_1] \times \ldots [0, l_{i-1}] \times \{0\} \times [0, l_{i+1}] \ldots \times [0, l_d] \neq \emptyset, \gamma \cap [0, l_1] \times \ldots [0, l_{i-1}] \times \{l_i\} \times [0, l_{i+1}] \ldots \times [0, l_d] \neq \emptyset$. It is easy to see that the event there exists a crossing of $Q_{m,3m,\ldots,3m}$ in the shortest direction is equivalent to the event $\{\sigma_m(\Phi, S) \geq 1\}$. The above notion of crossing in terms of a continuous curve can be extended to give a definition of a path in $C(\Phi, S)$ from 0 to ∂Q_m as well. Such a definition will again be consistent with ours as in that existence of a path will be equivalent to $\{N_m(\Phi, S) \geq 1\}$. Our definitions can be seen as a way of enumerating the crossings. We have associated a continuous curve γ in $C(\Phi, S)$ to the set of points in Φ through whose grains the curve passes. This consistency between the two definitions will allow us to talk of the critical radii defined below as critical radii for percolation as known in the literature (see [Meester and Roy 1996, Eqns. (3.20) and (3.41)]).

Definition 6.2.9. *For a Boolean model $C(\Phi, S)$, define the critical radii as follows :*

$$\begin{aligned}
r_c(\Phi,S) &:= \inf\{r : \liminf_{m\to\infty} \mathsf{P}\left(N_m(\Phi, S^{(r)}) \geq 1\right) > 0\}\,,\\
r_s(\Phi,S) &:= \inf\{r : \liminf_{m\to\infty} \mathsf{P}\left(\sigma_m(\Phi, S^{(r)}) \geq 1\right) > 0\}\,,\\
\underline{r}_c(\Phi,S) &:= \inf\{r : \liminf_{m\to\infty} \mathsf{E}\Big(N_m(\Phi, S^{(r)})\Big) > 0\}\,,\\
\underline{r}_s(\Phi,S) &:= \inf\{r : \liminf_{m\to\infty} \mathsf{E}\Big(\sigma_m(\Phi, S^{(r)})\Big) > 0\}\,,\\
\overline{r}_c(\Phi,S) &:= \inf\{r : \forall n, \sum_{\gamma\in\Gamma_n} \mathsf{P}\left(C(\Phi, S^{(r)}) \cap Q_\gamma = \emptyset\right) < \infty\}\,.
\end{aligned}$$

When $S \equiv \{O\}$, we shall abbreviate $r_c(\Phi, S)$ by $r_c(\Phi)$. Before going any further let us comment on the defined critical radii.

Remark 6.2.10.

- *We have $C(\Phi,S)^{(r)} = C(\Phi, S^{(r)})$ and so $r_c(C(\Phi,S)) = r_c(\Phi,S)$, the critical radii for the Boolean model as defined above is equivalent to the definition of the critical radii as a racs (see Definition 6.2.1).*

- *If Φ is a stationary pp, it is straightforward to see that*

$$\begin{aligned}
r_c(\Phi,S) &= \inf\{r : \mathsf{P}\left(0 \text{ percolates in } C(\Phi, S^{(r)})\right) > 0\}\,,\\
&= \inf\{r : \mathsf{P}\left(C(\Phi, S^{(r)}) \text{ percolates}\right) > 0\}\,,
\end{aligned}$$

  i.e., that the critical radius corresponding to the appearance of open paths from 0 to ∂Q_m for arbitrarily large m is equal to the actual critical radius for the percolation of the Boolean model when viewed as a racs (Section 6.2.1).

- *If Φ is a Poisson pp and ρ is a.s. bounded positive random variable, then $r_c(\Phi, B_O(\rho)) = r_s(\Phi, B_O(\rho))$ (see Theorem 3.5, [Meester and Roy 1996]), i.e., the actual critical radius of the Boolean model corresponds also to the appearance of open crossings of $Q_{m,3m,\ldots,3m}$ for arbitrarily large m.*

- *The two new critical radii $\underline{r}_c(\Phi,S)$ and $\underline{r}_s(\Phi,S)$ correspond to phase transitions in the Boolean model when the* expected *number of these paths and crossings become positive for arbitrarily large m. By Markov's inequality, we get easily that $\underline{r}_c(\Phi,S) \leq r_c(\Phi,S)$ and $\underline{r}_s(\Phi,S) \leq r_s(\Phi,S)$.*

- *The last critical radius $\overline{r}_c(\Phi, S)$ defined above is equivalent to the critical radius defined in Definition 6.2.1, when the Boolean model is viewed as a racs. As a consequence $\overline{r}_c(\Phi, S) \geq r_c(\Phi, S)$ by Lemma 6.2.2.*

Remark 6.2.11. *Given a marked stationary pp $\tilde{\Phi} \subset \mathbb{R}^d \times \mathbb{K}_d$ with i.i.d. marks, consider the random field $\mathcal{X}_r = \{X_r(x)\}_{x \in \mathbb{R}^d}$ where $X_r(x) = \sup_{(X,S_X)\in\tilde{\Phi}} \mathbf{1}[x \in X + S_X^{(r)}]$, $x \in \mathbb{R}^d$. The random field $\mathcal{X}_r$ is an example of extremal shot-noise field as seen in Definition 4.3.4 and $S_{>0}(\mathcal{X}_r) = \overline{S}_{>0}(\mathcal{X}_r) = C(\Phi, S^{(r)})$. Let us elaborate more on the relation between the critical radii for Boolean model and the critical levels for its associated extremal shot-noise field.*

$$\begin{aligned}
r > r_c(\Phi, S) &\Rightarrow h_c^u(\mathcal{X}_r) = 1 \ ; \ r < r_c(\Phi, S) \Rightarrow h_c^u(\mathcal{X}_r) = 0 \\
r \geq r_c(\Phi, S) &\Leftrightarrow \overline{h}_c^u(\mathcal{X}_r) = 1 \ ; \ r < r_c(\Phi, S) \Leftrightarrow \overline{h}_c^u(\mathcal{X}_r) = 0 \\
r > \overline{r}_c(\Phi, S) &\Rightarrow \underline{h}_c^u(\mathcal{X}_r) = 1 \ ; \ r < \overline{r}_c(\Phi, S) \Rightarrow \underline{h}_c^u(\mathcal{X}_r) = 0.
\end{aligned}$$

The last line follows from (6.11) (See proof of Theorem 6.2.15). Thus $h_c^u(\mathcal{X}_r)$ and $\overline{h}_c^u(\mathcal{X}_r)$ differ only at the critical radius for the Boolean model. Further, if $\mathcal{X}_r^i, i = 1, 2$ are the extremal shot-noise fields associated to the Boolean models of two pp $\Phi_i, i = 1, 2$ respectively, then we have that ,

$$\begin{aligned}
r_c(\Phi_1, S) \leq r_c(\Phi_2, S) &\Leftrightarrow \overline{h}_c^u(\mathcal{X}_r^1) \leq \overline{h}_c^u(\mathcal{X}_r^2) \ \ \forall r > 0. \\
r_c(\Phi_1, S) < r_c(\Phi_2, S) &\Rightarrow h_c^u(\mathcal{X}_r^1) \leq h_c^u(\mathcal{X}_r^2) \ \ \forall r > 0. \\
\overline{r}_c(\Phi_1, S) \leq \overline{r}_c(\Phi_2, S) &\Rightarrow \underline{h}_c^u(\mathcal{X}_r^1) \leq \underline{h}_c^u(\mathcal{X}_r^2) \ \ \forall r > 0.
\end{aligned}$$

So, for extremal shot-noise fields of Boolean models (though it is not a LSC random field), Theorem 6.2.8 follows as a corollary from Theorem 6.2.15. This consistency is due to Proposition 4.3.3.

Remark 6.2.12. *In the above remark, set $\Phi = \Phi_\lambda$, Poisson pp and $S = B_O(\rho)$ with ρ a.s. bounded random variable. In $d = 2$, from [Meester and Roy 1996, Theorem 4.4] we know that for $r < r_c(\Phi, S)$ $C(\Phi_\lambda, S)$ does not percolate and $\mathbb{R}^d \setminus C(\Phi_\lambda, S)$ percolates ; For $r > r_c(\Phi, S)$ $C(\Phi_\lambda, S)$ percolates and $\mathbb{R}^d \setminus C(\Phi_\lambda, S)$ does not. Thus, we have that*

$$r > r_c(\Phi, S) \Rightarrow h_c^l(\mathcal{X}_r) = 1 \ ; \ r < r_c(\Phi, S) \Rightarrow h_c^l(\mathcal{X}_r) = 0.$$

However in $d \geq 3$, by [Sarkar 1997] we have that there exists λ, ρ such that both $C(\Phi_\lambda, \rho)$ and $\mathbb{R}^d \setminus C(\Phi_\lambda, \rho)$ percolate. Thus there exists λ, ρ such that $h_c^l(\mathcal{X}_\rho) = 0 = 1 - h_c^u(\mathcal{X}_\rho)$.

Our first comparison result on the lower critical radii uses comparison of their joint intensities (see Section 2.1) and hence this result holds for a wide class of pp apart from *dcx* ordered pp. We shall now make a small detour into these pp and their properties that allow us to make such a comparison.

For Cox pp, associativity makes it possible to proceed with the proofs of their comparison results bypassing *dcx* ordering. A random measure $\Lambda(.)$ is said to be associated if for any finite collection of Borel subsets $B_1, \ldots, B_n$ and $f, g : \mathbb{R}^n \to [0, 1]$ continuous and increasing,

$$\mathsf{Cov}\left(f(\Lambda(B_1), \ldots, \Lambda(B_n))g(\Lambda(B_1), \ldots, \Lambda(B_n))\right) \geq 0. \tag{6.1}$$

The above definition is the natural extension of association of random vectors and random fields. From [Burton and Waymire 1985, Theorem 5.5], we know that Cox pp are associated.

Another class of pp whose joint intensities have neat expressions are determinantal pp. For a quick introduction refer [Ben Hough et al. 2006] and for a more elaborate reading, see [Ben Hough et al. 2009].

Definition 6.2.13. *A determinantal pp is a simple pp whose joint intensities satisfy for all n,*

$$\rho^{(n)}(x_1, \ldots, x_n) = det\Big(K(x_i, x_j)\Big)_{1 \leq i,j \leq n},$$

for some locally square integrable, Hermitian and non-negative definite kernel $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ and where $\Big(a_{ij}\Big)_{1 \leq i,j \leq n}$ stands for a matrix with entries a_{ij} and $det\Big(.\Big)$ denotes the determinant of the matrix.

We have only briefly mentioned the conditions on the kernel K that suffice for our purposes and refer the unsatisfied reader to [Ben Hough et al. 2009, Section 4.2.1] for the precise conditions on K in terms of its associated integral operator $\mathcal{K}$. Our assumptions in particular imply that, a.e. $K(x, y) = \overline{K(y, x)}$ and $det\Big(K(x_i, x_j)\Big)_{1 \leq i,j \leq n} \geq 0$. More importantly for us, these assumptions allow us to use Hadamard's inequality :

$$det\Big(K(x_i, x_j)\Big)_{1 \leq i,j \leq n} \leq \prod_{i=1}^{n} K(x_i, x_i). \tag{6.2}$$

The pp is stationary if the kernel is stationary i.e, $K(x,y) = k(x-y)$ for all $x, y \in \mathbb{R}^d$ and some suitable function $k : \mathbb{R}^d \to \mathbb{C}$. See [Ben Hough et al. 2009, Chapter 6] for a great many examples of determinantal pp including zeros of Gaussian analytic functions. We have already seen in Section 5.1.7 that perturbed lattices which are approximations to zeros of Gaussian analytic functions are less in *dcx* order than Poisson pp. The results below gives further hope to prove that determinantal processes are lesser in *dcx* order than Poisson pp.

For a kernel K, we define the Φ_K to be the inhomogeneous Poisson pp with the intensity field $\{K(x,x)\}_{x\in\mathbb{R}^d}$. In the case of a stationary kernel, Φ_K is the Poisson pp $\Phi_{K(0,0)}$ with intensity $K(0,0)$.

Similar to determinantal pp, one can define permanental pp as simple pp whose joint intensities satisfy for all n, $\rho^{(n)}(x_1,\ldots,x_n) = per\Big(K(x_i,x_j)\Big)_{1\le i,j\le n}$ where $per\Big(.\Big)$ stands for the permanent of a matrix. From [Ben Hough et al. 2006, Proposition 35 and Remark 36], it follows that all permanental pp are Cox pp with intensity field $|F|^2$ where F is a complex Gaussian process on $\mathbb{R}^d$. Due to this nice representation, this special class has been studied in more detail (see [Mccullagh and Møller 2006]) and since our comparison result shall cover all Cox pp, permanental pp are also included in them as a special case.

The following theorems are the comparison of the lower and upper bounds of the critical radii of *dcx* ordered point processes.

Theorem 6.2.14. *Let S be an a.s. bounded path connected racs. By boundedness, we mean that there exists a $R < \infty$ such that $S \subset B_O(R)$ a.s..*

1. *Let (Φ_i, S) be two marked pp with joint intensities $\rho_i^{(n)}(.,\ldots,.)$ respectively. Let $\rho_1^{(n)}(x_1,\ldots,x_n) \le \rho_2^{(n)}(x_1,\ldots,x_n)$ a.e.. Then we have that*

$$\underline{r}_c(\Phi_1, S) \ge \underline{r}_c(\Phi_2, S) \text{ and } \underline{r}_s(\Phi_1, S) \ge \underline{r}_s(\Phi_2, S). \tag{6.3}$$

2. *The inequalities (6.3) hold for $\Phi_1 \le_{idcx} \Phi_2$.*

3. *The inequalities (6.3) hold also for Φ_1, a determinantal pp with kernel K and $\Phi_2 = \Phi_K$ (the inhomogeneous Poisson pp).*

4. *The inequalities (6.3) hold also for Φ_2, a Cox(Λ) pp and Φ_1, a Cox (α) pp ($\alpha(.) = \mathsf{E}(\Lambda(.))$ i.e, the mean intensity measure of the Cox pp)).*

Though the reversal of the inequality for the critical radii than the heuristical prediction is surprising, it says that in a clustered pp whenever there is a path from origin to the ∂Q_m (which should happen with a smaller probability for a large m than for a unclustered pp) there are far too many paths to balance out that the non-existence of paths for other realizations. The proof of the result also can be extended to yield comparison result for critical radius defined with respect to N_n^*, defined as the number of paths of length n in the Boolean model.

Theorem 6.2.15. *Let Let S be an a.s. bounded path connected racs and$\Phi_i, i = 1, 2$ be stationary pp such that $\Phi_1 \leq_{ddcx} \Phi_2$. Then $\overline{r}_c(\Phi_1, S) \leq \overline{r}_c(\Phi_2, S)$. The inequality holds also if Φ_2 is a Cox(Λ) pp and Φ_1 is a Cox (α) pp and $S = \{O\}$.*

Remark 6.2.16. *Thus for $\Phi_1 \leq_{dcx} \Phi_2$ (or Φ_1 is a Cox (α) and Φ_2 is Cox(Λ) with $S = \{O\}$), combining Theorems 6.2.14, 6.2.15 and Remark 6.2.10, we have that*

$$\underline{r}_c(\Phi_2, S) \leq \underline{r}_c(\Phi_1, S) \leq r_c(\Phi_1, S) \leq \overline{r}_c(\Phi_1, S) \leq \overline{r}_c(\Phi_2, S).$$

Proof. (THEOREM 6.2.14) (1) : Assume that Φ_i are pp such that their joint intensities satisfy $\rho_1^{(n)}(x_1, \ldots, x_n) \leq \rho_2^{(n)}(x_1, \ldots, x_n)$ a.e. for all $n \geq 1$ respectively. This implies that for any disjoint bBs $B_1, \ldots, B_k$ we have that

$$\begin{aligned} \mathsf{E}\left(\prod_{i=1}^{k} \Phi_1(B_i)\right) &= \int_{\prod_{i=1}^n B_i} \rho_1^{(n)}(x_1, \ldots, x_k) \mathrm{d}x_1, \ldots, \mathrm{d}x_k \\ &\leq \int_{\prod_{i=1}^n B_i} \rho_2^{(n)}(x_1, \ldots, x_k) \mathrm{d}x_1, \ldots, \mathrm{d}x_k = \mathsf{E}\left(\prod_{i=1}^{k} \Phi_2(B_i)\right). \quad (6.4) \end{aligned}$$

Our proof will be driven by this simple inequality.

We will now first prove that $\underline{r}_c(\Phi_1, S) \geq \underline{r}_c(\Phi_2, S)$. For the same, it suffices to show that

$$\mathsf{E}\Big(N_m(\Phi_1, S^{(r)})\Big) \leq \mathsf{E}\Big(N_m(\Phi_2, S^{(r)})\Big), \quad \text{for } r \neq r_m^* \quad (6.5)$$

where $r_m^* := \inf\{r : \mathsf{E}\big(N_m(\Phi_2, S^{(r)})\big) = \infty\}, m \geq 1$. Let us first argue that (6.5) is enough to conclude this part of the proof. Indeed, suppose that $\underline{r}_c(\Phi_2, S) > \underline{r}_c(\Phi_1, S)$, then choose a $r \notin \{r_m^*\}_{m \in \mathbb{N}}$ such that $\underline{r}_c(\Phi_2, S) > r > \underline{r}_c(\Phi_1, S)$. Then $0 < \liminf_m \mathsf{E}\big(N_m(\Phi_1, S^{(r)})\big) \leq \liminf_m \mathsf{E}\big(N_m(\Phi_2, S^{(r)})\big)$ implying that $r \geq \underline{r}_c(\Phi_2, S)$, a contradiction.

We work with a fixed m for rest of the proof. Let $\{S_{n,z}\}_{n\geq 1, z\in\mathbb{Z}^d_n}$ be i.i.d random variables distributed as S. Let n_0 be the least n such that $\Phi_2(Q^n(z)) \leq 1, \forall z \in Q_{m+R}\cap\mathbb{Z}^d_n$. For a simple point process, such a n_0 exists a.s.. Now define $\{S^*_{n,z}\}_{n\geq 1, z\in\mathbb{Z}^d_n}$ as follows : $S^*_{n,z} = S_{n,z}$ if $z \notin Q_{m+R}\cap\mathbb{Z}^d_n$ or $n < n_0$ or $\Phi_2(Q^n(z)) = 0$; Else $S^*_{n,z} = S_{n_0,z(n_0)}$, where $z \in Q^{n_0}(z(n_0))$. Let $\mathfrak{S}_n = \{S_{n,z}\}_{z\in\mathbb{Z}^d_n}$ and $\mathfrak{S}_n = \{S^*_{n,z}\}_{z\in\mathbb{Z}^d_n}$ for $n \geq 1$. Also, we assign the marks $\mathfrak{S}$ to $\Phi_2\cap Q_{m+R}$ using $\mathfrak{S}_{n_0}$ i.e, for $X \in \Phi_2\cap Q^{n_0}(z)$, let $S_X = S_{n_0,z}$. Clearly, this marking yields the i.i.d. marked pp $\tilde{\Phi}_2 = (\Phi_2, S)$ in Q_m. We extend above construction can also be extended to the entire plane to assign i.i.d. marks to all the points of Φ_2 (see [Meester and Roy 1996, Sec 1.4]) as follows : If $X \notin Q_{m+R}$, then $S_X = S_{n_0,z)}$ where $n_0 := n_0(X)$ is chosen as the least integer for which there exists a $z := z(X)$ such that $Q^{n_0}(z)\cap\Phi = \{X\}$. For each $X \in \Phi_2$, $n_0(X) < \infty$ and also $z(X) \neq z(Y)$ for $X, Y \in \Phi$. Hence we have assigned i.i.d. marks to Φ_2. We shall use this marking scheme in the proof. This way of assigning marks is valid for Φ_1 as well but with a different n_0. Also for $n \geq 1$, let $\mathfrak{S}^{(r)}_n := \{S^{(r)}_{n,z}\}_{z\in\mathbb{Z}^d_n}$ and similarly for $\mathfrak{S}^{*(r)}_n$.

Recall that $\Upsilon^n_m(\mathfrak{S}^{(r)}_n)$ was the set of open path from the origin to ∂Q_m in the graph $\mathbb{L}^{*d}_n(\mathfrak{S}^{(r)}_n)$ (see Section 6.1). For a point process Φ and a path $\gamma \in \Upsilon^n_m(\mathfrak{S}^{(r)}_n)$, let $N_\gamma(\Phi) = \prod_{z\in\gamma}\Phi(Q^n(z))$, the number of paths through points of Φ lying in the boxes of the discrete path. Denote by $N^n_m(\Phi_1, \mathfrak{S}^{(r)}_n) = \sum_{\gamma\in\Upsilon^n_m(\mathfrak{S}^{(r)}_n)} N_\gamma(\Phi_1)$, $N^n_m(\Phi_1, \mathfrak{S}^{*(r)}_n) = \sum_{\gamma\in\Upsilon^n_m(\mathfrak{S}^{*(r)}_n)} N_\gamma(\Phi_1)$ respectively the number of paths of Φ_1 in $\Upsilon^n_m(\mathfrak{S}^{(r)}_n)$ and $\Upsilon^n_m(\mathfrak{S}^{*(r)}_n)$ (paths from the origin to ∂Q_m) in this discrete model induced by $\Phi_1, \mathfrak{S}^{(r)}_n, \mathfrak{S}^{*(r)}_n$. Note that $\mathsf{E}\Big(N^n_m(\Phi_1, \mathfrak{S}^{(r)}_n)\Big) = \mathsf{E}\Big(N^n_m(\Phi_1, \mathfrak{S}^{*(r)}_n)\Big), \forall n \geq 1$. The

following observation regarding the discrete models is crucial for our proof.

$$\begin{aligned}
\mathsf{E}\Big(N_m^n(\Phi_1, \mathfrak{S}_n^{(r)})|\mathfrak{S}_n\Big) &= \sum_{\gamma\in\Upsilon_m^n(\mathfrak{S}_n^{(r)})} \mathsf{E}(N_\gamma(\Phi_1)) \\
&= \sum_{\gamma=\langle 0,z_0,z_1,\ldots,z_k\rangle\in\Upsilon_m^n(\mathfrak{S}_n^{(r)})} \mathsf{E}\left(\prod_{i=1}^k \Phi_1(Q^n(z_i))\right) \\
&\le \sum_{\gamma=\langle 0,z_0,z_1,\ldots,z_k\rangle\in\Upsilon_m^n(\mathfrak{S}_n^{(r)})} \mathsf{E}\left(\prod_{i=1}^k \Phi_2(Q^n(z_i))\right) \\
&= \sum_{\gamma\in\Upsilon_m^n(\mathfrak{S}_n^{(r)})} \mathsf{E}(N_\gamma(\Phi_2)) \\
&= \mathsf{E}\Big(N_m^n(\Phi_2, \mathfrak{S}_n^{(r)})|\mathfrak{S}_n\Big) \qquad (6.6)
\end{aligned}$$

where the inequality is due to (6.4). By unconditioning over $\mathfrak{S}_n$, we get that

$$\mathsf{E}\Big(N_m^n(\Phi_1, \mathfrak{S}_n^{(r)})\Big) \le \mathsf{E}\Big(N_m^n(\Phi_2, \mathfrak{S}_n^{(r)})\Big). \qquad (6.7)$$

We will prove first that

$$\lim_{n\to\infty} \mathsf{E}\Big(N_m^n(\Phi_2, \mathfrak{S}_n^{(r)})\Big) = \lim_{n\to\infty} \mathsf{E}\Big(N_m^n(\Phi_2, \mathfrak{S}_n^{*(r)})\Big) = \mathsf{E}\Big(N_m(\Phi_2, S^{(r)})\Big) \text{ for } r \ne r_m^*. \quad (6.8)$$

For all $n > n_0, N_m(\Phi_2, S^{(r)}) \le N_m^n(\Phi_2, \mathfrak{S}_n^{*(r)})$. Thus $N_m(\Phi_2, S^{(r)}) \le \liminf_{n\to\infty} N_m^n(\Phi_2, \mathfrak{S}_n^{*(r)})$ and by Fatou's lemma we get that

$$\mathsf{E}\Big(N_m(\Phi_2, S^{(r)})\Big) \le \mathsf{E}\Big(\liminf_{n\to\infty} N_m^n(\Phi_2, \mathfrak{S}_n^{*(r)})\Big) \le \liminf_{n\to\infty} \mathsf{E}\Big(N_m^n(\Phi_2, \mathfrak{S}_n^{*(r)})\Big). \qquad (6.9)$$

This proves (6.8) for $r > r_m^*$. For $r < r_m^*$, we need to show the converse inequality to (6.9) as well to complete the proof of (6.8).

Choose n_1 such that for all $n \ge n_1$, $r + \frac{\sqrt{d}}{2n} < r_m^*$. For $n > n_1$, clearly there is a surjective mapping from $\Upsilon_m(\Phi_2, S^{(r+\frac{\sqrt{d}}{2n})})$ to $\Upsilon_m^n(\mathfrak{S}_n^{*(r)})$ and thus, $N_m^n(\Phi_2, \mathfrak{S}_n^{*(r)}) \le N_m(\Phi_2, S^{(r+\frac{\sqrt{d}}{2n})})$ for all n. This is so because, in $N_m^n(\Phi_2, \mathfrak{S}_n^{*(r)})$ we are counting the paths of $C(\Phi_2, S^{(r+\frac{\sqrt{d}}{2n})})$ from O to ∂Q_m that do not have edges between points in the same box $Q^n(z)$ for a $z \in \mathbb{Z}_n^d$. Hence $\limsup_{n\to\infty} N_m^n(\Phi_2, \mathfrak{S}_n^{*(r)}) \le N_m(\Phi_2, S^{(r)})$. Further, $N_m^n(\Phi_2, \mathfrak{S}_n^{*(r)}) \le N_m(\Phi_2, S^{(r+\frac{\sqrt{d}}{2n_1})})$ for all $n > n_1$ and $\mathsf{E}\Big(N_m(\Phi_2, S^{(r+\frac{\sqrt{d}}{2n_1})})\Big) < \infty$. So, we can use reverse

Fatou's lemma to get

$$\limsup_{n\to\infty} \mathsf{E}\Big(N_m^n(\Phi_2, \mathfrak{S}_n^{*(r)})\Big) \leq \mathsf{E}\bigg(\limsup_{n\to\infty} N_m^n(\Phi_2, \mathfrak{S}_n^{*(r)})\bigg) \leq \mathsf{E}\Big(N_m(\Phi_2, S^{(r)})\Big) \ . \tag{6.10}$$

Thus we have shown (6.8).

We are ready now to prove (6.5). The inequality is trivial for $r > r_m^*$. Now consider $r < r_m^*$. Observe that (6.9) holds for Φ_1 by same arguments as for Φ_2. Thus, by discrete approximation inequality (6.7) and (6.8) we get that

$$\begin{aligned} \mathsf{E}\Big(N_m(\Phi_1, S^{(r)})\Big) &\leq \liminf_{n\to\infty} \mathsf{E}\Big(N_m^n(\Phi_1, \mathfrak{S}_n^{*(r)})\Big) \leq \liminf_{n\to\infty} \mathsf{E}\Big(N_m^n(\Phi_2, \mathfrak{S}_n^{*(r)})\Big) \\ &= \mathsf{E}\Big(N_m(\Phi_2, S^{(r)})\Big) \ . \end{aligned}$$

This completes the proof of the first statement.

The proof of the second inequality in (6.3) $\underline{r}_s(\Phi_1) \geq \underline{r}_s(\Phi_2)$ goes along the same lines except that the $\Upsilon_m^n(\mathfrak{S}_n)$ is replaced by the set of all crossings of $Q_{m,3m,\ldots,3m}$ in the discrete lattice instead of paths from origin to ∂Q_m. Since a.s. $S \subset B_O(R)$, the possible crossings in the discrete model lie within $Q^{(R)}_{m,3m,\ldots,3m}$ and hence the number of such crossings is finite and the summation is well defined.

(2): If $\Phi_1 \leq_{idcx} \Phi_2$, then from Corollary 4.2.5 we have ordering of the joint intensities as required in the assumption of the theorem.

(3): If Φ is a determinantal pp, then by Hadamard's inequality (6.2), we have that $\rho^{(n)}_{\Phi}(x_1, \ldots, x_n) \leq \prod_i K(x_i, x_i) = \rho^{(n)}_{\Phi_K}(x_1 \ldots, x_n)$ and the result follows as above.

(4): For the case of Cox(Λ) pp Φ_2, by associativity of the pp (see (6.1)) we have (6.4) for Φ_α and Φ_2 and the proof follows as above. □

Proof. (Theorem 6.2.15)

(1): The proof follows similar to Lemma 6.2.3 by noting that due to Proposition 4.3.5, we have,

$$\mathsf{P}\left(C(\Phi_1, S) \cap Q_\gamma = \emptyset\right) \leq \mathsf{P}\left(C(\Phi_2, S) \cap Q_\gamma = \emptyset\right) \ . \tag{6.11}$$

(2): For any Borel set B, by Jensen's inequality and concavity of $(1-e^{-x})$,

$$\mathbf{T}_{C(\Phi_2,r)}(B) = \mathbf{T}_{\Phi_2}(B^{(r)}) = \mathsf{E}\Big(1-\exp\{-\Lambda(B^{(r)})\}\Big) \leq 1-\exp\{-\alpha(B^{(r)})\} = \mathbf{T}_{C(\Phi_1,r)}(B).$$

The rest of the proof is similar to Lemma 6.2.3 or the first part. □

6.3 Non-trivial phase transition for percolation models on sub-Poisson point processes

In the earlier section, we compared critical radii for Boolean model and crucial to the proof was the extremal shot-noise field representation of the Boolean model and the *lo* ordering of its capacity functionals. This technique does not extend to percolation models defined by additive shot-noise fields as we know that additive shot-noise fields are *dcx* ordered and not *lo* ordered. But, still we wish to say something about percolation models driven by additive shot-noise fields. For this, we need to get bounds on certain probabilities so as to couple these fields with a sub-critical branching process or use Peierls argument. The rough idea is as follows : Even though the probabilities of interest are not *dcx* functions of the shot-noise fields, they can be bounded by exponential estimates (namely via Laplace transforms) which are *dcx* function of the shot-noise fields. For sub-Poisson pp (pp that are lesser than Poisson in *dcx*), this can further be bounded by Laplace transform of the corresponding Poisson shot-noise field. Laplace transforms of the Poisson shot-noise field have closed-form expressions. For 'nice' response functions, one would expect the Laplace transform of Poisson shot-noise fields to be amenable enough to deduce asymptotic bounds for application of Lemmas 2.2.2, 2.2.3 and 2.2.4 on suitable discretisations. Then, we can ascertain existence of a non-trivial phase transition for the percolation model. We shall carry out such a program for various percolation models.

6.3.1 Bounds in discrete models

We shall again start with a generic bound on a discrete model which shall be used to prove bounds in the continuum models. Recall that $V_\Phi(x) := \sum_{X\in\Phi} \ell(x,X)$ is the additive shot-noise field (Definition 4.1.2) generated by a pp Φ and a non-negative response function $\ell(.,.)$. Define $\mathbb{Z}^d_r(V_\Phi, \leq h) := \{z \in r\mathbb{Z}^d : V_\Phi(z) \leq h\}$ and $\mathbb{Z}^d_r(V_\Phi, \geq h) := \{z \in r\mathbb{Z}^d : V_\Phi(z) \geq h\}$. Percolation of $\mathbb{Z}^d_r(V_\Phi, \leq h), \mathbb{Z}^d_r(V_\Phi, \geq h)$ is understood in the sense of

percolation as subsets of $\mathbb{L}^{*d}(r)$. We will derive generic bounds that shall be required for the use of Lemmas 2.2.2, 2.2.3 and 2.2.4 in all of the future subsections of the chapter.

Lemma 6.3.1. *Let Φ be a stationary pp and $V_\Phi(.)$, $\mathbb{Z}^d_r(V_\Phi, \leq h)$ and $\mathbb{Z}^d_r(V_\Phi, \geq h)$ be as defined above.*

1. *If $\Phi \leq_{idcx} \Phi_\lambda$ for some $\lambda > 0$, then for any $s > 0$,*

$$\mathsf{P}\left(V_\Phi(z_i) \geq h, 1 \leq i \leq n\right) \leq \exp\{-snh\} \exp\{\lambda \int_{\mathbb{R}^d} (e^{s\sum_{i=1}^n \ell(x,z_i)} - 1) dx\}. \quad (6.12)$$

2. *If $\Phi \leq_{ddcx} \Phi_\lambda$ for some $\lambda > 0$, then for any $s > 0$,*

$$\mathsf{P}\left(V_\Phi(z_i) \leq h, 1 \leq i \leq n\right) \leq \exp\{snh\} \exp\{\lambda \int_{\mathbb{R}^d} (e^{-s\sum_{i=1}^n \ell(x,z_i)} - 1) dx\}. \quad (6.13)$$

Proof. The proofs will use the following representation of a the Laplace transform of a functional of Poisson pp. For a function $f : \mathbb{R}^d \to \mathbb{R}_+$, $s \in \mathbb{R}$, (see [Kingman 1993, Pg. 28] and [Baccelli and Błaszczyszyn 2009a, Proposition 1.5])

$$\mathsf{E}\left(\exp\{s \sum_{X \in \Phi_\lambda} f(X)\}\right) = \exp\{-\lambda \int_{\mathbb{R}^d} (1 - e^{sf(x)}) \mathrm{d}x\} \quad (6.14)$$

(1): For any $s > 0$,

$$\begin{aligned}
\mathsf{P}\left(V_\Phi(z_i) \geq h, 1 \leq i \leq n\right) &\leq \mathsf{P}\left(\sum_{i=1}^n V_\Phi(z_i) \geq nh\right) \\
&\leq \exp\{-snh\} \mathsf{E}\left(\exp\{s \sum_{i=1}^n V_\Phi(z_i)\}\right) \text{ (by Chernoff's inequality)} \\
&\leq \exp\{-snh\} \mathsf{E}\left(\exp\{s \sum_{i=1}^n V_{\Phi_\lambda}(z_i)\}\right) \text{ (by } idcx \text{ ordering)} \\
&= \exp\{-snh\} \exp\{-\lambda \int_{\mathbb{R}^d} (1 - e^{s\sum_{i=1}^n \ell(x,z_i)}) \mathrm{d}x\} \text{ (by Eqn. 6.14).}
\end{aligned}$$

(2): The proof follows as above by noting that for any random variable X and $a \in \mathbb{R}, s > 0$,

$$\mathsf{P}\left(X \leq a\right) = \mathsf{P}\left(\exp\{-sX\} \geq \exp\{-sa\}\right) \leq \exp\{sa\} \mathsf{E}(\exp\{-sX\}).$$

□

The following lemma is useful when we are interested in percolation and non-percolation of multiple shot-noise fields simultaneously. It says that if we can use Peierls argument on each of the shot-noise fields individually, we shall also be able to use them collectively.

Lemma 6.3.2. *Let $\{X_{ij}\}_{1\le i\le k, 1\le j\le n}$ be Bernoulli random variables (not necessarily independent) such that for all $I \subset [n]$, $\mathsf{E}\Big(\prod_{j\in I} \overline{X}_{ij}\Big) \le q_i^{|I|}$ for $1 \le i \le k$ with $q_i < 1$ and $\overline{X}_{ij} = 1 - X_{ij}$. Then*

$$\mathsf{E}\left(\prod_{j=1}^{n} \overline{\prod_{i=1}^{k} X_{ij}}\right) \le (\sum_{i=1}^{k} q_i^{\frac{1}{k}})^n.$$

Proof. In the following we adopt the following convention $\prod_{i\in\emptyset} x_i \equiv 1$ for any collection of x_i's.

$$\begin{aligned}
\mathsf{E}\left(\prod_{j=1}^{n} \overline{\prod_{i=1}^{k} X_{ij}}\right) &\le \mathsf{E}\left(\prod_{j=1}^{n} (\sum_{i=1}^{k} \overline{X}_{ij})\right) \\
&= \sum_{(k_j)_{j=1}^n \in [k]^n} \mathsf{E}\left(\prod_{j=1}^{n} \overline{X}_{k_j j}\right) = \sum_{(k_j)_{j=1}^n \in [k]^n} \mathsf{E}\left(\prod_{i=1}^{k} \prod_{j:k_j=i} \overline{X}_{ij}\right) \\
&\le \sum_{(k_j)_{j=1}^n \in [k]^n} \prod_{i=1}^{k} \Big(\mathsf{E}\left(\prod_{j:k_j=i} \overline{X}_{ij}^k\right)\Big)^{\frac{1}{k}} \text{ (by Cauchy-Schwartz inequality)} \\
&= \sum_{(k_j)_{j=1}^n \in [k]^n} \prod_{i=1}^{k} \Big(\mathsf{E}\left(\prod_{j:k_j=i} \overline{X}_{ij}\right)\Big)^{\frac{1}{k}} \\
&\le \sum_{(k_j)_{j=1}^n \in [k]^n} \prod_{i=1}^{k} \prod_{j:k_j=i} (q_i)^{\frac{1}{k}} \\
&= \sum_{(k_j)_{j=1}^n \in [k]^n} \prod_{j=1}^{n} (q_{k_j})^{\frac{1}{k}} = (\sum_{i=1}^{k} q_i^{\frac{1}{k}})^n.
\end{aligned}$$

□

6.3.2 k-percolation

The aim of this subsection is to show that for sub-Poisson pp (i.e, less in *dcx*) than Poisson pp (denoted by Φ_λ), the critical intensity for k-percolation is non-degenerate.

Definition 6.3.3. *Given a pp Φ and $\{S_X\}_{X\in\Phi} \subset \mathbb{K}_d$ where S_X are i.i.d. copies of a racs $S \in \mathbb{K}_d$, define a shot-noise field $V_\Phi(x) := \sum_{X\in\Phi} \mathbf{1}[x \in X + S_X]$. The k-covered set is defined as*

$$C_k(\Phi, S) := \{x : V_\Phi(x) \geq k\}.$$

Note that $C_k(\Phi, S)^{(r)}$ percolates iff $C_k(\Phi, S^{(r)})$ percolates. Thus the critical radii can be expressed as

$$r_c^k(\Phi, S) := r_c(C_k(\Phi, S)) = \inf\{r : \mathsf{P}\left(C_k(\Phi, S^{(r)}) \text{ percolates}\right) > 0\}.$$

Clearly, $r_c^1(\Phi, S) = r_c(\Phi, S) \leq r_c^k(\Phi, S)$. We shall abbreviate $C_k(\Phi, B_O(r))$ by $C_k(\Phi, r)$ and $r_c^k(\Phi, \{O\})$ by $r_c^k(\Phi)$.

Theorem 6.3.4. *For $k \geq 1, \lambda > 0$, there exist $c(\lambda)$ and $c(\lambda, k)$ (i.e, depending only on k, λ and not on Φ) such that*

$$\begin{aligned} \Phi \leq_{idcx} \Phi_\lambda &\Rightarrow 0 < c(\lambda) \leq r_c^1(\Phi). \\ \Phi \leq_{ddcx} \Phi_\lambda &\Rightarrow r_c^k(\Phi) \leq c(\lambda, k) < \infty. \end{aligned}$$

Thus, for $\Phi \leq_{dcx} \Phi_\lambda$ combining both the above statements, we have that

$$0 < c(\lambda) \leq r_c^1(\Phi) \leq r_c^k(\Phi) \leq c(\lambda, k) < \infty.$$

Remark 6.3.5.

1. *More simply, the theorem gives an upper and lower bound for the critical radius of a sub-Poisson pp dependent only on its mean measure (as this determines the λ in Φ_λ) and not on the finer structure.*

2. *The following extensions follow by obvious coupling arguments. For a racs $S \subset B_O(R)$ a.s. with $R < c(\lambda)$, we have that $r_c(\Phi, S) \geq r_c(\Phi, B_O(R)) \geq c(\lambda) - R$ and for a racs $S \supset B_O(R)$ a.s. with $R < c(\lambda, k)$, $r_c^k(\Phi, S) \leq r_c^k(\Phi, B_O(R)) \leq c(\lambda, k) - R$.*

3. *For Poisson pp, we know that with probability* 1, $\|\mathbb{R}^d \setminus C(\Phi_\lambda, G)\| = 0$ *iff* $\mathsf{E}(\|G\|) = \infty$ *([Hall 1988, Theorem 3.1]) and the same result holds for any ergodic pp but restricting* G *to be a ball of random radius ([Meester and Roy 1996, Proposition 7.3]). Thus, the most general version of Theorem 6.3.4 would assume* $\mathsf{E}(\|G\|) < \infty$ *instead of a.s. bounded grains. But we would like to call attention to fact that this assumption of a.s. bounded grains is not uncommon in percolation theory even when the underlying pp is a Poisson pp.*

Proof. Let $\Phi \leq_{idcx} \Phi_\lambda$ and $r > 0$. Consider the close packed lattice $\mathbb{L}^{*d}(2r)$. Define the response function $l_r(x, y) := \mathbf{1}[x \in Q_r(y)]$ and the corresponding shot-noise field $V^r_\Phi(z)$ on $\mathbb{L}^{*d}(2r)$. Note that $C(\Phi, r)$ percolates only if $\mathbb{Z}^d_{2r}(V^r_\Phi, \geq 1)$ percolates. We shall now show that there exist $r > 0$ such that $\mathbb{Z}^d_{2r}(V^r_\Phi, \geq 1)$ does not percolate.

For any n and $z_i \in r\mathbb{Z}^d, 1 \leq i \leq n$, $\sum_{i=1}^n l_r(x, z_i) = 1$ iff $x \in \cup_{i=1}^n Q_r(z_i)$ and else 0. Thus, from Lemma 6.3.1, we have that

$$\begin{aligned}
\mathsf{P}\left(V^r_\Phi(z_i) \geq 1, 1 \leq i \leq n\right) &\leq \exp\{-sn\} \exp\{\lambda \int_{\mathbb{R}^d} (e^{s\sum_{i=1}^n l(x,z_i)} - 1)\mathrm{d}x\}, \\
&= \exp\{-sn\} \exp\{\lambda \| \cup_{i=1}^n Q_r(z_i)\|(e^s - 1)\}\mathrm{d}x, \\
&= (\exp\{-(s + (1 - e^s)\lambda(2r)^d)\})^n. \qquad (6.15)
\end{aligned}$$

Choosing s large enough that $e^{-s} < (3^d - 1)^{-1}$ and then by continuity of $(s+(1-e^s)\lambda(2r)^d)$ in r, we can choose a $c(\lambda) > 0$ such that for all $r < c(\lambda)$, $\exp\{-(s + (1 - e^s)\lambda(2r)^d))\} < (3^d - 1)^{-1}$. Now as in Lemma 2.2.4, we can show non-percolation of $\mathbb{Z}^d_{2r}(V^r_\Phi, \geq 1)$ for $r < c(\lambda)$. Hence for all $r < c(\lambda)$, $C(\Phi, r)$ does not percolate and so $c(\lambda) \leq r_c(\Phi)$.

Let $\Phi \leq_{ddcx} \Phi_\lambda$. Consider the close packed lattice $\mathbb{L}^{*d}(\frac{r}{\sqrt{d}})$. Define the response function $l_r(x, y) := \mathbf{1}[x \in Q_{\frac{r}{2\sqrt{d}}}(y)]$ and the corresponding additive shot-noise field $V^r_\Phi(z)$ on $\mathbb{L}^{*d}(\frac{r}{\sqrt{d}})$. Note that $C_k(\Phi, r)$ percolates if $\mathbb{Z}^d_{\frac{r}{\sqrt{d}}}(V^r_\Phi, \geq \lceil k/2 \rceil)$ percolates. We shall now show that there exist $r < \infty$ such that $\mathbb{Z}^d_{\frac{r}{\sqrt{d}}}(V^r_\Phi, \geq \lceil k/2 \rceil)$ percolates.

For any n and $z_i, 1 \leq i \leq n$, from Lemma 6.3.1, we have that

$$\begin{aligned}
\mathsf{P}\left(V^r_\Phi(z_i) \leq \lceil k/2 \rceil - 1, 1 \leq i \leq n\right) &\leq \exp\{sn(\lceil k/2 \rceil - 1)\} \exp\{\lambda \int_{\mathbb{R}^d} (e^{-s\sum_{i=1}^n l(x,z_i)} - 1)\mathrm{d}x\}, \\
&= \exp\{sn(\lceil k/2 \rceil - 1)\} \exp\{\lambda \| \cup_{i=1}^n Q_{\frac{r}{2\sqrt{d}}}(z_i)\|(e^{-s} - 1)\}, \mathrm{d}x \\
&= (\exp\{-((1 - e^{-s})\lambda(\frac{r}{\sqrt{d}})^d - s(\lceil k/2 \rceil - 1))\})^n. \qquad (6.16)
\end{aligned}$$

For any s, there exists $c(\lambda,k,s) < \infty$ such that for all $r > c(\lambda,k,s)$, the last term in the above equation is strictly less than $(3^d-1)^{-n}$. Thus one can use a Peierls argument (Lemma 2.2.2) to show the finiteness of the number of closed contours around origin and so $\mathbb{Z}^d_{\frac{r}{\sqrt{d}}}(V^r_\Phi, \geq \lceil k/2 \rceil)$ percolates. Further defining $c(\lambda,k) := \inf_{s>0} c(\lambda,k,s)$, we have that $C_k(\Phi,r)$ percolates for all $r > c(\lambda,k)$. Thus $r^k_c(\Phi) \leq c(\lambda,k)$. □

We have seen earlier in Theorem 6.2.15 that, by using the representation of the Boolean model as an extremal shot-noise field, we avoided the usage of exponential estimates as in the above theorem. We shall use such a representation to get an improved bound than $C(\lambda,1)$.

Proposition 6.3.6. *Let $\Phi \leq_{ddcx} \Phi_\lambda$ for some $\lambda > 0$. Then $r_c(\Phi) \leq \tilde{C}(\lambda) := \sqrt{d}\sqrt[\frac{1}{d}]{\frac{\log(3^d-2)^{-1}}{\lambda}} \leq C(\lambda,1) < \infty$.*

Proof. As in the second part of the proof of Theorem 6.3.4, consider the close packed lattice $\mathbb{L}^{*d}(\frac{r}{\sqrt{d}})$. Define the response function $l_r(x,y) := \mathbf{1}[x \in Q_{\frac{r}{2\sqrt{d}}}(y)]$ and the corresponding extremal shot-noise field $U^r_\Phi(z)$ on $\mathbb{L}^{*d}(\frac{r}{\sqrt{d}})$. Now, note that $C(\Phi,r)$ percolates if $\mathbb{Z}^d_{\frac{r}{2\sqrt{d}}}(U^r_\Phi, \geq 1)$ percolates. We shall now show that $\mathbb{Z}^d_{\frac{r}{2\sqrt{d}}}(V^r_\Phi, \geq k)$ percolates for $r > \tilde{C}(\lambda)$.

From Proposition 4.3.3, we have that $\mathbf{T}_{C(\Phi,r)}(B) \geq \mathbf{T}_{C(\Phi_\lambda,r)}(B)$ for all bBs B. Thus,

$$\begin{aligned} \mathsf{P}\left(U^r_\Phi(z_i) = 0, 1 \leq i \leq n\right) &= \mathsf{P}\left(\Phi \cap \cup_{i=1}^n Q_{\frac{r}{2\sqrt{d}}}(z_i) = \emptyset\right) \\ &\leq \mathsf{P}\left(\Phi_\lambda \cap \cup_{i=1}^n Q_{\frac{r}{2\sqrt{d}}}(z_i) = \emptyset\right) \\ &= (\exp\{-\lambda(\frac{r}{\sqrt{d}})^d\})^n. \end{aligned} \tag{6.17}$$

Clearly for $r > \tilde{C}(\lambda)$, $\mathbb{Z}^d_{\frac{r}{2\sqrt{d}}}(V^r_\Phi, \geq k)$ percolates by Peierls argument and hence $C(\Phi,r)$. It is easy to see that for any $s > 0$, $\exp\{-\lambda(\frac{r}{\sqrt{d}})^d\} \leq \exp\{-(1-e^{-s})\lambda(\frac{r}{\sqrt{d}})^d\}$ and hence $\tilde{C}(\lambda) \leq C(\lambda,1)$. □

Non-percolation in a Boolean model can be proved for more general class of pp by using the ordering of joint intensities as in Theorem 6.2.14. Thus, in particular, this is true for stationary determinantal pp. From Proposition 4.2.4, we know that the ordering of joint intensities is weaker assumption than

Proposition 6.3.7. *Let Φ be a stationary pp such that all its joint intensities satisfy $\rho^{(k)}(x_1,\ldots,x_k) \leq \lambda^k$ a.e. for some λ. Then $r_c(\Phi)^d \geq \frac{3^d-1}{\lambda 2^d} > 0$.*

Proof. We shall use the same method as in the first part of Theorem 6.3.4 but just that we will bound the level crossing probabilties by using the joint intensities. As in Theorem 6.3.4, consider the close packed lattice $\mathbb{L}^{*d}(2r)$, the response function $l_r(x,y) := \mathbf{1}[x \in Q_r(y)]$ and the corresponding shot-noise field $V^r_\Phi(z)$ on $\mathbb{L}^{*d}(2r)$. We know that $C(\Phi, r)$ percolates only if $\mathbb{Z}^d_{2r}(V^r_\Phi, \geq 1)$ percolates. Let us disprove the latter for $r < \frac{3^d-1}{\lambda 2^d}$.

$$\begin{aligned}
\mathsf{P}\left(V^r_\Phi(z_i) \geq 1, 1 \leq i \leq n\right) &= \mathsf{P}\left(\Phi(Q_r(z_i)) \geq 1, \forall i\right) = \mathsf{P}\left(\prod_{i=1}^{n} \Phi(Q_r(z_i)) \geq 1\right) \\
&\leq \mathsf{E}\left(\prod_i \Phi(Q_r(z_i))\right) \leq \mathsf{E}\left(\prod_i \Phi_\lambda(Q_r(z_i))\right) \\
&= \prod_i \mathsf{E}(\Phi_\lambda(Q_r(z_i))) = (\lambda(2r)^d)^n
\end{aligned}$$

Now using Lemma 2.2.4, we can see that for $r < \frac{3^d-1}{\lambda 2^d}$, the expected number of infinite paths in $\mathbb{Z}^d_{2r}(V^r_\Phi, \geq 1)$ is 0. This disproves percolation in this discrete model and so $C(\Phi, r)$ also does not percolate for this value of r. □

6.3.3 Word percolation

Word percolation has been already studied for Poisson pp in Chapter 3. Here we give the obvious extended definition for a general pp and prove the existence of a phase transition for sub-Poisson pp. In particular, it also proves phase transition for AB percolation on sub-Poisson pp.

A *k-word* ($k \in \mathbb{N}$) is an infinite sequence $w := \{w_i\}_{i\geq 1} \in [k]^\infty$. Let $\mathbf{p} := (p_i)_{i=1}^k$ be a probability distribution such that $p_* := \min_i p_i > 0$. Given a pp Φ, let $\tilde{\Phi} := \{(X, I_X)\}_{X\in\Phi}$ be the marked pp with the marks $\{I_X\}_{X\in\Phi}$ being i.i.d. distributed in $[k]$ with distribution $\mathbf{p}$. *A k-word w occurs* in $(\tilde{\Phi}, r)$ if there exists an infinite sequence $\{X_i\}_{i\geq 1}$ such that for all i, $B_{X_i}(r) \cap B_{X_{i+1}}(r) \neq \emptyset$ and $I_{X_i} = w_i$.

Theorem 6.3.8. *Let $k \geq 1$ and $\Phi \leq_{ddcx} \Phi_\lambda$ for some $\lambda > 0$. Then there exists a $C(\lambda, p_*)$ such that $\forall r > C(\lambda, p_*)$, $\mathsf{P}\left(\text{all } k\text{-words occur in } (\tilde{\Phi}, r)\right) = 1$.*

Proof. Let $\Phi_i(.) = \tilde{\Phi}(. \times i)$ be the projected pp on $\mathbb{R}^d$, $l(x,y) = \mathbf{1}[x \in Q_{\frac{r}{2\sqrt{d}}}(y)]$ the response function and $V^r_{\Phi_i}(z)$ be the corresponding additive shot-noise field on $\mathbb{Z}^d(\frac{r}{\sqrt{d}})$. Define the following Bernoulli random fields on $\mathbb{Z}^d(\frac{r}{\sqrt{d}})$: $X^r_i(z) := \mathbf{1}[V^r_{\Phi_i}(z) \geq 1]$, $X^r(z) :=$

$\prod_{i=1}^{k} X_i^r(z)$. If $X^r(.)$ percolates in $\mathbb{L}^{*d}(\frac{r}{\sqrt{d}})$, then all words occur in $(\tilde{\Phi}, r)$. We shall now show that the former event has probability 1.

Set $\overline{X}_i^r(z) = 1 - X_i^r(z)$ for $i \in [k]$. By Proposition 4.2.3 and our assumption, $\forall i \in [k]$, $\Phi_i \leq_{ddcx} \Phi_{\lambda p_i}$. From the proof of Proposition 6.3.6 (or from (6.17)) and that the upper bound is decreasing in λ, we have that there exists a $C(\lambda, p_*)$ such that for all $r > C(\lambda, p_*)$,

$$\forall n, \mathsf{E}\left(\prod_{j=1}^{n} \overline{X}_i^r(z_j)\right) \leq q^n, \quad 1 \leq i \leq k,$$

for a $q < (k(3^d - 1))^{-k}$. From Lemma 6.3.2, for $r > C(\lambda, p_*)$,

$$\mathsf{E}\left(\prod_{j=1}^{n} \overline{X}^r(z_j)\right) \leq (kq^{\frac{1}{k}})^n < (3^d - 1)^{-n}.$$

By Peierls argument (Lemma 2.2.2), $X^r(z)$ percolates a.s. in $\mathbb{L}^{*d}(\frac{r}{\sqrt{d}})$ and so all words occur in $(\tilde{\Phi}, r)$ a.s.. □

6.3.4 Percolation in SINR graphs

SINR graphs have been introduced informally in Section 1.1.2 and capacity of a link and connectivity probability have been studied in Section 5.2.3. Since we need to impose a graph based on SINR on the pp, we modify our framework for this section. In this subsection, we shall work only in $\mathbb{R}^2$. The parameters of the model are non-negative numbers P(signal power), N(environmental noise), γ, T(SINR threshold) and an attenuation function $\ell : \mathbb{R}_+^2 \to \mathbb{R}_+$ satisfying the following assumptions :

$$\begin{aligned}
\ell(x, y) &= l(|x - y|) \text{ for some function } l : \mathbb{R}_+ \to \mathbb{R}_+; && (6.18)\\
l(x) &\leq 1; && (6.19)\\
l(.) \text{ is continuous} &\ ; \ l(.) \text{ strictly decreasing on its support;} && (6.20)\\
l(0) &\geq \frac{TN}{P}; && (6.21)\\
\int_0^\infty x l(x) \mathrm{d}x &< \infty. && (6.22)
\end{aligned}$$

The first three assumptions are from the physics of the wave-propagation and the last two are to ensure that the model does not degenerate. In particular, if Assumption (6.21) is

violated, SINR ratio (see (6.24)) will never be greater than the threshold T. For a Poisson pp, the interference (see (6.23)) is *a.s.* finite iff Assumption (6.22) holds ([Daley 1971]).

Given a point process Φ, the interference generated due to the point process at a location x is defined as

$$I_\Phi(x) := \sum_{X \in \Phi \setminus \{x\}} l(|X - x|). \tag{6.23}$$

The interference is an additive shot-noise field (see Definition 4.1.2) of the pp. The SINR from x to y with interference from Φ is defined as

$$SINR(x, y, \Phi, \gamma) := \frac{Pl(|x - y|)}{N + \gamma P I_{\Phi \setminus \{x\}}(y)}. \tag{6.24}$$

Definition 6.3.9. *Let Φ_B and Φ_I be two pp. Let $P, N, T > 0$ and $\gamma \geq 0$. The SINR graph is defined as $G(\Phi_B, \Phi_I, \gamma) := (\Phi_B, E(\Phi_B, \Phi_I, \gamma))$ where $E(\Phi_B, \Phi_I, \gamma) := \{\langle X, Y\rangle \in \Phi_B^2 : SINR(Y, X, \Phi_I, \gamma) > T, SINR(X, Y, \Phi_I, \gamma) > T\}$. The SNR graph(i.e, the graph without interference, $\gamma = 0$) is defined as $G(\Phi_B) := (\Phi_B, E(\Phi_B))$ where $E(\Phi_B) := \{\langle X, Y\rangle \in \Phi_B^2 : SINR(X, Y, \Phi_B, 0) > T\}$.*

Observe that the SNR graph $G(\Phi)$ is the same as the graph $C(\Phi, r_l)$ with $2r_l = l^{-1}(\frac{TN}{P})$. Also, when $\Phi_I = \emptyset$, we shall omit it from the parameters of the SINR graph. Recall that percolation in the above graphs is existence of an infinite connected component in the graph-theoretic sense.

Poissonian back-bone nodes

Firstly, we consider the case when the backbone nodes (Φ_B) form a Poisson pp and in the next section, we shall relax this assumption. When $\Phi_B = \Phi_\lambda$, the Poisson pp, we shall use $G(\lambda, \Phi_I, \gamma)$ and $G(\lambda)$ to denote the SINR and SNR graphs respectively. Recall that $\lambda_c(r)$ is the critical intensity for percolation of $C(\lambda, r)$. The following result guarantees the existence of a $\gamma > 0$ such that for any sub-Poisson pp $\Phi = \Phi_I$, $G(\lambda, \Phi, \gamma)$ will percolate provided $G(\lambda)$ percolates i.e, the SINR graph percolates for small interference values when the corresponding SNR graph percolates.

Theorem 6.3.10. *Let $\lambda > \lambda_c(r_l)$ and $\Phi \leq_{idcx} \Phi_\mu$ for some $\mu > 0$. Then $\exists \gamma > 0$ such that $G(\lambda, \Phi, \gamma)$ percolates.*

Note that we have not assumed the independence of Φ and Φ_λ. In particular, Φ could

be $\Phi_\lambda \cup \Phi_0$ where Φ_0 is an independent sub-Poisson pp. The case $\Phi_0 = \emptyset$ was proved in [Dousse et al. 2006]. Our proof follows a similar idea of coupling the continuum model with a discrete model. Such a coupling technique has been already used in the thesis for the proof of Theorem 3.1.5. As in [Dousse et al. 2006], it is clear that in case of $N \equiv 0$, the above result holds with $\lambda_c(r_l) = 0$.

Proof. By rescaling, we get that the critical radius for $C(\lambda, r)$ is $r(\lambda) = \sqrt{\frac{\lambda_c(r_l)}{\lambda}} r_l < r_l$ and thus $C(\lambda, r)$ with $r(\lambda) < r < r_l$ is still super-critical i.e, $C(\lambda, r)$ percolates.

For $b > 0$, recall that $\mathbb{L}^2(b)$ is the graph with vertex set $\mathbb{Z}^2(b)$, the expanded two-dimensional integer lattice, and endowed with the usual graph structure, that is, $x, y \in b\mathbb{Z}^2$ share an edge if $|x - y| = b$. Denote the edge-set by $\mathbb{E}^2(b)$. For any edge $e \in \mathbb{E}^2(b)$ denote the mid-point of e by $z_e = (x_e, y_e)$. For every horizontal edge e, define three rectangles $R_{ei}, i = 1, 2, 3$ as follows : R_{e1} is the rectangle $[x_e - 3b/4, x_e - b/4] \times [y_e - b/4, y_e + b/4]$; R_{e2} is the rectangle $[x_e - b/4, x_e + b/4] \times [y_e - b/4, y_e + b/4]$ and R_{e3} is the rectangle $[x_e + b/4, x_e + 3b/4] \times [y_e - b/4, y_e - b/4]$. Let $R_e = \cup_i R_{ei}$. The corresponding rectangles for vertical edges are defined similarly. These rectangles are depicted in Figure 6.1.

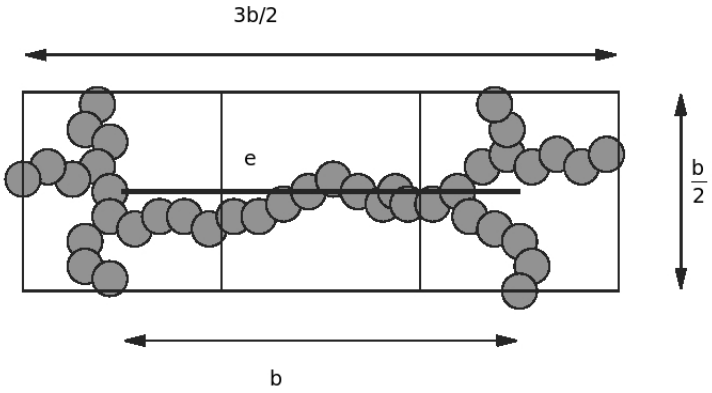

Figure 6.1: An horizontal edge e that satisfies the condition for $X_e = 1$. The balls are of radius $2r$, centered at points of Φ_λ. The figure is reproduced from [Dousse et al. 2006, Fig. 1]

We shall now define two Bernoulli random fields indexed by the edges of the lattice and show that for suitable choice of parameters the product random field percolates This will then be used to conclude that the SINR graph percolates. For ease of exposition, we shall break the proof into three steps.

Step 1: Let X_e be the indicator random variable for the event that there exists a left-right crossing of R_e by a component of $C(\lambda, r)$ and top-down crossings of R_{e1} and R_{e3} by a component of $C(\lambda, r)$. Refer Figure 6.1 for a pictorial representation of this event. Given $\epsilon > 0$, from [Meester and Roy 1996, Corollary 4.1] and 1-dependence of the random field, we can and do choose a b large enough such that (see also derivation of Eqn. (3.13) in the proof of Theorem 3.1.5), for all n

$$\mathsf{P}\left(X_{e_i} = 0, 1 \le i \le n\right) \le q_X^n, \tag{6.25}$$

where $q_X \le \epsilon$ and $e_i, 1 \le i \le n$ are collection of n distinct edges.

Step 2: Now, given a b, we define a second random field Y_e indexed by the edges. Define the following shifted version $\tilde{l}(.)$ of $l(.)$ as follows :

$$\tilde{l}(x) := \begin{cases} l(0) & x \le \frac{\sqrt{10}b}{4} \\ l(x - \frac{\sqrt{10}b}{4}) & x > \frac{\sqrt{10}b}{4} \end{cases}$$

Let $\tilde{I}(.) := \tilde{I}_\Phi(.)$ be the interference (Eqn. (6.23)) generated by a pp Φ with attenuation function $\tilde{l}(.)$. Define $Y_e = \mathbf{1}[\tilde{I}(z_e) \le M]$. By the choice of $R(e)$, we have that for all $z \in R(e), x \in \mathbb{R}^d$, $|z_e - x| \le |z - x| + \frac{\sqrt{10}b}{4}$ and thus from the definition of $\tilde{I}(.)$, it follows that $I_\Phi(z) \le \tilde{I}(z_e)$ for all $z \in R(e)$. Hence $Y_e = 1$ implies that $I(z) \le M$ for all $z \in R(e)$.

Now we show that for all $q_Y > 0$, there exists a $M < \infty$ such that

$$\mathsf{P}\left(Y_{e_i} = 0, 1 \le i \le n\right) = \mathsf{P}\left(\tilde{I}_\Phi(z_{e_i}) > M, 1 \le i \le n\right) \le q_Y^n, \tag{6.26}$$

where $e_i, 1 \le i \le n$ are any collection of n distinct edges. Then we have from Lemma 6.3.1 that

$$\mathsf{P}\left(\tilde{I}_\Phi(z_{e_i}) \ge M, 1 \le i \le n\right) \le \exp\{-snM\} \exp\{\mu \int_{\mathbb{R}^2} (e^{s\sum_{i=1}^n \tilde{l}(|x - z_{e_i}|)} - 1) \mathrm{d}x\}. \tag{6.27}$$

Thus, it is enough to show that the latter term in the above equations can be made as small as possible by increasing M. We shall first bound $\sum_{i=1}^n \tilde{l}(|x - z_{e_i}|)$ uniformly in n. The contribution to $\sum_{i=1}^n \tilde{l}(|x - z_{e_i}|)$ from the square in which x is located is atmost 4 as each of the corners contribute atmost 1 ($\tilde{l}(.) \le 1$). Inductively, one can calculate that

there are $4+8k$ nodes of the lattice at a distance of at least $\frac{kb}{\sqrt{2}}$. So,

$$\sum_{i=1}^{n} \tilde{l}(|x-z_{e_i}|) \leq \sum_{e\in\mathbb{Z}^2(b)} \tilde{l}(|x-z_e|) \leq 4+\sum_{k=1}^{\infty}(4+8k)\tilde{l}(\frac{kb}{\sqrt{2}}) \equiv K.$$

Since Assumption (6.22) on $l(.)$ implies the same for $\tilde{l}(.)$ also, we have that

$$\int_y^{\infty} x\tilde{l}(x)\mathrm{d}x < \infty \text{ for some } y > 0.$$

Thus, we have that $K < \infty$.

Now choose $s = \frac{1}{K}$ and so $s\sum_{i=1}^{n} \tilde{l}(|x-z_{e_i}|) \leq 1$ for all x. As $e^x - 1 \leq 2x$ for all $x \leq 1$, from equation (6.27), we have that for Φ and a $q > 0$,

$$\begin{aligned}
\mathsf{P}\left(\tilde{I}_\Phi(z_{e_i}) \geq M, 1 \leq i \leq n\right) &\leq \exp\{-\frac{nM}{K}\}\exp\{\frac{2\mu}{K}\int_{\mathbb{R}^2}\sum_{i=1}^{n}\tilde{l}(|x-z_{e_i}|)\mathrm{d}x\} \\
&= \exp\{-\frac{nM}{K}\}\exp\{\frac{2n\mu}{K}\int_{\mathbb{R}^2}\tilde{l}(|x|)\mathrm{d}x\} \\
&= \exp\{-\frac{1}{K}(M-2\mu\int_{\mathbb{R}^2}\tilde{l}(|x|)\mathrm{d}x)\}^n \leq q^n. \text{ (for } M \text{ large enough)}
\end{aligned}$$

Thus we have shown (6.26).

Step 3: Now, from (6.25), (6.26) and Lemma 6.3.2, we have that for any given $q > 0$, there exists b, M such that for all n,

$$\mathsf{P}\left(X_{e_i}Y_{e_i} = 0, 1 \leq i \leq n\right) \leq q^n,$$

where $e_i, 1 \leq i \leq n$ are collection of n distinct edges. Thus by Peierls argument (Lemma 2.2.3), the random field $W_e = X_eY_e$ percolates. Choose $\gamma = \frac{N}{PM}(\frac{l(2r)}{l(2r_l)} - 1) > 0$. Suppose $Y_e = 1$ (and so $I_\Phi(z) \leq M$ for all $z \in R(e)$), then for $x, y \in R(e)$ and $\|x-y\| \leq 2r$ we have that

$$\begin{aligned}\frac{Pl(|x-y|)}{N+\gamma I_{(\Phi)\setminus\{x\}}(y)} &\geq \frac{Pl(|x-y|)}{N+\gamma PM}\\ &\geq \frac{Pl(2r)}{N+\gamma PM}\\ &= \frac{Pl(2r_l)}{N} = T.\end{aligned}$$

Thus, there exists $\gamma > 0$ such that $\langle X, Y\rangle \in E(\Phi_\lambda, \Phi, \gamma)$ for any $X, Y \in R(e) \cap \Phi_\lambda$ with $\|X - Y\| \leq 2r$. So all edges of $C(\lambda, r)$ in $R(e)$ are preserved in the SINR graph also when $Y_e = 1$. Crossings in adjacent edges e and f overlap and hence they are in the same connected component. Consider the crossings in the infinite component induced by W_e (i.e, crossings in those boxes with $W_e = 1$). These overlap and form an infinite component of $C(\lambda, r)$. Since any two adjacent vertices in such an infinite component also will be in the same component of SINR graph, it follows that the infinite component will be preserved in the SINR graph i.e, the SINR graph percolates.

□

Non-Poissonian back-bone nodes

We shall now consider the case when the backbone nodes are formed by a sub-Poisson pp. In this case, we can give a weaker result, namely that with an increased signal power (i.e, much greater than the critical power), the SINR graph will percolate for small intereference parameter $\gamma > 0$.

Theorem 6.3.11. *Let $\Phi \leq_{ddcx} \Phi_\lambda$ for some $\lambda > 0$ and $\Phi_I \leq_{idcx} \Phi_\mu$ for some $\mu > 0$ and also assume that $l(x) > 0$ for all $x \in \mathbb{R}_+$. Then there exists $P, \gamma > 0$ such that $G(\Phi, \Phi_I, \gamma)$ percolates.*

As in Theorem 6.3.10, we have not assumed the independence of Φ_I and Φ. For example, $\Phi_I = \Phi \cup \Phi_0$ where Φ and Φ_0 are independent sub-Poisson pp. As can be seen from the proof, the above theorem is valid in higher dimensions provided that $\sum_{z\in\mathbb{Z}^d(r)} l(|x-z|)$ is a.s. bounded for all $x \in \mathbb{R}^d$ and any discretisation $\mathbb{Z}^d(r)$. Let us also justify the assumption of unbounded support for $l(.)$. Suppose that $r = \sup\{x : l(x) > 0\} < \infty$. Then if $C(\Phi, r)$ is sub-critical, $G(\Phi, \Phi_I, \gamma)$ will be sub-critical for any Φ_I, P, γ.

Proof. Consider the lattice $\mathbb{L}^2(\frac{r}{\sqrt{2}})$. Define random fields $X_z := \mathbf{1}[\Phi(Q_{\frac{r}{2\sqrt{2}}}(z)) \geq 1]$ and

$Y_z := \mathbf{1}[\tilde{I}(z) \leq M]$ on $\mathbb{Z}^2(\frac{r}{\sqrt{2}})$ where $\tilde{I}(.)$ is defined using $\tilde{l}(.)$ as in Equation (6.26) in the proof of Theorem 6.3.8. From Propositon 6.3.6, for a $q > 0$ we have $r(q) < \infty$ such that for all $r > r(q)$,

$$\mathsf{P}\left(X_{z_i} = 0, 1 \leq i \leq n\right) \leq q^n, \tag{6.28}$$

where $z_i, 1 \leq i \leq n$ are distinct collection of n vertices in the lattice. Since $l(.)$ has unbounded support, we can choose P large enough such that $2r_l = l^{-1}(\frac{TN}{P(q)}) > 2r(q)$. Choose r such that $r_l > r > r(q)$. Fixing the lattice $\mathbb{L}^2(\frac{r}{\sqrt{2}})$ and proceeding as in the proof of Eqn. (6.26), we get that for a $q > 0$, there exists $M(q) < \infty$ such that for $M > M(q)$

$$\mathsf{P}\left(Y_{z_i} = 0, 1 \leq i \leq n\right) \leq q^n. \tag{6.29}$$

Now from Equations (6.28), (6.29) and Lemma 6.3.2, we have that for a q sufficiently small, the Bernoulli random field $W_z = X_z Y_Z$ percolates on $\mathbb{L}^2(\frac{r}{\sqrt{2}})$ for $2r(q) < 2r < 2r_l = l^{-1}(\frac{TN}{P(q)})$ and $M > M(q)$. Suppose that for two adjacent sites z, z_*, we have that $W_z = 1, W_{z_*} = 1$. Let $X, Y \in \Phi \cap (Q_{\frac{r}{2\sqrt{2}}}(z)) \cap Q_{\frac{r}{2\sqrt{2}}}(z_*))$. Since $X_z X_{z_*} = 1$, we can always choose two such points. Then we have that $|X - Y| \leq 2r \leq 2r_l$ and also $I_{\Phi_I}(X) \leq M, I_{\Phi_I}(Y) \leq M$ as $Y_z \leq M, Y_{z_*} \leq M$. Now for $\gamma = \frac{N}{PM}(\frac{l(2r)}{l(2r_l)} - 1) > 0$,

$$\begin{aligned}
\frac{Pl(|X-Y|)}{N + \gamma I_{\Phi_I \setminus \{X\}}(Y)} &\geq \frac{Pl(|X-Y|)}{N + \gamma PM} \\
&\geq \frac{Pl(2r)}{N + \gamma PM} \\
&= \frac{Pl(2r_l)}{N} = T.
\end{aligned}$$

Thus whenever an edge exists between z, z_* in the sub-graph induced by the Bernoulli random field $W_.$, the points of the pp Φ that lie in the cubes centered at z, z_* are in the same component. Arguing as in the proof of the Theorem 6.3.10, we have that there exists $P < \infty$ and $\gamma > 0$ such that $G(\Phi, \Phi_I, \gamma)$ percolates. □

6.3.5 Percolation in a parametric family of point processes

In Sections 6.3.2 and 6.3.3, we gave sufficient conditions on the pp for non-degeneracy of the critical radius for k-percolation and word percolation. In this section, we will derive similar results for a parametric family of sub-Poisson pp as a corollary of Lemma 6.3.1. If the family has the same finite range dependence, then these results as well as non-

percolation of clustered pp can be deduced as a corollary of [Liggett et al. 1997, Theorem 1.3].

For a family of stationary pp, $\{\Phi^\lambda\}_{\lambda>0}$, define

$$\lambda_c^k(r) := \inf\{\lambda : \mathsf{P}\left(C_k(\Phi^\lambda, r) \text{ percolates}\right) > 0\}.$$

As nothing is known about the monotonicity of Φ^λ, it is not necessarily the case that $C_k(\Phi^\lambda, r)$ percolates for $\lambda > \lambda_c^k(r)$.

Proposition 6.3.12. *Let $\{\Phi^\lambda\}_{\lambda>0}$ be a family of pp and $r > 0$. Then there exist constants $\underline{\lambda}(r), \overline{\lambda}(r,k)$ such that*

$$\forall \lambda > 0, \Phi^\lambda \leq_{idcx} \Phi_\lambda \quad \Rightarrow \quad \lambda_c(r) \geq \underline{\lambda}(r) > 0. \tag{6.30}$$

$$\forall \lambda > 0, \Phi^\lambda \leq_{ddcx} \Phi_\lambda \quad \Rightarrow \quad \forall k, \lambda_c^k(r) \leq \overline{\lambda}(r,k) < \infty. \tag{6.31}$$

A pp is said to be R-independent if for bBs $A_i, 1 \leq i \leq n$ such that $d(A_i, A_j) > R$ for all i, j we have that the random variables $\Phi(A_i), 1 \leq i \leq n$ are independent.

Proposition 6.3.13. *Let $\{\Phi^\lambda\}_{\lambda>0}$ be a family of stationary R-independent pp. Suppose that*

$$\lim_{\lambda \to 0} \mathsf{P}\left(\Phi^\lambda(Q_r) \geq 1\right) = 0. \tag{6.32}$$

Then $\lambda_c^1(r) > 0$. Now if

$$\lim_{\lambda \to \infty} \mathsf{P}\left(\Phi^\lambda(Q_{\frac{r}{2\sqrt{d}}}) \geq k\right) = 1, \tag{6.33}$$

then $\lambda_c^k(r) < \infty$ and further for any k, $C_k(\Phi^\lambda, r)$ percolates for λ large enough.

The following proposition combined with examples in Chapter 5 provides many examples satisfying the conditions of the above theorem.

Proposition 6.3.14. *Let $\{\Phi^\lambda\}_{\lambda>0}$ be a family of R-independent pp. Then*

$$\forall \lambda > 0, \Phi^\lambda \leq_{idcx} \Phi_\lambda \quad \Rightarrow \quad \lim_{\lambda \to 0} \mathsf{P}\left(\Phi^\lambda(Q_r) \geq 1\right) = 0. \tag{6.34}$$

$$\forall \lambda > 0, \Phi^\lambda \leq_{ddcx} \Phi_\lambda \quad \Rightarrow \quad \lim_{\lambda \to \infty} \mathsf{P}\left(\Phi^\lambda(Q_{\frac{r}{2\sqrt{d}}}) \geq k\right) = 1, \forall k. \tag{6.35}$$

$$\forall \lambda > 0, \Phi_\lambda \leq_{ddcx} \Phi^\lambda \quad \Rightarrow \quad \lim_{\lambda \to 0} \mathsf{P}\left(\Phi^\lambda(Q_r) \geq 1\right) = 0. \tag{6.36}$$

Further, (6.32) holds when Φ^λ is a Cox($\Lambda^\lambda(.)$) pp where $\mathsf{E}\left(\Lambda^\lambda(.)\right) = \lambda\|.\|$.

Before moving onto the proofs, we shall briefly contrast the two Propositions 6.3.12 and 6.3.13. The latter is the only result in the chapter that can be used to ascertain non-degeneracy of critical radius for clustered point processes. The main limitation as regards considering a parametric family of pp is that many pp depend on two or more parameters unlike the Poisson pp which is characterized by a single parameter. In such cases, it is not not natural to parametrize it in terms of a single parameter. The second problem with parametrization that can affect Proposition 6.3.13 alone is that the dependence structure of the pp can change with the parametrization. The more natural way of looking at comparing percolative properties of two pp is to look at the critical radius as we have done in Section 6.3.2. We think this is the better approach, especially when one wants to see the impact of the dependence structures of the pp on their percolative properties.

Proof. (Proposition 6.3.12.) For the first statement, we use the bound in (6.15) and observe that as in that proof, we can first choose a s large enough and then λ small enough such that $C(\Phi^\lambda, r)$ does not percolate. This choice is dependent on λ alone and hence the bound is uniform. For the second statement, we use the bound in (6.16) and deduce that for a s and large enough λ, $C_k(\Phi^\lambda, r)$ percolates through Peierls argument. The bound is again uniform in this case too. □

As in Proposition 6.3.6, we can get an improved bound for $\lambda_c^1(r)$ using the arguments as in that proposition. We shall state [Liggett et al. 1997, Theorem 1.3] for Bernoulli random fields on their dependent graphs and then use it for ascertaining non-trivial phase transitions in finite-range dependent pp (Proposition 6.3.13). Recall that the critical probability of independent site percolation on G is defined as $p_c(G) := \inf\{p : \mathsf{P}\,(G(p) \text{ percolates}) > 0\}$. It is known that for close-packed lattices, $0 < p_c(\mathbb{L}^{*d}) < 1$.

Theorem 6.3.15. *Let $G = (V, E)$ be a graph with countable vertex set V and maximum vertex degree k. Let $\{X_v\}_{v\in V}$ be a stationary Bernoulli random field with G as the dependency graph (for definition, see the paragraph before Theorem 3.4.3). Denote the induced measure on $\{0,1\}^V$ by $\nu(.)$ and let Π_p be the product measure on $\{0,1\}^V$ with p being the 'open' probability. Suppose we have,*

$$\mathsf{P}\,(X_v = 1) \geq p, \;\; q = 1 - p \leq \frac{k^k}{(k+1)^{k+1}}.$$

Then there exists $\rho(p) > 0$ such that $\Pi_{\rho(p)}(.) \leq_{st} \nu(.)$. Further, $\lim_{p\to 1} \rho(p) = 1$. Suppose

we have,

$$\mathsf{P}\left(X_v = 1\right) \leq p, \;\; p \leq \frac{k^k}{(k+1)^{k+1}}.$$

Then for the same $\rho(.)$ *as above,* $\Pi_{1-\rho(1-p)}(.) \geq_{st} \nu(.)$.

Proof. (PROPOSITION 6.3.13.) First assume $\lim_{\lambda \to 0} \mathsf{P}\left(\Phi^\lambda(Q_r) \geq 1\right) = 0$. Define the graph $G_r^d(R) := (\mathbb{Z}^d(2r), \mathbb{E}^d(R))$ where the edge-set $\mathbb{E}^d(R) := \{\langle z, z_* \rangle : d(Q_r(z), Q_r(z_*)) \leq R\}$. This graph satisfies the assumptions of Theorem 6.3.15. Define $X(z) := \mathbf{1}[\Phi^\lambda(Q_r(z))) \geq 1]$. Let $\nu_X(.)$ be the induced measure on $\mathbb{Z}^d$. Choose a p such that $1 - \rho(1-p) < p_c(d)$ and so the Bernoulli random field with distribution $\Pi_{1-\rho(1-p)}(.)$ does not percolate in $\mathbb{L}^{*d}(2r)$. Now choose λ large enough such that $\mathsf{P}\left(\Phi^\lambda(Q_r) \geq 1\right) \leq p$ and so $\nu_X(.) \leq \Pi_{1-\rho(1-p)}(.)$ and hence $X(z)$ does not percolate in $\mathbb{L}^{*d}(2r)$. This implies that $C(\Phi^\lambda, r)$ also does not percolate.

In the above one can replace r by $\frac{r}{\sqrt{d}}$ and define $X(z) := \mathbf{1}[\Phi^\lambda(Q_{\frac{r}{2\sqrt{d}}}(r)) \geq \lceil k/2 \rceil]$. If $X(z)$ percolates in $\mathbb{L}^{*d}(2r)$, then $C_k(\Phi^\lambda, r)$ also percolates. Now arguing as above, we have the required result. □

Proof. (PROPOSITION 6.3.14.) Note that from Proposition 4.3.5, we have that for any $k \geq 1, r > 0$ and two pp Φ_1, Φ_2

$$\Phi_1 \leq_{ddcx} \Phi_2 \Rightarrow \mathsf{P}\left(\Phi_1(Q_r) \geq k\right) \geq \mathsf{P}\left(\Phi_2(Q_r) \geq k\right).$$

By standard computations with Poisson pp, this proves Equations (6.36) and (6.35). As for Eqn. (6.34), note that

$$\mathsf{P}\left(\Phi^\lambda(Q_r) \geq 1\right) \leq \mathsf{E}\Big(\Phi^\lambda(Q_r)\Big) \leq \lambda(2r)^d.$$

□

Notations and Symbols

Basic Notation

- $A + B := \{x + y : x \in A, y \in B\}$, the Minowski sum of two sets $A, B \subset \mathbb{R}^d$.
- $A^{(r)} := A + B_O(r)$, the r^{th} parallel set of A.
- $B_x(r)$ - the closed Euclidean ball centred at x and of radius r.
- $\mathbb{B}(\mathbb{D})$ - the σ-ring of Borel subsets of $\mathbb{D}$.
- $\mathbb{B}_b(\mathbb{D})$ or bBS - the class of bounded Borel subsets of $\mathbb{D}$.
- $\mathcal{C}$ - the semi-ring of $\mathbb{D}$.
- $C(\Phi, G)$ - The Boolean model with germs as point process Φ and i.i.d. grains distributed as G.
- $\mathbb{D}, \mathbb{D}'$ - Polish spaces.
- $\mathbb{E}^d$ - edge set in the usual d-dimensional integer lattice
- $\mathbb{E}^{*d}$ - edge set in the close-packed d-dimensional integer lattice,
- $\mathfrak{F}$ - class of functions.
- G or S - generic grain in a Boolean model.
- $\mathbb{K}_d$ - class of compact sets in $\mathbb{R}^d$.
- $[k] = \{1, \ldots, k\}, k \in \mathbb{N}$.
- Λ - random measure.

- $\mathbb{L}^d$ - the usual d-dimensional integer lattice viewed as a graph with nearest neighbour bonds.
- $\mathbb{L}^{*d}$ - the close-packed d-dimensional integer lattice viewed as a graph with diagonal bonds as well.
- $\mathbb{M} = \mathbb{M}(\mathbb{D})$ be the space of non-negative Radon measures on $\mathbb{D}$.
- $\mathcal{M}$ - the canonical Borel σ-algebra on $\mathbb{M}$.
- $\mathbb{N}$ - set of Natural numbers.
- O - the origin in $\mathbb{R}^d$.
- Φ - point process.
- $\tilde{\Phi}$ - marked point process.
- Φ_λ - Poisson point process with intensity λ.
- $\mathbb{R}^d$ - d-dimensional Euclidean lattice.
- $\mathbb{R}_+$ - non-negative half of the real line. $\overline{\mathbb{R}_+} := \mathbb{R}_+ \cup \{\infty\}$.
- $\mathbb{Z}^d$ - d-dimensional integer lattice. $\mathbb{Z}_+$ - set of non-negative integers.
- $\inf \emptyset = \infty$ and $\sup \emptyset = 0$.

Other symbols

- $|\cdot|$ - the metric on $\mathbb{R}^d$ be given by the usual Euclidean norm
- $\|.\|$ - the Lebesgue measure on $\mathbb{R}^d$.
- $\xrightarrow{d}$ - convergence in distribution (weak convergence).
- $\leq_{\mathfrak{F}}$ - the stochastic order generated by $\mathfrak{F}$ functions.
- cx - convex.
- ccx - componentwise convex.

- *dcx* - directionally convex.
- *idcx* - increasing directionally convex.
- *ddcx* - decreasing directionally convex.
- *dcv* - directionally concave.
- *idcv* - increasing directionally concave.
- *ddcv* - decreasing directionally concave.
- *lo* - lower orthant order.
- *st* - strong.
- *sm* - supermodular.

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