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Nicolas Rougerie

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Nicolas Rougerie. La théorie de Gross-Pitaevskii pour un condensat de Bose-Einstein en rotation : vortex et transitions de phase. Mathématiques [math]. Université Pierre et Marie Curie - Paris VI, 2010. Français. NNT: . tel-00547404

HAL Id: tel-00547404

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La théorie de Gross-Pitaevskii pour un condensat de Bose-Einstein en rotation : vortex et transitions de phase

THÈSE DE DOCTORAT

présentée et soutenue publiquement le 9 Décembre 2010

pour l'obtention du grade de

Docteur de l'Université Pierre et Marie Curie

Spécialité Mathématiques Appliquées

par

Nicolas Rougerie

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Mis en page avec la classe thloria.

Remerciements

Je me tourne tout d'abord avec émotion vers mes directeurs de thèse Xavier Blanc et Sylvia Serfaty. Ils ont été tout au long de ces années deux sources constantes de conseils, d'idées et de soutien. Qu'ils en soient ici mille et mille fois remerciés.

Il est traditionnel au moment de s'asseoir pour mettre la touche finale à sa thèse de jeter un œil en arrière et de repenser à son parcours. Il y a de cela quelques années, Mme Fradin, ma professeur de mathématiques en terminale, Mrs Rauch et Patte, mes professeurs de mathématiques en classe préparatoire me donnaient le goût des mathématiques. Je ne serais bien sûr pas en train d'écrire cette thèse sans ce don précieux.

Je veux aussi saisir l'occasion pour me remémorer que c'est lors de ma scolarité aux Mines de Nancy qu'est née mon envie de faire des mathématiques appliquées et de me lancer dans une thèse. Je veux donc remercier ceux que j'ai eu la chance d'avoir pour professeurs de mathématiques à cette époque : Samuel Herrmann, Xavier Antoine, Céline Lacaux et surtout Antoine Henrot. C'est surtout grâce à eux que mon passage par notre très particulier système français de 'grandes' écoles de management (pardon, d'ingénieurs!) me fut profitable.

Bernard Helffer et Robert Jerrard m'ont fait l'honneur d'accepter le rôle difficile de rapporteurs, ils méritent pour cela toute ma gratitude. C'est un grand plaisir pour moi que Fabrice Béthuel ait accepté de présider mon jury et je suis également extrêmement reconnaissant à Maria Esteban, Mathieu Lewin, Giandomenico Orlandi et Jean-Claude Saut d'avoir accepté d'en faire partie.

Mes collaborateurs Michele Correggi et Jakob Yngvason ont largement participé à l'élaboration de ma pensée scientifique. En particulier, avoir bénéficié du regard de Jakob sur le monde de la recherche et la science en général sera une inspiration pour de nombreuses années à venir. Notre collaboration n'aurait pas été possible sans l'hospitalité de l'institut Erwin Schrödinger de Vienne. Chacun des séjours que j'ai eu la chance d'y faire me laisse le souvenir d'un accueil chaleureux et d'un cadre de travail parfait.

Pour divers conseils et/ou discussions stimulantes je veux adresser mes remerciements à Radu Ignat, Vincent Millot, Ionut Danaila, Parimah Kazemi, Duvan Henao, Nicolas Raymond, Analisa Panati, Stan Alama et Lia Bronsard.

Je mets la dernière main à ce manuscrit dans un bureau de l'université d'Aarhus où je me trouve grâce à l'invitation de Søren Fournais. Qu'il soit chaleureusement remercié de son accueil.

J'ai bénéficié au laboratoire Jacques-Louis Lions d'un environnement scientifique et humain que je n'hésiterais pas à qualifier d'exceptionnel. Khashayar, Liliane, Danielle, Salima, Florence et Christian m'ont facilité la vie tout au long de ces trois années plus que je ne saurais le dire. Yvon Maday et Edwige Godlewski ont su se montrer des directeurs de laboratoire attentifs et ont toujours été de bon conseil.

Je tiens à remercier plus particulièrement les thésards ou jeunes docteurs que j'ai croisé pour des temps plus ou moins long. Les nombreuses discussions que nous avons eues ont contribué à m'élever l'esprit : Cuc, Baver, Vincent D, Etienne, Rim, Nicole, Evelyne, Paolo, Joëlle, Mathieu G, Alexandra F, Benjamin B, Benjamin A, Rachida, Raphaël, Nicolas L,

Jean-Marie, Maya, Vincent P, Thomas, Paul, Pierre, Giacomo, Mathieu L, Ange, Khaled, Marie, Laurent, Sepideh, André, Sylvain, Jean-Paul ... Une mention spéciale revient à Alexis et Alexandra C, mes dynamiques et charmantes collaboratrices pour l'organisation du groupe de travail des thésards.

Toute ma famille a participé à sa manière à ce que je puisse me retrouver à écrire ces lignes. Elle m'a doté de biens beaucoup trop précieux pour que je tente de les décrire ici. Je remercie plus particulièrement mes parents Pascale et François, mes grand-parents Jacques, Anne-Marie, Yves et Marie-Thérèse, mon frère Laurent et ma soeur Clémentine qui ont fait de moi un être humain.

Puisque ceci est après tout un manuscrit universitaire, je veux y remercier plus particulièrement mon grand-père Jacques Rougerie qui n'a jamais manqué de stimuler ma réflexion sur l'université, la recherche et l'enseignement.

Les membres de l'atelier Délicatement Pouêt m'ont offert l'hospitalité de nombreuses fois quand la malveillance du RER B, le froid, la pluie, la neige, la chaleur, la flemme ou autre raison hautement valable m'empêchaient de rejoindre le laboratoire. Nous avons je pense démontré le bon ménage que peuvent faire dessinateurs et mathématiciens, sinon dessin et mathématiques. Un grand merci donc à Alexis, Camille et Léonidas.

Une bonne partie de cette thèse a été rédigée avec la chatte Praline vautreée sur mes genoux. Je ne sais pas si je dois l'en remercier. Je crois qu'elle s'en fiche. Merci quand même.

S'il était fatigant physiquement, le rythme des répétitions hebdomadaires avec les Pamplemousses Ethyliques a été un grand stimulant spirituel pour moi. Jouer une musique si spontanée a été un contrepoint parfait à la rigueur mathématique de mon quotidien. Mille démonstrations de virile amitié et complicité musicale à Alexis "Jean Pierre Baguette", Camille "Cam' Guitar Magic" et François "Danger" Parmentier, de la part de "Crazy Nicko".

A tous mes amis j'adresse moult gesticulations et autres manifestations d'extrême affection. Chanter vos louanges rempliraient plus de pages que mes mathématiques, vous comprendrez donc que je ne peux prendre le risque de le faire ici.

Olivier, Simon, Mélodie, François, Judy, Vivien, Marie, Maxime, Pascal, Sven, Alexis, Camille, Léonidas, Billitch, merci à vous. Koju je crois que le moment est approprié pour te remercier de toutes les discussions mathématiques que nous avons eues malgré l'hostilité incompréhensible qu'elles rencontraient chez nos compagnons de soirée.

Je remercie enfin tous ceux que j'ai pu malencontreusement oublier sans qui cette thèse n'aurait pas été possible. Et surtout Anne-Laure sans qui rien ne saurait possible.

A Anne-Laure, pour son amour, sa patience, son soutien et ses grands yeux bruns.

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Introduction générale

Cette thèse est consacrée à l'étude asymptotique de la fonctionnelle de Gross-Pitaevskii dans différents régimes de paramètres décrivant des situations physiquement intéressantes pour des condensats de Bose-Einstein en rotation. Il s'agira en particulier de caractériser finement les tourbillons quantiques (vortex) que la rotation crée dans un tel système physique. Notre but principal sera l'identification mathématiquement rigoureuse de certaines transitions de phase séparant des régions de l'espace des paramètres où les tourbillons du condensat modélisé s'organisent de manière radicalement différente.

La présente introduction se divise en trois parties : nous commençons par rappeler le contexte physique dans lequel notre étude s'insère et mentionner des problèmes mathématiques reliés.

Les résultats obtenus sont ensuite rassemblés dans une seconde partie où nous les commentons et donnons quelques idées de leurs preuves. Une troisième partie est consacrée à une récapitulation des résultats obtenus. Nous y indiquerons également certaines perspectives possibles.

1 Contexte physique et mathématique

1.1 La condensation de Bose-Einstein

1.1.1 La condensation dans un gaz parfait

L'existence du phénomène que l'on connaît de nos jours sous le nom de *condensation de Bose-Einstein* a été prédite théoriquement par Einstein dans les articles [53]. En se basant sur des idées de Bose [34], Einstein considère un gaz idéal monoatomique constitué de particules indiscernables en contact avec un thermostat. L'argument original est de nature statistique : les particules étant supposées non-interagissantes, il s'agit de déterminer en fonction de la température du thermostat la répartition la plus probable des atomes dans les niveaux d'énergie qui leur sont accessibles. L'état macroscopique le plus probable est celui qui est réalisé par le plus grand nombre d'états microscopiques. L'hypothèse que les particules sont indiscernables (on saura plus tard que seules des particules bosoniques peuvent se condenser) est bien sûr cruciale dans le dénombrement des états du système et mène à la conclusion que les particules sont réparties suivant la statistique dite de Bose-Einstein. Celle-ci est très différente de la statistique de Maxwell-Boltzmann que l'on obtient si on considère les particules comme discernables.

En particulier, Einstein constate dans ses articles que cette distribution de particules a un comportement très singulier quand la température du thermostat devient petite. Il existe pour cette température une valeur critique en dessous de laquelle une proportion macroscopique des atomes (en termes mathématiques, un nombre d'atomes proportionnel au nombre total d'atomes N quand N tend vers l'infini) doit se trouver dans l'état fon-

damental de plus basse énergie. C'est ce phénomène d'occupation macroscopique de l'état fondamental d'un système à basse température que l'on appelle maintenant condensation de Bose-Einstein. Les particules se trouvant dans l'état fondamental du système forment un *condensat de Bose-Einstein* (BEC, en abrégant le terme anglais), et on appelle parfois les éventuels atomes peuplant les états excités du système un *nuage thermique*.

Il est utile de remarquer que des particules bosoniques non interagissantes sont nécessairement toutes dans le même état d'énergie à température nulle. Il n'est donc pas contre-intuitif de penser qu'à une température suffisamment basse ce phénomène puisse subsister. Toutefois, le raisonnement statistique qui mène à la notion de température de condensation n'a rien à voir avec l'effet plus simple qui serait que le thermostat ne fournit pas assez d'énergie aux particules pour qu'elles peuplent les états excités du système. La température obtenue par ce genre de raisonnement énergétique est largement inférieure à la température de condensation.

1.1.2 Un long chemin expérimental et théorique

A l'époque de sa parution, l'idée de Einstein rencontra beaucoup de scepticisme, et on répugna longtemps à la considérer comme autre chose qu'un intéressant exercice académique, une 'gedanken experiment'. Einstein lui même écrit à Ehrenfest : "c'est une belle théorie mais contient-elle une part de vérité?"

Une objection de taille apparaît très rapidement : la température critique de condensation est extrêmement basse. Outre qu'elle était totalement inatteignable avec les moyens de l'époque, tous les composés connus sont à l'état solide, et non gazeux, à une telle température.

Une autre objection, de nature peut-être plus théorique, est que dans un gaz réel les particules interagissent bel et bien. Il faudra un long travail pour simplement définir ce qu'il faut entendre par condensation de Bose-Einstein dans un système de particules en interaction. Par opposition à ce qu'il se passe dans un gaz idéal, même à température nulle les particules n'ont pas de raison évidente d'être toutes dans l'état fondamental du système. L'existence du phénomène de condensation est donc un problème non trivial même à température nulle dès lors qu'on ne néglige plus les interactions entre particules. En fait, le formalisme de la physique quantique, qui était encore en gestation à l'époque où Einstein écrit ses articles fondateurs, s'avère indispensable pour formuler rigoureusement ce problème, comme nous y reviendrons plus bas.

Il faudra attendre les années 90 pour que les températures nécessaires à la condensation soient atteintes en laboratoire, grâce notamment aux techniques de refroidissement par laser (prix Nobel de physique 1997 attribué à Chu, Phillips et Cohen-Tannoudji), de piégeage magnétique et de refroidissement par évaporation. En 1995 les premiers condensats sont obtenus dans des gaz alcalins : à Boulder (Colorado) à partir d'atomes de rubidium et au MIT à partir d'atomes de sodium. Cornell, Wieman et Ketterle obtiennent en 2001 le prix Nobel de physique pour ces expériences. Les systèmes dans lesquels on peut obtenir une condensation sont typiquement très dilués, de manière à prévenir la formation de molécules puis de composés solides. La condensation est donc réalisée dans des gaz métastables à longue durée de vie et c'est l'extrême dilution du système qui empêche la formation du solide qui est la phase stable à de telles températures.

Ainsi, 70 ans de recherche ont été nécessaires pour confirmer expérimentalement l'existence du phénomène de condensation. La condensation de Bose-Einstein est devenue depuis un

vaste champ d'études physique, et plus récemment mathématique.

Quant au problème conceptuel posé par les interactions entre particules, il ne devint formulable comme nous l'avons dit que lorsque la physique quantique fut solidement installée. La définition correcte de la condensation de Bose-Einstein pour des particules interagissantes a été donnée par Penrose et Onsager en 1956, et le travail sur ce phénomène est un champ théorique très actif. Au début des années 2000 paraissent sous l'impulsion notamment de Lieb, Seiringer, Solovej et Yngvason (voir le livre [98] et sa liste de références) les premières preuves mathématiquement rigoureuses de l'existence de la condensation dans des systèmes de particules interagissantes.

1.2 La superfluidité, le modèle de Gross-Pitaevskii

Une des raisons qui font des condensats de Bose-Einstein un champ d'études si intéressant est leur superfluidité, c'est à dire leur capacité à s'écouler sans viscosité, sans frottements entre les atomes. Ainsi un BEC peut être le siège de *courants permanents* qui ne s'amortissent pas comme ils le feraient dans un fluide normal.

Cette propriété prédite théoriquement a été confirmée spectaculairement par l'observation de vortex (tourbillons) dans des condensats mis en rotation. En effet, un fluide normal tournant assez longtemps atteindra un régime d'équilibre dans lequel il tournera exactement comme un solide. Un superfluide, et en particulier un condensat, a un comportement complètement différent : à l'équilibre et pour des rotations assez fortes, des tourbillons apparaissent et restent stables sur de longues périodes de temps. Des expériences typiques (voir par exemple [41, 63] pour de bonnes listes de références) consistent à créer un condensat, maintenu par un piège (champ) magnétique. On met ensuite ce piège en rotation et le mouvement est communiqué au condensat. Si la fréquence de rotation est assez grande, des tourbillons apparaissent. Un BEC étant un fluide quantique, la circulation autour de ces tourbillons est quantifiée, au sens que nous verrons plus bas.

Le lien entre superfluidité et condensation semble fort mais n'est pas entièrement compris de nos jours. En effet, la superfluidité a été d'abord observée en 1937 par Kapitza, Allen et Misener dans l'hélium 4 liquide dont on ne sait pas si il constitue un condensat. En revanche, un BEC est toujours superfluide. Du point de vue de l'étude de la superfluidité, l'avantage des BECs est que de nombreux paramètres physiques peuvent être manipulés en laboratoire, plus facilement que pour l'hélium 4 par exemple. De plus, la taille des tourbillons rapporté à la taille du système est plus grande dans un condensat, ce qui permet d'observer directement un vortex sur une image expérimentale comme la Figure 1.

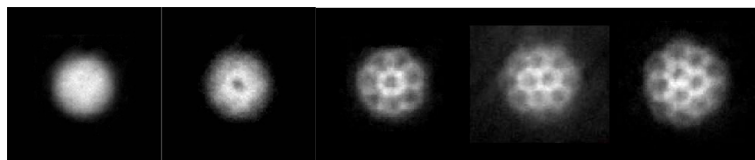


FIGURE 1 – Vortex dans un condensat en rotation, voir [144]

1.2.1 Le modèle de Gross-Pitaevskii

Le modèle le plus utilisé pour décrire un superfluide, en particulier un condensat, en rotation a été introduit en 1961 par Gross [72, 73] et Pitaevskii [122]. On suppose que

la condensation est complète, c'est à dire que tous les atomes sont dans le même état quantique. On peut alors décrire le condensat par une *fonction d'onde macroscopique* à valeurs complexes ψ . On peut voir cette fonction d'onde comme décrivant l'unique état quantique dans lequel se trouvent tous les atomes.

Pour une fonction d'onde ordinaire, $|\psi|$ donnerait la densité de probabilité de présence de la particule décrite. Pour le système macroscopique que constitue un condensat, $|\psi|^2(x)$ donne la densité de matière au point x . Un tourbillon est alors une courbe (ou un point dans un modèle bi-dimensionnel) où $|\psi|$ s'annule et autour duquel la phase ϕ de la fonction d'onde a une circulation non nulle (c'est à dire un degré topologique non nul) :

$$\int_{\mathcal{C}} \partial_{\tau} \phi = 2\pi d, \quad d \in \mathbb{Z}^*$$

si $\psi = \rho e^{i\phi}$ sur \mathcal{C} , une courbe fermée autour du vortex avec vecteur tangent τ . Ici d est le degré topologique (winding number) du vortex. Notons que dans cette description, la quantification du courant superfluide (porté par la phase de ψ) apparaît très naturellement puisque c'est une condition nécessaire pour que ψ soit bien définie.

Un 'prototype' de vortex bidimensionnel de degré d situé au point de coordonnées polaires (r_0, θ_0) est donné par une fonction de la forme

$$\psi(r, \theta) = f(r - r_0) e^{id(\theta - \theta_0)} = f(r - r_0) \left(\frac{z - z_0}{|z - z_0|} \right)^d$$

avec $z_0 = r_0 e^{i\theta_0}$ et f une fonction radiale positive, vérifiant $f(0) = 0$ pour compenser la singularité de la phase de ψ en (r_0, θ_0) .

Selon le modèle de Gross-Pitaevskii (GP) que nous allons étudier dans cette thèse, un condensat à l'équilibre minimise l'énergie dite de Gross-Pitaevskii. Elle s'exprime, dans le repère en rotation et avec un choix d'unités approprié,

$$\mathcal{E}_{3D}^{\text{GP}}[\psi] := \int_{\mathbb{R}^3} |\nabla \psi|^2 - 2\vec{\Omega} \times \vec{r} \cdot (i\psi, \nabla \psi) + V(\vec{r})|\psi|^2 + G|\psi|^4. \quad (0.1.1)$$

Pour refléter le fait que le nombre d'atomes est fixé, on impose une contrainte de masse

$$\int_{\mathbb{R}^3} |\psi|^2 = 1 \quad (0.1.2)$$

et le problème variationnel donnant l'état fondamental du système est donc le suivant

$$E_{3D}^{\text{GP}} := \inf \left\{ \mathcal{E}_{3D}^{\text{GP}}[\psi], \int_{\mathbb{R}^3} |\psi|^2 = 1 \right\} = \mathcal{E}_{3D}^{\text{GP}}[\Psi_{3D}^{\text{GP}}]. \quad (0.1.3)$$

L'existence d'un minimiseur pour ce problème sous des hypothèses convenables est un exercice classique, en revanche, il n'y a pas unicité a priori, sauf lorsque $\Omega = 0$. Nous y reviendrons par la suite.

Décrivons maintenant plus en détail la fonctionnelle d'énergie (0.1.1) que nous venons d'introduire.

- Le premier terme représente l'énergie cinétique des atomes.

- Le deuxième terme reflète le fait que l'énergie est exprimée dans le repère en rotation à vitesse $\vec{\Omega} = \Omega \vec{e}_z$ (on définit la verticale du système comme la direction du vecteur rotation). Le courant superfluide apparaissant dans ce terme est défini comme suit :

$$(i\psi, \nabla\psi) := \frac{1}{2}i(\psi \overline{\nabla\psi} - \overline{\psi} \nabla\psi).$$

- Le troisième terme a une influence cruciale : il s'agit d'un terme de confinement qui assure que les atomes restent localisés dans l'espace. D'un point de vue mathématique c'est lui qui, sous une hypothèse simple que nous donnons plus bas, garantit que le problème est bien posé. Il décrit l'énergie potentielle des atomes du condensat dans le champ magnétique $V(\vec{r})$ qui les confine.
- Le quatrième terme modélise les interactions entre les atomes. Nous reviendrons en détail dans la section 1.3 sur l'interprétation du paramètre G .

A cause de la contrainte de masse (0.1.2), l'équation d'Euler-Lagrange associée au problème de minimisation (0.1.3) fait apparaître un potentiel chimique (i.e. un multiplicateur de Lagrange) μ_{3D}^{GP} :

$$-\Delta \Psi_{3D}^{\text{GP}} - 2i\vec{\Omega} \times \vec{r} \cdot \nabla \Psi_{3D}^{\text{GP}} + V(\vec{r})\Psi_{3D}^{\text{GP}} + 2G|\Psi_{3D}^{\text{GP}}|^2 \Psi_{3D}^{\text{GP}} = \mu_{3D}^{\text{GP}} \Psi_{3D}^{\text{GP}}. \quad (0.1.4)$$

En multipliant cette équation par $\overline{\Psi_{3D}^{\text{GP}}}$ et en intégrant sur \mathbb{R}^3 , on s'aperçoit que le potentiel chimique est fixé par la contrainte de masse :

$$\mu_{3D}^{\text{GP}} = E_{3D}^{\text{GP}} + G \int_{\mathbb{R}^3} |\Psi_{3D}^{\text{GP}}|^2. \quad (0.1.5)$$

Pour mémoire, et bien que dans cette thèse nous nous intéresserons seulement au fondamental de l'énergie de Gross-Pitaevskii, écrivons l'équation de Schrödinger décrivant l'évolution temporelle d'un tel système :

$$i\partial_t \psi - \Delta \psi - 2i\vec{\Omega} \times \vec{r} \cdot \nabla \psi + V(\vec{r})\psi + 2G|\psi|^2 \psi = 0. \quad (0.1.6)$$

Il est instructif de reconnaître dans les deux premiers termes de la fonctionnelle (0.1.1) le début du développement d'un carré parfait et de s'en servir pour réécrire l'énergie

$$\mathcal{E}_{3D}^{\text{GP}}[\psi] = \int_{\mathbb{R}^3} \left| \nabla \psi - i\vec{\Omega} \times \vec{r} \psi \right|^2 + (V(\vec{r}) - \Omega^2 r^2) |\psi|^2 + G|\psi|^4. \quad (0.1.7)$$

Cette réécriture correspond à séparer

- les forces de Coriolis, apparentes dans le premier terme de la formule ci-dessus et qui sont responsables de la formation des tourbillons,
- les forces centrifuges, apparentes dans le deuxième terme de l'énergie et qui modifient le potentiel de piégeage effectivement ressenti par les atomes.

Sous cette forme apparaissent de manière simple les conditions pour que le problème de minimisation (0.1.3) ait une solution. On définit le *potentiel effectif*

$$V_{\text{eff}}(\vec{r}) := V(\vec{r}) - \Omega^2 r^2 \quad (0.1.8)$$

et on suppose en général que

- V_{eff} est borné inférieurement sur \mathbb{R}^3 ,
- $V_{\text{eff}}(\vec{r})$ tend vers $+\infty$ quand $r \rightarrow \infty$.

Se convaincre que ces conditions sont suffisantes pour assurer l'existence d'un minimiseur de l'énergie est aisé en utilisant la méthode directe du calcul des variations. Il s'agit essentiellement de se rappeler que sous notre hypothèse l'injection de $H^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3, V_{\text{eff}}(\vec{r})d\vec{r})$ dans $L^2(\mathbb{R}^3)$ est compacte. Ces conditions étant également assez générales pour couvrir tous les cas physiquement intéressants nous supposons qu'elles sont vérifiées dans toute la suite.

1.2.2 Superfluidité et supraconductivité

Il est bien connu des physiciens (on pourra par exemple consulter [145]) que la superfluidité a énormément de points communs avec un autre phénomène fondamental, la supraconductivité. Ce dernier consiste en la capacité d'un matériau à supporter des courants électriques sans aucune résistivité. En conséquence, le modèle de Ginzburg-Landau (GL) qui est le plus communément accepté pour décrire des supraconducteurs montre de grandes similarités avec le modèle de Gross-Pitaevskii que nous venons d'introduire.

Un supraconducteur est décrit par une fonction d'onde macroscopique (appelée aussi paramètre d'ordre dans ce contexte) ψ à valeurs complexes et un potentiel vecteur A à valeurs dans \mathbb{R}^3 . La densité d'électrons supraconducteurs (paires de Cooper) est donnée par $|\psi|^2$ ($|\psi| = 1$ correspond à une zone supraconductrice et $|\psi| = 0$ correspond à une zone normale) et le champ magnétique dans l'échantillon par

$$h := \text{curl } A.$$

Pour un échantillon supraconducteur représenté par un domaine $\mathfrak{D} \subset \mathbb{R}^3$ soumis à un champ magnétique extérieur h_{ex} , l'énergie de Ginzburg-Landau peut s'écrire dans un système convenable d'unités

$$\mathcal{E}^{\text{GL}}[\psi, A] := \int_{\mathfrak{D}} |\nabla\psi - iA\psi|^2 + |\text{curl } A - h_{ex}|^2 + \frac{1}{\varepsilon^2} (1 - |\psi|^2)^2. \quad (0.1.9)$$

Notons que nous avons choisi d'écrire la fonctionnelle de Ginzburg-Landau comme dans [135]. Certains auteurs considèrent d'autres fonctionnelles, voir par exemple [67].

Ici ε est un paramètre dépendant du matériau, correspondant à l'inverse du paramètre de Ginzburg-Landau usuellement noté κ . Les points communs entre les fonctionnelles (0.1.1) et (0.1.9) sont bien connus et largement exploités pour l'analyse mathématique des vortex dans les condensats de Bose-Einstein.

En effet, les supraconducteurs de type II (i.e. avec $\kappa = \varepsilon^{-1} > 2^{-1/2}$) exhibent également des vortex quantifiés quand ils sont soumis à un champ magnétique assez fort. Un vortex est ici une zone où la densité d'électrons supraconducteurs est nulle, i.e. $|\psi|^2 = 0$. Les vortex présents dans les superfluides et les supraconducteurs ont de nombreux points communs. Par exemple :

- Plusieurs vortex de degré 1 sont en général plus stables qu'un seul vortex de degré supérieur à 1.

- Des vortex dont le degré a même signe se repoussent alors que des vortex de degrés de signe opposé s'attirent et ont tendance à fusionner. En conséquence on observe en général des échantillons ne comprenant que des vortex de degré 1 ou que des vortex de degré -1 .
- La répulsion entre les vortex mène à la formation de réseaux hexagonaux, dits réseaux d'Abrikosov, du nom du physicien qui a prédit leur existence dans [2] (ce qui lui a valu le prix Nobel en 2003, en même temps que Ginzburg, récompensé également pour ses travaux sur la supraconductivité). Ces réseaux sont observés à la fois dans les supraconducteurs (cf Figure 2) et dans les BECs (par exemple Figure 1, image de droite).

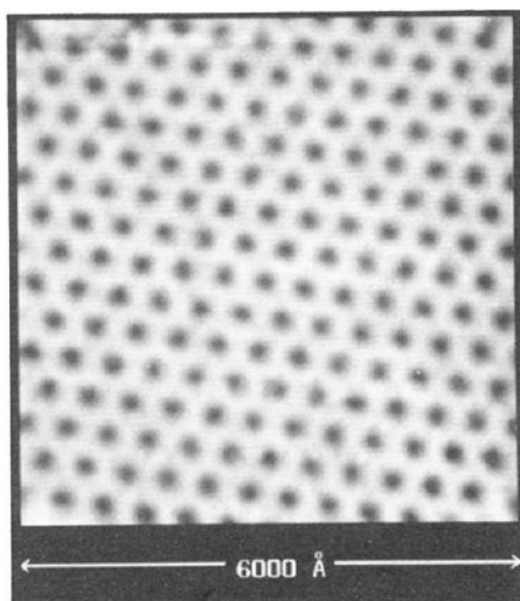


FIGURE 2 – Réseau de vortex dans un supraconducteur de type II [78]

Beaucoup de techniques mathématiques pour étudier la formation de vortex ont été développées (voir [29, 135]) pour comprendre la fonctionnelle (0.1.9) ou une de ses simplifications (dite fonctionnelle de Ginzburg-Landau sans champ magnétique). Puisqu'on fera dans cette thèse un grand usage de ces outils, il est utile de rappeler la similarité entre les deux fonctionnelles d'énergie, apparente en comparant les équations (0.1.7) et (0.1.9). L'intensité du champ magnétique h_{ex} joue le rôle de Ω et ce sont ces paramètres qui gouvernent l'apparition des vortex. Il existe des champs critiques pour l'apparition de vortex dans un supraconducteur (voir les livres [135, 67] et leurs listes de références), ainsi que des vitesses critiques pour l'apparition de vortex dans un superfluide (voir [4] et la suite de la présente thèse). Le paramètre ε^{-2} est l'équivalent dans le modèle de GL de G dans le modèle de GP.

Soulignons également des différences entre les deux problèmes dont l'importance s'est révélée cruciale au cours de nos travaux et sur lesquelles nous reviendrons amplement par la suite :

- Le potentiel vecteur A est un paramètre de la minimisation dans le modèle de Ginzburg-Landau, alors que son équivalent dans le modèle de Gross-Pitaevskii est fixé égal à $\vec{\Omega} \times \vec{r}$.
- L'énergie de Gross-Pitaevskii contient un terme d'énergie potentielle qui permet de poser le problème sur tout l'espace et a une grosse influence sur la densité de matière. Alors que pour un supraconducteur on a typiquement un profil homogène ($|\psi| \equiv 1$ sauf près des vortex), pour un condensat dans un piège magnétique la densité de matière varie suivant un profil fixé par le potentiel effectif. On verra notamment l'importance cruciale que peuvent prendre les forces centrifuges.
- La minimisation de l'énergie de Gross-Pitaevskii se fait sous une contrainte de masse car un condensat n'a pas d'état normal vers lequel relaxer : $|\psi|^2$ représente dans le modèle de GP une densité d'atomes, et ces atomes ne peuvent disparaître, contrairement aux paires d'électrons supraconducteurs qui peuvent se dissocier.

1.3 L'énergie de Gross-Pitaevskii comme limite du problème quantique à N corps

Les deux modèles que nous venons de décrire ont été introduits de manière phénoménologique pour expliquer certaines expériences. Il faut toutefois noter que la superfluidité et la supraconductivité s'inscrivent toutes deux dans le cadre plus large de la mécanique quantique, et il est donc intéressant d'étudier les liens entre les modèles de GP et GL et le modèle fondamental de la physique quantique, le problème quantique à N corps. Pour le cas de la supraconductivité, ce lien passe par la théorie microscopique de BCS (Bardeen-Cooper-Schrieffer, prix Nobel de physique 1972) mais n'est pas entièrement compris mathématiquement. Nous avons toutefois appris pendant la rédaction de cette thèse que des avancées avaient été faites sur la dérivation de la théorie de GL à partir de celle de BCS [70].

Un avantage de la théorie de Gross-Pitaevskii est que son rapport au problème quantique à N corps est maintenant bien connu mathématiquement grâce aux travaux de Lieb, Seiringer et Yngvason dans le cas statique et Erdős, Schlein et Yau dans le cas dynamique.

Bien que l'étude du problème à N corps ne soit pas l'objet de cette thèse, nous allons maintenant décrire brièvement la façon dont le modèle basé sur l'énergie de Gross-Pitaevskii se déduit du modèle fondamental de la mécanique quantique. Ceci nous permettra de préciser l'interprétation du paramètre d'interaction G du modèle de GP.

Le lecteur pourra trouver dans [98] de plus amples développements. Au sujet de la dérivation de l'aspect dynamique de la théorie et notamment de l'équation (0.1.6), mentionnons les travaux [55, 56, 57, 58], la revue [137] et les développements récents [120, 121].

1.3.1 Le problème quantique à N corps

Un système de N particules à température nulle est décrit par une fonction d'onde Ψ à N variables $x_1, \dots, x_N \in \mathbb{R}^3$. Le carré du module de cette fonction au point (x_1, \dots, x_N) donne la densité de probabilité que les particules soient simultanément aux positions res-

pectives x_1, \dots, x_N . On impose donc la normalisation

$$\int_{\mathbb{R}^{3N}} |\Psi(x_1, \dots, x_N)|^2 dx_1 \dots dx_N = 1. \quad (0.1.10)$$

Une distinction doit être faite suivant que l'on considère des fermions ou des bosons. Comme son nom l'indique la condensation de Bose-Einstein n'est possible que pour des bosons, c'est à dire des particules pour lesquelles la fonction d'onde Ψ est symétrique par rapport à l'échange de deux particules. Les fermions obéissent au principe d'exclusion de Pauli qui les empêche de se condenser.

Si on note $X = (x_1, \dots, x_N)$ le vecteur à N particules et σX un autre vecteur obtenu via la permutation σ des coordonnées (x_1, \dots, x_N) , il faut imposer à une fonction d'onde bosonique la condition

$$\Psi(\sigma X) = \Psi(X) \text{ pour toute permutation } \sigma \text{ des variable } x_1, \dots, x_N \quad (0.1.11)$$

Pour comparaison, rappelons que la condition à imposer pour des fermions est une condition d'antisymétrie : pour toute permutation σ agissant sur les variables du vecteur $X = (x_1, \dots, x_N)$ on doit avoir

$$\Psi(\sigma X) = \epsilon(\sigma)\Psi(X)$$

où $\epsilon(\sigma)$ est la signature de la permutation σ .

La fonctionnelle d'énergie du problème quantique à N corps que nous considérons est la suivante (les initiales QM renvoient à 'Quantum Mechanics') :

$$\mathcal{E}^{\text{QM}}[\Psi] := \int_{\mathbb{R}^{3N}} \bar{\Psi} H^{\text{QM}} \Psi \quad (0.1.12)$$

où H^{QM} est le Hamiltonien pour N particules en rotation à vitesse $\vec{\Omega}$ dans le potentiel de piégeage V et interagissant via le potentiel de paire v . Il s'exprime dans le repère en rotation

$$H^{\text{QM}} := \sum_{k=1}^N \left(-\Delta_k + V(x_k) - 2i\nabla_k \cdot \vec{\Omega} \times x_k \right) + \sum_{k \neq j} v(x_k - x_j). \quad (0.1.13)$$

Les opérateurs différentiels avec un indice k (comme Δ_k et ∇_k) symbolisent des différentiations par rapport à la variable x_k . Dans ce Hamiltonien, V est le potentiel de piégeage des atomes apparaissant déjà dans l'énergie de Gross-Pitaevskii (0.1.1).

Le potentiel de paire v modélise les interactions entre les atomes, que l'on suppose n'interagissant que deux à deux (plus précisément, on suppose que la probabilité que trois particules se rencontrent est faible). Pour l'exposé simple que nous voulons faire nous supposons que

$$v \text{ est positif et radial.}$$

Avec ces notations, l'état fondamental de N bosons à température nulle est donné par le problème variationnel

$$E^{\text{QM}} := \inf \{ \mathcal{E}^{\text{QM}}[\Psi], \Psi \text{ vérifiant (0.1.10) et (0.1.11)} \} = \mathcal{E}^{\text{QM}}[\Psi^{\text{QM}}]. \quad (0.1.14)$$

L'existence d'un minimiseur pour ce problème n'est pas difficile, son unicité peut devenir fautive si $\Omega \neq 0$.

Une quantité liée à v jouant un rôle crucial est la *longueur de diffusion* (scattering length) a (voir [98, Appendice C] et [141]) définie par

$$4\pi a := \inf \left\{ \int_{\mathbb{R}^3} \left(|\nabla \phi|^2 + \frac{1}{2} v |\phi|^2 \right) dx, \phi \text{ radiale, positive avec } \phi(x) \rightarrow 1 \text{ quand } |x| \rightarrow +\infty \right\}. \quad (0.1.15)$$

Un exemple simple mais important de potentiel v est le potentiel de sphère dure de rayon a donné formellement par

$$v(x) = \begin{cases} 0 & \text{si } |x| \geq a \\ +\infty & \text{si } |x| < a. \end{cases} \quad (0.1.16)$$

Pour ce potentiel la longueur de diffusion est égale au rayon a , c'est la raison pour laquelle nous n'avons pas introduit de notation supplémentaire. D'autres potentiels, notamment des potentiels plus réguliers, sont possibles. La quantité a définie plus haut s'appelle *longueur* de scattering car une solution ϕ_0 du problème variationnel (0.1.15) se comporte asymptotiquement comme

$$\phi_0(x) = 1 - \frac{a}{|x|} \quad (0.1.17)$$

lorsque $|x|$ tend vers l'infini.

1.3.2 Définition de la condensation de Bose-Einstein pour des particules en interaction

Avec ces (brefs) rappels de mécanique quantique en main nous pouvons revenir sur une question laissée en suspens plus haut (et laissée longtemps en suspens par les physiciens) : que faut-il entendre par condensation de Bose-Einstein pour des particules en interaction ? Dans ce formalisme, la fonction d'onde d'un condensat pur a la forme

$$\Psi(x_1, \dots, x_N) = \prod_{k=1}^N \psi(x_k) \quad (0.1.18)$$

ce qui correspond à ce que toutes les particules soient dans l'état ψ . Nous retrouvons la remarque que nous avons faite auparavant : si les particules n'interagissent pas (i.e. $v \equiv 0$), l'état fondamental du système est un condensat pur. En effet, dans ce cas le Hamiltonien prend la forme d'une somme de Hamiltoniens à une particule

$$H^{\text{QM}} := \sum_{k=1}^N \left(-\Delta_k + V(x_k) - 2i \nabla_k \cdot \vec{\Omega} \times x_k \right)$$

et il est clair que l'infimum dans (0.1.14) est atteint par

$$\Psi(x_1, \dots, x_N) = \prod_{k=1}^N \psi_0(x_k) \quad (0.1.19)$$

où ψ_0 minimise l'énergie pour une particule

$$\int_{\mathbb{R}^3} |\nabla \psi|^2 - 2i \vec{\Omega} \times x \cdot (i\psi, \nabla \psi) + V|\psi|^2,$$

c'est-à-dire une énergie de Gross-Pitaevskii où le terme d'interaction est nul ($G = 0$ dans (0.1.1)).

Pour des particules en interaction ($v \neq 0$) les solutions de (0.1.14) n'ont pas la structure (0.1.18) et l'existence d'un condensat est une question hautement non triviale. On s'attend à ce que la réponse soit asymptotiquement positive dans le cas où le nombre de particules N devient grand. Une difficulté apparaît alors : on voudrait écrire la convergence de Ψ^{QM} vers un état de la forme (0.1.18), mais la dimension de l'espace auquel appartient Ψ^{QM} varie avec N . Ceci complique la tâche si l'on désire considérer la limite $N \rightarrow +\infty$ du problème.

La définition que Penrose et Onsager ont donné en 1956 [119] permet de pallier à cette difficulté. Elle utilise la *matrice de densité à une particule* définie pour une fonction d'onde à N particules Ψ par

$$\gamma(x, x') = N \int_{\mathbb{R}^{3(N-1)}} \Psi(x, x_2, \dots, x_N) \bar{\Psi}(x', x_2, \dots, x_N) dx_2 \dots dx_N. \quad (0.1.20)$$

La matrice de densité γ , considérée comme un noyau (kernel), définit naturellement un opérateur sur $L^2(\mathbb{R}^3)$. Cette espace ne dépend plus de N et on lira donc l'apparition de la condensation plus aisément sur γ que sur Ψ . Selon la définition de Penrose et Onsager, la condensation de Bose-Einstein est la propriété pour cet opérateur d'avoir une valeur propre d'ordre N quand N tend vers l'infini. On notera que les valeurs propres d'une matrice de densité associée à une fonction d'onde normalisée sont nécessairement plus petites que N . On dira qu'il y a condensation complète si une de ces valeurs propres se comporte comme N quand N tend vers l'infini.

Remarquons que pour un condensat pur de la forme (0.1.18), l'opérateur associé à γ est le projecteur orthogonal sur l'espace vectoriel engendré par ψ , il y a donc bien condensation complète puisque il n'y a qu'une valeur propre non nulle, égale à N .

On s'attend depuis les premiers travaux utilisant l'énergie de GP (par exemple [72, 73]) à ce que la matrice de densité associée à un minimiseur Ψ^{QM} de (0.1.12) converge quand $N \rightarrow +\infty$ vers le projecteur sur Ψ_{3D}^{GP} , minimiseur de l'énergie de Gross-Pitaevskii (0.1.1) avec

$$G = 4\pi Na \quad (0.1.21)$$

où a est la longueur de diffusion définie par (0.1.15). Pour pouvoir écrire une limite à proprement parler, il est alors clair que la limite la plus naturelle à considérer est celle dite de Gross-Pitaevskii dans laquelle tous les termes de l'énergie (0.1.1) ont le même ordre de grandeur :

$$N \rightarrow +\infty, \quad Na \text{ fixé}, \quad \Omega \text{ fixé.} \quad (0.1.22)$$

Le choix de fixer Na (et donc de faire tendre a vers 0 quand $N \rightarrow +\infty$) garantit que Ψ_{3D}^{GP} , fondamental de l'énergie de Gross-Pitaevskii avec $G = 4\pi Na$, est bien une fonction indépendante de N . Le projecteur sur l'espace vectoriel qu'il engendre est donc également indépendant de N et on peut espérer que la matrice de densité associée à Ψ^{QM} converge en un certain sens vers ce projecteur.

Ces considérations font apparaître une nouvelle difficulté : si l'on prend un condensat pur de forme (0.1.18) comme fonction test pour la fonctionnelle (0.1.12), on obtient dans la limite (0.1.22)

$$\mathcal{E}^{\text{QM}}[\Psi] \sim N \mathcal{E}_{3D}^{\text{GP}}[\psi]$$

mais le paramètre G apparaissant dans $\mathcal{E}_{3D}^{\text{GP}}$ est alors

$$G = 4\pi N \int_{\mathbb{R}^3} v$$

où v est le potentiel d'interaction des atomes. Cette valeur de G est systématiquement plus grande que celle attendue (0.1.21) comme on le voit facilement grâce à la définition (0.1.15). Elle peut même être infinie comme dans l'exemple important d'un potentiel de sphère dure (0.1.16). On voit donc combien la définition de Penrose et Onsager est utile pour appréhender le concept de condensation. Il peut y avoir en effet condensation au sens de Penrose et Onsager sans que la fonction d'onde soit factorisée sous la forme (0.1.18). De plus, comme nous le verrons plus bas, la condensation est en général un corollaire de la convergence

$$\frac{1}{N} E^{\text{QM}} \rightarrow E^{\text{GP}}$$

où le paramètre d'interaction dans E^{GP} est donné par (0.1.21). Une convergence comme celle ci-dessus est bien sûr un premier argument fort en faveur de l'utilisation du modèle de GP.

1.3.3 La limite de Gross-Pitaevskii du problème quantique pour N bosons

Les premières avancées mathématiquement rigoureuses vers la compréhension de la condensation datent de [52]. Notamment, le mécanisme microscopique qui fait apparaître comme énergie limite l'énergie de Gross-Pitaevskii avec $G = 4\pi Na$ y est élucidé dans un cadre plus simple que celui que nous venons de décrire (nommément, celui d'un gaz de Bose homogène avec potentiel de sphère dure). En termes mathématiques, une méthode pour construire une fonction test donnant à la limite l'énergie correcte est développée. Cette fonction test contient une structure que nous ne détaillerons pas ici mais qui permet de prendre en compte les modifications que la présence d'interactions entre les atomes apporte à un condensat pur. Cette méthode fournit la borne supérieure attendue sur l'énergie. Par contre, la borne inférieure démontrée dans [52] est trop petite d'un facteur environ 14.

Les résultats complets, incluant notamment une borne inférieure asymptotiquement égale à la borne supérieure de Dyson sont dus à Lieb, Seiringer et Yngavson (voir [98] ou [99, 103] pour les références originales) dans le cas $\Omega = 0$ et à Lieb et Seiringer [97, 140] dans le cas $\Omega \neq 0$. Le second cas est plus complexe que le premier et demande des méthodes plus sophistiquées. L'énoncé du résultat lui-même est compliqué car le terme de rotation brise la symétrie sphérique du problème : le minimiseur de l'énergie (0.1.1) cesse d'être unique à une constante multiplicative près, donc la convergence de la matrice de densité peut au mieux se faire vers une combinaison convexe de projecteurs orthogonaux sur des minimiseurs de l'énergie de Gross-Pitaevskii. Mais l'ensemble de ces minimiseurs n'est pas discret, ce qui complique considérablement l'énoncé. Notons qu'il ne s'agit pas d'un problème uniquement technique puisque cette perte d'unicité est essentielle : elle peut notamment être due à la présence de vortex qui brisent la symétrie sphérique du minimiseur alors que le Hamiltonien est lui-même à symétrie sphérique.

Une autre difficulté, résolue dans [97] en utilisant le formalisme de la seconde quantification et (entre autres) les idées de [101], est que lorsque $\Omega \neq 0$, le fondamental dans le secteur bosonique de l'énergie (0.1.12) n'est pas égal au fondamental absolu, c'est à dire sans contrainte de symétrie. Une conséquence de ce fait est que, alors que dans le cas $\Omega = 0$ les deux fondamentaux et les énergies associées ont les mêmes limites, quand $\Omega \neq 0$ l'apparition

de l'énergie de Gross-Pitaevskii est une conséquence de la contrainte de symétrie imposée à la fonction d'onde. Sans cette contrainte, le problème limite est complètement différent, comme démontré dans [140].

Pour donner au lecteur une idée des résultats de Lieb, Seiringer et Yngavson justifiant l'emploi du modèle de Gross-Pitaevskii en évitant ces difficultés nous nous contenterons de citer un théorème obtenu dans le cas $\Omega = 0$. Le résultat ci-dessous est une combinaison des Théorèmes 6.1 et 7.1 de [98] auquel nous renvoyons pour la preuve, de plus amples détails et de nombreuses références.

Théorème 0.1.1 (Limite de Gross-Pitaevskii du problème à N corps, cas $\Omega = 0$). Soient Ψ^{QM} et Ψ_{3D}^{GP} les uniques solutions positives des problèmes (0.1.14) et (0.1.3) avec G donné par (0.1.21) et $\Omega = 0$. On note γ^{QM} la matrice de densité associée à Ψ^{QM} par la formule (0.1.20).

On suppose que N tend vers l'infini avec le scaling (0.1.22). On a alors

$$\frac{1}{N}\gamma^{\text{QM}}(x, x') \rightarrow \Psi_{3D}^{\text{GP}}(x)\Psi_{3D}^{\text{GP}}(x') \quad (0.1.23)$$

en norme de trace, i.e. l'opérateur de noyau $\frac{1}{N}\gamma^{\text{QM}}$ converge vers P^{GP} , le projecteur orthogonal sur Ψ_{3D}^{GP} .

De plus

$$\frac{1}{N}E^{\text{QM}} \rightarrow E_{3D}^{\text{GP}} \quad (0.1.24)$$

où le paramètre G dans la définition (0.1.3) de E_{3D}^{GP} est donné par (0.1.21).

On entend ici par 'norme de trace' d'un opérateur K agissant sur un espace de Hilbert \mathfrak{H} la quantité

$$\text{Tr}(|K|) := \sum_n \langle u_n, |K| u_n \rangle_{\mathfrak{H}}$$

où

$$|K| := \sqrt{K^*K}$$

et $(u_n)_{n \in \mathbb{N}}$ est une base Hilbertienne de \mathfrak{H} . Il est bien connu que cette définition est indépendante de la base choisie. Dans le théorème ci-dessus l'espace \mathfrak{H} est $L^2(\mathbb{R}^3)$.

Le résultat (0.1.24) reste vrai lorsque $\Omega \neq 0$ (voir [140] pour la borne supérieure et [97] pour la borne inférieure) est fixé. Cela justifie le modèle phénoménologique de Gross-Pitaevskii pour l'étude des condensats de Bose-Einstein en rotation. Remarquons que le cas compris mathématiquement correspond au régime (0.1.22). Dans cette thèse nous allons considérer différents régimes asymptotiques pour l'énergie de Gross-Pitaevskii où G et Ω ne seront plus fixés. Une question légitime est alors de savoir si de telles limites et (0.1.22) peuvent être prises l'une après l'autre et donner le même résultat que donnerait la variation simultanée de tous les paramètres au niveau du problème à N corps. Ceci est un difficile problème ouvert sur lequel peu d'éléments rigoureux ont été fournis.

Pour conclure ce paragraphe, mentionnons l'article [37] où l'on trouve des résultats partiels sur la limite du problème quantique à N corps dans le cas où G et Ω ne sont pas fixés mais peuvent varier et en particulier tendre vers l'infini. L'application des méthodes pour $\Omega \neq 0$ développées dans [97] n'est pas aisée dans ce cas : les différents termes de l'énergie de Gross-Pitaevskii n'ont plus nécessairement la même contribution. Dans [37] il est démontré que dans différents régimes l'énergie de GP donne au moins le premier ordre de l'énergie (0.1.14). Ces résultats partiels ont une importance puisqu'il existe des

régimes de paramètres où même à température nulle on s'attend à ce que l'énergie de Gross-Pitaevskii ne soit plus une bonne description du gaz de Bose. C'est en particulier le cas (nous y reviendrons) lorsque le potentiel de piégeage V est quadratique et que la vitesse de rotation Ω approche une certaine valeur critique. Dans ce cas on s'attend à ce que les atomes soient fortement corrélés (il ne peut donc pas y avoir de condensation). Nous renvoyons à [32, 41, 63] pour de nombreuses références physiques sur ce phénomène et à [95] pour des résultats mathématiques dans cette direction. Précisons simplement que dans les régimes asymptotiques que nous allons considérer dans cette thèse, la réduction au modèle de Gross-Pitaevskii n'est pas entièrement comprise mathématiquement. Elle est toutefois largement acceptée par les physiciens (avec une certaine réserve pour la situation considérée dans la Section 2.1.1, voir la Section 3.2) et les résultats mathématiques partiels de [37] confortent cette vue.

1.4 Phénoménologie d'un condensat de Bose-Einstein bi-dimensionnel en rotation

Pour donner une idée des phénomènes dont la compréhension mathématique est le but de cette thèse nous allons maintenant décrire dans les grandes lignes différents comportements que peut exhiber un BEC mis en rotation. Nous n'avons aucune prétention d'exhaustivité et le lecteur pourra s'orienter vers les revues [32, 41, 63] pour une vision plus large de ce riche domaine physique.

1.4.1 Le modèle de Gross-Pitaevskii en deux dimensions d'espace

A partir de maintenant et dans toute cette thèse nous nous restreindrons à des modèles bi-dimensionnels. Une manière de justifier une telle restriction est de supposer que le potentiel de piégeage V impose un fort confinement dans la direction verticale, i.e. parallèle au vecteur de rotation $\vec{\Omega}$. Notons que cette hypothèse est suffisante pour justifier l'emploi de modèles bi-dimensionnels, mais elle n'est pas toujours nécessaire. En effet, les forces centrifuges présentes dans le potentiel effectif (0.1.8) modifient déjà le confinement ressenti par un BEC en rotation. Suivant la valeur de Ω , le potentiel de piégeage effectif peut être suffisamment modifié pour justifier l'emploi de modèles 2D (voir [7] pour l'étude d'une situation de ce genre). Nous ne développerons pas plus avant sur ce point dans cette introduction, et supposerons que les situations que nous considérons sont bien décrites par une énergie de Gross-Pitaevskii 2D où le condensat est confiné dans le plan perpendiculaire au vecteur de rotation :

$$\mathcal{E}^{\text{GP}}[\psi] := \int_{\mathbb{R}^2} \left(|\nabla\psi|^2 - 2\Omega x^\perp \cdot (i\psi, \nabla\psi) + V(x)|\psi|^2 + G|\psi|^4 \right) dx. \quad (0.1.25)$$

Ici $x^\perp = (-x_2, x_1)$ si $x = (x_1, x_2)$. On peut également factoriser les deux premiers termes comme dans (0.1.7) pour obtenir la forme équivalente

$$\mathcal{E}^{\text{GP}}[\psi] := \int_{\mathbb{R}^2} \left(\left| \nabla\psi - i\Omega x^\perp \psi \right|^2 + (V(x) - \Omega^2 |x|^2) |\psi|^2 + G|\psi|^4 \right) dx. \quad (0.1.26)$$

Nous conserverons une contrainte de masse unité

$$\int_{\mathbb{R}^2} |\psi|^2 = 1 \quad (0.1.27)$$

et le problème variationnel donnant l'état fondamental du système sera

$$E^{\text{GP}} := \inf \left\{ \mathcal{E}^{\text{GP}}[\psi], \int_{\mathbb{R}^2} |\psi|^2 = 1 \right\} = \mathcal{E}^{\text{GP}}[\Psi^{\text{GP}}]. \quad (0.1.28)$$

Signalons que la valeur de la constante d'interaction G ci-dessus est essentiellement déterminée par la façon dont on déduit le modèle réduit à partir de modèles plus généraux, qu'ils soient de type Gross-Pitaevskii 3D [7] ou quantique à N corps. Pour l'obtention de modèles de type GP bi-dimensionnels à partir du problème quantique à N corps en trois dimensions d'espace, différentes approches sont possibles qui ne sont pas toutes physiquement équivalentes. Nous renvoyons à [98] et [138] pour des développements dans cette direction et prendrons dans toute la suite (0.1.28) comme point de départ et G comme paramètre sans en donner d'expression explicite de type (0.1.21). Dans la plupart des situations physiquement pertinentes G doit toutefois rester proportionnel à Na (où N est le nombre d'atomes et a la longueur de scattering) mais d'autres scalings sont envisageables. Des résultats sur les modèles de Gross-Pitaevskii en trois dimensions d'espace peuvent être trouvés dans le livre [4] et les articles [24, 86]. La forme tri-dimensionnelle des tourbillons peut notamment être non triviale et son étude d'importance, ce que nous négligerons dans la présentation qui suit.

De fortes différences apparaissent suivant le type de potentiel de piégeage V qui est utilisé. Pour la plupart des expériences le potentiel croît quadratiquement avec la distance à l'axe de rotation : il est de forme

$$V(x) = a_1 x_1^2 + a_2 x_2^2. \quad (0.1.29)$$

Avec un tel potentiel (que nous appellerons potentiel harmonique), Ω ne peut pas excéder une certaine limite, en effet si

$$\Omega^2 \geq \min(a_1, a_2)$$

on voit sur (0.1.26) que l'énergie \mathcal{E}^{GP} n'est pas bornée inférieurement et donc le problème de minimisation n'a pas de sens. Physiquement, si Ω est trop grand, la force centrifuge due à la rotation du condensat est supérieure à la force de confinement fournie par le piège, et les atomes sont expulsés.

Pour palier à ce défaut, de nouveaux types de piège ont été proposés, notamment un piège 'quadratique plus quartique'

$$V_k(x) = |x|^2 + k|x|^4 \quad (0.1.30)$$

proposé par Fetter [61]. D'autres possibilités sont des pièges homogènes

$$V_s(x) = |x|^s, \quad s > 2 \quad (0.1.31)$$

et un modèle simplifié ('flat trap'), qui correspond à restreindre la minimisation au disque de rayon 1 :

$$V(x) = \begin{cases} +\infty & \text{pour } |x| > 1 \\ 0 & \text{pour } |x| \leq 1. \end{cases} \quad (0.1.32)$$

Formellement le piège (0.1.32) est la limite quand $s \rightarrow +\infty$ du piège (0.1.31). Ces trois types de pièges (auxquels nous nous référerons par l'expression consacrée mais peu précise de 'pièges anharmoniques') ont en commun de permettre en théorie une vitesse de rotation Ω arbitrairement grande. L'objet de cette thèse est l'étude de l'énergie (0.1.25) avec des

potentiels des types décrits ci-dessus. En particulier, on vise à étudier des situations qui ne sont pas accessibles avec les pièges plus simples de type harmonique, où les structures de tourbillons sont encore mal connues.

Le modèle (0.1.30) décrit bien un type de potentiel obtenu au laboratoire Kastler Brossel de l'ENS Ulm [35, 146] en imposant en plus du traditionnel piège magnétique fournissant un potentiel quadratique un laser bleu créant un potentiel de type gaussien. La forme (0.1.30) apparaît en développant ce potentiel gaussien en série entière et négligeant des termes d'ordre inférieur. Remarquons que très récemment [131, 77] des types de piège très généraux ont été réalisés mais qu'à notre connaissance la rotation d'un condensat a été principalement mise en oeuvre dans des pièges quadratiques et dans le piège décrit dans les articles [35, 146].

1.4.2 Condensat en rotation dans un potentiel harmonique

Dans ce paragraphe nous donnons un bref aperçu des phénomènes observés dans un BEC en rotation dans un piège harmonique isotrope, i.e. de type (0.1.29) avec $a_1 = a_2$. Nous reproduisons d'abord la figure 1 qui représente la rotation d'un BEC dans un piège quadratique isotrope ($a_1 = a_2$). Il s'agit d'une image expérimentale obtenue au laboratoire Kastler Brossel de l'ENS Ulm [108, 109].

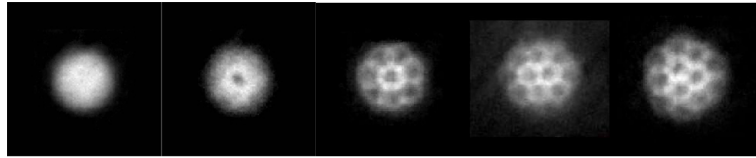


FIGURE 3 – Vortex dans un condensat en rotation dans un piège quadratique isotrope

Dans cette situation, le gaz mis en rotation à faible vitesse ne contient aucun tourbillon (image de gauche). Il y a ensuite une vitesse critique pour l'apparition d'un tourbillon, puis de plus en plus de tourbillons apparaissent au fur et à mesure que la vitesse de rotation augmente. Quand de nombreux vortex sont présents, ils tendent à s'organiser en réseaux hexagonaux (appelés réseaux d'Abrikosov par analogie avec la physique des supraconducteurs de type II [3]) pour minimiser leur répulsion. Lorsque la vitesse de rotation s'approche de la vitesse limite imposée par le confinement (i.e. $\Omega \rightarrow a_1 = a_2$ pour un piège isotrope), des centaines de vortex peuvent être observés, comme sur les spectaculaires images expérimentales de la Figure 4, tirées de [1]. Notons que tous les vortex sont ici de degré 1.

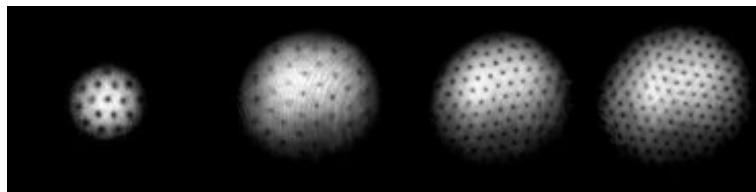


FIGURE 4 – Réseaux de vortex dans un condensat en rotation rapide

La littérature physique motivée par ces expériences est vaste et nous renvoyons encore une fois à l'excellente revue [63] et à son immense liste de références (voir également celle

de [4]). Plus récemment, certains phénomènes ont été étudiés de manière mathématiquement rigoureuse : mentionnons [81, 82] pour l'étude de l'apparition des premiers vortex et de leur organisation, ainsi que [6, 11, 12] pour l'étude des réseaux de vortex. Notons également que par simplicité nous avons seulement décrit les phénomènes observés dans un piège isotrope. Un piégeage anisotrope peut significativement modifier les phénomènes observés, voir par exemple [9, 10, 62].

Comme nous le mentionnions à la fin de la section 1.3, lorsque la vitesse de rotation approche de trop près la valeur critique fixée par le piège, la description par le modèle de Gross-Pitaevskii cesse d'être valide et on doit revenir au problème à N corps brièvement présenté ci-dessus. Ce phénomène donne lieu à des phases fortement corrélées de type 'état de Laughlin' [94] et est en général appelé 'effet Hall fractionnaire quantique' par analogie avec certains phénomènes observés lorsque des électrons sont soumis à un champ magnétique. Ce régime n'a pas encore pu être observé expérimentalement et correspondrait à une vitesse de rotation encore plus élevée que celles permettant d'obtenir la Figure 4. Le critère généralement retenu [41] pour déterminer la vitesse à laquelle les phénomènes de corrélation apparaissent est l'égalité du nombre de vortex prévu par la théorie GP et du nombre de particules. Une telle égalité pose évidemment un problème conceptuel et indique que la description GP ne devrait plus être utilisée. Plus précisément on s'attend à ce que la théorie de GP cesse d'être valide quand le nombre de vortex est du même ordre de grandeur que le nombre de particules. Puisque le nombre de vortex croît avec la vitesse de rotation, une vitesse seuil apparaît à laquelle ce critère est rempli et donc la fonction d'onde du problème à N corps devient fortement corrélée. Nous renvoyons à [102, 95] pour des résultats mathématiques sur cette vitesse critique, et notamment son expression en fonction des paramètres du problème.

1.4.3 Condensat en rotation dans un potentiel anharmonique

Dans cette thèse nous nous sommes intéressés aux nouveaux phénomènes que peut provoquer un potentiel de piégeage croissant plus vite que le carré de la distance à l'axe de rotation.

Dans [64, 65, 91], des arguments formels et numériques sont fournis en faveur du schéma suivant :

- Tant que la vitesse de rotation reste inférieure à un certain seuil, la séquence expérimentale ne diffère pas de celle décrite ci-dessus pour un piège quadratique. Des vortex de degré 1 apparaissent lorsqu'une première vitesse critique est franchie puis forment un réseau hexagonal.
- La grande spécificité du piégeage anharmonique apparaît lorsque la force centrifuge induite par la rotation devient conséquente. Les atomes ne sont pas expulsés du piège comme dans le cas d'un potentiel quadratique mais la force centrifuge creuse un trou central lorsque la vitesse de rotation franchit une seconde valeur critique. Un réseau hexagonal de vortex reste présent dans le condensat annulaire et le trou central créé par la force centrifuge agit comme un vortex multi-quantifié, i.e. il y a un degré supérieur à 1 autour.
- Lorsque la vitesse de rotation franchit un troisième seuil tous les vortex se ras-

semblent dans le trou central créé par la force centrifuge, laissant un condensat annulaire sans vortex mais avec une circulation de phase macroscopique (multiquantifiée). On appellera souvent par la suite un tel condensat un *vortex géant*.

Malheureusement, les expériences réalisées [35] n'ont pas atteint des vitesses suffisantes pour observer toutes les transition de phase que nous venons de décrire. Les observations expérimentales qui ont pu être faites (voir la Figure 5) confirment l'apparition d'une déplétion de la densité au centre du condensat mais pas au point qu'elle puisse être qualifiée de 'trou'. En d'autres termes la densité est plus faible au centre du condensat mais elle reste notable, alors qu'on s'attend à ce qu'elle soit exponentiellement petite dans le 'trou central' d'un condensat annulaire.

Outre la difficulté intrinsèque qu'il y a à obtenir des vitesses de rotation trop grandes, il est probable que le condensat créé en laboratoire n'est pas assez fortement confiné verticalement pour que les arguments théoriques généralement utilisés et basés sur des modèles 2D soient complètement valides (voir [49] pour des simulations numériques 3D de la séquence expérimentale et une discussion sur ce point).

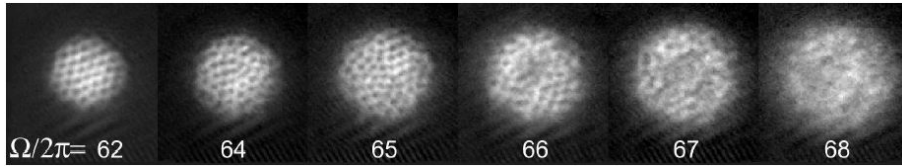


FIGURE 5 – Condensat en rotation dans un piège anharmonique, voir [35, 144]

Si les expériences n'atteignent pas les deux vitesses critiques décrites ci-dessus, des simulations numériques sont disponibles. Nous reproduisons ici celles de [64] (voir Figure 6).

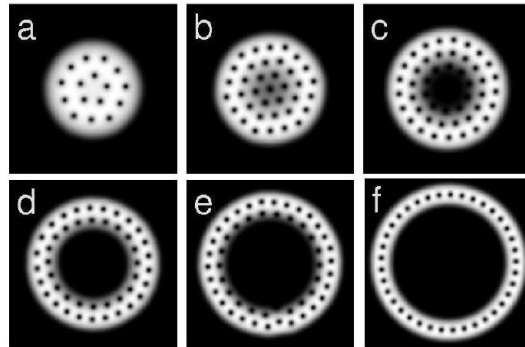


FIGURE 6 – Simulations numériques d'un condensat en rotation dans un piège de type (0.1.30)

Comme on peut le voir, le condensat annulaire supportant un réseau de vortex est clairement observé numériquement. En revanche, même les simulations numériques deviennent difficiles au voisinage de la troisième vitesse critique, celle conduisant à un vortex géant. Les simulations numériques que nous reproduisons ci-dessous (Figure 7, également tirée de [64]) sont faites pour un jeu de paramètres éloigné de celui requis par la théorie

(notamment un paramètre d'interaction G trop petit). Elles ont toutefois le mérite de montrer l'apparition du vortex géant même si celle-ci n'est pas précédée par la formation d'un réseau de vortex à proprement parler.

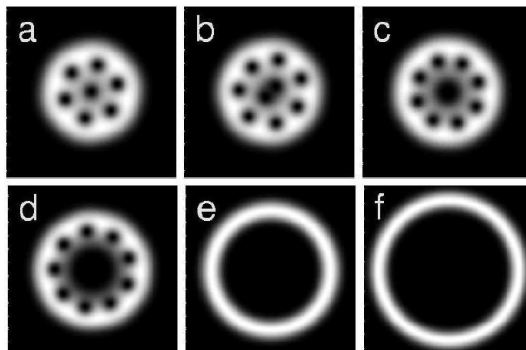


FIGURE 7 – Simulation numérique de l'apparition d'un vortex géant

Notons également que les simulations numériques des Figures 6 et 7 suggèrent qu'une phase comportant un vortex géant central entouré d'un cercle de vortex apparaît avant que la troisième vitesse critique ne soit franchie.

Dans cette thèse nous nous intéressons aux phénomènes décrits ci-dessus, apparaissant notamment dans des pièges de type (0.1.30), (0.1.31) ou (0.1.32), et principalement aux condensats annulaires créés par la force centrifuge. Avant de présenter ces résultats nous voulons signaler qu'il est possible [13] d'imaginer des pièges imposant au condensat une forme annulaire même en l'absence de rotation et donc de force centrifuge. Citons l'article [5] pour l'étude mathématique d'un condensat en rotation dans un tel piège. Des potentiels du genre considéré dans cet article sont maintenant réalisables [131, 77] mais à notre connaissance les condensats créés n'ont pas été mis en rotation. Dans [131, 148] d'autres techniques sont utilisées pour créer une circulation de phase.

La Section suivante est consacrée à une présentation des résultats obtenus dans cette thèse. Ils portent sur l'étude de l'état fondamental de la fonctionnelle (0.1.25) dans différents régimes asymptotiques, avec comme potentiel de piégeage soit (0.1.30) soit le modèle simplifié (0.1.32). Les phénomènes que nous avons cherché à comprendre via une analyse mathématique rigoureuse dans certains régimes asymptotiques sont les suivants.

- La structure du réseau de vortex présent dans un condensat annulaire. On donne en particulier des arguments indiquant que ce réseau devrait être déformé au bord de l'anneau et observons cette déformation numériquement.
- L'apparition du vortex géant. Nous obtenons en particulier une preuve rigoureuse de son apparition pour des grandes vitesses de rotation.
- La transition de phase amenant au vortex géant. Nous obtenons une expression asymptotique de la vitesse critique à laquelle la transition de phase a lieu. Nous sommes également capables de fournir une estimation du degré du vortex géant.

- La formation d'un cercle de vortex lorsque la vitesse de rotation approche par valeur inférieure la vitesse de transition vers le vortex géant. Nous montrons rigoureusement l'apparition de ce cercle de vortex et évaluons le nombre de vortex qu'il contient. Nous montrons également que ces vortex sont uniformément répartis le long du cercle.

2 Présentation des résultats

Nous allons maintenant décrire les résultats obtenus dans cette thèse. Nous avons choisi de les classer suivant le modèle utilisé : les résultats pour le modèle de GP avec comme potentiel (0.1.30) sont présentés dans la section 2.1 et les résultats pour le modèle de GP sur un disque (c'est-à-dire formellement avec le piège (0.1.32)) sont présentés dans la section 2.2.

Nos résultats portent sur le problème variationnel (0.1.28) et son comportement asymptotique (propriétés de l'énergie fondamentale et des minimiseurs associés) dans différents régimes de paramètres. Nous utiliserons dans la section 2.1.1 une hypothèse simplificatrice menant à un problème variationnel simplifié, mais dans toutes les autres sections nos résultats portent sur le problème complet.

2.1 Résultats pour le piège quadratique plus quartique

Dans cette section nous nous intéressons au modèle

$$\mathcal{E}^{\text{GP}}[\psi] := \int_{\mathbb{R}^2} \left(|\nabla\psi|^2 - 2\Omega x^\perp \cdot (i\psi, \nabla\psi) + (|x|^2 + k|x|^4) |\psi|^2 + G|\psi|^4 \right) dx \quad (0.2.1)$$

et faisons varier les paramètres k, Ω et G . Nous considérerons deux régimes asymptotiques distincts, l'un donnant lieu à l'apparition d'un condensat annulaire contenant un réseau de vortex et l'autre à un vortex géant.

2.1.1 Régime 'plus bas niveau de Landau' : réseau de vortex dans un condensat annulaire

Les résultats de cette section ont été obtenus en collaboration avec Xavier Blanc et sont parus sous une forme mathématiquement résumée dans [31]. Pour être consistant avec cet article nous écrirons l'énergie avec des paramètres légèrement modifiés :

$$\mathcal{E}^{\text{GP}}[\psi] = \int_{\mathbb{R}^2} \left(\frac{1}{2} |\nabla\psi - i\Omega x^\perp \psi|^2 + \left(\frac{1 - \Omega^2}{2} |x|^2 + \frac{k}{4} |x|^4 \right) |\psi|^2 + \frac{G}{2} |\psi|^4 \right) dx, \quad (0.2.2)$$

avec $x = (x_1, x_2)$, $x^\perp = (-x_2, x_1)$. Rappelons la contrainte de masse

$$\int_{\mathbb{R}^2} |\psi|^2 = 1. \quad (0.2.3)$$

L'hypothèse simplificatrice que nous ferons ici est que la minimisation de \mathcal{E}^{GP} peut être restreinte au plus bas niveau de Landau (LLL en abrégé le terme anglais). Il s'agit du premier espace propre de l'opérateur de Ginzburg-Landau $-(\nabla - i\Omega x^\perp)^2$ sur \mathbb{R}^2 , correspondant à la valeur propre Ω .

L'utilisation d'une telle hypothèse dans le contexte des BECs en rotation et sa justification

formelle datent de [79]. Le LLL a d'abord été utilisé dans le cas de potentiels de piégeage harmonique, pour la fonctionnelle d'énergie

$$\mathcal{E}^{\text{GP}}[\psi] = \int_{\mathbb{R}^2} \left(\frac{1}{2} |\nabla\psi - i\Omega x^\perp \psi|^2 + \left(\frac{1-\Omega^2}{2} |x|^2 \right) |\psi|^2 + \frac{G}{2} |\psi|^4 \right) dx, \quad (0.2.4)$$

i.e. avec $k = 0$ dans (0.2.2). L'utilisation de la restriction au plus bas niveau de Landau est alors justifiée dans le cas où $\Omega \rightarrow 1$. En effet le premier terme de l'énergie ci-dessus est au moins égal à $\Omega \sim 1$ puisque Ω est la première valeur propre de $-(\nabla - i\Omega x^\perp)^2$. En revanche le deuxième terme va visiblement tendre vers 0. Par conséquent, en minimisant la somme du deuxième et du troisième terme on obtient également un terme négligeable devant Ω .

Ceci indique que l'énergie devrait rester proportionnelle à Ω quand $\Omega \rightarrow 1$. Or, le gap spectral entre la première et la deuxième valeur propre de l'opérateur de Ginzburg-Landau est égal à Ω . Pour obtenir une énergie d'ordre Ω il faut donc que la projection d'un minimiseur de la fonctionnelle (1.1.7) sur les niveaux d'énergie excités de l'opérateur $-(\nabla - i\Omega x^\perp)^2$ soit négligeable.

Cette justification formelle de la restriction au LLL dans le régime $\Omega \rightarrow 1$ a servi de base à plusieurs travaux (par exemple [8, 42, 147]), et a été validée mathématiquement dans [7]. Pour d'autres travaux mathématiques sur les BECs avec piège quadratique dans le régime $\Omega \rightarrow 1$ (régime de Landau), voir [6, 12].

Dans ce travail nous étudions un régime de paramètres dans lequel la restriction au LLL est physiquement justifiée pour le modèle (0.2.2). Nous n'avons pas de preuve complète que la restriction est valide mais nos résultats fournissent un argument a posteriori pour la pertinence de l'utilisation du modèle simplifié : nous montrons que l'énergie du problème restreint au LLL est négligeable devant le gap entre le LLL et le premier niveau excité de l'opérateur de Ginzburg-Landau. L'approche de [7] combinée avec une telle estimation permettrait probablement de montrer rigoureusement la validité de la réduction. Nous étudions la minimisation de (0.2.2) en la restreignant au LLL. La forme très particulière de cet espace permet d'obtenir des résultats analytiques puis numériques dans l'esprit de [8, 12].

Le plus bas niveau de Landau est constitué (voir par exemple [107]) des fonctions de la forme $\psi(z) = f(z)e^{-\Omega|z|^2/2}$ où z est la variable complexe $x_1 + ix_2$ et f une fonction holomorphe. Pour une fonction ψ de cette forme, l'énergie de Gross-Pitaevskii devient

$$\mathcal{E}^{\text{GP}}[\psi] = \Omega + \int_{\mathbb{C}} \left(\left(\frac{1-\Omega^2}{2} |z|^2 + \frac{k}{4} |z|^4 \right) |\psi|^2 + \frac{G}{2} |\psi|^4 \right) dz.$$

Ici nous allons plutôt considérer des fonctions de la forme $\psi(z) = f(z)e^{-|z|^2/2}$ de façon à avoir un espace indépendant de Ω . Comme nous allons le voir, notre analyse est valide seulement si Ω est proche de 1, ce qui justifie intuitivement cette modification. En utilisant [7, Lemme 3.1] on montre que si ψ est de la forme $\psi(z) = f(z)e^{-|z|^2/2}$ avec f holomorphe, alors

$$\mathcal{E}^{\text{GP}}[\psi] = \Omega + \mathcal{E}^{\text{LLL}}[\psi] \quad (0.2.5)$$

$$\mathcal{E}^{\text{LLL}}[\psi] = \int_{\mathbb{C}} \left(\left((1-\Omega) |z|^2 + \frac{k}{4} |z|^4 \right) |\psi|^2 + \frac{G}{2} |\psi|^4 \right) dz. \quad (0.2.6)$$

Nous nous intéresserons donc à la minimisation de \mathcal{E}^{LLL} parmi les fonction de la forme $\psi(z) = f(z)e^{-|z|^2/2}$ avec f holomorphe et sous la contrainte de masse

$$\int_{\mathbb{R}^2} |\psi|^2 = 1$$

Le cadre mathématique que nous allons utiliser à été développé dans [11, 12].

Nous décrivons maintenant le régime de paramètres que nous considérerons dans cette section. Nous définissons le petit paramètre

$$\varepsilon = k^{1/3} \tag{0.2.7}$$

et étudions le comportement asymptotique du modèle lorsque $\varepsilon \rightarrow 0$. Nous prendrons une vitesse de rotation de forme

$$\Omega = 1 + \beta k^{2/3} = 1 + \beta \varepsilon^2 \tag{0.2.8}$$

et considérerons β et G comme des constantes fixées. On verra plus bas (0.2.17) que ces choix permettent d'obtenir une énergie réduite dont tous les termes ont le même ordre de grandeur. De plus, puisque k et $1 - \Omega$ tendent vers 0 il est intuitivement clair que l'énergie (0.2.6) sera négligeable devant Ω .

Pour que le condensat reste confiné dans une région bornée de l'espace il est nécessaire de changer d'échelle : nous définissons

$$\phi(z) = \varepsilon^{-1/2} \psi(\varepsilon^{-1/2} z). \tag{0.2.9}$$

Par définition du plus bas niveau de Landau, pour chaque ϕ dans le LLL, la fonction

$$f(z) = \phi(z) e^{|z|^2/2\varepsilon} \tag{0.2.10}$$

appartient à l'espace de Fock-Bargmann [27]

$$\mathfrak{F}_\varepsilon := \left\{ f \text{ holomorphe}, \int_{\mathbb{C}} |f|^2 e^{-|z|^2/\varepsilon} dz < \infty \right\} \tag{0.2.11}$$

qui est un espace de Hilbert pour le produit scalaire

$$\langle f, g \rangle := \int_{\mathbb{C}} \overline{f(z)} g(z) e^{-|z|^2/\varepsilon} dz. \tag{0.2.12}$$

Le projecteur orthogonal de $L^2(\mathbb{C}, e^{-|z|^2/\varepsilon} dz)$ dans \mathfrak{F}_ε (appelé le projecteur de Szegö) a une expression explicite [66, 110] :

$$\Pi_\varepsilon(g)(z) = \frac{1}{\pi\varepsilon} \int_{\mathbb{C}} e^{z\bar{z}'/\varepsilon} e^{-|z'|^2/\varepsilon} g(z') dz'. \tag{0.2.13}$$

Nous utiliserons les espaces

$$\mathfrak{F}_\varepsilon^s := \left\{ f \text{ holomorphe}, \int_{\mathbb{C}} (1 + |z|^2)^s |f|^2 e^{-|z|^2/\varepsilon} dz < \infty \right\} \tag{0.2.14}$$

munis des normes

$$\|f\|_{\mathfrak{F}_\varepsilon^s} := \left(\int_{\mathbb{C}} (1 + |z|^2)^s |f|^2 e^{-|z|^2/\varepsilon} dz \right)^{1/2}. \quad (0.2.15)$$

Après nos changements d'échelle (0.2.9) et de fonction (0.2.10), nous sommes amenés à considérer l'énergie

$$\mathcal{F}^{\text{LLL}}[f] = \varepsilon^{-1} \mathcal{E}^{\text{LLL}}[\psi]. \quad (0.2.16)$$

Explicitement :

$$\mathcal{F}^{\text{LLL}}[f] = \int_{\mathbb{C}} \left(\left(-\beta|z|^2 + \frac{1}{4}|z|^4 \right) |f|^2 e^{-|z|^2/\varepsilon} + \frac{G}{2}|f|^4 e^{-2|z|^2/\varepsilon} \right) dz. \quad (0.2.17)$$

Le problème de minimisation est le suivant

$$F^{\text{LLL}} := \inf \{ \mathcal{F}^{\text{LLL}}[f], f \in \mathfrak{F}_\varepsilon^1, \|f\|_{\mathfrak{F}_\varepsilon} = 1 \}. \quad (0.2.18)$$

Nous montrons d'abord que le problème (0.2.18) est bien posé :

Théorème 0.2.1 (Aspect bien posé du problème (0.2.18) et équation d'Euler–Lagrange).

Pour tout $\varepsilon > 0$, le problème (0.2.18) admet une solution dans $\mathfrak{F}_\varepsilon^2$. Toute solution du problème de minimisation satisfait l'équation

$$-\beta M_\varepsilon f + \frac{1}{4} (M_\varepsilon^2 f + \varepsilon M_\varepsilon f) + G \Pi_\varepsilon (e^{-|z|^2/\varepsilon} |f|^2 f) = \mu f \quad (0.2.19)$$

où μ est un multiplicateur de Lagrange fixé par la contrainte de masse et M_ε est l'opérateur défini par

$$M_\varepsilon f = \varepsilon \partial_z (zf). \quad (0.2.20)$$

L'équation (0.2.19) peut se réécrire

$$\left(-\beta + \frac{1}{4}\varepsilon \right) \Pi_\varepsilon (|z|^2 f) + \frac{1}{4} \Pi_\varepsilon (|z|^2 \Pi_\varepsilon (|z|^2 f)) + G \Pi_\varepsilon (e^{-|z|^2/\varepsilon} |f|^2 f) = \mu f \quad (0.2.21)$$

ou

$$-\beta M_\varepsilon f + \frac{1}{4} (M_\varepsilon^2 f + \varepsilon M_\varepsilon f) + \frac{G}{2} \bar{f}(\varepsilon \partial_z)[f^2(z/2)] = \mu f. \quad (0.2.22)$$

Si $f(z) = \sum a_k z^k$, l'opérateur $\bar{f}(\varepsilon \partial_z)$ est défini par

$$\bar{f}(\varepsilon \partial_z)[g] := \sum_{k=0}^{+\infty} \bar{a}_k (\varepsilon \partial_z)^k g.$$

Notre deuxième résultat montre qu'une solution du problème variationnel (0.2.18) ne peut pas avoir un nombre fini de zéros si ε est assez petit. En particulier, le minimum ne peut pas être atteint par un polynôme.

Théorème 0.2.2 (Nombre infini de zéros).

Si ε est suffisamment petit, toute solution du problème (0.2.18) a un nombre infini de zéros.

Notre troisième résultat concerne la construction de points quasi-critiques pour l'énergie (0.2.17), c'est à dire de fonctions qui sont quasiment des solutions de l'équation d'Euler-Lagrange (0.2.21). Inspirés par [12], nous procédons comme suit.

Soient les fonctions

$$u_\tau(z) = e^{-|z|^2/2\varepsilon} f_\tau(z), \quad f_\tau(z) = e^{z^2/2\varepsilon} \Theta\left(\sqrt{\frac{\tau_I}{\pi\varepsilon}} z, \tau\right) \quad (0.2.23)$$

où $\tau = \tau_R + i\tau_I \in \mathbb{C}$ et

$$\Theta(v, \tau) = \frac{1}{i} \sum_{n=-\infty}^{+\infty} (-1)^n e^{i\pi\tau(n+1/2)^2} e^{(2n+1)\pi iv} \quad (0.2.24)$$

est la fonction Θ de Jacobi (plus précisément, la fonction Thêta numéro 0 selon la classification de [116]). Elle a la propriété [40]

$$\Theta(v + k + l\tau, \tau) = (-1)^{k+l} e^{-2i\pi lv} e^{-i\pi l\tau} \Theta(v, \tau). \quad (0.2.25)$$

La fonction f_τ est une solution du problème d'Abrikosov (voir par exemple [3] pour l'énoncé du problème et [12, Section 4] pour la preuve de l'équation ci-dessous) et satisfait

$$\Pi_\varepsilon(|f_\tau|^2 e^{-|z|^2/\varepsilon} f_\tau) = \lambda_\tau f_\tau, \quad \text{avec } \lambda_\tau = \int |u_\tau|^2 b(\tau), \quad (0.2.26)$$

avec

$$b(\tau) = \frac{\int |u_\tau|^4}{(\int |u_\tau|^2)^2} = \sum_{k,l \in \mathbb{Z}} e^{-\pi|k\tau - l|^2/\tau_I}. \quad (0.2.27)$$

Nous avons noté $f v$ la moyenne d'une fonction périodique v . L'équation (0.2.26) est similaire à (0.2.21) sans le terme de confinement et avec $\mu = \lambda_\tau$. Ceci va nous permettre d'obtenir une quasi-solution de (0.2.21) en modifiant f_τ .

La fonction u_τ a été construite par Abrikosov [2] pour que $|u_\tau|$ soit périodique sur le réseau $\sqrt{\frac{\pi\varepsilon}{\tau_I}}\mathbb{Z} \oplus \sqrt{\frac{\pi\varepsilon}{\tau_I}}\mathbb{Z}\tau$ et s'annule avec racines simples aux points du réseau. En ajoutant la contrainte que $g = v e^{|z|^2/2\varepsilon}$ soit holomorphe, c'est en fait le seul choix possible (voir [40]). La constante $b(\tau)$ dépend du choix du réseau et est minimum ($b(\tau) \sim 1.16$) pour le réseau hexagonal, c'est-à-dire pour $\tau = e^{2i\pi/3}$. Nous renvoyons à [11, 12] pour de plus amples détails sur ce point et fixerons dans cette thèse

$$\tau = e^{2i\pi/3}, \quad b = b(e^{2i\pi/3}).$$

Maintenant que nous avons construit une fonction correspondant à un réseau hexagonal infini de vortex, il nous faut prendre en compte les modifications dues au terme de confinement, qui agit essentiellement sur le profil de densité à une échelle supérieure au pas du réseau (qui est proportionnel à $\sqrt{\varepsilon}$). Nous multiplions f_τ par un profil normalisé α à support compact. Afin d'avoir une fonction test admissible pour le problème (0.2.18) nous projetons αf_τ dans l'espace de Fock-Bargmann et normalisons la fonction obtenue :

$$f_{\alpha,\tau} = \frac{\Pi_\varepsilon(\alpha f_\tau)}{\|\Pi_\varepsilon(\alpha f_\tau)\|_{\mathfrak{F}_\varepsilon}}. \quad (0.2.28)$$

Nous définissons également

$$u_{\alpha,\tau} = f_{\alpha,\tau} e^{-|z|^2/2\varepsilon}. \quad (0.2.29)$$

Il nous reste à choisir α . Comme nous le verrons, le meilleur choix possible est déterminé par la fonctionnelle de Thomas-Fermi

$$\mathcal{E}_b^{\text{TF}}[\rho] = \int_{\mathbb{C}} \left(\left(-\beta|z|^2 + \frac{1}{4}|z|^4 \right) \rho + \frac{bG}{2} \rho^2 \right) dz. \quad (0.2.30)$$

Le problème de minimisation associé est

$$E_b^{\text{TF}} := \inf \left\{ \mathcal{E}_b^{\text{TF}}[\rho], \rho \geq 0, \int_{\mathbb{R}^2} |z|^2 \rho < +\infty, \int_{\mathbb{R}^2} \rho = 1 \right\}. \quad (0.2.31)$$

Son unique solution est donnée par

$$\rho_b^{\text{TF}} := \max \left(\frac{\mu_b^{\text{TF}} + \beta|z|^2 - \frac{1}{4}|z|^4}{bG}, 0 \right) \quad (0.2.32)$$

où le potentiel chimique (multiplicateur de Lagrange) μ_b^{TF} est fixé par la contrainte

$$\int_{\mathbb{R}^2} \rho_b^{\text{TF}} = 1.$$

Nous pouvons maintenant énoncer le théorème

Théorème 0.2.3 (Points quasi-critiques pour le problème (0.2.18)).

Soit α un profil satisfaisant

$$|\alpha(z)|^2 = \rho_b^{\text{TF}} \quad (0.2.33)$$

et $f_{\alpha,\tau}$ la fonction test associée par la formule (0.2.28). On a

$$\left(-\beta + \frac{1}{4}\varepsilon \right) \Pi_\varepsilon(|z|^2 f_{\alpha,\tau}) + \frac{1}{4} \Pi_\varepsilon(|z|^2 \Pi_\varepsilon(|z|^2 f_{\alpha,\tau})) + G \Pi_\varepsilon(e^{-|z|^2/\varepsilon} |f_{\alpha,\tau}|^2 f_{\alpha,\tau}) = \mu_b^{\text{TF}} f_{\alpha,\tau} + R_\varepsilon \quad (0.2.34)$$

avec $\|R_\varepsilon\|_{\mathfrak{F}_\varepsilon} \leq C\varepsilon^{1/4}$. De plus

$$\mathcal{F}^{\text{LLL}}[f_{\alpha,\tau}] = \int_{\mathbb{C}} \left(\left(-\beta|z|^2 + \frac{1}{4}|z|^4 \right) |\alpha|^2 + \frac{bG}{2} |\alpha|^4 \right) dz + O(\varepsilon^{1/4}) \quad (0.2.35)$$

et donc

$$F^{\text{LLL}} \leq \mathcal{F}^{\text{LLL}}[f_{\alpha,\tau}] = E_b^{\text{TF}} + O(\varepsilon^{1/4}). \quad (0.2.36)$$

On pourra trouver de plus amples commentaires sur ces résultats dans le Chapitre 1 ou l'article [31]. Nous voulons toutefois faire quelques remarques dans cette introduction :

1. Le Théorème 0.2.2 est un argument fort en faveur de l'existence d'un réseau de vortex s'étendant à l'infini. D'un point de vue physique un réseau 'infini' n'a pas grand sens, et la conclusion est plutôt que tous les vortex ne peuvent pas être localisés dans la zone de forte densité de matière (correspondant au support de α dans la construction ci-dessus). L'existence de 'vortex invisibles' dans les zones de faible densité de matière (plus précisément leur nécessité pour expliquer certaines propriétés des condensats) a fait débat dans la communauté physique [8, 42, 147], et des théorèmes tels que celui-ci sont un des arguments que l'on peut avancer en faveur de leur existence. Notons

que la preuve du Théorème 0.2.2 est essentiellement basée sur l'équation d'Euler-Lagrange obtenue dans le Théorème 0.2.1 et donc sur la structure de l'espace de Fock-Bargmann. Ainsi il semble que les vortex invisibles sont une caractéristique intrinsèque des condensats dans des régimes où la minimisation de l'énergie peut être restreinte aux fonctions du plus bas niveau de Landau (voir aussi [11, 12]).

2. Nous sommes convaincus que la construction menant au Théorème 0.2.3 est optimale, mais une preuve complète constitue un défi conséquent. La preuve d'une borne inférieure asymptotiquement égale à (0.2.36) nécessiterait probablement de montrer que les vortex d'un minimiseur de l'énergie sont proches des sites d'un réseau régulier. Ceci est une question difficile liée au problème de cristallisation et la réponse reste inconnue pour différents problèmes reliés [136].
3. Une borne inférieure non optimale est obtenue en minimisant $\mathcal{F}^{\text{LLL}}[f]$ sans la contrainte que f soit holomorphe (E_1^{TF} est définie en prenant $b = 1$ dans (0.2.31)) :

$$E_1^{\text{TF}} \leq F^{\text{LLL}}. \quad (0.2.37)$$

Comme b est une constante, cette borne inférieure montre que E_b^{TF} donne au moins l'ordre de grandeur de l'énergie. La différence entre la borne supérieure et la borne inférieure est due au coefficient b qui prend en compte la contribution du réseau de vortex à l'énergie via un effet d'homogénéisation, voir le Chapitre 1.

4. Calculant E_b^{TF} on obtient

$$E_b^{\text{TF}} = c_1 G^{2/3} + c_2 \beta^2 \quad (0.2.38)$$

où c_1 et c_2 sont bornés indépendamment des autres paramètres du problème. Ceci fournit a posteriori le régime dans lequel notre analyse est valable. La réduction au LLL est justifiée si et seulement si F^{LLL} est négligeable devant le gap spectral entre le plus bas niveau de Landau et le premier niveau excité. Avec notre choix d'unités, ce gap est ¹ $\propto \varepsilon^{-1} = k^{-1/3}$, et donc (0.2.36) et (0.2.38) indiquent que la réduction est valable si

$$G \ll k^{-1/2}, \quad \beta \ll k^{-1/6}. \quad (0.2.39)$$

5. Une autre condition est 'cachée' dans la preuve du Théorème 0.2.3. Celle-ci utilise un argument d'homogénéisation élémentaire basé sur le fait que le pas du réseau de vortex est très petit devant la taille du support de α . Ceci fournit la condition

$$k \ll G. \quad (0.2.40)$$

En combinant la première condition dans (0.2.39) et (0.2.40), on voit que nécessairement

$$k \ll 1,$$

ce qui justifie a posteriori notre choix d'un régime où k est petit. Notons finalement que nous avons pour simplifier considéré β et G comme fixés dans cette étude, ce qui implique que les conditions ci-dessus sont remplies.

Concluons cette section en remarquant que si, comme nous l'avons dit, le Théorème 0.2.2 suggère l'existence de vortex invisibles localisés dans les zones de faible densité de matière, nos résultats analytiques donnent peu d'informations sur la localisation exacte de

1. Dans toute cette thèse, le symbole \propto signifie 'asymptotiquement proportionnel à'. Par $a \propto b$ on entend que le rapport a/b tend vers une constante fixée.

ces vortex.

Si le lecteur est disposé à croire avec nous que la construction du Théorème 0.2.3 est optimale, nous pouvons affirmer en utilisant un résultat de [12] que les vortex localisés dans la zone de forte densité de matière, c'est-à-dire le support du profil α , sont disposés suivant un réseau hexagonal régulier (voir l'introduction du chapitre 1).

En revanche, la localisation des vortex invisibles en dehors du support de α n'est pas facilement devinée d'après nos résultats. On peut à la rigueur (voir [11, 31]) suggérer que la projection de αf_τ dans l'espace de Fock-Bargmann résulte en une déformation du réseau de vortex au bord du condensat. Pour confirmer cette hypothèse, nous avons procédé à des simulations numériques (un résultat typique est présenté dans la Figure 8).

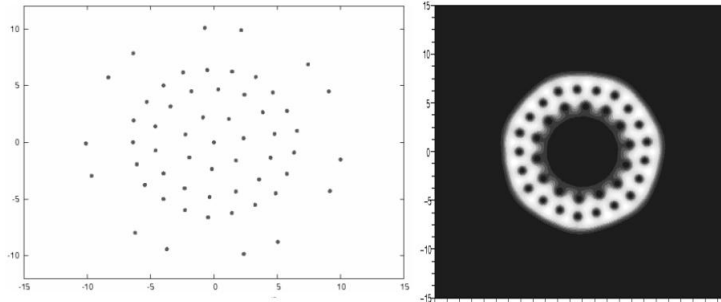


FIGURE 8 – Localisation des vortex (à gauche) et densité de matière (à droite) pour un condensat annulaire dans le régime LLL. Le vortex central contient onze unités de circulation de phase.

Le cas où le condensat forme un disque peut être considéré comme complètement élucidé par les simulations de [8]. Nous nous sommes donc intéressés au cas où le condensat devient annulaire. Nous renvoyons au Chapitre 1 pour la description de la méthode suivie et la présentation des résultats. Au niveau de cette introduction nous nous contenterons de signaler que les simulations montrent bien une déformation du réseau de vortex aux deux bords du condensat et répondent par l'affirmative à une question posée dans la littérature physique : il y a-t-il un vortex multi-quantifié au centre du piège lorsque le condensat est annulaire ? Ceci est visible sur la Figure 8.

2.1.2 Régime de rotation extrême et vortex géant

Nous continuons notre étude de la fonctionnelle d'énergie (0.2.1) que nous rappelons :

$$\mathcal{E}^{\text{GP}}[\psi] = \int_{\mathbb{R}^2} \left(\left| \nabla \psi - i\Omega x^\perp \psi \right|^2 + ((1 - \Omega^2) |x|^2 |\psi|^2 + k|x|^4) |\psi|^2 + G|\psi|^4 \right) dx. \quad (0.2.41)$$

Dans cette section nous cherchons à justifier l'apparition d'un vortex géant dans la limite où $\Omega \rightarrow +\infty$. Nous présentons ici les résultats principaux du preprint [127], reproduits dans le Chapitre 2.

Dans le régime $\Omega \rightarrow +\infty$, le rayon du condensat n'est pas borné en fonction de Ω et il est donc utile de faire le changement d'échelle (défini pour $\Omega > 1$)

$$u(x) = R\psi(Rx), \quad R = \sqrt{\frac{\Omega^2 - 1}{2k}}. \quad (0.2.42)$$

Nous avons alors

$$\mathcal{E}^{\text{GP}}[\psi] = \mathcal{F}_\Omega[u] \quad (0.2.43)$$

avec

$$\mathcal{F}_\Omega[u] = \frac{2k}{\Omega^2 - 1} \int_{\mathbb{R}^2} \left| \nabla u - i\Omega \frac{\Omega^2 - 1}{2k} x^\perp u \right|^2 + G|u|^4 + \frac{(\Omega^2 - 1)^3}{8k^2} (|x|^4 - 2|x|^2) |u|^2. \quad (0.2.44)$$

A cause de la contrainte de masse

$$\int_{\mathbb{R}^2} |u|^2 dx = 1, \quad (0.2.45)$$

nous pouvons ajouter à \mathcal{F}_Ω n'importe quel multiple de $\int_{\mathbb{R}^2} |u|^2 dx$ sans changer le problème de minimisation. Il est donc équivalent de minimiser

$$\tilde{\mathcal{F}}_\Omega[u] = \frac{2k}{\Omega^2 - 1} \int_{\mathbb{R}^2} \left| \nabla u - i\Omega \frac{\Omega^2 - 1}{2k} x^\perp u \right|^2 + G|u|^4 + \frac{(\Omega^2 - 1)^3}{8k^2} (|x|^2 - 1)^2 |u|^2. \quad (0.2.46)$$

Nous définissons maintenant de nouveaux paramètres

$$\begin{aligned} \omega &= \Omega \frac{\Omega^2 - 1}{2k} \\ D_\Omega &= \frac{\Omega^2 - 1}{\Omega^2} \end{aligned} \quad (0.2.47)$$

et réécrivons l'énergie

$$\tilde{\mathcal{F}}_\Omega[u] = \frac{2k}{\Omega^2 - 1} F_\omega[u] \quad (0.2.48)$$

avec

$$F_\omega[u] = \int_{\mathbb{R}^2} \left(|\nabla u - i\omega x^\perp u|^2 + D_\Omega \frac{\omega^2}{2} (|x|^2 - 1)^2 |u|^2 + G|u|^4 \right) dx. \quad (0.2.49)$$

Nous allons étudier le comportement asymptotique de F_ω lorsque $\omega \rightarrow +\infty$, avec des conditions sur les autres paramètres que nous précisons plus bas (régime de rotation extrême). Les résultats pour l'énergie originelle se déduisent simplement en utilisant les changements d'échelle et de fonction ci-dessus.

Nous noterons

$$I_\omega = \inf \left\{ F_\omega[u], u \in H^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2, x^4 dx), \int_{\mathbb{R}^2} |u|^2 = 1 \right\} = F_\omega[u_\omega]. \quad (0.2.50)$$

C'est un fait classique que I_ω est atteint par un certain u_ω (a priori non unique) qui vérifie l'équation

$$-\left(\nabla - i\omega x^\perp\right)^2 u_\omega + D_\Omega \frac{\omega^2}{2} (|x|^2 - 1)^2 u_\omega + 2G|u_\omega|^2 u_\omega = \mu_\omega u_\omega, \quad (0.2.51)$$

où μ_ω est le multiplicateur de Lagrange associé à la contrainte de masse.

Nous nous fixons pour but de justifier dans la limite $\omega \rightarrow +\infty$ la transformation (en un sens à préciser) de u_ω en un état de type 'vortex géant' de la forme

$$(r, \theta) \mapsto f(r)e^{in\theta}, \quad (0.2.52)$$

où r, θ sont les coordonnées polaires, n un entier et f est concentrée et strictement positive dans un anneau. De manière plus générale, on dira qu'il y a 'transition vers un vortex géant' si les conditions suivantes sont remplies :

- La masse de u_ω est concentrée dans un certain anneau, disons \mathcal{A} (en pratique cet anneau dépendra de ω), au sens

$$\int_{\mathcal{A}} |u_\omega|^2 \rightarrow 1.$$

- Cet anneau ne contient aucun vortex, c'est à dire que $|u_\omega|$ y est strictement positif.

Pour formuler nos résultats, les notations suivantes sont nécessaires : pour tout $n \in \mathbb{Z}$ nous définissons la fonctionnelle d'énergie F_n

$$F_n[f] = 2\pi \int_{\mathbb{R}^+} (|f'(r)|^2 + V_n(r)|f(r)|^2) r dr \quad (0.2.53)$$

où le potentiel V_n est donné par

$$V_n(r) = \frac{n^2}{r^2} - 2n\omega + (1 - D_\Omega)\omega^2 r^2 + D_\Omega \frac{\omega^2}{2} + D_\Omega \frac{\omega^2}{2} r^4. \quad (0.2.54)$$

Les énergies ci-dessus apparaissent naturellement lorsque l'on développe u_ω en série de Fourier dans la variable angulaire (voir (0.2.72) ci-dessous).

Nous noterons $g_{1,n}$ le fondamental (unique à un facteur multiplicatif de module 1 près, que nous fixons de manière à avoir $g_{1,n} \geq 0$) de l'énergie (0.2.53) sous la contrainte de masse

$$2\pi \int_{\mathbb{R}^+} g_{1,n}^2(r) r dr = 1 \quad (0.2.55)$$

et

$$\lambda_{1,n} = F_n[g_{1,n}] = \inf \left\{ F_n[f], f \in H^1(\mathbb{R}^+, r dr) \cap L^2(\mathbb{R}^+, r V_n(r) dr), 2\pi \int_{\mathbb{R}^+} f^2(r) r dr = 1 \right\}. \quad (0.2.56)$$

De même nous introduisons

$$E_n[f] = 2\pi \int_{\mathbb{R}^+} (|f'(r)|^2 + V_n(r)|f(r)|^2 + G|f(r)|^4) r dr \quad (0.2.57)$$

avec

$$\gamma_n = E_n[\Psi_n] = \inf \left\{ E_n[f], f \in H^1(\mathbb{R}^+, r dr) \cap L^2(\mathbb{R}^+, r V_n(r) dr), 2\pi \int_{\mathbb{R}^+} f^2(r) r dr = 1 \right\}. \quad (0.2.58)$$

Comme pour $g_{1,n}$, nous imposons l'unicité de Ψ_n en choisissant $\Psi_n \geq 0$. Notons que l'énergie E_n apparaît en testant F_ω contre une fonction de la forme (0.2.52).

Nous pouvons prouver l'apparition d'un vortex géant dans la limite suivante

$$\omega \rightarrow +\infty, \quad G \text{ est une constante fixée, } \quad D_\Omega \in (0, 1), \quad D_\Omega \rightarrow D \in (0, 1] \text{ constante fixée.} \quad (0.2.59)$$

Le théorème que nous formulons maintenant est une version simplifiée du Théorème 2.1.3 du Chapitre 2 de la présente thèse.

Théorème 0.2.4 (Vortex géant dans le cas G fixé).

Dans le régime (0.2.59), il existe un certain $n^* = \omega + O(1)$ tel que, à une sous-suite près

$$\| |u_\omega| - |\Psi_{n^*}| \|_{L^\infty(\mathbb{R}^2)} \leq C_\varepsilon \omega^{3/16+\varepsilon} \ll \|\Psi_{n^*}\|_{L^\infty(\mathbb{R}^2)} \propto \omega^{1/4} \quad (0.2.60)$$

pour tout $\varepsilon > 0$ fixé. En particulier, en définissant pour toute constante δ l'anneau

$$\mathcal{A}_\delta = \left\{ (r, \theta), |r - 1| \leq \delta \omega^{-1/2} \right\} \quad (0.2.61)$$

il existe un certain δ tel que

1. $\int_{\mathcal{A}_\delta} |u_\omega|^2 = 1 - o(1)$ quand ω tend vers l'infini,
2. u_ω ne s'annule pas sur \mathcal{A}_δ .

Les propriétés 1 et 2 dans le théorème ci-dessus se déduisent facilement des propriétés de Ψ_{n^*} que nous expliciterons dans le Chapitre 2. Elles traduisent bien ce que nous entendons par un vortex géant : la masse de u_ω est concentrée dans un certain anneau (propriété 1), et cet anneau ne contient aucun vortex (propriété 2).

Nous n'avons pas de preuve qu'il existe une circulation de phase autour de cet anneau puisque (0.2.60) ne porte que sur le module de u_ω . Cependant, l'absence d'une telle circulation est incompatible avec les estimations d'énergie que nous donnons maintenant, et sur lesquelles est basée la preuve du Théorème 0.2.4.

Théorème 0.2.5 (Asymptotiques d'énergie et de densité dans le cas G fixé).

Dans le régime (0.2.59) on a, pour un certain $n^* = \omega + O(1)$ et tout $\varepsilon > 0$

$$I_\omega = F_\omega[u_\omega] = \gamma_{n^*} + O(\omega^{-1/4+\varepsilon}). \quad (0.2.62)$$

De plus, il existe un certain $\alpha \in \mathbb{R}$ tel que, à une sous-suite près

$$\|u_\omega - e^{i\alpha} \Psi_{n^*} e^{in^*\theta}\|_{L^2(\mathbb{R}^2)} \leq C_\varepsilon \omega^{-1/16+\varepsilon}. \quad (0.2.63)$$

Le Théorème 0.2.4 confirme que le vortex géant est la phase favorisée par le condensat quand la vitesse de rotation devient très grande mais ne donne pas d'estimation de la vitesse critique à laquelle la transition de phase devrait avoir lieu (nous reviendrons plus bas sur la traduction du scaling dans les paramètres originaux). Ceci est dû au fait que nous avons considéré G comme fixé alors que cette vitesse critique est une fonction croissante de G . Pour évaluer une vitesse critique par une analyse asymptotique comme la nôtre, il est nécessaire de pouvoir considérer un régime où G tend vers l'infini également.

Malheureusement, les techniques que nous utilisons pour démontrer le Théorème 0.2.4 ne permettent pas d'étendre ce résultat au cas où G est grand, ce qui serait nécessaire pour estimer la vitesse critique. Nous présentons ci-dessous les résultats partiels que nous pouvons obtenir dans ce cas, plus précisément dans le régime de rotation extrême

$$\omega \rightarrow +\infty, \quad \omega \gg G^2 \geq C > 0, \quad D_\Omega \in (0, 1), \quad D_\Omega \rightarrow D \in (0, 1] \quad (0.2.64)$$

où C et D sont des constantes. Notons que l'hypothèse $G^2 \geq C$ est posée uniquement pour simplifier les énoncés des théorèmes. Nos méthodes s'appliquent également dans le cas où G tend vers 0, qui est en fait plus simple.

Théorème 0.2.6 (Asymptotiques d'énergie et de densité dans le régime (0.2.64)).
 Il existe un entier $n^* = \omega + O(1)$ tel que, dans le régime (0.2.64) on ait

$$I_\omega = F_\omega[u_\omega] = \lambda_{1,n^*} + 2\pi G \int_{\mathbb{R}^+} g_{1,n^*}(r)^4 r dr + O(G^2) = \gamma_{n^*} + O(G^2). \quad (0.2.65)$$

De plus

$$\| |u_\omega|^2 - g_{1,n^*}^2 \|_{L^2(\mathbb{R}^2)} \leq CG^{1/2} \ll \|g_{1,n^*}^2\|_{L^2(\mathbb{R}^2)} \propto \omega^{1/4} \quad (0.2.66)$$

et $|u_\omega|^2$ converge au sens des mesures vers une fonction delta de Dirac en $|x| = 1$.

Nous renvoyons au Théorème 2.2.1 de cette thèse pour plus de détails sur $g_{1,n}$ et $\lambda_{1,n}$. On a en particulier

$$\lambda_{1,n^*} \sim (2D_\Omega + 4)^{1/2} \omega$$

et

$$2\pi \int_{\mathbb{R}^+} g_{1,n^*}(r)^4 \sim \frac{(2D_\Omega + 4)^{1/4}}{\sqrt{2\pi}} \omega^{1/2} \int_{\mathbb{R}} \xi_1^4(x) dx,$$

où ξ_1 est une certaine gaussienne (voir ci-dessous). Ainsi, le Théorème 0.2.6 indique que le premier et le deuxième ordre de l'énergie sont donnés par l'étude du problème simplifié (0.2.56).

L'énergie E_n est obtenue en restreignant la minimisation de F_ω aux fonction de la forme $f(r)e^{in\theta}$, donc la deuxième égalité dans (0.2.65) est une première justification de l'apparition d'un vortex géant dans le régime (0.2.64). Toutefois, l'asymptotique de densité (0.2.66) n'est pas suffisante pour conclure comme dans le Théorème 0.2.4 qu'il n'y a pas de vortex dans la zone de forte densité de matière. Essentiellement, notre approche ne donne pas d'estimation d'énergie assez précise pour obtenir une telle conclusion. L'hypothèse que G est une constante permet de raffiner notre méthode et de déboucher sur les estimations beaucoup plus précises présentées dans le Théorème 0.2.5.

Nous présentons maintenant quelques idées des preuves des résultats ci-dessus. Il est naturel pour un régime où on s'attend à l'apparition d'un vortex géant de considérer une première fonction test de la forme

$$\begin{aligned} v_{test}(r, \theta) &= f_{test}(r)e^{in\theta} \\ f_{test}(r) &= c_{test} \xi \left(\frac{r-1}{\eta} \right). \end{aligned} \quad (0.2.67)$$

En faisant les choix appropriés pour η , n et c on obtient

$$F_\omega[u_\omega] \leq F_\omega[v_{test}] \leq (2D_\Omega + 4)^{1/2} \omega \int_{\mathbb{R}} (\xi^2(x)^2 + x^2 \xi^2(x)) dx + O(G\omega^{1/2}) \quad (0.2.68)$$

donc quand $\omega \gg G^2$ il est naturel de prendre $\xi = \xi_1$ la première fonction propre normalisées dans $L^2(\mathbb{R})$ de l'oscillateur harmonique

$$-\frac{d^2}{dx^2} + x^2, \quad (0.2.69)$$

associée à la valeur propre 1. Cette fonction est une gaussienne et sa symétrie permet de raffiner l'estimation pour obtenir

$$F_\omega[u_\omega] \leq F_\omega[v_{test}] \leq (2D_\Omega + 4)^{1/2} \omega + G\omega^{1/2} \frac{(2D_\Omega + 4)^{1/4}}{\sqrt{2\pi}} \int_{\mathbb{R}} \xi_1^4(x) dx + O(1). \quad (0.2.70)$$

Ce calcul suggère que, dans le régime de rotation extrême, les termes dominants de l'énergie seront le terme potentiel et le terme cinétique, menant à un problème limite linéaire. Il est alors intéressant de développer u_ω en série de Fourier par rapport à la variable angulaire

$$u_\omega(r, \theta) = \sum_{n \in \mathbb{Z}} f_n(r) e^{in\theta}. \quad (0.2.71)$$

Prouver que u_ω converge vers une fonction de la forme (0.2.52) revient alors à montrer que f_n converge vers 0 pour tous les n sauf 1.

La preuve se décompose alors grossièrement en les étapes suivantes :

1. On constate d'abord que la partie quadratique de l'énergie se découple en une somme d'énergie 1D définies en (0.2.53) :

$$F_\omega[u_\omega] = \sum_{n \in \mathbb{Z}} F_n(f_n) + G \int_{\mathbb{R}^2} |u_\omega|^4, \quad (0.2.72)$$

La preuve du théorème 0.2.6 est basée sur une étude détaillée des énergies (0.2.53) : en particulier, $\lambda_{1,n}$ est en général (i.e. sauf pour un ensemble discret de paramètres) minimisé par un unique entier n^* .

Nous prouvons que à cause de la forme des potentiels V_n , certains modes f_n portent trop d'énergie par rapport à notre borne supérieure (0.2.70) et peuvent donc être éliminés.

2. Une borne inférieure sur le gap spectral entre l'énergie fondamentale et le premier niveau excité de (0.2.53) permet ensuite de montrer que pour tous les autres modes on a $f_n \propto g_{1,n}$ quand ω tend vers l'infini. En effet, ce gap est d'ordre ω , donc, dans le régime (0.2.64), beaucoup plus grand que l'énergie d'interaction. Celle-ci est d'ordre $G\omega^{1/2}$ comme les calculs ci-dessus le montrent.
3. Il s'agit ensuite d'exploiter l'information $f_n \propto g_{1,n}$. Après blow-up autour du minimum du potentiel V_n à une échelle convenable, la fonction $g_{1,n}$ converge vers la gaussienne ξ_1 . Cette forme très particulière permet grâce à un lemme élémentaire et des calculs quelque peu fastidieux de fournir une borne inférieure de l'énergie d'interaction (terme quartique).
4. En combinant l'étude du fondamental des énergies 1D et la borne inférieure de l'énergie d'interaction, on est en mesure d'évaluer précisément le coût énergétique résultant du déplacement d'une unité de masse du mode n^* vers un autre mode. Deux effets entrent en jeu : concentrer la masse de u_ω sur le mode $f_{n^*} \propto g_{1,n^*}$ favorise la partie quadratique de l'énergie car λ_{1,n^*} est le minimum de $\lambda_{1,n}$ par rapport à n , mais une telle concentration augmente le terme quartique. Des calculs précis montrent que le premier effet est plus important que le second dans le régime de rotation extrême.
5. A ce stade, on a essentiellement prouvé le Théorème 0.2.6, mais les termes d'erreur sont encore compatibles avec la cohabitation de nombreux modes. Comme les énergies et les profils de densité correspondants à ces modes sont relativement proches, on arrive tout de même à montrer (0.2.65) et (0.2.66).
6. Les ingrédients additionnels requis pour obtenir les résultats plus précis dans le régime où G est fixé reposent essentiellement sur une utilisation systématique de l'équation d'Euler-Lagrange (0.2.51). La partie dominante de cette équation étant linéaire, il est crucial de se servir pleinement de son caractère elliptique. Notamment,

les estimations pour l'opérateur de Ginzburg-Landau introduits par Lu-Pan dans [106] (voir aussi [7] et [67, Chapitre 11]) sont utilisés et permettent entre autres de conclure à l'estimation (0.2.60). Une autre remarque essentielle est que l'injection du résultat (0.2.66) dans (0.2.51) permet d'obtenir des équations pour chacun des modes f_n et donc de raffiner leur estimation.

Nous revenons maintenant aux paramètres originaux de l'énergie (0.2.41). Le régime de rotation extrême peut être atteint essentiellement de deux façons. Soit en prenant $\Omega \rightarrow +\infty$ avec

$$\Omega \rightarrow +\infty, \quad \Omega^3 \gg k, \quad \Omega \gg G^{2/3}k. \quad (0.2.73)$$

Soit en gardant Ω de l'ordre d'une constante strictement plus grande que 1 (appelons la Ω_0) et en prenant $k \rightarrow 0$ avec

$$\Omega \rightarrow \Omega_0 > 1, \quad k \rightarrow 0, \quad k \ll G^{-2}. \quad (0.2.74)$$

Si Ω_0 est autorisé à tendre vers 1 dans (0.2.74), alors $D_\Omega \rightarrow 0$ et le terme potentiel de l'énergie devient négligeable devant le terme cinétique. Il est alors légitime d'imaginer que la minimisation de (0.2.49) peut être restreinte aux fonctions du plus bas niveau de Landau comme dans la Section 2.1.

Dans la série d'articles [43, 44, 47], la limite $\Omega \rightarrow +\infty$ a été étudiée pour des pièges de type (0.1.31) et (0.1.32). Nous voulons signaler que leur analyse s'applique à notre problème dans des régimes opposés à ceux indiqués ci-dessus, c'est-à-dire si $\Omega \ll G^{2/3}k$ (pour le régime opposé à (0.2.73)) ou si $k \gg G^{-2}$ (pour le régime opposé à (0.2.74)). Dans ce genre de régimes, les asymptotiques d'énergie et de densité sont donnés par une énergie limite de type Thomas-Fermi, obtenu en négligeant le terme cinétique de la fonctionnelle (0.2.49). Ce modèle réduit est celui le plus souvent utilisé dans la littérature physique, où l'hypothèse que le terme d'interaction domine l'énergie est souvent faite. Nous avons montré ici que cette approximation n'est plus valable pour des vitesses de rotation trop grandes. Le profil de densité du condensat change en conséquence lorsque le régime de rotation extrême est atteint : on passe d'un profil de type Thomas-Fermi à un profil gaussien (voir (0.2.66), g_{1,n^*} est proche d'une gaussienne après un changement d'échelle approprié).

Comme nous l'avons vu, nous n'avons pas été capable avec la méthode brièvement décrite ci-dessus de prouver l'apparition du vortex géant dans le régime de rotation extrême. Nous n'obtenons un tel résultat que dans le cas où G est fixé. Soulignons que même si les preuves pouvaient être améliorées pour fonctionner dans le régime (0.2.64) tout entier, il est probable que nous n'aurions pas identifié la vitesse de transition correcte, mais seulement une borne supérieure à cette vitesse.

En effet, en combinant l'analyse de [47] et les scalings de [44] il est logique d'imaginer (même si une preuve complète reste à être établie) que le réseau de vortex présent dans l'anneau de forte densité de matière disparaît lorsque $\Omega \propto G^{2/3}(\log G)^{-1}$ (nous considérons k comme fixé ici), c'est-à-dire avant que le régime de rotation extrême soit atteint.

Dans la section suivante nous identifions précisément la vitesse de transition dans le cas du piège plat (0.1.32).

2.2 Vitesses critiques pour l'énergie de Gross-Pitaevskii sur un disque

Dans cette section nous allons décrire les principales transitions de phase pour le modèle de Gross-Pitaevskii avec un potentiel donné formellement par (0.1.32). Nous entendons par là que nous considérerons une énergie sur le disque unité \mathcal{B} , sans terme de potentiel :

$$\begin{aligned} \mathcal{E}^{\text{GP}}[\Psi] &= \int_{\mathcal{B}} |\nabla \Psi|^2 - 2\Omega x^\perp (i\Psi, \nabla \Psi) + \varepsilon^{-2} |\Psi|^4 \\ &= \int_{\mathcal{B}} \left| \nabla \Psi - i\Omega x^\perp \Psi \right|^2 - \Omega^2 |x|^2 |\Psi|^2 + \varepsilon^{-2} |\Psi|^4. \end{aligned} \quad (0.2.75)$$

Nous avons noté

$$G = \varepsilon^{-2}$$

pour faciliter la comparaison avec la fonctionnelle de Ginzburg-Landau (0.1.9) et son équivalent 2D. Le paramètre ε est le rapport de la ‘healing length’ du condensat à son extension spatiale (voir par exemple l’introduction [4]). Pour la plupart des expériences ce paramètre est petit, ce qui justifie le fait que nous allons considérer la limite $\varepsilon \rightarrow 0$ (limite de Thomas-Fermi, équivalent de la limite de London pour les supraconducteurs).

Strictement parlant, le modèle que nous considérons correspond à un condensat en rotation dans une enceinte cylindrique avec parois rigides. Bien qu’un tel problème ne soit pas absurde d’un point de vue théorique, il est très éloigné de ce qu’il est possible de réaliser en laboratoire, c’est pourquoi nous préférons le voir comme une approximation des problèmes physiques avec potentiels de piégeage.

Comme nous l’avons déjà signalé, le piège plat est la limite formelle d’un piège homogène, et on peut donc le considérer comme le piège le plus ‘fortement anharmonique’ imaginable, i.e. comme un potentiel infiniment raide. D’autre part, les résultats que nous obtiendrons peuvent être généralisés à des situations plus réalistes (voir [45]), de la même manière que [44] a généralisé [43].

Avant de présenter les résultats obtenus dans cette thèse (Sections 2.2.3 et 2.2.4), qui portent essentiellement sur l’apparition du vortex géant et du cercle de vortex qui le précède, il est utile de rappeler les résultats attendus lorsque Ω augmente en tant que fonction de ε .

Dans la Section 2.2.1 nous nous basons sur [65] qui contient des arguments formels en faveur du scénario que nous avons présenté dans la Section 1.4.3. En particulier, trois vitesses critiques sont introduites et évaluées, la troisième constituant le seuil pour l’apparition du vortex géant.

Pour que l’analyse asymptotique du problème (0.2.76) dans la limite $\varepsilon \rightarrow 0$ présentée dans cette thèse soit complète, nous rappellerons dans la Section 2.2.2 certains résultats de Correggi, Rindler-Daller et Yngvason à propos des deux premières vitesses critiques.

Les résultats obtenus dans cette thèse portent sur la troisième vitesse critique. Nous obtenons d’abord une borne supérieure sur la vitesse critique en confirmant l’apparition d’un vortex géant au dessus d’un certain seuil. Ce travail, en collaboration avec Michele Correggi et Jakob Yngvason est présenté dans la Section 2.2.3.

Dans la Section 2.2.4 nous considérons des vitesses de rotation dans le voisinage inférieur de ce seuil, et montrons que dans cette situation un cercle de vortex est présent dans le condensat. Ceci fournit une borne inférieure à la troisième vitesse critique qui confirme l’optimalité de la borne supérieure obtenue dans la Section 2.2.3. Enfin, nous décrirons

certains des outils et méthodes mathématiques que nous avons utilisés dans la Section 2.2.5.

2.2.1 Présentation générale

Le problème de minimisation que nous allons considérer dans toute cette section est le suivant

$$E^{\text{GP}} := \inf \left\{ \mathcal{E}^{\text{GP}}[\Psi], \int_{\mathcal{B}} |\Psi|^2 = 1 \right\} = \mathcal{E}^{\text{GP}}[\Psi^{\text{GP}}]. \quad (0.2.76)$$

Un fondamental Ψ^{GP} (non unique a priori) de l'énergie ci-dessus satisfait l'équation d'Euler-Lagrange

$$-\Delta \Psi^{\text{GP}} - 2i\Omega x^\perp \cdot \nabla \Psi^{\text{GP}} + 2\varepsilon^{-2} |\Psi^{\text{GP}}|^2 \Psi^{\text{GP}} = \mu^{\text{GP}} \Psi^{\text{GP}}, \quad (0.2.77)$$

avec condition de Neumann

$$\partial_n \Psi^{\text{GP}} = 0 \text{ sur } \partial\mathcal{B} \quad (0.2.78)$$

et un multiplicateur de Lagrange donné par

$$\mu^{\text{GP}} := E^{\text{GP}} + \frac{1}{\varepsilon^2} \int_{\mathcal{B}} d\vec{r} |\Psi^{\text{GP}}|^4. \quad (0.2.79)$$

La condition de Neumann homogène (0.2.78) vient du fait que nous ne fixons pas le comportement de Ψ^{GP} sur $\partial\mathcal{B}$ dans la minimisation (0.2.76). Alternativement, et de manière plus réaliste au vu des profils de densité obtenus en laboratoire et par simulation numérique, on pourrait imposer des conditions de Dirichlet homogènes. La généralisation des résultats que nous allons présenter à ce cas est possible mais requiert quelques ingrédients supplémentaires, voir [45].

Nous décrivons maintenant les différentes phases successives que l'on s'attend à observer dans un condensat en rotation dans un piège anharmonique. Plus précisément nous complétons le tableau brossé à la Section 1.4.3 en donnant les ordres de grandeur des trois vitesses de transition pour un condensat dans un piège plat, tels que conjecturés dans [65] (bien que de manière moins explicite que ci-dessous).

On notera que cette description ne tient compte que de transitions de phases 'majeures' où l'aspect du condensat change radicalement. Nous ne mentionnerons par exemple pas les transitions entre phases à un vortex, deux vortex ...

Pour de faibles vitesses de rotation, il n'y a pas de vortex dans le condensat. Lorsque la rotation augmente, trois transitions de phase sont attendues :

- Une première vitesse critique Ω_{c1} sépare une phase sans vortex ('vortex-free state') d'une phase contenant un réseau de vortex ('vortex-lattice state'). Selon [65], (voir aussi [39, 14]) :

$$\Omega_{c1} \propto |\log \varepsilon|.$$

- Lorsqu'une seconde vitesse critique Ω_{c2} est franchie, les forces centrifuges entrent en compétition avec la force de piégeage, creusant un trou dans le condensat. Toutefois, un réseau de vortex reste présent dans le condensat annulaire ('vortex-lattice-plus-hole state') jusqu'à ce qu'une troisième vitesse critique soit franchie. On s'attend à

$$\Omega_{c2} \propto \frac{1}{\varepsilon}.$$

- La troisième vitesse critique, sur laquelle portent les travaux de cette thèse, est d'ordre

$$\Omega_{c3} \propto \frac{1}{\varepsilon^2 |\log \varepsilon|}.$$

Lorsque la vitesse de rotation franchit ce seuil, le réseau de vortex disparaît de l'anneau où la masse est concentrée et l'état résultant est un vortex géant pur ('giant vortex state').

Les états intermédiaires qui apparaissent au voisinage des vitesses de transition sont un peu plus subtils à décrire, nous y reviendrons.

Avant de présenter l'état de l'art mathématique sur les conjectures ci-dessus, c'est à dire leurs preuves rigoureuse et l'expression précise des vitesses critique dans la limite $\varepsilon \rightarrow 0$, nous voulons faire une comparaison avec les champs critiques en théorie de Ginzburg-Landau. Ce que nous allons en dire est maintenant bien compris mathématiquement dans le cadre de l'équivalent 2D du modèle (0.1.9) et on pourra en trouver une présentation synthétique dans les livres [135] pour ce qui concerne le premier champ critique et [67] pour les deuxième et troisième champs critiques (voir aussi [19, 68, 69, 134] pour des résultats non inclus dans [67]).

Il existe en théorie de Ginzburg-Landau trois champs critiques. Par ordre croissant,

- $H_{c1} \propto |\log \varepsilon|$ qui sépare une phase purement supraconductrice (sans vortex) d'une phase mixte contenant un réseau de vortex
- $H_{c2} \sim \varepsilon^{-2}$ qui sépare la phase mixte avec réseau de vortex d'une phase mixte avec supraconductivité de surface, où toute la supraconductivité est concentrée dans une bande fine le long de la frontière de l'échantillon
- $H_{c3} \sim \Theta_0^{-1} \varepsilon^{-2}$ avec $\Theta_0 < 1$, qui sépare la phase exhibant une supraconductivité de surface d'une phase purement normale.

Comme nous l'avons déjà dit, la vitesse de rotation Ω joue en théorie de Gross-Pitaevskii le rôle du champ magnétique extérieur h_{ex} en théorie de Ginzburg-Landau. Cette analogie est très efficace lorsque des champs ou des vitesses de rotation d'ordre $|\log \varepsilon|$ sont considérés, et elle a été utilisée avec succès dans les travaux [5, 81, 82, 142]. Dans cette perspective, la vitesse critique Ω_{c1} peut être vue comme l'équivalent du premier champ critique H_{c1} en théorie de Ginzburg-Landau.

Lorsque la force centrifuge (qui pourrait être interprétée comme un champ électrique en théorie de Ginzburg-Landau) devient importante, les deux modèles donnent des comportements qualitatifs très différents. Par exemple, les vitesses critiques Ω_{c2} et Ω_{c3} ont peu à voir avec les champs critiques H_{c2} et H_{c3} . Tout d'abord ces deux champs sont d'ordre ε^{-2} , avec une constante multiplicative différente. Ensuite, les phases qu'ils séparent sont assez différentes des phases décrites ci-dessus pour un condensat. On pourrait être tenté de voir Ω_{c3} comme l'équivalent de H_{c2} mais le fait qu'un cercle de vortex (voir Section 2.2.4) subsiste dans le régime de transition pour un condensat semble être une différence marquante avec le comportement d'un supraconducteur : le fait que la zone de supraconductivité de surface soit purement supraconductrice (sans vortex) n'a pas été prouvé mathématiquement mais, autant que nous sachions, semble faire consensus.

D'un point de vue mathématique, les outils que nous utiliserons (voir Section 2.2.5) pour analyser Ω_{c3} ont d'ailleurs plus à voir avec les outils utilisés en théorie de GL près de H_{c1} . Nous renvoyons à l'introduction du Chapitre 3 (Section 3.1.3) pour une explication heuristique de ce fait quelque peu surprenant. Très rapidement, le condensat annulaire dont l'épaisseur est infiniment fine au voisinage de Ω_{c3} peut être vu après changement d'échelle comme un condensat annulaire de rayon infiniment grand au voisinage de Ω_{c1} .

Signalons à nouveau que l'origine de la différence entre théorie de GP et théorie de GL en ce qui concerne les vitesses critiques est due à la combinaison de la force centrifuge et de la contrainte de masse. Cette dernière empêche d'ailleurs a priori l'existence d'une vitesse critique analogue à H_{c3} puisqu'un condensat n'a pas de phase normale vers laquelle relaxer.

2.2.2 Travaux antérieurs : les deux premières vitesses critiques

Dans cette section nous rappelons brièvement certains résultats mathématiques portant sur la première et la deuxième vitesse critique.

Pour l'expression de la première vitesse critique et les méthodes adéquates à son voisinage nous renvoyons à [81, 82, 125] et nous ne reproduisons pas les résultats précis. Les spécificités du piégeage anharmonique n'apparaissant pas dans le régime où Ω est proche de Ω_{c1} et l'approche mathématique adéquate est celle développée dans [5, 81, 82]. Le cas du piège plat est en fait plus simple que ceux considérés dans ces références et on obtient

$$\Omega_{c1} = |\log \varepsilon|(1 + o(1)). \quad (0.2.80)$$

Nous citons maintenant des résultats obtenus par Correggi et Yngvason dans [47] (voir aussi [43]). Nous aurons besoin de la fonctionnelle de Thomas-Fermi suivante, qui donne le premier ordre de l'énergie et de la densité :

$$\mathcal{E}^{\text{TF}}[\rho] = \int_{\mathcal{B}} \left(-\Omega^2 |x|^2 \rho + \frac{1}{\varepsilon^2} \rho^2 \right) dx, \quad (0.2.81)$$

où ρ joue le rôle d'une densité de matière, c'est à dire du carré du module d'une fonction d'onde. On définit l'énergie fondamentale par

$$E^{\text{TF}} = \inf \left\{ \mathcal{E}^{\text{TF}}[\rho], \rho \geq 0, \int_{\mathcal{B}} \rho = 1 \right\} = \mathcal{E}^{\text{TF}}[\rho^{\text{TF}}]. \quad (0.2.82)$$

L'état fondamental ρ^{TF} associé est unique, nous reviendrons sur ses propriétés par la suite.

Le résultat que nous voulons citer concerne l'énergie E^{GP} (voir Théorème 3.2 dans [47]) dans le régime $\Omega_{c1} \ll \Omega \ll \Omega_{c3}$.

Théorème 0.2.7 (Asymptotiques d'énergie pour $|\log \varepsilon| \ll \Omega \ll \varepsilon^{-2} |\log \varepsilon|^{-1}$).

- Si $|\log \varepsilon| \ll \Omega \lesssim \varepsilon^{-1}$ et ε est assez petit, alors

$$E^{\text{GP}} = E^{\text{TF}} + \Omega |\log(\varepsilon^2 \Omega)| (1 + o(1)). \quad (0.2.83)$$

- Si $\varepsilon^{-1} \lesssim \Omega \ll \varepsilon^{-2} |\log \varepsilon|^{-1}$ et ε est assez petit, alors

$$E^{\text{GP}} = E^{\text{TF}} + \Omega |\log \varepsilon| (1 + o(1)). \quad (0.2.84)$$

Trois commentaires sont de mise pour interpréter ce théorème.

1. L'énergie E^{TF} a une expression explicite que nous ne donnons pas ici (voir [43, 47]) mais qui confirme que les seconds termes de (0.2.83) et (0.2.84) sont bien des corrections par rapport à E^{TF} .
2. Le premier ordre de l'énergie implique dans les deux cas du théorème que

$$\| |\psi^{\text{GP}}|^2 - \rho^{\text{TF}} \|_{L^2} \ll \| \rho^{\text{TF}} \|_{L^2}, \quad (0.2.85)$$

ce qui donne le profil de densité du condensat et permettra d'identifier Ω_{c2} (voir ci-dessous).

3. Le deuxième ordre de l'énergie correspond à la contribution d'un réseau de vortex uniforme dans le support de ρ^{TF} . Son évaluation précise permet de démontrer que la vorticité de Ψ^{GP} est uniforme dans ce domaine (voir le Théorème 3.3 de [47]).

La densité ρ^{TF} est calculable explicitement. En particulier

- si

$$\Omega \leq \frac{2}{\sqrt{\pi\varepsilon}}$$

alors $\text{supp}(\rho^{\text{TF}}) = \mathcal{B}$. Le condensat a alors la forme d'un disque en vertu de (0.2.85).

- si

$$\Omega > \frac{2}{\sqrt{\pi\varepsilon}}$$

alors $\text{supp}(\rho^{\text{TF}}) = \mathcal{A}^{\text{TF}} = \{R_h \leq |x| \leq 1\}$ avec $R_h > 0$. Le condensat est alors annulaire, toujours à cause de (0.2.85). Plus précisément (des résultats plus forts sont formulés dans [43, 47])

$$\int_{\mathcal{A}^{\text{TF}}} |\Psi^{\text{GP}}|^2 = 1 - o(1)$$

i.e. la masse du condensat est concentrée dans \mathcal{A}^{TF} .

La seconde vitesse critique est celle à laquelle le condensat devient annulaire. Elle a donc l'expression asymptotique

$$\Omega_{c2} = \frac{2}{\sqrt{\pi\varepsilon}} (1 + o(1)). \quad (0.2.86)$$

Nous ne reproduirons pas ici le théorème de [47] portant sur la vorticité de Ψ^{GP} . Signalons simplement que les estimations d'énergie (0.2.83) et (0.2.84), correspondant à une fonction test exhibant un réseau de vortex uniforme dans le support de ρ^{TF} , permettent de définir une mesure de vorticité associée à Ψ^{GP} et de montrer que cette mesure est proche de la mesure uniforme sur le support de ρ^{TF} (stricto sensu ceci est démontré uniquement pour $|\log \varepsilon| \ll \Omega \lesssim \varepsilon^{-1}$ dans [47], mais leur résultat peut être étendu au régime $|\log \varepsilon| \ll \Omega \ll \varepsilon^{-2} |\log \varepsilon|^{-1}$ au prix de difficultés techniques supplémentaires, voir [45]). Ceci constitue une première étape dans la preuve de l'apparition d'un réseau de vortex. Notons que les formules (0.2.83) et (0.2.84) ne sont pas assez précises pour distinguer la géométrie du réseau optimal. On s'attend à ce que le réseau optimal soit hexagonal, mais

une preuve de l'apparition d'un tel réseau constitue un problème ouvert difficile comme nous l'avons déjà mentionné (voir [136] pour les résultats les plus récents que l'on pourrait espérer adapter à ce contexte).

On peut considérer que le Théorème 0.2.7 et ses corollaires concluent la question de l'identification de Ω_{c1} et Ω_{c2} (avec une petite réserve pour Ω_{c1} dont l'expression nécessite l'apport des outils de [81, 82]). Nous nous sommes donc intéressés dans cette thèse à l'identification de Ω_{c3} .

2.2.3 La troisième vitesse critique : borne supérieure et vortex géant

Nous exposons dans cette partie nos premiers résultats sur la vitesse critique Ω_{c3} . Ils ont été obtenus en collaboration avec Michele Correggi et Jakob Yngvason. Le texte du preprint [46] est reproduit dans le Chapitre 3.

Dans cette section nous nous plaçons dans le régime où la transition de phase est attendue : nous considérons des vitesses de rotation de la forme

$$\Omega = \frac{\Omega_0}{\varepsilon^2 |\log \varepsilon|} \quad (0.2.87)$$

avec Ω_0 constant. Nous prouvons que si Ω_0 est assez grand, une transition de phase a lieu : les vortex disparaissent de l'anneau où la masse de Ψ^{GP} est concentrée.

Notons d'abord que dans ce régime, la densité de Thomas-Fermi ρ^{TF} prend la forme

$$\rho^{\text{TF}}(r) = \max \left(\frac{(\varepsilon \Omega)^2}{2} (r^2 - R_h^2), 0 \right). \quad (0.2.88)$$

En particulier, elle s'annule pour $r \leq R_h$ avec

$$1 - R_h^2 = (2/\sqrt{\pi})(\varepsilon \Omega)^{-1} \sim \Omega_0^{-1} \varepsilon |\log \varepsilon|, \quad (0.2.89)$$

Notre preuve que les vortex disparaissent du condensat est valable dans une région où ρ^{TF} est assez grand. Nous définissons donc l'anneau

$$\tilde{\mathcal{A}} := \{\vec{r} : R_h + \varepsilon |\log \varepsilon|^{-1} \leq r \leq 1\} \quad (0.2.90)$$

sur lequel $\rho^{\text{TF}}(r) \gtrsim \Omega_0 \varepsilon^{-1} |\log \varepsilon|^{-3}$. Notre résultat sur l'apparition du vortex géant peut être formulé ainsi (voir le Théorème 3.1.1 et les commentaires associés dans le Chapitre 3) :

Théorème 0.2.8 (Apparition d'un vortex géant si $\Omega_0 > 2(3\pi)^{-1}$).

Si $\Omega_0 > 2(3\pi)^{-1}$ dans (0.2.87), alors pour tout $\vec{r} \in \tilde{\mathcal{A}}$

$$|\Psi^{\text{GP}}(\vec{r})|^2 - \rho^{\text{TF}}(r) \leq C \frac{|\log \varepsilon|^3}{\varepsilon^{7/8}} \ll \rho^{\text{TF}}(r). \quad (0.2.91)$$

En particulier :

1. $\int_{\tilde{\mathcal{A}}} |\Psi^{\text{GP}}|^2 = 1 - o(1)$,
2. Ψ^{GP} ne s'annule pas sur $\tilde{\mathcal{A}}$.

Comme nous l'avons déjà dit dans la Section 2.1.2, les propriétés 1 et 2 dans le théorème ci-dessus sont le signe de l'apparition d'un vortex géant : la masse de Ψ^{GP} est concentrée dans l'anneau $\tilde{\mathcal{A}}$ qui ne contient aucun vortex.

La preuve du Théorème 0.2.8 est basée sur des estimations d'énergie. En particulier nous fournissons une borne inférieure asymptotiquement égale à la borne supérieure obtenue en testant la fonctionnelle d'énergie contre une fonction test optimale de type vortex géant. Cette dernière a la forme

$$\Psi(\vec{r}) = f(\vec{r}) \exp(i\hat{\Omega}\theta) \quad (0.2.92)$$

où f est à valeur réelle et $\hat{\Omega} \in \mathbb{N}$. On notera dans la suite ($[\Omega]$ désigne la partie entière de Ω)

$$\hat{\Omega} = [\Omega] - \omega \quad (0.2.93)$$

avec $\omega \in \mathbb{Z}$. On définit les fonctionnelles d'énergie 'vortex géant' par

$$\hat{\mathcal{E}}_{\omega}^{\text{GP}}[f] := \mathcal{E}^{\text{GP}}[f \exp(i\hat{\Omega}\theta)] = \int_{\mathcal{B}} d\vec{r} \left\{ |\nabla f|^2 + (\hat{\Omega}^2 r^{-2} - 2\Omega\hat{\Omega})f^2 + \varepsilon^{-2}f^4 \right\}. \quad (0.2.94)$$

On notera g_{ω} l'unique minimiseur positif de l'énergie ci-dessus (sous contrainte de masse unité bien sûr). Il est radial et strictement positif pour $r > 0$.

Il est clair que

$$E^{\text{GP}} \leq \mathcal{E}^{\text{GP}}[g_{\omega} \exp(i\hat{\Omega}\theta)] = \hat{E}_{\omega}^{\text{GP}} = \inf \left\{ \hat{\mathcal{E}}_{\omega}^{\text{GP}}[f], \int_{\mathcal{B}} |f|^2 = 1 \right\} \quad (0.2.95)$$

pour tout ω , et donc

$$E^{\text{GP}} \leq \hat{E}^{\text{GP}} := \inf_{\omega \in \mathbb{Z}} \hat{E}_{\omega}^{\text{GP}}. \quad (0.2.96)$$

Nous montrons dans le théorème suivant que la borne supérieure (0.2.96) est optimale à de petites erreurs près et estimons le ω qui minimise $\hat{E}_{\omega}^{\text{GP}}$.

Théorème 0.2.9 (Energie du fondamental dans le régime $\Omega_0 > 2(3\pi)^{-1}$).

Pour $\Omega_0 > 2(3\pi)^{-1}$ et $\varepsilon > 0$ assez petit on a

$$E^{\text{GP}} = \hat{E}^{\text{GP}} - O\left(\frac{|\log \varepsilon|^{3/2}}{\varepsilon^{1/2}(\log |\log \varepsilon|)^2}\right). \quad (0.2.97)$$

De plus $\hat{E}^{\text{GP}} = \hat{E}_{\omega_{\text{opt}}}^{\text{GP}}$, avec

$$\omega_{\text{opt}} = \frac{2}{3\sqrt{\pi}\varepsilon} (1 + O(|\log \varepsilon|^{-1})). \quad (0.2.98)$$

Nous renvoyons aux Remarques 3.1.3 et 3.1.4 du Chapitre 3 pour des détails sur les ordres de grandeur en jeu dans \hat{E}^{GP} . Mentionnons ici que le terme de reste dans (0.2.97) est négligeable devant le coût énergétique d'un vortex dans $\tilde{\mathcal{A}}$, ce qui permet de prouver le Théorème 0.2.8.

Comme Ψ^{GP} ne s'annule pas sur l'anneau $\tilde{\mathcal{A}}$, son degré topologique sur $\partial\mathcal{B}$ est bien défini. Nous sommes capables d'en fournir une estimation en accord avec la phase de la fonction test optimale $g_{\omega_{\text{opt}}} e^{i([\Omega] - \omega_{\text{opt}})\theta}$:

Théorème 0.2.10 (Circulation de phase d'un minimiseur pour $\Omega_0 > 2(3\pi)^{-1}$).
Si $\Omega_0 > 2(3\pi)^{-1}$ et $\varepsilon > 0$ assez petit, le degré topologique de Ψ^{GP} sur $\partial\mathcal{B}$ satisfait

$$\deg \{\Psi^{\text{GP}}, \partial\mathcal{B}\} = [\Omega] - \omega_{\text{opt}}(1 + o(1)) \quad (0.2.99)$$

avec ω_{opt} comme dans (0.2.98).

Le Théorème 0.2.10 confirme qu'il y a bien une circulation de phase macroscopique autour du vortex géant. Ce résultat nous semble assez original : à notre connaissance il s'agit du seul théorème dans la littérature permettant d'évaluer rigoureusement la circulation de phase totale d'un condensat ou d'un supraconducteur. Cette affirmation est toutefois à pondérer par le fait que la preuve du Théorème 0.2.10 utilise de manière cruciale la condition de Neumann (0.2.78) qui est comme nous l'avons déjà mentionné assez discutable physiquement.

Il était bien connu auparavant (par des arguments heuristiques, voir [64, 65, 91]) que le premier ordre de la circulation de phase du minimiseur devait être égal à $[\Omega]$. L'importance du terme correctif que nous identifions dans (0.2.98) et (0.2.99) ne semble par contre pas avoir été soulignée dans la littérature physique. Nous trouvons par notre analyse rigoureuse que tenir compte de ce terme correctif est en fait crucial pour l'étude du vortex géant : sa contribution énergétique est nettement supérieure à celle d'un vortex individuel dans l'anneau (voir l'introduction du Chapitre 3 pour plus de détails).

Nous fournirons quelques idées des preuves des théorèmes ci-dessus dans la Section 2.2.5.

A ce stade, nous avons montré que dans la limite $\varepsilon \rightarrow 0$

$$\Omega_{c3} \leq \frac{2}{3\pi\varepsilon^2|\log \varepsilon|}(1 + o(1)). \quad (0.2.100)$$

En combinant cette information avec les résultats de [47] nous savons d'ores et déjà que la transition de phase a bien lieu dans le régime $\Omega \propto \varepsilon^{-2}|\log \varepsilon|^{-1}$ puisque si $\Omega \ll \varepsilon^{-2}|\log \varepsilon|^{-1}$ des vortex sont présents dans le condensat. Dans la section suivante nous fournirons une borne inférieure confirmant l'optimalité de (0.2.100) et donnerons des détails sur la phase qui sert d'intermédiaire entre le réseau de vortex et le vortex géant. Cette phase contient un cercle de vortex.

2.2.4 La troisième vitesse critique : borne inférieure et cercles de vortex

Dans cette section, nous présentons les résultats du preprint [129], reproduit au Chapitre 4. Nous allons montrer que si, dans la limite $\varepsilon \rightarrow 0$

$$\Omega < \frac{2}{3\pi\varepsilon^2|\log \varepsilon|}, \quad (0.2.101)$$

alors il y a des vortex dans le condensat. Plus précisément, nous considérons des vitesses de rotation de la forme

$$\Omega = \frac{2}{3\pi\varepsilon^2|\log \varepsilon|} - \frac{\Omega_1}{\varepsilon^2|\log \varepsilon|} \quad (0.2.102)$$

avec

$$\Omega_1 > 0 \tag{0.2.103}$$

$$|\Omega_1| \ll 1 \tag{0.2.104}$$

$$|\Omega_1| \gg \frac{\log |\log \varepsilon|}{|\log \varepsilon|}. \tag{0.2.105}$$

Les deux premières hypothèses ci-dessus sont naturelles : elles garantissent que nous nous situons sous le seuil (0.2.101) mais dans son voisinage. La troisième est technique. Nous pensons qu'elle peut être relaxée mais nos preuves devraient être améliorées pour ce faire.

Les résultats que nous allons présenter confirment que dans le régime (0.2.102), sous les hypothèses (0.2.103), (0.2.104) et (0.2.105), il y a des vortex dans l'anneau où la masse du condensat est concentrée. Ils sont uniformément distribués le long d'un cercle de rayon R_* et nous pouvons estimer leur nombre, qui est proportionnel à $\Omega_1 \varepsilon^{-1}$. Ceci complète la preuve que la troisième vitesse critique satisfait

$$\Omega_{c3} = \frac{2}{3\pi\varepsilon^2|\log \varepsilon|}(1 + o(1)) \tag{0.2.106}$$

dans la limite où $\varepsilon \rightarrow 0$. Plus précisément, ceci prouve la borne inférieure correspondant à (0.2.100). Une formule telle que (0.2.106) ne semble pas exister dans la littérature physique : celle donnée dans [65] diffère de (0.2.106) car ces auteurs négligent l'apparition du cercle de vortex dans leur mode de calcul. Une méthode semi-numérique pour le calcul de la vitesse critique est proposée dans [92] (voir aussi [71]), mais elle ne fournit pas de formule explicite.

Une formulation précise des résultats nécessite de prendre en compte les considérations suivantes : la masse de Ψ^{GP} est exponentiellement petite dans le complémentaire de \mathcal{A}^{TF} , donc les propriétés asymptotiques du modèle complet seront bien rendues par des fonctionnelles réduites à cet anneau. Pour des raisons technique il est toutefois nécessaire de considérer un anneau légèrement plus grand

$$\mathcal{A} := \{\vec{r} : R_{<} \leq r \leq 1\}, \tag{0.2.107}$$

avec

$$R_{<} := R_{\text{h}} - \varepsilon^{8/7}. \tag{0.2.108}$$

Il est ensuite nécessaire (ceci est classique dans ce genre de contexte, voir par exemple [23, Section 5]) d'extraire de Ψ^{GP} le profil de densité et la vorticit   dues au vortex g  ant. En effet, malgr   la pr  sence d'un cercle de vortex individuels, c'est le vortex g  ant central qui donne le premier ordre de la vorticit  . Nous proc  dons comme suit.

Comme en (0.2.94) nous introduisons des fonctionnelles d'  nergie de type 'vortex g  ant', mais restreintes    l'anneau \mathcal{A} : pour tout $\omega \in \mathbb{Z}$ et toute fonction ϕ    valeurs r  elles nous d  finissons

$$\hat{\mathcal{E}}_{\mathcal{A},\omega}^{\text{GP}}[\phi] := \mathcal{E}_{\mathcal{A}}^{\text{GP}}[\phi e^{i([\Omega]-\omega)\theta}] = \int_{\mathcal{A}} |\nabla \phi|^2 + \frac{1}{\varepsilon^2} |\phi|^4 + \left(\frac{([\Omega] - \omega)^2}{r^2} - 2\Omega([\Omega] - \omega) \right) |\phi|^2. \tag{0.2.109}$$

Dans la suite $g_{\mathcal{A},\omega}$ désigne l'unique minimiseur positif de cette fonctionnelle sous contrainte de masse unité et $\hat{E}_{\mathcal{A},\omega}^{\text{GP}}$ l'énergie correspondante. Pour extraire de Ψ^{GP} l'information pertinente, il nous faut considérer la plus basse parmi ces énergies :

$$\hat{E}_{\mathcal{A},\omega_0}^{\text{GP}} = \min_{\omega \in \mathbb{Z}} \hat{E}_{\mathcal{A},\omega}^{\text{GP}}. \quad (0.2.110)$$

L'existence d'un ω_0 réalisant le minimum ci-dessus et certaines de ses propriétés sont établies dans la Proposition 3.3.2 du Chapitre 3 . En particulier on a

$$\omega_0 = \frac{2}{3\sqrt{\pi}\varepsilon} (1 + O(|\log \varepsilon|^{-1})).$$

Dans toute la suite de cette partie nous considérerons un ω_0 minimisant $\hat{E}_{\mathcal{A},\omega}^{\text{GP}}$ par rapport à $\omega \in \mathbb{Z}$ (nous suivons les notations du Chapitre 3 et emploierons une conventions un peu différente au Chapitre 4). Nous notons

$$g = g_{\mathcal{A},\omega_0} \quad (0.2.111)$$

pour alléger la notation.

Nous extrayons maintenant de Ψ^{GP} la phase et la densité correspondant au vortex géant en définissant sur \mathcal{A}

$$u := \frac{\Psi^{\text{GP}}}{g e^{i([\Omega] - \omega_0)\theta}}. \quad (0.2.112)$$

Le résultat principal de cette section porte sur la *mesure de vorticité* associée à u

$$\mu := \text{curl}(iu, \nabla u). \quad (0.2.113)$$

Nous prouvons qu'elle est proche de la mesure de Dirac portée par un certain cercle de rayon R_* .

Il nous faut maintenant anticiper sur la Section 2.2.5 et introduire un des outils importants de nos preuves, la *fonction de coût*

$$H(r) := \frac{1}{2} g^2(r) |\log \varepsilon| + F(r). \quad (0.2.114)$$

Le premier terme donne le coût en énergie cinétique d'un vortex de degré 1 situé au point \vec{r} dans le condensat. Le deuxième terme donne le gain énergétique qu'un tel vortex apporte en interagissant avec le champ de rotation. Il est défini par

$$F(r) := 2 \int_{R_<}^r g^2(s) \left(\Omega s - ([\Omega] - \omega_0) \frac{1}{s} \right) ds. \quad (0.2.115)$$

Le rayon du cercle près duquel les vortex se concentrent (noté R_* ci-dessous) est essentiellement le point de minimum de H . Les vortex ne peuvent être détectés avec notre méthode énergétique que dans les régions où la densité de matière est suffisante, c'est-à-dire dans l'anneau

$$\mathcal{A}_{\text{bulk}} := \{\vec{r} \mid R_{\text{bulk}} \leq r \leq 1\}, \quad R_{\text{bulk}} := R_h + \varepsilon |\log \varepsilon| \Omega_1^{1/2}. \quad (0.2.116)$$

La notation $\mathcal{A}_{\text{bulk}}$ est justifiée par l'estimation

$$\int_{\mathcal{A}_{\text{bulk}}} |\Psi^{\text{GP}}|^2 = 1 - O(\Omega_1^{1/2}),$$

qui garantit que la masse de Ψ^{GP} est concentrée dans $\mathcal{A}_{\text{bulk}}$ lorsque $\Omega_1 \ll 1$. Dans le théorème ci-dessous nous estimons la mesure de vorticité μ à l'aide de la norme $\|\cdot\|_g$ définie pour toute mesure de Radon ν par

$$\|\nu\|_g := \sup_{\phi \in C_c^1(\mathcal{A}_{\text{bulk}})} \frac{\left| \int_{\mathcal{A}_{\text{bulk}}} \nu \phi \right|}{\left(\int_{\mathcal{A}_{\text{bulk}}} \frac{1}{g^2} |\nabla \phi|^2 \right)^{1/2} + \varepsilon |\log \varepsilon| \|\nabla \phi\|_{L^\infty(\mathcal{A}_{\text{bulk}})}}. \quad (0.2.117)$$

Théorème 0.2.11 (Comportement asymptotique de la mesure de vorticité).

Il existe un certain rayon R_* (voir l'Appendice A du Chapitre 4) satisfaisant

$$R_* = \sqrt{R_h^2 + (\varepsilon \Omega)^{-1} \left(\frac{1}{\sqrt{\pi}} + O(\Omega_1) \right)} \quad (0.2.118)$$

tel que, notant δ_* la mesure de Dirac normalisée portée par le cercle de rayon R_* , on ait

$$\left\| \mu + \frac{H(R_*)}{2I_*} \delta_* \right\|_g \ll \frac{\Omega_1}{\varepsilon} \leq C \left\| \frac{H(R_*)}{2I_*} \delta_* \right\|_g. \quad (0.2.119)$$

I_* est une quantité positive borné supérieurement et inférieurement indépendamment de ε que nous définissons ci-dessous.

Les deux remarques suivantes ont pour but d'expliquer en quoi le Théorème ci-dessus permet de déduire la structure de vortex portée par u et donc par Ψ^{GP} (on pourra trouver un troisième argument, explicitant les ordres de grandeur en jeu dans (0.2.119) dans l'introduction du Chapitre 4) :

1. La mesure de vorticité μ est une grandeur capturant des informations précises sur les vortex de u . En effet, on s'attend à ce que $|u| \sim 1$ (ce qui entraîne $\mu = \text{curl}(iu, \nabla u) \sim 0$) sauf près des vortex, donc pour tout domaine dont la frontière σ ne rencontre aucun vortex

$$2\pi \deg\{u, \sigma\} \sim \int_{\sigma} (iu, \partial_{\tau} u)$$

est donnée par l'intégrale de μ sur le domaine (ceci est une conséquence de la formule de Stokes). On aura alors $\mu \sim 2\pi \sum_j d_j \delta_{a_j}$ où les d_j sont les degrés des vortex et les a_j leurs positions. Ceci est rendu rigoureux par une estimation dite de Jacobienne, voir le point 4 dans la Section 2.2.5. On remarquera que par définition les vortex de u sont identiques à ceux de Ψ^{GP} .

2. Si la norme $\left(\int_{\mathcal{A}_{\text{bulk}}} g^{-2} |\nabla \phi|^2 \right)^{1/2}$ pouvait être contrôlée par $\varepsilon |\log \varepsilon| \|\nabla \phi\|_{L^\infty(\mathcal{A})}$ pour toute fonction test, alors (0.2.119) serait équivalent à

$$\left\| \mu + \frac{H(R_*)}{2I_*} \delta_* \right\|_{(C_c^1(\mathcal{A}_{\text{bulk}}))^*} \ll \Omega_1 |\log \varepsilon|. \quad (0.2.120)$$

Notons que la norme dans $(C_c^1(\mathcal{A}_{\text{bulk}}))^*$ des deux termes du membre de droite de (0.2.119) est $\gtrsim \Omega_1 |\log \varepsilon|$. Un tel contrôle n'est cependant pas possible car g^2 n'est pas borné inférieurement par $C(\varepsilon |\log \varepsilon|)^{-1}$ sur $\mathcal{A}_{\text{bulk}}$. La norme $\|\cdot\|_g$ prend donc en compte l'inhomogénéité du profil de densité (voir le Chapitre 4, Remarque 4.1.1 pour plus de commentaires sur $\|\cdot\|_g$).

3. Une conséquence de (0.2.119) est la suivante : si nous définissons un changement de coordonnées transformant \mathcal{A} en un anneau fixe de rayon 1 et d'épaisseur par exemple 1/2 et notons \tilde{R}_* l'image de R_* par ce changement de coordonnées, $\tilde{\mu}$ la mesure image de μ nous avons

$$-2I_* \frac{\tilde{\mu}}{H(R_*)} \rightharpoonup \tilde{\delta}_* \quad (0.2.121)$$

au sens des mesures, avec $\tilde{\delta}_*$ la mesure de Dirac normalisée portée par le cercle de rayon \tilde{R}_* . Nous n'utiliserons pas un tel changement de coordonnées dans la suite : appliqué à la fonctionnelle d'énergie il conduirait à une énergie cinétique hautement anisotrope en raison de l'épaisseur infiniment fine de l'anneau \mathcal{A} . La convergence (0.2.121) est toutefois utile pour déterminer le nombre de vortex : I_* est de l'ordre d'une constante, $-H(R_*)$ de l'ordre de $\Omega_1 \varepsilon^{-1}$ donc on déduit de (0.2.121) que le nombre de vortex est asymptotiquement égal à $-H(R_*)(2I_*)^{-1} \propto \Omega_1 \varepsilon^{-1}$ comme nous l'avions annoncé.

Rappelons que des superfluides ou des supraconducteurs dans des domaines simplement connexes montrent en général une concentration de vortex autour de points. Ici il y a concentration le long d'une courbe, au sens de [23]. Ce phénomène apparaît naturellement bien que notre domaine original soit simplement connexe et est dû aux forces centrifuges qui imposent une forme annulaire au condensat. Dans [23], un résultat de Γ -convergence est prouvé pour un modèle générique où la concentration le long d'une courbe est imposée (on peut obtenir des modèles proches de celui de [23] en prenant la limite 'film mince' du modèle de Ginzburg-Landau 3D dans certains cas particuliers, voir [22]).

Les autres situations connues donnant lieu à une concentration de vortex le long d'une courbe incluent

- le cas d'un condensat annulaire au voisinage de Ω_{c1} (voir [5], dans cet article la forme annulaire est directement imposée par le potentiel de piégeage),
- celui d'un supraconducteur avec inclusions normales au voisinage de H_{c1} [20, 21],
- celui de deux supraconducteurs aux propriétés physiques différentes en contact le long d'un cercle, au voisinage de H_{c1} ([89], voir aussi [26]).

Dans toutes ces situations, il a été prouvé que les vortex se rassemblaient bien le long d'une courbe mais la question de la répartition le long de la courbe que nous résolvons dans cette thèse est laissée ouverte. En particulier, notre résultat est autant que nous sachions le premier où une concentration de vortex le long d'une courbe plongée dans un milieu inhomogène est élucidée. La densité de matière (très!) inhomogène g^2 est source de difficultés n'apparaissant pas dans [23].

Des difficultés semblables apparaissent dans [5]. On peut probablement les surmonter avec notre approche et montrer que la répartition des vortex est uniforme le long du cercle où ils se rassemblent.

Nous explicitons maintenant la quantité I_* apparaissant dans le Théorème 0.2.11. Elle est définie via un problème électrostatique qui se trouve être intimement lié à l'asymptotique d'énergie du problème, comme le montre le Théorème 0.2.12 ci-dessous. Pour toute

mesure de Radon ν à support dans \mathcal{A} nous définissons h_ν comme l'unique solution du problème

$$\begin{cases} -\nabla \left(\frac{1}{g^2} \nabla h_\nu \right) = \nu \text{ dans } \mathcal{A} \\ h_\nu = 0 \text{ sur } \partial\mathcal{A}. \end{cases} \quad (0.2.122)$$

Nous introduisons ensuite l'énergie associée à ν

$$I(\nu) := \int_{\mathcal{A}} \frac{1}{g^2} |\nabla h_\nu|^2. \quad (0.2.123)$$

Alors

$$I_* := \inf \left\{ I(\nu), \nu \in \mathcal{D}_*, \int_{\mathcal{A}} \nu = 1 \right\} = I(\delta_*) \quad (0.2.124)$$

où \mathcal{D}_* est l'ensemble des mesures de Radon positives à support sur le cercle de rayon R_* . Nous fournirons plus bas (cf Proposition 4.3.5) la preuve que δ_* est bien une solution (en fait, la seule solution) du problème ci-dessus. La fonction h_ν peut être vue comme le champ électrique généré par la distribution de charge ν dans une enceinte \mathcal{A} dont la conductivité inhomogène serait décrite par g^{-2} . L'énergie $I(\nu)$ est alors l'énergie électrostatique associée.

Le problème de minimisation qui apparaît naturellement dans notre contexte est le suivant :

$$\inf_{\nu \in \mathcal{D}_*} \left(\int_{\mathcal{A}} \frac{1}{g^2} |\nabla h_\nu|^2 + H(R_*) \int_{\mathcal{A}} \nu \right) = -\frac{H(R_*)^2}{4I_*}. \quad (0.2.125)$$

Le problème ci-dessus est très proche de (0.2.124) : au lieu de fixer la masse de ν , on pénalise son opposé (deuxième terme dans l'équation ci-dessus, rappelons que $H(R_*)$ est négatif). On déduit donc facilement que l'unique minimiseur de (0.2.125) est $-(2I_*)^{-1}H(R_*)\delta_*$, ce qui explique pourquoi μ doit être proche de cette mesure et permet de calculer la valeur apparaissant au membre de droite de (0.2.125).

On peut penser à (0.2.125) comme à une énergie renormalisée, par analogie avec [29, 135]. Le premier terme représente le coût énergétique de l'interaction des vortex, calculé grâce au champ qu'ils génèrent. Le second terme prend en compte le gain énergétique qu'apportent les vortex en compensant partiellement le champ de rotation. Ce genre d'énergie est caractéristique de situations où des vortex se concentrent le long d'une courbe, remarquons en particulier sa similitude avec celle obtenue dans [23, Théorème 1.1]. La différence vient essentiellement du poids g^{-2} qui prend en compte l'inhomogénéité du profil de densité du condensat.

Nous formulons maintenant le théorème mettant en évidence le lien qui existe dans le régime de paramètres qui nous occupe entre les problèmes de minimisation (0.2.76) et (0.2.125) :

Théorème 0.2.12 (Asymptotiques d'énergie dans le régime $\Omega \sim \Omega_{c3}$).

Si Ω_1 satisfait (0.2.103), (0.2.104) et (0.2.105), on a, pour ε assez petit

$$E^{\text{GP}} = \hat{E}_{\mathcal{A}, \omega_0}^{\text{GP}} - \frac{H(R_*)^2}{4I_*} (1 + o(1)). \quad (0.2.126)$$

Nous présenterons dans la section suivante certaines des idées mathématiques entrant dans les preuves de nos résultats, mais nous voulons d'abord mentionner ici les principales intuitions qui peuvent guider le lecteur :

- Le champ de rotation est dirigé selon \vec{e}_θ avec une amplitude positive, ce qui implique que seuls des vortex de degré positif peuvent le compenser. Il ne devrait donc pas il y avoir de vortex de degré négatif.
- Les vortex (et donc la mesure de vortacité μ) se concentrent sur le cercle \mathcal{C}_{R_*} de rayon R_* car c'est là que la fonction coût (0.2.114) est minimale. Cette fonction représente le 'coût énergétique intrinsèque' d'un vortex (par opposition au coût dû à son interaction avec d'autres vortex) de degré 1 en fonction de sa position. Le minimum est négatif et donc favorise l'apparition de vortex si et seulement si $\Omega_1 > 0$.
- Comme nous l'avons rappelé dans la Section 1.2.2, des vortex de même signe se repoussent. Comme ici tous les vortex sont de degré positif et situés sur le cercle \mathcal{C}_{R_*} , cet effet mène à l'uniforme répartition le long de \mathcal{C}_{R_*} qui est la conclusion du Théorème 0.2.11.

2.2.5 Principales méthodes utilisées

Nous faisons ici une liste non exhaustive des outils mathématiques requis dans les preuves des résultats de la Section 2.2. Nous suivons plus ou moins l'ordre dans lequel ils seront utilisés dans la suite, le lecteur peut donc se faire une idée assez précise du déroulement des preuves dont nous présenterons les détails dans les Chapitres 3 et 4 en lisant ce résumé. Des résumés plus complets sont présentés dans les introductions des chapitres concernés.

La plupart des équations ci-dessous sont à comprendre en un sens heuristique : nous négligeons tous les termes de reste pour ne pas alourdir outre mesure ce panorama général. Nous passerons également sous silence un certain nombre de subtilités techniques pour nous concentrer sur l'essentiel.

1. Analyse des fonctionnelles d'énergie de type vortex géant :

Le point de départ est une bonne connaissance des propriétés des profils de densité obtenus en minimisant les fonctionnelles donnant l'énergie d'un vortex géant. Il faut en particulier de bonnes propriétés de décroissance dans la région $\mathcal{B} \setminus \mathcal{A}^{\text{TF}}$ pour confirmer que la densité de matière est faible dans le trou central. Ceci assurera entre autres que les fonctionnelles définies sur \mathcal{A} et sur \mathcal{B} tout entier sont très proches. Pour obtenir l'estimation de Ω_{c3} il est important de relier par une estimation L^∞ les profils de densité au profil de Thomas-Fermi explicite. Identifier et caractériser finement le degré donnant l'énergie minimale parmi tous les choix possibles est également crucial. Cette étape utilise les équations aux dérivées partielles associées aux fonctionnelles d'énergie et un certain nombre de techniques standards pour l'analyse des équations elliptiques.

2. Découplage d'énergie et fonctionnelle d'énergie réduite :

En utilisant une technique bien connue de découplage d'énergie (utilisée dans plusieurs contextes, voir par exemple [93, 99]) et une propriété de décroissance expo-

nentielle de Ψ^{GP} , on obtient (avec u défini en (0.2.112))

$$E^{\text{GP}} = \hat{E}_{\mathcal{A}, \omega_0}^{\text{GP}} + \mathcal{E}[u] + O(\varepsilon^\infty). \quad (0.2.127)$$

On est donc ramené à l'étude d'une fonctionnelle d'énergie réduite définie sur l'anneau \mathcal{A} (cf (0.2.107)) :

$$\mathcal{E}[v] := \int_{\mathcal{A}} g^2 |\nabla v|^2 - 2g^2 \vec{B} \cdot (iv, \nabla v) + \frac{g^4}{\varepsilon^2} (1 - |v|^2)^2 \quad (0.2.128)$$

avec

$$\vec{B}(r) := B(r) \vec{e}_\theta = (\Omega r - ([\Omega] - \omega_0) r^{-1}) \vec{e}_\theta \quad (0.2.129)$$

et

$$(iv, \nabla v) = \frac{1}{2} i (v \overline{\nabla v} - \overline{v} \nabla v).$$

La fonctionnelle $\mathcal{E}[u]$ prend en compte l'énergie qui s'ajoute à celle du vortex géant lorsque des vortex sont présents dans la zone de forte densité de matière (remarquons que g^2 apparaît comme un poids dans la fonctionnelle). En particulier, elle s'annule lorsque $v \equiv 1$. Une valeur strictement négative de $\mathcal{E}[u]$ indique la présence de vortex. Notons que cette fonctionnelle rappelle celles utilisées dans [5, 81, 82] pour l'étude de condensats au voisinage de Ω_{c1} . La différence majeure est que dans notre cas le domaine \mathcal{A} et la densité g^2 dépendent fortement de ε . L'épaisseur de \mathcal{A} est d'ordre $\varepsilon |\log \varepsilon|$ et donc (à cause de la contrainte de masse), le maximum de g^2 est d'ordre $(\varepsilon |\log \varepsilon|)^{-1}$. Nous renvoyons à l'introduction du Chapitre 3 pour une discussion plus détaillée sur ce point.

La fonction F introduite en (0.2.115) est contruite pour vérifier

$$\nabla^\perp(F) = 2g^2 \vec{B}, \quad F(R_<) = 0,$$

i.e., F est la 'primitive' de $g^2 \vec{B}$ qui s'annule au bord intérieur de \mathcal{A} . Nous pouvons donc réécrire l'énergie

$$\mathcal{E}[u] = \int_{\mathcal{A}} g^2 |\nabla u|^2 + \int_{\mathcal{A}} F \text{curl}(iu, \nabla u) - F(1) \int_{\partial \mathcal{B}} (iu, \partial_\tau u) + \int_{\mathcal{A}} \frac{g^4}{\varepsilon^2} (1 - |u|^2)^2 \quad (0.2.130)$$

en intégrant par parties le second terme de (0.2.128).

3. Méthode des boules vortex :

L'étude de $\mathcal{E}[u]$ commence par la construction de boules vortex : on recouvre l'ensemble où u pourrait s'annuler par des disques de rayon petit devant l'épaisseur de \mathcal{A} . La méthode de 'growth and merging' de Sandier [133] et Jerrard [85] fournit alors des bornes inférieures de la forme

$$\int_{B_i} g^2 |\nabla u|^2 \geq \pi |d_i| g^2(a_i) |\log \varepsilon| \quad (0.2.131)$$

où $B_i = B_i(a_i, r_i)$ est un disque générique dans la collection et d_i le degré de u autour du disque B_i .

4. Estimées de Jacobienne :

L'étape suivante consiste en l'adaptation de la méthode dite d'estimée de Jacobienne due à Jerrard et Soner [88] pour obtenir

$$\mu = \text{curl}(iu, \nabla u) \sim 2\pi \sum_i d_i \delta_{a_i} \quad (0.2.132)$$

dans un sens approprié. Ici δ_{a_i} est la masse de Dirac en a_i . On déduit

$$\int_{\mathcal{A}} F \operatorname{curl}(iu, \nabla u) \sim 2\pi \sum_i d_i F(a_i). \quad (0.2.133)$$

5. Découpage du domaine en cellules :

Appliquer les étapes 3 et 4 suppose un bon contrôle sur l'énergie

$$\mathcal{F}[u] = \int_{\mathcal{A}} g^2 |\nabla u|^2 + \frac{g^4}{\varepsilon^2} (1 - |u|^2)^2.$$

C'est en effet ce contrôle qui permet la construction des boules vortex. Jusqu'ici nous avons implicitement supposé qu'il allait de soi que nous pouvions déduire un contrôle suffisant des bornes supérieures sur $\mathcal{E}[u]$ qui sont notre seule information a priori.

Ce serait le cas si nous nous trouvions dans un régime où Ω est proche de Ω_{c1} , comme dans [5, 81, 82]. Dans notre situation, l'ordre de grandeur de Ω rend cette étape délicate, notamment à cause de la géométrie particulière que la force centrifuge impose à \mathcal{A} : rayon fixe mais épaisseur infiniment fine (d'ordre $\varepsilon |\log \varepsilon|$).

Il s'avère nécessaire de découper l'anneau en cellules de côté $\sim \varepsilon |\log \varepsilon|$ et de distinguer entre les 'bonnes cellules' où le contrôle auquel nous faisons référence est possible et les 'mauvaises cellules' où il ne l'est pas. On peut montrer qu'il y a relativement peu de mauvaises cellules, ce qui permet de démarrer l'analyse en les négligeant en première approximation.

La gestion de l'interface entre les bonnes et les mauvaises cellules est source de complications techniques que nous négligeons dans cette présentation heuristique.

6. Estimation à la frontière :

Le terme de bord (troisième terme dans (0.2.130)) est en fait négligeable. C'est ici qu'il est crucial de faire un choix judicieux pour la phase du vortex géant que nous extrayons. Il se trouve que avec notre choix de ω_0 , $F(1)$ est relativement petit. On combine ce fait avec une estimation de la circulation de u sur $\partial\mathcal{B}$. Celle-ci est obtenue en dérivant d'abord une EDP satisfaite par u et en utilisant une astuce à la Pohozaev [123] sur cette équation. Cette estimation mène également au Théorème (0.2.10).

On notera que cette étape n'apparaît pas dans [81, 82] en raison de la simple connexité du domaine considéré. Une alternative est utilisée dans [5] pour contrôler un terme de bord similaire, mais cette technique ne s'applique pas dans notre contexte à cause du découpage du domaine en cellules à l'étape 5.

7. Énergie cinétique longue portée :

Le terme que nous appelons 'énergie cinétique longue portée' est localisé en dehors des boules vortex, c'est à dire relativement loin des vortex. Il prend en compte l'interaction entre les vortex et est responsable de l'apparition du premier terme dans l'énergie renormalisée (0.2.125). Pour l'évaluer, on s'est inspiré de [23]. Notons toutefois que le domaine et le profil de densité variables que nous considérons empêchent l'utilisation de certains arguments de compacité de cette référence.

L'idée est que l'énergie cinétique est essentiellement due au *courant superfluide* associé à u :

$$j := (iu, \nabla u). \quad (0.2.134)$$

Notons que

$$\mu = \operatorname{curl}(j).$$

Puisque l'énergie cinétique dans les boules vortex a déjà été évaluée à l'étape 3, il est utile d'introduire un courant superfluide modifié

$$j_{\text{mod}} := \begin{cases} j & \text{in } \mathcal{A} \setminus \cup_i B_i \\ 0 & \text{in } \cup_i B_i. \end{cases} \quad (0.2.135)$$

et la vorticit  associ e

$$\mu_{\text{mod}} := \text{curl}(j_{\text{mod}}). \quad (0.2.136)$$

Cette op ration peut ˆtre vue comme une r gularisation : on ˆte de j la partie singuli re au voisinage des vortex. Par d finition des boules vortex on a $|u| \sim 1$ sauf dans $\cup_i B_i$. Ceci implique

$$\int_{\mathcal{A} \setminus \cup_i B_i} g^2 |\nabla u|^2 \sim \int_{\mathcal{A} \setminus \cup_i B_i} g^2 |j|^2 = \int_{\mathcal{A}} g^2 |j_{\text{mod}}|^2.$$

En utilisant la d finition (0.2.122) satisfait par $h_{\mu_{\text{mod}}}$ on obtient une  galit  du genre

$$j_{\text{mod}} = -\frac{1}{g^2} \nabla^\perp h_{\mu_{\text{mod}}} + \nabla f \quad (0.2.137)$$

o  $f \in H^1(\mathcal{A})$ (plus pr cis ment, on obtiendrait exactement (0.2.137) si \mathcal{A}  tait simplement connexe). On d duit ensuite

$$\int_{\mathcal{A} \setminus \cup_i B_i} g^2 |\nabla u|^2 \sim \int_{\mathcal{A}} g^2 |j_{\text{mod}}|^2 \geq \int_{\mathcal{A}} \frac{1}{g^2} |\nabla h_{\mu_{\text{mod}}}|^2. \quad (0.2.138)$$

car $h_{\mu_{\text{mod}}} = 0$ on $\partial\mathcal{A}$, ce qui explique l'apparition de l' nergie  lectrostatique (0.2.123) dans notre analyse.

8. Fonction de Green et construction de bornes sup rieures :

La borne sup rieure optimale dans le cas du vortex g ant pur est facile   obtenir (0.2.96). Dans le cas o  on s'attend   voir un cercle de vortex apparaˆtre il faut construire une fonction test correspondante. La phase de cette fonction est d finie en utilisant une fonction de Green pour le probl me (0.2.122)

$$\begin{cases} -\nabla_x \left(\frac{1}{g^2(x)} \nabla_x G(x, y) \right) = \delta_y(x) \text{ pour } x \in \mathcal{A} \\ G(x, y) = 0 \text{ pour } x \in \partial\mathcal{A}. \end{cases} \quad (0.2.139)$$

Cette technique a  t  utilis e par un certain nombre d'auteurs [5, 18, 20, 21, 23, 135] et se r v le tr s efficace pour le calcul de l' nergie de la fonction test. Une  tude assez pouss e de la fonction de Green est essentielle et dans le cas que nous traitons elle est compliqu e par la g om trie particuli re de \mathcal{A} . On prendra avantage de la sym trie radiale du probl me   plusieurs reprises.

Le r sultat des  tapes ci-dessus est une borne inf rieure de type

$$\mathcal{E}[u] \geq \int_{\mathcal{A}} \frac{1}{g^2} |\nabla h_{\mu_{\text{mod}}}|^2 + \int_{\mathcal{A}} H \mu_{\text{mod}} \quad (0.2.140)$$

o  H est la fonction de coˆt d finie en (0.2.114). Nous utilisons l'estim e de Jacobienne et le fait que les boules vortex ont des rayons relativement petits pour obtenir

$$\mu \sim \mu_{\text{mod}} \sim 2\pi \sum_i d_i \delta_{a_i}. \quad (0.2.141)$$

De plus on peut construire une borne supérieure montrant que (0.2.140) est en fait optimal.

Les raisonnements qui entrent en jeu sont alors les suivants :

- On revient d'abord à (0.2.133) : F est négative donc, tous les autres termes que nous avons évalué étant positifs ou négligeables, les vortex de degré négatif ne peuvent jamais être favorables. La plupart des vortex doivent être de degré positif, ce qui implique que μ_{mod} est une mesure positive en vertu de (0.2.141). En toute rigueur c'est un fait nécessaire pour pouvoir écrire (0.2.140).

- Lorsque

$$\Omega > \frac{2}{3\pi\varepsilon^2|\log\varepsilon|}$$

la fonction coût H est positive (cette fonction est relativement explicite : en première approximation elle se calcule essentiellement à partir de ρ^{TF} , voir ...). Le deuxième terme de (0.2.140) est alors positif, ce qui indique que les vortex ne sont pas favorables énergétiquement. Ce fait est la marque de la transition vers une phase vortex géant pur et la clé de la preuve du Théorème 0.2.8.

- Lorsque Ω est choisi comme dans la Section 2.2.4, la fonction coût est négative seulement au voisinage de son minimum, en $r = R_*$. En exploitant le deuxième terme de (0.2.140) on comprend alors que μ_{mod} est asymptotiquement à support sur \mathcal{C}_{R_*} . Le membre de droite de (0.2.140) se transforme donc en l'énergie renormalisée (0.2.125) qui est définie pour des mesures à support sur \mathcal{C}_{R_*} . Il reste à utiliser une borne supérieure adéquate sur l'énergie pour montrer que μ_{mod} est proche du minimiseur de cette énergie. Revenant à (0.2.141) on a alors essentiellement tous les ingrédients de la preuve du Théorème 0.2.11.

Le lecteur a probablement remarqué que d'après notre présentation rapide, les étapes 7 et 8 ci-dessus ne sont pas nécessaires pour obtenir les résultats de la Section 2.2.3. C'est en effet le cas et en conséquence le Chapitre 3 de la présente thèse correspond uniquement aux étapes 1 à 6. Les raffinements des étapes 7 et 8 sont obtenus dans la Chapitre 4 et ouvrent la voie aux résultats présentés dans la Section 2.2.4.

3 Conclusion

3.1 Discussion

Nous allons maintenant résumer les principales prédictions de la théorie de Gross-Pitaevskii pour un condensat en rotation à vitesse croissante. Dans cette section nous nous éloignons quelque peu de la rigueur qui a prévalu dans la section précédente pour brosser un tableau plus qualitatif et mettre nos résultats en perspective.

On distingue trois vitesses critiques principales en théorie de Gross-Pitaevskii, Ω_{c1} , Ω_{c2} et Ω_{c3} , caractérisées par des changements radicaux de la structure de vortex du condensat modélisé. Leur évaluation par des techniques mathématiques rigoureuses a principalement été effectuée à ce jour dans la limite de forte interaction $G \rightarrow +\infty$ (ou $\varepsilon \rightarrow 0$). Ce régime est appelé 'limite de Thomas-Fermi' car le profil de densité du condensat y est le plus souvent donné par une fonctionnelle d'énergie analogue à celle rencontrée dans la théorie

de Thomas-Fermi du gaz d'électrons. Cependant nous avons vu à la Section 2.1.2 que ce profil devient gaussien pour de très grandes vitesses de rotation. Nous appellerons Ω_{c4} dans la suite la vitesse à laquelle ce changement a lieu. On notera que Ω_{c4} n'a pas le même statut que Ω_{c1}, Ω_{c2} et Ω_{c3} puisque la transition est caractérisée par le profil du condensat et non par ses vortex. De plus il est probable que la transition entre le profil de type Thomas-Fermi et le profil gaussien soit graduelle et donc que Ω_{c4} ne soit pas un nombre bien défini mais plutôt un ordre de grandeur.

Dans le Tableau 1 nous rassemblons les principales caractéristiques d'un condensat en fonction de la vitesse de rotation dans la limite de Thomas-Fermi $G \rightarrow \infty$ (ou $\varepsilon \rightarrow 0$).

Vitesse de rotation	Profil de densité	Forme	Structure de vortex
$0 \leq \Omega \ll \Omega_{c1}$	Thomas-Fermi	Disque	Pas de vortex
$\Omega_{c1} \ll \Omega \ll \Omega_{c2}$	Thomas-Fermi	Disque	Réseau de vortex
$\Omega_{c2} \ll \Omega \ll \Omega_{c3}$	Thomas-Fermi	Anneau	Réseau de vortex + vortex géant
Ω 'proche de' Ω_{c3}	Thomas-Fermi	Anneau	Cercle de vortex + vortex géant
$\Omega_{c3} \ll \Omega \ll \Omega_{c4}$	Thomas-Fermi	Anneau	Vortex géant
$\Omega_{c4} \ll \Omega$	Gaussien	Anneau	Vortex géant

TABLE 1 – Principales caractéristiques d'un condensat en rotation en fonction de Ω .

Bien sûr le tableau ci-dessus résume la situation de manière informelle et les différents phénomènes sont compris à des degrés de rigueur mathématique variés. On pourra aussi consulter [139] pour des résultats dans un esprit différent.

La première vitesse critique est maintenant bien identifiée dans le cas de potentiels de piégeages généraux [5, 81, 82, 125, 16]. Dans le cas du piège plat, ces études s'adaptent et fournissent l'estimation

$$\Omega_{c1} = |\log \varepsilon|(1 + o(1)).$$

Le régime $\Omega_{c1} \ll \Omega \ll \Omega_{c3}$ a été étudié dans les articles [43, 47] pour le cas du piège plat et l'article [44] contient des résultats préliminaires sur les pièges homogènes. On a en particulier, dans le cas du piège plat

$$\Omega_{c2} = \frac{2}{\sqrt{\pi\varepsilon}}(1 + o(1)).$$

Les contributions de la présente thèse portent principalement sur le régime

$$\Omega \gtrsim \Omega_{c3}.$$

Nous avons confirmé l'apparition du cercle de vortex et du vortex géant pur dans le cas du piège plat ainsi que l'apparition du vortex géant et du profil gaussien dans le cas du piège quadratique plus quartique. Dans le cas du piège plat on conclut

$$\Omega_{c3} = \frac{2}{3\pi\varepsilon^2|\log\varepsilon|}(1 + o(1)).$$

Dans le cas du piège quadratique plus quartique (0.1.30), si k n'est plus fixé mais est pris comme paramètre, la situation est modifiée. On a par exemple vu à la Section 2.1.2 que l'équivalent du régime de rotation rapide peut être obtenu en diminuant k plutôt qu'en augmentant Ω .

Une autre possibilité est de prendre k petit et Ω proche de 1, ce qui mène au régime 'plus bas niveau de Landau' étudié à la Section 2.1. Dans ce cas nous avons donné des arguments en faveur de la présence d'un réseau de vortex déformé aux bords du condensat. Cette contribution se situerait plutôt dans le voisinage de Ω_{c2} selon la nomenclature précédente, mais l'étude ne portant pas sur la limite de Thomas-Fermi, on prendra ceci avec prudence.

3.2 Perspectives

Quelques prolongements de ce travail pourraient être les suivants :

- Le problème simplifié considéré dans la Section 2.2 est discutable physiquement, en particulier à cause de la condition de Neumann à laquelle il mène sur le bord du domaine. A minima, être capable de considérer une condition de Dirichlet est important car plus réaliste physiquement. Il serait également souhaitable de traiter le problème avec piège homogène (0.1.31). Une collaboration sur ce sujet est en cours avec Michele Correggi, Florian Pinsker et Jakob Yngason.
- Un autre de nos projets est la relaxation de l'hypothèse $\Omega_1 \ll 1$ de la Section 2.2.4. Dans ce cas il faut s'attendre à l'apparition d'un anneau inclus dans le condensat où des vortex forment un réseau. L'épaisseur de cet anneau tendrait probablement vers celle du condensat pour Ω_1 croissant, ce qui éluciderait la transition entre le réseau de vortex de [47] et le cercle de vortex décrit dans cette thèse.
- Dans toute cette thèse nous avons considéré des pièges à symétrie radiale (isotropes). Un piège anisotrope peut changer radicalement le comportement du condensat, voir par exemple [10] pour le cas d'un piège harmonique et les simulations numériques [80] pour un piège anharmonique. La raison principale pour laquelle nous nous sommes limités aux pièges isotropes est que dans les situations auxquelles nous nous sommes intéressés la force centrifuge joue un grand rôle et que celle-ci est à symétrie radiale. Si des pièges plus généraux sont considérés, il faut s'attendre à de nouveaux effets apportés par la compétition de la géométrie du piège et des forces centrifuges. Nous avons sur ce sujet un projet de collaboration avec Michele Correggi, Florian Pinsker et Jakob Yngason.

- Comme nous l'avons déjà mentionné, aucune preuve rigoureuse n'existe à ce jour de l'apparition des réseaux d'Abrikosov, que ce soit dans les supraconducteurs où les superfluides. Les résultats les plus complets concernent des problèmes de minimisation où on suppose a priori que les vortex sont disposés suivant un réseau régulier. On est alors capable de prouver que le meilleur réseau est le réseau hexagonal. Pour le problème d'Abrikosov, la preuve se déduit de l'article [113] (voir aussi [118] et [12]). Dans l'article récent [136] il est montré que le réseau hexagonal minimise parmi les réseaux réguliers une certaine énergie renormalisée apparaissant naturellement dans l'étude des supraconducteurs.

Résoudre ce problème dans toute sa généralité est probablement une question très difficile et correspond à un problème de théorie des nombres (voir l'introduction de [136] pour une discussion sur ce point). Un développement plus modeste dans cette direction serait l'adaptation des résultats récents de [136] au domaine des condensats de Bose-Einstein.

- Il pourrait être intéressant d'aller au delà de [37] et de donner plus d'éléments sur la validité de la théorie de Gross-Pitaevskii dans les régimes de forte rotation que nous avons considéré. Une difficulté majeure est qu'en dehors de la limite de Gross-Pitaevskii (0.1.22), les termes de l'énergie GP ont des ordres de grandeur différents, ce qui reflète la diversité des forces subies par les atomes. Evaluer toutes ces forces au niveau du problème à N corps est un défi conséquent.

Même si une validation complète est peut-être hors d'atteinte pour ces raisons, dégager plus explicitement les régimes où elle est au moins attendue serait souhaitable, voir ci-dessous.

- Dans certains régimes de paramètres la théorie de Gross-Pitaevskii n'est plus valable et il faut revenir au problème à N corps. Un exemple célèbre est l'effet Hall quantique. Les résultats mathématiques les plus avancés dans cette direction [95, 102] concernent des pièges harmoniques. Or, on peut s'attendre à ce que l'effet Hall joue un rôle dans les régimes considérés à la Section 2.1.1.

Deux directions sont alors envisageables : selon certains auteurs [114] le piègeage anharmonique facilite l'apparition des phases exotiques que l'on s'attend à observer pour un piège harmonique, dont le fameux état de Laughlin. On pourrait tenter de le confirmer mathématiquement. A l'opposé on pourrait profiter pleinement du piègeage anharmonique pour prédire l'existence de phases spécifiques, impossibles à créer avec un piège harmonique.

- Enfin, la recherche en physique dans ce domaine est plus active que jamais. En particulier, de nouveaux types de pièges sont développés (voir par exemple [77, 131]), et certaines expériences utilisant des réseaux optiques ont déjà inspiré des travaux mathématiques [15]. D'autres techniques que la mise en rotation sont proposées (par exemple dans [74], voir aussi la revue [48]) pour l'obtention de vortex, certaines ont déjà été mises en oeuvre [131, 148]. Il ne fait pas de doute que ce foisonnement pourra inspirer d'autres travaux mathématiques.

1

Annular Bose-Einstein condensates in the Lowest Landau Level

Ce chapitre reprend un article à paraître dans Applied Mathematics Research Express. Les résultats ont fait l'objet d'une publication résumée en collaboration avec Xavier Blanc [31]

1.1 Introduction

A Bose-Einstein condensate (BEC) is an object exhibiting quantum properties on a macroscopic scale. The effects of such phenomena as superfluidity can be observed experimentally on these systems, in particular the nucleation of quantized vortices in a rotating BEC (see for example the review [61] or [111, 108, 1, 124, 75, 54] for reports on original experiments). The most common description for a BEC is through a macroscopic wave function $\psi : \mathbb{R}^d \mapsto \mathbb{C}$ which modulus squared will give the matter density profile. Here $d = 2, 3$ and in this paper we will focus on a situation where it is physically justified to consider a two-dimensional BEC. For a condensate at equilibrium in the rotating frame, this wave function should minimize the following *Gross-Pitaevskii functional*

$$\mathcal{E}^{\text{GP}}[\psi] = \int_{\mathbb{R}^2} \left(\frac{1}{2} \left| \nabla \psi - i\Omega x^\perp \psi \right|^2 + \left(V(x) - \frac{\Omega^2}{2} |x|^2 \right) |\psi|^2 + \frac{G}{2} |\psi|^4 \right) dx, \quad (1.1.1)$$

under the mass constraint

$$\int_{\mathbb{R}^2} |\psi|^2 = 1. \quad (1.1.2)$$

Here $x = (x_1, x_2) \in \mathbb{R}^2$, $x^\perp = (-x_2, x_1)$, Ω is the angular velocity at which the condensate is rotated around the axis perpendicular to the plane x_1, x_2 and G is a dimensionless coefficient characterizing the strength of atomic interactions. An essential feature of the experimental set-ups is the trapping potential, denoted by V in (1.1.1). It is this potential that confines the atoms in a bounded region and prevents them from flying apart under the action of the centrifugal forces. For most experiments it takes the form (in the appropriate units that we have implicitly chosen in (1.1.1))

$$V(x) = \frac{1}{2} |x|^2. \quad (1.1.3)$$

Such a potential sets a limit to the rotational speed Ω that one can impose to the condensate. Indeed, one can see that with V chosen as above the second term in (1.1.1) is not

bounded below if $\Omega > 1$. Consequently the minimization problem makes no sense. Physically this corresponds to the fact that the centrifugal force overcomes the trapping force and drives the condensate out of the trap.

To prevent such a singular behavior in the limit $\Omega \rightarrow 1$, Fetter [61] proposed to use instead a potential of the form

$$V(x) = \frac{1}{2}|x|^2 + \frac{k}{4}|x|^4 \quad (1.1.4)$$

which, at least theoretically, allows for arbitrarily large rotation speeds Ω . Such a potential is a good approximation for those that have been used in the experiments ([35, 146]) and this is the one we shall consider in this paper.

An important experimental test for the superfluidity of a BEC is the observation of *quantized vortices*, which are described in Gross-Pitaevskii theory as zeros of the wave function carrying a positive *topological degree* or winding number (corresponding to the quantized phase circulation around the vortex). When a BEC is rotated, it will contain more and more vortices as the rotation speed is increased, until they become densely packed in the condensate when Ω approaches 1. Vortices repel each other, and to minimize this effect they will arrange on regular hexagonal lattices when $\Omega \rightarrow 1$, for both the harmonic trap (1.1.3) and the ‘anharmonic’ one (1.1.4). These Abrikosov lattices (named by analogy with the physics of type II superconductors, [3]) have been observed experimentally, see e.g. [1] for experiments with the trap (1.1.3) and [35] for experiments with the trap (1.1.4). It should be noted that all vortices in these lattices are simply quantized, i.e. they have degree 1.

A striking new feature of the trap (1.1.4) is the possibility to create annular condensates by increasing the rotation speed beyond the barrier that would be imposed by a weaker confinement. Indeed, for sufficiently large rotation speeds, the centrifugal forces will dip a hole in the center of the condensate. Interesting questions then arise about the vortex structure of the condensate. For example : does the hexagonal vortex lattice survive the formation of the hole? When the hole has just formed the answer is yes, see [64, 65, 91] for references in the physics literature and [47] for a mathematically rigorous treatment. At very large rotation speeds however, the vortices disappear from the bulk of the condensate, resulting in a so-called giant vortex phase, a situation that we do not address here and for which we refer to [64, 65, 91, 92] for physical references and [46, 127] for a mathematical analysis.

The locations of vortices in the central hole of low matter density can also be investigated. It is commonly known (see e.g. [64, 91, 90]) that there should be a phase circulation around this hole, indicating that vortices lie within it. In regions of high matter density, it is most favorable for the vortices to have degree 1, but it is far from obvious that it should also be the case in regions of low matter density. In particular, a question arising naturally [8, 64] is whether the vortices in the central hole of an annular Bose-Einstein condensate gather in a single multiply quantized vortex or form a pattern of simply quantized ones.

In this paper we address the case of a two-dimensional annular Bose-Einstein condensate rotating in a potential of the form (1.1.4). We will study the formation of the annular condensate by taking the limit $k \rightarrow 0$ of the problem, with the rotation speed Ω properly scaled so as to capture the regime where the central hole is created under the influence of the centrifugal force. In such a parameter regime it is physically justified to consider a simplified variational problem that we describe in the next section. The very particular

structure of the variational space considered allows to obtain interesting theoretical results. It also suggests a very natural numerical minimization method that we have used to confirm and extend our analytical results. The results we are going to present first appeared in a note intended for physicists [31] to which we refer for further physical discussion (see also [17]).

1.1.1 The GP energy and the LLL reduction

We consider the GP energy functional

$$\mathcal{E}^{\text{GP}}[\psi] = \int_{\mathbb{R}^2} \left(\frac{1}{2} |\nabla\psi - i\Omega x^\perp \psi|^2 + \left(\frac{1 - \Omega^2}{2} |x|^2 + \frac{k}{4} |x|^4 \right) |\psi|^2 + \frac{G}{2} |\psi|^4 \right) dx, \quad (1.1.5)$$

to be minimized under the mass constraint

$$\int_{\mathbb{R}^2} |\psi|^2 = 1. \quad (1.1.6)$$

The main idea of our analysis is to restrict the minimization of \mathcal{E}^{GP} to the first eigenspace of the Ginzburg-Landau operator $-(\nabla - i\Omega x^\perp)^2$, corresponding to the eigenvalue Ω . This is the Lowest Landau Level (LLL), introduced in the context of Bose-Einstein condensation by Ho [79].

The use of such a simplification has originally been introduced for condensates rotating in traps of the form (1.1.3), i.e. for the functional

$$\mathcal{E}^{\text{GP}}[\psi] = \int_{\mathbb{R}^2} \left(\frac{1}{2} |\nabla\psi - i\Omega x^\perp \psi|^2 + \frac{1 - \Omega^2}{2} |x|^2 |\psi|^2 + \frac{G}{2} |\psi|^4 \right) dx. \quad (1.1.7)$$

Restricting the minimization to the LLL is then justified in the limit $\Omega \rightarrow 1$. Indeed, the first term in the energy is at least equal to $\Omega \sim 1$ since Ω is the first eigenvalue of $-(\nabla - i\Omega x^\perp)^2$. On the other hand, the second term will obviously be small in the same limit. Thus, minimizing the sum of the second and the third term one also obtains something much smaller than Ω .

This indicates that the energy must stay close to Ω when $\Omega \rightarrow 1$. But the spectral gap between the first and the second eigenvalue of the Ginzburg-Landau operator is equal to Ω . To obtain an energy of order Ω the projection of a minimizer of the functional (1.1.7) on the excited energy levels of $-(\nabla - i\Omega x^\perp)^2$ must then be negligible.

This formal justification has been the basis of several works on harmonically trapped BECs in the regime $\Omega \rightarrow 1$, e.g. [8, 42, 147]. A rigorous justification has been given in [7]. For other mathematical developments on the functional (1.1.7) in the regime $\Omega \rightarrow 1$ (Landau regime), we refer to [6, 12].

The LLL consists (see e.g [107]) of functions of the form $\psi(z) = f(z)e^{-\Omega|z|^2/2}$ where z is the complex variable $x_1 + ix_2$ and f is a holomorphic function. For an LLL function ψ , the Gross-Pitaevskii energy reduces to (we denote $dz = dx_1 dx_2$)

$$\mathcal{E}^{\text{GP}}[\psi] = \Omega + \int_{\mathbb{C}} \left(\left(\frac{1 - \Omega^2}{2} |z|^2 + \frac{k}{4} |z|^4 \right) |\psi|^2 + \frac{G}{2} |\psi|^4 \right) dz.$$

In this paper we will rather consider functions of the form $\psi(z) = f(z)e^{-|z|^2/2}$ (thus we effectively restrict the minimization to the first eigenspace of $-(\nabla - ix^\perp)^2$) to have a

functional space independent of Ω . This approximation is harmless because, as we will see, our analysis is concerned with rotation speeds Ω close to 1. As for the energy, the difference is very small as showed by [7, Lemma 3.1] : with ψ of the form $\psi(z) = f(z)e^{-|z|^2/2}$ with f holomorphic we have

$$\mathcal{E}^{\text{GP}}[\psi] = \Omega + \mathcal{E}^{\text{LLL}}[\psi] \quad (1.1.8)$$

$$\mathcal{E}^{\text{LLL}}[\psi] = \int_{\mathbb{C}} \left(\left((1 - \Omega) |z|^2 + \frac{k}{4} |z|^4 \right) |\psi|^2 + \frac{G}{2} |\psi|^4 \right) dz, \quad (1.1.9)$$

so that we will minimize the energy \mathcal{E}^{LLL} amongst functions of the form $\psi(z) = f(z)e^{-|z|^2/2}$ with f holomorphic, under the mass constraint $\int |\psi|^2 = 1$. For this purpose we will use the mathematical framework introduced in [11, 12]. Note that the validity of such a reduction could be investigated with the tools of [7] (as well as the validity of the reduction to a two-dimensional model), but in this paper we shall be concerned only with the study of the reduced energy functional \mathcal{E}^{LLL} . We will provide in Subsection 1.1.3 an a posteriori criterion for the validity of the reduction.

We now describe the parameter regime that we shall consider. Let us define a small parameter

$$\varepsilon = k^{1/3} \quad (1.1.10)$$

corresponding to a small anharmonicity regime and study the asymptotics of the problem as $\varepsilon \rightarrow 0$. We take Ω satisfying

$$\Omega = 1 + \beta k^{2/3} \quad (1.1.11)$$

and will consider β and G as fixed. This choice will lead to a functional with bounded coefficients (see (1.1.20) below). In this regime the radius of the condensate is not bounded as a function of ε . We thus rescale distances by making the change of variables

$$\phi(z) = \varepsilon^{-1/2} \psi(\varepsilon^{-1/2} z) \quad (1.1.12)$$

and for every ϕ we define the function

$$f(z) = \phi(z) e^{|z|^2/2\varepsilon}. \quad (1.1.13)$$

By definition of the LLL, f belongs to the Fock-Bargmann space

$$\mathfrak{F}_\varepsilon := \left\{ f \text{ holomorphic, } \int_{\mathbb{C}} |f|^2 e^{-|z|^2/\varepsilon} dz < \infty \right\}. \quad (1.1.14)$$

The space \mathfrak{F}_ε is a Hilbert space for the scalar product

$$\langle f, g \rangle := \int_{\mathbb{C}} \overline{f(z)} g(z) e^{-|z|^2/\varepsilon} dz. \quad (1.1.15)$$

The point of introducing such a space is that the orthogonal projector from $L^2(\mathbb{C}, e^{-|z|^2/\varepsilon} dz)$ onto \mathfrak{F}_ε is explicitly known [110, 66] and called the Szegő projector :

$$\Pi_\varepsilon(g)(z) = \frac{1}{\pi\varepsilon} \int_{\mathbb{C}} e^{z\bar{z}'/\varepsilon} e^{-|z'|^2/\varepsilon} g(z') dz'. \quad (1.1.16)$$

We will also use the spaces

$$\mathfrak{F}_\varepsilon^s := \left\{ f \text{ holomorphic, } \int_{\mathbb{C}} (1 + |z|^2)^s |f|^2 e^{-|z|^2/\varepsilon} dz < \infty \right\} \quad (1.1.17)$$

that we equip with the norms

$$\|f\|_{\mathfrak{F}_\varepsilon^s} := \left(\int_{\mathbb{C}} (1 + |z|^2)^s |f|^2 e^{-|z|^2/\varepsilon} dz \right)^{1/2}. \quad (1.1.18)$$

After the change of scale (1.1.12) and the change of function (1.1.13), the energy becomes

$$\mathcal{E}^{\text{LLL}}[\psi] = \varepsilon \mathcal{F}^{\text{LLL}}[f] \quad (1.1.19)$$

with

$$\mathcal{F}^{\text{LLL}}[f] = \int_{\mathbb{C}} \left(\left(-\beta|z|^2 + \frac{1}{4}|z|^4 \right) |f|^2 e^{-|z|^2/\varepsilon} + \frac{G}{2} |f|^4 e^{-2|z|^2/\varepsilon} \right) dz \quad (1.1.20)$$

and we thus have to minimize (1.1.20) under the mass constraint

$$\|f\|_{\mathfrak{F}_\varepsilon} = 1. \quad (1.1.21)$$

It is this minimization problem that we will analyze in the sequel, namely we look at the problem

$$F^{\text{LLL}} := \inf \{ \mathcal{F}^{\text{LLL}}[f] \mid f \in \mathfrak{F}_\varepsilon^1, \|f\|_{\mathfrak{F}_\varepsilon} = 1 \}. \quad (1.1.22)$$

In particular we will see that this problem admits a solution, i.e. the infimum above is actually a minimum. We are able, using the Szegő projector (1.1.16) to derive the Euler-Lagrange equation satisfied by any solution of the minimization problem. Using this equation and the very particular structure of the Fock-Bargmann space (in particular the constraint that f has to be analytic) we are able to derive that any minimizer must have an infinite number of zeros (vortices) if ε is small enough. We also construct almost critical points, functions that are solutions to the Euler-Lagrange equation up to a small remainder term. Evaluating the energy of these almost critical points yields an upper bound to the energy that we believe is optimal although we are not able to prove it. This upper bound will be useful to discuss the validity of the LLL reduction in Subsection 1.1.3 below.

1.1.2 Main Analytical Results

Let us start by showing that the problem we want to consider is actually well-posed :

Theorem 1.1.1 (Well-posedness and Euler-Lagrange equation).

For any $\varepsilon > 0$ fixed, the problem (1.1.22) admits a solution in $\mathfrak{F}_\varepsilon^2$. Any minimizer is a solution to

$$-\beta M_\varepsilon f + \frac{1}{4} (M_\varepsilon^2 f + \varepsilon M_\varepsilon f) + G \Pi_\varepsilon (e^{-|z|^2/\varepsilon} |f|^2 f) = \mu f \quad (1.1.23)$$

where μ is the Lagrange multiplier coming from the mass constraint and M_ε is the operator defined by

$$M_\varepsilon = \varepsilon \partial_z z. \quad (1.1.24)$$

Alternatively, the Euler-Lagrange equation (1.1.23) may be written

$$\left(-\beta + \frac{1}{4} \varepsilon \right) \Pi_\varepsilon (|z|^2 f) + \frac{1}{4} \Pi_\varepsilon (|z|^2 \Pi_\varepsilon (|z|^2 f)) + G \Pi_\varepsilon (e^{-|z|^2/\varepsilon} |f|^2 f) = \mu f \quad (1.1.25)$$

or

$$-\beta M_\varepsilon f + \frac{1}{4} (M_\varepsilon^2 f + \varepsilon M_\varepsilon f) + \frac{G}{2} \bar{f}(\varepsilon \partial_z)[f^2(z/2)] = \mu f, \quad (1.1.26)$$

where the operator $\bar{f}(\varepsilon \partial_z)$ is defined as

$$\bar{f}(\varepsilon \partial_z)[g] := \sum_{k=0}^{+\infty} \bar{a}_k (\varepsilon \partial_z)^k g$$

if $f(z) = \sum_{k=0}^{+\infty} a_k z^k$.

Next we show that a minimizer of (1.1.22) cannot have a finite number of zeroes if ε is small enough. In particular, the minimum cannot be achieved by a polynomial.

Theorem 1.1.2 (Infinite number of zeroes).

If ε is sufficiently small, any solution to the minimization problem (1.1.22) has an infinite number of zeroes.

We will discuss this result in further details in Subsection 1.1.3 and now describe the procedure we follow to construct approximate solutions to the Euler-Lagrange equation (1.1.25).

As is the case for harmonically trapped condensates in fast rotation, we expect that in the range of parameters we explore the scales of the problem will decouple. Namely we expect any minimizer of (1.1.20) to be of the form αu where u varies on the scale of the vortex pattern (which is small compared to the size of the condensate) and α is a slow varying profile giving the general shape of the condensate. The kind of profile α we have in mind is a Thomas-Fermi distribution. More precisely, α should look like a minimizer of the energy (1.1.20) without the holomorphy constraint. Such a function has compact support and can thus not be in the LLL, but one can approach it by an LLL function which is an almost critical point for the energy (1.1.20). We proceed as follow. Let us introduce

$$u_\tau(z) = e^{-|z|^2/2\varepsilon} f_\tau(z), \quad f_\tau(z) = e^{z^2/2\varepsilon} \Theta\left(\sqrt{\frac{\tau_I}{\pi\varepsilon}} z, \tau\right) \quad (1.1.27)$$

where $\tau = \tau_R + i\tau_I$ is any complex number and

$$\Theta(v, \tau) = \frac{1}{i} \sum_{n=-\infty}^{+\infty} (-1)^n e^{i\pi\tau(n+1/2)^2} e^{(2n+1)\pi i v} \quad (1.1.28)$$

is the Jacobi Θ function (*citer Nier, Seminaire X et donner le numero de la fonction theta*). It has the property (see [40] for more details)

$$\Theta(v + k + l\tau, \tau) = (-1)^{k+l} e^{-2i\pi l v} e^{-i\pi l \tau} \Theta(v, \tau). \quad (1.1.29)$$

The interest of introducing such functions is twofold. Firstly it is known [40] that any function v whose modulus is periodic over the lattice $\sqrt{\frac{\pi\varepsilon}{\tau_I}}\mathbb{Z} \oplus \sqrt{\frac{\pi\varepsilon}{\tau_I}}\mathbb{Z}\tau$, vanishes exactly on the points of the lattice with simple zeroes and such that $g = v e^{|z|^2/2\varepsilon}$ is holomorphic must be proportional to u_τ . Secondly, the function f_τ is a solution to the Abrikosov problem (see e.g. [3] and [12, Section 4] for a detailed discussion)

$$\Pi_\varepsilon(|f_\tau|^2 e^{-|z|^2/\varepsilon} f_\tau) = \lambda_\tau f_\tau, \quad \text{with } \lambda_\tau = \int |u_\tau|^2 b(\tau), \quad (1.1.30)$$

and

$$b(\tau) = \frac{\overline{f} |u_\tau|^4}{(\overline{f} |u_\tau|^2)^2} = \sum_{k,l \in \mathbb{Z}} e^{-\pi |k\tau - l|^2 / \tau_I}. \quad (1.1.31)$$

Here we denote $\overline{f} v$ the average of a periodic function v . Equation (1.1.30) is similar to (1.1.25) without the potential term and with $\mu = \lambda_\tau$ so that one can expect to obtain a solution of (1.1.25) by a slight modification of f_τ . We refer to [11, 12] and references therein for details on the quantity $b(\tau)$. Let us just mention that it is minimum ($b(\tau) \sim 1.16$) for $\tau = e^{2i\pi/3}$, which corresponds to a hexagonal lattice. From now on we take

$$\tau = e^{2i\pi/3}$$

and denote $b = b(e^{2i\pi/3})$.

To obtain our approximate solutions we first multiply f_τ by a slow varying profile α with compact support and $\int_{\mathbb{R}^2} |\alpha|^2 = 1$. This trial state would correspond to having a regular hexagonal lattice of vortices in the bulk of the condensate (that is, the support of α). To obtain an admissible test function for the problem (1.1.22) we project αf_τ onto the Fock-Bargmann space and normalize it :

$$f_{\alpha,\tau} = \frac{\Pi_\varepsilon(\alpha f_\tau)}{\|\Pi_\varepsilon(\alpha f_\tau)\|_{\mathfrak{F}_\varepsilon}}. \quad (1.1.32)$$

Accordingly

$$u_{\alpha,\tau} = f_{\alpha,\tau} e^{-|z|^2/2\varepsilon}. \quad (1.1.33)$$

As for the choice of the appropriate α , it turns out to be related to the following Thomas-Fermi energy functional :

$$\mathcal{E}_b^{\text{TF}}[\rho] = \int_{\mathbb{C}} \left(\left(-\beta |z|^2 + \frac{1}{4} |z|^4 \right) \rho + \frac{bG}{2} \rho^2 \right) dz. \quad (1.1.34)$$

The associated minimization problem is

$$E_b^{\text{TF}} := \inf \left\{ \mathcal{E}_b^{\text{TF}}[\rho], \rho \geq 0, \int_{\mathbb{R}^2} |z|^2 \rho < +\infty, \int_{\mathbb{R}^2} \rho = 1 \right\}. \quad (1.1.35)$$

Its solution is unique and given as

$$\rho_b^{\text{TF}} := \max \left(\frac{\mu_b^{\text{TF}} + \beta |z|^2 - \frac{1}{4} |z|^4}{bG}, 0 \right) \quad (1.1.36)$$

where the chemical potential μ_b^{TF} appears as a Lagrange multiplier in the equation associated to the minimization (1.1.35) and is thus fixed by the constraint $\int_{\mathbb{R}^2} \rho_b^{\text{TF}} = 1$. Note that there exists a critical value β_c such that the support of such a function becomes an annulus when $\beta > \beta_c$ (see the discussion in Section 1.1.3). This is in contrast with the situation where $k = 0$ (purely quadratic trap) where the support of the TF minimizer is always a disc.

We have the following result :

Theorem 1.1.3 (Almost critical points).

Let α be a profile satisfying

$$|\alpha(z)|^2 = \rho_b^{\text{TF}} \quad (1.1.37)$$

and $f_{\alpha,\tau}$ be the associated function via formula (1.1.32). There holds

$$\left(-\beta + \frac{1}{4}\varepsilon\right)\Pi_\varepsilon(|z|^2 f_{\alpha,\tau}) + \frac{1}{4}\Pi_\varepsilon(|z|^2 \Pi_\varepsilon(|z|^2 f_{\alpha,\tau})) + G\Pi_\varepsilon(e^{-|z|^2/\varepsilon}|f_{\alpha,\tau}|^2 f_{\alpha,\tau}) = \mu_b^{\text{TF}} f_{\alpha,\tau} + R_\varepsilon \quad (1.1.38)$$

where $\|R_\varepsilon\|_{\mathfrak{F}_\varepsilon} \leq C\varepsilon^{1/4}$. Moreover

$$\mathcal{F}^{\text{LLL}}[f_{\alpha,\tau}] = \int_{\mathbb{C}} \left(\left(-\beta|z|^2 + \frac{1}{4}|z|^4 \right) |\alpha|^2 + \frac{bG}{2} |\alpha|^4 \right) dz + O(\varepsilon^{1/4}) \quad (1.1.39)$$

and thus

$$F^{\text{LLL}} \leq \mathcal{F}^{\text{LLL}}[f_{\alpha,\tau}] = E_b^{\text{TF}} + O(\varepsilon^{1/4}). \quad (1.1.40)$$

1.1.3 Discussion

In this subsection we discuss some of the physical insights that one can obtain from our theorems, and some of the questions that can not be answered analytically, for which we will rely on numerical simulations.

Let us first comment a little bit more the ideas behind the construction leading to Theorem 1.1.3. It is known from the experiments that in the regime we consider, the condensate should contain a very large number of vortices. Theorem 1.1.2 also suggests to use a trial function containing as many vortices as possible, although it does not give information on their locations. On the other hand the trapping potential will force the condensate to live essentially in a bounded region, the complement of which will be a region of very low matter density. It is natural to think that, at least in the bulk, the vortices will form a regular hexagonal lattice. This is widely observed in experiments and numerical simulations.

Combining these two requirements, we arrive at a trial function of the form $c\alpha f_\tau$ where f_τ is defined in (1.1.27), α is a function with compact support and c a normalization factor. Such a function is of course not in the Fock-Bargmann space : it has compact support and thus cannot be holomorphic. To obtain an admissible trial state we project it onto \mathfrak{F}_ε using the Szegő projector. Computing the energy of such a trial function consists of essentially two steps (see Subsection 1.2.3 for details). First, using a remarkable result from [12] we have (see Lemma 1.2.1 below for a precise statement)

$$\Pi_\varepsilon(\alpha f_\tau) \sim \alpha f_\tau$$

when $\varepsilon \rightarrow 0$. Thus the projection onto the Fock space has no effect on the energy to leading order and we obtain

$$\begin{aligned} \mathcal{F}^{\text{LLL}}[f_{\alpha,\tau}] \sim & \|\Pi_\varepsilon(\alpha f_\tau)\|_{\mathfrak{F}_\varepsilon}^{-2} \int_{\mathbb{C}} \left(-\beta|z|^2 + \frac{1}{4}|z|^4 \right) |\alpha|^2 |f_\tau|^2 e^{-|z|^2/\varepsilon} dz \\ & + \|\Pi_\varepsilon(\alpha f_\tau)\|_{\mathfrak{F}_\varepsilon}^{-4} \frac{G}{2} \int_{\mathbb{C}} |\alpha|^4 |f_\tau|^4 e^{-2|z|^2/\varepsilon} dz. \end{aligned}$$

Now, $|f_\tau|e^{-|z|^2/2\varepsilon} = |u_\tau|$ is periodic over a lattice whose period is very small ($\propto \varepsilon^{1/2}$), whereas α is chosen with a support of fixed size. Therefore one should expect a homogenization effect leading to

$$\mathcal{F}^{\text{LLL}}[f_{\alpha,\tau}] \sim \|\Pi_\varepsilon(\alpha f_\tau)\|_{\mathfrak{F}_\varepsilon}^{-2} \int |u_\tau|^2 \int_{\mathbb{C}} \left(-\beta|z|^2 + \frac{1}{4}|z|^4 \right) |\alpha|^2 dz + \|\Pi_\varepsilon(\alpha f_\tau)\|_{\mathfrak{F}_\varepsilon}^{-4} \int |u_\tau|^4 \frac{G}{2} \int_{\mathbb{C}} |\alpha|^4 dz.$$

Computing the norm of $\Pi_\varepsilon(\alpha f_\tau)$ uses the same ideas and leads to

$$\|\Pi_\varepsilon(\alpha f_\tau)\|_{\mathfrak{F}_\varepsilon} \sim \left(\int |u_\tau|^2 \right)^{1/2}.$$

Finally, using (1.1.31),

$$\mathcal{F}^{\text{LLL}}[f_{\alpha,\tau}] \sim \int_{\mathbb{C}} \left(\left(-\beta|z|^2 + \frac{1}{4}|z|^4 \right) |\alpha|^2 + \frac{bG}{2} |\alpha|^4 \right) dz$$

and one immediately sees that the optimal choice for α is the one we made in (1.1.37). One should note that the only contribution of the vortex structure to the leading order of the energy is through the coefficient b , which depends on the type of lattice we choose and is minimum for the hexagonal lattice.

We do believe that this construction is optimal, but a proof of a lower bound matching (1.1.40) would probably require to show that the vortices of a true minimizer of the energy are located close to the sites of a regular lattice. This is a difficult question, linked to the crystallization problem, and remains a challenging open problem in more than one context, see [12, 135] and references therein.

We have however the easy but non-optimal lower bound

$$E_1^{\text{TF}} \leq F^{\text{LLL}} \tag{1.1.41}$$

where E_1^{TF} is defined by taking $b = 1$ in (1.1.35). This follows by minimizing $\mathcal{F}^{\text{LLL}}[f]$ with respect to $f e^{-|z|^2/2\varepsilon}$, dropping the constraint that this function should be in the LLL. As b is a constant, this lower bound confirms that E_b^{TF} gives at least the order of magnitude of the energy. The gap between the upper and the lower bound lies in the coefficient b that takes into account the density modification due to the presence of a large number of vortices. Filling this gap would be a first step towards a rigorous proof of the onset of the Abrikosov lattice in this regime.

Although we have no rigorous proof that the trial state (1.1.32) accurately describes the true state of affairs, it is useful for a physical discussion to give some details on this function. The analysis of a profile such as (1.1.36) has already been carried out in [64, 8], so we only adapt and summarize their results.

The critical rotation speed for the condensate to develop a central hole is

$$\Omega_c = 1 + \left(\frac{3k^2 bG}{8\pi} \right)^{1/3}. \tag{1.1.42}$$

For subcritical velocities, the behavior of the condensate is not qualitatively different from that of a harmonically trapped condensate, so we focus on velocities $\Omega \geq \Omega_c$. In our scaling

(see (1.1.11)) this is equivalent to the requirement $\beta \geq (3bG)^{1/3}(8\pi)^{-1/3}$. Then the inner and outer radii of the condensate R_{\pm} are given by the relations

$$R_+^2 + R_-^2 = 4\beta, \quad R_+^2 - R_-^2 = (24bG)^{1/3}, \quad (1.1.43)$$

the chemical potential is

$$\mu_b^{\text{TF}} = \left(\frac{3bG}{8\pi}\right)^{2/3} - \beta^2 \quad (1.1.44)$$

and the energy is

$$E_b^{\text{TF}} = \frac{3}{5} \left(\frac{3bG}{8\pi}\right)^{2/3} - \beta^2. \quad (1.1.45)$$

In the case of a disc-shaped condensate, the order of magnitudes of the corresponding quantities is the same as above.

This information allows to check a posteriori the validity of our analysis. First, the LLL reduction should be valid if the energy F^{LLL} is much smaller than the spectral gap between the Lowest Landau Level and the first excited level. In our scaling, this gap is $\propto \varepsilon^{-1} = k^{-1/3}$, and thus from (1.1.40) and (1.1.45) we deduce that the reduction is valid if

$$G \ll k^{-1/2}, \quad \beta \ll k^{-1/6}. \quad (1.1.46)$$

One could probably show rigorously that the LLL reduction is valid under these conditions, working in the spirit of [7]. Note that we have considered β and G as fixed in this paper so a rigorous proof would require to track the dependence on these parameters of our remainder terms. At least one condition will appear for sure if one wants to use our upper bound : we have used a (elementary) homogeneization argument which requires the number of vortices in the support of ρ_b^{TF} to be large compared to 1. Given that the vortices lie on a lattice of period $\propto \varepsilon^{1/2} = k^{1/6}$ and using (1.1.43) we arrive at the condition

$$k \ll G. \quad (1.1.47)$$

It is interesting to remark (tracking the dependence on β and G in the proof of Theorem 1.1.2) that under the above condition the number of vortices for the true minimizer should be infinite. We also note that, as we require both $k \ll G$ and $G \ll \frac{1}{\sqrt{k}}$, necessarily $k \ll 1$, which justifies our study of a small anharmonicity regime.

Finally, interesting questions arise about the vortex pattern displayed by a minimizer of the energy. If one believes that a minimizer is close to our trial state (1.1.32), then the property

$$\Pi_{\varepsilon}(\alpha f_{\tau}) \sim \alpha f_{\tau}$$

that holds true in L^{∞} (amongst other topologies, see Lemma 1.2.1) indicates that the vortices should lie close to the sites of an hexagonal lattice in the region of significant matter density, that is the support of ρ_b^{TF} . Very little information however can be obtained about vortices lying in the region of low density.

These ‘invisible vortices’ have nevertheless a contribution : as shown in [6, 8], they help create the average Thomas-Fermi density profile. Indeed, if the vortex pattern was regular in the whole complex plane, one would obtain a gaussian density profile. More precisely it is to be expected that the regular vortex lattice will be distorted close to the edge of the

condensate, resulting in invisible vortices that will help shape the condensate. An interesting question (see e.g. [8, 64, 90]) is then, when Ω is above Ω_c and the condensate has an annular shape, what is the vortex distribution inside the hole? In particular : do vortices gather in a central multi-quantized vortex?

A nice feature of the LLL regime is that the condensate is completely determined by the location of its vortices, and one can thus minimize the energy numerically by varying the locations of the zeroes of the wave-function. This gives a direct access to the optimal vortex pattern and allows to spot precisely the ‘invisible vortices’. A direct minimization of the Gross-Pitaevskii energy would not give such an information because one should a posteriori look for density dips in a region where the density is already very small.

We have adapted the numerical method of [8] to the present setting and focused on the case where the condensate is annular. We refer to Section 1.3 for details. In particular we find invisible vortices both outside the outer edge of the condensate and in the central hole of low density, with multiply quantized vortices appearing at the center of the trap. We also find a good agreement between the optimal value of the energy numerically computed and the upper bound (1.1.40), thus giving another argument for the optimality of our construction.

The rest of the paper is organized as follows : in Section 1.2 we prove our analytical results, the proof of each theorem occupying a subsection. In Section 1.3 we describe our numerical method and present our results.

1.2 Proofs

1.2.1 The Minimization Problem

We prove Theorem 1.1.1. In this subsection, all parameters are considered as fixed.

Proof of Theorem 1.1.1. We first recall (see (1.1.41)) that the energy functional \mathcal{F}^{LLL} is indeed bounded below in the minimization domain we have chosen (1.1.22).

Our energy is defined only for $f \in \mathfrak{F}_\varepsilon^2$, so we take the convention that $\mathcal{F}^{\text{LLL}}[f] = +\infty$ if $f \notin \mathcal{F}_\varepsilon^2$. Let $(f_n)_{n \in \mathbb{N}}$ be a minimizing sequence for (1.1.22) and $u_n = f_n e^{-|z|^2/\varepsilon}$. We note that

$$\int_{\mathbb{C}} |z|^2 |u_n|^2 \leq \left(\int_{\mathbb{C}} |u_n|^2 \right)^{1/2} \left(\int_{\mathbb{C}} |z|^4 |u_n|^2 \right)^{1/2} = \left(\int_{\mathbb{C}} |z|^4 |u_n|^2 \right)^{1/2}$$

because of the mass constraint. We deduce that

$$\mathcal{F}^{\text{LLL}}[f_n] \geq \frac{1}{4} \int_{\mathbb{C}} |z|^4 |u_n|^2 dz - |\beta| \left(\int_{\mathbb{C}} |z|^4 |u_n|^2 dz \right)^{1/2}$$

and thus the sequence $(|z|^2 u_n)_n$ is bounded in L^2 whatever the sign of β . This implies that $(f_n)_n$ is bounded in $\mathfrak{F}_\varepsilon^2$. In [12] it is proved by using the Bargmann transform that $\mathfrak{F}_\varepsilon^1$ is compactly embedded in \mathfrak{F}_ε . Similarly $\mathfrak{F}_\varepsilon^2$ is compactly embedded² in $\mathfrak{F}_\varepsilon^1$. We thus have (possibly after extraction of a subsequence) the convergence of f_n to some limit analytic function f , strongly in $\mathfrak{F}_\varepsilon^1$, strongly in \mathfrak{F}_ε and weakly in $\mathfrak{F}_\varepsilon^2$. We deduce that

$$\inf_n \int_{\mathbb{C}} \left(-\beta |z|^2 + \frac{1}{4} |z|^4 \right) |f_n|^2 e^{-|z|^2/\varepsilon} \geq \int_{\mathbb{C}} \left(-\beta |z|^2 + \frac{1}{4} |z|^4 \right) |f|^2 e^{-|z|^2/\varepsilon}.$$

2. The Bargmann transform maps $\mathfrak{F}_\varepsilon^s$ onto the space $(1 - \partial_x^2 + x^2)^{-s} L^2(\mathbb{R})$.

and

$$\|f\|_{\mathfrak{F}_\varepsilon} = 1.$$

On the other hand, denoting

$$\|f\|_{A_\varepsilon^p} := \left(\frac{1}{\pi h} \int_{\mathbb{C}} |f(z)|^p e^{-|z|^2/\varepsilon} dz \right)^{1/p} \quad (1.2.1)$$

it is proved in [38, Theorem 4] (see also [12, Section 2.2]) that for any entire function f

$$\|f(e^{-t}\cdot)\|_{A_\varepsilon^q} \leq \|f\|_{A_\varepsilon^p}, \quad 0 < p < q, \quad e^{-t} \leq \frac{p^{1/2}}{q^{1/2}}. \quad (1.2.2)$$

where $f(e^{-t}\cdot)$ is the function associating $f(e^{-t}z)$ to z . Taking $t = \frac{\log 2}{2}$, $p = 2$ and $q = 4$ we deduce that for any entire function f

$$\left(\int_{\mathbb{C}} |f|^4 e^{-2|z|^2/\varepsilon} dz \right)^{1/4} \leq C \|f\|_{\mathfrak{F}_\varepsilon}.$$

Thus our sequence $(f_n)_n$ is also bounded in A_ε^4 and we can assume (after a possible further extraction) that the convergence of f_n to f holds also in the weak topology of A_ε^4 . In particular, by convexity, there holds

$$\inf_n \int_{\mathbb{C}} |f_n|^4 e^{-2|z|^2/\varepsilon} dz \geq \int_{\mathbb{C}} |f|^4 e^{-2|z|^2/\varepsilon} dz$$

which concludes the proof that

$$\mathcal{F}^{\text{LLL}}[f] \leq \inf_n \mathcal{F}^{\text{LLL}}[f_n]$$

and thus the proof that f minimizes \mathcal{F}^{LLL} under the mass constraint.

We turn to the derivation of the Euler-Lagrange equation. The weak derivative of the functional \mathcal{F}^{LLL} at f along g is given by

$$D\mathcal{F}^{\text{LLL}}(f) \cdot g = \int_{\mathbb{C}} \left(\left(-\beta|z|^2 + \frac{1}{4}|z|^4 \right) \bar{f} g e^{-|z|^2/\varepsilon} + \frac{G}{2} |f|^2 \bar{f} g e^{-2|z|^2/\varepsilon} \right) dz.$$

Using an integration by parts and $\partial_{\bar{z}} f = \partial_{\bar{z}} g = 0$ on the first term and the fact that $\Pi_\varepsilon(g) = g$ on the second term, we obtain (1.1.23).

Equation (1.1.26) is obtained from (1.1.23) exactly as in [12, Proposition 3.2] with some algebra on the non-linear term. To get (1.1.25) we use an integration by parts to show that $M_\varepsilon f = \Pi_\varepsilon(|z|^2 f)$, then $M_\varepsilon^2 f = \Pi_\varepsilon(|z|^2 \Pi_\varepsilon(|z|^2 f))$ and we get the result.

□

1.2.2 Infinite Number of Zeros

We now prove that any minimizer of (1.1.22) has an infinite number of zeroes. The argument is by contradiction and in two steps, using the two equivalent forms of the Euler-Lagrange equation (1.1.23) and (1.1.26).

Proof of Theorem 1.1.2.

Step 1. Suppose f has a finite number of zeroes. Then one may write $f(z) = P(z)e^{\varphi(z)}$ where P is a polynomial and φ is a holomorphic function. Now $f \in \mathfrak{F}_\varepsilon$ and the condition $\int_{\mathbb{C}} |f|^2 e^{-|z|^2/\varepsilon} dz < \infty$ imply that $Re(\varphi(z)) \leq |z|^2/(2\varepsilon)$. It is well-known (see [33] for example) that a holomorphic function can satisfy this condition only if it is a polynomial of degree less than 2. Therefore we know that

$$f(z) = P(z)e^{\alpha_1 z + \alpha_2 z^2} \quad (1.2.3)$$

and the integrability condition on f implies $\alpha_2 \leq 1/(2\varepsilon)$. Injecting (1.2.3) in (1.1.23) and comparing the exponential growth of the different terms of (1.1.23) as in [12] yields $\alpha_1 = \alpha_2 = 0$. So, if f has a finite number of zeroes, it is a polynomial.

Step 2. Now, suppose f is a polynomial of degree n

$$f(z) = \sum_{k=0}^n a_k z^k$$

and inject this in (1.1.26). The term

$$-\beta M_\varepsilon f + \frac{1}{4} (M_\varepsilon^2 f + \varepsilon M_\varepsilon f)$$

is a polynomial of degree n , therefore (1.1.26) implies that the term

$$\frac{G}{2} \bar{f}(\varepsilon \partial_z) [f^2(z/2)] = \sum_{k=0}^n (\varepsilon \partial_z)^k [f^2(z/2)]$$

is also of degree n . But $(\varepsilon \partial_z)^k [f^2(z/2)]$ is of degree $2n - k$, so that it must be that $a_k = 0$ for any $k < n$. Then f is of the form

$$f(z) = a_n z^n,$$

with

$$|a_n|^2 = (\pi \varepsilon^{n+1} n!)^{-1}$$

because of the normalization of f in \mathfrak{F}_ε . Injecting this a last time in (1.1.26) yields

$$-\beta(n+1)\varepsilon + \frac{G}{2} \frac{(2n)!}{\pi \varepsilon 2^{2n+1} (n!)^2} - \mu + \frac{1}{4} (2\varepsilon^2 + n\varepsilon^2 + n(n-1)\varepsilon^2) = 0.$$

Using the improved Stirling formula [126] we obtain a condition on n :

$$\mu + \beta\varepsilon \geq -\beta n\varepsilon + \frac{Ge^{-1/12}}{2\pi\varepsilon\sqrt{n}} + \frac{\varepsilon^2}{2} + \frac{n^2\varepsilon^2}{4} + \frac{3n\varepsilon^2}{4}. \quad (1.2.4)$$

We now need to bound the chemical potential μ : taking the \mathcal{F}_ε -scalar product of each side of (1.1.23) with f yields

$$\mu \leq 2\mathcal{F}^{\text{LLL}}[f] + \beta^2 - \frac{\beta\varepsilon}{2} + \frac{\varepsilon^2}{16}. \quad (1.2.5)$$

Here we used the fact that the spectrum of $-\beta M_\varepsilon + \frac{1}{4}(M_\varepsilon^2 + \varepsilon M_\varepsilon)$ is bounded below by $-\beta^2 + \frac{\beta\varepsilon}{2} - \frac{\varepsilon^2}{16}$. This follows by noting that the spectrum of M_ε is constituted by the eigenvalues $(n+1)h$ with (non-normalized) eigenvectors z^n . We use the upper bound on $\mathcal{F}^{\text{LLL}}[f]$ of Theorem 1.1.3 (see Subsection 1.2.3 for the proof) to deduce

$$\frac{6}{5} \left(\frac{3bG}{8\pi} \right)^{2/3} - \beta^2 - \frac{\beta\varepsilon}{2} - \frac{7\varepsilon^2}{16} \geq -\beta n\varepsilon + \frac{Ge^{-1/12}}{2\pi\varepsilon\sqrt{n}} + \frac{n^2\varepsilon^2}{4} + \frac{3n\varepsilon^2}{4}. \quad (1.2.6)$$

Minimizing the right-hand side of (1.2.6) with respect to n (taken as a continuous variable as it should be very large when ε is small) for fixed ε yields

$$n \sim \left(\frac{Ge^{-1/12}}{2\pi} \right)^{2/5} \varepsilon^{-6/5} \quad (1.2.7)$$

and we deduce

$$G^{2/3} \geq C \left(G^{4/5} \varepsilon^{-2/5} - \beta G^{2/5} \varepsilon^{-1/5} + \beta^2 \right)$$

which is a contradiction if ε is small enough. \square

1.2.3 Construction of Critical Points

The proof of Theorem 1.1.3 requires the following result taken from [12], that we quote for convenience. Here we abuse notations by taking the $\mathfrak{F}_\varepsilon^s$ norm of non-holomorphic functions.

Lemma 1.2.1 ([12] **Estimates for $f_{\alpha,\tau}$**).

Let $\gamma \in C^{0,1/2}(\mathbb{C}, \mathbb{C})$ have compact support in B_{R_0} . If $g \in \mathfrak{F}_\varepsilon^s$ for some s , then $\Pi_\varepsilon(\gamma g) \in \mathfrak{F}_\varepsilon^{s'}$ for any $s' \in \mathbb{R}$ and

$$\|\Pi_\varepsilon(\gamma g) - \gamma g\|_{\mathfrak{F}_\varepsilon^{s'}} \leq C_{s,s',R_0} \|\gamma\|_{C^{0,1/2}} \|g\|_{\mathfrak{F}_\varepsilon^s} \varepsilon^{1/4}. \quad (1.2.8)$$

If $v = e^{-|z|^2/\varepsilon} g \in L^p(B_{R_0})$ for $1 \leq p \leq +\infty$ then

$$\left\| (\Pi_\varepsilon(\gamma g) - \gamma g) e^{-|z|^2/\varepsilon} \right\|_{L^p} \leq C_{p,R_0} \|\gamma\|_{C^{0,1/2}} \|v\|_{L^p(B_{R_0})}. \quad (1.2.9)$$

We now provide the

Proof of Theorem 1.1.3. Let $\alpha \in C^{0,1/2}(\mathbb{C}, \mathbb{C})$ have a fixed compact support and satisfy

$$\int_{\mathbb{C}} |\alpha|^2 = 1. \quad (1.2.10)$$

We first claim that for any $\nu \in \mathbb{R}$

$$\begin{aligned} & (-\beta + \frac{1}{4}\varepsilon)\Pi_\varepsilon(|z|^2 f_{\alpha,\tau}) + \frac{1}{4}\Pi_\varepsilon(|z|^2 \Pi_\varepsilon(|z|^2 f_{\alpha,\tau})) + G\Pi_\varepsilon(e^{-|z|^2/\varepsilon} |f_{\alpha,\tau}|^2 f_{\alpha,\tau}) - \nu f_{\alpha,\tau} \\ &= \|\Pi_\varepsilon(\alpha f_\tau)\|_{\mathfrak{F}_\varepsilon}^{-1} \Pi_\varepsilon \left(\left(-\beta|z|^2 + \frac{1}{4}|z|^4 + Gb|\alpha|^2 - \nu \right) \alpha f_\tau \right) + R_\varepsilon, \end{aligned} \quad (1.2.11)$$

with $\|R_\varepsilon\|_{\mathfrak{F}_\varepsilon} \leq C\varepsilon^{1/4}$. The proof consists, as in [12] in a repeated use of Lemma 1.2.1. Let p be a polynomial. We estimate, using Cauchy-Schwarz and (1.2.8) applied with $g = f_\tau$ and $\gamma = \alpha$,

$$\left| \int_{\mathbb{C}} (|z|^2 \alpha f_\tau - |z|^2 \Pi_\varepsilon(\alpha f_\tau)) \bar{p} e^{-|z|^2/\varepsilon} dz \right| \leq C \|p\|_{\mathfrak{F}_\varepsilon} \|\Pi_\varepsilon(\alpha f_\tau) - \alpha f_\tau\|_{\mathfrak{F}_\varepsilon^2} \leq C\varepsilon^{1/4} \|p\|_{\mathfrak{F}_\varepsilon}.$$

Thus, by density of polynomial in \mathfrak{F}_ε

$$\|\Pi_\varepsilon(|z|^2 \Pi_\varepsilon(\alpha f_\tau)) - \Pi_\varepsilon(|z|^2 \alpha f_\tau)\|_{\mathfrak{F}_\varepsilon} \leq C\varepsilon^{1/4}. \quad (1.2.12)$$

Similarly, applying (1.2.8) with $\gamma = \alpha|z|^2$

$$\left| \int_{\mathbb{C}} (|z|^4 \alpha f_\tau - |z|^2 \Pi_\varepsilon(\alpha|z|^2 f_\tau)) \bar{p} e^{-|z|^2/\varepsilon} dz \right| \leq C \|p\|_{\mathfrak{F}_\varepsilon} \|\Pi_\varepsilon(\alpha|z|^2 f_\tau) - \alpha|z|^2 f_\tau\|_{\mathfrak{F}_\varepsilon^2} \leq C\varepsilon^{1/4} \|p\|_{\mathfrak{F}_\varepsilon}$$

and

$$\begin{aligned} \left| \int_{\mathbb{C}} (|z|^2 \Pi_\varepsilon(\alpha|z|^2 f_\tau) - |z|^2 \Pi_\varepsilon(|z|^2 \Pi_\varepsilon(\alpha f_\tau))) \bar{p} e^{-|z|^2/\varepsilon} dz \right| &\leq \|p\|_{\mathfrak{F}_\varepsilon} \|\Pi_\varepsilon(\alpha|z|^2 f_\tau - |z|^2 \Pi_\varepsilon(\alpha f_\tau))\|_{\mathfrak{F}_\varepsilon^2} \\ &\leq \|p\|_{\mathfrak{F}_\varepsilon} \|\alpha f_\tau - \Pi_\varepsilon(\alpha f_\tau)\|_{\mathfrak{F}_\varepsilon^4} \leq C\varepsilon^{1/4} \|p\|_{\mathfrak{F}_\varepsilon} \end{aligned}$$

which implies

$$\|\Pi_\varepsilon(|z|^2 \Pi_\varepsilon(|z|^2 \Pi_\varepsilon(\alpha f_\tau))) - \Pi_\varepsilon(|z|^4 \alpha f_\tau)\|_{\mathfrak{F}_\varepsilon} \leq C\varepsilon^{1/4}. \quad (1.2.13)$$

We turn to the non linear term in the equation. Using (1.1.30) we have for any polynomial p

$$\begin{aligned} b \int_{\mathbb{C}} |u_\tau|^2 \int_{\mathbb{C}} |\alpha|^2 \alpha \Pi_\varepsilon(e^{-|z|^2/\varepsilon} |f_\tau|^2 f_\tau) \bar{p} e^{-|z|^2/\varepsilon} dz &= \int_{\mathbb{C}} |\alpha|^2 \alpha f_\tau \bar{p} e^{-|z|^2/\varepsilon} dz \\ &= \int_{\mathbb{C}} |\alpha|^2 \alpha e^{-|z|^2/\varepsilon} |f_\tau|^2 f_\tau \bar{p} e^{-|z|^2/\varepsilon} dz \\ &\quad + O(\varepsilon^{1/4}). \end{aligned} \quad (1.2.14)$$

The second line is a consequence of (1.2.8) applied with $\gamma = |\alpha|^2 \alpha$ and $p = g$. On the other hand, using Lemma 1.2.1 again

$$\begin{aligned} \left| \int_{\mathbb{C}} \left(e^{-|z|^2/\varepsilon} |\Pi_\varepsilon(\alpha f_\tau)|^2 \Pi_\varepsilon(\alpha f_\tau) - |\alpha|^2 \alpha e^{-|z|^2/\varepsilon} |f_\tau|^2 f_\tau \right) \bar{p} e^{-|z|^2/\varepsilon} dz \right| \\ \leq \|p\|_{\mathfrak{F}_\varepsilon} \|\Pi_\varepsilon(\alpha f_\tau)\|_{\mathfrak{F}_\varepsilon} \|\alpha|^2 |u_\tau|^2 - |\Pi_\varepsilon(\alpha f_\tau)|^2 e^{-|z|^2/\varepsilon}\|_{L^\infty} \\ \quad + \|p\|_{\mathfrak{F}_\varepsilon} \|\alpha\|_{L^\infty}^2 \|u_\tau\|_{L^\infty}^2 \|\alpha f_\tau - \Pi_\varepsilon(\alpha f_\tau)\|_{\mathfrak{F}_\varepsilon} \leq \varepsilon^{1/4} \|p\|_{\mathfrak{F}_\varepsilon} \end{aligned}$$

and thus

$$\left\| \Pi_\varepsilon(e^{-|z|^2/\varepsilon} |\Pi_\varepsilon(\alpha f_\tau)|^2 \Pi_\varepsilon(\alpha f_\tau) - b \left(\int |u_\tau|^2 \right) |\alpha|^2 \alpha f_\tau) \right\|_{\mathfrak{F}_\varepsilon} \leq C\varepsilon^{1/4}. \quad (1.2.15)$$

Recall that

$$f_{\alpha, \tau} = \frac{\Pi_\varepsilon(\alpha f_\tau)}{\|\Pi_\varepsilon(\alpha f_\tau)\|_{\mathfrak{F}_\varepsilon}}.$$

We claim that

$$\|\Pi_\varepsilon(\alpha f_\tau)\|_{\mathfrak{F}_\varepsilon}^2 = \int |u_\tau|^2 \left(1 + O(\varepsilon^{1/4})\right). \quad (1.2.16)$$

Indeed, from Lemma 1.2.1 we have

$$\|\Pi_\varepsilon(\alpha f_\tau)\|_{\mathfrak{F}_\varepsilon}^2 = \int_{\mathbb{C}} |\alpha|^2 |u_\tau|^2 + O(\varepsilon^{1/4}).$$

On the other hand $|u_\tau|$ is periodic over a lattice of period proportional to $\varepsilon^{1/2}$, thus for any $\gamma \in C^\infty(\mathbb{C})$

$$\left| \int_{\mathbb{C}} |\gamma|^2 |u_\tau|^2 - \int_{\mathbb{C}} |u_\tau|^2 \int_{\mathbb{C}} |\gamma|^2 \right| \leq C \|\gamma\|_{C^0(\mathbb{C})}$$

and

$$\left| \int_{\mathbb{C}} |\gamma|^2 |u_\tau|^2 - \int_{\mathbb{C}} |u_\tau|^2 \int_{\mathbb{C}} |\gamma|^2 \right| \leq C \varepsilon^{1/2} \|\gamma\|_{C^1(\mathbb{C})}.$$

Using $\alpha \in C^{0,1/2}$ and interpolating between the two previous estimates we obtain

$$\int_{\mathbb{C}} |\alpha|^2 |u_\tau|^2 = \int_{\mathbb{C}} |u_\tau|^2 \int_{\mathbb{C}} |\alpha|^2 + O(\varepsilon^{1/4}) \quad (1.2.17)$$

and thus (1.2.16) recalling (1.2.10). Gathering (1.2.12),(1.2.13),(1.2.15) and (1.2.16) we deduce that (1.2.11) holds.

There only remains to choose α satisfying (1.1.37) and $\nu = \mu_b^{\text{TF}}$ to have

$$\left(-\beta |z|^2 + \frac{1}{4} |z|^4 + Gb |\alpha|^2 - \nu \right) \alpha = 0$$

and thus (1.1.38).

The proof of the energy result (1.1.39) uses the same tools. By definition

$$\begin{aligned} \mathcal{F}^{\text{LLL}}[f_{\alpha,\tau}] &= \|\Pi_\varepsilon(\alpha f_\tau)\|_{\mathfrak{F}_\varepsilon}^{-2} \int_{\mathbb{C}} \left(-\beta |z|^2 + \frac{1}{4} |z|^4 \right) |\Pi_\varepsilon(\alpha f_\tau)|^2 e^{-|z|^2/\varepsilon} \\ &\quad + \|\Pi_\varepsilon(\alpha f_\tau)\|_{\mathfrak{F}_\varepsilon}^{-4} \frac{G}{2} \int_{\mathbb{C}} |\Pi_\varepsilon(\alpha f_\tau)|^4 e^{-2|z|^2/\varepsilon} \end{aligned}$$

working as above, with the L^p estimate (1.2.9) instead of (1.2.8) we deduce

$$\mathcal{F}^{\text{LLL}}[f_{\alpha,\tau}] = \|\Pi_\varepsilon(\alpha f_\tau)\|_{\mathfrak{F}_\varepsilon}^{-2} \int_{\mathbb{C}} \left(-\beta |z|^2 + \frac{1}{4} |z|^4 \right) |\alpha u_\tau|^2 + \|\Pi_\varepsilon(\alpha f_\tau)\|_{\mathfrak{F}_\varepsilon}^{-4} \frac{G}{2} \int_{\mathbb{C}} |\alpha u_\tau|^4 + O(\varepsilon^{1/4}). \quad (1.2.18)$$

Next, using that $|u_\tau|$ is periodic over a lattice of period proportional to $\varepsilon^{1/2}$ as above (see the proof of (1.2.17)), we obtain

$$\begin{aligned} \mathcal{F}^{\text{LLL}}[f_{\alpha,\tau}] &= \|\Pi_\varepsilon(\alpha f_\tau)\|_{\mathfrak{F}_\varepsilon}^{-2} \int_{\mathbb{C}} |u_\tau|^2 \int_{\mathbb{C}} \left(-\beta |z|^2 + \frac{1}{4} |z|^4 \right) |\alpha|^2 \\ &\quad + \|\Pi_\varepsilon(\alpha f_\tau)\|_{\mathfrak{F}_\varepsilon}^{-4} \int_{\mathbb{C}} |u_\tau|^4 \frac{G}{2} \int_{\mathbb{C}} |\alpha|^4 + O(\varepsilon^{1/4}). \end{aligned} \quad (1.2.19)$$

There only remains to recall (1.1.31) and (1.2.16) to conclude that (1.1.39) holds. \square

1.3 Numerical Simulations

1.3.1 Method

We want to construct a numerical approximation of the minimizer ϕ of \mathcal{E}^{LLL} in the LLL. We write

$$\phi(z) = P(z)e^{-\Omega|z|^2/2} \quad (1.3.1)$$

with P a holomorphic function. Since polynomials are dense in \mathfrak{F}_ε , it is reasonable to fix an integer n and to restrict the analysis to functions ϕ where P is a polynomial of degree less than n (see [12, Section 6] for a discussion on this point). We write our trial functions as in [8] :

$$\phi(z) = A \prod_{j=1}^n (z - z_j) e^{-\Omega|z|^2/2}, \quad (1.3.2)$$

where

$$A = \|\phi\|_{L^2}^{-1/2}$$

is the normalization factor. We will numerically vary the locations z_j of vortices. An alternative method (used for example in [42] for a harmonically trapped condensate) would be to take

$$\phi(z) = A \left(\sum_{j=0}^n b_j z^j \right) e^{-\Omega|z|^2/2}$$

and vary the coefficients b_j . The interest of our approach is to give a direct access to the exact repartition of vortices, whereas the alternative method would require to compute the roots of a polynomial of degree n , which is a delicate task for large n . In particular, varying the coefficients could probably not give the precise locations of invisible vortices located in regions of low matter density. Note also that there exist some good methods to directly minimize the GP energy (see [50, 51] for recent developments), but they probably could not give access to the locations of invisible vortices neither. Indeed, one would have to spot zeros or phase singularities in a region where the density is very small.

The point here is that these vortices, although invisible, have a crucial influence on both the energy and the density profile of the condensate. One can be skeptical about the existence of density dips in a region where there is so to say no matter at all. But a vortex is also carrying a phase circulation (or superfluid current) and thus one can interpret the presence of invisible vortices as the existence of currents in the condensate that are equivalent to those generated by vortices in the low density region. Taking invisible vortex into account with a trial function of the form (1.3.2) is thus a way to evaluate more precisely the effect of superfluid currents in the condensate.

We used a conjugate gradient method with a Goldstein and Price line-search. The integrals are computed using the Gauss-Hermite method, and we take enough Gauss points for the computations to be exact. This results in quite expensive calculations, but we have been able to numerically construct condensates with up to ~ 120 vortices.

1.3.2 Results

We show in Fig. 1.1 and Fig. 1.2 typical examples of configurations we numerically computed. The qualitative features of the vortex patterns and atomic densities confirm

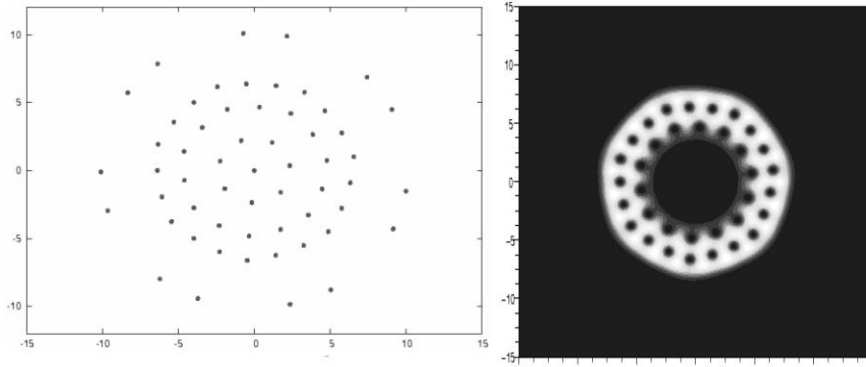


FIGURE 1.1 – Vortex structure and atomic density for $G = 3, k = 10^{-4}, \beta = 1$ ($\Omega - 1 = 2.2 \cdot 10^{-3}$). There are 67 vortices in total, the central vortex is constituted of 11 single vortices.

our theoretical results and are in good agreement with existing theoretical and numerical studies [64, 91]. Note however that the numerics become quite intricate for large number of vortices, which accounts for the relative lack of symmetry of Fig. 1.2.

As was expected, the condensate develops a central hole and visible vortices are regularly

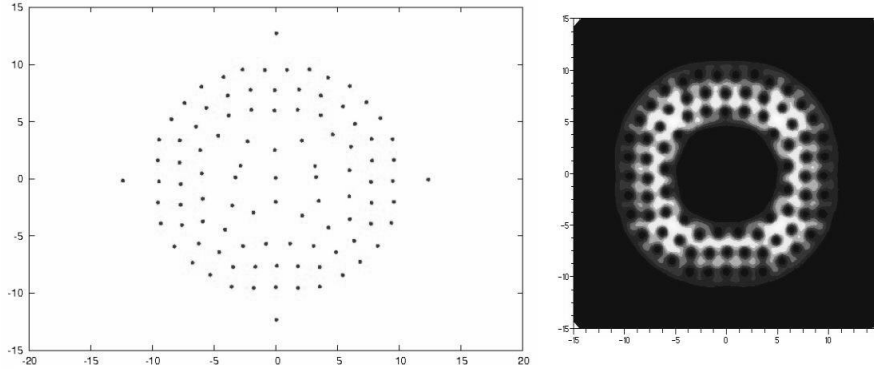


FIGURE 1.2 – Vortex structure and atomic density for $G = 3, k = 10^{-5}, \beta = 1$ ($\Omega - 1 = 4.6 \cdot 10^{-4}$). There are 119 vortices in total, the central vortex is constituted of 20 single vortices.

distributed and densely packed in the annular region of significant atomic density. Our computations also show a distortion of the vortex pattern near the external radius of the condensate, resulting in invisible vortices in the exterior region of low atomic density as is the case for harmonically trapped condensates [8]. Some vortices also lie in the central hole as theoretically predicted and we can get information on their precise locations : we observe a distortion of the regular lattice near the inner radius of the condensate, resulting in isolated singly-quantized vortices encircling a central multiply quantized vortex. We computed configurations for which this central vortex has up to 20 units of circulation while there is a total of 32 units of circulation in the entire hole and 83 visible vortices. The number of vortices (both visible and invisible) increases with increasing Ω or β , but since our scaling does not allow to explore a large domain of Ω when G is fixed, we mainly

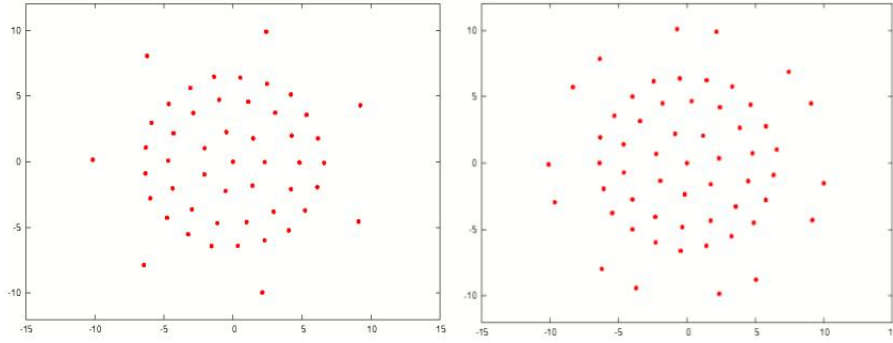


FIGURE 1.3 – Two example of vortex configurations for $G = 3, k = 10^{-4}, \beta = 1$ ($\Omega - 1 = 2.2 \cdot 10^{-3}$), respectively with $n = 60$ and $n = 67$ vortices.

varied k . The total vorticity of the system increases with decreasing k .

All vortices do not have the same contribution to the energy : as n increases, the vortex pattern in the annular region of significant atomic density remains the same up to possible rotations, with the additional vortices first gathering in the central vortex, then constituting the distorted lattice near the inner boundary of the condensate and finally occupying the distorted sites beyond the external radius. With increasing n (see Fig. 1.4), the energy reaches a first plateau when the central figure is formed, constituted of the visible vortices and the vortices in the central hole. A second plateau is reached when enough distorted sites beyond the external radius are occupied. For example, for the parameters corresponding to Fig. 1.1 and Fig. 1.4 ($k = 10^{-4}, G = 3, \beta = 1$), the energy varies by $\sim \pm 10^{-6}$ in relative value when n is increased from 60 to 67 (Fig. 1.3) and the atomic density does not vary significantly. We find a good agreement between our numerical results and the upper bound to the energy computed in Theorem 1.1.3. The energy of our analytical trial function typically differs by $\sim 10^{-3}$ to $\sim 10^{-2}$ in relative value from the energy numerically computed. The agreement between the radii theoretically predicted (1.1.43) and extracted from the numerical simulations is typically a little worse but still satisfactory (relative errors ranging from 10^{-2} to 10^{-1}). Again, obtaining a better agreement would probably require to consider a thicker annulus containing more vortices. This would result in more difficult simulations.

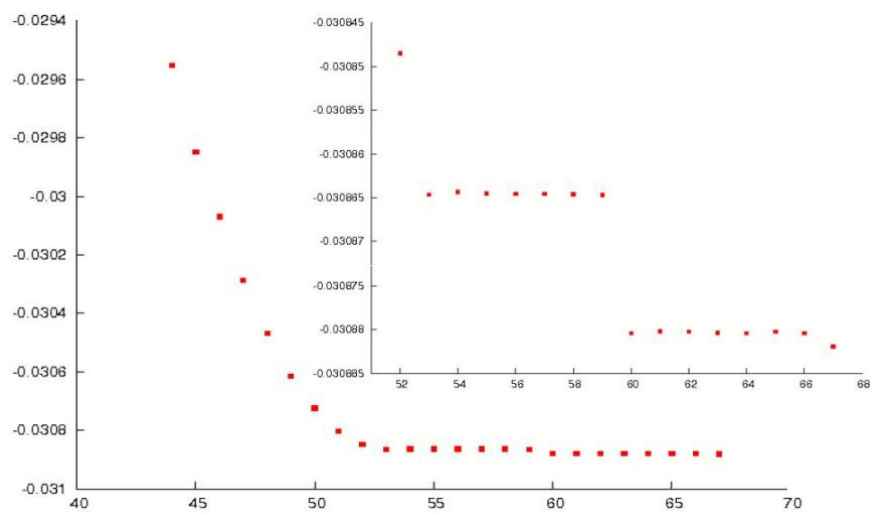


FIGURE 1.4 – Minimum energy as a function of the number of vortices in the trial wave function ($G = 3$, $k = 10^{-4}$, $\beta = 1$).

2

The giant vortex state for a Bose-Einstein condensate in a rotating anharmonic trap : extreme rotation regimes

Ce chapitre reprend un article à paraître au Journal de Mathématiques Pures et Appliquées.

2.1 Introduction

2.1.1 Physical background

Since the first experimental achievement of a Bose-Einstein condensate by the Jila and MIT groups in 1995 (2001 Nobel prize in physics attributed to Cornell, Wieman and Ketterle), these systems have been the subject of many studies from the condensed matter community. A reason (among others) for this is the fact that a Bose-Einstein condensate is a good system to study superfluidity issues, such as the existence of quantized vortices. These vortices can be generated by rotation of the container (generally a magnetic trap) enclosing the condensate. They are the subject of an ever increasing number of experimental and theoretical papers, see e.g. the review [63] or the monograph [4] for extensive lists of references.

Recently, mathematical contributions have studied certain issues arising from the physics of rotating Bose-Einstein condensates. Let us cite some of these issues :

- the appearance of the first vortices when increasing the rotational speed of the trap in the strong coupling regime [81, 82] ;
- the formation of a vortex lattice characteristic of regimes when the rotational speed nearly deconfines the atoms [6, 7, 12] ;
- the interaction of many condensates in an optical lattice [15] ;
- the symmetry breaking of the ground state of a condensate in certain parameter regimes [139] ;
- the energy and density asymptotics of strongly interacting condensates in anharmonic traps [43, 44, 47] .

Most of the available mathematical studies on Bose-Einstein condensates are made in the framework of the Gross-Pitaevskii energy. Here we will use a two-dimensional energy although the actual energy should be three-dimensional (see the discussion in Section 1.4 below). The condensate is described by a complex macroscopic wave-function ψ minimizing

the energy given (in the rotating frame) by

$$E_{GP}(\psi) = \int_{\mathbb{R}^2} |\nabla\psi - i\Omega x^\perp \psi|^2 dx + \int_{\mathbb{R}^2} (V(x)|\psi|^2 - \Omega^2|x|^2|\psi|^2) dx + G \int_{\mathbb{R}^2} |\psi|^4 dx, \quad (2.1.1)$$

under the mass constraint

$$\int_{\mathbb{R}^2} |\psi|^2 dx = 1. \quad (2.1.2)$$

The matter density at some point x is given by $|\psi|^2(x)$. A vortex is a zero of the wavefunction ψ around which there is a quantized phase circulation (ie topological degree or winding number).

Here we note $x = (x_1, x_2)$, $x^\perp = (-x_2, x_1)$, $V(x)$ is the potential confining the atoms (generally representing a magnetic trap), Ω is the speed at which the trap is rotated around the axis perpendicular to the $x_1 - x_2$ plane and $G > 0$ is the coupling constant modelling the atomic interactions which we assume to be repulsive. The different terms in the energy represent the kinetic energy (including the effect of Coriolis forces due to the transformation to the rotating frame), the potential energy (including the effect of centrifugal forces) and the energy due to the interatomic interactions.

The model based on the Gross-Pitaevskii energy is an approximation of the quantum mechanical many-body problem for N bosons at zero temperature. We refer to [99] and references therein for a discussion of the connection between this type of energies and the N -body problem. We remark that the Gross-Pitaevskii energy in the rotating case ($\Omega \neq 0$) was rigorously derived from the N -body problem in [97], but only for fixed rotation (ie fixed Ω) and fixed interactions (ie fixed G). In the regimes we are going to consider (Ω and/or G going to infinity), the derivation of the Gross-Pitaevskii energy from the quantum mechanical many-body problem is still a mostly open problem. To the best of the author's knowledge, the only rigorous results available are those of [37], where it is proved that the Gross-Pitaevskii energy is a correct first-order description in regimes similar to the ones we are going to study in this paper.

We remark that our results provide an a posteriori criterium for the justification of the reduction to the Gross-Pitaevskii energy in some regime of parameters. However, it is far from a rigorous proof, see Section 1.4 below.

In most experimental situations, the potential V is harmonic, of the form

$$V(x) = a_1 x_1^2 + a_2 x_2^2. \quad (2.1.3)$$

Such a potential does not allow to take arbitrarily large rotation speeds, as the so-called effective potential

$$V(x) - \Omega^2|x|^2 \quad (2.1.4)$$

is not bounded below for $\Omega^2 > \min(a_1, a_2)$ and consequently neither is the energy (2.1.1). Physically, this corresponds to the fact that the centrifugal force overcomes the magnetic trapping force when $\Omega^2 > \min(a_1, a_2)$, thus the condensate is no longer confined and the atoms fly apart. Fetter suggested [61] to use instead a potential with a growth steeper than harmonic, for example of the form

$$V(x) = |x|^2 + k|x|^4. \quad (2.1.5)$$

The nice feature of this potential is that the centrifugal force is always compensated by the trapping force, and thus one can in theory take arbitrarily large rotation speeds. Experiments with this type of potentials have been realized by the ENS group [35, 146],

using a blue-detuned laser beam to create the quartic part of the potential. The experiments motivated numerous theoretical studies [13, 31, 49, 64, 71, 83, 84, 90, 91, 92, 105] revealing the very rich vortex structure one can expect to be displayed by such systems. When the rotational speed of a condensate trapped by a potential of this kind is increased from zero, many phase transitions are expected to happen. Firstly, vortices are expected to appear one by one as is the case for a harmonically trapped condensate, but eventually with multiply quantized vortices becoming stable [83, 84, 105], which is never the case with a purely harmonic trap.

When the centrifugal force begins to compensate the trapping force corresponding to the quadratic part in the potential, a triangular lattice of singly quantized vortices (Abrikosov lattice) similar to that which is observed in purely quadratic traps appears [31], but a new feature of the quadratic plus quartic trap is the existence of a critical speed for the centrifugal force to create a central hole in the condensate. The resulting state is an annular condensate with a vortex lattice encircling a central giant hole carrying a macroscopic phase circulation [13, 31, 49, 64, 91]. At even larger rotation speeds, a new transition is expected to happen, all the individual vortices present in the annulus retreating in the central hole and gathering in a single multiply quantized vortex at the center of the trap. This is what we will refer to as the *giant vortex state* : an annular condensate with pure irrotational flow encircling a central multi-quantized vortex [64, 71, 90, 91, 92].

In this paper we aim at justifying rigorously the appearance of the giant vortex state in the limit $\Omega \rightarrow +\infty$. The existing mathematical results on giant vortices in Bose-Einstein condensates (mainly those of [5] to our knowledge) focused on the case when the trapping potential is taken so that the condensate has an annular shape even at slow rotation speeds and studied the effect of the central hole on the vortex structure in the annulus. Here we study the case where, as the rotational speed increases, a central hole is created in the condensate by the centrifugal force.

2.1.2 Model

Our model is the following : we consider a Gross-Pitaevskii energy of the form

$$E_{GP}(\psi) = \int_{\mathbb{R}^2} \left(|\nabla\psi - i\Omega x^\perp \psi|^2 + (1 - \Omega^2)|x|^2|\psi|^2 + k|x|^4|\psi|^2 + G|\psi|^4 \right) dx \quad (2.1.6)$$

to be minimized under the mass-constraint (2.1.2). In order to study the asymptotics of the problem when $\Omega \rightarrow +\infty$, it is more convenient to change scales, setting for $\Omega > 1$

$$u(x) = R\psi(Rx), \quad R = \sqrt{\frac{\Omega^2 - 1}{2k}}. \quad (2.1.7)$$

We then have

$$E_{GP}(\psi) = \mathcal{F}_\Omega(u) \quad (2.1.8)$$

with

$$\mathcal{F}_\Omega(u) = \frac{2k}{\Omega^2 - 1} \int_{\mathbb{R}^2} \left| \nabla u - i\Omega \frac{\Omega^2 - 1}{2k} x^\perp u \right|^2 + G|u|^4 + \frac{(\Omega^2 - 1)^3}{8k^2} (|x|^4 - 2|x|^2) |u|^2. \quad (2.1.9)$$

Let us emphasize that formal calculations [64] suggest that the ground state of (2.1.6) should be confined on an annulus of radius $\sim R$ so that our change of scales is quite

natural in this setting. Because of the mass constraint

$$\int_{\mathbb{R}^2} |u|^2 dx = 1, \quad (2.1.10)$$

we do not change the physics of the problem by adding to \mathcal{F}_Ω any multiple of $\int_{\mathbb{R}^2} |u|^2 dx$. Hence it is equivalent to minimize

$$\tilde{\mathcal{F}}_\Omega(u) = \frac{2k}{\Omega^2 - 1} \int_{\mathbb{R}^2} \left| \nabla u - i\Omega \frac{\Omega^2 - 1}{2k} x^\perp u \right|^2 + G|u|^4 + \frac{(\Omega^2 - 1)^3}{8k^2} (|x|^2 - 1)^2 |u|^2. \quad (2.1.11)$$

Defining the parameters

$$\begin{aligned} \omega &= \Omega \frac{\Omega^2 - 1}{2k} \\ D_\Omega &= \frac{\Omega^2 - 1}{\Omega^2} \end{aligned} \quad (2.1.12)$$

we have

$$\tilde{\mathcal{F}}_\Omega(u) = \frac{2k}{\Omega^2 - 1} F_\omega(u) \quad (2.1.13)$$

with

$$F_\omega(u) = \int_{\mathbb{R}^2} \left(|\nabla u - i\omega x^\perp u|^2 + D_\Omega \frac{\omega^2}{2} (|x|^2 - 1)^2 |u|^2 + G|u|^4 \right) dx, \quad (2.1.14)$$

and we are going to study the asymptotics of F_ω . Results for $\tilde{\mathcal{F}}_\Omega(u)$ are straightforward modifications of those we will present.

The potential $D_\Omega \omega^2 (|x|^2 - 1)^2$ is positive and has a degenerate minimum for $|x| = 1$, so that we expect the condensate to be tightly confined on an annulus centered on the circle $|x| = 1$ in the limit $D_\Omega \omega^2 \rightarrow +\infty$. This is reminiscent of semi-classical studies of Hamiltonians with potential wells (see [76] and the references therein), although our analysis will be quite different.

We denote

$$I_\omega = \inf \left\{ F_\omega(u), u \in H^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2, x^4 dx), \int_{\mathbb{R}^2} |u|^2 = 1 \right\} = F_\omega(u_\omega). \quad (2.1.15)$$

It is classical that I_ω is achieved by some (a priori non unique) u_ω satisfying the Euler-Lagrange equation

$$-\left(\nabla - i\omega x^\perp \right)^2 u_\omega + D_\Omega \frac{\omega^2}{2} (|x|^2 - 1)^2 u_\omega + 2G|u_\omega|^2 u_\omega = \mu_\omega u_\omega, \quad (2.1.16)$$

where μ_ω is the Lagrange multiplier associated with the mass constraint. We aim at studying the asymptotics of I_ω and u_ω when $\omega \rightarrow +\infty$. In particular we want to confirm rigorously the predicted transition of u_ω to a state with pure irrotational flow of the form

$$(r, \theta) \mapsto f(r) e^{in\theta}, \quad (2.1.17)$$

where r, θ are the polar coordinates and n an integer.

More precisely, it is to be expected that u_ω will be close to a function of the form (2.1.17) with $n \sim \omega$ and f a function tightly confined on a shrinking annulus centered on the potential well.

2.1.3 Main results

In order to state our main results, let us introduce some notation. For every $n \in \mathbb{Z}$ we define the one dimensional energy F_n

$$F_n(f) = 2\pi \int_{\mathbb{R}^+} (|f'(r)|^2 + V_n(r)|f(r)|^2) r dr \quad (2.1.18)$$

defined for every $f \in H^1(\mathbb{R}^+, r dr) \cap L^2(\mathbb{R}^+, r V_n(r) dr)$. The potential V_n is given by :

$$V_n(r) = \frac{n^2}{r^2} - 2n\omega + (1 - D_\Omega)\omega^2 r^2 + D_\Omega \frac{\omega^2}{2} + D_\Omega \frac{\omega^2}{2} r^4. \quad (2.1.19)$$

The energies F_n appear when decomposing the function u_ω as a Fourier series in the angular variable (see (2.1.43) below). We will denote $g_{1,n}$ the ground-state (it is unique up to a multiplicative constant of modulus 1, that we fix so as to have $g_{1,n} \geq 0$) of the energy (2.1.18) satisfying the mass constraint

$$2\pi \int_{\mathbb{R}^+} g_{1,n}^2(r) r dr = 1 \quad (2.1.20)$$

and

$$\lambda_{1,n} = F_n(g_{1,n}) = \inf \left\{ F_n(f), f \in H^1(\mathbb{R}^+, r dr) \cap L^2(\mathbb{R}^+, r V_n(r) dr), 2\pi \int_{\mathbb{R}^+} f^2(r) r dr = 1 \right\}. \quad (2.1.21)$$

Similarly we introduce

$$E_n(f) = 2\pi \int_{\mathbb{R}^+} (|f'(r)|^2 + V_n(r)|f(r)|^2 + G|f(r)|^4) r dr \quad (2.1.22)$$

with

$$\gamma_n = E_n(\Psi_n) = \inf \left\{ E_n(f), f \in H^1(\mathbb{R}^+, r dr) \cap L^2(\mathbb{R}^+, r V_n(r) dr), 2\pi \int_{\mathbb{R}^+} f^2(r) r dr = 1 \right\}. \quad (2.1.23)$$

Similarly to $g_{1,n}$ we can ensure the unicity of Ψ_n by requiring that $\Psi_n \geq 0$.

We will denote ξ_1 and ξ_2 the first two eigenfunctions normalized in $L^2(\mathbb{R})$ of the harmonic oscillator

$$-\frac{d^2}{dx^2} + x^2, \quad (2.1.24)$$

associated to the eigenvalues 1 and 3.

Our results concern the energy (2.1.14) in the limit $\omega \rightarrow +\infty$. They hold when the interaction energy is small compared to the potential and kinetic terms so that the first order of the energy is a quadratic term, leading to a linear equation. We show that this condition is fulfilled if

$$\omega \rightarrow +\infty, \quad \omega \gg G^2 \geq g^2, \quad D_\Omega \in (0, 1), \quad D_\Omega \rightarrow D \in (0, 1] \quad (2.1.25)$$

where D and g are fixed constants. The assumption that G^2 is larger than some constant is only a matter of convenience for writing the results. The opposite case where $G \ll 1$ can be dealt with using the tools that we develop in this paper, and is actually easier than

the case where G stays bounded below. We will refer to the limit (2.1.25) as the *extreme rotation regime*.

We have the following result (the notation $a \propto b$ has the usual meaning that a/b converges to some constant.) :

Theorem 2.1.1 (Energy and density asymptotics in the extreme rotation regime).

Let u_ω be any solution of (2.1.15) and suppose that (2.1.25) holds. There is an integer $n^* = \omega + O(1)$ so that

$$I_\omega = F_\omega(u_\omega) = \lambda_{1,n^*} + 2\pi G \int_{\mathbb{R}^+} g_{1,n^*}(r)^4 r dr + O(G^2) = \gamma_{n^*} + O(G^2). \quad (2.1.26)$$

Moreover

$$\| |u_\omega|^2 - g_{1,n^*}^2 \|_{L^2(\mathbb{R}^2)} \leq CG^{1/2} \ll \|g_{1,n^*}^2\|_{L^2(\mathbb{R}^2)} \propto \omega^{1/4} \quad (2.1.27)$$

and $|u_\omega|^2$ converges to a Dirac delta function at $|x| = 1$ in the weak sense of measures.

Note that $|g_{1,n^*}|^2$ converges to a Dirac delta function at $|x| = 1$ as a consequence of Theorem 2.2.1 below, to which we refer for more details on $\lambda_{1,n}$ and $g_{1,n}$. In particular we have

$$\lambda_{1,n^*} \sim (2D_\Omega + 4)^{1/2} \omega$$

and

$$2\pi \int_{\mathbb{R}^+} g_{1,n^*}(r)^4 \sim \frac{(2D_\Omega + 4)^{1/4}}{\sqrt{2\pi}} \omega^{1/2} \int_{\mathbb{R}} \xi_1^4(x) dx,$$

thus Theorem 2.1.1 says that the energy of the condensate is described to subleading order by a simplified one-dimensional problem, which also gives the density profile. The second equality in (2.1.26) holds because the energies F_n and E_n are actually very close in the parameter range (2.1.25), see Theorems 2.2.1 and 2.4.1 below. Similarly (2.1.27) could be stated with Ψ_{n^*} replacing g_{1,n^*} .

The energy E_n is obtained by restricting F_ω to wave functions of the form $f(r)e^{in\theta}$, so Theorem 2.1.1 is a first step towards the understanding of the giant vortex. However, a result such as (2.1.27) is not sufficient to conclude that there are no vortices in the bulk of the condensate. Actually our method does not give precise enough energy estimates to do so in the whole regime (2.1.25).

A special case where we can prove much more detailed results is that of a fixed coupling constant :

Theorem 2.1.2 (Refined asymptotics in the case of fixed G).

Let u_ω be any solution of (2.1.15) and suppose that there holds

$$\omega \rightarrow +\infty, \quad G \text{ is a fixed constant}, \quad D_\Omega \in (0, 1), \quad D_\Omega \rightarrow D \in (0, 1]. \quad (2.1.28)$$

Then there is an integer $n^* = \omega + O(1)$ so that for any $\varepsilon > 0$

$$I_\omega = F_\omega(u_\omega) = \gamma_{n^*} + O(\omega^{-1/4+\varepsilon}). \quad (2.1.29)$$

Moreover there exists $\alpha \in \mathbb{R}$ such that, along some subsequence

$$\|u_\omega - e^{i\alpha} \Psi_{n^*} e^{in^*\theta}\|_{L^2(\mathbb{R}^2)} \leq C_\varepsilon \omega^{-1/16+\varepsilon}. \quad (2.1.30)$$

Theorem 2.1.2 is still mainly concerned with energy and density asymptotics, although (2.1.30) allows to identify a global limiting phase. We now state our result about the appearance of the giant vortex, that is an annular condensate with no vortices in the bulk :

Theorem 2.1.3 (The giant vortex state in the case of fixed G).

Under the assumptions of Theorem 2.1.2

1. *There is a constant σ so that the following estimate holds true pointwise along some subsequence*

$$\| |u_\omega| - |\Psi_{n^*}| \| \leq C_\varepsilon \omega^{3/16+\varepsilon} e^{-\sigma\omega(|x|-1)^2} + C_\varepsilon \omega^{-1/32+\varepsilon}. \quad (2.1.31)$$

In particular

$$\| |u_\omega| - |\Psi_{n^*}| \|_{L^\infty(\mathbb{R}^2)} \leq C_\varepsilon \omega^{3/16+\varepsilon} \ll \| \Psi_{n^*} \|_{L^\infty(\mathbb{R}^2)} \propto \omega^{1/4}. \quad (2.1.32)$$

2. *$|u_\omega(x)| \rightarrow 0$ uniformly on*

$$\mathcal{H}_\omega^\delta = \left\{ x \in \mathbb{R}^2, \left| |x| - 1 \right|^2 \geq \delta \omega^{-1} \ln(\omega) \right\} \quad (2.1.33)$$

for every $\delta > \frac{1}{4\sigma}$.

3. *Suppose that there is some x_ω so that*

$$u_\omega(x_\omega) \rightarrow 0 \quad (2.1.34)$$

then, if ω is large enough, we have for every $\delta < \frac{1}{4\sigma}$

$$\left| |x_\omega| - 1 \right|^2 \geq \delta \omega^{-1} \ln(\omega). \quad (2.1.35)$$

We remark that whereas there can be a great number (proportional to G) of integers n^* satisfying (2.1.26) and (2.1.27), there are at most two integers satisfying (2.1.29) and (2.1.30). More precisely, two cases can occur : either γ_n is minimized at a single \hat{n} and the whole sequence u_ω satisfies (2.1.30) and (2.1.31) with $n^* = \hat{n}$ or γ_n is minimized at two integers \hat{n} and $\hat{n} + 1$. This case is very particular and happens only for special values of the parameters. For these particular values of the parameters, the sequence u_ω could in principle oscillate indefinitely between $\Psi_{\hat{n}} e^{i\hat{n}\theta}$ and $\Psi_{\hat{n}+1} e^{i(\hat{n}+1)\theta}$, so that we have to extract a subsequence to state the result.

We will discuss these results in further details in Section 1.4 and now present some ideas of the proofs. It is natural (see [64] for example) to construct a first trial function of the form

$$\begin{aligned} v_{test}(r, \theta) &= f_{test}(r) e^{i\hat{n}\theta} \\ f_{test}(r) &= c_{test} \xi \left(\frac{r-1}{\eta} \right) \end{aligned} \quad (2.1.36)$$

where $\xi : \mathbb{R} \rightarrow \mathbb{R}$ is chosen with $\int_{\mathbb{R}} \xi^2 = 1$, $\hat{n} = [\omega]$ is the closest integer to ω and η a small parameter to be chosen later on. We want the mass constraint (2.1.10) to be satisfied, which implies

$$c_{test}^2 \sim (\sqrt{2\pi\eta})^{-1} \quad (2.1.37)$$

when $\eta \rightarrow 0$ under some conditions on ξ , see Remark 2.2 in Section 2 below.

Expanding the potential $D_\Omega \omega^2 (|x|^2 - 1)^2$ allows to compute the energy of v_{test} . Minimizing it with respect to η gives the main scales of the problem : taking

$$\eta = (2D_\Omega + 4)^{-1/4} \omega^{-1/2} \quad (2.1.38)$$

we get

$$F_\omega(u_\omega) \leq F_\omega(v_{test}) \leq (2D_\Omega + 4)^{1/2} \omega \int_{\mathbb{R}} (\xi'(x)^2 + x^2 \xi^2(x)) dx + O(G\omega^{1/2}) \quad (2.1.39)$$

so that when $\omega \gg G^2$ the natural choice is to take for ξ the Gaussian ξ_1 achieving the infimum

$$1 = \inf_{\int_{\mathbb{R}} \xi^2 = 1} \int_{\mathbb{R}} (\xi'(x)^2 + x^2 \xi^2(x)) dx. \quad (2.1.40)$$

The symmetry of ξ_1 allows to improve the remainder term in (2.1.39) :

$$F_\omega(u_\omega) \leq F_\omega(v_{test}) \leq (2D_\Omega + 4)^{1/2} \omega + G\omega^{1/2} \frac{(2D_\Omega + 4)^{1/4}}{\sqrt{2\pi}} \int_{\mathbb{R}} \xi_1^4(x) dx + O(1). \quad (2.1.41)$$

This computation suggests that the dominant terms in the energy will be the kinetic and potential contributions, whose sum will be of the order of ω . The interaction energy will appear only at second order, with a contribution proportional to $G\omega^{1/2}$, which is much smaller than ω in the extreme rotation regime.

Thus, it is natural to expand u_ω in a Fourier series in the angular variable

$$u_\omega(r, \theta) = \sum_{n \in \mathbb{Z}} f_n(r) e^{in\theta}. \quad (2.1.42)$$

Proving that u_ω converges to a function of the form (2.1.17) then amounts to showing that f_n converges to 0 for all n except one, and the crucial fact will be the following decoupling of the quadratic part of the energy (2.1.11) :

$$F_\omega(u_\omega) = \sum_{n \in \mathbb{Z}} F_n(f_n) + G \int_{\mathbb{R}^2} |u_\omega|^4, \quad (2.1.43)$$

with the energies F_n being defined by (2.1.18). The proof of Theorem 2.1.1 is based on the detailed study of the ground states of these energies. In particular, for most values of the parameters, $\lambda_{1,n}$ is minimized at a unique integer, n^* .

We prove that because of the properties of the potentials V_n , some modes f_n carry too much energy to match our upper bound (2.1.41) and therefore can be eliminated. For the other ones, we use a lower bound on the gap between the ground-state energy of (2.1.18) and the first excited level. This gap is of order ω , which is much larger than the interaction energy if $\omega \gg G^2$, as shown by the preceding computation. We use this fact to prove that $f_n \propto g_{1,n}$ when ω goes to infinity.

This very particular form allows one to bound the quartic part of the energy from below. Combining the lower bound with the analysis of the one dimensional energies gives an estimate of the total energy contribution of each mode. Two effects come into play : concentration of the mass of u_ω on the mode $f_{n^*} \propto g_{1,n^*}$ is favorable for the quadratic part of the energy because λ_{1,n^*} is the minimum of $\lambda_{1,n}$ with respect to n , but it increases

the quartic part. We then show, confirming the energy scaling of (2.1.41) that, in order to match the subleading order of the energy, it is more favorable for u_ω to have its mass concentrated on only one mode.

However, the remainder terms that we obtain are not small enough to confirm this rigorously in the whole range of parameters (2.1.25). The energy expansion (2.1.26) still allows for many modes (a number proportional to G) not to be asymptotically 0 in the limit. We get the density asymptotics (2.1.27) nevertheless because the moduli of these modes are very close to one another.

If we fix G we can prove (2.1.30), which is much more precise than (2.1.27) and requires a refinement of the method described above. In particular, using the Euler-Lagrange equation for u_ω and the result (2.1.27) we refine the estimate $f_n \propto g_{1,n}$, finding a second order correction. This in turn allows to refine the remainder terms in the energy expansions and prove (2.1.30).

The dominant part of the Euler-Lagrange equation (terms corresponding to the variations of the kinetic and potential energies) is linear in the situation we consider. We are thus able, using elliptic estimates for the Ginzburg-Landau operator due to Lu and Pan [106], to get estimates in stronger norms. The uniform estimate (2.1.31) is obtained by this method. It implies that u_ω is confined to a shrinking annulus in which there are no zeroes, because so is Ψ_{n^*} (see Theorem 2.4.1).

We remark that a Fourier expansion such as (2.1.42) has been used in [28] to compute the lowest eigenvalue of the Ginzburg-Landau operator on a disc with an applied field going to infinity, but with no potential term in the energy. In that paper the minimizer gets confined close to the boundary of the disc, whereas in our work it is the potential that forces the confinement to a shrinking annulus. In [28], a rescaling similar to (2.1.36) leads to an energy expansion in powers of the small characteristic length of the problem. We perform a similar analysis in Section 2, with the difference that our original problem is posed on the whole space \mathbb{R}^2 , leading to rescaled eigenvalue problems on a line whereas in [28] a compact domain is considered, so that after rescaling one gets eigenvalue problems on a half line.

2.1.4 Discussion

Theorems 2.1.2 and 2.1.3 confirm the expected qualitative features at high rotation speeds : the condensate is tightly confined on a shrinking annulus in which there are no vortices. We identify a simplified limiting profile with a Gaussian shape (see Theorems 2.2.1 and 2.4.1 for details on the functions g_{1,n^*} and Ψ_{n^*} respectively). We are also able to identify a global limiting phase and prove that all vortices gather in the central low-density hole, confirming that a giant vortex state appears when ω goes to infinity with G fixed.

We prove that the width of the annulus where the mass is concentrated is of order $\omega^{-1/2} \sim \Omega^{-3/2} k^{1/2}$ in the extreme rotation limit. This is larger than the width predicted in former physical studies [64, 71, 92], and our limiting density is more regular than the Thomas-Fermi type profiles they suggest. This is due to the fact that these studies are made assuming that the interaction energy is the dominant part of the energy, leading to a profile solution of a nonlinear equation. Here we study a different regime of parameters and in particular we show that the Thomas-Fermi approximation for strong interactions breaks down in the extreme rotation limit, where u_ω converges to the solution of some

linear reduced problem.

We used a two-dimensional Gross-Pitaevskii energy instead of the complete 3D energy. A rather natural question would then be to find in which situations the reduction to the two-dimensional model is justified. One can always assume that the condensate is tightly confined in the direction of the rotation axis, so that the problem is essentially 2D, but in such fast rotation regimes it is to be expected that the reduction is valid independently of the strength of the vertical confinement (see [7] where this is shown for the case of a fast rotating condensate in a harmonic trap).

The extreme rotation regime can be reached in two ways : either by letting Ω tend to infinity with

$$\Omega \rightarrow +\infty, \quad \Omega^3 \gg k, \quad \Omega \gg G^{2/3}k. \quad (2.1.44)$$

or by keeping Ω of the order of a constant (strictly larger than 1) and letting k tend to zero with

$$\Omega \rightarrow \Omega_0 > 1, \quad k \rightarrow 0, \quad k \ll G^{-2} \quad (2.1.45)$$

where Ω_0 is a constant.

In a recent series of papers [43, 44, 47], the large Ω limit has been studied for traps related to the one we study (homogeneous traps, $V(x) = |x|^s$ with $s > 2$, and flat trap $V(x) = 0$ for $|x| \leq 1$ and $V(x) = +\infty$ for $|x| > 1$). Their analysis applies to our problem in regimes “opposite” to the two described above, namely under the condition that either $\Omega \ll G^{2/3}k$ (for the regime opposite to (2.1.44)) or with $k \gg G^{-2}$ (for the regime opposite to (2.1.45)). In this kind of regimes, the energy and density asymptotics are given to leading order by a limiting problem of Thomas-Fermi type. The limiting energy depends on the density only and is obtained from (2.1.14) by neglecting the kinetic (first) term.

If Ω_0 is allowed to be 1 in (2.1.45), we have $D_\Omega \rightarrow 0$ and so the potential term in the energy becomes negligible compared to the kinetic one. A solution u_ω should then converge to its projection on the first eigenspace of the Ginzburg-Landau operator $-(\nabla - i\omega x^\perp)^2$. This space is called the lowest Landau level and has been widely used for the study of rapidly rotating Bose-Einstein condensates in harmonic traps (see [12] and references therein). For the case $\Omega_0 = 1$ we refer to [31] where we have studied a regime of this kind in the lowest Landau level.

The main drawback of the method we present here is that it confirms the appearance of the giant vortex state but does not give the critical speed at which the transition would be expected to happen. Finding such a critical speed is an important issue for the condensed matter community (see for example [71, 92]). For simplicity and comparison with other results we will consider k as fixed in the following discussion.

At rotation speeds much smaller than those we have considered, the preferred state is known [47] to be a vortex lattice encircling a central low-density hole. The transition between this state and the giant vortex is not expected to happen in the regime $\Omega \sim G^{2/3}$. Indeed, the results in [47] (see also [65]) allow to conjecture that in the case of a flat trap ($V(x) = 0$ for $|x| \leq 1$ and $V(x) = +\infty$ for $|x| > 1$) the transition should happen when $\Omega \sim G(\log G)^{-1}$. Combined with the scalings in [44] this suggests that for our trap (2.1.5) the vortices will disappear from the condensate when $\Omega \sim G^{2/3}(\log G)^{-1}$, before the regime we have studied here is reached.

In a forthcoming work [46] ([Chapter 3]) we study, in the case of a flat trap, the transition

regime $\Omega \sim G(\log G)^{-1}$ (corresponding to $\Omega \sim G^{2/3}(\log G)^{-1}$ for the trap (2.1.5)) using tools from the Ginzburg-Landau theory [29, 135]. This completes the analysis in [47] where it is proved that the vorticity is uniform in the bulk for $\log G \ll \Omega \ll G(\log G)^{-1}$ and bridges between the situations considered in [43, 44, 47] and the present paper. We do believe that statements such as Theorem 2.1.2 and 2.1.3 are valid in the whole regime (2.1.25), but the proofs should rely on the tools used in [46] ([Chapter 3]) that allow for a better understanding of the energetic cost of the vortices.

We make one last remark before turning to the proofs of our results, regarding the validity of the Gross-Pitaevskii description. It is expected to be valid when the number of particles is much larger than the number of occupied states. In that case a significant fraction of these particles must occupy the same energy state, which is the phenomenon of Bose-Einstein condensation. In Section 2 we analyze in some detail the eigenstates of the single-particle Hamiltonian (corresponding to the linear part in (2.1.16)). In particular, we show (Corollary 2.2.1) that the energy splitting between two neighboring single particle states is bounded below by a constant. On the other hand, the interaction energy per particle (remark that we have taken a unit mass constraint, so that F_ω actually represents the energy per particle of the condensate, and $G \int_{\mathbb{R}^2} |u_\omega|^4$ the interaction energy per particle) is of order $G\omega^{1/2}$ as a consequence of Theorem 2.1.1. The number of occupied states can be computed by dividing the interaction energy by the energy splitting between two neighboring single particle states, so that one should expect the Gross-Pitaevskii description to be valid when

$$G\omega^{1/2} \ll N$$

where N is the number of particles.

This is interesting because the Gross-Pitaevskii description for rotating bosons is known to break down in some regimes of fast rotation, even at zero temperature. It is the case for a condensate trapped by an harmonic potential of the kind (2.1.3) when the rotation speed is too close (in some sense related to the number of particles considered) to the limit value $\min(a_1, a_2)$. In that situation, one must go back to the Hamiltonian for N bosons and the system exhibits strongly correlated states (fractional quantum Hall effect). We refer to [95] and references therein for details on this phenomenon.

The fractional quantum Hall effect happens in a regime where the minimization is restricted to the lowest Landau level. The corresponding situation in our setting would be a regime like (2.1.45) but with $\Omega_0 = 1$, which we do not allow in this paper. We are thus not in a regime where one should expect to get strongly correlated states.

The article is organized as follows : in Section 2 we first gather the main notation that is to be used in the paper, then we study the one-dimensional energies (2.1.18), providing the lower bounds on their ground states and first excited levels that we use in our analysis. In Section 3 we show how the results of Section 2 allow to bound the quartic term in (2.1.11) from below and prove Theorem 2.1.1. The proofs of Theorems 2.1.2 and 2.1.3 are then presented in Sections 4, with each main step occupying a subsection.

2.2 The one dimensional energies

We recall the definition of the one dimensional energies we are interested in :

$$F_n(f) = 2\pi \int_{\mathbb{R}^+} (|f'(r)|^2 + V_n(r)|f(r)|^2) r dr \quad (2.2.1)$$

for every $f \in H^1(\mathbb{R}^+, r dr) \cap L^2(\mathbb{R}^+, r V_n(r) dr)$. The potential V_n is given by :

$$\begin{aligned} V_n(r) &= V(r) + \left| \frac{n}{r} - \omega r \right|^2 \\ &= \frac{n^2}{r^2} - 2n\omega + (1 - D_\Omega)\omega^2 r^2 + D_\Omega \frac{\omega^2}{2} + D_\Omega \frac{\omega^2}{2} r^4. \end{aligned} \quad (2.2.2)$$

Where

$$V(r) = D_\Omega \frac{\omega^2}{2} (r^2 - 1)^2 \quad (2.2.3)$$

is the potential in the energy (2.1.14).

A straightforward computation shows that V_n has a unique minimum point for $r > 0$, which we shall note R_n . It is uniquely defined by the equation

$$R_n^6 + \frac{1 - D_\Omega}{D_\Omega} R_n^4 = \frac{n^2}{D_\Omega \omega^2} \quad (2.2.4)$$

and it is interesting to note that if $n = \omega$ then $R_n = 1$, which is the minimum point of the original potential V .

It is classical to show that there is an increasing sequence $(\lambda_{j,n})_{j=1 \dots +\infty}$ of eigenvalues for the operator associated to F_n . The corresponding sequence of normalized eigenfunctions $(g_{j,n})_{j=1 \dots +\infty}$ is a Hilbert basis for the space $H^1(\mathbb{R}^+, r dr) \cap L^2(\mathbb{R}^+, r V_n(r) dr)$. Considering the one-dimensional functions $g_{j,n}$ as radial functions defined on \mathbb{R}^2 , one has

$$-\Delta g_{j,n} + V_n g_{j,n} = \lambda_{j,n} g_{j,n} \quad (2.2.5)$$

and

$$\int_{\mathbb{R}^2} g_{j,n}^2(x) dx = 2\pi \int_{\mathbb{R}^+} g_{j,n}^2(r) r dr = 1. \quad (2.2.6)$$

We want to study the asymptotics of the first two eigenvalues $\lambda_{1,n}$ and $\lambda_{2,n}$ of the operator associated to F_n . We remark that $g_{1,n}$ is unique up to a multiplicative constant of modulus 1, that we fix so as to have

$$g_{1,n} \geq 0. \quad (2.2.7)$$

$\lambda_{1,n}$ satisfies

$$\begin{aligned} \lambda_{1,n} &= F_n(g_{1,n}) \\ &= \inf \left\{ F_n(f), 2\pi \int_{\mathbb{R}^+} f^2(r) r dr = 1 \right\} \end{aligned} \quad (2.2.8)$$

and $\lambda_{2,n}$

$$\begin{aligned} \lambda_{2,n} &= F_n(g_{2,n}) \\ &= \inf \left\{ F_n(f), 2\pi \int_{\mathbb{R}^+} f^2(r) r dr = 1, \int_{\mathbb{R}^+} f(r) g_{1,n}(r) r dr = 0 \right\}. \end{aligned} \quad (2.2.9)$$

Note that $g_{2,n}$ is not uniquely defined a priori.

In fact it is not necessary to carry out the analysis for every n . Indeed, we have the following result :

Lemma 2.2.1 (Lower bound to $\lambda_{1,n}$ for $|n - \omega|$ large).

Let $\lambda_{1,n}$ be defined by (2.2.8). There are constants $a, b > 0$ and $c > \sqrt{6}$ so that for

$$|n - \omega| > a \omega^{1/2}$$

or equivalently

$$|R_n - 1| > b \omega^{-1/2}$$

we have

$$\lambda_{1,n} > c \omega \tag{2.2.10}$$

Proof. For $|R_n - 1| > c_1 \omega^{-1/2}$, one has

$$\lambda_{1,n} \geq 2\pi \int_{\mathbb{R}^+} V_n(r) |g_{1,n}(r)|^2 r dr \geq 2\pi V_n(R_n) \int_{\mathbb{R}^+} |g_{1,n}(r)|^2 r dr = V_n(R_n) \geq V(R_n) > c_2 \omega$$

for two constants c_1, c_2 .

On the other hand one can see from formula (2.2.4) that $\omega \gg |n - \omega|$ is equivalent to $1 \gg |R_n - 1|$. For such values of n and R_n we have $|R_n - 1| \sim C \omega^{-1} |n - \omega|$, for a certain constant C , thus $\omega \gg |n - \omega| > c_3 \omega^{1/2}$ is equivalent to $1 \gg |R_n - 1| > c_2 \omega^{-1/2}$. We then note that the function mapping R_n to n is strictly increasing and has a strictly increasing inverse, which allows to deduce that $|n - \omega| > c_3 \omega^{1/2}$ is equivalent to $|R_n - 1| > c_2 \omega^{-1/2}$ for any value of n and R_n . There only remains to tune the constants c_1, c_2, c_3 to conclude the proof. \square

We will denote

$$\mathcal{N}_{1/2} = \left\{ n \in \mathbb{Z}, \quad |n - \omega| \leq a \omega^{1/2} \right\} \tag{2.2.11}$$

and study in detail the ground state and first excited level of the energy (2.2.1) under the mass constraint (2.2.6) for $n \in \mathcal{N}_{1/2}$. For $n \notin \mathcal{N}_{1/2}$, the simple lower bound (2.2.10) matched with the upper bound (2.1.41) will be enough for our purpose (note that $(2D_\Omega + 4)^{1/2} \leq \sqrt{6}$).

We now describe the main results concerning the modes $n \in \mathcal{N}_{1/2}$. As we will see $g_{1,n}$ and $g_{2,n}$ are asymptotically confined on a domain centered on R_n which size is of the order of $\omega^{-1/2}$. More precisely we denote

$$h_n = \left(\frac{2}{V_n''(R_n)} \right)^{1/4}. \tag{2.2.12}$$

From (2.2.2) and (2.2.4) it is straightforward to see that there is constant h so that for any $n \in \mathcal{N}_{1/2}$

$$h_n = h \omega^{-1/2} (1 + o(1)). \tag{2.2.13}$$

We define the rescaled functions $\xi_{i,n}$, $i \in \{1, 2\}$ by

$$\xi_{i,n}(x) = c_{i,n}^{-1} g_{i,n}(R_n + h_n x) \tag{2.2.14}$$

where $\|\xi_{i,n}\|_{L^2} = 1$ and

$$c_{i,n}^2 = \frac{1}{2\pi h_n R_n + 2\pi h_n^2 \int_{-\frac{R_n}{h_n}}^{+\infty} x \xi_{1,n}(x)^2 dx} \sim \frac{1}{2\pi h_n R_n}. \tag{2.2.15}$$

Remark 2.2.1. In order for the asymptotics in equations (2.1.37) and (2.2.15) to be justified, we need that ξ and $\xi_{1,n}$ decay faster than any polynomial when x goes to infinity. A typical example is to take for ξ a Gaussian, or more generally some function with an exponential decay. Very often in the rest of the paper we will have to deal with quantities of the form

$$\int_{-d_\omega}^{+\infty} F(x) dx$$

with $F : \mathbb{R} \rightarrow \mathbb{R}$ an exponentially decreasing function ($F(x) = x^k e^{-Cx^2}$, $k \in \mathbb{N}$ for example), and $d_\omega = O(\omega^\kappa)$, $\kappa > 0$. We always replace such quantities by the corresponding

$$\int_{-\infty}^{+\infty} F(x) dx$$

which is justified because the decay rate of F guarantees that the difference between the two will be exponentially small as ω goes to infinity, whereas all the other remainder terms that we will encounter will behave like powers of ω .

Note that the decay of $\xi_{1,n}$ is proved in Proposition (2.2.2).

Notation. In the rest of the paper, we will always mean by $O(\omega^k)$ a quantity bounded by ω^k uniformly with respect to D_Ω and n . C will be a generic positive constant depending neither on ω or D_Ω nor on n . When writing that a function is a $O_{L^2}(\omega^k)$ for example we mean that the function's L^2 norm is bounded by ω^k uniformly with respect to D_Ω and n . For the rescaled functions of the type (2.2.14) we will often use the notation $O_{HO}(\omega^k)$, the HO symbol meaning that the $O(\omega^k)$ is taken in the norm associated with the harmonic oscillator (2.1.24), namely the norm in $H^1(\mathbb{R}) \cap L^2(\mathbb{R}, x^2 dx)$.

We recall that the notation $a \propto b$ has the usual meaning that a/b converges to some constant.

From now on and in the rest of this section, we always implicitly consider only those n that belong to $\mathcal{N}_{1/2}$, which have the following important properties deduced from (2.2.2) and (2.2.4) :

- $|n - \omega| \leq a\omega^{1/2}$
- R_n and $\frac{1}{R_n}$ are bounded uniformly with respect to ω and D_Ω , more precisely $|R_n - 1| \leq b\omega^{-1/2}$. Moreover we have

$$|R_p - R_q| \propto \omega^{-1} |p - q| \text{ for any } p, q \in \mathcal{N}_{1/2}. \quad (2.2.16)$$

- $h_n \propto \omega^{-1/2}$. Moreover we have

$$|h_p - h_q| \propto \omega^{-3/2} |p - q| \text{ for any } p, q \in \mathcal{N}_{1/2}. \quad (2.2.17)$$

- $V_n^{(k)}(R_n) \propto \omega^2$ for any $k \geq 2$.

We are able, using the localization property of $g_{1,n}$ and $g_{2,n}$, to expand the corresponding eigenvalues in powers of h_n . We have the followings results :

Theorem 2.2.1 (Asymptotics for the ground states of the energies (2.2.1)).

Let $\lambda_{1,n}$ be defined by (2.2.8).

Suppose that $n \in \mathcal{N}_{1/2}$ as defined in equation (2.2.11) and let R_n be defined in equation (2.2.4).

We have as $\omega \rightarrow \infty$ and $D_\Omega \rightarrow D$

$$\lambda_{1,n} = V_n(R_n) + \sqrt{\frac{V_n''(R_n)}{2}} + K_n \quad (2.2.18)$$

where $K_n = O(1)$. Let ξ_1 be the normalized ground state of the harmonic oscillator (2.1.24) and $\xi_{1,n}$ be defined by equation (2.2.14). We have

$$\xi_{1,n} = \xi_1 + h_n P_n \xi_1 + h_n^2 Q_n \xi_1 + O_{HO}(\omega^{-3/2}) \quad (2.2.19)$$

where h_n is defined in equation (2.2.12), P_n and Q_n are two polynomials whose coefficients are bounded independently of ω and D_Ω and depend continuously on n in the following sense :

$$\begin{aligned} P_p \xi_1 &= P_q \xi_1 + O_{HO}(\omega^{-1}|p-q|) \\ Q_p \xi_1 &= Q_q \xi_1 + O_{HO}(\omega^{-1}|p-q|) \end{aligned} \quad (2.2.20)$$

for any $p, q \in \mathcal{N}_{1/2}$.

Note that

$$V_n(R_n) + \sqrt{\frac{V_n''(R_n)}{2}} = O(\omega).$$

From this theorem we deduce as a corollary

Corollary 2.2.1 (Variations of $\lambda_{1,n}$ with respect to n).

Under the assumptions of Theorem 2.2.1 and for ω large enough, $\lambda_{1,n}$ is minimized for at most two integers $n^* = \omega + O(1)$ and $n^* + 1$. For any $n \neq n^*, n^* + 1$ one has

$$\lambda_{1,n} = \lambda_{1,n^*} + C(n - n^*)^2 \left(1 + O(\omega^{-1/2})\right). \quad (2.2.21)$$

Equation (2.2.21) points out the dependence of $\lambda_{1,n}$ with respect to n , which is the key point to prove Theorem 2.1.1. We now state the result giving the gap between $\lambda_{2,n}$ and $\lambda_{1,n}$:

Proposition 2.2.1 (Lower bound on the first excited levels of the energies (2.2.1)).

Let $\lambda_{2,n}$ be defined by (2.2.9).

For every $n \in \mathcal{N}_{1/2}$, we have as $\omega \rightarrow +\infty$ and $D_\Omega \rightarrow D$

$$\lambda_{2,n} \geq V_n(R_n) + 3\sqrt{\frac{V_n''(R_n)}{2}} - C\omega^{1/2} \geq V_n(R_n) + 2\sqrt{\frac{V_n''(R_n)}{2}}. \quad (2.2.22)$$

Note that it is feasible to expand $\xi_{1,n}$ further and to make of Proposition 2.2.1 a statement as precise as Theorem 2.2.1, two improvements that we do not need in the sequel.

For convenience we sum up some properties of $g_{1,n}$ that will be useful in the rest of the paper in the following Corollary.

Corollary 2.2.2 (L^p bounds for $g_{1,n}$).

For any $n \in \mathcal{N}_{1/2}$ and any $1 \leq p \leq +\infty$ there is a constant C_p depending only on p such that

$$\|g_{1,n}\|_{L^p(\mathbb{R}^+, r dr)} \leq C_p \omega^{1/4 - \frac{1}{2p}} \quad (2.2.23)$$

with the convention that $\frac{1}{p} = 0$ if $p = +\infty$.

Proof. These estimates are simple consequences of (2.2.14), (2.2.15) and (2.2.19). The integrals are computed using the change of variables $r = R_n + h_n x$. Remark that $c_{1,n} \leq C\omega^{1/4}$ with C independent of ω , D_Ω and n . \square

We proceed to the proof of Theorem 2.2.1, assuming that its assumptions hold true for the rest of the section. We begin with an upper bound on $\lambda_{1,n}$:

Lemma 2.2.2 (Upper bound for $\lambda_{1,n}$).

Under the assumptions of Theorem 2.2.1 we have

$$\lambda_{1,n} \leq V_n(R_n) + \sqrt{\frac{V_n''(R_n)}{2}} + \frac{V_n^{(4)}(R_n)}{12V_n''(R_n)} \int_{\mathbb{R}} x^4 \xi_1^2(x) dx + O(\omega^{-1/2}). \quad (2.2.24)$$

Proof. We proceed as for the proof of (2.1.41), taking as a trial function

$$g_n^{test}(r) = c \xi_1(h_n^{-1}(r - R_n)) \quad (2.2.25)$$

where c is chosen so that $2\pi \|g_n^{test}\|_{L^2(\mathbb{R}^+, r dr)}^2 = 1$ with

$$h_n = \left(\frac{2}{V_n''(R_n)} \right)^{1/4}. \quad (2.2.26)$$

An important fact for the calculation is that for the range of n we are considering we always have $h_n = O(\omega^{-1/2})$. We also use that ξ_1 is an even function. \square

We now show the exponential decay of $g_{1,n}$ and its derivative outside of a shrinking region :

Proposition 2.2.2 (Pointwise estimates for $g_{1,n}$).

Let $g_{1,n}$ be defined by (2.2.8).

There are positive constants σ_1 and σ_2 so that for every $n \in \mathcal{N}_{1/2}$, every $r \in \mathbb{R}^+$ and for every $\varepsilon > 0$

$$|g_{1,n}(r)| \leq C_\varepsilon \omega^{1/4 + \varepsilon} e^{-\sigma_1 \left(\frac{r - R_n}{h_n} \right)^2} \quad (2.2.27)$$

and

$$|g'_{1,n}(r)| \leq C_\varepsilon \omega^{3/4 + \varepsilon} e^{-\sigma_2 \left(\frac{r - R_n}{h_n} \right)^2}. \quad (2.2.28)$$

Moreover, for any $\alpha < 1$ there is a constant C_α so that

$$|g_{1,n}(r)| \leq C e^{-C\omega r^\alpha} \quad (2.2.29)$$

for any $r \leq C_\alpha$.

Proof. The proof is done in four steps.

Step 1. We first need a global L^∞ bound for $g_{1,n}$. We claim that

$$\forall \eta > 0, \quad \forall \varepsilon > 0, \quad \exists C_\varepsilon > 0 \text{ so that } \|g_{1,n}\|_{L^\infty(] \eta, +\infty[)} \leq C_\varepsilon \omega^{1/4+\varepsilon}. \quad (2.2.30)$$

We first note that (2.2.24) implies

$$\lambda_{1,n} = F_n(g_{1,n}) \leq V_n(R_n) + \sqrt{V_n''(R_n)} \quad (2.2.31)$$

because both $V_n''(R_n) \propto \omega^2$ and $V_n^{(4)}(R_n) \propto \omega^2$ when $n \in \mathcal{N}_{1/2}$. Thus, using (2.2.31) and (2.2.6) :

$$2\pi \int_{\mathbb{R}^+} (|g'_{1,n}(r)|^2 + (V_n(r) - V_n(R_n)) |g_{1,n}(r)|^2) r dr \leq \sqrt{V_n''(R_n)}$$

but $V_n(r) - V_n(R_n) \geq 0$ so

$$\begin{aligned} 2\pi\eta \int_{] \eta, +\infty[} g'_{1,n}(r)^2 dr &\leq 2\pi \int_{] \eta, +\infty[} |g'_{1,n}(r)|^2 r dr \\ &\leq 2\pi \int_{\mathbb{R}^+} |g'_{1,n}(r)|^2 r dr \\ &\leq \sqrt{V_n''(R_n)} \end{aligned}$$

and recalling that $V_n''(R_n) = O(\omega^2)$ for $n \in \mathcal{N}_{1/2}$, using again (2.2.6)

$$\|g_{1,n}\|_{H^1(] \eta, +\infty[)} \leq C\omega^{1/2}. \quad (2.2.32)$$

Now, an interpolation argument, using (2.2.6) and (2.2.32) yields

$$\forall \varepsilon > 0 \quad \|g_{1,n}\|_{H^{\frac{1+3\varepsilon}{2}}(] \eta, +\infty[)} \leq C_\varepsilon \omega^{1/4+\varepsilon} \quad (2.2.33)$$

which implies (2.2.30) by Sobolev imbedding.

Step 2. We have

$$V_n(r) \geq V(r) = \frac{D_\Omega \omega^2}{2} (r^2 - 1)^2$$

and $V_n(R_n) + \sqrt{V_n''(R_n)} = O(\omega)$, thus there is a $\gamma > 0$ so that

$$V_n(r) - V_n(R_n) - \sqrt{V_n''(R_n)} > 0 \text{ for } |r - 1| > \gamma \omega^{-1/2}. \quad (2.2.34)$$

Using that $R_n = 1 + O(\omega^{-1/2})$ and $h_n \propto \omega^{-1/2}$ we deduce that there is a $\delta > 0$ so that

$$V_n(r) - V_n(R_n) - \sqrt{V_n''(R_n)} > 0 \text{ for } |r - R_n| > \delta h_n. \quad (2.2.35)$$

We denote

$$S_n = \{r \in \mathbb{R}^+, \quad |r - R_n| \leq \delta h_n\} \quad (2.2.36)$$

and prove an exponential decay property for $g_{1,n}$ on S_n^c . We recall the Euler-Lagrange equation for $g_{1,n}$ (considered as a radial function defined on \mathbb{R}^2)

$$-\Delta g_{1,n} + V_n g_{1,n} = \lambda_{1,n} g_{1,n}. \quad (2.2.37)$$

Using (2.2.31) we have for ω large enough

$$-\Delta g_{1,n} + \left(V_n(r) - V_n(R_n) - \sqrt{V_n''(R_n)} \right) g_{1,n} \leq 0. \quad (2.2.38)$$

We now apply the comparison principle on S_n^c : the right-hand side of (2.2.27) is a supersolution for (2.2.38) on S_n^c with a proper choice of σ_1 , the boundary condition being fulfilled thanks to (2.2.30). Thus the result holds on S_n^c . On the other hand, on S_n the inequality is true for C_ε large enough. Indeed, we have the global upper bound (2.2.30) and on S_n the function $e^{-\sigma_1 \left(\frac{r-R_n}{h_n} \right)^2}$ is bounded below by some constant.

Step 3. We prove (2.2.29). It is straightforward to see that the function on the right-hand side of (2.2.29) is a supersolution for equation (2.2.38) on B_{C_α} for some constant C_α depending only on α . We then use the maximum principle on $B_{\min(C_\alpha, \frac{1}{2})}$ to conclude that (2.2.29) holds. The boundary condition is fulfilled thanks to (2.2.27).

Step 4. We now prove (2.2.28) by an interpolation argument. Using (2.2.37), (2.2.24), the definition of V_n , (2.2.27) and (2.2.29) we have

$$|\Delta g_{1,n}(x)| \leq C\omega^{5/4+\varepsilon} e^{-s_3 \left(\frac{|x|-R_n}{h_n} \right)^2} \quad (2.2.39)$$

for some s_3 . Using Gagliardo-Nirenberg's inequality (see for example [117])

$$\|\nabla g_{1,n}\|_{L^\infty} \leq C \|\Delta g_{1,n}\|_{L^\infty}^{1/2} \|g_{1,n}\|_{L^\infty}^{1/2}$$

on B_R for any $0 \leq R < R_n$ and on B_R^c for any $R > R_n$, we get from (2.2.27) and (2.2.39)

$$|\nabla g_{1,n}(x)| \leq C\omega^{3/4+\varepsilon} e^{-s_2 \left(\frac{|x|-R_n}{h_n} \right)^2}$$

for some s_2 , which implies (2.2.28). □

We recall the definition of the blow-up function $\xi_{1,n}$:

$$g_{1,n}(r) = c_{1,n} \xi_{1,n} \left(h_n^{-1} (r - R_n) \right) \quad (2.2.40)$$

where h_n is defined in equation (2.2.12).

Note that this function is only defined on $] -\frac{R_n}{h_n}, +\infty[$, but as $-\frac{R_n}{h_n} \rightarrow -\infty$ and $g_{1,n}$ decreases exponentially fast in this limit, we may (remembering Remark 1) abuse notation and consider it as a function defined on \mathbb{R} .

The next lemma states a few estimates that will prove useful to expand $\lambda_{1,n}$. They are consequences of Proposition 2.2.2 :

Lemma 2.2.3 (Estimates for $\xi_{1,n}$).

Let n be in $\mathcal{N}_{1/2}$ and $C_{k,\varepsilon}$ a generic constant depending only on k and ε . We have, for ω large enough and every $\varepsilon > 0$

$$1. \left| c_{1,n}^2 h_n \int_{\mathbb{R}} x^k \xi_{1,n}(x)^2 (R_n + h_n x) dx \right| \leq C_{k,\varepsilon} \omega^\varepsilon.$$

$$2. c_{1,n}^2 h_n = \frac{1}{2\pi R_n} + O(\omega^{-1/2+\varepsilon})$$

$$3. \left| h_n \int_{\mathbb{R}} x^k \xi_{1,n}(x)^2 dx \right| \leq C_{k,\varepsilon} \omega^{\varepsilon-1/2}.$$

$$4. \left| h_n \int_{\mathbb{R}} x^k \xi_{1,n}'(x) dx \right| \leq C_{k,\varepsilon} \omega^{\varepsilon-1/2}.$$

Proof. We start with the proof of 1. Using (2.2.27) we have for every $\varepsilon > 0$

$$\begin{aligned} \left| c_{1,n}^2 h_n \int_{\mathbb{R}} x^k \xi_{1,n}(x)^2 (R_n + h_n x) dx \right| &= \left| \int_{\mathbb{R}^+} \left(\frac{r - R_n}{h_n} \right)^k g_{1,n}^2(r) r dr \right| \\ &\leq C_\varepsilon \int_{\mathbb{R}^+} \left| \frac{r - R_n}{h_n} \right|^k \omega^{1/2+\varepsilon} e^{-2s_1 \left(\frac{r-R_n}{h_n} \right)^2} r dr. \end{aligned}$$

A change of variables $r = R_n + h_n x$ then yields

$$\left| \int_{\mathbb{R}^+} \left(\frac{r - R_n}{h_n} \right)^k g_{1,n}^2(r) r dr \right| \leq C_\varepsilon \omega^\varepsilon \int_{\mathbb{R}} x^k e^{-2s_1 x^2} dx \leq C_{k,\varepsilon} \omega^\varepsilon. \quad (2.2.41)$$

We turn to the proof of 2. We have

$$c_{1,n}^2 h_n = c_{1,n}^2 h_n \int_{\mathbb{R}} \xi_{1,n}^2 = \int_{\mathbb{R}^+} g_{1,n}^2(r) dr \quad (2.2.42)$$

but

$$\begin{aligned} \int_{\mathbb{R}^+} g_{1,n}^2(r) dr &= \frac{1}{R_n} \int_{\mathbb{R}^+} g_{1,n}^2(r) r dr + \frac{1}{R_n} \int_{\mathbb{R}^+} (R_n - r) g_{1,n}^2(r) dr \\ &= \frac{1}{2\pi R_n} + \frac{1}{R_n} \int_{S_n} (R_n - r) g_{1,n}^2(r) dr \\ &\quad + \frac{1}{R_n} \int_{S_n^c} (R_n - r) g_{1,n}^2(r) dr. \end{aligned}$$

Using (2.2.27) again one gets

$$\left| \int_{S_n^c} (R_n - r) g_{1,n}^2(r) dr \right| \leq C \omega^{-1/2+\varepsilon}. \quad (2.2.43)$$

Now

$$-h_n \int_{S_n} g_{1,n}^2(r) dr \leq \int_{S_n} (R_n - r) g_{1,n}^2(r) dr \leq h_n \int_{S_n} g_{1,n}^2(r) dr \quad (2.2.44)$$

and gathering (2.2.43) and (2.2.44) we get

$$\left| \int_{\mathbb{R}^+} g_{1,n}^2(r) dr - \frac{1}{2\pi R_n} \right| \leq C \omega^{-1/2+\varepsilon}$$

for some constant C . Using (2.2.42) we have proved 2.

The proof of 3 uses the same kind of computations :

$$\begin{aligned} \left| h_n \int_{\mathbb{R}} x^k \xi_{1,n}^2(x) dx \right| &= \left| \frac{1}{c_{1,n}^2} \int_{\mathbb{R}^+} \left(\frac{r - R_n}{h_n} \right)^k g_{1,n}^2(r) dr \right| \\ &\leq \frac{C_{k,\varepsilon} \omega^\varepsilon}{c_{1,n}^2} \\ &\leq C_{k,\varepsilon} \omega^\varepsilon h_n. \end{aligned}$$

We have used (2.2.27) for the first inequality and the point 2 for the last one. We prove 4 exactly like 3, using (2.2.28) instead of (2.2.27). \square

We are now able to present the

Proof of Theorem 2.2.1. We split the proof in four steps.

Step 1. We use the blow-up (2.2.40) and expand $V_n(r)$ around R_n in the expression of $\lambda_{1,n}$. Using the fact that

$$\forall j \in \mathbb{N}, \quad \forall n \in \mathcal{N}_{1/2} \quad V^{(j)}(R_n) = O(\omega^2) \quad (2.2.45)$$

and $h_n = O(\omega^{-1/2})$ we get

$$\begin{aligned} \lambda_{1,n} = F_n(g_{1,n}) &= 2\pi \frac{c_{1,n}^2}{h_n} \int_{\mathbb{R}} \xi'_{1,n}(x)^2 (R_n + h_n x) dx \\ &+ 2\pi c_{1,n}^2 \sum_{k=0}^N \frac{V_n^{(k)}(R_n)}{k!} \int_{\mathbb{R}} h_n^{k+1} x^k \xi_{1,n}(x)^2 (R_n + h_n x) dx \\ &+ O(\omega^{2-N/2}) \int_{\mathbb{R}} |x^N \xi_{1,n}(x)^2 (R_n + h_n x)| dx \end{aligned} \quad (2.2.46)$$

where

$$c_{1,n}^2 = \frac{1}{2\pi h_n \int \xi_{1,n}(x)^2 (R_n + h_n x) dx} = \frac{1}{2\pi h_n R_n + 2\pi h_n^2 \int_{\mathbb{R}} x \xi_{1,n}(x)^2 dx} \quad (2.2.47)$$

Then we use the estimates of Lemma 2.2.3 and (2.2.45) again to bound the coefficients of $\lambda_{1,n}$ in its expansion in powers of h_n (2.2.46). Recalling the definition of h_n (2.2.12) we obtain

$$\lambda_{1,n} = V_n(R_n) + \sqrt{\frac{V_n''(R_n)}{2}} \int_{\mathbb{R}} (\xi'_{1,n}(x)^2 + x^2 \xi_{1,n}(x)^2) dx + O(\omega^{1/2+\varepsilon}). \quad (2.2.48)$$

for every $\varepsilon > 0$. Combining this estimate with (2.2.24) and dividing by $\sqrt{\frac{V_n''(R_n)}{2}} = O(\omega)$ we find

$$\int_{\mathbb{R}} (\xi'_1(x)^2 + x^2 \xi_1(x)^2) + O(\omega^{-1/2+\varepsilon}) \geq \int_{\mathbb{R}} (\xi'_{1,n}(x)^2 + x^2 \xi_{1,n}(x)^2) \geq \int_{\mathbb{R}} (\xi'_1(x)^2 + x^2 \xi_1(x)^2), \quad (2.2.49)$$

but ξ_1 is the unique normalized minimizer of the energy associated to (2.1.24), so we deduce

$$\xi_{1,n} = \xi_1 + O_{HO}(\omega^{-1/4+\varepsilon}) \quad (2.2.50)$$

for every $\varepsilon > 0$. The next steps consist in improving (2.2.50), using first a sharper energy expansion then an equation satisfied by $\xi_{1,n}$.

Step 2. We progressively improve the remainder term in (2.2.48) by a bootstrap argument. Going back to (2.2.46) and using again the results of Lemma 2.2.3 we may be more explicit :

$$\begin{aligned} \lambda_{1,n} &\geq V_n(R_n) + \sqrt{\frac{V_n''(R_n)}{2}} \int_{\mathbb{R}} (\xi'_{1,n}(x)^2 + x^2 \xi_{1,n}(x)^2) dx \\ &+ C\omega^{1/2+\varepsilon} \left(\int_{\mathbb{R}} x \xi_{1,n}(x)^2 dx + \int_{\mathbb{R}} x^3 \xi_{1,n}(x)^2 dx + \int_{\mathbb{R}} x \xi'_{1,n}(x)^2 dx \right) - C\omega^\varepsilon \end{aligned} \quad (2.2.51)$$

for any $\varepsilon > 0$. If, for some β we have

$$\xi_{1,n} = \xi_1 + O_{HO}(\omega^\beta), \quad (2.2.52)$$

then it is easy to show that for every $\varepsilon > 0$

$$\left| \int_{\mathbb{R}} x^3 \xi_{1,n}(x)^2 dx - \int_{\mathbb{R}} x^3 \xi_1(x)^2 dx \right| \leq C\omega^{\beta+\varepsilon}, \quad (2.2.53)$$

using (2.2.52), (2.2.27) and the fact that ξ_1 is a Gaussian. Now, ξ_1 being an even function we deduce

$$\left| \int_{\mathbb{R}} x^3 \xi_{1,n}(x)^2 dx \right| \leq C\omega^{\beta+\varepsilon}, \quad (2.2.54)$$

and arguing likewise to bound $\left| \int_{\mathbb{R}} x \xi_{1,n}(x)^2 dx \right|$ and $\left| \int_{\mathbb{R}} x \xi'_{1,n}(x)^2 dx \right|$ we improve (2.2.51) into

$$\lambda_{1,n} \geq V_n(R_n) + \sqrt{\frac{V_n''(R_n)}{2}} \int_{\mathbb{R}} (\xi'_{1,n}(x)^2 + x^2 \xi_{1,n}(x)^2) dx + C\omega^{1/2+\varepsilon+\beta} \quad (2.2.55)$$

and the same argument as that used with (2.2.49) to get (2.2.50) yields

$$\xi_{1,n} = \xi_1 + O_{HO}(\omega^{-\frac{1/2+\varepsilon+\beta}{2}}), \quad (2.2.56)$$

for every $\varepsilon > 0$. The fixed point of the function

$$\beta \mapsto -\frac{1}{4} + \frac{\beta}{2} + \frac{\varepsilon}{2}$$

being $\beta = -\frac{1}{2} + \varepsilon$, we deduce by induction that for every $\varepsilon > 0$

$$\xi_{1,n} = \xi_1 + O_{HO}(\omega^{-1/2+\varepsilon}), \quad (2.2.57)$$

using (2.2.50) as a starting point for the induction.

Step 3. We are going to improve the expansion (2.2.57) using the equation satisfied by $\xi_{1,n}$. We claim that

$$\xi_{1,n} = \xi_1 + h_n P_n \xi_1 + O_{HO}(\omega^{-1+\varepsilon}) \quad (2.2.58)$$

where P_n is an odd polynomial of degree 3. Let us write the equation for $\xi_{1,n}$. We write (2.2.37) in radial coordinates :

$$-g''_{1,n}(r) - \frac{g'_{1,n}(r)}{r} + V_n(r)g_{1,n}(r) = \lambda_{1,n}g_{1,n}(r). \quad (2.2.59)$$

Then, making the change of variables $r = R_n + h_n x$, expanding V_n and $\frac{1}{r}$ and multiplying by $h_n^2 c_{1,n}^{-1}$ we get

$$\begin{aligned} -\xi''_{1,n} - \frac{h_n}{R_n} \xi'_{1,n} + h_n^2 V_n(R_n) \xi_{1,n} + \frac{V_n''(R_n)}{2} h_n^4 x^2 \xi_{1,n} + \frac{V_n'''(R_n)}{6} h_n^5 x^3 \xi_{1,n} \\ = \lambda_{1,n} h_n^2 \xi_{1,n} + O_{L^2}(\omega^{-1+\varepsilon}). \end{aligned} \quad (2.2.60)$$

We have used the estimates of Lemma 2.2.3 again to evaluate the remainder. Taking $\beta = -1/2 + \varepsilon$ in (2.2.55) we have

$$\lambda_{1,n} = V_n(R_n) + \sqrt{\frac{V_n''(R_n)}{2}} + O(\omega^\varepsilon),$$

so, using the definition of h_n (2.2.12), (2.2.60) reduces to

$$-\xi_{1,n}'' - \frac{h_n}{R_n} \xi_{1,n}' + x^2 \xi_{1,n} + \frac{V_n'''(R_n)}{6} h_n^5 x^3 \xi_{1,n} = \xi_{1,n} + O_{L^2}(\omega^{-1+\varepsilon}). \quad (2.2.61)$$

Note that $\frac{V_n''(R_n)}{2} h_n^4 = 1$ by definition. Now, if we write $\xi_{1,n}$ as

$$\xi_{1,n} = \xi_1 + h_n \varphi_n \quad (2.2.62)$$

and insert this in (2.2.61), using the Euler-Lagrange equation for ξ_1 and the estimate

$$\varphi_n = O_{HO}(\omega^\varepsilon) \quad \forall \varepsilon > 0,$$

deduced from (2.2.57) and (2.2.13), we get an equation satisfied by φ_n , namely

$$-\varphi_n'' + x^2 \varphi_n - \varphi_n = \frac{1}{R_n} \xi_1' - \frac{V_n'''(R_n)}{6} h_n^4 x^3 \xi_1 + O_{L^2}(\omega^{-1/2+\varepsilon}). \quad (2.2.63)$$

On the other hand, the mass constraint for $\xi_{1,n}$ allows us to deduce from (2.2.62)

$$\left| \int \xi_1 \varphi_n \right| \leq C \omega^{-1/2+\varepsilon} \quad (2.2.64)$$

Using an expansion on the basis of normalized eigenfunctions of the harmonic oscillator (2.1.24), one can see that the problem

$$\begin{cases} -u'' + x^2 u - u = \frac{1}{R_n} \xi_1' - \frac{V_n'''(R_n)}{6} h_n^4 x^3 \xi_1 \\ \int \xi_1 u = 0 \end{cases} \quad (2.2.65)$$

has a unique solution in $H^1(\mathbb{R}) \cap L^2(\mathbb{R}, x^2 dx)$, which is of the form $u = P_n \xi_1$ where P_n is an odd polynomial of degree 3. Subtracting (2.2.63) from the first line of (2.2.65) and using (2.2.64) it is easy to show that

$$\varphi_n = P_n \xi_1 + O_{HO}(\omega^{-1/2+\varepsilon}) \quad (2.2.66)$$

thus proving (2.2.58). The first line of (2.2.20) follows by inspection of the right-hand side of (2.2.63).

Step 4. We complete the proof of (2.2.19). The argument is similar to that of Step 3. We can compute from (2.2.58) that

$$\lambda_{1,n} = V_n(R_n) + \sqrt{\frac{V_n''(R_n)}{2}} + K_n, \quad (2.2.67)$$

where $K_n = O(1)$. We write

$$\xi_{1,n} = \xi_1 + h_n P_n \xi_1 + h_n^2 \psi_n \quad (2.2.68)$$

and arguing as in Step 3 we get an equation for ψ_n :

$$\begin{aligned} -\psi_n'' + x^2\psi_n - \psi_n &= K_n\xi_1 - \frac{x}{R_n^2}\xi_1' - \frac{V_n^{(4)}(R_n)}{4!}h_n^4x^4\xi_1 \\ &\quad + \frac{(P_n\xi_1)'}{R_n} - \frac{V_n^{(3)}(R_n)}{4!}h_n^4x^3P_n\xi_1 + O_{L^2}(\omega^{-1/2+\varepsilon}) \end{aligned} \quad (2.2.69)$$

and the condition

$$\int \xi_1\psi_n = -\frac{1}{2} \int P_n^2\xi_1^2 + O(\omega^{-1/2}). \quad (2.2.70)$$

Note that, multiplying (2.2.69) by ξ_1 , integrating and using (2.2.70) we get

$$K_n = K_n' + O(\omega^{-1/2+\varepsilon}) \quad (2.2.71)$$

where

$$K_n' := \int \xi_1 \left(\frac{x}{R_n^2}\xi_1' + \frac{V_n^{(4)}(R_n)}{4!}h_n^4x^4\xi_1 - \frac{(P_n\xi_1)'}{R_n} + \frac{V_n^{(3)}(R_n)}{4!}h_n^4x^3P_n\xi_1 \right). \quad (2.2.72)$$

Solving (2.2.69) we find

$$\psi_n = Q_n\xi_1 + O_{HO}(\omega^{-1/2+\varepsilon}) \quad (2.2.73)$$

for every $\varepsilon > 0$, where Q_n is a polynomial and $Q_n\xi_1$ is the unique solution of the problem

$$\begin{cases} -u'' + x^2u - u = K_n'\xi_1 - \frac{x}{R_n^2}\xi_1' - \frac{V_n^{(4)}(R_n)}{4!}h_n^4x^4\xi_1 + \frac{(P_n\xi_1)'}{R_n} - \frac{V_n^{(3)}(R_n)}{4!}h_n^4x^3P_n\xi_1 \\ \int \xi_1u = -\frac{1}{2} \int P_n^2\xi_1^2. \end{cases} \quad (2.2.74)$$

This concludes the proof of (2.2.19). We deduce (2.2.18) by straightforward calculations. \square

We present the

Proof of Corollary 2.2.1. At this stage, we know that

$$\lambda_{1,n} = \mathcal{C}(R_n) + K_n$$

where the function \mathcal{C} is given by

$$\mathcal{C}(R_n) = V_n(R_n) + \sqrt{\frac{V_n''(R_n)}{2}} \quad (2.2.75)$$

and $K_n = O(1)$. Note that $K_n - K_m = O(\omega^{-1/2})$ for any $m, n \in \mathcal{N}_{1/2}$, as a consequence of (2.2.20), which allows to focus on \mathcal{C} . It is a function of R_n only, remembering the one-to-one relation (2.2.4) between R_n and n . Recalling (2.2.2) we have explicitly :

$$\begin{aligned} \mathcal{C}(R_n) &= D_\Omega \frac{\omega^2}{2} R_n^4 + (1 - D_\Omega) \omega^2 R_n^2 + D_\Omega \omega^2 - 2\omega^2 \sqrt{D_\Omega} R_n^2 \sqrt{R_n^2 + \frac{1 - D_\Omega}{D_\Omega}} \\ &\quad + \sqrt{12D_\Omega \omega^2 R_n^2 + 12\omega^2(1 - D_\Omega)}. \end{aligned} \quad (2.2.76)$$

Studying the variations of $\mathcal{C}(r)$ with respect to $r \in \mathbb{R}^+$ we can see that \mathcal{C} is minimized at a unique $R \in \mathbb{R}^+$ depending continuously on D_Ω and ω . We have no explicit expression for

R but one can see that $R = 1 + O(\omega^{-1})$. Expanding \mathcal{C} around R we have for any $n \in \mathcal{N}_{1/2}$

$$\mathcal{C}(R_n) = \mathcal{C}(R) + \frac{\mathcal{C}''(R)}{2} (R_n - R)^2 \left(1 + O(\omega^{-1/2})\right) \quad (2.2.77)$$

because $\mathcal{C}^{(k)}(R) = O(\omega^2)$ for $k \geq 2$ and $R_n - R = O(\omega^{-1/2})$ when $n \in \mathcal{N}_{1/2}$. On the other hand, using (2.2.4) we get that for any $n \in \mathcal{N}_{1/2}$, $\omega|R_n - R| \propto |N - n|$ where $N = D_\Omega^{1/2} \omega \left(R^6 + \frac{1-D_\Omega}{D_\Omega} R^4\right)^{1/2}$ so

$$\mathcal{C}(R_n) - \mathcal{C}(R) \propto \frac{\mathcal{C}''(R)}{2\omega^2} (n - N)^2 \left(1 + O(\omega^{-1/2})\right). \quad (2.2.78)$$

We want to define n^* as the minimizer in $\mathcal{N}_{1/2}$ of $(n - N)^2$. Only two cases can occur : either N is exactly half an integer, then $[N]$ and $[N] + 1$ both minimize $(n - N)^2$ in $\mathcal{N}_{1/2}$, or there is exactly one minimizer.

In both cases we pick one minimizer n^* and (2.2.21) follows by inspection of (2.2.78). An important point is that $|n^* - N| \leq 1/2$. □

We now turn to the proof of Proposition 2.2.1. We begin with an upper bound on $\lambda_{2,n}$:

Lemma 2.2.4 (Upper bound for $\lambda_{2,n}$).

Let $\lambda_{2,n}$ and $g_{2,n}$ be defined by (2.2.9).

We have for every $n \in \mathcal{N}_{1/2}$, as $\omega \rightarrow \infty$ and $D_\Omega \rightarrow D$

$$\lambda_{2,n} = F_n(g_{2,n}) \leq V_n(R_n) + 3\sqrt{\frac{V_n''(R_n)}{2}} + O(\omega^{1/2}). \quad (2.2.79)$$

Proof. We begin with a test function constructed as that in the proof of Lemma 2.2.2 :

$$G_n^{test}(r) = c\xi_2(h_n^{-1}(r - R_n)) \quad (2.2.80)$$

where c is chosen so that $2\pi\|G_n^{test}\|_{L^2}^2 = 1$. Then we define

$$\tilde{G}_n^{test} = G_n^{test} - \left(\int_{\mathbb{R}^+} g_{1,n}(r)G_n^{test}(r)rdr\right)g_{1,n} \quad (2.2.81)$$

and, as $\int_{\mathbb{R}^+} \tilde{G}_n^{test} g_{1,n} r dr = 0$ we have

$$F_n(g_{2,n}) \leq \frac{1}{\|\tilde{G}_n^{test}\|_{L^2(\mathbb{R}^+, r dr)}^2} F_n(\tilde{G}_n^{test}). \quad (2.2.82)$$

Using the Euler-Lagrange equation (2.2.37) for $g_{1,n}$ one proves that

$$F_n(\tilde{G}_n^{test}) = F_n(G_n^{test}) - \left(\int_{\mathbb{R}^+} g_{1,n}(r)G_n^{test}(r)rdr\right)^2 \lambda_{1,n} \leq F_n(G_n^{test}). \quad (2.2.83)$$

Computing $F_n(G_n^{test})$ as (2.1.41), we obtain

$$\begin{aligned} F_n(G_n^{test}) &= V_n(R_n) + \sqrt{\frac{V_n''(R_n)}{2}} \int_{\mathbb{R}} (\xi_2'(x)^2 + x^2\xi_2(x)^2) dx + O(\omega^{1/2}) \\ &= V_n(R_n) + 3\sqrt{\frac{V_n''(R_n)}{2}} + O(\omega^{1/2}). \end{aligned} \quad (2.2.84)$$

because ξ_2 is the second normalized eigenfunction of the harmonic oscillator (2.1.24), associated to the eigenvalue 3. On the other hand

$$\begin{aligned} \|\tilde{G}_n^{test}\|_{L^2(\mathbb{R}^+, r dr)}^2 &= 1 - \left(\int_{\mathbb{R}^+} g_{1,n}(r) G_n^{test}(r) r dr \right)^2 \\ &\geq 1 - C \left(\int_{\mathbb{R}} \xi_{1,n}(x) \xi_2(x) (R_n + h_n x) dx \right)^2 \geq 1 - C\omega^{-1} \end{aligned} \quad (2.2.85)$$

using (2.2.19) and $\int_{\mathbb{R}} \xi_1 \xi_2 = 0$. Gathering equations (2.2.82) to (2.2.85) we get the result, remembering that $V_n(R_n) + 3\sqrt{\frac{V_n''(R_n)}{2}} = O(\omega)$. \square

Proof of Proposition 2.2.1. Starting from the upper bound (2.2.79), we can prove the equivalents of Proposition 2.2.2 and Lemma 2.2.3 for $g_{2,n}$. We omit the detailed calculations since they are easy modifications of those we used for $g_{1,n}$. Just note that $g_{2,n}$ is not positive so we have to rely on the inequality

$$-\Delta |g_{2,n}|^2 + 2V_n |g_{2,n}|^2 \leq 2\lambda_{2,n} |g_{2,n}|^2 \quad (2.2.86)$$

to prove the equivalents of (2.2.27) and (2.2.29). (2.2.86) is a consequence of (2.2.5). Then we can argue as in Step 1 of the proof of Theorem 2.2.1 and obtain

$$\lambda_{2,n} = V_n(R_n) + \sqrt{\frac{V_n''(R_n)}{2}} \int_{\mathbb{R}} (\xi_{2,n}'(x)^2 + x^2 \xi_{2,n}(x)^2) dx + O(\omega^{1/2+\varepsilon}). \quad (2.2.87)$$

But

$$\int_{\mathbb{R}^+} g_{1,n}(r) g_{2,n}(r) r dr = 0$$

and this implies, using (2.2.19),

$$\int_{\mathbb{R}} \xi_1(x) \xi_{2,n}(x) dx = O(\omega^{-1/2}) \quad (2.2.88)$$

We obtain (2.2.22) from (2.2.87) and (2.2.88). The second inequality holds because $V_n''(R_n) = O(\omega^2)$. \square

2.3 Lower bound on the interaction energy of u_ω , proof of Theorem 2.1.1

In this section we show how the estimates of Section 2 on the one-dimensional problems allow one to bound the quartic term in the energy (2.1.11) from below. We then deduce the energy and density asymptotics of Theorem 2.1.1.

We begin with a property of exponential decay for u_ω :

Proposition 2.3.1 (Exponential decay for u_ω).

There is a constant σ so that

$$|u_\omega(x)| \leq C\omega^{1/2} e^{-\sigma\omega||x|-1|^2} \text{ for any } x \in \mathbb{R}^2. \quad (2.3.1)$$

Proof. We recall the Euler-Lagrange equation for u_ω :

$$-\left(\nabla - i\omega x^\perp\right)^2 u_\omega + D_\Omega \frac{\omega^2}{2} (|x|^2 - 1)^2 u_\omega + 2G|u_\omega|^2 u_\omega = \mu_\omega u_\omega \quad (2.3.2)$$

where μ_ω is the Lagrange multiplier associated with the mass constraint. From this we deduce an equation satisfied by $U := |u_\omega|^2$,

$$-\Delta U + 4GU^2 + 2\left|\nabla u_\omega - i\omega x^\perp u_\omega\right|^2 + 2U\left(D_\Omega \frac{\omega^2}{2} (|x|^2 - 1)^2 - \mu_\omega\right) = 0. \quad (2.3.3)$$

As in the proof of Proposition 2.2.2, we need first a global L^∞ bound. We claim that

$$\|U\|_{L^\infty} \leq C\omega, \quad (2.3.4)$$

which is proved using a method due to Alberto Farina [60]. We introduce

$$\tilde{U} := U - \frac{\mu_\omega}{2G}.$$

We use Kato's inequality and (2.3.3)

$$\begin{aligned} \Delta(\tilde{U}^+) &\geq \text{Id}_{\tilde{U} \geq 0} \Delta \tilde{U} \\ &= \text{Id}_{\tilde{U} \geq 0} \left(4GU^2 + 2\left|\nabla u_\omega - i\omega x^\perp u_\omega\right|^2 + 2U\left(D_\Omega \frac{\omega^2}{2} (|x|^2 - 1)^2 - \mu_\omega\right) \right) \\ &\geq \text{Id}_{\tilde{U} \geq 0} (4GU^2 - 2\mu_\omega U) \\ &= \text{Id}_{\tilde{U} \geq 0} 4G\tilde{U}^2 \geq 4G(\tilde{U}^+)^2. \end{aligned}$$

We have $-\Delta(\tilde{U}^+) + 4G(\tilde{U}^+)^2 \leq 0$ which implies $\tilde{U}^+ \equiv 0$ (see [36]). We multiply the Euler-Lagrange equation (2.1.16) by u_ω^* and integrate over \mathbb{R}^2 to get

$$\mu_\omega = I_\omega + G \int_{\mathbb{R}^2} |u_\omega|^4 \quad (2.3.5)$$

and thus

$$\mu_\omega \leq 2I_\omega \leq C\omega \quad (2.3.6)$$

because of the upper bound (2.1.41). Using (2.3.6) and the fact that G is larger than some constant, the claim is proved.

The rest of the proof is an application of the maximum principle. We define

$$\mathcal{A}_{\omega,\delta} = \left\{ x \in \mathbb{R}^2, \left| |x| - 1 \right|^2 \leq \delta\omega^{-1} \right\}. \quad (2.3.7)$$

We have

$$-\Delta U + 2U\left(D_\Omega \frac{\omega^2}{2} (|x|^2 - 1)^2 - \mu_\omega\right) \leq 0 \quad (2.3.8)$$

and $\left(D_\Omega \frac{\omega^2}{2} (|x|^2 - 1)^2 - \mu_\omega\right) > 0$ on $\mathcal{A}_{\omega,\delta}^c$ if δ is large enough. We use two comparison functions :

$$F_1(x) := C_1 \omega e^{-\sigma_1 \omega (|x|^2 - 1)^2}$$

which is a supersolution to (2.3.8) on $\mathcal{A}_{\omega,\delta}^c \cap \{|x| \leq 1\}$ if σ_1 is chosen small enough and

$$F_2(x) := C_2 \omega e^{-\sigma_2 \omega (|x| - 1)^2}$$

which is a supersolution to (2.3.8) on $\mathcal{A}_{\omega,\delta}^c \cap \{|x| \geq 1\}$ if σ_2 is chosen small enough. The boundary conditions for the application of the comparison principle are fulfilled thanks to (2.3.4) by taking C_1 and C_2 large enough. There remains to note that

$$F_1(x) \leq C_1 \omega e^{-\sigma_1 \omega (|x|-1)^2}$$

to show that (2.3.1) holds on $\mathcal{A}_{\omega,\delta}^c$ with $\sigma = \frac{1}{2} \min(\sigma_1, \sigma_2)$ and $C = \max(\sqrt{C_1}, \sqrt{C_2})$. On the other hand (2.3.4) implies that (2.3.1) is true (for a large enough constant C) on $\mathcal{A}_{\omega,\delta}$ because $e^{-\sigma \omega (|x|-1)^2}$ is bounded below there, which concludes the proof. \square

Let us recall that we have the Fourier expansion

$$u_\omega(r, \theta) = \sum_{n \in \mathbb{Z}} f_n(r) e^{in\theta}.$$

We denote $\langle \cdot, \cdot \rangle$ the scalar product in $L^2(\mathbb{R}^+, r dr)$:

$$\langle u, v \rangle = \int_{\mathbb{R}^+} u(r) v^*(r) r dr$$

and introduce

$$\tilde{u}_\omega(r, \theta) := \sum_{n \in \mathcal{N}_{1/2}} \langle f_n, g_{1,n} \rangle g_{1,n}(r) e^{in\theta}. \quad (2.3.9)$$

As a consequence of our analysis in Section 2, the modes corresponding to $n \notin \mathcal{N}_{1/2}$ carry very little mass, whereas for $n \in \mathcal{N}_{1/2}$ we have $f_n \propto g_{1,n}$ in the L^2 sense. This can be summed up in the

Proposition 2.3.2 (First estimate on $\|u_\omega - \tilde{u}_\omega\|_{L^2(\mathbb{R}^2)}$).

Let u_ω and \tilde{u}_ω be defined respectively by (2.1.15) and (2.3.9). Under the assumptions of Theorem 2.1.1 we have

$$\|u_\omega - \tilde{u}_\omega\|_{L^2(\mathbb{R}^2)} \leq CG^{1/2} \omega^{-1/4} = o(1). \quad (2.3.10)$$

Proof. We have to prove

$$\sum_{n \in \mathcal{N}_{1/2}^c} \int_{\mathbb{R}^+} |f_n|^2 r dr \leq CG \omega^{-1/2} \quad (2.3.11)$$

and

$$\sum_{n \in \mathcal{N}_{1/2}} \int_{\mathbb{R}^+} |f_n - \langle f_n, g_{1,n} \rangle g_{1,n}|^2 r dr \leq CG \omega^{-1/2}. \quad (2.3.12)$$

For $n \in \mathcal{N}_{1/2}$ we can write f_n as

$$f_n = \langle f_n, g_{1,n} \rangle g_{1,n} + \left(\int_{\mathbb{R}^+} |f_n - \langle f_n, g_{1,n} \rangle g_{1,n}|^2 r dr \right)^{1/2} g_n^\perp \quad (2.3.13)$$

where

$$\int_{\mathbb{R}^+} g_{1,n}(r) g_n^\perp(r) r dr = 0$$

and

$$2\pi \int_{\mathbb{R}^+} |g_n^\perp(r)|^2 r dr = 1,$$

thus we have

$$F_n(f_n) \geq \lambda_{1,n} |\langle f_n, g_{1,n} \rangle|^2 + \lambda_{2,n} \int_{\mathbb{R}^+} |f_n - \langle f_n, g_{1,n} \rangle g_{1,n}|^2 r dr \quad (2.3.14)$$

for $n \in \mathcal{N}_{1/2}$. Using (2.1.43) and (2.3.14), ignoring the interaction energy, we bound $F_\omega(u_\omega)$ from below in the following way

$$\begin{aligned} F_\omega(u_\omega) \geq \sum_{n \in \mathcal{N}_{1/2}} \lambda_{1,n} |\langle f_n, g_{1,n} \rangle|^2 + \sum_{n \in \mathcal{N}_{1/2}} \lambda_{2,n} \int_{\mathbb{R}^+} |f_n - \langle f_n, g_{1,n} \rangle g_{1,n}|^2 r dr \\ + \sum_{n \in \mathcal{N}_{1/2}^c} \lambda_{1,n} 2\pi \int_{\mathbb{R}^+} |f_n|^2 r dr. \end{aligned}$$

But the energy of $g_{1,n^*} e^{in^*\theta}$ is an upper bound for $F_\omega(u_\omega)$ and $\lambda_{1,n} \geq \lambda_{1,n^*}$ for any n . Combined with the results of Proposition 2.2.1 and (2.2.10) we get

$$\begin{aligned} \lambda_{1,n^*} + 2\pi G \int_{\mathbb{R}^+} g_{1,n^*}^4(r) r dr \geq \sum_{n \in \mathcal{N}_{1/2}} \left(|\langle f_n, g_{1,n} \rangle|^2 \lambda_{1,n^*} \right. \\ \left. + \left(\lambda_{1,n^*} + \sqrt{\frac{V_n''(R_n)}{2}} \right) \int_{\mathbb{R}^+} |f_n - \langle f_n, g_{1,n} \rangle g_{1,n}|^2 r dr \right) \\ + \sum_{n \in \mathcal{N}_{1/2}^c} c \omega 2\pi \int_{\mathbb{R}^+} |f_n|^2 r dr \quad (2.3.15) \end{aligned}$$

with $c > \sqrt{6}$. On the other hand, we know from the L^2 normalization of u_ω that

$$\begin{aligned} 1 = \sum_{n \in \mathbb{Z}} 2\pi \int_{\mathbb{R}^+} |f_n|^2 r dr &= \sum_{n \in \mathcal{N}_{1/2}} |\langle f_n, g_{1,n} \rangle|^2 \\ &+ \sum_{n \in \mathcal{N}_{1/2}} \int_{\mathbb{R}^+} |f_n - \langle f_n, g_{1,n} \rangle g_{1,n}|^2 r dr \\ &+ \sum_{n \in \mathcal{N}_{1/2}^c} 2\pi \int_{\mathbb{R}^+} |f_n|^2 r dr. \quad (2.3.16) \end{aligned}$$

Combining (2.3.15) and (2.3.16), using that $\lambda_{1,n^*} - \sqrt{6}\omega < 0$ for ω large enough we get

$$\begin{aligned} 2\pi G \int_{\mathbb{R}^+} g_{1,n^*}^4(r) r dr \geq \\ \sum_{n \in \mathcal{N}_{1/2}} \int_{\mathbb{R}^+} |f_n - \langle f_n, g_{1,n} \rangle g_{1,n}|^2 r dr \sqrt{\frac{V_n''(R_n)}{2}} + \sum_{n \in \mathcal{N}_{1/2}^c} C \omega 2\pi \int_{\mathbb{R}^+} |f_n|^2 r dr \end{aligned}$$

and this implies the desired result because $\sqrt{\frac{V_n''(R_n)}{2}} > C\omega$ for $n \in \mathcal{N}_{1/2}$ and

$$2\pi \int_{\mathbb{R}^+} g_{1,n^*}^4(r) r dr \leq C\omega^{1/2}$$

which is a consequence of (2.2.19). \square

We now prove a simple lemma which states that the total interaction energy of u_ω is an upper bound to the pairwise interactions of the modes f_n . We state the lemma for a general $\Phi \in L^4(\mathbb{R}^2)$:

Lemma 2.3.1 (Generic lower bound on $\int_{\mathbb{R}^2} |\Phi|^4$).

Let Φ be any $L^4(\mathbb{R}^2)$ function with Fourier decomposition

$$\Phi(r, \theta) = \sum_{n \in \mathbb{Z}} \phi_n(r) e^{in\theta}.$$

We have

$$\int_{\mathbb{R}^2} |\Phi|^4 \geq 2\pi \sum_{p,q} \int_{\mathbb{R}^+} |\phi_p|^2 |\phi_q|^2 r dr. \quad (2.3.17)$$

Proof. We expand $|\Phi|^4$:

$$\int_{\mathbb{R}^2} |\Phi|^4 = \sum_{n,p,q,r} \int_{\mathbb{R}^+} \int_0^{2\pi} \phi_n \phi_p \overline{\phi_q \phi_r} e^{i(n+p-q-r)\theta} d\theta r dr \quad (2.3.18)$$

and the only integrals with respect to θ that are not zero are those for which $n+p-q-r = 0$ so

$$\begin{aligned} \int_{\mathbb{R}^2} |\Phi|^4 &= 2\pi \sum_{n,p,q} \int_{\mathbb{R}^+} \phi_{n+q} \phi_p \overline{\phi_q \phi_{n+p}} r dr \\ &= 2\pi \sum_n \int_{\mathbb{R}^+} \left| \sum_p \phi_p \overline{\phi_{n+p}} \right|^2 r dr \\ &\geq 2\pi \int_{\mathbb{R}^+} \left| \sum_p |\phi_p|^2 \right|^2 r dr \\ &= 2\pi \sum_{p,q} \int_{\mathbb{R}^+} |\phi_p|^2 |\phi_q|^2 r dr \end{aligned}$$

□

We are now able to prove the lower bound on $\int_{\mathbb{R}^2} |u_\omega|^4$, combining the results of Proposition 2.3.2 and Lemma 2.3.1. A very important fact in the computation is that for $n \in \mathcal{N}_{1/2}$, $f_n \propto g_{1,n}$, and that the appropriate blow-up of $g_{1,n}$ converges to a gaussian (equation (2.2.19)).

Proposition 2.3.3 (Lower bound on the interaction energy).

Let u_ω be a solution to (2.1.15) with Fourier expansion

$$u_\omega = \sum_{n \in \mathbb{Z}} f_n(r) e^{in\theta}.$$

Set

$$\tilde{u}_\omega(r, \theta) := \sum_{n \in \mathcal{N}_{1/2}} \langle f_n, g_{1,n} \rangle g_{1,n}(r) e^{in\theta}.$$

where $g_{1,n}$ is defined by equation (2.1.21). We have

$$\begin{aligned} \int_{\mathbb{R}^2} |u_\omega|^4 &\geq 2\pi \int_{\mathbb{R}^+} |g_{1,n^*}|^4(r) r dr \\ &\quad - C\omega^{-1/2} \sum_{n \in \mathcal{N}_{1/2}} |n - n^*|^2 |\langle f_n, g_{1,n} \rangle|^2 \\ &\quad + O(\omega^{1/2} \|u_\omega - \tilde{u}_\omega\|_{L^2}) + O(1). \end{aligned} \quad (2.3.19)$$

Proof. We begin by using Lemma 2.3.1 :

$$\int_{\mathbb{R}^2} |u_\omega|^4 \geq 2\pi \sum_{p,q} \int_{\mathbb{R}^+} |f_p|^2 |f_q|^2 r dr.$$

A consequence of Proposition 2.3.2 is that it is enough to consider only the pairwise interactions of the modes labeled by $n \in \mathcal{N}_{1/2}$ (see equation (2.3.38) below), so that we use

$$\int_{\mathbb{R}^2} |u_\omega|^4 \geq 2\pi \sum_{p,q \in \mathcal{N}_{1/2}} \int_{\mathbb{R}^+} |f_p|^2 |f_q|^2 r dr$$

and evaluate the right-hand side.

We note that, using (2.3.1)

$$\begin{aligned} \sum_{p,q \in \mathcal{N}_{1/2}} \int_{r \geq 3/2} |f_p|^2 |f_q|^2 r dr &\leq C\omega^4 \sum_{p,q \in \mathcal{N}_{1/2}} \int_{r \geq 3/2} e^{-C\omega^2 |r-1|^2} r dr \\ &\leq C \sum_{p,q \in \mathcal{N}_{1/2}} e^{-C\omega} \int_{r \geq 3/2} e^{-C\omega^2 |r-1|^2} r dr \\ &\leq C(\#\mathcal{N}_{1/2})^2 e^{-C\omega} \leq e^{-C\omega}. \end{aligned} \quad (2.3.20)$$

We have used that $\#\mathcal{N}_{1/2} \leq C\omega^{1/2}$. A similar estimate holds true for $\sum_{p,q \in \mathcal{N}_{1/2}} \int_{r \leq 1/2} |f_p|^2 |f_q|^2 r dr$ so that it is sufficient to consider integration domains where $1/2 \leq r \leq 3/2$, which we implicitly do in the rest of the proof.

It is convenient to introduce the following quantities :

$$\alpha_n = R_n - R_{n^*} \quad (2.3.21)$$

and the rescaled functions ζ_p defined for $p \in \mathcal{N}_{1/2}$ by

$$f_p(r) = d_p \zeta_p(h_p^{-1}(r - R_p)) \quad (2.3.22)$$

where h_p is defined by equation (2.2.12),

$$d_p^2 = \frac{\int |f_p|^2 r dr}{h_p \int |\zeta_p(x)|^2 (R_p + h_p x) dx} \quad (2.3.23)$$

and $\|\zeta_p\|_{L^2} = 1$. In this proof we denote β_n the n -th term of a generic sequence satisfying

$$\sum_{n \in \mathbb{Z}} \beta_n^2 \leq C \quad (2.3.24)$$

where C is independent of ω and D_Ω . The actual value of the quantity β_n may thus change from line to line.

Using (2.3.10), we have

$$\begin{aligned} \sum_{p \in \mathcal{N}_{1/2}} \int |d_p \zeta_p - c_{1,p} \langle f_p, g_{1,p} \rangle \xi_{1,p}|^2 (R_p + h_p x) dx &= \sum_{p \in \mathcal{N}_{1/2}} h_p^{-1} \int_{\mathbb{R}^+} |f_p - \langle f_p, g_{1,p} \rangle g_{1,p}|^2 r dr \\ &\leq C \omega^{1/2} \|u_\omega - \tilde{u}_\omega\|_{L^2}^2 \end{aligned}$$

thus

$$\forall p \in \mathcal{N}_{1/2} \quad d_p \zeta_p = c_{1,p} \langle f_p, g_{1,p} \rangle \xi_{1,p} + O_{L^2((R_p+h_p x) dx)}(\beta_p \omega^{1/4} \|u_\omega - \tilde{u}_\omega\|_{L^2}) \quad (2.3.25)$$

where $\sum_p \beta_p^2$ is bounded uniformly with respect to ω .

On the other hand, using the bounds $\sum_n F_n(f_n) \leq F_\omega(u_\omega) \leq C\omega$ we have

$$\|d_p \zeta_p\|_{H^1((R_p+h_p x) dx)} \leq \beta_p \omega^{1/4}. \quad (2.3.26)$$

Also

$$\|c_{1,p} \xi_{1,p}\|_{H^1((R_p+h_p x) dx)} \leq \omega^{1/4} \quad (2.3.27)$$

follows from (2.2.19).

A change of variables $r = R_{n^*} + h_{n^*} x$ yields (α_n is defined by (2.3.21))

$$\begin{aligned} \sum_{p,q \in \mathcal{N}_{1/2}} 2\pi \int |f_p|^2 |f_q|^2 r dr &= \\ \sum_{p,q} h_{n^*} d_p^2 d_q^2 \left(\int |\zeta_p|^2 \left(\frac{h_{n^*}}{h_p} x + \frac{\alpha_p}{h_p} \right) |\zeta_q|^2 \left(\frac{h_{n^*}}{h_q} x + \frac{\alpha_q}{h_q} \right) (R_{n^*} + h_{n^*} x) dx \right). \end{aligned} \quad (2.3.28)$$

Then we have, using (2.3.25), (2.3.26), (2.3.27) and Sobolev imbeddings

$$\begin{aligned} \sum_{p,q} h_{n^*} d_p^2 d_q^2 \int |\zeta_p|^2 \left(\frac{h_{n^*}}{h_p} x + \frac{\alpha_p}{h_p} \right) |\zeta_q|^2 \left(\frac{h_{n^*}}{h_q} x + \frac{\alpha_q}{h_q} \right) (R_{n^*} + h_{n^*} x) dx &= \\ \sum_{p,q} |\langle f_p, g_{1,p} \rangle|^2 |\langle f_q, g_{1,q} \rangle|^2 h_{n^*} c_{1,p}^2 c_{1,q}^2 \int \xi_{1,p}^2 \left(\frac{h_{n^*}}{h_p} x + \frac{\alpha_p}{h_p} \right) \xi_{1,q}^2 \left(\frac{h_{n^*}}{h_q} x + \frac{\alpha_q}{h_q} \right) (R_{n^*} + h_{n^*} x) dx \\ &\quad + O(\omega^{1/2} \|u_\omega - \tilde{u}_\omega\|_{L^2}). \end{aligned} \quad (2.3.29)$$

Remark that it is clear from the definition of $c_{1,n}$ (2.2.15) and (2.2.19) that

$$c_{1,n} = \frac{1}{2\pi h_n R_n} + O(\omega^{-1/2}) \quad (2.3.30)$$

because ξ_1 is an even function. Using (2.2.19) and (2.3.30) we obtain

$$\begin{aligned} \sum_{p,q} h_{n^*} d_p^2 d_q^2 \int |\zeta_p|^2 \left(\frac{h_{n^*}}{h_p} x + \frac{\alpha_p}{h_p} \right) |\zeta_q|^2 \left(\frac{h_{n^*}}{h_q} x + \frac{\alpha_q}{h_q} \right) (R_{n^*} + h_{n^*} x) dx &= \\ \sum_{p,q} |\langle f_p, g_{1,p} \rangle|^2 |\langle f_q, g_{1,q} \rangle|^2 \frac{h_{n^*}}{h_p R_p h_q R_q} \int \xi_1^2 \left(\frac{h_{n^*}}{h_p} x + \frac{\alpha_p}{h_p} \right) \xi_1^2 \left(\frac{h_{n^*}}{h_q} x + \frac{\alpha_q}{h_q} \right) (R_{n^*} + h_{n^*} x) dx \\ &\quad + O(\omega^{1/2} \|u_\omega - \tilde{u}_\omega\|_{L^2}) + O(1). \end{aligned} \quad (2.3.31)$$

But for any $p \in \mathcal{N}_{1/2}$ we have from (2.2.17) that $\frac{h_{n^*}}{h_p} = 1 + O(\omega^{-1}|n^* - p|) = 1 + O(\omega^{-1/2})$ because $n^* = \omega + O(1)$, so

$$\begin{aligned} & \int \xi_1^2 \left(\frac{h_{n^*}}{h_p} x + \frac{\alpha_p}{h_p} \right) \xi_1^2 \left(\frac{h_{n^*}}{h_q} x + \frac{\alpha_q}{h_q} \right) (R_{n^*} + h_{n^*} x) dx \\ &= \int \xi_1^2 \left(x + \frac{\alpha_p}{h_p} \right) \xi_1^2 \left(x + \frac{\alpha_q}{h_q} \right) (R_{n^*} + h_{n^*} x) dx + O(\omega^{-1/2}) \end{aligned} \quad (2.3.32)$$

and using that ξ_1 is a Gaussian we have

$$\begin{aligned} & \int \xi_1^2 \left(x + \frac{\alpha_p}{h_p} \right) \xi_1^2 \left(x + \frac{\alpha_q}{h_q} \right) (R_{n^*} + h_{n^*} x) dx \\ &= e^{-\left(\frac{\alpha_p}{h_p} - \frac{\alpha_q}{h_q}\right)^2} \int \xi_1^4 \left(x + \frac{\alpha_p}{2h_p} + \frac{\alpha_q}{2h_q} \right) (R_{n^*} + h_{n^*} x) dx \\ &\geq \left(1 - C\omega(\alpha_p - \alpha_q)^2\right) \int \xi_1^4(x) \left(R_{n^*} + h_{n^*} \left(x - \frac{\alpha_p}{2h_p} - \frac{\alpha_q}{2h_q} \right) \right) dx \\ &= \left(1 - C\omega(\alpha_p - \alpha_q)^2\right) \int \xi_1^4(x) (R_{n^*} + h_{n^*} x) dx + O(\omega^{-1/2}) \end{aligned} \quad (2.3.33)$$

where we have used that $h_n = O(\omega^{-1/2})$ and $|\alpha_n| = |R_n - R_{n^*}| \propto \omega^{-1}|n - n^*| = O(\omega^{-1/2})$ (see (2.2.16)).

On the other hand we have

$$\forall p, q \in \mathcal{N}_{1/2}, \quad \frac{1}{h_p h_q} = \frac{1}{h_{n^*}^2} + O(\omega^{1/2}) \quad (2.3.34)$$

using that $\frac{h_{n^*}}{h_p} = 1 + O(\omega^{-1}|n^* - p|) = 1 + O(\omega^{-1/2})$ and

$$\forall p, q \in \mathcal{N}_{1/2}, \quad \frac{1}{R_p R_q} = \frac{1}{R_{n^*}^2} + O(\omega^{-1/2}) \quad (2.3.35)$$

using that $|R_p - R_{n^*}| \propto \omega^{-1}|p - n^*| = O(\omega^{-1/2})$. Gathering equations (2.3.28) to (2.3.35) we have

$$\begin{aligned} \sum_{p,q} 2\pi \int |f_p|^2 |f_q|^2 r dr &\geq \left(\sum_{p,q \in \mathcal{N}_{1/2}} |\langle f_p, g_{1,p} \rangle|^2 |\langle f_q, g_{1,q} \rangle|^2 \right) \frac{1}{2\pi h_{n^*} R_{n^*}^2} \\ &\times \left(\left(1 - C\omega(\alpha_p - \alpha_q)^2\right) \int \xi_1^4(x) (R_{n^*} + h_{n^*} x) dx \right) \\ &+ O(\omega^{1/2} \|u_\omega - \tilde{u}_\omega\|_{L^2}) + O(1). \end{aligned} \quad (2.3.36)$$

But (2.2.19) gives

$$2\pi \int_{\mathbb{R}} g_{1,n^*}(r)^4 r dr = \frac{1}{2\pi h_{n^*} R_{n^*}^2} \int_{\mathbb{R}} \xi_1(x)^4 (R_{n^*} + h_{n^*} x) dx + O(\omega^{-1/2}) \quad (2.3.37)$$

We also have, as a consequence of (2.3.10) and (2.1.10),

$$\begin{aligned} \sum_{p,q \in \mathcal{N}_{1/2}} |\langle f_p, g_{1,p} \rangle|^2 |\langle f_q, g_{1,q} \rangle|^2 &= \left(\sum_{p \in \mathcal{N}_{1/2}} |\langle f_p, g_{1,p} \rangle|^2 \right)^2 \\ &= (1 - \|u_\omega - \tilde{u}_\omega\|_{L^2})^2 \geq 1 - C\|u_\omega - \tilde{u}_\omega\|_{L^2} \end{aligned} \quad (2.3.38)$$

and

$$\begin{aligned}
 \sum_{p,q \in \mathcal{N}_{1/2}} (\alpha_p - \alpha_q)^2 |\langle f_p, g_{1,p} \rangle|^2 |\langle f_q, g_{1,q} \rangle|^2 &\leq 2 \sum_{p,q \in \mathcal{N}_{1/2}} (\alpha_p^2 + \alpha_q^2) |\langle f_p, g_{1,p} \rangle|^2 |\langle f_q, g_{1,q} \rangle|^2 \\
 &\leq 4 \left(\sum_{p \in \mathcal{N}_{1/2}} \alpha_p^2 |\langle f_p, g_{1,p} \rangle|^2 \right) \left(\sum_{q \in \mathcal{N}_{1/2}} |\langle f_q, g_{1,q} \rangle|^2 \right) \\
 &\leq 4 \sum_{p \in \mathcal{N}_{1/2}} \alpha_p^2 |\langle f_p, g_{1,p} \rangle|^2 \tag{2.3.39}
 \end{aligned}$$

because $\sum_{q \in \mathcal{N}_{1/2}} |\langle f_q, g_{1,q} \rangle|^2 \leq 1$. Combining (2.3.36), (2.3.37), (2.3.38) and (2.3.39) we obtain

$$\int_{\mathbb{R}^2} |u_\omega|^4 \geq 2\pi \int_{\mathbb{R}^+} |g_{1,n^*}|^4(r) r dr - C \sum_{p \in \mathcal{N}_{1/2}} \omega^{3/2} \alpha_p^2 |\langle f_p, g_{1,p} \rangle|^2 + O(\omega^{1/2} \|u_\omega - \tilde{u}_\omega\|_{L^2}) + O(1).$$

We conclude the proof by recalling that $|\alpha_n| = |R_n - R_{n^*}| \propto \omega^{-1} |n - n^*|$ using (2.2.4), so that

$$\omega^{3/2} \alpha_n^2 \geq C \omega^{-1/2} |n - n^*|^2.$$

□

Note that the n -th term of the second line of (2.3.19) can be interpreted as an upper bound to the gain in interaction energy per unit mass carried by the mode f_n . On the other hand Corollary 2.2.1 provides a lower bound to the loss in kinetic and potential energy per unit mass carried by the mode f_n . The comparison of this two quantities is the crucial argument of the

Proof of Theorem 2.1.1. We proceed in three steps

Step 1. We first claim that the following improvement of (2.3.10) holds

$$\|u_\omega - \tilde{u}_\omega\|_{L^2} \leq CG\omega^{-1/2}. \tag{2.3.40}$$

Combining the results of Theorem 2.2.1 and Proposition 2.3.3 we get

$$\begin{aligned}
 F_\omega(u_\omega) &\geq 2\pi G \int_{\mathbb{R}^+} |g_{1,n^*}|^4(r) r dr \\
 &+ \sum_{n \in \mathcal{N}_{1/2}} |\langle f_n, g_{1,n} \rangle|^2 \left(\lambda_{1,n} - CG\omega^{-1/2} |n - n^*|^2 \right) \\
 &+ \sum_{n \in \mathcal{N}_{1/2}} \lambda_{2,n} \int_{\mathbb{R}^+} |f_n - \langle f_n, g_{1,n} \rangle g_{1,n}|^2 r dr \\
 &+ \sum_{n \in \mathcal{N}_{1/2}^c} \lambda_{1,n} 2\pi \int_{\mathbb{R}^+} |f_n|^2 r dr + O(G\omega^{1/2} \|u_\omega - \tilde{u}_\omega\|_{L^2}) + O(G). \tag{2.3.41}
 \end{aligned}$$

We now use $g_{1,n^*} e^{in^*\theta}$ as a test function and proceed as in the proof of Proposition 2.3.2.

With the result of Corollary 2.2.1 we get

$$\begin{aligned}
 & C(G\omega^{1/2}\|u_\omega - \tilde{u}_\omega\|_{L^2} + G + G\omega^{-1/2}) \\
 & \geq \sum_{n \in \mathcal{N}_{1/2}, n \neq n^*, n^*+1} |\langle f_n, g_{1,n} \rangle|^2 \left(C|n - n^*|^2 - G\omega^{-1/2}|n - n^*|^2 \right) \\
 & \quad + \sum_{n \in \mathcal{N}_{1/2}} C\omega \int_{\mathbb{R}^+} |f_n - \langle f_n, g_{1,n} \rangle g_{1,n}|^2 r dr \\
 & \quad \quad \quad + \sum_{n \in \mathcal{N}_{1/2}^c} C\omega \int_{\mathbb{R}^+} |f_n|^2 r dr. \quad (2.3.42)
 \end{aligned}$$

Now, for $n \in \mathcal{N}_{1/2}$, one has

$$C|n - n^*|^2 - G\omega^{-1/2}|n - n^*|^2 \geq 0$$

because $G \ll \omega^{1/2}$, so equation (2.3.42) yields

$$\begin{aligned}
 C \left(G\omega^{1/2}\|u_\omega - \tilde{u}_\omega\|_{L^2} + G + G\omega^{-1/2} \right) & \geq C\omega \sum_{n \in \mathcal{N}_{1/2}} \int_{\mathbb{R}^+} |f_n - \langle f_n, g_{1,n} \rangle g_{1,n}|^2 r dr \\
 & \quad + C\omega \sum_{n \in \mathcal{N}_{1/2}^c} \int_{\mathbb{R}^+} |f_n|^2 r dr \\
 & \geq C\omega \|u_\omega - \tilde{u}_\omega\|_{L^2}^2
 \end{aligned}$$

which implies (2.3.40).

Remark also that we can replace $\lambda_{2,n} \int_{\mathbb{R}^+} |f_n - \langle f_n, g_{1,n} \rangle g_{1,n}|^2 r dr$ by $\sum_{j \geq 2} \lambda_{j,n} |\langle f_n, g_{j,n} \rangle|^2$ in (2.3.41). Indeed, writing

$$f_n = \sum_{j \geq 1} \langle f_n, g_{j,n} \rangle g_{j,n} \quad (2.3.43)$$

one has

$$F_n(f_n) = \sum_{j \geq 1} |\langle f_n, g_{j,n} \rangle|^2 \lambda_{j,n}. \quad (2.3.44)$$

This yields

$$C(G^2 + G + G\omega^{-1/2}) \geq \sum_{n \in \mathcal{N}_{1/2}} \sum_{j \geq 2} \lambda_{j,n} |\langle f_n, g_{j,n} \rangle|^2 \quad (2.3.45)$$

which will be useful in Section 4.

Step 2. We prove (2.1.26). The upper bound is obtained by taking $g_{1,n^*} e^{in^*\theta}$ as a test function. For the lower bound we first prove the following essential estimate :

$$\sum_{n \in \mathcal{N}_{1/2}} \|f_n\|_{L^2(\mathbb{R}^+, r dr)}^2 |n - n^*|^2 \leq CG^2. \quad (2.3.46)$$

We recall that we assume $G^2 \geq g^2$ for some constant g so it suffices to prove

$$\sum_{n \in \mathcal{N}_{1/2}, n \neq n^*, n^*+1} \|f_n\|_{L^2(\mathbb{R}^+, r dr)}^2 |n - n^*|^2 \leq CG^2. \quad (2.3.47)$$

Coming back to (2.3.42) and using (2.3.40) we get

$$C \left(G^2 + G + G\omega^{-1/2} \right) \geq \sum_{n \in \mathcal{N}_{1/2}, n \neq n^*, n^*+1} |\langle f_n, g_{1,n} \rangle|^2 \left(|n - n^*|^2 - G\omega^{-1/2}|n - n^*|^2 \right). \quad (2.3.48)$$

But $G\omega^{-1/2} \ll 1$, so if $n \neq n^*, n^* + 1$ we have, for some constant $C' < C$,

$$C|n - n^*|^2 - G\omega^{-1/2}|n - n^*|^2 \geq C'|n - n^*|^2.$$

This allows to deduce from (2.3.48) that

$$\sum_{n \in \mathcal{N}_{1/2}, n \neq n^*, n^*+1} |\langle f_n, g_{1,n} \rangle|^2 |n - n^*|^2 \leq CG^2. \quad (2.3.49)$$

On the other hand we already have as a consequence of (2.3.40) that

$$\sum_{n \in \mathcal{N}_{1/2}} |n - n^*|^2 \int_{\mathbb{R}^+} |f_n - \langle f_n, g_{1,n} \rangle|^2 r dr \leq C\omega \sum_{n \in \mathcal{N}_{1/2}} \int_{\mathbb{R}^+} |f_n - \langle f_n, g_{1,n} \rangle|^2 r dr \leq CG^2 \quad (2.3.50)$$

so (2.3.49) implies (2.3.47) and thus (2.3.46).

Now, according to Proposition 2.3.3 and (2.3.40) we have

$$F_\omega(u_\omega) \geq \lambda_{1,n^*} + 2\pi G \int_{\mathbb{R}^+} g_{1,n^*}^4(r) r dr - CG\omega^{-1/2} \sum_{n \in \mathcal{N}_{1/2}} |n - n^*|^2 \|f_n\|_{L^2(\mathbb{R}^+, r dr)}^2 - CG^2. \quad (2.3.51)$$

Using (2.3.46) and $G \ll \omega^{1/2}$ we get

$$F_\omega(u_\omega) \geq \lambda_{1,n^*} + 2\pi G \int_{\mathbb{R}^+} g_{1,n^*}^4(r) r dr - CG^2$$

which was the missing lower bound to prove (2.1.26). Recall that $G \ll \omega^{1/2}$ and $\int_{\mathbb{R}^+} g_{1,n^*}^4(r) r dr \propto \omega^{1/2}$.

Step 3. We prove (2.1.27). We have

$$\int_{\mathbb{R}^2} (|u_\omega|^2 - g_{1,n^*}^2)^2 = \int_{\mathbb{R}^2} |u_\omega|^4 + g_{1,n^*}^4 - 2g_{1,n^*}^2 |u_\omega|^2 \leq 2 \int_{\mathbb{R}^2} (g_{1,n^*}^4 - g_{1,n^*}^2 |u_\omega|^2). \quad (2.3.52)$$

Indeed

$$\lambda_{1,n^*} + G \int_{\mathbb{R}^2} |g_{1,n^*}|^4 \geq F_\omega(u_\omega) \geq \lambda_{1,n^*} + G \int_{\mathbb{R}^2} |u_\omega|^4$$

so

$$\int_{\mathbb{R}^2} |u_\omega|^4 \leq \int_{\mathbb{R}^2} |g_{1,n^*}|^4. \quad (2.3.53)$$

Now, using the same calculations as those in the proof of Proposition 2.3.3 and (2.3.40) we can prove

$$\begin{aligned} \int_{\mathbb{R}^2} g_{1,n^*}^2 |u_\omega|^2 &\geq \sum_{n \in \mathcal{N}_{1/2}} 2\pi \int_{\mathbb{R}^+} |f_n(r)|^2 g_{1,n^*}^2(r) r dr \\ &\geq \int_{\mathbb{R}^2} g_{1,n^*}^4 - C \sum_{n \in \mathcal{N}_{1/2}} \omega^{-1/2} |n - n^*|^2 \|f_n\|_{L^2(\mathbb{R}^+, r dr)}^2 - CG \end{aligned} \quad (2.3.54)$$

Then plugging (2.3.54) and (2.3.46) into (2.3.52) we get

$$\int_{\mathbb{R}^2} (|u_\omega|^2 - g_{1,n^*}^2)^2 \leq C \sum_{n \in \mathcal{N}_{1/2}} \omega^{-1/2} |n - n^*|^2 \|f_n\|_{L^2(\mathbb{R}^+, r dr)}^2 + CG \leq C(G^2 \omega^{-1/2} + G) \leq CG$$

and the proof is complete. \square

2.4 The giant vortex state for a fixed coupling constant

In this section we present the proofs of Theorems 2.1.2 and 2.1.3. They require improvements of the method presented in Sections 2 and 3. In particular we will need an energy expansion with a remainder term going to 0 as ω goes to infinity. We know from (2.1.26) that

$$F_\omega(u_\omega) = \lambda_{1,n^*} + 2\pi G \int_{\mathbb{R}^+} g_{1,n^*}^4 r dr + O(1)$$

in the regime where G is a fixed constant. But, for any n , by the definition (2.4.2) of γ_n

$$\lambda_{1,n} + 2\pi G \int_{\mathbb{R}^+} g_{1,n}^4 r dr \geq \gamma_n$$

and one can realize from (2.2.19) and (2.4.7) below that the difference between these two quantities is of order 1. We thus need to analyze in some details the nonlinear problem defining γ_n , and use the results to provide better upper and lower bounds to $F_\omega(u_\omega)$. We first present the analysis of the nonlinear one dimensional problems in Subsection 4.1 below.

2.4.1 Nonlinear one dimensional problems

We recall the definition of the one-dimensional energies

$$E_n(f) = 2\pi \int_{\mathbb{R}^+} (|f'(r)|^2 + V_n(r)|f(r)|^2 + G|f(r)|^4) r dr \quad (2.4.1)$$

with

$$\gamma_n = E_n(\Psi_n) = \inf \left\{ E_n(f), f \in H^1(\mathbb{R}^+, r dr) \cap L^2(\mathbb{R}^+, r V_n(r) dr), 2\pi \int_{\mathbb{R}^+} f^2(r) r dr = 1 \right\}. \quad (2.4.2)$$

Note that Ψ_n is uniquely defined up to a multiplicative constant of modulus 1, that we fix by requiring that

$$\Psi_n \geq 0. \quad (2.4.3)$$

We also define the rescaled functions ρ_n by

$$\rho_n(x) = c_n^{-1} \Psi_n(R_n + h_n x) \quad (2.4.4)$$

where c_n is chosen so that $\|\rho_n\|_{L^2} = 1$:

$$c_n^2 = \frac{1}{2\pi h_n R_n + 2\pi h_n^2 \int_{-\frac{R_n}{h_n}}^{+\infty} x \rho_n(x)^2 dx}. \quad (2.4.5)$$

We have

Theorem 2.4.1 (Asymptotics for the ground states of the energies (2.4.1)).

Let γ_n be defined by (2.4.2).

Suppose that $n \in \mathcal{N}_{1/2}$ as defined in (2.2.11) and let R_n be defined in (2.2.4). Let ξ_1 be the normalized ground state of the harmonic oscillator (2.1.24).

We have as $\omega \rightarrow \infty$ and $D_\Omega \rightarrow D$

$$\gamma_n = V_n(R_n) + \sqrt{\frac{V_n''(R_n)}{2}} + \frac{G}{2\pi h_n R_n} \int_{\mathbb{R}} \xi_1(x)^4 dx + J_n \quad (2.4.6)$$

where $J_n = O(1)$.

Let ρ_n be defined by equation (2.4.4). We have

$$\rho_n = \xi_1 + h_n \tau_n + h_n^2 v_n + O_{HO}(\omega^{-3/2}) \quad (2.4.7)$$

where h_n is defined in (2.2.12), τ_n and v_n are solutions to linear second order ODEs. They are bounded in harmonic oscillator norm, uniformly with respect to n . Also, for any $p, q \in \mathcal{N}_{1/2}$

$$\begin{aligned} \tau_p &= \tau_q + O_{HO}(\omega^{-1}|p - q|) \\ v_p &= v_q + O_{HO}(\omega^{-1}|p - q|). \end{aligned} \quad (2.4.8)$$

From this theorem we deduce as a corollary

Corollary 2.4.1 (Variations of γ_n with respect to n).

Let n^* and $n^* + 1$ be defined as in Corollary 2.2.1. Under the assumptions of Theorem 2.4.1 and for ω large enough, for any $n \neq n^*, n^* + 1$ one has

$$\gamma_n = \gamma_{n^*} + C(n - n^*)^2 \left(1 + O(\omega^{-1/2})\right). \quad (2.4.9)$$

Moreover

$$\gamma_{n^*+1} \geq \gamma_{n^*} - C\omega^{-1/2}. \quad (2.4.10)$$

We also state the equivalent of Corollary 2.2.2 :

Corollary 2.4.2 (L^p bounds for Ψ_n).

For any $n \in \mathcal{N}_{1/2}$ and any $1 \leq p \leq +\infty$ there is a constant C_p depending only on p such that

$$\|\Psi_n\|_{L^p(\mathbb{R}^+, r dr)} \leq C_p \omega^{1/4 - \frac{1}{2p}} \quad (2.4.11)$$

with the convention that $\frac{1}{p} = 0$ if $p = +\infty$.

The proofs are exactly similar to those of the corresponding results in Section 2, so we only give their main steps.

Proof of Theorem 2.4.1. We begin with an upper bound on γ_n using a test function of the form

$$\Psi_n^{\text{test}} = c_n^{\text{test}} \xi_1 \left(\frac{r - R_n}{h_n} \right).$$

We obtain

$$\gamma_n \leq V_n(R_n) + \sqrt{\frac{V_n''(R_n)}{2}} + \frac{G}{2\pi h_n R_n} \int_{\mathbb{R}} \xi_1(x)^4 dx + O(1) \quad (2.4.12)$$

We then prove an exponential decay property for Ψ_n and its first derivative exactly similar to that of Proposition 2.2.2. Step 1 of the proof of this Proposition needs no modification to apply to Ψ_n .

We use the maximum principle on the equation for Ψ_n which reads

$$-\Delta\Psi_n + V_n(r)\Psi_n + 2G\Psi_n^3 = \left(\gamma_n + 2\pi G \int_{\mathbb{R}^+} |\Psi_n(r)|^4 r dr \right) \Psi_n. \quad (2.4.13)$$

Using (2.4.12) we get easily

$$-\Delta\Psi_n + \left(V_n(r) - V_n(R_n) - \sqrt{V_n''(R_n)} \right) \Psi_n \leq 0$$

and we proceed as in the proof of Proposition 2.2.2. We then prove the equivalent of Lemma 2.2.3, which allows to get an expansion of γ_n . We use this expansion as in Step 1 of the proof of Theorem 2.2.1 and get

$$\rho_n = \xi_1 + O_{HO}(\omega^{-1/4+\varepsilon})$$

for any $\varepsilon > 0$, and follow the method of Step 2 of the same proof to improve the remainder term and get

$$\rho_n = \xi_1 + O_{HO}(\omega^{-1/2+\varepsilon}). \quad (2.4.14)$$

We deduce an equation for ρ_n starting from (2.4.13) :

$$\begin{aligned} -\rho_n'' - \frac{h_n}{R_n} \rho_n' + \frac{V_n''(R_n)}{2} h_n^4 x^2 \rho_n + 2Gc_n^2 h_n^2 \rho_n^3 + \frac{V_n^{(3)}(R_n)}{6} h_n^5 x^3 \rho_n \\ = h_n^2 \left(\gamma_n + 2\pi G \int_{\mathbb{R}^+} |\Psi_n(r)|^4 r dr \right) \rho_n + O_{L^2}(h_n^2). \end{aligned} \quad (2.4.15)$$

We then deduce from this equation that

$$\rho_n = \xi_1 + h_n \tau_n + h_n^2 v_n + O_{HO}(\omega^{-3/2}). \quad (2.4.16)$$

We have denoted τ_n the solution of the problem

$$\begin{cases} -u'' + x^2 u - u = \frac{1}{R_n} \xi_1' - \frac{V_n^{(3)}(R_n)}{6} h_n^4 x^3 \xi_1 + \frac{G}{\pi R_n} \left(\int \xi_1^4 \right) \xi_1 - \frac{G}{\pi R_n} \xi_1^3 \\ \int \xi_1 u = 0 \end{cases}$$

and v_n the solution of the problem

$$\begin{cases} -u'' + x^2 u - u = J_n' \xi_1 - \frac{x}{R_n^2} \xi_1' - \frac{V_n^{(4)}}{4!} h_n^4 x^4 \xi_1 + \frac{\tau_n'}{R_n} - \frac{V_n^{(3)}(R_n)}{6} h_n^4 x^3 \tau_n \\ - \frac{3G}{\pi R_n} \tau_n \xi_1^3 + \frac{2G}{\pi R_n} \left(\int \xi_1^4 \right) \tau_n \\ \int \xi_1 u = -\frac{1}{2} \int \tau_n^2 \end{cases}$$

where $J_n' = O(1)$ is chosen so that the right-hand side of the above equation is a function orthogonal to ξ_1 , which implies that the system indeed has a unique solution. In particular one has $J_n' = J_n + O(\omega^{-1/2})$.

This follows exactly Steps 3 and 4 of the proof of Theorem 2.2.1. Remark that the coefficients of the ODEs satisfied by τ_n and v_n depend continuously on R_n seen as a continuous variable for the range of n we consider. The estimates (2.4.8) are a consequence of this fact. We deduce (2.4.6) from (2.4.7). \square

Proof of Corollary 2.4.1. At this stage, we know that

$$\gamma_n = \mathcal{C}(R_n) + \mathcal{D}(R_n) + J_n$$

where the function \mathcal{C} is defined in equation (2.2.75), \mathcal{D} is given by

$$\mathcal{D}(R_n) = \frac{G}{2\pi h_n R_n} \int_{\mathbb{R}} \xi_1(x)^4 dx \quad (2.4.17)$$

and $J_n = O(1)$. We have $J_n - J_m = O(\omega^{-1/2})$ for any $m, n \in \mathcal{N}_{1/2}$, as a consequence of (2.4.8), and from the explicit expression of \mathcal{D} one can compute that

$$|\mathcal{D}'(R_n)| \leq C\omega^{1/2}$$

for any $n \in \mathcal{N}_{1/2}$. We thus have, using Corollary 2.2.1

$$\begin{aligned} \gamma_n &= \gamma_{n^*} + C|n - n^*|^2 + O(\omega^{1/2})|R_n - R_{n^*}| + O(\omega^{-1/2}) \\ &= \gamma_{n^*} + C|n - n^*|^2 + O(\omega^{-1/2})|n - n^*| + O(\omega^{-1/2}) \\ &= \gamma_{n^*} + C|n - n^*|^2(1 + O(\omega^{-1/2})) \end{aligned}$$

for any $n \neq n^*, n^* + 1$.

Equation (2.4.10) also follows from this discussion. \square

2.4.2 Some improved estimates

In this subsection we aim at obtaining a better L^∞ bound for u_ω , which will be crucial in our analysis. Indeed, the first bound that we obtained (see Proposition 2.3.1) is far from being optimal as one can realize by taking the L^2 norm of both sides of (2.3.1). The right-hand side has a L^2 norm proportional to $\omega^{1/4}$ whereas $\|u_\omega\|_{L^2(\mathbb{R}^2)} = 1$. A result of the analysis below is the improved bound

$$\|u_\omega\|_{L^\infty(\mathbb{R}^2)} \leq C\omega^{1/4}. \quad (2.4.18)$$

This will follow from (2.4.20) and (2.4.26) below and implies (this is a simple modification of the proof of Proposition 2.3.1) that (2.3.1) can be improved as

$$|u_\omega(x)| \leq C\omega^{1/4} e^{-\sigma\omega||x|-1|^2} \text{ for any } x \in \mathbb{R}^2. \quad (2.4.19)$$

The right-hand side of this inequality is now bounded in L^2 .

We begin with a estimates for \tilde{u}_ω :

Proposition 2.4.1 (Estimates for \tilde{u}_ω).

There exist a constant $\tilde{\sigma}$ so that the following holds on \mathbb{R}^2

$$|\tilde{u}_\omega(x)| \leq C\omega^{1/4} e^{-\tilde{\sigma}\omega||x|-1|^2}. \quad (2.4.20)$$

Moreover, for any $0 \leq p \leq +\infty$

$$\|\tilde{u}_\omega\|_{L^p(\mathbb{R}^2)} \leq C\omega^{1/4-1/2p} \quad (2.4.21)$$

with the convention that $\frac{1}{p} = 0$ if $p = +\infty$.

Proof. We recall that

$$\tilde{u}_\omega(r, \theta) := \sum_{n \in \mathcal{N}_{1/2}} \langle f_n, g_{1,n} \rangle g_{1,n}(r) e^{in\theta}$$

so, using (2.2.23) we get

$$\begin{aligned} |\tilde{u}_\omega(x)| &\leq C\omega^{1/4} \sum_{n \in \mathcal{N}_{1/2}} |\langle f_n, g_{1,n} \rangle| \\ &\leq C\omega^{1/4} \left(\sum_{n \in \mathcal{N}_{1/2}} |\langle f_n, g_{1,n} \rangle|^2 |n - n^*|^2 \right)^{1/2} \left(\sum_{n \in \mathcal{N}_{1/2}} |n - n^*|^{-2} \right)^{1/2} \\ &\leq C\omega^{1/4} \end{aligned} \tag{2.4.22}$$

where we have used (2.3.46) to pass to the third line. The estimate (2.4.20) is proved using the same kind of computations : We remark that if $||x| - 1| \geq \tilde{C}\omega^{-1/2}$, then $(|x| - R_n)^2 \geq (1 - C\tilde{C})(|x| - 1)^2$ for some constant C . This is a consequence of the fact that if $n \in \mathcal{N}_{1/2}$ then $R_n - 1 = O(\omega^{-1/2})$. With a proper choice of \tilde{C} we thus have, using (2.2.19), that for any $r \in \mathbb{R}$ so that $|r - 1| \geq \tilde{C}\omega^{-1/2}$ and any $n \in \mathcal{N}_{1/2}$

$$|g_{1,n}(r)| \leq C\omega^{1/4} e^{-ch_n^2|r-R_n|^2} \leq C\omega^{1/4} e^{-\tilde{\sigma}\omega|r-1|^2}. \tag{2.4.23}$$

We use this fact and the same trick as in (2.4.22) to conclude that (2.4.20) holds on the domain $\{|x| - 1| \geq \tilde{C}\omega^{-1/2}\}$. On the complement of this domain $e^{-\tilde{\sigma}\omega|r-1|^2}$ is bounded below, so (2.4.20) is a consequence of (2.4.22).

Finally (2.4.21) follows from (2.4.20). \square

We now aim at improving (2.3.40), giving estimates in stronger norms. As already emphasized the most important result is (2.4.26), but we also state estimates (2.4.24) and (2.4.25) because they actually imply (2.4.26) (see Step 5 of the proof below).

Proposition 2.4.2 (Estimates in stronger norms).

Recall that \tilde{u}_ω is defined in equation (2.3.9). The following estimates hold true when $\omega \rightarrow +\infty$ and $D_\Omega \rightarrow D$:

$$\|u_\omega - \tilde{u}_\omega\|_{H^1(\mathbb{R}^2)} \leq C_\varepsilon \omega^\varepsilon \text{ for any } \varepsilon > 0 \tag{2.4.24}$$

$$\|u_\omega - \tilde{u}_\omega\|_{H^2(\mathbb{R}^2)} \leq C_\varepsilon \omega^{1+\varepsilon} \text{ for any } \varepsilon > 0 \tag{2.4.25}$$

$$\|u_\omega - \tilde{u}_\omega\|_{L^\infty(\mathbb{R}^2)} \leq C_\varepsilon \omega^\varepsilon \text{ for any } \varepsilon > 0. \tag{2.4.26}$$

The proof uses mainly the Euler-Lagrange equation for u_ω , combined with elliptic estimates for the Ginzburg-Landau operator. We state the result we are going to use for convenience and refer to [106] or [7] for a proof.

Lemma 2.4.1.

Let $A \in (W^{2,\infty}(\mathbb{R}^2))^2$ be a divergence-free map, $g \in L^2(\mathbb{R}^2)$ and w a solution of

$$-(\nabla - iA)^2 w = g \text{ in } \mathbb{R}^2. \tag{2.4.27}$$

Then for any $R > 0$ there exists a constant $C_R > 0$ so that

$$\sum_{j,k=1}^2 \int_{B_R} |(\partial_j - iA_j)(\partial_k - iA_k)w|^2 \leq C_R (\|\Delta A\|_{L^\infty(B_{2R})} + \|\text{curl} A\|_{L^\infty(B_{2R})}) \left(\int_{B_{2R}} |g|^2 + |w|^2 \right). \quad (2.4.28)$$

Moreover, C_R remains bounded as R goes to infinity.

Proof of Proposition 2.4.2. Step 1. We claim that

$$F_\omega(u_\omega - \tilde{u}_\omega) \leq C\omega^{1/2} \quad (2.4.29)$$

which implies

$$\|u_\omega - \tilde{u}_\omega\|_{H^1(\mathbb{R}^2)} \leq C\omega^{1/4}. \quad (2.4.30)$$

We begin by multiplying the Euler-Lagrange equation (2.1.16) by \tilde{u}_ω^* . Integrating and injecting the result into $F_\omega(u_\omega - \tilde{u}_\omega)$, we obtain

$$\begin{aligned} F_\omega(u_\omega - \tilde{u}_\omega) &= F_\omega(u_\omega) + F_\omega(\tilde{u}_\omega) - \mu_\omega \int_{\mathbb{R}^2} (u_\omega \tilde{u}_\omega^* + u_\omega^* \tilde{u}_\omega) \\ &\quad + G \int_{\mathbb{R}^2} (2|u_\omega|^2 |\tilde{u}_\omega|^2 + u_\omega |u_\omega|^2 \tilde{u}_\omega^* + u_\omega^* |\tilde{u}_\omega|^2 \tilde{u}_\omega - u_\omega \tilde{u}_\omega^3 - u_\omega^* \tilde{u}_\omega^3). \end{aligned} \quad (2.4.31)$$

We note that a consequence of (2.3.53) is

$$\int_{\mathbb{R}^2} |u_\omega|^4 \leq CG\omega^{1/2}. \quad (2.4.32)$$

The terms in the second line of (2.4.31) are also bounded by $CG\omega^{1/2}$ using Hölder inequalities, (2.4.21) and (2.4.32). On the other hand, equations (2.1.26), (2.3.46) and (2.4.20) yield

$$F_\omega(u_\omega) = F_\omega(\tilde{u}_\omega) + O(G\omega^{1/2}) = \lambda_{1,n^*} + O(G\omega^{1/2}). \quad (2.4.33)$$

From (2.3.5) and (2.4.32) we also obtain

$$\mu_\omega = \lambda_{1,n^*} + O(G\omega^{1/2}). \quad (2.4.34)$$

Then, using (2.3.40) we deduce (2.4.29) from (2.4.31), remembering that in this section we assume that G is a fixed constant.

Step 2. Using (2.4.21) together with (2.4.30) and a Sobolev imbedding implies

$$\|u_\omega\|_{L^p(\mathbb{R}^2)} \leq C_\varepsilon \omega^{1/4+\varepsilon} \quad (2.4.35)$$

for any $1 \leq p \leq +\infty$. We then interpolate between L^2 (remember that $\|u_\omega\|_{L^2(\mathbb{R}^2)} = 1$) and L^p , make $p \rightarrow +\infty$ to get

$$\|u_\omega\|_{L^6(\mathbb{R}^2)} \leq C_\varepsilon \omega^{1/6+\varepsilon} \text{ for any } \varepsilon > 0. \quad (2.4.36)$$

Step 3. We now turn to the proof of (2.4.25). The Euler-Lagrange equation for $g_{1,n}$ can be written

$$-\left(\nabla - i\omega x^\perp\right)^2 g_{1,n} e^{in\theta} + D_\Omega \frac{\omega^2}{2} (|x|^2 - 1)^2 g_{1,n} e^{in\theta} = \lambda_{1,n} g_{1,n} e^{in\theta}, \quad (2.4.37)$$

so that we get for \tilde{u}_ω

$$-\left(\nabla - i\omega x^\perp\right)^2 \tilde{u}_\omega + D_\Omega \frac{\omega^2}{2} (|x|^2 - 1)^2 \tilde{u}_\omega = \sum_{n \in \mathcal{N}_{1/2}} \lambda_{1,n} \langle f_n, g_{1,n} \rangle g_{1,n} e^{in\theta}. \quad (2.4.38)$$

We subtract this equation from the equation for u_ω to obtain

$$\begin{aligned} -\left(\nabla - i\omega x^\perp\right)^2 (u_\omega - \tilde{u}_\omega) &= \mu_\omega u_\omega - \sum_{n \in \mathcal{N}_{1/2}} \lambda_{1,n} \langle f_n, g_{1,n} \rangle g_{1,n} e^{in\theta} \\ &\quad - D_\Omega \frac{\omega^2}{2} (|x|^2 - 1)^2 (u_\omega - \tilde{u}_\omega) - 2G|u_\omega|^2 u_\omega. \end{aligned} \quad (2.4.39)$$

We denote

$$\delta_\omega := \mu_\omega u_\omega - \sum_{n \in \mathcal{N}_{1/2}} \lambda_{1,n} \langle f_n, g_{1,n} \rangle g_{1,n} e^{in\theta} - D_\Omega \frac{\omega^2}{2} (|x|^2 - 1)^2 (u_\omega - \tilde{u}_\omega) - 2G|u_\omega|^2 u_\omega \quad (2.4.40)$$

and provide a bound in $L^2(B_{2R})$ to this quantity, for some $R > 0$. First, using (2.3.40) and the exponential decay results of Propositions 2.3.1 and 2.4.1 we get

$$\int_{B_{2R}} D_\Omega^2 \omega^4 (|x|^2 - 1)^4 |u_\omega - \tilde{u}_\omega|^2 \leq C_\varepsilon \omega^{1+\varepsilon}. \quad (2.4.41)$$

On the other hand

$$\begin{aligned} \int_{B_{2R}} \left| \mu_\omega u_\omega - \sum_{n \in \mathcal{N}_{1/2}} \lambda_{1,n} \langle f_n, g_{1,n} \rangle g_{1,n} e^{in\theta} \right|^2 &\leq C \mu_\omega^2 \left(\sum_{n \in \mathcal{N}_{1/2}} \int_{B_{2R}} |f_n - \langle f_n, g_{1,n} \rangle g_{1,n}|^2 + \sum_{n \in \mathcal{N}_{1/2}^c} \int_{B_{2R}} |f_n|^2 \right) \\ &\quad + C \sum_{n \in \mathcal{N}_{1/2}} (\lambda_{1,n} - \mu_\omega)^2 |\langle f_n, g_{1,n} \rangle|^2. \end{aligned}$$

We know that $|\mu_\omega| \leq C\omega$ and $\|u_\omega - \tilde{u}_\omega\|_{L^2(\mathbb{R}^2)} \leq C\omega^{-1/2}$, whereas comparing (2.1.26) and (2.2.21) we get $|\lambda_{1,n} - \mu_\omega| \leq C \max(\omega^{1/2}, |n - n^*|^2)$. Using (2.3.46) and recalling that for $n \in \mathcal{N}_{1/2}$, $|n - n^*| \leq C\omega^{1/2}$ we thus have

$$\begin{aligned} \int_{B_{2R}} \left| \mu_\omega u_\omega - \sum_{n \in \mathcal{N}_{1/2}} \lambda_{1,n} \langle f_n, g_{1,n} \rangle g_{1,n} e^{in\theta} \right|^2 &\leq C\omega^2 \|u_\omega - \tilde{u}_\omega\|_{L^2(\mathbb{R}^2)}^2 + C\omega \sum_{n \in \mathcal{N}_{1/2}} |\langle f_n, g_{1,n} \rangle|^2 \\ &\quad + C\omega \sum_{n \in \mathcal{N}_{1/2}} |n - n^*|^2 |\langle f_n, g_{1,n} \rangle|^2 \leq C\omega. \end{aligned} \quad (2.4.42)$$

Gathering (2.4.36), (2.4.41) and (2.4.42) we have for any $\varepsilon > 0$

$$\int_{B_{2R}} |\delta_\omega|^2 \leq C_\varepsilon \omega^{1+\varepsilon}.$$

Applying Lemma 2.4.1 with $A = \omega x^\perp$ we find for any R

$$\sum_{j,k=1}^2 \int_{B_R} |(\partial_j - iA_j)(\partial_k - iA_k)(u_\omega - \tilde{u}_\omega)|^2 \leq C_\varepsilon \omega^{2+\varepsilon} \quad (2.4.43)$$

with C_ε independent of R . Using the diamagnetic inequality (see [96]) twice, we get

$$\|\nabla|u_\omega - \tilde{u}_\omega|\|_{H^1(B_R)} \leq C_\varepsilon \omega^{1+\varepsilon} \quad (2.4.44)$$

with C_ε independent of R . This concludes the proof of (2.4.25).

Step 4. Let us now prove (2.4.24). We multiply (2.4.39) by $(u_\omega - \tilde{u}_\omega)^*$, integrate over \mathbb{R}^2 and use the diamagnetic inequality to obtain

$$\begin{aligned} \| |u_\omega - \tilde{u}_\omega| \|_{H^1(\mathbb{R}^2)}^2 &\leq 2G \int_{\mathbb{R}^2} |u_\omega|^3 |u_\omega - \tilde{u}_\omega| \\ &\quad + \int_{\mathbb{R}^2} \left| \mu_\omega u_\omega - \sum_{n \in \mathcal{N}_{1/2}} \lambda_{1,n} \langle f_n, g_{1,n} \rangle g_{1,n} e^{in\theta} \right| |u_\omega - \tilde{u}_\omega|. \end{aligned} \quad (2.4.45)$$

For both terms on the right-hand side of (2.4.45) we use the Cauchy-Schwarz inequality and (2.3.40), combined with (2.4.36) for the first one and (2.4.42) for the second. This yields (2.4.24).

Step 5. We interpolate between H^2 and $H^{1+\eta}$, make $\eta \rightarrow 0$ and use a Sobolev imbedding to deduce (2.4.26) from (2.4.24) and (2.4.25). □

2.4.3 A refined lower bound on the interaction energy

The last essential ingredient of our analysis is a refinement of Proposition 2.3.3 that we present in Proposition 2.4.4 below. Its proof uses the same ideas as that of Proposition 2.3.3 and requires to first improve the asymptotics for the modes f_n , $n \in \mathcal{N}_{1/2}$. We know from (2.3.40) (G is now a fixed constant) that for $n \in \mathcal{N}_{1/2}$

$$f_n = \langle f_n, g_{1,n} \rangle g_{1,n} + O_{L^2}(\omega^{-1/2}). \quad (2.4.46)$$

We now have to be more precise and evaluate the term proportional to $\omega^{-1/2}$ in the equation above. This is what we do in Proposition 2.4.3. The key observation is that (2.1.27) and (2.4.18) allow to linearize the Euler-Lagrange equation for u_ω , at the price of a relatively small remainder term. Thus, inserting the Fourier expansion of u_ω we obtain equations for each mode f_n that allow to improve (2.4.46). It then turns out that the term proportional to $\omega^{-1/2}$ in (2.4.46) has to be the solution of an elliptic problem that we describe in Lemma 2.4.2.

In this section we will often consider $g_{1,n}$ and f_n (as well as any function defined on \mathbb{R}^+) as radial functions defined on \mathbb{R}^2 .

We denote by $\Pi_{1,n}$ and $\Pi_{1,n}^\perp$ the orthogonal projectors on the space spanned by $g_{1,n}$ and its orthogonal in $H^1(\mathbb{R}^+, r dr) \cap L^2(\mathbb{R}^+, V_n(r) r dr)$, respectively.

In the following Lemma we introduce some functions Γ_n that will be useful in the rest of the paper because they appear naturally when writing expansions for the modes f_n (see Proposition 2.4.3 below). We also state the main properties of these functions that we will need in our analysis. In particular the estimate (2.4.50) will be essential in the proof of Proposition 2.4.4.

Lemma 2.4.2 (Properties of Γ_n).

Let $n \in \mathcal{N}_{1/2}$. The problem

$$\begin{cases} -\Delta\Gamma_n + V_n\Gamma_n - \lambda_{1,n}\Gamma_n = \Pi_{1,n}^\perp (-2G\omega^{1/2}g_{1,n}^2) \\ \int_{\mathbb{R}^2} g_{1,n}\Gamma_n = 0. \end{cases} \quad (2.4.47)$$

has a unique radial solution in $H^1(\mathbb{R}^+, r dr) \cap L^2(\mathbb{R}^+, rV_n(r)dr)$. It satisfies the bounds

$$\|\Gamma_n\|_{L^2(\mathbb{R}^2)} \leq C, \quad \|\Gamma_n\|_{H^1(\mathbb{R}^2)} \leq C\omega^{1/2} \quad (2.4.48)$$

and, for any $\varepsilon > 0$,

$$\|\Gamma_n\|_{L^4(\mathbb{R}^2)} \leq C_\varepsilon\omega^{1/8+\varepsilon}, \quad \|\Gamma_n\|_{L^6(\mathbb{R}^2)} \leq C_\varepsilon\omega^{1/6+\varepsilon}, \quad \|\Gamma_n\|_{L^\infty(\mathbb{R}^2)} \leq C_\varepsilon\omega^{1/4+\varepsilon}. \quad (2.4.49)$$

Moreover, for any $n, m \in \mathcal{N}_{1/2}$ and any $\varepsilon > 0$

$$\|\Gamma_n - \Gamma_m\|_{L^2(\mathbb{R}^2)} \leq C_\varepsilon\omega^{-1/2+\varepsilon}|n - m|. \quad (2.4.50)$$

Proof. It is easy to show that problem (2.4.47) has a unique radial solution in the energy space, for example by expanding both sides of the equation on the basis $(g_{j,n})_{j=1\dots+\infty}$. Using the expansion

$$\Gamma_n = \sum_{j \geq 1} \langle \Gamma_n, g_{j,n} \rangle g_{j,n} \quad (2.4.51)$$

combined with (2.2.5) the first line in (2.4.47) can be written

$$\sum_{j \geq 2} (\lambda_{j,n} - \lambda_{1,n}) \langle \Gamma_n, g_{j,n} \rangle g_{j,n} = \Pi_{1,n}^\perp (-2G\omega^{1/2}g_{1,n}^2). \quad (2.4.52)$$

Thus, taking the L^2 norm of both sides of (2.4.52) we obtain

$$\sum_{j \geq 2} (\lambda_{j,n} - \lambda_{1,n})^2 |\langle \Gamma_n, g_{j,n} \rangle|^2 \leq C\omega \|g_{1,n}^2\|_{L^2(\mathbb{R}^2)}^2 \leq C\omega^2 \quad (2.4.53)$$

because $g_{1,n}^2 \leq C\omega^{1/2}$ and $g_{j,n}$ is normalized in $L^2(\mathbb{R}^2)$ for any j . Now, using (2.2.18) and (2.2.22), one can find a constant $0 < \alpha < 1$ such that

$$\lambda_{j,n} - \lambda_{1,n} = \alpha\lambda_{j,n} + (1-\alpha)\lambda_{j,n} - \lambda_{1,n} \geq \alpha\lambda_{j,n} + (1-\alpha)\lambda_{2,n} - \lambda_{1,n} \geq \alpha\lambda_{j,n} \geq \alpha\lambda_{1,n} \geq C\omega, \quad (2.4.54)$$

for any $j \geq 2$. Thus, (2.4.53) implies

$$\sum_{j \geq 2} |\langle \Gamma_n, g_{j,n} \rangle|^2 \leq C. \quad (2.4.55)$$

This yields the L^2 estimate in (2.4.48) because of the expansion (2.4.51) (recall that by definition $\langle \Gamma_n, g_{1,n} \rangle = 0$). We also obtain from (2.4.53) and (2.4.54)

$$F_n(\Gamma_n) = \sum_{j \geq 2} \lambda_{j,n} |\langle \Gamma_n, g_{j,n} \rangle|^2 \leq C \sum_{j \geq 2} (\lambda_{j,n} - \lambda_{1,n}) |\langle \Gamma_n, g_{j,n} \rangle|^2 \leq C\omega,$$

thus the H^1 bound in (2.4.49) holds. We have used again $\langle \Gamma_n, g_{1,n} \rangle = 0$.

We now claim that Γ_n satisfies the pointwise estimates

$$|\Gamma_n(r)| \leq C_\varepsilon\omega^{1/4+\varepsilon} e^{-\hat{\sigma}\left(\frac{r-R_n}{h_n}\right)^2} \quad (2.4.56)$$

for some constant $\hat{\sigma}$, and for any $\alpha < 1$ and $r \leq C_\alpha$

$$|\Gamma_n(r)| \leq C e^{-C\omega r^\alpha}. \quad (2.4.57)$$

The proofs are very similar to that of Proposition 2.2.2, so we give only the main ideas. First we remark that Γ_n is radial, so for any $\eta > 0$

$$C\omega^{1/2} \geq \|\Gamma_n(x)\|_{H^1(\mathbb{R}^2, dx)} = \sqrt{2\pi} \|\Gamma_n(r)\|_{H^1(\mathbb{R}^+, r dr)} \geq \sqrt{2\pi\eta} \|\Gamma_n(r)\|_{H^1([\eta, +\infty[, dr)}$$

and thus the bound

$$\|\Gamma_n(r)\|_{L^\infty([\eta, +\infty[)} \leq C_\varepsilon \eta^{-1/2} \omega^{1/4+\varepsilon}$$

follows by interpolation and Sobolev imbeddings as in Step 1 of the proof of Proposition 2.2.2. We deduce from (2.4.47) that $|\Gamma_n|^2$ satisfies

$$-\Delta|\Gamma_n|^2 + 2|\Gamma_n|^2(V_n - \lambda_{1,n}) \leq \Gamma_n \Pi_{1,n}^\perp \left(-2G\omega^{1/2} g_{1,n}^2 g_{1,n} \right). \quad (2.4.58)$$

Using $\Pi_{1,n}^\perp(f) = f - \Pi_{1,n}(f)$ and $g_{1,n}^2 \leq C\omega^{1/2}$ we have

$$\left| \Pi_{1,n}^\perp \left(-2G\omega^{1/2} g_{1,n}^2 g_{1,n} \right) \right| \leq C \left| \omega^{1/2} g_{1,n}^2 g_{1,n} \right| + C \left| \int_{\mathbb{R}^2} \omega^{1/2} g_{1,n}^2 g_{1,n}^2 \right| |g_{1,n}| \leq C\omega g_{1,n}.$$

Defining the domain

$$I_n = \{x \in \mathbb{R}^2, \quad |\Gamma_n| \geq g_{1,n}\}$$

we thus deduce from (2.4.58) that

$$-\Delta|\Gamma_n|^2 + 2|\Gamma_n|^2(V_n - C\omega) \leq 0$$

on I_n . We prove that (2.4.56) and (2.4.57) hold in I_n by using the maximum principle as in the proof of Proposition 2.2.2. On the other hand, (2.4.56) and (2.4.57) hold on I_n^c by definition, recalling (2.2.27) and (2.2.29). The L^p bounds (2.4.49) follow from (2.4.56).

We now turn to the proof of (2.4.50). We begin by noting that because of (2.2.19), a change of variables $r = R_n + h_n x$ and a Taylor expansion yield

$$\|g_{1,n} - g_{1,m}\|_{L^2(\mathbb{R}^2)} \leq C\omega^{-1/2} |n - m|. \quad (2.4.59)$$

Thus

$$\left| \int_{\mathbb{R}^2} \Gamma_m g_{1,n} \right| = \left| \int_{\mathbb{R}^2} \Gamma_m (g_{1,m} - g_{1,n}) \right| \leq C\omega^{-1/2} |n - m|$$

because $\int_{\mathbb{R}^2} \Gamma_m g_{1,m} = 0$ and Γ_m is bounded in L^2 . This implies

$$\left| \int_{\mathbb{R}^2} (\Gamma_n - \Gamma_m) g_{1,n} \right| \leq C\omega^{-1/2} |n - m|. \quad (2.4.60)$$

On the other hand, denoting

$$S_{n,m} := \Gamma_n - \Gamma_m \quad (2.4.61)$$

we get from (2.4.47) the equation

$$\begin{aligned} -\Delta S_{n,m} + V_n S_{n,m} - \lambda_{1,n} S_{n,m} &= \Pi_{1,m}^\perp \left(2G\omega^{1/2} g_{1,n}^2 g_{1,m} \right) - \Pi_{1,n}^\perp \left(2G\omega^{1/2} g_{1,n}^2 g_{1,n} \right) \\ &\quad + (V_m - V_n) \Gamma_m + (\lambda_{1,m} - \lambda_{1,n}) \Gamma_m \\ &= \Pi_{1,m}^\perp \left(2G\omega^{1/2} g_{1,n}^2 (g_{1,m} - g_{1,n}) \right) + \left(\Pi_{1,m}^\perp - \Pi_{1,n}^\perp \right) \left(2G\omega^{1/2} g_{1,n}^2 g_{1,n} \right) \\ &\quad + (V_m - V_n) \Gamma_m + (\lambda_{1,m} - \lambda_{1,n}) \Gamma_m. \end{aligned} \quad (2.4.62)$$

We estimate the L^2 norm of the right-hand side of this equation. First remark that, using the definition (2.2.2) of the potential V_n , we have for $n, m \in \mathcal{N}_{1/2}$ and $|r-1| \leq C_\varepsilon \omega^{-1/2+\varepsilon}$

$$|V_m(r) - V_n(r)| = \left| \frac{m^2 - n^2}{r} - 2\omega(m-n) \right| = |n-m| \left| m+n-2\omega + (m+n)O(\omega^{-1/2+\varepsilon}) \right| \leq C_\varepsilon \omega^{1/2+\varepsilon} |m-n|. \quad (2.4.63)$$

We then write

$$\int_{\mathbb{R}^2} (V_m - V_n)^2 \Gamma_m^2 = \int_{|r-1| \leq C\omega^{-1/2+\varepsilon}} (V_m - V_n)^2 \Gamma_m^2 + \int_{|r-1| > C\omega^{-1/2+\varepsilon}} (V_m - V_n)^2 \Gamma_m^2.$$

We estimate the first term using (2.4.63) and the fact that Γ_m is bounded in L^2 . The second one is exponentially small because of (2.4.56) and (2.4.57) (recall that $R_n = 1 + O(\omega^{-1/2})$ and $h_n \propto \omega^{-1/2}$). The result is

$$\| (V_m - V_n) \Gamma_m \|_{L^2(\mathbb{R}^2)} \leq C_\varepsilon \omega^{1/2+\varepsilon} |n-m|. \quad (2.4.64)$$

Also, using (2.2.21), we have for $n, m \in \mathcal{N}_{1/2}$

$$|\lambda_{1,m} - \lambda_{1,n}| \leq C\omega^{1/2} |n-m|. \quad (2.4.65)$$

On the other hand

$$\left\| \Pi_{1,m}^\perp \left(2G\omega^{1/2} g_{1,n}^2 (g_{1,m} - g_{1,n}) \right) \right\|_{L^2(\mathbb{R}^2)} \leq \left\| 2G\omega^{1/2} g_{1,n}^2 (g_{1,m} - g_{1,n}) \right\|_{L^2(\mathbb{R}^2)} \leq C\omega^{1/2} |n-m| \quad (2.4.66)$$

because of (2.4.59). We now use the fact that $\Pi_{1,m}^\perp - \Pi_{1,n}^\perp = \Pi_{1,n} - \Pi_{1,m}$ to obtain

$$\left(\Pi_{1,m}^\perp - \Pi_{1,n}^\perp \right) \left(2G\omega^{1/2} g_{1,n}^2 g_{1,n} \right) = \left(\int_{\mathbb{R}^2} 2Gg_{1,n}^2 g_{1,n}^2 \right) (g_{1,n} - g_{1,m}) + \left(\int_{\mathbb{R}^2} 2Gg_{1,n}^2 g_{1,n} (g_{1,n} - g_{1,m}) \right) g_{1,m}$$

and thus, using (2.4.59) again, plus the Cauchy-Schwarz inequality for the second term

$$\left\| \left(\Pi_{1,m}^\perp - \Pi_{1,n}^\perp \right) \left(2G\omega^{1/2} g_{1,n}^2 g_{1,n} \right) \right\|_{L^2(\mathbb{R}^2)} \leq C\omega^{1/2} |n-m|. \quad (2.4.67)$$

Arguing exactly as when we obtained (2.4.53) from (2.4.47), the combination of (2.4.62) and the estimates (2.4.64) to (2.4.67) yield

$$\sum_{j \geq 2} (\lambda_{j,n} - \lambda_{1,n})^2 |\langle S_{n,m}, g_{j,n} \rangle|^2 \leq C_\varepsilon \omega^{1+\varepsilon} |n-m|^2. \quad (2.4.68)$$

Using (2.4.54) we thus have

$$\sum_{j \geq 2} |\langle S_{n,m}, g_{j,n} \rangle|^2 \leq C_\varepsilon \omega^{-1+\varepsilon} |n-m|^2$$

and there only remains to recall (2.4.60) to conclude the proof because $(g_{j,n})_{j=1 \dots +\infty}$ is a Hilbert basis for $H^1(\mathbb{R}^+, r dr) \cap L^2(\mathbb{R}^+, V_n r dr)$. \square

We now show that the second order in (2.4.46) is given, after an appropriate normalization, by the function Γ_n that we have introduced in Lemma 2.4.2.

Let us first introduce some notation :

$$\mathcal{N}_{1/4} = \left\{ n \in \mathbb{Z}, |n - \omega| \leq \omega^{1/4} \right\} \quad (2.4.69)$$

$$\mathcal{N}_{1/4,+} = \left\{ n \in \mathcal{N}_{1/4}, \langle f_n, g_{1,n} \rangle \neq 0 \right\} \quad (2.4.70)$$

$$\mathcal{N}_{1/2,+} = \left\{ n \in \mathcal{N}_{1/2}, \langle f_n, g_{1,n} \rangle \neq 0 \right\} \quad (2.4.71)$$

In the rest of the paper we will always denote β_n the n -th term of a generic sequence satisfying

$$\sum_{n \in \mathbb{Z}} \beta_n^2 \leq C. \quad (2.4.72)$$

The actual value of the quantity β_n may thus change from line to line. Note that β_n can depend on ω and D_Ω as long as (2.4.72) is satisfied.

Proposition 2.4.3 (Refined asymptotics for the mode f_n).

Let $n \in \mathcal{N}_{1/2,+}$. We have

$$\left\| f_n - \langle f_n, g_{1,n} \rangle \left(g_{1,n} + \omega^{-1/2} \Gamma_n \right) \right\|_{L^2(\mathbb{R}^+, r dr)} \leq \beta_n \max \left(\omega^{-3/4}, |n - n^*|^2 \omega^{-3/2} \right) \quad (2.4.73)$$

and

$$F_n \left(f_n - \langle f_n, g_{1,n} \rangle \left(g_{1,n} + \omega^{-1/2} \Gamma_n \right) \right) \leq \beta_n^2 \max \left(\omega^{-1/2}, |n - n^*|^4 \omega^{-2} \right) \quad (2.4.74)$$

where Γ_n is defined in Lemma 2.4.2.

Proof of Proposition 2.4.3. We need an equation satisfied by f_n . First recall that (2.4.20) and (2.4.26) imply that

$$\|u_\omega\|_{L^\infty(\mathbb{R}^2)} \leq C\omega^{1/4}. \quad (2.4.75)$$

Then, using (2.1.27) (G is now a fixed constant) we get

$$\int_{\mathbb{R}^2} (|u_\omega|^2 u_\omega - g_{1,n^*}^2 u_\omega)^2 \leq C\omega^{1/2} \int_{\mathbb{R}^2} (|u_\omega|^2 - g_{1,n^*}^2)^2 \leq C\omega^{1/2}.$$

This allows to rewrite the Euler-Lagrange equation for u_ω as follows :

$$-\left(\nabla - i\omega x^\perp \right)^2 u_\omega + D_\Omega \frac{\omega^2}{2} (|x|^2 - 1)^2 u_\omega + 2Gg_{1,n^*}^2 u_\omega = \mu_\omega u_\omega + L, \quad (2.4.76)$$

where

$$L = 2G \left(g_{1,n^*}^2 u_\omega - |u_\omega|^2 u_\omega \right) \quad (2.4.77)$$

satisfies $\|L\|_{L^2} \leq C\omega^{1/4}$. Using the Fourier expansion (2.1.42) of u_ω we can write (2.4.76) as

$$\sum_n e^{in\theta} \left(-\Delta f_n + V_n(r) f_n + 2Gg_{1,n^*}^2 f_n \right) = \sum_n e^{in\theta} \mu_\omega f_n + L \quad (2.4.78)$$

We multiply this equation by $e^{-in\theta}$ and integrate over θ to get

$$-\Delta f_n + V_n(r) f_n + 2Gg_{1,n^*}^2 f_n = \mu_\omega f_n + L_n \quad (2.4.79)$$

where

$$L_n(r) = \int_0^{2\pi} e^{-in\theta} L(r, \theta) d\theta, \quad (2.4.80)$$

so that

$$\sum_n \|L_n\|_{L^2(\mathbb{R}^+, r dr)}^2 \leq C \|L\|_{L^2(\mathbb{R}^2)}^2 \leq C\omega^{1/2}. \quad (2.4.81)$$

For $n \in \mathcal{N}_{1/2,+}$ we write f_n as

$$f_n = \langle f_n, g_{1,n} \rangle \left(g_{1,n} + \omega^{-1/2} \kappa_n \right) \quad (2.4.82)$$

where

$$\langle \kappa_n, g_{1,n} \rangle = 0. \quad (2.4.83)$$

Remembering (2.3.40) we have

$$\sum_{n \in \mathcal{N}_{1/2,+}} |\langle f_n, g_{1,n} \rangle|^2 \|\kappa_n\|_{L^2(\mathbb{R}^+, r dr)}^2 \leq C. \quad (2.4.84)$$

We claim that

$$\|\langle f_n, g_{1,n} \rangle (\kappa_n - \Gamma_n)\|_{L^2(\mathbb{R}^2)} \leq \beta_n \max\left(\omega^{-1/4}, |n - n^*|^2 \omega^{-1}\right) \quad (2.4.85)$$

and

$$F_n(\langle f_n, g_{1,n} \rangle (\kappa_n - \Gamma_n)) \leq \beta_n \max\left(\omega^{1/2}, |n - n^*|^4 \omega^{-1}\right) \quad (2.4.86)$$

with β_n satisfying (2.4.72). Let us denote

$$D_n := \langle f_n, g_{1,n} \rangle (\kappa_n - \Gamma_n) \quad (2.4.87)$$

Inserting (2.4.82) into (2.4.79), using the fact that $g_{1,n}$ is a normalized eigenfunction of $-\Delta + V_n$ for the eigenvalue $\lambda_{1,n}$, we obtain an equation for $\langle f_n, g_{1,n} \rangle \kappa_n$:

$$\begin{aligned} -\Delta \langle f_n, g_{1,n} \rangle \kappa_n + V_n \langle f_n, g_{1,n} \rangle \kappa_n - \lambda_{1,n} \langle f_n, g_{1,n} \rangle \kappa_n &= (\mu_\omega - \lambda_{1,n}) \langle f_n, g_{1,n} \rangle \kappa_n - 2G g_{1,n}^2 \langle f_n, g_{1,n} \rangle \kappa_n \\ &\quad - 2G g_{1,n}^2 \langle f_n, g_{1,n} \rangle \omega^{1/2} g_{1,n} + (\mu_\omega - \lambda_{1,n}) \omega^{1/2} \langle f_n, g_{1,n} \rangle g_{1,n} + \omega^{1/2} L_n. \end{aligned} \quad (2.4.88)$$

We multiply this equation by $g_{1,n}$ and integrate to get the useful relation

$$\begin{aligned} \omega^{1/2} \langle f_n, g_{1,n} \rangle \left(2G \int_{\mathbb{R}^2} g_{1,n}^2 g_{1,n}^2 - (\mu_\omega - \lambda_{1,n}) \right) &= \omega^{1/2} \int_{\mathbb{R}^2} L_n g_{1,n} - 2G \langle f_n, g_{1,n} \rangle \int_{\mathbb{R}^2} g_{1,n}^2 \kappa_n g_{1,n} \\ &= O(\beta_n \omega^{3/4}). \end{aligned} \quad (2.4.89)$$

We have used Cauchy-Schwarz, the bound $g_{1,n}^2 \leq C\omega^{1/2}$, (2.4.81) and (2.4.84). Subtracting the equation for $\langle f_n, g_{1,n} \rangle \Gamma_n$ from (2.4.88), we obtain

$$\begin{aligned} -\Delta D_n + V_n D_n - \lambda_{1,n} D_n &= (\mu_\omega - \lambda_{1,n}) \langle f_n, g_{1,n} \rangle \kappa_n - 2G \langle f_n, g_{1,n} \rangle g_{1,n}^2 \kappa_n \\ &\quad + \langle f_n, g_{1,n} \rangle \Pi_{1,n} \left((\mu_\omega - \lambda_{1,n}) \omega^{1/2} g_{1,n} - 2G g_{1,n}^2 \omega^{1/2} g_{1,n} \right) + \omega^{1/2} L_n. \end{aligned} \quad (2.4.90)$$

Using (2.1.26), (2.2.21), (2.3.5) and (2.4.32) we obtain $|\mu_\omega - \lambda_{1,n}| \leq C \max(\omega^{1/2}, |n - n^*|^2)$ and thus

$$\|(\mu_\omega - \lambda_{1,n}) \langle f_n, g_{1,n} \rangle \kappa_n\|_{L^2(\mathbb{R}^2)} \leq \beta_n \max\left(\omega^{1/2}, |n - n^*|^2\right) \quad (2.4.91)$$

for any $n \in \mathcal{N}_{1/2,+}$. We have used (2.4.84) and $n^* = \omega + O(1)$. Also, using $g_{1,n^*}^2 \leq C\omega^{1/2}$ and (2.4.84),

$$\|2G \langle f_n, g_{1,n} \rangle g_{1,n^*}^2 \kappa_n\|_{L^2(\mathbb{R}^2)} \leq \omega^{1/2} \beta_n. \quad (2.4.92)$$

On the other hand (2.4.89) implies

$$\begin{aligned} & \left\| \langle f_n, g_{1,n} \rangle \Pi_{1,n} \left((\mu_\omega - \lambda_{1,n}) \omega^{1/2} g_{1,n} - 2G g_{1,n^*}^2 \omega^{1/2} g_{1,n} \right) \right\|_{L^2(\mathbb{R}^2)} = \\ & \left| \omega^{1/2} \langle f_n, g_{1,n} \rangle \left((\mu_\omega - \lambda_{1,n}) - 2G \int_{\mathbb{R}^2} g_{1,n^*}^2 g_{1,n}^2 \right) \right| \|g_{1,n}\|_{L^2(\mathbb{R}^2)} \leq \beta_n \omega^{3/4} \end{aligned} \quad (2.4.93)$$

We compute the L^2 norm of both sides of (2.4.90) after having expanded the left-hand side on the basis $(g_{j,n})_{j=1,\dots,+\infty}$ as in (2.4.52). Gathering equations (2.4.91) to (2.4.93), recalling (2.4.81) we obtain

$$\sum_{j \geq 2} (\lambda_{j,n} - \lambda_{1,n})^2 |\langle D_n, g_{j,n} \rangle|^2 \leq C \beta_n^2 \max \left(\omega^{3/2}, |n - n^*|^4 \right) \quad (2.4.94)$$

where $\lambda_{j,n}$ and $g_{j,n}$ are the eigenvalues and eigenfunctions of the restriction of $-\Delta + V_n$ to the space of radial functions. Recalling (2.4.54), (2.4.94) implies

$$\sum_{j \geq 2} |\langle D_n, g_{j,n} \rangle|^2 \leq C \beta_n^2 \omega^{-2} \max \left(\omega^{3/2}, |n - n^*|^4 \right)$$

which is (2.4.85) because $\langle D_n, g_{1,n} \rangle = 0$ (see the second equation in (2.4.47) and (2.4.83)). We also obtain from (2.4.94) and (2.4.54)

$$F_n(D_n) = \sum_{j \geq 2} \lambda_{j,n} |\langle D_n, g_{j,n} \rangle|^2 \leq C \sum_{j \geq 2} (\lambda_{j,n} - \lambda_{1,n}) |\langle D_n, g_{j,n} \rangle|^2 \leq C \beta_n^2 \omega^{-1} \max \left(\omega^{3/2}, |n - n^*|^4 \right),$$

thus (2.4.86) holds. We have used again $\langle D_n, g_{1,n} \rangle = 0$. \square

We are now able to present the main result of this section, a refinement of the lower bound on the interaction energy. Its proof uses the same as ideas as that of Proposition 2.3.3, but the next order in the lower bound is now computed. This is made possible because of the refined asymptotics of Proposition 2.4.3.

Note however that the right-hand side of (2.4.73) increases with $|n - n^*|$ so the refinement is not equally efficient for all modes. On the other hand (2.3.46) states that the larger $|n - n^*|$ is, the smaller is the mass of the corresponding mode, so the smaller is its contribution to the interaction energy. It turns out that the best lower bound available with our results can be computed by using the refined results of Proposition 2.4.3 only when $n \in \mathcal{N}_{1/4,+}$.

Proposition 2.4.4 (Refined lower bound on the interaction energy).

We have, for any $\varepsilon > 0$,

$$\begin{aligned} \int_{\mathbb{R}^2} |u_\omega|^4 & \geq \sum_{n \in \mathcal{N}_{1/4,+}} |\langle f_n, g_{1,n} \rangle|^2 \int_{\mathbb{R}^+} \left(g_{1,n} + \omega^{-1/2} \Gamma_n \right)^4 r dr \\ & \quad + \sum_{n \in \mathcal{N}_{1/2} \setminus \mathcal{N}_{1/4,+}} |\langle f_n, g_{1,n} \rangle|^2 \int_{\mathbb{R}^+} g_{1,n}^4 r dr - C_\varepsilon \omega^{-1/4+\varepsilon}. \end{aligned} \quad (2.4.95)$$

Proof. We begin by using Lemma 2.3.1 :

$$\int_{\mathbb{R}^2} |u_\omega|^4 \geq 2\pi \sum_{p,q \in \mathcal{N}_{1/2}} \int_{\mathbb{R}^+} |f_p|^2 |f_q|^2 r dr \quad (2.4.96)$$

and recall that we can restrict the integration domains to $1/2 \leq r \leq 3/2$ at the price of an exponentially small remainder (see (2.3.20)). We implicitly do this restriction in the rest of the proof.

We continue to denote β_n the n -th term of a generic sequence satisfying (2.4.72) and note that for any $n \in \mathcal{N}_{1/2}$ we have

$$\|f_n\|_{L^2([1/2,3/2],dr)} \leq C \|f_n\|_{L^2(\mathbb{R}^+,rdr)} \leq \beta_n$$

because of the L^2 normalization of u_ω . Moreover

$$\|f_n\|_{H^1([1/2,3/2],dr)} \leq C \|f_n\|_{H^1(\mathbb{R}^+,rdr)} \leq \beta_n \omega^{1/2}$$

follows from the bounds $\sum_n F_n(f_n) \leq F_\omega(u_\omega) \leq C\omega$. Interpolating between these two estimates we get

$$\|f_n\|_{H^{1/2}([1/2,3/2],dr)} \leq \beta_n \omega^{1/4}.$$

Then we obtain, for any $\varepsilon > 0$

$$\|f_n\|_{L^6([1/2,3/2],dr)} \leq C_\varepsilon \beta_n \omega^{1/6+\varepsilon} \quad (2.4.97)$$

by the Sobolev imbedding of $H^{1/2}$ in L^p for any p and interpolation between L^2 and L^p with $p \rightarrow +\infty$.

If $n \notin \mathcal{N}_{1/4}$ this estimate is improved :

$$\|f_n\|_{L^6([1/2,3/2],dr)} \leq C_\varepsilon \beta_n \omega^{-1/12+\varepsilon}. \quad (2.4.98)$$

Indeed, for $n \notin \mathcal{N}_{1/4}$ we have

$$\|f_n\|_{L^2([1/2,3/2],dr)} \leq C \|f_n\|_{L^2(\mathbb{R}^+,rdr)} \leq \beta_n \omega^{-1/4} \quad (2.4.99)$$

using (2.3.46). Moreover the expansion

$$F_n(f_n) = \sum_{j \geq 1} \lambda_{j,n} |\langle f_n, g_{j,n} \rangle|^2 \quad (2.4.100)$$

combined with (2.3.45), $|\langle f_n, g_{1,n} \rangle|^2 \leq \beta_n^2 \omega^{-1/2}$ and $\lambda_{1,n} = O(\omega)$ yields

$$\|f_n\|_{H^1([1/2,3/2],dr)} \leq C \|f_n\|_{H^1(\mathbb{R}^+,rdr)} \leq C \sqrt{F_n(f_n)} \leq \beta_n \omega^{1/4}. \quad (2.4.101)$$

Interpolating between (2.4.99) and (2.4.101) we obtain (2.4.98). We note that there also holds, for any $n \in \mathcal{N}_{1/2}$,

$$\|\langle f_n, g_{1,n} \rangle g_{1,n}\|_{L^6([1/2,3/2],dr)} \leq C \beta_n \omega^{1/6} \quad (2.4.102)$$

and for any $n \notin \mathcal{N}_{1/4,+}$

$$\|\langle f_n, g_{1,n} \rangle g_{1,n}\|_{L^6([1/2,3/2],dr)} \leq C \beta_n \omega^{-1/12}. \quad (2.4.103)$$

These last two estimates are basic consequences of the properties of $g_{1,n}$, see Corollary 2.2.2.

We now bound from below the p -th term of the sum (2.4.96), distinguishing two cases :

Case 1 : $p \notin \mathcal{N}_{1/4,+}$

We have, for any $q \in \mathcal{N}_{1/2}$

$$\begin{aligned} \left| \int \left(|f_p|^2 |f_q|^2 - |\langle f_p, g_{1,p} \rangle|^2 |\langle f_q, g_{1,q} \rangle|^2 g_{1,p}^2 g_{1,q}^2 \right) r dr \right| \leq \\ C \|f_q - \langle f_q, g_{1,q} \rangle g_{1,q}\|_{L^2} \|f_p\|_{L^6}^2 (\|f_q\|_{L^6} + \|\langle f_q, g_{1,q} \rangle g_{1,q}\|_{L^6}) \\ + C \|f_p - \langle f_p, g_{1,p} \rangle g_{1,p}\|_{L^2} \|\langle f_q, g_{1,q} \rangle g_{1,q}\|_{L^6}^2 (\|f_p\|_{L^6} + \|\langle f_p, g_{1,p} \rangle g_{1,p}\|_{L^6}) \\ \leq C_\varepsilon \omega^{-1/4+\varepsilon} \beta_p^2 \beta_q^2 \end{aligned} \quad (2.4.104)$$

using (2.3.40) and the estimates (2.4.97) to (2.4.103). We then use exactly the same technique as in the proof of Proposition 2.3.3 to obtain that for any $q \in \mathcal{N}_{1/2}$

$$\int g_{1,p}^2 g_{1,q}^2 r dr \geq \int g_{1,p}^4 r dr - C \omega^{-1/2} |p - q|^2 - C. \quad (2.4.105)$$

We conclude from (2.4.104) and (2.4.105) that for any $p \notin \mathcal{N}_{1/4,+}$

$$\begin{aligned} \sum_{q \in \mathcal{N}_{1/2}} 2\pi \int |f_p|^2 |f_q|^2 r dr \geq \sum_{q \in \mathcal{N}_{1/2}} |\langle f_p, g_{1,p} \rangle|^2 |\langle f_q, g_{1,q} \rangle|^2 \left(2\pi \int g_{1,p}^4 r dr - C \omega^{-1/2} |p - q|^2 - C \right) \\ - C_\varepsilon \beta_p^2 \omega^{-1/4+\varepsilon}. \end{aligned} \quad (2.4.106)$$

Case 2 : $p \in \mathcal{N}_{1/4,+}$ We again distinguish two cases.

Case 2.1 : $q \notin \mathcal{N}_{1/4,+}$ We argue as in Case 1. Using estimates (2.4.97) to (2.4.103),

$$\left| \int \left(|f_p|^2 |f_q|^2 - |\langle f_p, g_{1,p} \rangle|^2 |\langle f_q, g_{1,q} \rangle|^2 g_{1,p}^2 g_{1,q}^2 \right) r dr \right| \leq C_\varepsilon \omega^{-1/4+\varepsilon} \beta_p^2 \beta_q^2 \quad (2.4.107)$$

because $q \notin \mathcal{N}_{1/4,+}$. Also

$$\int g_{1,p}^2 g_{1,q}^2 r dr \geq \int g_{1,p}^4 r dr - C \omega^{-1/2} |p - q|^2 - C. \quad (2.4.108)$$

and it is clear from (2.2.19) and the estimates in Lemma 2.4.2 that for any $\varepsilon > 0$

$$\int g_{1,p}^4 r dr = \int \left(g_{1,p} + \omega^{-1/2} \Gamma_p \right)^4 r dr + O(\omega^\varepsilon). \quad (2.4.109)$$

Gathering (2.4.107), (2.4.108) and (2.4.109) we thus have for any $p \in \mathcal{N}_{1/4,+}$

$$\begin{aligned} \sum_{q \notin \mathcal{N}_{1/4,+}} 2\pi \int |f_p|^2 |f_q|^2 r dr \geq \sum_{q \notin \mathcal{N}_{1/4,+}} |\langle f_p, g_{1,p} \rangle|^2 |\langle f_q, g_{1,q} \rangle|^2 \left(2\pi \int \left(g_{1,p} + \omega^{-1/2} \Gamma_p \right)^4 r dr \right. \\ \left. - C \omega^{-1/2} |p - q|^2 - C_\varepsilon \omega^\varepsilon \right) - C_\varepsilon \beta_p^2 \omega^{-1/4+\varepsilon}. \end{aligned} \quad (2.4.110)$$

Case 2.2 : $q \in \mathcal{N}_{1/4,+}$ This is the case where the refined asymptotics of Proposition 2.4.3 will be crucial. Note that for $n \in \mathcal{N}_{1/4,+}$ (2.4.73) becomes

$$\left\| f_n - \langle f_n, g_{1,n} \rangle \left(g_{1,n} + \omega^{-1/2} \Gamma_n \right) \right\|_{L^2(\mathbb{R}^+, r dr)} \leq \beta_n \omega^{-3/4}. \quad (2.4.111)$$

We begin with

$$\begin{aligned} & \left| \int \left(|f_p|^2 |f_q|^2 - |\langle f_p, g_{1,p} \rangle|^2 |\langle f_q, g_{1,q} \rangle|^2 \left(g_{1,p} + \omega^{-1/2} \Gamma_p \right)^2 \left(g_{1,q} + \omega^{-1/2} \Gamma_q \right)^2 \right) r dr \right| \leq \\ & C \left\| f_q - \langle f_q, g_{1,q} \rangle \left(g_{1,q} + \omega^{-1/2} \Gamma_q \right) \right\|_{L^2} \|f_p\|_{L^6}^2 \left(\|f_q\|_{L^6} + \left\| \langle f_q, g_{1,q} \rangle \left(g_{1,q} + \omega^{-1/2} \Gamma_q \right) \right\|_{L^6} \right) \\ & + C \left\| f_p - \langle f_p, g_{1,p} \rangle \left(g_{1,p} + \omega^{-1/2} \Gamma_p \right) \right\|_{L^2} \left\| \langle f_q, g_{1,q} \rangle \left(g_{1,q} + \omega^{-1/2} \Gamma_q \right) \right\|_{L^6}^2 \\ & \times \left(\|f_p\|_{L^6} + \left\| \langle f_p, g_{1,p} \rangle \left(g_{1,p} + \omega^{-1/2} \Gamma_p \right) \right\|_{L^6} \right) \leq C_\varepsilon \omega^{-1/4+\varepsilon} \beta_p^2 \beta_q^2. \quad (2.4.112) \end{aligned}$$

We have used (2.4.97) and (2.4.102) along with (2.4.111) and the estimates of Lemma 2.4.2.

Now we use the bounds $\|g_{1,p}\|_{L^4} \leq C\omega^{1/8}$ (see Corollary 2.2.2) and $\|\Gamma_p\|_{L^4} \leq C_\varepsilon \omega^{1/8+\varepsilon}$ (see Lemma 2.4.2), to obtain

$$\begin{aligned} \int \left(g_{1,p} + \omega^{-1/2} \Gamma_p \right)^2 \left(g_{1,q} + \omega^{-1/2} \Gamma_q \right)^2 r dr & \geq \int g_{1,p}^2 g_{1,q}^2 r dr + 2\omega^{-1/2} \int g_{1,p}^2 \Gamma_q g_{1,q} r dr \\ & + 2\omega^{-1/2} \int g_{1,q}^2 \Gamma_p g_{1,p} r dr - C_\varepsilon \omega^{-1/2+\varepsilon}. \quad (2.4.113) \end{aligned}$$

Using (2.2.14) and (2.2.19) we compute

$$2\pi \int g_{1,p}^2 g_{1,q}^2 r dr \geq h_p c_{1,p}^2 c_{1,q}^2 I_{p,q} \quad (2.4.114)$$

with a change of variables $r = R_p + h_p x$ where

$$I_{p,q} = J_{p,q} + 2K_{p,q} + 2L_{p,q} + O(\omega^{-1})$$

and

$$\begin{aligned} J_{p,q} & := \int_{\mathbb{R}^+} \xi_1^2(x) \xi_1^2 \left(\frac{h_p}{h_q} x + \frac{R_p - R_q}{h_q} \right) (R_p + h_p x) dx \\ K_{p,q} & := \int_{\mathbb{R}^+} \xi_1^2(x) h_q P_q \left(\frac{h_p}{h_q} x + \frac{R_p - R_q}{h_q} \right) \xi_1^2 \left(\frac{h_p}{h_q} x + \frac{R_p - R_q}{h_q} \right) (R_p + h_p x) dx \\ L_{p,q} & := \int_{\mathbb{R}^+} h_p \xi_1^2(x) P_q(x) \xi_1^2 \left(\frac{h_p}{h_q} x + \frac{R_p - R_q}{h_q} \right) (R_p + h_p x) dx. \end{aligned}$$

We recall that for any $p, q \in \mathcal{N}_{1/2}$

$$|h_p - h_q| \propto \omega^{-3/2} |p - q|$$

and

$$|R_p - R_q| \propto \omega^{-1} |p - q|.$$

Then arguing exactly as in the proof of Proposition 2.3.3 we get

$$J_{p,q} = J_{p,p} + O(\omega^{-1}|p - q|^2)$$

whereas simple Taylor inequalities coupled with the estimates (2.2.20) yield

$$K_{p,q} = K_{p,p} + O(\omega^{-1}|p - q|) \quad L_{p,q} = L_{p,p} + O(\omega^{-1}|p - q|).$$

Moreover, (2.2.19) and (2.2.20) imply that for any $p, q \in \mathcal{N}_{1/2}$

$$|c_{1,p}^2 - c_{1,q}^2| \leq C\omega^{-1}|p - q|.$$

All in all,

$$2\pi \int g_{1,p}^2 g_{1,q}^2 r dr \geq 2\pi \int g_{1,p}^4 - C\omega^{-1/2}|p - q|^2 - C\omega^{-1/2}|p - q| - C\omega^{-1/2}. \quad (2.4.115)$$

On the other hand

$$\begin{aligned} \left| \int g_{1,p}^2 \Gamma_q g_{1,q} - \int g_{1,p}^3 \Gamma_p \right| &\leq \left(\int g_{1,p}^4 \Gamma_q^2 \right)^{1/2} \left(\int (g_{1,q} - g_{1,p})^2 \right)^{1/2} \\ &\quad + \left(\int g_{1,p}^6 \right)^{1/2} \left(\int (\Gamma_p - \Gamma_q)^2 \right)^{1/2} \\ &\leq \|g_{1,p}\|_{L^6}^2 \|\Gamma_q\|_{L^6} \|g_{1,p} - g_{1,q}\|_{L^2} + \|g_{1,p}\|_{L^6}^3 \|\Gamma_p - \Gamma_q\|_{L^2} \\ &\leq C_\varepsilon \omega^\varepsilon |p - q| \end{aligned} \quad (2.4.116)$$

using Corollary 2.2.2, (2.4.49), (2.4.50) and (2.4.59).

We conclude from (2.4.112), (2.4.113), (2.4.115) and (2.4.116) that for any $p \in \mathcal{N}_{1/4,+}$

$$\begin{aligned} \sum_{q \in \mathcal{N}_{1/4,+}} 2\pi \int |f_p|^2 |f_q|^2 r dr &\geq \sum_{q \in \mathcal{N}_{1/4,+}} |\langle f_p, g_{1,p} \rangle|^2 |\langle f_q, g_{1,q} \rangle|^2 \left(2\pi \int (g_{1,p} + \omega^{-1/2} \Gamma_p)^4 r dr \right. \\ &\quad \left. - C\omega^{-1/2}|p - q|^2 - C\omega^{-1/2}|p - q| - C\omega^{-1/2} \right) - C_\varepsilon \beta_p^2 \omega^{-1/4+\varepsilon}. \end{aligned} \quad (2.4.117)$$

To conclude the proof we sum (2.4.106), (2.4.110) and (2.4.117) and use the following estimates that we obtain from (2.3.40) and (2.3.46) :

$$\begin{aligned} \sum_{p,q \in \mathcal{N}_{1/2}} |\langle f_p, g_{1,p} \rangle|^2 |\langle f_q, g_{1,q} \rangle|^2 |p - q|^2 &\leq C \left(\sum_{p,q \in \mathcal{N}_{1/2}} |\langle f_p, g_{1,p} \rangle|^2 |\langle f_q, g_{1,q} \rangle|^2 |p - n^*|^2 \right. \\ &\quad \left. + \sum_{p,q \in \mathcal{N}_{1/2}} |\langle f_p, g_{1,p} \rangle|^2 |\langle f_q, g_{1,q} \rangle|^2 |q - n^*|^2 \right) \\ &\leq C \sum_{p \in \mathcal{N}_{1/2}} \|f_p\|_{L^2(\mathbb{R}^+, r dr)}^2 |p - n^*|^2 \leq C \end{aligned}$$

and

$$\begin{aligned} \sum_{p,q \in \mathcal{N}_{1/2}} |\langle f_p, g_{1,p} \rangle|^2 |\langle f_q, g_{1,q} \rangle|^2 |p - q| &\leq \left(\sum_{p,q \in \mathcal{N}_{1/2}} \|f_p\|_{L^2(\mathbb{R}^+, r dr)}^2 \|f_q\|_{L^2(\mathbb{R}^+, r dr)}^2 |p - q|^2 \right)^{1/2} \\ &\quad \times \left(\sum_{p,q \in \mathcal{N}_{1/2}} \|f_p\|_{L^2(\mathbb{R}^+, r dr)}^2 \|f_q\|_{L^2(\mathbb{R}^+, r dr)}^2 \right)^{1/2} \leq C. \end{aligned}$$

Also,

$$\sum_{q \notin \mathcal{N}_{1/4,+}} |\langle f_q, g_{1,q} \rangle|^2 \leq C\omega^{-1/2}.$$

Finally

$$\sum_{n \in \mathcal{N}_{1/2}} |\langle f_p, g_{1,p} \rangle|^2 = 1 - C\omega^{-1}.$$

□

2.4.4 Proofs of Theorems 2.1.2 and 2.1.3

Proof of (2.1.29). The upper bound is obtained by taking $\Psi_{n^*} e^{in^*\theta}$ as a test function in F_ω . For the lower bound we first write, using Proposition 2.4.4

$$\begin{aligned} F_\omega(u_\omega) &\geq \sum_{n \in \mathbb{Z}} F_n(f_n) + \sum_{n \in \mathcal{N}_{1/4,+}} 2\pi G |\langle f_n, g_{1,n} \rangle|^2 \int_{\mathbb{R}^+} \left(g_{1,n} + \omega^{-1/2} \Gamma_n \right)^4 r dr \\ &+ \sum_{n \in \mathcal{N}_{1/2} \setminus \mathcal{N}_{1/4,+}} 2\pi G |\langle f_n, g_{1,n} \rangle|^2 \int_{\mathbb{R}^+} g_{1,n}^4 r dr - C_\varepsilon \omega^{-1/4+\varepsilon}. \end{aligned} \quad (2.4.118)$$

Now, using Lemma 2.2.1, (2.4.74) and the definitions of $\lambda_{1,n}$, $\lambda_{2,n}$ and E_n we obtain from (2.4.118)

$$\begin{aligned} F_\omega(u_\omega) &\geq \sum_{n \in \mathcal{N}_{1/4,+}} |\langle f_n, g_{1,n} \rangle|^2 \left(F_n \left(g_{1,n} + \omega^{-1/2} \Gamma_n \right) + 2\pi G \int_{\mathbb{R}^+} \left(g_{1,n} + \omega^{-1/2} \Gamma_n \right)^4 r dr \right) \\ &+ \sum_{n \in \mathcal{N}_{1/2} \setminus \mathcal{N}_{1/4,+}} |\langle f_n, g_{1,n} \rangle|^2 \left(F_n(g_{1,n}) + 2\pi G \int_{\mathbb{R}^+} g_{1,n}^4 r dr \right) \\ &+ \int_{\mathbb{R}^+} (f_n - \langle f_n, g_{1,n} \rangle g_{1,n})^2 r dr \lambda_{2,n} \\ &+ \sum_{n \in \mathcal{N}_{1/2}^c} 2\pi \|f_n\|_{L^2(\mathbb{R}^+, r dr)}^2 \lambda_{1,n} - C_\varepsilon \omega^{-1/4+\varepsilon} \\ &\geq \sum_{n \in \mathcal{N}_{1/4,+}} |\langle f_n, g_{1,n} \rangle|^2 E_n \left(g_{1,n} + \omega^{-1/2} \Gamma_n \right) + \sum_{n \in \mathcal{N}_{1/2} \setminus \mathcal{N}_{1/4,+}} \|f_n\|_{L^2(\mathbb{R}^+, r dr)}^2 E_n(g_{1,n}) \\ &+ \sum_{n \in \mathcal{N}_{1/2}^c} 2\pi \|f_n\|_{L^2(\mathbb{R}^+, r dr)}^2 c\omega - C_\varepsilon \omega^{-1/4+\varepsilon} \end{aligned} \quad (2.4.119)$$

where c is the constant appearing in Lemma 2.2.1. We have used the following :

$$\lambda_{2,n} \geq \lambda_{1,n} + C\omega \geq E_n(g_{1,n}) + C\omega - C'\omega^{1/2} \geq E_n(g_{1,n})$$

which is proved by computing $E_n(g_{1,n})$ with the results of Theorem 2.2.1 and Proposition 2.2.1 in mind. The normalization of $g_{1,n}$ and the orthogonality of $g_{1,n}$ and Γ_n imply

$$2\pi \left\| g_{1,n} + \omega^{-1/2} \Gamma_n \right\|_{L^2(\mathbb{R}^+, r dr)}^2 \geq 1$$

so

$$E_n \left(g_{1,n} + \omega^{-1/2} \Gamma_n \right) \geq 2\pi \left\| g_{1,n} + \omega^{-1/2} \Gamma_n \right\|_{L^2(\mathbb{R}^+, r dr)}^2 \gamma_n. \quad (2.4.120)$$

On the other hand, because of (2.4.73) we have

$$2\pi\|f_n\|_{L^2(\mathbb{R}^+,rdr)}^2 = 2\pi|\langle f_n, g_{1,n} \rangle|^2 \left\| g_{1,n} + \omega^{-1/2}\Gamma_n \right\|_{L^2(\mathbb{R}^+,rdr)}^2 + \beta_n^2\omega^{-3/2} \quad (2.4.121)$$

for any $n \in \mathcal{N}_{1/4,+}$. Using that $\gamma_n = O(\omega)$ we finally obtain from (2.4.119), (2.4.120) and (2.4.121)

$$F_\omega(u_\omega) \geq \sum_{n \in \mathcal{N}_{1/2}} 2\pi\|f_n\|_{L^2(\mathbb{R}^+,rdr)}^2 \gamma_n + \sum_{n \in \mathcal{N}_{1/2}^c} 2\pi\|f_n\|_{L^2(\mathbb{R}^+,rdr)}^2 (\gamma_{n^*} + C\omega) - C_\varepsilon\omega^{-1/4+\varepsilon}. \quad (2.4.122)$$

Note that for $n \notin \mathcal{N}_{1/2}$ one has $\lambda_{1,n} \geq c\omega \geq \gamma_{n^*} + C\omega$ because $c > \sqrt{6}$ and $\gamma_{n^*} \leq \sqrt{6}\omega$ which can be computed from (2.4.6). The lower bound in (2.1.29) follows from (2.4.122) and Corollary 2.4.1, recalling

$$\sum_{n \in \mathbb{Z}} 2\pi\|f_n\|_{L^2(\mathbb{R}^+,rdr)}^2 = 1.$$

□

Proof of (2.1.30). Using

$$F_\omega(u_\omega) \leq \gamma_{n^*} = \sum_{n \in \mathbb{Z}} \gamma_{n^*} 2\pi\|f_n\|_{L^2(\mathbb{R}^+,rdr)}^2$$

we obtain from (2.4.122)

$$C_\varepsilon\omega^{-1/4+\varepsilon} \geq \sum_{n \in \mathcal{N}_{1/2}} 2\pi\|f_n\|_{L^2(\mathbb{R}^+,rdr)}^2 (\gamma_n - \gamma_{n^*}) + \sum_{n \in \mathcal{N}_{1/2}^c} 2\pi\|f_n\|_{L^2(\mathbb{R}^+,rdr)}^2 C\omega$$

with $C > 0$, thus, Corollary 2.4.1 implies

$$C_\varepsilon\omega^{-1/4+\varepsilon} \geq \sum_{n \in \mathcal{N}_{1/2}, n \neq n^*, n^*+1} 2\pi\|f_n\|_{L^2(\mathbb{R}^+,rdr)}^2 |n - n^*|^2 + \sum_{n \in \mathcal{N}_{1/2}^c} 2\pi\|f_n\|_{L^2(\mathbb{R}^+,rdr)}^2 C\omega$$

This immediately yields

$$\sum_{n \in \mathcal{N}_{1/2}, n \neq n^*, n^*+1} 2\pi\|f_n\|_{L^2(\mathbb{R}^+,rdr)}^2 |n - n^*|^2 \leq C_\varepsilon\omega^{-1/4+\varepsilon}. \quad (2.4.123)$$

and

$$\sum_{n \in \mathcal{N}_{1/2}^c} 2\pi\|f_n\|_{L^2(\mathbb{R}^+,rdr)}^2 \leq C_\varepsilon\omega^{-5/4+\varepsilon}. \quad (2.4.124)$$

We now introduce

$$\dot{u}_\omega = \alpha\Psi_{n^*}e^{in^*\theta} + \beta\Psi_{n^*+1}e^{i(n^*+1)\theta} \quad (2.4.125)$$

where

$$\begin{aligned} \alpha &:= \sqrt{2\pi}\|f_{n^*}\|_{L^2(\mathbb{R}^+,rdr)} \\ \beta &:= \sqrt{2\pi}\|f_{n^*+1}\|_{L^2(\mathbb{R}^+,rdr)}. \end{aligned} \quad (2.4.126)$$

We note that (2.2.19) and (2.4.7) imply

$$g_{1,n} = \Psi_n + O_{L^2(\mathbb{R}^2)}(\omega^{-1/2}) \quad (2.4.127)$$

so, using equation (2.4.123) and (2.4.124) combined with (2.4.73) we see that after extraction of a subsequence

$$\|u_\omega - c\dot{u}_\omega\|_{L^2(\mathbb{R}^2)} \leq C_\varepsilon \omega^{-1/8+\varepsilon}$$

with c a constant of modulus 1, so that we have, recalling (2.4.11) and (2.4.36)

$$\begin{aligned} \int_{\mathbb{R}^2} \left| |u_\omega|^4 - |\dot{u}_\omega|^4 \right| &\leq C \left(\|u_\omega\|_{L^6(\mathbb{R}^2)}^3 + \|\dot{u}_\omega\|_{L^6(\mathbb{R}^2)}^3 \right) \|u_\omega - c\dot{u}_\omega\|_{L^2(\mathbb{R}^2)} \\ &\leq C_\varepsilon \omega^{1/2+\varepsilon} \|u_\omega - c\dot{u}_\omega\|_{L^2(\mathbb{R}^2)} \leq C_\varepsilon \omega^{3/8+\varepsilon}. \end{aligned} \quad (2.4.128)$$

The estimate (2.4.7) yields by a change of variables similar to those we used in the proofs of Propositions 2.3.3 and 2.4.4

$$\int_{\mathbb{R}^2} |\Psi_{n^*}|^4 = \int_{\mathbb{R}^2} |\Psi_{n^*+1}|^4 + O(\omega^{-1/2}) = \int_{\mathbb{R}^2} |\Psi_{n^*}|^2 |\Psi_{n^*+1}|^2 + O(\omega^{-1/2})$$

Thus, computing the interaction energy of \dot{u}_ω , using (2.4.128) and the fact that $\alpha^2 + \beta^2 = 1 + O(\omega^{-1/8+\varepsilon})$ for any $\varepsilon > 0$ we have

$$\begin{aligned} \gamma_{n^*} &\geq F_\omega(u_\omega) \geq \gamma_{n^*} + 2\alpha^2\beta^2 \int_{\mathbb{R}^2} |\Psi_{n^*}|^4 - C_\varepsilon \omega^{3/8+\varepsilon} \\ C_\varepsilon \omega^{3/8+\varepsilon} &\geq 2\alpha^2\beta^2 \int_{\mathbb{R}^2} |\Psi_{n^*}|^4 \geq C\omega^{1/2}\alpha^2\beta^2. \end{aligned}$$

But, according to our results (2.4.123) and (2.4.124), either α^2 or β^2 is bounded below. Then, along some subsequence we have either $\alpha^2 \leq C_\varepsilon \omega^{-1/8+\varepsilon}$ or $\beta^2 \leq C_\varepsilon \omega^{-1/8+\varepsilon}$. Renaming n^* if necessary, we have proved that along some subsequence

$$\sum_{n \in \mathcal{N}_{1/2}, n \neq n^*} \|f_n\|_{L^2(\mathbb{R}^+, r dr)}^2 \leq C_\varepsilon \omega^{-1/8+\varepsilon} \quad (2.4.129)$$

which concludes the proof of (2.1.30), recalling (2.4.73) and (2.4.127). \square

Finally, we present the

Proof of Theorem 2.1.3. Let us define

$$\mathcal{N}_{7/16,+} := \left\{ n \in \mathbb{Z}, \quad |n - n^*| \leq \omega^{7/16}, \langle f_n, g_{1,n} \rangle \neq 0 \right\}. \quad (2.4.130)$$

and

$$\hat{u}_\omega(r, \theta) := \sum_{n \in \mathcal{N}_{7/16,+}} \langle f_n, g_{1,n} \rangle \phi_n e^{in\theta} \quad (2.4.131)$$

where we have denoted

$$\phi_n := g_{1,n} + \omega^{-1/2} \Gamma_n \quad (2.4.132)$$

for short. We claim that

$$\|u_\omega - \hat{u}_\omega\|_{L^\infty(\mathbb{R}^2)} \leq C_\varepsilon \omega^{-1/32+\varepsilon}. \quad (2.4.133)$$

Firstly we obtain

$$\begin{aligned} \|u_\omega - \hat{u}_\omega\|_{L^2(\mathbb{R}^2)}^2 &\leq \sum_{n \in \mathcal{N}_{7/16,+}^c} 2\pi \|f_n\|_{L^2}^2 + \sum_{n \in \mathcal{N}_{7/16,+}} \|f_n - \langle f_n, g_{1,n} \rangle \phi_n\|_{L^2}^2 \\ &\leq C_\varepsilon \omega^{-9/8+\varepsilon} + C\omega^{-5/4} \leq C_\varepsilon \omega^{-9/8+\varepsilon} \end{aligned} \quad (2.4.134)$$

from (2.4.73), (2.4.123) and (2.4.124). We then note that ϕ_n satisfies

$$-\Delta \phi_n + V_n \phi_n = \lambda_{1,n} \phi_n + \Pi_{1,n}^\perp \left(-2G\omega^{1/2} g_{1,n}^2 g_{1,n} \right). \quad (2.4.135)$$

Multiplying (2.4.135) by $\langle f_n, g_{1,n} \rangle e^{in\theta}$ and summing over $n \in \mathcal{N}_{7/16,+}$ we deduce that

$$\begin{aligned} -\left(\nabla - i\omega x^\perp\right)^2 \hat{u}_\omega + V(r)\hat{u}_\omega &= \sum_{n \in \mathcal{N}_{7/16,+}} \lambda_{1,n} \langle f_n, g_{1,n} \rangle \phi_n e^{in\theta} \\ &+ \sum_{n \in \mathcal{N}_{7/16,+}} \langle f_n, g_{1,n} \rangle \Pi_{1,n}^\perp \left(-2G\omega^{1/2} g_{1,n}^2 g_{1,n} \right) e^{in\theta}. \end{aligned} \quad (2.4.136)$$

Subtracting this equation from the Euler-Lagrange equation (2.1.16) we get an equation for $u_\omega - \hat{u}_\omega$:

$$\begin{aligned} -\left(\nabla - i\omega x^\perp\right)^2 (u_\omega - \hat{u}_\omega) + V(r)(u_\omega - \hat{u}_\omega) &= \mu_\omega u_\omega - 2G|u_\omega|^2 u_\omega \\ &- \sum_{n \in \mathcal{N}_{7/16,+}} \lambda_{1,n} \langle f_n, g_{1,n} \rangle \phi_n e^{in\theta} - \sum_{n \in \mathcal{N}_{7/16,+}} \langle f_n, g_{1,n} \rangle \Pi_{1,n}^\perp \left(-2G\omega^{1/2} g_{1,n}^2 g_{1,n} \right) e^{in\theta}. \end{aligned} \quad (2.4.137)$$

The same technique as in Steps 3, 4 and 5 of the proof of Proposition 2.4.2 allows to deduce (2.4.133) from (2.4.134) and (2.4.137).

Proof of Item 1.

We write, using (2.4.133)

$$\| |u_\omega| - |\phi_{n^*}| \| \leq |1 - \langle f_{n^*}, g_{1,n^*} \rangle| |\phi_{n^*}| + \sum_{n \in \mathcal{N}_{7/16,+}, n \neq n^*} |\langle f_n, g_{1,n} \rangle| |\phi_n| + C_\varepsilon \omega^{-1/32+\varepsilon}, \quad (2.4.138)$$

A consequence of Lemma 2.4.2 (more precisely the L^∞ estimate in (2.4.49)), (2.2.19) and (2.4.7) is

$$\phi_n = g_{1,n} + O_{L^\infty}(\omega^{-1/4}) = \Psi_n + O_{L^\infty}(\omega^{-1/4}) = c_n e^{-d\left(\frac{r-R_n}{h_n}\right)^2} + O_{L^\infty}(\omega^{-1/4}) \quad (2.4.139)$$

where c_n is defined by (2.4.4) and d is the standard deviation of ξ_1 . A consequence of (2.4.7) is

$$c_n^2 = \frac{1}{2\pi h_n R_n} + O(\omega^{-1/2}) \quad (2.4.140)$$

because ξ_1 is an even function. We recall that, for some constant h

$$R_n = 1 + o(\omega^{-1/2}) \quad (2.4.141)$$

$$h_n = h\omega^{-1/2}(1 + o(1)) \quad (2.4.142)$$

for any $n \in \mathcal{N}_{7/16}$, so arguing as in the proof of Proposition 2.4.1 one deduce from (2.4.138) that

$$\begin{aligned} \| |u_\omega| - |\Psi_{n^*}| \| \leq C \left(|1 - |\langle f_{n^*}, g_{1,n^*} \rangle|| + \sum_{n \in \mathcal{N}_{7/16,+}, n \neq n^*} |\langle f_n, g_{1,n} \rangle| \right) \omega^{1/4} e^{-\sigma\omega(|x|-1)^2} \\ + C_\varepsilon \omega^{-1/32+\varepsilon} \end{aligned} \quad (2.4.143)$$

with

$$\sigma := \frac{\hbar^2}{4d}. \quad (2.4.144)$$

On the other hand (2.4.129) and (2.4.73) imply

$$|1 - |\langle f_{n^*}, g_{1,n^*} \rangle|| \leq C_\varepsilon \omega^{-1/8+\varepsilon}$$

and

$$\begin{aligned} \sum_{n \in \mathcal{N}_{7/16,+}, n \neq n^*} |\langle f_n, g_{1,n} \rangle| &\leq C \left(\sum_{n \in \mathcal{N}_{7/16,+}, n \neq n^*} \|f_n\|_{L^2(\mathbb{R}^+, r dr)}^2 |n - n^*|^2 \right)^{1/2} \\ &\times \left(\sum_{n \in \mathcal{N}_{7/16,+}, n \neq n^*} |n - n^*|^{-2} \right)^{1/2} \\ &\leq C_\varepsilon \omega^{-1/16+\varepsilon}, \end{aligned}$$

so we obtain (2.1.31) from (2.4.143).

Proof of Items 2 and 3. These final results are consequences of (2.1.31). Using the facts that

$$\Psi_{n^*} = c_{n^*} e^{-d\left(\frac{r-R_{n^*}}{\hbar_{n^*}}\right)^2} + O_{L^\infty}(\omega^{-1/4})$$

and $c_{n^*} \propto \omega^{1/4}$ one can see from (2.1.31) that $u_\omega(x_\omega) \rightarrow 0$ uniformly on any region where

$$||x_\omega| - 1| \geq \delta \omega^{-1} \ln(\omega)$$

with $\delta > \sigma$. On the other hand, if $x_\omega = r_\omega e^{i\theta_\omega}$ is a point such that

$$u_\omega(x_\omega) \rightarrow 0$$

as $\omega \rightarrow \infty$, (2.1.31) implies

$$\omega^{1/4} e^{-d\hbar_{n^*}^{-2}(r-R_{n^*})^2} \leq C$$

for any constant C as $r \rightarrow r_\omega$ and thus

$$|r_\omega - R_{n^*}|^2 \geq \frac{1}{4d} \hbar_{n^*}^{-2} (\ln(\omega) - C) \geq \sigma \omega^{-1} (\ln(\omega) - C) \quad (2.4.145)$$

which completes the proof because

$$R_{n^*} = 1 + O(\omega^{-1}).$$

□

3

The Transition to a Giant Vortex Phase in a Fast Rotating Bose-Einstein Condensate

Ce chapitre est une collaboration avec Michele Correggi et Jakob Yngvason. Il reprend un article à paraître dans Communications in Mathematical Physics.

3.1 Introduction

An ultracold Bose gas in a magneto-optical trap exhibits remarkable phenomena when the trap is set in rotational motion. In the ground state the gas is a superfluid and responds to the rotation by the creation of quantized vortices whose number increases with the angular velocity. The literature on this subject is vast but there exist some excellent reviews [4, 41, 63]. The mathematical analysis is usually carried out in the framework of the (time independent) Gross-Pitaevskii (GP) equation that has been derived from the many-body quantum mechanical Hamiltonian in [99] for the non-rotating case and in [97] for a rotating system at *fixed* coupling constant and rotational velocity. When these parameters tend to infinity the leading order approximation was established in [37] but it is still an open problem to derive the full GP equation rigorously in this regime. The present paper, however, is not concerned with the many-body problem and the GP description will be assumed throughout.

When studying gases in rapid rotation a distinction has to be made between the case of harmonic traps, where the confining external potential increases quadratically with the distance from the rotation axis, and anharmonic traps where the confinement is stronger. In the former case there is a limiting angular velocity beyond which the centrifugal forces drive the gas out of the trap. In the latter case the angular velocity can in principle be as large as one pleases [61]. Using variational arguments it was predicted in [65] that at sufficiently large angular velocities in such traps a phase transition changing the character of the wave function takes place : Vortices disappear from the bulk of the density and the vorticity resides in a ‘giant vortex’ in the central region of the trap where the density is very low. This phenomenon was also discussed and studied numerically in both earlier and later works [64, 71, 91, 92, 90], but proving theorems that firmly establish it mathematically has remained quite challenging.

The emergence of a giant vortex state at large angular velocity with the interaction parameter *fixed* was proved in [127] for a quadratic plus quartic trap potential. The present

paper is concerned with the combined effects of a large angular velocity *and* a large interaction parameter on the distribution of vorticity in an anharmonic trap. In particular, we shall establish rigorous estimates on the relation between the interaction strength and the angular velocity required for creating a giant vortex. The physical regime of interest and hence the mathematical problem treated is rather different from that in [127] but a common feature is that the annular shape of the condensate is created by the centrifugal forces at fast rotation. By contrast, the paper [5] focuses on a situation where, even at slow rotation speeds, the condensate has a fixed annular shape due to the choice of the trapping potential. The basic methodology of the present paper is related to that of [5] and [81, 82] but with important novel aspects that will become apparent in the sequel.

As in the papers [43, 47], that deal mainly with rotational velocities below the giant vortex transition, the mathematical model we consider is that of a two-dimensional ‘flat’ trap in two dimensions. The angular velocity vector points in the direction orthogonal to the plane and the energy functional in the rotating frame of reference is³

$$\mathcal{E}^{\text{GP}}[\Psi] := \int_{\mathcal{B}} d\vec{r} \left\{ |\nabla\Psi|^2 - 2\vec{\Omega} \cdot \Psi^* \vec{L}\Psi + \varepsilon^{-2} |\Psi|^4 \right\}, \quad (3.1.1)$$

where we have denoted the angular velocity by $2\vec{\Omega}$, the integral domain is the unit disc $\mathcal{B} = \{\vec{r} \in \mathbb{R}^2 : r \leq 1\}$, and $\vec{L} = -i\vec{r} \wedge \nabla$ is the angular momentum operator. It is also useful to write the functional in the form

$$\mathcal{E}^{\text{GP}}[\Psi] = \int_{\mathcal{B}} d\vec{r} \left\{ \left| (\nabla - i\vec{A}) \Psi \right|^2 - \Omega^2 r^2 |\Psi|^2 + \varepsilon^{-2} |\Psi|^4 \right\}, \quad (3.1.2)$$

where the vector potential \vec{A} is given by

$$\vec{A} := \vec{\Omega} \wedge \vec{r} = \Omega r \vec{e}_\vartheta. \quad (3.1.3)$$

Here (r, ϑ) are two-dimensional polar coordinates and \vec{e}_ϑ a unit vector in the angular direction. The complex valued Ψ is normalized in $L^2(\mathcal{B})$ and the ground state energy is defined as

$$E^{\text{GP}} := \inf_{\|\Psi\|_2=1} \mathcal{E}^{\text{GP}}[\Psi]. \quad (3.1.4)$$

The minimization in (3.1.4) leads to Neumann boundary conditions on $\partial\mathcal{B}$. Alternatively we could have imposed Dirichlet boundary conditions, or considered the case of a homogeneous trapping potential as in [44]. Our general strategy should apply to those cases, but this requires further investigation (see Section 3.1.2).

We denote by Ψ^{GP} a minimizer of (3.1.2), i.e., any normalized function such that $\mathcal{E}^{\text{GP}}[\Psi^{\text{GP}}] = E^{\text{GP}}$.

The minimizer is in general not unique because vortices can break the rotational symmetry [43, Proposition 2.2] but any minimizer satisfies the variational equation (GP equation)

$$-\Delta\Psi^{\text{GP}} - 2\vec{\Omega} \cdot \vec{L}\Psi^{\text{GP}} + 2\varepsilon^{-2} |\Psi^{\text{GP}}|^2 \Psi^{\text{GP}} = \mu^{\text{GP}} \Psi^{\text{GP}}, \quad (3.1.5)$$

with Neumann boundary conditions and the chemical potential

$$\mu^{\text{GP}} := E^{\text{GP}} + \frac{1}{\varepsilon^2} \int_{\mathcal{B}} d\vec{r} |\Psi^{\text{GP}}|^4. \quad (3.1.6)$$

3. The notation “ $a := b$ ” means that a is *by definition* equal to b .

The subsequent analysis concerns the asymptotic behavior of Ψ^{GP} and E^{GP} as $\varepsilon \rightarrow 0$ with Ω tending to ∞ in a definite way.

As discussed in [43] the centrifugal term $-\Omega^2 r^2 |\Psi|^2$ in (3.1.2) creates for $\Omega \gtrsim \varepsilon^{-1}$ a ‘hole’ around the center where the density $|\Psi^{\text{GP}}|^2$ is exponentially small while the mass is concentrated in an annulus of thickness $\sim \varepsilon \Omega$ at the boundary. Moreover, by establishing upper and lower bounds on E^{GP} , it was shown in [47] that in the asymptotic parameter regime

$$|\log \varepsilon| \ll \Omega \ll (\varepsilon^2 |\log \varepsilon|)^{-1} \quad (3.1.7)$$

the ground state energy is to subleading order correctly reproduced by the energy of a trial function exhibiting a lattice of vortices reaching all the way to the boundary of the disc. It can even be shown that in the whole regime (3.1.7) the vorticity of a true minimizer Ψ^{GP} in the annulus is uniformly distributed⁴ with density Ω/π .

3.1.1 Main Results

In this paper we investigate the case

$$\Omega = \frac{\Omega_0}{\varepsilon^2 |\log \varepsilon|} \quad (3.1.8)$$

with Ω_0 a fixed constant and prove that for Ω_0 sufficiently large a phase transition takes place : Vortices disappear from the bulk of the density and all vorticity is contained in a hole where the density is low.

To state this result precisely, we first recall from [43] that the leading order in the asymptotic expansion of the GP ground state energy is given by the minimization of the ‘Thomas-Fermi’ (TF) functional obtained from \mathcal{E}^{GP} by simply neglecting the kinetic term :

$$\mathcal{E}^{\text{TF}}[\rho] := \int_B d\vec{r} \{ -\Omega^2 r^2 \rho + \varepsilon^{-2} \rho^2 \}, \quad (3.1.9)$$

where ρ plays the role of the density $|\Psi|^2$. The minimizing normalized density, denoted by ρ^{TF} , can be computed explicitly (see Appendix). For $\Omega > 2\pi^{-1/2}\varepsilon^{-1}$ it vanishes for $r < R_h$ with

$$1 - R_h^2 = (2/\sqrt{\pi})(\varepsilon\Omega)^{-1} \sim \Omega_0^{-1}\varepsilon|\log \varepsilon|, \quad (3.1.10)$$

while for $R_h \leq r \leq 1$ it is given by

$$\rho^{\text{TF}}(r) = \frac{(\varepsilon\Omega)^2}{2}(r^2 - R_h^2). \quad (3.1.11)$$

Our proof of the disappearance of vortices is based on estimates that require the TF density to be sufficiently large. For this reason we consider an annulus

$$\tilde{\mathcal{A}} := \{ \vec{r} : R_h + \varepsilon |\log \varepsilon|^{-1} \leq r \leq 1 \} \quad (3.1.12)$$

where $\rho^{\text{TF}}(r) \gtrsim \Omega_0 \varepsilon^{-1} |\log \varepsilon|^{-3}$. Our result on the disappearance of vortices from the essential support of the density is as follows :

4. Cf. [47, Theorem 3.3]. This theorem is stated only for the case $\Omega \lesssim \varepsilon^{-1}$ but it holds in fact true in the whole region (3.1.7) [128].

Theorem 3.1.1 (Absence of vortices in the bulk).

If Ω is given by (3.1.8) with $\Omega_0 > 2(3\pi)^{-1}$, then Ψ^{GP} does not have any zero in $\tilde{\mathcal{A}}$ for $\varepsilon > 0$ small enough. More precisely, for $\vec{r} \in \tilde{\mathcal{A}}$,

$$||\Psi^{\text{GP}}(\vec{r})|^2 - \rho^{\text{TF}}(r)| \leq C \frac{|\log \varepsilon|^3}{\varepsilon^{7/8}} \ll \rho^{\text{TF}}(r). \quad (3.1.13)$$

Remark 3.1.1. (Bulk of the condensate).

The annulus $\tilde{\mathcal{A}}$ contains the bulk of the mass. Indeed, because $\int_{\tilde{\mathcal{A}}} \rho^{\text{TF}} = 1 - o(1)$ and Ψ^{GP} is normalized, (3.1.13) implies that also $\int_{\tilde{\mathcal{A}}} |\Psi^{\text{GP}}|^2 = 1 - o(1)$.

The proof of Theorem 1.1 is based on precise energy estimates that involve a comparison of the energy of the restriction of Ψ^{GP} to $\tilde{\mathcal{A}}$ with the energy of a giant vortex trial function. By the latter we mean a function of the form

$$\Psi(\vec{r}) = f(\vec{r}) \exp(i\hat{\Omega}\vartheta) \quad (3.1.14)$$

with f real valued and $\hat{\Omega} \in \mathbb{N}$. It is convenient to write⁵

$$\hat{\Omega} = [\Omega] - \omega \quad (3.1.15)$$

with $\omega \in \mathbb{Z}$ and use ω as label for the $\hat{\Omega}$ -dependent quantities in the sequel. We define a functional of f by

$$\hat{\mathcal{E}}_{\omega}^{\text{GP}}[f] := \mathcal{E}^{\text{GP}}[f \exp(i\hat{\Omega}\vartheta)] = \int_{\mathcal{B}} d\vec{r} \left\{ |\nabla f|^2 + (\hat{\Omega}^2 r^{-2} - 2\Omega\hat{\Omega})f^2 + \varepsilon^{-2}f^4 \right\}. \quad (3.1.16)$$

Minimizing the functional $\hat{\mathcal{E}}_{\omega}^{\text{GP}}[f]$ (see Proposition 3.2.3), that is convex in f^2 , over L^2 -normalized f for fixed $\hat{\Omega}$, we obtain an energy denoted by $\hat{E}_{\omega}^{\text{GP}}$ and a minimizer that is rotationally symmetric, without a zero for $r > 0$, and unique up to sign⁶. We denote the unique positive minimizer by g_{ω} . If $0 \leq \omega \leq C\varepsilon^{-1}$ the corresponding density g_{ω}^2 is close to ρ^{TF} (see Remark 1.3 below).

It is clear that

$$E^{\text{GP}} \leq \mathcal{E}^{\text{GP}}[g_{\omega} \exp(i\hat{\Omega}\vartheta)] = \hat{E}_{\omega}^{\text{GP}} \quad (3.1.17)$$

for any value of ω and hence

$$E^{\text{GP}} \leq \hat{E}^{\text{GP}} := \inf_{\omega \in \mathbb{Z}} \hat{E}_{\omega}^{\text{GP}}. \quad (3.1.18)$$

Our second main result is a lower bound that matches (3.1.18) up to small errors (see Remark 1.4 below) and an estimate of the phase that optimizes $\hat{E}_{\omega}^{\text{GP}}$.

Theorem 3.1.2 (Ground state energy).

For $\Omega_0 > 2(3\pi)^{-1}$ and $\varepsilon > 0$ small enough the ground state energy is

$$E^{\text{GP}} = \hat{E}^{\text{GP}} - O\left(\frac{|\log \varepsilon|^{3/2}}{\varepsilon^{1/2}(\log |\log \varepsilon|)^2}\right). \quad (3.1.19)$$

Moreover $\hat{E}^{\text{GP}} = \hat{E}_{\omega_{\text{opt}}}^{\text{GP}}$, with $\omega_{\text{opt}} \in \mathbb{N}$ satisfying

$$\omega_{\text{opt}} = \frac{2}{3\sqrt{\pi\varepsilon}} (1 + O(|\log \varepsilon|^{-1})). \quad (3.1.20)$$

5. We use the notation $[\cdot]$ for the integer part of a real number.

6. Instead of requiring f to be real we could have required f to be radial in which case the minimizer is unique up to a constant phase.

Because Ψ^{GP} does not have any zeros in the annulus $\tilde{\mathcal{A}}$ the winding number (degree) of Ψ^{GP} around the unit disc is well defined. Our third main result is that the giant vortex phase $[\Omega] - \omega_{\text{opt}}$ is, up to eventual small errors, equal to this winding number.

Theorem 3.1.3 (Degree of a minimizer).

For $\Omega_0 > 2(3\pi)^{-1}$ and $\varepsilon > 0$ small enough, the winding number of Ψ^{GP} is

$$\deg \{\Psi^{\text{GP}}, \partial\mathcal{B}\} = [\Omega] - \omega_{\text{opt}}(1 + o(1)) \quad (3.1.21)$$

with ω_{opt} as in (3.1.20).

The following remarks are intended to elucidate the phase difference ω_{opt} and to justify the claim that the remainder in (3.1.19) is, indeed, a small correction to \hat{E}^{GP} .

Remark 3.1.2. (Giant vortex phase).

For a fixed trial function f the energy (3.1.16) is minimal for

$$\hat{\Omega} = \Omega \left(\int d\vec{r} r^{-2} f^2 \right)^{-1} + O(1). \quad (3.1.22)$$

If f^2 is close to ρ^{TF} and thus essentially concentrated in an annulus of width $\sim (\varepsilon\Omega) = O(\varepsilon|\log \varepsilon|)$, it follows that

$$0 < (\Omega - \hat{\Omega}) = O(\varepsilon^{-1}). \quad (3.1.23)$$

The leading term $2/(3\sqrt{\pi})\varepsilon^{-1}$ in (3.1.20) can, indeed, be computed from (3.1.22) with $f^2 = \rho^{\text{TF}}$, or more simply, for $f^2(r)$ increasing linearly from 0 at $r = R_h$ to its maximal value at $r = 1$. It is worth noting that if $f^2(r)$ were constant in the interval $[R_h, 1]$, the leading term for ω according to (3.1.22) would be $2/(\sqrt{\pi}\varepsilon)$ and the phase $\hat{\Omega}$ therefore approximately equal to $(\Omega/\pi) \times \text{area of the hole}$. The optimal $\hat{\Omega}$ is closer to Ω because of the inhomogeneity of the density in the annulus.

Remark 3.1.3. (Giant vortex density functional).

The functional (3.1.16) can also be written

$$\hat{\mathcal{E}}_{\omega}^{\text{GP}}[f] = \int_{\mathcal{B}} d\vec{r} \{ |\nabla f|^2 - \Omega^2 r^2 f^2 + B_{\omega}^2 f^2 + \varepsilon^{-2} f^4 \} \quad (3.1.24)$$

with

$$B_{\omega}(r) := \hat{\Omega}/r - \Omega r. \quad (3.1.25)$$

If $\omega = O(\varepsilon^{-1})$, then $B_{\omega}(r) = O(\varepsilon^{-1})$ close to the boundary of the disc. Neglecting the term $B_{\omega}^2 f^2$ as well as the gradient term in (3.1.24) in comparison with the centrifugal and interaction terms leads to the TF functional evaluated at f^2 . This makes plausible the assertion above that g_{ω}^2 is close to ρ^{TF} if $\omega = O(\varepsilon^{-1})$ and this will indeed be proved in Section 3.2.

Remark 3.1.4. (Composition of the ground state energy).

If one drops the gradient term in (3.1.24) but retains the term with B_{ω}^2 , one obtains a modified (and ω -dependent) TF functional

$$\hat{\mathcal{E}}_{\omega}^{\text{TF}}[\rho] := \int_{\mathcal{B}} d\vec{r} \{ -\Omega^2 r^2 \rho + B_{\omega}^2 \rho + \varepsilon^{-2} \rho^2 \}. \quad (3.1.26)$$

Its minimizer and minimizing energy $\hat{E}_\omega^{\text{TF}}$ can be computed explicitly (see the Appendix) and one sees that the energy is minimal for $\omega = 2/(3\sqrt{\pi}\varepsilon)(1+o(1))$. Denoting the minimal value by \hat{E}^{TF} , we have

$$\hat{E}^{\text{TF}} = E^{\text{TF}} + O(\varepsilon^{-2}), \quad (3.1.27)$$

while

$$E^{\text{TF}} = -\Omega^2 - 4/(3\sqrt{\pi})\varepsilon^{-1}\Omega = O(\varepsilon^{-4}|\log \varepsilon|^{-2}) + O(\varepsilon^{-3}|\log \varepsilon|^{-1}). \quad (3.1.28)$$

The term $O(\varepsilon^{-2})$ in (3.1.27) is the angular kinetic energy corresponding to the third term in (3.1.24). The difference between \hat{E}^{GP} and \hat{E}^{TF} is the radial kinetic energy of the order $(\varepsilon\Omega)^2 = O(\varepsilon^{-2}|\log \varepsilon|^{-2})$. The remainder in (3.1.19) is thus much smaller than all terms in \hat{E}^{GP} . It is also smaller than the energy a vortex in $\tilde{\mathcal{A}}$ would have, which is $|\log \varepsilon| \times \text{density} = O(\varepsilon^{-1})$.

3.1.2 Proof Strategy

We now explain the general strategy for the proof of the main results.

The first step is an energy splitting as in [93]. We write a generic, normalized wave function as $\Psi(\vec{r}) = g_\omega(r)w(\vec{r})$ with a complex valued function w . The identity

$$\int_{\mathcal{B}} d\vec{r} |\nabla \Psi|^2 = \int_{\mathcal{B}} d\vec{r} (-\Delta g_\omega) g_\omega |w|^2 + \int_{\mathcal{B}} d\vec{r} g_\omega^2 |\nabla w|^2 \quad (3.1.29)$$

holds by partial integration and the boundary condition on g_ω . Using the variational equation for g_ω and writing $w(\vec{r}) = \exp(i\hat{\Omega}\vartheta)v(\vec{r})$ and $\hat{\Omega}$ as in (3.1.15), we obtain a splitting of the energy functional :

$$\mathcal{E}^{\text{GP}}[\Psi] = \hat{E}_\omega^{\text{GP}} + \mathcal{E}_\omega[v] \quad (3.1.30)$$

with

$$\mathcal{E}_\omega[v] := \int_{\mathcal{B}} d\vec{r} \left\{ g_\omega^2 |\nabla v|^2 - 2g_\omega^2 \vec{B}_\omega \cdot (iv, \nabla v) + \frac{g_\omega^4}{\varepsilon^2} (1 - |v|^2)^2 \right\} \quad (3.1.31)$$

where

$$\vec{B}_\omega(r) := B_\omega(r)\vec{e}_\vartheta = (\Omega r - ([\Omega] - \omega) r^{-1}) \vec{e}_\vartheta \quad (3.1.32)$$

and we have used the notation

$$(iv, \nabla v) := \frac{1}{2}i(v\nabla\bar{v} - \bar{v}\nabla v). \quad (3.1.33)$$

Since the divergence of the two-dimensional vector field $g_\omega^2 \vec{B}_\omega$ vanishes we can write

$$2g_\omega^2 \vec{B}_\omega = \nabla^\perp F_\omega \quad (3.1.34)$$

with a scalar potential F_ω and the dual gradient $\nabla^\perp = (-\partial_y, \partial_x)$. Stokes theorem then gives

$$\mathcal{E}_\omega[v] = \int_{\mathcal{B}} d\vec{r} \{ g_\omega^2 |\nabla v|^2 + F_\omega \text{curl}(iv, \nabla v) \} - \int_{\partial\mathcal{B}} d\sigma F_\omega (iv, \partial_\tau v) + \int_{\mathcal{B}} d\vec{r} \frac{g_\omega^4}{\varepsilon^2} (1 - |v|^2)^2. \quad (3.1.35)$$

Here and in the rest of the paper ‘curl $(iv, \nabla v)$ ’ stands for the 3-component of $\nabla \wedge (iv, \nabla v)$, ∂_τ is the tangential derivative and $d\sigma$ the Lebesgue measure on the circle, i.e, given a ball \mathcal{B}_R of radius R centered at the origin, $\partial_\tau := R^{-1}\partial_\vartheta$ and $d\sigma := R d\vartheta$ on $\partial\mathcal{B}_R$.

The giant vortex trial function $g_\omega \exp(i\hat{\Omega}\vartheta)$ gives an upper bound to the energy and hence we see from (3.1.30) that

$$\mathcal{E}_\omega[u] \leq 0 \quad (3.1.36)$$

where u is the remaining factor of Ψ^{GP} after the giant vortex trial function has been extracted, e.g., $\Psi^{\text{GP}} = g_\omega \exp(i\hat{\Omega}\vartheta) u$. The proof of Theorem 3.1.1 is based on a lower bound on $\mathcal{E}_\omega[u]$ that, for Ω_0 large enough, would be positive and hence contradict (3.1.36) if u had zeros in $\tilde{\mathcal{A}}$. The main steps are as follows :

- *Concentration of the density* : There is an annulus \mathcal{A} , slightly larger than $\tilde{\mathcal{A}}$, so that the supremum of $|\Psi^{\text{GP}}|^2$ over $\mathcal{B} \setminus \mathcal{A}$ is⁷ $O(\varepsilon^\infty)$. The same holds for g_ω^2 provided $\omega \leq C\varepsilon^{-1}$. This implies that for $v = u$ the integrations in (3.1.31) can be restricted to \mathcal{A} up to errors that are $O(\varepsilon^\infty)$.
- *Optimization of the phase* : Choice of $\omega = O(\varepsilon^{-1})$ so that F_ω , chosen to vanish on the inner boundary of \mathcal{A} , is small on $\partial\mathcal{B}$.
- *Boundary estimate* : Using the variational equations for Ψ^{GP} and g_ω and the smallness of F_ω on the boundary it is proved that the boundary term in (3.1.35) is small. This step uses the Neumann boundary conditions on $\partial\mathcal{B}$ in a strong way and its extension to the case of Dirichlet boundary conditions is a nontrivial open problem.
- *Vortex balls* : Isolation of the possible zeros of u in ‘vortex balls’ and an estimate of the first term in (3.1.35) from below in terms of the infimum of g_ω^2 on the vortex balls and the winding numbers of u around the centers. For the construction of vortex balls a suitable upper bound on the last term in (3.1.35) is essential.
- *Jacobian estimates* : Approximation of the integral of F_ω against the vorticity $\text{curl}(iu, \nabla u)$ by a sum of the values of F_ω at the centers of the vortex balls multiplied by the corresponding winding numbers.
- *Gradient estimates* : Estimates on ∇u leading to a lower bound on the last term in (3.1.35) that, for Ω_0 large enough, excludes zeros of u .

A key point to notice is that the vortex ball construction in combination with the jacobian estimates leads to a *cost function* defined as

$$H_\omega(r) := \frac{1}{2}g_\omega^2(r)|\log \varepsilon| + F_\omega(r). \quad (3.1.37)$$

The first term is, to leading approximation, the kinetic energy of a vortex of unit strength at radius r , the second term is the gain due to the potential energy of the vortex in the field $2g_\omega^2\vec{B}_\omega$. If H_ω is negative at some point, the energy may be lowered by inserting a vortex at this point. A positive value of H_ω means that the cost of the kinetic energy outweighs the gain. Note that H_ω depends on Ω_0 through F_ω ; in fact it is not difficult to see that⁸

$$|F_\omega(r)| \leq \frac{C}{\Omega_0}g_\omega^2(r)|\log \varepsilon|. \quad (3.1.38)$$

Hence, if Ω_0 is large enough, the cost function is positive everywhere on the annulus and vortices are energetically unfavorable. An upper bound, $2(3\pi)^{-1}$, on the critical value for Ω_0 is computed in the Appendix. We believe it to be optimal, i.e., that there will be vortices in the bulk if one takes $\Omega_0 < 2(3\pi)^{-1}$ but this requires further investigation.

7. We use the symbol $O(\varepsilon^\infty)$ to denote a remainder which is smaller than any positive power of ε , e.g., exponentially small.

8. Here, as in the rest of the paper, C stands for a finite, positive constant whose value may vary from one formula to another.

The construction of vortex balls is a technique introduced independently by Jerrard [85] and Sandier [133] in Ginzburg-Landau (GL) theory, whereas the jacobian estimates originate from the work [88]. Both techniques are described in details in the monograph [135]. This method has been applied in [5] and [81, 82] to functionals that at first sight look exactly like (3.1.31). There is, however, an essential difference : While in [5] and [81, 82] the size of the relevant integral domain is fixed, the weight function g_ω^2 is in our case concentrated in an annulus whose width tends to zero as $\varepsilon \rightarrow 0$. This shrinking of the width of the annulus and the large gradient of g_ω^2 are the main reasons for the complications encountered in the proof of our main theorems.

To appreciate the difficulty the following consideration is helpful. As already noted, the concentration of g_ω^2 on the annulus $\tilde{\mathcal{A}}$ of width $O(\varepsilon|\log \varepsilon|)$ implies that $B_\omega = O(\varepsilon^{-1})$ on this annulus. From (3.1.36) and the Cauchy-Schwarz inequality applied to the second term in (3.1.31), one obtains the bound

$$\int_{\tilde{\mathcal{A}}} d\vec{r} \frac{g_\omega^4}{\varepsilon^2} (1 - |u|^2)^2 \leq \frac{C}{\varepsilon^2}. \quad (3.1.39)$$

Now, although $g_\omega^4 \sim (\varepsilon\Omega)^2 \sim \varepsilon^{-2} |\log \varepsilon|^{-2}$ on $\tilde{\mathcal{A}}$, the estimate (3.1.39) is not sufficient for the construction of vortex balls on $\tilde{\mathcal{A}}$. In fact, (3.1.39) is compatible with the vanishing of u on a ball of radius $O(\varepsilon|\log \varepsilon|)$, i.e., comparable to the width of $\tilde{\mathcal{A}}$, while for a construction of vortex balls one must be able to isolate the possible zeros of u in balls of much smaller radius.

The solution of this problem, elaborated in Section 3.4, involves a division of the annulus into cells of area $\sim (\varepsilon\Omega)^2$ with the upshot that a local version of (3.1.39) holds in every cell. The local version,

$$\int_{\text{Cell}} d\vec{r} \frac{g_\omega^4}{\varepsilon^2} (1 - |u|^2)^2 \leq \frac{C}{\varepsilon^2} \times (\text{number of cells})^{-1} \sim \frac{|\log \varepsilon|}{\varepsilon} \quad (3.1.40)$$

means that, in the cell, u can only vanish in a region of area $O(\varepsilon^3 |\log \varepsilon|^3)$ that is much smaller than the area of the cell, i.e., $\sim \varepsilon^2 |\log \varepsilon|^2$. Hence the vortex ball technique applies in the cells where (3.1.40), or a sufficiently close approximation to it, holds. The gist of the proof of a global lower bound, that eventually leads to Theorems 3.1.1 and 3.1.2, is a stepwise increase in the number of cells where an estimate close to (3.1.40) is valid until all potential zeros in the annulus are included in vortex balls. The proof of Theorem 3.1.3 involves in addition an estimate on the winding number of u .

In the sketch of the proof strategy above we have for simplicity deviated slightly from the actual procedure that will be followed in the sequel. Namely, for technical reasons, we find it necessary to restrict the considerations to a problem on the annulus *before* the splitting of the GP energy functional as in (3.1.30). This means that, instead of the function $g_{\omega_{\text{opt}}}$ and the optimal phase ω_{opt} defined above, we shall work with corresponding quantities for a functional like (3.1.24) but with the integration restricted to \mathcal{A} . The main reason for this complication is lack of precise information about the behavior of Ψ^{GP} in a neighborhood of the origin. In fact, the distribution of the ‘giant vorticity’ of Ψ^{GP} in the hole is unknown. In particular it is not known whether Ψ^{GP} vanishes at the origin like g_ω , which has a zero there of order $\hat{\Omega}$.

3.1.3 Heuristic Considerations

A heuristic argument for the giant vortex transition, based on an analogy with an electrostatic problem, has been given in [47] and goes as follows : Writing a wave function as $\Psi(\vec{r}) = |\Psi(\vec{r})| \exp(i\varphi(\vec{r}))$ with a real phase φ the kinetic energy term in (3.1.2) is

$$\int_{\mathcal{B}} d\vec{r} \left\{ |\nabla|\Psi||^2 + |\Psi|^2 |\nabla\varphi - \vec{A}|^2 \right\} \quad (3.1.41)$$

with $\vec{A}(\vec{r}) = \Omega r \vec{e}_\vartheta$. We focus on the second term and consider the situation that the phase φ contains, besides the giant vortex, also a possible contribution from a vortex of unit degree at a point \vec{r}_0 in the annulus of thickness $\sim (\Omega\varepsilon)^{-1}$. The phase can thus be written

$$\varphi(\vec{r}) := \hat{\Omega}\vartheta + \arg(\vec{r} - \vec{r}_0). \quad (3.1.42)$$

where $\arg(x, y) = \arctan(y/x)$. The modulus $|\Psi|$ vanishes at \vec{r}_0 and is small in a disc ('vortex core') of radius $\sim \sqrt{\varepsilon/\Omega} \sim \varepsilon^{3/2} |\log \varepsilon|^{1/2}$ around \vec{r}_0 . The phase $\arg(\vec{r} - \vec{r}_0)$ can be regarded as the imaginary part of the complex logarithm if we identify \mathbb{R}^2 with \mathbb{C} . The corresponding real part, i.e., conjugate harmonic function, is $\log |\vec{r} - \vec{r}_0|$ which is the 2d electrostatic potential of a unit point charge localized in \vec{r}_0 . Likewise, the conjugate harmonic function of $\hat{\Omega}\vartheta$ is $\hat{\Omega} \log r$, i.e., the electrostatic potential of a charge $\hat{\Omega}$ placed at the origin. By the Cauchy-Riemann equations for the complex logarithm we can write

$$\nabla\varphi = \partial_r\varphi \vec{e}_r + r^{-1}\partial_\vartheta\varphi \vec{e}_\vartheta = \partial_r\chi \vec{e}_\vartheta - r^{-1}\partial_\vartheta\chi \vec{e}_r \quad (3.1.43)$$

with

$$\chi(\vec{r}) := \hat{\Omega} \log r + \log |\vec{r} - \vec{r}_0|. \quad (3.1.44)$$

After a rotation by $\pi/2$, i.e., replacement of \vec{e}_ϑ by \vec{e}_r at every point, the vector potential has also an electrostatic interpretation : $\Omega r \vec{e}_r$ is the electrostatic field of a uniform charge distribution with charge density Ω/π . Employing (3.1.43) we now have

$$|\nabla\varphi - \vec{A}|^2 = |\nabla\chi - \Omega r \vec{e}_r|^2 \quad (3.1.45)$$

and $\vec{E}(\vec{r}) := \nabla\chi - \Omega r \vec{e}_r$ is the electric field generated by the point charges and the uniform background.

We can now apply *Newton's theorem* to argue that the effect of the giant vortex, i.e., the first term in (3.1.44), is to neutralize in the annulus the field generated by the uniform charge distribution in the 'hole', i.e., the second term in (3.1.45), provided $\hat{\Omega}$ is chosen to match the area of the hole times the uniform charge density. By the same argument the point charge at \vec{r}_0 neutralizes, outside a disc of radius $\sim 1/\sqrt{\Omega}$, the effect of one unit of the continuous charge.

The total charge of the continuous distribution in the annulus is $Q \sim \Omega(\Omega\varepsilon)^{-1} \sim \varepsilon^{-1}$. Inserting the vortex reduces the effective charge by one unit so the corresponding energy gain is $\sim \varepsilon^{-1}$. On the other hand, the energy associated with the electrostatic field from the point charge outside the vortex core (that is cut off by the modulus of the wave function) is $\sim |\Psi|^2 |\log \varepsilon| \sim (\Omega\varepsilon) |\log \varepsilon|$. The condition for the cost outweighing the gain is $\Omega\varepsilon |\log \varepsilon|^{-1} \gtrsim \varepsilon^{-1}$, i.e.,

$$\Omega \gtrsim \frac{1}{\varepsilon^2 |\log \varepsilon|}, \quad (3.1.46)$$

marking the transition to the giant vortex phase.

It is also instructive to consider a version of the functional (3.1.31) where the variables have been scaled so that the width of the annulus becomes $O(1)$. We define

$$\ell := (\varepsilon\Omega)^{-1} = \Omega_0^{-1}\varepsilon|\log\varepsilon|, \quad \vec{s} := \ell^{-1}\vec{r}, \quad \check{g}(\vec{s}) := g_\omega(\ell\vec{s}), \quad \check{v}(\vec{s}) := v(\ell\vec{s}), \quad \vec{\check{B}}(\vec{s}) := \ell\vec{B}_\omega(\ell\vec{s}). \quad (3.1.47)$$

Then (3.1.31) is equal to

$$\check{\mathcal{E}}[\check{v}] := \int_{\vec{\check{B}}} d\vec{s} \check{g}^2 \left\{ |\nabla\check{v}|^2 - 2\vec{\check{B}} \cdot (i\check{v}, \nabla\check{v}) + \frac{\check{g}^2\ell^2}{\varepsilon^2} (1 - |\check{v}|^2)^2 \right\} \quad (3.1.48)$$

with $\vec{\check{B}} := \mathcal{B}_{\ell^{-1}}$ a ball of radius $\ell^{-1} = \Omega_0 \varepsilon^{-1} |\log\varepsilon|^{-1}$. For the sake of a heuristic consideration we now assume that g_ω^2 is constant, $\sim \ell^{-1}$, on the annulus of width ℓ and zero otherwise. We define a new small parameter

$$\check{\varepsilon} := \varepsilon\ell^{-1/2} \sim \varepsilon^{1/2} |\log\varepsilon|^{-1/2} \quad (3.1.49)$$

and note that

$$|\vec{\check{B}}| \sim \varepsilon^{-1}\ell \sim \Omega_0^{-1} (|\log\check{\varepsilon}| + O(\log|\log\check{\varepsilon}|)). \quad (3.1.50)$$

We are thus led to consider the functional

$$\int_{\vec{\check{A}}} d\vec{s} \left\{ |\nabla\check{v}|^2 - 2\vec{\check{B}} \cdot (i\check{v}, \nabla\check{v}) + \frac{1}{\check{\varepsilon}^2} (1 - |\check{v}|^2)^2 \right\} \quad (3.1.51)$$

on an annulus $\vec{\check{A}}$ of width $O(1)$ and an effective vector potential of strength $O(\Omega_0^{-1} |\log\check{\varepsilon}|)$. This is reminiscent of the situation considered in [5] and [81, 82] where the domain is fixed and the rotational velocity is proportional to the logarithm of the small parameter. Moreover, increasing Ω_0 *decreases* the coefficient in front of the logarithm. Hence for large Ω_0 this coefficient is small and *if* the analysis of [5] and [81, 82] would apply, one could conclude that vortices are absent. This reduction of the problem to known results is, however, too simplistic because the annulus $\vec{\check{A}}$ is not fixed : Although its width stays constant, its diameter and hence the area increases as $\check{\varepsilon}^{-2/3} |\log\check{\varepsilon}|^{-2/3}$. A new ingredient is needed, and in our approach this is the division of the annulus into cells as mentioned in the previous subsection. In the scaled version (3.1.48) of the energy functional these cells are (essentially) of fixed size. When writing the actual proofs we prefer to use the original unscaled variables but the picture provided by the scaling is still helpful, in particular for comparison with [5] and [81, 82].

Remark 3.1.5. (*Alternative approach*).

After the submission of this paper we learned [87] of a possible alternative approach to prove the absence of vortices in the bulk, relying on [86, Lemma 8] (see also [16, Lemma 4.1]). This method could replace some of the arguments in Section 3.4 and lead to a shorter proof of our Theorem 2.1.3 but is likely to yield worse remainder terms in the energy (Theorem 3.1.2). We also stress that the tools developed in Section 3.4 of the present paper are an essential input in the paper [129]([Chapter 4]), where it is proved that vortices *do* appear in the bulk if $\Omega_0 < 2(3\pi)^{-1}$. For this, Lemma 8 in [86] would not be sufficient.

3.1.4 Organization of the Paper

The paper is organized as follows. In Section 3.2 we gather some definitions and notation that are to be used in the rest of the paper. We also prove useful estimates on

the matter densities, both for the actual GP minimizer $|\Psi^{\text{GP}}|^2$ and for the ‘giant vortex profiles’ g_ω . In Section 3.3 we introduce the auxiliary problem that we are going to study on the annulus \mathcal{A} and prove that it indeed captures the main energetic features of the full problem. Section 3.4 is devoted to the study of the auxiliary problem via tools from the GL theory. We conclude the proof of our main results in Section 3.5. Finally an Appendix gathers important facts about the TF functionals that we use in our analysis, as well as the analysis of the cost function that leads to an upper bound on the critical speed.

3.2 Useful Estimates

In this Section we state some useful estimates which are going to be used in the rest of the paper. Some of them are simple consequences of energy considerations and in particular energy upper bounds, whereas others depend in a crucial way on the variational equations solved by Ψ^{GP} , etc., and apply therefore only to (global) minimizers.

In the first part of the Section we investigate the properties of any GP minimizer : Starting from simple energy estimates, we prove the concentration of Ψ^{GP} on an annulus of width $O(\varepsilon|\log \varepsilon|)$ close to the boundary of the trap and in particular its exponential smallness inside the hole. This is the major result about Ψ^{GP} .

The second part of the Section is devoted to the analysis of the densities associated with the giant vortex energy. We shall introduce first some notation and then, exploiting energy considerations, discuss the properties of those minimizers : Most of them are very similar to the ones proven for Ψ^{GP} , as, e.g., the exponential smallness, but we also need other general properties as, for instance, an a priori bound on the gradient of the densities and an estimate of their difference from the TF density ρ^{TF} .

3.2.1 Estimates for GP Minimizers

We briefly recall the main notation : Given the GP energy functional $\mathcal{E}^{\text{GP}}[\Psi]$ defined in (3.1.2), we denote by E^{GP} its infimum over L^2 -normalized wave functions. Any GP minimizer is denoted by Ψ^{GP} and solves the variational equation (3.1.5).

The TF functional $\mathcal{E}^{\text{TF}}[\rho]$ was introduced in (3.1.9) and E^{TF} and ρ^{TF} stand for its ground state energy and density respectively (see also the Appendix).

Any further label to the functionals \mathcal{E}^{GP} and \mathcal{E}^{TF} , as, e.g., $\mathcal{E}_{\mathcal{D}}^{\text{GP}}$, denotes a restriction of the integration in the functional to the domain \mathcal{D} . The same convention is used for the corresponding ground state energies and minimizers.

Finally we use the notation $\mathcal{B}(\vec{r}, \varrho)$ for a ball of radius ϱ centered at \vec{r} , whereas \mathcal{B}_R is a ball with radius R centered at the origin.

The starting point is a simple GP energy upper bound proven for instance in [47, Proposition 4.2], i.e.,

$$E^{\text{GP}} \leq E^{\text{TF}} + \Omega|\log \varepsilon| \leq E^{\text{TF}} + C\varepsilon^{-2}. \quad (3.2.1)$$

A straightforward consequence of such an upper bound is that the L^2 norm of Ψ^{GP} is concentrated in the support $\mathcal{D}^{\text{TF}} := \{\vec{r} : R_h \leq r \leq 1\}$ of the TF minimizer. At the same time the bound implies an useful upper bound on $|\Psi^{\text{GP}}|$:

Proposition 3.2.1 (Preliminary estimates for Ψ^{GP}).

As $\varepsilon \rightarrow 0$,

$$\| |\Psi^{\text{GP}}|^2 - \rho^{\text{TF}} \|_{L^2(\mathcal{B})} \leq O(1), \quad \|\Psi^{\text{GP}}\|_{L^\infty(\mathcal{B})}^2 \leq \|\rho^{\text{TF}}\|_{L^\infty(\mathcal{B})} \left(1 + O(\sqrt{\varepsilon |\log \varepsilon|})\right). \quad (3.2.2)$$

Proof. In order to prove the first statement it is sufficient to use the fact that ρ^{TF} is the positive part of the function $(\varepsilon^2/2)(\mu^{\text{TF}} + \Omega^2 r^2)$ together with the normalization of ρ^{TF} and Ψ^{GP} and estimate

$$\begin{aligned} \int_{\mathcal{B}} d\vec{r} [|\Psi^{\text{GP}}|^2 - \rho^{\text{TF}}]^2 &= \|\Psi^{\text{GP}}\|_4^4 + \|\rho^{\text{TF}}\|_2^2 - 2 \int_{\mathcal{B}} d\vec{r} |\Psi^{\text{GP}}|^2 \rho^{\text{TF}} \leq \\ &\|\Psi^{\text{GP}}\|_4^4 + \|\rho^{\text{TF}}\|_2^2 - \varepsilon^2 \mu^{\text{TF}} - \varepsilon^2 \Omega^2 \int_{\mathcal{B}} d\vec{r} r^2 |\Psi^{\text{GP}}|^2 = \varepsilon^2 (\mathcal{E}^{\text{TF}} [|\Psi^{\text{GP}}|^2] - E^{\text{TF}}) \leq \\ &\varepsilon^2 (E^{\text{GP}} - E^{\text{TF}}) \leq C, \end{aligned} \quad (3.2.3)$$

by (3.2.1).

The proof of the second inequality is similar to the proof of Lemma 5.1 in [47] and involves the variational equation (3.1.5). We define

$$\rho^{\text{GP}} := |\Psi^{\text{GP}}|^2. \quad (3.2.4)$$

The crucial point is that at any maximum point of ρ^{GP} in the closed ball $\bar{\mathcal{B}}$, it has to be $\Delta \rho^{\text{GP}} \leq 0$. This is trivially true in the open ball \mathcal{B} but can be extended to the boundary thanks to Neumann boundary conditions : Since $\partial_r \Psi^{\text{GP}} = 0$ at the boundary $\partial \mathcal{B}$, which implies $\partial_r \rho^{\text{GP}} = 0$ there, ρ^{GP} can have a maximum at $\vec{r}_0 \in \partial \mathcal{B}$ only if $\nabla \rho^{\text{GP}}(\vec{r}_0) = 0$ and therefore $\Delta \rho^{\text{GP}}(\vec{r}_0) \leq 0$.

From the variational equation (3.1.5) solved by Ψ^{GP} , one can estimate

$$-\Delta \rho^{\text{GP}} \leq 4\varepsilon^{-2} (\varepsilon^2 \mu^{\text{GP}} + \varepsilon^2 \Omega^2 r^2 - 2\rho^{\text{GP}}) \rho^{\text{GP}} \leq 4\varepsilon^{-2} (\varepsilon^2 \mu^{\text{GP}} + \varepsilon^2 \Omega^2 - 2\rho^{\text{GP}}) \rho^{\text{GP}}, \quad (3.2.5)$$

by using the properties

$$\begin{aligned} -\Delta \rho^{\text{GP}} &= -\Psi^{\text{GP}*} \Delta \Psi^{\text{GP}} - \Psi^{\text{GP}} \Delta \Psi^{\text{GP}*} - 2|\nabla \Psi^{\text{GP}}|^2, \\ |\Psi^{\text{GP}*} 2\vec{\Omega} \cdot \vec{L} \Psi^{\text{GP}}| &\leq |\nabla \Psi^{\text{GP}}|^2 + \Omega^2 r^2 |\Psi^{\text{GP}}|^2. \end{aligned} \quad (3.2.6)$$

Now since $\Delta \rho^{\text{GP}} \leq 0$ at any maximum point of ρ^{GP} , one immediately has

$$\|\Psi^{\text{GP}}\|_\infty^2 \leq (\varepsilon^2 \mu^{\text{GP}} + \varepsilon^2 \Omega^2)/2 \leq \rho^{\text{TF}}(1) + C\varepsilon^2 |\mu^{\text{GP}} - \mu^{\text{TF}}|.$$

Now

$$\rho^{\text{TF}}(1) = \|\rho^{\text{TF}}\|_\infty = C\varepsilon\Omega = C\varepsilon^{-1} |\log \varepsilon|^{-1}, \quad (3.2.7)$$

and the difference between the chemical potentials (see (3.1.6) and (3.6.2) for the definitions) can be estimated as follows

$$\begin{aligned} |\mu^{\text{GP}} - \mu^{\text{TF}}| &\leq E^{\text{GP}} - E^{\text{TF}} + \varepsilon^{-2} \left(\|\rho^{\text{TF}}\|_\infty^{1/2} + \|\Psi^{\text{GP}}\|_\infty \right) \| |\Psi^{\text{GP}}|^2 - \rho^{\text{TF}} \|_2 \leq \\ &C\varepsilon^{-2} \left[1 + \|\rho^{\text{TF}}\|_\infty^{1/2} (1 + \varepsilon^3 |\log \varepsilon| |\mu^{\text{GP}} - \mu^{\text{TF}}|)^{1/2} \right] \leq \\ &C\varepsilon^{-5/2} |\log \varepsilon|^{-1/2} \left(1 + \varepsilon^{3/2} |\log \varepsilon|^{1/2} |\mu^{\text{GP}} - \mu^{\text{TF}}|^{1/2} \right), \end{aligned}$$

which yields

$$|\mu^{\text{GP}} - \mu^{\text{TF}}| \leq C\varepsilon^{-5/2} |\log \varepsilon|^{-1/2} \quad (3.2.8)$$

and thus the result. \square

A consequence of the L^2 estimate in (3.2.2) is that the L^2 norm of Ψ^{GP} is concentrated inside the support of ρ^{TF} or, in other words, the mass of Ψ^{GP} inside the hole \mathcal{B}_{R_h} is small. Indeed one can easily realize that the first inequality in (3.2.2) implies

$$\|\Psi^{\text{GP}}\|_{L^2(\mathcal{B}_{R_h})}^2 \leq O(\sqrt{\varepsilon |\log \varepsilon|}), \quad (3.2.9)$$

since, due to the normalization of ρ^{TF} and its support in \mathcal{D}^{TF} ,

$$\begin{aligned} 1 = \|\Psi^{\text{GP}}\|_{L^2(\mathcal{B}_{R_h})}^2 + \|\Psi^{\text{GP}}\|_{L^2(\mathcal{D}^{\text{TF}})}^2 &= \|\Psi^{\text{GP}}\|_{L^2(\mathcal{B}_{R_h})}^2 + \int_{\mathcal{D}^{\text{TF}}} d\vec{r} \left(|\Psi^{\text{GP}}|^2 - \rho^{\text{TF}} \right) + 1 \geq \\ &\|\Psi^{\text{GP}}\|_{L^2(\mathcal{B}_{R_h})}^2 + 1 - O(\sqrt{\varepsilon |\log \varepsilon|}). \end{aligned}$$

This simple estimate can be refined (see, e.g., [43, Proposition 2.5]) and one can actually show that Ψ^{GP} is exponentially small in ε inside \mathcal{B}_{R_h} . Next Proposition is devoted to the proof of two pointwise inequalities of this kind :

Proposition 3.2.2 (Exponential smallness of Ψ^{GP} inside the hole).

As $\varepsilon \rightarrow 0$ and for any $\vec{r} \in \mathcal{B}$,

$$|\Psi^{\text{GP}}(\vec{r})|^2 \leq C\varepsilon^{-1} |\log \varepsilon|^{-1} \exp \left\{ -\frac{1-r^2}{1-R_h^2} \right\}. \quad (3.2.10)$$

Moreover there exist a strictly positive constant c such that for any $1/2 \leq r \leq R_h - O(\varepsilon^{7/6})$,

$$|\Psi^{\text{GP}}(\vec{r})|^2 \leq C\varepsilon^{-1} |\log \varepsilon|^{-1} \exp \left\{ -\frac{c}{\varepsilon^{1/6}} \right\}. \quad (3.2.11)$$

Remark 3.2.1. (Comparison between (3.2.10) and (3.2.11)).

At first sight the pointwise estimates in (3.2.10) and (3.2.11) look very similar. In fact, since $1 - R_h^2 = O(\varepsilon |\log \varepsilon|)$, one can easily realize that the first one yields a much better upper bound than the latter as soon as $1 - r^2 \gg O(\varepsilon^{5/6})$, i.e., in particular for $1 - r^2 = O(1)$. The main drawback of the first inequality is however that it becomes much weaker and even useless if one gets closer to the radius R_h . More precisely as soon as $1 - r^2 \lesssim O(\varepsilon)$ the bound is no longer exponentially small in ε . On the opposite the second inequality has no r dependence and a worse coefficient in the exponential function but it holds true and yields some exponential smallness up to a distance of order $\varepsilon^{7/6}$ from the boundary $\partial\mathcal{B}_{R_h}$. Because of this fact the second inequality will be crucial in the reduction of the original problem to another one on an annulus close to the support of the TF minimizer. The first inequality, on the contrary, will be useful in a region far from the boundary of the hole $\partial\mathcal{B}_{R_h}$ and close to the origin, where the second bound does not apply.

Note also that the factor in front of the exponential in both (3.2.10) and (3.2.11) is essentially given by the sup of $|\Psi^{\text{GP}}|^2$ over the domain \mathcal{B} . In the case of the first inequality this is actually needed in order to make it meaningful on the whole of \mathcal{B} .

Proof. The starting point of the proof of (3.2.10) is the inequality (3.2.5) together with the estimate (3.2.8), which yield

$$-\Delta\rho^{\text{GP}} \leq 4\varepsilon^{-2} \left[\tilde{\rho}^{\text{TF}}(r) + C\varepsilon^{-\frac{1}{2}} |\log \varepsilon|^{-\frac{1}{2}} - \rho^{\text{GP}} \right] \rho^{\text{GP}}, \quad (3.2.12)$$

where we have set

$$\tilde{\rho}^{\text{TF}}(r) := \frac{\varepsilon^2}{2} [\mu^{\text{TF}} + \Omega^2 r^2], \quad (3.2.13)$$

which coincides with the TF density ρ^{TF} inside \mathcal{D}^{TF} and is negative everywhere else. More precisely, for any \vec{r} such that $r^2 \leq R_h^2 - \varepsilon$, one has

$$-\Delta\rho^{\text{GP}} + 2\varepsilon^{-3} |\log \varepsilon|^{-2} \rho^{\text{GP}} \leq 0, \quad (3.2.14)$$

since in that region

$$\tilde{\rho}^{\text{TF}}(r) + C\varepsilon^{-\frac{1}{2}} |\log \varepsilon|^{-\frac{1}{2}} \leq -\varepsilon^3 \Omega^2 (1 - o(1)) \leq -\frac{1}{2\varepsilon |\log \varepsilon|^2}.$$

On the other hand the function

$$W(r) := \exp \left\{ \frac{r^2 - 1}{1 - R_h^2 - \varepsilon} \right\},$$

satisfies for any $r \leq 1$

$$-\Delta W + 2\varepsilon^{-3} |\log \varepsilon|^{-2} W \geq \frac{C}{\varepsilon^2 |\log \varepsilon|^2} [-\varepsilon |\log \varepsilon| - r^2 + \varepsilon^{-1}] W \geq 0.$$

If we then multiply W by $\|\rho^{\text{GP}}\|_\infty$ we get then a supersolution to (3.2.14), so that by the maximum principle (see, e.g., [59, Theorem 1 at p. 508])

$$|\Psi^{\text{GP}}(\vec{r})|^2 \leq \|\Psi^{\text{GP}}\|_\infty^2 W(r),$$

for any $r \leq R_h - \varepsilon$. However $W(r)$ is an increasing function and $W(\sqrt{R_h^2 - \varepsilon}) = \|\rho^{\text{GP}}\|_\infty$ by construction, which implies that the estimate trivially holds true for any $\vec{r} \in \mathcal{B}$. The estimate (3.2.10) is a consequence of (3.2.2) and the inequality $1 - R_h^2 = \varepsilon |\log \varepsilon| \gg \varepsilon$.

Concerning the proof of the second inequality (3.2.11), we first notice that (3.2.12) implies that, for any $\vec{r} \in \mathcal{B}_{R_h}$ such that $r \leq R_h - O(\varepsilon^{7/6})$,

$$-\Delta\rho^{\text{GP}} \leq -C\varepsilon^{-17/6} |\log \varepsilon|^{-2} \rho^{\text{GP}}, \quad (3.2.15)$$

since in that region

$$\tilde{\rho}^{\text{TF}}(r) = \frac{\varepsilon^2 \Omega^2}{2} (r^2 - R_h^2) \leq -\frac{C}{\varepsilon^{5/6} |\log \varepsilon|^2}. \quad (3.2.16)$$

For any \vec{r}_0 such that $r_0 \leq R_h - O(\varepsilon^{7/6})$ we can thus consider the annulus $\mathcal{I} = \{\vec{r} \in \mathcal{B} : r \in [r_0 - \varrho, r_0 + \varrho]\}$ for some $\varrho = O(\varepsilon^{7/6})$, such that $r_0 + \varrho \leq R_h - O(\varepsilon^{7/6})$, i.e., the outer boundary of the annulus is still at a distance of order $\varepsilon^{7/6}$ from $\partial\mathcal{B}_{R_h}$, and it is straightforward to verify that the function

$$U(r) := \|\rho^{\text{GP}}\|_\infty \exp \left\{ \frac{\varrho^2 - (r - r_0)^2}{\varepsilon^{5/2}} \right\}, \quad (3.2.17)$$

satisfies

$$\begin{aligned}
 -\Delta U &= -\left[\frac{4(r-r_0)^2}{\varepsilon^5} + \frac{2}{\varepsilon^{5/2}} + \frac{2(r-r_0)}{\varepsilon^{5/2}r}\right]U(r) \geq -C\left[\frac{\varrho^2}{\varepsilon^5} + \frac{1}{\varepsilon^{5/2}}\right]U(r) \geq \\
 &\quad -C\varepsilon^{-8/3}U(r) \gg -C\varepsilon^{-17/6}|\log \varepsilon|^{-2}U(r), \quad (3.2.18)
 \end{aligned}$$

where we have used the fact that $|r-r_0| \leq \varrho$ inside \mathcal{I} and $\varrho = O(\varepsilon^{7/6})$. Denoting now $V(\vec{r}) = \rho^{\text{GP}}(\vec{r}) - U(r)$, we aim at proving that $V \leq 0$, but one has, for any $\vec{r} \in \mathcal{I}$,

$$-\Delta V(\vec{r}) \leq -C\varepsilon^{-17/6}|\log \varepsilon|^{-2}V(\vec{r}).$$

Now at any maximum point of V in the interior of \mathcal{I} it must be $\Delta V \leq 0$, which implies $V \leq 0$ because of the above inequality. Hence it remains only to prove that $V \leq 0$ at boundary $\partial\mathcal{I}$ since the function might have a positive maximum there. However by construction $V \leq 0$ at $\partial\mathcal{I}$ since $U(r_0 - \varrho) = U(r_0 + \varrho) = \|\rho^{\text{GP}}\|_\infty$ and thus $V(\vec{r})$ is negative everywhere. In particular $\rho^{\text{GP}}(\vec{r}) \leq U(r)$ for all \vec{r} such that $r = r_0$ and $\varrho = O(\varepsilon^{7/6})$, which yields (3.2.11). \square

3.2.2 The Giant Vortex Densities

We now discuss some basic properties of the energy functional (3.1.16) together with those of an analogous functional where the integration is restricted to an annulus $\mathcal{A} = \{\vec{r} : R_< \leq r \leq 1\} \subset \mathcal{B}$ with $R_< < R_h$ and $|R_< - R_h| \ll \varepsilon|\log \varepsilon|$. A precise choice for $R_<$ will be made at the beginning of Section 3.3 (see (3.3.2)) but the results contained in this Section applies to any $R_<$ satisfying the above conditions.

As indicated at the end of Section 3.1.2 the restricted functional is for technical reasons actually more useful for the proof of the main results than the original functional. It is defined in analogy with (3.1.16) and (3.1.24) for *real* valued functions f on \mathcal{A} by

$$\begin{aligned}
 \hat{\mathcal{E}}_{\mathcal{A},\omega}^{\text{GP}}[f] &:= \int_{\mathcal{A}} d\vec{r} \{|\nabla f|^2 + ([\Omega] - \omega)^2 r^{-2} f^2 - 2\Omega([\Omega] - \omega)f^2 + \varepsilon^{-2}f^4\} = \\
 &\quad \int_{\mathcal{A}} d\vec{r} \{|\nabla f|^2 - \Omega^2 r^2 f^2 + B_\omega^2 f^2 + \varepsilon^{-2}f^4\}, \quad (3.2.19)
 \end{aligned}$$

with $B_\omega(r)$ given by (3.1.25). We use \mathcal{A} as a subindex to label quantities associated with the annulus and thus denote the infimum of the functional (3.2.19) by

$$\hat{E}_{\mathcal{A},\omega}^{\text{GP}} := \inf_{\|f\|_2=1} \hat{\mathcal{E}}_{\mathcal{A},\omega}^{\text{GP}}[f] \quad (3.2.20)$$

while $\hat{E}_\omega^{\text{GP}}$ stands for the corresponding infimum of (3.1.16).

We shall use the short-hand notation $\hat{\mathcal{E}}_\star^{\text{GP}}$, $\hat{E}_\star^{\text{GP}}$, g_\star , etc., for quantities related either to (3.1.16) or (3.2.19), e.g., a statement about g_\star is meant to apply to both g_ω and $g_{\mathcal{A},\omega}$.

Proposition 3.2.3 (Minimization of $\hat{\mathcal{E}}_\star^{\text{GP}}$).

For any $\omega \in \mathbb{Z}$ such that $|\omega| \leq O(\varepsilon^{-1})$, the ground state energies $\hat{E}_\star^{\text{GP}}$ satisfy the estimates

$$E^{\text{TF}} \leq \hat{E}_\omega^{\text{TF}} \leq \hat{E}_\star^{\text{GP}} \leq \hat{E}_\omega^{\text{TF}} + O(\varepsilon^{-2}|\log \varepsilon|^{-2}) \leq E^{\text{TF}} + O(\varepsilon^{-2}). \quad (3.2.21)$$

The minimizers g_ω and $g_{\mathcal{A},\omega}$ exist, are radially symmetric and unique up to a global sign, which can be chosen so that they are given by positive functions solving the variational equation

$$-\Delta g_\star + \frac{([\Omega] - \omega)^2}{r^2} g_\star - 2\Omega([\Omega] - \omega)g_\star + 2\varepsilon^{-2}g_\star^3 = \hat{\mu}_\star^{\text{GP}} g_\star, \quad (3.2.22)$$

where the chemical potentials $\hat{\mu}_\star^{\text{GP}}$ are fixed by the L^2 normalization of g_\star , i.e., $\hat{\mu}_\star^{\text{GP}} = \hat{E}_\star^{\text{GP}} + \varepsilon^{-2}\|g_\star\|_4^4$.

In addition g_\star are smooth and increasing and satisfy Neumann conditions at the boundary $\partial\mathcal{B}$, i.e., $\partial_r g_\star(1) = 0$; $g_{\mathcal{A},\omega}$ satisfies an identical condition at the inner boundary as well, i.e., $\partial_r g_{\mathcal{A},\omega}(R_<) = 0$.

Proof. The lower bounds to the ground state energies are simply obtained by neglecting positive terms (the kinetic energies) in the functionals : In the case of $\hat{\mathcal{E}}_\omega^{\text{GP}}$ (the other case is identical), one has

$$\hat{\mathcal{E}}_\omega^{\text{GP}}[f] \geq \hat{\mathcal{E}}^{\text{TF}}[f^2] \geq \hat{E}^{\text{TF}} \geq E^{\text{TF}},$$

where we refer to the Appendix for the simple proof of the last inequality.

The upper bound can be easily obtained by testing the functionals on suitable regularizations of $\sqrt{\hat{\rho}_\omega^{\text{TF}}}$ (see (3.6.6) for its definition and [47], where such regularizations are performed on $\sqrt{\rho^{\text{TF}}}$) : The main correction to the energy is due to the radial kinetic energy of the regularization and one can easily realize that this energy can be made of order $O(\varepsilon^{-2}|\log \varepsilon|^{-2})$, since $\sqrt{\hat{\rho}_\omega^{\text{TF}}}$ is a monotone function going from 0 to $O(\varepsilon^{-1}|\log \varepsilon|^{-1})$ in an interval of size $O(\varepsilon|\log \varepsilon|)$.

The last inequality is also discussed in the Appendix but it is basically due to the estimate

$$|B_\omega(\vec{r})| \leq \Omega |r^{-1} - r| + C|\omega| + O(1) \leq C\varepsilon^{-1}, \quad (3.2.23)$$

for any ω such that $|\omega| \leq O(\varepsilon^{-1})$ and $\vec{r} \in \mathcal{A}$.

We remark that the restriction of the integration to the annulus \mathcal{A} has no effect because the support of the trial function can be assumed to be contained inside the support of $\hat{\rho}_\omega^{\text{TF}}$, i.e., the region where $r \geq \hat{R}$. On the other hand by (3.6.9), $\hat{R} \geq R_h - O(\varepsilon^2|\log \varepsilon|^2)$, for any $\omega \in \mathbb{Z}$ such that $|\omega| \leq C\varepsilon^{-1}$, which implies that the support of $\hat{\rho}_\omega^{\text{TF}}$ is contained inside \mathcal{A} .

Existence and uniqueness of the minimizers trivially follow from strict convexity of the functionals with respect to the density f^2 , which can be made clearer by writing them as

$$\hat{\mathcal{E}}_\star^{\text{GP}}[\sqrt{\rho}] = \int d\vec{r} \left\{ |\nabla \sqrt{\rho}|^2 + ([\Omega] - \omega)^2 r^{-2} \rho + \varepsilon^{-2} \rho^2 \right\} - 2\Omega([\Omega] - \omega), \quad (3.2.24)$$

where $\rho = f^2$ and we have used the L^2 normalization of f . Similarly the radial symmetry of g_\star can be proven by averaging over the angular variable and exploiting the convexity of the functional.

All the other properties of g_\star , including the variational equations (3.2.22), are trivial consequences of the minimization : Positivity can be proven by noticing that the minimizers g_\star are actually ground states of suitable one-dimensional Schrödinger operators and therefore can not vanish except at the origin (see, e.g., [96, Theorem 11.8]). Smoothness follows from (3.2.22) by a simple bootstrap argument, etc.

The only property which requires a brief discussion is the monotonicity and we state it in a separate Lemma. \square

Lemma 3.2.1 (Monotonicity of the density).

Let $\rho(r) \geq 0$ be the L^1 -normalized minimizer of

$$\int_R^1 dr r \left\{ \left| \frac{d\sqrt{\rho}}{dr} \right|^2 + \frac{a}{r^2} \rho + b\rho^2 \right\}$$

in $H^1(\mathcal{B} \setminus \mathcal{B}_R) \cap L^2(\mathcal{B} \setminus \mathcal{B}_R)$ with $a, b > 0$ and $0 \leq R < 1$. Then ρ is monotonously increasing in r .

Proof. After a transformation of variables $r^2 \mapsto s$ and considering ρ as a function of s , the functional takes the form

$$\int_{R^2}^1 ds \left\{ s \left| \frac{d\sqrt{\rho}}{ds} \right|^2 + \frac{a}{s} \rho + b\rho^2 \right\}$$

and the normalization condition is

$$\int_{R^2}^1 ds \rho = \text{const.}$$

Suppose the assertion is false. Then ρ has a maximum at $s = s_1$ for some $R \leq s_1 < 1$ and a local minimum at some s_2 with $s_1 < s_2 \leq 1$. For $0 < \epsilon < \rho(s_1) - \rho(s_2)$ consider the set $\mathcal{I}_\epsilon = \{s < s_2 : \rho(s_1) - \epsilon \leq \rho(s) \leq \rho(s_1)\}$. Then, because ρ is continuous,

$$\Phi(\epsilon) := \int_{\mathcal{I}_\epsilon} ds \rho(s)$$

is strictly positive and $\Phi(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. Likewise, for $\delta > 0$ we consider $\mathcal{J}_\delta = \{s > s_1 : \rho(s_2) \leq \rho(s) \leq \rho(s_2) + \delta\}$ and the function

$$\Gamma(\delta) := \int_{\mathcal{J}_\delta} ds \rho(s)$$

that has the same properties as Φ .

Since both Φ and Γ are continuous, strictly positive and tend to zero when their arguments tends to zero, there exist $\bar{\epsilon}, \bar{\delta} > 0$ such that $\rho(s_2) + \bar{\delta} < \rho(s_1) - \bar{\epsilon}$ and $\Phi(\bar{\epsilon}) = \Gamma(\bar{\delta})$: Because ρ is continuous one can, for any given $\bar{\delta} > 0$, always find an $\bar{\epsilon} > 0$ such that the equality $\Phi(\bar{\epsilon}) = \Gamma(\bar{\delta})$ holds and $\bar{\epsilon} \rightarrow 0$ as $\bar{\delta} \rightarrow 0$. The inequality $\rho(s_2) + \bar{\delta} < \rho(s_1) - \bar{\epsilon}$ is fulfilled if $\bar{\delta}$, and hence also $\bar{\epsilon}$, are small enough.

We now define a new function $\bar{\rho}$ by putting

$$\bar{\rho}(s) := \begin{cases} \rho(s_1) - \bar{\epsilon}, & \text{if } s \in \mathcal{I}_{\bar{\epsilon}}, \\ \rho(s_2) + \bar{\delta}, & \text{if } s \in \mathcal{J}_{\bar{\delta}}, \\ \rho(s), & \text{otherwise.} \end{cases}$$

We now compute the energy of the new density $\bar{\rho}$. Note that it belongs by definition to the minimization domain and it is normalized in L^1 because $\Phi(\bar{\epsilon}) = \Gamma(\bar{\delta})$.

The kinetic energy of $\bar{\rho}$ vanishes in the intervals $\mathcal{I}_{\bar{\epsilon}}$ and $\mathcal{J}_{\bar{\delta}}$ and the potential term for $\bar{\rho}$

is strictly smaller than for ρ because $1/s$ is strictly decreasing and the value of $\bar{\rho}$ on $\mathcal{I}_{\bar{\varepsilon}}$ is larger than on \mathcal{J}_{δ} . Finally

$$\int_{R^2} ds \bar{\rho}^2 < \int_{R^2} ds \rho^2$$

because when modifying ρ to define $\bar{\rho}$, mass is moved from $\mathcal{I}_{\bar{\varepsilon}}$ to \mathcal{J}_{δ} where the density is lower.

Altogether the functional evaluated on $\bar{\rho}$ is strictly smaller than on ρ and this contradicts the assumption that ρ is a minimizer. \square

As next we compare the densities g_{\star}^2 with the TF density and prove exponential smallness in the hole. The analogue of Proposition 3.2.1 is

Proposition 3.2.4 (Preliminary estimates for g_{\star}).

As $\varepsilon \rightarrow 0$ and for any $\omega \in \mathbb{Z}$ such that $|\omega| \leq O(\varepsilon^{-1})$,

$$\|g_{\star}^2 - \rho^{\text{TF}}\|_{L^2(B)} = O(1), \quad \|g_{\star}\|_{L^{\infty}(B)}^2 \leq \|\rho^{\text{TF}}\|_{L^{\infty}(B)} \left(1 + O(\sqrt{\varepsilon} |\log \varepsilon|)\right). \quad (3.2.25)$$

Proof. The proof of the L^2 estimate is exactly the same as for the first inequality in (3.2.2). The only difference occurs in the energy remainder on the r.h.s. of (3.2.3) which can now be bounded by $\varepsilon^2(\hat{E}_{\star}^{\text{GP}} - E^{\text{TF}}) \leq C$ because of (3.2.21). Here the condition on ω , i.e., $|\omega| \leq C\varepsilon^{-1}$, is used as in Proposition 3.2.3.

The sup estimate can be proven by applying the same argument used to prove the second inequality in (3.2.2) to the variational equations (3.2.22) solved by g_{\star} , exploiting as well Neumann boundary conditions : Indeed thanks to the monotonicity of the potential $([\Omega] - \omega)^2 r^{-2}$, one obtains the inequality

$$-\Delta g_{\star}^2 \leq 4\varepsilon^{-2} (\varepsilon^2 \hat{\mu}_{\star}^{\text{GP}} + 2\varepsilon^2 \Omega([\Omega] - \omega) - \varepsilon^2([\Omega] - \omega)^2 - 2g_{\star}^2) g_{\star}^2,$$

which as in the proof of Proposition 3.2.1 implies

$$\|g_{\star}\|_{\infty}^2 \leq \frac{\varepsilon^2 \hat{\mu}_{\star}^{\text{GP}} + 2\varepsilon^2 \Omega([\Omega] - \omega) - \varepsilon^2([\Omega] - \omega)^2}{2}. \quad (3.2.26)$$

The difference between the chemical potential can be estimated as in (3.2.8) :

$$|\hat{\mu}_{\star}^{\text{GP}} - \mu^{\text{TF}}| \leq O(\varepsilon^{-5/2} |\log \varepsilon|^{-1/2}), \quad (3.2.27)$$

so that $\|g_{\star}\|_{\infty}^2 \leq \rho^{\text{TF}}(1)(1 + o(1))$. \square

As for Ψ^{GP} the above estimates imply the exponential smallness of the densities g_{\star} inside the hole. Moreover the L^2 estimate (3.2.2) can be refined to a pointwise estimate inside \mathcal{D}^{TF} and one can actually prove that there is a region where the difference between g_{\star} and ρ^{TF} is much smaller than ρ^{TF} :

Proposition 3.2.5 (Exponential smallness of g_{\star} inside the hole).

As $\varepsilon \rightarrow 0$ and for any $\vec{r} \in \mathcal{B}$

$$g_{\star}^2(r) \leq C\varepsilon^{-1} |\log \varepsilon|^{-1} \exp \left\{ -\frac{1 - r^2}{1 - R_{\text{h}}^2} \right\}. \quad (3.2.28)$$

Moreover there exists a strictly positive constant c such that for any $1/2 \leq r \leq R_{\text{h}} - O(\varepsilon^{7/6})$,

$$g_{\star}^2(r) \leq C\varepsilon^{-1} |\log \varepsilon|^{-1} \exp \left\{ -\frac{c}{\varepsilon^{1/6}} \right\}. \quad (3.2.29)$$

Proof. The proof can be easily reduced to the proof of Proposition 3.2.2 by using (3.2.27). \square

Proposition 3.2.6 (Pointwise estimate for g_\star).

As $\varepsilon \rightarrow 0$ and for any $\omega \in \mathbb{Z}$ with $|\omega| \leq O(\varepsilon^{-1})$

$$|g_\star^2(r) - \rho^{\text{TF}}(r)| \leq C\varepsilon^2 |\log \varepsilon|^2 (r^2 - R_h^2)^{-3/2} \rho^{\text{TF}}(r) \quad (3.2.30)$$

for any $\vec{r} \in \mathcal{D}^{\text{TF}}$ such that $r \geq R_h + O(\varepsilon^{3/2} |\log \varepsilon|^2)$.

Proof. The proof is similar to the proof of Proposition 1 in [5] but for the sake of completeness we bring here all the details.

The variational equation (3.2.22) can be rewritten in the following form

$$-\Delta g_\star = \frac{2}{\varepsilon^2} [\tilde{\rho}_\star(r) - g_\star^2] g_\star, \quad (3.2.31)$$

where the function $\tilde{\rho}_\star$ is given by

$$\tilde{\rho}_\star(r) := \frac{1}{2} (\varepsilon^2 \hat{\mu}_\star^{\text{GP}} + \varepsilon^2 \Omega^2 r^2 - \varepsilon^2 B_\omega^2(r)). \quad (3.2.32)$$

Moreover by (3.2.23) and (3.2.27)

$$\|\tilde{\rho}_\star - \rho^{\text{TF}}\|_{L^\infty(\mathcal{D}^{\text{TF}})} \leq \varepsilon^2 |\hat{\mu}_\star^{\text{GP}} - \mu^{\text{TF}}| + O(1) \leq O(\varepsilon^{-1/2} |\log \varepsilon|^{-1/2}). \quad (3.2.33)$$

On the other hand, for any $\vec{r} \in \mathcal{D}^{\text{TF}}$ such that $r - R_h \geq O(\varepsilon^{3/2} |\log \varepsilon|^2)$, $\rho^{\text{TF}}(r) \geq O(\varepsilon^{-1/2})$ and therefore

$$\tilde{\rho}_\star(r) \geq \rho^{\text{TF}}(r)(1 - C|\log \varepsilon|^{-1/2}) > C\varepsilon^{-1/2} \quad (3.2.34)$$

and in particular $\tilde{\rho}_\star$ is strictly positive for such \vec{r} , which is crucial in order to apply the maximum principle. The pointwise estimates are indeed proven by providing local super- and subsolutions to (3.2.31).

For the upper bound, we consider an interval $[r_0 - \delta, r_0 + \delta]$, where $R_h + C\varepsilon^{3/2} |\log \varepsilon|^2 + \delta < r_0 < 1 - \delta$ with $\delta \ll 1$, and the function

$$W(r) := \sqrt{\tilde{\rho}_\star(r_0 + \delta)} \coth \left[\coth^{-1} \left(\sqrt{\frac{\tilde{\rho}_\star(1)}{\tilde{\rho}_\star(r_0 + \delta)}} \right) + \frac{\delta^2 - |r - r_0|}{3\delta\varepsilon} \sqrt{2\tilde{\rho}_\star(r_0 + \delta)} \right]. \quad (3.2.35)$$

One has (see [25, Proof of Proposition 2.1]) for any $r \in [r_0 - \delta, r_0 + \delta]$

$$-\Delta W \geq \frac{2}{\varepsilon^2} (\tilde{\rho}_\star(r_0 + \delta) - W^2) W \geq \frac{2}{\varepsilon^2} (\tilde{\rho}_\star(r) - W^2) W, \quad (3.2.36)$$

where we have used the fact that $\tilde{\rho}_\star(r)$ is an increasing function of r . Moreover at the boundary of the interval $W(r_0 - \delta) = W(r_0 + \delta) = \sqrt{\tilde{\rho}_\star(1)}$ which is not smaller than g_\star thanks to the upper bound (3.2.26), which reads $g_\star^2 \leq \tilde{\rho}_\star(1)$. Therefore $W(r)$ is a supersolution to (3.2.31) in the interval $[r_0 - \delta, r_0 + \delta]$ and by the maximum principle

$$g_\star(r_0) \leq W(r_0) = \sqrt{\tilde{\rho}_\star(r_0 + \delta)} \coth \left[\frac{\delta}{3\varepsilon} \sqrt{2\tilde{\rho}_\star(r_0 + \delta)} \right] \quad (3.2.37)$$

where we have used the fact that $\coth(x)$ is a non-increasing function.

By the explicit expression (3.2.32) and the inequality (3.2.34), one has

$$|\tilde{\rho}_\star(r_0 + \delta) - \tilde{\rho}_\star(r_0)| \leq C\varepsilon^2 \Omega^2 \delta, \quad (3.2.38)$$

$$\tilde{\rho}_*(r_0) \geq \rho^{\text{TF}}(r_0) \left(1 - O(|\log \varepsilon|^{-1/2})\right) \geq C\varepsilon^2\Omega^2(r_0^2 - R_h^2), \quad (3.2.39)$$

so that (3.2.37) becomes

$$g_*(r_0) \leq \sqrt{\tilde{\rho}_*(r_0)} \left(1 + \frac{C\delta}{r_0^2 - R_h^2}\right) \coth \left[\frac{\delta}{3\varepsilon} \sqrt{2\tilde{\rho}_*(r_0 + \delta)}\right]. \quad (3.2.40)$$

When the argument of \coth tends to ∞ , i.e.,

$$\frac{\delta}{\varepsilon} \sqrt{2\tilde{\rho}_*(r_0 + \delta)} \gg 1,$$

we can bound

$$\coth(x) = \frac{1 + e^{-2x}}{1 - e^{-2x}} \leq (1 + Ce^{-2x}),$$

and obtain the inequality

$$g_*(r_0) \leq \sqrt{\tilde{\rho}_*(r_0)} \left(1 + \frac{C\delta}{r_0^2 - R_h^2}\right) \left(1 + C \exp \left\{-\frac{2\delta}{3\varepsilon} \sqrt{2\tilde{\rho}_*(r_0 + \delta)}\right\}\right). \quad (3.2.41)$$

By (3.2.39) the second error term on the r.h.s. of the above expression is bounded from above by

$$\exp \left\{-C\delta\Omega\sqrt{r_0^2 - R_h^2}\right\},$$

so that by taking

$$\delta = C\varepsilon^2|\log \varepsilon|^2\sqrt{r_0^2 - R_h^2}, \quad (3.2.42)$$

such an error can be made smaller than any power of ε , since one can choose the constant coefficient in δ arbitrarily large. With such a choice the other error term becomes

$$C\varepsilon^2|\log \varepsilon|^2(r_0^2 - R_h^2)^{-3/2} > C\varepsilon^{1/2}|\log \varepsilon|^{1/2},$$

since by definition $r^2 - R_h^2 \leq 1 - R_h^2 \leq O(\varepsilon|\log \varepsilon|)$. Hence the second factor on the r.h.s. of (3.2.41) can be absorbed in the above remainder.

Moreover for any $r \geq R_h$, $\delta \leq \varepsilon^{5/2}|\log \varepsilon|^{1/2} \ll \varepsilon^{3/2}$ and one can extend the estimate to any $r \geq R_h + O(\varepsilon^{3/2}|\log \varepsilon|^2)$. For the same reason the estimate applies also to the region $[1 - 2\delta, 1]$: There one can use (3.2.26) and the fact that $\tilde{\rho}_*(1) - \tilde{\rho}_*(1 - 2\delta) \leq C\Omega^2\varepsilon^2\delta \leq C\varepsilon^{1/2}$, which is much smaller than the error term above.

The final estimate is then

$$g_*(r) \leq \sqrt{\tilde{\rho}_*(r)} \left[1 + C\varepsilon^2|\log \varepsilon|^2(r^2 - R_h^2)^{-3/2}\right], \quad (3.2.43)$$

for any $R_h + C\varepsilon^{3/2}|\log \varepsilon|^2 \leq r \leq 1$.

The next step is the replacement of $\tilde{\rho}_*$ with ρ^{TF} by means of (3.2.33), which yields an additional remainder given by

$$\begin{aligned} \rho^{\text{TF}}(r)^{-1} |\tilde{\rho}_*(r) - \rho^{\text{TF}}(r)| &\leq C(\varepsilon\Omega)^{-2}(r^2 - R_h^2)^{-1} \|\tilde{\rho}_* - \rho^{\text{TF}}\|_{L^\infty(\mathcal{D}^{\text{TF}})} \leq \\ &C\varepsilon^2|\log \varepsilon|^2(r^2 - R_h^2)^{-1}, \end{aligned} \quad (3.2.44)$$

which can however be absorbed in the error term in (3.2.40), since $r^2 - R_h^2 \leq \varepsilon |\log \varepsilon|$.

In order to prove a corresponding lower bound, we fix some r_0 and δ as before, i.e., such that $R_h + C\varepsilon^{3/2} |\log \varepsilon|^2 + \delta < r_0 < 1 - \delta$ and $\delta \ll 1$. Since $\tilde{\rho}_*$ is an increasing function of r and g_* is positive

$$-\Delta g_* \geq \frac{2}{\varepsilon^2} [\tilde{\rho}_*(r_0 - \delta) - g_*^2] g_*. \quad (3.2.45)$$

Then we denote by $h(r)$ the function solving for $\tilde{r} \in \mathcal{B}$

$$-\Delta h = \tilde{\varepsilon}^{-2} (1 - h^2) h,$$

with Dirichlet boundary condition $h(1) = 0$ and $\tilde{\varepsilon} \rightarrow 0$. In [142] it was proven that h satisfies the bound

$$1 - c \exp \left\{ -\frac{1 - r^2}{2\tilde{\varepsilon}} \right\} \leq h(r) \leq 1.$$

If we now set

$$\tilde{h}(r) := \sqrt{\tilde{\rho}_*(r_0 - \delta)} h \left(\frac{|r - r_0|}{\delta} \right), \quad (3.2.46)$$

$$\tilde{\varepsilon} := \frac{\varepsilon}{\delta \sqrt{2\tilde{\rho}_*(r_0 - \delta)}}, \quad (3.2.47)$$

then \tilde{h} solves in $[r_0 - \delta, r_0 + \delta]$ the equation

$$-\Delta \tilde{h} = \frac{2}{\varepsilon^2} [\tilde{\rho}_*(r_0 - \delta) - \tilde{h}^2] \tilde{h},$$

with Dirichlet conditions at the boundary $r = r_0 \pm \delta$. Thanks to (3.2.45), g_* is a supersolution for the same problem, so that by the maximum principle $g_*(r) \geq \tilde{h}(r)$ inside the interval and in particular

$$g_*(r_0) \geq \tilde{h}(r_0) \geq \sqrt{\tilde{\rho}_*(r_0 - \delta)} \left[1 - c \exp \left\{ -\frac{1}{2\tilde{\varepsilon}} \right\} \right], \quad (3.2.48)$$

for any $R_h + C\varepsilon^{3/2} |\log \varepsilon|^2 + \delta < r_0 < 1 - \delta$.

Note that by choosing δ as in (3.2.42) the remainder in the above expression can be made smaller than any power of ε . However the estimate of $\tilde{\rho}_*(r - \delta)$ in terms of $\tilde{\rho}_*(r)$ provides the same remainder as in the upper bound proof, i.e.,

$$g_*(r) \geq \sqrt{\tilde{\rho}_*(r)} \left[1 - C\varepsilon^2 |\log \varepsilon|^2 (r^2 - R_h^2)^{-3/2} \right].$$

The extension of the estimate to the whole region $[R_h + O(\varepsilon^{3/2} |\log \varepsilon|^2), 1]$ as well as the replacement of $\tilde{\rho}_*$ with ρ^{TF} can be done exactly as in the upper bound and the remainders included in the above error term. \square

We conclude the Section with an useful estimate of the gradient of the densities g_* :

Proposition 3.2.7 (Gradient estimate for g_*).

As $\varepsilon \rightarrow 0$ and for any $R > R_h + O(\varepsilon^{3/2} |\log \varepsilon|^2)$, one has

$$\|\nabla g_*\|_{L^\infty(\mathcal{B} \setminus \mathcal{B}_R)} \leq C\varepsilon^{-1} (R^2 - R_h^2)^{-1/4} \|g_*\|_{L^\infty(\mathcal{B})}, \quad (3.2.49)$$

which inside $\tilde{\mathcal{A}} = \{\tilde{r} : R_h + \varepsilon |\log \varepsilon|^{-1} \leq r \leq 1\}$ becomes

$$\|\nabla g_*\|_{L^\infty(\tilde{\mathcal{A}})} \leq C\varepsilon^{-7/4} |\log \varepsilon|^{-3/4}. \quad (3.2.50)$$

Proof. We first exploit the fact that any g_\star is radial to rewrite (3.2.22) as

$$-g_\star'' - r^{-1}g_\star' + ([\Omega] - \omega)^2 r^{-2}g_\star + 2\varepsilon^{-2}g_\star^3 = (\hat{\mu}_\star^{\text{GP}} + 2\Omega[\Omega - \omega])g_\star \quad (3.2.51)$$

and take the L^∞ norm inside $\mathcal{B} \setminus \mathcal{B}_R$, for some $R > R_h + O(\varepsilon^{3/2}|\log \varepsilon|^2)$, of both sides to obtain (see (3.2.32))

$$\begin{aligned} \|g_\star''\|_{L^\infty(\mathcal{B} \setminus \mathcal{B}_R)} &\leq C \|r^{-1}g_\star'\|_{L^\infty(\mathcal{B} \setminus \mathcal{B}_R)} + C\varepsilon^{-2} \|(\tilde{\rho}_\star - g_\star^2)g_\star\|_{L^\infty(\mathcal{B} \setminus \mathcal{B}_R)} \leq \\ &C \left[\|g_\star'\|_{L^\infty(\mathcal{B} \setminus \mathcal{B}_R)} + \varepsilon^{-2} (R^2 - R_h^2)^{-1/2} \|g_\star\|_{L^\infty(\mathcal{B})} \right], \end{aligned} \quad (3.2.52)$$

by (3.2.33) and the pointwise estimate (3.2.30) :

$$\begin{aligned} \|\tilde{\rho}_\star - g_\star^2\|_{L^\infty(\mathcal{B} \setminus \mathcal{B}_R)} &\leq \|\rho^{\text{TF}} - g_\star^2\|_{L^\infty(\mathcal{B} \setminus \mathcal{B}_R)} + O(\varepsilon^{-1/2}|\log \varepsilon|^{-1/2}) \leq \\ &C (R^2 - R_h^2)^{-1/2} + O(\varepsilon^{-1/2}|\log \varepsilon|^{-1/2}) \leq C (R^2 - R_h^2)^{-1/2}, \end{aligned}$$

since $R^2 - R_h^2 \leq C\varepsilon|\log \varepsilon|$.

On the other hand by the Gagliardo-Nirenberg inequality (see, e.g., [117, Theorem at p. 125])

$$\begin{aligned} \|g_\star'\|_{L^\infty(\mathcal{B} \setminus \mathcal{B}_R)} &\leq C \|g_\star''\|_{L^\infty(\mathcal{B} \setminus \mathcal{B}_R)}^{1/2} \|g_\star\|_{L^\infty(\mathcal{B})}^{1/2} \leq \\ &C \left[\varepsilon^{-1} (R^2 - R_h^2)^{-1/4} \|g_\star\|_{L^\infty(\mathcal{B})} + \|g_\star'\|_{L^\infty(\mathcal{B} \setminus \mathcal{B}_R)}^{1/2} \|g_\star\|_{L^\infty(\mathcal{B})}^{1/2} \right], \end{aligned} \quad (3.2.53)$$

which implies the result. \square

3.3 Reduction to an Auxiliary Problem on an Annulus

The first step towards the proof of the main results is the reduction of the original GP energy functional to an analogous functional on a suitable annulus. The main ingredients of such a reduction are, on the one hand, the exponential smallness of the GP minimizer proven in last Section (see, e.g., (3.2.10), (3.2.11), (3.2.28), etc.), which intuitively implies that all the L^2 mass and therefore the energy are concentrated in a annulus close to the boundary, and on the other a decoupling of the GP energy functional, which allows the extraction of the giant vortex energies introduced in (3.1.16).

The second part of the Section is devoted to the discussion of the optimal giant vortex phase : Whereas both the energy splitting and upper bound hold true for any reasonable phase ω , they are useful only for specific choices of the phase. As we are going to see, the optimal phase ω_0 can be defined as the minimizer of a suitable coupled problem in the annulus.

We also discuss the existence of the analogous minimizer associated with the original problem in the ball \mathcal{B} , i.e., the phase ω_{opt} occurring in Theorems 3.1.2 and 3.1.3, as well as some relevant properties of it.

3.3.1 Energy Decoupling

Before stating the main result of this Section, we first recall and introduce some notation.

The main object of the reduction is an annulus

$$\mathcal{A} := \{\vec{r} : R_< \leq r \leq 1\}, \quad (3.3.1)$$

where the inner radius $R_{<} < R_{\text{h}}$ has to be chosen in a proper way so that two conditions are simultaneously fulfilled : The radius $R_{<}$ should be sufficiently far from R_{h} in such a way that the exponential smallness proven in Propositions 3.2.2 and 3.2.5 applies inside the complement of \mathcal{A} . At the same time, $R_{<}$ must not be too far from R_{h} ; in fact we shall in Section 3.4 need that $|R_{<} - R_{\text{h}}| \ll \varepsilon |\log \varepsilon|^{-1}$ (see (3.4.89) and the use of (3.4.22) in the proof of Proposition 3.4.1).

All these conditions are satisfied if one chooses

$$R_{<} := R_{\text{h}} - \varepsilon^{8/7}. \quad (3.3.2)$$

The apparently strange power of ε occurring in the above expression is essentially motivated by the pointwise estimates (3.2.11) and (3.2.29), which hold true for any $r \leq R_{\text{h}} - O(\varepsilon^{7/6})$: In the definition of the inner radius of the annulus we could have chosen any $R_{<}$ satisfying such a condition in addition to $|R_{<} - R_{\text{h}}| \ll \varepsilon |\log \varepsilon|^{-1}$. In particular any remainder given by a power of ε smaller than $7/6$ but larger than 1 would have been all right. However for the sake of simplicity we make an explicit choice among the allowed remainders and take $\varepsilon^{8/7}$.

The auxiliary problem on the annulus is associated with the estimate of the energy functional

$$\mathcal{E}_{\mathcal{A},\omega}[v] := \int_{\mathcal{A}} d\vec{r} g_{\mathcal{A},\omega}^2 \left\{ |\nabla v|^2 - 2\vec{B} \cdot (iv, \nabla v) + \varepsilon^{-2} g_{\mathcal{A},\omega}^2 (1 - |v|^2)^2 \right\}, \quad (3.3.3)$$

where \vec{B} is defined in (3.1.32). According to the convention used in the rest of the paper, \mathcal{E}_{ω} without the label \mathcal{A} stands for the same energy as above but with the integral extended to the whole of \mathcal{B} .

A key ingredient in the proof of the GP energy asymptotics is a lower estimate of (3.3.3). In fact in the next Proposition, which is the main result proven in this Section, we show that the energy (3.3.3) provides a lower bound to $E^{\text{GP}} - \hat{E}_{\mathcal{A},\omega}^{\text{GP}}$ when v is suitably linked to Ψ^{GP} :

Proposition 3.3.1 (Reduction to an annulus).

For any $\omega \in \mathbb{Z}$ such that $|\omega| \leq O(\varepsilon^{-1})$ and for ε sufficiently small

$$\hat{E}_{\mathcal{A},\omega}^{\text{GP}} + \mathcal{E}_{\mathcal{A},\omega}[u_{\omega}] - O(\varepsilon^{\infty}) \leq E^{\text{GP}} \leq \hat{E}_{\mathcal{A},\omega}^{\text{GP}} + O(\varepsilon^{\infty}), \quad (3.3.4)$$

where the function $u_{\omega} \in H^1(\mathcal{A})$ is given by the decomposition

$$\Psi^{\text{GP}}(\vec{r}) =: g_{\mathcal{A},\omega}(r) u_{\omega}(\vec{r}) \exp \{ i([\Omega] - \omega)\vartheta \}. \quad (3.3.5)$$

Proof. The proof of (3.3.4) is done by proving suitable upper and lower bounds to the GP ground state energy.

The upper bound is obtained by testing \mathcal{E}^{GP} on a trial function of the form $\tilde{g}(r) \exp\{i([\Omega] - \omega)\vartheta\}$, where \tilde{g} is an appropriate regularization of $g_{\mathcal{A},\omega}$, and using the definition (3.2.19). Since $g_{\mathcal{A},\omega}$ does not vanish at $R_{<}$, it is not in the minimization domain of \mathcal{E}^{GP} and one has to regularize it at the boundary $\partial R_{<}$, e.g., taking

$$\tilde{g}(r) := c \begin{cases} g_{\mathcal{A},\omega}(r) & \text{if } \vec{r} \in \mathcal{A}, \\ \varepsilon^{-n} g_{\mathcal{A},\omega}(R_{<}) (R_{<} - \varepsilon^n - r) & \text{if } R_{<} - \varepsilon^n \leq r \leq R_{<}, \\ 0 & \text{if } r \leq R_{<} - \varepsilon^n, \end{cases} \quad (3.3.6)$$

where c is a normalization constant and n some arbitrary power greater than 0. Thanks to the exponential smallness (3.2.29), which implies $g_{\mathcal{A},\omega}(R_<) = O(\varepsilon^\infty)$, the normalization constant is $c = 1 - O(\varepsilon^\infty)$ and the energy of \tilde{g} in $[R_< - \varepsilon^n, R_<]$ is exponentially small as well. The upper bound trivially follows.

The lower bound is mainly a consequence of a classical result of energy decoupling, which has been already used in different contexts, e.g., in [93]. The starting point is however a reduction to the annulus \mathcal{A} of the GP energy functional : Exploiting the exponential smallness (3.2.11) of Ψ^{GP} , it is very easy to show that

$$E^{\text{GP}} = \mathcal{E}^{\text{GP}}[\Psi^{\text{GP}}] \geq \mathcal{E}_{\mathcal{A}}^{\text{GP}}[\Psi^{\text{GP}}] - O(\varepsilon^\infty), \quad (3.3.7)$$

where we have used the symbol $\mathcal{E}_{\mathcal{A}}^{\text{GP}}$ to denote the restriction of the integration to \mathcal{A} . Note that in the above inequality one can avoid the estimate of the gradient of Ψ^{GP} by simply rewriting the functional as in (3.1.2) and neglecting all the positive terms given by the integration over $\mathcal{B} \setminus \mathcal{A}$; the only negative term (centrifugal energy) can then be estimated by (3.2.11).

Thanks to the exponential smallness, one also has that the L^2 mass of Ψ^{GP} outside \mathcal{A} is very small, i.e.,

$$\|\Psi^{\text{GP}}\|_{L^2(\mathcal{A})} \geq 1 - O(\varepsilon^\infty). \quad (3.3.8)$$

Now inside \mathcal{A} the decomposition (3.3.5) is well defined and therefore one can calculate

$$\begin{aligned} \mathcal{E}_{\mathcal{A}}^{\text{GP}}[\Psi^{\text{GP}}] = \mathcal{E}_{\mathcal{A},\omega}[u_\omega] + \int_{\mathcal{A}} d\vec{r} \left\{ |u_\omega|^2 |\nabla g_{\mathcal{A},\omega}|^2 + g_{\mathcal{A},\omega} \nabla g_{\mathcal{A},\omega} \cdot \nabla |u_\omega|^2 + ([\Omega] - \omega)^2 r^{-2} g_{\mathcal{A},\omega}^2 |u_\omega|^2 - \right. \\ \left. 2\Omega([\Omega] - \omega) g_{\mathcal{A},\omega}^2 |u_\omega|^2 + \varepsilon^{-2} g_{\mathcal{A},\omega}^4 (2|u_\omega|^2 - 1) \right\}. \end{aligned}$$

A simple integration by parts then yields

$$\int_{\mathcal{A}} d\vec{r} g_{\mathcal{A},\omega} \nabla g_{\mathcal{A},\omega} \cdot \nabla |u_\omega|^2 = - \int_{\mathcal{A}} d\vec{r} \left\{ |u_\omega|^2 |\nabla g_{\mathcal{A},\omega}|^2 + g_{\mathcal{A},\omega} |u_\omega|^2 \Delta g_{\mathcal{A},\omega} \right\},$$

since the boundary terms vanish because of the Neumann conditions satisfied by $g_{\mathcal{A},\omega}$ at the boundaries $\partial\mathcal{B}_{R_<}$ and $\partial\mathcal{B}$. Then one can replace in the above expression $\Delta g_{\mathcal{A},\omega}$ by means of the variational equation (3.2.22) and the result is

$$\begin{aligned} \mathcal{E}_{\mathcal{A}}^{\text{GP}}[\Psi^{\text{GP}}] = \mathcal{E}_{\mathcal{A},\omega}[u_\omega] + \hat{\mu}_{\mathcal{A},\omega}^{\text{GP}} \int_{\mathcal{A}} d\vec{r} g_{\mathcal{A},\omega}^2 |u_\omega|^2 - \varepsilon^{-2} \int_{\mathcal{A}} d\vec{r} g_{\mathcal{A},\omega}^4 = \\ \mathcal{E}_{\mathcal{A},\omega}[u_\omega] + \hat{\mu}_{\mathcal{A},\omega}^{\text{GP}} \|\Psi^{\text{GP}}\|_{L^2(\mathcal{A})}^2 - \varepsilon^{-2} \|g_{\mathcal{A},\omega}\|_{L^4(\mathcal{A})}^4 \geq \mathcal{E}_{\mathcal{A},\omega}[u_\omega] + \hat{E}_{\mathcal{A},\omega}^{\text{GP}} - O(\varepsilon^\infty), \quad (3.3.9) \end{aligned}$$

by (3.3.8) and the definition of the chemical potential $\hat{\mu}_{\mathcal{A},\omega}^{\text{GP}}$ (see Proposition 3.2.3). \square

3.3.2 Optimal Phases and Densities

The idea behind the decomposition (3.3.5) is that, after the extraction from Ψ^{GP} of a density g_ω and a giant vortex phase, i.e., the phase factor $\exp\{i([\Omega] - \omega)\vartheta\}$, what is left is a function u_ω , which contains all the remaining vorticity in Ψ^{GP} . Therefore in the giant vortex regime, one would like to prove that $|u_\omega| \sim 1$ inside a suitable annulus at the boundary of the trap, and the key tool to prove such a behavior is a detailed analysis of the reduced energy $\mathcal{E}_{\mathcal{A},\omega}$.

However, in order to prove such a result, both the phase and the associated density $g_{\mathcal{A},\omega}$ have to be chosen in an appropriate way : The result stated in Proposition 3.3.1 is basically independent of ω , i.e., it applies to any reasonable giant vortex phase ω . Nevertheless the leading order term in the GP energy asymptotics is given in (3.3.4) by $\hat{E}_{\mathcal{A},\omega}^{\text{GP}}$ which depends in crucial way on ω and it is clear that, in order to extract some delicate information about u_ω like the absence of zeros, the estimate (upper bound) of the reduced energy $\mathcal{E}_{\mathcal{A},\omega}[u_\omega]$ through (3.3.4) has to be very precise. This leads to the definition of a giant vortex *optimal phase* (associated with the annulus \mathcal{A}), which is nothing but the minimizer of the energy $\hat{E}_{\mathcal{A},\omega}^{\text{GP}}$ with respect to ω .

In the next Proposition we show that there exists at least one ω_0 minimizing $\hat{E}_{\mathcal{A},\omega}^{\text{GP}}$ as well as some properties which are going to be crucial in the rest of the proof.

Proposition 3.3.2 (Properties of the optimal phase ω_0 and density $g_{\mathcal{A},\omega_0}$).

For every $\varepsilon > 0$ there exists an $\omega_0 \in \mathbb{Z}$ minimizing $\hat{E}_{\mathcal{A},\omega}^{\text{GP}}$. It satisfies

$$\omega_0 = \frac{2}{3\sqrt{\pi\varepsilon}} (1 + O(|\log \varepsilon|^{-1})). \quad (3.3.10)$$

Moreover the minimizer $g_{\mathcal{A},\omega_0}$ of $\hat{\mathcal{E}}_{\mathcal{A},\omega_0}^{\text{GP}}$ satisfies the bound

$$\int_{\mathcal{A}} d\vec{r} g_{\mathcal{A},\omega_0}^2 \left(\Omega - \frac{[\Omega] - \omega_0}{r^2} \right) = O(1). \quad (3.3.11)$$

Proof. The existence of a minimizing $\omega_0 \in \mathbb{Z}$ can be easily proven by noticing that $\lim_{\omega \rightarrow \pm\infty} \hat{E}_{\mathcal{A},\omega}^{\text{GP}} = +\infty$ since by (3.2.19)

$$\hat{E}_{\mathcal{A},\omega}^{\text{GP}} \geq ([\Omega] - \omega)^2 - 2\Omega([\Omega] - \omega),$$

which implies that, for given $\varepsilon > 0$, only finitely many $\omega \in \mathbb{Z}$ can minimize the energy.

The main ingredient for the proof of (3.3.10) is the energy bound (3.2.21), which implies, for any $\omega \in \mathbb{Z}$ such that $|\omega| \leq C\varepsilon^{-1}$,

$$\hat{E}_{\omega_0}^{\text{TF}} \leq \hat{E}_{\mathcal{A},\omega_0}^{\text{GP}} \leq \hat{E}_{\mathcal{A},\omega}^{\text{GP}} \leq \hat{E}_{\omega}^{\text{TF}} + O(\varepsilon^{-2} |\log \varepsilon|^{-2}), \quad (3.3.12)$$

by definition of ω_0 . Choosing now $\omega = [2/(3\sqrt{\pi\varepsilon})]$ in the r.h.s. of the above expression and using (3.6.10) for $\hat{E}_{\omega}^{\text{TF}}$, we get the inequality

$$\begin{aligned} E^{\text{TF}} + \left[\omega_0 - \frac{2}{3\sqrt{\pi\varepsilon}} \right]^2 + \frac{2}{9\pi\varepsilon^2} - 2\Omega([\Omega] - \Omega) - C\varepsilon^{-2} |\log \varepsilon|^{-2} &\leq \hat{E}_{\omega_0}^{\text{TF}} \\ &\leq E^{\text{TF}} + \frac{2}{9\pi\varepsilon^2} - 2\Omega([\Omega] - \Omega) + C\varepsilon^{-2} |\log \varepsilon|^{-2} \end{aligned} \quad (3.3.13)$$

which yields the result.

We now prove (3.3.11). A simple estimate yields

$$\begin{aligned} \hat{\mathcal{E}}_{\mathcal{A},\omega_0 \pm 1}^{\text{GP}}[g_{\mathcal{A},\omega_0}] &= \hat{\mathcal{E}}_{\mathcal{A},\omega_0}^{\text{GP}}[g_{\mathcal{A},\omega_0}] \pm 2 \int_{\mathcal{A}} d\vec{r} [\Omega - ([\Omega] - \omega_0) r^{-2}] g_{\mathcal{A},\omega_0}^2 + \int_{\mathcal{A}} d\vec{r} r^{-2} g_{\mathcal{A},\omega_0}^2 \leq \\ &\hat{E}_{\mathcal{A},\omega_0}^{\text{GP}} \pm 2 \int_{\mathcal{A}} d\vec{r} [\Omega - ([\Omega] - \omega_0) r^{-2}] g_{\mathcal{A},\omega_0}^2 + R_{<}^{-2}. \end{aligned} \quad (3.3.14)$$

Now suppose that

$$\left| \int_{\mathcal{A}} d\vec{r} [\Omega - ([\Omega] - \omega_0) r^{-2}] g_{\mathcal{A}, \omega_0}^2 \right| > \frac{2}{3},$$

then, since $R_{<} = 1 - o(1)$, (3.3.14) would imply that there exists some $\bar{\omega} \in \mathbb{Z}$ equal to $\omega_0 \pm 1$ such that

$$\hat{E}_{\mathcal{A}, \bar{\omega}}^{\text{GP}} \leq \hat{\mathcal{E}}_{\mathcal{A}, \bar{\omega}}^{\text{GP}} [g_{\mathcal{A}, \omega_0}] < \hat{E}_{\mathcal{A}, \omega_0}^{\text{GP}},$$

which contradicts the fact that ω_0 minimizes $\hat{E}_{\mathcal{A}, \omega}^{\text{GP}}$. Note that the above argument implies that the constant on the r.h.s. of (3.3.11) is actually smaller than 1, i.e.,

$$\left| \int_{\mathcal{A}} d\vec{r} [\Omega - ([\Omega] - \omega_0) r^{-2}] g_{\mathcal{A}, \omega_0}^2 \right| < 1. \quad (3.3.15)$$

□

As anticipated above and in the Introduction, there is another giant vortex phase that naturally emerges in the study of the above functionals, i.e., the one associated with the energy in the whole ball \mathcal{B} : More precisely it is defined as the minimizer of

$$\hat{E}_{\omega}^{\text{GP}} := \inf_{\|f\|_2=1} \hat{\mathcal{E}}_{\omega}^{\text{GP}} [f]$$

with respect to $\omega \in \mathbb{Z}$ and denoted by ω_{opt} :

Proposition 3.3.3 (Optimal phase ω_{opt}).

For every $\varepsilon > 0$ there exists an $\omega_{\text{opt}} \in \mathbb{Z}$ minimizing $\hat{E}_{\omega}^{\text{GP}}$. It satisfies

$$\omega_{\text{opt}} = \frac{2}{3\sqrt{\pi\varepsilon}} (1 + O(|\log \varepsilon|^{-1})). \quad (3.3.16)$$

Proof. The proof is basically identical to the one of (3.3.10): For instance it is sufficient to replace (3.3.12) with the corresponding version in \mathcal{B} (see (3.2.21)). □

3.4 Estimates of the Reduced Energy

In this Section we study the auxiliary problem introduced in Section 3. From now on we shall simplify notations by dropping some subscripts:

$$g_{\mathcal{A}, \omega_0} := g,$$

also

$$\vec{B}(r) := \vec{B}_{\omega_0}(r) = \left(\Omega r - \frac{[\Omega] - \omega_0}{r} \right) \vec{e}_{\vartheta}, \quad (3.4.1)$$

and

$$\mathcal{E}[v] := \int_{\mathcal{A}} d\vec{r} \left\{ g^2 |\nabla v|^2 - 2g^2 \vec{B} \cdot (iv, \nabla v) + \frac{g^4}{\varepsilon^2} (1 - |v|^2)^2 \right\}.$$

The following energy is crucial in our analysis:

$$\mathcal{F}[v] := \int_{\mathcal{A}} d\vec{r} \left\{ g^2 |\nabla v|^2 + \frac{g^4}{\varepsilon^2} (1 - |v|^2)^2 \right\}. \quad (3.4.2)$$

We recall that $u := u_{\omega_0}$ is defined by

$$\Psi^{\text{GP}}(\vec{r}) =: g(r)u(\vec{r}) \exp \{i([\Omega] - \omega_0)\vartheta\} \quad (3.4.3)$$

and that from Proposition 3.3.1 we have

$$\mathcal{E}[u] \leq O(\varepsilon^\infty). \quad (3.4.4)$$

We also need to define a reduced annulus

$$\tilde{\mathcal{A}} := \{\vec{r} : R_{>} \leq r \leq 1\} \quad (3.4.5)$$

with

$$R_{>} := R_{\text{h}} + \varepsilon |\log \varepsilon|^{-1}. \quad (3.4.6)$$

An important point is that from (3.2.30) we have the lower bound

$$g^2(r) \geq \frac{C}{\varepsilon |\log \varepsilon|^3} \text{ on } \tilde{\mathcal{A}}. \quad (3.4.7)$$

The main result of this Section is

Proposition 3.4.1 (Bounds for the reduced energies).

Let u be defined by (3.4.3). If $\Omega_0 > 2(3\pi)^{-1}$ we have for ε small enough

$$\mathcal{F}[u] \leq C \frac{|\log \varepsilon|^{5/2}}{\varepsilon^{1/2} (\log |\log \varepsilon|)^2} \quad (3.4.8)$$

$$\mathcal{E}[u] \geq -C \frac{|\log \varepsilon|^{3/2}}{\varepsilon^{1/2} \log |\log \varepsilon|}. \quad (3.4.9)$$

Since the proof of Proposition 3.4.1 is rather involved we sketch the main ideas before going into the details.

It is useful to introduce a potential function F defined as follows :

$$F(r) := 2 \int_{R_{<}}^r ds g^2(s) \left(\Omega s - ([\Omega] - \omega_0) \frac{1}{s} \right) = 2 \int_{R_{<}}^r ds g^2(s) \vec{B}(s) \cdot \vec{e}_\vartheta. \quad (3.4.10)$$

We have

$$\nabla^\perp F = 2g^2 \vec{B}, \quad F(R_{<}) = 0, \quad (3.4.11)$$

i.e., F is the ‘‘primitive’’ of $g^2 \vec{B}$ vanishing at $R_{<}$. We refer to Subsection 4.1 for further properties of F . Integrating by parts we have

$$\int_{\mathcal{A}} d\vec{r} \left\{ g^2 |\nabla u|^2 - 2g^2 \vec{B} \cdot (iu, \nabla u) \right\} = \int_{\mathcal{A}} d\vec{r} \left\{ g^2 |\nabla u|^2 + F \text{curl}(iu, \nabla u) \right\} - \int_{\partial \mathcal{B}} d\sigma F(1)(iu, \partial_\tau u) \quad (3.4.12)$$

and thus the energy $\mathcal{E}[u]$ can be rewritten as follows

$$\mathcal{E}[u] = \int_{\mathcal{A}} d\vec{r} g^2 |\nabla u|^2 + \int_{\mathcal{A}} d\vec{r} F \text{curl}(iu, \nabla u) - \int_{\partial \mathcal{B}} d\sigma F(1)(iu, \partial_\tau u) + \int_{\mathcal{A}} d\vec{r} \frac{g^4}{\varepsilon^2} (1 - |u|^2)^2. \quad (3.4.13)$$

It is in this form that the energy is best bounded from below.

The boundary term (third term in (3.4.13)) is estimated in the following way : The property

of ω_0 given in (3.3.11) implies that $F(1) = O(1)$. We combine this fact with an estimate of the circulation of u on the boundary of the unit ball that we provide in Subsection 4.2. Proving this estimate requires to derive a PDE satisfied by u , and use it in much the same way as in the proof of the Pohozaev identity [123].

The two first terms can be estimated in terms of the vorticity of u . Indeed, suppose that $|u| \sim 1$ except in some balls (that we identify with vortices) whose radii are much smaller than the width of \mathcal{A} . Let us denote this balls by $\{\mathcal{B}(\vec{a}_j, t)\}_{j \in J}$ with $J \subset \mathbb{N}$ and $t \ll \varepsilon |\log \varepsilon|$, i.e., much smaller than the width of the annulus. Then by Stokes theorem, if the degree of u around \vec{a}_j is d_j (we systematically neglect remainder terms in this sketch)

$$\int_{\mathcal{A}} d\vec{r} F \operatorname{curl}(iu, \nabla u) \simeq \sum_{j \in J} 2\pi F(a_j) d_j. \quad (3.4.14)$$

Minimizing the sum of the first and the last term with respect to t yields $t \propto \varepsilon^{3/2} |\log \varepsilon|^{1/2}$ (see also [47] for heuristics about the optimal size of the vortex core), which implies an estimate of the form

$$\int_{\mathcal{A}} d\vec{r} g^2 |\nabla u|^2 \gtrsim \sum_{j \in J} 2\pi g^2(a_j) |d_j| \log \left(\frac{\varepsilon |\log \varepsilon|}{t} \right) \gtrsim \sum_{j \in J} \pi g^2(a_j) |d_j| |\log \varepsilon|. \quad (3.4.15)$$

In Subsections 4.3 and 4.4 we give a rigorous version of this heuristic analysis. The main tools were originally introduced in the context of GL theory (we refer to [30, 135] and references therein). The method of growth and merging of vortex balls introduced independently by Sandier [133] and Jerrard [85] provides a lower bound of the form (3.4.15) (see Subsection 4.3). On the other hand, the jacobian estimates of Jerrard and Soner [88] allow to deduce in Subsection 4.4 that

$$\operatorname{curl}(iu, \nabla u) \simeq \sum_{j \in J} 2\pi d_j \delta(\vec{r} - \vec{a}_j), \quad (3.4.16)$$

i.e., the vorticity measure of u is close to a sum of Dirac masses. This implies an estimate of the form (3.4.14).

Having performed this analysis, the role of the critical velocity becomes clear : We have essentially (note that the boundary term is negligible in a first approach)

$$\mathcal{E}[u] \gtrsim \sum_{j \in J} 2\pi |d_j| \left(\frac{1}{2} g^2(a_j) |\log \varepsilon| + F(a_j) \right). \quad (3.4.17)$$

If $\Omega_0 > 2(3\pi)^{-1}$ the sum in the parenthesis is positive for any \vec{a}_j in the bulk (see the Appendix), which means that vortices become energetically unfavorable.

So far we have assumed that the zeros of u were isolated in small ‘vortex’ balls. One can show that this is indeed the case by exploiting upper bounds to $\mathcal{F}[u]$ (this energy controls, via the co-area formula the size of the set where $|u|$ is no close to 1). We derive a first bound from our a priori estimate on $\mathcal{E}[u]$ (3.4.4) in Subsection 4.1. We emphasize however that this first bound is not strong enough to construct vortex balls in the whole annulus \mathcal{A} . To get around this point we split in Subsection 4.3 the annulus into cells, and distinguish between two type of cells. In ‘good’ cells we have the proper control and perform locally the vortex balls construction. On the other hand we are able to show that there are relatively few ‘bad’ cells where the construction is not possible. Thus the analysis in the good cells

allows to get lower bounds for $\mathcal{E}[u]$. These bounds can in turn be used to improve our control on $\mathcal{F}[u]$ and reduce the number of bad cells. The analysis can then be repeated, but now on a larger set. Finally we see that there are no bad cells. Such an induction process is ‘hidden’ at the end of the proof of Proposition 3.4.1 in Subsection 3.4.5 (see in particular the discussion after (3.4.101)).

3.4.1 Preliminaries

We first give some elementary estimates on B and F that we need in our analysis :

Lemma 3.4.1 (Useful properties of \vec{B} and F).

Let \vec{B} and F be defined in (3.4.1) and (3.4.10) respectively. We have

$$\|B\|_{L^\infty(\mathcal{A})} \leq C\varepsilon^{-1}, \quad (3.4.18)$$

$$\|F\|_{L^\infty(\mathcal{A})} \leq C\varepsilon^{-1}, \quad (3.4.19)$$

$$\|\nabla F\|_{L^\infty(\mathcal{A})} \leq C\varepsilon^{-2}|\log \varepsilon|^{-1}, \quad (3.4.20)$$

$$|F(1)| = 2 \left| \int_{R_<}^1 ds g^2(s) \vec{B}(s) \cdot \vec{e}_\vartheta \right| \leq C. \quad (3.4.21)$$

Moreover there is a constant C such that

$$|F(r)| \leq C \min \left(\frac{|r - R_<|}{\varepsilon} g^2(r), 1 + \frac{C}{\varepsilon^2 |\log \varepsilon|} |r - 1| \right) \quad (3.4.22)$$

for any $\vec{r} \in \mathcal{A}$.

Proof. We have

$$\vec{B}(r) = ((\Omega - [\Omega])r + [\Omega](r - r^{-1}) + \omega_0 r^{-1}) \vec{e}_\vartheta$$

so (3.4.18) is a consequence of $|\omega_0| \leq C\varepsilon^{-1}$ (see (3.3.10)) and the fact that $|\mathcal{A}| \propto \varepsilon |\log \varepsilon|$ (see (3.3.1) and (3.3.2)) and $\Omega \propto \varepsilon^{-2} |\log \varepsilon|^{-1}$.

We obtain (3.4.20) from (3.2.25), (3.4.11) and (3.4.18). On the other hand $F(R_<) = 0$, which implies that (3.4.19) follows from (3.4.20) and $|\mathcal{A}| \propto \varepsilon |\log \varepsilon|$. Moreover (3.4.21) is exactly the same as (3.3.11) in Proposition 3.3.2.

To prove (3.4.22) we prove that $|F(r)|$ is smaller than both terms on the right-hand side. From equation (3.4.10) we have

$$|F(r)| \leq 2 \|B\|_{L^\infty(\mathcal{A})} \int_{R_<}^r ds g^2(s) \leq C\varepsilon^{-1} \int_{R_<}^r ds g^2(s)$$

but g^2 is increasing so

$$\int_{R_<}^r ds g^2(s) \leq g^2(r) |r - R_<|.$$

We deduce that $|F(r)|$ is smaller than the first term on the right-hand side of (3.4.22). On the other hand, using (3.4.20) and (3.4.21) we have immediately

$$|F(r)| \leq C + \frac{C}{\varepsilon^2 |\log \varepsilon|} |r - 1|$$

for some finite constant C . Thus (3.4.22) is proven. \square

We now prove the energy bounds that are the starting point for the vortex balls construction in Subsection 3.4.3.

Lemma 3.4.2 (Preliminary energy bounds).

We have, for $\varepsilon > 0$ small enough,

$$\mathcal{F}[u] \leq \frac{C}{\varepsilon^2}, \quad (3.4.23)$$

$$\mathcal{E}[u] \geq -\frac{C}{\varepsilon^2}. \quad (3.4.24)$$

Proof. We have, using (3.4.18) together with the normalization of Ψ^{GP} ,

$$\left| 2 \int_{\mathcal{A}} d\vec{r} g^2 \vec{B} \cdot (iu, \nabla u) \right| \leq \frac{1}{2} \int_{\mathcal{A}} d\vec{r} g^2 |\nabla u|^2 + \frac{1}{2} \int_{\mathcal{A}} d\vec{r} g^2 B^2 |u|^2 \leq \frac{1}{2} \int_{\mathcal{A}} d\vec{r} g^2 |\nabla u|^2 + \frac{C}{\varepsilon^2}. \quad (3.4.25)$$

Equation (3.4.24) immediately follows. We obtain (3.4.23) from (3.4.4) :

$$\begin{aligned} \mathcal{F}[u] = \mathcal{E}[u] + 2 \int_{\mathcal{A}} d\vec{r} g^2 \vec{B} \cdot (iu, \nabla u) &\leq O(\varepsilon^\infty) + 2 \left| \int_{\mathcal{A}} d\vec{r} g^2 \vec{B} \cdot (iu, \nabla u) \right| \leq \\ &\frac{1}{2} \int_{\mathcal{A}} d\vec{r} g^2 |\nabla u|^2 + \frac{C}{\varepsilon^2} \leq \frac{1}{2} \mathcal{F}[u] + \frac{C}{\varepsilon^2}. \end{aligned} \quad (3.4.26)$$

Subtracting $\frac{1}{2}\mathcal{F}[u]$ from both sides of the last inequality we get (3.4.23). \square

3.4.2 Equation for u and Boundary Estimate

We first derive from the equations for Ψ^{GP} and g an equation satisfied by u :

Lemma 3.4.3 (Equation for u).

Let u be defined by (3.4.3). We have on \mathcal{A}

$$-\nabla(g^2 \nabla u) - 2ig^2 \vec{B} \cdot \nabla u + 2\frac{g^4}{\varepsilon^2} (|u|^2 - 1) u = \lambda g^2 u \quad (3.4.27)$$

where $\lambda \in \mathbb{R}$ satisfies the estimate

$$|\lambda| \leq C \left(|\mathcal{E}[u]| + \frac{1}{\varepsilon^{3/2} |\log \varepsilon|^{1/2}} \mathcal{F}[u]^{1/2} \right). \quad (3.4.28)$$

Moreover u satisfies the boundary condition

$$\partial_r u = 0 \text{ on } \partial\mathcal{B}. \quad (3.4.29)$$

Proof. In this proof we use the short hand notation

$$\vec{\Omega} = \hat{\Omega} \vec{e}_r := ([\Omega] - \omega_0) \vec{e}_r. \quad (3.4.30)$$

The equation for u is a consequence of the variational equation (3.1.5) satisfied by Ψ^{GP} ,

$$-\Delta \Psi^{\text{GP}} - 2\vec{\Omega} \cdot \vec{L} \Psi^{\text{GP}} + 2\varepsilon^{-2} |\Psi^{\text{GP}}|^2 \Psi^{\text{GP}} = \mu^{\text{GP}} \Psi^{\text{GP}}, \quad (3.4.31)$$

and that solved by g (see (3.2.22)),

$$-\Delta g + \left(\Omega r - \frac{\hat{\Omega}}{r} \right)^2 g - \Omega^2 r^2 + 2\varepsilon^{-2} g^3 = \hat{\mu}^{\text{GP}} g. \quad (3.4.32)$$

Direct computations starting from (3.4.3) show that

$$-\Delta \Psi^{\text{GP}} = \left(-u \Delta g - g \Delta u - 2 \nabla u \cdot \nabla g + \frac{\hat{\Omega}^2}{r^2} g u + \frac{2}{r^2} g \vec{\Omega} \cdot \vec{L} u \right) e^{i\hat{\Omega}\vartheta} \quad (3.4.33)$$

and

$$-2\vec{\Omega} \cdot \vec{L} \Psi^{\text{GP}} = \left(-2g\vec{\Omega} \cdot \vec{L} u - 2\Omega \hat{\Omega} g u \right) e^{i\hat{\Omega}\vartheta}. \quad (3.4.34)$$

Plugging the equation for g in (3.4.33) we obtain

$$-\Delta \Psi^{\text{GP}} = \left(-g \Delta u - 2 \nabla u \cdot \nabla g + \hat{\mu}^{\text{GP}} g u + 2\Omega \hat{\Omega} g u - 2\varepsilon^{-2} g^3 u + 2g\vec{\Omega} \cdot \vec{L} u \right) e^{i\hat{\Omega}\vartheta}. \quad (3.4.35)$$

Next, combining (3.4.31), (3.4.34) and (3.4.35), we have

$$-g \Delta u - 2 \nabla u \cdot \nabla g + \frac{2}{\varepsilon^2} g^3 (|u|^2 - 1) u + \frac{2}{r^2} g \vec{\Omega} \cdot \vec{L} u - 2g\vec{\Omega} \cdot \vec{L} u = \lambda g u \quad (3.4.36)$$

where

$$\lambda = \mu^{\text{GP}} - \hat{\mu}^{\text{GP}}. \quad (3.4.37)$$

There only remains to multiply (3.4.36) by g and reorganize the terms to obtain (3.4.27). The Neumann condition (3.4.29) is a straightforward consequence of the corresponding boundary conditions for g and Ψ^{GP} .

We now turn to the proof of (3.4.28). Multiplying (3.4.27) by u^* and integrating over \mathcal{A} , we obtain, recalling (3.3.8),

$$|\lambda| (1 - O(\varepsilon^\infty)) \leq |\lambda| \int_{\mathcal{A}} d\vec{r} |\Psi^{\text{GP}}|^2 = |\lambda| \int_{\mathcal{A}} d\vec{r} g^2 |u|^2 = \mathcal{E}[u] + \int_{\mathcal{A}} d\vec{r} \frac{g^4}{\varepsilon^2} (|u|^4 - 1). \quad (3.4.38)$$

Using Cauchy-Schwarz inequality

$$\left| \int_{\mathcal{A}} \frac{g^4}{\varepsilon^2} (|u|^4 - 1) \right| \leq \left(\int_{\mathcal{A}} \frac{g^4}{\varepsilon^2} (|u|^2 - 1)^2 \right)^{1/2} \left(\int_{\mathcal{A}} \frac{1}{\varepsilon^2} (g^2 |u|^2 + g^2)^2 \right)^{1/2}$$

so (3.4.28) follows using the upper bounds (3.2.2) and (3.2.25) on $g^2 |u|^2 = |\Psi^{\text{GP}}|^2$ and g^2 respectively together with $|\mathcal{A}| \propto \varepsilon |\log \varepsilon|$. \square

We remark that the equation satisfied by u is exactly of the form that we would have obtained if u had been a minimizer of \mathcal{E} under a mass constraint for $g u$.

We are now able to estimate the circulation of u at the boundary of the unit ball :

Lemma 3.4.4 (Boundary estimate).

We have, for $\varepsilon > 0$ small enough,

$$\int_{\partial \mathcal{B}} d\sigma g^2 |\partial_\tau u|^2 + \int_{\partial \mathcal{B}} d\sigma \frac{g^4}{\varepsilon^2} (1 - |u|^2)^2 \leq \frac{C}{\varepsilon^{3/2} |\log \varepsilon|^{1/2}} \mathcal{F}[u] + \frac{C}{\varepsilon} |\mathcal{E}[u]|. \quad (3.4.39)$$

Proof. The main idea is a Pohozaev-like trick, like in [30, Theorem 3.2]. We multiply both sides of (3.4.27) by $(r - R_{>}) \vec{e}_r \cdot \nabla u^*$, then integrate over $\tilde{\mathcal{A}} \subset \mathcal{A}$ (see (3.4.5) for the definition of $\tilde{\mathcal{A}}$) and take the real part. Recall that $1 - R_{>} \propto \varepsilon |\log \varepsilon|$.

The first term is given by

$$\begin{aligned} & -\frac{1}{2} \int_{\tilde{\mathcal{A}}} d\vec{r} \left\{ (r - R_{>}) \vec{e}_r \cdot \nabla u^* \cdot \nabla (g^2 \nabla u) + (r - R_{>}) \vec{e}_r \cdot \nabla u \cdot \nabla (g^2 \nabla u^*) \right\} = \\ & \int_{\tilde{\mathcal{A}}} d\vec{r} \left\{ \left(1 - \frac{R_{>}}{2r}\right) g^2 |\nabla u|^2 + \frac{1}{2} g^2 (r - R_{>}) \vec{e}_r \cdot \nabla |\nabla u|^2 \right\} = \\ & \frac{1}{2} \int_{\partial \mathcal{B}} d\sigma g^2 (1 - R_{>}) |\partial_\tau u|^2 - \int_{\tilde{\mathcal{A}}} d\vec{r} g (r - R_{>}) \vec{e}_r \cdot \nabla g |\nabla u|^2, \end{aligned} \quad (3.4.40)$$

where we have integrated by parts twice, remembering the Neumann boundary conditions for g and u on $\partial \mathcal{B}$.

The third terms yields

$$\begin{aligned} & \frac{1}{\varepsilon^2} \int_{\tilde{\mathcal{A}}} d\vec{r} \left\{ (r - R_{>}) \vec{e}_r \cdot \nabla u^* g^4 (|u|^2 - 1) u + (r - R_{>}) \vec{e}_r \cdot \nabla u g^4 (|u|^2 - 1) u^* \right\} = \\ & \frac{1}{2\varepsilon^2} \int_{\tilde{\mathcal{A}}} d\vec{r} g^4 (r - R_{>}) \vec{e}_r \cdot \nabla (1 - |u|^2)^2 = \frac{1}{2\varepsilon^2} \int_{\partial \mathcal{B}} d\sigma (1 - R_{>}) g^4 (1 - |u|^2)^2 - \\ & \frac{1}{\varepsilon^2} \int_{\tilde{\mathcal{A}}} d\vec{r} \left\{ \left(1 - \frac{R_{>}}{2r}\right) g^4 (1 - |u|^2)^2 + 2g^3 (r - R_{>}) \vec{e}_r \cdot \nabla g (1 - |u|^2)^2 \right\}. \end{aligned} \quad (3.4.41)$$

Altogether we thus obtain

$$\begin{aligned} & \frac{1}{2} (1 - R_{>}) \int_{\partial \mathcal{B}} d\sigma \left\{ g^2 |\partial_\tau u|^2 + \frac{g^4}{\varepsilon^2} (1 - |u|^2)^2 \right\} = \\ & \int_{\tilde{\mathcal{A}}} d\vec{r} \left\{ g (r - R_{>}) \vec{e}_r \cdot \nabla g |\nabla u|^2 + 2ig^2 (r - R_{>}) \vec{e}_r \cdot \nabla u^* \vec{B} \cdot \nabla u \right\} + \\ & \frac{1}{\varepsilon^2} \int_{\tilde{\mathcal{A}}} d\vec{r} \left\{ \left(1 - \frac{R_{>}}{2r}\right) g^4 (1 - |u|^2)^2 + 2g^3 (r - R_{>}) \vec{e}_r \cdot \nabla g (1 - |u|^2)^2 \right\} + \\ & \frac{\lambda}{2} \int_{\tilde{\mathcal{A}}} d\vec{r} g^2 (r - R_{>}) \vec{e}_r \cdot \nabla |u|^2. \end{aligned} \quad (3.4.42)$$

We then estimate the moduli of the terms of the r.h.s.. Obviously

$$\int_{\tilde{\mathcal{A}}} d\vec{r} \frac{g^4}{\varepsilon^2} \left(1 - \frac{R_{>}}{2r}\right) (1 - |u|^2)^2 \leq \mathcal{F}[u]. \quad (3.4.43)$$

Using (3.4.6) and (3.4.18) we obtain

$$\left| \int_{\tilde{\mathcal{A}}} d\vec{r} ig^2 (r - R_{>}) \vec{e}_r \cdot \nabla u^* \vec{B} \cdot \nabla u \right| \leq C |\log \varepsilon| \int_{\tilde{\mathcal{A}}} d\vec{r} g^2 |\nabla u|^2 \leq C |\log \varepsilon| \mathcal{F}[u], \quad (3.4.44)$$

whereas, using the normalization of Ψ^{GP} , (3.4.23) and (3.4.28),

$$\begin{aligned} & \left| \frac{\lambda}{2} \int_{\tilde{\mathcal{A}}} d\vec{r} g^2 (r - R_{>}) \vec{e}_r \cdot \nabla |u|^2 \right| = \left| \lambda \int_{\tilde{\mathcal{A}}} d\vec{r} g^2 u (r - R_{>}) \vec{e}_r \cdot \nabla u^* \right| \leq \\ & C \varepsilon |\log \varepsilon| \left(|\mathcal{E}[u]| + \frac{1}{\varepsilon^{3/2} |\log \varepsilon|^{1/2}} \mathcal{F}[u]^{1/2} \right) \left(\int_{\tilde{\mathcal{A}}} d\vec{r} g^2 |u|^2 \right)^{1/2} \left(\int_{\tilde{\mathcal{A}}} d\vec{r} g^2 |\nabla u|^2 \right)^{1/2} \leq \\ & C |\log \varepsilon| |\mathcal{E}[u]| + \frac{C |\log \varepsilon|^{1/2}}{\varepsilon^{1/2}} \mathcal{F}[u]. \end{aligned} \quad (3.4.45)$$

Combining (3.2.50) with (3.4.7), we obtain

$$\left| \frac{1}{\varepsilon^2} \int_{\tilde{\mathcal{A}}} d\vec{r} g^3 (r - R_{>}) \vec{e}_r \cdot \nabla g (1 - |u|^2)^2 \right| \leq \frac{C |\log \varepsilon|^{9/4}}{\varepsilon^{1/4}} \int_{\tilde{\mathcal{A}}} d\vec{r} \frac{1}{\varepsilon^2} g^4 (1 - |u|^2)^2 \leq \frac{C |\log \varepsilon|^{9/4}}{\varepsilon^{1/4}} \mathcal{F}[u]. \quad (3.4.46)$$

By similar arguments

$$\left| \int_{\tilde{\mathcal{A}}} d\vec{r} g (r - R_{>}) \vec{e}_r \cdot \nabla g |\nabla u|^2 \right| \leq \frac{C |\log \varepsilon|^{9/4}}{\varepsilon^{1/4}} \mathcal{F}[u]. \quad (3.4.47)$$

Gathering equations (3.4.42) to (3.4.47), we have

$$\frac{1}{2} (1 - R_{>}) \int_{\partial \mathcal{B}} d\sigma \left\{ g^2 |\partial_\tau u|^2 + \frac{g^4}{\varepsilon^2} (1 - |u|^2)^2 \right\} \leq C |\log \varepsilon| |\mathcal{E}[u]| + \frac{C |\log \varepsilon|^{1/2}}{\varepsilon^{1/2}} \mathcal{F}[u]$$

and there only remains to divide by $1 - R_{>} \propto \varepsilon |\log \varepsilon|$ to get the result. \square

3.4.3 Cell Decomposition and Vortex Ball Construction

In this Subsection we aim at constructing vortex balls for u . Namely, we want to construct a collection of balls whose radii are much smaller than the width of \mathcal{A} and that cover the set where $|u|$ is not close to 1. The usual way of performing this task is to exploit bounds on $\mathcal{F}[u]$. Unfortunately, the bound (3.4.23) is not sufficient for our purpose. It only implies that the area of the set where u can possibly vanish is of order $\varepsilon^2 |\log \varepsilon|^2$, whereas the vortex balls method requires to cover it by balls of radii much smaller than the width of \mathcal{A} , which is $O(\varepsilon |\log \varepsilon|)$. However, the estimates of Lemma 3.4.2 are definitely not optimal and can be improved by using a procedure of local vortex balls construction. The idea is the following : If the bound (3.4.23) could be localized (in a sense made clear below), we could construct vortex balls. As this is not the case we split the annulus \mathcal{A} into regions where the bound can be localized and therefore vortex balls can be constructed as usual, and regions where this is not the case.

Definition 3.4.1 (Good and bad cells).

We cover \mathcal{A} with (almost rectangular) cells of side length $C\varepsilon |\log \varepsilon|$, using a corresponding division of the angular variable. We denote by $N \propto \varepsilon^{-1} |\log \varepsilon|^{-1}$ the total number of cells and label the cells as $\mathcal{A}_n, n \in \{1, \dots, N\}$. Let $0 \leq \alpha < \frac{1}{2}$ be a parameter to be fixed later on.

- We say that \mathcal{A}_n is an α -good cell if

$$\int_{\mathcal{A}_n} d\vec{r} \left\{ g^2 |\nabla u|^2 + \frac{g^4}{\varepsilon^2} (1 - |u|^2)^2 \right\} \leq \frac{|\log \varepsilon|}{\varepsilon} \varepsilon^{-\alpha}. \quad (3.4.48)$$

We denote by N_α^G the number of α -good cells and GS_α the (good) set they cover.

- We say that \mathcal{A}_n is an α -bad cell if

$$\int_{\mathcal{A}_n} d\vec{r} \left\{ g^2 |\nabla u|^2 + \frac{g^4}{\varepsilon^2} (1 - |u|^2)^2 \right\} > \frac{|\log \varepsilon|}{\varepsilon} \varepsilon^{-\alpha}. \quad (3.4.49)$$

We denote by N_α^B the number of α -bad cells and BS_α the (bad) set they cover.

Note that the annulus \mathcal{A} has a width $\ell \propto \varepsilon |\log \varepsilon|$ (which implies that $N \propto \varepsilon^{-1} |\log \varepsilon|^{-1}$) so that we are actually dividing it into bad cells where there is much more energy than what would be expected from the localization of the bound (3.4.23) (namely $C\ell\varepsilon^{-2} \propto \varepsilon^{-1} |\log \varepsilon|$) and regions (good cells) of reasonably small energy. A first consequence of this is, neglecting the good cells and using (3.4.23),

$$N_\alpha^B \leq \frac{\varepsilon}{|\log \varepsilon|} \varepsilon^\alpha \mathcal{F}[u] \leq \frac{C}{\varepsilon |\log \varepsilon|} \varepsilon^\alpha \ll N, \quad (3.4.50)$$

i.e., there are very few α -bad cells. A consequence of the refined bound (3.4.8) that we are aiming at is that there are actually no α -bad cells at all.

We now construct the vortex balls in the good set. The proof is merely sketched because it is an adaptation of well-established methods (see [135] and references therein). Note that the construction is possible only in the subdomain $\tilde{\mathcal{A}}$ where the density is large enough.

Proposition 3.4.2 (Vortex ball construction in the good set).

Let $0 \leq \alpha < \frac{1}{2}$. There is a certain ε_0 so that, for $\varepsilon \leq \varepsilon_0$ there exists a finite collection $\{\mathcal{B}_i\}_{i \in I} := \{\mathcal{B}(\vec{a}_i, \varrho_i)\}_{i \in I}$ of disjoint balls with centers \vec{a}_i and radii ϱ_i such that

1. $\{\vec{r} \in GS_\alpha \cap \tilde{\mathcal{A}} : ||u| - 1| > |\log \varepsilon|^{-1}\} \subset \cup_{i=1}^I \mathcal{B}_i$,
2. for any α -good cell \mathcal{A}_n , $\sum_{i, \mathcal{B}_i \cap \mathcal{A}_n \neq \emptyset} \varrho_i = \varepsilon |\log \varepsilon|^{-5}$.

Setting $d_i := \deg\{u, \partial \mathcal{B}_i\}$, if $\mathcal{B}_i \subset \tilde{\mathcal{A}} \cap GS_\alpha$, and $d_i = 0$ otherwise, we have the lower bounds

$$\int_{\mathcal{B}_i} d\vec{r} g^2 |\nabla u|^2 \geq 2\pi \left(\frac{1}{2} - \alpha\right) |d_i| g^2(a_i) |\log \varepsilon| \left(1 - C \frac{\log |\log \varepsilon|}{|\log \varepsilon|}\right). \quad (3.4.51)$$

Proof. We begin by covering the subset of $GS_\alpha \cap \tilde{\mathcal{A}}$ where u can possibly vanish by balls of much smaller radii than those announced in the Proposition. Let \mathcal{A}_n be an α -good cell. We have, using (3.4.7) and (3.4.48),

$$\begin{aligned} \int_{\mathcal{A}_n \cap \tilde{\mathcal{A}}} d\vec{r} \left\{ |\nabla u|^2 + \frac{1}{\varepsilon^3 |\log \varepsilon|^3} (1 - |u|^2)^2 \right\} &\leq \\ C\varepsilon |\log \varepsilon|^3 \int_{\mathcal{A}_n} d\vec{r} \left\{ g^2 |\nabla u|^2 + \frac{g^4}{\varepsilon^2} (1 - |u|^2)^2 \right\} &\leq C |\log \varepsilon|^4 \varepsilon^{-\alpha}. \end{aligned}$$

Then, using the Cauchy-Schwarz inequality,

$$\int_{\mathcal{A}_n \cap \tilde{\mathcal{A}}} d\vec{r} \frac{|\nabla |u|| |1 - |u|^2|}{\varepsilon^{3/2} |\log \varepsilon|^{3/2}} \leq C |\log \varepsilon|^4 \varepsilon^{-\alpha}$$

and the coarea formula implies

$$\int_{t \in \mathbb{R}^+} dt \frac{|1 - t^2|}{\varepsilon^{3/2} |\log \varepsilon|^{3/2}} \mathcal{H}^1 \left(\left\{ \vec{r} \in \mathcal{A}_n \cap \tilde{\mathcal{A}} : |u(\vec{r})| = t \right\} \right) \leq C |\log \varepsilon|^4 \varepsilon^{-\alpha},$$

where \mathcal{H}^1 stands for the 1-dimensional Hausdorff measure. We argue as in [135, Propositions 4.4 and 4.8] to deduce from this that the set $\{\vec{r} \in \mathcal{A}_n \cap \tilde{\mathcal{A}} : ||u| - 1| > |\log \varepsilon|^{-1}\}$ can be covered by a finite number of disjoint balls $\{\mathcal{B}(\vec{a}_j, \tilde{\varrho}_j)\}_{j \in J}$ with $\sum_{j \in J} \tilde{\varrho}_j \leq$

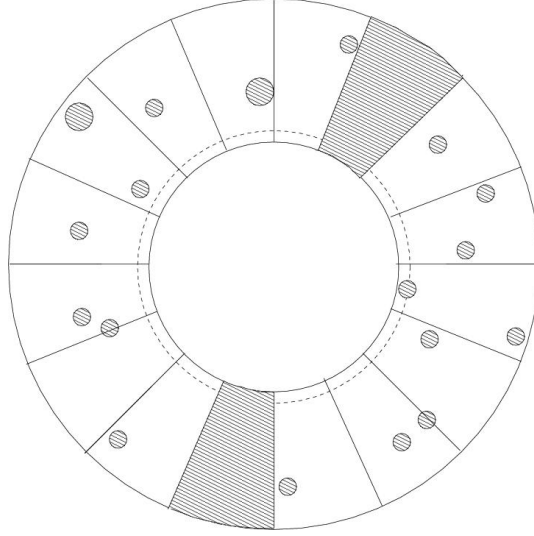


FIGURE 3.1 – Initial collection of vortex balls

$C\varepsilon^{3/2-\alpha}|\log \varepsilon|^{13/2}$. Doing likewise on each α -good cell, we get a collection $\{\mathcal{B}(\vec{a}_i, \tilde{\varrho}_i)\}_{i \in I}$, covering $\{\vec{r} \in GS_\alpha \cap \tilde{\mathcal{A}} : ||u| - 1| > |\log \varepsilon|^{-1}\}$ and satisfying

$$\sum_{i \in I, \mathcal{B}(\vec{a}_i, \tilde{\varrho}_i) \cap \mathcal{A}_n \neq \emptyset} \tilde{\varrho}_i \leq C\varepsilon^{3/2-\alpha}|\log \varepsilon|^{13/2}, \text{ for any } \alpha\text{-good cell } \mathcal{A}_n. \quad (3.4.52)$$

The balls in this collection may overlap because we have constructed them locally in the cells. However, by merging the balls that intersect as in [135, Lemma 4.1], we can construct a finite collection of disjoint balls (still denoted by $\mathcal{B}(\vec{a}_i, \tilde{\varrho}_i)$) satisfying the same bounds on their radii and still covering $\{\vec{r} \in GS_\alpha \cap \tilde{\mathcal{A}} : ||u| - 1| > |\log \varepsilon|^{-1}\}$.

The collection $\mathcal{B}(\vec{a}_i, \tilde{\varrho}_i)$ is represented in Figure 3.1. The gray regions are those where u could vanish, i.e., bad cells and vortex balls. The dashed circle is the inner boundary of $\tilde{\mathcal{A}}$.

We now let the balls in our initial collection grow and merge using the method described in [135, Section 4.2], adding lower bounds on the conformal annuli constructed. In brief, the idea is to introduce a process parametrized by a variable t (interpreted as ‘time’) so that, as t increases, the balls will grow all with the same dilatation factor. There are two phases in the process : When balls grow independently their radii get multiplied by some factor, say $m(t)$, and we add lower bounds to the kinetic energy of u on the annuli between the initial and grown balls. When two or more balls hit, we merge them into larger balls (see [135, Lemma 4.1]), and continue with the growing phase. Figure 3.2 shows this process. The lower bounds on the annuli are obtained by integrating the kinetic energy over circles centered at the balls centers \vec{a}_i during the growth phases :

$$\int_{\mathcal{B}(\vec{a}_i, m(t)\varrho_i) \setminus \mathcal{B}(\vec{a}_i, \varrho_i)} d\vec{r} |\nabla u|^2 \geq 2\pi d_i^2 \log(m(t)) (1 - |\log \varepsilon|^{-1}), \quad (3.4.53)$$

where $d_i := \deg\{u, \partial\mathcal{B}(\vec{a}_i, \varrho_i)\}$.

The main point in this computation is that $||u| - 1| < |\log \varepsilon|^{-1}$ (in particular u can not vanish) between the circles $\partial\mathcal{B}(\vec{a}_i, \varrho_i)$ and $\partial\mathcal{B}(\vec{a}_i, m(t)\varrho_i)$. Thus the degree of u is d_i on any circle $\partial\mathcal{B}(\vec{a}_i, \varrho)$ with $\varrho_i < \varrho < m(t)\varrho_i$.

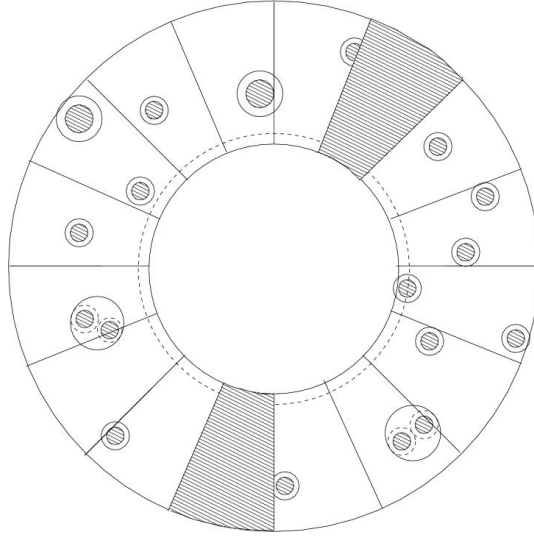


FIGURE 3.2 – Growth and merging process

To add the lower bounds obtained during the different growth phases we use that if two balls, say $\mathcal{B}(\vec{a}_p, \varrho_p)$ and $\mathcal{B}(\vec{a}_q, \varrho_q)$ merge into a new ball $\mathcal{B}(\vec{a}_s, \varrho_s)$ one has always $d_p^2 + d_q^2 \geq |d_p| + |d_q| \geq |d_p + d_q| = |d_s|$. This accounts for the fact that we get a factor $|d_i|$ in (3.4.51), whereas the factor in (3.4.53) is d_i^2 .

We stop the process when we have a collection of disjoint balls $\{\mathcal{B}_i\}_{i \in I} := \{\mathcal{B}(\vec{a}_i, \varrho_i)\}_{i \in I}$ satisfying the condition 2 of Proposition 3.4.2 (the property 1 is satisfied by construction, since it holds true for the initial family of balls), i.e., the balls are large enough, so that we have

$$\int_{\mathcal{B}(\vec{a}_i, \varrho_i)} d\vec{r} |\nabla u|^2 \geq 2\pi \left(\frac{1}{2} - \alpha \right) |d_i| |\log \varepsilon| \left(1 - C \frac{\log |\log \varepsilon|}{|\log \varepsilon|} \right), \quad (3.4.54)$$

where $d_i := \deg\{u, \partial \mathcal{B}_i\}$, if $\mathcal{B}_i \subset GS_\alpha \cap \tilde{\mathcal{A}}$, and $d_i = 0$ otherwise. The final configuration is drawn in Figure 3.3.

The logarithmic factor $(\frac{1}{2} - \alpha) |\log \varepsilon| \left(1 - C \frac{\log |\log \varepsilon|}{|\log \varepsilon|} \right)$ comes from the logarithm of the dilatation factor of the collections of balls, i.e.,

$$\frac{\sum_{i \in I} \varrho_i}{\sum_{i \in \tilde{I}} \tilde{\varrho}_i} \geq C \varepsilon^{\alpha-1/2} |\log \varepsilon|^{-23/2}.$$

On the other hand, using (3.2.30) and the fact that $|\nabla \rho^{\text{TF}}| \leq C \varepsilon^{-2} |\log \varepsilon|^{-2}$ (see (3.1.11)), we have

$$\left| \min_{\vec{r} \in \mathcal{B}_i} g^2(r) - g^2(a_i) \right| \leq C \left(\varepsilon^{1/2} |\log \varepsilon|^{3/2} \|\rho^{\text{TF}}\|_{L^\infty(\mathcal{A})} + \left| \min_{\vec{r} \in \mathcal{B}_i} \rho^{\text{TF}}(r) - \rho^{\text{TF}}(a_i) \right| \right) \leq C \left(\varepsilon^{-1/2} |\log \varepsilon|^{1/2} + \varepsilon^{-2} |\log \varepsilon|^{-2} \varrho_i \right) \leq C \varepsilon^{-1} |\log \varepsilon|^{-7} \leq C |\log \varepsilon|^{-4} g^2(a_i), \quad (3.4.55)$$

because $\vec{a}_i \in \tilde{\mathcal{A}}$ and $\varrho_i \leq C \varepsilon |\log \varepsilon|^{-5}$. We conclude from (3.4.54) and (3.4.55) that the lower bound (3.4.51) holds on each ball we have constructed by bounding below g^2 with its minimum on the ball \mathcal{B}_i . The error $\min_{\vec{r} \in \mathcal{B}_i} g^2(\vec{r}) - g^2(a_i)$ can then be absorbed into the $g^2(a_i) \log |\log \varepsilon|$ term. \square

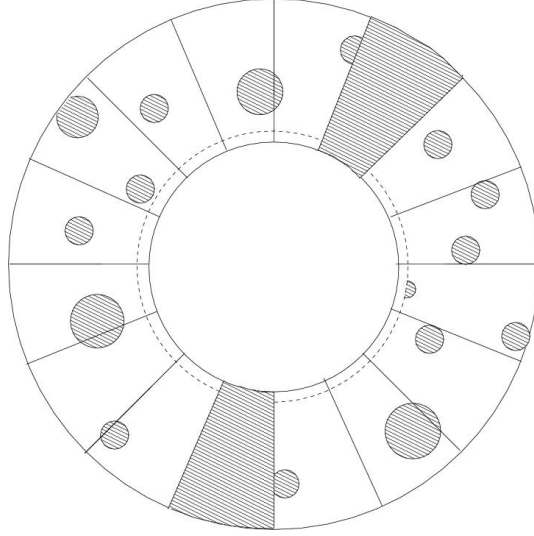


FIGURE 3.3 – Final configuration of cells and vortex balls.

3.4.4 Jacobian Estimate

We now turn to the jacobian estimate. With the vortex balls that we have constructed, it is to be expected that in the α -good set the vorticity measure of u will be close to a sum of Dirac masses

$$\operatorname{curl}(iu, \nabla u) \simeq 2\pi \sum_{i \in I} d_i \delta(\vec{r} - \vec{a}_i),$$

where $\delta(\vec{r} - \vec{a}_i)$ stands for the Dirac mass at \vec{a}_i . Indeed, outside of the balls $|u| \simeq 1$, so $\operatorname{curl}(iu, \nabla u) \simeq 0$, and the balls have very small radii compared to the size of \mathcal{A} .

Proposition 3.4.3 gives a rigorous statement of this fact. Note that we can quantify the difference between $\operatorname{curl}(iu, \nabla u)$ and $\sum_i d_i \delta(\vec{r} - \vec{a}_i)$ in terms of the energy only in the domain $\tilde{\mathcal{A}}$ where the density is large enough.

Proposition 3.4.3 (Jacobian estimate).

Let $0 \leq \alpha < \frac{1}{2}$ and ϕ be any piecewise- C^1 test function with compact support

$$\operatorname{supp}(\phi) \subset \tilde{\mathcal{A}} \cap GS_\alpha.$$

Let $\{\mathcal{B}_i\}_{i \in I} := \{\mathcal{B}(\vec{a}_i, \rho_i)\}_{i \in I}$ be a disjoint collection of balls as in Proposition 3.4.2. Setting $d_i := \deg\{u, \partial \mathcal{B}_i\}$, if $\mathcal{B}_i \subset \tilde{\mathcal{A}} \cap GS_\alpha$, and $d_i = 0$ otherwise, one has

$$\left| \sum_{i \in I} 2\pi d_i \phi(\vec{a}_i) - \int_{GS_\alpha \cap \tilde{\mathcal{A}}} d\vec{r} \phi \operatorname{curl}(iu, \nabla u) \right| \leq C \|\nabla \phi\|_{L^\infty(GS_\alpha)} \varepsilon^2 |\log \varepsilon|^{-2} \mathcal{F}[u]. \quad (3.4.56)$$

Proof. We argue as in [135, Chapter 6]. We first introduce a function $\xi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ as follows : $\xi(x) = 2x$, if $x \in [0, 1/2]$, and $\xi(x) = 1$, if $x \in [1/2, +\infty[$. This function satisfies

- $\xi(t) \leq 2t$ and $\xi'(t) \leq 2$,
- $|\xi(t) - t| \leq |1 - t|$ and $|\xi(t) - 1| \leq |1 - t|$,
- $|\xi(t)^2 - t^2| \leq 3t|1 - t|$,

and we define w as a regularization of u (in the sense that $|w| = 1$ and therefore $\text{curl}(iw, \nabla w) = 0$ when $|u|$ is far enough from 0) :

$$w = \frac{\xi(|u|)}{|u|}u. \quad (3.4.57)$$

Remark that we have $(iw, \nabla w) = \frac{|w|^2}{|u|^2}(iu, \nabla u)$ and this has a meaning even if u vanishes. By integrating by parts on $GS_\alpha \cap \tilde{\mathcal{A}}$ and using the assumptions on ϕ , one has

$$\int_{GS_\alpha \cap \tilde{\mathcal{A}}} d\vec{r} [\text{curl}(iu, \nabla u) - \text{curl}(iw, \nabla w)] \phi = - \int_{GS_\alpha \cap \tilde{\mathcal{A}}} d\vec{r} [(iu, \nabla u) - (iw, \nabla w)] \nabla^\perp \phi. \quad (3.4.58)$$

Now,

$$\begin{aligned} \left| \int_{GS_\alpha \cap \tilde{\mathcal{A}}} d\sigma [(iu, \nabla u) - (iw, \nabla w)] \nabla^\perp \phi \right| &\leq \|\nabla \phi\|_{L^\infty(GS_\alpha)} \int_{\tilde{\mathcal{A}}} d\vec{r} |(iu, \nabla u) - (iw, \nabla w)| \leq \\ &\|\nabla \phi\|_{L^\infty(GS_\alpha)} \int_{\tilde{\mathcal{A}}} d\vec{r} \frac{(|u|^2 - |w|^2)}{|u|} |\nabla u| \leq \|\nabla \phi\|_{L^\infty(GS_\alpha)} \|1 - |u|\|_{L^2(\tilde{\mathcal{A}})} \|\nabla u\|_{L^2(\tilde{\mathcal{A}})} \leq \\ &C \|\nabla \phi\|_{L^\infty(GS_\alpha)} \varepsilon^{5/2} |\log \varepsilon|^{9/2} \mathcal{F}[u] \end{aligned} \quad (3.4.59)$$

using the properties of ξ , the Cauchy-Schwarz inequality, the definition (3.4.2) of $\mathcal{F}[u]$, $g^2 \geq C\varepsilon^{-1} |\log \varepsilon|^{-3}$ on $\tilde{\mathcal{A}}$ and $(1 - |u|)^2 \leq (1 - |u|^2)^2$.

We now evaluate

$$\int_{GS_\alpha \cap \tilde{\mathcal{A}}} d\vec{r} \text{curl}(iw, \nabla w) \phi = \sum_{i \in I} \int_{\mathcal{B}_i \cap \tilde{\mathcal{A}}} d\vec{r} \text{curl}(iw, \nabla w) \phi, \quad (3.4.60)$$

which follows from the fact that $||u| - 1| < |\log \varepsilon|^{-1}$ outside $\cup_{i \in I} \mathcal{B}_i$, so that $|w| = 1$ and $\text{curl}(iw, \nabla w) = 0$ outside $\cup_{i \in I} \mathcal{B}_i$. If $\mathcal{B}_i \subset \tilde{\mathcal{A}} \cap GS_\alpha$, we have

$$\int_{\mathcal{B}_i} d\vec{r} |\text{curl}(iw, \nabla w)| |\phi(\vec{r}) - \phi(\vec{a}_i)| \leq C \|\nabla \phi\|_{L^\infty(\mathcal{B}_i)} \varrho_i \int_{\mathcal{B}_i} d\vec{r} |\nabla u|^2 \quad (3.4.61)$$

and

$$\int_{\mathcal{B}_i} d\vec{r} \text{curl}(iw, \nabla w) = 2\pi \text{deg}\{u, \partial \mathcal{B}_i\} = 2\pi d_i \quad (3.4.62)$$

by definition of w and d_i . On the other hand, if $\mathcal{B}_i \not\subset \tilde{\mathcal{A}} \cap GS_\alpha$ then $\mathcal{B}_i \cap \partial(\tilde{\mathcal{A}} \cap GS_\alpha) \neq \emptyset$ and thus

$$\begin{aligned} \left| \int_{\mathcal{B}_i \cap \tilde{\mathcal{A}}} d\vec{r} \text{curl}(iw, \nabla w) \phi \right| &\leq \int_{\mathcal{B}_i \cap \tilde{\mathcal{A}} \cap GS_\alpha} d\vec{r} |\phi| |\nabla w|^2 \leq 4 \int_{\mathcal{B}_i \cap \tilde{\mathcal{A}} \cap GS_\alpha} d\vec{r} |\phi| |\nabla u|^2 \leq \\ &C \|\nabla \phi\|_{L^\infty(\mathcal{B}_i)} \varrho_i \int_{\mathcal{B}_i \cap \tilde{\mathcal{A}} \cap GS_\alpha} d\vec{r} |\nabla u|^2, \end{aligned} \quad (3.4.63)$$

because $|\nabla w| \leq 2|\nabla u|$ and ϕ is supported in the interior of $\tilde{\mathcal{A}} \cap GS_\alpha$. Gathering equations (3.4.58) to (3.4.63), we obtain

$$\begin{aligned} \left| \sum_{i \in I} 2\pi d_i \phi(\vec{a}_i) - \int_{GS_\alpha \cap \tilde{\mathcal{A}}} d\vec{r} \phi \text{curl}(iu, \nabla u) \right| &\leq C \|\nabla \phi\|_{L^\infty(GS_\alpha)} \varepsilon^{5/2} |\log \varepsilon|^{9/2} \mathcal{F}[u] + \\ &C \sum_i \|\nabla \phi\|_{L^\infty(\mathcal{B}_i)} \varrho_i \int_{\mathcal{B}_i \cap \tilde{\mathcal{A}} \cap GS_\alpha} d\vec{r} |\nabla u|^2 \end{aligned} \quad (3.4.64)$$

and the result follows because, using $\varrho_i \leq \varepsilon |\log \varepsilon|^{-5}$ and (3.4.7),

$$\begin{aligned} \sum_i \|\nabla \phi\|_{L^\infty(\mathcal{B}_i)} \varrho_i \int_{\mathcal{B}_i \cap \tilde{\mathcal{A}} \cap GS_\alpha} d\vec{r} |\nabla u|^2 &\leq C \|\nabla \phi\|_{L^\infty(GS_\alpha)} \varepsilon^2 |\log \varepsilon|^{-2} \sum_i \int_{\mathcal{B}_i \cap \tilde{\mathcal{A}} \cap GS_\alpha} d\vec{r} g^2 |\nabla u|^2 \leq \\ &C \|\nabla \phi\|_{L^\infty(GS_\alpha)} \varepsilon^2 |\log \varepsilon|^{-2} \mathcal{F}[u]. \end{aligned} \quad (3.4.65)$$

□

Note that (3.4.56) is equivalent to saying that the $\left(C_c^1(\tilde{\mathcal{A}} \cap GS_\alpha)\right)^*$ norm of $\text{curl}(iu, \nabla u) - \sum_i d_i \delta(\vec{r} - \vec{a}_i)$ is controlled by the energy.

3.4.5 Completion of the Proof of Proposition 3.4.1

We now complete the proof of Proposition 3.4.1, collecting the estimates of the preceding Subsections.

We want to avoid any unwanted boundary term when performing integrations by parts in the proof below. Indeed, our radial frontiers between the good set and the bad set are somewhat artificial and have no physical interpretation. Therefore it is difficult to estimate integrals on this boundaries.

To get around this point we need to introduce an azimuthal partition of unity on the annulus in order to ‘smooth’ the radial boundaries appearing in our construction. This requires new definitions :

Definition 3.4.2 (Pleasant and unpleasant cells).

Recall the covering of the annulus \mathcal{A} by cells \mathcal{A}_n , $n \in \{1, \dots, N\}$. We say that \mathcal{A}_n is

- an α -pleasant cell if \mathcal{A}_n and its two neighbors are good cells. We note PS_α the union of all α -pleasant cells and N_α^P their number,
- an α -unpleasant cell if either \mathcal{A}_n is a bad cell, or \mathcal{A}_n is a good cell but its two neighbors are bad cells. We note UPS_α the union of all α -unpleasant cells and N_α^{UP} their number,
- an α -average cell if \mathcal{A}_n is a good cell but exactly one of its neighbors is not. We note AS_α the union of all α -average cells and N_α^A their number.

Remark that one obviously has, recalling (3.4.50),

$$N_\alpha^{UP} \leq \frac{3}{2} N_\alpha^B \ll N \quad (3.4.66)$$

and

$$N_\alpha^A \leq 2N_\alpha^B \ll N. \quad (3.4.67)$$

The average cells will play the role of transition layers between the pleasant set, where we will use the tools of Subsections 3.4.3 and 3.4.4, and the unpleasant set, where we have little information and therefore have to rely on more basic estimates (like those we used in the proof of Lemma 3.4.2). To make this precise we now introduce the azimuthal partition of unity we have announced.

Let us label UPS_α^l , $l \in \{1, \dots, L\}$, and PS_α^m , $m \in \{1, \dots, M\}$, the connected components of the α -unpleasant set and α -pleasant set respectively. We construct azimuthal positive

functions, bounded independently of ε , denoted by χ_l^U and χ_m^P (the labels U and P stand for “pleasant set” and “unpleasant set”) so that

$$\begin{aligned} \chi_l^U &:= 1 \text{ on } UPS_\alpha^l, \\ \chi_l^U &:= 0 \text{ on } PS_\alpha^m, \forall m \in \{1, \dots, M\}, \text{ and on } UPS_\alpha^{l'}, \forall l' \neq l, \\ \chi_m^P &:= 1 \text{ on } PS_\alpha^m, \\ \chi_m^P &:= 0 \text{ on } UPS_\alpha^l, \forall l \in \{1, \dots, L\}, \text{ and on } PS_\alpha^{m'}, \forall m' \neq m, \\ \sum_m \chi_m^P + \sum_l \chi_l^U &= 1 \text{ on } \mathcal{A}. \end{aligned} \quad (3.4.68)$$

It is important to note that each function so defined varies from 0 to 1 in an average cell. A crucial consequence of this is that we can take functions satisfying

$$|\nabla \chi_l^U| \leq \frac{C}{\varepsilon |\log \varepsilon|}, \quad |\nabla \chi_m^P| \leq \frac{C}{\varepsilon |\log \varepsilon|}, \quad (3.4.69)$$

because the side length of a cell is $\propto \varepsilon |\log \varepsilon|$. For example one can choose this partition of unity to be constituted of piecewise affine functions of the angle.

We will use the short-hand notation

$$\chi_{\text{in}} := \sum_{m=1}^M \chi_m^P, \quad (3.4.70)$$

$$\chi_{\text{out}} := \sum_{l=1}^L \chi_l^U. \quad (3.4.71)$$

The subscripts ‘in’ and ‘out’ refer to ‘in the pleasant set’ and ‘out of the pleasant set’ respectively.

We would like to use the jacobian estimate of Proposition 4.3.3 with $\phi = \chi_{\text{in}}^F$, which support is not included in $\tilde{\mathcal{A}}$ but only in \mathcal{A} (moreover it does not vanish on $\partial\mathcal{B}$). We will thus need a radial partition of unity : We introduce two radii R_{cut}^+ and R_{cut}^- as

$$R_{\text{cut}}^+ := 1 - \varepsilon |\log \varepsilon|^{-1}, \quad (3.4.72)$$

$$R_{\text{cut}}^- := R_{>} + \varepsilon |\log \varepsilon|^{-1}. \quad (3.4.73)$$

Let $\xi_{\text{in}}(r)$ and $\xi_{\text{out}}(r)$ be two positive radial functions satisfying

$$\begin{aligned} \xi_{\text{in}}(r) &:= 1 \text{ for } R_{\text{cut}}^- \leq r \leq R_{\text{cut}}^+, \\ \xi_{\text{in}}(r) &:= 0 \text{ for } R_{<} \leq r \leq R_{>} \text{ and for } r = 1, \\ \xi_{\text{out}}(r) &:= 1 \text{ for } R_{<} \leq r \leq R_{>}, \\ \xi_{\text{out}}(r) &:= 0 \text{ for } R_{\text{cut}}^- \leq r \leq R_{\text{cut}}^+, \\ \xi_{\text{in}} + \xi_{\text{out}} &= 1 \text{ on } \mathcal{A}. \end{aligned} \quad (3.4.74)$$

For example ξ_{in} and ξ_{out} can be defined as piecewise affine functions of the radius. Moreover, because of (3.4.72) and (3.4.73), we can impose

$$|\nabla \xi_{\text{in}}| \leq \frac{C |\log \varepsilon|}{\varepsilon}, \quad |\nabla \xi_{\text{out}}| \leq \frac{C |\log \varepsilon|}{\varepsilon}. \quad (3.4.75)$$

The subscripts ‘in’ and ‘out’ refer to ‘inside $\tilde{\mathcal{A}}$ ’ and ‘outside of $\tilde{\mathcal{A}}$ ’ respectively.

In the sequel $\{\mathcal{B}_i\}_{i \in I} := \{\mathcal{B}(\tilde{a}_i, \varrho_i)\}_{i \in I}$ is a collection of disjoint balls as in Proposition 3.4.2. For the sake of simplicity we label \mathcal{B}_j , $j \in J \subset I$, the balls such that $\mathcal{B}_j \subset \tilde{\mathcal{A}} \cap GS_\alpha$.

Proof of Proposition 3.4.1.

Recall the properties of F (3.4.11). By integration by parts, we have

$$\int_{\mathcal{A}} d\vec{r} \left\{ g^2 |\nabla u|^2 - 2g^2 \vec{B} \cdot (iu, \nabla u) \right\} = \int_{\mathcal{A}} d\vec{r} \left\{ g^2 |\nabla u|^2 + F \operatorname{curl}(iu, \nabla u) \right\} - \int_{\partial \mathcal{B}} d\sigma F(1)(iu, \partial_\tau u) \quad (3.4.76)$$

and we are going to evaluate the three terms using our previous results. We begin with a lower bound on the kinetic term in (3.4.76), using Proposition 3.4.2. We introduce a parameter γ to be fixed later in the proof and estimate

$$\int_{\mathcal{A}} d\vec{r} g^2 |\nabla u|^2 \geq (1 - \gamma) \sum_{j \in J} \int_{\mathcal{B}_j} d\vec{r} \xi_{\text{in}} g^2 |\nabla u|^2 + (1 - \gamma) \int_{\mathcal{A}} d\vec{r} \xi_{\text{out}} g^2 |\nabla u|^2 + \gamma \int_{\mathcal{A}} d\vec{r} g^2 |\nabla u|^2. \quad (3.4.77)$$

Using the lower bound (3.4.51), we have

$$\begin{aligned} \int_{\mathcal{B}_j} d\vec{r} \xi_{\text{in}} g^2 |\nabla u|^2 &\geq \xi_{\text{in}}(a_j) \int_{\mathcal{B}_j} d\vec{r} g^2 |\nabla u|^2 + \left(\inf_{\vec{r} \in \mathcal{B}_j} \xi_{\text{in}}(r) - \xi_{\text{in}}(a_j) \right) \int_{\mathcal{B}_j} d\vec{r} g^2 |\nabla u|^2 \geq \\ &\left(\frac{1}{2} - \alpha \right) |d_j| \xi_{\text{in}}(a_j) g^2(a_j) |\log \varepsilon| \left(1 - C \frac{\log |\log \varepsilon|}{|\log \varepsilon|} \right) - \frac{C}{|\log \varepsilon|^4} \int_{\mathcal{B}_j} d\vec{r} g^2 |\nabla u|^2. \end{aligned} \quad (3.4.78)$$

The estimate of $\inf_{\mathcal{B}_j} \xi_{\text{in}} - \xi_{\text{in}}(a_j)$ is a consequence of (3.4.75) combined with $\varrho_j \leq C\varepsilon |\log \varepsilon|^{-5}$.

We now compute

$$\int_{\mathcal{A}} d\vec{r} F \operatorname{curl}(iu, \nabla u) = \int_{\mathcal{A}} d\vec{r} [\xi_{\text{in}} \chi_{\text{in}} F \operatorname{curl}(iu, \nabla u) + \xi_{\text{out}} \chi_{\text{in}} F \operatorname{curl}(iu, \nabla u) + \chi_{\text{out}} F \operatorname{curl}(iu, \nabla u)]. \quad (3.4.79)$$

We can use Proposition 4.3.3 to estimate the first term because $\xi_{\text{in}} \chi_{\text{in}} F$ is a piecewise- C^1 function with support included in $\tilde{\mathcal{A}} \cap GS_\alpha$. We obtain

$$\begin{aligned} \int_{\mathcal{A}} d\vec{r} \xi_{\text{in}} \chi_{\text{in}} F \operatorname{curl}(iu, \nabla u) &\geq \\ &2\pi \sum_{j \in J} d_j F(a_j) \xi_{\text{in}}(a_j) \chi_{\text{in}}(\vec{a}_j) - C \|\nabla(\xi_{\text{in}} \chi_{\text{in}} F)\|_{L^\infty(GS_\alpha)} \varepsilon^2 |\log \varepsilon|^{-2} \mathcal{F}[u]. \end{aligned} \quad (3.4.80)$$

Now, using (3.4.19), (3.4.20), (4.3.17) and (3.4.75), we have

$$\|\nabla(\xi_{\text{in}} \chi_{\text{in}} F)\|_{L^\infty(\mathcal{A})} \leq \frac{C |\log \varepsilon|}{\varepsilon^2},$$

so that

$$\int_{\mathcal{A}} d\vec{r} \xi_{\text{in}} \chi_{\text{in}} F \operatorname{curl}(iu, \nabla u) \geq 2\pi \sum_{j \in J} d_j F(a_j) \xi_{\text{in}}(a_j) \chi_{\text{in}}(\vec{a}_j) - C |\log \varepsilon|^{-1} \mathcal{F}[u]. \quad (3.4.81)$$

The second term in the r.h.s. of (3.4.79) is simply bounded below as follows

$$\int_{\mathcal{A}} d\vec{r} \xi_{\text{out}} \chi_{\text{in}} F \operatorname{curl}(iu, \nabla u) \geq - \int_{\mathcal{A}} d\vec{r} \xi_{\text{out}} |F| |\nabla u|^2. \quad (3.4.82)$$

We now estimate the third term in the r.h.s. of (3.4.79) : We integrate by parts back to get

$$\int_{\mathcal{A}} d\vec{r} \chi_{\text{out}} F \operatorname{curl}(iu, \nabla u) \geq - \int_{\mathcal{A}} d\vec{r} \nabla^{\perp}(\chi_{\text{out}} F) \cdot (iu, \nabla u) - C \int_{\partial \mathcal{B}} d\sigma |F(1)| |(iu, \partial_{\tau} u)|, \quad (3.4.83)$$

but

$$\int_{\mathcal{A}} d\vec{r} \nabla^{\perp}(\chi_{\text{out}} F) \cdot (iu, \nabla u) = \int_{\mathcal{A}} d\vec{r} \left\{ F \nabla^{\perp}(\chi_{\text{out}}) \cdot (iu, \nabla u) + 2\chi_{\text{out}} g^2 \vec{B} \cdot (iu, \nabla u) \right\} \quad (3.4.84)$$

and the second term can be bounded using the same computations as in the proof of Lemma 3.4.2 :

$$\int_{\mathcal{A}} d\vec{r} \chi_{\text{out}} g^2 \vec{B} \cdot (iu, \nabla u) \leq \delta \int_{\mathcal{A}} d\vec{r} \chi_{\text{out}} g^2 |\nabla u|^2 + C\delta^{-1} \int_{\mathcal{A}} d\vec{r} \chi_{\text{out}} g^2 B^2 |u|^2,$$

where δ is a parameter to be fixed later. For the first term in (3.4.84) we use (3.4.22) :

$$\left| \int_{\mathcal{A}} d\vec{r} \nabla^{\perp}(\chi_{\text{out}}) F \cdot (iu, \nabla u) \right| \leq C\varepsilon^{-1} \int_{\mathcal{A}} d\vec{r} \left| \nabla^{\perp} \chi_{\text{out}} \right| |r - R_{<}| g^2 |u| |\nabla u| \leq \delta \int_{\{\nabla \chi_{\text{out}} \neq 0\}} d\vec{r} g^2 |\nabla u|^2 + \frac{C}{\delta \varepsilon^2} \int_{\{\nabla \chi_{\text{out}} \neq 0\}} d\vec{r} g^2 |u|^2.$$

The second inequality uses $|1 - R_{<}| \propto \varepsilon |\log \varepsilon|$ and (4.3.17).

We inject the preceding computations in (3.4.83), taking into account that $B \leq C\varepsilon^{-1}$. We also note that we have $\chi_{\text{out}} \neq 0$ and/or $\nabla \chi_{\text{out}} \neq 0$ only in the unpleasant set and the average set, so

$$\int_{\mathcal{A}} d\vec{r} \chi_{\text{out}} F \operatorname{curl}(iu, \nabla u) \geq - C\delta \int_{UPS_{\alpha} \cup AS_{\alpha}} d\vec{r} g^2 |\nabla u|^2 - \frac{C}{\delta \varepsilon^2} \int_{UPS_{\alpha} \cup AS_{\alpha}} d\vec{r} g^2 |u|^2 - C \int_{\partial \mathcal{B}} d\sigma |F(1)| |(iu, \partial_{\tau} u)|. \quad (3.4.85)$$

We gather equations (3.4.77), (3.4.78), (3.4.79), (3.4.81), (3.4.82) and (3.4.85) to obtain (recalling that $|\chi_{\text{in}}| \leq 1$)

$$\begin{aligned} & \int_{\mathcal{A}} d\vec{r} \left\{ g^2 |\nabla u|^2 + F \operatorname{curl}(iu, \nabla u) \right\} \geq \\ & \sum_{j \in J} 2\pi \xi_{\text{in}}(a_j) |d_j| \left[(1 - \gamma) \left(\frac{1}{2} - \alpha \right) g^2(a_j) |\log \varepsilon| \left(1 - C \frac{\log |\log \varepsilon|}{|\log \varepsilon|} \right) - |F(a_j)| \right] + \\ & (1 - \gamma) \int_{\mathcal{A}} d\vec{r} \xi_{\text{out}} g^2 |\nabla u|^2 - \int_{\mathcal{A}} d\vec{r} \xi_{\text{out}} |F| |\nabla u|^2 - C \int_{\partial \mathcal{B}} d\sigma |F(1)| |(iu, \partial_{\tau} u)| + \\ & (\gamma - \delta) \int_{\mathcal{A}} d\vec{r} g^2 |\nabla u|^2 - \frac{C}{\delta \varepsilon^2} \int_{UPS_{\alpha} \cup AS_{\alpha}} d\vec{r} g^2 |u|^2 - C |\log \varepsilon|^{-1} \mathcal{F}[u]. \end{aligned} \quad (3.4.86)$$

We choose now the parameters in (3.4.86) as follows :

$$\gamma = 2\delta = \frac{\log |\log \varepsilon|}{|\log \varepsilon|}, \quad \alpha = \tilde{\alpha} \frac{\log |\log \varepsilon|}{|\log \varepsilon|}, \quad (3.4.87)$$

where $\tilde{\alpha}$ is a large enough constant (see below). This choice allows to bound the terms in (3.4.86) from below : Indeed, if $\Omega_0 > 2(3\pi)^{-1}$, we have from Proposition 3.6.2

$$\frac{1}{2}g^2(a_j) |\log \varepsilon| - |F(a_j)| \geq \frac{C}{\varepsilon |\log \varepsilon|^{-2}}$$

for any $\tilde{a}_j \in \tilde{\mathcal{A}}$ and thus

$$(1 - \gamma) \left(\frac{1}{2} - \alpha \right) g^2(a_j) |\log \varepsilon| \left(1 - C \frac{\log |\log \varepsilon|}{|\log \varepsilon|} \right) - |F(a_j)| \geq \frac{1}{2}g^2(a_j) |\log \varepsilon| \left(1 - C \frac{\log |\log \varepsilon|}{|\log \varepsilon|} \right) - |F(a_j)| \geq \frac{C}{\varepsilon |\log \varepsilon|^{-2}} > 0 \quad (3.4.88)$$

where we have used (3.2.25).

On the other hand, by the definition of ξ_{out} , for any $\vec{r} \in \text{supp}(\xi_{\text{out}})$, we have either

$$|r - R_{<}| \leq C\varepsilon |\log \varepsilon|^{-1} \quad (3.4.89)$$

or

$$|r - 1| \leq C\varepsilon |\log \varepsilon|^{-1}. \quad (3.4.90)$$

Therefore, using (3.4.22), we have in the first case

$$|F(r)| \leq C \frac{g^2(r)}{|\log \varepsilon|}.$$

In the second case (3.4.22) yields

$$|F(r)| \leq \frac{C}{\varepsilon |\log \varepsilon|^2}$$

but (3.2.30) shows that, if \vec{r} satisfies (3.4.90),

$$g^2(r) \geq \frac{C}{\varepsilon |\log \varepsilon|}.$$

We conclude that

$$g^2(r) \geq C |\log \varepsilon| |F(r)| \gg |F(r)| \quad (3.4.91)$$

for any $\vec{r} \in \text{supp}(\xi_{\text{out}})$ and thus

$$(1 - \gamma) \int_{\mathcal{A}} d\vec{r} \xi_{\text{out}} g^2 |\nabla u|^2 - \int_{\mathcal{A}} d\vec{r} \xi_{\text{out}} |F| |\nabla u|^2 \geq 0. \quad (3.4.92)$$

Finally we have from (3.4.86), (3.4.88) and (3.4.92)

$$\begin{aligned} \int_{\mathcal{A}} d\vec{r} \left\{ g^2 |\nabla u|^2 + F \text{curl}(iu, \nabla u) \right\} &\geq C \frac{\log |\log \varepsilon|}{|\log \varepsilon|} \int_{\mathcal{A}} d\vec{r} g^2 |\nabla u|^2 - \\ &C \frac{|\log \varepsilon|}{\varepsilon^2 \log |\log \varepsilon|} \int_{UPS_\alpha \cup AS_\alpha} d\vec{r} g^2 |u|^2 - C \int_{\partial \mathcal{B}} d\sigma |F(1)| |(iu, \partial_\tau u)| - C |\log \varepsilon|^{-1} \mathcal{F}[u]. \end{aligned} \quad (3.4.93)$$

Adding

$$\int_{\mathcal{A}} d\vec{r} \frac{g^4}{\varepsilon^2} (1 - |u|^2)^2 - \int_{\partial\mathcal{B}} d\sigma F(1)(iu, \partial_\tau u)$$

to both sides of (3.4.93) and using (3.4.76), we get the lower bound

$$\mathcal{E}[u] \geq C \left\{ \frac{\log |\log \varepsilon|}{|\log \varepsilon|} \mathcal{F}[u] - \frac{|\log \varepsilon|}{\varepsilon^2 \log |\log \varepsilon|} \int_{UPS_\alpha \cup AS_\alpha} d\vec{r} g^2 |u|^2 - \int_{\partial\mathcal{B}} d\sigma F(1)(iu, \partial_\tau u) \right\}, \quad (3.4.94)$$

valid for ε small enough and $\Omega_0 > 2(3\pi)^{-1}$. But $g^2|u|^2 = |\Psi^{\text{GP}}|^2 \leq C\varepsilon^{-1}|\log \varepsilon|^{-1}$, whereas the side length of a cell is $O(\varepsilon|\log \varepsilon|)$, thus

$$\int_{UPS_\alpha \cup AS_\alpha} d\vec{r} g^2 |u|^2 \leq \frac{C}{\varepsilon |\log \varepsilon|} |UPS_\alpha \cup AS_\alpha| \leq C\varepsilon |\log \varepsilon| (N_\alpha^{\text{UP}} + N_\alpha^{\text{A}}). \quad (3.4.95)$$

Using (3.4.50), (4.3.14) and (3.4.67), we deduce

$$\int_{UPS_\alpha \cup AS_\alpha} d\vec{r} g^2 |u|^2 \leq C\varepsilon |\log \varepsilon| N_\alpha^{\text{B}} \leq C\varepsilon^2 \varepsilon^\alpha \mathcal{F}[u]. \quad (3.4.96)$$

On the other hand (3.2.30) implies that $g^2(1) \geq C\varepsilon^{-1}|\log \varepsilon|^{-1}$. Combining this fact with the upper bound (3.2.2) yields

$$|u| \leq C \text{ on } \partial\mathcal{B} \quad (3.4.97)$$

and thus, using (3.4.21) and Cauchy-Schwarz inequality,

$$\left| \int_{\partial\mathcal{B}} d\sigma F(1)(iu, \partial_\tau u) \right| \leq C \left(\int_{\partial\mathcal{B}} d\sigma |\partial_\tau u|^2 \right)^{1/2}. \quad (3.4.98)$$

Using Lemma 3.4.4 we conclude

$$\left| \int_{\partial\mathcal{B}} d\sigma F(1)(iu, \partial_\tau u) \right| \leq C \left(|\log \varepsilon|^{1/2} |\mathcal{E}[u]|^{1/2} + \frac{|\log \varepsilon|^{1/4}}{\varepsilon^{1/4}} \mathcal{F}[u]^{1/2} \right). \quad (3.4.99)$$

Combining (3.4.94), (3.4.96) and (3.4.99), we have

$$\mathcal{E}[u] \geq C \left\{ \frac{\log |\log \varepsilon|}{|\log \varepsilon|} \mathcal{F}[u] - \frac{|\log \varepsilon|}{\log |\log \varepsilon|} \varepsilon^\alpha \mathcal{F}[u] - |\log \varepsilon|^{1/2} |\mathcal{E}[u]|^{1/2} - \frac{|\log \varepsilon|^{1/4}}{\varepsilon^{1/4}} \mathcal{F}[u]^{1/2} \right\}. \quad (3.4.100)$$

Recall the choice of α in (3.4.87) : We choose now a constant $\tilde{\alpha} > 2$. Then

$$\frac{|\log \varepsilon|}{\log |\log \varepsilon|} \varepsilon^\alpha = \frac{|\log \varepsilon|^{1-\tilde{\alpha}}}{\log |\log \varepsilon|} \ll \frac{\log |\log \varepsilon|}{|\log \varepsilon|}$$

and thus there exists a finite constant c such that

$$O(\varepsilon^\infty) \geq \mathcal{E}[u] \geq c \left(\frac{\log |\log \varepsilon|}{|\log \varepsilon|} \mathcal{F}[u] - |\log \varepsilon|^{1/2} |\mathcal{E}[u]|^{1/2} - \frac{|\log \varepsilon|^{1/4}}{\varepsilon^{1/4}} \mathcal{F}[u]^{1/2} \right) \quad (3.4.101)$$

where the upper bound comes from (3.4.4).

Since the sign of $\mathcal{E}[u]$ is not known, we might have two possible cases : If $\mathcal{E}[u] \geq 0$, (3.4.4) implies that $|\mathcal{E}[u]| \leq O(\varepsilon^\infty)$, which can be plugged in (3.4.101) yielding

$$O(\varepsilon^\infty) \geq \mathcal{E}[u] \geq c \left(\frac{\log |\log \varepsilon|}{|\log \varepsilon|} \mathcal{F}[u] - \frac{|\log \varepsilon|^{1/4}}{\varepsilon^{1/4}} \mathcal{F}[u]^{1/2} \right) \quad (3.4.102)$$

This implies first

$$\mathcal{F}[u] \leq C \frac{|\log \varepsilon|^{5/2}}{\varepsilon^{1/2} \log |\log \varepsilon|^2}$$

and, going back to (3.4.102),

$$\mathcal{E}[u] \geq -C \frac{|\log \varepsilon|^{3/2}}{\varepsilon^{1/2} \log |\log \varepsilon|}$$

which concludes the proof of Proposition 3.4.1, if $\mathcal{E}[u] \geq 0$.

On the opposite, if $\mathcal{E}[u] < 0$, either

$$0 \geq \mathcal{E}[u] + c |\log \varepsilon|^{1/2} |\mathcal{E}[u]|^{1/2} \geq c \left(\frac{\log |\log \varepsilon|}{|\log \varepsilon|} \mathcal{F}[u] - \frac{|\log \varepsilon|^{1/4}}{\varepsilon^{1/4}} \mathcal{F}[u]^{1/2} \right),$$

which implies the result as before, or $\mathcal{E}[u] \leq C |\log \varepsilon|$, which gives

$$C |\log \varepsilon|^{3/2} \geq c \left(\frac{\log |\log \varepsilon|}{|\log \varepsilon|} \mathcal{F}[u] - \frac{|\log \varepsilon|^{1/4}}{\varepsilon^{1/4}} \mathcal{F}[u]^{1/2} \right),$$

and thus again (3.4.8) and (3.4.9). \square

As already noted, the end of the proof could be formulated as an induction. Plugging the estimates of Lemma 3.4.2 in (3.4.100) and using the upper bound on $\mathcal{E}[u]$ would yield improved estimates of $\mathcal{F}[u]$ and $|\mathcal{E}[u]|$, thus reducing the number of bad cells and improving the boundary estimate. The process could then be repeated a large number of times, proving that there are no bad cells at all. The second term in the lower bound (3.4.100) would then vanish and the process stop when the first term would reach the order of magnitude of the last one (coming from the boundary estimate), thus giving the results of Proposition 3.4.1.

3.5 Energy Asymptotics and Absence of Vortices

In this Section we conclude the proofs of our main results. The proof of the energy asymptotics is a straightforward combination of the results of Sections 3.3 and 3.4 :

Proof of Theorem 3.1.2.

From Proposition 3.3.1 we have, using the simplified notation of Section 3.4,

$$E^{\text{GP}} \geq \hat{E}_{\mathcal{A}, \omega_0}^{\text{GP}} + \mathcal{E}[u] - O(\varepsilon^\infty)$$

which reduces to

$$E^{\text{GP}} \geq \hat{E}_{\mathcal{A}, \omega_0}^{\text{GP}} - C \frac{|\log \varepsilon|^{3/2}}{\varepsilon^{1/2} \log |\log \varepsilon|}$$

thanks to (3.4.9). Using a regularization of $g_{\mathcal{A}, \omega_0}$ as a trial function for the functional $\hat{\mathcal{E}}_{\omega_0}^{\text{GP}}$ as in the proof of Proposition 3.3.1, we get $\hat{E}_{\mathcal{A}, \omega_0}^{\text{GP}} \geq \hat{E}_{\omega_0}^{\text{GP}} - O(\varepsilon^\infty)$ and thus

$$E^{\text{GP}} \geq \hat{E}_{\omega_0}^{\text{GP}} - C \frac{|\log \varepsilon|^{3/2}}{\varepsilon^{1/2} \log |\log \varepsilon|}.$$

We conclude the proof of the lower bound recalling that, by definition,

$$\hat{E}^{\text{GP}} = \inf_{\omega} \hat{E}_{\omega}^{\text{GP}} \leq \hat{E}_{\omega_0}^{\text{GP}}.$$

For the upper bound

$$E^{\text{GP}} \leq \hat{E}^{\text{GP}} = \hat{E}_{\omega_{\text{opt}}}^{\text{GP}}$$

we simply use $g_{\omega_{\text{opt}}}(r) \exp\{i([\Omega] - \omega_{\text{opt}})\vartheta\}$ as a trial function for \mathcal{E}^{GP} . \square

The proof of the absence of vortices requires an additional ingredient :

Lemma 3.5.1 (Estimate for the gradient of u_{ω_0}).

Recall the definition of u_{ω_0} in (3.3.5). If $\Omega_0 > 2(3\pi)^{-1}$, then there is a finite constant C such that

$$\|\nabla u_{\omega_0}\|_{L^\infty(\tilde{\mathcal{A}})} \leq C \frac{|\log \varepsilon|^{3/2}}{\varepsilon^{3/2}}. \quad (3.5.1)$$

Proof. We use the short-hand notation defined at the beginning of Section 3.4 (in particular $u = u_{\omega_0}$). Recall the variational equation (3.4.27)

$$-\nabla (g^2 \nabla u) - 2ig^2 \vec{B} \cdot \nabla u + \frac{2}{\varepsilon^2} g^4 (|u|^2 - 1) u = \lambda g^2 u. \quad (3.5.2)$$

From this equation we get the pointwise estimate

$$|\Delta u| \leq C \left(\frac{|\nabla g|}{g} |\nabla u| + |B| |\nabla u| + \frac{1}{\varepsilon^2} |g^2 (|u|^2 - 1) u| + |\lambda| |u| \right), \quad (3.5.3)$$

holding true on \mathcal{A} . Recalling that $|\Psi^{\text{GP}}| \leq \varepsilon^{-1/2} |\log \varepsilon|^{-1/2}$ and

$$g \geq \frac{C}{\varepsilon^{1/2} |\log \varepsilon|^{3/2}} \text{ on } \tilde{\mathcal{A}}, \quad (3.5.4)$$

we have

$$|u| \leq C |\log \varepsilon| \text{ on } \tilde{\mathcal{A}}. \quad (3.5.5)$$

Thus, using (3.2.25),

$$\frac{1}{\varepsilon^2} |g^2 (|u|^2 - 1) u| \leq C \frac{|\log \varepsilon|^2}{\varepsilon^3}. \quad (3.5.6)$$

On the other hand, estimating the chemical potential with (3.4.28) and plugging in the results of Proposition 3.4.1, we have

$$|\lambda u| \leq C |u| \left(|\mathcal{E}[u]| + \frac{1}{\varepsilon^{3/2} |\log \varepsilon|^{1/2}} \mathcal{F}[u]^{1/2} \right) \leq \frac{C |\log \varepsilon|^{7/4}}{\varepsilon^{7/4} |\log \varepsilon|} \ll \frac{|\log \varepsilon|^2}{\varepsilon^3} \quad (3.5.7)$$

on $\tilde{\mathcal{A}}$.

Combining (3.2.25), (3.2.50), (3.4.7) and (3.4.18) with (3.5.6) and (3.5.7), we deduce from (3.5.3)

$$\|\Delta u\|_{L^\infty(\tilde{\mathcal{A}})} \leq C \left(\frac{|\log \varepsilon|^{3/4}}{\varepsilon^{5/4}} \|\nabla u\|_{L^\infty(\tilde{\mathcal{A}})} + \frac{|\log \varepsilon|^2}{\varepsilon^3} \right).$$

From Gagliardo-Nirenberg inequality [117, Theorem at pg. 125], we deduce

$$\|\Delta u\|_{L^\infty(\tilde{\mathcal{A}})} \leq C \left(\frac{|\log \varepsilon|^{3/4}}{\varepsilon^{5/4}} \|\Delta u\|_{L^\infty(\tilde{\mathcal{A}})}^{1/2} \|u\|_{L^\infty(\tilde{\mathcal{A}})}^{1/2} + \frac{|\log \varepsilon|^2}{\varepsilon^3} \right). \quad (3.5.8)$$

Inserting (3.5.5), we conclude

$$\|\Delta u\|_{L^\infty(\tilde{\mathcal{A}})} \leq C \frac{|\log \varepsilon|^2}{\varepsilon^3}$$

and we get (3.5.1) by using (3.5.5) and the Gagliardo-Nirenberg inequality again. \square

We are now in position to complete the

Proof of Theorem 3.1.1.

The proof relies on a combination of (3.4.8) and (3.5.1), as in [29].

Suppose that at some point \vec{r}_0 such that

$$R_h + \frac{1}{2}\varepsilon |\log \varepsilon|^{-1} \leq r_0 \leq 1,$$

we have

$$||u(\vec{r}_0)| - 1| \geq \varepsilon^{1/8} |\log \varepsilon|^3.$$

Then, using (3.5.1), there is a constant C such that, for any $\vec{r} \in B_0 := \mathcal{B}(\vec{r}_0, C\varepsilon^{13/8} |\log \varepsilon|^{3/2})$, we have

$$||u(\vec{r})| - 1| \geq \frac{1}{2}\varepsilon^{1/8} |\log \varepsilon|^3.$$

This implies (recall (3.4.7))

$$\int_{B_0} d\vec{r} \frac{g^4}{\varepsilon^2} (1 - |u|^2)^2 \geq \frac{C |\log \varepsilon|^3}{\varepsilon^{1/2}},$$

and thus (note that by the initial condition on r_0 , $B_0 \subset \mathcal{A}$)

$$\mathcal{F}[u] \geq \frac{C |\log \varepsilon|^3}{\varepsilon^{1/2}}, \tag{3.5.9}$$

which is a contradiction with (3.4.8).

We have thus proven that (recall (3.5.5))

$$||\Psi^{\text{GP}}|^2 - g^2| \leq g^2 ||u|^2 - 1| \leq C \frac{|\log \varepsilon|^3}{\varepsilon^{7/8}} \tag{3.5.10}$$

on $\tilde{\mathcal{A}} = \{R_h + \varepsilon |\log \varepsilon|^{-1} \leq r \leq 1\}$. The result then follows by a combination of (3.2.30) and (3.5.10). \square

Remark 3.5.1. (*Absence of vortices in a larger domain*).

By direct inspection of the proof of Theorem 3.1.1, one can easily realize that we could have proven the main result, i.e., the absence of vortices, in a domain larger than $\tilde{\mathcal{A}}$, i.e., there is some freedom in the choice of the bulk of the condensate.

More precisely the choice of a larger domain would have implied a worse lower bound on g^2 via (3.2.30) and in turn a worse remainder in (3.5.10), but at the same time this would have allowed the extension of the no vortex result up to a distance of order $\varepsilon |\log \varepsilon|^{-a}$ from R_h for some power $a > 1$.

We have however chosen to state the main result in $\tilde{\mathcal{A}}$ for the sake of simplicity.

The proof of the result about the degree of Ψ^{GP} is a corollary of the main result proven above :

Proof of Theorem 3.1.3.

We first note that the pointwise estimate in (3.5.10) implies that Ψ^{GP} does not vanish on $\partial\mathcal{B}$, so that its degree is indeed well defined. We then compute

$$2\pi \deg\{\Psi^{\text{GP}}, \partial\mathcal{B}\} = -i \int_{\partial\mathcal{B}} d\sigma \frac{|\Psi^{\text{GP}}|}{\Psi^{\text{GP}}} \partial_\tau \left(\frac{\Psi^{\text{GP}}}{|\Psi^{\text{GP}}|} \right) = -i \int_{\partial\mathcal{B}} d\sigma \frac{|u|}{u} \partial_\tau \left(\frac{u}{|u|} e^{i([\Omega] - \omega_0)\vartheta} \right) e^{-i([\Omega] - \omega_0)\vartheta} = 2\pi([\Omega] - \omega_0) - i \int_{\partial\mathcal{B}} d\sigma \frac{|u|}{u} \partial_\tau \left(\frac{u}{|u|} \right). \quad (3.5.11)$$

Then

$$\left| \int_{\partial\mathcal{B}} d\sigma \frac{|u|}{u} \partial_\tau \left(\frac{u}{|u|} \right) \right| \leq \int_{\partial\mathcal{B}} d\sigma \left| \partial_\tau \left(\frac{u}{|u|} \right) \right| \leq C \int_{\partial\mathcal{B}} d\sigma |\partial_\tau u|, \quad (3.5.12)$$

where we have used that $|u|$ is bounded below by a constant on $\partial\mathcal{B}$.

Finally, combining (3.4.39) and the results of Proposition 3.4.1, we obtain (recall that $g^2 \geq C\varepsilon^{-1}|\log \varepsilon|^{-1}$ on $\partial\mathcal{B}$)

$$\int_{\partial\mathcal{B}} |\partial_\tau u|^2 \leq \frac{C|\log \varepsilon|^3}{\varepsilon(\log |\log \varepsilon|)^2}. \quad (3.5.13)$$

Using the Cauchy-Schwarz inequality, we thus conclude from (3.5.11), (3.5.12) and (3.5.13) that

$$\deg\{\Psi^{\text{GP}}, \partial\mathcal{B}\} = [\Omega] - \omega_0 + O\left(\varepsilon^{-1/2}|\log \varepsilon|^{3/2}(\log |\log \varepsilon|)^{-1}\right). \quad (3.5.14)$$

There only remains to recall that (see (3.3.10))

$$\omega_0 = \frac{2}{3\sqrt{\pi}\varepsilon} + O(\varepsilon^{-1}|\log \varepsilon|^{-1})$$

and that an identical estimate applies to ω_{opt} (see (3.3.16)). \square

Remark 3.5.2. (*Degree of a GP minimizer.*)

According to (3.5.14), we could have stated the result (3.1.21) about the degree of Ψ^{GP} in terms of ω_0 , i.e., the optimal giant vortex phase when the minimization problem is restricted to the annulus \mathcal{A} , instead of ω_{opt} , i.e., the optimal giant vortex phase in the whole of \mathcal{B} . Moreover the remainder in (3.5.14), i.e., $O(\varepsilon^{-1/2}|\log \varepsilon|^{3/2}(\log |\log \varepsilon|)^{-1})$, is much better than the one contained in the final result (3.1.21), i.e., $O(\varepsilon^{-1}|\log \varepsilon|^{-1})$, which is inherited from (3.3.10) and (3.3.16). Note however that the latter remainder is the best precision to which one can estimate the giant vortex phase in terms of the explicit quantity $2/(3\sqrt{\pi})\varepsilon^{-1}$. For this reason and the fact that ω_{opt} occurs more naturally in the analysis, we have used it in (3.1.21).

3.6 Appendix A

In this Appendix we discuss some useful properties of the TF-like functionals involved in the analysis as well as the critical angular velocity Ω_c for the emergence of the giant vortex phase.

3.6.1 The TF Functionals

We start by considering the TF functional defined in (3.1.9) :

$$\mathcal{E}^{\text{TF}}[\rho] := \int_{\mathcal{B}} d\vec{r} \{ -\Omega^2 r^2 \rho + \varepsilon^{-2} \rho^2 \}.$$

Its minimizer over positive functions in $L^1(\mathcal{B})$ is unique and is given by the radial density

$$\rho^{\text{TF}}(r) := \frac{1}{2} [\varepsilon^2 \mu^{\text{TF}} + \varepsilon^2 \Omega^2 r^2]_+ = \frac{\varepsilon^2 \Omega^2}{2} [r^2 - R_{\text{h}}^2]_+, \quad (3.6.1)$$

where the chemical potential is fixed by normalizing ρ^{TF} in $L^1(\mathcal{B})$, i.e.,

$$\mu^{\text{TF}} := E^{\text{TF}} + \varepsilon^{-2} \|\rho^{\text{TF}}\|_2^2 = -\Omega^2 R_{\text{h}}^2. \quad (3.6.2)$$

Note that, if $\Omega \geq 2/(\sqrt{\pi}\varepsilon)$, the TF minimizer is a compactly supported function, since it vanishes outside \mathcal{D}^{TF} , i.e., for $r \leq R_{\text{h}}$, where

$$R_{\text{h}} := \sqrt{1 - \frac{2}{\sqrt{\pi}\varepsilon\Omega}}. \quad (3.6.3)$$

The corresponding ground state energy can be explicitly evaluated and is given by

$$E^{\text{TF}} = -\Omega^2 \left(1 - \frac{4}{3\sqrt{\pi}\varepsilon\Omega} \right). \quad (3.6.4)$$

By (3.6.3) and (3.6.4) the annulus \mathcal{D}^{TF} has a shrinking width of order $\varepsilon |\log \varepsilon|$ and the leading order term in the ground state energy asymptotics is $-\Omega^2$, which is due to the convergence of ρ^{TF} to a delta function supported at the boundary of the trap.

In other Sections of the paper we often consider the restrictions of the functionals to domains \mathcal{D} strictly contained inside \mathcal{B} (denoted by $\mathcal{E}_{\mathcal{D}}^{\text{TF}}$). However in the case of the TF functional there is no need to make a distinction between \mathcal{E}^{TF} and $\mathcal{E}_{\mathcal{D}}^{\text{TF}}$ since all the ground state properties are basically independent of the integration domain, provided $\mathcal{D}^{\text{TF}} \subset \mathcal{D}$.

Another important TF-like functional is defined in (3.1.26) and includes the giant vortex energy contribution, i.e.,

$$\hat{\mathcal{E}}_{\omega}^{\text{TF}}[\rho] := \int_{\mathcal{B}} d\vec{r} \{ -\Omega^2 r^2 \rho + B_{\omega}^2(r) \rho + \varepsilon^{-2} \rho^2 \} = \int_{\mathcal{B}} d\vec{r} \{ ([\Omega] - \omega)^2 r^{-2} \rho + \varepsilon^{-2} \rho^2 \} - 2\Omega[\Omega - \omega], \quad (3.6.5)$$

where the potential \vec{B} is defined in (3.1.25), $\omega \in \mathbb{Z}$ and we have used the normalization in $L^1(\mathcal{B})$ of the density in the last term.

The minimization is essentially the same as for (3.1.9) : The normalized minimizer is

$$\hat{\rho}_{\omega}^{\text{TF}}(r) := \frac{\varepsilon^2}{2} \left[\hat{\mu}_{\omega}^{\text{TF}} - ([\Omega] - \omega)^2 r^{-2} \right]_+, \quad (3.6.6)$$

and the normalization condition becomes

$$\frac{1 - \hat{R}^2}{\hat{R}^2} + \log \hat{R}^2 = \frac{2}{\pi \varepsilon^2 ([\Omega] - \omega)^2}, \quad (3.6.7)$$

where we have denoted

$$\hat{R}^2 := \frac{([\Omega] - \omega)^2}{\hat{\mu}_\omega^{\text{TF}}}. \quad (3.6.8)$$

With such a definition the minimizer (3.6.6) can be rewritten in a form very close to the TF minimizer (3.6.1), i.e.,

$$\tilde{\rho}_\omega^{\text{TF}}(r) = \frac{\varepsilon^2([\Omega] - \omega)^2}{2\hat{R}^2 r^2} \left[r^2 - \hat{R}^2 \right]_+.$$

In order to make a comparison it would then be useful to evaluate the radius \hat{R} but the equation (3.6.7) has no explicit solution. However, since the right hand side of (3.6.7) tends to zero as $\varepsilon \rightarrow 0$, we can expand the left hand side assuming $\hat{R}^{-2} = 1 + \delta$ for some $\delta \ll 1$:

$$\frac{1}{2}\delta^2 - \frac{1}{3}\delta^3 + O(\delta^4) = \frac{2}{\pi\varepsilon^2([\Omega] - \omega)^2},$$

which yields

$$\frac{1}{\hat{R}^2} - 1 = \delta = \frac{2}{\sqrt{\pi\varepsilon}([\Omega] - \omega)} \left[1 + \frac{2}{3\sqrt{\pi\varepsilon}([\Omega] - \omega)} + \frac{1}{9\pi\varepsilon^2([\Omega] - \omega)^2} + O(\varepsilon^{-3}\Omega^{-3}) \right].$$

We thus have

$$\hat{R}^2 = 1 - \frac{2}{\sqrt{\pi\varepsilon}([\Omega] - \omega)} - \frac{4}{3\pi\varepsilon^2([\Omega] - \omega)^2} + O(\varepsilon^{-3}\Omega^{-3}) = R_h^2 + \frac{2\omega}{\sqrt{\pi\varepsilon}\Omega^2} - \frac{4}{3\pi\varepsilon^2\Omega^2} + O(\varepsilon^3|\log\varepsilon|^3), \quad (3.6.9)$$

and whether \hat{R} is larger or smaller than R_h depends in a crucial way on the phase ω : In particular in the case of the giant vortex phase ω_0 (see Proposition 3.3.2), the sum of the two last terms in the above expression vanishes to the leading order (see (3.3.10)), i.e., it is much smaller than $O(\varepsilon^2|\log\varepsilon|^2)$.

The ground state energy $\hat{E}_\omega^{\text{TF}}$ is easy to compute :

$$\begin{aligned} \hat{E}_\omega^{\text{TF}} &= -2\Omega([\Omega] - \omega) + \frac{\pi\varepsilon^2([\Omega] - \omega)^4}{4} (\hat{R}^{-2} - 1)^2 \\ &= -\Omega^2 + \frac{4\Omega}{3\sqrt{\pi\varepsilon}} + \omega^2 - \frac{4\omega}{3\sqrt{\pi\varepsilon}} + \frac{2}{3\pi\varepsilon^2} - 2\Omega([\Omega] - \Omega) + O(\varepsilon^{-2}|\log\varepsilon|^{-2}), \end{aligned}$$

and, assuming that $|\omega| \leq O(\varepsilon^{-1})$, one can easily recognize that the leading term and the first remainder coincide with (3.6.4), i.e., the energy $\hat{E}_\omega^{\text{TF}}$ is equal to E^{TF} up to second order corrections :

$$\hat{E}_\omega^{\text{TF}} = E^{\text{TF}} + \left[\omega - \frac{2}{3\sqrt{\pi\varepsilon}} \right]^2 + \frac{2}{9\pi\varepsilon^2} - 2\Omega([\Omega] - \Omega) + O(\varepsilon^{-2}|\log\varepsilon|^{-2}). \quad (3.6.10)$$

This formula implies that $\hat{E}_\omega^{\text{TF}}$ is minimized by a phase which is given up to corrections of order $\varepsilon^{-1}|\log\varepsilon|^{-1}$ by

$$\omega^{\text{TF}} := \frac{2}{3\sqrt{\pi\varepsilon}}, \quad (3.6.11)$$

and the same is true for the giant vortex phases ω_0 (see Proposition 3.3.2 and (3.3.10)) and ω_{opt} (see Proposition 3.3.3 and (3.3.16)).

3.6.2 The Critical Angular Velocity and the Vortex Energy

The last part of this Appendix is devoted to the study of the critical velocity Ω_c , which is defined as the angular velocity at which vortices disappear from the bulk of the condensate. To estimate this velocity we have, according to the discussion in Section 3.4, to compare the vortex energy cost $\frac{1}{2}g^2(r)|\log \varepsilon|$ and the vortex energy gain $|F(r)|$ (see (3.4.10)) : This leads to the definition of the function

$$H(r) := \frac{1}{2}g_{\mathcal{A},\omega_0}^2(r)|\log \varepsilon| - |F_{\omega_0}(r)|, \quad (3.6.12)$$

which yields the overall energy contribution of a vortex at a radius \vec{r} inside the bulk : If H is positive in some region, then a vortex is energetically unfavorable there, and, if this holds true in the whole of the bulk, the condensate is in the giant vortex phase.

Before studying the behavior of the above function H , it is however convenient to obtain an explicit approximative value for the critical velocity and to this purpose we replace the density g^2 with ρ^{TF} and study the function

$$H^{\text{TF}}(r) := \frac{1}{2}|\log \varepsilon|\rho^{\text{TF}}(r) - |F^{\text{TF}}(r)|, \quad (3.6.13)$$

where the function F^{TF} is explicitly given by

$$F^{\text{TF}}(r) := 2 \int_{R_h}^r ds \vec{B}_{\omega^{\text{TF}}}(r) \cdot \vec{e}_\vartheta \rho^{\text{TF}}(r) = \varepsilon^2 \Omega^2 \int_{R_h}^r ds [\Omega s - ([\Omega] - \omega^{\text{TF}}) s^{-1}] (s^2 - R_h^2), \quad (3.6.14)$$

with ω^{TF} defined in (3.6.11).

In order to investigate the behavior of the infimum of H^{TF} inside the bulk, it is convenient to rescale the quantities and set

$$z := \varepsilon \Omega (r^2 - R_h^2), \quad (3.6.15)$$

so that z varies on a scale of order one, i.e., more precisely $z \in [0, 2/\sqrt{\pi}]$ (see (3.6.3)). With such a choice the gain function can be easily estimated :

$$\begin{aligned} F^{\text{TF}}(r) &= \frac{\varepsilon^2 \Omega^2}{2} \int_0^{r^2 - R_h^2} dt t [\Omega(t + R_h^2) - [\Omega] + \omega^{\text{TF}}] (t + R_h^2)^{-1} = \\ &= \frac{\varepsilon^2 \Omega^2}{2} \int_0^{r^2 - R_h^2} dt t \left(\Omega t - \frac{4}{3\sqrt{\pi}\varepsilon} + O(1) \right) \left(1 - \frac{2}{\sqrt{\pi}\varepsilon\Omega} + t \right)^{-1} = \\ &= \frac{1}{2\varepsilon} \int_0^z ds s \left(s - \frac{4}{3\sqrt{\pi}} + O(\varepsilon) \right) \left(1 - \frac{2}{\sqrt{\pi}\varepsilon\Omega} + \frac{s}{\varepsilon\Omega} \right)^{-1} = \\ &= \frac{1}{2\varepsilon} \int_0^z ds s \left(s - \frac{4}{3\sqrt{\pi}} \right) + O(|\log \varepsilon|) = \frac{z^2}{6\varepsilon} \left(z - \frac{2}{\sqrt{\pi}} \right) + O(|\log \varepsilon|), \end{aligned}$$

where we have used the approximation $[1 - O((\varepsilon\Omega)^{-1})]^{-1} = 1 + O((\varepsilon\Omega)^{-1})$.

Applying the same rescaling to the energy cost function, we thus obtain

$$H^{\text{TF}}(r) := \frac{z \tilde{H}^{\text{TF}}(z)}{12\varepsilon}, \quad (3.6.16)$$

where

$$\tilde{H}^{\text{TF}}(z) = 3\Omega_0 - 2z \left| z - \frac{2}{\sqrt{\pi}} + O(\varepsilon|\log \varepsilon|) \right| = 3\Omega_0 - 2z \left(\frac{2}{\sqrt{\pi}} - z \right) - O(\varepsilon|\log \varepsilon|), \quad (3.6.17)$$

since $z \leq 2/\sqrt{\pi}$ by the definition of the scaling. Now it is very easy to see that

$$\tilde{H}^{\text{TF}}(z) \geq 3\Omega_0 - 2\pi^{-1} - O(\varepsilon|\log \varepsilon|). \quad (3.6.18)$$

The above considerations lead to the following

Proposition 3.6.1 (TF vortex energy).

For any $\Omega_0 > 2(3\pi)^{-1}$ and ε small enough, there exists a finite constant C such that

$$H^{\text{TF}}(r) \geq C\varepsilon^{-1}|\log \varepsilon|^{-2} > 0$$

for any \vec{r} such that $r \geq R_h + \varepsilon|\log \varepsilon|^{-1}$.

Proof. It is sufficient to collect (3.6.16), (3.6.18) and recall the rescaling (3.6.15). \square

We now go back to the original function H and show that the above property holds true as well, i.e., $\Omega_c = 2(3\pi)^{-1}\varepsilon^{-2}|\log \varepsilon|^{-1}$ is the critical velocity for the disappearance of vortices from the bulk of the condensate :

Proposition 3.6.2 (Critical angular velocity).

If $\Omega_0 > 2(3\pi)^{-1}$ and ε is small enough, there exists a finite constant C such that

$$H(r) \geq C\varepsilon^{-1}|\log \varepsilon|^{-2} > 0$$

for any \vec{r} such that $r \geq R_h + \varepsilon|\log \varepsilon|^{-1}$.

Proof. The result basically follows from what is proven about H^{TF} : We are going to show that

$$\sup_{\vec{r} \in \bar{\mathcal{A}}} |H^{\text{TF}}(r) - H(r)| \leq C\varepsilon^{-1}|\log \varepsilon|^{-1}. \quad (3.6.19)$$

In order to prove the above inequality, we use the estimates (3.2.30) and (3.2.25) to get

$$\sup_{\vec{r} \in \bar{\mathcal{A}}} |\rho^{\text{TF}}(r) - g_{\mathcal{A},\omega_0}^2(r)| \leq C\varepsilon^{1/2}|\log \varepsilon|^{7/2} \|\rho^{\text{TF}}\|_{\infty} \leq C\varepsilon^{-1/2}|\log \varepsilon|^{5/2}, \quad (3.6.20)$$

and

$$\begin{aligned} \sup_{\vec{r} \in \bar{\mathcal{A}}} |F^{\text{TF}}(r) - F_{\omega_0}(r)| &\leq 2 \int_{R_<}^{R_>} ds |B_{\omega_0}(s)| g_{\mathcal{A},\omega_0}^2(s) + \\ &C\varepsilon^2 \Omega^2 |\omega_0 - \omega^{\text{TF}}| \int_{R_h}^1 ds s^{-1}(s^2 - R_h^2) + \sup_{\vec{r} \in \bar{\mathcal{A}}} 2 |\rho^{\text{TF}}(r) - g_{\mathcal{A},\omega_0}^2(r)| \int_{R_h}^1 ds |B_{\omega_0}(s)| \leq \\ &C \left[\varepsilon^{-1} |R_> - R_<| g_{\mathcal{A},\omega_0}^2(R_>) + \varepsilon^3 \Omega^2 |\log \varepsilon| + \varepsilon^{-1/2} |\log \varepsilon|^{7/2} \right] \leq \\ &C [|\log \varepsilon|^{-1} \rho^{\text{TF}}(R_>) + \varepsilon^{-1} |\log \varepsilon|^{-1}] \leq C\varepsilon^{-1} |\log \varepsilon|^{-1}, \end{aligned} \quad (3.6.21)$$

where we have used (3.6.20), the monotonicity of $g_{\mathcal{A},\omega_0}(r)$ (see Proposition 3.2.3) and the estimate (3.3.10).

Hence one obtains (3.6.19) and the final result follows from Proposition 3.6.1 if $r \geq R_h + \varepsilon|\log \varepsilon|^{1/2}$. Indeed, using (3.6.15), (A.6) and (3.6.18) we have

$$H^{\text{TF}}(r) \geq C\varepsilon^{-1}|\log \varepsilon|^{-1/2}$$

in this case.

On the other hand, if $R_h + \varepsilon|\log \varepsilon|^{-1} \leq r \leq R_h + \varepsilon|\log \varepsilon|^{1/2}$ it follows from (3.4.22) that

$$H(r) \geq C|\log \varepsilon|g_{\mathcal{A},\omega_0}^2(r) \geq C\frac{1}{\varepsilon|\log \varepsilon|^2}.$$

The second inequality comes from (3.4.7). This concludes the proof. \square

4

Vortex Rings in Fast Rotating Bose-Einstein Condensates

4.1 Introduction

A Bose gas trapped in a magnetic potential exhibits the remarkable property that, at sufficiently low temperatures, a macroscopic fraction of the atoms are in the same quantum state. This phenomenon is referred to as *Bose Einstein Condensation* and has been first observed experimentally by the JILA and MIT groups in 1995 (2001 Nobel prize in physics attributed to Cornell, Wieman and Ketterle). Particularly interesting experiments consist in creating a BEC by cooling atomic gases in a magnetic trap, and setting the trap into rotation. Indeed, a BEC is a superfluid and responds to rotation by the nucleation of quantized vortices. There is a rich literature on this subject, both in physics and in mathematics. We refer the reader to [4, 63] for extensive lists of references.

When dealing with rotating BECs, a strong distinction has to be made according to the type of trapping potential that is being used. If the potential is harmonic, there exists a critical speed at which the centrifugal forces overcome the trapping force and drive the gas out of the trap. By contrast, in the experiments of [35, 146], a blue-detuned laser is applied to the condensate, resulting in a stronger confinement that prevents such a behavior. To illustrate the difference, let us recall briefly what is observed or expected when rotating a BEC at increasing speeds.

If the trapping potential is harmonic, as in most experiments, the gas set into rotation nucleates more and more vortices as the rotation speed is increased. Eventually, when vortices become densely packed in the condensate, they form a hexagonal lattice, called Abrikosov lattice by analogy with the physics of type II superconductors (see [4, Chapter 5] for a mathematical study of this phenomenon). If the rotation is very close to the limiting speed set by the confinement it is believed that the gas will enter a regime where the atoms are strongly correlated. Such a phenomenon has not yet been observed, see [41] for a review of the physical literature on this subject and [97] for recent mathematical results.

If the trapping potential increases faster than the square of the distance from the center of the trap, it is theoretically possible to explore regimes of arbitrary rotation rates, as first noted in [61]. For a potential of this type, variational arguments have been proposed in [65] to support the following picture : three successive phase transitions should occur when increasing the rotation speed, at which the ground state and its vortex structure

change drastically. For slow rotation speeds, there are no vortices and the condensate is at rest in the rotating frame. When reaching a first critical speed, vortices start to appear in the condensate and organize themselves in order to minimize their repulsion. This leads to a vortex-lattice state where a dense hexagonal lattice of vortices is observed. There is then a second critical speed where the centrifugal force dips a hole in the center of the condensate. This results in a vortex-lattice-plus-hole state where the condensate is annular and still supports a dense lattice of vortices. When the third critical speed, that will be our main interest in this paper, is reached, vortices are expected to retreat from the annular bulk of the BEC. There should however remain a macroscopic phase circulation (circular superflow) in the annulus. These phenomena have motivated numerous theoretical and numerical studies [64, 71, 91, 92, 90]. In particular, it is observed numerically (see e.g. [64]) that in the transition regime between the vortex-lattice-plus-hole state and the giant vortex state, an other phase appears which displays a single vortex circle around the central hole.

In this paper we aim at providing an estimate of the third critical speed in the limit where the coupling constant measuring the inter-atomic interactions tends to infinity (Thomas-Fermi regime). More precisely we build on the results of [46] ([Chapter 3]) where a rigorous upper bound to the critical speed has been obtained and aim at proving the corresponding lower bound. We also want to prove rigorously that when the rotation speed is close but below the critical speed, a circle of vortices is present in the annular bulk of the condensate.

Our setting is the same as in [43, 47, 46] : we consider the Gross-Pitaevskii energy for a Bose-Einstein condensate in a two-dimensional ‘flat’ trap, i.e. a potential equal to 0 in the unit disc \mathcal{B} and $+\infty$ outside, leading to a problem posed on the unit disc. Formally this is the limit as $s \rightarrow +\infty$ of a homogeneous trap of the form $V(r) = r^s$ (see [44]). In the rotating frame, the energy reads

$$\mathcal{E}^{\text{GP}}[\Psi] := \int_{\mathcal{B}} \left| \left(\nabla - i\Omega x^\perp \right) \Psi \right|^2 - \Omega^2 r^2 |\Psi|^2 + \varepsilon^{-2} |\Psi|^4, \quad (4.1.1)$$

and it should be minimized under the mass constraint

$$\int_{\mathcal{B}} |\Psi|^2 = 1. \quad (4.1.2)$$

The terms in the energy have the following interpretation. The first one is the kinetic energy, including the contribution of Coriolis forces due to the transformation to the rotating frame. The second term takes into account the centrifugal force while the third one models the interactions between the atoms. We denote by E^{GP} and Ψ^{GP} respectively the ground-state energy and a ground state (a priori not unique) of (4.1.1)

$$E^{\text{GP}} := \inf_{\int_{\mathcal{B}} |\Psi|^2 = 1} \mathcal{E}^{\text{GP}}[\Psi] = \mathcal{E}^{\text{GP}}[\Psi^{\text{GP}}]. \quad (4.1.3)$$

Any GP minimizer satisfies the Euler-Lagrange equation

$$- \left(\nabla - i\Omega x^\perp \right)^2 \Psi^{\text{GP}} - \Omega^2 r^2 \Psi^{\text{GP}} + 2\varepsilon^{-2} |\Psi^{\text{GP}}|^2 \Psi^{\text{GP}} = \mu^{\text{GP}} \Psi^{\text{GP}}, \quad (4.1.4)$$

where μ^{GP} is the chemical potential associated with the mass constraint (4.1.2).

To evaluate the critical speed for the appearance of the giant vortex we consider the asymptotic behavior of Ψ^{GP} and E^{GP} as $\varepsilon \rightarrow 0$. For $|\log \varepsilon| \ll \Omega \ll \frac{1}{\varepsilon^2 |\log \varepsilon|}$ it has been proved

[47] that many vortices are present and evenly distributed in the bulk of the condensate. Also the critical speed for the condensate to develop a hole (second critical speed in the above terminology) is estimated. It behaves as $\Omega \sim \Omega_h \varepsilon^{-1}$ where Ω_h is a fixed constant. In a recent work [46] ([Chapter 3]) (see also [127]([Chapter 2]) where a related problem is treated) we have proved that if

$$\Omega = \frac{\Omega_0}{\varepsilon^2 |\log \varepsilon|} \quad (4.1.5)$$

where Ω_0 is a fixed constant chosen above the critical value $2(3\pi)^{-1}$ then the support of any GP minimizer is essentially vortex free. We have thus confirmed that a transition occurs in the regime where Ω scales as (4.1.5) from a vortex lattice plus hole state (as described in [47]) to a pure giant vortex state. We have also provided an upper bound to the critical speed at which the transition is expected to occur :

$$\Omega_{\text{GV}} \leq \frac{2}{3\pi\varepsilon^2 |\log \varepsilon|}. \quad (4.1.6)$$

In the present paper we aim at providing the corresponding lower bound. Namely, we want to show that if

$$\Omega < \frac{2}{3\pi\varepsilon^2 |\log \varepsilon|} \quad (4.1.7)$$

then there are individual vortices in the bulk of the condensate. More precisely we consider rotation speeds of the form

$$\Omega = \frac{2}{3\pi\varepsilon^2 |\log \varepsilon|} - \frac{\Omega_1}{\varepsilon^2 |\log \varepsilon|} \quad (4.1.8)$$

where

$$\Omega_1 > 0 \quad (4.1.9)$$

$$|\Omega_1| \ll 1 \quad (4.1.10)$$

$$|\Omega_1| \gg \frac{\log |\log \varepsilon|}{|\log \varepsilon|}. \quad (4.1.11)$$

The two first assumptions on Ω_1 are natural : we want to study slightly subcritical speeds, namely we want to be below the threshold $2(3\pi\varepsilon^2 |\log \varepsilon|)^{-1}$ by a relatively small amount. The third assumption is technical. We do believe that the results we are going to present stay true without this assumption but the proofs require some improvements to apply to a regime where Ω_1 is allowed to be extremely small.

Stated loosely, our result is that in the regime (4.1.8), with Ω_1 satisfying (4.1.9), (4.1.10) and (4.1.11), there are vortices in the annular bulk of the condensate. They are uniformly distributed on a particular circle of radius R_* and we are able to estimate their number, which is of order $\Omega_1 \varepsilon^{-1}$. This completes the proof started in [46] ([Chapter 3]) that the third critical speed is asymptotically equal to $2(3\pi\varepsilon^2 |\log \varepsilon|)^{-1}$ in the limit of small ε :

$$\Omega_{\text{GV}} \sim \frac{2}{3\pi\varepsilon^2 |\log \varepsilon|}. \quad (4.1.12)$$

4.1.1 Statement of the Main Results

We first recall some notation that was introduced in [46] ([Chapter 3]). The following Thomas-Fermi energy functional will be of importance in our analysis because it gives the leading order of the GP energy when ε becomes small with the above scaling of the rotation speed :

$$\mathcal{E}^{\text{TF}}[\rho] := \int_{\mathcal{B}} (-\Omega^2 r^2 \rho + \varepsilon^{-2} \rho^2). \quad (4.1.13)$$

Here ρ is a matter density, it thus plays the role of $|\Psi|^2$. In particular it is positive. Minimizing ρ^{TF} with respect to all positive ρ 's normalized in L^1 we find a unique ground-state given by the radial density

$$\rho^{\text{TF}}(r) = \frac{1}{2} [\varepsilon^2 \mu^{\text{TF}} + \varepsilon^2 \Omega^2 r^2]_+ = \frac{\varepsilon^2 \Omega^2}{2} [r^2 - R_{\text{h}}^2]_+, \quad (4.1.14)$$

where the chemical potential is fixed by normalizing ρ^{TF} in $L^1(\mathcal{B})$, i.e.,

$$\mu^{\text{TF}} = E^{\text{TF}} + \varepsilon^{-2} \|\rho^{\text{TF}}\|_2^2. \quad (4.1.15)$$

Note that the TF minimizer is a compactly supported function, since it vanishes outside \mathcal{A}^{TF} , i.e., for $r \leq R_{\text{h}}$, where

$$R_{\text{h}} = \sqrt{1 - \frac{2}{\sqrt{\pi\varepsilon\Omega}}}, \quad \mathcal{A}^{\text{TF}} = \{R_{\text{h}} \leq r \leq 1\}. \quad (4.1.16)$$

The corresponding ground state energy can be explicitly evaluated and is given by

$$E^{\text{TF}} = -\Omega^2 \left(1 - \frac{4}{3\sqrt{\pi\varepsilon\Omega}}\right). \quad (4.1.17)$$

We stress that the annulus \mathcal{A}^{TF} has a shrinking width of order $\varepsilon|\log \varepsilon|$ and that the leading order term in the ground state energy asymptotics is $-\Omega^2$, which is due to the convergence of ρ^{TF} to a delta function supported at the boundary of the trap.

When $\varepsilon \rightarrow 0$ with the scaling (4.1.5), the leading order of the energy is given by the TF energy and the GP density $|\Psi^{\text{GP}}|^2$ is very close to ρ^{TF} . In particular, the mass of Ψ^{GP} becomes exponentially small in the central hole $\mathcal{B} \setminus \mathcal{A}^{\text{TF}}$ and the properties of the ground state and ground state energy are well approximated by a functional restricted to the TF annulus. For technical reasons however it is necessary to consider a slightly larger annulus

$$\mathcal{A} := \{\vec{r} : R_{<} \leq r \leq 1\}, \quad (4.1.18)$$

with the choice

$$R_{<} := R_{\text{h}} - \varepsilon^{8/7}. \quad (4.1.19)$$

In order to give a precise statement, we need to be more specific. Namely we need to take into account the leading order (i.e. E^{TF}) and the sub-leading order of the GP energy via a simplified functional : For any $a \in \mathbb{Z}$ and any **real-valued** wave-function ϕ we introduce the ‘giant vortex energy’ :

$$\hat{\mathcal{E}}_{\mathcal{A},a}^{\text{GP}}[\phi] := \mathcal{E}_{\mathcal{A}}^{\text{GP}}[\phi e^{i([\Omega]-a)\theta}] = \int_{\mathcal{A}} |\nabla \phi|^2 + \frac{1}{\varepsilon^2} |\phi|^4 + \left(\frac{([\Omega] - a)^2}{r^2} - 2\Omega([\Omega] - a) \right) |\phi|^2. \quad (4.1.20)$$

Let us denote $g_{\mathcal{A},a}$ and $\hat{E}_{\mathcal{A},a}^{\text{GP}}$ the ground state (unique up to a constant phase factor that we fix by requiring that $g_{\mathcal{A},a} > 0$) and ground state energy of this functional. We recall the following result

Proposition 4.1.1 ([46] **Properties of the optimal phase and associated density**).

There exists at least one $\omega \in \mathbb{Z}$ minimizing $\hat{E}_{\mathcal{A},a}^{\text{GP}}$ with respect to a . Moreover we have

$$\omega = \frac{2}{3\sqrt{\pi\varepsilon}} (1 + O(|\log \varepsilon|^{-1})) \quad (4.1.21)$$

and

$$\int_{\mathcal{A}} g_{\omega}^2 (\Omega - ([\Omega] - \omega) r^{-2}) = O(1) \quad (4.1.22)$$

where g_{ω} is the unique positive normalized minimizer of $\hat{\mathcal{E}}_{\mathcal{A},\omega}^{\text{GP}}$.

The key to the results of [46] ([Chapter 3]) in the super-critical case is that the asymptotics of the ground state and ground state energy of (4.1.1) are described with a very good precision by $g_{\mathcal{A},\omega} e^{i([\Omega] - \omega)\theta}$ (we will denote $g_{\mathcal{A},\omega}$ by g for short) and $\hat{E}_{\mathcal{A},\omega}^{\text{GP}}$ respectively. In particular, a minimizer of (4.1.1) contains a large vorticity similar to that generated by a giant vortex located at the origin, which is apparent in the phase factor $e^{i([\Omega] - \omega)\theta}$. To identify individual vortices in the annulus where the mass is concentrated we thus have to look at a reduced function as in Section 5 of [23]. More precisely we extract the giant-vortex density and phase factor by introducing

$$u := \frac{\Psi^{\text{GP}}}{g e^{i([\Omega] - \omega)\theta}} \quad (4.1.23)$$

which is defined only on \mathcal{A} . Our main result deals with the vortices of u and is stated, as is usual, in terms of the *vorticity measure* of u

$$\mu := \text{curl}(iu, \nabla u) = \frac{1}{2} \text{curl}(u \nabla \bar{u} - \bar{u} \nabla u) \quad (4.1.24)$$

We prove that the vorticity measure is close to a Delta function concentrated on some particular circle of radius R_* (see Appendix A).

We have to deal with a technical point in the statement of the Theorem. In [46]([Chapter 3]) we had identified the ‘short-range’ energetic cost (by opposition to a ‘long-range’ cost that we will identify in the present paper) of a vortex of degree 1 at some point \vec{r} . It is given in terms of the following *cost function*

$$H(r) := \frac{1}{2} g^2(r) |\log \varepsilon| + F(r) \quad (4.1.25)$$

where F is defined as

$$F(r) := 2 \int_{R_{<}}^r g^2(s) \left(\Omega s - ([\Omega] - \omega) \frac{1}{s} \right) ds. \quad (4.1.26)$$

The dependence on g makes it very difficult to identify vortices close to the inner boundary of \mathcal{A} . Indeed, ρ^{TF} vanishes on ∂B_{R_h} , so g is very small there (compared for example with its value on $\partial \mathcal{B}$) and the energetic cost of a vortex too close to ∂B_{R_h} can not be taken into

account with our method. To avoid this difficulty we limit ourselves to a statement about the asymptotics of μ on a smaller domain $\mathcal{A}_{\text{bulk}}$ where the bulk of the mass is concentrated and the density is large enough :

$$\mathcal{A}_{\text{bulk}} := \{\bar{r} \mid R_{\text{bulk}} \leq r \leq 1\}, \quad R_{\text{bulk}} := R_h + \varepsilon |\log \varepsilon| \Omega_1^{1/2}. \quad (4.1.27)$$

Note that, using the results in [46, Section 2.1](Section 3.2.1)

$$\int_{\mathcal{A}_{\text{bulk}}} |\Psi^{\text{GP}}|^2 = 1 - O(\Omega_1^{1/2}).$$

This justifies the notation : $\mathcal{A}_{\text{bulk}}$ indeed contains the bulk of the mass when $\Omega_1 \ll 1$.

In the following theorem, we use the norm

$$\|\nu\|_g := \sup_{\phi \in C_c^1(\mathcal{A}_{\text{bulk}})} \frac{\left| \int_{\mathcal{A}_{\text{bulk}}} \nu \phi \right|}{\left(\int_{\mathcal{A}_{\text{bulk}}} \frac{1}{g^2} |\nabla \phi|^2 \right)^{1/2} + \varepsilon |\log \varepsilon| \|\nabla \phi\|_{L^\infty(\mathcal{A}_{\text{bulk}})}}, \quad (4.1.28)$$

where ν is a Radon measure, to estimate μ :

Theorem 4.1.1 (Asymptotics for the vorticity).

Let Ω_1 , u and μ be defined as above. Let R_* be the radius defined in (A.10), which satisfies

$$R_* = \sqrt{R_h^2 + (\varepsilon \Omega)^{-1} \left(\frac{1}{\sqrt{\pi}} + O(\Omega_1) \right)}. \quad (4.1.29)$$

We denote δ_* the normalized arclength measure on the circle of radius R_* .

There holds

$$\left\| \mu + \frac{H(R_*)}{2I_*} \delta_* \right\|_g \ll \frac{\Omega_1}{\varepsilon} \leq C \left\| \frac{H(R_*)}{2I_*} \delta_* \right\|_g \quad (4.1.30)$$

where H is defined as in (4.1.25) and I_* is a positive quantity, bounded above and below independently of ε , to be defined below.

Remark 4.1.1.[Comments on Theorem 4.1.1]

1. The vorticity measure μ is really the intrinsic vorticity quantity associated with u . Indeed, we expect that $|u| \sim 1$ (which implies $\text{curl}(iu, \nabla u) \sim 0$) outside of the vortex cores, so formally, for any domain which boundary σ does not intersect any vortex core,

$$2\pi \deg\{u, \sigma\} \sim \int_{\sigma} (iu, \partial_\tau u)$$

is given by the integral of μ over the domain. Thus, if the vortex cores are small, we should have $\mu \sim 2\pi \sum_j d_j \delta_{a_j}$ where the d_j are the degrees of the vortices and a_j their locations. Note that by definition the vortices of u are identical to those of Ψ^{GP} .

2. Let us explain a bit why (4.1.30) indeed captures the leading order of the vorticity. The radius R_* is such that

$$-C \frac{\Omega_1}{\varepsilon} < H(R_*) < 0.$$

Thus, taking a positive test function (or rather a sequence of test functions) in $C_c^1(\mathcal{A})$ we have

$$-\frac{H(R_*)}{2I_*} \int_{\mathcal{A}} \phi \delta_* \geq -C \frac{H(R_*)}{I_*} \int_{r=R_*} \phi(r, \theta) d\theta.$$

If in addition ϕ is radial and $\max \phi = \phi(R_*)$ then

$$-\frac{H(R_*)}{2I_*} \int_{\mathcal{A}} \phi \delta_* \geq -C \frac{H(R_*)}{I_*} \phi(R_*).$$

Since the thickness of the annulus \mathcal{A} is of order $\varepsilon |\log \varepsilon|$, one can obviously construct ϕ such that

$$\phi(R_*) \geq \|\nabla \phi\|_{L^\infty(\mathcal{A})} \varepsilon |\log \varepsilon|.$$

If in addition the support of ϕ is included in a region where $g^2 \geq C(\varepsilon |\log \varepsilon|^{-1})$ we have

$$\left(\int_{\mathcal{A}_{\text{bulk}}} \frac{1}{g^2} |\nabla \phi|^2 \right)^{1/2} \leq C \varepsilon |\log \varepsilon| \|\nabla \phi\|_{L^\infty(\mathcal{A})} \leq C(\varepsilon |\log \varepsilon|)^2 \|\phi\|_{L^\infty(\mathcal{A})}.$$

Thus

$$\begin{aligned} -\frac{H(R_*)}{2I_*} \int_{\mathcal{A}} \phi \delta_* &\geq -C \varepsilon |\log \varepsilon| \frac{H(R_*)}{I_*} \|\nabla \phi\|_{L^\infty(\mathcal{A})} \\ &> C \Omega_1 |\log \varepsilon| \|\nabla \phi\|_{L^\infty(\mathcal{A})} > C \frac{\Omega_1}{\varepsilon} \left(\int_{\mathcal{A}_{\text{bulk}}} \frac{1}{g^2} |\nabla \phi|^2 \right)^{1/2} \end{aligned} \quad (4.1.31)$$

which proves

$$\frac{\Omega_1}{\varepsilon} \leq C \left\| \frac{H(R_*)}{2I_*} \delta_* \right\|_g.$$

3. Let us comment on the norm (4.1.28) that we use in the theorem. In the denominator of the definition (4.1.28), the first term is rather natural in view of the energy estimates below. Indeed, the quantity $\left(\int_{\mathcal{A}_{\text{bulk}}} g^{-2} |\nabla \phi|^2 \right)^{1/2}$ defines a norm that is associated with the problem (4.1.36). The second term appears when regularizing μ in the course of the proof.

Of course if the norm $\left(\int_{\mathcal{A}_{\text{bulk}}} g^{-2} |\nabla \phi|^2 \right)^{1/2}$ could be controlled by $\varepsilon |\log \varepsilon| \|\nabla \phi\|_{L^\infty(\mathcal{A})}$ for any test function, then (4.1.30) would be equivalent to

$$\left\| \mu + \frac{H(R_*)}{2I_*} \delta_* \right\|_{(C_c^1(\mathcal{A}_{\text{bulk}}))^*} \ll \Omega_1 |\log \varepsilon|, \quad (4.1.32)$$

whereas the $(C_c^1(\mathcal{A}_{\text{bulk}}))^*$ norms of both terms in the above right-hand side are $\gtrsim \Omega_1 |\log \varepsilon|$ as demonstrated by (4.1.31). However, such a control is not possible because g^2 is not uniformly bounded below by $C(\varepsilon |\log \varepsilon|)^{-1}$ on $\mathcal{A}_{\text{bulk}}$. The norm $\|\cdot\|_g$ thus takes into account the effect of the inhomogeneous density background.

4. If we define a change of coordinates mapping \mathcal{A} to a fixed annulus of radius 1 and width say 1/2 and denote \tilde{R}_* the image of R_* under this change of coordinates and $\tilde{\mu}$ the push-forward of μ we will have

$$-2I_* \frac{\tilde{\mu}}{H(R_*)} \rightharpoonup \tilde{\delta}_* \quad (4.1.33)$$

in the weak sense of measures. Here $\tilde{\delta}_*$ is the normalized arc-length measure on the circle of radius \tilde{R}_* . We do not want to be too precise here in order not to mislead the reader : such a change of coordinates will not be used in the paper. The reason is that, applied to the energy functional, it would lead to a highly anisotropic kinetic energy term because of the shrinking width of the original annulus \mathcal{A} . The convergence (4.1.33) is useful however to identify the optimal number of vortices : I_* is of the order of a constant, $-H(R_*)$ of the order of $\Omega_1 \varepsilon^{-1}$ so from (4.1.33) the number of vortices is to leading order equal to $-H(R_*)(2I_*)^{-1} \propto \Omega_1 \varepsilon^{-1}$, as we announced.

The quantity I_* appearing in Theorem 4.1.1 is defined as follows. For any Radon measure ν supported in \mathcal{A} we define h_ν as the unique solution to the elliptic problem

$$\begin{cases} -\nabla \left(\frac{1}{g^2} \nabla h_\nu \right) = \nu \text{ in } \mathcal{A} \\ h_\nu = 0 \text{ on } \partial\mathcal{A}. \end{cases} \quad (4.1.34)$$

Note that g^2 is bounded below on \mathcal{A} because it is the solution of a problem with Neumann (natural) boundary conditions (see Lemma 4.2.1 below). There is thus no difficulty in defining h_ν as above. Next we introduce an ‘electrostatic energy’ associated to ν

$$I(\nu) := \int_{\mathcal{A}} \frac{1}{g^2} |\nabla h_\nu|^2. \quad (4.1.35)$$

Then

$$I_* := \inf_{\nu \in \mathcal{D}_*, \int \nu = 1} I(\nu) = I(\delta_*) \quad (4.1.36)$$

where the infimum is taken over the set \mathcal{D}_* of positive Radon measures with support on the circle of radius R_* . The fact that δ_* is a solution (actually, the unique solution) of the above minimization problem will be proved in Proposition 4.3.5 below. The energy (4.1.35) is similar to that of a charge distribution ν in a shell with inhomogeneous conductivity described by g^{-2} .

The problem (4.1.36) naturally appears in our analysis because it is linked in a crucial way to the asymptotics of the energy E^{GP} , as demonstrated in the

Theorem 4.1.2 (Energy asymptotics).

Let Ω_1 be as above and ε be small enough. There holds

$$E^{\text{GP}} = \hat{E}_{\mathcal{A}, \omega}^{\text{GP}} - \frac{H(R_*)^2}{4I_*} (1 + o(1)). \quad (4.1.37)$$

The way the value $\frac{H(R_*)^2}{4I_*}$ appears is through the following minimization problem :

$$-\frac{H(R_*)^2}{4I_*} = \inf_{\nu \in \mathcal{D}_*} \left(\int_{\mathcal{A}} \frac{1}{g^2} |\nabla h_\nu|^2 + H(R_*) \int_{\mathcal{A}} \nu \right). \quad (4.1.38)$$

The above functional of ν describes the energy of a vortex distribution corresponding to a vorticity measure ν concentrated on \mathcal{C}_{R_*} . The right-hand side can be thought of as a renormalized energy (in analogy with [29, 135]). The first term represents the interaction of the vortices, which is computed through the field h_ν that they create (other formulae, including a particularly useful Green representation will be used in the sequel). The second

term represents the energy gain of having a vortex partly compensating the rotation field. The unique minimizer of (4.1.38) is given by $-(2I_*)^{-1}H(R_*)\delta_*$, which explains why the vorticity measure of u has to be close, at least in the weak sense of Theorem 4.1.1, to this particular measure. Note the close analogy between our renormalized energy and that obtained in [23, Theorem 1.1], the difference being essentially apparent in the weight g^{-2} .

We recall that superfluids or superconductors in simply-connected geometries generically exhibit vortex concentration around isolated points (see [4, 135] and references therein). Here, although the original domain \mathcal{B} is simply connected, the strong centrifugal forces impose an annular form to the condensate. As a consequence, *vortex concentration along a curve* occurs for this model at rotation rates $\Omega \propto \varepsilon^{-2}|\log \varepsilon|^{-1}$. In [23], a model case for vortex concentration along a curve in the regime $\Omega \propto |\log \varepsilon|$ is considered and the limiting vorticity measure is identified via Γ -convergence.

Other physical situations where concentration along a curve occurs include superconductors with normal inclusions [20, 21], the case described in [5] of a BEC whose trapping potential impose an annular shape in the regime $\Omega \propto |\log \varepsilon|$, and that of two superconductors with different physical properties in contact along a circle [89]. In those three cases the question of the distribution of the vortices along the curve that we solve here is left open. In particular our result is as far as we know the only one in the literature where the limiting vorticity measure is computed for a case where there is vortex concentration along a curve against an inhomogeneous background. Indeed, the inhomogeneous density profile g^2 is a new feature compared to the situation in [23]. Our method could be used in the context of [5] where a similar inhomogeneity prevents from using directly the analysis of [23].

A formula such as (4.1.12) seems to be absent from the physics literature : in [65] the critical speed is estimated by comparing the energy of the giant vortex state to that of the vortex-lattice-plus-hole state. One then argues that the critical speed is that at which the former is smaller than the latter. This is too rough an estimate, for it does not take into account the fact that a circle of vortices appears at the transition. As a consequence, the formula we find by rigorous analysis differs from that given in [65]. We note that for a problem in a slightly different setting (namely, for a condensate trapped in a harmonic plus quartic trap) the papers [71, 92] propose methods to numerically compute the critical speed. This is done by comparing the giant vortex energy to the energy of a condensate containing a single ring of vortices. Varying the parameters defining the latter state (size of the vortex cores, number of vortices ...) to find the best possible choice yields the critical speed. It is again defined as the one above which the giant vortex energy is the smaller of the two. This method does not provide an explicit formula such as (4.1.12) and of course neither does a direct numerical minimization of the GP energy.

The response of type II superconductors to imposed external magnetic fields and that of superfluids to rotation of the container bear some striking similarities. This analogy between superfluidity (usually described by the Gross-Pitaevskii theory) and superconductivity (usually described by the Ginzburg-Landau theory) has been well-known to physicists for tens of years. More recently it turned out that mathematical tools originally developed for the GL theory could be successfully used in GP theory. Also there is an analogy between our (somewhat informal) terminology about critical speeds and that of critical fields in GL theory. In particular, the analogy between the first critical speed and

the field H_{c1} is well-known and of great use in the papers [5, 81, 82].

We want to emphasize however that the Gross-Pitaevskii theory in the regime we consider largely deviates from the Ginzburg-Landau theory. This is due to the presence of a mass constraint (which reflects the fact that a BEC has no normal state to relax to, contrary to a superconductor) and of centrifugal forces (which could be interpreted as electrical fields in GL theory). In particular, the second and third critical speeds we have informally defined have little to do with the second and third critical fields in GL theory (to our knowledge this has been first noticed in [65]). The discrepancy between GP theory and GL theory arises when the centrifugal forces can no longer be neglected, namely close to the second critical speed. We refer to [43, 44, 47] for this aspect of the theory.

As for the third critical speed, that is our main concern here, it turns out that it bears more similarities with H_{c1} than with H_{c2} or H_{c3} . In fact, the transition happening there can be seen as a H_{c1} type transition but *backwards*, which is apparent in the fact that vortices *disappear* from the bulk of the condensate when the rotation speed is increased. As a consequence, many mathematical tools characteristic of the study of type II superconductors (strictly) between H_{c1} and H_{c2} will be of great use in this paper, as they have been in [46]([Chapter 3]). We refer to Section 1.3 of that paper for a heuristic explanation of this surprising fact.

4.1.2 Formal Derivations

The main intuitions in the proofs of our results are :

- The rotation field being along \vec{e}_θ with positive amplitude, a vortex of negative degree can only create energy. All the vortices should then be of positive degree.
- μ should necessarily have its support on \mathcal{C}_{R_*} because the cost function (4.1.25) is minimum there. Indeed, the cost function gives the short range energetic cost (self-interacting kinetic energy plus interaction with the rotation potential) of a vortex situated at a particular location.
- It is well-known that two vortices of same degree should repel each other while two vortices of opposite degree should attract each other. The vortices, all of positive degrees and located on \mathcal{C}_{R_*} should organize in order to minimize their repulsion. This effect leads to the uniform distribution on the circle.

Let us explain how the renormalized energy (4.1.38) appears in our setting. A large part of the analysis, in particular the introduction of the cost function (4.1.25), has been carried out in [46]([Chapter 3]). A first step is to extract from E^{GP} the contribution of the giant vortex profile and phase factor. This uses a classical method of energy decoupling and an exponential decay property for Ψ^{GP} in $\mathcal{B} \setminus \mathcal{A}$ (see [46, Propositions 2.2 and 3.1])(Propositions 3.2.2 and 3.3.1). We obtain

$$E^{\text{GP}} = \hat{E}_{\mathcal{A}, \omega}^{\text{GP}} + \mathcal{E}[u] + O(\varepsilon^\infty). \quad (4.1.39)$$

The reduced energy functional \mathcal{E} is defined as follows

$$\mathcal{E}[v] := \int_{\mathcal{A}} g^2 |\nabla v|^2 - 2g^2 \vec{B} \cdot (iv, \nabla v) + \frac{g^4}{\varepsilon^2} (1 - |v|^2)^2 \quad (4.1.40)$$

where

$$\vec{B}(r) := B(r) \vec{e}_\theta = (\Omega r - ([\Omega] - \omega) r^{-1}) \vec{e}_\theta \quad (4.1.41)$$

and we have used the notation

$$(iv, \nabla v) := \frac{1}{2}i(v\nabla\bar{v} - \bar{v}\nabla v).$$

The energy $\mathcal{E}[u]$ effectively takes into account the energy added to the giant vortex contribution $\hat{E}_{\mathcal{A},\omega}^{\text{GP}}$ when individual vortices are present in the annular bulk of the condensate. Note in particular that in the case where $v \equiv 1$ the energy is 0. We are thus interested in estimates for $\mathcal{E}[u]$. A strictly negative value of this energy will indicate the presence of vortices. We note that this energy functional is very similar to a functional appearing in [5] where an annular condensate at slow rotation speeds is considered (see also [81, 82]). The major difference is that the domain \mathcal{A} depends on ε in a crucial way : its width tends to zero proportionally to $\varepsilon|\log \varepsilon|$ when $\varepsilon \rightarrow 0$. We refer to [46, Section 1.3](Section 3.1.3) for a more detailed discussion on this point, but stress that even in the fixed domain context of [5] the question of the uniform repartition of vortices along the circle where their energetic cost is minimum was left open.

The study of the energy $\mathcal{E}[u]$ starts with the construction of vortex balls : we isolate the set where u could possibly vanish and cover it with a collection of disjoint balls with relatively small radii. The growth and merging method described e.g. in [135, Chapter 4] yields lower bounds of the form

$$\int_{B_i} g^2 |\nabla u|^2 \geq \pi |d_i| g^2(a_i) |\log \varepsilon| \quad (4.1.42)$$

where $B_i = B_i(a_i, r_i)$ is a generic ball in the collection and d_i the degree of u around the ball B_i . We neglect remainder terms for the sake of clarity. We stress that strictly speaking we do not cover all possible zeroes of u by this method : there is first a layer close to the inner boundary of \mathcal{A} where the density g^2 is too small to use this method. More importantly, the shrinking width of \mathcal{A} makes it difficult to obtain the estimates needed for the implementation of the method. There is thus a region where we have basically no information on the vortices of u . We will neglect this fact in this sketch since it produces only remainder terms in the energy.

The next step is to use a Jacobian Estimate (see [135, Chapter 6]) to obtain

$$\mu \sim 2\pi \sum_i d_i \delta_{a_i} \quad (4.1.43)$$

where δ_{a_i} is the Dirac mass at a_i . We then note that the function F in (4.1.26) is constructed to satisfy

$$\nabla^\perp(F) = 2g^2 \vec{B}, \quad F(R_<) = 0.$$

Thus, integrating by parts the second term in (4.1.40) we have

$$-2 \int_{\mathcal{A}} g^2 \vec{B} \cdot (iv, \nabla v) = \int_{\mathcal{A}} F \mu - \int_{\partial \mathcal{B}} F(iu, \partial_\tau u). \quad (4.1.44)$$

The boundary term has been shown to be negligible in [46]([Chapter 3]). At this stage we thus have essentially, gathering (4.1.42), (4.1.44) and (4.1.43),

$$\int_{\mathcal{A}} g^2 |\nabla u|^2 - 2g^2 \vec{B} \cdot (iu, \nabla u) \geq \sum_i 2\pi \left(\frac{1}{2} |d_i| g^2(a_i) |\log \varepsilon| + d_i F(a_i) \right) + \int_{\mathcal{A} \setminus \cup_i B_i} g^2 |\nabla u|^2. \quad (4.1.45)$$

In [46]([Chapter 3]) the first two terms in the above right-hand side were enough to conclude. Indeed, F is negative in the bulk, thus we can already read on the above formula

that negative degree vortices can only increase the energy. All vortices should then be of positive degree and we can write

$$\int_{\mathcal{A}} g^2 |\nabla u|^2 - 2g^2 \vec{B} \cdot (iu, \nabla u) \geq \sum_i 2\pi d_i H(a_i) + \int_{\mathcal{A} \setminus \cup_i B_i} g^2 |\nabla u|^2 \quad (4.1.46)$$

where H is the (radial) cost function (4.1.25). Above the critical speed, H is positive everywhere in the bulk, thus no vortices can be present in the condensate. By contrast, when the rotation speed is sub-critical, the cost function has a negative minimum at R_* (actually, close to R_* , see Appendix A for the precise definition of R_*). Positive degree vortices are thus favorable in this regime, and they will gather close to the circle \mathcal{C}_{R_*} .

We have now to understand what mechanism limits the nucleation of vortices and leads to their uniform distribution on \mathcal{C}_{R_*} . This is where the third term in the right-hand side (4.1.45) comes into play. In [46] ([Chapter 3]) this term was neglected, but in the present context it is crucial to bound it from below.

Let us introduce the *superfluid current* of u

$$j := (iu, \nabla u). \quad (4.1.47)$$

Note that by definition $\mu = \text{curl}(j)$. The idea is now that the kinetic (first) term in (4.1.40) is essentially due to the presence of this supercurrent. The energetic contribution of the current inside vortex balls is taken into account in (4.1.42). The next step is to estimate the energetic contribution of the part of the current that is located outside vortex balls (this is the long-range energetic cost of the vortices we were alluding to before). To this end we define a modified superfluid current

$$j_{\text{mod}} := \begin{cases} j & \text{in } \mathcal{A} \setminus \cup_i B_i \\ 0 & \text{in } \cup_i B_i. \end{cases} \quad (4.1.48)$$

and the associated vorticity

$$\mu_{\text{mod}} := \text{curl}(j_{\text{mod}}). \quad (4.1.49)$$

It turns out that

$$\mu_{\text{mod}} \sim \mu. \quad (4.1.50)$$

Then, combining (4.1.43) and (4.1.50) we can rewrite (4.1.46) in the form

$$\int_{\mathcal{A}} g^2 |\nabla u|^2 - 2g^2 \vec{B} \cdot (iu, \nabla u) \geq \int_{\mathcal{A}} H \mu_{\text{mod}} + \int_{\mathcal{A} \setminus \cup_i B_i} g^2 |\nabla u|^2. \quad (4.1.51)$$

Let us now estimate the last term in the above expression. By definition of the vortex balls, outside the set $\cup_i B_i$ we have $|u| \sim 1$, thus

$$\int_{\mathcal{A} \setminus \cup_i B_i} g^2 |\nabla u|^2 \sim \int_{\mathcal{A} \setminus \cup_i B_i} g^2 |j|^2 = \int_{\mathcal{A}} g^2 |j_{\text{mod}}|^2.$$

Recalling the equation (4.1.34) satisfied by $h_{\mu_{\text{mod}}}$ we have

$$\text{curl} \left(j_{\text{mod}} + \frac{1}{g^2} \nabla^\perp h_{\mu_{\text{mod}}} \right) = 0 \text{ in } \mathcal{A}. \quad (4.1.52)$$

If we were working in a simply connected domain this would imply that there exists a function f such that

$$j_{\text{mod}} = -\frac{1}{g^2} \nabla^\perp h_{\mu_{\text{mod}}} + \nabla f. \quad (4.1.53)$$

The argument has to be modified because we work on an annulus, but since this does not produce significant modifications of the energy, we assume that (4.1.53) holds. Then, using $h_{\mu_{\text{mod}}} = 0$ on $\partial\mathcal{A}$, we obtain

$$\int_{\mathcal{A} \setminus \cup_i B_i} g^2 |\nabla u|^2 \sim \int_{\mathcal{A}} g^2 |j_{\text{mod}}|^2 \geq \int_{\mathcal{A}} \frac{1}{g^2} |\nabla h_{\mu_{\text{mod}}}|^2. \quad (4.1.54)$$

Gathering (4.1.51) and (4.1.54) we obtain, up to the (many!) remainder terms we have neglected

$$\int_{\mathcal{A}} g^2 |\nabla u|^2 - 2g^2 \vec{B} \cdot (iu, \nabla u) \geq \int_{\mathcal{A}} \frac{1}{g^2} |\nabla h_{\mu_{\text{mod}}}|^2 + \int_{\mathcal{A}} H \mu_{\text{mod}}. \quad (4.1.55)$$

The right-hand side of the above equation is equal to the renormalized energy of μ_{mod} defined in (4.1.38) if this measure has its support on \mathcal{C}_{R_*} . Intuitively this is justified because the vortices of u will want to be close to \mathcal{C}_{R_*} where the cost function is minimum. Mathematically, showing that the right-hand side of (4.1.51) is bounded below by the infimum in (4.1.38) uses essentially two ingredients. One is the rigorous version of the Jacobian Estimate making precise the property (4.1.43), the other is the information that the field associated (via (4.1.34)) to the measure minimizing (4.1.36) is constant on the support of the measure. This is a classical fact in potential theory. In our context this means that $h_{\delta_*} = h_*$ is constant on \mathcal{C}_{R_*} . We refer to Section 4.3.3 for details on the way we use this information to complete the lower bound, and accept for the present sketch that μ_{mod} has support on \mathcal{C}_{R_*} so that (4.1.51) yields the renormalized energy (4.1.38) of μ_{mod} as a lower bound to $\mathcal{E}[u]$. It is then intuitively clear, if we believe not to have lost too much information on the way, that μ_{mod} should ‘almost’ minimize the renormalized energy. Thus it should be close to $-H(R_*)(2I_*)^{-1} \delta_*$ and by (4.1.50) so should μ .

What remains to be proved is that ‘we have not lost too much information on the way’, i.e. we want an upper bound to the energy matching our lower bound. We thus construct a test function displaying the optimal number of vortices evenly distributed on the circle \mathcal{C}_{R_*} . To evaluate the energy of such a test function we adapt the well-established technique used e.g in [5, 18, 20, 21, 23, 135]. It involves in particular a Green representation of the electrostatic energy (4.1.35). In our case the computation is significantly complicated by the fact that the matter density g^2 vanishes at the inner boundary of \mathcal{A} and more importantly by the particular geometry of the annulus (fixed radius and shrinking width). We have to rely on the periodicity (in the angular variable) of our test function to obtain the estimates allowing to conclude that the appropriate upper bound holds true.

The paper is organized as follows. We first construct our trial function and prove our upper bound in Section 4.2. Section 4.3 is then devoted to the proof of the lower bound. We will collect many estimates on the way that will allow us to conclude the proof of Theorem 4.1.1 in Section 4.4. Appendix A is concerned with the analysis of the cost function and contains mainly technical computations. As the reader will see, our proofs use a lot of auxiliary notation. We have found convenient to gather this in Appendix B.

4.2 Energy Upper Bound

In this section we provide the upper bound part of the energy asymptotics of Theorem 4.1.2 by constructing a trial function for \mathcal{E}^{GP} which displays the expected vortex structure. The result we are aiming at is the following :

Proposition 4.2.1 (Upper bound to the energy).

Let \mathcal{N} be a number satisfying

$$\mathcal{N} = -\frac{H(R_*)}{4\pi I_*}(1 + o(1)) \quad (4.2.1)$$

where I_* is defined in (4.1.36) and H is the cost function (4.1.25). We have, for ε small enough

$$\begin{aligned} E^{\text{GP}} &\leq \hat{E}_{\mathcal{A},\omega}^{\text{GP}} + 4\pi^2(\mathcal{N})^2(1 + o(1))I_* + 2\pi\mathcal{N}H(R_*) + O\left(\frac{\mathcal{N} \log |\log \varepsilon|}{\varepsilon |\log \varepsilon|}\right) \\ &= \hat{E}_{\mathcal{A},\omega}^{\text{GP}} - \frac{H(R_*)^2}{4I_*}(1 + o(1)). \end{aligned} \quad (4.2.2)$$

4.2.1 The trial function

We start by dividing \mathcal{A} into $N \propto \frac{1}{\varepsilon |\log \varepsilon|}$ identical cells. The cell \mathcal{A}_i is defined as

$$\mathcal{A}_i = \mathcal{A} \cap \{\theta_i \leq \theta \leq \theta_{i+1}\} \quad (4.2.3)$$

with the convention that $\theta_{N+1} = \theta_1$. We require $\theta_{i+1} - \theta_i$ to be independent from i (that is the cells are identical) and proportional to $\varepsilon |\log \varepsilon|$ (which is the thickness of the annulus \mathcal{A}). We construct a structure of vortices periodic in the angular variable, of period $\theta_{i+1} - \theta_i$.

We introduce a parameter t which will be the size of the vortex cores and M points $p_{i,1}, \dots, p_{i,M}$ in each cell, which will be the locations of the vortices. We require that for any i, j

$$B(p_{i,j}, t) \subset \mathcal{A}_i \quad (4.2.4)$$

and that the collection of points $(p_{i,j})_{i=1\dots N, j=1\dots M}$ is evenly distributed on the circle of radius R_* (which we denote \mathcal{C}_{R_*}). The distance between two adjacent vortices is thus $2\pi R_*(MN)^{-1}$. The number of vortices will be fixed (recall that I_* is of the order of a constant and that $H(R_*)$ is negative and proportional to $\Omega_1 \varepsilon^{-1}$) as

$$\mathcal{N} := MN = -\frac{H(R_*)}{4\pi I_*}(1 + o(1)) \propto \frac{\Omega_1}{\varepsilon} \propto \frac{\Omega_1}{\varepsilon}. \quad (4.2.5)$$

Note that we leave some freedom (in the $o(1)$) in the choice of the number of vortices. This will be required below to ensure that it is possible to construct a vortex configuration having the symmetries we want.

The natural vorticity measure associated with our ring of (degree one) vortices would be a sum of Dirac masses $\sum \delta_{p_{i,j}}$, which would lead to an infinite energy. We thus introduce a regularized vorticity measure

$$f := \sum_{i,j} \frac{2}{t^2} \text{Id}_{B(p_{i,j},t)} \quad (4.2.6)$$

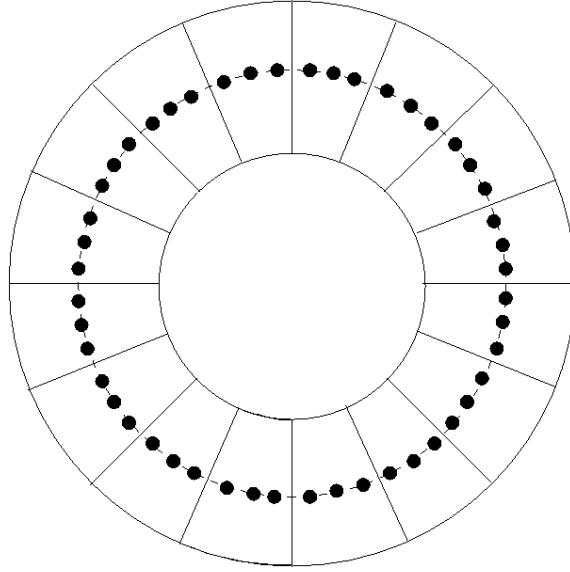


FIGURE 4.1 – Vortex configuration in the trial function.

where $\text{Id}_{B(p_{i,j},t)}$ is the characteristic function of the ball $B(p_{i,j},t)$. The measure f is normalized to have total mass $2\pi MN$ (i.e. 2π times the number of vortices). We require in our construction that this function be invariant by some reflections of the angular variable. More precisely, for any i , we require

$$f(r, \theta) = f(r, 2\theta_i - \theta) \tag{4.2.7}$$

$$f(r, \theta) = f(r, \theta_i + \theta_{i+1} - \theta). \tag{4.2.8}$$

Clearly this conditions imply the periodicity of f :

$$f_i(r, \theta) := f_{|\mathcal{A}_i}(r, \theta) = f_{|\mathcal{A}_j}(r, \theta + \theta_i - \theta_j). \tag{4.2.9}$$

Our trial function is given by

$$\Psi = \begin{cases} ge^{i([\Omega]-\omega)\theta}v & \text{in } \mathcal{A} \\ 0 & \text{in } \mathcal{B} \setminus \mathcal{A} \end{cases} \tag{4.2.10}$$

with

$$v = c\xi e^{i\phi}. \tag{4.2.11}$$

The real function ξ is a cut-off function ensuring that

- v vanishes in small discs (of radius t) around the vortex locations, where the phase ϕ will have a singularity
- v vanishes continuously at the inner boundary of \mathcal{A} , so that Ψ is indeed in the energy space.

The constant c is chosen so that $\int_{\mathcal{B}} |\Psi|^2 = 1$. As for the phase ϕ , we define it in Lemma 4.2.2 in such a way that v has degree one around each vortex in the collection.

Let us introduce functions $\xi_{i,j}$, $i = 1 \dots N$, $j = 1 \dots M$ satisfying

$$\xi_{i,j} := \begin{cases} 0 & \text{in } B(p_{i,j}, t) \\ 1 & \text{in } \mathcal{A} \setminus B(p_{i,j}, 2t) \end{cases} \quad (4.2.12)$$

and

$$0 \leq \xi_{i,j} \leq 1.$$

Obviously one has

$$\nabla \xi_{i,j} = 0 \text{ in } B(p_{i,j}, t) \cup B(p_{i,j}, 2t)^c \quad (4.2.13)$$

and we can impose

$$|\nabla \xi_{i,j}| \leq \frac{C}{t} \text{ in } B(p_{i,j}, 2t) \setminus B(p_{i,j}, t). \quad (4.2.14)$$

Next we define, for some radius \tilde{R} larger than $R_{<}$,

$$\xi_{\text{BL}}(\vec{r}) = \begin{cases} 1 & \text{if } r \geq \tilde{R} \\ 1 + \frac{r-R_{<}}{\tilde{R}-R_{<}} & \text{if } r \leq \tilde{R}. \end{cases} \quad (4.2.15)$$

The subscript ‘‘BL’’ stands for boundary layer and is justified because our choice of \tilde{R} will satisfy

$$\tilde{R} = R_{<} + \varepsilon^n < R_h - \frac{1}{2}\varepsilon^{8/7} \quad (4.2.16)$$

for some large power n (say $n = 10000$ for definiteness). The inequality above is a consequence of (4.1.19) and allows one to use the exponential smallness of [46, Equation (2.29)](Equation (3.2.29)) in the region $R_{<} \leq r \leq \tilde{R}$.

The final cut-off is given as

$$\xi = \xi_{\text{BL}} \prod_{i,j} \xi_{i,j}. \quad (4.2.17)$$

The choice of ξ_{BL} vanishing continuously at $r = R_{<}$ ensures that $\Psi \in H^1(\mathcal{B})$. We now choose

$$t = \varepsilon^{3/2} |\log \varepsilon|^{1/2}. \quad (4.2.18)$$

This is the optimal choice and allows to estimate the L^2 norm of Ψ : it is easily seen that for any i and any j

$$\int_{B(p_{i,j}, t)} g^2 = O(\varepsilon^2),$$

because $g^2 \leq C(\varepsilon |\log \varepsilon|)^{-1}$ (see [46]([Chapter 3])). Moreover, using the exponential smallness result [46, Equation (2.29)](Equation (3.2.29)) one can see that

$$\int_{R_{<} \leq r \leq \tilde{R}} g^2 = O(\varepsilon^\infty)$$

with n large enough in (4.2.16). With these two estimates in hand, using the normalization of g^2 we conclude that

$$\int_{\mathcal{A}} g^2 \xi^2 \geq 1 - CMN \varepsilon^2. \quad (4.2.19)$$

On the other hand

$$\int_{\mathcal{A}} g^2 \xi^2 \leq \int_{\mathcal{A}} g^2 = 1$$

and thus one can normalize Ψ by taking a constant c satisfying

$$c^2 = 1 + O(MN\varepsilon^2). \quad (4.2.20)$$

We now turn to the definition of the phase of v . The following modified density ρ will be used in the definition :

$$\rho(r) = \begin{cases} \rho^{\text{TF}}(r) & \text{if } r \geq \bar{R} \\ g^2(r) & \text{if } r < \bar{R} \end{cases} \quad (4.2.21)$$

where

$$R_{<} < \bar{R}$$

with the actual choice specified in Lemma 4.2.1 below. This function is constructed to satisfy two properties that we shall need in the sequel. First its gradient has to be bounded by $C(\varepsilon|\log \varepsilon|)^{-2}$ for $r \geq \bar{R}_+$, which is insured by the explicit form of ρ^{TF} . This is the main reason why we do not use simply g^2 , the bound that is available on the gradient of this function [46, Proposition 2.7](Proposition 3.2.7) is not sufficient for our purpose. Second ρ has to stay close to g^2 in L^∞ norm, which is insured by the following

Lemma 4.2.1 (Properties of ρ).

Let us define

$$\bar{R} := R_h + \varepsilon^{5/6}. \quad (4.2.22)$$

Then for any $r \in \mathcal{A}$

$$|g^2(r) - \rho(r)| \leq C\varepsilon^{3/4} |\log \varepsilon|^2 \rho(r). \quad (4.2.23)$$

Moreover ρ is bounded below by a positive constant in \mathcal{A} (the constant depends on ε).

Proof. The estimate is of course trivial when $r \leq \bar{R}$. When $r \geq \bar{R}$ it is a consequence of Proposition 2.6 in [46]([Chapter 3]). The fact that ρ is bounded below is a consequence of the corresponding result for g^2 . Indeed, g satisfies the equation

$$-\Delta g + \frac{([\Omega] - \omega)^2}{r^2} g - 2\Omega([\Omega] - \omega)g + 2\varepsilon^{-2}g^3 = \hat{\mu}^{\text{GP}}g, \quad (4.2.24)$$

with Neumann boundary conditions on $\partial\mathcal{A}$, where $\hat{\mu}^{\text{GP}}$ is the Lagrange multiplier associated with the mass constraint. It is well known that such a function, ground-state of a one-dimensional Schrödinger operator, cannot vanish except at the origin (see e.g. [96, Theorem 11.8]). A possible proof is through the Harnack inequality [115]. \square

Note that ρ has a jump discontinuity. We could have constructed a regular function instead, but since the discontinuity has no consequence in the sequel, we stick to the simple definition (4.2.21).

We now define \bar{h}_f as the unique solution of

$$\begin{cases} -\nabla \cdot \left(\frac{1}{\rho} \nabla \bar{h}_f \right) = f & \text{in } \mathcal{A} \\ \bar{h}_f = 0 & \text{on } \partial\mathcal{A}. \end{cases} \quad (4.2.25)$$

Note that the elliptic operators appearing in (4.1.34) and (4.2.25) are similar up to the replacement of the weight g^{-2} by ρ . Thanks to Lemma 4.2.1 these two operators are close in some sense. We denote

$$\kappa = \int_{\partial\mathcal{B}} \frac{1}{\rho} \frac{\partial \bar{h}_f}{\partial n} - 2\pi \left[\frac{1}{2\pi} \int_{\partial\mathcal{B}} \frac{1}{\rho} \frac{\partial \bar{h}_f}{\partial n} \right]. \quad (4.2.26)$$

Also we introduce Γ as the solution of

$$\begin{cases} -\nabla \left(\frac{1}{\rho} \nabla \Gamma \right) = 0 \text{ in } \mathcal{A} \\ \Gamma = 1 \text{ on } \partial\mathcal{B} \\ \Gamma = 0 \text{ on } \partial B_{R_{<}}. \end{cases} \quad (4.2.27)$$

Note that Γ is radial and has the explicit expression

$$\Gamma(\vec{r}) = \frac{\int_{R_{<}}^r \rho(s) s^{-1} ds}{\int_{R_{<}}^1 \rho(s) s^{-1} ds}. \quad (4.2.28)$$

The denominator in the above equation is a $O(1)$ thanks to (4.2.23) and the normalization of g^2 .

We prove the following lemma, which defines the phase of our trial function.

Lemma 4.2.2 (Phase of the trial function).

The formula

$$\nabla \phi = \frac{1}{\rho} \nabla^\perp \bar{h}_f \quad (4.2.29)$$

where

$$\bar{h}_f = \bar{h}_f - \frac{\kappa}{\int_{\mathcal{A}} \frac{1}{\rho} |\nabla \Gamma|^2} \Gamma \quad (4.2.30)$$

defines a phase, i.e. $e^{i\phi}$ is well-defined, in $\mathcal{A} \setminus \cup_{i,j} B(p_{i,j}, t)$. Moreover

$$\int_{\mathcal{A}} \rho \xi^2 |\nabla \phi|^2 \leq \int_{\mathcal{A}} \frac{1}{\rho} |\nabla \bar{h}_f|^2 \leq \int_{\mathcal{A}} \frac{1}{\rho} |\nabla \bar{h}_f|^2 + O(1). \quad (4.2.31)$$

Proof. By definition of \bar{h}_f and Γ the right-hand side of (4.2.29) is irrotational in $\mathcal{A} \setminus \cup_{i,j} B(p_{i,j}, t)$, thus ϕ is well-defined locally in this set. To see that $e^{i\phi}$ is well defined, we must check that for any closed curve σ included in $\mathcal{A} \setminus \cup_{i,j} B(p_{i,j}, t)$

$$\int_{\sigma} \nabla \phi \cdot \tau \in 2\pi\mathbb{Z}. \quad (4.2.32)$$

Clearly, it is sufficient to consider two cases :

1. σ winds around at most one ball $B(p_{i,j}, t)$
2. σ is a contour enclosing all the balls

and one deduces the general case from these two.

In case 1, (4.2.32) is a simple consequence of the quantification of the mass of f . One shows easily that (4.2.32) is satisfied using (4.2.25), (4.2.27) and integrations by parts. In case 2, equations (4.2.25) and (4.2.27) give

$$\int_{\sigma} \nabla \phi \cdot \tau = \int_{\partial\mathcal{B}} \left(\frac{1}{\rho} \frac{\partial \bar{h}_f}{\partial n} - \frac{\kappa}{\int_{\mathcal{A}} \frac{1}{\rho} |\nabla \Gamma|^2} \frac{1}{\rho} \frac{\partial \Gamma}{\partial n} \right).$$

But, using again (4.2.27)

$$\int_{\partial\mathcal{B}} \frac{1}{\rho} \frac{\partial\Gamma}{\partial n} = \int_{\partial\mathcal{B}} \frac{1}{\rho} \Gamma \frac{\partial\Gamma}{\partial n} = \int_{\mathcal{A}} \frac{1}{\rho} |\nabla\Gamma|^2$$

and thus it is clear from (4.2.29) that

$$\int_{\partial\mathcal{B}} \left(\frac{1}{\rho} \frac{\partial\bar{h}_f}{\partial n} - \frac{\kappa}{\int_{\mathcal{A}} \frac{1}{\rho} |\nabla\Gamma|^2} \frac{1}{\rho} \frac{\partial\Gamma}{\partial n} \right) \in 2\pi\mathbb{Z}.$$

To prove (4.2.31) one remarks that

$$\int_{\mathcal{A}} \rho \xi^2 |\nabla\phi|^2 \leq \int_{\mathcal{A}} \frac{1}{\rho} |\nabla\bar{h}_f|^2 = \int_{\mathcal{A}} \frac{1}{\rho} \left| \nabla\bar{h}_f - \frac{\kappa}{\int_{\mathcal{A}} \frac{1}{\rho} |\nabla\Gamma|^2} \nabla\Gamma \right|^2.$$

Then, using the equation for Γ and the boundary condition on \bar{h}_f

$$\int_{\mathcal{A}} \frac{1}{\rho} \nabla\bar{h}_f \cdot \nabla\Gamma = 0.$$

Finally one uses that $\kappa = O(1)$ and

$$\int_{\mathcal{A}} \frac{1}{\rho} |\nabla\Gamma|^2 \geq C \tag{4.2.33}$$

which follows easily from (4.2.28). □

We can now start the computation of the energy of our trial function :

Proof of Proposition 4.2.1. A classical technique of energy decoupling (see e.g. the proof of Proposition 3.1 in [46]([Chapter 3])) uses the variational equation satisfied by g to show that

$$\mathcal{E}^{\text{GP}}[\Psi] = \hat{E}_{\mathcal{A},\omega}^{\text{GP}} + \mathcal{E}[v] \tag{4.2.34}$$

where

$$\mathcal{E}[v] := \int_{\mathcal{A}} g^2 |\nabla v|^2 - 2g^2 \vec{B} \cdot (iv, \nabla v) + \frac{g^4}{\varepsilon^2} (1 - |v|^2)^2 \tag{4.2.35}$$

and

$$\vec{B}(r) := B(r)\vec{e}_\theta = (\Omega r - ([\Omega] - \omega) r^{-1}) \vec{e}_\theta. \tag{4.2.36}$$

We have also used the notation

$$(iv, \nabla v) := \frac{1}{2}i(v\nabla\bar{v} - \bar{v}\nabla v).$$

We now have to estimate energy $\mathcal{E}[v]$. We first note that this energy can be decomposed into the contribution of the density profile ξ and the contribution of the phase, i.e. the energy generated by the vortices :

$$\mathcal{E}[v] = c^2 \int_{\mathcal{A}} \left(g^2 |\nabla\xi|^2 + \frac{g^4}{\varepsilon^2} (1 - c^2\xi^2)^2 \right) + \int_{\mathcal{A}} \left(c^2 g^2 \xi^2 |\nabla\phi|^2 - g^2 \vec{B} \cdot (iv, \nabla v) \right). \tag{4.2.37}$$

The contribution of the profile (first term in the above equation) is readily estimated by using the definition of ξ . We separate the boundary layer $\{r \leq \tilde{R}\}$ where $\xi = \xi_{\text{BL}}$ from the bulk where $\xi = \prod_{i,j} \xi_{i,j}$:

$$\int_{\mathcal{A}} \left(g^2 |\nabla \xi|^2 + \frac{g^4}{\varepsilon^2} (1 - c^2 \xi^2)^2 \right) = \int_{\mathcal{A} \cap \{r \leq \tilde{R}\}} \left(g^2 |\nabla \xi|^2 + \frac{g^4}{\varepsilon^2} (1 - c^2 \xi^2)^2 \right) + \int_{\mathcal{A} \cap \{r \geq \tilde{R}\}} \left(g^2 |\nabla \xi|^2 + \frac{g^4}{\varepsilon^2} (1 - c^2 \xi^2)^2 \right).$$

In the boundary layer we use the exponential smallness of g proved in [46, Equation (2.29)](Equation (3.2.29)) and the definition of ξ_{BL} (4.2.15) to obtain

$$\int_{\mathcal{A} \cap \{r \leq \tilde{R}\}} \left(g^2 |\nabla \xi|^2 + \frac{g^4}{\varepsilon^2} (1 - c^2 \xi^2)^2 \right) = O(\varepsilon^\infty).$$

For the bulk term we observe that (4.2.13) and (4.2.14) imply

$$\int_{\mathcal{A} \cap \{r \geq \tilde{R}\}} g^2 |\nabla \xi|^2 \leq C \frac{MN}{\varepsilon |\log \varepsilon|}.$$

We then write

$$\begin{aligned} \int_{\mathcal{A} \cap \{r \geq \tilde{R}\}} \frac{g^4}{\varepsilon^2} (1 - c^2 \xi^2)^2 &= \int_{\mathcal{A} \cap \{r \geq \tilde{R}\}} \frac{g^4}{\varepsilon^2} (1 - \xi^2)^2 \\ &+ (c^2 - 1) \left(\frac{c^2 + 1}{\varepsilon^2} \int_{\mathcal{A} \cap \{r \geq \tilde{R}\}} g^4 \xi^4 - \frac{1}{\varepsilon^2} \int_{\mathcal{A} \cap \{r \geq \tilde{R}\}} g^4 \xi^2 \right) \end{aligned} \quad (4.2.38)$$

We remark that $\xi = 1$ outside of $\cup_{i,j} B(p_{i,j}, 2t)$, thus

$$\int_{\mathcal{A} \cap \{r \geq \tilde{R}\}} \frac{g^4}{\varepsilon^2} (1 - \xi^2)^2 \leq C \frac{MN}{\varepsilon |\log \varepsilon|}$$

using (4.2.18) and $g^2 \leq C(\varepsilon |\log \varepsilon|^{-1})$. For the second term in the right-hand side of (4.2.38) it is sufficient to recall that $\xi \leq 1$, $|\mathcal{A}| \propto \varepsilon |\log \varepsilon|$ and use (4.2.20) to obtain

$$(c^2 - 1) \left(\frac{c^2 + 1}{\varepsilon^2} \int_{\mathcal{A} \cap \{r \geq \tilde{R}\}} g^4 \xi^4 - \frac{1}{\varepsilon^2} \int_{\mathcal{A} \cap \{r \geq \tilde{R}\}} g^4 \xi^2 \right) \leq C \frac{MN}{\varepsilon |\log \varepsilon|}$$

We can thus conclude that (using again (4.2.20))

$$\mathcal{E}[v] \leq (1 + O(MN\varepsilon^2)) \int_{\mathcal{A}} \left(g^2 \xi^2 |\nabla \phi|^2 - g^2 \vec{B} \cdot (iv, \nabla v) \right) + C \frac{MN}{\varepsilon |\log \varepsilon|}. \quad (4.2.39)$$

We thus need to compute

$$\int_{\mathcal{A}} g^2 \xi^2 |\nabla \phi|^2 \quad (4.2.40)$$

which is the term taking into account the kinetic energy of the vortices, and

$$- \int_{\mathcal{A}} g^2 \vec{B} \cdot (iv, \nabla v) \quad (4.2.41)$$

which represents the interaction of the vortices with the rotation field. These two terms will be estimated in the next two subsections respectively. To conclude the proof it is sufficient to gather (4.2.34), (4.2.39), (4.2.42) and the results of Proposition 4.2.2 and 4.2.3 below. \square

4.2.2 Kinetic energy of the vortices

We first relate (4.2.40) to the energy of \bar{h}_f : using (4.2.23), $|\xi| \leq 1$ and (4.2.31) we have

$$\begin{aligned} \int_{\mathcal{A}} g^2 \xi^2 |\nabla \phi|^2 &= \left(1 + O(\varepsilon^{3/4} |\log \varepsilon|^2)\right) \int_{\mathcal{A}} \rho \xi^2 |\nabla \phi|^2 \\ &\leq \left(1 + O(\varepsilon^{3/4} |\log \varepsilon|^2)\right) \int_{\mathcal{A}} \frac{1}{\rho} |\nabla \bar{h}_f|^2 + O(1). \end{aligned} \quad (4.2.42)$$

The required estimate is then given in the following proposition

Proposition 4.2.2 (Kinetic energy generated by the vortices).

Let \mathcal{N} be as in (4.2.5) and I_* defined by (4.1.36). For ε small enough there holds

$$\int_{\mathcal{A}} \frac{1}{\rho} |\nabla \bar{h}_f|^2 \leq 4\pi^2 (\mathcal{N})^2 (1 + o(1)) I_* + \pi \mathcal{N} g^2(R_*) |\log \varepsilon| + O\left(\frac{\mathcal{N} \log |\log \varepsilon|}{\varepsilon |\log \varepsilon|}\right). \quad (4.2.43)$$

Note that the third term in the right-hand side of the above equation is always much smaller than the second one because $g^2(R_*) \propto (\varepsilon |\log \varepsilon|)^{-1}$. It is also a remainder compared to the first one because of the specific choice (4.2.5).

The proof requires several constructions. The term we want to compute can be estimated using a Green function defined as follows

$$\begin{cases} -\nabla_x \left(\frac{1}{\rho(x)} \nabla_x G(x, y)\right) = \delta_y(x) \text{ for } x \in \mathcal{A} \\ G(x, y) = 0 \text{ for } x \in \partial \mathcal{A}. \end{cases} \quad (4.2.44)$$

The existence and symmetry ($G(x, y) = G(y, x)$) of such a function is classical, because at fixed ε , ρ is bounded above and below in \mathcal{A} (see Lemma 4.2.1). Then, using the Green representation of \bar{h}_f

$$\int_{\mathcal{A}} \frac{1}{\rho} |\nabla \bar{h}_f|^2 = \int_{x \in \mathcal{A}} \int_{y \in \mathcal{A}} G(x, y) f(x) dx f(y) dy. \quad (4.2.45)$$

Before going to the technical implementation, let us explain the ideas behind the computation. We have

$$\int_{x \in \mathcal{A}} \int_{y \in \mathcal{A}} G(x, y) f(x) dx f(y) dy = \sum_{i,j,k,l} \iint_{B(p_{i,j},t) \times B(p_{k,l},t)} G(x, y) \frac{4 dx dy}{t^4}. \quad (4.2.46)$$

As is well-known, the Green function has a logarithmic singularity at $x = y$. More precisely, for x close to y we expect the behavior (see Lemma 4.2.4 where this is proved for slightly different Green functions)

$$G(x, y) \sim \frac{\rho(y)}{2\pi} \log \left(\frac{\varepsilon |\log \varepsilon|}{|x - y|} \right) \sim \frac{g^2(y)}{2\pi} \log \left(\frac{\varepsilon |\log \varepsilon|}{|x - y|} \right).$$

The second estimate is a consequence of Lemma 4.2.1. We replace G by the above expression in the diagonal terms (i.e. $i = j$ and $k = l$) of the sum (4.2.46). A direct computation yields the second term in (4.2.43). For the off-diagonal terms we simply note that G is regular when x is not too close to y , in particular when $x \in B(p_{i,j}, t)$ and $y \in B(p_{k,l}, t)$

with $i \neq k$ or $j \neq l$. Then the off-diagonal sum can be seen as a Riemann sum and yields to leading order

$$\begin{aligned} \sum_{(i,j) \neq (k,l)} \iint_{B(p_{i,j},t) \times B(p_{k,l},t)} G(x,y) \frac{4dx dy}{t^4} &\sim \sum_{(i,j) \neq (k,l)} G(p_{i,j}, p_{k,l}) \\ &\sim (4\pi\mathcal{N})^2 \iint_{\mathcal{A} \times \mathcal{A}} G(x,y) \delta_*(dx) \delta_*(dy) = (4\pi\mathcal{N})^2 \int_{\mathcal{A}} \frac{1}{\rho} |\nabla \bar{h}_{\delta_*}|^2. \end{aligned}$$

There remains to note that because $\rho \sim g^2$ (Cf Lemma 4.2.1)

$$\int_{\mathcal{A}} \frac{1}{\rho} |\nabla \bar{h}_{\delta_*}|^2 \sim \int_{\mathcal{A}} \frac{1}{g^2} |\nabla h_{\delta_*}|^2 \sim I_*.$$

For this heuristic presentation we have deviated from the actual proof procedure that we present below.

We note that several authors (see e.g. [5, 18, 20, 21, 23, 135]) have already successfully used the representation (4.2.45) for the computation of similar quantities. However, in our case, the particular geometry of \mathcal{A} (fixed radius but shrinking width) makes it difficult to obtain the properties of G required in the computation. We thus prefer to take advantage of the symmetry of the vorticity measure f to obtain another expression of (4.2.45).

Let us introduce for any i the function \bar{h}_f^i defined on the cell \mathcal{A}_i as follows

$$\begin{cases} -\nabla \left(\frac{1}{\rho} \nabla \bar{h}_f^i \right) = f_i \text{ in } \mathcal{A}_i \\ \bar{h}_f^i = 0 \text{ on } \partial \mathcal{A}_i \cap \partial \mathcal{A} \\ \frac{1}{\rho} \frac{\partial \bar{h}_f^i}{\partial n} = 0 \text{ on } \partial \mathcal{A}_i \setminus \partial \mathcal{A}. \end{cases} \quad (4.2.47)$$

See (4.2.9) for the definition of f_i . Note that we impose Dirichlet conditions only on the azimuthal part of the boundary of \mathcal{A}_i (that is, the part that coincides with the boundary of \mathcal{A}), while the behavior on the radial parts of the boundary is left free, which leads to the Neumann condition in (4.2.47). Because of the symmetry of f , the following holds

Lemma 4.2.3 (Alternate definition of \bar{h}_f).

We have

$$\bar{h}_f = \sum_i \bar{h}_f^i \text{ on } \mathcal{A} \quad (4.2.48)$$

i.e. for any $i \in \{1, \dots, N\}$

$$\bar{h}_f = \bar{h}_f^i \text{ on } \mathcal{A}_i.$$

As a consequence

$$\int_{\mathcal{A}} \frac{1}{\rho} |\nabla \bar{h}_f|^2 = \sum_i \int_{\mathcal{A}_i} \frac{1}{\rho} |\nabla \bar{h}_f^i|^2. \quad (4.2.49)$$

Proof. Using (4.2.8) and the fact that ρ is radial, we obtain that \bar{h}_f^i is symmetric with respect to a reflection of the angular variable, that is

$$\bar{h}_f^i(r, \theta) = \bar{h}_f^i(r, \theta_i + \theta_{i+1} - \theta). \quad (4.2.50)$$

On the other hand, using (4.2.9) we have also

$$\bar{h}_f^i(r, \theta) = \bar{h}_f^j(r, \theta + \theta_i - \theta_j). \quad (4.2.51)$$

Using the last two properties, it is clear that $\sum_i \bar{h}_f^i$ is well-defined, continuous on \mathcal{A} and satisfies

$$\int_{\mathcal{A}} \left| \nabla \left(\sum_i \bar{h}_f^i \right) \right|^2 < +\infty.$$

A simple computation shows that

$$-\nabla \left(\frac{1}{\rho} \nabla \left(\sum_i \bar{h}_f^i \right) \right) = f$$

on \mathcal{A} , in the sense of distributions. The conclusion follows by uniqueness of the solution to the elliptic problem (4.2.25). \square

The point of using this expression of \bar{h}_f is that we are now reduced to the computation of the terms in the right-hand side of (4.2.49). We introduce new Green functions, defined on each cell

$$\begin{cases} -\nabla_x \left(\frac{1}{\rho(x)} \nabla_x G_i(x, y) \right) = \delta_y(x) \text{ for } x \in \mathcal{A}_i \\ G_i(x, y) = 0 \text{ for } x \in \partial \mathcal{A}_i \cap \partial \mathcal{A} \\ \frac{1}{\rho} \frac{\partial G_i(x, y)}{\partial n} = 0 \text{ for } x \in \partial \mathcal{A}_i \setminus \partial \mathcal{A}. \end{cases} \quad (4.2.52)$$

Then, using the Green representation of \bar{h}_f^i

$$\int_{\mathcal{A}_i} \frac{1}{\rho} |\nabla \bar{h}_f^i|^2 = \int_{x \in \mathcal{A}_i} \int_{y \in \mathcal{A}_i} G_i(x, y) f(x) dx f(y) dy. \quad (4.2.53)$$

Now we are reduced to a computation on each cell. What makes the computation easier compared to what it would have been using (4.2.45) is that each cell can be mapped by a dilation of the variables to a fixed domain. The difficulty of the shrinking width of \mathcal{A} is thus avoided.

Note also that instead of computing in (4.2.45) the interaction through the Green potential G of each pair of vortices in our collection, we can now simply compute the interaction through G_i of each pair of vortices in \mathcal{A}_i , and sum the contributions of each cell. Thus each vortex effectively interacts only with $M \propto \Omega_1 |\log \varepsilon|$ vortices through a modified Green function instead of $NM \propto \Omega_1 \varepsilon^{-1}$ vortices through the original Green function.

We now prove useful properties of the Green functions defined in (4.2.52). It is mostly here that our choice of ρ will prove useful, in particular the fact that the gradient of this function is properly bounded above close to \mathcal{C}_{R_*} . Since f has its support close to the circle of radius R_* we are mainly concerned with properties of G_i in that region for the computation of (4.2.53).

Lemma 4.2.4 (Properties of the Green functions).

Let G_i be the Green function defined in (4.2.52). Let C_G be some constant, small enough and independent of ε . There holds

1. for any $y \in \mathcal{A}_i$, $G(\cdot, y) \in W^{1,p}(\mathcal{A}_i)$ for any $p < 2$
2. $G_i(x, y) \geq 0$ for any $(x, y) \in \mathcal{A}_i \times \mathcal{A}_i \setminus \{x = y\}$
3. G_i is symmetric, i.e. $G_i(x, y) = G_i(y, x)$

4. Let $y \in \mathcal{A}_i$ be such that $||y| - R_*| \leq C_G \varepsilon |\log \varepsilon|$. For any compact set $K \subset \subset \mathcal{A}_i$ satisfying

$$|K| \leq C_G \varepsilon |\log \varepsilon|, \quad (4.2.54)$$

and $y \in K$, there exists a constant C_K such that, for any $x \in K$

$$\left| G_i(x, y) + \frac{\rho(y)}{2\pi} \log \left(\frac{|x - y|}{\varepsilon |\log \varepsilon|} \right) \right| \leq C_K \frac{1}{\varepsilon |\log \varepsilon|}. \quad (4.2.55)$$

5. for any $y \in \mathcal{A}_i$, $G_i(x, y)$ is continuous, uniformly in ε , on

$$\mathcal{A}_i \cap \{|x| - R_*| \leq C_G \varepsilon |\log \varepsilon|\} \cap \{|x - y| \geq C_G \varepsilon |\log \varepsilon|\}.$$

Before giving the proof of this result we stress that we are going to use some results of [143] (see also [104]), where the main emphasis is on homogeneous Dirichlet boundary conditions. However, a careful inspection of the proofs shows that the results apply as well to our case where the boundary conditions are mixed (Neumann/Dirichlet), as long as we are only concerned with interior estimates, as we are.

Proof. The first three properties are classical, because the operator we consider is uniformly elliptic in \mathcal{A}_i (note however that the ellipticity is not uniform with respect to ε). To prove items 4 and 5, we introduce a dilation of the variables

$$x = \varepsilon |\log \varepsilon| \tilde{x}, \quad y = \varepsilon |\log \varepsilon| \tilde{y} \quad (4.2.56)$$

mapping \mathcal{A}_i to $(\varepsilon |\log \varepsilon|)^{-1} \mathcal{A}_i$, which is a domain of fixed size. We define

$$\tilde{\rho}(\tilde{x}) := \varepsilon |\log \varepsilon| \rho(x) \quad (4.2.57)$$

$$\tilde{G}(\tilde{x}, \tilde{y}) := \varepsilon |\log \varepsilon| G_i(x, y) \quad (4.2.58)$$

$$w(\tilde{x}, \tilde{y}) := \tilde{G}(\tilde{x}, \tilde{y}) + \frac{\tilde{\rho}(\tilde{y})}{2\pi} \log |\tilde{x} - \tilde{y}|. \quad (4.2.59)$$

We have from (4.2.52)

$$-\nabla_{\tilde{x}} \left(\frac{1}{\tilde{\rho}(\tilde{x})} \nabla_{\tilde{x}} \tilde{G}(\tilde{x}, \tilde{y}) \right) = \delta_{\tilde{y}}(\tilde{x}) \quad (4.2.60)$$

and

$$-\nabla_{\tilde{x}} \left(\frac{1}{\tilde{\rho}(\tilde{x})} \nabla_{\tilde{x}} w(\tilde{x}, \tilde{y}) \right) = \frac{\tilde{\rho}(\tilde{y}) \nabla_{\tilde{x}} \tilde{\rho}(\tilde{x})}{2\pi \tilde{\rho}^2(\tilde{x})} \nabla_{\tilde{x}} (\log |\tilde{x} - \tilde{y}|) := F(\tilde{x}, \tilde{y}) \quad (4.2.61)$$

on $(\varepsilon |\log \varepsilon|)^{-1} \mathcal{A}_i$ in the sense of distributions. We now proceed as in [18, Lemma 3.1] : Let $y \in \mathcal{A}_i$ be such that $||y| - R_*| \leq C_G \varepsilon |\log \varepsilon|$ for a small enough constant C_G and K be a compact set included in \mathcal{A}_i satisfying (4.2.54) and $y \in K$. We have, for any $x \in K$

$$||x| - R_*| \leq C_G \varepsilon |\log \varepsilon| \quad (4.2.62)$$

for a small enough constant C_G . Let \tilde{K} be the image of K under the dilation of variables (4.2.56). Using Theorem 2 of [112], there exists $p > 2$, $p' < 2$ and a constant C_K such that (here we fix \tilde{y} and consider the functions as depending only on \tilde{x})

$$\|\nabla w\|_{L^p(\tilde{K})} \leq C_K \left(\|\nabla w\|_{L^{p'}(\tilde{K})} + \|F\|_{W^{-1,p}(\tilde{K})} \right). \quad (4.2.63)$$

Using a Sobolev embedding, we can take some $1 < q < 2$ such that

$$\|F\|_{W^{-1,p}(\tilde{K})} \leq C \|F\|_{L^q(\tilde{K})}. \quad (4.2.64)$$

But, using the definition of ρ and the explicit expression of ρ^{TF} we see that as long as x satisfies (4.2.62) one has $\rho(x) \geq C(\varepsilon|\log \varepsilon|)^{-1}$ and thus

$$\frac{\tilde{\rho}(\tilde{y})}{\tilde{\rho}^2(\tilde{x})} \leq C$$

because $\rho(y) \leq C(\varepsilon|\log \varepsilon|)^{-1}$ for any $y \in \mathcal{A}$. Also, if C_G is small enough, $\rho(x) = \rho^{\text{TF}}(x)$ for any $x \in K$, and thus

$$|\nabla \rho(x)| \leq \frac{C}{\varepsilon^2 |\log \varepsilon|^2}$$

on K , which implies that on the domain we are considering

$$\frac{\tilde{\rho}(\tilde{y}) \nabla_{\tilde{x}} \tilde{\rho}(\tilde{x})}{2\pi \tilde{\rho}^2(\tilde{x})} \leq C.$$

Hence, for any $q < 2$

$$\|F\|_{L^q(\tilde{K})} \leq C. \quad (4.2.65)$$

On the other hand, using Theorems 9.1 of [143] (note that the value of the constant in the right-hand side of (9.6) in [143] is given by Theorem 4.2 and 4.3 of the same paper) to estimate the $W^{1,p'}$ norm of \tilde{G} we have, for any $p' < 2$

$$\|\nabla w\|_{L^{p'}(\tilde{K})} \leq \|w\|_{W^{1,p'}(\tilde{K})} \leq C |(\varepsilon|\log \varepsilon|)^{-1} \mathcal{A}_i|^{1/2-1/p'} \int_{(\varepsilon|\log \varepsilon|)^{-1} \mathcal{A}_i} \delta_{\tilde{y}} \leq C \quad (4.2.66)$$

because $(\varepsilon|\log \varepsilon|)^{-1} \mathcal{A}_i$ has a fixed size. Plugging (4.2.64), (4.2.65) and (4.2.66) in (4.2.63) and using a Sobolev embedding (recall that $p > 2$) we obtain

$$\|w\|_{L^\infty(\tilde{K})} \leq C \|w\|_{W^{1,p}(\tilde{K})} \leq C. \quad (4.2.67)$$

There only remains to change variables to conclude that (4.2.55) holds.

To prove Item 5 it is sufficient to fix $\tilde{y} \in (\varepsilon|\log \varepsilon|)^{-1} \mathcal{A}_i$ and show that $\tilde{G}(\tilde{x}, \tilde{y})$ is continuous on

$$(\varepsilon|\log \varepsilon|)^{-1} \mathcal{A}_i \cap \{|\tilde{x}| - (\varepsilon|\log \varepsilon|)^{-1} R_* \leq C_G\} \cap \{|\tilde{x} - \tilde{y}| \geq C_G\}.$$

This follows from Theorem 7.1 of [143]. Note that the constant in the right-hand side of (7.5) in that paper does not depend on the domain, so that the continuity is indeed uniform with respect to ε . \square

We are now able to present the proof of Proposition 4.2.2. With the above constructions in hand there is now no difficulty to adapt a method which has already been used several times in the literature [5, 18, 20, 21, 23, 135], we will thus be a little sketchy in places.

Proof of Proposition 4.2.2. Step 1. Clearly, from (4.2.51) we have for any $i, j \in \{1, \dots, N\}$

$$\int_{\mathcal{A}_i} \frac{1}{\rho} |\nabla \bar{h}_f^i|^2 = \int_{\mathcal{A}_j} \frac{1}{\rho} |\nabla \bar{h}_f^j|^2.$$

It is thus sufficient to compute one of the terms in the sum (4.2.49) and the result will follow.

We use the Green representation of \bar{h}_f^1 for the computation :

$$\begin{aligned}
 \int_{\mathcal{A}_1} \frac{1}{\rho} |\nabla \bar{h}_f^1|^2 &= \iint_{(x,y) \in \mathcal{A}_1 \times \mathcal{A}_1} G_1(x,y) f(x) dx f(y) dy \\
 &= \sum_i \iint_{(x,y) \in B(p_{1,i},t) \times B(p_{1,i},t)} G_1(x,y) \frac{4}{t^4} dx dy \\
 &\quad + \sum_{j \neq i, |p_{1,i} - p_{1,j}| < \beta \varepsilon |\log \varepsilon|} \iint_{(x,y) \in B(p_{1,i},t) \times B(p_{1,j},t)} G_1(x,y) \frac{4}{t^4} dx dy \\
 &\quad + \iint_{(x,y) \in \mathcal{A}_1 \times \mathcal{A}_1 \cap \{|x-y| \geq \beta \varepsilon |\log \varepsilon|\}} G_1(x,y) f(x) dx f(y) dy \quad (4.2.68)
 \end{aligned}$$

where β is a small parameter (see below). On $B(p_{1,i},t) \times B(p_{1,i},t)$, according to (4.2.55) there is a constant (independent of i) so that (recall that $t = \varepsilon^{3/2} |\log \varepsilon|^{1/2}$ and $|p_{1,i}| = R_*$)

$$\left| G_i(x,y) + \frac{\rho(y)}{2\pi} \log \left(\frac{|x-y|}{\varepsilon |\log \varepsilon|} \right) \right| \leq \frac{C_K}{\varepsilon |\log \varepsilon|}.$$

Also, on $B(p_{1,i},t)$, $\rho \equiv \rho^{\text{TF}}$ which gradient is bounded by $C(\varepsilon |\log \varepsilon|)^{-2}$, thus on $B(p_{1,i},t) \times B(p_{1,i},t)$

$$\left| G_i(x,y) + \frac{\rho(R_*)}{2\pi} \log \left(\frac{|x-y|}{\varepsilon |\log \varepsilon|} \right) \right| \leq \frac{C_K}{\varepsilon |\log \varepsilon|}$$

which allows to compute

$$\sum_i \iint_{(x,y) \in B(p_{1,i},t) \times B(p_{1,i},t)} G_1(x,y) \frac{4}{t^4} dx dy = 2\pi M \rho(R_*) \log \left(\frac{\varepsilon |\log \varepsilon|}{t} \right) + O \left(\frac{M}{\varepsilon |\log \varepsilon|} \right). \quad (4.2.69)$$

Using (4.2.18) and (4.2.23) we thus obtain

$$\sum_i \iint_{(x,y) \in B(p_{1,i},t) \times B(p_{1,i},t)} G_1(x,y) \frac{4}{t^4} dx dy = \pi M g^2(R_*) |\log \varepsilon| + O \left(\frac{M \log |\log \varepsilon|}{\varepsilon |\log \varepsilon|} \right). \quad (4.2.70)$$

Now the estimate (4.2.55) combined with the upper bound $\rho \leq C(\varepsilon |\log \varepsilon|)^{-1}$ and a Riemann sum (recall that $|p_{1,i} - p_{1,i+1}| = 2\pi R_* (MN)^{-1}$) yields

$$\begin{aligned}
 \sum_{j \neq i, |p_{1,i} - p_{1,j}| < \beta \varepsilon |\log \varepsilon|} \iint_{(x,y) \in B(p_{1,i},t) \times B(p_{1,j},t)} G_1(x,y) \frac{4}{t^4} dx dy \\
 \leq C \frac{MN}{\varepsilon |\log \varepsilon|} \sum_{i=1}^M \int_{x \in \mathcal{C}_{R_*}^i} \left| \log \frac{|x - p_i|}{\varepsilon |\log \varepsilon|} \right| dx \quad (4.2.71)
 \end{aligned}$$

where we denote

$$\mathcal{C}_{R_*}^i := \{x \in \mathcal{C}_{R_*} \mid 2\pi R_* (MN)^{-1} < |x - p_i| < \beta \varepsilon |\log \varepsilon|\}.$$

We conclude from (4.2.71), using $MN \propto \Omega_1 \varepsilon^{-1}$ that

$$\sum_{j \neq i, |p_{1,i} - p_{1,j}| < \beta \varepsilon |\log \varepsilon|} \iint_{(x,y) \in B(p_{1,i},t) \times B(p_{1,j},t)} G_1(x,y) \frac{4}{t^4} dx dy \leq C \beta M^2 N. \quad (4.2.72)$$

The term on the fourth line of (4.2.68) is also estimated using a Riemann sum. Using the fact that G_1 is continuous uniformly in ε on $\mathcal{A}_i \times \mathcal{A}_i \setminus \{x = y\}$ we obtain

$$\begin{aligned} & \iint_{(x,y) \in \mathcal{A}_1 \times \mathcal{A}_1 \cap \{|x-y| \geq \beta \varepsilon |\log \varepsilon|\}} G_1(x,y) f(x) dx f(y) dy \\ &= 4\pi^2 (MN)^2 \iint_{(x,y) \in \mathcal{A}_1 \times \mathcal{A}_1 \cap \{|x-y| \geq \beta \varepsilon |\log \varepsilon|\}} G_1(x,y) \delta_*(x) \delta_*(y) (1 + o(1)) \\ &\leq 4\pi^2 (MN)^2 \iint_{(x,y) \in \mathcal{A}_1 \times \mathcal{A}_1} G_1(x,y) \delta_*(x) \delta_*(y) (1 + o(1)). \end{aligned} \quad (4.2.73)$$

Let us denote \bar{h}_*^i the function defined in \mathcal{A}_i satisfying

$$\begin{cases} -\nabla \left(\frac{1}{\rho} \nabla \bar{h}_*^i \right) = \delta_* \text{ in } \mathcal{A}_i \\ \bar{h}_*^i = 0 \text{ on } \partial \mathcal{A}_i \cap \partial \mathcal{A} \\ \frac{1}{\rho} \frac{\partial \bar{h}_*^i}{\partial n} = 0 \text{ on } \partial \mathcal{A}_i \setminus \partial \mathcal{A} \end{cases} \quad (4.2.74)$$

and \bar{h}_* satisfying in \mathcal{A}

$$\begin{cases} -\nabla \left(\frac{1}{\rho} \nabla \bar{h}_* \right) = \delta_* \text{ in } \mathcal{A} \\ \bar{h}_* = 0 \text{ on } \partial \mathcal{A}. \end{cases} \quad (4.2.75)$$

Recall that δ_* is the normalized arclength measure on the circle \mathcal{C}_{R_*} of radius R_* . Up to now we have proved, letting $\beta \rightarrow 0$ in (4.2.72)

$$\int_{\mathcal{A}} \frac{1}{\rho} |\nabla \bar{h}_f|^2 \leq 4\pi^2 (MN)^2 (1 + o(1)) \sum_i \int_{\mathcal{A}_i} \frac{1}{\rho} |\nabla \bar{h}_*^i|^2 + \pi MN g^2(R_*) |\log \varepsilon| + O\left(\frac{MN \log |\log \varepsilon|}{\varepsilon |\log \varepsilon|}\right)$$

and arguing as in the proof of Lemma 4.2.3 we deduce

$$\int_{\mathcal{A}} \frac{1}{\rho} |\nabla \bar{h}_f|^2 \leq 4\pi^2 (MN)^2 (1 + o(1)) \int_{\mathcal{A}} \frac{1}{\rho} |\nabla \bar{h}_*|^2 + \pi MN g^2(R_*) |\log \varepsilon| + O\left(\frac{MN \log |\log \varepsilon|}{\varepsilon |\log \varepsilon|}\right). \quad (4.2.76)$$

Compared to (4.2.43) there only remains to estimate the difference

$$\int_{\mathcal{A}} \frac{1}{\rho} |\nabla \bar{h}_*|^2 - \int_{\mathcal{A}} \frac{1}{g^2} |\nabla h_*|^2 = \int_{\mathcal{A}} \frac{1}{\rho} |\nabla \bar{h}_*|^2 - I_*$$

where we have denoted $h_* = h_{\delta_*}$ for short (see (4.1.34) for the definition of h_{δ_*}).

Step 2. To estimate the above difference we note that

$$-\nabla \left(\frac{1}{\rho} (\nabla \bar{h}_* - \nabla h_*) \right) = \nabla \left(\left(\frac{1}{\rho} - \frac{1}{g^2} \right) \nabla h_* \right)$$

in \mathcal{A} . Multiplying by $\bar{h}_* - h_*$, integrating over \mathcal{A} and using Cauchy-Schwarz, we obtain

$$\int_{\mathcal{A}} \frac{1}{\rho} |\nabla \bar{h}_* - \nabla h_*|^2 \leq \left(\int_{\mathcal{A}} \frac{1}{\rho} |\nabla \bar{h}_* - \nabla h_*|^2 \right)^{1/2} \left(\int_{\mathcal{A}} \rho \left(\frac{1}{\rho} - \frac{1}{g^2} \right)^2 |\nabla h_*|^2 \right)^{1/2}.$$

Using the definition of ρ and Lemma 4.2.1 it is straightforward to deduce that

$$\left(\int_{\mathcal{A}} \frac{1}{\rho} |\nabla \bar{h}_* - \nabla h_*|^2 \right)^{1/2} \leq \sup_{\mathcal{A}} \left(\rho g^2 \left(\frac{1}{\rho} - \frac{1}{g^2} \right)^2 \right)^{1/2} \left(\int_{\mathcal{A}} \frac{1}{g^2} |\nabla h_*|^2 \right)^{1/2} \ll \left(\int_{\mathcal{A}} \frac{1}{g^2} |\nabla h_*|^2 \right)^{1/2}.$$

On the other hand, by Lemma 4.2.1 again

$$\int_{\mathcal{A}} \frac{1}{g^2} |\nabla h_*|^2 = \int_{\mathcal{A}} \frac{1}{\rho} |\nabla h_*|^2 (1 + o(1)).$$

We conclude that

$$\int_{\mathcal{A}} \frac{1}{\rho} |\nabla \bar{h}_*|^2 = I_*(1 + o(1)) \quad (4.2.77)$$

and thus that (4.2.43) follows from (4.2.76). □

4.2.3 Interaction with the rotation potential

In this subsection we estimate the interaction of the vortices with the rotation potential (4.2.41) :

Proposition 4.2.3 (Interaction of the vortices with the rotation potential).

Let v be defined in (4.2.11) and MN be as in (4.2.5). There holds

$$-2 \int_{\mathcal{A}} g^2 \vec{B} \cdot (iv, \nabla v) = 2\pi \mathcal{N} F(R_*) + O\left(\frac{\mathcal{N}}{\varepsilon^{1/2} |\log \varepsilon|^{1/2}} \right) \quad (4.2.78)$$

where F is the potential function defined in (4.1.26).

Note that $F(R_*)$ is negative and proportional to ε^{-1} so that the second term above is really a remainder.

As in the preceding subsection, the proof below will require many technical estimates. We thus find useful to give the core idea before going into the rigorous proof : Integrating by parts

$$-2 \int_{\mathcal{A}} g^2 \vec{B} \cdot (iv, \nabla v) = \int_{\mathcal{A}} F \operatorname{curl}(iv, \nabla v) - \int_{\partial \mathcal{B}} F(iv, \partial_{\tau} v).$$

Now, F (see (4.1.26)) is defined in such a way that $F(1) = O(1)$. This has been proved in [46]([Chapter 3]) to be a consequence of (4.1.22). Since on the other hand $F(R_*) \propto \varepsilon^{-1}$, it is natural to neglect the boundary term above in a first approach. Then

$$-2 \int_{\mathcal{A}} g^2 \vec{B} \cdot (iv, \nabla v) = \int_{\mathcal{A}} F \operatorname{curl}(iv, \nabla v) = \sum_{i,j} \int_{B(p_{i,j}, 2t)} F \operatorname{curl}(iv, \nabla v)$$

because $|v| \sim 1$ outside $\cup_{i,j} B(p_{i,j}, 2t)$, which implies $\operatorname{curl}(iv, \nabla v) = 0$ there. Note that we neglect the thin boundary layer where $|v|$ is small for the sake of clarity. Then we use that t is very small to obtain

$$\int_{B(p_{i,j}, 2t)} F \operatorname{curl}(iv, \nabla v) \sim F(p_{i,j}) \int_{B(p_{i,j}, 2t)} \operatorname{curl}(iv, \nabla v).$$

Finally, using the fact that F is radial and by definition of the phase of v

$$F(p_{i,j}) \int_{B(p_{i,j}, 2t)} \operatorname{curl}(iv, \nabla v) \sim 2\pi F(R_*) \operatorname{deg}\{v, \partial B(p_{i,j}, 2t)\} = 2\pi F(R_*).$$

Collecting the above heuristics justifies (4.2.78), recalling that there are \mathcal{N} balls in the collection.

That being said, let us go into details. We need to introduce a new potential function. Indeed, since the phase of our trial function is defined using the modified density ρ it is useful for the proof to define also a potential function accordingly. We denote

$$\bar{F}(r) := 2 \int_{R_<}^r \rho(s) \left(\Omega s - ([\Omega] - \omega) \frac{1}{s} \right) ds = 2 \int_{R_<}^r \rho(s) \vec{B}(s) \cdot \vec{e}_\theta ds. \quad (4.2.79)$$

The function \bar{F} satisfies

$$\nabla^\perp \bar{F} = 2\rho \vec{B}, \quad \bar{F}(R_<) = 0 \quad (4.2.80)$$

and

$$-\nabla \left(\frac{1}{\rho} \nabla \bar{F} \right) = -2\nabla (B\vec{e}_r) \text{ in } \mathcal{A}.$$

We can thus decompose \bar{F} as follows :

$$\bar{F} = \bar{F}_{\text{in}} + \bar{F}_{\text{out}} \quad (4.2.81)$$

where \bar{F}_{in} and \bar{F}_{out} are the solutions to the following elliptic problems :

$$\begin{cases} -\nabla \left(\frac{1}{\rho} \nabla \bar{F}_{\text{in}} \right) = -2\nabla (B\vec{e}_r) \text{ in } \mathcal{A} \\ \bar{F}_{\text{in}} = 0 \text{ on } \partial\mathcal{A} \end{cases} \quad (4.2.82)$$

and

$$\begin{cases} -\nabla \left(\frac{1}{\rho} \nabla \bar{F}_{\text{out}} \right) = 0 \text{ in } \mathcal{A} \\ \bar{F}_{\text{out}} = \bar{F}(1) \text{ on } \partial\mathcal{B} \\ \bar{F}_{\text{out}} = 0 \text{ on } \partial B_{R_<}. \end{cases} \quad (4.2.83)$$

Both \bar{F}_{in} and \bar{F}_{out} are radial. The maximum principle implies

$$|\bar{F}_{\text{out}}| \leq |\bar{F}(1)| \quad (4.2.84)$$

and more precisely we have

$$\bar{F}_{\text{out}} = \bar{F}(1) \Gamma \quad (4.2.85)$$

where Γ is defined in (4.2.27). We sum up some useful properties of \bar{F} in the following Lemma :

Lemma 4.2.5 (Properties of \bar{F} and \bar{F}_{in}).

Let \bar{F} , \bar{F}_{in} and \bar{F}_{out} be defined as above and F in (4.1.26). The following properties hold

$$|F(r) - \bar{F}(r)| \leq C \frac{|\log \varepsilon|^2}{\varepsilon^{1/4}} \quad (4.2.86)$$

$$|\bar{F}(1)| \leq C \frac{|\log \varepsilon|^2}{\varepsilon^{1/4}} \quad (4.2.87)$$

$$|F(r) - \bar{F}_{\text{in}}(r)| \leq C \frac{|\log \varepsilon|^2}{\varepsilon^{1/4}} \quad (4.2.88)$$

$$|\nabla \bar{F}_{\text{in}}| \leq \frac{C}{\varepsilon^2 |\log \varepsilon|}. \quad (4.2.89)$$

Proof. It is easy using the definitions of F and \bar{F} to deduce (4.2.86) from (4.2.23). As $F(1) = O(1)$ (see [46, Lemma 4.1](Lemma 3.4.1)), (4.2.87) follows. A combination of (4.2.86), (4.2.85) and (4.2.87) yields (4.2.88). Using the definitions of \bar{F} and \bar{F}_{out} , Lemma 4.2.1 and (4.2.87), we obtain

$$|\nabla \bar{F}| \leq \frac{C}{\varepsilon^2 |\log \varepsilon|}$$

and

$$|\nabla \bar{F}_{\text{out}}| \leq \frac{C |\log \varepsilon|^2}{\varepsilon^{3/2}}.$$

The estimate (4.2.89) then follows from (4.2.81). \square

We now present the

Proof of Proposition 4.2.3. Step 1. We first replace g^2 by ρ in (4.2.41) : Using $|\bar{B}| \leq C\varepsilon^{-1}$

$$\begin{aligned} \left| \int_{\mathcal{A}} (g^2 - \rho) \bar{B} \cdot (iv, \nabla v) \right| &\leq \frac{C}{\varepsilon} \int_{\mathcal{A}} |g^2 - \rho| \xi^2 |\nabla \phi| \\ &\leq \frac{C}{\varepsilon} \left(\int_{\mathcal{A}} \rho \xi^2 |\nabla \phi|^2 \right)^{1/2} \left(\int_{\mathcal{A}} \frac{(g^2 - \rho)^2}{\rho} \right)^{1/2}. \end{aligned} \quad (4.2.90)$$

From (4.2.42), (4.2.43) and (4.2.5) we have

$$\int_{\mathcal{A}} \rho \xi^2 |\nabla \phi|^2 \leq C(MN)^2 + C \frac{MN}{\varepsilon} \leq C \frac{MN}{\varepsilon}.$$

We also note that from the definition of ρ

$$\int_{\mathcal{A}} \frac{(g^2 - \rho)^2}{\rho} = \int_{\mathcal{A} \cap \{r \geq \bar{R}_-\}} \frac{(g^2 - \rho)^2}{\rho} \leq C\varepsilon^{1/4} \int_{\mathcal{A} \cap \{r \geq \bar{R}_-\}} (g^2 - \rho)^2 \leq C\varepsilon^{1/4}$$

where we have used Propositions 2.4 and 2.6 from [46]([Chapter 3]) for the estimate of $\int_{\mathcal{A} \cap \{r \geq \bar{R}_-\}} (g^2 - \rho)^2$ and the lower bound to ρ respectively. We conclude

$$\int_{\mathcal{A}} g^2 \bar{B} \cdot (iv, \nabla v) = \int_{\mathcal{A}} \rho \bar{B} \cdot (iv, \nabla v) + O\left(\frac{(MN)^{1/2}}{\varepsilon^{11/8}}\right). \quad (4.2.91)$$

We now compute, using (4.2.80), (4.2.81) and Stokes' formula,

$$-2 \int_{\mathcal{A}} \rho \bar{B} \cdot (iv, \nabla v) = - \int_{\mathcal{A}} \nabla^\perp (\bar{F}_{\text{in}} + \bar{F}_{\text{out}}) (iv, \nabla v) = \int_{\mathcal{A}} \bar{F}_{\text{in}} \text{curl}(iv, \nabla v) - \int_{\mathcal{A}} \nabla^\perp (\bar{F}_{\text{out}}) (iv, \nabla v). \quad (4.2.92)$$

The second term is estimated as follows :

$$\int_{\mathcal{A}} \nabla^\perp (\bar{F}_{\text{out}}) (iv, \nabla v) = \int_{\mathcal{A}} \nabla^\perp (\bar{F}_{\text{out}}) \xi^2 \nabla \phi = \int_{\mathcal{A}} \bar{F}(1) \frac{\xi^2}{\rho} \nabla \Gamma \cdot \nabla \bar{h}_f. \quad (4.2.93)$$

We then note that

$$\left| \bar{F}(1) \int_{\mathcal{A}} \frac{(\xi^2 - 1)}{\rho} \nabla \Gamma \cdot \nabla \bar{h}_f \right| \leq |\bar{F}(1)| \sup_{\mathcal{A}} \frac{|\nabla \Gamma|}{\rho^{1/2}} \left(\int_{\mathcal{A}} (\xi^2 - 1)^2 \right)^{1/2} \left(\int_{\mathcal{A}} \frac{1}{\rho} |\nabla \bar{h}_f|^2 \right)^{1/2}. \quad (4.2.94)$$

But by definition (see (4.2.28))

$$\sup_{\mathcal{A}} \frac{|\nabla\Gamma|}{\rho^{1/2}} \leq C \sup_{\mathcal{A}} \rho^{1/2} \leq C(\varepsilon|\log\varepsilon|)^{-1/2}.$$

Also, $\xi \leq 1$ with equality everywhere except in a boundary layer $R_{<} \leq r \leq \tilde{R}$ and in $\cup_{i,j} B(p_{i,j}, t)$. The former domain has an area of order ε^n with n large (we can choose it as large as needed) and the latter an area of order $MN\varepsilon^3|\log\varepsilon|$. Thus

$$\int_{\mathcal{A}} (\xi^2 - 1)^2 \leq CMN\varepsilon^3|\log\varepsilon|.$$

Using also (4.2.87), Lemma 4.2.2 and Proposition 4.2.2 we obtain from (4.2.94) the estimate

$$\left| \bar{F}(1) \int_{\mathcal{A}} \frac{(\xi^2 - 1)}{\rho} \nabla\Gamma \cdot \nabla\bar{h}_f \right| \leq C(MN)^{3/2}\varepsilon^{1/2}|\log\varepsilon|^2. \quad (4.2.95)$$

On the other hand

$$\int_{\mathcal{A}} \bar{F}(1) \frac{1}{\rho} \nabla\Gamma \cdot \nabla\bar{h}_f = \bar{F}(1) \left(\int_{\mathcal{A}} \frac{1}{\rho} \nabla\Gamma \cdot \nabla\bar{h}_f - \frac{\kappa}{\int_{\mathcal{A}} \frac{1}{\rho} |\nabla\Gamma|^2} \int_{\mathcal{A}} \frac{1}{\rho} |\nabla\Gamma|^2 \right) = O\left(\frac{|\log\varepsilon|^2}{\varepsilon^{1/4}}\right)$$

because $\kappa = O(1)$, $\bar{F}(1) = O\left(\frac{|\log\varepsilon|^2}{\varepsilon^{1/4}}\right)$ and

$$\int_{\mathcal{A}} \frac{1}{\rho} \nabla\Gamma \cdot \nabla\bar{h}_f = 0$$

which follows from the definition of Γ and $\bar{h}_f = 0$ on $\partial\mathcal{A}$.

At this stage we have, gathering equations (4.2.91) to (4.2.95) and using (4.2.5) to estimate the remainders

$$-2 \int_{\mathcal{A}} g^2 \vec{B} \cdot (iv, \nabla v) = \int_{\mathcal{A}} \bar{F}_{in} \text{curl}(iv, \nabla v) + O\left(\frac{(MN)^{1/2}}{\varepsilon^{11/8}}\right). \quad (4.2.96)$$

Step 2. To compute the remaining term we separate the contribution of the boundary layer from that of the bulk :

$$\int_{\mathcal{A}} \bar{F}_{in} \text{curl}(iv, \nabla v) = \int_{\mathcal{A} \cap \{r \leq \tilde{R}\}} \bar{F}_{in} \text{curl}(iv, \nabla v) + \int_{\mathcal{A} \cap \{r \geq \tilde{R}\}} \bar{F}_{in} \text{curl}(iv, \nabla v). \quad (4.2.97)$$

To estimate the boundary layer contribution we remark that for any $r \leq \tilde{R}$

$$|\bar{F}(r)| \leq \frac{C}{\varepsilon} (r - R_{<}) g^2(r) \quad (4.2.98)$$

because $|\vec{B}| \leq C\varepsilon^{-1}$, $\rho(r) = g^2(r)$ for any $r \leq \tilde{R}$ and g^2 is an increasing function. Also

$$|\bar{F}_{out}(r)| \leq C \frac{|\log\varepsilon|^2}{\varepsilon^{1/4}} (r - R_{<}) g^2(r)$$

for any $r \leq \tilde{R}$, using the definition of Γ and the same arguments as above. We conclude that

$$|\bar{F}_{in}(r)| \leq C \frac{|\log\varepsilon|^2}{\varepsilon^{1/4}} (r - R_{<}) g^2(r) \quad (4.2.99)$$

if $r \leq \tilde{R}$. We now compute :

$$\begin{aligned} \left| \int_{\mathcal{A} \cap \{r \leq \tilde{R}\}} \bar{F}_{in} \operatorname{curl}(iv, \nabla v) \right| &\leq C \int_{\mathcal{A} \cap \{r \leq \tilde{R}\}} |\bar{F}_{in}| |\nabla v|^2 \\ &\leq C \int_{\mathcal{A} \cap \{r \leq \tilde{R}\}} |\bar{F}_{in}| |\nabla \xi|^2 + C \int_{\mathcal{A} \cap \{r \leq \tilde{R}\}} |\bar{F}_{in}| |\xi|^2 |\nabla \phi|^2 \leq C \varepsilon^\infty \end{aligned} \quad (4.2.100)$$

where we have used (4.2.99) combined with the exponential smallness of g^2 in the boundary layer [46, Proposition 2.5] (Proposition 3.2.5). For the bulk term we use that in $\mathcal{A} \cap \{r \geq \tilde{R}\} \setminus \cup_{i,j} B(p_{i,j}, 2t)$, $|v| \equiv c$ which is a constant, thus $\operatorname{curl}(iv, \nabla v) = 0$ there. This yields

$$\int_{\mathcal{A} \cap \{r \geq \tilde{R}\}} \bar{F}_{in} \operatorname{curl}(iv, \nabla v) = \sum_{i,j} \int_{B(p_{i,j}, 2t)} \bar{F}_{in} \operatorname{curl}(iv, \nabla v). \quad (4.2.101)$$

But, because of (4.2.18) and (4.2.89)

$$\begin{aligned} \left| \sum_{i,j} \int_{B(p_{i,j}, 2t)} (\bar{F}_{in} - \bar{F}_{in}(p_{i,j})) \operatorname{curl}(iv, \nabla v) \right| &\leq \frac{C}{\varepsilon^{1/2} |\log \varepsilon|^{1/2}} \sum_{i,j} \int_{B(p_{i,j}, 2t)} |\nabla v|^2 \\ &\leq C \varepsilon^{1/2} |\log \varepsilon|^{1/2} \int_{\mathcal{A}} g^2 |\nabla v|^2 \\ &\leq C \frac{MN}{\varepsilon^{1/2} |\log \varepsilon|^{1/2}}. \end{aligned} \quad (4.2.102)$$

Indeed, on $B(p_{i,j}, 2t)$ $g^2 \geq C(\varepsilon |\log \varepsilon|)^{-1}$ and $\int_{\mathcal{A}} g^2 |\nabla v|^2$ has been shown in the preceding subsections to be a $O\left(\frac{MN}{\varepsilon}\right)$. Finally

$$\begin{aligned} \bar{F}_{in}(p_{i,j}) \int_{B(p_{i,j}, 2t)} \operatorname{curl}(iv, \nabla v) &= 2\pi \bar{F}_{in}(p_{i,j}) \operatorname{deg}\{v, \partial B(p_{i,j}, 2t)\} = 2\pi \bar{F}_{in}(p_{i,j}) \\ &= 2\pi F(R_*) + O\left(\frac{|\log \varepsilon|^2}{\varepsilon^{1/4}}\right) \end{aligned} \quad (4.2.103)$$

by definition of the phase of v and (4.2.88). The conclusion follows by gathering (4.2.96), (4.2.97) and equations (4.2.100) to (4.2.103). \square

4.3 Energy Lower Bound

In this Section we provide the lower bound announced in Theorem 4.1.2. The key to a lower bound matching the upper bound of Proposition 4.2.1 is the identification, in the energy of Ψ^{GP} , of terms representing the kinetic energy of the vortices and their interaction with the rotation potential. These terms should then be bounded from below to show that our construction in the preceding Section is optimal. Ultimately we obtain the following

Proposition 4.3.1 (Lower bound to the energy).

Recall the decomposition

$$\Psi^{\text{GP}} = u g e^{i([\Omega] - \omega)\theta} \quad (4.3.1)$$

valid in \mathcal{A} and the definition of the reduced energy (4.2.35). Let I_* be defined in (4.1.36) and H be the cost function (4.1.25). For ε small enough there holds :

$$E^{\text{GP}} \geq \hat{E}_{\mathcal{A}, \omega}^{\text{GP}} + \mathcal{E}[u] - O(\varepsilon^\infty) \geq \hat{E}_{\mathcal{A}, \omega}^{\text{GP}} - \frac{H(R_*)^2}{4I_*} (1 + o(1)). \quad (4.3.2)$$

The proof of this result will occupy the rest of the section. The main new ingredient with respect to [46]([Chapter 3]) is a lower bound to the kinetic energy located outside the vortex balls, that is, relatively far from the vortex cores. In a first subsection we recall constructions from [46]([Chapter 3]) and deduce some basic bounds that will be our starting point for the evaluation of the kinetic energy outside vortex balls. We refer to that paper for detailed explanations and comments. In a second subsection we show that this energy can be bounded below using a variational problem related to electrostatics. We also present the analysis of the problem (4.1.36) as well as some important properties of the field h_* associated to the minimizing measure δ_* . We conclude the proof in a final subsection.

4.3.1 Preliminary Constructions

The first inequality in (4.3.2) has actually been proved in [46, Proposition 3.1](Proposition 3.3.1). The method is to combine the variational equation satisfied by g to obtain an energy decoupling, and the exponential smallness of Ψ^{GP} in the complement of \mathcal{A} . Our goal is now to bound $\mathcal{E}[u]$ from below to obtain the second inequality.

To begin with, we define the energy

$$\mathcal{F}[v] := \int_{\mathcal{A}} g^2 |\nabla v|^2 + \frac{g^4}{\varepsilon^2} (1 - |v|^2)^2 \quad (4.3.3)$$

which plays a crucial role here. Indeed, a control on this energy allows to construct the vortex balls that are our main tool in this Section. The bounds that we use as starting point are the following (Lemma 4.2 of [46]([Chapter 3]))

Lemma 4.3.1 (First energy bounds).

Let u be defined in (4.1.23). We have

$$\mathcal{F}[u] \leq C\varepsilon^{-2} \quad (4.3.4)$$

$$\mathcal{E}[u] \geq -C\varepsilon^{-2}. \quad (4.3.5)$$

Our analysis uses the same covering of \mathcal{A} by (almost rectangular) cells as in the upper bound Section, but we distinguish different types of cells :

Definition 4.3.1 (Good and Bad Cells).

We cover \mathcal{A} with (almost rectangular) cells of side length $C\varepsilon|\log \varepsilon|$, using a corresponding division of the angular variable as in Section 2.1. We note $N \propto \frac{1}{\varepsilon|\log \varepsilon|}$ the total number of cells and label the cells $\mathcal{A}_n, n \in \{1, \dots, N\}$. Let $0 \leq \alpha < \frac{1}{2}$ be a parameter to be chosen below.

- We say that \mathcal{A}_n is an α -good cell if

$$\int_{\mathcal{A}_n} g^2 |\nabla u|^2 + \frac{g^4}{\varepsilon^2} (1 - |u|^2)^2 \leq \frac{|\log \varepsilon|}{\varepsilon} \varepsilon^{-\alpha} \quad (4.3.6)$$

We will denote N_α^G the number of α -good cells and GS_α the (good) set they cover.

- We say that \mathcal{A}_n is an α -bad cell if

$$\int_{\mathcal{A}_n} g^2 |\nabla u|^2 + \frac{g^4}{\varepsilon^2} (1 - |u|^2)^2 > \frac{|\log \varepsilon|}{\varepsilon} \varepsilon^{-\alpha} \quad (4.3.7)$$

We will denote N_α^B the number of α -bad cells and BS_α the (bad) set they cover.

Note that the annulus \mathcal{A} has a width $\ell \propto \varepsilon |\log \varepsilon|$ (which implies that $N \propto (\varepsilon |\log \varepsilon|)^{-1}$) so that we are dividing it into cells where there is much more energy than what would be expected from the localization of the bound (4.3.4) (namely $\frac{C\ell}{\varepsilon^2} \propto \frac{|\log \varepsilon|}{\varepsilon}$) and regions of reasonably small energy. A first consequence of this is, using (4.3.4) and neglecting the good cells

$$N_\alpha^B \leq \frac{\varepsilon}{|\log \varepsilon|} \varepsilon^\alpha \frac{C}{\varepsilon^2} \leq \frac{C}{\varepsilon |\log \varepsilon|} \varepsilon^\alpha \ll N, \quad (4.3.8)$$

i.e. there are (relatively) very few α -bad cells.

The construction of vortex balls is feasible only in the regions of sufficient density. We thus introduce a reduced annulus

$$\tilde{\mathcal{A}} := \{\vec{r} : R_> \leq r \leq 1\} \quad (4.3.9)$$

with

$$R_> := R_h + \varepsilon |\log \varepsilon|^{-1}. \quad (4.3.10)$$

An important point is that from [46, Proposition 2.6] (Proposition 3.2.6) we have the lower bound

$$g^2(r) \geq \frac{C}{\varepsilon |\log \varepsilon|^3} \text{ on } \tilde{\mathcal{A}}. \quad (4.3.11)$$

We can now recall the vortex balls construction (see [46, Proposition 4.2] for the proof) :

Proposition 4.3.2 (Vortex ball construction in the good set).

Let $0 \leq \alpha < \frac{1}{2}$. There is a certain ε_0 so that, for $\varepsilon \leq \varepsilon_0$ there exists a finite collection $\{B_i\}_{i \in I} := \{B(\vec{a}_i, r_i)\}_{i \in I}$ of disjoint balls with centers \vec{a}_i and radii ϱ_i such that

1. $\{\vec{r} \in GS_\alpha \cap \tilde{\mathcal{A}} : ||u| - 1| > |\log \varepsilon|^{-1}\} \subset \cup_{i=1}^I B_i$,
2. for any α -good cell \mathcal{A}_n , $\sum_{i, B_i \cap \mathcal{A}_n \neq \emptyset} \varrho_i = \varepsilon |\log \varepsilon|^{-5}$.

Setting $d_i := \deg\{u, \partial B_i\}$, if $B_i \subset \tilde{\mathcal{A}} \cap GS_\alpha$, and $d_i = 0$ otherwise, we have the lower bounds

$$\int_{B_i} d\vec{r} g^2 |\nabla u|^2 \geq 2\pi \left(\frac{1}{2} - \alpha\right) |d_i| g^2(a_i) |\log \varepsilon| \left(1 - C \frac{\log |\log \varepsilon|}{|\log \varepsilon|}\right). \quad (4.3.12)$$

The second main tool that we need to introduce is the so-called Jacobian Estimate. For convenience we recall the result of [46, Proposition 4.3] :

Proposition 4.3.3 (Jacobian estimate).

Let $0 \leq \alpha < \frac{1}{2}$ and ϕ be any piecewise- C^1 test function with compact support

$$\text{supp}(\phi) \subset \tilde{\mathcal{A}} \cap GS_\alpha.$$

Let $\{B_i\}_{i \in I} := \{B(\vec{a}_i, r_i)\}_{i \in I}$ be a collection of disjoint balls as in Proposition 4.3.2. Setting $d_i := \deg\{u, \partial B_i\}$, if $B_i \subset \tilde{\mathcal{A}} \cap GS_\alpha$, and $d_i = 0$ otherwise, one has

$$\left| 2\pi \sum_{i \in I} d_i \phi(\vec{a}_i) - \int_{GS_\alpha \cap \tilde{\mathcal{A}}} d\vec{r} \phi \text{curl}(iu, \nabla u) \right| \leq C \|\nabla \phi\|_{L^\infty(GS_\alpha)} \varepsilon^2 |\log \varepsilon|^{-2} \mathcal{F}[u]. \quad (4.3.13)$$

We refine our classification of cells :

Definition 4.3.2 (Pleasant and unpleasant cells).

Recall the covering of the annulus \mathcal{A} by cells \mathcal{A}_n , $n \in \{1, \dots, N\}$. We say that \mathcal{A}_n is

- an α -pleasant cell if \mathcal{A}_n and its two neighbors are good cells. We denote PS_α the union of all α -pleasant cells and N_α^P their number,
- an α -unpleasant cell if either \mathcal{A}_n is a bad cell, or \mathcal{A}_n is a good cell but its two neighbors are bad cells. We denote UPS_α the union of all α -unpleasant cells and N_α^{UP} their number,
- an α -average cell if \mathcal{A}_n is a good cell but exactly one of its neighbors is not. We denote AS_α the union of all α -average cells and N_α^A their number.

Remark that one obviously has, recalling (3.4.50),

$$N_\alpha^{UP} \leq \frac{3}{2} N_\alpha^B \ll N \quad (4.3.14)$$

and

$$N_\alpha^A \leq 2N_\alpha^B \ll N. \quad (4.3.15)$$

The average cells will play the role of transition layers between the pleasant set, where we will use the tools described above, and the unpleasant set, where we have little information and therefore have to rely on more basic estimates.

We now introduce an azimuthal partition of unity that will allow us to avoid some boundary terms when integrating by parts in the sequel. Let us label UPS_α^l , $l \in \{1, \dots, L\}$, and PS_α^m , $m \in \{1, \dots, M\}$, the connected components of the α -unpleasant set and α -pleasant set respectively. We construct azimuthal positive functions, bounded independently of ε , denoted by χ_l^U and χ_m^P (the labels U and P stand for ‘‘pleasant set’’ and ‘‘unpleasant set’’) so that

$$\begin{aligned} \chi_l^U &:= 1 \text{ on } UPS_\alpha^l, \\ \chi_l^U &:= 0 \text{ on } PS_\alpha^m, \forall m \in \{1, \dots, M\}, \text{ and on } UPS_\alpha^{l'}, \forall l' \neq l, \\ \chi_m^P &:= 1 \text{ on } PS_\alpha^m, \\ \chi_m^P &:= 0 \text{ on } UPS_\alpha^l, \forall l \in \{1, \dots, L\}, \text{ and on } PS_\alpha^{m'}, \forall m' \neq m, \\ \sum_m \chi_m^P + \sum_l \chi_l^U &= 1 \text{ on } \mathcal{A}. \end{aligned} \quad (4.3.16)$$

It is important to note that each function so defined varies from 0 to 1 in an average cell. A crucial consequence of this is that we can take functions satisfying

$$|\nabla \chi_l^U| \leq \frac{C}{\varepsilon |\log \varepsilon|}, \quad |\nabla \chi_m^P| \leq \frac{C}{\varepsilon |\log \varepsilon|}, \quad (4.3.17)$$

because the side length of a cell is $\propto \varepsilon |\log \varepsilon|$.

We will use the short-hand notation

$$\chi_{\text{in}} := \sum_{m=1}^M \chi_m^P, \quad (4.3.18)$$

$$\chi_{\text{out}} := \sum_{l=1}^L \chi_l^U. \quad (4.3.19)$$

The subscripts ‘in’ and ‘out’ refer to ‘in the pleasant set’ and ‘out of the pleasant set’ respectively.

We want to use the Jacobian estimate of Proposition 4.3.3 with $\phi = \chi_{\text{in}} F$, which does not vanish on $\partial\mathcal{B}$ and has its support included in \mathcal{A} which is larger than $\tilde{\mathcal{A}}$. We will need one more construction to make this possible :

We introduce two radii R_{cut}^+ and R_{cut}^- as

$$R_{\text{cut}}^+ := 1 - \varepsilon |\log \varepsilon|^{-1}, \quad (4.3.20)$$

$$R_{\text{cut}}^- := R_{>} + \varepsilon |\log \varepsilon|^{-1}. \quad (4.3.21)$$

Let $\xi_{\text{in}}(r)$ and $\xi_{\text{out}}(r)$ be two positive radial functions satisfying

$$\begin{aligned} \xi_{\text{in}}(r) &:= 1 \text{ for } R_{\text{cut}}^- \leq r \leq R_{\text{cut}}^+, \\ \xi_{\text{in}}(r) &:= 0 \text{ for } R_{<} \leq r \leq R_{>} \text{ and for } r = 1, \\ \xi_{\text{out}}(r) &:= 1 \text{ for } R_{<} \leq r \leq R_{>}, \\ \xi_{\text{out}}(r) &:= 0 \text{ for } R_{\text{cut}}^- \leq r \leq R_{\text{cut}}^+, \\ \xi_{\text{in}} + \xi_{\text{out}} &= 1 \text{ on } \mathcal{A}. \end{aligned} \quad (4.3.22)$$

Moreover, because of (4.3.20) and (4.3.21), we can impose

$$|\nabla \xi_{\text{in}}| \leq \frac{C |\log \varepsilon|}{\varepsilon}, \quad |\nabla \xi_{\text{out}}| \leq \frac{C |\log \varepsilon|}{\varepsilon}. \quad (4.3.23)$$

The subscripts ‘in’ and ‘out’ refer to ‘inside $\tilde{\mathcal{A}}$ ’ and ‘outside of $\tilde{\mathcal{A}}$ ’ respectively.

In the sequel $\{B_i\}_{i \in I} := \{B(\vec{a}_i, r_i)\}_{i \in I}$ is a collection of disjoint balls as in Proposition 4.3.2. For the sake of simplicity we label B_j , $j \in J \subset I$, the balls such that $B_j \subset \tilde{\mathcal{A}} \cap GS_\alpha$.

An important step towards a lower bound to $\mathcal{E}[u]$ is to remark that, integrating by parts,

$$-2 \int_{\mathcal{A}} g^2 \vec{B} \cdot (iu, \nabla u) = \int_{\mathcal{A}} F \text{curl}(iu, \nabla u) - F(1) \int_{\partial\mathcal{B}} (iu, \partial_\tau u) \quad (4.3.24)$$

and to note that the boundary term above can be neglected. In [46] (see equation (4.100) therein, equation 3.4.100 of Chapter 3) we have proved

$$\left| \int_{\partial\mathcal{B}} F(1)(iu, \partial_\tau u) \right| \leq C \left(|\log \varepsilon|^{1/2} |\mathcal{E}[u]|^{1/2} + \frac{|\log \varepsilon|^{1/4}}{\varepsilon^{1/4}} \mathcal{F}[u]^{1/2} \right)$$

and thus we deduce as in [46]([Chapter 3]) that

$$\left| \int_{\partial\mathcal{B}} F(1)(iu, \partial_\tau u) \right| \leq C \frac{|\log \varepsilon|^{1/4}}{\varepsilon^{5/4}}. \quad (4.3.25)$$

Note that this term is much smaller (in absolute value) than the lower bound we are aiming at.

Our first lower bound is the intermediate result (4.87) in [46]([Chapter 3]), that we reorganize to obtain :

$$\begin{aligned}
 & \int_{\mathcal{A}} g^2 |\nabla u|^2 + F \operatorname{curl}(iu, \nabla u) \geq (1 - 2\gamma) \int_{\mathcal{A} \setminus \cup_{j \in J} B_j} \xi_{\text{in}} g^2 |\nabla u|^2 \\
 & + 2\pi \sum_{j \in J} \chi_{\text{in}}(a_j) \left[|d_j| (1 - 2\gamma) \left(\frac{1}{2} - \alpha \right) g^2(a_j) |\log \varepsilon| \left(1 - C \frac{\log |\log \varepsilon|}{|\log \varepsilon|} \right) + d_j \xi_{\text{in}}(a_j) F(a_j) \right] \\
 & + \gamma \int_{\mathcal{A}} d\vec{r} \xi_{\text{out}} g^2 |\nabla u|^2 - \int_{\mathcal{A}} d\vec{r} \xi_{\text{out}} |F| |\nabla u|^2 - C \int_{\partial B} d\sigma |F(1)| |(iu, \partial_\tau u)| \\
 & + (\gamma - \delta) \int_{\mathcal{A}} d\vec{r} g^2 |\nabla u|^2 - \frac{C}{\delta \varepsilon^2} \int_{UPS_\alpha \cup AS_\alpha} d\vec{r} g^2 |u|^2 - C |\log \varepsilon|^{-1} \mathcal{F}[u]. \quad (4.3.26)
 \end{aligned}$$

We emphasize that we have kept the kinetic energy contained outside the vortex balls (first term in the right-hand side, on the first line) that was neglected in [46]([Chapter 3]). The parameters in (4.3.26) are chosen as follows :

$$\gamma = 2\delta = \frac{\log |\log \varepsilon|}{|\log \varepsilon|}, \quad \alpha = \tilde{\alpha} \frac{\log |\log \varepsilon|}{|\log \varepsilon|}, \quad (4.3.27)$$

where $\tilde{\alpha}$ is a large enough constant (see below).

Using the estimate on the number of bad cells (4.3.8) and the upper bound on $g^2 |u|^2 = |\Psi^{\text{GP}}|^2$ [46, Proposition 2.1](Proposition 3.2.1) we have

$$\frac{1}{\delta \varepsilon^2} \int_{UPS_\alpha \cup AS_\alpha} g^2 |u|^2 \leq C \frac{|\log \varepsilon| \varepsilon^\alpha}{\varepsilon^2 \log |\log \varepsilon|}.$$

Also, using [46, Equation (4.22)](Equation 3.4.22),

$$\gamma \int_{\mathcal{A}} g^2 |\nabla u|^2 - \int_{\mathcal{A}} \xi_{\text{out}} |F| |\nabla u|^2 \geq C \frac{\log |\log \varepsilon|}{|\log \varepsilon|} \int_{\mathcal{A}} g^2 |\nabla u|^2.$$

Plugging these estimates and (4.3.25) in (4.3.26) we obtain

$$\begin{aligned}
 & \int_{\mathcal{A}} g^2 |\nabla u|^2 + F \operatorname{curl}(iu, \nabla u) \geq \left(1 - C \frac{\log |\log \varepsilon|}{|\log \varepsilon|} \right) \int_{\mathcal{A} \setminus \cup_{j \in J} B_j} \xi_{\text{in}} g^2 |\nabla u|^2 \\
 & + \sum_{j \in J} 2\pi \chi_{\text{in}}(a_j) \left[|d_j| \frac{1}{2} g^2(a_j) |\log \varepsilon| \left(1 - C \frac{\log |\log \varepsilon|}{|\log \varepsilon|} \right) + d_j \xi_{\text{in}}(a_j) F(a_j) \right] - C \frac{|\log \varepsilon| \varepsilon^\alpha}{\varepsilon^2 \log |\log \varepsilon|}. \quad (4.3.28)
 \end{aligned}$$

The critical speed $(3\pi \varepsilon^2 |\log \varepsilon|)^{-1}$ is (roughly speaking) defined as the first speed at which the terms on the second line of (3.4.93) all become positive. This corresponds to the speed at which the vortices cease to be energetically favorable. In [46]([Chapter 3]) we were above the threshold $\Omega_0 > (3\pi)^{-1}$, so the preceding lower bound was enough for our purpose, because the terms on the second line were positive. Here we are in the opposite situation where the vortices can become energetically favorable if they are suitably located in the annulus. Thus, a lower bound to $\mathcal{E}[u]$ requires an upper bound to the number of vortices (more precisely, to the sum of their degrees). We will provide this upper bound in the sequel.

As is standard in such problems, we need to distinguish between different types of vortex balls. First we need to distinguish the vortices lying close to the inner boundary of \mathcal{A} from those in the bulk. The vortices close to the inner boundary of \mathcal{A} have positive energy and can thus be neglected in the lower bound. However their energetic cost is not large enough to show that there are few vortices of this type. Also, vortices in the bulk can be energetically favorable only if their degrees are positive. On the other hand, in the range of Ω that we consider even a vortex of positive degree can lower the energy only if it is close to the circle \mathcal{C}_{R_*} .

Thus we first divide J into two subsets

$$J_{\text{in}} = \left\{ j \in J, |a_j| \geq R_{\text{bulk}} = R_h + \varepsilon |\log \varepsilon| \Omega_1^{1/2} \right\} \quad (4.3.29)$$

$$J_{\text{out}} = J \setminus J_{\text{in}}. \quad (4.3.30)$$

We next divide J_{in} into three subsets :

$$J_- = \{j \in J_{\text{in}}, d_j < 0\} \quad (4.3.31)$$

$$J_+ = \left\{ j \in J_{\text{in}}, d_j \geq 0 \text{ and } ||a_j| - R_*| > \varepsilon |\log \varepsilon| \Omega_1^{1/4} \right\} \quad (4.3.32)$$

$$J_* = J_{\text{in}} \setminus J_- \setminus J_+. \quad (4.3.33)$$

With these definition we can state our lower bound to the kinetic energy contained ‘far’ from the vortex cores :

Proposition 4.3.4 (Lower bound to the kinetic energy outside vortex balls).

Recall the definition of I_* (4.1.36). Let J_* be as above. There holds

$$\int_{\mathcal{A} \setminus \cup_{j \in J} B_j} \xi_{\text{in}} g^2 |\nabla u|^2 \geq \left(2\pi \sum_{j \in J_*} \chi_{\text{in}}(a_j) d_j \right)^2 I_*(1 + O(\Omega_1^{1/4})) - C \frac{\sum_{j \in J_*} \chi_{\text{in}}(a_j) d_j}{\varepsilon |\log \varepsilon|^3}. \quad (4.3.34)$$

The energy evaluated in Proposition 4.3.4 is associated with the superfluid currents that the vortices induce in the condensate. As is well known the current generated by vortices of negative degrees could compensate that generated by vortices of positive degrees. In the sequel we prove that there are relatively few vortices of negative degrees (see (4.3.42) below). We will thus deduce that this phenomenon does not affect the leading order of the energy in the situation we consider.

Note also that the minimization problem appearing in our lower bound involves vorticity measures with support on \mathcal{C}_{R_*} . Such a restriction of the set of admissible measures will be proved to be favorable because most vortices have to be located close the circle \mathcal{C}_{R_*} where they are energetically favorable (see (4.3.43) below).

We begin the proof of Proposition 4.3.4 with the following lemma that gives lower bounds to the energetic cost of the vortices, depending on their degrees and locations in the annulus. The proof is postponed to Appendix A.

Lemma 4.3.2 (Energetic cost of the different types of vortices).

For any $j \in J_{\text{out}}$

$$|d_j| \frac{1}{2} g^2(a_j) |\log \varepsilon| \left(1 - C \frac{\log |\log \varepsilon|}{|\log \varepsilon|} \right) + d_j \xi_{\text{in}}(a_j) F(a_j) \geq 0. \quad (4.3.35)$$

For any $j \in J_{\text{in}}$

$$\frac{1}{2}|d_j|g^2(a_j)|\log \varepsilon| + d_j\xi_{\text{in}}(a_j)F(a_j) \geq |d_j|H(R_*)(1 + O(\Omega_1)) \geq -C|d_j|\frac{\Omega_1}{\varepsilon}. \quad (4.3.36)$$

Moreover, if $j \in J_- \cup J_+$

$$\frac{1}{2}|d_j|g^2(a_j)|\log \varepsilon| + d_j\xi_{\text{in}}(a_j)F(a_j) \geq |d_j|\frac{C\Omega_1^{1/2}}{\varepsilon}. \quad (4.3.37)$$

Recalling that

$$\frac{\log |\log \varepsilon|}{|\log \varepsilon|} \ll \Omega_1 \ll 1$$

we can use Lemma 4.3.2 to simplify further (3.4.93)

$$\begin{aligned} \int_{\mathcal{A}} g^2 |\nabla u|^2 + F \text{curl}(iu, \nabla u) &\geq \left(1 - \frac{\log |\log \varepsilon|}{|\log \varepsilon|}\right) \int_{\mathcal{A} \setminus \cup_{j \in J} B_j} \xi_{\text{in}} g^2 |\nabla u|^2 \\ &+ 2\pi H(R_*) (1 + o(1)) \sum_{j \in J_*} \chi_{\text{in}}(a_j) d_j + \frac{C\Omega_1^{1/2}}{\varepsilon} \sum_{j \in J_+ \cup J_-} \chi_{\text{in}}(a_j) |d_j| - C \frac{|\log \varepsilon| \varepsilon^\alpha}{\varepsilon^2 \log |\log \varepsilon|}. \end{aligned} \quad (4.3.38)$$

Note that by definition of ξ_{in} , $\xi_{\text{in}}(a_j) = 1$ for any $j \in J_*$. Adding $\int_{\mathcal{A}} \frac{g^4}{\varepsilon^2} (1 - |u|^2)^2 - F(1) \int_{\partial B} (iu, \partial_\tau u)$ to both sides of (4.3.38) and using (4.3.25) we obtain

$$\begin{aligned} \mathcal{E}[u] &\geq \left(1 - \frac{\log |\log \varepsilon|}{|\log \varepsilon|}\right) \int_{\mathcal{A} \setminus \cup_{j \in J} B_j} \xi_{\text{in}} g^2 |\nabla u|^2 + 2\pi H(R_*) (1 + o(1)) \sum_{j \in J_*} \chi_{\text{in}}(a_j) d_j \\ &+ \frac{C\Omega_1^{1/2}}{\varepsilon} \sum_{j \in J_+ \cup J_-} \chi_{\text{in}}(a_j) |d_j| - C \frac{|\log \varepsilon| \varepsilon^\alpha}{\varepsilon^2 \log |\log \varepsilon|}. \end{aligned} \quad (4.3.39)$$

On the other hand, combining the upper bound to the GP energy of Proposition 4.2.1 and the first inequality in (4.3.2) we have

$$\mathcal{E}[u] \leq -C \frac{\Omega_1^2}{\varepsilon^2}. \quad (4.3.40)$$

We deduce from the above

$$\sum_{j \in J_*} \chi_{\text{in}}(a_j) |d_j| \geq C \frac{\varepsilon}{\Omega_1} \left(\frac{\Omega_1^2}{\varepsilon^2} - \frac{|\log \varepsilon|^{1-\tilde{\alpha}}}{\varepsilon^2 \log |\log \varepsilon|} \right) \geq C \frac{\Omega_1}{\varepsilon}. \quad (4.3.41)$$

The second inequality holds true if we choose $\tilde{\alpha} > 1$, which we now do. From (4.3.39) and (4.3.40) we also deduce (note that the last term in (4.3.39) is a remainder because of (4.3.41))

$$\sum_{j \in J_-} \chi_{\text{in}}(a_j) |d_j| \leq C\Omega_1^{1/2} \sum_{j \in J_*} \chi_{\text{in}}(a_j) |d_j| \quad (4.3.42)$$

$$\sum_{j \in J_+} \chi_{\text{in}}(a_j) |d_j| \leq C\Omega_1^{1/2} \sum_{j \in J_*} \chi_{\text{in}}(a_j) |d_j| \quad (4.3.43)$$

One can interpret (4.3.41), (4.3.42) and (4.3.43) as follows (recall that $\Omega_1 \ll 1$): There are at least $C \frac{\Omega_1}{\varepsilon}$ essential vortices in \mathcal{A} (meaning vortices with nonzero degree). Most of them are of positive degree and close to the circle \mathcal{C}_{R_*} .

4.3.2 The electrostatic problem

A difficulty in the proof below is to define a vorticity with support on \mathcal{C}_{R_*} starting from u . A possible track, following [23], would be to use the Jacobian estimate and a first rough upper bound to the number of vortices to obtain some compactness for the vorticity measure of u . Using (4.3.43) one would then show that the limit measure has its support on the circle of interest and obtain the lower bound by a lower semi-continuity argument. Such a strategy is difficult to adapt to our setting because the geometry of our domain strongly depends on ε . To obtain non trivial limits one should rescale the annulus to work on fixed domains. Also the weights appearing in the energy would complicate the argument (recall that g^2 is very small close to $\partial B_{R_<}$).

We follow another route that does not require any compactness argument. In this subsection we prove that the left-hand side of (4.3.34) can be bounded below using the electrostatic energy of a suitable modification of the vorticity of u . The method that we use to conclude the proof of Proposition 4.3.4 in Subsection 4.3.3 requires precise informations on the field $h_* = h_{\delta_*}$ generated by δ_* according to (4.1.34). We provide this information in Proposition 4.3.5 below.

Let us describe the electrostatic energy that will serve as intermediate lower bound. For technical reasons it is necessary to reduce the domain on which we work to $\mathcal{A}_{\text{bulk}}$ where the density is large enough.

For any Radon measure ν supported in $\mathcal{A}_{\text{bulk}}$ we define \check{h}_ν as the unique solution to the elliptic problem

$$\begin{cases} -\nabla \left(\frac{1}{g^2} \nabla \check{h}_\nu \right) = \nu \text{ in } \mathcal{A}_{\text{bulk}} \\ \check{h}_\nu = 0 \text{ on } \partial \mathcal{A}_{\text{bulk}}. \end{cases} \quad (4.3.44)$$

and introduce

$$\check{I}(\nu) := \int_{\mathcal{A}_{\text{bulk}}} \frac{1}{g^2} |\nabla \check{h}_\nu|^2. \quad (4.3.45)$$

We will later be interested in the minimization problem

$$\check{I}_* := \inf_{\nu \in \mathcal{D}_*, \int \nu = 1} \check{I}(\nu) = \check{I}(\delta_*) \quad (4.3.46)$$

where the infimum is taken over the set \mathcal{D}_* of positive Radon measures with support on the circle of radius R_* . The fact that δ_* is the unique solution of (4.3.46) will be proven below (Proposition 4.3.5).

The definition of the measure whose energy will be used in the lower bound goes as follows : Let \tilde{j} be the modified superfluid current

$$\tilde{j} = \begin{cases} (iu, \nabla u) \text{ in } \mathcal{A}_{\text{bulk}} \cap GS_\alpha \setminus \cup_{j \in J} B_j \\ 0 \text{ otherwise .} \end{cases} \quad (4.3.47)$$

The measure $\tilde{\mu}$ is the vorticity associated to \tilde{j}

$$\tilde{\mu} := \text{curl}(\tilde{j}). \quad (4.3.48)$$

The following lemma is a key ingredient in our analysis :

Lemma 4.3.3 (Lower bound via an electrostatic energy).

With the above definitions, there holds for ε small enough :

$$\int_{\mathcal{A} \setminus \cup_{j \in J} B_j} \xi_{\text{in}} g^2 |\nabla u|^2 \geq (1 - C |\log \varepsilon|^{-1}) \check{I}(\tilde{\mu}). \quad (4.3.49)$$

Proof of Lemma 4.3.3. We need to define Γ as the solution to

$$\begin{cases} -\nabla \left(\frac{1}{g^2} \nabla \Gamma \right) = 0 & \text{in } \mathcal{A}_{\text{bulk}} \\ \Gamma = 0 & \text{on } \partial B_{R_{\text{bulk}}} \\ \Gamma = 1 & \text{on } \partial \mathcal{B}. \end{cases} \quad (4.3.50)$$

Explicitly :

$$\Gamma(\vec{r}) = \frac{\int_{R_{\text{bulk}}}^r g^2(s) s^{-1} ds}{\int_{R_{\text{bulk}}}^1 g^2(s) s^{-1} ds}. \quad (4.3.51)$$

Note that this function is not the same as that defined in (4.2.27), which is not confusing because the latter will not be used again in the sequel. We recall that in $\mathcal{A}_{\text{bulk}} \setminus \cup_{j \in J} B_j$, $\xi_{\text{in}} = 1$ and $|u|$ is close to 1 according to Item 1 in Proposition 4.3.2. Thus

$$\begin{aligned} \int_{\mathcal{A} \setminus \cup_{j \in J} B_j} \xi_{\text{in}} g^2 |\nabla u|^2 &\geq \int_{\mathcal{A}_{\text{bulk}} \setminus \cup_{j \in J} B_j} g^2 |\nabla u|^2 \\ &\geq (1 - C |\log \varepsilon|^{-1}) \int_{\mathcal{A}_{\text{bulk}} \setminus \cup_{j \in J} B_j} g^2 |u|^2 |\nabla u|^2 \\ &\geq (1 - C |\log \varepsilon|^{-1}) \int_{\mathcal{A}_{\text{bulk}} \setminus \cup_{j \in J} B_j} g^2 |(iu, \nabla u)|^2 \\ &= (1 - C |\log \varepsilon|^{-1}) \int_{\mathcal{A}_{\text{bulk}}} g^2 |\tilde{j}|^2. \end{aligned} \quad (4.3.52)$$

We set

$$f_{\tilde{\mu}} := \check{h}_{\tilde{\mu}} - \frac{1}{\int_{\mathcal{A}_{\text{bulk}}} \frac{1}{g^2} |\nabla \Gamma|^2} \left(\int_{\partial \mathcal{B}} \tilde{j} \cdot \tau + \frac{1}{g^2} \frac{\partial \check{h}_{\tilde{\mu}}}{\partial n} \right) \Gamma. \quad (4.3.53)$$

By definition

$$\begin{aligned} \text{curl} \left(\tilde{j} + \frac{1}{g^2} \nabla^\perp f_{\tilde{\mu}} \right) &= 0 \text{ in } H^{-1}(\mathcal{A}_{\text{bulk}}) \\ \int_{\partial \mathcal{B}} \left(\tilde{j} + \frac{1}{g^2} \nabla^\perp f_{\tilde{\mu}} \right) \cdot \tau &= 0. \end{aligned} \quad (4.3.54)$$

Hence there exists $f \in H^1(\mathcal{A}_{\text{bulk}})$ such that

$$\tilde{j} = -\frac{1}{g^2} \nabla^\perp f_{\tilde{\mu}} + \nabla f.$$

Using the fact that $f_{\tilde{\mu}}$ is constant on the boundary of $\mathcal{A}_{\text{bulk}}$, we have $\int_{\mathcal{A}_{\text{bulk}}} \nabla^\perp f_{\tilde{\mu}} \cdot \nabla f = 0$ and thus

$$\int_{\mathcal{A}_{\text{bulk}}} g^2 |\tilde{j}|^2 \geq \int_{\mathcal{A}_{\text{bulk}}} \frac{1}{g^2} |\nabla f_{\tilde{\mu}}|^2.$$

Next we note that

$$\int_{\mathcal{A}_{\text{bulk}}} \frac{1}{g^2} \nabla \check{h}_{\check{\mu}} \cdot \nabla \Gamma = 0$$

because $\check{h}_{\check{\mu}} = 0$ on $\partial \mathcal{A}_{\text{bulk}} = \partial B_{R_{\text{bulk}}} \cup \partial \mathcal{B}$. We thus have

$$\int_{\mathcal{A}_{\text{bulk}}} g^2 |\check{j}|^2 \geq \int_{\mathcal{A}_{\text{bulk}}} \frac{1}{g^2} |\nabla g_{\check{\mu}}|^2 \geq \int_{\mathcal{A}_{\text{bulk}}} \frac{1}{g^2} |\nabla \check{h}_{\check{\mu}}|^2.$$

Combining with (4.3.52) we conclude that (4.3.49) holds. \square

Our next task in this subsection is to give some details on the minimization problems (4.1.36) and (4.3.46). In particular we prove that the minimizing measure is in both cases the normalized arclength measure on \mathcal{C}_{R_*} and we compute the associated field explicitly. We also show that considering the problem on the reduced annulus $\mathcal{A}_{\text{bulk}}$ does not change significantly the energy.

The following proposition contains probably only facts known from potential theory. Indeed, apart from the weight g^{-2} , the minimization problems we are considering fall in the general context of [132] (see e.g. Theorem II.5.12 there and the discussion in [23, Example 5.1]). We nevertheless provide a short proof for the sake of completeness.

Proposition 4.3.5 (The electrostatic problems).

1. *The minimization problems (4.1.36) and (4.3.46) both admit the normalized arclength measure on \mathcal{C}_{R_*} (denoted δ_*) for unique solution. Moreover there exists two positive constants C_1 and C_2 independent of ε such that*

$$C_1 \leq I_* \leq C_2. \tag{4.3.55}$$

2. *We denote $h_* = h_{\delta_*}$ and $\check{h}_* = \check{h}_{\delta_*}$ the fields associated to δ_* by equations (4.1.34) and (4.3.44). Both are radial piecewise C^1 functions. Explicitly we have*

$$h_*(r) = h_*(R_*) \begin{cases} \frac{\int_{R_<}^r g^2(s) s^{-1} ds}{\int_{R_<}^{R_*} g^2(s) s^{-1} ds} & \text{for } R_< \leq r \leq R_* \\ \frac{\int_r^1 g^2(s) s^{-1} ds}{\int_{R_*}^1 g^2(s) s^{-1} ds} & \text{for } R_* \leq r \leq 1 \end{cases} \tag{4.3.56}$$

and

$$\check{h}_*(r) = \check{h}_*(R_*) \begin{cases} \frac{\int_{R_{\text{bulk}}}^r g^2(s) s^{-1} ds}{\int_{R_{\text{bulk}}}^{R_*} g^2(s) s^{-1} ds} & \text{for } R_{\text{bulk}} \leq r \leq R_* \\ \frac{\int_r^1 g^2(s) s^{-1} ds}{\int_{R_*}^1 g^2(s) s^{-1} ds} & \text{for } R_* \leq r \leq 1 \end{cases} \tag{4.3.57}$$

where

$$h_*(R_*) = I_* = \frac{\int_{R_<}^1 g^2(s) s^{-1} ds}{2\pi \left(\int_{R_<}^{R_*} g^2(s) s^{-1} ds \right) \left(\int_{R_*}^1 g^2(s) s^{-1} ds \right)} \tag{4.3.58}$$

$$\check{h}_*(R_*) = \check{I}_* = \frac{\int_{R_{\text{bulk}}}^1 g^2(s) s^{-1} ds}{2\pi \left(\int_{R_{\text{bulk}}}^{R_*} g^2(s) s^{-1} ds \right) \left(\int_{R_*}^1 g^2(s) s^{-1} ds \right)}. \tag{4.3.59}$$

3. The following estimate holds :

$$|I_* - \check{I}_*| \leq C\Omega_1^{1/2}. \quad (4.3.60)$$

Proof. The results for the problem (4.3.46) are exactly similar to those for (4.1.36), we thus only prove the later.

Let us start with the general problem of minimizing $I(\nu)$ over the set \mathcal{D}_E of Radon measures supported on E , a compact subset of \mathcal{A} :

$$I_E = \inf_{\nu \in \mathcal{D}_E, \int \nu = 1} I(\nu). \quad (4.3.61)$$

The existence of a minimizer ν_E to such a problem is classical (see e.g. [132]). We denote h_E the function associated to ν_E by formula (4.1.34). Next, computing the first variation of I , we observe that there holds, for any $\nu \in \mathcal{D}_E$

$$\int_E \nu h_E = \lambda_E \int_E \nu \quad (4.3.62)$$

where λ_E is a Lagrange multiplier. We deduce that h_E is constant on E . Using the above equation and (4.1.34) with $\nu = \nu_E$ we see that

$$h|_E = \lambda_E = \int_{\mathcal{A}} \frac{1}{g^2} |\nabla h_E|^2 = I_E. \quad (4.3.63)$$

We turn to the proof of Item 1. Let h_* be a solution to the minimization problem. Using (4.3.63) and the definition of h_* we have

$$\begin{cases} -\nabla(\frac{1}{g^2} \nabla h_*) = 0 & \text{in } \mathcal{A} \cap B_{R_*} \\ h_* = 0 & \text{on } \partial B_{R_*} \\ h_* = I_* & \text{on } \partial B_{R_*}. \end{cases}$$

This implies that h_* is radial on $\mathcal{A} \cap B_{R_*}$. A similar argument yields that h_* must be radial also on $\mathcal{A} \setminus B_{R_*}$ and thus radial in \mathcal{A} . Then the associated minimizing measure $\nu_{\mathcal{C}_{R_*}}$ is radial also. We conclude that δ_* is the unique solution to the problem (4.1.36). The bounds (4.3.55) are proved by noting that, using (4.1.34)

$$I_* = \int_{\mathcal{A}} \frac{1}{g^2} |\nabla h_*|^2 = \sup_{\phi \in C_c^1(\mathcal{A})} \frac{\left| \int_{\mathcal{A}} \frac{1}{g^2} \nabla h_* \cdot \nabla \phi \right|^2}{\int_{\mathcal{A}} \frac{1}{g^2} |\nabla \phi|^2} = \frac{1}{4\pi^2 R_*^2} \sup_{\phi \in C_c^1(\mathcal{A})} \frac{\left| \int_{\mathcal{C}_{R_*}} \phi \right|^2}{\int_{\mathcal{A}} \frac{1}{g^2} |\nabla \phi|^2}.$$

The upper bound follows because $g^2 \leq C(\varepsilon |\log \varepsilon|)^{-1}$ on \mathcal{A} whose thickness is of order $\varepsilon |\log \varepsilon|$. A small computation shows that the above supremum is bounded. The lower bound is proved via a trial function for the maximization problem. For example one can use the test function used in Remark 4.1.1, Item 2.

Proving that the function defined in the right-hand side of (4.3.56) and (4.3.58) is a solution to (4.1.34) with $\nu = \delta_*$ is a straightforward computation. By uniqueness we deduce that it must be equal to h_* .

Finally (4.3.60) follows from the formulas (4.3.58) and (4.3.59) once one has noted that

$$\int_{R_*}^{R_{\text{bulk}}} g^2(s) s^{-1} ds \leq C \frac{|R_{\text{bulk}} - R_*|}{\varepsilon |\log \varepsilon|} \leq C\Omega_1^{1/2}.$$

□

4.3.3 Completion of the proofs of Propositions 4.3.1 and 4.3.4

In what follows we denote

$$D := 2\pi \sum_{j \in J_{\text{in}}} \chi_{\text{in}}(a_j) d_j \quad (4.3.64)$$

and

$$\mu_* := D\delta_*. \quad (4.3.65)$$

We also decompose $\tilde{\mu}$ as

$$\tilde{\mu} = \mu_* + \check{\mu} = D\delta_* + \check{\mu}. \quad (4.3.66)$$

It is useful to recall that from (4.3.41) (4.3.42) and (4.3.43) we have

$$D = 2\pi \left(1 - C\Omega_1^{1/2}\right) \sum_{j \in J_*} \chi_{\text{in}}(a_j) d_j \geq C \frac{\Omega_1}{\varepsilon}. \quad (4.3.67)$$

We are now ready to finish the

Proof of Proposition 4.3.4. Starting from Lemma 4.3.3 there remains to evaluate $\check{I}(\tilde{\mu})$. The two main ingredients will be the Jacobian Estimate and the properties of \check{h}_* proved in Proposition 4.3.5.

We have

$$\check{I}(\tilde{\mu}) = \int_{\mathcal{A}_{\text{bulk}}} \frac{1}{g^2} |\nabla \check{h}_{\tilde{\mu}}|^2 = \int_{\mathcal{A}_{\text{bulk}}} \frac{1}{g^2} |\nabla \check{h}_{\mu_*}|^2 + 2 \int_{\mathcal{A}_{\text{bulk}}} \frac{1}{g^2} \nabla \check{h}_{\tilde{\mu}} \cdot \nabla \check{h}_{\mu_*} + \int_{\mathcal{A}_{\text{bulk}}} \frac{1}{g^2} |\nabla \check{h}_{\tilde{\mu}}|^2. \quad (4.3.68)$$

Now,

$$\int_{\mathcal{A}_{\text{bulk}}} \frac{1}{g^2} |\nabla \check{h}_{\mu_*}|^2 = D^2 \check{I}(\delta_*) = D^2 I_*(1 + O(\Omega_1^{1/2})) \quad (4.3.69)$$

by definition and use of (4.3.60). To obtain an appropriate lower bound it is thus sufficient to compute the second term in the right-hand side of (4.3.68). Using (4.3.44) and Proposition 4.3.5 we have

$$\int_{\mathcal{A}_{\text{bulk}}} \frac{1}{g^2} \nabla \check{h}_{\tilde{\mu}} \cdot \nabla \check{h}_{\mu_*} = \int_{\mathcal{A}_{\text{bulk}}} \check{h}_{\mu_*} \check{\mu} = D \left(\int_{\mathcal{A}_{\text{bulk}}} \check{h}_* \tilde{\mu} - D \int_{\mathcal{A}_{\text{bulk}}} \check{h}_* \delta_* \right) = D \left(\int_{\mathcal{A}_{\text{bulk}}} \check{h}_* \tilde{\mu} - D \check{h}_*(R_*) \right). \quad (4.3.70)$$

We now compute from the definition (4.3.47) of \tilde{j}

$$\begin{aligned} \int_{\mathcal{A}_{\text{bulk}}} \check{h}_* \tilde{\mu} &= - \int_{\mathcal{A}_{\text{bulk}}} \tilde{j} \cdot \nabla^\perp \check{h}_* = - \int_{GS_\alpha} (iu, \nabla u) \cdot \nabla^\perp (\chi_{\text{in}} \check{h}_*) - \int_{GS_\alpha} (iu, \nabla u) \cdot \nabla^\perp (\chi_{\text{out}} \check{h}_*) \\ &+ \sum_{j \in J} \int_{B_j \cap \tilde{\mathcal{A}}} (iu, \nabla u) \cdot \nabla^\perp \check{h}_* \\ &= \int_{GS} \mu \chi_{\text{in}} \check{h}_* - \int_{GS_\alpha} (iu, \nabla u) \cdot \nabla^\perp (\chi_{\text{out}} \check{h}_*) \\ &+ \sum_{j \in J} \int_{B_j \cap \mathcal{A}_{\text{bulk}}} (iu, \nabla u) \cdot \nabla^\perp \check{h}_*. \end{aligned} \quad (4.3.71)$$

Let us first show how to estimate the second term. The integral is actually located on $\mathcal{A}_{\text{bulk}} \cap AS_\alpha$ (see the definitions of χ_{out} and \check{h}_*). Recalling that $g^2 \geq C\varepsilon^{-1}|\log \varepsilon|^{-3}$ and $|\nabla \chi_{\text{out}}| \leq C(\varepsilon|\log \varepsilon|)^{-1}$ there we have

$$\begin{aligned} \left| \int_{GS_\alpha} (iu, \nabla u) \cdot \nabla^\perp (\chi_{\text{out}} \check{h}_*) \right| &\leq C \left(\frac{\|\check{h}_*\|_{L^\infty}}{\varepsilon|\log \varepsilon|} + \|\nabla \check{h}_*\|_{L^\infty} \right) \int_{AS_\alpha \cap \mathcal{A}_{\text{bulk}}} |u| |\nabla u| \\ &\leq C |\log \varepsilon|^2 \int_{AS_\alpha} g^2 |u| |\nabla u| \\ &\leq C |\log \varepsilon|^2 \left(\eta \int_{AS_\alpha} g^2 |u|^2 + \frac{1}{\eta} \int_{AS_\alpha} g^2 |\nabla u|^2 \right) \end{aligned} \quad (4.3.72)$$

where η is a parameter that we fix below and we have used the explicit formula (4.3.57) for the bounds on \check{h}_* and $|\nabla \check{h}_*|$. We recall the basic estimates

$$\int_{AS_\alpha} g^2 |\nabla u|^2 \leq \frac{C}{\varepsilon^2}$$

and

$$\int_{AS_\alpha} g^2 |u|^2 \leq C\varepsilon^\alpha,$$

coming respectively from (3.4.23) and (4.3.15). Choosing

$$\eta = \varepsilon^{-1-\alpha/2}$$

and using (4.3.27) we obtain

$$\left| \int_{GS_\alpha} (iu, \nabla u) \cdot \nabla^\perp (\chi_{\text{out}} \check{h}_*) \right| \leq C \frac{|\log \varepsilon|^2}{\varepsilon |\log \varepsilon|^{\bar{\alpha}}}. \quad (4.3.73)$$

For the third term in (4.3.71) we use exactly the same kind of argument. We now use the smallness of the set covered by the vortex balls instead of the smallness of AS_α . Indeed, using the division of \mathcal{A} into $N \propto (\varepsilon|\log \varepsilon|)^{-1}$ cells and Item 2 of Proposition 4.3.2

$$|\cup_j B_j| \leq C \sum_{n=1}^N |\cup_{j \in J} B_j \cap \mathcal{A}_n| \leq C \sum_{n=1}^N \sum_{j, B_j \cap \mathcal{A}_n \neq \emptyset} \varrho_j^2 \leq C\varepsilon |\log \varepsilon|^{-11}. \quad (4.3.74)$$

It follows that (recall that $g^2 |u|^2 \leq C(\varepsilon|\log \varepsilon|)^{-1}$)

$$\int_{\cup_j B_j} g^2 |u|^2 \leq C |\log \varepsilon|^{-12}.$$

Using this fact and arguing as in (4.3.72) above we obtain

$$\left| \sum_{j \in J} \int_{B_j} (iu, \nabla u) \cdot \nabla^\perp \check{h}_* \right| \leq C \frac{1}{\varepsilon |\log \varepsilon|^4}. \quad (4.3.75)$$

The first term in (4.3.71) is estimated using the Jacobian Estimate : The function $\chi_{\text{in}} \check{h}_*$ satisfies the assumptions of Proposition 4.3.3. In particular, its support is included in

$\mathcal{A}_{\text{bulk}} \cap GS_\alpha \subset \tilde{\mathcal{A}} \cap GS_\alpha$. We thus have, using again (4.3.57) to bound the gradient of \check{h}_* and (3.4.23),

$$\int_{GS_\alpha} \mu \chi_{\text{in}} \check{h}_* = 2\pi \sum_{j \in J_{\text{in}}} \chi_{\text{in}}(a_j) d_j \check{h}_*(a_j) + O\left(\frac{1}{\varepsilon |\log \varepsilon|^3}\right). \quad (4.3.76)$$

Provided $\tilde{\alpha}$ is large enough (which we are free to decide) we thus finally have, inserting (4.3.73), (4.3.75) and (4.3.76) in (4.3.71)

$$\int_{\mathcal{A}_{\text{bulk}}} \check{h}_* \tilde{\mu} = 2\pi \sum_{j \in J_{\text{in}}} \chi_{\text{in}}(a_j) d_j \check{h}_*(a_j) + O\left(\frac{1}{\varepsilon |\log \varepsilon|^3}\right). \quad (4.3.77)$$

Then, going back to (4.3.70)

$$\int_{\mathcal{A}_{\text{bulk}}} \frac{1}{g^2} \nabla \check{h}_{\tilde{\mu}} \cdot \nabla \check{h}_{\mu_*} = D \left(2\pi \sum_{j \in J_{\text{in}}} \chi_{\text{in}}(a_j) d_j (\check{h}_*(a_j) - \check{h}_*(R_*)) \right) + O\left(\frac{1}{\varepsilon |\log \varepsilon|^3}\right) \quad (4.3.78)$$

On the other hand by definition of J_* and the explicit formula for \check{h}_*

$$|\check{h}_*(a_j) - \check{h}_*(R_*)| \leq C\Omega_1^{1/4} \text{ for any } j \in J_*.$$

Recalling that $\check{h}_*(R_*) = \max_{\mathcal{A}_{\text{bulk}}} \check{h}_*$ we have for any $j \in J_-$

$$d_j (\check{h}_*(a_j) - \check{h}_*(R_*)) \geq 0.$$

Also, for any $j \in J_+$,

$$d_j (\check{h}_*(a_j) - \check{h}_*(R_*)) \geq -d_j \check{h}_*(R_*) \geq -C d_j.$$

It follows that

$$\begin{aligned} \int_{\mathcal{A}_{\text{bulk}}} \frac{1}{g^2} \nabla \check{h}_{\tilde{\mu}} \cdot \nabla \check{h}_{\mu_*} &\geq -CD \left(\Omega_1^{1/4} \sum_{j \in J_*} \chi_{\text{in}}(a_j) d_j + \sum_{j \in J_+} \chi_{\text{in}}(a_j) d_j \right) \\ &\geq -CD \left(D\Omega_1^{1/4} + \frac{1}{\varepsilon |\log \varepsilon|^3} \right) \end{aligned} \quad (4.3.79)$$

where we have used (4.3.67). Going back to (4.3.68) and (4.3.69), combining with (4.3.49) and (4.3.67), the result is proved. \square

With the result of Proposition 4.3.4 in hand it is an easy task to complete the

Proof of Proposition 4.3.1. Collecting (4.3.34) and (4.3.39) we have

$$\begin{aligned}
 \mathcal{E}[u] &\geq \left(2\pi \sum_{j \in J_*} \chi_{\text{in}}(a_j) d_j \right)^2 I_* \left(1 - C\Omega_1^{1/4} \right) \\
 &+ 2\pi H(R_*) (1 + o(1)) \sum_{j \in J_*} \chi_{\text{in}}(a_j) d_j \\
 &- C \frac{\sum_{j \in J_*} \chi_{\text{in}}(a_j) d_j}{\varepsilon |\log \varepsilon^3|} - C \frac{|\log \varepsilon| \varepsilon^\alpha}{\varepsilon^2 \log |\log \varepsilon|} \\
 &\geq \left(2\pi \sum_{j \in J_*} \chi_{\text{in}}(a_j) d_j \right)^2 I_* \left(1 - C\Omega_1^{1/4} \right) \\
 &+ 2\pi H(R_*) (1 + o(1)) \sum_{j \in J_*} \chi_{\text{in}}(a_j) d_j - C \frac{\log |\log \varepsilon|}{\varepsilon^2 |\log \varepsilon|^{1+\tilde{\alpha}}}. \tag{4.3.80}
 \end{aligned}$$

We have used (4.1.11) and (4.3.27) for the second inequality. There only remains to minimize the sum of the first two terms above with respect to $2\pi \sum_{j \in J_*} \chi_{\text{in}}(a_j) d_j$ and use (4.1.10) and (4.1.11) to obtain

$$\mathcal{E}[u] \geq -\frac{H(R_*)^2}{4I_*} (1 + o(1)) - C \frac{\log |\log \varepsilon|}{\varepsilon^2 |\log \varepsilon|^{1+\tilde{\alpha}}} \geq -\frac{H(R_*)^2}{4I_*} (1 + o(1)). \tag{4.3.81}$$

The second inequality holds true because we choose $\tilde{\alpha}$ large enough and $H(R_*) \propto -\Omega_1^2 \varepsilon^{-2}$. This concludes the proof of the second inequality in (4.3.2). \square

4.4 Asymptotics for the vorticity

In this Section we provide the proof of Theorem 4.1.1. Actually, most of the ingredients are scattered in the preceding subsections. In particular it is rather straightforward from the proofs in Subsection 3.3 to deduce

Proposition 4.4.1 (Asymptotics for the modified vorticity).

Let $\tilde{\mu}$ be defined as in (4.3.48). We have for any test function $\phi \in C_c^1(\mathcal{A}_{\text{bulk}})$

$$\left| \int_{\mathcal{A}_{\text{bulk}}} \tilde{\mu} \phi + \frac{H(R_*)}{2I_*} \int_{\mathcal{A}_{\text{bulk}}} \delta_* \phi \right| \ll \frac{\Omega_1}{\varepsilon} \left(\int_{\mathcal{A}_{\text{bulk}}} \frac{1}{g^2} |\nabla \phi|^2 \right)^{1/2} \tag{4.4.1}$$

Proof. We first remark that, combining (4.2.2), (4.3.2) and (4.3.80)

$$\begin{aligned}
 -\frac{H(R_*)^2}{4I_*} (1 + o(1)) &\geq \left(2\pi \sum_{j \in J_*} \chi_{\text{in}}(a_j) d_j \right)^2 I_* (1 + o(1)) \\
 &+ 2\pi H(R_*) (1 + o(1)) \sum_{j \in J_*} \chi_{\text{in}}(a_j) d_j - o\left(\frac{\Omega_1}{\varepsilon^2} \right). \tag{4.4.2}
 \end{aligned}$$

We deduce that

$$2\pi \sum_{j \in J_*} \chi_{\text{in}}(a_j) d_j = -\frac{H(R_*)}{2I_*} (1 + o(1)) \tag{4.4.3}$$

and thus, going back to (4.3.67)

$$D = -\frac{H(R_*)}{2I_*}(1 + o(1)). \quad (4.4.4)$$

Next we note that, up to now, we have neglected one useful term in the proof of the lower bound, namely the third term in (4.3.68). Keeping this term and using (4.4.3), Equation (4.3.80) reads

$$\mathcal{E}[u] \geq -\frac{H(R_*)^2}{4I_*}(1 + o(1)) + \int_{\mathcal{A}_{\text{bulk}}} \frac{1}{g^2} |\nabla \check{h}_{\check{\mu}}|^2 (1 + o(1)) - o\left(\frac{\Omega_1}{\varepsilon^2}\right). \quad (4.4.5)$$

Thus (recall that $\check{\mu} = \tilde{\mu} - D\delta_*$), using (4.2.2) and the first inequality in (4.3.2)

$$\int_{\mathcal{A}_{\text{bulk}}} \frac{1}{g^2} |\nabla \check{h}_{\check{\mu}}|^2 \ll \frac{\Omega_1^2}{\varepsilon^2}$$

which implies the result via (4.3.44) and (4.4.4). \square

Recall that $\tilde{\mu}$ is defined by neglecting the superfluid current in a region where no convenient lower bound to the energetic contribution of μ is available and considering the vorticity associated to the remaining current. One can see this procedure as a regularization of μ , because we actually neglect the regions where we expect the phase of u to be singular (therefore $|u|$ to go to zero). Proposition 4.4.1 states that, modulo this regularization, one can estimate the vorticity in the dual norm of $(\int_{\mathcal{A}} g^{-2} |\nabla \phi|^2)^{1/2}$. This is the natural norm associated with the minimization problem (4.1.36) defining the contribution of the vortices to the energy. Note that such a control is not exactly stronger than a control in $(C_c^1(\mathcal{A}))^*$ norm because of the strong inhomogeneity of the weight g^{-2} (see Remark 4.1.1). It would be stronger for example if g^2 was a constant because of the embedding (in two dimensions) of H^{-1} in $(C_c^1(\mathcal{A}))^*$.

The necessity to use the L^∞ norm of the gradient of the test function ϕ to bound the left-hand side of (4.1.30) appears when approximating μ by $\tilde{\mu}$ in the proof below (that is when justifying (4.1.50) rigorously). The main task in the proof below is to estimate the effect of the currents that we have neglected when defining $\tilde{\mu}$.

Proof of Theorem 4.1.1. Let $\phi \in C_c^1(\mathcal{A}_{\text{bulk}})$. We compute

$$\begin{aligned} \int_{\mathcal{A}} \mu \phi &= - \int_{\mathcal{A}} (iu, \nabla u) \nabla^\perp (\chi_{\text{in}} \phi + \chi_{\text{out}} \phi) \\ &= - \int_{\mathcal{A}_{\text{bulk}} \setminus \cup_{j \in J} B_j} (iu, \nabla u) \nabla^\perp (\chi_{\text{in}} \phi) + \int_{\mathcal{A}_{\text{bulk}} \cap \cup_{j \in J} B_j} (iu, \nabla u) \nabla^\perp (\chi_{\text{in}} \phi) \\ &\quad - \int_{\mathcal{A}} (iu, \nabla u) \nabla^\perp (\chi_{\text{out}} \phi). \end{aligned} \quad (4.4.6)$$

The first term is estimated using Proposition 4.4.1 (note that $\chi_{\text{in}} \check{\xi}_{\text{in}} \phi$ has its support

included in $\mathcal{A}_{\text{bulk}}$). By definition

$$\begin{aligned}
 & - \int_{\mathcal{A}_{\text{bulk}} \setminus BS_\alpha \setminus \cup_{j \in J} B_j} (iu, \nabla u) \nabla^\perp (\chi_{\text{in}} \phi) = \int_{\mathcal{A}_{\text{bulk}}} \tilde{\mu} \chi_{\text{in}} \phi = \\
 & \quad - \frac{H(R_*)}{2I_*} \int_{\mathcal{A}} \delta_* \chi_{\text{in}} \phi + o\left(\frac{\Omega_1}{\varepsilon}\right) \left(\int_{\mathcal{A}} \frac{1}{g^2} |\nabla \phi|^2 \right)^{1/2} \\
 & = - \frac{H(R_*)}{2I_*} \int_{\mathcal{A}} \delta_* \phi + o\left(\frac{\Omega_1}{\varepsilon}\right) \left(\int_{\mathcal{A}} \frac{1}{g^2} |\nabla \phi|^2 \right)^{1/2} + O\left(N_\alpha^B \varepsilon |\log \varepsilon| \|\phi\|_{L^\infty(\mathcal{A})} \frac{\Omega_1}{\varepsilon}\right) \\
 & = - \frac{H(R_*)}{2I_*} \int_{\mathcal{A}_{\text{bulk}}} \phi + o\left(\frac{\Omega_1}{\varepsilon}\right) \left(\int_{\mathcal{A}} \frac{1}{g^2} |\nabla \phi|^2 \right)^{1/2} + o(\Omega_1 |\log \varepsilon| \|\nabla \phi\|_{L^\infty(\mathcal{A})}). \quad (4.4.7)
 \end{aligned}$$

We have used the fact that $\chi_{\text{in}} = 1$ on $\mathcal{C}_{R_*} \cap PS_\alpha$. To pass to the second line it is thus sufficient to note that the length of $\mathcal{C}_{R_*} \setminus PS_\alpha = \mathcal{C}_{R_*} \cap AS_\alpha$ is of order $N_\alpha^B \varepsilon |\log \varepsilon|$. Also, we have

$$\|\phi\|_{L^\infty(\mathcal{A})} \leq C \varepsilon |\log \varepsilon| \|\nabla \phi\|_{L^\infty(\mathcal{A})}$$

because $|\mathcal{A}| \propto \varepsilon |\log \varepsilon|$ and ϕ vanishes on $\partial \mathcal{A}$. Recalling (4.3.8) we obtain the third line of (4.4.7).

We now show that the other terms in (4.4.6) are remainders, arguing as when dealing with (4.3.71) in the proof of Proposition 4.3.4.

The second term in the right-hand side is estimated exactly as the sum of the second and the third term in (4.3.71), using the small area of the region covered by the vortex balls. The result is

$$\left| \int_{\mathcal{A}_{\text{bulk}} \cap \cup_{j \in J} B_j} (iu, \nabla u) \nabla^\perp (\chi_{\text{in}} \phi) \right| \leq C |\log \varepsilon|^{-1} \|\nabla \phi\|_{L^\infty(\mathcal{A})}.$$

Finally, the last term in (4.4.6) is estimated exactly as in (4.3.72), (4.3.73). We obtain

$$\left| \int_{\mathcal{A}} (iu, \nabla u) \nabla^\perp (\chi_{\text{out}} \phi) \right| \leq C \frac{|\log \varepsilon|^3}{|\log \varepsilon|^{\tilde{\alpha}}} \|\nabla \phi\|_{L^\infty}$$

which concludes the proof, taking $\tilde{\alpha}$ large enough. \square

4.5 Appendix A : The Cost Function and the Vortex Energy

In this appendix we study the cost function (4.1.25) and provide the proof of Lemma 4.3.2.

It is convenient to define

$$\omega^{\text{TF}} := \frac{2}{\sqrt{\pi} \varepsilon} \quad (\text{A.1})$$

and study the related function

$$H^{\text{TF}}(r) := \frac{1}{2} |\log \varepsilon| \rho^{\text{TF}}(r) + F^{\text{TF}}(r), \quad (\text{A.2})$$

where

$$F^{\text{TF}}(r) := 2 \int_{R_h}^r ds \vec{B}_{\omega^{\text{TF}}}(r) \cdot \vec{e}_\theta \rho^{\text{TF}}(r) = \varepsilon^2 \Omega^2 \int_{R_h}^r ds [\Omega s - ([\Omega] - \omega^{\text{TF}}) s^{-1}] (s^2 - R_h^2). \quad (\text{A.3})$$

In order to investigate the behavior of the infimum of H^{TF} inside the bulk, it is convenient to rescale the quantities and set

$$z := \varepsilon\Omega(r^2 - R_h^2), \quad (\text{A.4})$$

so that z varies on a scale of order one, i.e., more precisely $z \in [0, 2/\sqrt{\pi}]$. With such a choice the gain function can be easily estimated :

$$\begin{aligned} F^{\text{TF}}(r) &= \frac{\varepsilon^2\Omega^2}{2} \int_0^{r^2 - R_h^2} dt t [\Omega(t + R_h^2) - [\Omega] + \omega^{\text{TF}}] (t + R_h^2)^{-1} = \\ &= \frac{\varepsilon^2\Omega^2}{2} \int_0^{r^2 - R_h^2} dt t \left(\Omega t - \frac{4}{3\sqrt{\pi}\varepsilon} + O(1) \right) \left(1 - \frac{2}{\sqrt{\pi}\varepsilon\Omega} + t \right)^{-1} = \\ &= \frac{1}{2\varepsilon} \int_0^z ds s \left(s - \frac{4}{3\sqrt{\pi}} + O(\varepsilon) \right) \left(1 - \frac{2}{\sqrt{\pi}\varepsilon\Omega} + \frac{s}{\varepsilon\Omega} \right)^{-1} = \\ &= \frac{1}{2\varepsilon} \int_0^z ds s \left(s - \frac{4}{3\sqrt{\pi}} \right) + O(|\log \varepsilon|) = \frac{z^2}{6\varepsilon} \left(z - \frac{2}{\sqrt{\pi}} \right) + O(|\log \varepsilon|), \quad (\text{A.5}) \end{aligned}$$

where we have used the approximation $[1 - O((\varepsilon\Omega)^{-1})]^{-1} \equiv 1 + O((\varepsilon\Omega)^{-1})$. Applying the same rescaling to the energy cost function, we thus obtain

$$H^{\text{TF}}(r) := \frac{\tilde{H}^{\text{TF}}(z)}{12\varepsilon}, \quad (\text{A.6})$$

where (recall that $\Omega = (2(3\pi)^{-1} - \Omega_1) \varepsilon^{-2} |\log \varepsilon|^{-1}$)

$$\tilde{H}^{\text{TF}}(z) = z \left(\frac{2}{\pi} - 3\Omega_1 - 2z \left(\frac{2}{\sqrt{\pi}} - z \right) \right) - O(\varepsilon |\log \varepsilon|). \quad (\text{A.7})$$

Let us denote, for $z \in [0, \frac{2}{\sqrt{\pi}}]$

$$k(z) := z \left(\frac{2}{\pi} - 3\Omega_1 - 2z \left(\frac{2}{\sqrt{\pi}} - z \right) \right). \quad (\text{A.8})$$

It is straightforward to see that this function takes two local maxima and two local minima in $[0, \frac{2}{\sqrt{\pi}}]$. The maxima are at $\frac{2}{\sqrt{\pi}}$ and at

$$z_1 = \frac{2}{3\sqrt{\pi}} - \sqrt{\frac{1}{9\pi} + \frac{\Omega_1}{2}}.$$

The minima are at $z = 0$ with $k(0) = 0$ and at

$$z_2 = \frac{2}{3\sqrt{\pi}} + \sqrt{\frac{1}{9\pi} + \frac{\Omega_1}{2}}.$$

Computing this minimum we obtain that $k(z_2) < 0$ if and only if $\Omega_1 > 0$, thus $k(z_2)$ is the absolute minimum in this regime. More precisely, for $|\Omega_1| \ll 1$ we have

$$k(z_2) = -\frac{3}{\sqrt{\pi}}\Omega_1 + O(\Omega_1^2) \quad (\text{A.9})$$

and thus, defining $R_* > 0$ by

$$R_*^2 := R_h^2 + (\varepsilon\Omega)^{-1} z_2 \quad (\text{A.10})$$

we obtain, for $(\varepsilon|\log \varepsilon|)^{1/2} \ll |\Omega_1| \ll 1$

$$H^{\text{TF}}(R_*) = -\frac{\Omega_1}{4\sqrt{\pi\varepsilon}} + O\left(\frac{\Omega_1^2}{\varepsilon}\right). \quad (\text{A.11})$$

Also

$$k''(z_2) = \frac{1}{3\sqrt{\pi}} + O(\Omega_1)$$

and

$$k'(0) = \frac{1}{\pi} - \Omega_1.$$

Recalling that $k(z)$ increases from 0 to z_1 and from z_2 to 1 and decreases from z_1 to z_2 , we have for any z and an appropriate choice of constants k_1, k_2, k_3 such that $|z - z_2| > k_1\Omega_1^{1/4}$ and $|z| > k_2C\Omega_1^{1/2}$

$$k(z) > k_3\Omega_1^{1/2}.$$

Collecting the preceding facts we have proved

Proposition A.1 (TF vortex energy).

Let Ω be of the form

$$\Omega = \frac{2}{3\pi\varepsilon^2|\log \varepsilon|} - \frac{\Omega_1}{\varepsilon^2|\log \varepsilon|} \quad (\text{A.12})$$

with $(\varepsilon|\log \varepsilon|)^{1/2} \ll \Omega_1 \ll 1$. Let $\vec{r} \in \mathcal{A}^{\text{TF}}$. We have

$$H^{\text{TF}}(r) \geq -\frac{\Omega_1}{4\sqrt{\pi\varepsilon}} + O\left(\frac{\Omega_1^2}{\varepsilon}\right) \quad (\text{A.13})$$

$$H^{\text{TF}}(r) \geq C\frac{\Omega_1^{1/2}}{\varepsilon} \text{ if } r \geq R_h + C\varepsilon|\log \varepsilon|\Omega_1^{1/2} \text{ and } |r - R_*| \geq C\varepsilon|\log \varepsilon|\Omega_1^{1/4} \quad (\text{A.14})$$

We now compare the cost TF function H^{TF} to the original function appearing in our analysis, H . The following result is a part of the proof of Proposition A.2 in [46] ([Chapter 3]) and is sufficient for our purpose.

Proposition A.2 (Comparison of the cost functions).

Let ω and $g = g_{A,\omega}$ be defined as in Proposition 4.1.1. Let Ω be as above and H be the cost function defined in (4.1.25). For any $\vec{r} \in \tilde{\mathcal{A}}$ (see definition (4.3.9)) there holds

$$|g^2(r) - \rho^{\text{TF}}(r)| \leq \frac{C|\log \varepsilon|^{5/2}}{\varepsilon^{1/2}} \quad (\text{A.15})$$

$$|F(r) - F^{\text{TF}}(r)| \leq \frac{C}{\varepsilon|\log \varepsilon|} \quad (\text{A.16})$$

$$(\text{A.17})$$

and thus

$$|H(r) - H^{\text{TF}}(r)| \leq \frac{C}{\varepsilon|\log \varepsilon|}. \quad (\text{A.18})$$

We are now equipped to present the

Proof of Lemma 4.3.2. The proof of (4.3.35) is based on a simple estimate from [46, Lemma 4.1](Lemma 3.4.1) that we recall

$$|F(r)| \leq C \min \left(\frac{|r - R_{<}|}{\varepsilon} g^2(r), 1 + \frac{C|r - 1|}{\varepsilon^2 |\log \varepsilon|} \right). \quad (\text{A.19})$$

We then write

$$\begin{aligned} |d_j| \frac{1}{2} g^2(a_j) |\log \varepsilon| \left(1 - C \frac{\log |\log \varepsilon|}{|\log \varepsilon|} \right) + d_j \xi_{\text{in}}(a_j) F(a_j) \\ \geq C |d_j| \left(g^2(a_j) |\log \varepsilon| \left(1 - C \frac{\log |\log \varepsilon|}{|\log \varepsilon|} \right) - |F(a_j)| \right). \end{aligned}$$

On the other hand, we have from the definition (4.3.30)

$$||a_j| - R_{<}| \leq C \varepsilon |\log \varepsilon| \Omega_1^{1/2} \ll \varepsilon |\log \varepsilon|$$

for any $j \in J_{\text{out}}$. The result (4.3.35) follows using (A.19).

We turn to the energetic cost of the vortices in J_{in} . First we deduce from (A.16) that

$$F(a_j) = F^{\text{TF}}(a_j) + O \left(\frac{1}{\varepsilon |\log \varepsilon|} \right)$$

for any $j \in J_{\text{in}}$. It is then straightforward (recall that $\Omega_1 \gg \log |\log \varepsilon| |\log \varepsilon|^{-1}$) from the computation (A.5) to obtain

$$F(a_j) < -C \frac{\Omega_1^{1/2}}{\varepsilon} < 0$$

for any $j \in J_{\text{in}}$ such that $|a_j| \leq 1 - \varepsilon |\log \varepsilon|^{1/2}$. On the other hand, if $|a_j| \geq 1 - \varepsilon |\log \varepsilon|^{1/2}$, $g^2(a_j) \geq C(\varepsilon |\log \varepsilon|)^{-1}$ and thus, using (A.19)

$$|F(a_j)| \ll g^2(a_j).$$

We deduce that for any $j \in J_-$

$$\frac{1}{2} g^2(a_j) |\log \varepsilon| |d_j| + d_j \xi_{\text{in}}(a_j) F(a_j) \geq C |d_j| g^2(a_j) \geq C |d_j| \frac{\Omega_1^{1/2}}{\varepsilon}$$

using (A.15). On the other hand, if $j \in J_+$

$$\frac{1}{2} g^2(a_j) |\log \varepsilon| |d_j| + d_j \xi_{\text{in}}(a_j) F(a_j) = |d_j| \left(\frac{1}{2} g^2(a_j) |\log \varepsilon| + \xi_{\text{in}}(a_j) F(a_j) \right)$$

and thus (4.3.37) follows from Propositions A.1 and A.2 in the case where $j \in J_+$. We also use $\xi_{\text{in}} \leq 1$.

There remains to show that (4.3.36) holds when $j \in J_*$, which is a consequence of (A.13) and (A.18) once one has recalled that $\xi_{\text{in}}(a_j) = 1$ for any $j \in J_*$. \square

4.6 Appendix B : Summary of Notation

4.6.1 Radii and Annuli

The first annulus we defined is \mathcal{A}^{TF} :

$$\mathcal{A}^{\text{TF}} = \{R_h \leq r \leq 1\}, \quad R_h = \sqrt{1 - \frac{2}{\sqrt{\pi}\varepsilon\Omega}}. \quad (4.6.1)$$

The annulus where most of the analysis is actually performed is

$$\mathcal{A} := \{\tilde{r} : R_{<} \leq r \leq 1\}, \quad R_{<} = R_h + \varepsilon|\log \varepsilon|^{-1}. \quad (4.6.2)$$

We then used the notation $\mathcal{A}_{\text{bulk}}$ for the annulus where we could prove the vorticity asymptotics

$$\mathcal{A}_{\text{bulk}} := \{\tilde{r} | R_{\text{bulk}} \leq r \leq 1\}, \quad R_{\text{bulk}} := R_h + \varepsilon|\log \varepsilon|\Omega_1^{1/2}. \quad (4.6.3)$$

In the proof of the upper bound, we needed to define a boundary layer by using, for a large integer n ,

$$\tilde{R} = R_{<} + \varepsilon^n. \quad (4.6.4)$$

We also introduced a density functional with a jump discontinuity at

$$\bar{R} := R_h + \varepsilon^{5/6}. \quad (4.6.5)$$

For the lower bound we used another reduced annulus where the density g^2 satisfies a suitable lower bound

$$\tilde{\mathcal{A}} := \{\tilde{r} : R_{>} \leq r \leq 1\}, \quad R_{>} := R_h + \varepsilon|\log \varepsilon|^{-1}. \quad (4.6.6)$$

The construction of a radial partition of unity required the radii

$$R_{\text{cut}}^+ := 1 - \varepsilon|\log \varepsilon|^{-1}, \quad (4.6.7)$$

$$R_{\text{cut}}^- := R_{>} + \varepsilon|\log \varepsilon|^{-1}. \quad (4.6.8)$$

And there was the radii defining where the vortices gathered, satisfying

$$R_* = \sqrt{R_h^2 + (\varepsilon\Omega)^{-1} \left(\frac{1}{\sqrt{\pi}} + O(\Omega_1) \right)}. \quad (4.6.9)$$

We recall that

$$R_{<} < \tilde{R} < R_h < \bar{R} < R_{>} < R_{\text{bulk}} < R_{\text{cut}}^- < R_* < R_{\text{cut}}^+ < 1.$$

All radii defined above are relatively close to R_h , except R_{cut}^- which is relatively close to 1 and R_* . By relatively close we mean that the distance is much smaller than the size of \mathcal{A}^{TF} , that is of order $(\varepsilon|\log \varepsilon|)^{-1}$.

4.6.2 Cut-offs, Partitions of Unity

The construction of the trial function uses cut-offs around the vortex cores to compensate the phase singularity

$$\xi_{i,j} := \begin{cases} 0 & \text{in } B(p_{i,j}, t) \\ 1 & \text{in } \mathcal{A} \setminus B(p_{i,j}, 2t) \end{cases} \quad (4.6.10)$$

and a cut-off near the inner boundary of \mathcal{A} to ensure that the trial function is indeed in $H^1(\mathcal{B})$

$$\xi_{\text{BL}}(\vec{r}) = \begin{cases} 1 & \text{if } r \geq \tilde{R} \\ 1 + \frac{r - R_{<}}{\tilde{R} - R_{<}} & \text{if } r \leq \tilde{R}. \end{cases} \quad (4.6.11)$$

These two types of cut-offs are combined in

$$\xi = \xi_{\text{BL}} \prod_{i,j} \xi_{i,j}. \quad (4.6.12)$$

It was important to note that

$$|\nabla \xi_{i,j}| \leq \frac{C}{t} \text{ in } B(p_{i,j}, 2t) \setminus B(p_{i,j}, t). \quad (4.6.13)$$

The lower bound used an azimuthal partition of unity (see Definitions 4.3.1 and 4.3.2)

$$\begin{aligned} \chi_l^{\text{U}} &:= 1 \text{ on } UPS_{\alpha}^l, \\ \chi_l^{\text{U}} &:= 0 \text{ on } PS_{\alpha}^m, \forall m \in \{1, \dots, M\}, \text{ and on } UPS_{\alpha}^{l'}, \forall l' \neq l, \\ \chi_m^{\text{P}} &:= 1 \text{ on } PS_{\alpha}^m, \\ \chi_m^{\text{P}} &:= 0 \text{ on } UPS_{\alpha}^{l'}, \forall l \in \{1, \dots, L\}, \text{ and on } PS_{\alpha}^{m'}, \forall m' \neq m, \\ \sum_m \chi_m^{\text{P}} + \sum_l \chi_l^{\text{U}} &= 1 \text{ on } \mathcal{A}. \end{aligned} \quad (4.6.14)$$

We shortened the notation to

$$\chi_{\text{in}} := \sum_{m=1}^M \chi_m^{\text{P}}, \quad \chi_{\text{out}} := \sum_{l=1}^L \chi_l^{\text{U}} \quad (4.6.15)$$

and it was crucial to remark that

$$|\nabla \chi_l^{\text{U}}| \leq \frac{C}{\varepsilon |\log \varepsilon|}, \quad |\nabla \chi_m^{\text{P}}| \leq \frac{C}{\varepsilon |\log \varepsilon|}. \quad (4.6.16)$$

A radial partition of unity was also used

$$\begin{aligned} \xi_{\text{in}}(r) &:= 1 \text{ for } R_{\text{cut}}^- \leq r \leq R_{\text{cut}}^+, \\ \xi_{\text{in}}(r) &:= 0 \text{ for } R_{<} \leq r \leq R_{>} \text{ and for } r = 1, \\ \xi_{\text{out}}(r) &:= 1 \text{ for } R_{<} \leq r \leq R_{>}, \\ \xi_{\text{out}}(r) &:= 0 \text{ for } R_{\text{cut}}^- \leq r \leq R_{\text{cut}}^+, \\ \xi_{\text{in}} + \xi_{\text{out}} &= 1 \text{ on } \mathcal{A} \end{aligned} \quad (4.6.17)$$

where we imposed

$$|\nabla \xi_{\text{in}}| \leq \frac{C |\log \varepsilon|}{\varepsilon}, \quad |\nabla \xi_{\text{out}}| \leq \frac{C |\log \varepsilon|}{\varepsilon}. \quad (4.6.18)$$

All these cut-offs and partitions of unity are positive and bounded above by 1.

4.6.3 Functionals and Energies

Apart from the original GP energy we worked with

$$\mathcal{E}^{\text{TF}}[\rho] = \int_{\mathcal{B}} -\Omega^2 r^2 \rho + \varepsilon^{-2} \rho^2. \quad (4.6.19)$$

that was to be minimized amongst positive and normalized in $L^1(\mathcal{B})$ densities ρ . The minimization yields a ground state

$$\rho^{\text{TF}}(r) = \frac{1}{2} [\varepsilon^2 \mu^{\text{TF}} + \varepsilon^2 \Omega^2 r^2]_+ = \frac{\varepsilon^2 \Omega^2}{2} [r^2 - R_{\text{h}}^2]_+, \quad (4.6.20)$$

and a ground state energy

$$E^{\text{TF}} = -\Omega^2 \left(1 - \frac{4}{3\sqrt{\pi}\varepsilon\Omega} \right). \quad (4.6.21)$$

For any $a \in \mathbb{Z}$ there was a giant vortex functional

$$\hat{\mathcal{E}}_{\mathcal{A},a}^{\text{GP}}[\phi] := \mathcal{E}_{\mathcal{A}}^{\text{GP}}[\phi e^{i([\Omega]-a)\theta}] = \int_{\mathcal{A}} |\nabla\phi|^2 + \frac{1}{\varepsilon^2} |\phi|^4 + \left(\frac{([\Omega]-a)^2}{r^2} - 2\Omega([\Omega]-a) \right) |\phi|^2 \quad (4.6.22)$$

defined for real-valued functions ϕ . Associated to $\hat{\mathcal{E}}_{\mathcal{A},a}^{\text{GP}}$ are a ground state $g_{\mathcal{A},a}$ and a ground state energy $\hat{E}_{\mathcal{A},a}^{\text{GP}}$ that we minimized with respect to a :

$$\hat{E}_{\mathcal{A},\omega}^{\text{GP}} = \min_{a \in \mathbb{Z}} \hat{E}_{\mathcal{A},a}^{\text{GP}}. \quad (4.6.23)$$

The matter density that we used most often was g^2 , which was a short-hand notation for $g_{\mathcal{A},\omega}$:

$$g = g_{\mathcal{A},\omega}. \quad (4.6.24)$$

The following reduced energy was central in the analysis :

$$\mathcal{E}[v] := \int_{\mathcal{A}} g^2 |\nabla v|^2 - 2g^2 \vec{B} \cdot (iv, \nabla v) + \frac{g^4}{\varepsilon^2} (1 - |v|^2)^2. \quad (4.6.25)$$

The vector potential in there is

$$\vec{B}(r) := B(r) \vec{e}_\theta = (\Omega r - ([\Omega] - \omega) r^{-1}) \vec{e}_\theta. \quad (4.6.26)$$

Finally, two energy functionals whose arguments are Radon measures with support in \mathcal{A} have been used. For such a measure ν we defined h_ν by requiring

$$\begin{cases} -\nabla \left(\frac{1}{g^2} \nabla h_\nu \right) = \nu \text{ in } \mathcal{A} \\ h_\nu = 0 \text{ on } \partial\mathcal{A}. \end{cases} \quad (4.6.27)$$

The first energy functional was then

$$I(\nu) := \int_{\mathcal{A}} \frac{1}{g^2} |\nabla h_\nu|^2 \quad (4.6.28)$$

and the associated variational problem was

$$I_* := \inf_{\nu \in \mathcal{D}_*, \int \nu = 1} I(\nu) = I(\delta_*) \quad (4.6.29)$$

where the infimum is taken over the set \mathcal{D}_* of positive Radon measures with support on the circle \mathcal{C}_{R_*} of radius R_* .

The renormalized energy appearing in the analysis was (see below where the definition of H is recalled)

$$-\frac{H(R_*)^2}{4I_*} = \inf_{\nu \in \mathcal{D}_*} \left(\int_{\mathcal{A}} \frac{1}{g^2} |\nabla h_\nu|^2 + H(R_*) \int_{\mathcal{A}} \nu \right). \quad (4.6.30)$$

In the proof of the lower bound we had to introduce a modified electrostatic problem

$$\check{I}(\nu) := \int_{\mathcal{A}_{\text{bulk}}} \frac{1}{g^2} |\nabla \check{h}_\nu|^2. \quad (4.6.31)$$

where

$$\begin{cases} -\nabla \left(\frac{1}{g^2} \nabla \check{h}_\nu \right) = \nu \text{ in } \mathcal{A}_{\text{bulk}} \\ \check{h}_\nu = 0 \text{ on } \partial \mathcal{A}_{\text{bulk}}. \end{cases} \quad (4.6.32)$$

Similarly to I_* we defined

$$\check{I}_* := \inf_{\nu \in \mathcal{D}_*, \int \nu = 1} \check{I}(\nu) = \check{I}(\delta_*). \quad (4.6.33)$$

We recall that there are two positive constants C_1, C_2 such that

$$C_1 \leq I_*, \check{I}_* \leq C_2.$$

4.6.4 Some Orders of Magnitude

We want to recall that typically (that is, sufficiently far from the inner boundary of the annulus)

$$\begin{aligned} \rho^{\text{TF}}(r) &\propto (\varepsilon |\log \varepsilon|)^{-1} \\ g^2(r) &\propto (\varepsilon |\log \varepsilon|)^{-1} \\ F(r) &\propto -\varepsilon^{-1} \\ H(r) &\propto \varepsilon^{-1} \end{aligned}$$

where

$$\begin{aligned} F(r) &= 2 \int_{R_<}^r g^2(s) \left(\Omega s - ([\Omega] - \omega) \frac{1}{s} \right) ds \\ H(r) &= \frac{1}{2} g^2(r) |\log \varepsilon| + F(r). \end{aligned}$$

The only region where H is negative is close to the radius R_* and we have

$$H(R_*) \propto -\frac{\Omega_1}{\varepsilon}.$$

This leads to the conclusion that

$$\mathcal{E}[u] \propto -\frac{\Omega_1^2}{\varepsilon^2}$$

where u is defined in (4.1.23).

Finally we recall that both in the upper and lower bound proofs we used a partition of the annulus into

$$N \propto \frac{1}{\varepsilon |\log \varepsilon|}$$

cells of side-lengths $\varepsilon |\log \varepsilon|$. The trial function we constructed had

$$M \propto \Omega_1 |\log \varepsilon|$$

vortices per cell, and thus

$$\mathcal{N} = MN \propto \frac{\Omega_1}{\varepsilon}$$

vortices in total.

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Résumé

Lorsqu'un gaz de bosons est suffisamment refroidi, une transition de phase apparaît : toutes les particules se concentrent dans le même état d'énergie. On appelle l'objet résultant de ce phénomène un condensat de Bose-Einstein. On peut le décrire par une fonction d'onde macroscopique, minimisant à l'équilibre la fonctionnelle d'énergie de Gross-Pitaevskii.

Une des propriétés remarquables des condensats est leur superfluidité. Elle peut se manifester par l'apparition de vortex (tourbillons) dans un condensat mis en rotation.

Dans cette thèse nous étudions le comportement asymptotique des minimiseurs de la fonctionnelle de Gross-Pitaevskii bi-dimensionnelle et des énergies associées dans différents régimes de paramètres rendant compte de situations physiquement intéressantes. Nous cherchons à identifier certaines transitions de phase caractérisées par l'organisation des vortex du condensat.

Dans une première partie nous étudions une situation où il est justifié physiquement de considérer un problème simplifié. La minimisation de la fonctionnelle d'énergie est alors restreinte au premier espace propre de l'opérateur de Ginzburg-Landau (le plus bas niveau de Landau, lié à l'espace de Fock-Bargmann). Nous étudions théoriquement et numériquement le modèle simplifié dans un régime où le condensat est annulaire et contient un réseau de vortex déformé.

Une seconde partie est consacrée à un régime de rotation extrême où le problème limite devient linéaire. Nous montrons que ce problème limite décrit correctement les asymptotiques d'énergie et de densité de matière. Sous une hypothèse supplémentaire nous démontrons qu'un vortex géant se forme, c'est-à-dire un condensat annulaire dont tous les vortex se rassemblent dans la zone centrale de faible densité de matière.

Les deux dernières parties de la thèse sont consacrées à l'évaluation de la vitesse critique pour l'apparition du vortex géant. Nous montrons d'abord que le vortex géant apparaît au dessus d'un certain seuil que nous calculons en fonction des autres paramètres du problème, ce qui fournit une borne supérieure de la vitesse critique. Dans une quatrième partie nous montrons que cette borne supérieure est en fait optimale en considérant des vitesses proches du seuil par valeur inférieure. Nous montrons alors que des vortex sont présents dans le condensat et qu'ils se répartissent uniformément le long d'un cercle.

Abstract

When a Bose gas is cooled below a certain critical temperature, all particles gather in the same energy state, forming a Bose-Einstein condensate. Such an object can be described using a macroscopic wave function that minimizes the Gross-Pitaevskii energy functional. A remarkable property of Bose-Einstein condensates is that they are superfluid. In particular they can nucleate quantized vortices when set into rotation.

In this thesis we study the behavior of the ground states and ground state energies of the two-dimensional Gross-Pitaevskii energy functional in certain asymptotic regimes describing interesting physical situations. Our purpose is to identify phase transitions characterized by the organization of vortices in the condensate.

In a first chapter we consider a situation where it is physically justified to simplify the problem by restricting the variational space. We minimize the Gross-Pitaevskii functional amongst the functions of the first eigenspace of the Ginzburg-Landau operator (that is the lowest Landau level, linked to the Fock-Bargmann space). We study theoretically and numerically the simplified problem in a regime where the condensate is annular and contains a distorted vortex lattice.

A second chapter is devoted to an extreme rotation regime where the limit problem becomes linear. We prove that this limit problem accurately describes the energy and matter density asymptotics. Under an additional assumption we prove that a giant vortex appears. That is, an annular condensate whose vortices gather in the central low density hole.

In the third and fourth chapters we aim at evaluating the critical speed for the appearance of the giant vortex. We first prove that the giant vortex forms above a certain threshold that we compute as a function of the other parameters in the problem. This yields an upper bound to the critical speed. We then prove that this upper bound is optimal by considering slightly subcritical speeds. We prove that in this case vortices do occur in the condensate and are uniformly distributed along a particular circle.