

Adaptive and anisotropic finite element approximation: Theory and algorithms

Parameters

Position
Area
Aspect ratio and
orientation
Angles

Questions raised

Equivalence of
meshes and
metrics
Arbitrary degree
Smoothness
classes
Hierarchical
triangulations

Conclusion

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University Paris VI
Laboratoire Jacques Louis Lions

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Ph.D directed by Albert Cohen

Motivation : anisotropic phenomena

The solutions of many PDE's exhibit a **strongly anisotropic** behavior.

- ▶ **Boundary layers** in fluid simulation.
- ▶ **Spikes and edges** of metallic objects in electromagnetism.
- ▶ **Shockwaves** in transport equations.

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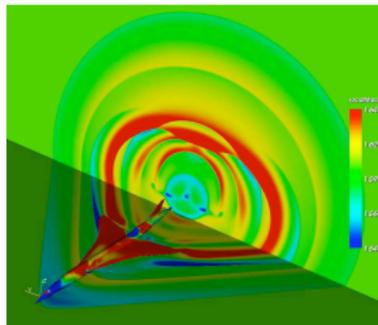
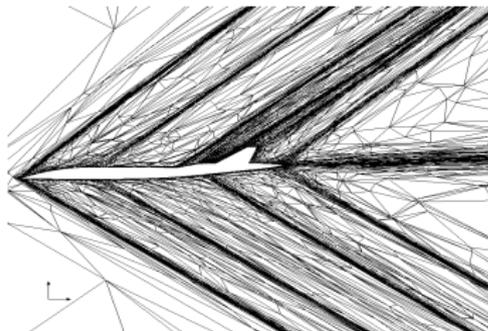


Figure: Fluid simulation around a supersonic plane (F. Alauzet).

Mesh optimization

Adaptive
anisotropic
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A general objective: reduce at best the trade-off between accuracy and numerical complexity.

- ▶ **Accuracy:** for example, the error between the solution and its approximation in some given norm.
- ▶ **Complexity:** typically tied to the cardinality of the mesh.

An appetizer: given a function $f : \Omega \rightarrow \mathbb{R}$ and an integer N , construct a triangulation \mathcal{T} of Ω which minimizes

$$\|\nabla(f - I_{\mathcal{T}} f)\|_{L^2(\Omega)},$$

over all triangulations such that $\#(\mathcal{T}) \leq N$, with $I_{\mathcal{T}}$ the piecewise linear interpolant.

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In the numerical examples $N = 500$ and

$$f(x, y) := \tanh(10(\sin(5y) - 2x)) + x^2y + y^3.$$

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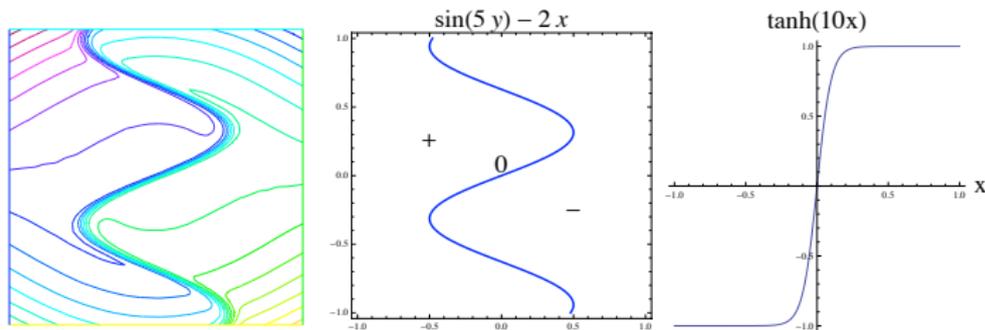


Figure: Sharp transition along the curve $\sin(5y) = 2x$, of width $1/10$.

A classical result

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain, let $f \in H^2(\Omega)$ and let \mathcal{T} be a triangulation.

Ciarlet-Raviart

On each $T \in \mathcal{T}$, the local error satisfies

$$\|\nabla(f - I_{\mathcal{T}} f)\|_{L^2(T)} \leq C_0 \frac{h_T^2}{r_T} \|d^2 f\|_{L^2(T)},$$

where $h_T := \text{diam}(T)$ and r_T is the radius of the largest disc inscribed in T , and C_0 is an absolute constant.

Consequence: with $h = \max_{T \in \mathcal{T}} h_T$

$$\|\nabla(f - I_{\mathcal{T}} f)\|_{L^2(\Omega)} \leq C(\mathcal{T}) h \|d^2 f\|_{L^2(\Omega)},$$

with $C(\mathcal{T}) = C_0 \max_{T \in \mathcal{T}} \frac{h_T}{r_T}$ that remains bounded for isotropic triangulations.

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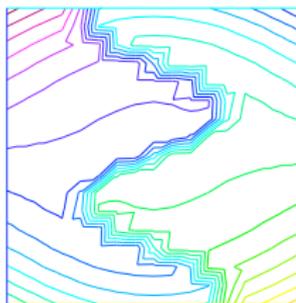
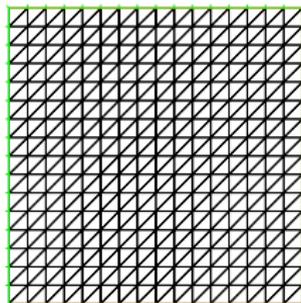
with $C(\mathcal{T}) = C_0 \max_{T \in \mathcal{T}} \frac{h_T}{r_T}$ that remains bounded for isotropic triangulations.

In terms of $N = \#(\mathcal{T})$, this gives

$$\sqrt{N} \|\nabla(f - I_{\mathcal{T}} f)\|_{L^2(\Omega)} \leq C'(\mathcal{T}) \|d^2 f\|_{L^2(\Omega)},$$

where $C'(\mathcal{T}) = C_0 \sqrt{|\Omega|} \frac{\max_{T \in \mathcal{T}} h_T^2 / r_T}{\min_{T \in \mathcal{T}} \sqrt{|T|}}$, that remains bounded for uniform triangulations:

$$h \sim h_T \sim r_T \sim \sqrt{|T|} \Rightarrow h \sim N^{-1/2}.$$



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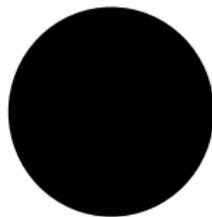
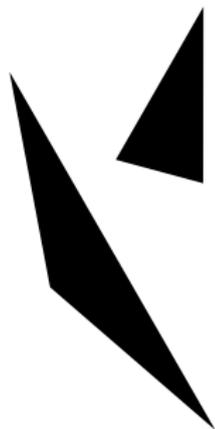
The parameters of a triangle.

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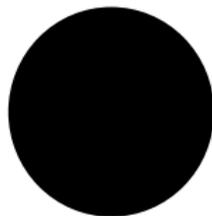
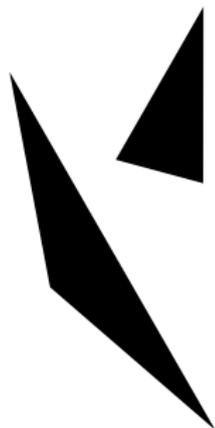
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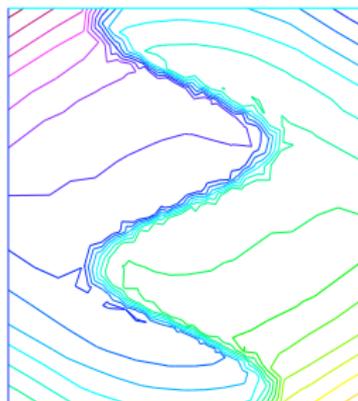
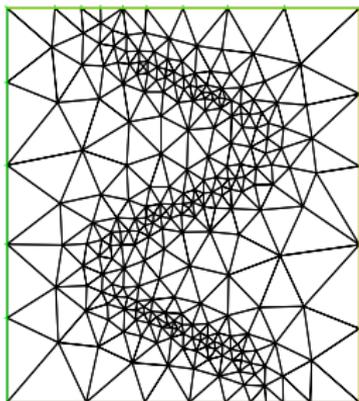
Isotropic meshes: the triangle seen as a disk.

Theorem (Adaptive approximation: DeVore-Yu)

For any $f \in W^{2,1}(\Omega)$, $\Omega =]0, 1[^2$, there exists a sequence $(\mathcal{T}_N)_{N \geq 2}$ of (isotropic) triangulations of Ω , $\#(\mathcal{T}_N) \leq N$, such that

$$\sqrt{N} \|\nabla(f - I_{\mathcal{T}_N} f)\|_{L^2(\Omega)} \leq C \|M(d^2 f)\|_{L^1(\Omega)}$$

$M(g)$: Hardy-Littlewood maximal function of g .



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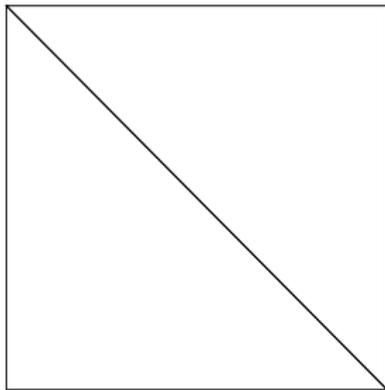
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Key principle : error equidistribution

Such sequences of triangulations may be obtained by a hierarchical refinement algorithm, starting from a coarse mesh.

- ▶ Refine the triangle with **largest local error**
 $\|\nabla(f - I_T f)\|_{L^2(T)}$.
- ▶ Propagate the refinement to preserve conformity.
- ▶ Iterate until prescribed number of triangles is met.



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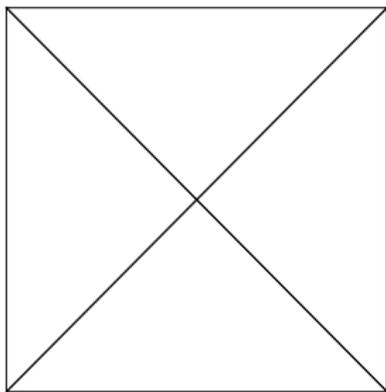
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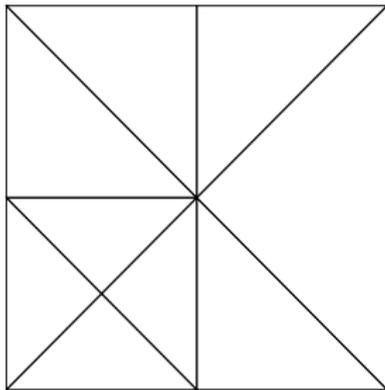
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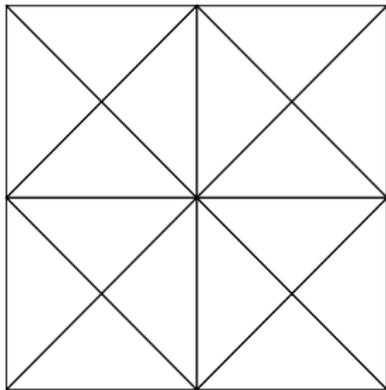
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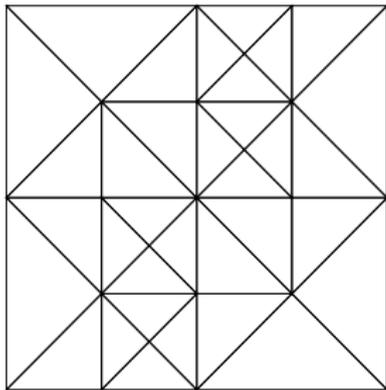
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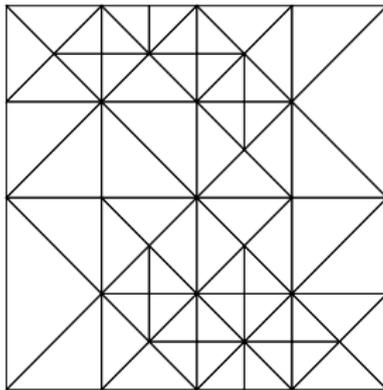
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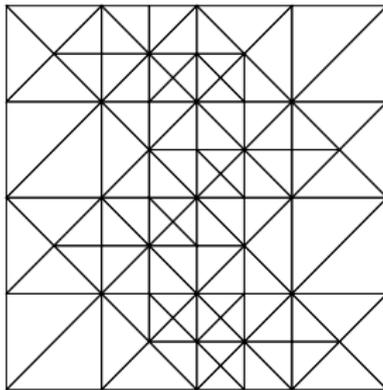
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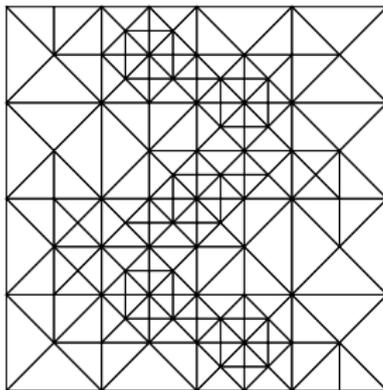
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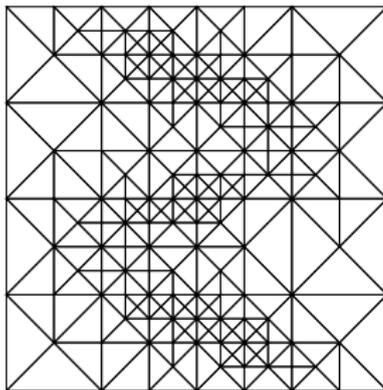
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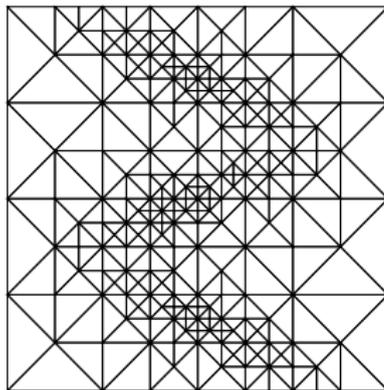
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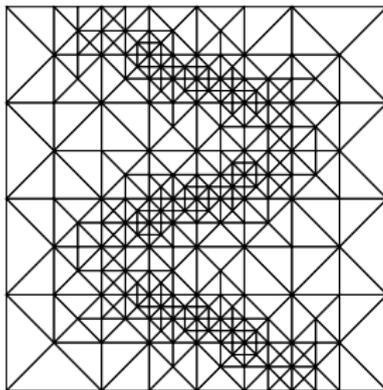
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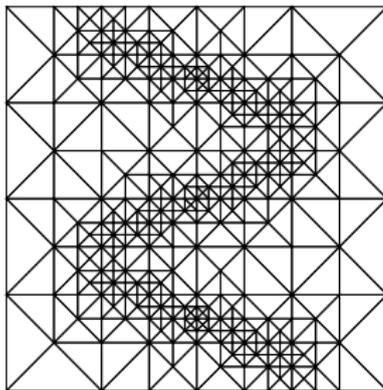
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Such sequences of triangulations may be obtained by a **hierarchical refinement algorithm**, starting from a coarse mesh.

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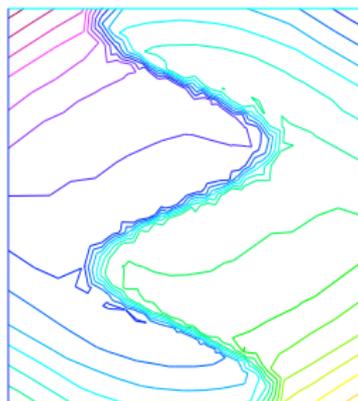
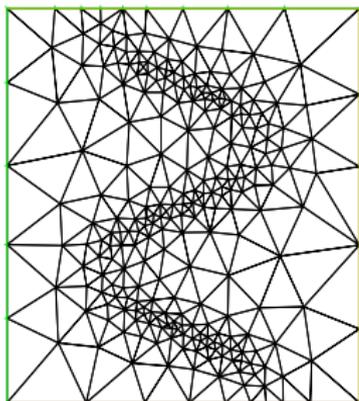
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Aspect ratio: the triangle seen as an ellipse.

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The ellipse of minimal area containing a triangle T is defined by

$$(z - z_T)^T \mathcal{H}_T (z - z_T) \leq 1,$$

where \mathcal{H}_T is a symmetric positive definite matrix and z_T is the barycenter of T .



Anisotropic mesh generation

Given a metric $H : \Omega \rightarrow S_2^+$ produce a triangulation \mathcal{T} such that : for any $T \in \mathcal{T}$ and any $z \in T$,

$$H(z) \simeq \mathcal{H}_T$$

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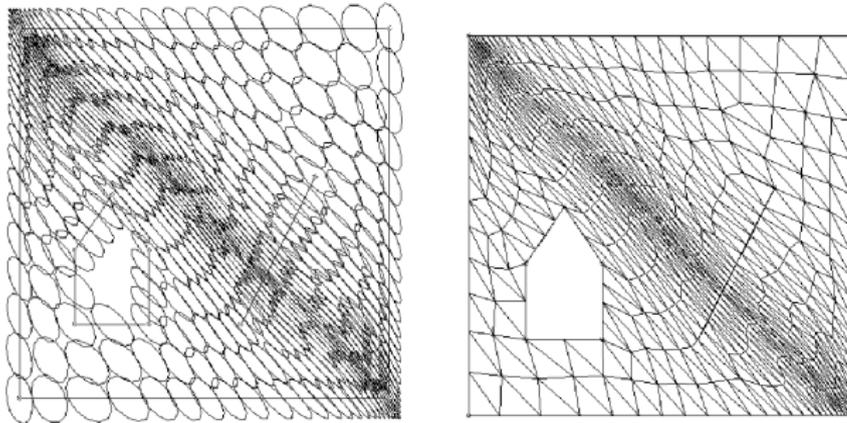


Figure: A metric and an adapted triangulation (credit: J. Schoen)

Theoretical results by Boissonat & al, Shewchuk & al.

$\pi = ax^2 + 2bxy + cy^2$: homogeneous quadratic polynomial.

$$L_G(\pi) := \inf_{\det H=1} \sup_{\mathcal{H}_T=H} \|\nabla(\pi - I_T \pi)\|_{L^2(T)}.$$

(Near) Minimizing matrix H

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Figure: Level lines of π (red, dashed), ellipse (blue, thick) associated to (near) optimal H which is proportional to the absolute value of the matrix associated to π .

Explicit equivalent of L_G

$$L_G(\pi) \simeq \sqrt{\|\pi\|} \sqrt[4]{|\det \pi|},$$

where $\|\pi\|$ and $\det \pi$ are the norm and determinant of the symmetric matrix associated to π .

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(Near) Minimizing matrix H

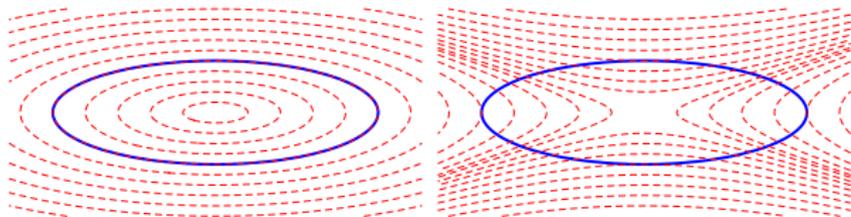


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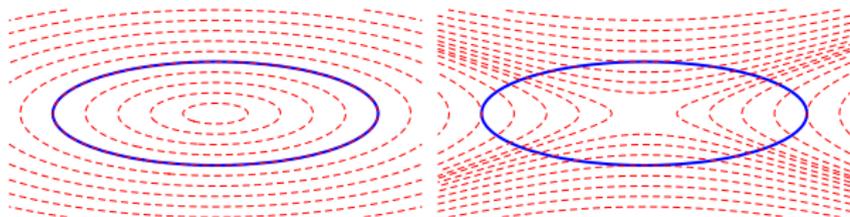


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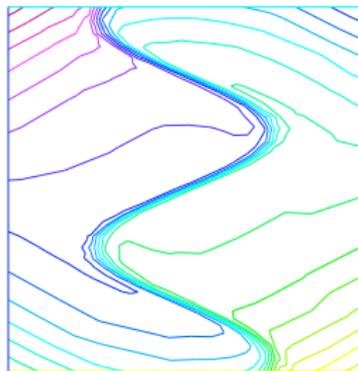
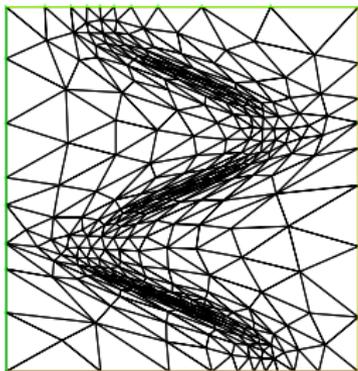
Local model :

$$f(x_0+x, y_0+y) = \alpha + (\beta x + \gamma y) + \underbrace{(ax^2 + 2bxy + cy^2)}_{\pi = \frac{1}{2} d^2 f(x_0, y_0)} + \mathcal{O}(|x|^3 + |y|^3)$$

Theorem

For any bounded polygonal domain Ω and any $f \in C^2(\bar{\Omega})$ there exists a sequence $(\mathcal{T}_N)_{N \geq N_0}$ of triangulations of Ω , $\#(\mathcal{T}_N) \leq N$, such that

$$\limsup_{N \rightarrow \infty} \sqrt{N} \|\nabla(f - I_{\mathcal{T}_N} f)\|_{L^2(\Omega)} \leq C \|L_G(d^2 f)\|_{L^1(\Omega)}$$



An unusual estimate

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$$\limsup_{N \rightarrow \infty} \sqrt{N} \|\nabla(f - I_{\mathcal{T}_N} f)\|_{L^2(\Omega)} \leq C \|L_G(d^2 f)\|_{L^1(\Omega)}$$

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- ▶ The quantity $L_G(d^2 f(z)) \simeq \sqrt{\|d^2 f(z)\|} \sqrt[4]{|\det(d^2 f(z))|}$ depends **nonlinearly** on f .

- ▶ Defining $A(f) := \|L_G(d^2 f)\|_{L^1}$ we generally do not have

$$A(f + g) \leq C(A(f) + A(g)).$$

- ▶ The estimate holds asymptotically as $N \rightarrow \infty$.

An earlier estimate of this type was known for the L^p -norm.

Theorem (Chen, Sun, Xu; Babenko)

If $f \in C^2(\bar{\Omega})$ and $1 \leq p \leq \infty$, then there exists a sequence $(\mathcal{T}_N)_{N \geq N_0}$ of triangulations of Ω , $\#(\mathcal{T}_N) \leq N$, such that

$$\limsup_{N \rightarrow \infty} N \|f - I_{\mathcal{T}_N} f\|_{L^p(\Omega)} \leq C \left\| \sqrt{|\det d^2 f|} \right\|_{L^{\frac{p}{p-1}}(\Omega)}, \quad \frac{1}{\tau} := 1 + \frac{1}{p}.$$

An unusual estimate

Adaptive
anisotropic
approx:
Theory and
algorithms

Jean-Marie
Mirebeau

$$\limsup_{N \rightarrow \infty} \sqrt{N} \|\nabla(f - I_{\mathcal{T}_N} f)\|_{L^2(\Omega)} \leq C \|L_G(d^2 f)\|_{L^1(\Omega)}$$

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Questions raised

Equivalence of
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Angles: the triangle seen as a triangle (!)

$\pi = ax^2 + 2bxy + cy^2$: homogeneous quadratic polynomial.

$$L_A(\pi) := \inf_{|T|=1} \|\nabla(\pi - I_T \pi)\|_{L^2(T)}.$$

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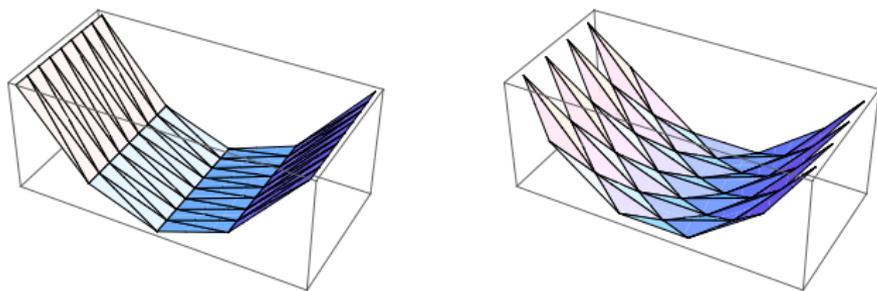


Figure: Interpolation of a parabola on a **acute** or **obtuse** mesh.

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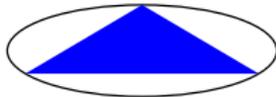
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(Near) Minimizing triangle

$$L_G \Rightarrow \frac{h_T}{r_T} \simeq \frac{1}{\sqrt{\epsilon}}$$



$$\pi = x^2 \epsilon + y^2$$

$$L_A \Rightarrow \frac{h_T}{r_T} \simeq \frac{1}{\epsilon}$$



Figure: The minimizing triangle for L_A has acute angles and is more anisotropic than the minimizing ellipse for L_G .

Explicit equivalent of L_A

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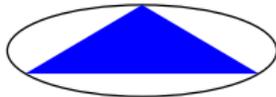
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Theorem

For any bounded polygonal domain Ω and any $f \in C^2(\overline{\Omega})$ there exists a sequence $(\mathcal{T}_N)_{N \geq N_0}$ of triangulations of Ω , $\#(\mathcal{T}_N) \leq N$, such that

$$\limsup_{N \rightarrow \infty} \sqrt{N} \|\nabla(f - I_{\mathcal{T}_N} f)\|_{L^2(\Omega)} \leq \left\| L_A \left(\frac{d^2 f}{2} \right) \right\|_{L^1(\Omega)}.$$

Furthermore for any admissible sequence $(\mathcal{T}_N)_{N \geq N_0}$ of triangulations, $\#(\mathcal{T}_N) \leq N$, one has

$$\liminf_{N \rightarrow \infty} \sqrt{N} \|\nabla(f - I_{\mathcal{T}_N} f)\|_{L^2(\Omega)} \geq \left\| L_A \left(\frac{d^2 f}{2} \right) \right\|_{L^1(\Omega)}.$$

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$$\sup_{N \geq N_0} \left(N^{\frac{1}{2}} \sup_{T \in \mathcal{T}_N} \text{diam}(T) \right) < \infty.$$

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Guideline of the upper estimate (heuristic)

The **asymptotically optimal** sequence is built using a two-scale local patching strategy. (Not suited for applications)

Parameters

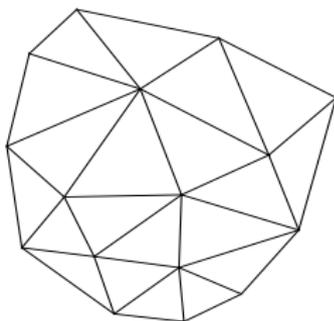
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- ▶ **Initial triangulation** of the domain.
- ▶ The interior of each cell is tiled with a triangle “optimally adapted” in size and shape to the Taylor development of f .
- ▶ Additional triangles at the interfaces ensure conformity.

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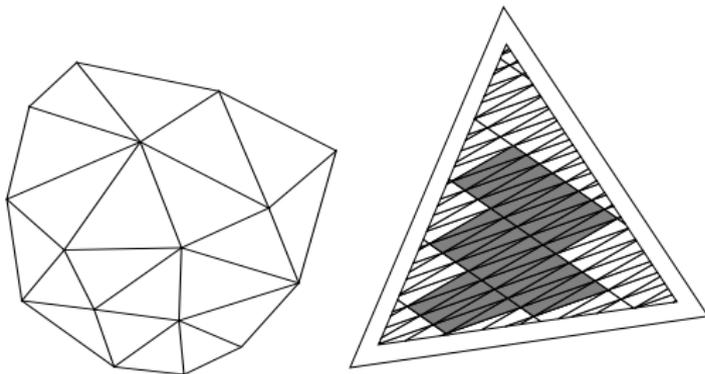
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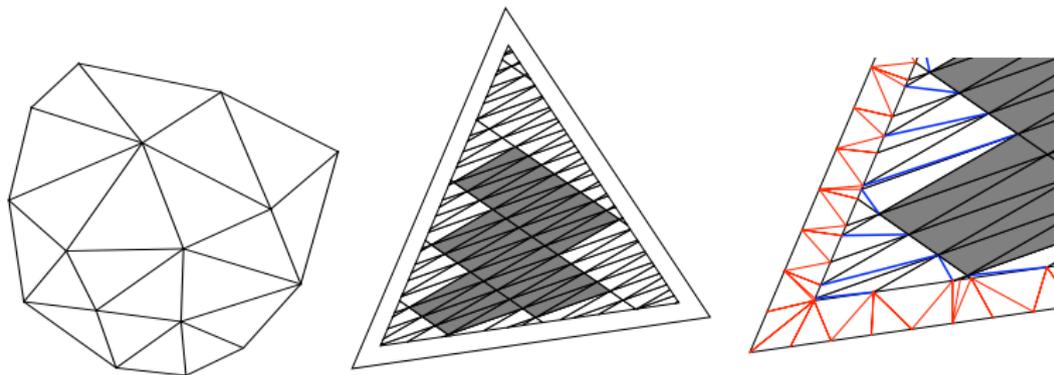
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Introduction : Parameters of a triangle

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Questions raised in the thesis

Equivalence of meshes and metrics

Finite elements of arbitrary degree

Anisotropic smoothness classes

Hierarchical anisotropic triangulations

Conclusion and perspectives

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Equivalence of meshes and metrics

Metrics and triangulations on \mathbb{R}^2

Definition (Equivalence triangulation/metric)

A (conforming) triangulation \mathcal{T} of \mathbb{R}^2 is C -equivalent to a metric $H \in C^0(\mathbb{R}^2, S_2^+)$ if for all $T \in \mathcal{T}$ and $z \in T$ one has

$$C^{-1}H(z) \leq \mathcal{H}_T \leq CH(z).$$

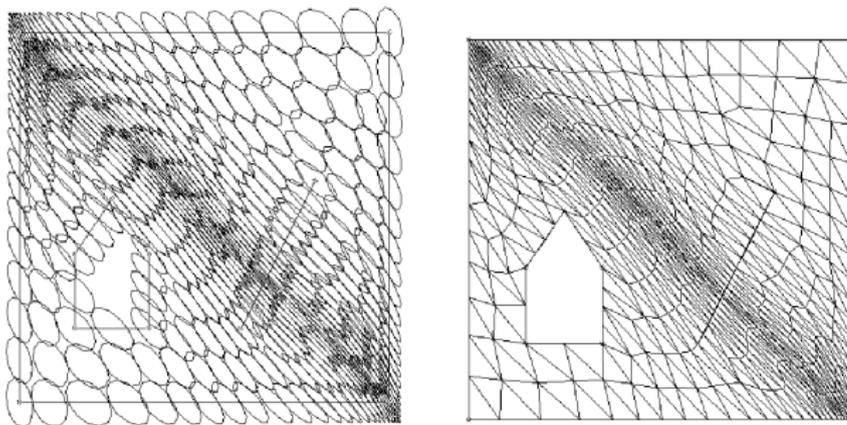


Figure: A metric and an equivalent triangulation, Credit : J. Schoen

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Definition (Equivalence collection of triangulations/ collection of metrics)

A collection \mathbb{T} of triangulations of \mathbb{R}^2 is equivalent to a collection $\mathbb{H} \subset C^0(\mathbb{R}^2, S_2^+)$ of metrics if there exists C such that

- ▶ $\forall T \in \mathbb{T}, \exists H \in \mathbb{H}$, such that T and H are C -equivalent.
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Isotropic triangulations

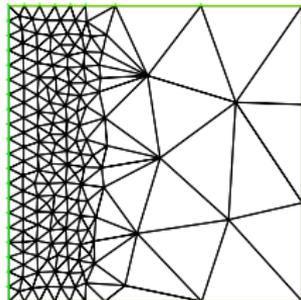
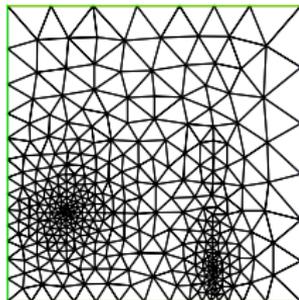
Theorem (reformulation of earlier work)

The collection \mathbb{T} of all triangulations \mathcal{T} satisfying for each $T \in \mathcal{T}$

$$\text{diam}(T)^2 \leq 4|T|$$

is equivalent to the collection \mathbb{H} of metrics H of the form

$$H(z) = \frac{\text{Id}}{s(z)^2} \quad \text{where} \quad |s(z) - s(z')| \leq |z - z'|$$



Triangulations
produced by
FreeFem

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From isotropic to anisotropic metrics

Isotropic "Lipschitz" metrics

$H(z) = s(z)^{-2} \text{Id}$. Two equivalent properties:

▶ (d) $\forall z, z' \in \mathbb{R}^2, |s(z) - s(z')| \leq |z - z'|$

▶ (r) $\forall z, z' \in \mathbb{R}^2, \left| \ln \left(\frac{s(z')}{s(z)} \right) \right| \leq d_H(z, z')$

where d_H denotes the Riemannian distance

$$d_H(z, z') := \inf_{\substack{\gamma(0)=z \\ \gamma(1)=z'}} \int_0^1 \sqrt{\gamma'(t)^T H(\gamma(t)) \gamma'(t)} dt.$$

Anisotropic "Lipschitz" metrics

$H(z) = S(z)^{-2}$. Two natural (but non-equivalent) generalizations:

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▶ (R)

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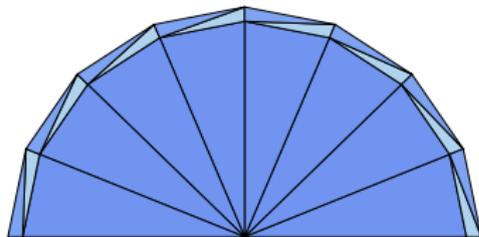
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Graded Triangulations

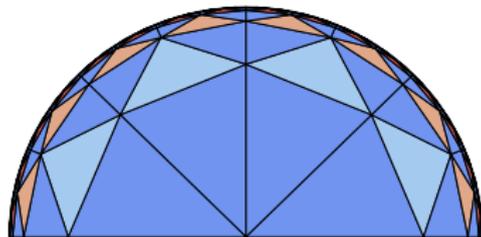
Definition

A triangulation \mathcal{T} of \mathbb{R}^2 is K -graded if for all $T, T' \in \mathcal{T}$,

$$T \text{ intersects } T' \Rightarrow K^{-1} \mathcal{H}_T \leq \mathcal{H}_{T'} \leq K \mathcal{H}_T.$$



Non Graded



Graded

Theorem

For any $K \geq K_0$ the collection \mathbb{T} of K -graded triangulations is equivalent to the collection \mathbb{H} of metrics satisfying (R).

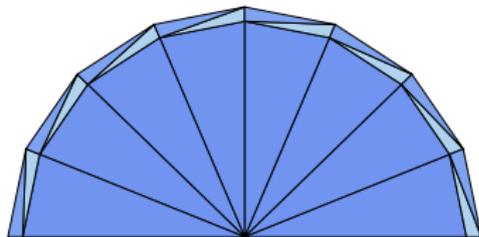
Key ingredient : mesh generation results by Labelle, Shewchuk.

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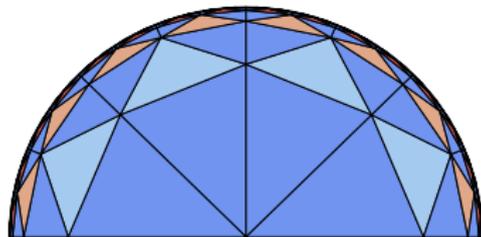
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Heuristic of the construction of \mathcal{T} from H

Construct a collection $\mathcal{V} \subset \mathbb{R}^2$ of sites which satisfies:

covering For all $z \in \mathbb{R}^2$, $d_H(z, \mathcal{V}) := \min_{v \in \mathcal{V}} d(z, v) \leq 1$.

separation For all $v \neq w \in \mathcal{V}$, $d_H(v, w) \geq 1$. (or $\geq \delta_0 > 0$).

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Conclusion

Connect sites when Anisotropic Voronoi regions intersect.

Euclidean case $\text{Vor}(v) := \{z; |z - v| = \min_{w \in \mathcal{V}} |z - w|\}$.

Peyré, & al $\text{Vor}(v) := \{z; d_H(z, v) = \min_{w \in \mathcal{V}} d_H(z, w)\}$.

Connect sites when $\text{Vor}(v) \cap \text{Vor}(w) \neq \emptyset$.

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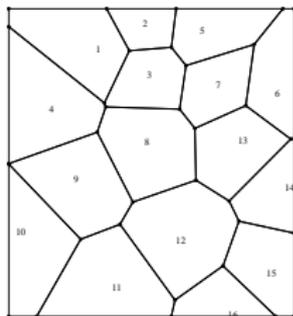
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Heuristic of the construction of \mathcal{T} from H

Construct a collection $\mathcal{V} \subset \mathbb{R}^2$ of sites which satisfies:

covering For all $z \in \mathbb{R}^2$, $d_H(z, \mathcal{V}) := \min_{v \in \mathcal{V}} d(z, v) \leq 1$.

separation For all $v \neq w \in \mathcal{V}$, $d_H(v, w) \geq 1$. (or $\geq \delta_0 > 0$).

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Conclusion

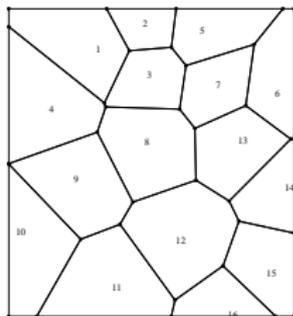
Connect sites when Anisotropic Voronoi regions intersect.

Euclidean case $\text{Vor}(v) := \{z ; |z - v| = \min_{w \in \mathcal{V}} |z - w|\}$.

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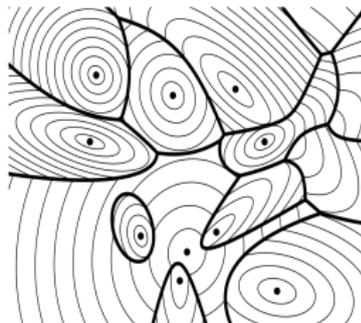
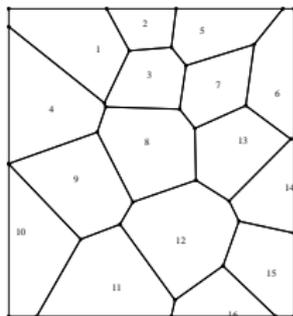
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QuasiAcute triangulations

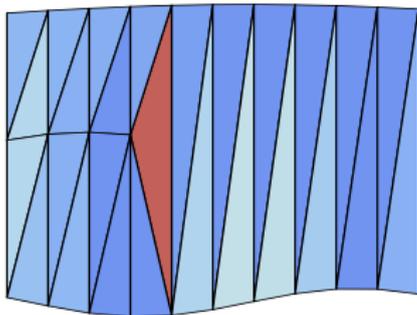
Definition

A triangulation \mathcal{T} is *K-QuasiAcute* if

▶ \mathcal{T} is *K-graded*.

▶ There exists a *K-refinement* \mathcal{T}' of \mathcal{T} such that any angle θ of any $T \in \mathcal{T}'$ satisfies

$$\theta \leq \pi - \frac{1}{K}.$$



\mathcal{T} : *K-QuasiAcute*

\mathcal{T}' : *K-refinement* of \mathcal{T} .

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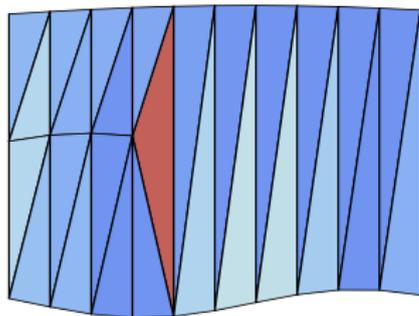
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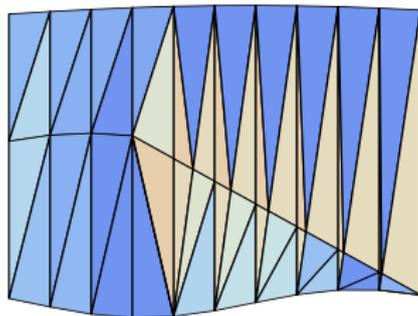
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Theorem

For all $K \geq K_0$ the collection \mathbb{T} of *K-QuasiAcute* triangulations is equivalent to the collection \mathbb{H} of metrics satisfying simultaneously (R) and (D).

A comparison: how to capture a curvilinear discontinuity.
Objective: layer of width δ of triangles covering a smooth curve, using an Isotropic, QuasiAcute or Graded triangulation.

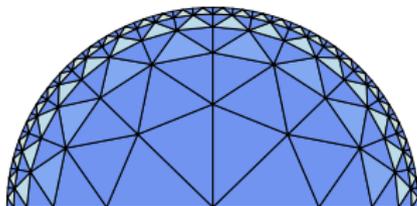
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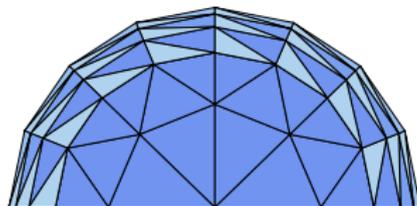
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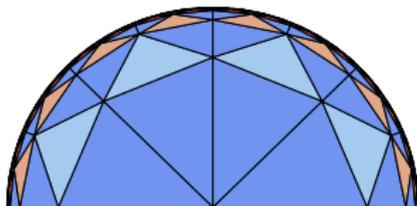
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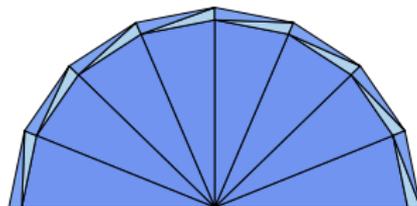
Isotropic
 $\#(\mathcal{T}) \simeq \delta^{-1}$



QuasiAcute
 $\#(\mathcal{T}) \simeq \delta^{-\frac{1}{2}} |\ln \delta|$



Graded
 $\#(\mathcal{T}) \simeq \delta^{-\frac{1}{2}}$



No restriction
 $\#(\mathcal{T}) \simeq \delta^{-\frac{1}{2}}$

Finite elements of arbitrary degree $m - 1$

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Measuring error in L^p :  J.-M. Mirebeau, *Optimal meshes for finite elements of arbitrary order*, Constructive Approximation, 2010.

Measuring error in $W^{1,p}$ semi-norm:  J.-M. Mirebeau, *Optimal meshes for finite elements of arbitrary order and $W^{1,p}$ norms*, to appear in Numerische Mathematik, 2011.

In this talk we only consider the $W^{1,2}$ semi-norm and L^2 norm.

For all $\pi \in \mathbf{H}_m$ (homogeneous polynomials of degree m)

$$L_A(\pi) := \inf_{|T|=1} \|\nabla(\pi - \mathbf{I}_T^{m-1} \pi)\|_{L^2(T)}.$$

\mathbf{I}_T^{m-1} : Lagrange interpolant of degree $m - 1$.

Theorem (Optimal asymptotic interpolation error)

For any bounded polygonal domain Ω , and any $f \in C^m(\overline{\Omega})$ there exists a sequence $(\mathcal{T}_N)_{N \geq N_0}$ of triangulations of Ω , $\#(\mathcal{T}_N) \leq N$, such that

$$\limsup_{N \rightarrow \infty} N^{\frac{m-1}{2}} \|\nabla(f - \mathbf{I}_{\mathcal{T}_N}^{m-1} f)\|_{L^2(\Omega)} \leq \left\| L_A \left(\frac{d^m f}{m!} \right) \right\|_{L^{\frac{2}{m}}(\Omega)}.$$

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Any *acute* triangle T , $|T| = 1$, such that \mathcal{H}_T is proportional to the following matrix is a (near) minimizer of the optimization defining $L_A(\pi)$.

► $m = 2$, $\pi = ax^2 + 2bxy + cy^2$, matrix: $\begin{pmatrix} a & b \\ b & c \end{pmatrix}^2$

► $m = 3$, $\pi = ax^3 + 3bx^2y + 3cxy^2 + dy^3$, matrix:

$$\mathcal{M}_A(\pi) := \sqrt{\begin{pmatrix} a & b \\ b & c \end{pmatrix}^2 + \begin{pmatrix} b & c \\ c & d \end{pmatrix}^2}.$$

Explicit minimizing triangle for $m > 3$: open problem.

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Optimal metric for the approximation of f (heuristic)

Set

$$H(z) := \lambda(\det M(z))^{\frac{-1}{2m}} M(z),$$

where $\lambda > 0$ is a sufficiently large constant and

- ▶ ($m = 2$), $M(z) \simeq [d^2 f(z)]^2$.
- ▶ ($m = 3$), $M(z) \simeq \mathcal{M}_A(d^3 f(z))$.

Mesh generation:

1. Produce a **QuasiAcute** triangulation \mathcal{T} which is C -equivalent to H .
2. **Interpolate** f on the refinement \mathcal{T}' on which angles are uniformly bounded.

⇒ Optimal estimate up to a fixed multiplicative constant.

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Polynomials on \mathbf{H}_m : $Q(\pi) = \tilde{Q}(a_0, \dots, a_m)$ if
 $\pi = a_0 x^m + a_1 x^{m-1} y + \dots + a_m y^m$.

Proposition (Explicit equivalent of L_A)

There exists a polynomial Q on \mathbf{H}_m , of degree r , such that

$$L_A(\pi) \simeq |Q(\pi)|^{1/r}$$

uniformly.

- ▶ $m = 2$, $L_A(\pi) \simeq \sqrt{|\det \pi|}$.
- ▶ $m = 3$, $L_A(\pi) \simeq \sqrt{|\det \mathcal{M}_A(\pi)|}$.
- ▶ $m \geq 4$: explicit polynomials Q are obtained using Hilbert's theory of invariants.

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- ▶ $m \geq 4$: *explicit polynomials Q are obtained using Hilbert's theory of invariants.*

Optimizing only the aspect ratio:

$$L_G(\pi) := \inf_{\det H=1} \sup_{\mathcal{H}_T=H} \|\nabla(\pi - \mathbb{I}_T^{m-1} \pi)\|_2.$$

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Proposition (Explicit minimizing ellipse, $m = 2$ or 3)

The matrix H such that $\det H = 1$ and which is proportional to the following is a (near) minimiser of the optimization problem defining $L_G(\pi)$.

- ▶ $m = 2$, $\pi = ax^2 + 2bxy + cy^2$, matrix : $\left| \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right|$.
- ▶ $m = 3$, $\pi = ax^3 + 3bx^2y + 3cxy^2 + dy^3$

$$\mathcal{M}_G(\pi) := \mathcal{M}_A(\pi) + \left(\frac{-\text{disc}(\pi)}{\|\pi\|} \right)_+^{\frac{1}{3}} \text{Id},$$

where $\text{disc } \pi = 4(ac - b^2)(bd - c^2) - (ad - bc)^2$.

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Interpolate f on a mesh \mathcal{T} which is C -equivalent to H
 \Rightarrow estimate in terms of L_G .

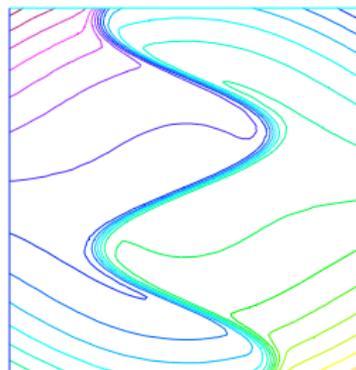
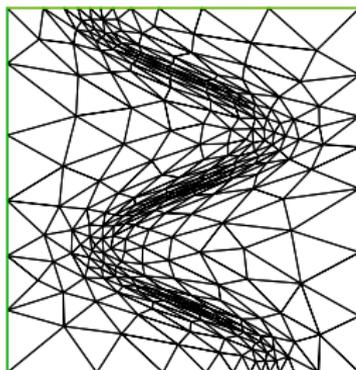


Figure: Interpolation with anisotropic \mathbf{P}_2 elements.

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Numerical experiments : $\|\nabla(f - I_{\mathcal{T}}^{m-1} f)\|_{L^2}$, with 500 triangles.

	Uniform	Isotropic	Based on L_G	Based on L_A
P_1	110	51	11	?
P_2	79	14	0.88	?

Anisotropic smoothness classes: from finite element approximation to image models

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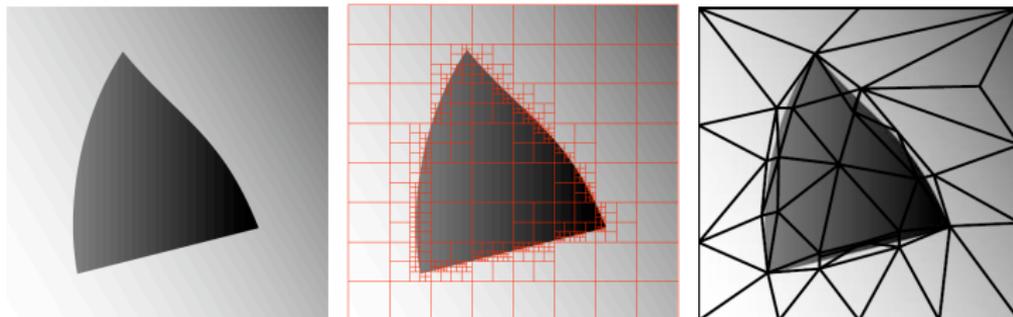


Figure: A [cartoon function](#), and an adapted triangulation. Picture :
Gabriel Peyré

📄 A. Cohen, J.-M. Mirebeau, *Anisotropic smoothness classes:
from finite element approximation to image models*, Journal of
Mathematical Imaging and Vision, 2010.

Approximation of cartoon functions

If $g = \sum_{1 \leq i \leq r} g_i \chi_{\Omega_i}$ where $g_i \in C^2(\overline{\Omega}_i)$ and $\partial\Omega_i$ is piecewise C^2 , then there exists a sequence $(\mathcal{T}_N)_{N \geq N_0}$ of triangulations such that

$$N \|g - I_{\mathcal{T}_N} g\|_{L^2(\Omega)} \leq C(g).$$

On the other hand, we have for smooth functions:

Theorem (Chen, Sun Xu; Babenko)

If $f \in C^2(\overline{\Omega})$ and $(\mathcal{T}_N)_{N \geq N_0}$ is an optimally adapted sequence then

$$\limsup_{N \rightarrow \infty} N \|f - I_{\mathcal{T}_N} f\|_{L^2(\Omega)} \leq C \left\| \sqrt{|\det d^2 f|} \right\|_{L^{\frac{2}{3}}(\Omega)}$$

How to connect these estimates ?

Does $\left\| \sqrt{|\det d^2 g|} \right\|_{L^{\frac{2}{3}}}$ make sense if g is a cartoon function ?

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Theorem (Chen, Sun Xu; Babenko)

If $f \in C^2(\overline{\Omega})$ and $(\mathcal{T}_N)_{N \geq N_0}$ is an optimally adapted sequence then

$$\limsup_{N \rightarrow \infty} N \|f - I_{\mathcal{T}_N} f\|_{L^2(\Omega)} \leq C \left\| \sqrt{|\det d^2 f|} \right\|_{L^{\frac{2}{3}}(\Omega)}$$

How to connect these estimates ?

Does $\left\| \sqrt{|\det d^2 g|} \right\|_{L^{\frac{2}{3}}}$ make sense if g is a cartoon function ?

For any $f \in C^2(\overline{\Omega})$

$$J(f) := \left\| \sqrt{|\det d^2 f|} \right\|_{L^{\frac{2}{3}}}.$$

If g is a cartoon function with discontinuity set E we define

$$J(g) := \lim_{\delta \rightarrow 0} J(g * \varphi_\delta),$$

where $\varphi_\delta := \delta^{-2} \varphi(\delta^{-1} \cdot)$ is a mollifier.

Proposition

$$J(g)^{\frac{2}{3}} = \left\| \sqrt{|\det d^2 g|} \right\|_{L^{\frac{2}{3}}(\Omega \setminus E)}^{\frac{2}{3}} + C(\varphi) \left\| [g] \sqrt{|\kappa|} \right\|_{L^{\frac{2}{3}}(E)}^{\frac{2}{3}}$$

where $[g]$ is the *jump* of g , and κ the *curvature* of E .

Compare with

$$TV(g) = \|\nabla g\|_{L^1(\Omega \setminus E)} + \|[g]\|_{L^1(E)}.$$

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Piecewise constant functions

$$TV(g) = \int_{\Gamma} |[g]|$$

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$$J(g)^{\frac{2}{3}} = \int_{\Gamma} |[g]|^{\frac{2}{3}} |\kappa|^{\frac{1}{3}}$$

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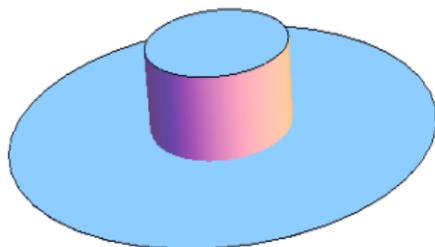
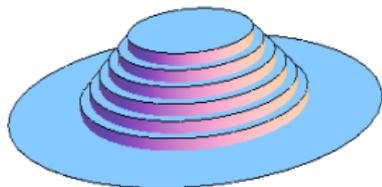


Figure: $TV(g) \simeq J(g)$

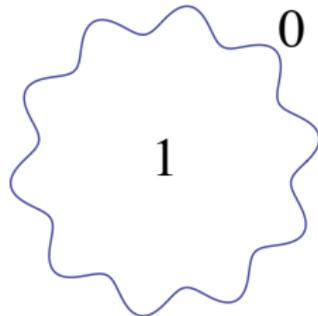


Figure: $TV(g) \ll J(g)$

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 A. Cohen, N. Dyn, F. Hecht, J.-M. Mirebeau, *Adaptive multiresolution analysis based on anisotropic triangulations*, accepted in Maths of Comp 2010

 A. Cohen, J.-M. Mirebeau, *Greedy bisection generates optimally adapted triangulations*, accepted in Maths of Comp 2010

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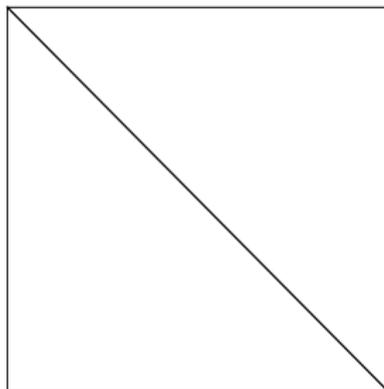


Figure: An isotropic hierarchical refinement algorithm, used for PDEs

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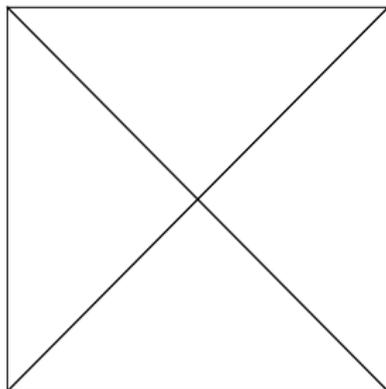


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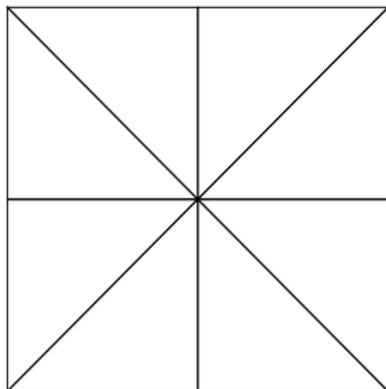


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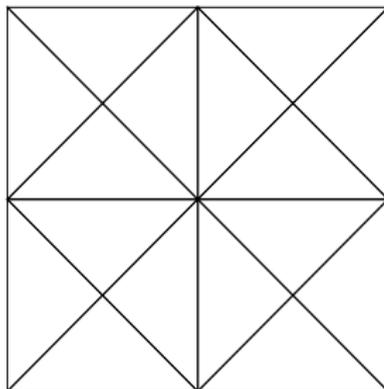


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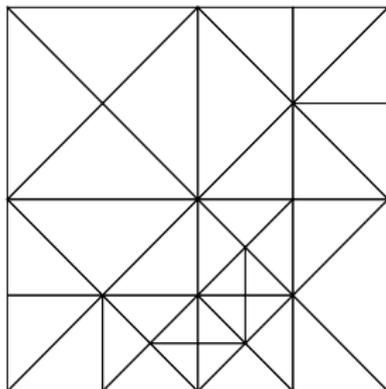


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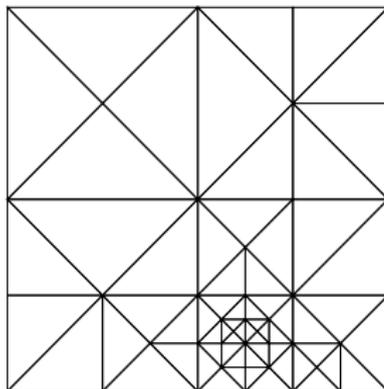


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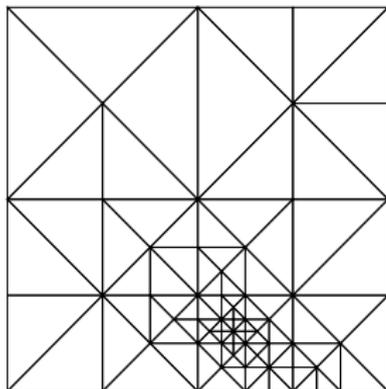


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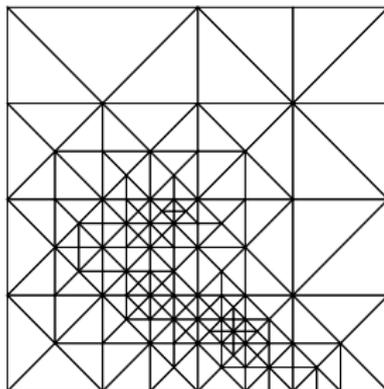


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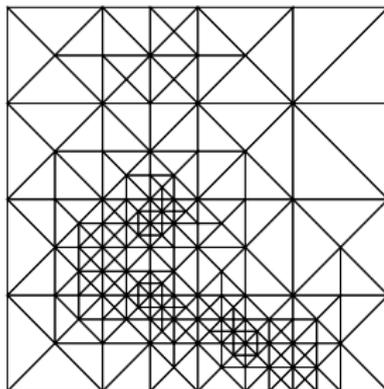


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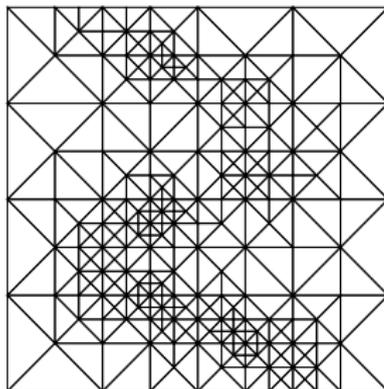


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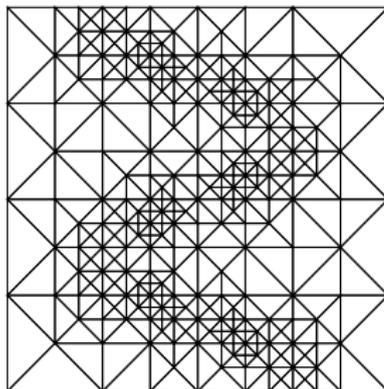


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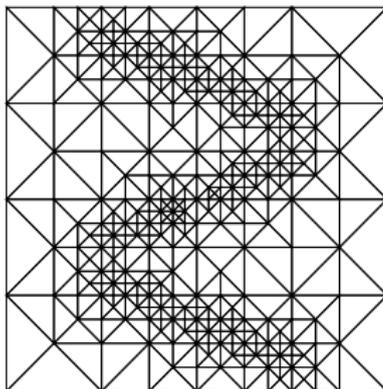


Figure: An isotropic hierarchical refinement algorithm, used for PDEs

Given a triangulation of a domain and a function f :

- ▶ Select the triangle on which the L^2 interpolation error is maximal $\|f - I_T f\|_{L^2(T)}$.
- ▶ Bisect it along one median, so as to minimize the resulting L^1 interpolation error.
- ▶ Repeat these steps until targeted number of triangles is met.

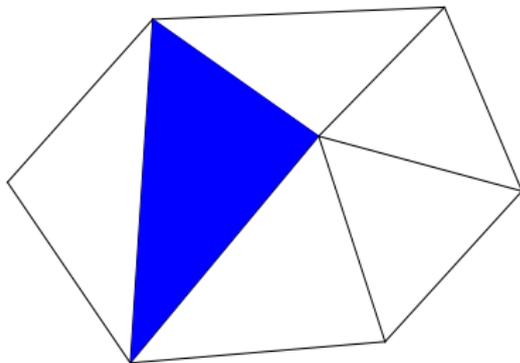


Figure: Algorithm proposed by : N. Dyn, F. Hecht, A. Cohen

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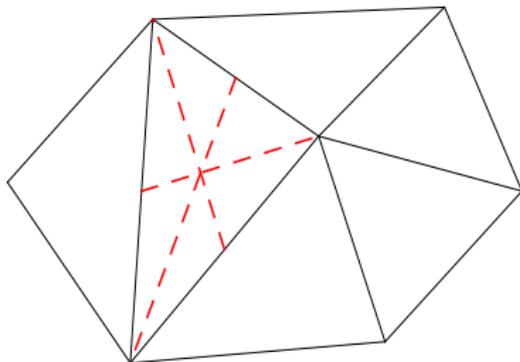


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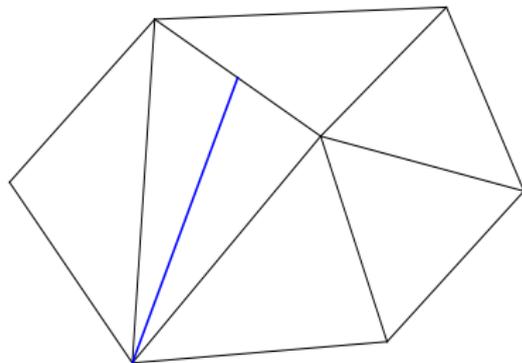


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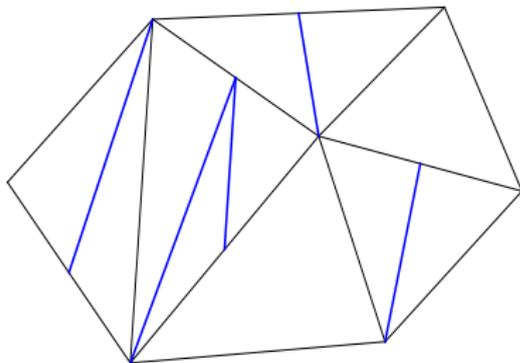
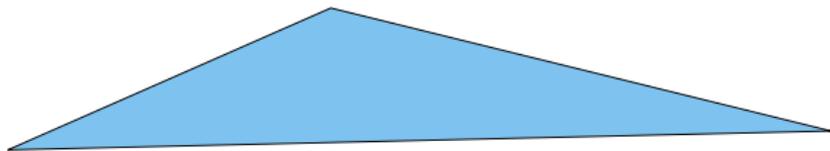
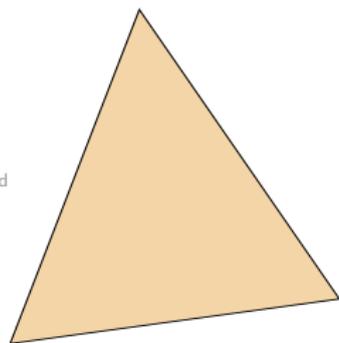


Figure: Algorithm proposed by : N. Dyn, F. Hecht, A. Cohen

Proposition (Identification of the bisection)

The algorithm applied to $f(x, y) = x^2 + y^2$ chooses to cut the **longest edge** of the selected triangle.



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Preserves isotropy.

Restores isotropy.

For any triangle T with edges $|a| \geq |b| \geq |c|$ we define

$$\sigma(T) := \frac{|b|^2 + |c|^2}{4|T|}.$$

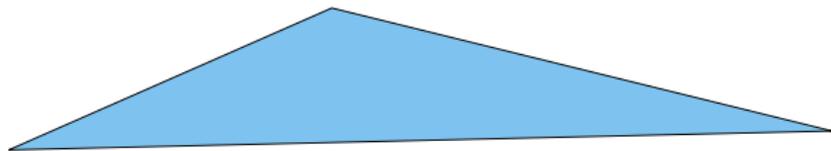
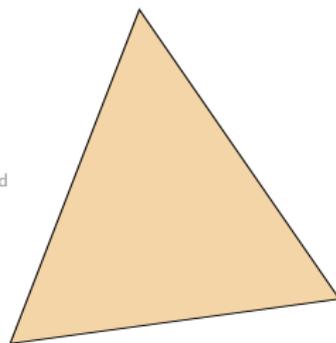
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If T_1, T_2 are obtained by bisecting the longest edge of T then

$$\max\{\sigma(T_1), \sigma(T_2)\} \leq \sigma(T).$$

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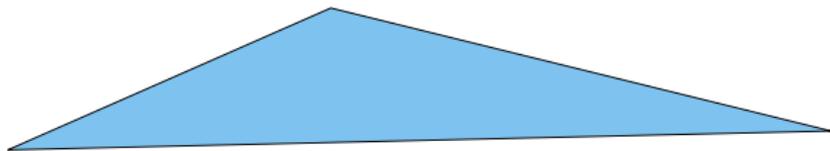
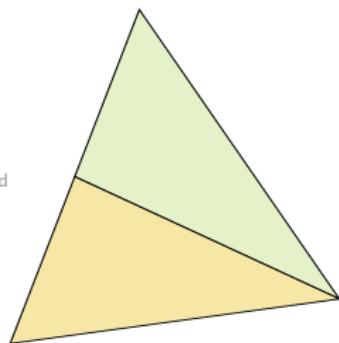
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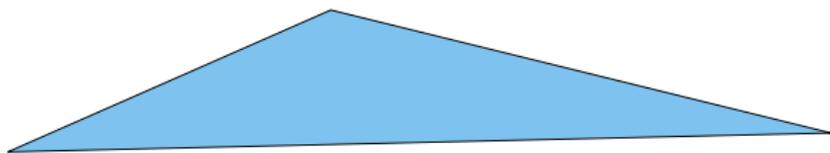
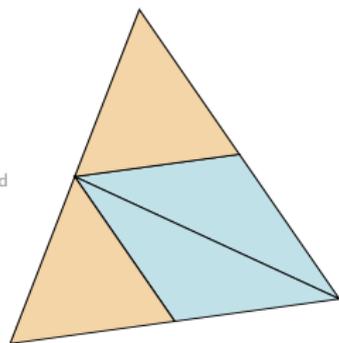
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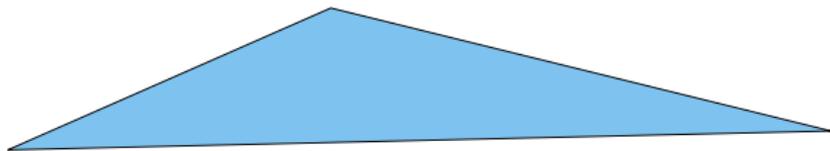
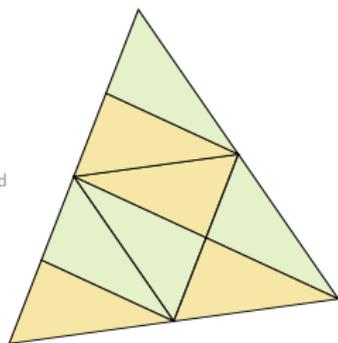
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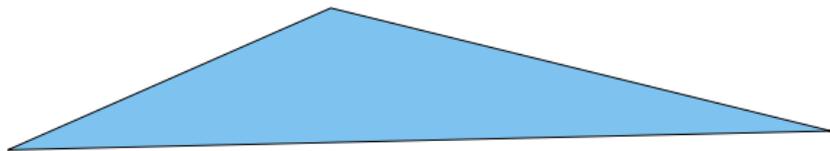
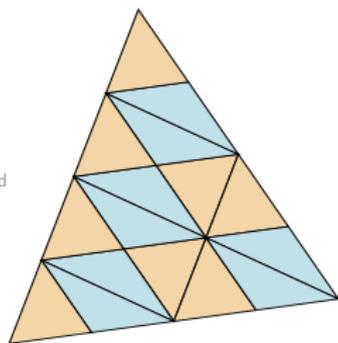
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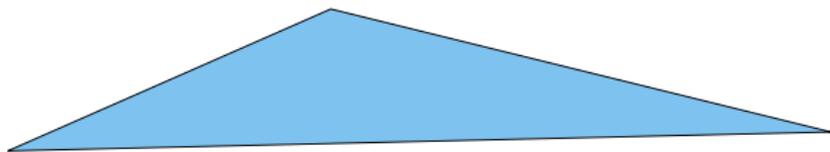
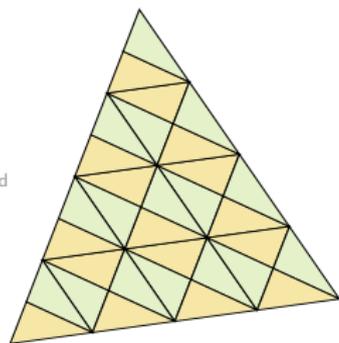
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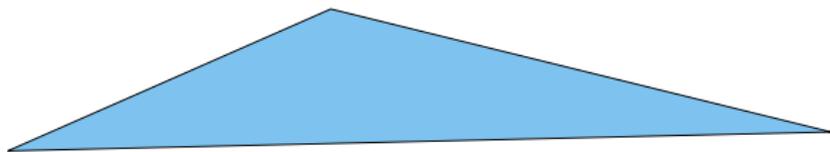
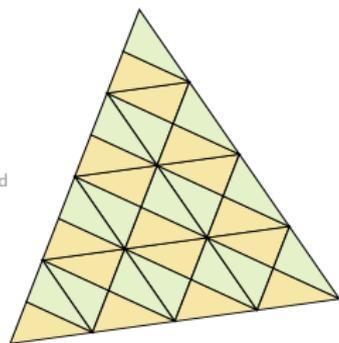
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Proposition

One of the eight children $(T_i)_{i=1}^8$ of T obtained by bisecting recursively three times the longest edge satisfies

$$\sigma(T_i) \leq 0.69\sigma(T) \text{ or } \sigma(T_i) \leq 5.$$

Restores isotropy.

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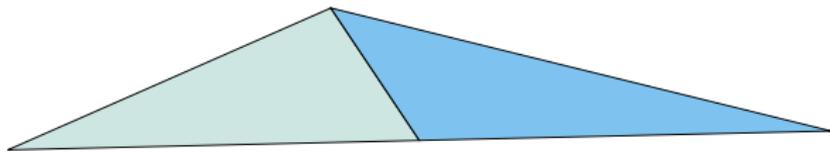
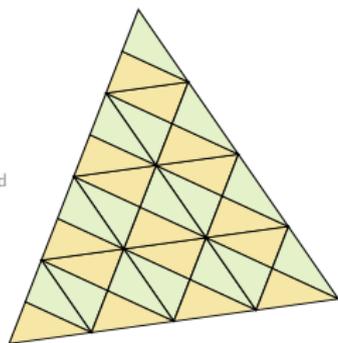
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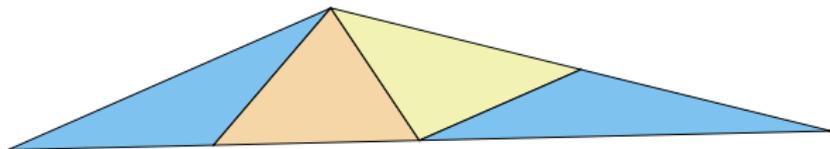
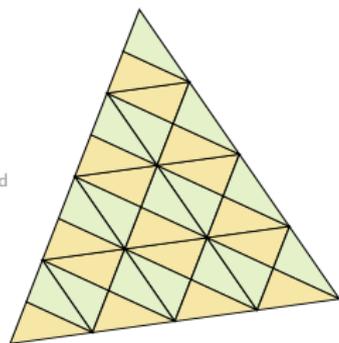
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One of the eight children $(T_i)_{i=1}^8$ of T obtained by bisecting recursively three times the longest edge satisfies

$$\sigma(T_i) \leq 0.69\sigma(T) \text{ or } \sigma(T_i) \leq 5.$$

Proposition (Identification of the bisection)

The algorithm applied to $f(x, y) = x^2 + y^2$ chooses to cut the **longest edge** of the selected triangle.



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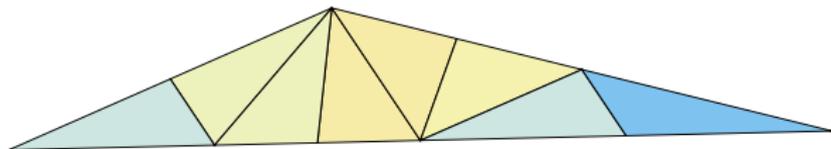
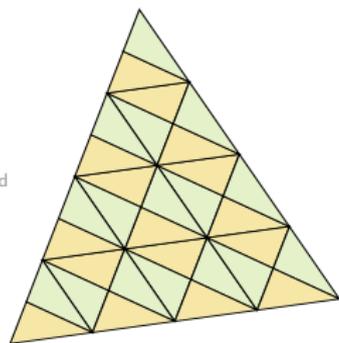
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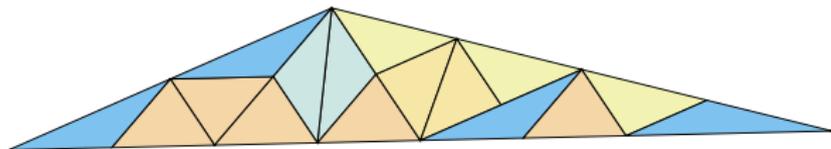
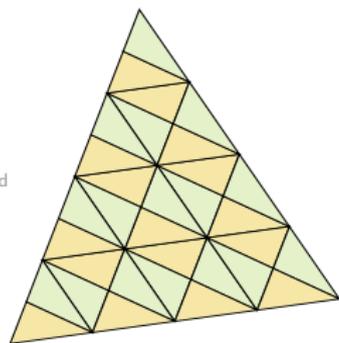
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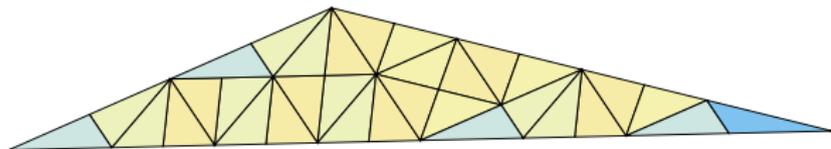
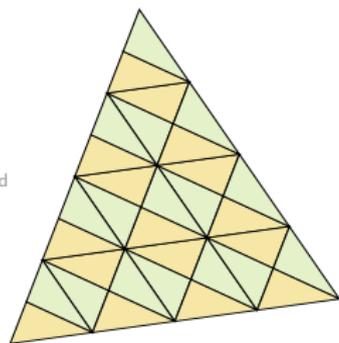
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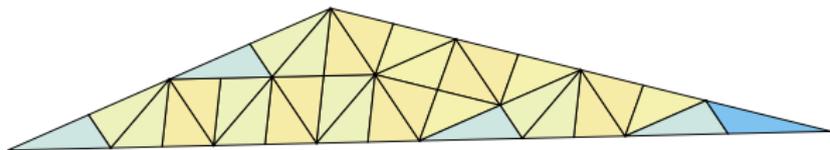
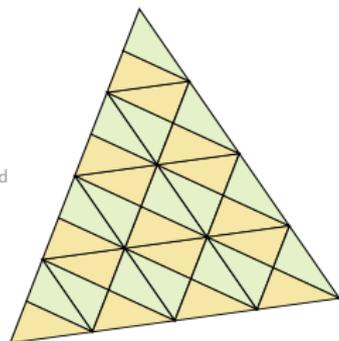
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Markov chain argument \Rightarrow almost all triangles adopt an **isotropic** aspect ratio.

Theorem

The algorithm applied to any **strongly convex** function $f \in C^2(\bar{\Omega})$ produces a sequence $(\mathcal{T}_N)_{N \geq N_0}$ of triangulations which satisfies the (optimal) estimate

$$\limsup_{N \rightarrow \infty} N \|f - I_{\mathcal{T}_N} f\|_{L^2(\Omega)} \leq C \|\sqrt{|\det d^2 f|}\|_{L^{\frac{2}{3}}(\Omega)}$$

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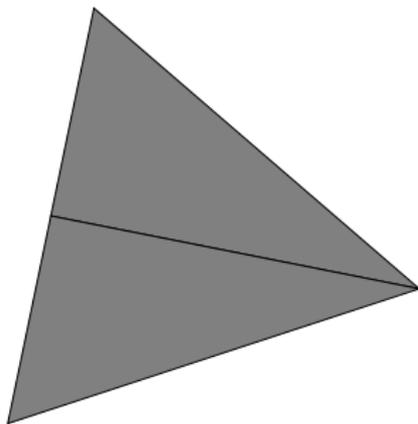


Figure: Algorithm applied to $f(x, y) = x^2 + 100y^2$

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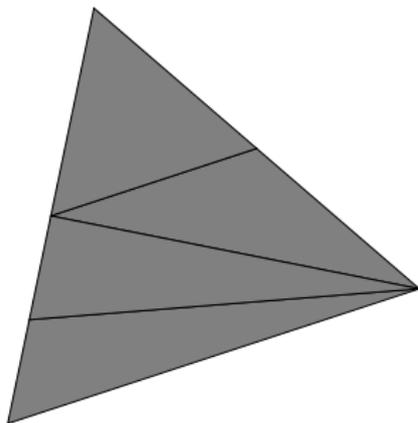


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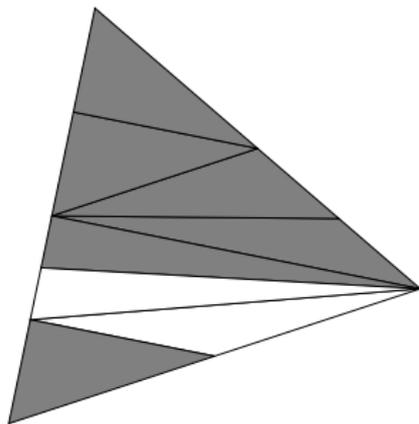


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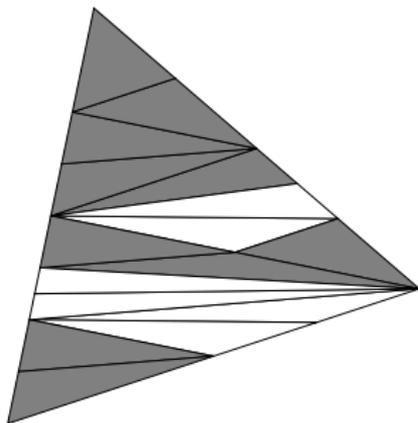


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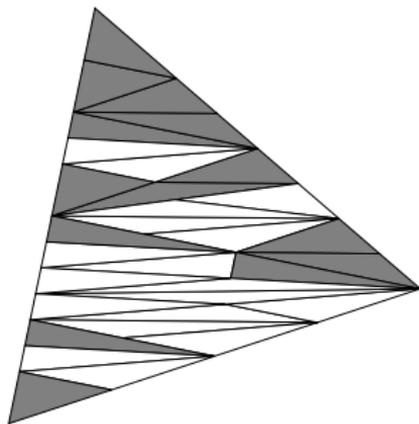


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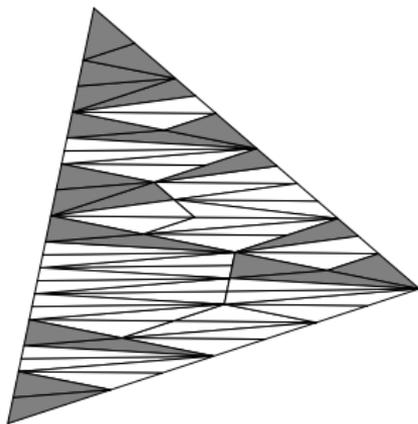


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Adaptive
anisotropic
approx:
Theory and
algorithms

Jean-Marie
Mirebeau

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Conclusion and perspectives

Conclusion:

- ▶ A result of **algorithmic geometry for QuasiAcute triangulations**.
- ▶ Sharp asymptotic estimates for \mathbb{P}_m interpolation error on optimal mesh, for H^1 but also L^p and $W^{1,p}$ norms.
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Perspectives:

- ▶ Numerical applications to \mathbb{P}_2 elements in PDEs.
- ▶ Realistic mesh generation algorithms for QuasiAcute triangulations.
- ▶ Extension to dimension $d > 2$.
- ▶ Non asymptotic error estimates.

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