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[1] This WiFi does not give you access to the WWW (not even DNS, sorry for IP-over-DNSers)
[2] No DOSes, please.

Fast Algorithms
for Towers of Finite Fields and Isogenies

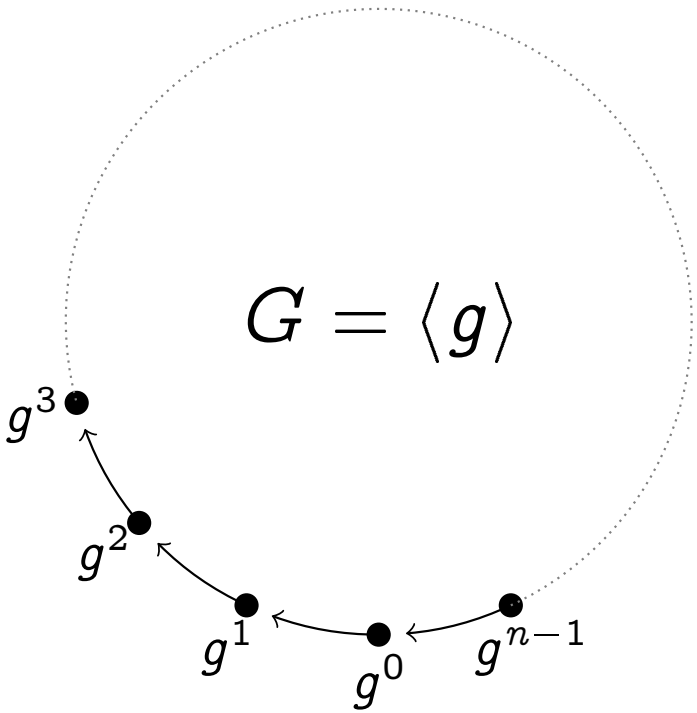
Luca De Feo

LIX, École Polytechnique & INRIA Saclay, Projet TANC

December 13, 2010
École Polytechnique, Palaiseau

Outline

The Discrete Logarithm Problem

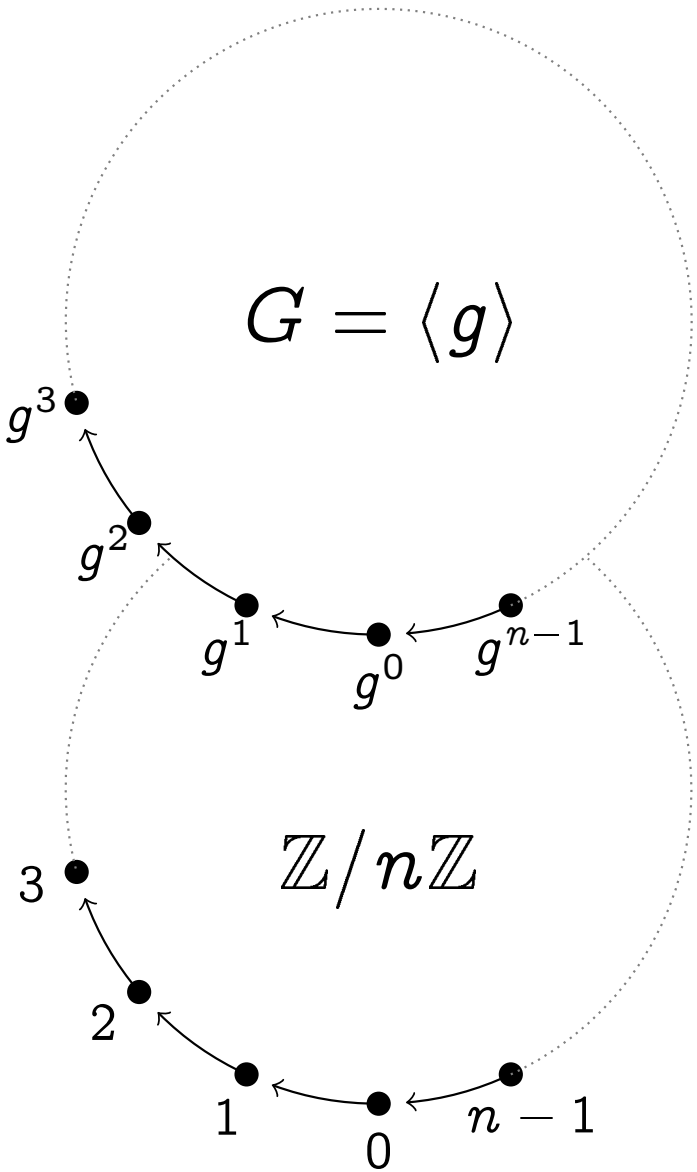


Exponentiation in a cyclic group

Let $G = \langle g \rangle$ be a cyclic group of order n. Define

$$\exp_g : \mathbb{Z}/n\mathbb{Z} \to G,$$
$$x \mapsto g^x.$$

The Discrete Logarithm Problem



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Discrete logarithm

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$$\log_g : G \to \mathbb{Z}/n\mathbb{Z},$$
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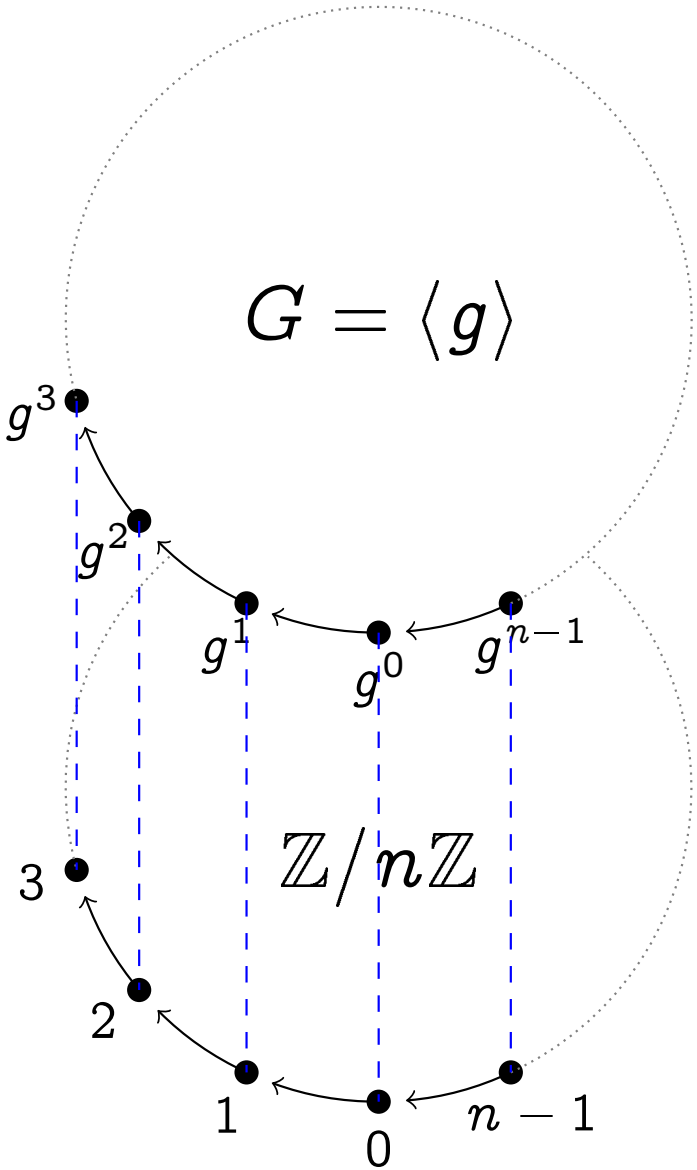

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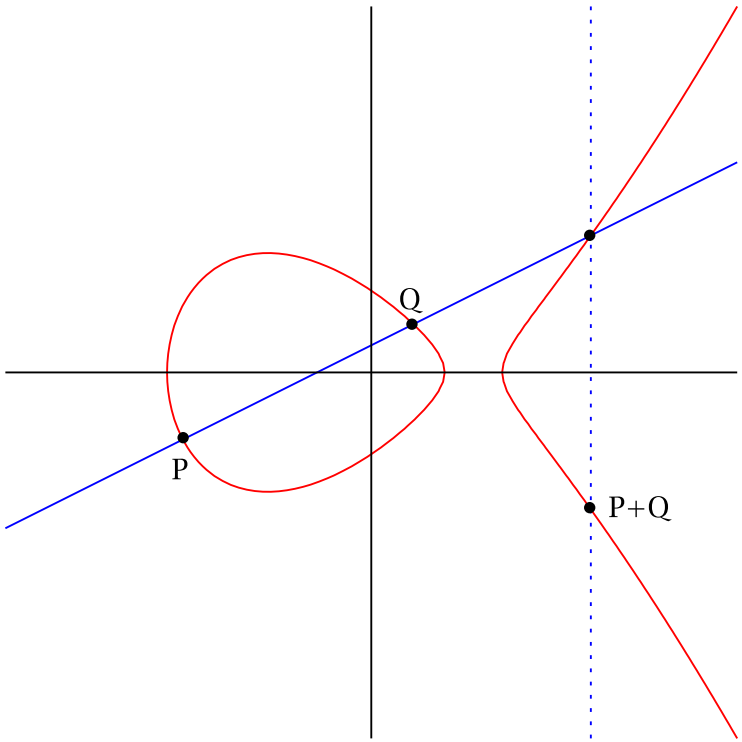
Computing it is called the Discrete Logarithm Problem (DLP) of G.

Elliptic curve cryptography



Elliptic curve (Weierstrass form)

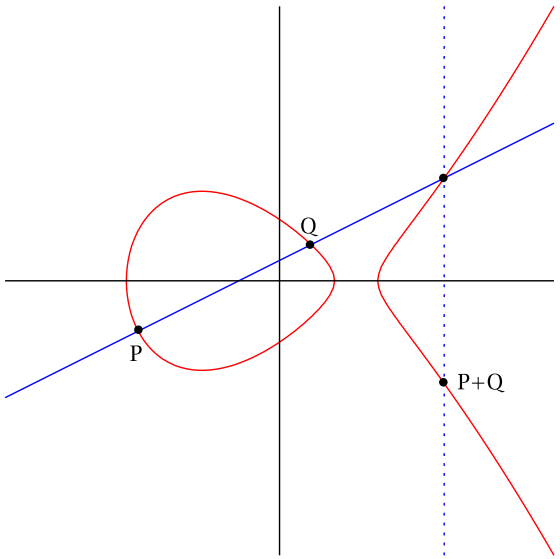
$$E : y^2 = x^3 + ax + b;$$

The set of points of an elliptic curve is endowed with a group law via the chord-and-tangent law.

Hasse bound: $\#E(\mathbb{F}_q) \sim q$.

	Finite field crypto	**Elliptic curve crypto**
Group	$\mathbb{F}_q^*$	$E(\mathbb{F}_q)$
Protocols	El Gamal, DSA, ...	ECDH, ECDSA, ECMQV, ...
Key sizes	1024, 2048, 3072	160, 225, 256

Group law and scalar multiplication



$$y^2 = x^3 + ax + b$$

$$P = (x_0, y_0), Q = (x_1, y_1)$$

$$\lambda = \frac{y_1 - y_0}{x_1 - x_0}$$

$$P + Q = (\lambda^2 - x_0 - x_1, (x_0 - x_2)\lambda - y_0)$$

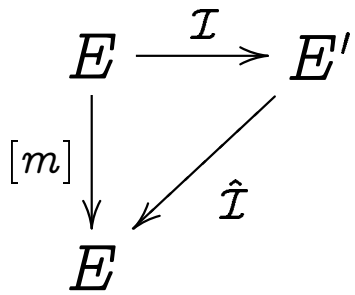
Multiplication: $[m]P = \overbrace{P + P + \cdots + P}^{m \text{ times}}$

m-torsion: $E[m] = \{P \in E(\bar{\mathbb{K}}) | [m]P = \mathcal{O}\} \cong (\mathbb{Z}/m\mathbb{Z})^2$

$$[m](x, y) = \left(\frac{\psi_m(x, y)}{\phi_m^2(x, y)}, \frac{\omega_m(x, y)}{\phi_m^3(x, y)}\right)$$

Division polynomials: ϕ_m can be computed with $O(\log m)$ polynomial multiplications, $\deg \phi_m = O(m^2)$.

Isogenies between elliptic curves



(Separable) isogeny: (separable) non-constant rational morphism preserving the identity.

Properties

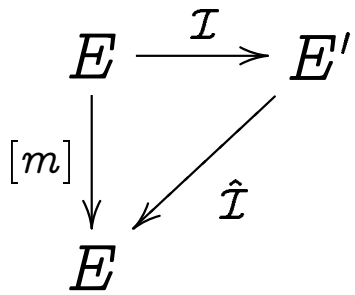
- Isogeny = rational map + group morphism;
- Finite kernel, surjective (in $\bar{\mathbb{K}}$);
- Dual isogeny theorem: they factor the multiplication map into two pieces.

Multiplication

$$[m] : E(\bar{\mathbb{K}}) \to E(\bar{\mathbb{K}})$$
$$P \mapsto [m]P$$

$\ker \mathcal{I} = E[m]$.

Isogenies between elliptic curves



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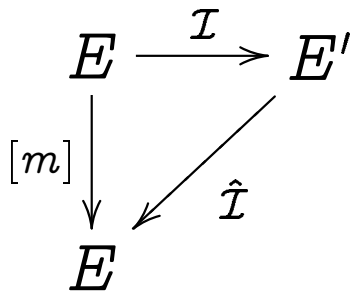
Frobenius endomorphism

$$\varphi : E(\bar{\mathbb{K}}) \to E(\bar{\mathbb{K}})$$
$$(x, y) \mapsto (x^q, y^q)$$

$\ker \varphi = \{\mathcal{O}\}$ (inseparable).

Isogenies between elliptic curves



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Separable isogeny (simplified Weierstrass model)

$$\mathcal{I}(x,y) = \left(\frac{g(x)}{h(x)}, cy\left(\frac{g(x)}{h(x)}\right)'\right)$$

h vanishes on the abscissas of $\ker \mathcal{I}$. $\deg \mathcal{I} = \#\ker \mathcal{I}$.

Why compute (large) isogenies over finite fields?

SEA algorithm (Schoof 1985; Elkies 1992; Atkin 1988)

Hasse bound $\#E(\mathbb{F}_q) = q - t + 1$;

- **Schoof** Compute t modulo small primes ℓ $\Leftrightarrow$ compute the action of φ_q on $E[\ell] \cong (\mathbb{Z}/\ell\mathbb{Z})^2$;
- **Atkin** Determine the order of the roots of $X^2 - tX + q$ by factoring the ℓ-th modular polynomial.
- **Elkies** Compute an ℓ-isogeny $\mathcal{I}$ and the action of φ_q on $\ker \mathcal{I} \cong \mathbb{Z}/\ell\mathbb{Z} \subset E[\ell]$;

Other cryptographic applications

- Transfer DLPs between curves (Gaudry, Hess, and Smart 2002; Smith 2009);
- Construct new cryptosystems (Teske 2006; Rostovtsev and Stolbunov 2006);
- Construct hash functions (Charles, Lauter, and Goren 2009);
- Compute modular polynomials (Broker, Lauter, and Sutherland 2010);
- Compute the endomorphism ring (Kohel 1996; Sutherland 2010).

Outline

Vélu's formulas

Compute an isogeny with given kernel (Vélu 1971)

Given the kernel H, computes $\mathcal{I} : E \to E/H$ given by

$$\mathcal{I}(\mathcal{O}_E) = \mathcal{I}(\mathcal{O}_{E/H}),$$

$$\mathcal{I}(P) = \left(x(P) + \sum_{Q \in H^*} x(P+Q) - x(Q), y(P) + \sum_{Q \in H^*} y(P+Q) - y(Q) \right).$$

In practice, given $h(x)$ vanishing on H

$$y^2 = f(x) \qquad t = \sum_{Q \in H^*} f'(Q), \quad u = \sum_{Q \in H^*} 2f(Q), \quad w = u + \sum_{Q \in H^*} x(Q)f'(Q),$$

$$\mathcal{I}(x,y) = \left(\frac{g(x)}{h(x)}, y \left(\frac{g(x)}{h(x)} \right)' \right) \quad \text{with} \quad \frac{g(x)}{h(x)} = x + t\frac{h'(x)}{h(x)} - u \left(\frac{h'(x)}{h(x)} \right)'$$

Isogeny computation

By Vélu's formulas: $\mathcal{I}(x,y) = \left(\frac{g(x)}{h(x)}, y\left(\frac{g(x)}{h(x)}\right)'\right)$, hence

$$(x^3+ax+b)\left(\frac{g(x)}{h(x)}\right)'^2 = \left(\frac{g(x)}{h(x)}\right)^3 + a'\frac{g(x)}{h(x)} + b'$$

Algorithms: Given E, E', ℓ, compute $\mathcal{I} : E \to E'$

- Stark 1973: First algorithm in characteristic 0, using continued fractions.
- Elkies 1992, 1998: Find a power series solution to the differential equation.
- Couveignes 1994: (p small) Compute morphisms of formal groups, look for one that corresponds to an isogeny.
- Lercier 1996: (only for $p = 2$) solve a linear system.
- Couveignes 1996: (p small) Interpolate over the p-torsion, look for a polynomial that corresponds to an isogeny.
- BMSS algorithm (Bostan, Morain, Salvy, and Schost 2008): Improve Elkies 1998 to run in quasi-linear time.
- Lercier and Sirvent 2008: Lift E and E' in the p-adics, then apply BMSS.

Couveignes' algorithm (Couveignes 1996)

Given E, E', ℓ, compute $\mathcal{I} : E \to E'$

Idea: Map $E[p^k]$ onto $E'[p^k]$

- Compute the extensions $\mathbb{U}_i/\mathbb{F}_q$ such that $E[p^i]$ is defined over $\mathbb{U}_i$; $\tilde{O}(\ell^2)$
- Pick k large enough ($k \sim \log_p 4\ell$);
- Compute P, a generator of $E[p^k]$; $\tilde{O}(\ell^2)$
- Compute P', a generator of $E'[p^k]$; $\tilde{O}(\ell^2)$
- Compute the polynomial T vanishing on $E[p^k]$; $\tilde{O}(\ell^2)$
- For $m \in \left(\mathbb{Z}/p^k\mathbb{Z}\right)^*$
  - Interpolate $A : x(P) \mapsto x([m]P')$; $\tilde{O}(\ell^2)$
  - Reconstruct a rational fraction $\frac{g}{h} \equiv A \bmod T$; $\tilde{O}(\mathsf{M}(\ell))$

  Stop when $\frac{g}{h}$ is an isogeny. ℓ times on average

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Stop when $\frac{g}{h}$ is an isogeny.

How to recognize an isogeny?

- **Degree:** $\frac{g}{h}$ with $\deg g = \ell$, $\deg h = \ell - 1$; $O(1)$
- **Square factor:** $h = \prod_{Q \in H^*}(X - x(Q)) = f^2$ if ℓ odd; $\tilde{O}(\mathsf{M}(\ell))$
- **Group action:** Test on random points: $\mathcal{I}(P + Q) = \mathcal{I}(P) + \mathcal{I}(Q)$; $O(\ell)$
- **Factor of the ℓ-division polynomial:** Check $\phi_\ell = 0 \bmod h$. $\tilde{O}(\mathsf{M}(\ell))$

How to recognize an isogeny?

T vanishes on $E[p^k]$, A interpolates $x(P) \mapsto x(P')$

$$A U_i + T V_i = R_i \qquad \Leftrightarrow \qquad A \equiv \frac{R_i}{U_i} \bmod T$$

$$\ell = 11$$

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$$AU_i + TV_i = R_i \quad \Leftrightarrow \quad A \equiv \frac{R_i}{U_i} \bmod T$$

$$\ell = 11$$

$\deg R_i$	$\deg U_i$
31415926535897932384626 43	0

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$\deg R_i$	$\deg U_i$
31415926535897932384626643	0
31415926535897932384626642	1

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$$\ell = 11$$

$\deg R_i$	$\deg U_i$
31415926535897932384626643	0
31415926535897932384626642	1
31415926535897932384626641	2

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$\deg R_i$	$\deg U_i$
31415926535897932384626643	0
31415926535897932384626642	1
31415926535897932384626641	2
$\vdots$	$\vdots$
31415926535897932384626634	9

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$\deg R_i$	$\deg U_i$
314159265358979323846264 3	0
314159265358979323846264 2	1
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$\vdots$	$\vdots$
314159265358979323846263 4	9
11	10

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$\vdots$	$\vdots$
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11	**10**
10	314159265358979323846263 3
$\vdots$	$\vdots$

Isogenies of unknown degree

It costs *only* $\tilde{O}(\ell)$

- This pattern is extremely rare.
- It can be detected in quasi-linear time using a fast XGCD algorithm (Khodadad and Monagan 2006).
- This is the only phase of Couveignes' algorithm that depends on ℓ.

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Actually, it does not even depend on ℓ

- This just depends on the bound p^k, not on the exact degree ℓ.
- If ℓ is not known in advance, it is enough to look for a gap.
- Thus, any isogeny of degree $\ll p^k$ can be computed with one single run of Couveignes' algorithm.

Comparison of isogeny algorithms

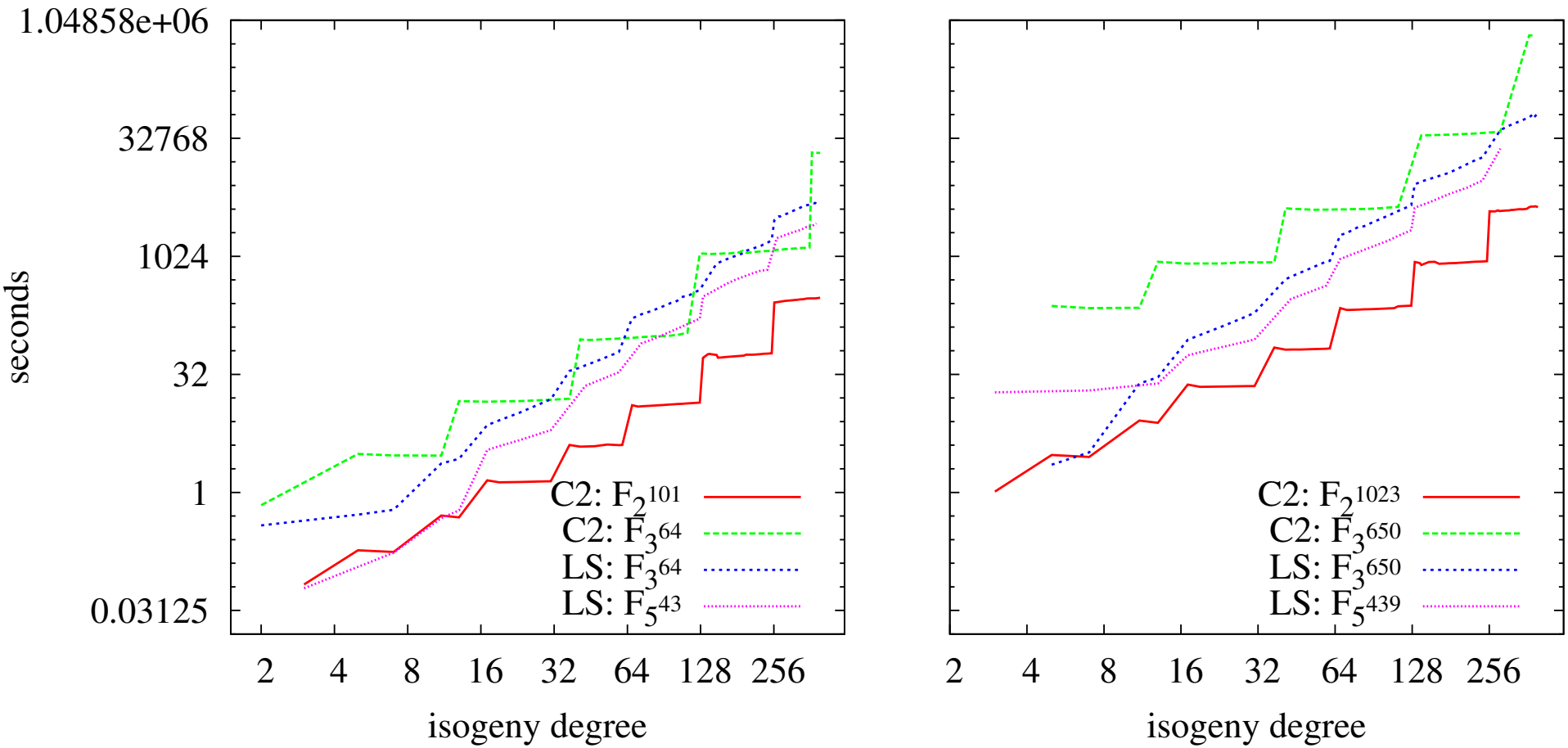

Figure: Comparative timings for Couveignes 1996 (C2) and Lercier and Sirvent 2008 (LS) over various curves. Plot in logarithmic scale.

Isogenies of unknown degree

- This variant of Couveignes 1996 is currently the fastest algorithm for this task (both in theory and in practice).
- We tested two curves over $\mathbb{F}_{2^{161}}$, isogenous of unknown degree, taken from Teske 2006: proven in 694 cpu-hours that no isogeny of degree less than 2^{13} exists.

Outline

The field of definition of $E[p^k]$

$$
\begin{array}{l}
\mathbb{U}_k \leftarrow\!\!\shortmid\!\!- - E[p^k] \\
\quad\Big| p \\
\mathbb{U}_{k-1} \leftarrow\!\!\shortmid\!\!- E[p^{k-1}] \\
\quad\vdots \\
\mathbb{U}_2 \leftarrow\!\!\shortmid\!\!- - E[p^2] \\
\quad\Big| p \\
\mathbb{U}_1 \leftarrow\!\!\shortmid\!\!- - - E[p] \\
\quad\Big| \\
\mathbb{F}_q
\end{array}
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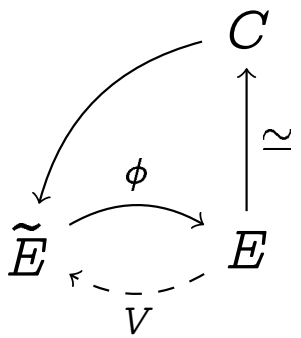

p-descent (Voloch 1990)

If $P_i = (x_i, y_i)$ generates $E[p^i]$, the solution to

$$
\begin{cases}
Z^p - Z & - \dfrac{\sqrt[p]{y_i \beta_E(x_i)}}{\sqrt[p-1]{H_E}} \\
X^p & - x_i \\
Y^p & - y_i
\end{cases}
$$

generates $C[p^{i+1}]$.
Then apply the isomorphism $C \cong E$.

The field of definition of $E[p^k]$

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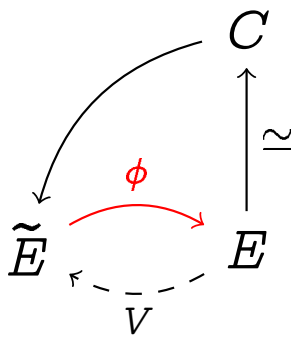

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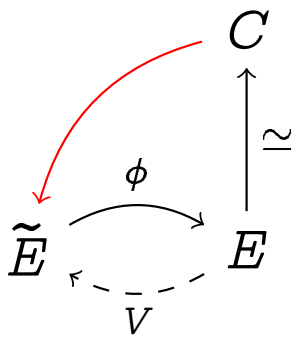

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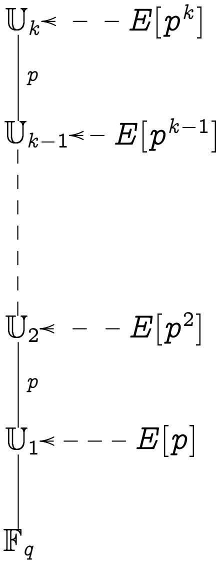

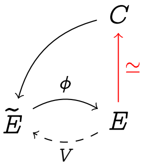

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generates $C[p^{i+1}]$.

Then apply the isomorphism $C \cong E$.

Artin-Schreier towers

$$\mathbb{U}_k = \frac{\mathbb{U}_{k-1}[X_k]}{X_k^p - X_k - \alpha_{k-1}}$$

$\Big| p$

$\mathbb{U}_{k-1}$

$\vdots$

$$\mathbb{U}_1 = \frac{\mathbb{U}_0[X_1]}{X_1^p - X_1 - \alpha_0}$$

$\Big| p$

$$\mathbb{U}_0 = \mathbb{F}_q = \frac{\mathbb{F}_p[X_0]}{Q(X_0)}$$

Artin-Schreier polynomials

Artin-Schreier polynomial:

$$X^p - X - \alpha \qquad \text{with } \alpha \in \mathbb{K}$$

Proposition

$X^p - X - \alpha$ is either irreducible or split in $\mathbb{K}$.

Artin-Schreier extensions

Defined by an irreducible Artin-Schreier polynomial

$$\mathbb{L} = \mathbb{K}[X]/(X^p - X - \alpha)$$

ANY separable extension of degree p can be expressed this way.

Arithmetic in towers of extensions

$$\mathbb{U}_k = \frac{\mathbb{U}_{k-1}[X_k]}{X_k^p - X_k - \alpha_{k-1}}$$

$$\Big| p$$

$$\mathbb{U}_{k-1}$$

$$\vdots$$

$$\mathbb{U}_1 = \frac{\mathbb{U}_0[X_1]}{X_1^p - X_1 - \alpha_0}$$

$$\Big| p$$

$$\mathbb{U}_0 = \mathbb{F}_q = \frac{\mathbb{F}_p[X_0]}{Q(X_0)}$$

Multiplication

$$\begin{aligned}
\mathsf{M}(\mathbb{U}_0) &= O\big(\mathsf{M}(d)\big) \\
\mathsf{M}(\mathbb{U}_1) &= O\big(p^2\mathsf{M}(d)\big) \\
\mathsf{M}(\mathbb{U}_2) &= O\big(p^4\mathsf{M}(d)\big) \\
&\vdots \\
\mathsf{M}(\mathbb{U}_k) &= O\big(p^{2k}\mathsf{M}(d)\big) \supsetneq \tilde{O}(p^k d)
\end{aligned}$$

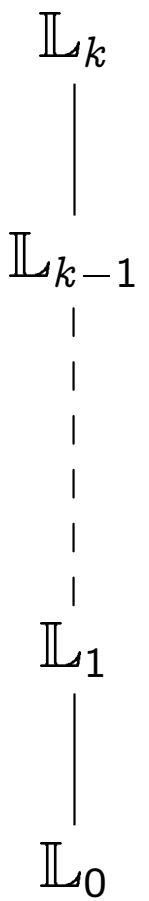
Even using Karatsuba or FFT multiplication, $\tilde{O}(p^k d)$ can't be attained.

Other operations

- Exponentiation, Inversion, GCD, all depend upon $\mathsf{M}(\mathbb{U}_k)$,
- The canonical injection $\mathbb{U}_{i-1} \to \mathbb{U}_i$ can be computed in $O(p^i)$.

One fast tower, many fast towers!

A Fast A-S tower



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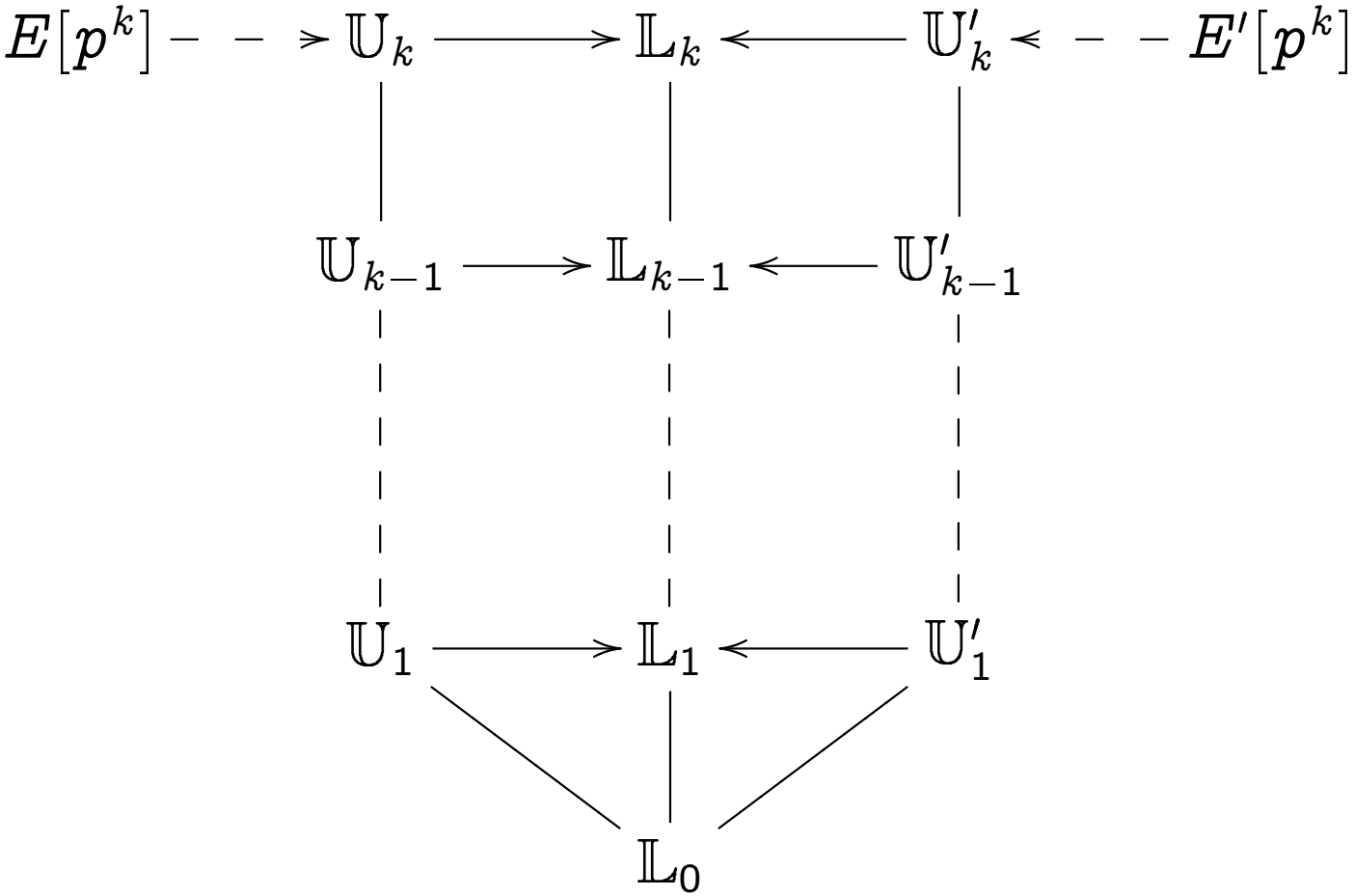

Theorem (Couveignes 2000): There exist an isomorphism algorithm that runs in $O(k^3 \mathsf{M}(\mathbb{L}_k))$ operations in $\mathbb{L}_0$. (Remember that $k = \log_p \#\mathbb{L}_k$)

A fast tower

$$\mathbb{L}_k = \frac{\mathbb{L}_{k-1}[X_k]}{X_k^p - X_k - \alpha_{k-1}}$$

$$\Big| \, p$$

$$\mathbb{L}_{k-1}$$

$$\vdots$$

$$\mathbb{L}_1 = \frac{\mathbb{L}_0[X_1]}{X_1^p - X_1 - \alpha_0}$$

$$\Big| \, p$$

$$\mathbb{L}_0 = \mathbb{F}_q = \frac{\mathbb{F}_p[X_0]}{Q(X_0)}$$

$$\Big| \, d$$

$$\mathbb{F}_p$$

Idea: Convert to a univariate basis over $\mathbb{F}_p$

A fast tower

$$\mathbb{L}_k = \frac{\mathbb{L}_{k-1}[X_k]}{X_k^p - X_k - \alpha_{k-1}}$$
$$\Big| p$$
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$$\vdots$$
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$$\Big| p$$
$$\mathbb{L}_0 = \mathbb{F}_q = \frac{\mathbb{F}_p[X_0]}{Q(X_0)}$$
$$\Big| d$$
$$\mathbb{F}_p$$

Idea: Convert to a univariate basis over $\mathbb{F}_p$

$\longleftrightarrow$

$$\mathbb{L}_k = \frac{\mathbb{F}_p[Y_k]}{Q_k(Y_k)}$$
$$\Big| p$$
$$\mathbb{L}_{k-1}$$
$$\vdots$$
$$\mathbb{L}_1 = \frac{\mathbb{F}_p[Y_1]}{Q_1(Y_1)}$$
$$\Big| p$$
$$\mathbb{L}_0 = \mathbb{F}_q = \frac{\mathbb{F}_p[X_0]}{Q(X_0)}$$
$$\Big| d$$
$$\mathbb{F}_p$$

A fast tower: the multivariate side

$$\mathbb{L}_k = \frac{\mathbb{L}_{k-1}[X_k]}{X_k^p - X_k - x_{k-1}^{2p-1}}$$

$$\Big| p$$

$$\mathbb{L}_{k-1}$$

$$\vdots$$

$$\mathbb{L}_1 = \frac{\mathbb{L}_0[X_1]}{X_1^p - X_1 - x_0}$$

$$\Big| p$$

$$\mathbb{L}_0 = \mathbb{F}_q = \frac{\mathbb{F}_p[X_0]}{Q(X_0)}$$

$$\Big| d$$

$$\mathbb{F}_p$$

Our construction (inspired by Cantor 1989)

$$\begin{cases} X_i^p - X_i - X_{i-1}^{2p-1} \\ \qquad\qquad \vdots \\ X_2^p - X_2 - X_1^{2p-1} \\ X_1^p - X_1 - X_0 \\ Q(X_0) \end{cases}$$

Theorem: Let x_i be the class of X_i in $\mathbb{L}_i$ (and thus in $\mathbb{L}_{i+1}, \ldots$). If $\mathrm{Tr}_{\mathbb{L}_0/\mathbb{F}_p}(x_0) \neq 0$, each line contains an irreducible polynomial and x_i generates $\mathbb{L}_i$ over $\mathbb{F}_p$.

A fast tower: the univariate side

$$\mathbb{L}_k = \frac{\mathbb{F}_p[Y_k]}{Q_k(Y_k)}$$

$$\Big| \, p$$

$$\mathbb{L}_{k-1}$$

$$\vdots$$

$$\mathbb{L}_1 = \frac{\mathbb{F}_p[Y_1]}{Q_1(Y_1)}$$

$$\Big| \, p$$

$$\mathbb{L}_0 = \mathbb{F}_q = \frac{\mathbb{F}_p[X_0]}{Q(X_0)}$$

$$\Big| \, d$$

$$\mathbb{F}_p$$

Univariate representation

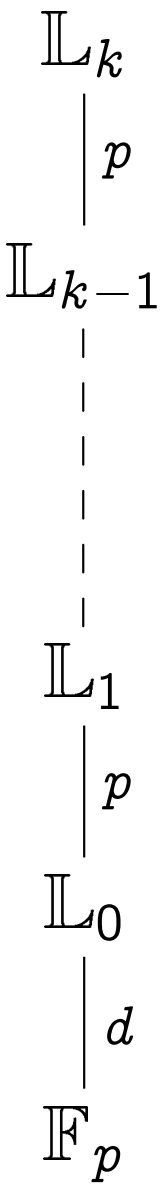
The minimal polynomials Q_i of x_i over $\mathbb{F}_p$ can be computed as follows.

- $Q_0 = Q$;
- $Q_1 = Q_0(Z^p - Z)$.

Let ω be a $(2p-1)$-th root of unity,

- $q_{i+1}(Z^{2p-1}) = \prod_{j=0}^{2p-2} Q_i(\omega^j Z)$,
- $Q_{i+1} = q_{i+1}(Z^p - Z)$.

A fast tower: change of basis



Two bases to represent $\mathbb{L}_k$

- Univariate: $\mathbb{L}_k = \mathbb{F}_p[Y_k]/Q_k(Y_k)$
- Multivariate: $\mathbb{L}_k = \mathbb{F}_p[X_0, X_1, \ldots, X_k]/(Q, \ldots)$

A fast change of basis is the key to fast arithmetic:

- Multiplication is faster in univariate;
- Field embeddings are faster in multivariate.

Change of basis

Univariate $\to$ multivariate

- Recursively split input in p slices, recombine by Horner's rule;
- Simple and fast: $O(\mathsf{M}(\ell) + \ell \log^2 \ell)$

Multivariate $\to$ univariate

- Trace formulas + duality + transposed algorithms;
- Same complexity: $O(\mathsf{M}(\ell) + \ell \log^2 \ell)$

Example: univariate → multivariate

$$\mathbb{F}_{3^9} = \mathbb{F}_3[X_1, X_2]/I = \mathbb{F}_3[Y_2]/Q_2(Y_2)$$

$$I = \begin{cases} X_2^3 - X_2 - X_1^5, \\ X_1^3 - X_1 - 1, \end{cases}$$

$$Q_2(Z) = Z^9 + Z^6 + Z^4 + Z^2 - Z - 1, \qquad Q_2(x_2) = 0,$$

Let a be given in the univariate basis:

$$a = \underbrace{X_2^8 + X_2^7 + X_2^6} + \underbrace{X_2^5} + \underbrace{X_2} =$$

Example: univariate → multivariate

$$\mathbb{F}_{3^9} = \mathbb{F}_3[X_1, X_2]/I = \mathbb{F}_3[Y_2]/Q_2(Y_2)$$

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Example: univariate → multivariate

$$\mathbb{F}_{3^9} = \mathbb{F}_3[X_1, X_2]/I = \mathbb{F}_3[Y_2]/Q_2(Y_2)$$

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Then, $a = \left(X_2 + X_1^5\right)\left(\left(X_2 + X_1^5\right)a_2 + a_1\right) + a_0$

Example: univariate → multivariate

$$\mathbb{F}_{3^9} = \mathbb{F}_3[X_1, X_2]/I = \mathbb{F}_3[Y_2]/Q_2(Y_2)$$

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Let a be given in the univariate basis:

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Then, $a = \left(X_2 + X_1^5\right)\left((X_2 + X_1^5)a_2 + a_1\right) + a_0 \mod (X_2^3 - X_2 - X_1^5) =$

$$(X_1^2 - X_1 - 1)X_2^2 + (X_1 - 1)X_2 - X_1^2 - X_1 - 1$$

Multivariate → univariate

Trace formulas (Rouillier 1999)

$$a(Z) = \frac{A(Z)}{Q_k'(Z)} \bmod Q_k(Z) \qquad \text{where} \qquad A(T) = Q_k(T) \sum_{i \geq 0} \frac{\operatorname{Tr} a x_k^i}{T^{i+1}}.$$

- Compute the *trace form* Tr,
- compute the form $a \cdot \operatorname{Tr} : x \mapsto \operatorname{Tr} ax$,
- compute $\sum_{i \geq 0} \frac{a \cdot \operatorname{Tr} x_k^i}{T^{i+1}}$,
- deduce $a(Z)$.

Example: Let $a = x_1$, then

$$A(T) = Q_k(T) \left(T^{-6} - T^{-8} - T^{-9} + O(T^{-10}) \right) = T^3 - T,$$

$$a = \frac{x_2^3 - x_2}{x_2^3 - x_2 - 1} = x_2^6 + x_2^4 - x_2^3 + x_2^2 + x_2 + 1.$$

Multivariate → univariate

Trace formulas (Rouillier 1999)

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Duality (Shoup 1995, 1999; Bostan, Salvy, and Schost 2003)

Polynomial evaluation: $k[Z] \to \mathbb{K}/k$

$$g \mapsto g(\sigma)$$

Power projection: $(\mathbb{K}/k)^* \to k[T]^* \simeq k[[1/T]]$

$$\ell \mapsto \sum_{i>0} \frac{\ell(\sigma^i)}{T^i}$$

Univariate $\to$ multivariate = Polynomial evaluation

If $Z \mapsto x_k$ (with $x_k \in \mathbb{U}_k$ written in the multivariate basis):

$$a(Z) = a_d Z^d + \cdots + a_1 Z + a_0 \quad \mapsto \quad a_d x_k^d + \cdots + a_1 x_k + a_0 = a(x_k) = a.$$

Transposed univariate $\to$ multivariate = Power projection

$$\begin{aligned} \mathrm{Tr} &\mapsto \sum_{i\geq 0} \frac{\mathrm{Tr}\, x_k^i}{T^i}, \\ a \cdot \mathrm{Tr} &\mapsto \sum_{i\geq 0} \frac{\mathrm{Tr}\, a x_k^i}{T^i}. \end{aligned}$$

The transposition principle

Theorem

Let $\mathcal{P}$ be an arbitrary set. To any R-algebraic algorithm A computing a family of linear functions $(f_p : M \to N)_{p \in \mathcal{P}}$ corresponds an R-algebraic algorithm A^* computing the dual family $(f_p^* : N^* \to M^*)_{p \in \mathcal{P}}$. The algebraic time and space complexities of A^* are bounded by the time complexity of A.

Change of basis

Univariate → multivariate

- Recursively split input in p slices, recombine by Horner's rule;
- Simple and fast: $O(\mathsf{M}(\ell) + \ell \log^2 \ell)$

Multivariate → univariate $O(\mathsf{M}(\ell) + \ell \log^2 \ell)$

By *trace formulas* (Rouillier 1999):

$$a(Z) = \frac{A(Z)}{Q_k'(Z)} \bmod Q_k(Z) \qquad \text{where} \qquad A(T) = Q_k(T) \sum_{i \geq 0} \frac{\operatorname{Tr} a x_k^i}{T^{i+1}}.$$

- Compute the *trace form* Tr and the form $a \cdot \operatorname{Tr} : x \mapsto \operatorname{Tr} ax$;
- compute $\sum_{i \geq 0} \frac{a \cdot \operatorname{Tr} x_k^i}{T^{i+1}}$ using a transposed univariate → multivariate;
- deduce $a(Z)$.

Outline

Implementations

Automatic transposition

- Algorithms are hard to transpose, transposed algorithms are hard or impossible to understand;
- How can we be confident that a transposed algorithm is well implemented if no one understands it?
- When proving programs with a proof assistant, why should we do the work twice?

`http://transalpyne.gforge.inria.fr/`

A Python-like ad-hoc language for automatic transposition, compiled/interpreted in Python.

Artin-Schreier towers and isogenies

- FAAST (Fast Arithmetic in Artin-Schreier towers): C++ with NTL implementation released under GPL:
  `http://www.lix.polytechnique.fr/~defeo/FAAST/`
- With F. Morain and É. Schost, writing a C++ library for isogenies over finite fields.

Outline

Perspectives

The future of `transalpyne`

- Finish the implementation, add more output languages.
- Integration in a Computer Algebra System (Sage? Mathemagix?).
- Development in a dependently typed system (Coq? Agda?).

Perspectives on isogenies

- All known algorithms to compute isogenies have suboptimal complexity $\Omega(\ell^2)$.
- Are there models, where algorithms *à la* BMSS and Lercier and Sirvent 2008 may be faster?
- Couveignes' first isogeny algorithm (Couveignes 1994) is very similar to Couveignes 1996 and may be modified in the same way. There is some hope that it may be faster in practice, because of the simpler arithmetic operations needed.
- Generalize Couveignes' algorithms to genus 2?

Thank you all for coming. . .