

Contributions to the theory of sliding mode control

Vincent Brégeault

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Contributions to the theory of sliding mode control

2

- 1 Introduction to sliding mode control
- 2 Adaptive sliding mode control
- 3 Introduction to higher order sliding mode control
- 4 Introduction to time optimal control
- 5 Variable structure and time optimal control
- 6 Conclusion

- 1 Introduction to sliding mode control
 - Considered system
 - Classical SMC
 - Equivalent control
 - Chattering
- 2 Adaptive sliding mode control
- 3 Introduction to higher order sliding mode control
- 4 Introduction to time optimal control
- 5 Variable structure and time optimal control
- 6 Conclusion

Considered system

SISO uncertain non linear system under canonical controllability form

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\vdots$$

$$\dot{x}_n \in \psi(\mathbf{x}, t) + [-C(\mathbf{x}, t), C(\mathbf{x}, t)] + [\Gamma_m(\mathbf{x}, t), \Gamma_M(\mathbf{x}, t)]u$$

$$y = x_1$$

$[-C(\mathbf{x}, t), C(\mathbf{x}, t)]$ is a matched perturbation,
 $[\Gamma_m(\mathbf{x}, t), \Gamma_M(\mathbf{x}, t)]$ an uncertainty on the gain.

- $0 < \Gamma_m(\mathbf{x}, t) \leq \Gamma_M(\mathbf{x}, t) < \infty$
- $C(\mathbf{x}, t) < \Gamma_m(\mathbf{x}, t)u_M$

Classical sliding mode control

[Utkin, 1992, Utkin et al., 1999]

5

- Design a sliding hyperplane defined by

$$\{\mathbf{x} \text{ so that } \sigma(\mathbf{x}) = x_n + a_{n-1}^* x_{n-1} + \dots + a_1^* x_1 = 0\}$$

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$$\{\mathbf{x} \text{ so that } \sigma(\mathbf{x}) = x_n + a_{n-1}^* x_{n-1} + \dots + a_1^* x_1 = 0\}$$

- and a control law $u = -u_M \text{sign}(\sigma(\mathbf{x}))$
which forces the system to the sliding surface in finite time

Classical sliding mode control

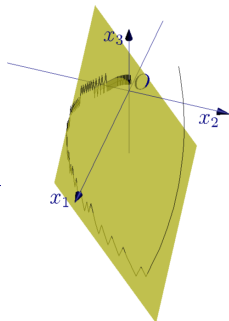
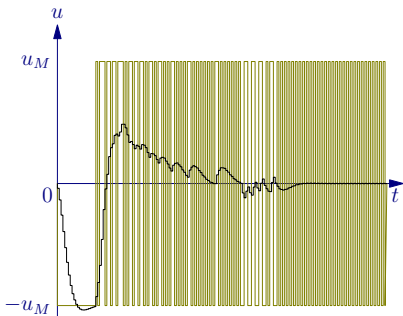
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- and a control law $u = -u_M \text{sign}(\sigma(\mathbf{x}))$
which forces the system to the sliding surface in finite time



Equivalent and nominal control

The actual control is discontinuous, but 2 useful continuous controls :

- **Nominal control** : If perturbations are not taken into account, u_{nom} can be computed *in advance* :

$$\dot{x}_n = \psi(\mathbf{x}, t) + \Gamma u_{nom} = - \sum_{i=1}^{n-1} a_i x_{i+1}$$

$$u_{nom}(\mathbf{x}) = \frac{\sum_{i=1}^{n-1} a_i x_i - \psi(\mathbf{x}, t)}{\Gamma(\mathbf{x}, t)}$$

Use : add nominal control to reduce amplitude of discontinuous control :

$$u = u_{nom} - u_M \text{sign}(\sigma)$$

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- **Equivalent control** : If perturbations are taken into account, u_{eq} can be known *afterwards*, by filtering.

Use : observer, suppress discontinuity, dimension amplitude of control

Chattering

7

Definition : Unwanted high frequency oscillation of the control and the output

Causes :

- time delay, neglected (fast) dynamics
- measurement/observation noise

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- using the super twisting algorithm
- adapting the amplitude

Conclusion

2 design steps = 2 questions :

- Sliding surface : It sets the dynamics of the system in sliding mode.
Which one to choose ?

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- Sliding surface : It sets the dynamics of the system in sliding mode.
Which one to choose ?
- Reaching law : Discontinuous = robust, but chattering.
How to reduce the chattering while keeping (most of) the robustness ?

- 1 Introduction to sliding mode control
- 2 Adaptive sliding mode control**
 - Control law
 - Worst case and enhancement
 - Electropneumatic benchmark
 - Test of the adaptive control
- 3 Introduction to higher order sliding mode control
- 4 Introduction to time optimal control
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Other adaptive approaches

10

- Fuzzy logic : do not guarantee precision
[Munoz and Sbarbaro, 2000, Tao et al., 2003]
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- This approach : boundary layer [Plestan et al., 2010, Plestan et al., ture]

Adaptive control law [Plestan et al., 2010]

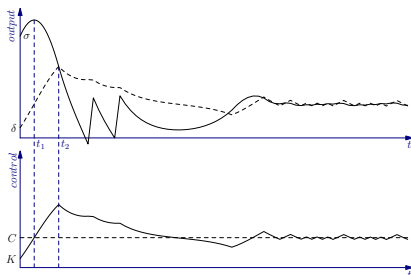
11

Theorem

The control law $u = -K(t) \text{sign}(\sigma)$ is stable
when amplitude $K(t)$ varies as

$$\dot{K} = \begin{cases} \bar{K}|\sigma| \text{sign}(|\sigma| - \delta(t)) & \text{if } K > 0 \text{ or } \text{sign}(|\sigma| - \delta(t)) > 0 \\ 0 & \text{if } K = 0 \text{ and } \text{sign}(|\sigma| - \delta(t)) \leq 0 \end{cases}$$

$K(t) \searrow$ if $\sigma \in [-\delta(t); \delta(t)]$ and
 $K(t) \nearrow$ outside.



Adaptive control law [Plestan et al., 2010]

11

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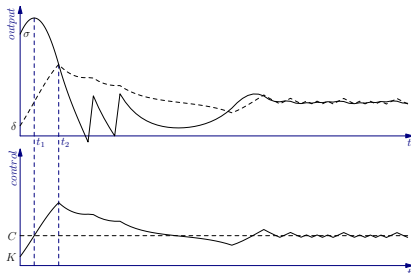
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$\delta(t)$ must be

- as small as possible, for precision
- bigger than amplitude of chattering (depending on $K(t)$)



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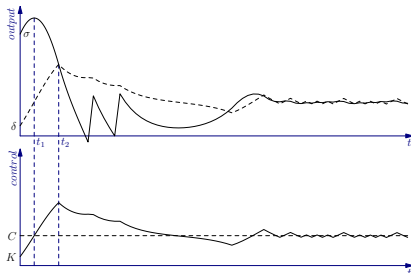
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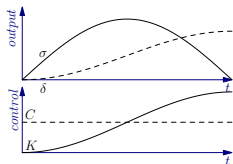
$$\delta(t) \geq 2\Gamma_M \theta K(t)$$



Worst case and enhancement

Consider the system as LTI, because $\text{sign}(\sigma)$ constant.

$$\begin{cases} \dot{\sigma} = -\Gamma_m K + C \\ \dot{K} = \bar{K}\sigma \end{cases} \Leftrightarrow \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \begin{bmatrix} C \\ 0 \end{bmatrix}$$



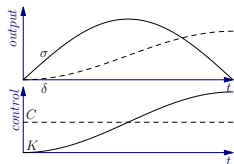
Worst case and enhancement

12

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Majorant : $\sigma_M = \frac{C}{\sqrt{\bar{K}\Gamma_m}}$

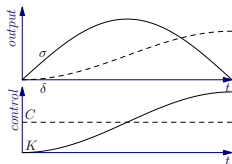


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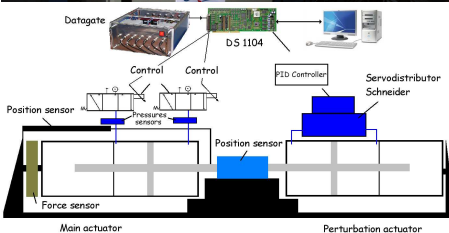
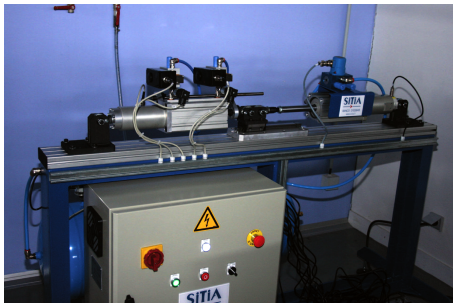


Add a linear term : $u = -K(t)\text{sign}(\sigma) - K_I\sigma$

The majorant of σ become : $\sigma_M = \frac{C}{\sqrt{\bar{K}\Gamma_m}} e^{-\frac{\kappa_I \pi}{\sqrt{4\bar{K}\Gamma_m - \kappa_I^2}}}$

Electropneumatic benchmark

13



Electropneumatic benchmark

13

System :

$$\dot{p}_P = \frac{krT}{V_P(y)} [\phi_P(p_P) + \psi_P(p_P, \text{sign}(u_P))u_P - \frac{S}{rT}p_P v]$$

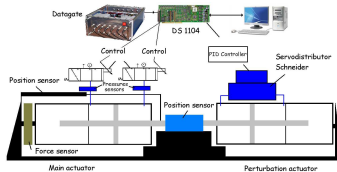
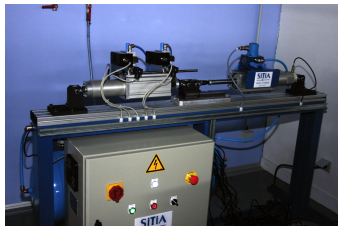
$$\dot{p}_N = \frac{krT}{V_N(y)} [\phi_N(p_N) + \psi_N(p_N, \text{sign}(u_N))u_N + \frac{S}{rT}p_N v]$$

$$\dot{v} = \frac{1}{M} [S(p_P - p_N) - F_f - F]$$

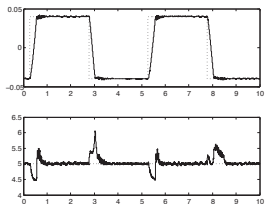
$$\dot{y} = v$$

Outputs :

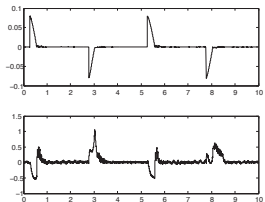
- position : y (relative degree 3)
- mean of pressures : $\frac{p_P + p_N}{2}$
(relative degree 1)
(control stiffness or consumption of air)



Test of the adaptive control [Brégeault et al., 2010]

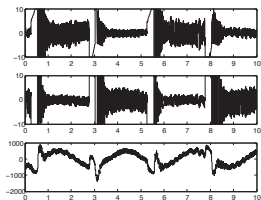
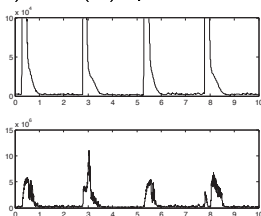


Position (m) and mean of pressures (bar)



Errors of position (m) and pressures (bar)

$$\bar{K}_y = 8000, \epsilon_1(t) = 2.5 K_y(t)T ; \bar{K}_p = 8000 \text{ and } \epsilon_p(t) = 10 K_2(t)T$$

 u_P (V), u_N (V), perturbation (N)amplitude K

- 1 Introduction to sliding mode control
- 2 Adaptive sliding mode control
- 3 Introduction to higher order sliding mode control**
 - Definitions
 - Errors
 - Homogeneity
 - Examples of second order algorithms
 - Proof of convergence of the super twisting
- 4 Introduction to time optimal control
- 5 Variable structure and time optimal control
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Definitions [Levant, 1993]

16

Definition (Ideal n -order sliding mode)

The sliding variable σ and its $n-1$ successive derivatives are continuous and reach 0, in the *absence* of chattering.

Definitions [Levant, 1993]

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Definition (Ideal n-order sliding mode)

The sliding variable σ and its $n-1$ successive derivatives are continuous and reach 0, in the *absence* of chattering.

Definition (Real n-order sliding mode)

Precision in $\mathcal{O}(\tau^n)$, in the *presence* of a source of chattering of amplitude majored by τ . Usually time delay, measurement error $\tau = \sqrt[n]{\epsilon}$.

Theorem

If there is a real n^{th} order sliding mode, then $\sigma = \mathcal{O}(\tau^n)$, $\dot{\sigma} = \mathcal{O}(\tau^{n-1})$, ..., $\sigma^{(n-1)} = \mathcal{O}(\tau)$

Error due to time delay

17

Theorem ([Levant, 1993])

If

- $\sigma^{(n)}$ is continuous and bounded on an interval τ
- σ remains in a vicinity of 0

then $\sigma = \mathcal{O}(\tau^n)$

Example

A classical sliding mode is

- first order with respect to the sliding variable σ
- n^{th} order with respect to x_1 for chattering due to pure time delay

Error due to measurement and observation noise

18

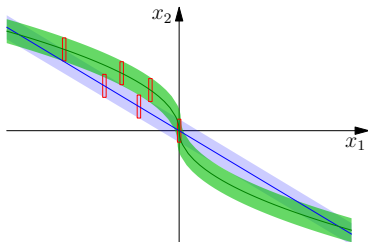
If the state is known with the precision

$$\epsilon = [\mathcal{O}(\epsilon) \ \mathcal{O}(\epsilon^{\frac{1}{2}}) \ \dots \ \mathcal{O}(\epsilon^{\frac{1}{n-1}}) \ \mathcal{O}(\epsilon^{\frac{1}{n}})]^T$$

for example, if it comes from a differentiator such as [Levant, 2003].

For a classical sliding mode, the error is

$$\begin{aligned} \sigma(\Delta \mathbf{x}) &= \lambda_1 \Delta x_1 + \dots + \lambda_{n-1} \Delta x_{n-1} + \Delta x_n \\ &= \lambda_1 \mathcal{O}(\epsilon) + \lambda_2 \mathcal{O}(\epsilon^{\frac{1}{2}}) + \dots \\ &\quad + \lambda_{n-1} \mathcal{O}(\epsilon^{\frac{1}{n-1}}) + \mathcal{O}(\epsilon^{\frac{1}{n}}) \\ &= \mathcal{O}(\epsilon^{\frac{1}{n}}) = \mathcal{O}(\tau) \end{aligned}$$



Homogeneity [Bacciotti and Rosier, 2001, Levant, 2005] 19

Example

- Linear system : homogeneous with degree 0 : $\mathbf{f}(\kappa\mathbf{x}) = \kappa\mathbf{f}(\mathbf{x})$

Homogeneity [Bacciotti and Rosier, 2001, Levant, 2005] 19

Definition

A vector field $\mathbf{f} \in \mathbb{R}^n$ is homogeneous of degree q by the dilation $d_\kappa(x_1, \dots, x_n) \rightarrow (\kappa^{m_1} x_1, \dots, \kappa^{m_n} x_n)$, with $m_i > 0$ and $\kappa > 0$ if

$$\mathbf{f}(d_\kappa \mathbf{x}) = \kappa^q d_\kappa \mathbf{f}(\mathbf{x})$$

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The differential equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

is invariant with respect to the transformation $(t, \mathbf{x}) \rightarrow (\kappa^{-q} t, d_\kappa \mathbf{x})$.

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Example

- Linear system : homogeneous with degree 0 : $\mathbf{f}(\kappa \mathbf{x}) = \kappa \mathbf{f}(\mathbf{x})$
- Pure chain of integrators : homogeneous with degree -1 and weights $n, n-1, \dots, 1$.
Homogeneity kept with a suitable homogeneous control

Properties of homogeneity

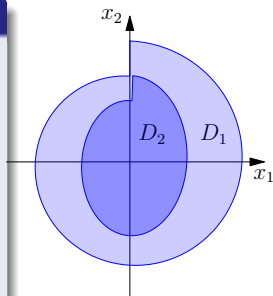
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20

Definition (Contractivity)

[Levant, 2005] A differential inclusion is contractive iff there exist 2 compacts D_1 and D_2 and a time $T > 0$ so that

- $d_\kappa D_1 \in D_1$ for $\kappa < 1$,
- D_2 belong to the interior of D_1 and contain the origin,
- all the trajectories starting in D_1 reach D_2 at the time T .



Properties of homogeneity

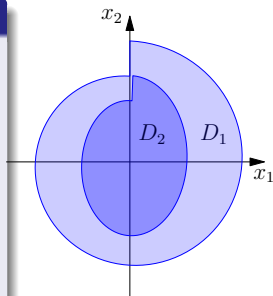
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Theorem

[Levant, 2005] For a homogeneous system with a negative degree, the following properties are equivalent :

Asymptotic stability \Leftrightarrow *finite time stability* \Leftrightarrow *Contractivity*.

Examples of second order algorithms

Twisting algorithm

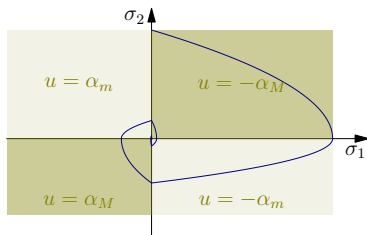
$$u = \begin{cases} -\alpha_m \operatorname{sign}(\sigma_1) & \text{si } \sigma_1 \sigma_2 < 0 \\ -\alpha_M \operatorname{sign}(\sigma_1) & \text{si } \sigma_1 \sigma_2 \geq 0 \end{cases}$$

with α_m and α_M so that

$$\alpha_M > 4 \frac{\Gamma_m}{\sigma_{max}}$$

$$\alpha_m > \frac{C}{\Gamma_m}$$

$$\gamma_m \alpha_M - C > \Gamma_M \alpha_m + C$$



Examples of second order algorithms

Suboptimal algorithm

$$u = \begin{cases} -\alpha_m \operatorname{sign}(\sigma_1) & \text{si } \sigma_1 \sigma_2 < 0 \\ -\alpha_M \operatorname{sign}(\sigma_1) & \text{si } \sigma_1 \sigma_2 \geq 0 \end{cases}$$

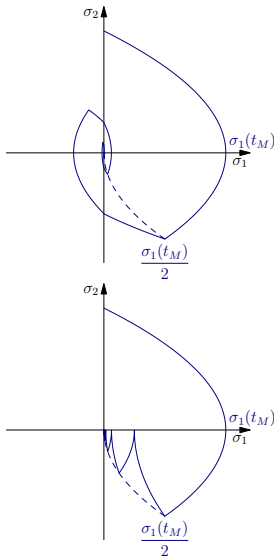
$$u(t) = \lambda(t) u_M \operatorname{sign}(\sigma_1(t) - \frac{\sigma_1(t_M)}{2})$$

$$\text{with } \lambda(t) = \begin{cases} 1 & \text{if } |\sigma_1(t)| \geq \sigma_1(t_M) \\ \lambda^* & \text{if } |\sigma_1(t)| < \sigma_1(t_M) \end{cases}$$

and t_M , the last moment the state reaches the σ_1 axis ($\sigma_2 = 0$).

$$\lambda^* \in]0; 1[\cap]0, \frac{3\Gamma_m}{\Gamma_M}[$$

$$u_M > \max\left(\frac{C}{\lambda^* \Gamma_m}, \frac{4C}{3\Gamma_m - \lambda^* \Gamma_M}\right)$$



Examples of second order algorithms

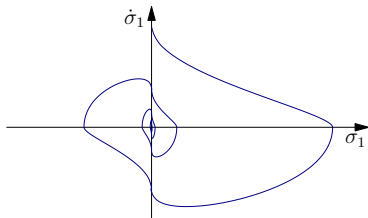
Super twisting algorithm

$$\begin{aligned} u(t) &= u_I(t) + \lambda_1 \sqrt{L} \sqrt{|\sigma_1|} \operatorname{sign}(\sigma_1) \\ \dot{u}_I(t) &= \lambda_2 L \operatorname{sign}(\sigma_1) \end{aligned} \quad (1)$$

with $L = \frac{C}{\Gamma_m}$ and

- $\lambda_2 > 1$
- $\lambda_1 > \sqrt{-2\lambda_2 + 2\sqrt{\lambda_2^2 + 2\lambda_2 + 2}}$

Common values : $\lambda_2 = 1.1$ et $\lambda_1 = 2$.



A new proof of convergence of the super twisting

24

Available proofs

- Original proof [Levant, 1998] : numerical, gives the smallest coefficients, but tied to super twisting

A new proof of convergence of the super twisting

24

Available proofs

- Original proof [Levant, 1998] : numerical, gives the smallest coefficients, but tied to super twisting
- Majorant based proof [Davila et al., 2005] : quite small coefficients, more easily extendable to similar control laws

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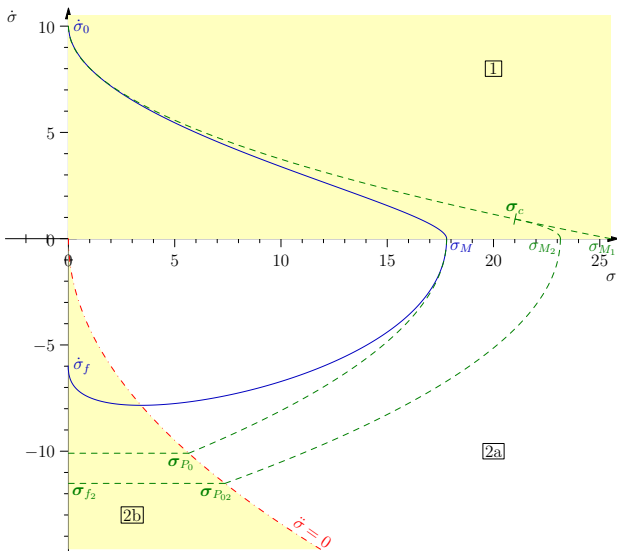
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- study only the worst case
- simplify the differential equation to obtain analytical results

A new proof of convergence of the super twisting

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- 1 Introduction to sliding mode control
- 2 Adaptive sliding mode control
- 3 Introduction to higher order sliding mode control
- 4 Introduction to time optimal control**
 - Open loop control
 - Closed loop control
 - Compute the implicit equation
- 5 Variable structure and time optimal control
- 6 Conclusion

Open loop time optimal control

26

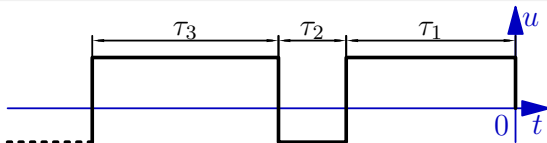
Perfectly known observable and controllable SISO LTI system with input $v \in [-v_M; v_M]$:

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Theorem (Pontryagin's theorem for SISO LTI systems)

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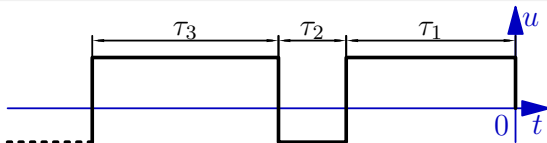
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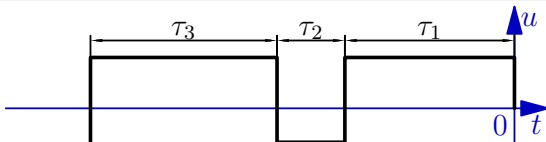
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Theorem (Feldbaum's theorem)

[Athans and Falb, 1966, Boltyanski and Gorelikova, 1997]

- *Order n system with n real poles \Rightarrow at most n phases ($n - 1$ switchings)*

Closed loop time optimal control for SISO LTI systems 27

Theorem

The time optimal closed loop control has the form

$$v = -v_M \operatorname{sign}(f_{v_M}(\mathbf{x}))$$

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Closed loop time optimal control for SISO LTI systems

27

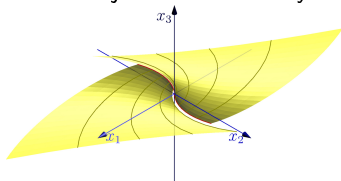
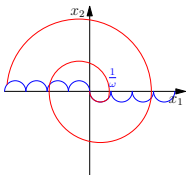
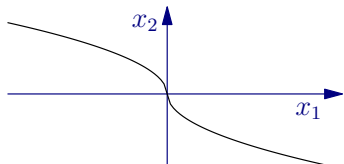
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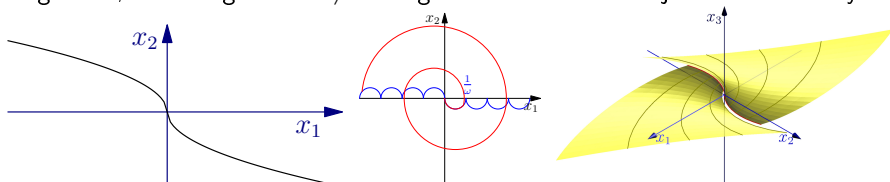
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Lemma ([Brégeault and Plestan, 2009])

For systems with real poles only,
switching surface \Leftrightarrow trajectories of the system driven by a time optimal control
with at most $n - 1$ phases

Computing the implicit equation for real poles systems 28

Function $[x_1, \dots, x_k]^T = \mathbf{f}_k(s, \tau_1, \dots, \tau_k)$

Theorem ([Brégeault and Plestan, 2009])

$\forall k \leq n$, \mathbf{f}_k , is a bijection between $\{-1; +1\} \times \mathbb{R}^{+k}$ and \mathcal{R}_k .

\Rightarrow Equation of the switching surface : $x_n - x_{n_S}(x_1, \dots, x_{n-1}, v_M) = 0$ for systems in canonical controllability form

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For pure chains of integrators :

The time optimal switching surface is homogeneous of degree -1 .

So, $f_{v_M}(\mathbf{x}) = \|\mathbf{x}\|_H f_{v_M}\left(\frac{\mathbf{x}}{\|\mathbf{x}\|_H}\right)$

\Rightarrow reduces the dimension of the set of points by 1, and increases precision near the origin.

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 - Parametrization of the system
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 - Proof of stability
 - Asymptotic precision
 - Examples
 - General case : VSS
 - Reduction of the chattering
 - Example

- 6 Conclusion

Parametrization of the system

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Reference system : totally known LTI system...

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{bv} \Leftrightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_n \in \sum_{i=1}^n a_i x_i + v \end{cases}$$

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... because all the uncertainties are in the new virtual control

$$v \in [-C'; C'] + [\Gamma_m; \Gamma_M]u$$

so that $-C' + \Gamma_m u_M > 0$.

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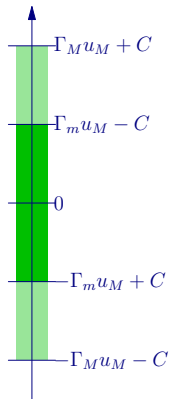
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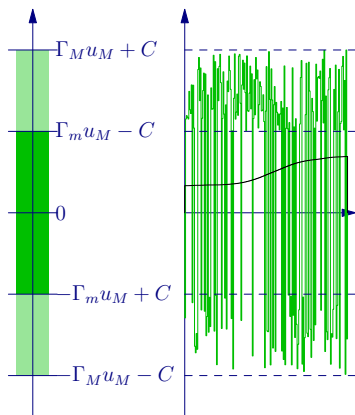
so that $-C' + \Gamma_m u_M > 0$.

If $u = \pm u_M$, $|v| \geq \Gamma_m u_M - C' > 0$



Parametrization of the system

30



We can theoretically generate any control within $[-(\Gamma_m u_M - C'); \Gamma_m u_M - C']$ thanks to high frequency switching (equivalent control).

Theorem

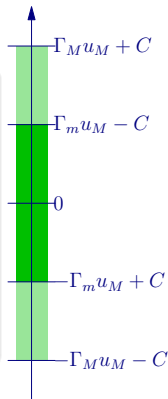
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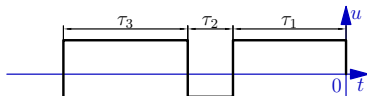
is a n^{th} order sliding mode control, provided the reference system has only real poles



Ideal sliding mode

32

Prove the stability of the switching surface :
Time optimal switching surface $\Leftrightarrow \tau_n(\mathbf{x}) = 0$



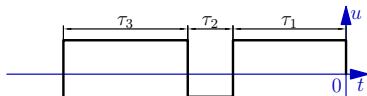
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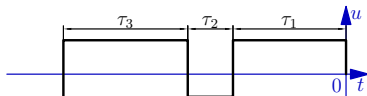
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$$\tau_n(\mathbf{x}(t)) = \tau_n(t=0) - t$$

$$\Rightarrow \dot{\tau}_n = -1$$

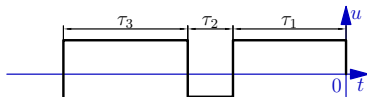
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The direction of the control $\mathbf{b}v_N \text{sign}(f_{v_N}(\mathbf{x}))$ is so that τ_n decreases as fast as possible.

As $\Gamma_m u_M - C' \geq v_N$, τ_n decreases at least as fast : $\dot{\tau}_n \leq -1$

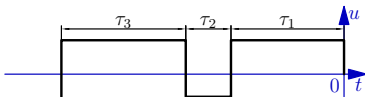
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\Rightarrow attractive surface \Rightarrow sliding mode

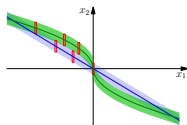
\Rightarrow the system behaves like a LTI system subject to a time optimal control

\Rightarrow stable n^{th} order ideal sliding mode.

Real sliding mode

33

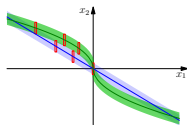
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Real sliding mode

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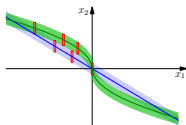
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parametric equation of the surface :
 $(s, k_1\tau, k_2\tau, \dots, k_{n-1}\tau, 0)$ with small $\tau > 0$
Integrating the system with this control yields

$$x_i = \alpha_i(s, v_N, k_2, \dots, k_{n-1})\tau^{n+1-i} + \mathcal{O}(\tau^{n+2-i})$$

$$\dot{x}_n = \sum_{i=1}^n a_i x_i + |v| \text{sign}(f_{v_N}(\mathbf{x})) = \mathcal{O}(\tau) + |v| \text{sign}(f_{v_N}(\mathbf{x}))$$



Real sliding mode

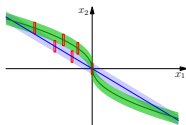
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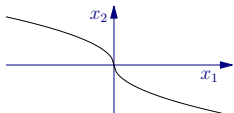
Example : double integrator

34

System:

$$\dot{\sigma}_1 = \sigma_2$$

$$\dot{\sigma}_2 \in [-C; C] + [\Gamma_m; \Gamma_M]u$$



Control:

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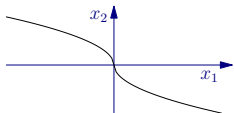
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with $v_N \leq \Gamma_m u_M - C$ 2nd order sliding mode control with prescribed convergence law:

$$u = -u_M \operatorname{sign} \left(\sigma_2 + \beta \sqrt{|\sigma_1|} \operatorname{sign}(\sigma_1) \right)$$

with $\frac{\beta^2}{2} \leq \Gamma_m u_M - C$ [Levant, 2007]

Example : triple integrator

35

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 \in [-\psi_M; \psi_M] + u$$

Equation from
[Pao and Franklin, 1993]

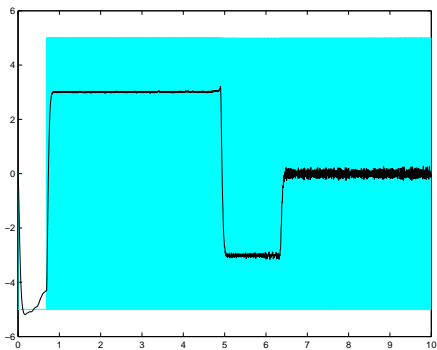


Figure: Control u and equivalent control of v : v_{eq}

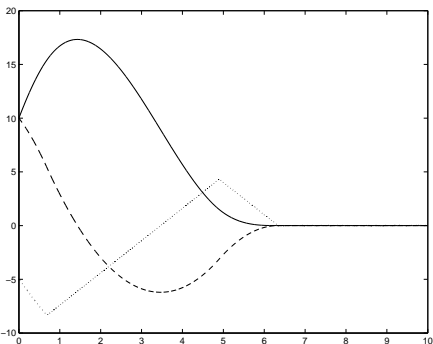


Figure: x_1 (—), x_2 (---), x_3 (···)

General LTI systems and VSS control

36

Theorem

The control

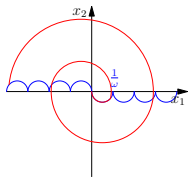
$$u = -u_M \operatorname{sign}(f_{v_N}(\mathbf{x}))$$

with v_N the amplitude of the reference control chosen so that:

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is a variable structure control which stabilizes the system in finite time.

The convergence time is no greater than the corresponding time optimal control law with amplitude v_N .



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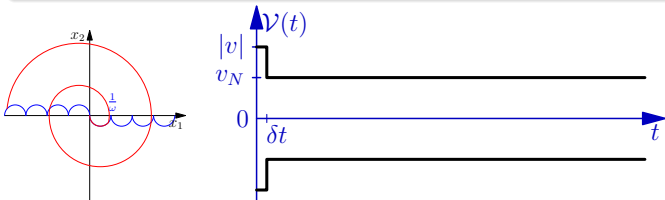
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⇒ use of saturation or nominal control do not work

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Theorem

The control law using

- *the nominal value of u (triangular)*
- *a super twisting of sliding variable $x_n - x_{n_{nom}}(x_1, \dots, x_{n-1})$,*

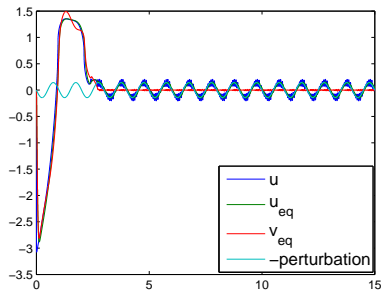
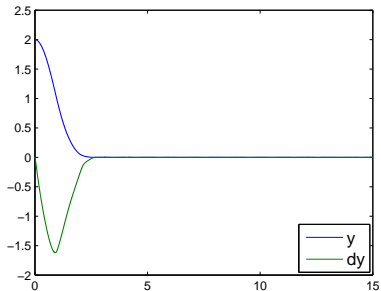
$$\text{coefficients } L = \frac{C + L_\gamma v_N}{\Gamma_m}, \lambda_1 = 1, \lambda_2 = 1.1$$

stabilizes the system in finite time

Example : double integrator

38

Neglected first order dynamic (time constant: 10ms), and sinusoidal matched perturbation.



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Conclusion







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- New control law : VSS+TOC (HOSMC for real poles, VSS for complex poles, reduction of chattering for real poles)

Perspectives:

- Adaptive control for higher order sliding mode
- Improve algorithms to compute the switching surfaces
- Take saturation into account in VSS+TOC smooth control laws
- Extend VSS+TOC algorithms to MIMO or nonlinear cases

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