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Quelques problèmes d'inspiration physique en théorie des probabilités

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THÈSE

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Titre :

Quelques problèmes d'inspiration physique en théorie des probabilités

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Préambule :
Mathématique et physique

le 12 novembre 2010

MATHÉMATIQUE et physique. Deux sciences aux objets d'étude bien distincts, la première s'intéressant aux objets abstraits tandis que la seconde étudie le monde réel. Pourtant, chacun sait confusément que les deux disciplines sont étroitement intriquées. Un grand nombre de savants des siècles passés, d'Archimède à Laplace en passant par Newton, se sont ainsi penchés simultanément sur ces deux domaines ; et aujourd'hui encore les deux matières sont enseignées de conserve jusqu'en premier cycle universitaire, collectivement qualifiées de « sciences exactes ».

QUELLES sont donc ces intrications entre mathématique et physique ? La plus évidente d'entr'elles tient à la notion même d'exactitude physique, en tant que mathématisation du réel : le physicien a fondamentalement besoin du formalisme mathématique pour exprimer ses résultats. Sans différentiation, pas de dynamique ; sans espaces de Hilbert, pas de mécanique quantique ; sans géométrie lorentzienne, pas de relativité générale.

MAIS le rôle de la mathématique en physique ne s'arrête pas au simple outillage. La prédiction du résultat d'une expérience à partir de la théorie, en effet, est un travail cent pour cent mathématique ! D'où l'exigence, par les écoles d'ingénieur, d'un solide bagage mathématique de la part de leurs élèves. D'où encore la présence, dans les laboratoires de physique, de théoriciens de haute volée mathématique.

BREF, la mathématique est indispensable à la physique ; cela n'est pas un scoop. En revanche, on oublie trop souvent que l'interaction entre mathématique et physique n'est pas à sens unique : le mathématicien a également beaucoup à gagner à rendre visite à ses confrères physiciens ! Je voudrais évoquer ici deux exemples de telles influences réciproques.

D'ABORD, il arrive que les problèmes mathématiques rencontrés par les physiciens dans leur travail soient récupérés par les mathématiciens pour leur intérêt intrinsèque. C'est ainsi que les équations de Navier – Stokes continuent aujourd'hui de susciter des recherches nombreuses et difficiles en théorie des équations aux dérivées partielles. Parfois même, les objets suggérés par la physique engendrent à leur tour de nouveaux développements mathématiques, comme le modèle d'Ising du ferromagnétisme qui a conduit de fil en aiguille à la FK-percolation et au processus SLE.

D'AUTRE part, une complémentarité très importante à mes yeux entre physique et mathématique tient à la différence entre les méthodes de travail des scientifiques des deux disciplines. Quand le physicien rencontre un problème mathématique dans ses travaux, en effet, il essaye de l'attaquer en restant guidé malgré tout par la réalité physique des objets étudiés ; en conséquence, son raisonnement visera à une approche intuitive du problème et se rattachera autant que possible à des notions physiques classiques : énergie, temps, loi des grands nombres, entropie, ... Cela est d'autant plus important en physique que les modèles mathématiques ne sont généralement qu'une approximation du monde réel, et que dans cette mesure il compte moins d'avoir une preuve parfaitement rigoureuse que de parvenir à des heuristiques donnant une compréhension suffisamment profonde du sujet pour en déduire des prédictions valides. L'exemple le plus frappant à cet égard est sans doute la théorie de Yang – Mills, qui sous-tend tout le modèle standard de la physique des particules, mais dont la démonstration mathématique est toujours mise à prix pour un million de dollars !

Si j'ai choisi d'intituler cette thèse « Quelques problèmes d'inspiration physique en théorie des probabilités », c'est donc dans les deux sens évoqués ci-dessus. *Primo*, parce que les questions abordées seront souvent motivées par des situations physiques, d'où l'emploi récurrent des termes « modèle », « particule », « interaction », « énergie », « temps », « équilibre »... tout au long de cette thèse. *Secundo*, parce que les arguments de mes démonstrations chercheront en permanence à rester aussi près que possible des motivations initiales : c'est ainsi par exemple que je proposerai des approches « élémentaires » de certaines inégalités démontrées à coups d'analyse spectrale.

Cela dit, qu'on ne se méprenne pas : cette thèse est bel et bien un travail de pure mathématique, et les théorèmes qu'elle contient y sont démontrés de manière totalement rigoureuse. Si j'ai cherché malgré tout à m'inspirer des raisonnements des physiciens, c'est pour une raison double. Esthétique d'une part, car j'estime que la démonstration d'un théorème compte au moins autant que son énoncé, et qu'une bonne preuve se doit d'avoir une structure globale naturellement compréhensible, d'être « morale ». Mais pragmatique également, car derrière cette volonté de cohérence physique se trouve l'ambition qu'une méthode « naturelle » sera plus fructueuse qu'une méthode « artificielle » et s'adaptera dans un cadre plus vaste.

Rémi Peyre

Remerciements

Au terme des quatre années de travail dont cette thèse marque l'accomplissement, je tiens à remercier tous celles et ceux sans qui ce doctorat aurait été mission impossible.

À tout seigneur, tout honneur ! Merci en premier lieu à Cédric Villani d'avoir accepté de me prendre comme thésard et d'avoir supervisé mon travail. Les thématiques qu'il m'a exposées et sa façon de les envisager ont joué une place importante dans les recherches que ce manuscrit présente. Si j'ai parfois eu du mal avec à la vision de l'encadrement de Cédric, qui se conçoit plus comme « *advisor* » que comme « directeur », au final cependant les méthodes de mon patron m'ont beaucoup apporté en autonomie et en confiance.

Sur le plan scientifique, de nombreux membres de la communauté mathématique m'ont fourni à l'occasion une aide précieuse, que ce soit pour m'indiquer des références bibliographiques, résoudre certains points techniques, ou suggérer des prolongements à mes recherches. La plupart de ces personnes sont remerciées plus en détail dans les chapitres concernés.

Mais une thèse ne saurait être réduite à son aspect scientifique. Ce sont aussi des années de vie, au long desquelles l'amitié humaine fut pour moi un carburant indispensable. À ce point de vue, j'ai particulièrement apprécié l'ambiance au sein de l'UMPA [*] où j'ai partagé moult moments conviviaux et conversations passionnantes. En-dehors des murs du laboratoire, j'ai aussi eu le plus grand plaisir à voir régulièrement mes amis de Lyon et d'ailleurs. Enfin bien sûr, j'ai énormément apprécié de pouvoir compter sur l'appui moral d'une famille unie et enjouée, notamment mes parents et mes sœurs "Chogosov", "L@urence" et "Crapatcha".

La rédaction de ce manuscrit à proprement parler doit un peu à mes grands-parents "Vatti" et "Morfar" dont les maisons de campagne respectives m'ont abrité de la fournaise lyonnaise au cours de mon travail d'écriture. Par ailleurs, je remercie spécialement ma collègue Alice qui a relu toute l'introduction de cette thèse.

*

À l'heure où cette soutenance marque mon entrée définitive dans le monde de la recherche mathématique, il ne faudrait toutefois pas croire que l'histoire a commencé à mon entrée à l'école doctorale. Cette vocation trouve en effet sa source auprès d'enseignants qui ont su me passionner pour les mathématiques, me stimuler dans mes études, et me conseiller dans mes choix : qu'ils en soient remerciés de tout cœur. Je voudrais également rendre hommage à un homme qui ne me connaît pas : Olivier Courcelle, dont l'humour m'a convaincu, au moment de choisir ma voie, que les mathématiciens pouvaient décidément être désopilants.

*

Pour finir, je remercie respectueusement mes rapporteurs, MM. Ledoux et Presutti, d'avoir pris le temps de lire ce manuscrit en détail et de lui avoir accordé leur caution scientifique. Je remercie également Mmes Méléard et Guionnet d'avoir accepté de siéger à mon jury de soutenance.

⁂

Arrêter une liste de personnes à remercier nommément est un exercice diplomatiquement déli-

[*]. Unité de Mathématiques Pures et Appliquées : le laboratoire de mathématiques de l'École Normale Supérieure de Lyon, au sein duquel j'ai travaillé pendant toute la durée de ma thèse.

cat et dont le résultat est nécessairement imparfait. Toutefois, s'il n'en fallait retenir que quarante, je voudrais exprimer ma gratitude particulière aux personnes suivantes :

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Comment lire cette thèse

Cette thèse de doctorat présente les recherches que j'ai menées à l'École Normale Supérieure de Lyon de novembre 2006 à mai 2010. Elle est constituée de quatre parties indépendantes correspondant à autant de travaux différents, que j'ai choisi de présenter dans l'ordre chronologique de leur réalisation :

1. Contrôle des probabilités de transition d'une chaîne de Markov ;
2. Limite de champ moyen pour un modèle de Boltzmann ;
3. Tensorisation des corrélations maximales ;
4. Brisure spontanée de symétrie en dimension infinie.

Afin de permettre au lecteur d'appréhender rapidement la teneur de ces recherches, une introduction générale a été rédigée en premier lieu. Cette introduction elle-même est constituée de deux volets :

- Je commence par présenter en quelques mots, sans utiliser de formule, les enjeux et les grandes lignes de mes travaux ;
- Suivent des version résumées de chacune des parties de la thèse.

À noter que, bien que l'introduction traite les quatre parties sur un pied d'égalité, celles-ci sont de longueurs très inégales ; en particulier, la troisième partie représente à elle seule plus de la moitié du contenu de la thèse.

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Introduction générale

Présentation de la thèse

Le premier sujet traité s'intéresse aux chaînes de Markov réversibles dont les transitions suivent les arêtes d'un graphe. En 1985, N. Varopoulos [80] et T. Carne [18] avaient démontré que, pour une telle chaîne de Markov, la probabilité de transition d'un point à un autre en temps fixé est contrôlée par une borne où intervient un facteur universel gaussien en la distance séparant les deux points. J'avais été interpellé par deux collègues sur l'aspect « miraculeux » de la preuve de Carne, où le facteur gaussien apparaissait *via* des techniques d'analyse spectrale puissantes : cette approche était en effet surprenante dans la mesure où les probabilistes sont plus habitués à voir les gaussiennes émerger de phénomènes de fluctuations (cf. théorème-limite central). Y avait-il une explication alternative à la démonstration de Carne ?

Je suis parvenu à montrer que oui, en réinterprétant le résultat de Carne & Varopoulos en termes de fluctuations de martingales. Pour ce faire, j'introduis une fonctionnelle sur l'espace des états qui mesure si on est plus proche du point de départ ou de l'arrivée. Bien que cette fonctionnelle ne soit pas une martingale *a priori*, l'hypothèse de réversibilité permet d'affirmer que sa tendance à croître lorsqu'on va du départ à l'arrivée sera compensée exactement par sa tendance à croître pour revenir de l'arrivée au départ. Comme les fluctuations de notre fonctionnelle sont contrôlées du fait que la chaîne de Markov doit suivre les arêtes du graphe, on peut appliquer des résultats de grandes déviations sur la valeur terminale de cette fonctionnelle pour en déduire la gaussienne — il s'agit en fait d'un analogue discret de la technique de décomposition *forward/backward* des martingales inventée par Lyons et Zheng [50].

Il y avait en outre, dans la borne de Carne, un facteur de nature *a priori* purement spectrale qui raffinait le résultat dans certains cas. Dans la mesure où ces cas correspondaient à une marche « fortement transitoire », j'ai eu l'idée d'interpréter ce facteur probabilistiquement comme une contrainte de conditionnement. En effet, imposer à la marche d'aller d'un point à un autre lui interdit en quelque sorte de partir à l'infini, de sorte qu'on peut alors se restreindre à la marche « conditionnée à être récurrente ». Cette marche étant réversible également, la borne simple de Carne s'y applique ; dès lors, le facteur « spectral » est simplement la probabilité que la chaîne reste récurrente pendant le temps imparti.

Au-delà de son intérêt esthétique, ma nouvelle démonstration présente surtout une souplesse qui lui permet de s'adapter à des distances différentes de la distance du graphe : l'idée est de pouvoir ajouter des transitions « exceptionnelles », qui ne suivraient pas les arêtes, sans changer pour autant la borne de façon conséquente — c'est en tout cas ce que l'intuition physique suggère. Alors que les arguments spectraux de Carne semblent impuissants dans de telles situations, mes techniques permettent au contraire d'obtenir une borne proche de la borne initiale. De fait, je donne des exemples concrets où mes nouvelles bornes améliorent nettement ce qui était connu jusqu'alors.

Le résultat de ces recherches a été publié [65] en 2008 dans la revue *Potential Analysis*.

❦

Dans le second volet de cette thèse, on s'intéresse au comportement hors équilibre d'un gaz constitué d'un grand nombre de particules interagissant par des collisions élastiques. On sait heuristiquement depuis les travaux de Boltzmann [8] que pour un nombre infini de particules, le système est régi par une équation intégro-différentielle dans laquelle le comportement microscopique des collisions se traduit macroscopiquement par une forme quadratique appelée « noyau de collision ». Je me suis placé plus précisément dans le cadre *spatialement homogène*, où on omet l'aspect cinétique du transport des particules — qui soulève d'importants défis techniques — pour se concentrer sur les seules collisions. Le modèle particulaire associé est de nature stochastique.

Quand le nombre de particules tend vers l'infini, on s'attend alors à une convergence vers l'équation de Boltzmann spatialement homogène. Chaque particule interagissant avec toutes les autres avec une intensité comparable, on parle de limite « de champ moyen » (par opposition aux limites « hydrodynamiques »). Ce type de limite a été étudié par Kac [43] dans un cas d'école, puis par divers autres [77, 56, 36] pour l'équation de Boltzmann.

Les arguments de « propagation du chaos » utilisés dans ces travaux présentaient à mes yeux l'inconvénient de ne pas dire grand-chose sur l'existence de bornes non asymptotiques ou sur la vitesse de cette convergence. Notamment, on donnait un résultat de convergence « différent » pour *chaque* fonctionnelle du système, plutôt qu'une convergence *simultanée* de toutes ces fonctionnelles — le même distinguo qu'entre le théorème-limite central « ordinaire » et les TLC uniformes [28]. Or ces enjeux importent dans les situations « appliquées » : à quel point l'équation de Boltzmann est-elle valide pour un nombre d'atomes de l'ordre du nombre d'Avogadro ? Si on lance une simulation informatique, combien d'atomes faut-il pour que le modèle particulaire ait de grandes chances d'être très proche de la limite continue ? Pour toutes ces raisons, j'ai tenté d'établir un résultat de limite de champ moyen dans un cadre aussi « concret que possible » : je voulais des bornes uniformes (contrôle simultané de toutes les fonctionnelles), quantitatives et non asymptotiques (résultats applicables numériquement pour un nombre arbitraire de particules).

Plusieurs difficultés ont dû être surmontées. D'abord, les travaux [7, 77] dont je m'inspirais utilisaient une technique de couplage du système à un ensemble de particules non linéaires, qui devenait inefficace ici à cause de l'évolution du système par chocs. J'ai donc préféré « oublier » l'existence individuelle de chaque particule pour étudier directement la mesure empirique du système. À partir de là, j'ai établi par des techniques de martingales un résultat « abstrait » susceptible d'être utilisé dans diverses situations, à commencer par celle qui m'intéressait.

Pour mettre en œuvre ce résultat abstrait, il fallait choisir dans quel espace de Banach mesurer la distance entre mesure empirique et mesure limite. Comme le choix classique des distances de Wasserstein aurait donné des vitesses de convergence « trop lentes » pour le TLC uniforme en dimension $\geqslant 2$, j'ai préféré prendre des espaces de Sobolev d'exposant négatif $\dot{H}^{-s}$ (pris homogènes pour des raisons tant physiques que mathématiques), plus grossiers, qui ne présentaient pas cet écueil.

Restait à vérifier les hypothèses du théorème abstrait. Le contrôle des fluctuations (stochastiques) du système discret s'obtient naturellement, mais la condition de contractivité sur l'équation limite (déterministe) est plus délicate. Dans le cas des espaces $\dot{H}^{-s}$, il s'avère que cette hypothèse ne peut être vérifiée que pour un modèle maxwellien, c.-à-d. quand la probabilité que deux particules se heurtent ne dépend pas de leur vitesse relative. Cette limitation est assez décevante (car physiquement non pertinente), mais des choix d'espaces fonctionnels plus subtils permettraient peut-être de l'éviter. Toujours est-il que dans le cas maxwellien, j'ai pu effectivement démontrer

l'hypothèse de contractivité et en déduire des estimations explicites pour la convergence du modèle particulaire.

Le résultat de ces recherches a été publié [66] en 2009 dans le *Journal of Statistical Physics*.

❦

Dans la troisième partie de la thèse, on s'intéresse à des systèmes de mécanique statistique où l'état de deux particules distinctes est corrélé, mais où cette corrélation tend vers zéro quand la distance entre les particules augmente — un exemple typique en est le modèle d'Ising sous-critique [61]. Deux particules distantes sont ainsi asymptotiquement décorrélées, mais qu'en est-il si on s'intéresse à des *groupes* de particules ? Par exemple, en dimension 2, y a-t-il décorrélation entre les particules situées sur deux droites parallèles distantes ? Il s'avère que la réponse dépend de la mesure de dépendance employée [11] : ainsi, la corrélation entre ces deux groupes de particules est maximale au sens du *β-mélange* ; mais au sens du *ρ-mélange*, ou *corrélation maximale*, il y a décroissance exponentielle des corrélations [57]. Cependant, on ne savait jusqu'ici montrer cette limitation de la propagation de l'information indépendamment de la taille de l'interface que dans des cas très particuliers, par l'utilisation d'analyse spectrale. Y avait-t-il une explication plus générale ?

Reprenons le cas des droites parallèles évoquées ci-dessus et considérons les paires de particules se faisant vis-à-vis. Si ces paires de particules étaient indépendantes entre elles, c'est un résultat relativement simple [21] que la corrélation maximale se *tensorise* : la corrélation entre les deux droites serait alors le supremum des corrélations entre deux spins d'une même paire. En pratique, les paires de particules ne sont certes pas indépendantes, néanmoins elles le deviennent asymptotiquement quand la distance entre deux paires augmente. C'est ainsi que j'ai cherché à tensoriser les corrélations maximales dans un cadre plus général en vue d'obtenir du ρ-mélange entre groupes de spins.

Le coefficient de ρ-mélange entre deux tribus est le cosinus de l'angle entre les espaces de fonctions L^2 centrées mesurables par rapport à chacune de ces tribus ; il s'agit donc d'un concept hilbertien. Pour des fonctions dépendant d'un ensemble de spins, une idée naturelle est donc de les décomposer en sommes de termes orthogonaux dont chacun exprimera la dépendance par rapport à un spin différent. Cette méthode redonne la tensorisation dans le cas indépendant, et permet d'évaluer simplement la corrélation entre un groupe de particules et une particule seule (ce que j'appelle la tensorisation « en N contre 1 »). Pour la corrélation entre deux groupes de particules (tensorisation « en N contre M »), l'idée de base est la même, mais il faut gérer la décomposition dans deux bases orthogonales différentes, ce qui donne à surmonter de nombreuses difficultés techniques. Je borne ainsi la corrélation en N contre M par la norme d'opérateur de la matrice des corrélations individuelles. Grâce à cette borne, on peut majorer finement la corrélation entre deux groupes de spins distants pour le modèle d'Ising, ainsi que pour des nombreux autres modèles de physique statistique, dans des cas beaucoup plus généraux que ce que permettait l'approche spectrale.

Cette partie de la thèse, de très loin la plus volumineuse, a été conçue comme un travail exhaustif et aborde de ce fait plusieurs questions annexes. Des résultats fins d'optimalité sont ainsi démontrés ; on s'intéresse également au ρ-mélange pour des géométries non planes, et pour l'espace-temps (ce qui donne des résultats d'hypocoercivité) ; on montre aussi comment appliquer les corrélations hilbertiennes à l'étude du théorème-limite central spatial, et du trou spectral pour la dynamique de Glauber. Le plus important de ces résultats secondaires est un nouveau critère pour obtenir le ρ-mélange à partir de conditions de type *α-mélange*, qui améliore ceux connus jusqu'alors [15] en atteignant la borne optimale.

Ce travail a été rédigé sous forme de monographie et soumis pour publication.

❦

La quatrième et dernière partie étudie un phénomène de transition de phase pour une équation aux dérivées partielles. L'équation en question, de type *Mc Kean – Vlasov*, décrit l'évolution d'une assemblée dense de particules soumises, en présence de bruit thermique, à un potentiel d'interaction attractif, à courte portée et non singulier ; il s'agit d'une ÉDP non linéaire. Les deux phénomènes auxquels sont soumises les particules agissent dans des directions opposées : la force d'attraction tend à agréger les particules, tandis que l'agitation thermique tend au contraire à les disperser. On comprend donc qu'à basse température l'évolution du système aura tendance à faire apparaître une densité de particules hétérogène, alors qu'à haute température la densité aura tendance à s'homogénéiser.

Le phénomène que nous voulons étudier dans ce contexte est la *brisure spontanée de symétrie*. Prenons pour situation initiale une distribution parfaitement homogène sur $\mathbb{R}^d$. Il est alors clair par symétrie que, quelle que soit la température, le système sera à l'équilibre ; mais en fonction de la température la stabilité de cet équilibre diffèrera : il y a une (unique) *transition de phase*, l'équilibre étant instable en-deçà et stable au-delà. Ce genre de transition est un analogue infinidimensionnel du phénomène de *flambage* d'une poutre, *i.e.* sa déformation transversale sous l'effet d'une compression longitudinale [29].

Pour notre modèle, un résultat d'Otto [59] énonce que l'évolution du système peut être vue comme une descente de gradient de la fonctionnelle d'énergie libre, de sorte qu'il nous suffit d'étudier les variations de cette fonctionnelle. La problématique est alors la suivante :

1. L'état homogène correspond-il à un minimum local de l'énergie libre ?
2. Le cas échéant, quelle est la profondeur du « puits » dans lequel loge ce minimum, *i.e.* quelle est l'*énergie d'activation* ?

Pour répondre à ces questions, il faut minorer l'énergie libre, en particulier son terme d'entropie. Comme minorer l'entropie quadratiquement est délicat dès lors que la densité n'est pas bornée, j'ai eu l'idée de « rendre la densité bornée » en la convolant par un noyau markovien — ce qui diminue l'entropie. Une difficulté technique vient de ce que l'espace naturellement adapté à l'étude de la distribution des particules est *non linéaire* : il s'agit de l'espace de transport associé à la métrique de Wasserstein W_2 — dont les éléments sont ici des mesures de masse infinie. Une partie importante de mon travail, que je trouve intéressante en soi, a donc consisté à plonger continûment cet espace métrique de Wasserstein dans un espace linéaire plus classique, à savoir le dual d'un espace de Sobolev inhomogène.

Grâce entre autres à ces techniques, je suis parvenu à calculer exactement la température de transition sous réserve que le potentiel d'interaction soit suffisamment régulier, transition qui est en fait identique à celle du système linéarisé. En raffinant mes minorations, j'ai en outre démontré qu'au-delà de la température de transition de phase, l'énergie d'activation est non nulle et croît au moins comme une certaine puissance de la différence entre température et température de transition.

Cette recherche est encore en cours et j'espère mener à bien un certain nombre d'améliorations. Je pense notamment que le résultat de plongement de l'espace de Wasserstein dans un espace linéaire peut être raffiné à des espaces plus petits, optimaux dans un certain sens. Par ailleurs, la version actuelle de mes théorèmes demande un supplément de régularité sur le potentiel d'interaction quand celui-ci n'est pas négatif partout, ce qui ne semble pas justifié physiquement et pourrait

être contourné par des outils judicieux d'analyse de Fourier. Enfin, un défi intéressant serait de déterminer l'exposant critique optimal qui régit l'apparition de l'énergie d'activation quand on dépasse la température de transition.

Résumé des travaux présentés

1 Contrôle des probabilités de transition d'une chaîne de Markov

☛ *Cette section est un résumé du travail présenté en détail dans la partie I de la thèse.*

1.a Présentation du problème

Soit V un ensemble de points au plus dénombrable, muni d'une mesure μ (dont la masse totale peut être infinie) appelée *mesure d'équilibre* ; nous noterons par commodité $\mu(x)$ pour $\mu(\{x\})$. On considère une chaîne de Markov homogène $(X_t)_{t\in\mathbb{N}}$ sur V, décrite par les probabilités de transition

$$p(x,y) := \mathbf{P}[X_1 = y | X_0 = x]. \tag{1A}$$

Pour $t \in \mathbb{N}$, on note p^t la t-ième puissance de convolution de p, c.-à-d.

$$p^t(x,y) = \mathbf{P}[X_t = y | X_0 = x]. \tag{1B}$$

La chaîne est supposée *réversible*, c.-à-d. que pour tous $x, y \in V$, on a

$$\mu(x)p(x,y) = \mu(y)p(y,x); \tag{1C}$$

en particulier, μ est une mesure invariante de la chaîne.

V peut alors être muni d'une structure canonique de *graphe* (non orienté), l'ensemble E de ses arêtes étant constitué par les paires $\{x,y\}$ d'éléments de V tels que $p(x,y) > 0$ — on supposera qu'on a ôté de V les points de mesure nulle, de sorte que cette condition est bien symétrique grâce à l'hypothèse de réversibilité. On note d la distance associée à ce graphe.

On définit enfin le *noyau de transition* P de la chaîne de Markov. Il s'agit d'un opérateur que nous considèrerons ici sur l'espace $L^2(\mu)$ des fonctions f vérifiant $\|f\|^2_{L^2(\mu)} := \sum_V f(x)^2\mu(x) < \infty$ — dans la suite, nous noterons simplement $\|f\|$ pour $\|f\|_{L^2(\mu)}$. P est défini par

$$(Pf)(x) := \mathbf{E}[f(X_1)|X_0 = x] = \sum_{y\in V} \mu(x)p(x,y). \tag{1D}$$

Comme μ est une mesure d'équilibre de la chaîne, l'inégalité de Jensen donne que pour toute f, $\|Pf\|^2 \leqslant \|f\|^2$. On peut alors définir la norme d'opérateur de P par

$$|P| := \sup_{f\in L^2(\mu)\setminus\{0\}} \frac{\|Pf\|}{\|f\|}, \tag{1E}$$

qui est un nombre compris entre 0 (strictement) et 1. Notons que dans bon nombre de cas on a en fait $|P| = 1$, notamment dès que la chaîne est récurrente.

Avec ces notations, Carne [18] a montré en 1985 le

1.1 Théorème (Carne)**.** *Pour tous $x, y \in V$, pour tout $t > 0$,*

$$p^t(x,y) \leqslant 2\left(\frac{\mu(y)}{\mu(x)}\right)^{1/2} |P|^t \exp\left(-\frac{d(x,y)^2}{2t}\right). \tag{1F}$$

♣

1.b L'argument spectral de Carne

Décrivons en quelques mots le fonctionnement de la preuve de Carne :

1. On commence par réécrire $(\mu(x)/\mu(y))^{1/2} p^t(x,y)$ comme le produit scalaire
$$\langle \mu(x)^{-1/2}\delta_x, P^t(\mu(y)^{-1/2}\delta_y)\rangle_{L^2(\mu)}, \tag{1G}$$
où les fonctions $\mu(x)^{-1/2}\delta_x$ et $\mu(y)^{-1/2}\delta_y$ sont normées dans $L^2(\mu)$.
2. La décomposition du polynôme Z^t dans la base $(Q_k(Z))_{k\in\mathbb{Z}}$ des polynômes de Tchebychev de première espèce donne
$$P^t = \frac{|P|^t}{2^t} \sum_{k\in\mathbb{Z}} \binom{t}{(t+k)/2} Q_k(|P|^{-1}P). \tag{1H}$$
3. La chaîne de Markov étant réversible, P est autoadjoint dans $L^2(\mu)$; il est donc normal, à valeurs propres réelles, et sa norme est égale à son rayon spectral. Ainsi $|P|^{-1}P$ est un opérateur normal de spectre contenu dans $[-1,1]$, de sorte que $Q_k(P)$ est un opérateur normal de spectre contenu dans $Q_k([-1,1])$. Mais les polynômes de Tchebychev envoient $[-1,1]$ dans lui-même, donc au final la norme d'opérateur de $Q_k(|P|^{-1}P)$ est bornée par 1.
4. Comme le polynôme Q_k est de degré $|k|$ et que l'opérateur $|P|^{-1}P$ a une portée de 1, $\langle \delta_x, Q_k(P)\delta_y\rangle$ est nul dès lors que $|k| < d(x,y)$; on peut donc restreindre la somme (1H) aux k tels que $|k| \geqslant d$.
5. En interprétant $2^{-t}\binom{t}{(t+k)/2}$ comme la probabilité qu'une marche aléatoire simple sur $\mathbb{Z}$ soit en k au temps t, on montre par des arguments de grandes déviations que pour tout $d \geqslant 0$,
$$\frac{1}{2^t} \sum_{|k|\geqslant d} \binom{t}{(t+k)/2} \leqslant 2e^{-d^2/2t}. \tag{1I}$$
6. Il ne reste plus qu'à mettre bout à bout les étapes 1 à 5 pour obtenir (1F).

1.c Mon argument probabiliste

Passons maintenant à ma preuve probabiliste. On définit la fonction $\xi\colon V \to \mathbb{R}$ par $\xi(\cdot) := d(x,\cdot)$, de sorte que ξ est 1-lipschitzienne pour la distance d et vérifie $\xi(x) = 0, \xi(y) = d(x,y)$. On définit pour tout $z \in V$

$$m(z) := \mathbf{E}[\xi(X_1)|X_0 = z] - \xi(z), \tag{1J}$$

et on introduit le processus

$$M_u := \xi(X_u) - \xi(X_0) - \sum_{s=0}^{u-1} m(X_s) \tag{1K}$$

qui est alors une martingale.

Maintenant, sur l'événement "$X_0 = x$ et $X_t = y$", on a

$$M_t \geqslant d(x,y) - \sum_{u=0}^{t-1} m(X_u); \tag{1L}$$

a contrario, sur l'événement "$X_0 = y$ et $X_t = x$", on a

$$M_t \leqslant -d(x,y) - \sum_{u=0}^{t-1} m(X_u). \tag{1M}$$

Or grâce à l'hypothèse de réversibilité, les points visités par la chaîne de Markov pour aller de x à y en t pas sont les mêmes (dans l'ordre inverse) que pour aller de y à x en t pas, de sorte que les derniers termes de (1L) et (1M) sont « pratiquement identiques ». Plus précisément, on a que

$$\mathbf{E}\Big[\sum_{u=1}^{t-1} m(X_u)\Big| X_0 = x \text{ et } X_t = y\Big] = \mathbf{E}\Big[\sum_{u=1}^{t-1} m(X_u)\Big| X_0 = y \text{ et } X_t = x\Big]. \tag{1N}$$

Définissons maintenant un processus $(X_u^x, X_u^y)_{0\leqslant u\leqslant t}$ de loi notée $\mathbf{P}_{x\otimes y}$, où X^x est la chaîne issue de x et X^y la chaîne issue de y, X^x et X^y étant indépendants, et introduisons les martingales M^x et M^y correspondant respectivement aux processus X^x et X^y. La combinaison de (1L) et (1M) donne alors, grâce à (1N) :

$$\mathbf{E}_{x\otimes y}[(M_t^x - M_1^x) - (M_t^y - M_1^y) | X_t^x = y \text{ et } X_t^y = x] \geqslant 2d(x,y) - 2. \tag{1O}$$

(Les premiers pas de chaque chaîne, qui ne se simplifiaient pas, ont été simplement bornés par 1).

Dans (1O), la variable aléatoire "$(M_t^x - M_1^x) - (M_t^y - M_1^y)$" peut être vue comme la valeur terminale d'une martingale (par rapport à une filtration judicieuse) issue de 0, à $(2t-2)$ pas, où chaque pas, conditionnellement aux précédents, est supporté par un intervalle de longueur 2. Dans de telles conditions, on sait qu'on peut appliquer des techniques de grandes déviations à une telle variable aléatoire : ainsi, l'inégalité d'Azuma donne que pour tout $C \geqslant 0$,

$$\mathbf{P}_{x\otimes y}[(M_t^x - M_1^x) - (M_t^y - M_1^y) \geqslant C] \leqslant \exp\left(-\frac{C^2}{2(2t-2)}\right). \tag{1P}$$

Ici nous allons utiliser une approche « duale » de ce genre de résultat : sachant que conditionner notre variable aléatoire par l'événement "$X_t^x = y$ et $X_t^y = x$" rend son espérance grande, on peut en déduire que l'événement en question a nécessairement une probabilité petite. On trouve ainsi que

$$\mathbf{P}_{x\otimes y}[X_t^x = y \text{ et } X_t^y = x] \leqslant \exp\left(-\frac{(d(x,y)-1)_+^2}{t-1}\right). \tag{1Q}$$

Pour finir, $\mathbf{P}_{x\otimes y}[X_t^x = y \text{ et } X_t^y = x]$ se réécrit immédiatement comme $p^t(x,y)p^t(y,x)$, où, vu que la chaîne de Markov est réversible, on a $p^t(y,x) = \frac{\mu(x)}{\mu(y)} p^t(x,y)$. Au bout du compte, (1Q) donne :

$$p^t(x,y) \leqslant \left(\frac{\mu(y)}{\mu(x)}\right)^{1/2} \exp\left(-\frac{(d(x,y)-1)_+^2}{2(t-1)}\right). \tag{1R}$$

On déduit facilement de cette borne la formule (1F) de Carne (où le préfacteur 2 peut même être remplacé par $\sqrt{e}$), à l'exclusion du facteur $|P|^t$ [voir § 1.d à ce sujet] dont nous avons toutefois fait remarquer qu'il était souvent égal à 1.

1.d Le facteur $|P|^t$

Maintenant nous allons voir qu'on peut en fait récupérer le facteur $|P|^t$ de la borne de Carne comme un corollaire de la « simple » borne gaussienne (1R). Je ne donne que la démarche générale de la preuve, en négligeant tous les aspects techniques : on supposera ainsi que la chaîne de Markov est dans le meilleur cas possible, à savoir que l'espace d'états est fini avec un point absorbant, et que pour le reste la chaîne est irréductible apériodique.

Rappelons que nous cherchons à borner $p^t(x,y)$, et que pour cela il suffit de borner $p^t(x,y) \times p^t(y,x)$, qui peut se réécrire comme $\mathbf{P}[X_t = y \text{ et } X_{2t} = x | X_0 = x]$.

Notons $\mathscr{R}_u$ l'événement "la chaîne revient en x après le temps u". Le rayon spectral $|P|$ peut alors être réinterprété par la propriété :

$$\mathbf{P}[\mathscr{R}_u | X_0 = x] \overset{u\to\infty}{\sim} c|P|^{-u} \qquad \text{pour un certain } 0 < c < \infty. \tag{1S}$$

Notons $\mathscr{A}$ l'événement "$X_t = y$ et $X_{2t} = x$". La propriété de Markov entraîne que, pour tout $u \geqslant 2t$,

$$\mathbf{P}[\mathscr{R}_u | X_0 = x \text{ et } \mathscr{A}] = \mathbf{P}[\mathscr{R}_{u-2t} | X_0 = x], \tag{1T}$$

d'où par la formule de Bayes :

$$\mathbf{P}[\mathscr{A} | X_0 = x] = \frac{\mathbf{P}[\mathscr{A} | X_0 = x \text{ et } \mathscr{R}_u]}{\mathbf{P}[\mathscr{R}_u | X_0 = x \text{ et } \mathscr{A}]} \mathbf{P}[\mathscr{R}_u | X_0 = x] = \frac{\mathbf{P}[\mathscr{R}_u | X_0 = x]}{\mathbf{P}[\mathscr{R}_{u-2t} | X_0 = x]} \mathbf{P}[\mathscr{A} | X_0 = x \text{ et } \mathscr{R}_u]. \tag{1U}$$

Voyons comment se comporte (1U) quand $u \to \infty$. D'abord, le quotient $\mathbf{P}[\mathscr{R}_u | X_0 = x] / \mathbf{P}[\mathscr{R}_{u-2t} | X_0 = x]$ tend vers $|P|^{2t}$ d'après la formule (1S), ce qui est précisément le facteur recherché.

Il reste à étudier le comportement de $\mathbf{P}[\mathscr{A} | X_0 = x \text{ et } \mathscr{R}_u]$. En fait, on peut montrer que la chaîne de Markov conditionnée à revenir en x au temps u, quand u tend vers l'infini, « tend vers » une nouvelle chaîne de Markov homogène qu'on pourrait appeler « chaîne conditionnée à être récurrente ». Cette chaîne possède pour probabilités de transition les $p'(z,v)$ définis de la manière suivante :

1.2 Définition. Notons τ_x le temps d'atteinte de x par la chaîne de Markov, c.-à-d. $\tau_x := \inf\{s \geqslant 0 : X_s = x\}$. Pour tout $z \in V$, notons

$$R(z) := \mathbf{E}[\mathbf{1}_{\tau_x < \infty} |P|^{-\tau_x} | X_0 = z]; \tag{1V}$$

on définit alors

$$p'(z,v) := \frac{R(v)p(z,v)}{\sum_{\{w,z\} \in E} R(w)p(z,w)}. \tag{1W}$$

◇

On montre facilement que la réversibilité de la chaîne de Markov de départ entraîne la réversibilité de la chaîne conditionnée à être récurrente (avec une mesure d'équilibre différente). Par conséquent, on peut appliquer la borne (1R) à la chaîne conditionnée pour trouver

$$\overline{\lim_{u\to\infty}} \mathbf{P}[\mathscr{A} | X_0 = x \text{ et } \mathscr{R}_u] \leqslant \exp\left(-\frac{(d(x,y)-1)_+^2}{t-1}\right), \tag{1X}$$

ce qui conclut l'argument. Notez bien qu'on utilise ici que la chaîne conditionnée est associée au même graphe que la chaîne initiale, et que donc la distance intervenant dans la borne de Carne est la *même* dans les deux cas.

1.e Généralisations

Revenons à notre démonstration de la « simple » borne (1R). Contrairement à la preuve spectrale de Carne, les arguments de martingale par lesquels nous sommes passés sont suffisamment souples pour être généralisés à d'autres distances que la distance du graphe. Notre objectif est ainsi de pouvoir utiliser dans la borne de Carne des distances pour lesquelles, quand la probabilité de passer d'un point à un autre est non nulle mais très faible, la distance entre ces deux points pourra être beaucoup plus grande que 1, de sorte que l'ajout de transitions très improbables modifiera peu la structure métrique de V.

Essentiellement, la condition que doit satisfaire une telle distance pour donner une borne de Carne avec notre preuve est de vérifier un analogue de l'inégalité de Hoeffding. Voici un exemple concret de telle généralisation :

1.3 Théorème (*Peyre*)**.** *Fixons un paramètre $\beta \in (0,1]$; pour $x \in V$, définissons*

$$H_\beta(x) := \sum_{y \in V} p(x,y)^{1-\beta}. \tag{1Y}$$

Fixons en outre un paramètre $\alpha > 0$; pour $x, y \in V$, posons

$$l(x,y) := 1 + \sqrt{2}\alpha\big(\beta \log\big(p(x,y)^{-1}\big) - \log H_\beta(x)\big)_+^{1/2}, \tag{1Z}$$

et donnons à l'arête $\{x,y\}$ de E la longueur $\min\{l(x,y), l(y,x)\}$, de manière à définir une nouvelle distance $\tilde{d}$ sur V. Il existe alors des constantes universelles $A(\alpha)$ et $B(\alpha)$ telles que, pour tous $x, y \in V$, $t \geqslant 0$:

$$p^t(x,y) \leqslant \left(\frac{\mu(y)}{\mu(x)}\right)^{1/2} \exp\left(-\frac{\big(\tilde{d}(x,y) - B(\alpha)\big)_+^{\,2}}{2A(\alpha)t}\right). \tag{1AA}$$

$A(\alpha)$ et $B(\alpha)$ peuvent en outre être calculées explicitement : ainsi pour $\alpha = 1$, $A = 23\frac{1}{2}$ et $B = 1 + \sqrt{\pi/2}$ conviennent. ♣

2 Limite de champ moyen pour un modèle de Boltzmann

☛ *Cette section est un résumé du travail présenté en détail dans la partie II de la thèse.*

2.a Les modèles particulaire et continu

On considère un modèle particulaire stochastique pour l'évolution d'un gaz spatialement homogène. Dans ce modèle, il y a N particules $i \in \{1, \ldots, N\}$, complètement décrites par leurs vitesses $v_i \in \mathbb{R}^d$. Les collisions entre deux particules de vitesses respectives v et w sont décrites par une mesure positive $\gamma_{v,w}$, où $N^{-1} d\gamma_{v,w}(v', w')\delta t$ donne la probabilité, pendant un intervalle de temps infinitésimal δt, que ces particules subissent un choc élastique et repartent instantanément avec les vitesses émergentes respectives v' et w'. On fera sur les $\gamma_{v,w}$ les hypothèses physiques habituelles de conservation et d'invariance. Le système particulaire évolue ainsi suivant le processus de Markov de générateur

$$\mathscr{L} f(v_1, \ldots, v_N) = \frac{1}{2N} \sum_{0 < i,j \leqslant N} \int_{(\mathbb{R}^d)^2} \big(-f(v_1, \ldots, v_N) + f(\ldots, v'_i, \ldots, v'_j, \ldots)\big) d\gamma_{v_i, v_j}(v'_i, v'_j). \tag{2A}$$

On peut associer à ce modèle particulaire un modèle continu, qui décrit l'évolution de la mesure empirique $\hat{\mu}^N := N^{-1}\sum_{i=1}^N \delta_{v_i}$ quand on fait tendre formellement N vers l'infini dans (2A). Celui-ci est régi par l'équation différentielle déterministe

$$\frac{d\mu}{dt} = Q(\mu_t, \mu_t), \tag{2B}$$

où le *noyau de collision* $Q(\cdot,\cdot)$ est la forme quadratique

$$Q(\mu,\nu) = \frac{1}{2}\int\Big(\int(-\delta_v - \delta_w + \delta_{v'} + \delta_{w'})d\gamma_{v,w}(v',w')\Big)\,d\mu(v)d\nu(w). \tag{2C}$$

Notre objectif est de quantifier l'idée que, quand N est grand, l'évolution du système microscopique est proche de celle du système macroscopique avec grande probabilité.

2.b Approche abstraite

Dans un premier temps nous raisonnons dans un cadre abstrait. On se place dans un espace affine A au-dessus d'un espace hilbertien H, et on considère un processus markovien $(\hat{X}_t)_{t\geqslant 0}$ à valeurs dans A évoluant par sauts, dont nous notons $\mathscr{L}$ le générateur. À ce processus est associé une équation différentielle de la façon suivante : pour un point-origine $o \in A$ arbitraire, on a un opérateur « identité » $I\colon\ A \to H$ défini par $I(x) := x - o$, et l'opérateur $\mathscr{L}I\colon\ A \to H$ ne dépend pas du choix de o. [Notez que générateur $\mathscr{L}$ est *a priori* défini pour les fonctions de A dans $\mathbb{R}$, mais on peut immédiatement le généraliser pour les fonctions de A dans n'importe quel espace de Banach, en l'occurrence pour I]. L'équation différentielle associée au processus Markovien est alors :

$$\frac{dX}{dt} = (\mathscr{L}I)(X_t). \tag{2D}$$

Pour contrôler la distance entre $\hat{X}_t$ et X_t, nous nous inspirons de la méthode de Cramér pour les grandes déviaitions, mais dans un cadre infini-dimensionel. On introduit à cette fin une fonction d'utilité exponentielle sur H :

$$\mathscr{U}(x) = e^{\|x\|} + e^{-\|x\|}, \tag{2E}$$

qui est de classe $\mathscr{C}^2$ avec $|||\nabla^2\mathscr{U}(x)||| \leqslant \mathscr{U}(x)$.

On peut alors contrôler $\mathscr{U}(\hat{X}_t - X_t)$ par des arguments de martingale. Le théorème abstrait est le suivant :

2.1 Théorème (*Peyre*)**.** *On se place sous les hypothèses suivantes :*

(i) Pour un certain $\kappa \in \mathbb{R}$, le semi-groupe de l'équation différentielle (2D) est « κ-contractif », au sens où pour tous $x \in A, h \in H$,

$$\langle \nabla[\mathscr{L}I](x)\cdot h, h\rangle \leqslant -\kappa\|h\|^2. \tag{2F}$$

(ii) Pour un certain $V < \infty$, les sauts du processus markovien ont le carré de leur longueur borné par V en espérance. Plus précisément,

$$\forall x \in A \quad \mathscr{L}(\|\cdot - x\|^2)(x) \leqslant V. \tag{2G}$$

(iii) Pour un certain $L < \infty$, tous les sauts ont leur longueur bornée par L, presque-sûrement.

Notons $\hat{X}_0$ la valeur initiale (aléatoire) du processus markovien et X_0 la valeur initiale (déterministe) de l'équation différentielle associée. Alors pour tout $T \geqslant 0$, pour tout $\lambda > 0$:

$$\log \mathbf{E}[\mathscr{U}(\lambda(\hat{X}_T - X_T))] \leqslant \log \mathbf{E}[\mathscr{U}(\lambda e^{-\kappa T}(\hat{X}_0 - X_0))] + \lambda^2 e_2(\lambda e^{2\kappa_- T} L) e_1(-2\kappa T) V T, \tag{2H}$$

où $e_1(\cdot)$ et $e_2(\cdot)$ notent les fonctions $e_1(z) := (e^z - 1)/z$, resp. $e_2(z) := (e^z - 1 - z)/z^2$, prolongées par continuité en 0 par $e_1(0) = 1$, resp. $e_2(0) = 1/2$. ♣

Ce théorème montre que pour tout T fixé, $\hat{X}_T$ converge vers X_T à vitesse $N^{-1/2}$ quand le nombre de particules augmente, pour peu qu'initialement $\hat{X}_0$ soit suffisamment proche de X_0. Toutefois, pour $\kappa < 0$ les constantes se dégradent très vite dès que $T \gtrsim 1/|\kappa|$.

Sans donner une démonstration complète du théorème, expliquons où se situe son argument central. Il s'agit montrer qu'une fonctionnelle de la forme

$$F(t) = e^{h(t)} \mathscr{U}(\lambda e^{\kappa(t-T)}(\hat{X}_t - X_t)) \tag{2I}$$

est une surmartingale. Pour cela, on regarde l'espérance de l'évolution de F sur un intervalle de temps infinitésimal $[t, t + \delta t]$, en supposant $\hat{X}_t$ connu. Par définition d'un générateur, on a alors

$$\mathbf{E}[\delta \hat{X}] = (\mathscr{L} I)(\hat{X}_t)\, \delta t + O(\delta t^2), \tag{2J}$$

ce qui suggère d'écrire la décomposition suivante, où $\hat{Y}_t$ note $(\hat{X}_t - X_t)$:

$$\begin{aligned} F(t+\delta t) - F(t) &= h'(t) F(t) \delta t && \text{(2K)} \\ &+ e^{h(t)} \lambda e^{\kappa(t-T)} \nabla \mathscr{U}(\lambda e^{\kappa(t-T)} \hat{Y}_t) \cdot (\mathscr{L} I(\hat{X}_t) - \mathscr{L} I(X_t) + \kappa \hat{Y}_t) \delta t && \text{(2L)} \\ &+ e^{h(t)} \big[\mathscr{U}(\lambda e^{\kappa(t-T)} \hat{Y}_{t+\delta t}) - \mathscr{U}\big(\lambda e^{\kappa(t-T)} \{\hat{Y}_t + [\mathscr{L} I(\hat{X}_t) - \mathscr{L} I(X_t)] \delta t\}\big) \big] && \text{(2M)} \\ &+ O(\delta t^2). \end{aligned}$$

Dans cette décomposition, chaque ligne est contrôlée de manière à assurer que la somme soit toujours négative :

- La ligne (2K), qui est déterministe, sera contrôlée par un choix judicieux de h.
- L'hypothèse (i) du théorème est précisément ce qu'il faut pour assurer que la ligne (2L) soit toujours négative.
- Dans la ligne (2M), réécrivons le terme entre crochets sous la forme $\mathscr{U}(A + b) - \mathscr{U}(A)$, où A est déterministe et b est aléatoire. Nous observons alors que $\mathbf{E}[b] = 0$ (à un terme $O(\delta t^2)$ près), de sorte que, par application de la formule de Taylor, on a moralement, sous réserve que les sauts ne soient pas trop grands — c'est là qu'intervient l'hypothèse (iii) — :

  $$\mathbf{E}[\mathscr{U}(A + b) - \mathscr{U}(A)] \simeq \mathbf{E}[\nabla^2 \mathscr{U}(A) \cdot (b, b)]. \tag{2N}$$

  Comme ici b est essentiellement égal au saut effectué par le processus, on voit ainsi que l'espérance de (2M) est contrôlée par l'espérance du carré de la longueur des sauts, que nous majorons par l'hypothèse (ii).

2.c Mise en œuvre dans le cas maxwellien

Voyons maintenant comment on peut appliquer le théorème 2.1 au modèle de Boltzmann. Pour le choix de l'espace de Hilbert H dans lequel on étudie de processus, j'ai pris l'espace de Sobolev homogène $\dot{H}^{-s}$ défini par

$$\|\mu\|_{\dot{H}^{-s}} := \left(\int_{\mathbb{R}^d} |\hat{\mu}(\xi)|^2 \, |\xi|^{-2s} \, d\xi \right)^{1/2}, \tag{2O}$$

ou encore, à un facteur constant près :

$$\|\mu\|_{\dot{H}^{-s}} = \left\| \mu * \left(|\cdot|^{-(d-s)}\right) \right\|_{L^2(\mathbb{R}^d)}. \tag{2P}$$

Posant $s =: d/2 + r$, on a que pour $r \in (0,1)$, la différence de deux mesures de probabilité ayant des moments polynômiaux de degré r sera dans $\dot{H}^{-s}$, de sorte que cet espace convient effectivement pour mesurer la distance entre μ_t et $\hat{\mu}_t^N$.

Dans $\dot{H}^{-s}$, la collisions entre deux particules de vitesse relative v correspond à un saut d'amplitude proportionnelle à $N^{-1}|v|^r$. Quand le nombre de particules augmente, il est raisonnable de supposer que cela se fait de façon que l'énergie cinétique totale du système croisse en $O(N)$; par conséquent on vérifie l'hypothèse (ii) du théorème 2.1 avec $V = O(N^{-1})$. Pour l'hypothèse (iii), il faut contrôler la vitesse de la plus rapide des particules : par conservation de l'énergie, on peut montrer que celle-ci est bornée par $O(N^{1/2})$, de sorte que $L = O(N^{-1+r/2})$ convient — on s'attend même à ce que sous des hypothèses raisonnables, on puisse en fait prendre $L = O(N^{-1+\eta})$ pour tout $\eta > 0$.

L'hypothèse du théorème abstrait la plus délicate à vérifier est (i). Commençons par observer que dans notre cas, $(\mathscr{L}I)(\mu)$ est la forme quadratique $Q(\mu,\mu)$, de sorte que $\langle \nabla[\mathscr{L}I](x)\cdot f, f\rangle = 2\langle Q(\mu,f), f\rangle$. Comme μ est une mesure de probabilité, en décomposant celle-ci sous la forme $\mu = \int_{\mathbb{R}^d} \delta_v d\mu(v)$, on s'aperçoit que pour vérifier l'hypothèse (i) il suffit de démontrer que pour tout $v \in \mathbb{R}^d$, l'opérateur $Q(\delta_v, \cdot)$ est $(-\kappa/2)$-dissipatif, *i.e.* que pour toute $f \in H$,

$$\langle Q(\delta_v, f), f\rangle \leqslant -\frac{\kappa}{2}\|f\|^2. \tag{2Q}$$

Pour établir (2Q) dans l'espace $\dot{H}^{-s}$, nous avons besoin de supposer que le modèle est *maxwellien*. Cela signifie que la probabilité par unité de temps que deux particules données subissent une collision présentant un angle de déviation θ est la même *quelles que soit la vitesse relative incidente entre ces deux particules*. Cette propriété est vérifiée par exemple par le modèle « de Kac », où on tire poissonniennement des paires de particules dont les vitesses émergentes sont redistribuées uniformément sur l'ensemble des couples (v', w') vérifiant la conservation de la quantité de mouvement et de l'énergie. Dans un tel contexte, l'opérateur $Q(\delta_v, \cdot)$ se décompose en une somme (continue) d'opérateurs « en cerceau » $\check{Q}_\theta$, par lesquels une masse de Dirac δ_w est envoyée uniformément sur un « parallèle » de la sphère de diamètre $[v, w]$ (voir figure 1). Le comportement dissipatif d'un tel opérateur peut être calculé explicitement. Considérons en effet l'ensemble $\mathscr{R}_\theta$ des similitudes de $\mathbb{R}^d$ qui fixent v, ont un rapport $\cos(\theta/2)$, et font tourner tous les points de $\mathbb{R}^d$ d'un angle $\theta/2$ autour de v. Nous supposerons ici que d est pair (le résultat reste vrai pour d impair, mais la démonstration est plus compliquée) ; dans ce cas, l'ensemble $\mathscr{R}_\theta$ est non vide et muni d'une mesure de probabilité naturelle [une sorte de mesure de Haar], que nous noterons π_θ, pour laquelle

$$\check{Q}_\theta(f) = \int_{\mathscr{R}_\theta} (R \# f)\, d\pi_\theta(R). \tag{2R}$$

On ramène ainsi $Q(\delta_v, \cdot)$ à une combinaison de similitudes de $\mathbb{R}^d$, dont le comportement dissipatif dans $\dot{H}^{-s}$ est facile à étudier grâce aux propriétés d'homogénéité de cet espace, en particulier la formule (2P).

In fine, dans le cas d'un modèle maxwellien étudié dans $\dot{H}^{-s}$, nous obtenons une valeur explicite pour κ. Cette valeur est toutefois strictement négative, ce qui est comme nous le verrons le cas le moins favorable.

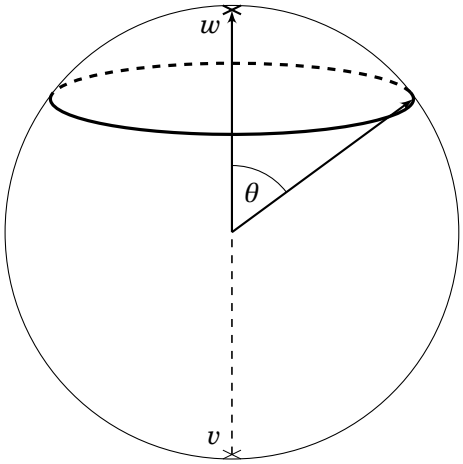

FIGURE 1 – L'opérateur « en cerceau » $\check{Q}_\theta$. Cet opérateur envoie une masse de Dirac en un point w sur la mesure de probabilité uniforme sur le « cerceau » dessiné ci-dessus [ici en dimension 3].

Il nous reste à contrôler l'écart initial entre la mesure limite μ_0 et la mesure empirique $\hat{\mu}_0^N$. Nous supposerons que μ_0 a un moment r-exponentiel (*i.e.* $\int e^{a|v|^r} d\mu_0(v) < \infty$ pour un $a > 0$), et nous prendrons pour condition initiale sur le modèle stochastique des particules i.i.d. selon la loi μ_0. On peut alors contrôler $\mathbf{E}[\mathscr{U}(\lambda e^{-\kappa T}(\hat{X}_0 - X_0))]$ par les mêmes méthodes de martingales que pour la démonstration du théorème 2.1.

En mettant bout-à-bout l'estimation sur la différence initiale, le théorème 2.1 et les estimations sur V, L et κ, nous obtenons alors une majoration explicite de la probabilité d'un événement de la forme $\mathbf{P}(\|\hat{\mu}_T^N - \mu_T\| \geqslant \varepsilon)$. Cette majoration est assez compliquée à écrire [son expression précise est donnée dans la partie II de cette thèse, formule (CJ)], mais peut aisément être appliquée numériquement. J'ai fait une telle application numérique dans le cas du modèle « de Kac » en dimension 3, avec $\mu_0 = \frac{1}{2}(\delta_{-1} + \delta_1)$ (on part loin de l'équilibre maxwellien) et $T = 3$ (chaque particule subit trois collisions en moyenne), le paramètre r de l'espace $\dot{H}^{-s}$ étant pris à 1/2. Je trouve alors que

$$N \geqslant 4 \times 10^5 \quad \Rightarrow \quad \mathbf{P}(\|\hat{\mu}_T^N - \mu_T\|_{\dot{H}^{-s}} \geqslant 10^{-2}) \leqslant 10^{-1}, \tag{2S}$$

ce qui montre qu'un nombre de particules assez faible (et largement à portée de simulation) assure avec grande probabilité une très bonne concordance (correspondant à des erreurs de vitesse de l'ordre de 10^{-3}) entre modèle particulaire et modèle continu.

D'un point de vue plus qualitatif, ma borne présente les comportements suivants :

- Quand on regarde les asymptotiques pour $N \to \infty$, on obtient :

$$\overline{\lim_{N\to\infty}}\, \mathbf{P}(\|\hat{\mu}_T^N - \mu_T\| \geqslant x N^{-1/2}) \leqslant 2 \exp\Big(\frac{-x^2}{2[e^{2|\kappa|T}\sigma^2 + e_1(2|\kappa|T)\omega T]}\Big), \tag{2T}$$

  où σ^2 est relié aux queues de la distribution initiale et ω à l'énergie du système, ces quantités étant calculables explicitement en fonction de μ_0. Cette borne montre une convergence à vitesse $N^{-1/2}$, avec un contrôle gaussien, ce qui est typique du théorème-limite central (uniforme).
- Quand on s'intéresse au comportement non asymptotique, mes bornes explicites donnent un contrôle gaussien des fluctuations tant que $xN^{1/2}$ reste plus petit que $O(N^{-r/2})$, c.-à-d. jusqu'à un régime intermédiaire entre les déviations standard et les grandes déviations ; au-delà, on garde encore un contrôle exponentiel des fluctuations.
- Toujours du point de vue non asymptotique, dans le cas $\kappa < 0$, mes bornes se dégradent très rapidement quand T augmente. En revanche, dans le cas $\kappa > 0$, les bornes obtenues restent

bonnes quand T devient grand, et on peut même obtenir de bonnes bornes sur la déviation uniforme $\sup_{t\in[0,T]} \|\hat{\mu}_t^N - \mu_t\|$.

3 Tensorisation des corrélations maximales

☛ *Cette section est un résumé du travail présenté en détail dans la partie III de la thèse.*

3.a β-mélange et ρ-mélange

En théorie des probabilités, quand on veut *quantifier* la dépendance entre deux tribus, deux méthodes concurrentes sont le β-mélange et le ρ-mélange :

3.1 Définition. Pour X, Y deux variables aléatoires définies sur un même espace de probabilités, à valeurs dans deux espaces arbitraires $\mathscr{X}, \mathscr{Y}$, le *coefficient de β-mélange* entre X et Y est

$$\beta(X,Y) := dist_{\mathrm{VT}}(Loi_X \otimes Loi_Y, Loi_{(X,Y)}), \tag{3A}$$

où $dist_{\mathrm{VT}}$ désigne la distance en variation totale. En d'autres termes,

$$\beta(X,Y) = \frac{1}{2}\int_{\mathscr{X}\times\mathscr{Y}} |d\mathbf{P}[(X,Y)=(x,y)] - d\mathbf{P}[X=x]d\mathbf{P}[Y=y]|. \tag{3B}$$

♢

3.2 Définition. Pour X, Y deux variables aléatoires définies sur un même espace de probabilités, à valeurs dans deux espaces arbitraires $\mathscr{X}, \mathscr{Y}$ le *coefficient de ρ-mélange* entre X et Y est

$$\{X:Y\} := \sup_{f,g} \frac{\mathrm{Cov}(f(X), g(Y))}{\mathrm{Sd}(f(X))\mathrm{Sd}(g(Y))}, \tag{3C}$$

où "$\mathrm{Sd}(f)$" note l'écart-type de f (*i.e.* $\mathrm{Sd}(f) := \sqrt{\mathrm{Var}(f)}$), f et g étant des fonctions *réelles* définies resp. sur $\mathscr{X}$ et $\mathscr{Y}$, supposées mesurables et de carrés sommables. Ce coefficient est encore appelé *corrélation maximale*, ou *hilbertienne*, entre X et Y. ♢

3.3 Proposition.

(i) $\beta(X,Y)$ *et* $\{X:Y\}$ *ne dépendent de* X *et* Y *que* via *les tribus que ces variables engendrent ;*

(ii) Il y a équivalence entre les affirmations suivantes :

1. $\beta(X,Y) = 0$ *;*
2. $\{X:Y\} = 0$ *;*
3. X *et* Y *sont indépendantes.*

♣

L'étude des corrélations est particulièrement intéressante en mécanique statistique, où l'état d'une particule peut influencer ses voisines de proche en proche. Un cas d'école est le modèle d'Ising des matériaux ferromagnétiques, dont nous rappelons la définition :

3.4 Définition. Dans le modèle d'Ising, l'état du système est décrit par une variable aléatoire $\vec{\omega} \in \{\pm 1\}^{\mathbb{Z}^n}$, $\omega_i \in \{\pm 1\}$ étant appelé le *spin* du site $i \in \mathbb{Z}^n$. Deux spins i et j sont dits *voisins* s'ils sont adjacents dans le réseau $\mathbb{Z}^n$, ce qu'on note "$i \sim j$". $\vec{\omega}$ suit alors la *mesure de Gibbs à paramètre β* définie (formellement) par

$$\mathbf{P}(\omega) \propto e^{-\beta H(\omega)}, \tag{3D}$$

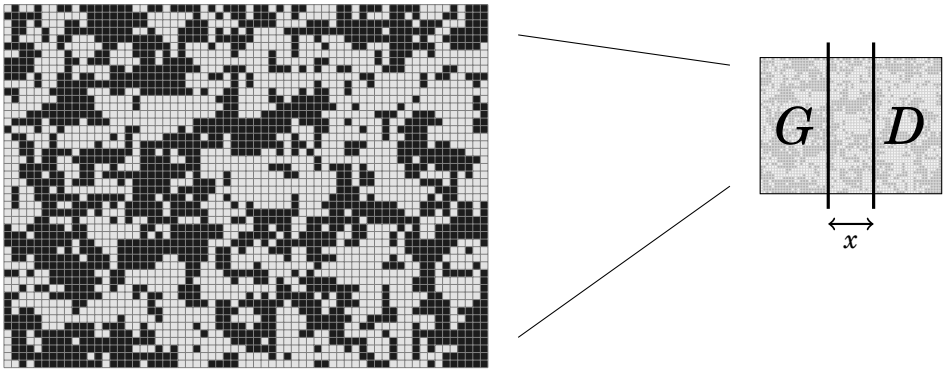

FIGURE 2 – Une réalisation du modèle d'Ising sous-critique. Chaque carré représente un nœud de $\mathbb{Z}^2$; les spins $+1$ sont représentés en clair et les spins -1 en foncé. On voit clairement que les spins proches sont corrélés mais que cette corrélation décroît avec la distance. La miniature définit les demi-espaces G et D.

où le *Hamiltonien* du système est (formellement)

$$H(\omega) := \sum_{i \sim j} \mathbf{1}_{\omega_i \neq \omega_j}. \tag{3E}$$

♢

Dans le cas du modèle d'Ising à petit paramètre, le théorème ci-dessous énonce qu'on a décroissance exponentielle des corrélations entre deux spins distants. Notez que la façon de mesurer ces corrélations (β- ou ρ-mélange, notamment) importe peu, dans la mesure où la loi jointe d'une paire de spins vit dans un espace de dimension finie.

3.5 Théorème (Peierls)**.** *Pour peu que le paramètre β soit suffisamment petit (régime* sous-critique*) :*

(i) La mesure de Gibbs est uniquement définie, et il n'y a pas d'aimantation globale, i.e. $\mathbf{E}[\omega_i] = 0 \ \ \forall i$.

(ii) Les spins proches présentent une corrélation positive, qui décroît exponentiellement avec la distance, i.e.

$$\exists \psi > 0, C < \infty \quad \forall i, j \in \mathbb{Z}^n \quad 0 < \mathbf{E}[\omega_i \omega_j] < C \exp\big(-\psi dist(i,j)\big). \tag{3F}$$

♣

En revanche, on peut montrer que, dans certains cas au moins, β- et ρ-mélange se comportent différemment quand on s'intéresse à des groupes *infinis* de spins :

3.6 Théorème (Minlos & Sinai)**.** *Définissons les ensembles G et D comme sur la figure 2, et notons resp. $\vec{\omega}_G$ et $\vec{\omega}_D$, l'état de l'ensemble des spins de G et de D. Alors :*

(i) Quelle que soit la valeur de x, $\beta(\vec{\omega}_G, \vec{\omega}_D) = 1$, ce qui est la valeur maximale pour un coefficient de β-mélange ;

(ii) $\{\vec{\omega}_G : \vec{\omega}_D\} \leqslant e^{-\psi x}$, où ψ est la constante introduite dans (3F).

♣

La démonstration classique du point (ii) repose toutefois sur des propriétés très particulières, tant du modèle d'Ising [on a besoin que le modèle présente certaines symétries, que la portée des interactions soit finie et que les corrélations décroissent exponentiellement] que des ensembles G et D [c'est pratiquement la seule forme qui convienne].

3.b Tensorisation des corrélations hilbertiennes

Le théorème 3.6 nous montre que les décorrélations hilbertiennes permettent de saisir certains phénomènes intéressants. Nous allons donc tenter d'étudier ces décorrélations « pour elles-mêmes », en espérant qu'on en tirera une démonstration alternative de ce théorème qui soit susceptible de généralisations.

Un aspect important des corrélations hilbertiennes est leur capacité à se *tensoriser* :

3.7 Théorème (Csáki & Fischer)**.** *Soit I un ensemble au plus dénombrable, et soient $(X_i, Y_i)_{i\in I}$ des couples indépendants de variables aléatoires ; alors :*

$$\{\vec{X}_I : \vec{Y}_I\} = \sup_{i\in I}\{\vec{X}_i : \vec{Y}_i\}. \tag{3G}$$

♣

En adaptant certaines techniques de preuve du théorème 3.7, on peut démontrer des théorèmes de tensorisation plus généraux. Je donne ci-dessous un exemple de tel théorème, avec sa démonstration. On a d'abord besoin d'une définition complémentaire sur les corrélations hilbertiennes :

3.8 Définition. Soient X, Y et Z trois variables aléatoires. On définit la *corrélation subjective* $\{X : Y\}_Z$ entre X et Y par rapport à Z comme le supremum (plus précisément, le supremum essentiel) des valeurs de $\{X : Y\}$ sous les différentes lois conditionnelles $\mathbf{P}[\cdot|Z = z]$.

En termes hilbertiens,

$$\{X : Y\}_Z = \sup_{\substack{f\in L^2(\sigma(X,Z)) \\ g\in L^2(\sigma(Y,Z)) \\ \mathbf{E}[f|Z],\mathbf{E}[g|Z]\equiv 0}} \frac{|\mathbf{E}[fg]|}{\mathrm{Sd}(f)\mathrm{Sd}(g)}. \tag{3H}$$

♢

3.9 Théorème (*Peyre*)**.** *Pour $I = \{1, \dots, N\}$, soient $(X_i)_{i\in I}$ et Y des variables aléatoires. On suppose que pour tout $i \in I$,*

$$\{X_i : Y\}_{\vec{X}_{\{1,\dots,i-1\}}} \leqslant \varepsilon_i; \tag{3I}$$

alors :

$$\{X_i : Y\} \leqslant \sqrt{\sum_{i\in I} \varepsilon_i^2}. \tag{3J}$$

♣

Démonstration. Soient f et g des fonctions L^2 centrées resp. $\vec{X}$- et Y-mesurables. L'enjeu est de borner $|\mathbf{E}[fg]|$.

Définissions, pour tout $i \in \{0, \dots, N\}$,

$$\mathscr{F}_i := \sigma(X_1, \dots, X_i), \tag{3K}$$

et pour tout $i \in \{1, \dots, N\}$,

$$f_i := \mathbf{E}[f|\mathscr{F}_i] - \mathbf{E}[f|\mathscr{F}_{i-1}]. \tag{3L}$$

On a alors $f = \sum_i f_i$, où les f_i sont orthogonales dans $L^2(\mathbf{P})$, d'où

$$|\mathbf{E}[fg]| \leqslant \sum_{i\in I} |\mathbf{E}[f_i g]| \tag{3M}$$

et

$$\mathrm{Var}(f) = \sum_{i \in I} \mathrm{Var}(f_i). \tag{3N}$$

Pour borner $|\mathbf{E}[f_i g]|$, on conditionne suivant la valeur de $\vec{X}_{\{1,\dots,i-1\}}$:

$$\mathbf{E}[f_i g] = \int \mathbf{E}\left[f_i g | X_1 = x_1, \dots, X_{i-1} = x_{i-1}\right] d\mathbf{P}[x_1, \dots, x_{i-1}]. \tag{3O}$$

Or sous la loi $d\mathbf{P}[\cdot | x_1, \dots, x_{i-1}]$, f_i est X_i-mesurable et centrée, et g est Y-mesurable, de sorte qu'en appliquant l'hypothèse (3I) :

$$\begin{aligned} |\mathbf{E}[f_i g]| &\leqslant \varepsilon_i \int \sqrt{\mathbf{E}[f_i^2 | x_1, \dots, x_{i-1}]} \sqrt{\mathbf{E}[g^2 | x_1, \dots, x_{i-1}]} d\mathbf{P}[x_1, \dots, x_{i-1}] \\ &\underset{\mathrm{CS}}{\leqslant} \varepsilon_i \sqrt{\int \mathbf{E}[f_i^2 | x_1, \dots, x_{i-1}] d\mathbf{P}[x_1, \dots, x_{i-1}]} \sqrt{\textit{idem pour } g} = \varepsilon_i \, \mathrm{Sd}(f_i) \mathrm{Sd}(g). \end{aligned} \tag{3P}$$

("$\underset{\mathrm{CS}}{\leqslant}$" signifie qu'on a utilisé l'inégalité de Cauchy – Schwarz). En sommant (3P) sur i :

$$|\mathbf{E}[fg]| \leqslant \sum_{i=1}^{N} \varepsilon_i \, \mathrm{Sd}(f_i) \mathrm{Sd}(g) \underset{\mathrm{CS}}{\leqslant} \sqrt{\sum_i \varepsilon_i^2} \sqrt{\sum_i \mathrm{Var}(f_i)} \, \mathrm{Sd}(g) = \sqrt{\sum_i \varepsilon_i^2} \, \mathrm{Sd}(f) \mathrm{Sd}(g). \tag{3Q}$$

(3Q) étant vraie pour toutes f, g, cela démontre le théorème. ♠

Le théorème précédent s'applique à la tensorisation de N spins contre un seul. Pour tensoriser N spins contre M autres, le théorème général est le suivant, qui redonne notamment le théorème 3.7 sur la tensorisation indépendante :

3.10 Théorème (*Peyre*)**.** *Soient I et J deux ensembles finis, et $(X_i)_{i \in I}, (Y_j)_{j \in J}$ des variables aléatoires. Sur chaque couple (i, j), on fait l'hypothèse suivante : uniformément en $I' \subset I \smallsetminus \{i\}$ et $J' \subset J \smallsetminus \{j\}$, la corrélation subjective entre X_i et Y_j par rapport à $(\vec{X}_{I'}, \vec{Y}_{J'})$ est majorée par un certain ε_{ij} — hypothèse qu'on notera de manière synthétique :*

$$\{X_i : Y_j\}_* \leqslant \varepsilon_{ij}. \tag{3R}$$

Alors on a :

$$\{\vec{X}_I : \vec{Y}_J\} \leqslant |||\boldsymbol{\varepsilon}|||, \tag{3S}$$

où "$|||\boldsymbol{\varepsilon}|||$" désigne la norme d'opérateur de l'opérateur linéaire $\boldsymbol{\varepsilon}$ entre espaces euclidiens défini par

$$\begin{array}{rccl} \boldsymbol{\varepsilon} : & L^2(J) & \to & L^2(I) \\ & (a_j)_{j \in J} & \mapsto & \left(\sum_{j \in J} \varepsilon_{ij} a_j\right)_{i \in I}. \end{array} \tag{3T}$$

♣

Grâce à la tensorisation des corrélations hilbertiennes, on peut donner de nouveaux résultats de décorrélation entre groupes de particules en mécanique statistique. Dans le cas du modèle d'Ising, on obtient ainsi le théorème suivant :

3.11 Théorème (*Peyre*)**.** *Pour le modèle d'Ising sur $\mathbb{Z}^n$ à paramètre β suffisamment petit, il existe des constantes $\psi' > 0$ et $k < 1$ telles que, pour I, J des ensembles de spins disjoints de tailles et de formes arbitraires, on a, uniformément en I et J :*

$$\{\vec{\omega}_I : \vec{\omega}_J\} \leqslant \left(\exp\left[-(\psi' + o(1)) dist(I, J)\right]\right) \wedge k. \tag{3U}$$

♣

En fait, Des théorèmes analogues peuvent également être prouvés dans des cadres beaucoup plus généraux que le modèle d'Ising : aucune symétrie n'est requise, la décroissance des corrélations peut être polynomiale, l'espace d'états peut être continu, le réseau qui supporte les spins peut avoir de la courbure, etc.

3.c Critère d'α-mélange pour les décorrélations hilbertiennes

J'ai par ailleurs démontré un nouveau critère de décorrélation hilbertienne. Remarquons d'abord que les corrélations hilbertiennes contrôlent la différence entre mesure produit et mesure jointe sur les rectangles — de telles estimations sont qualifiées d'estimations d'*α-mélange* :

3.12 Proposition. *Si $\mathscr{F}$ et $\mathscr{G}$ sont deux tribus avec*

$$\{\mathscr{F} : \mathscr{G}\} \leqslant \varepsilon, \tag{3V}$$

alors pour tous événements $A \in \mathscr{F}, B \in \mathscr{G}$ de probabilités respectives p et q,

$$|\mathbf{P}[A \cap B] - pq| \leqslant \varepsilon \sqrt{p(1-p)q(1-q)}. \tag{3W}$$

♣

Démonstration. Appliquer la définition (3C) des corrélations hilbertiennes avec $f = \mathbf{1}_A$ et $g = \mathbf{1}_B$.
♠

Il était déjà connu que la Propostion 3.12 admet des réciproques partielles, au sens où établir (3W) pour un ε suffisamment petit permet de majorer arbitrairement $\{\mathscr{F} : \mathscr{G}\}$. La forme optimale de cette réciproque était conjecturée, mais pas connue. C'est elle que je suis parvenu à obtenir :

3.13 Théorème (*Peyre*)**.** *Si $\mathscr{F}$ et $\mathscr{G}$ sont deux tribus telles que, pour tous événements $A \in \mathscr{F}, B \in \mathscr{G}$ de probabilités respectives p et q,*

$$|\mathbf{P}[A \cap B] - pq| \leqslant \varepsilon \sqrt{p(1-p)q(1-q)} \tag{3X}$$

pour un certain $\varepsilon \in (0,1]$, alors :

$$\{\mathscr{F} : \mathscr{G}\} \leqslant \varepsilon(1 + |\log \varepsilon|). \tag{3Y}$$

♣

4 Brisure spontanée de symétrie en dimension infinie

☛ *Cette section est un résumé du travail présenté en détail dans la partie IV de la thèse.*

4.a Modèle et problématique

On considère une assemblée dense de particules ponctuelles sur l'espace euclidien affine $\mathbb{R}^d$, dont la distribution est décrite par une mesure positive μ. Les particules sont soumises à un potentiel

d'interaction v symétrique, ainsi qu'à une agitation brownienne à température T, le tout en présence de forts frottements. La densité m de μ évolue alors selon une *équation de Mc Kean – Vlasov* :

$$\partial_t m = \nabla \cdot \big(T\nabla m + m\nabla(v * m)\big). \tag{4A}$$

Le potentiel v est supposé vérifier les hypothèses suivantes :

4.1 Hypothèse. La fonction $v\colon \mathbb{R}^d \to \mathbb{R}$ est dans l'espace de Schwartz (*i.e.* elle est infiniment différentiable et toutes ses dérivées sont intégrables), et est négative sur tout $\mathbb{R}^d$. ♢

Otto [59] a montré que l'équation (4A) pouvait s'interpréter comme une descente de gradient dans une « variété riemannienne ». La fonctionnelle de Lyapounov correspondant à cette descente est l'*énergie libre* de la mesure μ :

$$\mathscr{F}(\mu) := \mathscr{U}(\mu) + T\mathscr{S}(\mu), \tag{4B}$$

où $\mathscr{U}$ et $\mathscr{S}$ sont respectivement l'*énergie interne* et l'*entropie* :

$$\mathscr{U}(\mu) := \frac{1}{2}\int_{(\mathbb{R}^d)^2} v(x-y)\,d\mu(x)d\mu(y), \tag{4C}$$

$$\mathscr{S}(\mu) := \int_{\mathbb{R}^d} \log m(x)\,d\mu(x). \tag{4D}$$

La « variété riemannienne » dans laquelle a lieu la descente de de gradient est la structure associée à la métrique de Wasserstein W_2, dont nous rappelons la définition :

4.2 Définition. Pour μ, ν deux mesures positives σ-finies sur $\mathbb{R}^d$ de même masse totale, un *couplage* entre μ et ν est une mesure γ sur $\mathbb{R}^d \times \mathbb{R}^d$ dont les deux marginales sont respectivement μ et ν. Le *coût quadratique* de ce couplage est

$$I[\gamma] := \int_{(\mathbb{R}^d)^2} |y-x|^2\,d\gamma(x,y). \tag{4E}$$

Notant $\Gamma(\mu,\nu)$ l'ensemble des couplages entre μ et ν, la *distance de Wasserstein* $W_2(\mu,\nu)$ entre μ et ν est alors définie comme

$$W_2(\mu,\nu) := \inf_{\gamma\in\Gamma(\mu,\nu)} I[\gamma]^{1/2}. \tag{4F}$$

(Nous admettrons ici qu'il s'agit bien d'une distance). ♢

Nous allons étudier la fonctionnelle $\mathscr{F}$ pour les perturbations de la *mesure uniforme* λ sur $\mathbb{R}^d$ définie par $d\lambda(x) := R\,dx$, où $R \in (0,\infty)$ est un paramètre d'homogénéité fixé. Aussi travaillerons-nous dans l'espace défini ci-dessous :

4.3 Définition. L'*espace de Wassertein*, noté $\mathbf{F}$, est l'ensemble des mesures μ telles que $W_2(\lambda,\mu) < \infty$, muni de la distance de Wasserstein W_2. ♢

Pour μ dans $\mathbf{F}$, les formules (4C) et (4D) doivent être renormalisées pour faire sens : notant $\pi := \mu - \lambda$ et $p(x) := d\pi(x)/dx = m(x) - R$, on prendra en fait

$$\mathscr{U}(\mu) = \frac{1}{2}\int_{(\mathbb{R}^d)^2} v(y-x)\big(m(x)m(y) - R^2\big)\,dx\,dy = \frac{1}{2}\int_{\mathbb{R}^d} (v * p)(x)\,d\pi(x); \tag{4G}$$

$$\mathscr{S}(\mu) = \int_{\mathbb{R}^d} \log\big(R^{-1}m(x)\big)\,d\mu(x) = \int_{\mathbb{R}^d} \underbrace{\Big((R+p(x))\log\big(1+R^{-1}p(x)\big) - p(x)\Big)}_{=:\Phi(p(x))}\,dx, \tag{4H}$$

de sorte que $\mathscr{U}(\lambda), \mathscr{S}(\lambda) = 0$ — et donc $\mathscr{F}(\lambda) = 0$. On notera qu'on a alors $\mathscr{S} \geqslant 0$ sur tout $\mathbf{F}$.

4.4 Définition.

1. Pour une température T donnée, nous dirons que l'équilibre homogène est *stable* quand la fonctionnelle $\mathscr{F}$ sur $\mathbf{F}$ atteint un minimum local en λ.
2. Quand l'équilibre homogène est stable, son *énergie d'activation* E_{a} sera définie comme le supremum des valeurs E vérifiant la propriété suivante : sur la composante connexe de λ au sein du sous-ensemble de $\mathbf{F}$ constitué par les fonctions $\{\mu\colon\ \mathscr{F}(\mu) \leqslant \mathscr{F}(\lambda)+E\}$, $\mathscr{F}$ atteint un minimum global en λ.

$\diamondsuit$

L'énergie d'activation a deux interprétations :

1. E_{a} est l'énergie minimale qu'il faut fournir au système pour passer de l'état λ à un autre état plus stable ;
2. Quand on revient au système particulaire dont l'évolution (4A) n'est qu'une approximation, il peut arriver, en partant de la distribution homogène, que le système quitte cet état pour passer à un état plus stable sous le seul effet des flucutations. L'énergie d'activation caractérise alors le temps (très long) qu'il faut attendre pour que cela se produise, qui est de l'ordre de $\exp(\mathscr{N}E_{\mathrm{a}}/T)$, $\mathscr{N}$ étant le nombre d'Avogadro.

Notre problématique est la suivante : dans quelles conditions l'équilibre homogène est-il stable, et que vaut alors l'énergie d'activation ?

4.b Stratégie de minoration

Puisque la fonctionnelle d'énergie libre $\mathscr{U}$ est quadratique, on voudrait minorer $\mathscr{S}$ par une forme quadratique. Comme la fonction $\Phi(\cdot)$ qui intervient dans (4H) est sous-quadratique, on va considérer deux cas selon la valeur de $p(x)$. Introduisons donc un paramètre $\eta \in (0,\infty)$ moralement très petit, et décomposons $p(x)$ en $p_2(x)+p_1(x)$, où

$$\begin{cases} p_2(x) := \mathbf{1}_{\{|p(x)| \leqslant \eta\}} p(x); \\ p_1(x) := \mathbf{1}_{\{|p(x)| > \eta\}} p(x). \end{cases} \tag{4I}$$

On a alors pour tout p que

$$\Phi(p) \geqslant \frac{\Phi(\eta)}{\eta^2} p_2^2 + \frac{\Phi(\eta)}{\eta} |p_1|, \tag{4J}$$

d'où en intégrant :

$$\mathscr{S} \geqslant \frac{\Phi(\eta)}{\eta^2} \|p_2\|_{L^2}^2 + \frac{\Phi(\eta)}{\eta} \|p_1\|_{L^1}. \tag{4K}$$

Notons que dans (4K), quand $\eta \searrow 0$, on a $\Phi(\eta)/\eta^2 \to \frac{1}{2}R^{-1}$ et $\Phi(\eta)/\eta \to 0$.

Pour appliquer cela à la minoration de $\mathscr{F}$, posant

$$V := \int_{\mathbb{R}^d} \big(-v(x)\big)\, dx, \tag{4L}$$

on écrit que

$$\begin{aligned}\mathscr{F}(\mu) = \frac{1}{2}\langle p, v * p\rangle_{L^2} + T\mathscr{S}(\mu) &= \frac{1}{2}\langle p_2, v * p\rangle + \frac{1}{2}\langle p_1, v * p\rangle + T\mathscr{S}(\mu)\\ &\geqslant -\frac{1}{2}\|v * p\|_{L^2}\|p_2\|_{L^2} - \frac{1}{2}\|v * p\|_{L^\infty}\|p_1\|_{L^1} + T\mathscr{S}\\ &\underset{\text{Young}}{\geqslant} -\frac{1}{4V}\|v * p\|_2^2 \underbrace{-\frac{V}{4}\|p_2\|_2^2 - \frac{\|v * p\|_\infty}{2}\|p_1\|_1}_{\substack{\geqslant -(\eta^2 V/4\Phi(\eta))\mathscr{S}\\ \text{si } \|v * p\|_\infty \leqslant V\eta/2}} + T\mathscr{S}\\ &\geqslant -\frac{1}{4V}\|v * p\|_2^2 + \left(T - \frac{\eta^2 V}{4\Phi(\eta)}\right)\mathscr{S} \quad \text{si } \|v * p\|_\infty \leqslant V\eta/2. \qquad (4\mathtt{M})\end{aligned}$$

On a donc besoin, d'une part de majorer $\|v * p\|_\infty$ au voisinage de λ — ce que nous reportons au § 4.c —, d'autre part de minorer $\mathscr{S}$ par $\|v * p\|_2^2$.

Pour ce dernier point, nous introduisons la marche aléatoire sur $\mathbb{R}^d$ dont les pas sont distribués suivant la mesure de probabilité K définie par

$$dK(x) := \frac{-v(x)}{V}dx. \qquad (4\mathtt{N})$$

On utilise alors le résultat suivant sur l'entropie :

4.5 Théorème. *Si P est le noyau d'une chaîne de Markov sur $\mathbb{R}^d$ admettant λ pour mesure invariante, alors pour toute mesure μ sur Ω, l'entropie de μ est décroissante sous l'action de P :*

$$\mathscr{S}(\mu P) \leqslant \mathscr{S}(\mu). \qquad (4\mathtt{O})$$

♣

On en déduit que $\mathscr{S}(p) \geqslant \mathscr{S}(V^{-1}v * p)$. Sous réserve que $\|v * p\|_\infty$ soit suffisamment petit, on peut alors appliquer la formule (4K) qui donne :

$$\|v * p\|_\infty \leqslant V\eta \quad \Rightarrow \quad \mathscr{S} \geqslant \frac{\Phi(\eta)}{\eta^2}V^{-2}\|v * p\|_2^2. \qquad (4\mathtt{P})$$

Combinant (4M) et (4P), on obtient finalement :

$$\|v * p\|_\infty \leqslant \frac{V\eta}{2} \quad \Rightarrow \quad \mathscr{F} \geqslant \Big(T - \frac{\eta^2}{\Phi(\eta)}\frac{V}{2}\Big)\mathscr{S}. \qquad (4\mathtt{Q})$$

4.c Plongement de l'espace de Wasserstein dans un espace linéaire

Nous devons encore borner $\|v * p\|_\infty$ au voisinage de λ dans $\mathbf{F}$, ce pour quoi nous aurons besoin d'un résultat de plongement de l'espace de Wasserstein dans le dual d'un espace de Sobolev. Voici un tel résultat :

4.6 Théorème (*Peyre*)**.** *Pour f dans l'espace de Schwartz, notant $\|f\|_{1,2} := \|Df\|_{L^2}$, $\|f\|_{2,\infty} := \|D^2 f\|_{L^\infty}$, définissons*

$$\|f\|_{\mathbf{W}_0} := \|f\|_{1,2} \vee |f\|_{2,\infty} = \Big(\int_{\mathbb{R}^d} |Df(x)|^2\,dx\Big)^{1/2} \vee \sup_{x\in\mathbb{R}^d} |D^2 f(x)|, \qquad (4\mathtt{R})$$

et notons $\mathbf{W}_0$ l'espace obtenu par complétion de cette norme. Alors l'application $\mu \mapsto \pi$, considérée de $\mathbf{F}$ dans le dual $\mathbf{W}_0'$ de cet espace, est continue en λ. ♣

Démonstration. Soit f une fonction de $\mathbf{W}_0$ et soit T un plan de transport de λ vers $\mu \in \mathbf{F}$. Notons $u(x) := T(x)/|T(x)|$ le vecteur unitaire qui indique la direction dans laquelle le point x se déplace au cours de l'opération de transport.

On a

$$|\langle f,\pi\rangle| = \Big|\int_{\mathbb{R}^d} \big(f(x+T(x)) - f(x)\big)\,d\lambda(x)\Big| \leqslant R\int_{x\in\mathbb{R}^d}\int_{r=0}^{|T(x)|} |Df(x+ru(x))|\,dr\,dx \tag{4S}$$

et

$$I[T] = \int_{\mathbb{R}^d} |T(x)|^2\,d\lambda(x) = 2R\int_{x\in\mathbb{R}^d}\int_{r=0}^{|T(x)|} r\,dr\,dx. \tag{4T}$$

Nous introduisons alors l'espace $\Omega := \{(x,r):\ 0 \leqslant r \leqslant |T(x)|\}$, et sur Ω nous définissons la mesure $d\gamma := 2Rr\,dr\,dx$ et la fonction $e(x,r) := |Df(x+ru(x))|/2r$. Les formules (4T) et (4S) deviennent alors respectivement :

$$I[T] = \int_{(x,r)\in\Omega} d\gamma(x,r); \tag{4U}$$

$$|\langle f,\pi\rangle| \leqslant \int_{(x,r)\in\Omega} e(x,r)\,d\gamma(x,r). \tag{4V}$$

Cette position du problème suggère l'utilisation d'un lemme de couplage :

4.7 Lemme (*Peyre*)**.** *Soit γ une mesure (positive) σ-finie sur un espace mesurable Ω, et $e:\ \Omega \to \mathbb{R}_+$ une fonction mesurable. On suppose qu'il existe une fonction $Y_*:\ \mathbb{R}_+ \to \mathbb{R}_+$, de classe $\mathscr{C}^1$ et décroissante avec $Y_*(e) \overset{e\to\infty}{\to} 0$, telle que pour tout $\theta \in \mathbb{R}_+$,*

$$\int_{\{e(\omega)\geqslant\theta\}} e(\omega)\,d\gamma(\omega) \leqslant Y_*(\theta). \tag{4W}$$

Définissant à partir de Y_* la fonction décroissante $X_*:\ \mathbb{R}_+ \to \mathbb{R}_+$ par

$$X_*(\theta) := -\int_\theta^\infty \frac{Y_*'(\tau)}{\tau}\,d\tau, \tag{4X}$$

on a alors :

$$\int_\Omega e(\omega)\,d\gamma(\omega) \leqslant \big(Y_*\circ X_*^{-1}\big)\Big(\int_\Omega d\gamma(\omega)\Big). \tag{4Y}$$

♣

Pour appliquer le lemme 4.7 à notre situation, il faut majorer $\int_{e(x,r)\geqslant\theta} |Df(x+ru(x))|\,dr\,dx$ pour $\theta > \|f\|_{2,\infty}/2$ arbitraire. Or "$e(x,r) \geqslant \theta$" signifie que $|Df(x+ru(x))| \geqslant 2\theta r$, d'où $|Df(x)| \geqslant (2\theta - \|f\|_{2,\infty})r$, et donc $r \leqslant (2\theta - \|f\|_{2,\infty})^{-1}|Df(x)|$. On a en outre $|Df(x+ru(x))| \leqslant |Df(x)| + r\|f\|_{2,\infty}$, d'où la majoration :

$$\begin{aligned}\int_{e(x,r)\geqslant\theta} |Df(x+ru(x))|\,dx &\leqslant \int_{r\leqslant\frac{|Df(x)|}{2\theta-\|f\|_{2,\infty}}} \big(|Df(x)| + r\|f\|_{2,\infty}\big)\,dx\,dr \\ &= \Big(\frac{1}{2\theta-\|f\|_{2,\infty}} + \frac{\|f\|_{2,\infty}}{2(2\theta-\|f\|_{2,\infty})^2}\Big)\|f\|_{1,2}^2.\end{aligned} \tag{4Z}$$

Le lemme 4.7 donne alors $|\langle f,\pi\rangle| \leqslant \sqrt{R}\|f\|_{1,2}I[T]^{1/2} + \frac{1}{2}\|f\|_{2,\infty}I[T]$, et donc en prenant l'infimum sur les plans de transport de λ à μ :

$$|\langle f,\pi\rangle| \leqslant \sqrt{R}\|f\|_{1,2}W_2(\lambda,\mu) + \frac{1}{2}\|f\|_{2,\infty}W_2(\lambda,\mu)^2, \tag{4AA}$$

ce qui montre bien que le plongement de $\mathbf{F}$ dans $\mathbf{W}_0'$ est lipschitzien en λ. ♠

En raffinant les techniques présentées ci-dessus, on peut même obtenir un résultat de continuité globale :

4.8 Théorème (*Peyre*)**.** *Pour $\alpha \in [0, 2/(d+2))$, pour f dans l'espace de Schwartz, définissons formellement*

$$\|f\|_{\mathbf{W}_\alpha} := \|Df\|_2 \vee \|D^{2-\alpha} f\|_{2/\alpha}, \tag{4AB}$$

et notons $\mathbf{W}_\alpha$ l'espace obtenu par complétion de cette norme. Alors l'application $\mu \mapsto \pi$, considérée de $\mathbf{F}$ dans $\mathbf{W}'_\alpha$, est continue sur tout $\mathbf{F}$. ♣

4.d Quelques résultats

Grâce aux estimations présentées ci-dessus, on obtient le

4.9 Théorème (*Peyre*)**.** *Sous l'hypothèse 4.1, la transition de phase a lieu à la température RV : l'équilibre homogène est instable en-deçà de cette température, et stable au-delà.* ♣

Notons que cette température de transition coïncide avec celle du système linéarisé.

Pour minorer l'énergie d'activation, on peut utiliser une minoration de $\|v * p\|_2$ en fonction de $\|v * p\|_\infty$ — l'existence d'une telle minoration est rendue possible par la positivité de μ — :

4.10 Lemme (*Peyre*)**.** *Sous l'hypothèse 4.1, il existe une constante $C(v) < \infty$ telle que pour toute mesure* positive *μ,*

$$\|v * p\|_2 \geqslant C(v) \|v * m\|_\infty^{-d/4} \|v * p\|_\infty^{1+d/4}. \tag{4AC}$$

♣

4.11 Corollaire (*Peyre*)**.** *Pour $T > RV$, l'énergie d'activation de l'équilibre uniforme est non nulle, et minorée au voisinage de la température critique par $C(T - RV)^{3+d/2} + o((T - RV)^{3+d/2})$ pour un $C > 0$.* ♣

Finissons en évoquant quelques prolongements et conjectures :

- Le théorème 4.8 devrait rester valable pour tout $\alpha \in [0, 4/(d+2))$, ce qui serait alors l'intervalle optimal.
- Pour v non négatif, on étend la définition de V en posant

$$V := \sup_{\xi \in \mathbb{R}^d} \big(-\hat{v}(\xi) \big). \tag{4AD}$$

  Le théorème 4.9 reste alors valable pour tout v dans l'espace de Schwartz, même non négatif, ainsi que pour tout $v \in \mathbf{W}_\alpha$ si v est négatif. J'espère pouvoir l'étendre à tout v de $\mathbf{W}_\alpha$. Les hypothèses du théorème 4.11 devraient également pouvoir être affaiblies, mais j'ai plus de mal à cerner quelles seraient les conditions minimales.
- Dans le théorème 4.11, j'ignore si l'exposant $(3 + d/2)$ est optimal ; une conjecture plausible est qu'on pourrait abaisser cette valeur à 3.

Première partie

Une approche probabiliste de la borne de Carne

Résumé

La borne de Carne est une inégalité fine qui contrôle les probabilités de transition d'une chaîne de Markov discrète réversible [§ 1]. Sa preuve habituelle [§ 2] s'appuie sur des techniques spectrales efficaces mais d'apparence miraculeuse. Je présente ici une nouvelle preuve, dans laquelle on compare la « dérive » entre les chemins « aller » et « retour » pour récupérer la partie gaussienne de la borne [§ 3], et où on utilise une technique de conditionnement pour récupérer le facteur de dissipation [§ 5]. Je montre en outre que cette preuve est plus souple que celle de Carne et peut ainsi se généraliser [§ 4].

☛ *Les recherches présentées dans cette partie de la thèse ont été publiées dans [65].*

1 Introduction

1.a The Markov chain

Let V be a finite or countable set of points. Let us consider an irreducible Markov chain $(X_t)_{t\in\mathbb{N}}$ on V, with transition kernel $(p(x,y))_{x,y\in V}$, and whose law is denoted by $\mathbf{P}_x$ when starting at x. That chain is supposed to be *reversible*, i.e. we suppose that there exists a measure μ on V such that, for all $x\in V$ $0<\mu(x)<\infty$, and

$$\forall x,y\in V \quad \mu(x)p(x,y)=\mu(y)p(y,x). \tag{A}$$

By irreducibility, μ is then uniquely determined up to a multiplicative factor; in the sequel, we shall suppose it fixed. Note that we do not demand μ to be finite.

Then one may associate to the kernel a (non-oriented) graph (V,E) with vertices set V by defining the set of edges through

$$\{x,y\}\in E \Leftrightarrow p(x,y)\neq 0. \tag{B}$$

(*A priori* that definition should determine an *oriented* graph, but actually $p(x,y)\neq 0 \Leftrightarrow p(y,x)\neq 0$ by reversibility). As usual, we shall write $z\sim z'$ to mean that $\{z,z'\}\in E$ [†]. The graph distance, denoted by d, will stand for the length of the shortest path(s) in E joining two points. Speaking in terms of probability, one has:

$$d(x,y)=\inf\{t\in\mathbb{N}:\ p^t(x,y)\neq 0\}, \tag{C}$$

where p^t denotes the t-th convolution power of the kernel p.

This part of the thesis aims at explaining by probabilistic arguments an inequality due to Carne to sharply bound $p^t(x,y)$ above when $d(x,y)\gtrsim\sqrt{t}$. Indeed to the best of our knowledge, all the methods developed so far to get that kind of bounds used spectral analysis techniques [80, 18]. We shall also show how our probabilistic approach allows us to generalize Carne–Varopoulos type bounds for more "flexible" distances than the graph distance.

1.b Carne's bound and its history

In 1985, N. Th. Varopoulos [80] was the first to give a concentration result bounding above $p^t(x,y)$ for a reversible Markov chain, whose leading term was $\exp\big(-d(x,y)^2/Ct\big)$, $C>0$ being an explicit constant depending on the transition kernel p. His method introduced a time-continuous Markov process on the cabled graph associated to (V,E), and studied the spectral properties of that process in an L^2 space. Moreover that proof required extra assumptions about the transition kernel.

The same year, T. K. Carne [18], by a simpler spectral method, got a finer result under the general assumptions stated in § 1.a:

1.1 Theorem (Carne 1985)**.** *Suppose the hypotheses of § 1.a are satisfied. Denote by P the $L^2(\mu)$-operator associated to the transition kernel p and let $|P|$ stand for its norm, which is always $\leqslant 1$ (see a more precise definition in § 2.a). Then:*

$$p^t(x,y)\leqslant 2\left(\frac{\mu(y)}{\mu(x)}\right)^{1/2}|P|^t\exp\left(\frac{-d(x,y)^2}{2t}\right). \tag{D}$$

♣

[†]. Look out for the fact that $\sim$ is *not* an equivalence relation.

My work was motivated by two goals: first, find a proof of Theorem 1.1 which would be more natural than the original proof of Carne, then, adapt Carne–Varopoulos type bounds to distances which depend continuously on the transition kernel (see § 4.b).

2 Carne's proof

We give here the proof of [18] as it was exposed in [49, Theorem 13.4].

2.a Norm of the transition kernel

Let us first give a precise definition of P:

2.1 Definition. P is the operator induced by $\mathbf{P}$ on $L^2(\mu)$ through:

$$Pf(x) = \mathbf{E}_x[f(X_1)] = \sum_{y\sim x} p(x,y)f(y), \tag{E}$$

$\diamondsuit$

Then we define $|P|$ as the operator norm of P in $L^2(\mu)$, i.e. $|P| = \sup_{\|f\|_{L^2(\mu)}=1} \|Pf\|_{L^2(\mu)}$. Note that P is self-adjoint by reversibility of μ, and $|P| \leqslant 1$ by Jensen's inequality.

A more intrinsic defintion of $|P|$ is given by the following classical

2.2 Lemma ([49, Chap. 5-2])**.** *For any $x \in V$,*

$$|P| = \overline{\lim_{t\to\infty}} \left(p^t(x,x)\right)^{1/t} = \sup_{t\geqslant 1} \left(p^t(x,x)\right)^{1/t}. \tag{F}$$

$\clubsuit$

2.b Chebychev's polynomials

Since $P^n f(x) = \mathbf{E}_x[f(X_n)]$, one can write

$$p^t(x,y) = \left\langle \frac{\boldsymbol{\delta}_x}{\mu(x)}, P^t \boldsymbol{\delta}_y \right\rangle_{L^2(\mu)} = \frac{|P|^t}{\mu(x)} \left\langle \boldsymbol{\delta}_x, \left(\frac{P}{|P|}\right)^t \boldsymbol{\delta}_y \right\rangle_{L^2(\mu)}. \tag{G}$$

The trick then consists in decomposing the polynomial Z^t in the basis of Chebychev's polynomials. The following results are classical:

2.3 Lemma. *For any $k \in \mathbb{Z}$, there exists a unique polynomial $Q_k(Z)$ satisfying*

$$\forall \theta \in \mathbb{C} \quad Q_k(\cos\theta) = \cos(k\theta), \tag{H}$$

called the k-th (first type) Chebychev polynomial. *It satisfies:*

(i) $\deg Q_k = |k|$;
(ii) $|x| \leqslant 1 \quad \Rightarrow \quad |Q(x)| \leqslant 1$;
(iii) $\forall t \in \mathbb{N} \quad Z^t = \frac{1}{2^t} \sum_{k\in\mathbb{Z}} \binom{t}{(t+k)/2} Q_k(Z)$, *where by convention* $\binom{t}{p} = 0$ *whenever* $p \notin \{0,1,\dots,t\}$.

$\clubsuit$

By property (iii) in Lemma 2.3, formula (G) gives

$$p^t(x,y) = \frac{|P|^t}{2^t\mu(x)} \sum_{k\in\mathbb{Z}} \binom{t}{(t+k)/2} \left\langle \boldsymbol{\delta}_x, Q_k\left(\frac{P}{|P|}\right)\boldsymbol{\delta}_y \right\rangle_{L^2(\mu)}. \tag{I}$$

The linear operator $(|P|^{-1}P)$ on $L^2(\mu)$ is self-adjoint and its norm is 1 by construction; so it decomposes onto a countable orthonormal basis of eigenvectors as

$$\frac{P}{|P|}\left(\sum_{\lambda\in\mathrm{Spec}(P/|P|)} a_\lambda \mathbf{v}_\lambda\right) = \sum_{\lambda\in\mathrm{Spec}(P/|P|)} \lambda a_\lambda \mathbf{v}_\lambda, \tag{J}$$

where $\mathbf{v}_\lambda$ is the eigenvector associated to the eigenvalue λ, the eigenvalues being counted with multiplicity. By definition of $|P|$ we have $\mathrm{Spec}\left(|P|^{-1}P\right) \subset [-1,1]$. So

$$Q_k\left(\frac{P}{|P|}\right)\left(\sum_{\lambda\in\mathrm{Spec}(P/|P|)} a_\lambda \mathbf{v}_\lambda\right) = \sum_{\lambda\in\mathrm{Spec}(P/|P|)} Q_k(\lambda) a_\lambda \mathbf{v}_\lambda, \tag{K}$$

where the $Q_k(\lambda)$ are all of absolute value less than or equal to 1 by Lemma 2.3, property (ii). Then, the operator norm of $Q_k\left(|P|^{-1}P\right)$ on $L^2(\mu)$ is at most 1, so to write:

$$\left\langle \boldsymbol{\delta}_x, Q_k\left(\frac{P}{|P|}\right)\boldsymbol{\delta}_y \right\rangle_{L^2(\mu)} \leqslant \|\boldsymbol{\delta}_x\|_{L^2(\mu)} \cdot \|\boldsymbol{\delta}_y\|_{L^2(\mu)} = \sqrt{\mu(x)\mu(y)}. \tag{L}$$

Now, we notice that for $|k| < d(x,y)$, $Q_k\left(|P|^{-1}P\right)\boldsymbol{\delta}_y$ is a linear combination of the $P^u\boldsymbol{\delta}_y, 0 \leqslant u < d(x,y)$, by property (i) in Lemma 2.3; then $Q_k\left(|P|^{-1}P\right)\boldsymbol{\delta}_y$ is a function supported by the $z \in V$ satisfying $d(z,y) < d(x,y)$, in particular $\left\langle \boldsymbol{\delta}_x, Q_k\left(|P|^{-1}P\right)\boldsymbol{\delta}_y \right\rangle_{L^2(\mu)} = 0$. In the end,

$$\begin{aligned} p^t(x,y) &= \frac{|P|^t}{2^t\mu(x)} \sum_{|k|\geqslant d(x,y)} \binom{t}{(t+k)/2} \left\langle \boldsymbol{\delta}_x, Q_k\left(\frac{P}{|P|}\right)\boldsymbol{\delta}_y \right\rangle_{L^2(\mu)} \\ &\overset{(\mathrm{L})}{\leqslant} \frac{|P|^t}{2^t\mu(x)} \sqrt{\mu(x)\mu(y)} \sum_{|k|\geqslant d(x,y)} \binom{t}{(t+k)/2} \leqslant 2|P|^t \left(\frac{\mu(y)}{\mu(x)}\right)^{1/2} \frac{1}{2^t} \sum_{k\geqslant d(x,y)} \binom{t}{(t+k)/2} \end{aligned} \tag{M}$$

(where the last inequality is an equality as soon as $d(x,y) > 0$).

To conclude, it only remains to prove the relation

$$\frac{1}{2^t} \sum_{k\geqslant d(x,y)} \binom{t}{(t+k)/2} \leqslant \exp(-d(x,y)^2/2t). \tag{N}$$

To do that, we notice that, if X is a random variable equidistributed on $\{-1,1\}$, then, the law of X^{*t} (which denotes the t-th convolution power of X) is $\frac{1}{2^t}\sum_{k\in\mathbb{Z}} \binom{t}{(t+k)/2}\boldsymbol{\delta}_k$, so

$$\frac{1}{2^t} \sum_{k\geqslant d(x,y)} \binom{t}{(t+k)/2} = \mathbf{P}\left(X^{*t} \geqslant d(x,y)\right). \tag{O}$$

Now, we check by direct computation that for all $\lambda > 0$, $\mathbf{E}\left[e^{\lambda X}\right] \leqslant e^{\lambda^2/2}$, hence $\mathbf{E}\left[e^{\lambda X^{*t}}\right] \leqslant e^{t\lambda^2/2}$, and then by Chebychev's inequality:

$$\mathbf{P}\left(X^{*t} \geqslant d(x,y)\right) = \mathbf{P}\left(e^{\lambda X^{*t}} \geqslant e^{\lambda d(x,y)}\right) \leqslant \frac{e^{t\lambda^2/2}}{e^{\lambda d(x,y)}}, \tag{P}$$

hence we get (N) by taking $\lambda = d(x,y)/t$, which ends the proof.

3 The Gaussian factor

As told before, this work presents a new, probabilistic proof of Carne's bound. In this section, only the Gaussian part of the bound will be considered. The fundamental estimate is the

3.1 Theorem. *Let* $\mathbf{P}$ *be a Markov chain as described in § 1.a; let* $t \geqslant 2$*; let* $x \neq y \in V$*; then*

$$p^t(x,y) \leqslant \left(\frac{\mu(y)}{\mu(x)}\right)^{1/2} \exp\left(-\frac{(d(x,y)-1)^2}{2(t-1)}\right). \tag{Q}$$

♣

The following immediate corollary yields a more pleasant formula:

3.2 Corollary. *For* $t \geqslant 1$ *and* $x, y \in V$*,*

$$p^t(x,y) \leqslant \sqrt{e}\left(\frac{\mu(y)}{\mu(x)}\right)^{1/2} \exp\left(-\frac{d(x,y)^2}{2t}\right). \tag{R}$$

♣

3.3 Remark. The first factor in the bound (R) is slightly better than that of (D), but actually one could replace the 2 by a $\sqrt{e}$ in the proof of § 2 by refining the bound (N). ♡

Proof. Denote $d = d(x,y)$. First, note that by reversibility of the chain, one has

$$p^t(y,x) = \frac{\mu(x)}{\mu(y)} p^t(x,y); \tag{S}$$

so to prove (Q) it suffices to show:

$$p^t(x,y)\, p^t(y,x) \leqslant \exp\left(-\frac{(d-1)^2}{t-1}\right). \tag{T}$$

Now, rather than reasoning on the graph, which is a "complicated" object, we shall introduce a function $\xi : V \to \mathbb{R}$ which measures how much the random walk X is closer to x or to y. ξ must satisfy:

3.4 Assumption.
- $\xi(x) = 0$; $\xi(y) = d$;
- ξ is 1-Lipschitz, i.e. for $z \sim v$ we have $|\xi(v) - \xi(z)| \leqslant 1$.

♢

Such a map ξ always exists since, for instance, the map $d(x,\cdot)$ always satisfies Assumption 3.4. Each point of V tends to make ξ increase or decrease, depending on the values of the transition kernel. Let us denote by $m(z)$ the expected value for the variation of ξ after the particle having visited z, i.e.

$$m(z) = \mathbf{E}_z[\xi(X_1)] - \xi(z). \tag{U}$$

Now, let $(M_u)_{u \geqslant 1}$ be the process defined by:

$$M_u = \xi(X_u) - \xi(X_1) - \sum_{s=1}^{u-1} m(X_s) = \sum_{s=1}^{u-1} (\xi(X_{s+1}) - \xi(X_s) - m(X_s)); \tag{V}$$

obviously M is a martingale starting at 0. Let us look at the chain starting at x. On the event $\{X_t = y\}$, one trivially has $\xi(X_t) - \xi(X_1) \geqslant d-1$, hence

$$\mathbf{E}_x[M_t | X_t = y] \geqslant d - 1 - \mathbf{E}_x\left[\sum_{u=1}^{t-1} m(X_u) \middle| X_t = y\right]. \tag{W}$$

One may carry out the same reasoning starting at y, which gives:

$$\mathbf{E}_y[M_t | X_t = x] \leqslant -(d-1) - \mathbf{E}_y\left[\sum_{u=1}^{t-1} m(X_u) \middle| X_t = x\right]. \tag{X}$$

What can we see? If the terms $\mathbf{E}_x\left[\sum_{u=1}^{t-1} m(X_u) | X_t = y\right]$, resp. $\mathbf{E}_y\left[\sum_{u=1}^{t-1} m(X_u) | X_t = x\right]$ were not present in (W) and (X), these formulae would reduce to $\mathbf{E}_x[M_t | X_t = y] \geqslant d-1$, resp. $\mathbf{E}_y[M_t | X_t = x] \leqslant -(d-1)$, so that we would observe a large deviation phenomenon on martingales, which would yield a control respectively on $p^t(x,y)$ and $p^t(y,x)$. Unfortunately, that phenomenon seems to be wiped out because of the terms $m(X_s)$. The key idea then consists in noticing that, by reversibility, these $m(X_s)$ are the same for the "way out" as for the "way back"; subsequently, if the $m(X_s)$ tend to make the right hand side of (W) diminish (which would *damp* the large deviation phenomenon), then they tend to make the right hand side of (X) increase, which this time translates into a *strengthening* of the large deviation phenomenon. So, $p^t(x,y)$ and $p^t(y,x)$ cannot be large simultaneously, which will lead us to (T).

So, we consider X^x, X^y two independent chains with respective laws $\mathbf{P}_x$ and $\mathbf{P}_y$; let $\mathbf{P}_{x\otimes y}$ be their joint law. The respective realizations of $(M_u)_{u\geqslant 1}$ for the paths starting at x and at y are denoted by $(M_u^x)_{u\geqslant 1}$ and $(M_u^y)_{u\geqslant 1}$. By reversibility,

$$\forall u \in \{1,\dots,t-1\} \quad \mathbf{E}_x[m(X_u) | X_t = y] = \mathbf{E}_y[m(X_{t-u}) | X_t = x]. \tag{Y}$$

Hence by combining (W) and (X),

$$\mathbf{E}_{x\otimes y}[M_t^x - M_t^y | X_t^x = y \text{ and } X_t^y = x] \geqslant 2(d-1). \tag{Z}$$

It remains to control the deviations of $M_t^x - M_t^y$. We remark that this random variable may be interpreted as the final value of a $2(t-1)$ steps martingale, to which one can apply the following variant of Azuma's inequality:

3.5 Lemma. *Let $(\mathscr{F}_t)_{t\in\mathbb{N}}$ be a filtration; let $(X_t)_{t\geqslant 1}$ be an adapted real-valued process with $\mathbf{E}[X_{t+1}|\mathscr{F}_t] = 0$ [‡]. We suppose that, for all $t \in \mathbb{N}$, $Law(X_{t+1}|\mathscr{F}_t)$ is supported by an interval of length 2 almost surely. Then, letting $u \geqslant 0$ be a fixed time, we have for all $\lambda \in \mathbb{R}$:*

$$\mathbf{E}\left[\exp\left(\lambda \sum_{t=1}^{u} X_t\right)\right] \leqslant \exp\left(u\frac{\lambda^2}{2}\right). \tag{AA}$$

♣

Proof. The proof relies on Hoeffding's inequality, whose statement is recalled below:

3.6 Lemma (Hoeffding)**.** *Let X be a centered real-valued random variable, supported by an interval of length 2, then*

$$\forall \lambda \in \mathbb{R} \quad \mathbf{E}\left[e^{\lambda X}\right] \leqslant e^{\lambda^2/2}. \tag{AB}$$

♣

[‡]. In other words, the X_t's are the increments of a martingale.

That point being taken for granted, we prove Lemma 3.5 by induction on u:

- For $u = 0$ the result is trivial.
- Let $u \geqslant 1$; suppose the result to be true for $u-1$. Let $\lambda \in \mathbb{R}$; we write

$$\mathbf{E}\left[\exp\left(\lambda \sum_{t=1}^{u} X_t\right)\right] = \mathbf{E}\left[\exp\left(\lambda \sum_{t=1}^{u-1} X_t\right)\mathbf{E}\left[e^{\lambda X_u}\middle|\mathscr{F}_{u-1}\right]\right]$$
$$\leqslant \underbrace{\mathbf{E}\left[\exp\left(\lambda \sum_{t=1}^{u-1} X_t\right)\right]}_{\leqslant e^{(u-1)\lambda^2/2} \text{ by induction}} \cdot \underbrace{\left\|\mathbf{E}\left[e^{\lambda X_u}\middle|\mathscr{F}_{u-1}\right]\right\|_\infty}_{\leqslant e^{\lambda^2/2} \text{ by (AB)}} \leqslant e^{u\lambda^2/2}. \quad \text{(AC)}$$

♠

To conclude, it only remains to us to prove the following measure concentration lemma:

3.7 Lemma. *Let X be a centered real-valued random variable satisfying for some $k > 0$:*

$$\forall \lambda \in \mathbb{R} \quad \mathbf{E}[e^{\lambda X}] \leqslant e^{k\lambda^2/2}. \quad \text{(AD)}$$

If $\mathscr{A}$ is an event such that

$$\mathbf{E}[X|\mathscr{A}] \geqslant C \quad \text{(AE)}$$

for some $C \geqslant 0$, then

$$\mathbf{P}(\mathscr{A}) \leqslant \exp\left(-\frac{C^2}{2k}\right). \quad \text{(AF)}$$

♣

Proof. To lighten notations, let us denote $p = \mathbf{P}(\mathscr{A})$. Let us fix $\lambda > 0$, then we have

$$\mathbf{E}\left[e^{\lambda X}\middle|\mathscr{A}\right] = \frac{\mathbf{E}\left[\mathbf{1}_{\mathscr{A}} e^{\lambda X}\right]}{\mathbf{P}(\mathscr{A})} \overset{\text{(AD)}}{\leqslant} \frac{1}{p} e^{k\lambda^2/2}. \quad \text{(AG)}$$

It follows, by Jensen's inequality, that

$$\mathbf{E}[X|\mathscr{A}] \leqslant \frac{1}{\lambda} \log\left(\frac{1}{p} e^{k\lambda^2/2}\right), \quad \text{(AH)}$$

hence by assumption (AE)

$$\frac{1}{\lambda} \log\left(\frac{1}{p} e^{k\lambda^2/2}\right) \geqslant C, \quad \text{(AI)}$$

whence finally:

$$p \leqslant e^{-C\lambda + k\lambda^2/2}. \quad \text{(AJ)}$$

Then it suffices to take $\lambda = C/k$ to get the announced result. ♠

Now we conclude the proof of Theorem 3.1. Lemma 3.5 permits us to control the Laplace transform of $M_t^x - M_t^y$ under $\mathbf{P}_{x\otimes y}$:

$$\forall \lambda \in \mathbb{R} \quad \mathbf{E}_{x\otimes y}\left[e^{\lambda(M_t^x - M_t^y)}\right] \leqslant e^{(t-1)\lambda^2}, \quad \text{(AK)}$$

and formula (Z) then gives, *via* Lemma 3.7:

$$\mathbf{P}_{x\otimes y}(X_t^x = y \text{ and } X_t^y = x) \leqslant \exp\left(-\frac{(d-1)^2}{t-1}\right), \quad \text{(AL)}$$

which is formula (T). ♠

4 Generalization to a larger class of distances

4.a Statement of the generalized theorem

Now we will show that the reasoning made above can in fact adapt to a whole class of distances. So let us consider a new distance on V, which we will also call d —to avoid confusions, the graph distance that we had defined by (C) will be denoted d^{G} from now on. We have:

4.1 Theorem. *Suppose that d is built so that, if $\xi : V \to \mathbb{R}$ is any 1-Lipschitz function with respect to d, one has, for all $x \in V$:*

- $$|\mathbf{E}_x[\xi(X_1)] - \xi(x)| \leqslant B; \tag{AM}$$
- $$\forall \lambda \geqslant 0 \quad \mathbf{E}_x\left[e^{\lambda\{\xi(X_1)-\mathbf{E}_x[\xi(X_1)]\}}\right] \leqslant e^{A\lambda^2/2}, \tag{AN}$$

for some constants A and B independent of ξ. Then, for all $x, y \in V$, one has

$$p^t(x,y) \leqslant \left(\frac{\mu(y)}{\mu(x)}\right)^{1/2} \exp\left(-\frac{(d(x,y)-B)_+^{\,2}}{2At}\right) \leqslant e^{B/A}\left(\frac{\mu(y)}{\mu(x)}\right)^{1/2} \exp\left(-\frac{d(x,y)^2}{2At}\right). \tag{AO}$$

♣

4.2 Remark. We can already point out that, in the case when the distance is d^{G}, it is impossible to get anything better than $A = 1$ and $B = 1$. Subsequently, the result will be worsened by a $\sqrt{e}$ factor compared to (R)—which remains negligible compared to the exponential part of the bound—: as we will see later, it is due to the difference in treating the first steps, since the specific argument that we have used for d^{G} in the proof of Theorem 3.1 may not generalize. ♡

Proof. We follow again the proof of Theorem 3.1: denote $d = d(x,y)$, let ξ satisfy Assumption 3.4, define m by (U) and let $\mathbf{P}_{x\otimes y}$ be the joint law of two independent chains of respective laws $\mathbf{P}_x$ and $\mathbf{P}_y$; we want to bound $\mathbf{P}_{x\otimes y}[X_t^x = y \text{ and } X_t^y = x]$ above to conclude by formula (S).

The first difference lies in the definition of M: now, the martingale starts at time 0. So it is defined by:

$$M_u = \xi(X_u) - \xi(X_0) - \sum_{s=0}^{u-1} m(X_s) = \sum_{s=0}^{u-1} \big(\xi(X_{s+1}) - \xi(X_s) - m(X_s)\big). \tag{AP}$$

Then we get:

$$\begin{cases} \mathbf{E}_x[M_t | X_t = y] = d - \mathbf{E}_x\left[\sum_{u=0}^{t-1} m(X_u) \middle| X_t = y\right], \\ \mathbf{E}_y[M_t | X_t = x] = -d - \mathbf{E}_y\left[\sum_{u=0}^{t-1} m(X_u) \middle| X_t = x\right] \end{cases} \tag{AQ}$$

When we want to combine these two formulae as we did in (Z), we observe that all the terms $\mathbf{E}[m(X_u)]$ will cancel pairwise, *except* the terms corresponding to the first steps, i.e. to $u = 0$ in the two respective formulae. But we know exactly what these terms are, since under $\mathbf{P}_x$, we have $X_0 = x$ a.s. (by definition!), resp. $X_0 = y$ a.s. under $\mathbf{P}_y$. Thus

$$\mathbf{E}_{x\otimes y}\left[M_t^x - M_t^y \middle| X_t^x = y \text{ and } X_t^y = x\right] = 2d - m(x) + m(y) \geqslant 2d - 2B. \tag{AR}$$

Taking into account assumption (AN)—which plays here the role played before by Hoeffding's inequality—, we copy off the proof of Lemma 3.5 to get:

$$\forall \lambda \geqslant 0 \quad \mathbf{E}_{x\otimes y}\left[e^{\lambda(M_t^x + M_t^y)}\right] \leqslant e^{At\lambda^2}, \tag{AS}$$

and it only remains to conclude by Lemma 3.7 [§]. ♠

4.b More flexible distances

Now we will show how one may build distances statisfying Theorem 4.1, such that the metric structure of V continuously depends on the transition kernel. The method developed below is certainly neither the best nor the most elegant, but it has the advantage to be of relative pedagogical simplicity.

We keep on the principle of putting a length to each edge, but that time all the edges will not have the same size: indeed we will put a larger length to the edges that are more difficult to visit, in order to ensure that the metric structure of the graph will not be too much perturbed when we add a very "unlikely" edge.

Let $\alpha > 0$ be an arbitrary parameter. To each couple $(x,y) \in V \times V$, we associate a length $l(x,y)$ such that:

$$\forall a \geqslant 0 \quad \mathbf{P}_x(l(x,X_1) \geqslant 1+a) \leqslant e^{-a^2/2\alpha^2}, \tag{AT}$$

and we define the length of the edge $[xy]$ by $|[xy]| = \min\{l(x,y), l(y,x)\}$. Then, for any 1-Lipschitz function ξ on V, we have:

4.3 Proposition.

$$\forall x \in V \quad \forall a \geqslant 0 \quad \mathbf{P}_x(|\xi(X_1) - \xi(x)| \geqslant 1+a) \leqslant e^{-a^2/2\alpha^2}. \tag{AU}$$

♣

Now we give a formula for $l(x,y)$ satisfying (AT). First we define what we will call the β-entropy of a probability law:

4.4 Definition. Let $\beta \in (0,1]$; let p be a probability measure on a discrete space $\mathscr{X}$. We call β-entropy of p the (possibly infinite) number:

$$H_\beta(p) = \sum_{x \in \mathscr{X}} p(x)^{1-\beta} \text{ [¶]}. \tag{AV}$$

A transition kernel p on V being given, we will also denote, for $x \in V$, $H_\beta(x) = H_\beta(p(x,\cdot))$. ♢

The β-entropy permits us to control the probability that the observed event is rare:

4.5 Proposition. *Let $\beta \in (0,1]$; let p be a probability measure on a discrete state space $\mathscr{X}$. We suppose $H_\beta(p) < \infty$. Then under the law p, for all $q \in (0,1]$ one has:*

$$\mathbf{P}(p(x) \leqslant q) \leqslant H_\beta(p) q^\beta. \tag{AW}$$

♣

Proof. Use the identity $H_\beta(p) = \mathbf{E}[p(x)^{-\beta}]$ and the fact that the map $q \mapsto q^{-\beta}$ is decreasing, then apply Markov's inequality. ♠

[§]. If it occurs that $B \geqslant d$, Lemma 3.7 cannot be applied and we may only bound $p^t(x,y)p^t(y,x)$ above by 1; this explains the positive part appearing in (AO).

[¶]. For $\beta = 1$ we set by continuity $H_\beta(p) = \#\{x \in \mathscr{X} : \ p(x) > 0\}$.

So, one may choose the following expression for $l(x,y)$ to satisfy (AT), where we set that, for $a<0$, $a^{1/2}=0$:

$$l(x,y)=1+\sqrt{2}\alpha\big(\beta\log(p(x,y)^{-1})-\log H_\beta(x)\big)^{1/2}. \tag{AX}$$

Now, we want to show that (AU) permits us to get (AM) and (AN) indeed. Let us begin with an easy observation:

4.6 Proposition. *Let ξ be a 1-Lipchitz function on V for a distance d built as above. Then there exists a random variable Y whose repartition map satisfies $\forall a\geqslant 0$ $\mathbf{P}(Y\geqslant 1+a)=e^{-a^2/2\alpha^2}$, i.e. Y has a law with density*

$$d\mathbf{P}(Y=y)=\mathbf{1}_{y\geqslant 1}\frac{y-1}{\alpha^2}e^{-(y-1)^2/2\alpha^2}dy, \tag{AY}$$

such that one has:

$$\mathbf{P}_x\text{-p.s.}\quad |\xi(X_1)-\xi(x)|\leqslant Y. \tag{AZ}$$

♣

So, we easily find that we can take $B=\mathbf{E}[Y]=1+\sqrt{\pi/2}\,\alpha$ into (AM).

To get a formula for A in (AN), things are a bit more complicated. The tool which we will use is the

4.7 Lemma. *Let Y be a positive random variable whose Laplace tranform $\hat{Y}(\lambda)=\mathbf{E}\left[e^{\lambda Y}\right]$ is supposed to be finite for all $\lambda\geqslant 0$, and let us denote $\overline{Y}=\mathbf{E}[Y]$. Let X be a real-valued random variable satisfying $|X|\leqslant Y$ a.s.; let us denote $\overline{X}=\mathbf{E}[X]$ and $\tilde{X}=X-\mathbf{E}[X]$. Then the Laplace transform of $\tilde{X}$ satisfies:*

$$\forall\lambda\in\mathbb{R}\quad \hat{\tilde{X}}(\lambda)\leqslant e^{|\lambda|\overline{Y}}\hat{Y}(|\lambda|)-2|\lambda|\overline{Y}. \tag{BA}$$

♣

4.8 Remark. The bound (BA) is of quite poor quality close to 0, as it may be particularily striking in the case $Y\equiv 1$, where we get the bound $e^{2|\lambda|}-2|\lambda|$, while we know (Hoeffding's lemma 3.6) that $e^{\lambda^2/2}$ would work. In fact, we can compute that in a neighborhood of 0, the right hand side of (BA) takes the form:

$$1+\big(\mathbf{E}[Y^2]+3\,\mathbf{E}[Y]^2\big)\frac{\lambda^2}{2}+o(\lambda^2), \tag{BB}$$

while a variance calculation proves that in fact,

$$\hat{\tilde{X}}(\lambda)\leqslant 1+\mathbf{E}[Y^2]\frac{\lambda^2}{2}+o(\lambda^2). \tag{BC}$$

♡

Proof. We may restrict ourselves to the case $\lambda\geqslant 0$, the case $\lambda\leqslant 0$ being then treated by using the result for $-X$.
We write

$$\hat{\tilde{X}}(\lambda)=\mathbf{E}\left[e^{\lambda\tilde{X}}-\lambda\tilde{X}\right]. \tag{BD}$$

But, since $|X|\leqslant Y$, we have $|\overline{X}|\leqslant\overline{Y}$, hence $\tilde{X}\in[-Y-\overline{Y},Y+\overline{Y}]$. And since, on an interval of the form $[-a,a]$, the map $x\mapsto e^{\lambda x}-\lambda x$ takes its maximum at a, it follows that

$$\hat{\tilde{X}}(\lambda)\leqslant\mathbf{E}\left[e^{\lambda(Y+\overline{Y})}-\lambda(Y+\overline{Y})\right]=e^{\lambda\overline{Y}}\hat{Y}(\lambda)-2\lambda\overline{Y}. \tag{BE}$$

♠

Now we have the following control on the Laplace transform of the random variable Y defined by (AY):

4.9 Claim. *If Y is a random variable whose law is given by (AY), then for $\lambda \geqslant 0$ one has:*

$$\mathbf{E}\left[e^{\lambda Y}\right] \leqslant \left(1+\sqrt{\frac{\pi}{2}}\alpha\lambda+\frac{\alpha^2}{2}\lambda^2\right)e^{\lambda+\alpha^2\lambda^2/2}. \tag{BF}$$

Moreover, that bound is sharp close to 0*, by which we mean that, if we rewrite (BF) under the form $\mathbf{E}[e^{\lambda Y}] \leqslant f(\lambda)$, we have $\mathbf{E}[Y] = \frac{d}{d\lambda}_{|\lambda=0}\mathbf{E}[e^{\lambda Y}] = f'(0) = 1+\sqrt{\pi/2}\,\alpha$.* ♣

Proof. We begin with noticing that we can write $Y = 1+\alpha Z$, where Z is a random variable with law

$$d\mathbf{P}(Z=z) = \mathbf{1}_{z\geqslant 0} z e^{-z^2/2} dz. \tag{BG}$$

Then it suffices to prove that

$$\forall \lambda \geqslant 0 \quad \mathbf{E}\left[e^{\lambda Z}\right] \leqslant \left(1+\sqrt{\frac{\pi}{2}}\lambda+\frac{\lambda^2}{2}\right)e^{\lambda^2/2}. \tag{BH}$$

To do that, we write:

$$\mathbf{E}\left[e^{\lambda Z}\right] = e^{\lambda^2/2}\int_0^\infty z e^{-(z-\lambda)^2}dz \overset{t=z-\lambda}{=} e^{\lambda^2/2}\underbrace{\int_{-\lambda}^\infty (t+\lambda)e^{-t^2/2}dt}_{I(\lambda)}, \tag{BI}$$

where $I(0)=0$ and, by the theorem of differenciation under the integral,

$$I'(\lambda) = \int_{-\lambda}^\infty e^{-t^2/2}dt \leqslant \sqrt{\frac{\pi}{2}}+\frac{\lambda}{2}, \tag{BJ}$$

whence $I(\lambda) \leqslant 1+\sqrt{\pi/2}\,\lambda+\lambda^2/2$, and (BF). ♠

From that we deduce the existence of a suitable value for A:

4.10 Claim. *For all $\alpha > 0$, there exists a constant $A(\alpha) < \infty$ such that, if d satisfies condition (AU) for the value α, then (AN) is satisfied for the value $A(\alpha)$.* ♣

Proof. For Y with law (AY), denoting $\overline{Y} := \mathbf{E}[Y] = 1+\sqrt{\pi/2}\,\alpha$, Claim 4.9 shows that $2\lambda^{-2}\times \log\big(e^{\lambda\overline{Y}}\mathbf{E}[e^{\lambda Y}]-2\lambda\overline{Y}\big)$ is bounded for $\lambda \overset{\geqslant}{\to} 0$ and $\lambda\to\infty$. By continuity, this function is thus bounded on the whole half-line $[0,+\infty)$. Lemma 4.7 then gives the existence of A. ♠

4.11 Remark. We have not found any simple bound for $A(\alpha)$, but, for a given value of α, it is easy to compute numerically the maximum of the map $\lambda \mapsto 2\log(e^{\lambda\overline{Y}}\mathbf{E}[e^{\lambda Y}]-2\lambda\overline{Y})/\lambda^2$, which gives a suitable value for A. ♡

4.c A concrete example

I shall illustrate the preceding subsection by showing how Theorem 4.1 can give worthy results in cases where the usual Carne bound is irrelevant. [‖]

[‖]. This section has been entirely rewritten for its thesis version: while in [65] I took an example on a *hyperbolic* graph, I eventually found it more relevant to take an *amenable* example.

Let us take for V the *discrete Heisenberg group*, that is, the set of upper triangular matrices in $\mathbb{M}_3(\mathbb{Z})$ whose all diagonal entries are 1 —which is a multiplicative group. On V, the natural graph structure consists in connecting two matrices M and N by an edge as soon as

$$NM^{-1} \in \left\{ \begin{pmatrix} 1 & \pm 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \pm 1 \\ 0 & 0 & 1 \end{pmatrix} \right\}; \tag{BK}$$

we denote by d^{H} the distance associated to that natural graph, *which will not be the same as the graph distance* d^{G}. We consider as the Markov chain on V the process which, from point x, jumps on each neighbor of x with probability $(1-\varepsilon)/4$, and, with probability ε, chooses uniformly an arrival point in the (open) ball centered on x with radius L. That chain is clearly reversible, with the counting measure as invariant measure. So, our process looks much like the simple random walk on the Heisenberg group, except that sometimes (with probability $\simeq \varepsilon$) the mobile may make a larger jump (of length $\simeq L$). We would like to say that, even if L is large, it suffices for ε to be small enough to get a Gaussian bound where L does not appear.

If we naively apply formula (D) to this transition kernel, we will not get anything interesting: indeed, small as ε might be, the graph distance is the same:

$$\forall x, y \in G \quad d^{\mathrm{G}}(x,y) = \left\lceil \frac{d^{\mathrm{H}}(x,y)}{L-1} \right\rceil. \tag{BL}$$

Then, bounding below $d^{\mathrm{G}}(\cdot)$ by $d^{\mathrm{H}}(\cdot)/(L-1)$, [*] Carne's bound yields

$$p^t(x,y) \leqslant 2\exp\left(-\frac{d^{\mathrm{H}}(x,y)}{2t(L-1)^2}\right). \tag{BM}$$

Concretely, if $L = 105$ and $\varepsilon = 1/2^{2^{26}}$, it will give a bound with a $d^{\mathrm{H}}(x,y)/21{,}632t$ in the exponential, which is strongly worse than the $d^{\mathrm{H}}(x,y)/2t$ of the case $\varepsilon = 0$. Yet it is obvious that the influence of large jumps should be nearly zero: the bound (BM) thus must be improvable!

So we will apply the techniques exposed in § 4.b. Here we have chosen arbitrarily $\alpha = 1$ and $\beta = 1/2$. Let us denote by N the cardinality of a ball of radius L, which is easily bounded by $2L^4$ [†] For technical reasons we shall assume that $L > 2$, so that $N \geqslant 17$. We have:

$$H_{1/2}(x) = 4\left(\frac{1-\varepsilon}{4} + \frac{\varepsilon}{N}\right)^{1/2} + (N-4)\left(\frac{\varepsilon}{N}\right)^{1/2} \leqslant 2 + N^{1/2}\varepsilon^{1/2}, \tag{BN}$$

hence $H_{1/2}(x) \leqslant 3$ for $\varepsilon \leqslant 1/N$. So, if we bound above $H_{1/2}(x)$ by 3, we get that for $1 < d^{\mathrm{H}}(x,y) < L$, one has:

$$l(x,y) \geqslant 1 + \sqrt{2}\left(\tfrac{1}{2}\log(N/\varepsilon) - \log 3\right)^{1/2}, \tag{BO}$$

whence in particular $l(x,y) \geqslant 1 + \sqrt{\log(\varepsilon^{-1})}$.

Our observation is that, if ε is small enough, d coincides with d^{H}: indeed one has $l(x,y) = 1$ for $x \sim y$, $l(x,y) = \infty$ if $d^{\mathrm{H}}(x,y) \geqslant L$, and $l(x,y) \geqslant L$ for $2 \leqslant d^{\mathrm{H}}(x,y) < L$ as soon as:

$$\varepsilon \leqslant e^{-L^2} \tag{BP}$$

—which is however a quite strong condition. (Note that condition (BP) implies automatically that $\varepsilon \leqslant 1/N$.)

[*]. Here one has $|P| = 1$, due to Heisenberg's group's being amenable.

[†]. Actually it can be shown that when $L \to \infty$, N is equivalent to $C \cdot L^4$ for some explicit $C \in (0, +\infty)$.

Numerical computations for our choice of α give $A = 11.6\ldots$, resp. $B = 2.25\ldots$; so in the end we have, for $\varepsilon \leqslant e^{-L^2}$,

$$p^t(x,y) \leqslant \frac{10}{9} \exp\left(-\frac{d^{\mathrm{H}}(x,y)^2}{47t}\right). \tag{BQ}$$

Although that bound undoubtedly improves Carne's bound (BM) in "extreme" cases like that mentioned above, and even if it is certainly possible to get some better results by a more subtle choice of α and β, I find that bound rather disappointing in the sense that we still remain far from Carne's bound for the smallest values of ε. Anyway, Theorem 4.1 is theoretically interesting and may have better applications; in particular Lemma 4.7 can certainly be improved.

5 The flight factor

5.a Frame of the proof

The Gaussian bound (R) has got a disadvantage with respect to Carne's bound (D): in the case when $|P| < 1$, it does not show the exponential decreasing of $p^t(x,y)$ in the variable t. In fact, Lemma 2.2 implies $p^t(x,y)p^t(y,x) \leqslant p^{2t}(x,x) \leqslant |P|^{2t}$, whence

$$p^t(x,y) \leqslant \left(\frac{\mu(y)}{\mu(x)}\right)^{1/2} |P|^t, \tag{BR}$$

but that is not enough to get again (D). The present section precisely aims at doing this. Here we will exclusively focus on the case when $d = d^{\mathrm{G}}$, cf. Remark 5.8 below.

Let $x, y \in V$. For u a time devised to go to infinity, denote by

$$\mathscr{R}_u := \{\exists s \geqslant u : \ X_s = x\} \tag{BS}$$

the event which tells that the particle comes back at x at least once after time u. The strategy of our proof then consists in looking at our Markov chain *conditioned to the event* $\mathscr{R}_u$. Why this? Well, the fact that $|P| < 1$ expresses a possibility for the particle to "flee to infinity". That flight is responsible for the exponential decay with respect to t of the quantity $p^t(x,y)p^t(y,x)$ introduced in (T), which measures the probability that the particle, starting at x, goes to y at time t and then comes back to x at time $2t$. Conditioning with respect to $\mathscr{R}_u$ then aims, in a way, at preventing the particle from going to infinity, which will give us a Markov chain for which $|P| = 1$, where the bound (R) will be relevant. Then it will remain to show that this conditioning selects sufficiently well the cases when the particle makes a return trip to get back a factor $|P|^t$ in (R).

Our proof will use a kind of density argument: in a first step we will add some more assumptions on our Markov chain to carry out the reasoning, then in a second step we will prove that we can get rid of these extra assumptions by slightly perturbing the original Markov chain.

5.b Proof under extra assumptions

We will use the following notation:

5.1 Definition. We denote by τ_x the hitting time of x by a walk on V, i.e. $\tau_x := \inf\{t \geqslant 0 : \ X_t = x\}$. For all $z \in V$, we denote

$$R(z) := \mathbf{E}_z\left[\mathbf{1}_{\tau_x < \infty} |P|^{-\tau_x}\right]. \tag{BT}$$

$\diamondsuit$

In this first part of the proof, we add to the assumptions of § 1.a the following conditions:

5.2 Assumption.

(i) V is finite;
(ii) There exists a cemetery point $\partial \in V$ such that $p(\partial, \cdot) = \boldsymbol{\delta}_\partial$. We will denote $\tilde{V}$ for $V \smallsetminus \{\partial\}$;
(iii) The chain $\mathbf{P}$ is aperiodic on $\tilde{V}$.

♢

5.3 Remark. Under Assumption 5.2, the chain will just be required to be irreducible and reversible on $\tilde{V}$; moreover the definition of $|P|$ will be that given by the formula (F) of Lemma 2.2, for arbitrary $x \in \tilde{V}$. ♡

Assumption 5.2 permits to obtain sharp results about the recurrence behaviour of the chain:

5.4 Lemma. *Suppose we have an irreducible and reversible Markov chain satisfying Assumption 5.2. Then:*

(i) There exist two constants $0 < c_1 \leqslant c_2 < \infty$ such that

$$\forall t \geqslant 0 \quad c_1 |P|^t \leqslant \mathbf{P}_x(\mathscr{R}_t) \leqslant c_2 |P|^t. \tag{BU}$$

(ii) $\mathbf{P}_x(\mathscr{R}_{t+1})/\mathbf{P}_x(\mathscr{R}_t) \overset{t\to\infty}{\to} |P|$.
(iii) For all $z \in \tilde{V}$, $R(z) < \infty$.

♣

The proof of this lemma, which is rather technical, is postponed to Appendix 6.

Now we are armed to prove Carne's bound for a Markov chain satisfying Assumption 5.2. Let us fix $t \geqslant 0$; we have the key proposition:

5.5 Proposition. *The law of $(X_s)_{0\leqslant s\leqslant 2t}$ under $\mathbf{P}_x(\cdot|\mathscr{R}_u)$ converges when $u \to \infty$ (for the total variation norm on $V^{\{0,\dots,2t\}}$) to the law $\mathbf{P}'_x$ of the Markov chain on $\tilde{V}$ starting at x, with transition probabilities:*

$$\forall z, v \in \tilde{V} \quad p'(z,v) = \frac{R(v)p(z,v)}{\sum\limits_{w\sim z} R(w)p(z,w)}\,[\ddagger]. \tag{BV}$$

♣

Proof. It is true in a general framework that $\mathbf{P}(\cdot|\mathscr{R}_u)$ is a time-inhomogeneous Markov chain with

$$\mathbf{P}\left(X_{s+1} = v | X_s = z \text{ and } \mathscr{R}_u\right) = \frac{S_{s+1,u}(v)p(z,v)}{\sum\limits_{w\sim z} S_{s+1,u}(w)p(z,w)}, \tag{BW}$$

where we let $S_{s+1,u}(z) = \mathbf{P}(\mathscr{R}_u | X_{s+1} = z)$. Our attack will consist in proving that, for all $s \in \{0,\dots,2t-1\}$, for all $z \in V$, we have $S_{s+1,u}(z)/\mathbf{P}_x(\mathscr{R}_{u-(s+1)}) \overset{u\to\infty}{\to} R(z)$.

To begin with, let us notice that $S_{s+1,u}(z) = \mathbf{P}_z(\mathscr{R}_{u-(s+1)})$, which we shall denote by $S_{u-(s+1)}(z)$; then what we want to prove can be written:

$$\forall z \in \tilde{V} \quad \frac{S_u(z)}{\mathbf{P}_x(\mathscr{R}_u)} \overset{u\to\infty}{\to} R(z). \tag{BX}$$

[‡]. That defines a Markov chain on $\tilde{V}$ indeed because $R(\partial) = 0$.

The idea consists in splitting the probability space according to the value of τ_x, thanks to the strong Markov property:

$$S_u(z) = \sum_{s=0}^{u} \mathbf{P}_z(\tau_x = s)\mathbf{P}_x(\mathscr{R}_{u-s}) + \sum_{s \geqslant u+1} \mathbf{P}_z(\tau_x = s), \tag{BY}$$

whence

$$\frac{S_u(z)}{\mathbf{P}_x(\mathscr{R}_u)} = \sum_{s=0}^{u} \mathbf{P}_z(\tau_x = s)\frac{\mathbf{P}_x(\mathscr{R}_{u-s})}{\mathbf{P}_x(\mathscr{R}_u)} + \sum_{s \geqslant u+1} \frac{\mathbf{P}_z(\tau_x = s)}{\mathbf{P}_x(\mathscr{R}_u)}. \tag{BZ}$$

Let us fix an arbitrarily small $\varepsilon > 0$. Since $\sum_s \mathbf{P}_z(\tau_x = s)|P|^{-s}$ converges (cf. Lemma 5.4-(iii)), we may introduce a time u_0 for which one has $\sum_{s>u_0} \mathbf{P}_z(\tau_x = s) \leqslant \varepsilon$. Then, for $u \geqslant u_0$ one has on the one hand,

$$\sum_{s=u_0+1}^{u} \mathbf{P}_z(\tau_x = s)\frac{\mathbf{P}_x(\mathscr{R}_{u-s})}{\mathbf{P}_x(\mathscr{R}_u)} + \sum_{s>u} \frac{\mathbf{P}_z(\tau_x = s)}{\mathbf{P}_x(\mathscr{R}_u)} \overset{5.4\text{-(i)}}{\leqslant} \frac{c_2 \vee 1}{c_1} \sum_{s>u_0} \mathbf{P}_z(\tau_x = s)|P|^{-s} = \frac{c_2 \vee 1}{c_1}\varepsilon, \tag{CA}$$

on the other hand,

$$\sum_{s=0}^{u_0} \mathbf{P}_z(\tau_x = s)\frac{\mathbf{P}_x(\mathscr{R}_{u-s})}{\mathbf{P}_x(\mathscr{R}_u)} \overset{5.4\text{-(ii)}}{\underset{u\to\infty}{\to}} \sum_{s=0}^{u_0} \mathbf{P}_z(\tau_x = s)|P|^{-s}. \tag{CB}$$

It follows that

$$\overline{\lim_{u\to\infty}} \left| R(z) - \frac{S_u(v)}{\mathbf{P}_x(\mathscr{R}_u)} \right| \leqslant \left(1 + \frac{c_2 \vee 1}{c_1}\right)\varepsilon, \tag{CC}$$

hence (BX) by letting $\varepsilon \to 0$. ♠

Now we want to look at the chain $\mathbf{P}'$. First, $\mathbf{P}'$ is clearly irreducible. Then, one has:

5.6 Proposition. *The chain $\mathbf{P}'$ is reversible, and its invariant measure is:*

$$\forall z \in \tilde{V} \quad \mu'(z) = \begin{cases} R(z)^2\mu(z) & \text{if } z \neq x; \\ R(x)R^+(x)\mu(x) & \text{if } z = x \text{ [§]}, \end{cases} \tag{CD}$$

where $R^+(x)$ is defined by Definition 5.7 just below. ♣

5.7 Definition. We denote by τ_x^+ the *return time to x*, i.e.:

$$\tau_x^+ := \inf\{s \geqslant 1 : \ X_s = x\}. \tag{CE}$$

Then $R^+(x)$ is defined by

$$R^+(x) = \mathbf{E}_x\left[\mathbf{1}_{\tau_x^+ < \infty}|P|^{-\tau_x^+}\right]. \tag{CF}$$

♢

Proof. Let $z, v \in \tilde{V}$ with $z \neq x$. We can lighten the expression of $p'(z,v)$, since by Markov's property,

$$R(z) = \sum_{w\sim z} p(z,w)\mathbf{E}_w\left[\mathbf{1}_{\tau_x<\infty}|P|^{-(\tau_x+1)}\right] = |P|^{-1}\sum_{w\sim z} p(z,w)R(w), \tag{CG}$$

thus (BV) can be rewritten as

$$p'(z,v) = \frac{p(z,v)R(v)}{|P|R(z)}. \tag{CH}$$

In the case when $z = x$, the same argument leads to

$$p^*(x,v) = \frac{p(x,v)R(v)}{|P|R^+(x)}. \tag{CI}$$

So, it only remains to use (CH), (CI) and the reversibility of μ under $\mathbf{P}$ to get the reversibility of μ' under $\mathbf{P}'$. ♠

[§]. The careful reader may have noticed that $R(x) = 1$; we let that factor appear for ease of understanding.

Now we are ready to end the proof. We observe that, by Markov's property, $p^t(x,y)p^t(y,x) = \mathbf{P}_x(X_t = y \text{ and } X_{2t} = x)$, which we will denote by $\mathbf{P}_x(\mathscr{A})$. For $u \geqslant 2t$, Bayes' formula gives:

$$\frac{\mathbf{P}_x(\mathscr{A})}{\mathbf{P}_x(\mathscr{A}|\mathscr{R}_u)} = \frac{\mathbf{P}_x(\mathscr{R}_u)}{\mathbf{P}_x(\mathscr{R}_u|\mathscr{A})} \overset{\text{Markov}}{=} \frac{\mathbf{P}_x(\mathscr{R}_u)}{\mathbf{P}_x(\mathscr{R}_{u-2t})}, \tag{CJ}$$

hence, letting u go to infinity:

$$\frac{\mathbf{P}_x(\mathscr{A})}{\mathbf{P}'_x(\mathscr{A})} = |P|^{2t}. \tag{CK}$$

Now, the Markov chain $\mathbf{P}'$ satisfies the assumptions of Theorem 3.1, hence $\mathbf{P}'_x(\mathscr{A}) \leqslant e \cdot \exp\left(-d(x,y)^2/t\right)$, so $\mathbf{P}_x(\mathscr{A}) \leqslant e|P|^{2t}\exp\left(-d(x,y)^2/t\right)$, and finally we get the desired formula:

$$p^t(x,y) \leqslant \sqrt{e}\left(\frac{\mu(x)}{\mu(y)}\right)^{1/2} |P|^t \exp\left(-\frac{d(x,y)^2}{2t}\right). \tag{CL}$$

5.8 Remark. The reasoning carried out above cannot apply to other distances than d^{G}: indeed, the distance which appears in (CL) in fact comes as the distance *associated to the process* $\mathbf{P}'$. When one works with the graph distance, that distance is the same for $\mathbf{P}$ and for $\mathbf{P}'$, but this is no more true if d depends in a more subtle way on the transition kernel. ♡

5.c The density argument

Now, we want to get rid of Assumption 5.2. We will proceed in two steps: first we will just relax Assumption 5.2-(iii), then we will deal with the general case.

Relaxing the aperiodicity condition

We consider a finite set V with a transition law $(p(x,y))_{x,y\in V}$, such that there exists a cemetery point $\partial \in V$ satisfying Assumption 5.2-(ii). We suppose that the Markov chain defined by p is irreducible and reversible on $\widetilde{V} = V \smallsetminus \{\partial\}$, with a reversible measure μ. We denote by n the cardinality of $\widetilde{V}$, and by M the matrix $\left(\!\left(p(y,x)\right)\!\right)_{x,y\in\widetilde{V}}$. The following lemma gives an algebraic characterization of the value $|P|$ defined in (F):

5.9 Lemma ([49, Chap. 5-2]). *$|P|$ is the spectral radius of M.* ♣

For $\varepsilon \in [0,1)$, let p_ε be the transition kernel defined by:

$$\forall x,y \in V \quad p_\varepsilon(x,y) = \begin{cases} p(x,x) + \varepsilon(1-p(x,x)) & \text{if } y = x; \\ (1-\varepsilon)p(x,y) & \text{if } y \neq x. \end{cases} \tag{CM}$$

The Markov chain $\mathbf{P}^\varepsilon$ generated by p_ε is an irreducible reversible chain whose graph and reversible measure are the same as for $p = p_0$, and which satisfies thye whole of Assumption 5.2 as soon as $\varepsilon > 0$. Thus, for $x,y \in \widetilde{V}$, $t > 0$ and $\varepsilon > 0$, we have

$$p^t_\varepsilon(x,y) \leqslant \sqrt{e}\left(\frac{\mu(y)}{\mu(x)}\right)^{1/2} |P_\varepsilon|^t \exp\left(-\frac{d(x,y)^2}{2t}\right). \tag{CN}$$

To conclude, we just have to notice that $p^t_\varepsilon(x,y)$, resp. P_ε, are functions of ε continuous at 0. Indeed, the finite-sized matrix M_ε varies continuously with ε, thus its spectral radius $|P_\varepsilon|$ also varies continuously, as well as $p^t_\varepsilon(x,y)$ which is the coefficient number (y,x) of M^t_ε.

Infinite graphs

Now we turn to the general case, i.e. we consider a chain that merely satisfies the assumptions of § 1.a. Let us give a mark $\nu(z) > 0$ to each vertice z of V, in such a way that for all $\varepsilon > 0$, $\#\{z \in V : \nu(z) > 0\}$ is finite.

Let us fix $x, y \in V$, and let us take $\varepsilon > 0$ arbitrarily small (we shall always suppose $\varepsilon < \nu(x), \nu(y)$ to avoid certain problems). We define a finite set V_ε equipped with a transition kernel $(p_\varepsilon(z,v))_{z,v \in V_\varepsilon}$ through the following way:

5.10 Definition. V_ε is obtained by identifying all the points with ν-mass less than ε to a cemetery point ∂:

$$V_\varepsilon = \{z \in V : \ \nu(z) \geqslant \varepsilon\} \cup \{\partial\}. \tag{CO}$$

From now on we will denote the points of V in the same way as their images on V_ε. Then p_ε is the kernel p projected on V_ε, with the requirement that ∂ is a cemetery point:

$$\forall z, v \in V_\varepsilon \quad p_\varepsilon(z,v) = \begin{cases} p(z,v) & \text{if } z, v \in \widetilde{V}_\varepsilon; \\ 0 & \text{if } z = \partial \text{ and } v \in \widetilde{V}_\varepsilon; \\ 1 & \text{if } z = v = \partial; \\ \sum_{\nu(w) < \varepsilon} p(z,w) & \text{if } z \in \widetilde{V}_\varepsilon \text{ and } w = \partial. \end{cases} \tag{CP}$$

$\diamondsuit$

Then the chain $\mathbf{P}^\varepsilon$ satisfies points (i) and (ii) of Assumption 5.2, and it is reversible with measure $\mu_{|\widetilde{V}_\varepsilon}$. This chain may not be irreducible, but we can suppose that such is the case by keeping only the irreducible component of $\widetilde{V}_\varepsilon$ containing x.

So the relation (CL) is satisfied for V_ε equipped with $\mathbf{P}^\varepsilon$; it only remains to prove that $p_\varepsilon^t(x,y) \overset{\varepsilon \to 0}{\to} p^t(x,y)$, resp. $|P_\varepsilon| \overset{\varepsilon \to 0}{\to} |P|$.

Let us deal immediately with the operator norm. The very construction of p_ε ensures that for all $z, v \in \widetilde{V}_\varepsilon$, we have $p_\varepsilon^t(z,v) \leqslant p^t(z,v)$. Taking $z = v = x$, the characterization (F) of $|P|$ immediately gives that $|P_\varepsilon| \leqslant |P|$, which is enough for us (but convergence when $\varepsilon \to 0$ is also true).

Now, we observe that the law of the t first steps of the chain generated by p_ε converges to the law of the initial chain in the sense of total variation. Indeed, given the way how V_ε and p_ε are constructed respectively from V and p, we have a canonical map which associates a walk on $V_{(\varepsilon)}$ to a walk on V, so that the law $\mathbf{P}_x$ maps into the law $\mathbf{P}_x^\varepsilon$. That map is defined as follows: the points of the walk on V are sent onto their projections on V_ε until the image walk hits ∂, and from that time on the walk stays at ∂. So if a realization of the original chain stays in $\widetilde{V}_\varepsilon$ up to time t, its image by our map is kept safe on $\{0, \dots, t\}$, and thus

$$\left\| \mathbf{P}^\varepsilon_{x|V_\varepsilon^{\{0,\dots,t\}}} - \mathbf{P}_{x|V^{\{0,\dots,t\}}} \right\|_{\mathrm{TV}} \leqslant \mathbf{P}_x \left(\exists u \in \{0, \dots, t\} \ X_u \notin \widetilde{V}_\varepsilon \right) \leqslant \sum_{u=0}^{t} \mathbf{P}_x(\nu(X_u) < \varepsilon) \underset{\mathrm{DCV}}{\overset{\varepsilon \to 0}{\to}} 0. \tag{CQ}$$

In particular, $p_\varepsilon^t(x,y) \overset{\varepsilon \to 0}{\to} p^t(x,y)$, and so (CL) is satisfied for V equipped with p, QED.

6 Appendix: Finite sub-Markov chains

This appendix aims at proving Lemma 5.4. Let us recall that in that lemma, we consider a sub-Markov chain on a finite graph $\widetilde{V}$, given by a kernel p, which is irreducible and aperiodic (the

fact that the chain is reversible is not used in the proof of Lemma 5.4). Denote by n the cardinality of $\widetilde{V}$.

The study of the chain may be expressed into matricial terms: we introduce the matrix

$$M = \big((p(v,z))\big)_{z,v\in\widetilde{V}}. \tag{CR}$$

Then the aperiodicity condition translates into the existence of a time t_0 such that, for all $t \geqslant t_0$, M^{t_0} has strictly positive coefficients (actually $t_0 = n^2$ would always do). On the other hand, Lemma 5.9 above permits us to consider $|P|$ as the spectral radius of M.

So we have in hand all the assumptions to apply the strongest form of the Perron–Frobenius theorem, whose general statement and proof the reader can find in [75, Chap. 5]:

6.1 Proposition (Perron–Frobenius)**.** *$|P|$ is a simple eigenvalue of M, and all the other eigenvalues of M have an absolute value strictly less than $|P|$. Moreover, the eigenvector $\mathbf{v}$ associated to the eigenvalue $|P|$ has all its entries strictly positive.* ♣

Now, let us begin with proving point (i) of Lemma 5.4, and in a first step let us prove the second inequality. Markov's strong property gives us the over-muliplicativity relation:

$$\forall t,u \geqslant 0 \quad \mathbf{P}_x(\mathscr{R}_{t+u}) \geqslant \mathbf{P}_x(\mathscr{R}_t)\mathbf{P}_x(\mathscr{R}_u). \tag{CS}$$

We deduce that, for all $t \geqslant 1$, one has $\mathbf{P}_x(\mathscr{R}_t)^{1/t} \leqslant \overline{\lim}_{u\to\infty}\mathbf{P}_x(\mathscr{R}_u)^{1/u}$. Moreover, if $|P| < 1$, we have, for all $t \geqslant 1$,

$$\mathbf{P}(\mathscr{R}_t) \leqslant \sum_{u\geqslant t} p^u(x,x) \overset{(F)}{\leqslant} \sum_{u\geqslant t} |P|^u = \frac{|P|^t}{1-|P|}, \tag{CT}$$

hence $\overline{\lim}_{u\to\infty}\mathbf{P}_x(\mathscr{R}_u)^{1/u} \leqslant |P|$, that last relation being also trivially true in the case when $|P| = 1$. Finally, the second inequality of (BU) is satisfied for $t \geqslant 1$ with $c_2 = 1$, the case $t = 0$ being trivial.

For the lower bound, we will only show that there exists a constant $c_3 > 0$ such that one has, for t large enough:

$$p^t(x,x) \geqslant c_3|P|^t, \tag{CU}$$

the first inequality of (BU) then will follow for t large enough, and the case when t is small then will be dealt with by finiteness, thanks to noticing that irreducibility ensures that $\mathbf{P}_x(\mathscr{R}_t) > 0$ for all $t \in \mathbb{N}$. To prove (CU), let us consider the eigenvector $\mathbf{v} = (v_i)_{i\in\widetilde{V}}$ associated to the eigenvalue $|P|$. We shall keep in mind that, by Proposition 6.1, $v_x > 0$. Denote $\overline{v} = \max_i v_i$; the relation $M^t v = |P|^t v$ then gives for all $t \geqslant 0$:

$$|P|^t v_x = \sum_{z\in\widetilde{V}} p^t(z,x)v_z, \tag{CV}$$

hence

$$\max_{z\in\widetilde{V}} p^t(z,x) \geqslant \frac{v_x}{n\overline{v}}|P|^t. \tag{CW}$$

Let now t_0 be such that one has $\forall z,v \in \widetilde{V}\ p^{t_0}(z,v) > 0$ (such a t_0 exists by aperiodicity, as we noticed *supra*), and let us denote $\eta = \min_{z,v\in\widetilde{V}} p^{t_0}(z,v) > 0$. For $t \geqslant t_0$, by (CW) we can fix z_1 such that $p^{t-t_0}(z_1,x) \geqslant v_x|P|^{t-t_0}/n\overline{v}$. It follows that

$$p^t(x,x) \geqslant \frac{\eta v_x}{n\overline{v}}|P|^{t-t_0}, \tag{CX}$$

hence (CU) with $c_3 = \eta v_x/n\overline{v}|P|^{t_0}$.

Now, let us look at the fine behaviour of the sequence $\mathbf{P}_x(\mathscr{R}_t)$ when $t \to \infty$. By Markov's property,

$$\mathbf{P}_x(\mathscr{R}_t) = \sum_{z \in \widetilde{V}} p^t(x,z)\mathbf{P}_z(\mathscr{R}_0), \tag{CY}$$

subsequently to prove property (ii) in Lemma 5.4, we just have to show that we have $p^{t+1}(x,z)/p^t(x,z) \overset{t\to\infty}{\to} |P|$ for all $z \in \widetilde{V}$. More precisely, we will show that there exists a constant $c_4(z) > 0$ such that $p^t(x,z)/|P|^t \overset{t\to\infty}{\to} c_4(z)$.

$p^t(x,z)$ can be rewritten in matricial terms as $\boldsymbol{\delta}_z^\dagger M^t \boldsymbol{\delta}_x$. Now, by 6.1, if $\dot{M}$ stands for the matrix of the projection on $\mathbb{R}\mathbf{v}$ relatively to the sum of the characteristic spaces for the eigenvalues of M other than $|P|$:

$$\frac{1}{|P|^t} M^t \overset{t\to\infty}{\to} \dot{M}. \tag{CZ}$$

Subsequently, $p^t(x,z)/|P|^t$ tends to the value $c_4(z) = \boldsymbol{\delta}_y^\dagger \dot{M} \boldsymbol{\delta}_x$ when t tends to infinity. The non-nullity of $c_4(z)$ then is a consequence of point (i) which we have proved a few lines above: indeed, taking again the notations t_0 and η used above, we have by Markov's property:

$$\forall t \geqslant t_0 \quad p^t(x,z) \geqslant p^{t-t_0}(x,x)p^{t_0}(x,z) \geqslant c_2|P|^{t-t_0}\eta, \tag{DA}$$

hence $c_4 \geqslant c_2 > 0$.

Point (iii) may be the most subtle. In fact we will prove that for all z in $\widetilde{V}$, $\mathbf{P}_z(\tau_x = t)$ decreases exponentially *with an exponential decay factor strictly less than* $|P|$. More precisely, we will estimate the decay factor of $\mathbf{P}_z(\tau_x \geqslant t \text{ and } \tau_\partial \geqslant t) = \mathbf{P}_z\big((\forall u \in \{0,\dots,t-1\})(X_u \neq x, \partial)\big)$. In other words, we have to look at the decay speed of the sub-Markov chain associated to the transition kernel p, *but restricted to* $\widetilde{V} \smallsetminus \{x\}$. In matricial words, it is the spectral radius of the matrix M^*, which is the matrix M where the x-th line has been replaced by zeroes. Let us denote by $|P^*|$ its spectral radius. The weak form of Perron–Frobenius theorem (cf. [75]) claims that there exists a $|P^*|$-eigenvector $\mathbf{v}^*$ with positive or zero entries for M^*. Each entry of M^* is less than or equal to the corresponding entry of M, and moreover $M^* \neq M$; since M is the matrix of an irreducible aperiodic chain, it follows that, for t sufficiently large, each entry of $(M^*)^t$ is *strictly* less than the correponding entry of M^t. So, for t sufficiently large:

$$|P^*|^t\mathbf{v}^* = \big(M^*\big)^t\mathbf{v}^* < M^t\mathbf{v}^*, \tag{DB}$$

which means that each entry of $|P^*|^t\mathbf{v}^*$ is strictly less than the corresponding entry of $M^t\mathbf{v}^*$. Now let us reason by contradiction by supposing that $|P^*| \geqslant |P|$, then (DB) shows that we can find $t_1 > 0$ and $\rho_1 > |P|$ such that $M^{t_1}\mathbf{v}^* \geqslant \rho_1^{t_1}\mathbf{v}^*$, hence by iterating:

$$\varlimsup_{t\to\infty} \big|M^t\mathbf{v}^*\big|^{1/t} \geqslant \rho_1 > |P|. \tag{DC}$$

But that is absurd, since the spectral radius of M is actually $|P|$. This implies that $|P^*| < |P|$, as we wanted.

Further readings and acknowledgements

The use of the martingales introduced in § 3 can actually be seen as a discrete variant of the forward/backward martingale decomposition technique of Lyons and Zheng [50], as was pointed out to me by Laurent Saloff-Coste. In fact, paper [48] gives a bound for continuous diffusions whose spirit is quite close to that of (Q).

The present work was launched by informal discussions with my colleagues Yann Ollivier and Vincent Beffara, who usefully encouraged and helped me when necessary.

Deuxième partie

Convergence quantitative de modèles à chocs vers la limite de champ moyen

Résumé

Nous considérons un modèle discret à N particules pour l'évolution de Boltzmann spatialement homogène, et nous prouvons sa convergence vers l'équation différentielle associée quand $N \to \infty$ avec des estimées non asymptotiques : pour tout temps $T > 0$, nous majorons la distance, dans un espace de Hilbert adapté, entre la mesure empirique du système particulaire et la mesure prédite par l'équation de Boltzmann spatialement homogène. Le contrôle obtenu est gaussien, c.-à-d. que nous prouvons que la probabilité que cette distance soit plus grande que $xN^{-1/2}$ décroît en $O(e^{-x^2})$. Les deux principaux ingrédients de notre preuve sont, d'une part le contrôle des fluctuations dues à la nature discrète des collisions, d'autre part une hypothèse de type Lipschitz sur le noyau de Boltzmann. Nous étudions plus en détail le cas où l'espace de Hilbert est un espace de Sobolev homogène négatif $\dot{H}^{-s}$; dans ce cas nos résultats ne peuvent fonctionner que pour les modèles maxwelliens, néanmoins les applications numériques semblent indiquer que ces bornes sont utiles en pratique.

☛ *Les recherches présentées dans cette partie de la thèse ont été publiées dans [66].*

Introduction

The Boltzmann equation was written down by L. Boltzmann [8] in 1872, five years after Maxwell's seminal paper [55], to describe the behaviour of a large number of gas molecules interacting by pairwise collisions. Proving rigorously the heuristic arguments of Boltzmann to get some convergence of the N-particle model to the continuous Boltzmann equation when $N \to \infty$ is an extremely difficult challenge that mathematicians are still dealing with.

Here we are only going to handle the *spatially homogeneous* Boltzmann equation (also called *mean field Boltzmann equation*), in which one forgets the positions of the gas particles to concentrate only on the collision phenomenon. Then, proving the convergence of the N-particle system to the continuous equation is a typical mean field limit problem—a particle model is said to be *mean field* when each particle interacts with comparable strength with *all* the other ones. Such a problem, which was first proposed by Kac [43], is far more tractable than the original one, and convergence results, mostly qualitative, have already been obtained for it (see § 6.d).

Here however we are interested in a *quantitative* and *non-asymptotic* version of these results. We would also like to set our results in an *infinite-dimensional* setting, that is, to say that not only any reasonable functional of the particle model converges to the corresponding functional of the limit system, but moreover that all these functionals converge *uniformly*. The quantitative convergence we are going to prove will even have an $N^{-1/2}$ speed, typical of the uniform central limit theory (see [28] about it).

Concerning concrete Boltzmann models, in the present state of my work I am only able to use my results for Maxwellian systems, and moreover constants in convergence bounds deteriorate rapidly with time. However that does not seem to be a fundamental feature of my approach, and further improvements might overcome these issues.

0.2 ***Important Remark***. There are *two* sides in this part of the thesis. The first one, whose climax is Theorem 3.3, is abstract: it consists in showing how Hilbert spaces can be used to prove a new powerful type of convergence results for collision models like Boltzmann's. That work is *a priori* likely to be applied to a wide range of situations, but for each of them checking the hypotheses of the abstract theorem is a different challenge. The second side, which is more physical, consists in studying *one* particular case of application of my formulas, namely the Boltzmann model looked at in the $\dot{H}^{-s}$ space, for which I obtain precise numerical bounds (cf. § 6.c). Though the results got for that particular choice can be proved to be definitely limited in some way (cf. § 4.a), that may be not true any longer for a smarter choice of Hilbert space—which would however be more complicated to handle. So this work highlights a way of studying collision models, but remains at a simple level in the applications, which is the reason why the article [66] I wrote about it was entitled "Some ideas ...". ♡

Here is some notation which will be used throughout this part of the thesis:

- The space $\mathbb{R}^d$ is equipped with its Euclidean structure, whose norm is denoted by $|\cdot|$.
- For $f : E \to F$ a measurable function and μ a measure on E, the image measure of μ by f on F will be denoted $f \# \mu$.
- $\boldsymbol{\delta}_x$ denotes a Dirac mass at x.
- $\mathscr{S}(\mathbb{R}^d)$ is the Schwartz space on $\mathbb{R}^d$, i.e. the set of (complex-valued) $\mathscr{C}^\infty$ functions on $\mathbb{R}^d$ which tend to 0 at infinity faster than any $|x|^{-k}$, as well as all their derivatives.
- The Fourier transform of a function $f \in \mathscr{S}(\mathbb{R}^d)$ is denoted by $\hat{f}$, with the unitary convention $\hat{f}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-i\xi\cdot x} dx$.

- The notation $\|\cdot\|$ will be used to denote Hilbert norms in functional spaces. If Q is a linear operator between two Hilbert spaces, its operator norm $\sup_{\|x\|\leqslant 1}\|Qx\|$ will be denoted $|||Q|||$.
- For x, y and z three points of an affine Hilbert space with $y,z\neq x$, $\widehat{yxz}$ denotes the angle between $\overrightarrow{xy}$ and $\overrightarrow{xz}$, which is an element of $[0,\pi]$.
- The identity matrix of size d is denoted by $\mathbf{I}_d$.

1 The model

1.a The microscopic model

Let us describe the particle model for the spatially homogeneous Boltzmann evolution. Such models have been first proposed by Kac [43] and later thoroughly studied by Sznitman [78], Spohn [76] and others. There are N identical particles indexed by $1,\dots,N$, each particle i being completely described by its velocity $v_i\in\mathbb{R}^d$. One imposes random collision times, so that the microscopic evolution is a Markov process. The way two particles with respective velocities v and w hit each other is described by some positive measure $\gamma_{v,w}$ on $(\mathbb{R}^d)^2$, $N^{-1}d\gamma_{v,w}(v',w')$ being the collision rate from state (v,w) to state (v',w'). In other words, the generator $\mathscr{L}$ of the Markov process is

$$\mathscr{L}f(v_1,\dots,v_N)=\frac{1}{2N}\sum_{0<i,j\leqslant N}\int_{(\mathbb{R}^d)^2}\big(-f(v_1,\dots,v_N)+f(\dots,v_i',\dots,v_j',\dots)\big)\,d\gamma_{v_i,v_j}(v_i',v_j'). \tag{A}$$

We may add to this model some extra physical conditions. First, we will always suppose that the momentum and energy are conserved by collisions, and that the model is invariant by velocity translation or rotation, i.e. that for all $v,w\in\mathbb{R}^d$, for all (positive) isometries J of $\mathbb{R}^d$:

$$\gamma_{v,w}\text{-a.e.}\qquad v'+w'=v+w; \tag{B}$$

$$\gamma_{v,w}\text{-a.e.}\qquad |w'-v'|=|w-v|; \tag{C}$$

$$\gamma_{Jv,Jw}=(J,J)\#\gamma_{v,w}. \tag{D}$$

When conditions (B) to (D) are satisfied, the model is completely described by the family of measures $(\overline{\gamma}_u)_{u\in(0,\infty)}$ on $(0,\pi]$, where $d\overline{\gamma}_u(\theta)$ is the rate of particles with relative speed u which undergo a collision making them deviate by an angle θ in the collision referential.

Moreover, it is often assumed that the $\overline{\gamma}_u$ have a scale invariance property, in the sense that there exists a real parameter g such that for any $\lambda\in(0,+\infty)$,

$$\overline{\gamma}_{\lambda u}=\lambda^g\,\overline{\gamma}_u. \tag{E}$$

For instance, the hard sphere model is scale-invariant with $g=1$. Another very interesting particular case is when $g=0$ —then one says that the model is *Maxwellian*. In this work the concrete results obtained will actually concern Maxwellian models.

Before turning to the macroscopic model, let us make some remarks on the microscopic one:

1.1 Remark.

(i) The N^{-1} factor in Equation (A) is essential to get the mean field limit: it morally says that the global collision rate of one particle is independent of the total number of particles.

(ii) Strictly speaking, generator (A) allows a particle to collide with itself, which is physically absurd. Yet because of the conservation law (C), the auto-collision term is actually zero, so there is no problem.

(iii) The $\gamma_{v,w}$ have to satisfy some integrability conditions for the Markov process to be well-defined. For instance, if conditions (B) to (E) are statisfied, then it suffices that for an arbitrarily chosen $u \in (0,\infty)$, $\int_0^\pi \theta^{d-1} d\overline{\gamma}_u(\theta)$ is finite, cf. [77].

♡

1.b The macroscopic model

The macroscopic space-homogeneous Boltzmann equation [16] is obtained informally by letting N tend to infinity in the microscopic evolution. Then the particles' velocities are described by their empirical measure, which is a (non-atomic in general) probability measure μ_t on $\mathbb{R}^d$. The evolution of that measure is deterministic and is governed by the equation

$$D_t\mu = Q(\mu_t, \mu_t), \tag{F}$$

where Q is the *Boltzmann collision kernel* of the system, formally defined by:

$$Q(\mu,\nu) = \frac{1}{2}\int\Big(\int(-\boldsymbol{\delta}_v - \boldsymbol{\delta}_w + \boldsymbol{\delta}_{v'} + \boldsymbol{\delta}_{w'})\,d\gamma_{v,w}(v',w')\Big)\,d\mu(v)d\nu(w). \tag{G}$$

Equation (F) is an ordinary differential equation in an infinite-dimensional space; that equation is non-linear because of the quadratic term $Q(\mu,\mu)$. Unique existence of a solution to it has been thoroughly studied over the last decades [24, 81]. For our theory to work, we will need to consider a setting where that unique existence is achieved in some convenient space—which is quite logical altogether. Later we will see concrete examples where (F) behaves well for our purpose.

1.c Conservation laws, convergence to equilibrium

Because of the conservation laws (B) and (C), we get $d+1$ invariant functions for the microscopic system: the first d are synthetised in the momentum $P := \sum_{i=1}^N v_i$, and the last one is the energy $K := \frac{1}{2}\sum_{i=1}^N |v_i|^2$. In the macroscopic model, these invariants become $p := \int v\,d\mu(v)$ and $k := \frac{1}{2}\int |v|^2\,d\mu(v)$. Moreover, the fact that the macroscopic model derives from the description of an evolution of particles implies two extra properties for it: first *positivity* of Equation (F), which means that if μ_0 is a positive measure, then so are the μ_t for t positive; secondly *conservation of mass* which gives the $(d+2)$-nd invariant $m := \int d\mu(v)$ for the macroscopic equation.

Concerning equilibrium, if we impose some minimal non-degeneracy condition (see [81]), then it is a well-known beautiful result due to Boltzmann [8] that Equation (F) is dissipative for positive measures and converges to an equilibrium measure μ_{eq} depending only on p, k and m: for $m=1$ and $p=0$, it is

$$d\mu_{\text{eq}}(v) = \Big(\frac{d}{4\pi k}\Big)^{d/2} e^{-d|v|^2/4k}\,dv, \tag{H}$$

and it has the invariance properties $\mu_{\text{eq}}(p,k,1) = \tau_p\#\mu_{\text{eq}}(0,k-p^2/2,1)$, τ_p being the translation by vector p, and $\mu_{\text{eq}}(\lambda p,\lambda k,\lambda m) = \lambda\mu_{\text{eq}}(p,k,m)$. More recently a beautiful quantitative version of that convergence result has been proved by Carlen, Gabetta and Toscani [17].

For the microscopic model, there is also a unique ergodic equilibrium measure for each value of P and K (N being fixed), which is merely the uniform measure on the $(dN-d-1)$-dimensional

sphere [¶] of $(\mathbb{R}^d)^N$ made of the N-uples of vectors having these P and K. Note that for N particles with momentum Np and energy Nk, the marginals of that measure tend to the continuous equilibrium measure $\mu_{\text{eq}}(p,k,1)$ when $N \to \infty$.

Finally it is worth recalling that the microscopic process is reversible under its equilibrium measure, while on the contrary the macroscopic equation (F) exhibits a dissipative behaviour—a phenomenon which caused much trouble at Boltzmann's time, but has been well understood today.

2 Homogeneous Sobolev spaces

2.a Why homogeneous Sobolev spaces?

To be able to speak of quantitative convergence, we will work in some Banach space. Which one shall we take? As we want to compare the empirical measure of our particle system to its limit evolution, a natural choice would be to take some coupling distance between measures—say, the W_1 Wasserstein distance [82, § 7], defined for μ, ν two positive measures with the same mass by

$$W_1(\mu,\nu) := \sup_{f \text{ 1-Lip.}} \Big| \int f \, d(\nu - \mu) \Big|, \tag{I}$$

where "f 1-Lip." means that the supremum is taken over all 1-Lipschitz functions on $\mathbb{R}^d$. However it turns out that it is hopeless to get an $N^{-1/2}$ rate of convergence in such a space, because testing $(\nu - \mu)$ against so much test functions makes the uniform central limit theory fail: see [28, § 6.4] for more details. We also give a more intuitive, completely different explanation for that fact in § 7.

Thus, the idea is to test $(\nu - \mu)$ against a smaller space made of more regular functions. Sobolev spaces $W^{s,p}$, $s > 0$, are such natural test spaces; then $(\nu - \mu)$ will be seen as an element of the dual space $W^{-s,p/(p-1)}$. For our theory we will have to work in a Hilbert space, so we take $p = 2$ and work in $W^{-s,2} =: H^{-s}$; then we can take s fractional, which will turn out to be useful indeed. Yet since defining a norm for H^{-s} spaces requires to choose some aribtrary length, which is physically annoying, we will rather consider *homogeneous* $\dot{H}^{-s}$ spaces, which do have a canonical norm—plus other advantages. Note however, cf. Remark 0.2, that this choice is only one possibility—certainly particularly reasonable—among other ones, and that trickier choices might also be relevant.

2.b Definition and useful properties

Let us define properly the $\dot{H}^{-s}$ spaces.

2.1 Definition. Let $s \in \mathbb{R}$, and for $f \in \mathscr{S}(\mathbb{R}^d)$, set

$$\|f\|_{\dot{H}^{-s}} := \left(\int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \, |\xi|^{-2s} \, d\xi \right)^{1/2}. \tag{J}$$

Then those of the $f \in \mathscr{S}(\mathbb{R}^d)$ for which $\|f\|_{\dot{H}^{-s}} < \infty$, equipped with the norm $\|\cdot\|_{\dot{H}^{-s}}$, constitute a pre-Hilbert space with scalar product

$$\langle f, g \rangle_{\dot{H}^{-s}} = \int_{\mathbb{R}^d} \hat{f}(\xi) \overline{\hat{g}}(\xi) \, |\xi|^{-2s} \, d\xi. \tag{K}$$

[¶]. Possibly of radius 0.

The Hilbert space obtained by completing it is denoted by $\dot{H}^{-s}$. ♢

2.2 Remark. For a physicist, $f : \mathbb{R}^d \to \mathbb{C}$ has some homogeneity: say, the elements in $\mathbb{R}^d$ are measured in x (generally x is a unit of length, say meter) and the elements in $\mathbb{C}$ are measured in y (which will often be a density unit, say $\mathrm{kg}\cdot\mathrm{m}^{-d}$). Then $\|f\|_{\dot{H}^{-s}}$ is measured in $\mathrm{y}\cdot\mathrm{x}^{s+d/2}$ (in our example, $\|f\|_{\dot{H}^{-s}}$ would be measured in $\mathrm{kg}\cdot\mathrm{m}^{s-d/2}$). Equivalently, if μ is a measure on $\mathbb{R}^d$, the physical dimension of $\|\mu\|_{\dot{H}^{-s}}$ is $\mathrm{z}\cdot\mathrm{x}^{s-d/2}$, x being the physical dimension of the elements of $\mathbb{R}^d$ and z the physical dimension of μ (which in our example would be kilogram). ♡

As we told in § 2.a, bounding a function or a measure in $\dot{H}^{-s}$ means bounding uniformly its integral against some class of regular functions:

2.3 Proposition. *Define $\dot{H}^s$ in the same way as $\dot{H}^{-s}$. Then, for any f for which it makes sense:*

$$\|f\|_{\dot{H}^{-s}} = \sup_{\|g\|_{\dot{H}^s}\leqslant 1} \Big|\int_{\mathbb{R}^d} f(x)\overline{g}(x)\,dx\Big|. \tag{L}$$

♣

2.4 Proposition. *For $s \in (0,d)$, let φ_s be the locally integrable function*

$$\varphi_s(x) = |x|^{-(d-s)}, \tag{M}$$

then one has for all $f, g \in \mathscr{S}(\mathbb{R}^d)$:

$$\langle f, g\rangle = c(s,d)^2 \langle f * \varphi_s, g * \varphi_s\rangle_{L^2(\mathbb{R}^d)}, \tag{N}$$

with

$$c(s,d) = \frac{\Gamma((d-s)/2)}{(2\pi)^{d/2}\Gamma(s/2)}, \tag{O}$$

$\Gamma(\cdot)$ *being Euler's Gamma function.* ♣

Proof. Use that the Fourier transform of $|\xi|^{-s}$ is $(2\pi)^{d/2}c(s,d)\varphi_s(x)$, cf. [74, Exercise V-10]. ♠

2.5 Proposition. *Let J_λ be a similarity of $\mathbb{R}^d$ with dilation factor λ; then for any map $f \in \dot{H}^{-s}$,*

$$\|f \circ J_\lambda\|_{\dot{H}^{-s}} = \lambda^{s+d/2}\|f\|_{\dot{H}^{-s}}. \tag{P}$$

Equivalenty, for any measure $\mu \in \dot{H}^{-s}$,

$$\|J_\lambda \# \mu\|_{\dot{H}^{-s}} = \lambda^{s-d/2}\|\mu\|_{\dot{H}^{-s}}. \tag{Q}$$

♣

☛ *From now on, we will always write implicitly $s = d/2 + r$.*

2.6 Proposition. *For $d \geqslant 2$ [‖], $r \in (0,1)$, every signed measure on $\mathbb{R}^d$ of total mass zero having an r-th polynomial moment can be seen as an element of $\dot{H}^{-s}$.* ♣

[‖]. The proposition remains valid with $d = 1$, except that it must be demanded that $r < 1/2$.

Proof. First, we check that $\|\mu\|_{\dot{H}^{-s}}$ is finite for μ of the form $(\boldsymbol{\delta}_x - \boldsymbol{\delta}_y)$, by observing that in this case the function $(\mu * \psi_s)$ is L^2: indeed it increases like a polynomial of degree $-(d-s) > -d/2$ about its singularities at x and y, and it is bounded by a polynomial of degree $-(d-s)-1 < -d/2$ about infinity. Moreover, changing variables by a similarity shows that $\|\boldsymbol{\delta}_x - \boldsymbol{\delta}_y\|_{\dot{H}^{-s}}$ takes the form $C_r|y-x|^r$ for some absolute constant $C_r \in (0,\infty)$.

Now, let μ be a signed measure satisfying the general assumptions, and denote $\mu =: \mu_+ - \mu_-$ its Hahn decomposition, μ_+ and μ_- each having total mass M. Then, since μ has an r-th polynomial moment, the integral Minkowski inequality gives:

$$\|\mu\|_{\dot{H}^{-s}} \leqslant \frac{1}{M}\int_{(\mathbb{R}^d)^2} \|\boldsymbol{\delta}_x - \boldsymbol{\delta}_y\|_{\dot{H}^{-s}} d\mu_+(x)d\mu_-(y) = \frac{C_r}{M}\cdot\int_{(\mathbb{R}^d)^2} |x-y|^r d\mu_+(x)d\mu_-(y) < \infty, \tag{R}$$

which proves the result for the general case. ♠

2.7 Remark. The $\dot{H}^{-s}$ norm allows us to measure the *distance* between two (sufficiently integrable) probability measures, but speaking of the $\dot{H}^{-s}$ norm of a *single* probability measure would be nonsense! Note also that, by Sobolev imbedding, one can bound above $\|\nu - \mu\|_{\dot{H}^{-s}}$, for any two probability measures μ and ν, by (up to some explicit multiplicative constant)

$$W_{1,r}(\mu,\nu) = \sup\Big\{\Big|\int f d\mu - \int f d\nu\Big| \ ; \ \forall x,y \ |f(x)-f(y)| \leqslant |y-x|^r\Big\}. \tag{S}$$

♡

3 Dynamic control

3.a Abstract setting

Now let us study the evolution of our particle system along time. We first give our main result in an abstract setting to alleviate its proof; the reader more comfortable with physical settings may read Theorem 3.6 instead.

Let H be a Hilbert space, let A be an H-affine space and let $(\hat{X}_t)_{t\geqslant 0}$ be some jump Markov process on A with generator $\mathscr{L}$. Fix o an arbitrary point of A and define

$$\begin{aligned} I:\ A &\to H \\ x &\mapsto \overrightarrow{ox}\ : \end{aligned} \tag{T}$$

since I is defined up to an additive constant, the operator $(\mathscr{L}I): A \to H$ [*] does not depend on the choice of o and we can therefore define $(X_t)_{t\geqslant 0}$ as the deterministic process on A following the differential equation

$$D_t X = (\mathscr{L}I)(X_t). \tag{V}$$

Our goal is to control the distance between $\hat{X}_t$ and X_t. Here what is important for us is to have a good control of large deviations for that distance. As Cramér's method cannot be applied directly

[*]. *Stricto sensu* $\mathscr{L}$ acts on some space of *real* functions on A, say the space of continuous bounded functions $\mathscr{C}_b(A,\mathbb{R})$. Yet we can straightforwardly extend $\mathscr{L}$ to the space $\mathscr{C}_b(A,E)$ for any Banach space E by defining the operator $\mathscr{L}^{(E)}: \mathscr{C}_b(A,E) \to \mathscr{C}_b(A,E)$ through:

$$\forall \varphi \in E' \ \ \forall f \in \mathscr{C}_b(A,E) \qquad \langle \varphi, \mathscr{L}^{(E)} f\rangle = \mathscr{L}(\langle \varphi, f\rangle). \tag{U}$$

That is what we do here: I is a function from A to H, so $\mathscr{L}I$ actually denotes $\mathscr{L}^{(H)}I$.

because of the infinite-dimensional setting, we introduce an exponential utility function $\mathscr{U} : H \to \mathbb{R}$ defined by:

$$\mathscr{U}(x) = e^{\|x\|} + e^{-\|x\|}. \tag{W}$$

The following proposition gathers the properties of $\mathscr{U}$ we will use in our work:

3.1 Proposition.

(i) For all $x \in H$, $\mathscr{U}(x) \geqslant e^{\|x\|}$;
(ii) $\mathscr{U}(0) = 2$;
(iii) For all $x, h \in H$, $\mathscr{U}(x+h) \leqslant e^{\|h\|}\mathscr{U}(x)$;
(iv) $\mathscr{U}$ is of class $\mathscr{C}^\infty$ [†];
(v) For all $x \in H$, $\nabla\mathscr{U}(x)$ is positively colinear to x;
(vi) For all $x \in H$, $|||\nabla^2\mathscr{U}(x)||| \leqslant \mathscr{U}(x)$.

♣

Then one can state the theorem which will be our central tool. We first need some notation to alleviate our formulas:

3.2 Definition. We denote $e_1(z) := (e^z - 1)/z$, extended by $e_1(0) = 1$, resp. $e_2(z) := (e^z - 1 - z)/z^2$, extended by $e_2(0) = 1/2$. We also denote by κ_- the negative part of κ, i.e. $\kappa_- := \max\{-\kappa, 0\}$. ♢

3.3 Theorem. *Suppose that Equation (V) has a κ-contracting semigroup for some $\kappa \in \mathbb{R}$, in the sense that for all $x \in A, h \in H$:*

$$\langle D_x(\mathscr{L}I)\cdot h, h\rangle \leqslant -\kappa\|h\|^2. \tag{X}$$

Suppose moreover that the Markov process—which we recall to be a jump process—has the amplitude of all its jumps bounded above by some $L < \infty$, and satisfies:

$$\forall x \in A \quad \mathscr{L}(\|\cdot - x\|^2)(x) \leqslant V \tag{Y}$$

for some $V < \infty$.

Then, denoting by $\hat{X}_0$ the (random) initial value of the Markov process and by X_0 the (deterministic) initial value of the differential equation (V), one has for any $T \geqslant 0$, for any $\lambda > 0$:

$$\log\mathbf{E}[\mathscr{U}(\lambda(\hat{X}_T - X_T))] \leqslant \log\mathbf{E}[\mathscr{U}(\lambda e^{-\kappa T}(\hat{X}_0 - X_0))] + \lambda^2 e_2(\lambda e^{2\kappa_- T}L)e_1(-2\kappa T)VT. \tag{Z}$$

♣

Proof. The principle of the proof is to show that some time-depending functional

$$F_t(\hat{X}_t) := e^{h(t)}\mathscr{U}\big(\lambda e^{\kappa(t-T)}(\hat{X}_t - X_t)\big), \tag{AA}$$

for a well-chosen function h, is a supermartingale. To prove this, we will do some Itō calculus for jump processes.

☛ *To make our computations completely rigorous, throughout the proof we will assume that the expected number of collisions per unit of time is uniformly bounded, that is, that there is some $M < \infty$ such that $|(\mathscr{L}\mathbf{1}_{A'})(x)| \leqslant M$ for all $x \in A$ and all Borel subsets $A' \subset A$. Then the general result can be recovered by a standard truncation argument.*

[†]. To prove it, note that $\mathscr{U}(x) = f(\|x\|^2)$, where $f = 2\cosh(\sqrt{\cdot})$ is (the restriction to $[0, +\infty)$ of) an analytic function on $\mathbb{R}$.

Let us fix some $t \in [0,T]$ and suppose $(\hat{X}_{t'})_{t' \in [0,t]}$ is known. Let δt be a small amount of time devised to tend to 0; $O(\delta t^n)$ will denote any quantity bounded by some $C\delta t^n$ when δt tends to 0, where C is deterministic. With this notation, the law of $\hat{X}_{t+\delta t}$ depends on $(\hat{X}_{t'})_{t' \in [0,t]}$ only through $\hat{X}_t$, and our goal is to show that $\mathbf{E}[F(\hat{X}_{t+\delta t})] - F(\hat{X}_t)$, which is $O(\delta t)$, is nonpositive—more precisely, we only need to prove that $\mathbf{E}[F(\hat{X}_{t+\delta t})] - F(\hat{X}_t) \leqslant O(\delta t^2)$ [‡].

Set $\hat{Y} := \hat{X} - X$. Denote $\delta\hat{X} := \hat{X}_{t+\delta t} - \hat{X}_t$, resp. $\delta X := X_{t+\delta t} - X_t$, $\delta\hat{Y} := \hat{Y}_{t+\delta t} - \hat{Y}_t$, $\delta F := F(\hat{X}_{t+\delta t}) - F(\hat{X}_t)$. The fundamental observation is that

$$\mathbf{E}[\delta\hat{X}] = (\mathscr{L}I)(\hat{X}_t)\delta t + O(\delta t^2). \tag{AB}$$

Now, admitting temporarily that h will be of class $\mathscr{C}^2$, we write:

$$\begin{aligned}
\delta F &= h'(t)F(t)\delta t && \text{(AC)}\\
&+ e^{h(t)}\lambda e^{\kappa(t-T)}\nabla\mathscr{U}(\lambda e^{\kappa(t-T)}\hat{Y}_t)\cdot(\mathscr{L}I(\hat{X}_t) - \mathscr{L}I(X_t) + \kappa\hat{Y}_t)\delta t && \text{(AD)}\\
&+ e^{h(t)}\big[\mathscr{U}\big(\lambda e^{\kappa(t-T)}\hat{Y}_{t+\delta t}\big) - \mathscr{U}\big(\lambda e^{\kappa(t-T)}\{\hat{Y}_t + [\mathscr{L}I(\hat{X}_t) - \mathscr{L}I(X_t)]\delta t\}\big)\big] && \text{(AE)}\\
&+ O(\delta t^2).
\end{aligned}$$

In that sum we first see that the term (AD) is nonpositive: (X) implies indeed, for all $x \in A, y \in H$,

$$\langle(\mathscr{L}I)(x+y) - (\mathscr{L}I)(x) + \kappa y\ ,\ y\rangle \leqslant 0, \tag{AF}$$

which we apply here with $x = X_t$ and $y = \hat{Y}_t$, using that $\nabla\mathscr{U}(\lambda e^{\kappa(t-T)}\hat{Y}_t)$ is positively colinear to $\hat{Y}_t$ (Proposition 3.1-(v)).

Now let us look at term (AE). Because of (AB), the expectation of the random variable

$$\lambda e^{\kappa(t-T)}\big(\hat{Y}_{t+\delta t} - (\hat{Y}_t + [\mathscr{L}I(\hat{X}_t) - \mathscr{L}I(X_t)]\delta t)\big) \tag{AG}$$

is $O(\delta t^2)$. We will use it thanks to the following

3.4 Lemma. *Let $X \in H$; let y be an H-valued random variable with zero mean. Then one has:*

$$\mathbf{E}[\mathscr{U}(X+y)] \leqslant \mathscr{U}(X)(1 + \mathbf{E}[e_2(\|y\|)\|y\|^2]). \tag{AH}$$

♣

Proof of the lemma. Taylor's formula yields

$$\mathscr{U}(X+y) = \mathscr{U}(X) + \nabla\mathscr{U}(X)\cdot y + \left(\int_0^1 (1-\theta)\nabla^2\mathscr{U}(X+\theta y)d\theta\right)\cdot(y\otimes y). \tag{AI}$$

In that sum the third term is bounded above by

$$\|y\|^2\int_0^1 (1-\theta)\mathscr{U}(X+\theta y)d\theta \tag{AJ}$$

by Proposition 3.1-(vi), which in turn is bounded by

$$\|y\|^2\mathscr{U}(X)\int_0^1 (1-\theta)e^{\theta\|y\|}d\theta\ =\ e_2(\|y\|)\|y\|^2. \tag{AK}$$

by Proposition 3.1-(iii). Taking expectation gives the result since the second term in sum (AI) has zero mean by assumption. ♠

[‡]. Beware that "*expr.* $\leqslant O(\delta t^n)$" does not mean "*expr.* $= O(\delta t^n)$" but actually "$(expr.)_+ = O(\delta t^n)$".

What does it give for us? Let E be the event "some collision occurs between t and $t+\delta t$". E is an event of probability $O(\delta t)$; on E, the random variable (AG) is $O(1)$, and on E^{c} it is $O(\delta t)$. Hence, shorthanding temporarily this quantity into "$*$", $\mathbf{E}[\| * \|^2 e_2(\| * \|)]$, up to some $O(\delta t^2)$, is merely $\lambda^2 e^{2\kappa(t-T)}\mathbf{E}[\|\delta\hat{Y}\|^2 e_2(\|\lambda e^{\kappa(t-T)}\delta\hat{Y}\|)]$, which is bounded above by $\lambda^2 e^{2\kappa(t-T)} e_2(\lambda e^{2\kappa_- T}L)V$ uniformly in t.

Putting all things together, we get

$$\mathbf{E}[\delta F] \leqslant \big(h'(t) + \lambda^2 e^{2\kappa(t-T)} e_2(\lambda e^{2\kappa_- T}L)V\big)F(t)\delta t + O(\delta t^2), \tag{AL}$$

which will be $\leqslant O(\delta t^2)$ provided

$$h'(t) \leqslant -\lambda^2 e^{2\kappa(t-T)} e_2(\lambda e^{2\kappa_- T}L)V. \tag{AM}$$

To achieve that optimally with $h(T) = 0$, we choose

$$h(t) = \lambda^2 e_2(\lambda e^{2\kappa_- T}L) e_1(2\kappa(t-T))V(T-t), \tag{AN}$$

which is of class $\mathscr{C}^2$ indeed. Formula (Z) then follows by the supermartingale property. ♠

3.5 Remark. Strictly speaking our proof only shows that $F(\hat{X}_t)$ is a *local* supermartingale. But this local supermartingale is nonnegative, so it is actually a global supermartingale (see [69, § IV-1.5]). ♡

3.b Application to Boltzmann's model

Translation of Theorem 3.3 Let us see what Theorem 3.3 gives for the Boltzmann model. For the time being, according to Remark 0.2 I do not precise what H and A are: all you have to know is that H is some Hilbert space of measures and A the corresponding affine space, which is assumed to contain all the probability measures having certain moments.

Let $N \in \mathbb{N}^*$. The stochastic process $(\hat{X}_t)_{t\geqslant 0}$ on A will be the empirical measure $\hat{\mu}_t^N$ of our microscopic process $(v_1(t), \dots, v_N(t))$ on $(\mathbb{R}^d)^N$, which is a Markov process indeed; let us denote its generator by $\mathscr{L}^N$. Regardless of N, one has as expected:

$$\forall \mu \in A \qquad (\mathscr{L}^N I)(\mu) = Q(\mu,\mu), \tag{AO}$$

so the deterministic process $(X_t)_{t\geqslant 0}$ on A will be our macroscopic process following the Boltzmann equation (F). Finally, for $\mu \in A, \nu \in H$, $(D_\mu(\mathscr{L}I)) \cdot \nu = 2Q(\mu,\nu)$ [§]. So Theorem 3.3 becomes:

3.6 Theorem. *Let H be a Hilbert space of measures and A the corresponding affine space containing probability measures (or a convex subset of that space, cf. Remark 3.7). Consider our microscopic and macroscopic models for some $N \in \mathbb{N}^*$, with certain initial conditions* [¶] $(v_1, \dots, v_N)$, *resp.* μ_0.

Suppose that there exists some constants $\kappa \in \mathbb{R}, L < \infty, V < \infty$ such that:

(i) For all $\mu \in A, \nu \in H$,

$$\langle Q(\mu,\nu), \nu\rangle \leqslant -\frac{\kappa}{2}\|\nu\|^2; \tag{AP}$$

[§]. *Stricto sensu* $Q(\mu,\cdot)$ is an affine operator from A to H, not a linear operator on H: in fact here $Q(\mu,\nu)$ denotes $\overline{Q}(\mu,\nu)$, $\overline{Q}(\mu,\cdot)$ being the linear part of $Q(\mu,\cdot)$. Identifying notations is relevant because $\overline{Q}$, like Q, is formally defined by (G).

[¶]. The initial condition for the stochastic process can be random.

(ii) For all $\mu \in A$,

$$\mathbf{E}[\,\|\mu_{t+} - \mu_t\|^2 \,|\, \mu_t = \mu] \leqslant V; \tag{AQ}$$

(iii) The effect of collisions for the microscopic model in A is always bounded by L, i.e. one has almost surely

$$\forall t \geqslant 0 \qquad \|\hat{\mu}^N_{t+} - \hat{\mu}^N_t\| \leqslant L. \tag{AR}$$

Then for any $T \geqslant 0$, for any $\lambda > 0$,

$$\log \mathbf{E}[\mathscr{U}(\lambda(\hat{\mu}_T - \mu_T))] \leqslant \log \mathbf{E}[\mathscr{U}(\lambda e^{-\kappa T}(\hat{\mu}_0 - \mu_0))] + \lambda^2 e_2(\lambda e^{2\kappa_- T} L) e_1(-2\kappa T) V T. \tag{AS}$$

♣

3.7 Remark. Theorem 3.6 remains valid, with the same proof, if we replace A by any convex subset $\tilde{A} \subset A$ such that almost-surely $\forall t \geqslant 0 \quad X_t, \hat{X}^N_t \in \tilde{A}$. An important example is taking for $\tilde{A}$ the subset of the true probability measures of A (which subset is stable because of positivity and conservation of mass for the evolutions, cf. § 1.c), for which the properties of positive measures can be used. ♡

Constants for the Sobolev setting

☛ *From now on, when dealing with Boltzmann models we work in the space $\dot{H}^{-s}(\mathbb{R}^d)$ for some $r \in (0,1)$. We denote by C_r the $\dot{H}^{-s}$ norm of any $(\boldsymbol{\delta}_x - \boldsymbol{\delta}_y)$ for $|x-y| = 1$, which is some finite explicit function of d and r.*

To apply Theorem 3.6, we have to compute the values of L, V, κ and $\mathbf{E}[\mathscr{U}(\lambda e^{-\kappa T}(\hat{X}_0 - X_0))]$. Here let us just look at the first two quantities—the last two ones shall be the objects of separate sections.

Recall that K denotes the energy of the N-particle system, which is conserved along the stochastic evolution—note by the way that up to translating the origin of $\mathbb{R}^d$, we can replace K by the internal energy

$$\tilde{K} = K - \frac{|P|^2}{2N}. \tag{AT}$$

Then at any time no particle has speed greater than $\sqrt{2K}$, so the effect of a collision between two particles on the empirical measure cannot be more than $2 \cdot (8K)^{r/2} C_r N^{-1}$, which yields an admissible value for L.

3.8 Remark. To get the bound $L \leqslant 2 \cdot (8K)^{r/2} C_r N^{-1}$, we have used that the relative speed between two particles is at most $2\sqrt{2K}$ and that the effect of a collision with relative speed u is at most $2u^r N^{-1}$. Actually one can do slightly better: the relative speed between two particles is at most $2\sqrt{K}$ and the effect of a collision with relative speed u is at most $2\sqrt{2^{1-r} - 1}\, C_r u^r N^{-1}$ (corresponding to the deviation angle $\theta = \pi/2$), so we can take

$$L = 2^{1+r} \sqrt{2^{1-r} - 1}\, C_r K^{r/2} N^{-1}. \tag{AU}$$

It is that bound that we will use in the sequel. ♡

Anyway remember that, since K is going to be of order of magnitude $O(N)$, one has $L = O(N^{-1+r/2})$ when $N \to \infty$.

Now let us compute V: V is defined by (AQ), which is bounded above by

$$2C_r^2N^{-1}\int_{(\mathbb{R}^d)^2}|w-v|^{2r}d\mu(v)d\mu(w) \overset{\text{Jensen}}{\leqslant} 2C_r^2N^{-1}\left(\int_{(\mathbb{R}^d)^2}|w-v|^2d\mu(v)d\mu(w)\right)^r$$
$$= 2^{1+2r}C_r^2N^{-1}\left(\frac{\tilde{K}}{N}\right)^r \leqslant 2^{1+2r}C_r^2K^rN^{-(1+r)}. \quad \text{(AV)}$$

Taking into account Remark 3.8, we can even take

$$V = (2^{1-r}-1)2^{1+2r}C_r^2K^rN^{-(1+r)}. \quad \text{(AW)}$$

Anyway remember that $V = O(N^{-1})$ when $N \to \infty$.

3.c Comments on the results

☛ *All the computations in this subsection are heuristic, so we will drop lower order terms without wondering when we can do so. $C_1, C_2, \dots$ will denote constants depending only on κ, V, L and T, whose exact expression does not interest us.*

In the right-hand side of Formula (AS) there are two terms: the first one, $\log\mathbf{E}[\mathscr{U}(\lambda e^{-\kappa T}(\hat{X}_0 - X_0))]$, merely expresses the difference between the experimental initial condition and its continuous limit. There is obviously no surprise in getting such a term, whose study is deferred to § 5: for the time being just notice the presence of the factor $e^{-\kappa T}$ in front of $(\hat{X}_0 - X_0)$, which means that the effect of initial fluctuations will be quite large if $\kappa < 0$, and conversely quite small if $\kappa > 0$.

The actual dynamic effect in (AS) lies in the term $\lambda^2 e_2(\lambda e^{2\kappa_- T}L)e_1(-2\kappa T)VT$. Let us study it in the case of our Sobolev setting. We have noticed that, when N becomes large, one has $L = O(N^{r/2-1})$, resp. $V = O(N^{-1})$, so let us write $L \simeq lN^{r/2-1}$, resp. $V \simeq \omega N^{-1}$. Then the dynamic term of (AS) becomes

$$\lambda^2 e_2(\lambda e^{2\kappa_- T}L)e_1(-2\kappa T)VT \simeq \lambda^2 N^{-1}e_2(\lambda e^{2\kappa_- T}lN^{r/2-1})e_1(-2\kappa T)\omega T. \quad \text{(AX)}$$

The λ^2N^{-1} factor hints that the good order of magnitude for λ will be $\lambda = O(N^{1/2})$. So write $\lambda = yN^{1/2}$; then (AX) becomes

$$\lambda^2 e_2(\lambda e^{2\kappa_- T}L)e_1(-2\kappa T)VT \simeq e_2(ye^{2\kappa_- T}lN^{(r-1)/2})e_1(-2\kappa T)\omega y^2 T. \quad \text{(AY)}$$

In our case $(r-1)/2 < 0$ so, if N is sufficiently large, $ye^{2\kappa_- T}lN^{(r-1)/2}$ is very close to zero and the $e_2(*)$ term is very close to $e_2(0) = 1/2$, finally yielding

$$\lambda^2 e_2(\lambda e^{2\kappa_- T}L)e_1(-2\kappa T)VT \simeq \frac{1}{2}e_1(-2\kappa T)\omega y^2 T. \quad \text{(AZ)}$$

For a fixed T, (AZ) shows that the dynamic term in Formula (AS) is approximately C_1y^2. Moreover, as we will see in § 5, the static term $\log\mathbf{E}[\mathscr{U}(\lambda e^{-\kappa T}(\hat{X}_0 - X_0))]$ is approximately $C_2y^2 + C_3$. In the end, one gets

$$\log\mathbf{E}[\mathscr{U}(yN^{1/2}(\hat{X}_t - X_t))] \lesssim C_4y^2 + C_3, \quad \text{(BA)}$$

hence by Markov's inequality and Proposition 3.1-(i), for all $x > 0$,

$$\mathbf{P}\big(yN^{1/2}\|\hat{X}_T - X_T\| \geqslant x\big) \lesssim e^{C_4y^2+C_3-x}. \quad \text{(BB)}$$

Optimizing Formula (BB) for fixed x/y ratio, one finally finds:

$$\forall \varepsilon \geqslant 0 \quad \mathbf{P}(\|\hat{X}_T - X_T\| \geqslant \varepsilon) \lesssim \exp(C_3 - C_5 N \varepsilon^2). \tag{BC}$$

So Theorem 3.6 gives a Gaussian control for the fluctuations between $\hat{X}_T$ and X_T for any fixed value of T —provided the existence of some contractivity constant κ, which for $H = \dot{H}^{-s}$ will be proved in § 4 for the Maxwellian case. Moreover the order of magnitude of the fluctuations we get is $N^{-1/2}$, the typical deviation size in central limit theorems. So we can say that the bounds we have got are a kind of explicit dynamic central limit bound for the Boltzmann model.

3.9 Remark. Actually the approximations we made to get (BA) are sensible only if y is not too large, otherwise $\lambda e^{2\kappa_- T} l N^{r/2-1} \gtrsim 1$ and then the $e_2(*)$ term in (AX) can no longer be considered as close to 1/2. It follows that our computations are valid only for $\lambda \lesssim N^{1-r/2}/l$, i.e. for $y \lesssim N^{(1-r)/2}/l$. Tracking that constraint throughout our reasoning, it finally turns out that (BC) is only valid for $\varepsilon \lesssim \omega T N^{-r/2}/l$. So our Gaussian control does not hold up to large deviations but only till certain intermediate deviations [‖]. Fortunately, (BC) itself tells us that the probability of such intermediate deviations is bounded above by something like $e^{-C_6 N^{1-r}}$, which goes very fast to 0 anyway. Moreover, even for $\varepsilon \gg \omega T N^{-r/2}/l$ one can still use (BB) with $y \simeq N^{(1-r)/2}/l$ and $x = y N^{1/2} \varepsilon$, which gives an sharp exponential control of the tail of the law of $\|\hat{X}_T - X_T\|$ (whose half-life is $O(N^{-1+r/2})$) applicable up to large deviations. ♡

The behaviour of Formula (AZ) as T becomes large depends on the sign of κ: [*]

- If $\kappa < 0$ (the worst case), then the $e_1(-2\kappa T)$ factor becomes exponentially large as soon as $T \gtrsim 1/|\kappa|$. Thus the dynamic control given by Theorem 3.6 is relevant only for moderate values of T corresponding to durations for which each particle makes only a couple of collisions. Moreover, as we noticed in the beginning of that subsection, in that case the term due to the control of initial fluctuations will become huge as T increases. Note however that qualitatively we get a Gaussian control for *any* fixed T, only the constants in that control becoming bad.
- If $\kappa = 0$ the dynamic term of (AS) increases proportionally to T, so our bound remains good even for moderately large values of T, but ultimately becomes uninteresting.
- If $\kappa > 0$ (the best case) then $T e_1(-2\kappa T) \to 1/2\kappa$ when $T \to \infty$, so the right-hand side of (AS) remains bounded uniformly in T, implying that the N-particle model approximates well its continuous limit for *any* time [†]. Note that $\kappa > 0$ is tantamount to having an exponential convergence of (F) to equilibrium in A, so in that case our bound rather looks like a result of convergence to "equilibrium" for the empirical measure $\hat{\mu}_t^N$.

[‖]. Here it is worth noticing that our control (AU) on L was very coarse: in real situations indeed the maximal relative speed between two particles is $\sim \sqrt{\log N}$ with very large probability (think about the Maxwell distribution), so in most cases $L \sim (\log N)^{r/2} N^{-1}$. Therefore, provided we could control sharply the probability that L becomes larger than $N^{-1+\eta}$, i.e. that some particle becomes abnormally hot, we could have a Gaussian control of $\|\hat{X}_T - X_T\|$ up to deviations of order $N^{-\eta}$ for $\eta < r/2$ —but such a control would require another article.... Anyway, η could not be taken arbitrarily close to 0, because for η too small the probability of having a hot particle would become larger than the Gaussian bound (the threshold for the Maxwellian distribution is $\eta \gtrsim r/2(r+1)$).

[*]. In § 4.a we will see that for $H = \dot{H}^{-s}$, κ is actually always negative. Our the discussion is relevant nevertheless, because it remains valid for other applications of abstract Theorem 3.3, therefore highlighting the interest of choosing a Hilbert space better than $\dot{H}^{-s}$.

[†]. Beware: it does not mean that *one* random particle system has large probability to stay *always* close to the continuous limit—which is trivially false by ergodicity—but that at any *given* time, *most* of the particle systems will be close to the limit.

4 Contractivity of the collision kernel

4.a Limitations due to our settings

In this section we are going to look for a suitable constant κ for (X). Unfortunately it turns out that, for the choices I have made, our results shall unavoidably be limited, as I quickly explain in this foreword. Let me stress however that all the issues encoutered might be solved by working in a trickier space than the plain $\dot{H}^{-s}$ (cf. Remark 0.2).

First, κ can only be negative, which is the worst case (see page 62). Why that? Well, if κ were positive, as we said previously it would imply convergence of Equation (F) to a unique equilibrium for all measures in A. Yet there are several different equilibrium probability measures for the Boltzmann evolution (see Formula (H) and below), whose differences lie in $\dot{H}^{-s}$, which is a contradiction. So κ is nonpositive. Then we could prove, using that the model is nondegenerate, that κ cannot be zero and thus is negative. To have a chance to get positive values of κ, $\dot{H}^{-s}$ should be replaced by a Hilbert space containing only signed measures η such that $\int d\eta(x)$, $\int x\, d\eta(x)$, $\int |x|^2 d\eta(x) = 0$ —but which one?

Secondly, the only chance for κ to be finite is the case of Maxwellian models (recall their definition below (E)): this is due to a bad scale invariance property for non-Maxwellian models, cf. Remark 4.2. Though the Maxwellian case is often a useful first step for theoretists, the physical models encountered in real life do not have any reason for being so! To have a chance to get results for non-Maxwellian models, $\dot{H}^{-s}$ should be replaced by some non-homogeneous space—but non-homogeneous spaces are often less tractable than homogeneous spaces and more difficult to interpret physically.

4.b Principle for computing κ

To check Hypothesis (AP), according to Remark 3.7 we can consider our Markov processes restricted to the set of probability measures, and then by positive linearity it suffices to prove (AP) when μ is a Dirac mass:

4.1 Proposition. *If, for one arbitrary (then for all) $v \in \mathbb{R}^d$, the linear operator $Q(\delta_v, \cdot) : \dot{H}^{-s} \to \dot{H}^{-s}$ satisfies the "contractivity" property*

$$\forall f \in \dot{H}^{-s} \quad \langle Q(\delta_v, f), f\rangle \leqslant -\frac{\kappa}{2}\|f\|^2, \tag{BD}$$

then the restriction of Q to probability measures satisfies Hypothesis (AP). ♣

4.2 Remark. Now we can understand why κ cannot be finite for a non-Maxwellian model. Suppose indeed that the model satisfies (E) with $g \neq 0$; then, denoting by I_λ the homothety ($v \mapsto \lambda v$) for $\lambda \in (0, +\infty)$, one has

$$Q(\delta_0, I_\lambda \# \mu) = \lambda^g I_\lambda \# Q(\delta_0, \mu). \tag{BE}$$

If assumption (AP) were satisfied for some $\kappa < 0$ (which we told to be the only possible case), then $Q(\delta_0, \cdot)$ would be κ-contracting, hence also $(\lambda^g \kappa)$-contracting for all λ by (BE), thus 0-contracting, and thus (AP) would be satisfied with $\kappa = 0$ by Proposition 4.1, which is impossible. ♡

4.3 Proposition. *Recall the definition (M) of φ_s. Denote by $(*\varphi_s)$ the convolution operator*

$$\begin{aligned} (*\varphi_s) :\ \dot{H}^{-s} &\to L^2 \\ f &\mapsto f * \varphi_s. \end{aligned} \tag{BF}$$

Then $Q(\boldsymbol{\delta}_v,\cdot)\colon\ \dot{H}^{-s}\to\dot{H}^{-s}$ *satisfies the property (BD) if and only if*

$$(*\varphi_s)\circ Q(\boldsymbol{\delta}_v,\cdot)\circ(*\varphi_s)^{-1}\colon\ L^2\to L^2 \tag{BG}$$

satisfies the same property in the space $L^2(\mathbb{R}^d)$. ♣

Proof. It follows directly from the isomorphism formula (N). ♠

4.c Effective computation

4.4 Lemma. *Let* $\theta\in[0,\pi]$*; define the linear operator* $\check{Q}_\theta$ *on measures on* $\mathbb{R}^d$ *such that* $\check{Q}_\theta(\boldsymbol{\delta}_v)$ *is the uniform probability measure on the* $(d-2)$*-dimensional sphere* [‡] *of velocities* v' *such that* $|v'-v/2|=|v|/2$ *and* $\widehat{v\tfrac{v}{2}v'}=\theta$*. Then*

$$(*\varphi_s)\circ\check{Q}_\theta=\big(\cos(\theta/2)\big)^s\check{Q}_\theta\circ(*\varphi_s). \tag{BH}$$

♣

4.5 Remark. $\check{Q}_\theta(\boldsymbol{\delta}_v)$ respresents the post-collisional distribution of velocity of a particle at initial velocity v which collided with a particle at initial velocity 0 and underwent an angular deviation θ in the collision referential, the precise direction of that deviation being random. ♡

Proof. Let us give first a neat proof working when d is even. Call $\mathscr{R}_\theta$ the set of the rotations R of $\mathbb{R}^d$ satisfying $\widehat{v0(Rv)}=\theta/2$ for *all* $v\in\mathbb{R}^d$. If d is even, $\mathscr{R}_\theta$ is non-empty and has some canonical probability measure π_θ equipping it. Then we notice that

$$\check{Q}_\theta(\mu)=\int_{\mathscr{R}_\theta}[\cos(\theta/2)R]\#\,\mu\,d\pi_\theta(R). \tag{BI}$$

Because of the rotational invariance of φ_s, for any $R\in\mathscr{R}_\theta$,

$$(*\varphi_s)\circ(R\#)=(R\#)\circ(*\varphi_s); \tag{BJ}$$

similarly, the scale invariance of φ_s makes that for any $\lambda\in(0,\infty)$, denoting by I_λ the homothety $(v\mapsto\lambda v)$,

$$(*\varphi_s)\circ(I_\lambda\#)=\lambda^s(I_\lambda\#)\circ(*\varphi_s). \tag{BK}$$

The result then follows by applying Formulas (BJ) and (BK) to the integral (BI).

When d is odd unfortunately I have nothing better than a calculation—which by the way also works when d is even. Fix an arbitrary $v>0$; we will prove that $(\check{Q}_\theta\boldsymbol{\delta}_v)*\varphi_s=\check{Q}_\theta(\boldsymbol{\delta}_v*\varphi_s)$, where v also denotes the point $(v,0,\dots,0)\in\mathbb{R}^d$. Since these two functions are obviously invariant by any rotation around v, we will locate a point in $\mathbb{R}^d$ merely by its first coordinate z and its distance ρ to the z axis; we will also denote $Z=\sqrt{z^2+\rho^2}$ its distance to 0. In the following calculations $\mathbb{S}$ denotes the unit sphere in $\mathbb{R}^{d-1}$, equipped with its Lebesgue probability measure σ, and ρ also denotes the point $(\rho,0,\dots,0)\in\mathbb{R}^{d-1}$; points of $\mathbb{S}$ are denoted $y=(y_0,y_1)$ with $y_0\in\mathbb{R}$, $y_1\in\mathbb{R}^{d-2}$. Treating $(\check{Q}_\theta\boldsymbol{\delta}_v)*\varphi_s$ as a function, we find:

$$\begin{aligned}((\check{Q}_\theta\boldsymbol{\delta}_v)*\varphi_s)(z,\rho)&=\int_{\mathbb{S}}\big\{(\cos(\theta/2)v-z)^2+(\sin\theta\,y_0v/2-\rho)^2+(\sin\theta)^2|y_1|^2v^2/4\big\}^{-(d-s)/2}d\sigma(y_0,y_1)\\&=\int_{\mathbb{S}}\big\{Z^2+\cos(\theta/2)^2v^2-2\cos(\theta/2)^2vz-\sin\theta\,v\rho y_0\big\}^{-(d-s)/2}d\sigma(y_0,y_1).\end{aligned} \tag{BL}$$

[‡]. That sphere degenerates into a point if $\theta\in\{0,\pi\}$.

For $\check{Q}_\theta(\boldsymbol{\delta}_v * \varphi_s)$ it is more complicated since that case needs computing a expression of type $\check{Q}_\theta f$, f being a function. Usually that kind of computation raises no difficulty, but here the operator $\check{Q}_\theta$ has some singularity which makes it less tractable: in $\check{Q}_\theta f$, the "mass" (in the measure sense) received by the point $(z,0,\dots,0)$ comes only from a $(d-2)$-dimensional sphere in $\mathbb{R}^d$ —namely, the sphere of points (z,ρ), $\rho \in \mathbb{R}^{d-1}$, with $|\rho| = \tan(\theta/2)z$. That regularity problem can be overcome by an approximation technique, yielding:

$$(\check{Q}_\theta f)(z,0,\dots,0) = \frac{1}{\cos(\theta/2)^d} \int_{\mathbb{S}} f(z,[\tan(\theta/2)z]y)\, d\sigma(y) \tag{BM}$$

—this formula also allows to compute $\check{Q}_\theta f$ at points not located on the z axis by rotational invariance.

So

$$\begin{aligned}
&(\check{Q}_\theta(\boldsymbol{\delta}_v * \varphi_s))(z,\rho) = \\
&\cos(\theta/2)^{-d} \int_{\mathbb{S}} \left\{(z-\tan(\theta/2)\rho y_0 - v)^2 + (\rho + \tan(\theta/2) z y_0)^2 + \tan(\theta/2)^2 Z^2 |y_1|^2\right\}^{-(d-s)/2} d\sigma(y_0,y_1) \\
&\quad = \cos(\theta/2)^{-d} \int_{\mathbb{S}} \left\{(1+\tan(\theta/2)^2)Z^2 - 2v(z-\tan(\theta/2)\rho y_0) + v^2\right\}^{-(d-s)/2} d\sigma(y_0,y_1) \\
&\qquad = \cos(\theta/2)^{-s}((\check{Q}_\theta \boldsymbol{\delta}_v) * \varphi_s)(z,\rho).
\end{aligned} \tag{BN}$$

♠

4.6 Corollary. *Let $Q_\theta := \check{Q}_\theta + \check{Q}_{\pi-\theta} - \check{Q}_0 - \check{Q}_\pi$. Then, for all $f \in \dot{H}^{-s}$,*

$$\langle Q_\theta f, f\rangle \leqslant [(\cos(\theta/2))^r + (\sin(\theta/2))^r - 1]\, \|f\|^2. \tag{BO}$$

♣

Proof. Observe first that $\check{Q}_0$ is the identity and that $\check{Q}_\pi = 0$, so it suffices to prove that the operator norm of $\check{Q}_\theta$ in $\dot{H}^{-s}$ is bounded above by $(\cos(\theta/2))^r$. By isomorphism Formula (N), that is also the norm of $(*\varphi_s)\circ\check{Q}_\theta\circ(*\varphi_s)^{-1}$ in L^2, which operator is $\cos(\theta/2)^s\check{Q}_\theta$ by Lemma 4.4. So we just have to bound the norm of $\check{Q}_\theta$, *regarded as an operator in* L^2, by $\cos(\theta/2)^{-d/2}$. Now we note that one can write

$$\check{Q}_\theta f = I_{\cos(\theta/2)} \# (\tilde{Q}_\theta f), \tag{BP}$$

where $\tilde{Q}_\theta$ is the kernel of the Markov chain on $\mathbb{R}^d$ which sends x uniformly to the $(d-2)$-dimensional sphere of points y such that $|y| = |x|$ and $\widehat{x0y} = \theta/2$. But that Markov chain has the Lebesgue measure on $\mathbb{R}^d$ as reversible equilibrium measure, so $|||\tilde{Q}_\theta|||_{L^2} \leqslant 1$, thus $|||\check{Q}_\theta|||_{L^2} \leqslant \cos(\theta/2)^{-d/2}$, *quod erat demonstrandum.* ♠

Now we are ready to state the main result of this section:

4.7 Theorem. *In a Maxwellian model, calling $\overline{\gamma}$ the common value of all the measures $\overline{\gamma}_u$, the collision kernel Q, when restricted to the probability measures, satisfies hypothesis (X) with*

$$\kappa = \int_0^\pi [1 - \cos(\theta/2)^r - \sin(\theta/2)^r]\, d\overline{\gamma}(\theta). \tag{BQ}$$

♣

Proof. Note that

$$Q(\boldsymbol{\delta}_0, \cdot) = \frac{1}{2}\int_0^\pi Q_\theta \, d\overline{\gamma}(\theta) \tag{BR}$$

and apply all the previous work of this section (Lemmas 4.1, 4.3, 4.4 and 4.6). ♠

4.8 Example. The "Kac" model [§] is the case where the measure $\gamma_{v,w}$ always has total mass 1 and is uniform on the sphere supporting it, i.e. it is the Maxwellian model with

$$d\overline{\gamma}(\theta) = \frac{\Gamma(d-1)}{2^{d-2}\Gamma\big((d-1)/2\big)^2}(\sin\theta)^{d-2}d\theta. \tag{BS}$$

By Theorem 4.7, for this model one has $-\infty < \kappa < 0$ for any $r \in (0,1)$. ♡

4.9 Example. The model of Maxwellian potential corresponds to particles having a repulsive force with a radially symmetric potential decreasing like $\rho^{-(2d-2)}$ as the distance ρ between two particles increases. For that model $d\overline{\gamma} \sim \theta^{-3/2}d\theta$ when $\theta \to 0$ for any d [¶], so by Theorem 4.7 one also has $-\infty < \kappa < 0$ for any $r \in (0,1)$. ♡

5 Initial value

In Formula (AS) given by Theorem 3.6, as we saw, besides the dynamic term there is a term due to the fluctuations of the initial empirical measure. In this section we will control these fluctuations in the case of i.i.d. initial particles.

Let μ be a probability measure on $\mathbb{R}^d$ and let $r \in (0,1)$. We assume that μ has an r-th exponential moment, i.e. that there exists some $a > 0$ such that

$$\int_{\mathbb{R}^d} e^{a|v|^r} d\mu(v) < \infty. \tag{BT}$$

In the sequel we suppose that a fixed.

If v is a random variable in $\mathbb{R}^d$ with law μ, then $(\boldsymbol{\delta}_v - \mu)$ is a random variable in $\dot{H}^{-s}$, whose law will be denoted by $\mathfrak{D}_\mu$: $\mathfrak{D}_\mu$ is a centered probability measure on $\dot{H}^{-s}$. I claim that $\mathfrak{D}_\mu$ has an exponential moment with parameter a, i.e.

$$\int_{\dot{H}^{-s}} e^{a\|\nu\|} d\mathfrak{D}_\mu(\nu) < \infty : \tag{BU}$$

to prove it it suffices to note that

$$\|\boldsymbol{\delta}_v - \mu\| \leqslant \|\boldsymbol{\delta}_v - \boldsymbol{\delta}_0\| + \|\boldsymbol{\delta}_{v_0} - \mu\| = C_r|v - v_0|^r + \|\boldsymbol{\delta}_{v_0} - \mu\|, \tag{BV}$$

whose a-parameter exponential is integrable because of (BT).

So the law $\mathfrak{D}_\mu$ has a finite exponential moment, hence *a fortiori* finite variance. Let us denote it by σ^2:

$$\sigma^2 := \int_{\dot{H}^{-s}} \|\nu\|^2 d\mathfrak{D}_\mu(\nu). \tag{BW}$$

Now we have all the definitions at hand to state the main result of this section:

[§]. Actually this is not exactly the Kac model of [43], but the spirit is the same.

[¶]. Then the measure $\overline{\gamma}$ is not finite, however it remains possible to define both the N-particle and the limit models, cf. Remark 1.1-(iii).

5.1 Theorem. *Let $v_1,\dots,v_N$ be N i.i.d. random variables on $\mathbb{R}^d$ with law μ, and denote $\hat{\mu}^N = N^{-1}\sum_{i=1}^N \boldsymbol{\delta}_{v_i}$ their empirical measure. Then there exists an explicit constant $A(\mu)$, which is easy to bound, such that for all $\lambda \leqslant aN$:*

$$\mathbf{E}[\mathscr{U}(\lambda(\hat{\mu}^N - \hat{\mu}))] \leqslant 2\exp\Big(\frac{\lambda^2\sigma^2}{2N} + \frac{\lambda^3 A(\mu)}{N^2 a^3}\Big). \tag{BX}$$

♣

Before proving Theorem 5.1, let us examine Formula (BX) further: the term in the exponential remains bounded when $N \to \infty$ if λ increases as $N^{1/2}$, like in (AX). Thus, writing $\lambda = yN^{1/2}$ like in (BA):

$$\mathbf{E}[\mathscr{U}(yN^{1/2}(\hat{\mu}^N - \mu))] \leqslant 2\exp\Big(\frac{\sigma^2 y^2}{2} + \frac{A(\mu)y^3}{a^3}N^{-1/2}\Big). \tag{BY}$$

Though we will not use it in the sequel, note the following

5.2 Corollary. *For $S \geqslant \sigma^2$, for all $x \geqslant 0$, for all $N \geqslant N_0 := x^2/a^2S^2$:*

$$\mathbf{P}\big(\|\hat{\mu}^N - \mu\| \geqslant xN^{-1/2}\big) \leqslant \exp\Big(-\frac{x^2}{2S} + \log 2 + A(\mu){N_0}^{1/2}N^{-1/2}\Big). \tag{BZ}$$

♣

5.3 Remark. The condition "$N \geqslant x^2/a^2S^2$" is equivalent to "$x \leqslant aSN^{1/2}$", so the estimate (BZ) is valid up to the large deviations setting. ♡

Proof of Theorem 5.1. The principle of the proof is the same as for Theorem 3.3, except that here time will be discrete. Let $v_1,\dots,v_N$ be N i.i.d. random variables with law μ. Set $\hat{M}_i = \sum_{j=1}^i N^{-1}(\boldsymbol{\delta}_{v_i} - \mu)$; then $(\hat{M}_i)_i$ is a martingale and a Markov chain, and $\hat{M}_N$ has the same law as $\hat{\mu}^N$. So it suffices to prove that for all $0 \leqslant i < N$,

$$\mathbf{E}[\mathscr{U}(\lambda\hat{M}_{i+1})|\hat{M}_i] \leqslant \exp\Big(\frac{\lambda^2\sigma^2}{2N} + \frac{\lambda^3 A(\mu)}{a^3N^2}\Big)\mathscr{U}(\lambda\hat{M}_i). \tag{CA}$$

To get (CA), thanks to Lemma 3.4 it suffices to prove that

$$\int (e^{\lambda N^{-1}\|\nu\|} - N^{-1}\lambda N^{-1}\|\nu\|)\,d\mathfrak{D}_\mu(\nu) \leqslant \exp\Big(\frac{\lambda^2\sigma^2}{2N^2} + \frac{\lambda^3 A(\mu)}{a^3N^3}\Big). \tag{CB}$$

We set

$$A(\mu) = \int (e^{a\|\nu\|} - a\|\nu\| - 1)\,d\mathfrak{D}_\mu(\nu). \tag{CC}$$

The function $e_2(t) = (e^t - 1 - t)/t^2$ is convex on $\mathbb{R}_+$, so

$$\forall t \geqslant 0 \quad \forall \theta \in [0,1] \qquad e^{\theta t} - \theta t - 1 \leqslant \frac{1}{2}(1-\theta)\theta^2 t^2 + \theta^3(e^t - t - 1). \tag{CD}$$

Consequently

$$\int (e^{N^{-1}\lambda\|\nu\|} - N^{-1}\lambda\|\nu\| - 1)\,d\mathfrak{D}_\mu(\nu) \leqslant (1 - \lambda a N^{-1})\frac{\lambda^2\sigma^2}{2N^2} + \frac{\lambda^3 A(\mu)}{a^3N^3}, \tag{CE}$$

whence (CB). ♠

6 Discussion

6.a Examples of synthetic results

Until now in this part of the thesis I have just given separate results, mainly Theorem 3.6, Formulas (AU) and (AW), Theorem 4.7 and Theorem 5.1. Obviously all these results are to be put together to get synthetic results on the convergence of N-particle dynamic models to their mean field limit; yet I did not do it in the previous sections. There are several reasons why I have postponed the presentation of such synthetic results to the last section. The most obvious one is that these results would have been quite unreadable if put in the beginning of the article. More important, the different "bricks" of results given within the core of the paper are open to improvements different for each, some of which may work for some cases but not for others, so that there may be no ideal general result.

Let us however give some examples of formulas got by piling my theorems together—proofs will not be given since they really consist in plain gluing game:

6.1 Theorem. *Let $d \geqslant 2$, $r \in (0,1)$. Let μ_0 be a probability measure on $\mathbb{R}^d$ with finite r-exponential moments for all $r < 1$. Up to translating the origin of $\mathbb{R}^d$ we can suppose that $p := \int_{\mathbb{R}^d} v d\mu_0(v) = 0$; then let $k := \frac{1}{2}\int |v|^2 d\mu_0(v)$. Choose some $k_1 > k$ and define*

$$\kappa := 1 - \frac{\Gamma(d-1)}{2^{d-3}\Gamma((d-1)/2)^2}\int_0^\pi \sin(\theta/2)^r \sin(\theta)^{d-2} d\theta^{[\|]}, \tag{CF}$$

$$l := 2^{1+r}\sqrt{2^{1-r}-1}C_r k_1^{r/2}, \tag{CG}$$

$$\omega := (2^{1-r}-1)2^{1+2r}C_r^2 k_1^r, \tag{CH}$$

$$\sigma^2 := \int_{\mathbb{R}^d} \|\boldsymbol{\delta}_v - \mu_0\|_{H^{-s}}^2 d\mu_0(v). \tag{CI}$$

Let $N \geqslant 2$; let $v_1^0, \dots, v_N^0$ be N i.i.d. random variables with law μ_0 and let $\hat{\mu}_0^N$ be their empirical measure; denote $\hat{K}^N := \frac{1}{2}\sum_{i=1}^N |v_i^0|^2$. Let $\hat{\mu}_t^N$ be the empirical measure at time t of the Markov process with generator (A) for the "Kac" model (BS) and initial condition $(v_1^0, \dots, v_N^0)$. Let $(\mu_t)_{t\geqslant 0}$ be the deterministic evolution (F) for the same model with initial value μ_0.

Then for any $a > 0$, there is a (easily bounded) constant $A(a,\mu)$ such that, for any $T > 0$, as soon as $\lambda \leqslant a e^{-|\kappa|T} N$:

$$\log \mathbf{E}[\mathbf{1}_{\hat{K}^N \leqslant N k_1} \mathscr{U}(\lambda(\hat{\mu}_T^N - \mu_T))] \leqslant \log 2 + \frac{e^{2|\kappa|T}\lambda^2\sigma^2}{2N} + \frac{e^{2|\kappa|T}\lambda^3 A(a,\mu)}{N^2 a^3} + \frac{\lambda^2 \omega T}{N} e_1(2|\kappa|T) e_2(\lambda e^{2|\kappa|T} l N^{r/2-1}). \tag{CJ}$$

♣

6.2 Corollary. *For the same model, for any $y \geqslant 0$:*

$$\overline{\lim_{N\to\infty}} \log \mathbf{E}[\mathbf{1}_{\hat{K}^N \leqslant N k_1} \mathscr{U}(y N^{1/2}(\hat{\mu}_T^N - \mu_T))] \leqslant \log 2 + e^{2|\kappa|T}\frac{\sigma^2 y^2}{2} + e_1(2|\kappa|T)\frac{\omega T y^2}{2}. \tag{CK}$$

♣

[‖]. Warning, κ is negative.

6.3 Corollary. *Still for the same model, for any $x \geqslant 0$:*

$$\overline{\lim_{N\to\infty}}\, \mathbf{P}(\|\hat{\mu}_T^N - \mu_T\| \geqslant xN^{-1/2}) \leqslant 2\exp\Big(\frac{-x^2}{2[e^{2|\kappa|T}\sigma^2 + e_1(2|\kappa|T)\omega T]}\Big). \tag{CL}$$

♣

6.4 Remark. As (CL) is true for any value of k_1, we can make k_1 approach k in it, which allows to replace ω by $\omega_0 := (2^{1-r}-1)2^{1+2r}C_r^2 k^r$. ♡

6.b Optimality

Theorem 6.1 essentially gives convergence to the continuous limit at rate $N^{-1/2}$ with Gaussian control. Qualitatively it is the best result one could hope for, because it is the same way of convergence as for central limit theorems. Quantitatively however, is the parameter in the Gaussian bound optimal?

Here we will look at what happens for Theorem 5.1 (Theorem 3.3 exhibits the same behaviour, but it is harder to see). Through Corollary 5.2, Theorem 5.1 gives some Gaussian bound in an infinite-dimensional frame. Yet its proof, whose main ingredient is the use of the utility function $\mathscr{U}$, would work as well in a finite-dimensional setting. So let us imagine that we replace $\dot{H}^{-s}$ by $\mathbb{R}^d$ and $\mathfrak{D}_\mu$ by the centered normal law with variance $\mathbf{I}_d$, denoted by $\mathscr{N}$; then σ^2 becomes $\int_{\mathbb{R}^d} |x|^2\, d\mathscr{N}(x) = d$. In that case $\hat{\mu}^N$ turns into a random variable X^N on $\mathbb{R}^d$ which is centered normal with variance $N^{-1}\mathbf{I}_d$, and we get

$$\mathbf{P}(|X^N| \geqslant xN^{-1/2}) \leqslant \exp\left(-\frac{x^2}{2d} + \log 2 + A{N_0}^{1/2}N^{-1/2}\right) \tag{CM}$$

for some A and N_0 not depending on N, so making $N \to \infty$:

$$\mathscr{N}(|X| \geqslant x) \leqslant 2e^{-x^2/2d}, \tag{CN}$$

whereas the exact result is

$$\mathscr{N}(|X| \geqslant x) = \frac{2^{1-d/2}}{\Gamma(d/2)}\int_x^\infty y^{d-1}e^{-y^2/2}dy \overset{x\to\infty}{\approx} e^{-x^2/2}, \tag{CO}$$

where "$\approx$" means "having equivalent logarithms".

So, for $d > 1$ the parameter in the Gaussian bound is underestimeted by a factor d. Why that? Well, the proof of Theorem 5.1 uses the bound on the curvature of $\mathscr{U}$ given by Proposition 3.1-(vi). But as soon as x is reasonably large, the Hessian of $\mathscr{U}$ at x is much more curved in one direction than in all the other ones, so that Formula (AH) in Lemma 3.4 becomes strongly suboptimal since the factor $\|y\|^2$ in it should morally be replaced by the sole component of the variance of y in the direction along which $\nabla^2\mathscr{U}(x)$ is most curved.

That points out that the techniques involving $\mathscr{U}$ become poor as soon as the dimension in which the random phenomena occur is large. For our particle models we work in $\dot{H}^{-s}$, whose dimension is… infinite! Does that mean that our results are "infinitely bad"? Obviously not! My explanation for that paradox is the following: in the proof of Theorem 5.1, each increment of the martingale $\hat{M}$ is determined by the value of one v_i, so that the law of this increment may be seen as a probability on $\mathbb{R}^d$; that makes the "effective dimension" of $\dot{H}^{-s}$ in our theorems finite, lying somewhere in (d,∞). More precisely, I tried a heuristic derivation of value of the

"effective dimension" of $\dot{H}^{-s}$, based on studying the curvature of $\mathscr{U}$ along a random direction of $\dot{H}^{-s}$ parametrized by $\mathbb{S}^{d-1}$: my computations suggest that for large d, the effective dimension of $\dot{H}^{-s}$ would be about $4^{r-1}r^{-2}d$, which is actually d for $r = 1$ and increases to infinity as $r \searrow 0$. [*]

As a consequence, we had better not choose r too close to 0. On the other hand, the bigger s is, the more regular the test functions in the definition of $\|\cdot\|_{\dot{H}^{-s}}$ are (see Proposition 2.3), so the less small-scale details $\|\cdot\|_{\dot{H}^{-s}}$ catches [†]. So it should be advised to take medium values of r, e.g. $r = 1/2$.

6.c A numerical computation

One important side of my work is that it gives non-asymptotic results. The motivation behind that is that, to understand Boltzmann's evolution, we will not actually look at $N \to \infty$, but rather take some *fixed* large N and say that the behaviour of the N-particle system for that N is very close to the limit evolution with very large probability. In particular, think about the case of numerical simulation: we cannot afford dealing with 10^{24} particles on our computers!

Here I will compute numerical values for the following case: the collision kernel is the one of "Kac" model for $d = 3$, and we take $\mu_0 = \frac{1}{2}(\boldsymbol{\delta}_{-1} + \boldsymbol{\delta}_1)$. Physically speaking, it means that we crash together two same-sized sets of frozen particles with relative speed 2. Then the collisions between particles of differents sets will tend to scatter the distribution of velocities of the particles, which will morally converge to the law (H) with $k = 1/2$ in a few units of time—this is the behaviour of Boltzmann's equation (F) indeed. The question is, which N will we choose to be almost certain that the evolution of the particle system shall be fairly close to (F)?

Say we take $r = 1/2$ and we want to have $\|\hat{\mu}_T^N - \mu_T\|_{\dot{H}^{-s}}$ greater than $\varepsilon = 10^{-2}$ (this corresponds to the error made when moving a unit Dirac mass by about 1.3×10^{-3}) with probability less than $q = 10^{-1}$ for $T = 3$. As in our case $\hat{K}^N \leqslant Nk$ almost surely, we take $k_1 = k = 1/2$. Then one computes the following numerical values, which all are rounded above:

$$-\kappa \simeq 0.600; \tag{CP}$$

$$l \simeq 0.432; \tag{CQ}$$

$$\omega \simeq 0.0933; \tag{CR}$$

$$\sigma^2 \simeq 0.0398. \tag{CS}$$

A good choice for a for the measure μ_0 we have taken is $a = 11$; then (CC) gives $A(a, \mu) \simeq 5.78$. We have to take $\lambda \gtrsim |\log q|/\varepsilon$, so let us choose $\lambda = 400$. Then for $N = 4 \times 10^5$ we find by (CJ):

$$\log \mathbf{E}[\mathscr{U}(\lambda(\hat{\mu}_t^N - \mu_t))] \leqslant 1.620, \tag{CT}$$

thus

$$\mathbf{P}[\|\hat{\mu}_t^N - \mu_t\| \geqslant 10^{-2}] \leqslant 10^{-1} \tag{CU}$$

by Markov's inequality.

So, with a discrete system of 4×10^5 particles one will much probably find a quite good approximation of the Boltzmann mean field limit by running the particle system over 3 units of time. Simulating 4×10^5 particles is easy for today's computers, which shows that our bounds can actually be useful in practice. However there is little doubt that the true speed of convergence is much faster than what our computations suggest.

[*]. In my article [66], I used a looser heuristics which suggested rather an effective dimension d/r.

[†]. Remember however that homogeneous Sobolev spaces have no inclusion relations between them.

6.5 Remark. Here we have bypassed the problem of the $\mathbf{1}_{\hat{K}^N \leq Nk_1}$ factor in (CJ) by a specific argument. How can we do for it in the general case? Well, merely note that, as soon as one wants to have a result in terms of probability, they will just have to add $\mathbf{P}(\hat{K}^N > Nk_1)$ to the probability they get by forgetting the indicator. But the event $\{\hat{K}^N > Nk_1\}$ is a large deviations event, so as soon as μ_0 has some square-exponential moment its probability will decrease exponentially with N and thus cause actually no problem. ♡

6.d Comparison to older results

The usual method to tackle mean field limit problems relies on the concept of *propagation of chaos* devised by Kac [43]. Briefly speaking, that method consists in studying the (deterministic) evolution of $Law(v_1(t),\dots,v_N(t))$, which is a probability measure on $(\mathbb{R}^d)^N$ (the labelling of the particles will be taken random, so that this law is symmetric by permuting particles). As the states space of this law varies with N, the law is studied through its k-dimensional marginals, which are probability measures on the $(\mathbb{R}^d)^k$. One says that there is "chaos" when, for all k, the k-dimensional marginal tends to product law $u^{\otimes k}$ when $N \to \infty$. The goal is to prove that, if there is chaos at time 0, then this chaos "propagates" for all t.

Propagation of chaos has been proved by Sznitman [77] for spatially homogeneous Boltzmann models, and more recently by Graham and Méléard [36] for the more general *Povzner equation*. Actually, proving propagation of chaos is the same as proving the convergence of empirical distributions of the N-particle system to some deterministic distribution, but propagation of chaos emphasizes the individual behaviour of each particle, which is described by the *nonlinear particle* [78]. On the other hand, the finite-dimensional setting of that method makes that the quantitative results got thanks to it (for instance in [31, 51]) do not translate very well when one tries to control the difference between $\hat{\mu}^N$ and μ in some metric space.

My work was motivated by reading [7], in which Bolley, Guillin and Villani tackle some mean field limit problems in a quantitative way by working with W_1 Wasserstein distances for the empirical measures. They get an explicit control of the large deviations of the difference between the empirical measure of the N-particle system and its mean field limit for positive times. Yet there are two annoying shortcomings in their work:

- First, it seems to be limited to McKean–Vlasov models, that is, systems where the interactions between particles are due to forces rather than collisions. The proofs of [7] indeed fundamentally relie on a coupling technique (popularized by Sznitman [77]) in which one defines a coupling between the real assembly of particles and a virtual assembly of N independent nonlinear particles. Such a technique has little relevance when one deals with collisions, because these events imply *two* particles at the *same* moment each time they occur, so in this case there is no natural way of coupling with independent particles.
- Secondly, the results of [7] are good for large deviations, but the control they give for medium deviations is far too poor to get, as we would wish, some $N^{-1/2}$ convergence rate. As we explain in § 7, that is actually an intrinsic shortcoming of W_1 distances.

After writing my article I discovered some other papers sharing certain features with mine:

- The microscopic model exposed in this work is an example of Bird's direct simulation Monte-Carlo method [6], whose convergence for the Boltzmann equation was proved in [84, 67], with explicit L^1 estimates on the marginals.
- The first having looked at the empirical distribution of the particles in Hilbert spaces to bypass

the coupling problems were Fernandez and Méléard [30], who analysed the fluctuations of the particle distribution when $N \to \infty$ in a spirit close to uniform central limit theory [28].

- Aldéric Joulin pointed out to me that my convergence theorems could be interpreted as results of concentration of measure for a Markov process with positive curvature, according to the geometric notions for Markov chains introduced by Ollivier in [58]. Hypotheses (i), (ii) and (iii) in my Theorem 3.6 indeed correspond resp. to the hypotheses about the discrete Ricci curvature κ, the coarse diffusion constant $\sigma(x)$ and the granularity σ_∞ in Theorem 33 of [58].

6.e Uniform in time bounds

The results I have given work for some *fixed* T, i.e. they control $\|\hat{\mu}_T^N - \mu_T\|$. It may be more natural to control $\sup_{t\in[0,T]} \|\hat{\mu}_t^N - \mu_t\|$, i.e. to say that the system is *always* close to the Boltzmann mean field limit between times 0 and T, as [7] does for McKean–Vlasov models. We do not do it here, but note that, as we used martingale techniques, getting results valid for all $t \in [0,T]$ could be easily achieved from the previous work by using stopping times. Actually for $\kappa \leqslant 0$ it would turn out that uniform in time results are not much different from fixed time results (which is quite logical because then the control on $\|\hat{\mu}_t^N - \mu_t\|$ is worst for $t = T$). For $\kappa > 0$ yet, when T is large the maximum of the difference between $\hat{\mu}_t^N$ and μ_t is much less well controlled than its terminal value, as we already noticed in Footnote [†] on page 62.

7 Appendix: Why Wasserstein distances cannot yield $N^{-1/2}$ convergence

This appendix aims at explaining quickly why, in general situations, the W_1 distance *cannot* yield a $N^{-1/2}$ rate of convergence for the empirical distribution of an assembly of N particles to its continuous limit. As it is not the main matter of this work, I will remain at a heuristic level.

My explanation relies on the transportation interpretation of the W_1 distance. Recall that a *coupling* between two finite measures of same mass μ and ν on respective spaces X and Y is a measure π on $X \times Y$ whose marginals are resp. μ and ν, i.e. s.t. $\pi(A \times Y) = \mu(A)$ for all measurable $A \subset X$, resp. $\pi(X \times B) = \nu(B)$. π is also called a *transportation plan* because it describes a way to transport a mass distributed according to μ into the mass distribution ν. The set of couplings between μ and ν is always non-void; we denote it by $\Pi(\mu,\nu)$. When $X = Y = \mathbb{R}^d$, for $\pi \in \Pi(\mu,\nu)$ we define the *transportation cost*

$$I[\pi] := \int_{(\mathbb{R}^d)^2} |y - x| \, d\pi(x,y), \tag{CV}$$

which represents the total effort you have to put in to transform μ into ν following the plan π. Then the *optimal transportation cost* is merely

$$W_1(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)} c(\pi). \tag{CW}$$

It is a deep result due to Kantorovitch [44] that the definitions (I) and (CW) actually coincide. For more details on all that, see [82, § 1].

Now, consider a N-particle system whose empirical measure $\hat{\mu}^N$ is expected to converge to some density measure μ, say the Lebesgue measure on $[0,1]^d$, and look at the Wasserstein distance $W_1(\mu,\hat{\mu}^N)$. Let π be a coupling between $\hat{\mu}^N$ and μ. Write $\hat{\mu}^N = N^{-1}\sum_{i=1}^N \boldsymbol{\delta}_{v_i}$,

and call $A_i \subset \mathbb{R}^d$ the image of $N^{-1}\boldsymbol{\delta}_{v_i}$ by the transportation plan π. A_i has Lebesgue measure N^{-1}, so its observable diameter, which is minimal when A_i is a ball, is at least $\sim N^{-1/d}$, therefore $\int_{A_i} |v - v_i| d\mu(v) \gtrsim N^{-1}N^{-1/d}$. Thus the total transportation cost between μ and $\hat{\mu}^N$ is $I[\pi] = \sum_i \int_{A_i} |v - v_i|\, d\mu(v) \gtrsim N^{-1/d}$, and since that is true for any transportation plan, in the end $W_1(\mu, \hat{\mu}^N) \gtrsim N^{-1/d}$. And that is always true, *however cleverly you might choose the* v_i —in other words, the phenomenon we describe is not due to *fluctuations* but to *discretization*—, so for $d > 2$ [‡] it is hopeless getting an $N^{-1/2}$ convergence rate of $\hat{\mu}^N$ to μ for the Wasserstein distance.

[‡]. For $d = 1$, the Wasserstein distance *does* typically yield $N^{-1/2}$ fluctuations, but this case is physically trivial when one studies Boltzmann gases. For the critical dimension $d = 2$ (for which the phenomena of discretization and fluctuations have the same order of magnitude $N^{-1/2}$), it turns out that the typical rate of convergence of empirical measures is $N^{-1/2}\log N$, so in fact there is no uniform central limit theorem either in the Wasserstein space.

Troisième partie

Décorrélations hilbertiennes

Résumé

Le *coefficient de corrélation maximale*, ou *hilbertienne*, entre deux variables aléatoires X et Y, noté $\{X:Y\}$, est le supremum des $|\mathrm{Corr}(f(X),g(Y))|$ pour f et g mesurables réelles, où "Corr" désigne le coefficient de corrélation de Pearson. Un résultat classique énonce que, pour des paires indépendantes de variables $(X_i,Y_i)_{i\in I}$, $\{\vec{X}_I:\vec{Y}_I\}$ est le supremum des $\{X_i:Y_i\}$. L'objectif principal de cette partie de la thèse sera d'établir des résultats de tensorisation similaires quand l'indépendance entre les (X_i,Y_i) n'est que partielle ; plus généralement, pour $(X_i)_{i\in I}$ et $(Y_j)_{j\in J}$ des v.a., on cherchera à obtenir une majoration de $\{\vec{X}_I:\vec{Y}_J\}$ à partir de majorations des $\{X_i:Y_j\}$, $i\in I, j\in J$.

Nos théorèmes de tensorisation nous permettront d'établir de nouveaux résultats de décorrélation pour des modèles de physique statistique avec indépendance asymptotique, comme le modèle d'Ising sous-critique. Nous prouverons que pour de tels modèles, deux groupes de spins distants sont décorrélés (au sens hilbertien) *uniformément en leurs tailles et leurs formes* : si I et J sont des ensembles de spins tels que $dist(i,j)\geqslant d$ pour tous $i\in I, j\in J$, alors on a une majoration non triviale de $\{\vec{X}_I:\vec{Y}_J\}$ qui ne dépend que de d.

Nous montrerons également comment on peut utiliser les décorrélations hilbertiennes pour obtenir le théorème-limite central spatial, ainsi que la stricte positivité du trou spectral de la dynamique de Glauber, pour des modèles du type d'Ising sous-critique — tout cela, toujours grâce aux techniques de tensorisation.

Enfin, nous établirons au passage un nouveau critère pour majorer la corrélation maximale $\{\mathscr{F}:\mathscr{G}\}$ entre deux tribus $\mathscr{F}$ et $\mathscr{G}$ à partir d'une borne uniforme sur les $|\mathbf{P}[A\cap B]-\mathbf{P}[A]\mathbf{P}[B]|/\sqrt{\mathbf{P}[A]\mathbf{P}[B]}$ pour $A\in\mathscr{F}, B\in\mathscr{G}$. Des critères de ce type étaient déjà connus, mais le nôtre les améliore strictement et, qui plus est, est optimal.

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Introduction

Overview of this part of the thesis

This part of my thesis is devoted to the study of Hilbertian correlations (also called "maximal correlations" or "ρ-mixing coefficients"), in particular to showing how this concept can be 'tensorized' to yield new results on systems of statistical mechanics exhibiting asymptotic independence. I have divided it into six chapters:

- The first chapter, numbered "0", aims at motivating the study of Hilbertian correlations and their tensorization. In this chapter, I will recall some classical results on the subcritical Ising model, which is a classical model showing asymptotic independence between pairs of spins. When one gets interested in very large 'bunches' of spins, it is known that asymptotic independence cannot be captured by β-mixing any more, but that, in certain cases at least, it still holds in terms of ρ-mixing. The techniques used so far to establish ρ-mixing for bunches of spins are strongly limited by technical assumptions looking somehow artificial, which will motivate studying ρ-mixing 'for itself' and trying to tensorize it.
- In Chapter 1, I shall recall the definition of the Hilbertian correlation coefficient; I shall also recall some classical facts about this concept and give some examples. This chapter can be seen as a 'crash course' on ρ-mixing for the non-specialist reader: almost nothing in it is new.
- In Chapter 2, I shall give some new criteria to bound the Hilbertian correlation between two σ-algebras, which criteria assume bounds on the $(\mathbf{P}[A \cap B] - \mathbf{P}[A]\mathbf{P}[B])$ for events A and B belonging to these respective σ-algebras. My "strong event sufficient condition", which improves previous results by several authors, shall even be shown to be optimal.
- Chapter 3 is the core of this work: in it I will handle tensorization of Hilbertian decorrelations. This chapter begins with a refined version of the concept of correlation, called "subjective correlation", which is necessary to write the subsequent tensorization results. Then I shall state and prove my three main tensorization theorems: Theorem 3.2.2 ('N against 1' theorem) bounds the correlation between a 'simple' and a 'vector' variable; Theorem 3.3.1 ('N against M' theorem) deals with correlation between two vector variables, and Theorem 3.3.10 ('$\mathbb{Z}$ against $\mathbb{Z}$' theorem) refines the previous one in the case where certain symmetries are present. Then, I will discuss some refinement and optimality statements about these theorems; in § 3.9, I will also present a geometric corollary of tensorization results which underlines quite well the Hilbertian aspect of maximal correlations.
- In Chapter 4, I will continue to use the tensorization techniques of Chapter 3, but that time instead of proving tensorization results *stricto sensu* I will turn to different types of results, namely the spatial central limit theorem and the presence of spectral gap for the Glauber dynamics.
- Finally, Chapter 5 will present some concrete applications of the results of this monograph.

For instance, I shall prove new results about decorrelation between distant bunches of spins in Ising's model (see Theorem 5.1.1); I will also give results of the same type for quite general models of statistical mechanics (see e.g. Theorems 5.2.10 and 5.3.7), also proving spatial CLT and spectral gap for the Glauber dynamics for these models. I will also show how tensorization of Hilbertian correlations can be used to get 'hypocoercivity' results [Theorem 5.4.6].

Conventions and notation

Notation will not always be perfectly rigorous: to make reading easier, it may occur sometimes that formalism is slightly loose, or that some writing conventions or assumptions are implicit. However this shall only be done in situations where adding the missing information by the reader is (hopefully) obvious.

Here is some notation used throughout this text:

Miscellaneous

- The symbol $\mathbb{N}$ denotes the set of nonnegative integers, including 0. The set of positive integers $\mathbb{N} \smallsetminus \{0\}$ is denoted by $\mathbb{N}^*$.
- For a, b real numbers, $a \wedge b$ denotes $\min\{a, b\}$, resp. $a \vee b$ denotes $\max\{a, b\}$; a_+ denotes the positive part of a, i.e. $a \vee 0$.
- For A a set, A^{c} denotes the complement set of A (the set of reference shall always be clear); $\mathbf{1}_A$ denotes the indicator function of A, that is, the function being 1 on A and 0 on A^{c}.
- For A, B sets, $A \triangle B$ denotes the symmetric difference of A and B, i.e. $(A \smallsetminus B) \uplus (B \smallsetminus A)$, where "$\uplus$" means the same as "$\cup$", but with underlining that the union is disjoint.
- The identity matrix in dimension n will be denoted by $\mathbf{I}_n$. The transpose of a matrix A will be denoted by A^{T}.
- If Θ is a set endowed with a metric $dist$, then for $I, J \subset \Theta$, $dist(I, J)$ denotes the distance between I and J, that is, $dist(I, J) := \inf\{dist(i, j) : \ i \in I, j \in J\}$.
- As is customary in physical literature, $\propto$ means "proportional to".
- Whenever I is a set and X a symbol, $\vec{X}_I$ will be a shorthand for "$(X_i)_{i \in I}$".

Probability

- We will always work on an implicit probability space $(\Omega, \mathscr{B})$ equipped with a probability measure $\mathbf{P}$. Sub-σ-algebras of $\mathscr{B}$ will be merely called "σ-algebras"; I will also often write "variable" for "random variable". Unless explicitly specified, variables on Ω can be valued in any set.
- If f is a real random variable, the expectation of f is denoted by $\mathbf{E}[f]$; its variance is denoted by $\mathrm{Var}(f)$; its standard deviation is denoted by $\mathrm{Sd}(f) := \sqrt{\mathrm{Var}(f)}$; if g is another real variable, the covariance between f and g is denoted by $\mathrm{Cov}(f, g) := \mathbf{E}[fg] - \mathbf{E}[f]\mathbf{E}[g]$. All that notation extends to the case where f and g are valued in some vector space $\mathbb{R}^N$, except that in that case it refers to vectors or matrices.
- If B is an event with $\mathbf{P}[B] > 0$, then $\mathbf{P}[A|B]$, $\mathbf{E}[f|B]$, $\mathrm{Var}(f|B)$, ... stand resp. for the probability of A, the expectation of f, the variance of f, ... under the conditional law $d\mathbf{P}[\cdot|B] := \mathbf{1}_B \, d\mathbf{P}[\cdot]/\mathbf{P}[B]$. Similarly, if $\mathscr{F}$ is a σ-algebra, $\mathbf{P}[A|\mathscr{F}]$, $\mathbf{E}[f|\mathscr{F}]$, ... stand for the conditional probability of A, the conditional expectation of f, ... w.r.t. $\mathscr{F}$.

- Concerning conditional expectations, I will actually use two different conventions: for $\mathscr{G}$ a σ-algebra, $\mathbf{E}[f|\mathscr{G}]$ can also be denoted by $f^{\mathscr{G}}$. Both conventions can be used inside the same formula. [§] [¶]
- If X is a variable on Ω, the σ-algebra generated by X (that is, the smallest σ-algebra w.r.t. which X is measurable) is denoted by $\sigma(X)$. If $\mathscr{F}$ and $\mathscr{G}$ are σ-algebras, the σ-algebra generated by $\mathscr{F}$ and $\mathscr{G}$ (that is, the smallest σ-algebra containing both $\mathscr{F}$ and $\mathscr{G}$) is denoted by $\mathscr{F} \vee \mathscr{G}$, and for an arbitrary number of σ-algebras this notation extends into the ∞-ary operator $\bigvee$.
- An event $A \in \mathscr{B}$ is said to have trivial probability, or to be trivial, if $\mathbf{P}[A] \in \{0,1\}$. A σ-algebra is said to be trivial if all its events are trivial. The σ-algebra $\{\emptyset, \Omega\}$, which is trivial under any law $\mathbf{P}$, will be denoted by $\mathscr{O}$ and refered to as "the" trivial sigma-algebra.
- The Lebesgue measure on $\mathbb{R}^n$ will be denoted by dx, "x" being the name of the integration variable. For a Borel set $A \subset \mathbb{R}^n$, $\int_{x \in A} dx$ will sometimes be denoted by $|A|$.
- For C a positive-semidefinite matrix (possibly of dimension 1, in which case it is identified with $\sigma^2 \in \mathbb{R}_+$), $\mathscr{N}(C)$ denotes the law of the centered Gaussian vector with covariance matrix C. I will write $\mathscr{N}(C) + m$ to denote the non-centered Gaussian vector with variance C and mean m.

Functional analysis

- Unless otherwise specified, all the functional spaces considered in this monograph shall be real.
- For I an open interval of $\mathbb{R}$ and $k \in \mathbb{N} \cup \{\infty\}$, $\mathscr{C}_0^k(I)$ denotes the subset of functions of $\mathscr{C}^k(I)$ with compact support.
- If μ is a nonnegative measure on some measurable space $(\Omega, \mathscr{B})$, $L^2(\mu)$ denotes the set of measurable functions f (up to μ-a.e. equality) such that $\int_\Omega f(\omega)^2 \, d\mu(\omega) < \infty$. If I is a countable set, $L^2(I)$ denotes the set of functions $f : I \to \mathbb{R}$ such that $\sum_{i \in I} f(i)^2 < \infty$. If $\mathscr{F}$ is a σ-algebra, $L^2(\mathscr{F})$ denotes the space of $\mathscr{F}$-measurable functions (up to a.s. equality) which are square-integrable w.r.t. $\mathbf{P}$. All these spaces are equipped with their natural Hilbertian product $\langle \cdot, \cdot \rangle$ and the associated norm $\|\cdot\|$.
- For μ a finite measure, in $L^2(\mu)$ the constant functions make a line which can be identified with $\mathbb{R}$; then, $\bar{L}^2(\mu)$ will denote the quotient $L^2(\mu)/\mathbb{R}$, equipped with its natural Hilbert structure. In other words, if $\bar{f} \in \bar{L}^2(\mu)$ is the projection of $f \in L^2(\mu)$, $\|\bar{f}\|_{\bar{L}^2} := \inf\{\|f - a\|_{L^2} : a \in \mathbb{R}\} = (\|f\|_{L^2}^2 - \langle f, 1/\|1\| \rangle_{L^2}^2)^{1/2}$. $\bar{L}^2(\mu)$ can also be seen as the subspace of centered functions of $L^2(\mu)$, i.e. as $\{f \in L^2(\mu) : \ \langle f, 1 \rangle = 0\}$; throughout the monograph we will implicitly switch between both interpretations.
- If $L : \ H_1 \to H_2$ is a linear operator between two Hilbert spaces, then $L^* : \ H_2 \to H_1$ denotes the adjoint operator of L, characterized by the relationship $\langle L^* y, x \rangle_{H_1} = \langle y, Lx \rangle_{H_2}$.
- If $L : \ E \to F$ is a linear operator between two Banach spaces (not necessarily Hilbert) with respective norms $\|\cdot\|_E$ and $\|\cdot\|_F$, the operator norm of f, denoted by $|||f|||$, is defined as $\sup\{\|Lx\|_F : \ \|x\|_E = 1\}$.
- If $L : \ E \to E$ is a bounded linear operator on a Banach space, then $\rho(L)$ denotes the spectral radius of f, that is, $\rho(L) := \lim_{k \to \infty} |||L^k|||^{1/k}$ —this limit always exists.

[§]. The use of the first or the second convention will depend on the way we prefer to see the conditional expectation of f w.r.t. $\mathscr{G}$: if it is rather seen as the expectation of f knowing the information of $\mathscr{G}$, notation $\mathbf{E}[f|\mathscr{G}]$ will be chosen, while if it is more seen like the $\mathscr{G}$-measurable function best approximating f, we will use the notation $f^{\mathscr{G}}$.

[¶]. One must not confuse $\mathrm{Var}(f|\mathscr{G})$, which is the variance of f under the law $\mathbf{P}[\cdot|\mathscr{G}]$, with $\mathrm{Var}(f^{\mathscr{G}})$ which is the (unconditioned) variance of the random variable $f^{\mathscr{G}}$. One has the well-known identity $\mathrm{Var}(f) = \mathrm{Var}(f^{\mathscr{G}}) + \mathbf{E}[\mathrm{Var}(f|\mathscr{G})]$, which I shall refer to as *associativity of variance*.

- A column vector $(a_i)_{i\in I}$ will automatically be identified with the corresponding element of $L^2(I)$. Likewise, a matrix $A = ((a_{ij}))_{(i,j)\in I\times J}$ will be identified with the corresponding linear operator from $L^2(J)$ to $L^2(I)$.
- In our computations we will often use the Cauchy - Schwarz inequality and its variants [‖]; when using such an inequality, we will indicate it by writing "CS" under the inequality sign concerned. Similarly, "IP" under an equality sign will mean that this equality follows from integrating by parts.

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This work owes to many people's help. First of all I must mention V. Beffara, who launched me on this topic incidentally. His relevant comments, as well of those of (among others) Y. Ollivier, C. Villani, Y. Velenik, T. Bodineau, C. Shalizi and R. Bradley, have also been the origin for several important improvements of this monograph.

Several colleagues provided me with some help on mathematical topics I was not familiar with. In particular, Y. Ollivier suggested to me the use of Lipschitz spaces to prove Lemma 2.2.13; S. Martineau pointed out how Gershgorin's Lemma solved a technical point in the complete proof of Theorem 3.5.2; V. Calvez had the idea of using Laplace transform to prove Lemma 5.5.5. Over the Internet, F. Martinelli and S. Shlosman also gave me precious bibliographic references on the state of the art about weak and strong mixing in statistical mechanics.

Most of the drawings in this monograph were made thanks to the excellent LaTeX extension Ti*k*Z, combined with computations in C language. The dice of Figure 10 have been kindly drawn for me by A. Alvarez, using POV-Ray.

[‖]. For instance, the discrete form $|\sum_{i=1}^N a_i b_i| \leqslant (\sum_{i=1}^N a_i^2)^{1/2}(\sum_{i=1}^N b_i^2)^{1/2}$, the probabilistic form $|\mathrm{Cov}[fg]| \leqslant \mathrm{Sd}(f)\mathrm{Sd}(g)$, etc..

Chapter 0

Motivation

0.1 Some results on Ising's model

In this subsection we recall the definition of Ising's model and give two classical results on it, namely Theorems 0.1.8 and 0.1.9. In § 0.2, considerations on these results will serve as a motivation to the work of this part of the thesis.

0.1.a Ising's model

Ising's celebrated model is a basic model of equilibrium thermodynamics, which represents a ferromagnetic material:

0.1.1 Definition. For n an integer, consider the lattice $\mathbb{Z}^n$ endowed with its usual graph structure (each vertex has $2n$ neighbours), and denote by $dist$ the graph distance. Define $\Omega = \{\pm 1\}^{\mathbb{Z}^n}$, and for $\vec{\omega} \in \Omega$, set formally:

$$H(\vec{\omega}) = -\frac{1}{2} \sum_{dist(i,j)=1} \omega_i \omega_j. \tag{A}$$

Then, for $T \geqslant 0$, the *Ising model on $\mathbb{Z}^n$ at temperature T* is, formally, a probability measure $\mathbf{P}$ on Ω such that $\mathbf{P}[\vec{\omega}] \propto \exp\big(-T^{-1}H(\vec{\omega})\big)$. In rigorous terms, saying that $\mathbf{P}$ is an equilibrium measure for Ising's model means that for all $i \in \mathbb{Z}^n$, for all $\vec{\hat{\omega}}_{\{i\}^c} \in \{\pm 1\}^{\{i\}^c}$,

$$\mathbf{P}[\omega_i = \hat{\omega}_i | \vec{\omega}_{\{i\}^c} = \vec{\hat{\omega}}_{\{i\}^c}] \propto \exp\Big(T^{-1} \sum_{dist(i,j)=1} \hat{\omega}_i \hat{\omega}_j\Big). \tag{B}$$

$\diamondsuit$

Ising's model and the phase transition it exhibits have been the subject of dozens of works; see [37] for an overview. Here we are interested in the subcritical regime:

0.1.2 Theorem (Subcritical regime, [61])**.** *There is a $T_c < \infty$ (the 'Curie temperature') such that the solution of (B) is unique for $T > T_c$.* ♣

For $T > T_c$ one says that they are in the *subcritical regime*. An interesting feature of this regime is that for distant i and j, the random variables ω_i and ω_j are 'almost independent'. That phenomenon, called *exponential decay of correlations*, is stated by the following theorem:

0.1.3 Theorem (Exponential decay of correlations, [2]). *For Ising's model on $\mathbb{Z}^n$ in the subcritical regime,*

(i) For all $i \in \mathbb{Z}^n$, $\mathbf{P}[\omega_i = -1] = \mathbf{P}[\omega_i = 1] = 1/2$.
(ii) There exists $\psi > 0$ and $C < \infty$ such that for all $i, j \in \mathbb{Z}^n$,

$$|\mathbf{E}[\omega_i \omega_j]| \leqslant C \exp\big(-\psi dist(i,j)\big). \tag{C}$$

♣

0.1.b Absence of β-mixing

Theorem 0.1.3 states that two distant spins i and j are exponentially decorrelated. However, it does not inform us about the dependence between 'bunches' of spins. The question is the following: if I and J are two disjoint, distant subsets of $\mathbb{Z}^n$, to what extent are $\vec{\omega}_I$ and $\vec{\omega}_J$ independent?

To answer such a question, the first thing to do is to define a way of measuring independence between 'complicated' variables having an arbitrarily large range like $\vec{\omega}_I$ and $\vec{\omega}_J$. The most common choice is the β-mixing coefficient:

0.1.4 Definition.

1. Recall that for μ, ν two probability measures on the same measurable space $(\Omega, \mathscr{F})$, the *total variation distance* between μ and ν is the total mass of both the positive and the negative parts of the signed measure $\nu - \mu$, that is, $dist_{\mathrm{TV}}(\mu, \nu) = \sup_{A \subset \mathscr{F}} |\nu(A) - \mu(A)|$.
2. If X and Y are two random variables (with arbitrary ranges) defined on the same space, then one defines the *β-mixing coefficient* between X and Y as

$$\beta(X,Y) := dist_{\mathrm{TV}}(Law_X \otimes Law_Y, Law_{(X,Y)}). \tag{D}$$

◇

Notice that $\beta(X,Y)$ actually only depends on the σ-algebras $\sigma(X)$ and $\sigma(Y)$ [11, Formula (1.5)]. The following proposition is immediate:

0.1.5 Proposition.

(i) One has always $\beta(X,Y) \in [0,1]$, and
(ii) $\beta(X,Y) = 0$ if and only if X and Y are independent ;
(iii) $\beta(X,Y) = 1$ if and only if $Law_X \otimes Law_Y$ and $Law_{(X,Y)}$ are mutually disjoint.
(iv) If X' is X-measurable and Y' is Y-measurable, then $\beta(X',Y') \leqslant \beta(X,Y)$.

♣

So, $\beta(X,Y)$ is a way of measuring 'how much X and Y are correlated'.

With that tool at hand, decorrelation between bunches of spins in statistical physics models has already been thoroughly studied. Concerning Ising's model, there are two well-known great results:

0.1.6 Theorem (Weak mixing property, [52]). *For Ising's model on $\mathbb{Z}^n$ in the subcritical regime, there exists $\psi > 0$ and $C < \infty$ (the same as in Theorem 0.1.3) such that for all disjoint $I, J \subset \mathbb{Z}^n$:*

$$\beta(\vec{\omega}_I, \vec{\omega}_J) \leqslant C \sum_{(i,j) \in I \times J} \exp\big(-\psi dist(i,j)\big). \tag{E}$$

♣

0.1.7 Theorem (Complete analyticity, [25]). *There exists some $T_c \leqslant T'_c < \infty$ [*] such that for $T > T'_c$, Ising's model is* completely analytical, *i.e. there exists $\psi' > 0$ and $C' > 0$ such that the following holds: for all $K \subset \mathbb{Z}^n$, for all 'boundary' conditions $\vec{\hat{\omega}}_K \in \{\pm 1\}^K$, denoting $\mathbf{P}_{\vec{\hat{\omega}}_K} := \mathbf{P}[\cdot|\vec{\omega}_K = \vec{\hat{\omega}}_K]$, Formula (E) holds with Law replaced by $Law_{\vec{\hat{\omega}}_K}$ and ψ, C replaced resp. by ψ' and C'.* ♣

Thanks to Theorem 0.1.6, we get an exponential decay of correlation between two bunches of spins of fixed size when the distance between these bunches increases. However, we cannot say much about decorrelation between bunches of *variable* size which are at fixed distance from each other. For instance, for $n = 2$ fix $x > 0$ and define $I_l := \{(0,y) : \ |y| \leqslant l\}$, resp. $J_l := \{(x,y) : \ |y| \leqslant l\}$. Then Theorem 0.1.6 gives us something like:

$$\beta(\vec{\omega}_{I_l}, \vec{\omega}_{J_l}) \lesssim C l e^{-\psi x}. \tag{F}$$

But recall that a β-mixing coefficient is always bounded by 1; so, for $l \gtrsim e^{\psi x}/C$, (F) tell us absolutely nothing about the decorrelation between I_l and J_l.

Though the bound (E) is not completely optimal, the previous point is an *intrinsic* shortcoming of β-mixing coefficients, in the sense that it can be proved that bounds like (F) *must* become trivial when $l \to \infty$:

0.1.8 Theorem. *For all $T_c < T < \infty$, for all $x > 0$, denoting $I := \{0\} \times \mathbb{Z}$ and $J := \{x\} \times \mathbb{Z}$, one has*

$$\beta(\vec{\omega}_I, \vec{\omega}_J) = 1. \tag{G}$$

♣

Proof. Denote $i_0 := (0,0)$, resp. $j_0 := (x,0)$. As we told in Theorem 0.1.3-(i), $\mathbf{E}[\omega_{i_0}], \mathbf{E}[\omega_{j_0}] = 0$. Interpretation of Ising's model as a random-cluster model [37, § 1.4] shows that $\mathbf{P}[\omega_{i_0} = \omega_{j_0}] > 1/2$, so we define

$$\gamma := \mathbf{E}[\omega_{i_0}\omega_{j_0}] > 0. \tag{H}$$

Now, let N be some large integer, fixed for the time being, let p be some large integer and define $i_1, \ldots, i_N$, resp. $j_1, \ldots, j_N$, by $i_k := (0, kp)$, resp. $j_k := (x, kp)$; denote by $P_{N,p}$ the joint law of $(\omega_{i_1}, \ldots, \omega_{i_N}, \omega_{j_1}, \ldots, \omega_{j_N})$. By translation invariance, for each k, $(\omega_{i_k}, \omega_{j_k})$ has the same law as $(\omega_{i_0}, \omega_{j_0})$, which law we denote by P_1. Then when $p \to \infty$, by Theorem 0.1.6, $P_{N,p}$ tends to the law $P_{N,\infty} := P_1^{\otimes N}$. [†] In other words, $P_{N,\infty}$ is the law such that all the $(\omega_{i_k}, \omega_{j_k})$ are independent with $P_{N,\infty}[\omega_{i_k} = \eta \text{ and } \omega_{j_k} = \theta] = (1 + \gamma\eta\theta)/4$ for all k. Therefore, the value of $\beta((\omega_{i_k})_k, (\omega_{j_k})_k)$ under the law $P_{N,p}$, which by Proposition 0.1.5-(iv) is a lower bound for $\beta(\vec{\omega}_I, \vec{\omega}_J)$, tends to its value under $P_{N,\infty}$ when $p \to \infty$. This is summed up by the following formula:

$$\beta(\vec{\omega}_I, \vec{\omega}_J) \geqslant \beta_{P_{N,\infty}}((\omega_{i_k})_{1 \leqslant k \leqslant N}, (\omega_{j_k})_{1 \leqslant k \leqslant N}). \tag{I}$$

To end the proof, we will bound the right-hand side of (I) below by a quantity which tends to 1 when $N \to \infty$. Denote by $\tilde{P}_{N,\infty}$ the product of two the marginals of $P_{N,\infty}$ relative resp. to the $(\omega_{i_k})_k$ and the $(\omega_{j_k})_k$, so that $\beta_{P_{N,\infty}}((\omega_{i_k})_k, (\omega_{j_k})_k) = dist_{\mathrm{TV}}(P_{N,\infty}, \tilde{P}_{N,\infty})$ by the very definition of the β-mixing coefficient. Obviously the expression of $\tilde{P}_{N,\infty}$ is the same as the expression of $\tilde{P}_{N,\infty}$, but

[*]. It is not known whether $T'_c = T_c$ today, but in general situations weak mixing does not always imply complete analyticity. A classical counterexample is Ising's model with external field [53, § 2].

[†]. Note that $P_{N,p}$ takes its values in a space of finite dimension, so there is no ambiguity when speaking of its convergence.

with γ replaced by 0 in the definition of $P_{N,\infty}$. Under $P_{N,\infty}$, $(\omega_{i_k}\omega_{j_k})_{1\leqslant k\leqslant N}$ is a sequence of i.i.d. random variables having a certain law with mean $\gamma > \gamma/2$, so that by the law of large numbers,

$$P_{N,\infty}\Big[N^{-1}\sum_{k=1}^{N}\omega_{i_k}\omega_{j_k} \leqslant \frac{\gamma}{2}\Big] \overset{N\to\infty}{\to} 0. \tag{J}$$

Similarly, since $0 < \gamma/2$,

$$P_{N,\infty}\Big[N^{-1}\sum_{k=1}^{N}\omega_{i_k}\omega_{j_k} \leqslant \frac{\gamma}{2}\Big] \overset{N\to\infty}{\to} 1, \tag{K}$$

so that

$$dist_{\mathrm{TV}}(P_{N,\infty},\tilde{P}_{N,\infty}) \geqslant \Big|\tilde{P}_{N,\infty}\Big[N^{-1}\sum_{k=1}^{N}\omega_{i_k}\omega_{j_k} \leqslant \frac{\gamma}{2}\Big] - P_{N,\infty}[\textit{the same}]\Big| \overset{N\to\infty}{\to} 1, \tag{L}$$

which proves our point. ♠

0.1.c Presence of ρ-mixing

So, Theorem 0.1.8 tells us that, for Ising's model on $\mathbb{Z}^2$, there is a 'full' correlation between $\vec{\omega}_I$ and $\vec{\omega}_J$ in the sense of β-mixing. Yet it is well known too that Theorem 0.1.6 nevertheless implies a Hilbertian form of decorrelation (called "ρ-mixing", cf. Remark 1.1.2) between these variables:

0.1.9 Theorem. *For Ising's model on $\mathbb{Z}^2$ in the subcritical regime, defining as before $I = \{0\}\times\mathbb{Z}$ and $J = \{x\}\times\mathbb{Z}$ for some $x > 0$, one has for all $f \in \bar{L}^2(\vec{\omega}_I)$ and $g \in \bar{L}^2(\vec{\omega}_J)$:*

$$|\mathbf{E}[fg]| \leqslant e^{-\psi x}\,\mathrm{Sd}(f)\,\mathrm{Sd}(g), \tag{M}$$

where ψ is the same as in Theorem 0.1.6. ♣

Proof. Define the operator

$$\begin{array}{rccc} P: & \bar{L}^2(\vec{\omega}_I) & \to & \bar{L}^2(\vec{\omega}_J) \\ & f & \mapsto & f^{\sigma(\vec{\omega}_J)}. \end{array} \tag{N}$$

(Recall that $f^{\sigma(\vec{\omega}_J)}$ is an alternative notation for $\mathbf{E}[f|\vec{\omega}_J]$, insisting on the its being a $\sigma(\vec{\omega}_J)$-measurable function). Then (M) is equivalent to proving that $|||P||| \leqslant e^{-\psi x}$ (see § 1.1.c). Now for all $t \in \{0,\dots,x\}$, denote $\omega_{(t)} := \vec{\omega}_{\{t\}\times\mathbb{Z}}$, and for all $t \in \{1,\dots,x\}$,

$$\begin{array}{rccc} \pi_t: & \bar{L}^2(\omega_{(t-1)}) & \to & \bar{L}^2(\omega_{(t)}) \\ & f & \mapsto & f^{\sigma(\omega_{(t)})}. \end{array} \tag{O}$$

Due to the fact that the interactions in Ising's model have only range 1, $\omega_{(0)} \to \omega_{(1)} \to \cdots \to \omega_{(x)}$ is a Markov chain, and therefore

$$P = \pi_x \circ \cdots \circ \pi_2 \circ \pi_1. \tag{P}$$

Now, by horizontal translation all the $\bar{L}^2(\omega_{(t)})$ can be identified with a common Hilbert space H. Then all the π_t are identified with operators on $\bar{L}^2(H)$, and by the translation invariance of the model all these operators are actually the same. P is also identified with an operator on $\bar{L}^2(H)$, and (P) becomes:

$$P = \pi^x. \tag{Q}$$

But π is self-adjoint because, as the model is invariant by translation *and by reflection*, the Markov chain $\omega_{(0)} \to \cdots \to \omega_{(x)}$ is stationary *and reversible*. In particular π is a normal operator, and thus

$|||P||| = |||\pi|||^x$. So, proving that $|||P||| \leqslant e^{-\psi x}$ is equivalent to proving that $|||\pi||| \leqslant e^{-\psi}$, which will be our new goal.

Take $C < \infty$ like in Theorem 0.1.6. For l an integer, denote I_l and J_l to be resp. $\{0\} \times \{-l, \dots, l\}$ and $\{x\} \times \{-l, \dots, l\}$. Let f be a bounded [‡] function of $\overline{L}^2(\vec{\omega}_{I_l})$ and denote $M := \|f\|_{L^\infty}$. By translation, f can also be identified with a function of $\overline{L}^2(\vec{\omega}_{J_l})$, which is also bounded by M. Now, since

$$\mathbf{E}[f(\vec{\omega}_{I_l})f(\vec{\omega}_{J_l})] = \mathrm{Cov}(f(\vec{\omega}_{I_l}), f(\vec{\omega}_{J_l})) = \int f(\vec{\omega}_{I_l})f(\vec{\omega}_{J_l})\, d(Law(\vec{\omega}_{I_l \uplus J_l}) - Law(\vec{\omega}_{I_l}) \otimes Law(\vec{\omega}_{J_l})), \tag{R}$$

we can apply (E) to I_l and J_l to obtain:

$$|\mathbf{E}[f(\vec{\omega}_{I_l})f(\vec{\omega}_{J_l})]| \leqslant M^2 \cdot 2C(2l+1)^2 e^{-\psi x}. \tag{S}$$

In terms of operators, (S) means that

$$|\langle f, Pf\rangle_{\overline{L}^2(H)}| \leqslant 2(2l+1)^2 M^2 C e^{-\psi x}. \tag{T}$$

As the value of x played no particular role to establish (T), that formula can be generalized into

$$|\langle f, \pi^t f\rangle_{\overline{L}^2(H)}| \leqslant 2(2l+1)^2 M^2 C e^{-\psi t} \tag{U}$$

for all $t \in \mathbb{N}^*$. Letting t tend to infinity, we obtain that for all l, for all $f \in \overline{L}^2(\vec{\omega}_{I_l}) \cap L^\infty$,

$$\overline{\lim_{t\to\infty}} \left(\log|\langle f, \pi^t f\rangle|\right)^{1/t} \leqslant e^{-\psi}. \tag{V}$$

But $\bigcup_{l\in\mathbb{N}}(\overline{L}^2(\vec{\omega}_{I_l}) \cap L^\infty)$ is a dense subset of $\overline{L}^2(\vec{\omega}_I)$, so by Lemma 0.3.1 set in appendix, we conclude that $|||\pi|||_{\overline{L}^2(H)} \leqslant e^{-\psi}$, which is what we wanted. ♠

0.1.10 Remark. Claim 0.1.8 and Theorem 0.1.9 adapt straighforwardly, with similar proofs, to any $n \geqslant 2$, replacing I by $\{0\} \times \mathbb{Z}^{n-1}$ and J by $\{x\} \times \mathbb{Z}^{n-1}$. ♡

0.2 Problematics

Thanks to Theorem 0.1.9, we see that the Hilbertian concept of ρ-mixing can reveal some independence between infinite bunches of lowly correlated variables in situations where the β-mixing coefficient does not show any independence at all. In the proof we gave, ρ-mixing appeared as a corollary of β-mixing for finite bunches of spins. What additional hypotheses did we need to get our corollary? We used at least the following:

- To introduce the Markov chain $\omega_{(0)} \to \cdots \to \omega_{(x)}$, we used that the interactions of our model had finite range.
- To identify all the spaces $\overline{L}^2(\omega_{(t)})$, we used that I and J had the same shape and that one could tile up $\mathbb{Z}^2$ with a sequence of tiles having that shape (namely, here, tiles of the form $\{t\} \times \mathbb{Z}$).
- To say that all the π_t were the same modulo that identification, we used the translation invariance of the model.
- To state that the stationary Markov chain $\omega_{(0)} \to \cdots \to \omega_{(x)}$ was reversible, we used the reflection invariance of the model.

[‡]. In fact here it is superfluous to impose that f is bounded since $\vec{\omega}_{I_l}$ can only take a finite number of values. I wrote the proof like this just to underline that the finiteness of the range of the ω_i does not play any role in the proof.

- To use Lemma 0.3.1, we used the exponential decay of correlations.

All these points make the proof of Theorem 0.1.9 we gave in § 0.1 quite difficult to generalize. What, for instance, if we take I and J with arbitrary shapes, just requiring that $dist(I,J) \geqslant x$? What if we consider statistical physics models with infinite-range interactions? Etc.. The above arguments would not work any more! Yet, we do not have the impression that the presence of ρ-mixing fundamentally relies on the peculiar symmetries of the case we treated...

So, here will be the goal of this work: *establishing ρ-mixing estimates by general methods*. To achieve this goal, I shall try to concentrate on the properties of ρ-mixing 'for itself', rather than on its links with other forms of decorrelation. I will carry out a thorough study of the ρ-mixing coefficient, in order to get ρ-mixing results for 'complicated' variables from decorrelation results *of the same type* for more 'basic' variables; in other words, I will *tensorize* Hilbertian decorrelations. It turns out that tensorization for such correlation coefficients gives results which are quite robust as the size of the bunches of variables increases. Thanks to this method, I shall obtain fairly new decorrelation theorems for various models of statistical physics.

This part of the thesis is intended to be complete in some sense. I mean, besides the core of this work—namely, tensorization results—, I have tried to answer several other questions which appeared natural to me concerning Hilbertian decorrelation. This includes studying many examples, finding sharp criteria for maximal decorrelation, looking at the optimality issues in the tensorization results or showing other applications of the tensorization techniques. Though these topics were initially thought as 'sidework', some of them may be quite interesting for themselves.

0.3 Appendix: On the norm of self-adjoint operators

In this appendix we prove the following

0.3.1 Lemma. *Let L be a self-adjoint operator on a real Hilbert space H, and let $C < \infty$. Then, to prove that $|||L||| \leqslant C$, it suffices to ensure that*

$$\{x \in H : \overline{\lim_{k\to\infty}} |\langle L^k x, x\rangle|^{1/k} \leqslant C\} \tag{W}$$

is a dense subset of H. ♣

Proof. Reasoning by contraposition, we have to show that, for L a self-adjoint operator on H, for all $C < |||L|||$, the set of the $x \in H$ such that

$$\overline{\lim_{k\to\infty}} |\langle L^k x, x\rangle|^{1/k} > C \tag{X}$$

contains a non-empty open subset of H.

Since L is self-adjoint, by the spectral theorem [85, Theorem 7.18], it is unitarily equivalent to the "multiplication by identity" operator M on a space $\bigoplus_{\alpha\in A} L^2(\rho_\alpha)$, for A some set and ρ_α some Radon measures on $\mathbb{R}$, that is [in the following equation, the variable λ is free, so that $f(\lambda)$ is synonymous with f]:

$$M\big(\sum_\alpha f_\alpha(\lambda)\big) = \sum_\alpha \lambda f_\alpha(\lambda). \tag{Y}$$

So we will assume L is of that form.

One has obviously:

$$|||L||| = \sup\{\lambda \geqslant 0 : \ (\exists \alpha \in A)\ (\rho_\alpha([-\lambda,\lambda]^c) > 0)\}; \tag{Z}$$

moreover, for all $f \in H$, $f = \sum_{\alpha \in A} f_\alpha$ with $f_\alpha \in L^2(\rho_\alpha)$,

$$\langle L^k f, f\rangle = \sum_{\alpha \in A} \int_{\mathbb{R}} \lambda^k |f_\alpha(\lambda)|^2 \, d\rho_\alpha(\lambda), \tag{AA}$$

so that (observing that, for k even, $\lambda^k \geqslant 0 \ \ \forall \lambda$)

$$\overline{\lim_{k\to\infty}} |\langle L^k f, f\rangle|^{1/k} = \sup\Big\{\lambda \geqslant 0 : \ (\exists \alpha \in A)\ \Big(\int_{[-\lambda,\lambda]^c} |f_\alpha(\lambda')|^2 \, d\rho_\alpha(\lambda') > 0\Big)\Big\}. \tag{AB}$$

Now, for $C < |||L|||$, the set

$$U = \Big\{f \in H : \ (\exists \alpha \in A)\ \Big(\int_{[-C,C]^c} |f_\alpha(\lambda)|^2 \, d\rho_\alpha(\lambda) > 0\Big)\Big\} \tag{AC}$$

is open because $\int_{[-C,C]^c} |f_\alpha(\lambda)|^2 \, d\rho_\alpha(\lambda)$ is a continuous function of f, and it is non-empty by (Z). But (X) is satisfied for all $x \in U$ by (AB), so U fulfills our quest. ♠

Chapter 1

A first approach to Hilbertian correlations

1.1 Definition and first properties

1.1.a Equivalent definitions

1.1.1 Definition. Let $(\Omega, \mathscr{B}, \mathbf{P})$ be a probability space. For $\mathscr{F}, \mathscr{G}$ two sub-σ-algebras of $\mathscr{B}$, the *Hilbertian correlation coefficient* (or merely "correlation") between $\mathscr{F}$ and $\mathscr{G}$ is defined as

$$\{\mathscr{F} : \mathscr{G}\} := \sup_{\substack{f \in \bar{L}^2(\mathscr{F}) \smallsetminus \{0\} \\ g \in \bar{L}^2(\mathscr{G}) \smallsetminus \{0\}}} \frac{|\mathbf{E}[fg]|}{\mathrm{Sd}(f)\mathrm{Sd}(g)}. \tag{AD}$$

If the supremum in (AD) is taken over an empty set, that is, if $\mathscr{F}$ or $\mathscr{G}$ is trivial, we define this supremum to be 0. ♢

1.1.2 Remark. $\{\mathscr{F} : \mathscr{G}\}$ is often called the "maximal correlation coefficient" or "ρ-mixing coefficient" between $\mathscr{F}$ and $\mathscr{G}$, and denoted by $\rho(\mathscr{F}, \mathscr{G})$ (see [11]). ♡

1.1.3 Remark. In other words, $\{\mathscr{F} : \mathscr{G}\}$ is the best $k \in \mathbb{R}_+$ such that the following refined Cauchy–Schwarz inequality holds in the Hilbert space $\bar{L}^2(\mathscr{B})$:

$$\forall f \in \bar{L}^2(\mathscr{F}) \;\; \forall g \in \bar{L}^2(\mathscr{G}) \qquad |\langle f, g \rangle| \leqslant k \|f\| \|g\|. \tag{AE}$$

Yet another formulation is that $\{\mathscr{F} : \mathscr{G}\}$ is the cosine of the angle between $\bar{L}^2(\mathscr{F})$ and $\bar{L}^2(\mathscr{G})$, seen as subspaces of $\bar{L}^2(\mathscr{B})$ —this angle being defined as the infimum angle between any two non-zero vectors of these respective subspaces.

If we speak in terms of L^2 spaces rather than $\bar{L}^2$ spaces, $\{\mathscr{F} : \mathscr{G}\}$ is the best $k \in \mathbb{R}_+$ such that for all non-constant square-integrable f, g resp. $\mathscr{F}$ and $\mathscr{G}$-measurable,

$$|\mathrm{Corr}(f, g)| \leqslant k, \tag{AF}$$

where $\mathrm{Corr}(f, g) := \mathrm{Cov}(f, g)/\mathrm{Sd}(f)\mathrm{Sd}(g)$ is the Pearson correlation coefficient between f and g. ♡

1.1.4 Definition. We say that $\mathscr{F}$ and $\mathscr{G}$ are *ε-decorrelated*, resp. *ε-correlated*, if $\{\mathscr{F} : \mathscr{G}\} \leqslant \varepsilon$, resp. $\{\mathscr{F} : \mathscr{G}\} \geqslant \varepsilon$. ♢

1.1.5 Definition. For X and Y random variables (with arbitrary ranges), we will denote $\{X:Y\}$ for $\{\sigma(X):\sigma(Y)\}$. ♢

1.1.6 Remark. One can rewrite Definition 1.1.5 as

$$\{X:Y\} = \sup_{f,g} \frac{\mathrm{Cov}(f(X),g(Y))}{\mathrm{Sd}(f(X))\mathrm{Sd}(g(Y))}, \tag{AG}$$

where it is implied that f and g have to be measurable, real, and such that $0 < \mathrm{Sd}(f(X)), \mathrm{Sd}(g(Y)) < \infty$. ♡

☛ *More generally, all the questions relative to Hilbertian correlations may be handled either in terms of σ-algebras or in terms of random variables. In the sequel, we will frequently switch implicitly between these two paradigms.*

It is natural to enquire what happens if one deals with *complex* $\bar{L}^2$ spaces. In fact it does not change anything:

1.1.7 Proposition ([86, Theorem 1.1])**.** *Let $\mathscr{F}$ and $\mathscr{G}$ be two σ-algebras and let f,g be two complex centered L^2 variables, measurable w.r.t. resp. $\mathscr{F}$ and $\mathscr{G}$. Then, with* $\mathrm{Sd}(f)$ *meaning* $\sqrt{\mathbf{E}[|f-\mathbf{E}[f]|^2]}$*, one has:*

$$|\mathbf{E}[fg]| \leqslant \{\mathscr{F}:\mathscr{G}\}\mathrm{Sd}(f)\mathrm{Sd}(g). \tag{AH}$$

♣

Proof. I recall the proof for the sake of completeness. Up to multiplying g by a well-chosen unit complex number, we can assume that $\mathbf{E}[fg] \in \mathbb{R}_+$. Then we can apply Definition 1.1.1 to the *real* $\bar{L}^2$ variables $\mathfrak{Re}\, f$ and $\mathfrak{Re}\, g$, resp. $\mathfrak{Im}\, f$ and $\mathfrak{Im}\, g$, getting:

$$\begin{aligned}
|\mathbf{E}[fg]| = \mathfrak{Re}\,\mathbf{E}[fg] &= \mathbf{E}[\mathfrak{Re}\, f\, \mathfrak{Re}\, g] - \mathbf{E}[\mathfrak{Im}\, f\, \mathfrak{Im}\, g] \\
&\leqslant \{\mathscr{F}:\mathscr{G}\}\big(\mathrm{Sd}(\mathfrak{Re}\, f)\mathrm{Sd}(\mathfrak{Re}\, g) + \mathrm{Sd}(\mathfrak{Im}\, f)\mathrm{Sd}(\mathfrak{Im}\, g)\big) \\
&\underset{\mathrm{CS}}{\leqslant} \{\mathscr{F}:\mathscr{G}\}\sqrt{\mathrm{Var}(\mathfrak{Re}\, f) + \mathrm{Var}(\mathfrak{Im}\, f)}\sqrt{\mathrm{Var}(\mathfrak{Re}\, g) + \mathrm{Var}(\mathfrak{Im}\, g)} \\
&= \{\mathscr{F}:\mathscr{G}\}\mathrm{Sd}(f)\mathrm{Sd}(g).
\end{aligned} \tag{AI}$$

♠

Now we turn to a different way of seeing correlation levels.

1.1.8 Definition. For $\mathscr{F},\mathscr{G}$ two σ-algebras, we denote by $\pi_{\mathscr{G}\mathscr{F}}$ the 'projection' operator

$$\begin{array}{rccc} \pi_{\mathscr{G}\mathscr{F}}: & \bar{L}^2(\mathscr{F}) & \to & \bar{L}^2(\mathscr{G}) \\ & f & \mapsto & f^{\mathscr{G}}. \end{array} \tag{AJ}$$

For $\mathscr{F},\dots,\mathscr{Z}$ σ-algebras, we denote $\pi_{\mathscr{Z}\mathscr{Y}\mathscr{X}\dots\mathscr{G}\mathscr{F}} := \pi_{\mathscr{Z}\mathscr{Y}} \circ \pi_{\mathscr{Y}\mathscr{X}} \circ \cdots \circ \pi_{\mathscr{G}\mathscr{F}}$. ♢

With this vocabulary at hand,

1.1.9 Proposition. *For $\mathscr{F},\mathscr{G}$ two σ-algebras, $\{\mathscr{F}:\mathscr{G}\} = |||\pi_{\mathscr{G}\mathscr{F}}|||$.* ♣

Proof. $\pi_{\mathscr{G}\mathscr{F}}$ is the orthogonal projection from $\bar{L}^2(\mathscr{F})$ to $\bar{L}^2(\mathscr{G})$ in the Hilbert space $\bar{L}^2(\mathscr{B})$, so its norm is the cosine of the angle between $\bar{L}^2(\mathscr{F})$ and $\bar{L}^2(\mathscr{G})$, i.e. $\{\mathscr{F}:\mathscr{G}\}$. ♠

1.1.10 Remark. One has $\pi_{\mathscr{F}\mathscr{G}} = \pi^*_{\mathscr{G}\mathscr{F}}$, since $\langle \pi_{\mathscr{G}\mathscr{F}} f, g\rangle = \mathbf{E}[fg] = \langle f, \pi_{\mathscr{F}\mathscr{G}} g\rangle$. Therefore the expression $|||\pi_{\mathscr{G}\mathscr{F}}|||$ in Proposition 1.1.9 can be rewritten into $\sqrt{|||\pi_{\mathscr{F}\mathscr{G}\mathscr{F}}|||}$, which is also $\rho(\pi_{\mathscr{F}\mathscr{G}\mathscr{F}})$ since $\pi_{\mathscr{F}\mathscr{G}\mathscr{F}}$ is self-adjoint. ♡

1.1.b Immediate properties

Having defined Hilbertian correlations, it is now time to study their behaviour.

The following properties are immediate from Definition 1.1.1:

1.1.11 Proposition. *For all σ-algebras $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{G}'$,*

(i) $\{\mathscr{G}:\mathscr{F}\} = \{\mathscr{F}:\mathscr{G}\}$;
(ii) $\mathscr{G} \subset \mathscr{G}' \Rightarrow \{\mathscr{F}:\mathscr{G}\} \leqslant \{\mathscr{F}:\mathscr{G}'\}$;
(iii) $\{\mathscr{F}:\mathscr{G}\} \in [0,1]$;
(iv) $\{\mathscr{F}:\mathscr{G}\} = 0$ *if and only if* $\mathscr{F}$ *and* $\mathscr{G}$ *are independent;*
(v) If $\mathscr{F}$ *is not trivial, then* $\{\mathscr{F}:\mathscr{F}\} = 1$.

♣

When one is concerned by correlation between variables, it often occurs that some of these variables are vector-valued. The following proposition means that it suffices to know the behaviour of finite-length vectors to understand the behaviour of all vectors:

1.1.12 Proposition. *Let I, J be possibly infinite sets and let $\vec{X}_I, \vec{Y}_J$ be vector-valued variables. Then, denoting "$I' \Subset I$" to mean that I' is a finite subset of I,*

$$\{\vec{X}_I : \vec{Y}_J\} = \sup_{I' \Subset I, J' \Subset J} \{\vec{X}_{I'} : \vec{Y}_{J'}\}. \tag{AK}$$

♣

Proof. This is because $\bigcup_{I' \Subset I} \bar{L}^2(\vec{X}_{I'})$, resp. $\bigcup_{J' \Subset J} \bar{L}^2(\vec{X}_{J'})$, is a dense subset of $\bar{L}^2(\vec{X}_I)$, resp. $\bar{L}^2(\vec{Y}_J)$. That property follows by classical approximation arguments like in the proof of [73, Theorem 3.14]. See [12, Theorem 3.16(II-3)] for a more detailed proof. ♠

1.1.c Operator interpretation

1.1.13 Proposition. *If $X \to Y \to Z$ is a Markov chain, then $\{X:Z\} \leqslant \{X:Y\}\{Y:Z\}$.* ♣

Proof. The Markov chain property is equivalent to meaning that $\pi_{ZX} = \pi_{ZYX}$, so the result is a consequence of the submultiplicativity of operator norms. See also [72, § VII-4]. ♠

There is a refined version of Proposition 1.1.13 which is particularly interesting for reversible chains:

1.1.14 Proposition. *If $X \to Y \to Z$ is a Markov chain, then $\{X:Z\} = \sqrt{\rho(\pi_{YZY} \circ \pi_{YXY})}$.* ♣

Proof. Because of the Markov chain property, $\pi_{XZ} = \pi_{XY} \circ \pi_{YZ}$ and $\pi_{ZX} = \pi_{ZY} \circ \pi_{YX}$. Using that for any pair of operators $\pi\colon H_1 \to H_2$ and $\tau\colon H_2 \to H_1$ one has $\rho(\pi \circ \tau) = \rho(\tau \circ \pi)$, we get that $\{X:Z\}^2 = \rho(\pi_{XY}) = \rho(\pi_{XYZYX}) = \rho(\pi_{XY} \circ \pi_{YZYX}) = \rho(\pi_{YZYX} \circ \pi_{XY}) = \rho(\pi_{YZY} \circ \pi_{YXY})$. ♠

1.1.15 Corollary. *If $\cdots \to X_{-1} \to X_0 \to X_1 \to \cdots$ is a stationary Markov chain so that $\pi_{X_1X_0}$ and $\pi_{X_0X_1}$ commute* [*], *then for all $k \in \mathbb{Z} \smallsetminus \{0\}$, $\{X_0 : X_k\} = \{X_0 : X_1\}^{|k|}$.* ♣

[*]. Reversible chains always satisfy this condition since then $\pi_{X_1X_0} = \pi_{X_0X_1}$.

Proof. Since the chain is stationary, all the X_n have the same law and thus all the $\overline{L}^2(X_n)$ can be identified; then the stationarity property is equivalent to saying that $\pi_{X_{n+1}X_n} = \pi_{X_1X_0}$ for all $n \in \mathbb{Z}$. Thanks to the commutation hypothesis, one can write for $k > 0$:

$$\{X_0 : X_k\} = \sqrt{\rho(\pi_{X_0X_kX_0})} = \sqrt{\rho(\pi^k_{X_0X_1} \circ \pi^k_{X_1X_0})} = \sqrt{\rho(\pi^k_{X_0X_1X_0})} = \sqrt{\rho(\pi_{X_0X_1X_0})^k} = \{X_0 : X_1\}^k. \tag{AL}$$

For the case $k < 0$, we use that $\{X_0 : X_k\} = \{X_{-k} : X_0\}$. ♠

1.1.d First criteria for decorrelation

Density sufficient condition

1.1.16 Proposition. *Let X and Y be two random variables valued resp. in E and F. Suppose that $Law(X,Y)$ has a density h w.r.t. the product probability $Law(X) \otimes Law(Y)$. Then:*

$$\{X : Y\} \leqslant \Big(\int_{E\times F} (h-1)^2 \, dLaw_X \, dLaw_Y\Big)^{1/2}. \tag{AM}$$

♣

1.1.17 Remark. The integral expression in (AM) is nothing but 2 times the bilinearized version of the mutual information

$$I(X;Y) := \int_{E\times F} h \log h \, dLaw_X \, dLaw_Y. \tag{AN}$$

Yet Example 1.3.3 will show that one does not have $\{X : Y\} \leqslant \sqrt{2}(X;Y)^{1/2}$ in general. ♡

Proof. To alleviate notation, denote resp. $\mathbf{P}_X, \mathbf{P}_Y, \mathbf{P}_{(X,Y)}$ for $Law(X), Law(Y), Law(X,Y)$. Let f and g be centered L^2 functions being resp. X- and Y-measurable. Observe first that

$$\int_{E\times F} f(x)g(y)\, d\mathbf{P}_X[x] d\mathbf{P}_Y[y] = \Big(\int_E f \, d\mathbf{P}_X\Big)\Big(\int_F g \, d\mathbf{P}_Y\Big) = 0 \times 0 = 0, \tag{AO}$$

so that

$$\mathbf{E}[fg] = \int fg \, d\mathbf{P}_{(X,Y)} = \int hfg \, d\mathbf{P}_X d\mathbf{P}_Y = \int (h-1)fg \, d\mathbf{P}_X d\mathbf{P}_Y \tag{AP}$$

and thus

$$|\mathbf{E}[fg]| \leqslant \Big(\int (h-1)^2 \, d\mathbf{P}_X d\mathbf{P}_Y\Big)^{1/2} \Big(\int f^2 g^2 \, d\mathbf{P}_X d\mathbf{P}_Y\Big)^{1/2} \tag{AQ}$$

by the Cauchy–Schwarz inequality. But the last factor in the right-hand side of (AQ) is

$$\Big(\int f^2(x)g^2(y)\, d\mathbf{P}_X[x] d\mathbf{P}_Y[y]\Big)^{1/2} = \Big(\int f^2 d\mathbf{P}_X\Big)^{1/2} \Big(\int g^2 d\mathbf{P}_Y\Big)^{1/2} = \mathrm{Sd}(f)\mathrm{Sd}(g), \tag{AR}$$

so that (AM) is proved.

You may also see [14, Theorem 2.5] for an analogous result. ♠

Event necessary condition

1.1.18 Proposition (event necessary condition)**.** *Let $\mathscr{F}$ and $\mathscr{G}$ be two σ-algebras. If $\{\mathscr{F} : \mathscr{G}\} \leqslant \varepsilon$, then for all events $A \in \mathscr{F}$ and $B \in \mathscr{G}$ with respective probabilities p and q,*

$$|\mathbf{P}[A \cap B] - pq| \leqslant \varepsilon \sqrt{p(1-p)q(1-q)}. \tag{AS}$$

In particular, if there exists two non-trivial events $A \in \mathscr{F}, B \in \mathscr{G}$ which are equivalent (in the sense that $\mathbf{P}[A \triangle B] = 0$), then $\{\mathscr{F} : \mathscr{G}\} = 1$. [†] ♣

Proof. It follows from (AF) applied to $\mathbf{1}_A$ and $\mathbf{1}_B$. ♠

1.1.e Independent tensorization

Now we are turning to the basic tensorization theorem, which will motivate § 3:

1.1.19 Theorem ([21, Theorem 6.2])**.** *Let I be a set and let $\vec{X}_I, \vec{Y}_I$ be vectorial variables. Suppose all the pairs (X_i, Y_i), $i \in I$, are independent, then*

$$\{\vec{X} : \vec{Y}\} = \sup_{i \in I} \{X_i : Y_i\}. \tag{AT}$$

♣

Proof. The simplest proof of Theorem 1.1.19 relies on the operator interpretation of correlations, see e.g. the proof of [87, Theorem 1]. Here however I shall give a proof based on decomposing functions of several variables into telescopic sums, for this kind of arguments will be used again in the proofs of the more general tensorization theorems of § 3.

First, observe that the "$\geqslant$" inequality of (AT) is trivial, so we only have to prove the "$\leqslant$" inequality. We denote $\varepsilon_i := \{X_i : Y_i\}$, and to alleviate notation, x_i will implicitly stand for an element in the range of X_i, resp. y_i for an element in the range of Y_i.

By Proposition 1.1.12, we may assume that I is finite, say $I = \{1, \dots, N\}$ for some $N \in \mathbb{N}$. Let f and g be resp. $\vec{X}_I$-measurable and $\vec{Y}_I$-measurable centered L^2 real functions; our goal is to bound above $|\mathbf{E}[fg]|$.

For $i \in \{0, \dots, N\}$, define $\mathscr{F}_i := \bigvee_{j \leqslant i} \sigma(X_j, Y_j)$. I claim that, because of the independence hypothesis, $f^{\mathscr{F}_i}$ only depends on the values of $X_1, \dots, X_i$ and not on $Y_1, \dots, Y_i$, and similarly that $g^{\mathscr{F}_i}$ only depends on the values of $Y_1, \dots, Y_i$: one can write indeed (in the case of f)

$$\begin{aligned} f^{\mathscr{F}_i}(x_1, y_1, \dots, x_i, y_i) &= \int f(x_1, \dots, x_i, x_{i+1}, \dots, x_n)\, d\mathbf{P}[x_{i+1}, \dots, x_n | x_1, y_1, \dots, x_i, y_i] \\ &= \int f(x_1, \dots, x_i, x_{i+1}, \dots, x_n)\, d\mathbf{P}[x_{i+1}, \dots, x_n]. \end{aligned} \tag{AU}$$

Now, for $i \in \{1, \dots, N\}$, define

$$f_i(x_1, \dots, x_i) := f^{\mathscr{F}_i}(x_1, \dots, x_i) - \mathbf{E}[f | x_1, \dots, x_{i-1}], \tag{AV}$$

with a similar definition for g. One has $f = \sum_i f_i$, resp. $g = \sum_i g_i$, and f_i and g_i are $\mathscr{F}_i$-measurable and centered w.r.t. $\mathscr{F}_{i-1}$ (that is, $\mathbf{E}[f_i | \mathscr{F}_{i-1}], \mathbf{E}[g_i | \mathscr{F}_{i-1}] \equiv 0$), so

$$\operatorname{Var} f = \sum_i \operatorname{Var} f_i, \tag{AW}$$

[†]. The converse is not true: it can occur that $\{\mathscr{F} : \mathscr{G}\} = 1$ but that no non-trivial events of $\mathscr{F}$ and $\mathscr{G}$ are equivalent. A counterexample is the following: let $(X_n)_{n \in \mathbb{N}}$ be independent *Bernoulli*(1/2) variables, and define independently $Y_n = 1 - X_n$ with probability ε_n and $Y_n = X_n$ otherwise, where $(\varepsilon_n)_{n \in \mathbb{N}}$ is a sequence of numbers such that $0 < \varepsilon_n \leqslant 1/2$ for all n and $\varepsilon_n \overset{n \to \infty}{\to} 0$. Then the vectorial variables $\vec{X}$ and $\vec{Y}$ obviously satisfy $\{\vec{X} : \vec{Y}\} = 1$, yet it is not hard to prove that no $\vec{X}$-measurable non-trivial event is equivalent to a $\vec{Y}$-measurable one.

resp. $\operatorname{Var} g = \sum_i \operatorname{Var} g_i$.

We expand:

$$\mathbf{E}[fg] = \sum_{(i,j)\in I\times I} \mathbf{E}[f_i g_j]. \tag{AX}$$

In the right-hand side of (AX), if $i \neq j$ then $\mathbf{E}[f_i g_j] = 0$ since if, say, $i < j$, f_i is $\mathscr{F}_i$-measurable while g_j is centered w.r.t. $\mathscr{F}_{j-1} \supset \mathscr{F}_i$. So (AX) turns into:

$$\mathbf{E}[fg] = \sum_{i\in I} \mathbf{E}[f_i g_i]. \tag{AY}$$

Writing the law of total expectation,

$$\mathbf{E}[f_i g_i] = \int \mathbf{E}[f_i g_i | x_1, y_1, \ldots, x_{i-1}, y_{i-1}]\, d\mathbf{P}[x_1, y_1, \ldots, x_{i-1}, y_{i-1}]. \tag{AZ}$$

But, as we noticed before, under $\mathbf{P}[\cdot | x_1, y_1, \ldots, x_{i-1}, y_{i-1}]$, f_i only depends on X_i and g_i only depends on Y_i. Moreover, because of the independence property, the law of (X_i, Y_i) is the same under $\mathbf{P}[\cdot | x_1, y_1, \ldots, x_{i-1}, y_{i-1}]$ as under $\mathbf{P}$, so under $\mathbf{P}[\cdot | x_1, y_1, \ldots, x_{i-1}, y_{i-1}]$, f_i and g_i are centered and ε_i-independent. Thus

$$\begin{aligned} |\mathbf{E}[f_i g_i]| &\leqslant \varepsilon_i \int \operatorname{Sd}(f_i | x_1, y_1, \ldots, x_{i-1}, y_{i-1}) \operatorname{Sd}(g_i | x_1, y_1, \ldots, x_{i-1}, y_{i-1})\, d\mathbf{P}[x_1, y_1, \ldots, x_{i-1}, y_{i-1}] \\ &\underset{\text{CS}}{\leqslant} \varepsilon_i \sqrt{\int \operatorname{Var}(f_i | x_1, y_1, \ldots, x_{i-1}, y_{i-1})\, d\mathbf{P}[x_1, y_1, \ldots, x_{i-1}, y_{i-1}]} \sqrt{\textit{the same for } g} \\ &\leqslant \varepsilon_i \operatorname{Sd}(f_i) \operatorname{Sd}(g_i)^{[\ddagger]}. \end{aligned} \tag{BA}$$

Summing over i,

$$\begin{aligned} |\mathbf{E}[fg]| &\leqslant \sum_{i\in I} \varepsilon_i \operatorname{Sd}(f_i) \operatorname{Sd}(g_i) \leqslant \sup_{i\in I} \varepsilon_i \cdot \sum_{i\in I} \operatorname{Sd}(f_i) \operatorname{Sd}(g_i) \\ &\underset{\text{CS}}{\leqslant} \sup_{i\in I} \varepsilon_i \cdot \sqrt{\sum_{i\in I} \operatorname{Var}(f_i)} \sqrt{\sum_{i\in I} \operatorname{Var}(g_i)} = \sup_{i\in I} \varepsilon_i \cdot \operatorname{Sd}(f) \operatorname{Sd}(g), \end{aligned} \tag{BB}$$

which is the desired bound. ♠

1.2 Examples

1.2.a Finite-ranged variables

1.2.1 Proposition. *Let X and Y be random variables with finite ranges resp. $\{1, \ldots, N\}$ and $\{1, \ldots, M\}$, and denote $p_a := \mathbf{P}[X = a]$, $p^b := \mathbf{P}[Y = b]$, $p_a^b := \mathbf{P}[X = a$ and $Y = b]$. Then $\{X : Y\} = |||\Pi|||$, where Π is the $N \times M$ matrix with general entry*

$$\Pi_{ab} = \frac{p_a^b - p_a p^b}{\sqrt{p_a p^b}}. \tag{BC}$$

♣

[‡]. The last inequality is actually an equality, because f_i and g_i are centered w.r.t. $\mathscr{F}_{i-1}$.

1.2.2 Remark. In particular, if both X and Y have range $\{1,2\}$, using the same notation as before, one has

$$\{X:Y\} = \frac{|p_a^b - p_a p^b|}{\sqrt{p_1 p_2 p^1 p^2}}, \tag{BD}$$

where the right-hand side of (BD) does not depend on the choice of $a,b \in \{1,2\}$. ♡

Proof of Proposition 1.2.1. By Proposition 1.1.9, $\{X:Y\}$ is the norm of the operator $\pi_{XY}: \bar{L}^2(Y) \to \bar{L}^2(X)$. Here it will be more convenient to work in L^2 spaces than in $\bar{L}^2$ spaces, so we rather compute the norm of

$$\begin{array}{rccl} \tilde{\pi}: & L^2(Y) & \to & L^2(X) \\ & g & \mapsto & g^X - \mathbf{E}[g], \end{array} \tag{BE}$$

which is obviously the same as $|||\pi_{XY}|||$.

A function $g \in L^2(Y)$ can be identified with a M-dimensional vector also denoted by g, and similarly $\tilde{\pi} g \in L^2(X)$ can be identified with a N-dimensional vector. Denote $P := ((p_a^b))_{a,b} \in \mathbb{R}^{N\times M}$, $I_X := ((\delta_{aa'} p_a))_{a,a'} \in \mathbb{R}^{N\times N}$, $I_Y := ((\delta_{bb'} p^b))_{b,b'} \in \mathbb{R}^{M\times M}$, $1_N := 1^{\{1,\dots,N\}} \in \mathbb{R}^N$. Applying Bayes' formula yields that

$$\tilde{\pi} g = I_X^{-1} P g - 1_N (1_N)^\top P g. \tag{BF}$$

Now, $\|g\|_{L^2(Y)} = \|I_Y^{1/2} g\|$, resp. $\|\tilde{\pi} g\|_{L^2(X)} = \|I_X^{1/2}(\tilde{\pi} g)\|$, so:

$$\{X:Y\} = \sup_{g\neq 0} \frac{\|\,(I_X^{-1/2} P - I_X^{1/2} 1_N (1_N)^\top P\,) g\,\|}{\|\,I_Y^{1/2} g\,\|}. \tag{BG}$$

Performing the change of variables $h = I_Y^{1/2} g$, (BG) becomes $\{X:Y\} = \sup_{h\neq 0} \|\Pi h\| / \|h\| = |||\Pi|||$, with

$$\Pi = I_X^{-1/2} P I_Y^{-1/2} - I_X^{1/2} 1_N (1_N)^\top P I_Y^{-1/2}, \tag{BH}$$

which is Equation (BC) indeed. ♠

1.2.3 Remark. With the same kind of proof, there is even a similar proposition to calculate $\{X,Y\}$ if *either* X or Y has finite range, provided you know (in the case it is X which has finite range) all the $\mathbf{P}[X=x]$ and all the

$$\int_y \frac{d\mathbf{P}[Y=y|X=x]\, d\mathbf{P}[Y=y|X=x']}{d\mathbf{P}[Y=y]}. \tag{BI}$$

♡

1.2.4 Remark. In the case X or Y has range of cardinality 2, applying Proposition (1.2.1) yields that $\{X:Y\}^2$ depends smoothly on $Law(X,Y)$. Yet this is not the case in general: in fact, maximal correlations are nothing more than a particular case of operator norms (cf. § 1.1.c), and thus they have the same behaviour—they are a continuous function of the parameters, but they can have some $\mathscr{C}^1$ singularity. The following example exhibits such a singularity. ♡

1.2.5 Example. Suppose both X and Y have range $\{1,2,3\}$ and

$$((\mathbf{P}[X=a \text{ and } Y=b]))_{a,b} = \begin{pmatrix} 2/9 & 1/18 & 1/18 \\ 1/18 & 2/9+\alpha & 1/18-\alpha \\ 1/18 & 1/18-\alpha & 2/9+\alpha \end{pmatrix} \tag{BJ}$$

for a parameter $\alpha \in [-2/9, 1/18]$. Then the matrix Π defined by (BC) is

$$\Pi = \begin{pmatrix} 1/3 & -1/6 & -1/6 \\ -1/6 & 1/3+3\alpha & -1/6-3\alpha \\ -1/6 & -1/6-3\alpha & 1/3+3\alpha \end{pmatrix} = U \begin{pmatrix} 1/2+6\alpha & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} U^{-1}, \tag{BK}$$

with

$$U = \begin{pmatrix} 0 & -2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{pmatrix} \tag{BL}$$

being orthogonal. So by Proposition 1.2.1, $\{X : Y\} = 1/2 + 6\alpha_+$. ♡

1.2.b Gaussian variables

The following theorem, which I will frequently use in the sequel, computes exactly the Hilbertian correlation between two jointly Gaussian variables:

1.2.6 Theorem ([46, 45])**.** *Let $(\vec{X}, \vec{Y})$ be an $(N+M)$-dimensional Gaussian vector whose covariance matrix writes blockwise*

$$\mathrm{Var}(\vec{X}, \vec{Y}) = \begin{pmatrix} \mathbf{I}_N & C \\ C^{\mathsf{T}} & \mathbf{I}_M \end{pmatrix}, \tag{BM}$$

then $\{\vec{X} : \vec{Y}\} = |||C|||$. ♣

1.2.7 Remark. In other words, Theorem 1.2.6 tells that in the Gaussian case, the supremum in (AG) defining $\{\vec{X} : \vec{Y}\}$ can be restricted to *linear* functions f and g. ♡

1.2.8 Remark. By a linear change of variables, Theorem 1.2.6 actually allows us to compute $\{\vec{X} : \vec{Y}\}$ for *any* Gaussian vector $(\vec{X}, \vec{Y})$. ♡

Proof of Theorem 1.2.6. I recall a (sketch of) proof for the sake of completeness. By the properties of Gaussian vectors, the law of Y knowing that $X = x$ [I dropped the vector arrows] is the normal law $\mathcal{N}(\mathbf{I}_M - C^{\mathsf{T}}C) + C^{\mathsf{T}}x$, and similarly the law of X knowing that $Y = y$ is the normal law $\mathcal{N}(\mathbf{I}_N - CC^{\mathsf{T}}) + Cy$. Consequently, the operator π_{XYX} is the generator of the following random walk on $\mathbb{R}^N$ (whose equilibrium measure is the standard Gaussian law): when one is at x, they jump to a point distributed according to the normal law $\mathcal{N}(\mathbf{I}_N - CC^{\mathsf{T}}CC^{\mathsf{T}}) + CC^{\mathsf{T}}x$. This walk is a multidimensional AR(1)-process (see. [63, § 2.6]), whose properties are perfectly known; in particular, the eigenvalue f of π_{XYX} responsible for its spectral radius will be a linear function, so we only have to consider linear f in the supremum (AG). For such f, the optimal g will also be linear by the Gaussian nature of the system, so in the end $\{\vec{X} : \vec{Y}\}$ is equal to $|||C|||$. ♠

1.2.c Miscellaneous examples

Random conditional laws

1.2.9 Example. Let $0 < p < n$ be integers. We consider a random variable (X, Y) for which Y has range $\mathscr{Y} := \{1, \dots, n\}$ and X has range $\mathscr{X} := \mathfrak{P}_p(\mathscr{Y})$, $\mathfrak{P}_p(\mathscr{Y})$ denoting the set of subsets $y \subset \mathscr{Y}$ with cardinality p —so, $\#\mathscr{X} = \binom{n}{p}$ and $\#\mathscr{Y} = n$ —, and we take the law of (X, Y) uniform on the pairs (x, y) such that $y \in x$: see Figure 3.

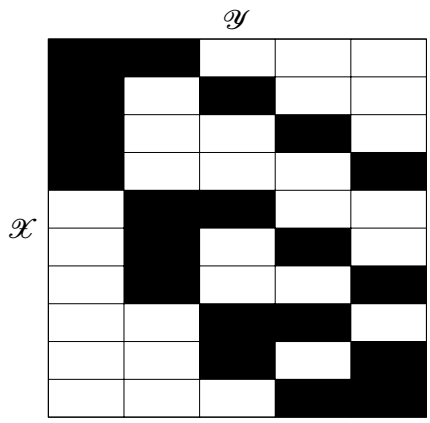

Figure 3: Schematic representation of Example 1.2.9 for $n = 5$ and $p = 2$.

When considered as operators on L^2 spaces, it is obvious that π_{XY} and π_{YX} are characterized by $(\pi_{XY} f)(x) = p^{-1} \sum_{y \in x} f(y)$, resp. $(\pi_{YX} g)(y) = \binom{n-1}{p-1}^{-1} \sum_{y \in x} g(x)$, so that

$$(\pi_{YXY} f)(y) = \frac{1}{p} f(y) + \sum_{y' \neq y} \frac{p-1}{p(n-1)} f(y'). \tag{BN}$$

Thus, on $\bar{L}^2(Y)$, π_{YXY} is nothing but the scalar operator $\frac{n-p}{p(n-1)}\mathbf{I}$, and therefore

$$\{X : Y\} = \sqrt{\frac{n-p}{p(n-1)}} \tag{BO}$$

by Proposition 1.1.9 and Remark 1.1.10. ♡

Weakly coupled particles

1.2.10 Proposition. *Let V_1 and V_2 be potentials on $\mathbb{R}^n$, $n \geqslant 1$, i.e. the V_i are real-valued measurable functions on $\mathbb{R}^n$ with $\int_{\mathbb{R}^n} e^{-V_i(x)} dx < \infty$. For $i \in \{1,2\}$, denote by $\mathbf{P}_i$ the probability measure on $\mathbb{R}^n$ proportional to $e^{-V_i(x)} dx$, which is to be thought as the law of the position X_i of a particle i subjected to the potential V_i. Denote $\mathbf{P}_\otimes := \mathbf{P}_1 \otimes \mathbf{P}_2$, which is the joint law of (X_1, X_2) in absence of interaction.*

Now let W be an interaction potential on $(\mathbb{R}^n)^2$ such that $e^{-[V_1(x_1) + V_2(x_2) + W(x_1,x_2)]}$ is integrable; denote by $\mathbf{P}$ the probability measure on $(\mathbb{R}^n)^2$ proportional to $e^{-[V_1+V_2+W]} dx_1 dx_2$, which is the joint law of (X_1, X_2) in presence of interaction potential W.

Then, under the law $\mathbf{P}$,

$$\{X_1 : X_2\} \leqslant \frac{\mathrm{Sd}_\otimes(e^{-W})}{\mathbf{E}_\otimes[e^{-W}]}. \tag{BP}$$

♣

Proof. The law $\mathbf{P}$ has density $h = e^{-W}/\mathbf{E}_\otimes[e^{-W}]$ w.r.t. $\mathbf{P}_\otimes$, whence the result by Proposition 1.1.16. ♠

1.2.11 Remark. Proposition 1.2.10 gives a rigorous sense to the intuition that two weakly coupled particles must have nearly independent positions. This is valid in a quite general setting, in particular, W does not have to be bounded. ♡

Non-reversible Markov chain

1.2.12 Example. Here is a example showing that the inequality in Proposition 1.1.13 is strict in general. Consider the stationary Markov chain on $\{1,2,3\}$ defined by

$$P = ((\mathbf{P}[X_{k+1} = a | X_k = b]))_{ab} = \begin{pmatrix} 0 & 1/2 & 1 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}, \tag{BQ}$$

which has equilibrium measure $(2/5, 2/5, 1/5)$. Diagonalizing P shows that

$$P^t = \begin{pmatrix} 2/5 & 2/5 & 2/5 \\ 2/5 & 2/5 & 2/5 \\ 1/5 & 1/5 & 1/5 \end{pmatrix} + O(2^{-t/2}), \tag{BR}$$

whence $\{X_k : X_{k+t}\} = O(2^{-t/2})$ when $t \to +\infty$ by Proposition 1.2.1. Yet $\{X_k : X_{k+1}\} = 1$, since the non-trivial events $\{X_k = 1\}$ and $\{X_{k+1} = 2\}$ are equivalent (cf. Proposition 1.1.18). ♡

Hyperplanes in Ising's model As I told in Chapter 0, the initial motivation of this work was to understand the presence of ρ-mixing in Ising's model (cf. § 0.1.a); in particular, I intended to re-get a result similar to Theorem 0.1.9 by a more 'natural' method. That shall be achieved indeed in § 5.1:

1.2.13 Theorem (Theorem 5.1.1-(i))**.** *For Ising's model on $\mathbb{Z}^n$ in the completely analytical regime, for all disjoint $I, J \subset \mathbb{Z}^n$,*

$$\{\vec{\omega}_I : \vec{\omega}_J\} \leqslant \exp\big[-(\psi' + o(1))\,dist(I,J)\big], \tag{BS}$$

where ψ' is the same as in Theorem 0.1.7 and where the "$o(1)$" (to be understood "as $dist(I,J) \to \infty$") is uniform in I, J. ♣

If we apply that result to the case of parallel 'hyperplanes' of $\mathbb{Z}^n$ (I mean, sets of the form $\{t\} \times \mathbb{Z}^{n-1}$), Formula (BS) looks far less neat than Formula (M) in Theorem 0.1.9.

That bound can however be improved by using Proposition 1.1.14. Indeed, as we noticed in § 0.1.c, the states of two parallel hyperplanes are elements of some reversible stationary Markov chain. Therefore, applying Corollary 1.1.15 (in which we let $k \to \infty$), we get a result exactly similar to (M), except that we have to replace ψ by ψ' —recall that it is not known whether $\psi' = \psi$.

1.3 Comparing ρ-mixing to other measures of dependence

The material of this section is classical; most of it can be found for instance in [12, §§ 3 & 5]. Here we will say that a sequence of pairs of σ-algebras $(\mathscr{F}^n, \mathscr{G}^n)$ is *ρ-mixing* to mean that $\{\mathscr{F}^n : \mathscr{G}^n\} \overset{n\to\infty}{\to} 0$.

1.3.a α-mixing

1.3.1 Definition. The *α-mixing* coefficient of two σ-algebras $\mathscr{F}$ and $\mathscr{G}$ is

$$\alpha(\mathscr{F}, \mathscr{G}) := \sup_{\substack{A \in \mathscr{A} \\ B \in \mathscr{B}}} |\mathbf{P}[A \cap B] - \mathbf{P}[A]\mathbf{P}[B]|. \tag{BT}$$

♢

Proposition 1.1.18 shows that 'ρ-mixing implies α-mixing', in the sense that one has $\alpha(\mathscr{F},\mathscr{G}) \leqslant A(\{\mathscr{F}:\mathscr{G}\})$ for some universal function $A:\ [0,1]\to[0,1]$ with $A(\rho)\overset{\rho\to 0}{\to} 0$.

1.3.2 Remark. Saying that the correlation of two variables tends to 0 means that their joint law tends in some sense to the product law. When the variables are ranged in Polish spaces, a common notion of convergence is weak convergence, that is, convergence against all bounded continuous function. [4, Theorem 2.2] states that weak convergence is implied by α-mixing, hence by ρ-mixing. The precise statement is the following: if $(X^n,Y^n)_{n\in\mathbb{N}}$ is a sequence of pairs of random variables such that all the X_n (resp. Y_n) have the same law $Law(X)$ (resp. $Law(Y)$) in some Polish space E (resp. F), then $\big(\alpha(X^n,Y^n)\overset{n\to\infty}{\to} 0\big) \Rightarrow \big(Law(X^n,Y^n)\overset{n\to\infty}{\to} Law(X)\otimes Law(Y)\big)$. ♡

On the other hand, the following example shows that α-mixing does not imply ρ-mixing:

1.3.3 Example. For $\varepsilon\in(0,1/2]$, define $(X^\varepsilon,Y^\varepsilon)$ in the following way:

- With probability ε, one samples X^ε and Y^ε independently with common law uniform on $[0,\varepsilon]$;
- With probability $(1-\varepsilon)$, one samples X^ε and Y^ε independently with common law uniform on $[\varepsilon,1]$.

Then for all $\varepsilon>0$ one has $\{X^\varepsilon:Y^\varepsilon\}=1$, since the non-trivial events $\{X^\varepsilon\leqslant\varepsilon\}$ and $\{Y^\varepsilon\leqslant\varepsilon\}$ are equivalent (cf. Proposition 1.1.18). However it is easy to show that $\alpha(X^\varepsilon,Y^\varepsilon)=\varepsilon-\varepsilon^2\overset{\varepsilon\to 0}{\to} 0$. ♡

1.3.b β-mixing

Recall the definition of the β-mixing coefficient from the previous chapter [Definition 0.1.4].

1.3.4 Example. For $\varepsilon\in(0,1)$, consider two random sequences $(X_i)_{i\in\mathbb{N}}$ and $(Y_i)_{i\in\mathbb{N}}$ defined in the following way: $(X_i)_{i\in\mathbb{N}}$ is a sequence of i.i.d. variables with uniform law on $\{\pm1\}$, and for each $i\in\mathbb{N}$, independently, one sets $Y_i=X_i$ with probability ε, and with probability $(1-\varepsilon)$ one chooses Y_i uniformly on $\{\pm1\}$. Then all the (X_i,Y_i) are i.i.d. with $\mathbf{P}[X_i=\eta \text{ and } Y_i=\theta]=(1+\eta\theta\varepsilon)/4$ for all $\eta,\theta\in\{\pm1\}$, thus $\{X_i:Y_i\}=\varepsilon$ by Remark 1.2.2, whence $\{\vec{X}:\vec{Y}\}=\varepsilon$ by Theorem 1.1.19. Yet $Law(\vec{X},\vec{Y})$ and $Law(\vec{X})\otimes Law(\vec{Y})$ are mutually singular for all $\varepsilon>0$.

This shows that ρ-mixing does not imply β-mixing, and *a fortiori* that there can be no kind of converse to Proposition 1.1.16. ♡

1.3.c Mutual information

Recall the definition (AN) of mutual information. [12, Theorem 5.3(III)] states that mutual information controls the β-mixing coefficient, so Example 1.3.4, which shows that ρ-mixing does not imply β-mixing in general, shows that it does not imply mutual information to tend to 0 either.

Proposition 1.1.16 suggests that, on the other hand, maximal correlation could be controlled by mutual information, but that is not true either: in Example 1.3.3 indeed, $\{X^\varepsilon:Y^\varepsilon\}=1$ for all $\varepsilon>0$, but

$$I(X^\varepsilon;Y^\varepsilon)=\varepsilon\log(\varepsilon^{-1})+(1-\varepsilon)\log\big((1-\varepsilon)^{-1}\big)\overset{\varepsilon\to 0}{\to} 0. \tag{BU}$$

Mutual information measures the quantity of information shared by two random variables, which explains intuitively the following property ([20, Theorem 2.5.2]): if $X\to Y\to Z$ is a Markov chain, then $I(Y;X,Z)\leqslant I(X;Y)+I(Y;Z)$. Does a similar inequality hold for Hilbertian correla-

tion? In the Gaussian case, the answer is "yes" thanks to Theorem 1.2.6: one gets that

$$\{Y:X,Z\}^2 \leqslant \frac{(1-\{Y:Z\}^2)\{X:Y\}^2+(1-\{X:Y\}^2)\{Y:Z\}^2}{1-\{X:Y\}^2\{Y:Z\}^2} \leqslant \{X:Y\}^2+\{Y:Z\}^2. \tag{BV}$$

But that property does not hold in general, as the following example shows:

1.3.5 Example. Consider a Markov chain $X \to Y \to Z$, where (Y,X) and (Y,Z) have the same law, which is the joint law described in Example 1.2.9—the role of "Y" in that example being played here by Y in both cases. Fix $y \in \mathscr{Y}$; define event A as "$Y = y$" and event B as "$y \in X \cap Z$". Then one computes that $\mathbf{P}[A] = n^{-1}$, while

$$\mathbf{P}[B] = \frac{1}{n} + \frac{(p-1)^2}{n(n-1)}. \tag{BW}$$

Since $A \subset B$, Proposition 1.1.18 then yields that

$$\{Y:X,Z\} \geqslant \sqrt{\mathbf{P}[B^c]\mathbf{P}[A]/\mathbf{P}[A^c]\mathbf{P}[B]} = \left(\frac{(n-1)^2-(p-1)^2}{(n-1)^2+(n-1)(p-1)^2}\right)^{1/2}. \tag{BX}$$

Comparing (BO) and (BX), one sees that taking $n \gg 1$ and $1 \ll p \ll n^{1/2}$ makes $\{X:Y\}$ and $\{Y:Z\}$ arbitrarily close to 0 while $\{Y:X,Z\}$ gets arbitrarily close to 1. ♡

1.3.6 Remark. There are similar examples with $(X,Y,Z) = (Y_{-1},Y_0,Y_1)$ for a reversible Markov process $(Y_t)_{t\in\mathbb{R}}$ [64]. ♡

Chapter 2

Event sufficient conditions

In § 1.1.d we saw that the maximal correlation coefficient $\{\mathscr{F}:\mathscr{G}\}$ controls the difference between $\mathbf{P}[A\cap B]$ and $\mathbf{P}[A]\mathbf{P}[B]$ for A and B two events resp. $\mathscr{F}$- and $\mathscr{G}$-measurable. A natural question is whether the converse is true, i.e. whether saying that $\mathbf{P}[A\cap B]$ is always close in some sense to $\mathbf{P}[A]\mathbf{P}[B]$ implies a control on $\{\mathscr{F}:\mathscr{G}\}$. We saw in § 1.3.a that α-mixing does not fit, but maybe stronger conditions of the same type would work.

In § 2.1 I will present a simple such condition (Theorem 2.1.3). This condition demands $|\mathbf{P}[A\cap B]-\mathbf{P}[A]\mathbf{P}[B]|$ to be bounded uniformly by $\zeta(\mathbf{P}[A])\theta(\mathbf{P}[B])$ for functions $\zeta,\theta\colon\ [0,1]\to\mathbb{R}_+$ sufficiently well behaved. This result, whose proof is rather simple, is apparently new.

Proposition 1.1.18, however, suggests that the natural condition on events would be a uniform control on $|\mathbf{P}[A\cap B]-\mathbf{P}[A]\mathbf{P}[B]|/\sqrt{\mathbf{P}[A^c]\mathbf{P}[A]}\sqrt{\mathbf{P}[B^c]\mathbf{P}[B]}$, which is out of the scope of Theorem 2.1.3. Bradley [9] proved in 1983 that that condition was indeed sufficient to get ρ-mixing. His result was improved in the next few years (see for instance the bound of [15]), but the optimal bound was remaining unknown, though its value was being conjectured. In § 2.2, I shall prove this optimal bound. My method, different from the techniques of [9, 15], relies on the analysis of the spectral properties of an operator linked to a law which I call the "Chogosov law", whose study is proceeded to in § 2.2.b.

2.1 Weak event sufficient condition

To state our next result we need some functional analysis reminders first:

2.1.1 Definition. On the space $\mathscr{C}_0^\infty(0,1)$ of compactly supported fuctions of $\mathscr{C}^\infty(0,1)$, one defines the scalar product

$$\langle\varphi,\psi\rangle_{H_0^1}=\int_0^1\varphi'(x)\psi'(x)\,dx. \tag{BY}$$

$\mathscr{C}_0^\infty(0,1)$ endowed with $\langle\cdot,\cdot\rangle_{H_0^1}$ is a pre-Hilbert space; its completion is denoted by $H_0^1(0,1)$. $\diamondsuit$

Recall that elements of $H_0^1(0,1)$ may be seen as ordinary functions:

2.1.2 Lemma (Sobolev, [1, Theorem 4.12])**.** *Any element $f\in H_0^1(0,1)$ can be identified with a unique function $\overline{f}\in\mathscr{C}_0^0[0,1]$, the space of continuous functions on $[0,1]$ with $\overline{f}(0),\overline{f}(1)=0$. Con-*

versely, a function $\overline{f} \in \mathscr{C}_0^0[0,1]$ *corresponds to an element of* $H_0^1(0,1)$ *if and only if*

$$\sup_{g \in \mathscr{C}_0^\infty(0,1)} \frac{|\int_0^1 f(x)g''(x)dx|}{\sqrt{\int_0^1 g'(x)^2 dx}} \tag{BZ}$$

is finite, and then there is a unique $f \in H_0^1(0,1)$ *associated to* $\overline{f}$*, whose norm is (BZ).* ♣

In accordance with Lemma 2.1.2, we will identify functions of $\mathscr{C}_0^0[0,1]$ with elements of $H_0^1(0,1)$ whenever it is possible. If $f \in \mathscr{C}_0^0[0,1]$ does not correspond to an element of $H_0^1(0,1)$, then we will set $\|f\|_{H_0^1} = +\infty$.

Now we can state the

2.1.3 Theorem (Weak event sufficient condition)**.** *Let* $\mathscr{F}$ *and* $\mathscr{G}$ *be two* σ*-algebras such that, for all* $A \in \mathscr{F}$ *and* $B \in \mathscr{G}$ *with respective probabilities* p *and* q*,*

$$\mathbf{P}[A \cap B] - pq \leqslant \zeta(p)\theta(q)^{[*]} \tag{CA}$$

for some $\zeta, \theta \in \mathscr{C}_0^0[0,1]$*. Then:*

$$\{\mathscr{F} : \mathscr{G}\} \leqslant \|\zeta\|_{H_0^1} \|\theta\|_{H_0^1}. \tag{CB}$$

♣

Proof. We begin with the following formula for covariance:

2.1.4 Lemma. *For* f *and* g *two real* L^2 *functions,*

$$\mathrm{Cov}(f,g) = \int_{\mathbb{R}\times\mathbb{R}} (\mathbf{P}[f \leqslant x \text{ and } g \leqslant y] - \mathbf{P}[f \leqslant x]\mathbf{P}[g \leqslant y])\, dxdy. \tag{CC}$$

♣

Proof of Lemma 2.1.4. Suppose in a first time that f and g are nonnegative. A classical Fubini argument (see [5, Problem 21.6]) shows that

$$\mathbf{E}[f] = \int_{\mathbb{R}_+} \mathbf{P}[f > x]\, dx, \tag{CD}$$

with a similar formula for g. By the same method,

$$\mathbf{E}[fg] = \int_{\mathbb{R}_+\times\mathbb{R}_+} \mathbf{P}[f > x \text{ and } g > y]\, dxdy, \tag{CE}$$

so that, using the computational formula $\mathrm{Cov}(f,g) = \mathbf{E}[fg] - \mathbf{E}[f]\mathbf{E}[g]$,

$$\mathrm{Cov}(f,g) = \int_{\mathbb{R}_+\times\mathbb{R}_+} (\mathbf{P}[f > x \text{ and } g > y] - \mathbf{P}[f > x]\mathbf{P}[g > y])\, dxdy. \tag{CF}$$

Observing that the integrand is also $(\mathbf{P}[f \leqslant x \text{ and } g \leqslant y] - \mathbf{P}[f \leqslant x]\mathbf{P}[g \leqslant y])$ and that it is zero for $(x,y) \notin \mathbb{R}_+ \times \mathbb{R}_+$, we get (CC) in the nonnegative case. By translation invariance, the formula remains true for all f, g bounded below, and then by approximation for all $f, g \in L^2$. ♠

[*]. Note that there is no need to put absolute values in the left-hand side of (CA).

Now, let f and g be L^2 variables resp. $\mathscr{F}$- and $\mathscr{G}$-mesurable, and denote by F and G the respective distribution functions of f and g. Up to a slight perturbation, F and G may be supposed to be diffeomorphisms from $\mathbb{R}$ onto $(0,1)$; denote by α and β their respective inverse maps. Then a change of variables in (CC) yields:

$$\mathrm{Cov}(f,g) = \int_{(0,1)^2} \big(\mathbf{P}[f \leqslant \alpha(p) \text{ and } g \leqslant \beta(q)] - pq\big)\alpha'(p)\beta'(q)\,dp\,dq, \tag{CG}$$

so by assumption (CA):

$$\mathrm{Cov}(f,g) \leqslant \Big(\int_0^1 \zeta(p)\alpha'(p)\,dp\Big) \times \Big(\int_0^1 \theta(q)\beta'(q)\,dq\Big). \tag{CH}$$

Then our theorem becomes equivalent to the claim stated and proved just below. ♠

2.1.5 Claim. *If f is a random variable whose repartition function F is a diffeomorphism of inverse α, then for $\zeta \in \mathscr{C}_0^0[0,1]$:*

$$\Big|\int_0^1 \zeta(p)\alpha'(p)dp\Big| \leqslant \|\zeta\|_{H_0^1}\,\mathrm{Sd}(f). \tag{CI}$$

♣

Proof. First note that, replacing g by f in (CG), one has:

$$\mathrm{Var}(f) = \int_{(0,1)^2} [p(1-q)\wedge q(1-p)]\alpha'(p)\alpha'(q)\,dp\,dq. \tag{CJ}$$

In fact, one can define a scalar product [†] $\langle\cdot,\cdot\rangle_V$ on $\mathscr{C}^0(0,1)$ by setting

$$\langle\varphi,\psi\rangle_V = \int_{(0,1)^2} [p(1-q)\wedge q(1-p)]\varphi(p)\psi(q)\,dp\,dq, \tag{CK}$$

so that if α is the inverse distribution fuction of a variable f, $\mathrm{Var}(f) = \|\alpha'\|_V^2$.

So, we are considering three scalar products on some subspaces of $\mathscr{C}^0(0,1)$: the ordinary L^2 product, which we denote by $\langle\cdot,\cdot\rangle_{L^2}$, the $H_0^1(0,1)$ product $\langle\cdot,\cdot\rangle_{H_0^1}$ and the variance product $\langle\cdot,\cdot\rangle_V$. Our goal is to show that for all $\varphi \in H_0^1(0,1), \psi \in \mathscr{C}^0(0,1)$,

$$|\langle\varphi,\psi\rangle_{L^2}| \leqslant \|\varphi\|_{H_0^1}\|\psi\|_V. \tag{CL}$$

By approximation we can suppose that $\varphi \in \mathscr{C}_0^\infty(0,1)$. A direct computation shows that

$$\langle\varphi,\psi\rangle_V = \langle L\varphi,\psi\rangle_{L^2}, \tag{CM}$$

where the operator $L\colon\ \mathscr{C}_0^\infty(0,1) \to \mathscr{C}^2(0,1)$ is defined by:

$$(L\varphi)(x) = x\int_0^x (1-y)\varphi(y)\,dy + (1-x)\int_x^1 y\varphi(y)\,dy. \tag{CN}$$

But we can make L appear thanks to the following formula: for $\varphi \in \mathscr{C}_0^2(0,1)$,

$$\varphi = L(-\varphi''), \tag{CO}$$

as one checks by integrating by parts twice. So,

$$|\langle\varphi,\psi\rangle_{L^2}| = |\langle L(-\varphi''),\psi\rangle_{L^2}| = |\langle -\varphi'',\psi\rangle_V| \underset{\mathrm{CS}}{\leqslant} \|\varphi''\|_V\|\psi\|_V, \tag{CP}$$

where

$$\|\varphi''\|_V^2 = \langle\varphi'',\varphi''\rangle_V = \langle L\varphi'',\varphi''\rangle_{L^2} = -\langle\varphi,\varphi''\rangle_{L^2} \underset{\mathrm{IP}}{=} \|\varphi'\|_{L^2}^2 = \|\varphi\|_{H_0^1}^2, \tag{CQ}$$

whence (CL). ♠

[†]. The positivity of $\langle\cdot,\cdot\rangle_V$ follows from the identity $\langle\varphi,\varphi\rangle_V = \int_{p<q}\big(\int_p^q \varphi(r)\,dr\big)^2\,dp\,dq$.

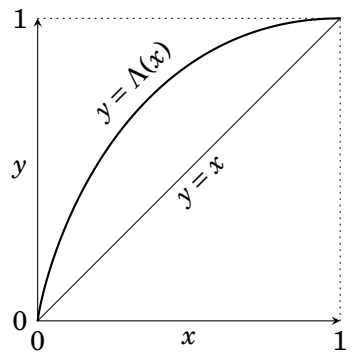

Figure 4: The function Λ.

2.2 Strong event sufficient condition

2.2.a The strong event sufficient condition

A natural choice for functions ζ and θ in Theorem 2.1.3 would be $\zeta(p) = \theta(p) = \varepsilon^{1/2} \times \sqrt{p(1-p)}$, since that would give a converse to Formula (AS) of Proposition 1.1.18. Unfortunately $\|\sqrt{p(1-p)}\|_{H_0^1} = +\infty$, so Theorem 2.1.3 does not work in this case. There is however a specific result then:

2.2.1 Theorem (Strong event sufficient condition)**.** *Let $\mathscr{F}$ and $\mathscr{G}$ be two σ-algebras such that, for all A and B resp. in $\mathscr{F}$ and $\mathscr{G}$ with respective probabilities p and q,*

$$\mathbf{P}[A \cap B] - pq \leqslant \varepsilon\sqrt{p(1-p)q(1-q)} \tag{CR}$$

for some $\varepsilon \in [0,1]$. Then

$$\{\mathscr{F} : \mathscr{G}\} \leqslant \Lambda(\varepsilon), \tag{CS}$$

where $\Lambda\colon\ [0,1] \to \mathbb{R}_+$ is defined by

$$\Lambda(\varepsilon) := \begin{cases} \varepsilon(1 + |\log \varepsilon|) & \text{if } \varepsilon > 0, \\ 0 & \text{if } \varepsilon = 0. \end{cases} \tag{CT}$$

♣

2.2.2 Remark. The function Λ is increasing on $[0,1]$ and satisfies $\Lambda(0) = 0$, $\Lambda(1) = 1$, and $\Lambda(\varepsilon) > \varepsilon$ for all $\varepsilon \in (0,1)$. Moreover it is continuous, in particular $\Lambda(\varepsilon) \searrow 0$ as $\varepsilon \searrow 0$ (see Figure 4). ♡

2.2.3 Remark. I called Theorems 2.1.3 and 2.2.1 resp. "weak" and "strong" event sufficient conditions; yet that vocabulary is a bit misleading, since the strong condition does not imply the weak one *stricto sensu*: with the hypotheses of Theorem 2.1.3 indeed, Theorem 2.2.1 only implies that

$$\{\mathscr{F} : \mathscr{G}\} \leqslant \Lambda\big(\|\zeta\|_{H_0^1}\|\theta\|_{H_0^1}\big). \tag{CU}$$

But the right-hand side of (CU) tends to 0 as soon the right-hand side of (CB) does, so it is relevant to say that Theorem 2.2.1 is 'qualitatively stronger' than Theorem 2.1.3. ♡

2.2.4 Remark. With the same informal vocabulary, Theorem 2.2.1 is a 'qualitative converse' of Proposition 1.1.18: maximal decorrelation is 'qualitatively equivalent' to decorrelation of events as defined by Formula (AS). ♡

2.2.5 Remark. One can prove that the bound $\Lambda(\varepsilon)$ in (CS) is the best possible: see § 2.2.c. ♡

Proof. The core principle of the proof is the same as for Theorem 2.1.3, except that we first perform a tricky refinement of the hypothesis: observing that, for A and B with respective probabilities p and q, one trivially has $\mathbf{P}[A \cap B] \leqslant p \wedge q$, the bound (CR) can be strengthened into:

$$\mathbf{P}[A \cap B] \leqslant \big(pq + \varepsilon\sqrt{p(1-p)q(1-q)}\big) \wedge p \wedge q. \tag{CV}$$

The right-hand side of (CV) will be denoted by $Z_\varepsilon(p,q)$.

Now, like in the proof of Theorem 2.1.3, if (CR) is satisfied, for f and g two L^2 real variables resp. $\mathscr{F}$- and $\mathscr{G}$-measurable, having respective distribution functions F and G with respective inverses maps α and β:

$$\mathrm{Cov}(f,g) \leqslant \int_{(0,1)^2} (Z_\varepsilon(p,q) - pq)\alpha'(p)\beta'(q)\,dp\,dq. \tag{CW}$$

Call $\langle \alpha', \beta' \rangle_{Z_\varepsilon}$ the right-hand side of (CW).

To bound $\langle \alpha', \beta' \rangle_{Z_\varepsilon}$, this time we are remaining on a random variable paradigm:

2.2.6 Definition. The *Chogosov law* [‡] is the (unique) probability measure μ on $(0,1)^2$ such that

$$\forall (p,q) \in (0,1)^2 \qquad \mu[(0,p) \times (0,q)] = Z_\varepsilon(p,q). \tag{CX}$$

♢

It shall be proved in § 2.2.b that the Chogosov law actually exists. μ is invariant under switching x_1 and x_2 as the function $Z(p,q)$ is, and its marginals are uniform on $(0,1)$ as $Z(p,1) \equiv p$.

The Chogosov law is linked to $\langle \cdot, \cdot \rangle_{Z_\varepsilon}$ by the operator defined next:

2.2.7 Definition. For $p \in (0,1)$, denote by μ_p the conditional law of x_2 knowing that $x_1 = p$ under μ: $(\mu_p)_{p \in (0,1)}$ is the family of probability laws on $(0,1)$ such that for all measurable $X \subset (0,1)^2$,

$$\mu[X] = \int_0^1 \mu_p[\{q : \ (p,q) \in X\}]\,dp. \text{[§]} \tag{CY}$$

♢

2.2.8 Definition. Let $\mathscr{L}$ be the operator on bounded functions on $(0,1)$ defined by

$$(\mathscr{L}\beta)(p) := \int_0^1 \beta(q)\,d\mu_p[q]; \tag{CZ}$$

in other words, $\mathscr{L}$ is the generator of the stationary Markow chain $\ldots \to r_0 \to r_1 \to \ldots$ with uniform equilibrium measure on $(0,1)$ such that the (r_i, r_{i+1}) have law μ. ♢

Then the very definition of $\langle \cdot, \cdot \rangle_{Z_\varepsilon}$ yields:

$$\langle \alpha', \beta' \rangle_{Z_\varepsilon} = \mathrm{Cov}(\alpha, \mathscr{L}\beta), \tag{DA}$$

[‡]. So called in honour of my dear friend M. K. Chogosov.

[§]. *Stricto sensu* the family of measures $(\mu_p)_{p \in (0,1)}$ is only defined up to a.s. equality; however, this family has a unique version such that $p \mapsto \mu_p$ is continuous for the weak convergence of measures, which is the one that we will consider in the sequel.

where by writing "$\mathrm{Cov}(\alpha, \mathscr{L}\beta)$" I consider functions α and $\mathscr{L}\beta$ as real random variables on the probability space $(0,1)$ endowed with the uniform measure.

By the Cauchy–Schwarz inequality, it is then enough to prove that $\mathrm{Var}(\mathscr{L}\beta) \leqslant \Lambda(\varepsilon)\mathrm{Var}(\beta)$, i.e. that the operator norm of $\mathscr{L}$ on $\overline{L}^2(0,1)$ is bounded above by $\Lambda(\varepsilon)$. That work is achieved by Lemma 2.2.13 in next subsection. ♠

2.2.b The Chogosov law

This subsection deals with the "Chogosov law", which we introduced in the proof of Theorem 2.2.1.

☛ *Throughout this subsection we suppose $\varepsilon \in (0,1)$ fixed and we write Λ for $\Lambda(\varepsilon)$, resp. Z for Z_ε. The drawings will be made for $\varepsilon = 1/2$.*

Recall Definition 2.2.6 of the Chogosov law. We first have to check that the Chogosov law actually exists:

2.2.9 Claim. *There exists a (unique) probability measure μ on $(0,1)^2$ such that*

$$\forall p,q \in [0,1] \quad \mu[\{(x_1,x_2) \in (0,1)^2 : \ x_1 \leqslant p \text{ and } x_2 \leqslant q\}] = Z(p,q), \tag{DB}$$

where we recall that

$$Z(p,q) := \big(pq + \varepsilon\sqrt{p(1-p)q(1-q)}\big) \wedge p \wedge q. \tag{DC}$$

♣

Proof. (DB) means that the density of μ on $(0,1)^2$ is equal to the distribution $\partial^2_{x_1x_2}Z$; the non-trivial point consists in proving that that distribution is nonnegative.

☛ *From now on in this subsection elements of $(0,1)^2$ will be automatically denoted by (p,q). Moreover, we will denote $\overline{p} := 1-p$ and $\tilde{p} := p - 1/2$, resp. $\overline{q} := 1-q$ and $\tilde{q} := q - 1/2$.*

The analytic formula defining $Z(p,q)$ depends on the zone of $(0,1)^2$ in which (p,q) lies (see Figure 5):

- If $p\overline{q}/q\overline{p} \leqslant \varepsilon^2$, then $Z(p,q) = p$ and we will say that we are in zone ①;
- If $\varepsilon^2 \leqslant p\overline{q}/q\overline{p} \leqslant \varepsilon^{-2}$, then $Z(p,q) = pq + \sqrt{p\overline{p}q\overline{q}}$ and we will say that we are in zone ②;
- If $\varepsilon^{-2} \leqslant p\overline{q}/q\overline{p}$, then $Z(p,q) = q$ and we will say that we are in zone ③.

So the expression of $\partial_p Z$ depends on the zone where one lies: in ① it is "1", in ② it is "$q - \varepsilon\tilde{p}\sqrt{q\overline{q}/p\overline{p}}$", and in ③ it is "0". Anyway it is defined and finite evererywhere, just having jumps at the borders between the zones, which borders we will denote respectively $\mathfrak{U}$ for the border between ① and ②, and $\mathfrak{D}$ for the border between ② and ③ (see Figure 5). To prove that the distribution $\partial^2_{pq}Z$ is nonnegative, we have to show that $\partial_p Z$ is increasing in q at p fixed. Let us check it:

- In ① and ③, $\partial_p Z$ is differentiable with $\partial_q(\partial_p Z) = 0 \geqslant 0$;
- In ②, $\partial_p Z$ is differentiable with $\partial_q(\partial_p Z) = 1 + \varepsilon\tilde{p}\tilde{q}/\sqrt{p\overline{p}q\overline{q}}$. Denoting by $\rho(p,q)$ that expression, let us prove that $\rho(p,q)$ is nonnegative (and even positive) in ②: either $\tilde{p}$ and $\tilde{q}$ have the same sign and then $\rho(p,q)$ is trivially $\geqslant 1$, or $\tilde{p}$ and $\tilde{q}$ have opposite signs. In the latter case, say for instance that ($\tilde{p} \geqslant 0$ and $\tilde{q} \leqslant 0$). Then $p \geqslant 1/2$ and $q \leqslant 1/2$, so $|\tilde{p}| = p - 1/2 < p$ and $|\tilde{q}| = 1/2 - q < \overline{q}$, which implies that

$$\varepsilon\frac{|\tilde{p}\tilde{q}|}{\sqrt{p\overline{p}q\overline{q}}} < \varepsilon\sqrt{\frac{p\overline{q}}{q\overline{p}}} \overset{②}{\leqslant} \varepsilon\sqrt{\varepsilon^{-2}} = 1, \tag{DD}$$

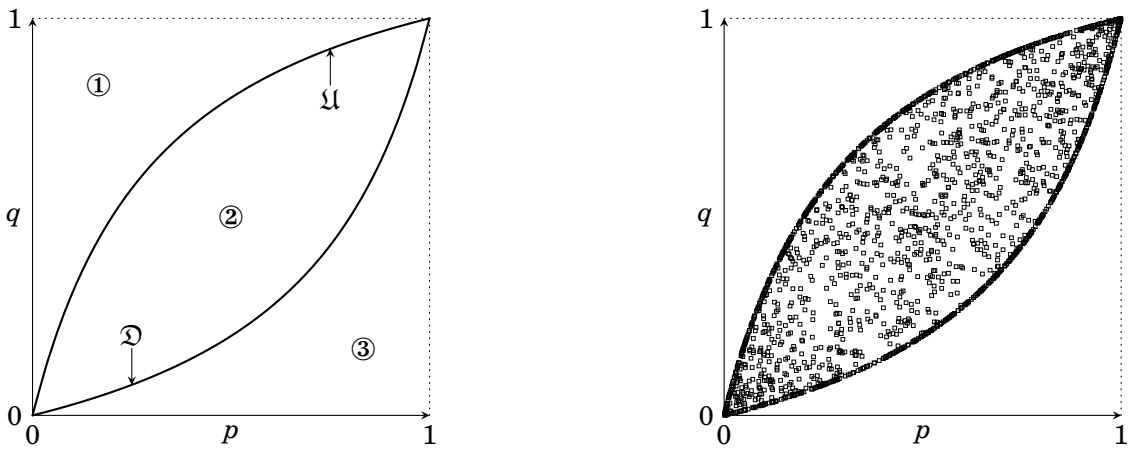

Figure 5: The Chogosov law μ. On the left are drawn the different zones relative to the support of the measure; on the right is a cloud of 2,048 independent points with law μ.

so that $\rho(p,q) > 0$.

- On $\mathfrak{D}$, $\partial_p Z$ makes a jump. Denote by $q_p^{\mathfrak{D}}$ the unique q such that $(p,q) \in \mathfrak{D}$. When q tends to $q_p^{\mathfrak{D}}$ by lower values, (p,q) is in ③, so $\partial_p Z(p, q_p^{\mathfrak{D}}-) = 0$, while when q tends to $q_p^{\mathfrak{D}}$ by upper values, (p,q) is in ②, so $\partial_p Z(p, q_p^{\mathfrak{D}}+) = q_p^{\mathfrak{D}} - \varepsilon\tilde{p}\sqrt{q_p^{\mathfrak{D}}\bar{q}_p^{\mathfrak{D}}/p\bar{p}}$. But on $\mathfrak{D}$, $q\bar{p} = \varepsilon^2 p\bar{q}$, so
$$q_p^{\mathfrak{D}} - \varepsilon\tilde{p}\sqrt{\frac{q_p^{\mathfrak{D}}\bar{q}_p^{\mathfrak{D}}}{p\bar{p}}} = q_p^{\mathfrak{D}} - \varepsilon\tilde{p}\sqrt{\frac{(q_p^{\mathfrak{D}})^2}{\varepsilon^2 p^2}} = q_p^{\mathfrak{D}}\Big(1 - \frac{\tilde{p}}{p}\Big) = \frac{q_p^{\mathfrak{D}}}{2p} > 0, \tag{DE}$$
so that the jump of $\partial_p Z(p,\cdot)$ at $q_p^{\mathfrak{D}}$ occurs in the increasing sense.
- Similarly we find that on $\mathfrak{U}$, with obvious notation, $\partial_p Z(p, q_p^{\mathfrak{U}}+) - \partial_p Z(p, q_p^{\mathfrak{U}}-) = \bar{q}_p^{\mathfrak{U}}/2\bar{p} > 0$.

So we have proved that $\partial_p Z(p,q)$ is increasing in q, which is what we wanted. ♠

2.2.10 Remark. The measure μ has a rather complicated structure: it is supported by zone ②; it has density $1 + \varepsilon\tilde{p}\tilde{q}/\sqrt{p\bar{p}q\bar{q}}$ w.r.t. the Lebesgue measure in the interior of that zone, and on its boundaries it has a *linear* density giving a mass $(q/2p)dp$ to the infinitesimal part of $\mathfrak{D}$ of abscissa p, resp. a mass $(\bar{q}/2\bar{p})dp$ to the infinitesimal part of $\mathfrak{U}$ of abscissa p. See Figure 5. ♡

Now that its existence is ensured, we notice a crucial property of the operator $\mathscr{L}$:

2.2.11 Proposition. *$\mathscr{L}$ is self-adjoint on $L^2(0,1)$.* ♣

2.2.12 Remark. As $\mathscr{L}1 \equiv 1$, we can also consider $\mathscr{L}$ as an operator on the quotient space $\bar{L}^2(0,1)$, on which it shall also be self-adjoint. ♡

Proof of Proposition 2.2.11. Indeed $\langle \alpha, \mathscr{L}\beta \rangle = \mathbf{E}_\mu[\alpha(p)\beta(q)]$, which is invariant under switching α and β as μ is invariant under switching p and q. ♠

Now we can turn to the main result of this subsection:

2.2.13 Lemma. *The operator norm of $\mathscr{L}$ on $\bar{L}^2(0,1)$ is bounded above by Λ.* ♣

Proof. Let $\eta \in (0,1/2)$, devised to tend to 0, and define the distance d_η on $(0,1)$ by:
$$\forall p_1 < p_2 \qquad d_\eta(p_1,p_2) := \int_{p_1}^{p_2} (p\bar{p})^{-3/2+\eta}\, dp. \tag{DF}$$

For continuous $f:\ (0,1)\to\mathbb{R}$, define

$$\|f\|_{Lip(\eta)} := \sup_{p_1\neq p_2} \frac{|f(p_2)-f(p_1)|}{d_\eta(p_1,p_2)}, \tag{DG}$$

and denote by $Lip(\eta)$ the set of functions f with $\|f\|_{Lip(\eta)}<\infty$. $Lip(\eta)$ is obviously complete for $\|\cdot\|_{Lip(\eta)}$, yet that semi-norm is not definite since it is zero for any constant function. We thus define $\overline{Lip}(\eta)$ as $Lip(\eta)/\mathbb{R}$, which is actually a Banach space. I claim that

2.2.14 Claim. *$\overline{Lip}(\eta)$ is continuously imbedded in $\overline{L}^2(0,1)$, i.e. there exists some $C<\infty$ (depending on η) such that for all $f\in Lip(\eta)$, $\mathrm{Sd}(f)\leqslant C\|f\|_{Lip(\eta)}$.* ♣

Proof of Claim 2.2.14. Fix some arbitrary $p_0\in(0,1)$. For $f\in Lip(\eta)$, denoting $y_0:=f(p_0)$, one has, for all $p\in(0,1)$,

$$|f(p)-y_0|\leqslant \|f\|_{Lip(\eta)}\Big|\int_{p_0}^{p}(q\overline{q})^{-3/2+\eta}dq\Big|, \tag{DH}$$

whence:

$$\mathrm{Sd}(f)\leqslant\sqrt{\int_0^1|f(p)-y_0|^2dp}\leqslant\|f\|_{Lip(\eta)}\sqrt{\int_0^1\Big(\int_{p_0}^{p}(q\overline{q})^{-3/2+\eta}dq\Big)^2dp}. \tag{DI}$$

Since $\eta>0$, the integral in the right-hand side of (DI) is finite, which proves the claim. ♠

Now, the cruxpoint is the following claim, whose proof is postponed:

2.2.15 Claim. *(i) There exists a constant $\Lambda_\eta<\infty$ such that for all $f\in\overline{Lip}(\eta)$, $\|\mathscr{L}f\|_{Lip(\eta)}\leqslant\Lambda_\eta\|f\|_{Lip(\eta)}$.*

(ii) It is possible to choose Λ_η so that $\underline{\lim}_{\eta\searrow 0}\Lambda_\eta\leqslant\Lambda$.

♣

Using Claims 2.2.14 and 2.2.15, one has for all $f\in\overline{Lip}(\eta)$, for $n\in\mathbb{N}$,

$$\mathrm{Sd}(\mathscr{L}^nf)\leqslant C\|\mathscr{L}^nf\|_{Lip(\eta)}\leqslant C\Lambda_\eta^n\|f\|_{Lip(\eta)}\overset{n\to\infty}{=}O(\Lambda_\eta^n). \tag{DJ}$$

By Lemma 0.3.1, since $\mathscr{L}$ is self-adjoint on $\overline{L}^2(0,1)$ and $\overline{Lip}(\eta)$ is a dense subset of $\overline{L}^2(0,1)$, (DJ) implies that $|||\mathscr{L}|||_{\overline{L}^2(0,1)}\leqslant\Lambda_\eta$. Making $\eta\searrow 0$, $|||\mathscr{L}|||_{\overline{L}^2(0,1)}\leqslant\Lambda$, QED. ♠

Proof of Claim 2.2.15. The proof relies on monotone rearrangement of measures (cf. [82, p. 75]). For $p\in(0,1)$, $\omega\in[0,1]$, define

$$Q(p,\omega):=\inf\{q\in(0,1):\ \partial_pZ(p,q)\geqslant\omega\} \tag{DK}$$

(see Figure 6), so that $Q(p,\omega)$ is nondecreasing in ω and that, for ω with uniform law on $[0,1]$, the law of $Q(p,\omega)$ is the conditioned version μ_p of the Chogosov law, therefore giving:

$$(\mathscr{L}f)(p)=\mathbf{E}[f(Q(p,\omega))]^{[\P]}. \tag{DL}$$

Then one has the following 'coupling formula':

$$(\mathscr{L}f)(p_2)-(\mathscr{L}f)(p_1)=\mathbf{E}[f(Q(p_2,\omega))-f(Q(p_1,\omega))]. \tag{DM}$$

From (DM) we deduce that

$$|(\mathscr{L}f)(p_2)-(\mathscr{L}f)(p_1)|\leqslant\|f\|_{Lip(\eta)}\mathbf{E}[d_\eta(Q(p_1,\omega),Q(p_2,\omega))]. \tag{DN}$$

[¶]. Beware that here the expectation is not taken w.r.t. p but w.r.t. ω.

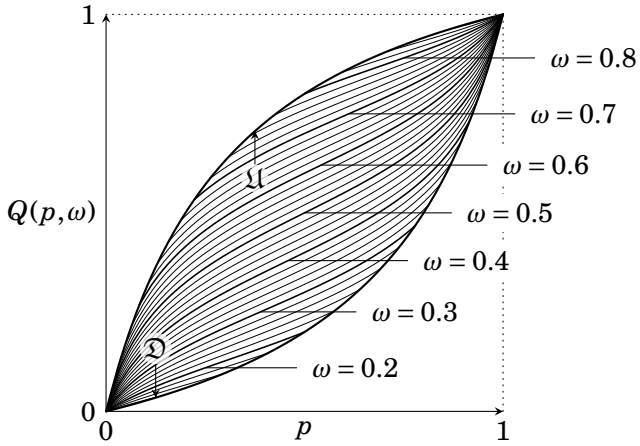

Figure 6: The function $Q(p,\omega)$. This drawing plots the functions $Q(\cdot,\omega)$ for values of ω running from 0 to 1 with step 0.02. Note that all these functions are defined on the whole $(0,1)$: in fact the graph of Q 'merges' with $\mathfrak{D}$ beyond a certain point for $\omega < 1/2$, resp. it merges with $\mathfrak{U}$ below a certain point for $\omega > 1/2$. For $\omega < \varepsilon^2/2$, resp. $\omega > 1-\varepsilon^2/2$ (which corresponds here to $\omega < 0.125$, resp. $\omega > 0.875$), the whole graph of $Q(p,\cdot)$ is actually equal to the curve $\mathfrak{D}$, resp. $\mathfrak{U}$.

So, if we can prove that for all $p_1 < p_2$,

$$\mathbf{E}[d_\eta(Q(p_1,\omega),Q(p_2,\omega))] \leqslant \Lambda_\eta d_\eta(p_1,p_2), \tag{DO}$$

then we are done.

Now I claim (it will be checked later) that Q is absolutely continuous w.r.t. p, i.e. that there exists an integrable function $Q'\colon (0,1)\times[0,1]\to\mathbb{R}$ such that for all ω, p_1, p_2 one has $Q(p_2,\omega) = Q(p_1,\omega) + \int_{p_1}^{p_2} Q'(p,\omega)\,dp$. Introducing that function, (DO) becomes:

$$\mathbf{E}\Big[\Big|\int_{p_1}^{p_2} (Q(p,\omega)\bar{Q}(p,\omega))^{-3/2+\eta} Q'(p,\omega)\,dp\Big|\Big] \leqslant \Lambda_\eta \mathbf{E}\Big[\int_{p_1}^{p_2} (p\bar{p})^{-3/2+\eta}\,dp\Big], \tag{DP}$$

so by Fubini's theorem (which is legal here since, as we will see later, Q' is bounded), proving (DP) for all $p_1 < p_2$ is tantamount to proving that, for all $p \in (0,1)$,

$$\mathbf{E}\big[(Q(p,\omega)\bar{Q}(p,\omega))^{-3/2+\eta}|Q'(p,\omega)|\big] \leqslant \Lambda_\eta (p\bar{p})^{-3/2+\eta}. \tag{DQ}$$

So we have to compute $Q'(p,\omega)$. Using the structure of the law μ (cf. Remark 2.2.10), we find the following (see Figure 6):

- First if $\omega < q_p^{\mathfrak{D}}/2p$, then $Q(p,\omega) = q_p^{\mathfrak{D}}$, whence $Q'(p,\omega) = dq_p^{\mathfrak{D}}/dp$. Differentiating the equality $q\bar{p} = \varepsilon^2 p\bar{q}$ defining $\mathfrak{D}$, one finds that $dq_p^{\mathfrak{D}}/dp = (q_p^{\mathfrak{D}} + \varepsilon^2\bar{q}_p^{\mathfrak{D}})/(\bar{p}+\varepsilon^2 p)$, which simplifies into $q_p^{\mathfrak{D}}\bar{q}_p^{\mathfrak{D}}/p\bar{p}$ using once again that $q\bar{p} = \varepsilon^2 p\bar{q}$.
- Similarly if $\omega > 1 - \bar{q}_p^{\mathfrak{U}}/2\bar{p}$, one has $Q'(p,\omega) = q_p^{\mathfrak{U}}\bar{q}_p^{\mathfrak{U}}/p\bar{p}$.
- If $q_p^{\mathfrak{D}}/2p < \omega < 1 - \bar{q}_p^{\mathfrak{U}}/2\bar{p}$, then $\partial_p Z(p,q) = q - \varepsilon\tilde{p}\sqrt{q\bar{q}/p\bar{p}}$, thus differentiating the equality $\partial_p Z(p,Q(p,\omega)) = \omega$, we get:

$$Q'(p,\omega) = \frac{\varepsilon\sqrt{Q(p,\omega)\bar{Q}(p,\omega)}}{4\sqrt{p\bar{p}}^3\Big(1+\varepsilon\frac{\tilde{p}\tilde{Q}(p,\omega)}{\sqrt{p\bar{p}Q(p,\omega)\bar{Q}(p,\omega)}}\Big)}. \tag{DR}$$

- Finally in the critical cases $\omega = q_p^{\mathfrak{D}}/2p, 1-\bar{q}_p^{\mathfrak{U}}/2\bar{p}$, there is no canonical value for $Q'(\omega)$ since at these points $Q(\cdot,\omega)$ is not $\mathscr{C}^1$, but that does not matter.

2.2.16 Remark. Note that one always has $Q'(p,\omega) > 0$, i.e. $Q(\cdot,\omega)$ is increasing. In other words, for $p_1 < p_2$, μ_{p_1} is stochastically smaller than μ_{p_2}. ♡

We have computed $Q'(p,\omega)$, so now we can tackle (DQ): we have to bound

$$\int_0^1 \Big(\frac{p\bar{p}}{Q(p,\omega)\bar{Q}(p,\omega)}\Big)^{3/2-\eta} Q'(p,\omega)d\omega, \tag{DS}$$

uniformly in p. We begin with noticing that

2.2.17 Claim. *For all $p \in (0,1)$, all $q \in [q_p^{\mathfrak{D}}, q_p^{\mathfrak{U}}]$, one has $q\bar{q}/p\bar{p} \leqslant \varepsilon^{-2}$.* ♣

Proof of Claim 2.2.17. The condition $q \in [q_p^{\mathfrak{D}}, q_p^{\mathfrak{U}}]$ means that $\varepsilon^2 \leqslant p\bar{q}/q\bar{p} \leqslant \varepsilon^{-2}$. Then we distinguish two cases:

- If $p \leqslant q$, then $q\bar{q}/p\bar{p} = (\bar{q}/\bar{p})^2 q\bar{p}/p\bar{q} \leqslant q\bar{p}/p\bar{q} = (p\bar{q}/q\bar{p})^{-1} \leqslant \varepsilon^{-2}$;
- If $p \geqslant q$, then $q\bar{q}/p\bar{p} = (q/p)^2 p\bar{q}/q\bar{p} \leqslant p\bar{q}/q\bar{p} \leqslant \varepsilon^{-2}$.

♠

Thanks to Claim 2.2.17, we bound (DS) by

$$\varepsilon^{-2\eta}\int_0^1 \Big(\frac{p\bar{p}}{Q(p,\omega)\bar{Q}(p,\omega)}\Big)^{3/2} Q'(p,\omega)d\omega, \tag{DT}$$

which we shorthand into $\varepsilon^{-2\eta}\lambda(p)$. Splitting the integral in (DT) according to the value of ω (resp. for $\omega \in (0, q_p^{\mathfrak{D}}/2p)$, $(q_p^{\mathfrak{D}}/2p, 1-\bar{q}_p^{\mathfrak{U}}/2\bar{p})$ and $(1-\bar{q}_p^{\mathfrak{U}}/2\bar{p}, 1)$), one finds:

$$\begin{aligned}
\lambda(p) &= \frac{q_p^{\mathfrak{D}}}{2p}\Big(\frac{p\bar{p}}{q_p^{\mathfrak{D}}\bar{q}_p^{\mathfrak{D}}}\Big)^{3/2}\frac{q_p^{\mathfrak{D}}\bar{q}_p^{\mathfrak{D}}}{p\bar{p}} && \text{(DU)}\\
&+ \int_{q_p^{\mathfrak{D}}/2p}^{1-\bar{q}_p^{\mathfrak{U}}/2\bar{p}} \Big(\frac{p\bar{p}}{Q(p,\omega)\bar{Q}(p,\omega)}\Big)^{3/2} \frac{\varepsilon\sqrt{Q(p,\omega)\bar{Q}(p,\omega)}}{4\sqrt{p\bar{p}}^3\Big(1+\varepsilon\frac{\tilde{p}\tilde{Q}(p,\omega)}{\sqrt{p\bar{p}Q(p,\omega)\bar{Q}(p,\omega)}}\Big)}d\omega && \text{(DV)}\\
&+ \frac{\bar{q}_p^{\mathfrak{U}}}{2\bar{p}}\Big(\frac{p\bar{p}}{q_p^{\mathfrak{U}}\bar{q}_p^{\mathfrak{U}}}\Big)^{3/2}\frac{q_p^{\mathfrak{U}}\bar{q}_p^{\mathfrak{U}}}{p\bar{p}}. && \text{(DW)}
\end{aligned}$$

(DU) simplifies into $\frac{1}{2}\sqrt{q_p^{\mathfrak{D}}\bar{p}/p\bar{q}_p^{\mathfrak{D}}} = \frac{1}{2}\sqrt{\varepsilon^2} = \varepsilon/2$; similarly (DW) $= \frac{1}{2}\sqrt{p\bar{q}_p^{\mathfrak{U}}/q_p^{\mathfrak{U}}\bar{p}} = \varepsilon/2$. Concerning term (DV), we make the change of variables $q = Q(p,\omega)$, for which $d\omega = (1+\varepsilon\tilde{p}\tilde{q}/\sqrt{p\bar{p}q\bar{q}})dq$ because of the expression of the density of μ in zone ② (cf. Remark 2.2.10). One gets:

$$\text{(DV)} = \frac{\varepsilon}{4}\int_{q_p^{\mathfrak{D}}}^{q_p^{\mathfrak{U}}}\frac{1}{q\bar{q}}dq = \frac{\varepsilon}{4}\Big[\log\frac{q}{\bar{q}}\Big]_{q_p^{\mathfrak{D}}}^{q_p^{\mathfrak{U}}} = \frac{\varepsilon}{4}\Big(\log\frac{\bar{p}}{\varepsilon^2 p} - \log\frac{\varepsilon^2\bar{p}}{p}\Big) = \frac{\varepsilon}{4}\log\frac{1}{\varepsilon^4} = \varepsilon|\log\varepsilon|. \tag{DX}$$

So in the end we have $\lambda(p) = \varepsilon/2 + \varepsilon/2 + \varepsilon|\log\varepsilon| = \Lambda$ for all p, thus $\Lambda_\eta \leqslant \varepsilon^{-2\eta}\Lambda$ (hence (i)), which tends to Λ as $\eta \searrow 0$ (hence (ii)). ♠

2.2.18 Remark. The simplifications in the computation of $\lambda(p)$ look rather miraculous… *A priori* I only expected that $\lambda(p) \leqslant \Lambda$ on $(0,1)$ with $\lambda(p) \overset{p\to 0,1}{\to} \Lambda$). That I found the exact quasi-eigenvector associated to the quasi-eigenvalue Λ (cf. Remark 2.2.19) is purely fortuitous; I have no simple explanation for why things work so well. ♡

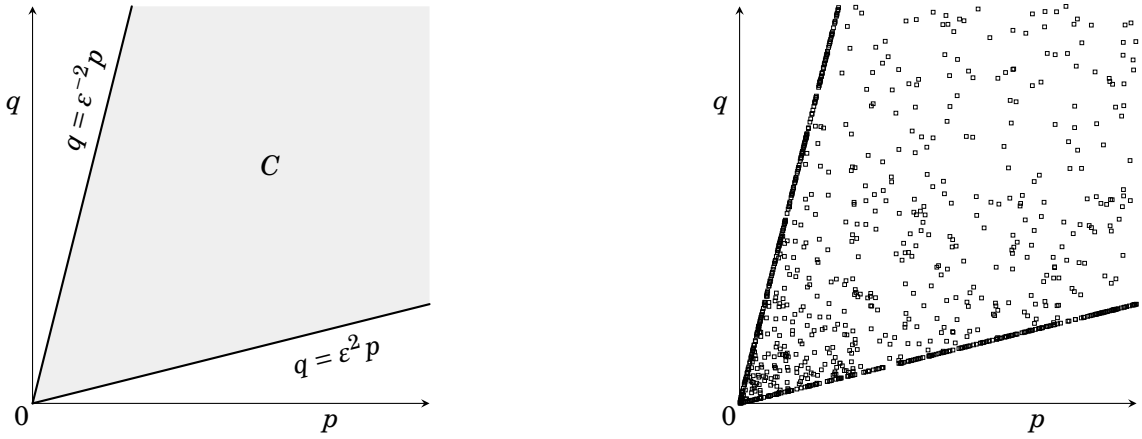

Figure 7: The measure μ^*. On the left, the different zones for the measure; on the right, a Poisson cloud of points with density μ^*. The scale and density of the cloud are consistent with Figure 5.

2.2.19 Remark. $\mathscr{L}$ is self-adjoint, hence normal, so its operator norm is also its spectral radius. Therefore there is some (*eigenvalue*,*eigenvector*) pair, or more precisely (since here the spectral radius of $\mathscr{L}$ is due to its continuous spectrum) some 'quasi-eigenvalue' and its 'quasi-eigenvector' (cf. [70, § 4]), which are responsible for the value of the operator norm.

Tracking this quasi-eigenvector throughout the proof of Lemma 2.2.13, we find that Λ is a quasi-eigenvalue of $\mathscr{L}$ and that the associated quasi-eigenvector is:

$$f_\Lambda : \; p \mapsto \int_{1/2}^{p} (p'\overline{p}')^{-3/2} dp'. \tag{DY}$$

Obviously f_λ is not in L^2, so it is not a true eigenvector; however one can perturb it slightly to get an element $\widetilde{f}_\Lambda \in \overline{L}^2(0,1) \smallsetminus \{0\}$ such that $\langle \mathscr{L}\widetilde{f}_\Lambda, \widetilde{f}_\Lambda \rangle_{\overline{L}^2(0,1)} / \|\widetilde{f}_\Lambda\|^2_{\overline{L}^2(0,1)}$ is arbitrarily close to Λ. ♡

2.2.20 Remark. An interesting feature of f_Λ is that its 'L^2 mass' is concentrated about 0 and 1, so that one needs only look at what happens near 0 and 1 to understand how f_Λ contributes to the operator norm of $\mathscr{L}$.

When one 'zooms' more and more to the point $(0,0)$ —the same behaviour would happen about $(1,1)$ —, μ 'looks more and more like' the measure μ^* on $(0,\infty)^2$ defined by (see Figure 7):

$$\forall p,q \in [0,\infty)^2 \quad \mu^*\big[\{(x_1,x_2) \in (0,\infty)^2 : \; x_1 \leqslant p \text{ and } x_2 \leqslant q\}\big] = \varepsilon\sqrt{pq} \wedge p \wedge q, \tag{DZ}$$

i.e.

$$d\mu^*(p,q) = \mathbf{1}_{\{\varepsilon^2 p < q < \varepsilon^{-2} p\}} \frac{\varepsilon}{4\sqrt{pq}} dp\,dq + \mathbf{1}_{\{q = \varepsilon^2 p\}} \frac{\varepsilon^2}{2} dp + \mathbf{1}_{\{q = \varepsilon^{-2} p\}} \frac{1}{2} dp. \tag{EA}$$

So, near 0, $\mathscr{L}$ behaves like the operator $\mathscr{L}^*$ on $L^2(0,\infty)$ defined by:

$$(\mathscr{L}^* f)(p) = \int_{\varepsilon^2 p}^{\varepsilon^{-2} p} \frac{\varepsilon}{4\sqrt{pq}} dq + \frac{\varepsilon^2}{2} f(\varepsilon^2 p) + \frac{1}{2} f(\varepsilon^{-2} p). \tag{EB}$$

$\mathscr{L}^*$ has scale invariance properties which make it easy to study. One finds that $\mathscr{L}^*$ is self-adjoint, that its spectral radius is Λ, and that it has Λ as a quasi-eigenvalue, associated with the quasi-eigenvector $(p \mapsto 1/\sqrt{p})$. So, you see that it suffices to study the 'local' operator $\mathscr{L}^*$ to compute the spectral radius of the 'global' operator $\mathscr{L}$; in other words, there is a phenomenon of 'localization of the spectral radius' for $\mathscr{L}$. ♡

2.2.c Optimality of the strong event sufficient condition

Now I will prove that Theorem 2.2.1 is optimal:

2.2.21 Theorem. *The factor $\Lambda(\varepsilon)$ in (CS) cannot be improved. In other words, for all $\Lambda' < \Lambda(\varepsilon)$ it is possible to find σ-fields $\mathscr{F}$ and $\mathscr{G}$ satisfying*

$$\forall A \in \mathscr{F}, B \in \mathscr{G} \quad \mathbf{P}[A \cap B] - \mathbf{P}[A]\mathbf{P}[B] \leqslant \varepsilon \sqrt{\mathbf{P}[A]\mathbf{P}[A^c]\mathbf{P}[B]\mathbf{P}[B^c]}, \tag{EC}$$

but such that $\{\mathscr{F} : \mathscr{G}\} \geqslant \Lambda'$. ♣

2.2.22 Remark. One can automatically add absolutes values in the left-hand side of the condition (EC), since $-\big(\mathbf{P}[A \cap B] - \mathbf{P}[A]\mathbf{P}[B]\big) = \mathbf{P}[A \cap B^c] - \mathbf{P}[A]\mathbf{P}[B^c]$. ♡

Actually I will rather prove the following statement, which is equivalent to the theorem by continuity of the function $\Lambda(\cdot)$:

2.2.23 Claim. *For all $\varepsilon' > \varepsilon$ it is possible to find σ-fields $\mathscr{F}$ and $\mathscr{G}$ satisfying $\{\mathscr{F} : \mathscr{G}\} \geqslant \Lambda(\varepsilon)$, but such that*

$$\forall A \in \mathscr{F}, B \in \mathscr{G} \quad \mathbf{P}[A \cap B] - \mathbf{P}[A]\mathbf{P}[B] \leqslant \varepsilon' \sqrt{\mathbf{P}[A]\mathbf{P}[A^c]\mathbf{P}[B]\mathbf{P}[B^c]}. \tag{ED}$$

♣

Proof. According to the proof of Theorem 2.2.1, the 'natural' proof would be to take for space $(\Omega, \mathscr{B}, \mathbf{P})$ the set $(0,1)^2$ equipped with its Borel σ-field and endowed with the Chogosov law μ, and to set $\mathscr{F} = \sigma(p)$ and $\mathscr{G} = \sigma(q)$. Though it seems to be true that that system satisfies (CR), the complicated structure of μ makes existence of a short proof for that property unlikely. Therefore I will rather adapt the previous idea to the nicer measure μ^* defined by (DZ), or more precisely to a 'truncation' of it.

My system is the following: $(\Omega, \mathscr{B}, \mathbf{P})$ is the set $(0,1)^2$ equipped with its Borel σ-field and endowed with a certain measure ν (specified just after), and I take $\mathscr{F} = \sigma(p)$, resp. $\mathscr{G} = \sigma(q)$. The measure ν, which depends on some parameter $x \in (0,1)$ morally close to 0, is a measure on $(0,1)^2$ having uniform marginals, which coincides with μ^* on $(0,x]^2$ and which is 'as uniform as possible' outside $(0,x]^2$ (see Figure 8). Technically:

$$\nu[A \times B] = \begin{cases} \mu^*(A \times B) & \text{if } A \subset (0,x] \text{ and } B \subset (0,x]; \\ 0 & \text{if } A \subset (0,\varepsilon^2 x] \text{ and } B \subset (x,1); \\ 0 & \text{if } A \subset (x,1) \text{ and } B \subset (0,\varepsilon^2 x]; \\ [\int_A (1 - \frac{\varepsilon}{2}\sqrt{\frac{x}{p}})dp]|B|/(1-x) & \text{if } A \subset (\varepsilon^2 x, x] \text{ and } B \subset (x,1); \\ |A|[\int_B (1 - \frac{\varepsilon}{2}\sqrt{\frac{x}{q}})dq]/(1-x) & \text{if } A \subset (x,1) \text{ and } B \subset (\varepsilon^2 x, x]; \\ [1-(2-\varepsilon)x]|A||B|/(1-x)^2 & \text{if } A \subset (x,1) \text{ and } B \subset (x,1). \end{cases} \tag{EE}$$

First step: Proof that $\{\mathscr{F} : \mathscr{G}\} \geqslant \Lambda$. Let $\Lambda' < \Lambda$. Since Λ is in the spectrum of the self-adjoint operator $\mathscr{L}^*$ on $L^2(0,\infty)$ (see (EB) and the lines just below), there exists $f \in L^2(0,\infty) \smallsetminus \{0\}$ such that $\langle \mathscr{L}^* f, f \rangle / \|f\|_{L^2(0,\infty)} > \Lambda'$. By a standard truncation argument, we can assume that f has bounded support, say that f is zero outside $(0,Y]$. Dividing f by its norm we can also assume that $\|f\|_{L^2} = 1$.

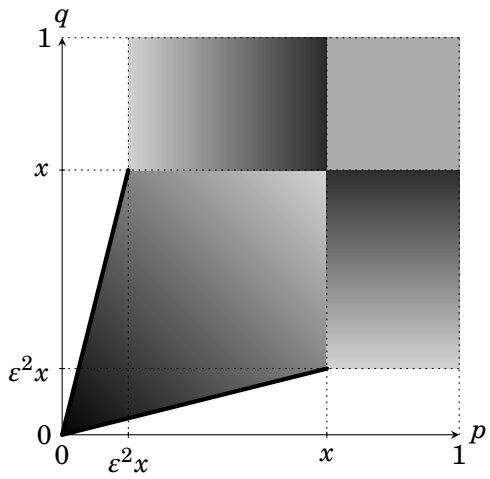

Figure 8: A schematic representation of the measure ν.

Now, for $y \in (0,x]$ define the function f_y by:

$$f_y(p) := \sqrt{\frac{Y}{y}} f\Big(\frac{Y}{y} p\Big). \tag{EF}$$

f_y is zero outside $(0,y] \subset (0,x]$; it satisfies $\|f_y\|_{L^2} = 1$ and

$$\langle \mathscr{L}^* f_y, f_y \rangle = \langle \mathscr{L}^* f, f \rangle > \Lambda' \tag{EG}$$

by the scale invariance properties of $\mathscr{L}^*$.

Denote $m := \int f(p)dp/\sqrt{Y}$, which is finite since f is L^2 with compact support; one has $\int f_y(p)dp = \sqrt{y}m$, so the projection of f_y on $\bar{L}^2(0,1)$ is the function $\bar{f}_y = f_y - \sqrt{y}m$. One has $\|\bar{f}_y\|_{\bar{L}^2(0,1)} \leqslant \|f_y\|_{L^2} = 1$, and

$$\mathbf{E}[\bar{f}_y(p)\bar{f}_y(q)] = \langle \mathscr{L}^* f_y, f_y \rangle - m^2 y > \Lambda' - m^2 y, \tag{EH}$$

so that $\{\mathscr{F} : \mathscr{G}\} > \Lambda' - m^2 y$. Making $y \to 0$ and then $\Lambda' \to \Lambda$, one finally gets $\{\mathscr{F} : \mathscr{G}\} \geqslant \Lambda$.

Second step: Proof of (ED). Let $\varepsilon' > \varepsilon$; we want to prove that, provided x is small enough, (ED) is satisfied.

Let A and B be resp. $\mathscr{F}$- and $\mathscr{G}$-measurable events. One can assume safely that $|A| \leqslant 1/2$, since replacing simultaneously A by A^c and B by B^c leaves both sides of (ED) unchanged. One can also assume that $|B| < 1/(1+\varepsilon^2)$, since for $|B| \geqslant 1/(1+\varepsilon^2)$, (ED) comes 'for nothing' by writing

$$\mathbf{P}[A \cap B] - \mathbf{P}[A]\mathbf{P}[B] \leqslant \mathbf{P}[A](1 - \mathbf{P}[B]) \leqslant \sqrt{\mathbf{P}[A]\mathbf{P}[A^c]} \times \varepsilon \sqrt{\mathbf{P}[B]\mathbf{P}[B^c]}. \tag{EI}$$

☛ *In the sequel of this proof we indentify A and B with Borel subsets of $(0,1)$, rewriting the p-measurable event A into the set $A \times (0,1)$, resp. the q-measurable event B into the set $(0,1) \times B$. Since both marginals of ν are uniform on $(0,1)$, one then has $\mathbf{P}[A] = |A|$, resp. $\mathbf{P}[B] = |B|$, so that our goal becomes proving:*

$$\nu[A \times B] - |A||B| \leqslant \varepsilon' \sqrt{|A||B||A^c||B^c|}. \tag{EJ}$$

Denote $\check{A} := A \cap (0,x]$, resp. $\check{B} := B \cap (0,x]$. Provided $x \leqslant \varepsilon/2$, the signed measure $d\nu(p,q) - dpdq$ is nonpositive on $(0,x] \times (x,1) \cup (x,1) \times (0,x]$, so that

$$\nu[A \times B] - |A||B| \leqslant \nu[\check{A} \times \check{B}] - |\check{A}||\check{B}| + \nu[(A \smallsetminus \check{A}) \times (B \smallsetminus \check{B})] - |A \smallsetminus \check{A}||B \smallsetminus \check{B}|. \tag{EK}$$

Now let us bound above the right-hand side of (EK):

- The second term is obviously nonpositive.
- The third term is $[1-(2-\varepsilon)x]|A\smallsetminus\breve{A}||B\smallsetminus\breve{B}|/(1-x)^2$, so the sum of the two last terms is $(\varepsilon x-x^2)|A\smallsetminus\breve{A}||B\smallsetminus\breve{B}|/(1-x)^2 \leqslant (\varepsilon x-x^2)|A||B|/(1-x)^2$. Since $|A|\leqslant 1/2$ and $|B|\leqslant 1/(1+\varepsilon^2)$, that quantity is in turn bounded by $\frac{\varepsilon x-x^2}{\varepsilon(1-x)^2}\times\sqrt{|A||B||A^c||B^c|}$.
- For the first term, by Lemma 2.2.24 stated just below, one has $\nu[\breve{A}\times\breve{B}]=\mu^*[\breve{A}\times\breve{B}]\leqslant \varepsilon\sqrt{|\breve{A}||\breve{B}|}\leqslant\varepsilon\sqrt{|\breve{A}||\breve{A}^c||\breve{B}||\breve{B}^c|}/(1-x)$, in which, provided $x\leqslant\varepsilon^2/(1+\varepsilon^2)$, one has $\sqrt{|\breve{B}||\breve{B}^c|}\leqslant\sqrt{B|B^c|}$ (because then $|\breve{B}|\leqslant|B|\wedge x$ and $|B|\leqslant 1-x$), and similarly $\sqrt{|\breve{A}||\breve{A}^c|}\leqslant\sqrt{A|A^c|}$, so that in the end the first term is bounded by $\frac{\varepsilon}{1-x}\sqrt{|A||B||A^c||B^c|}$.

Summing things up, we get:

$$\nu[A\times B]-|A||B|\leqslant\Big(\frac{\varepsilon}{1-x}+\frac{\varepsilon x-x^2}{\varepsilon(1-x)^2}\Big)\sqrt{|A||B||A^c||B^c|}. \tag{EL}$$

Taking x sufficiently close to 0, the first factor of the right-hand side of (EL) is $\leqslant\varepsilon'$, whence the second step of the proof. ♠

2.2.24 Lemma. *For all $A,B\subset(0,\infty)^2$ with Lebesgue measures $|A|,|B|<\infty$, $\mu^*[A\times B]\leqslant\varepsilon\sqrt{|A||B|}$.* ♣

Proof of Lemma 2.2.24. Recall that μ^* is the Radon measure on $(0,\infty)^2$ having density $\varepsilon/4\sqrt{pq}$ w.r.t. the Lebesgue measure inside the cone $C=\{(p,q)\colon\ \varepsilon^2p<q<\varepsilon^{-2}p\}$, being zero outside C, and giving to the borders of C a lineic mass defined by $\mu^*\{(p,\varepsilon^2p)\colon\ p\in A\}=\varepsilon^2|A|/2$, resp. $\mu^*\{(p,\varepsilon^{-2}p)\colon\ p\in A\}=|A|/2$ (see Figure 7). μ^* is invariant under switching p and q, and its marginals both are the Lebesgue measure on $(0,\infty)$. Let $A,B\subset(0,\infty)$ be Borel; our goal is to show that $\mu^*[A\times B]\leqslant\varepsilon\sqrt{|A||B|}$.

Step 1. If $|A|\leqslant\varepsilon^2|B|$ the result is trivially true, since then $\mu^*[A\times B]\leqslant\mu^*[A\times(0,\infty)]=|A|\leqslant\varepsilon\sqrt{|A||B|}$. Similarly the result is true if $|B|\leqslant\varepsilon^2|A|$. Therefore in our proof we will always assume that $\varepsilon^2|A|<|B|<\varepsilon^{-2}|A|$.

Step 2. As for the measure μ, decompose the support of μ^* into three parts $\mathfrak{U}$, ② and $\mathfrak{D}$, corresponding resp. to the line "$p=\varepsilon^2q$", the cone C and the line "$p=\varepsilon^{-2}q$" (see Figure 7). Write $\mu^*[A\times B]=m_U+m_2+m_D$, where $m_U=\mu^*[(A\times B)\cap\mathfrak{U}]$, etc..

Denote by μ^*_q the 'conditioned version' of μ knowing q, i.e. the probability measure such that

$$\mu^*[X]=\int_0^\infty\mu^*_q[\{p\colon\ (p,q)\in X\}]\,dq, \tag{EM}$$

which can be computed explicitly to be:

$$d\mu^*_q[p]=\mathbf{1}_{\{p=\varepsilon^2q\}}\frac{\varepsilon^2}{2}+\mathbf{1}_{\{\varepsilon^2q<p<\varepsilon^{-2}q\}}\frac{\varepsilon}{4\sqrt{pq}}dp+\mathbf{1}_{\{p=\varepsilon^{-2}q\}}\frac{1}{2}. \tag{EN}$$

The three terms of the right-hand side of (EN) are respectively due to $\mathfrak{U}$, ② and $\mathfrak{D}$, so that, integrating the first one, one finds:

$$m_U=\int_B\frac{\varepsilon^2}{2}\mathbf{1}_{\{A\ni\varepsilon^2q\}}dq\leqslant\frac{\varepsilon^2}{2}|B|. \tag{EO}$$

Switching the roles of p and q, one has similarly $m_D\leqslant\varepsilon^2|A|/2$. Then it only remains to bound m_2.

Step 3. Let us study further the measures μ^*_q. If $q\in\varepsilon^2A$, then $A\ni\varepsilon^{-2}q$ and thus $\mu^*_q[A]\geqslant\mu^*[\{\varepsilon^{-2}q\}]=1/2$, and conversely if $q\notin\varepsilon^2A$, then $A\not\ni\varepsilon^{-2}q$ and thus $\mu^*_q[A]\leqslant 1-\mu^*[\{\varepsilon^{-2}q\}]=1/2$. So, $\mu^*_q[A]$ is never smaller if $q\in\varepsilon^2A$ than if $q\notin\varepsilon^2A$.

As a consequence, let us show that we can always assume that $\varepsilon^2 A \subset B$. Since $|B| > \varepsilon^2|A|$, $|B \smallsetminus \varepsilon^2 A| > |\varepsilon^2 A \smallsetminus B|$, so we can fix some $B^- \subset B \smallsetminus \varepsilon^2 A$ such that $|B^-| = |\varepsilon^2 A \smallsetminus B|$. One has:

$$\mu^*[A \times B^-] = \int_{B^-} \mu_q^*(A)\,dq \leqslant \frac{|B^-|}{2} = \frac{|\varepsilon^2 A \smallsetminus B|}{2} \leqslant \int_{\varepsilon^2 A \smallsetminus B} \mu_q^*(A)\,dq = \mu^*[A \times (\varepsilon^2 A \smallsetminus B)]. \tag{EP}$$

Shorthanding "$(B \smallsetminus B^-) \cup \varepsilon^2 A$" into "$B'$", (EP) implies that replacing B by B' —which does not modify the value of $|B|$ —cannot make $\mu^*[A \times B]$ decrease. Consequently, if we prove that $\mu^*[A \times B'] \leqslant \varepsilon\sqrt{|A||B'|}$, then we will also have proved that $\mu^*[A \times B] \leqslant \varepsilon\sqrt{|A||B|}$. As $\varepsilon^2 A \subset B'$, we thus have demonstrated the statement at the beginning of this paragraph: one can always assume that $\varepsilon^2 A \subset B$.

☛ *Switching the roles of p and q, we will rather impose, instead of $\varepsilon^2 A \subset B$, that $\varepsilon^2 B \subset A$.*

Step 4. Call μ° the measure μ^* restricted to C, i.e. $d\mu^\circ = \mathbf{1}_C d\mu^*$, so that $m_2 = \mu^\circ[A \times B]$. μ° is absolutely continuous w.r.t. the Lebesgue measure; denote by μ_q° its 'conditioned version' for fixed q, i.e. the measure such that

$$\mu^\circ[X] = \int_0^\infty \mu_q^\circ[\{p\colon\ (p,q) \in X\}]\,dq, \tag{EQ}$$

which has the following explicit density w.r.t. the Lebesgue measure:

$$d\mu_q^\circ[p] = \mathbf{1}_{\{\varepsilon^2 q < p < \varepsilon^{-2} q\}} \frac{\varepsilon}{4\sqrt{pq}}\,dp. \tag{ER}$$

We perform a change of variables: for $y \in (0, |B|)$, define

$$\beta(y) = \inf\{q \in (0,\infty)\colon\ |B \cap (0,q)| \geqslant y\}; \tag{ES}$$

so that the push-forward $\beta \# dq$ of the Lebesgue measure on $(0,|B|)$ by the map β is equal to $\mathbf{1}_B dq$, the Lebesgue measure restricted to B; then

$$m_{②} = \int_B \mu_q^\circ[A]\,dq = \int_0^{|B|} \mu_{\beta(y)}^\circ[A]\,dy. \tag{ET}$$

Our strategy will consist in bounding $\mu_{\beta(y)}^\circ[A]$ for all y.

First, we observe that there is some portion of A which does not contribute to $\mu_{\beta(y)}^\circ[A]$. Denote indeed $A_y := \{\varepsilon^2 q\colon\ q \in B \cap (0, \beta(y))\}$; by the definition of β, $|A_y| = \varepsilon^2 y$, and one has $A_y \subset \varepsilon^2 B \subset A$. But $A_y \subset (0, \varepsilon^2 \beta(y))$, so $\mu_{\beta(y)}^\circ[A_y] = 0$, and thus $\mu_{\beta(y)}^\circ[A] = \mu_{\beta(y)}^\circ[A \smallsetminus A_y]$, where $|A \smallsetminus A_y| = |A| - |A_y| = |A| - \varepsilon^2 y$.

Now, for $q \in (0,\infty)$, the density of μ_q° is zero for $p \leqslant \varepsilon^2 q$ and it is nonincreasing for $p > \varepsilon^2 q$, so an immediate coupling argument shows that the maximal value of $\mu_q^\circ[X]$ under the constraint "$|X| = x$" is attained for $X = (\varepsilon^2 q, \varepsilon^2 q + x)$. Applying that result to the conclusion of the previous paragraph, we get that:

$$\mu_{\beta(y)}^\circ[A] \leqslant \mu_{\beta(y)}^\circ[(\varepsilon^2 \beta(y), \varepsilon^2 \beta(y) + |A| - \varepsilon^2 y)]. \tag{EU}$$

But for $x \geqslant 0$, the quantity $\mu_q^\circ[(\varepsilon^2 q, \varepsilon^2 q + x)]$ can be computed explicitly to be

$$\mu_q^\circ[(\varepsilon^2 q, \varepsilon^2 q + x)] = \begin{cases} (1-\varepsilon^2)/2 & \text{if } q \leqslant x/(\varepsilon^{-2} - \varepsilon^2); \\ (\varepsilon\sqrt{\varepsilon^2 + x/q} - \varepsilon^2)/2 & \text{if } q > x/(\varepsilon^{-2} - \varepsilon^2). \end{cases} \tag{EV}$$

In particular, that quantity is a nonincreasing function of q. Since, by the definition of β, one always has $\beta(y) \geqslant y$, it follows that (EU) can be improved into:

$$\mu^{\circ}_{\beta(y)}[A] \leqslant \mu^{\circ}_{y}[(\varepsilon^2 y\,,\,|A|)] = \begin{cases} (1-\varepsilon^2)/2 & \text{if } y \leqslant \varepsilon^2|A|; \\ (\varepsilon\sqrt{|A|/y} - \varepsilon^2)/2 & \text{if } y > \varepsilon^2|A|. \end{cases} \tag{EW}$$

Integrating, one finds finally:

$$\begin{aligned} m_2 &\leqslant \int_0^{\varepsilon^2|A|} \frac{1-\varepsilon^2}{2}dy + \int_{\varepsilon^2|A|}^{|B|} \Big(\frac{\varepsilon\sqrt{|A|}}{2\sqrt{y}} - \frac{\varepsilon^2}{2}\Big)dy \\ &= \frac{(1-\varepsilon^2)\varepsilon^2|A|}{2} + \Big[\varepsilon\sqrt{|A|y} - \frac{\varepsilon^2 y}{2}\Big]_{\varepsilon^2|A|}^{|B|} = \varepsilon\sqrt{|A||B|} - \frac{\varepsilon^2}{2}(|A|+|B|). \end{aligned} \tag{EX}$$

Step 5. We put our bounds together to get the lemma:

$$\mu^*[A\times B] \leqslant m_D + m_U + m_2 = \frac{\varepsilon^2}{2}(|A|+|B|) + \varepsilon\sqrt{|A||B|} - \frac{\varepsilon^2}{2}(|A|+|B|) = \varepsilon\sqrt{|A||B|}. \tag{EY}$$

♠

2.2.25 Remark. A careful reading of the proof above shows that the maximal value of $\mu^*[A\times B]$ is attained for $A = (0,|A|), B = (0,|B|)$, in which case, provided $\varepsilon^2|A| \leqslant |B| \leqslant \varepsilon^{-2}|A|$, one has equality in Lemma 2.2.24. ♡

Chapter 3

Tensorization

3.1 Subjective correlation

In this chapter we will need more advanced definitions for decorrelation.

3.1.1 Definition. Let X, Y and Z be random variables. For $\varepsilon \geqslant 0$, one says that X and Y are *subjectively ε-decorrelated w.r.t. Z* (or *ε-decorrelated seen from Z*) if X and Y are ε-decorrelated under the law $Law(X,Y|Z=z)$ for $Law(Z)$-almost-all z [*].

The smallest ε such that X and Y are ε-decorrelated seen from Z will be called the *subjective correlation level between X and Y w.r.t. Z* (or *correlation level between X and Y seen from Z*); we denote it $\{X:Y\}_Z$. ♢

In § 1.1, we had given the definitions in terms of σ-algebras rather than random variables. Of course there is also a σ-algebra definition for subjective correlation, though I find it harder to understand:

3.1.2 Definition. Let $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{H}$ be σ-algebras. For $\varepsilon \in [0,1]$, the expression "$\{\mathscr{F}:\mathscr{G}\}_{\mathscr{H}} \leqslant \varepsilon$" means that for all $f \in \bar{L}^2(\mathscr{F} \vee \mathscr{H})$ and all $g \in \bar{L}^2(\mathscr{G} \vee \mathscr{H})$ satisfying $\mathbf{E}[f|\mathscr{H}] \equiv 0$, resp. $\mathbf{E}[g|\mathscr{H}] \equiv 0$, one has:

$$|\mathbf{E}[fg]| \leqslant \varepsilon \,\mathrm{Sd}(f)\mathrm{Sd}(g). \tag{EZ}$$

♢

We let the reader check that with that definition, for X, Y and Z random variables, $\{X:Y\}_Z = \{\sigma(X):\sigma(Y)\}_{\sigma(Z)}$.

3.1.3 Remark. The ordinary correlation can be seen as a particular case of subjective correlation, since $\{\mathscr{F}:\mathscr{G}\} = \{\mathscr{F}:\mathscr{G}\}_{\mathscr{O}}$ for $\mathscr{O} = \{\emptyset, \Omega\}$ the trivial σ-field. ♡

3.1.4 Remark. Warning! Writing that $\{X:Y\}_Z \leqslant \varepsilon$ does *not* imply that for all subset C of the range of Z, X and Y are ε-decorrelated under $Law(X,Y|Z \in C)$: see Examples 3.1.8 and 3.1.9 below. ♡

3.1.5 Remark. Warning again! There is no general inequality between $\{X:Y\}$ and $\{X:Y\}_Z$: see Examples 3.1.7 and 3.1.8 below. ♡

[*]. The conditional laws $Law(\cdot|Z=z)$ are only defined up to $Law(Z)$-a.e. equality, whence the need to specify "for $Law(Z)$-almost-all z".

3.1.6 Example. Let $f:\ \mathbb{R}\to\mathbb{R}_+$ be a nonnegative continuous function with $\int_{\mathbb{R}} f(x)\,dx = 1$ and let (X,Y,Z) be a variable on $\mathbb{R}^3$ with density

$$d\mathbf{P}[(X,Y,Z)=(x,y,z)] = \frac{1}{2\pi} f(z)\exp\big(\sinh z\cdot xy - \tfrac{1}{2}\cosh z\cdot(x^2+y^2)\big)\,dxdydz. \tag{FA}$$

Then, conditionally to "$Z=z$", (X,Y) is a Gaussian vector with $\mathrm{Var}(X)=\mathrm{Var}(Y)=\cosh z$ and $\mathrm{Cov}(X,Y)=\sinh z$, so by Theorem 1.2.6, under the law $\mathbf{P}[\cdot|Z=z]$ one has $\{X:Y\}=|\tanh z|$. Consequently $\{X:Y\}_Z = \sup\{|\tanh z|:\ f(z)>0\}$. ♡

The three following examples show that subjective correlation may behave rather wildly, especially when one changes the σ-field of reference:

3.1.7 Example. Let X and Y be independent variables with uniform law on $\mathbb{R}/\mathbb{Z}$ and let $Z=X+Y$; then $\{X:Y\}_Z=1$: under $\mathbf{P}[\cdot|Z=z]$ indeed Y is X-measurable (and not constant), since $Y\equiv z-X$. ♡

3.1.8 Example. Let α,β,γ be three independent random variables uniform on $\{0,1\}$; define $X=(\gamma,\alpha), Y=(\gamma,\beta)$ and $Z=\gamma$. Then, conditionally to "$Z=0$", X and Y are independent with common law uniform on $\{(0,0),(0,1)\}$, and similarly X and Y are independent conditionally to "$Z=1$", so $\{X:Y\}_Z=0$. Yet X and Y are not independent since the events "$X\in\{(0,0),(0,1)\}$" and "$Y\in\{(0,0),(0,1)\}$", which are non-trivial under $\mathbf{P}$, are equivalent, so that $\{X:Y\}=1$. ♡

3.1.9 Example. Let $X=(X_1,X_2)$ and $Y=(Y_1,Y_2)$ be independent with uniform laws on $\{0,1\}^2$ and define $Z=(X_1,Y_1)$; then one easily checks that $\{X:Y\}_Z=0$. Now let $Z'=\mathbf{1}_{X_1=Y_1}$, which is Z-measurable; one has $\{X:Y\}_{Z'}=1$ since, for instance, under "$Z'=1$" the events "$X_1=0$" and "$Y_1=0$" are non-trivial and equivalent. ♡

Now we define a more restrictive concept of subjective correlation.

3.1.10 Definition. A *σ-metalgebra* $\mathscr{M}$ is a set $\{\mathscr{H}:\ \mathscr{H}\in\mathscr{M}\}$ of σ-algebras which is stable under the "$\bigvee$" operator, i.e. such that for any $\mathscr{M}'\subset\mathscr{M}$, $\bigvee_{\mathscr{H}\in\mathscr{M}'}\mathscr{H}\in\mathscr{M}$. ♢

One can speak of the 'σ-metalgebra spanned by some set of σ-algebras', as states the following immediate proposition:

3.1.11 Proposition. *If $(\mathscr{H}_k)_{k\in K}$ is a set of σ-algebras, then there is a smallest σ-metalgebra containing all the $\mathscr{H}_k$, which is*

$$\mathscr{M} = \Big\{\bigvee_{k\in K'}\mathscr{H}_k\ ;\ K'\subset K\Big\}. \tag{FB}$$

♣

When one deals with random variables rather than σ-algebras, one has the following variant of Proposition 3.1.11:

3.1.12 Proposition. *Let $(Z_k)_{k\in K}$ be a set of random variables, then the σ-metalgebra spanned by $\{\sigma(Z_k):\ k\in K\}$ is $\{\sigma(\vec{Z}_{K'}):\ K'\subset K\}$.* ♣

3.1.13 Definition. Let $\mathscr{F}$ and $\mathscr{G}$ be σ-algebras and $\mathscr{M}$ be a σ-metalgebra. We define the *correlation between $\mathscr{F}$ and $\mathscr{G}$ seen from $\mathscr{M}$* by:

$$\{\mathscr{F}:\mathscr{G}\}_{\mathscr{M}} = \sup_{\mathscr{H}\in\mathscr{M}}\{\mathscr{F}:\mathscr{G}\}_{\mathscr{H}}. \tag{FC}$$

♢

3.1.14 Remark. Speaking in terms of random variables, if X, Y and $(Z_k)_{k\in K}$ are variables, denoting by $\mathscr{M}$ the σ-metalgebra spanned by the Z_k, then $\{X:Y\}_{\mathscr{M}}$ is the supremum [†] of the $\{X:Y\}$ when taken under all the laws of kind $\mathbf{P}[\cdot|\vec{Z}_{K'}=\vec{z}_{K'}]$ for K' a subset of K and $z_k, k\in K'$ elements of the respective ranges of the Z_k. ♡

Finally, the following proposition gathers some easy properties of relative correlation w.r.t. a σ-metalgebra:

3.1.15 Proposition.

(i) Call $\mathscr{M}_\emptyset$ the trivial σ-metalgebra, that is, $\mathscr{M}_\emptyset=\{\mathscr{O}\}$; then for all σ-algebras $\mathscr{F}$ and $\mathscr{G}$, $\{\mathscr{F}:\mathscr{G}\}=\{\mathscr{F}:\mathscr{G}\}_{\mathscr{M}_\emptyset}$.

(ii) If $\mathscr{M}\subset\mathscr{M}'$, then $\{\mathscr{F}:\mathscr{G}\}_{\mathscr{M}}\leqslant\{\mathscr{F}:\mathscr{G}\}_{\mathscr{M}'}$.

(iii) Let $\mathscr{F}$ and $\mathscr{G}$ be σ-algebras, let $\mathscr{M}$ be a σ-metalgebra, and call $\tilde{\mathscr{M}}$ the σ-metalgebra spanned by $\mathscr{M}$, $\mathscr{F}$ and $\mathscr{G}$; then $\{\mathscr{F}:\mathscr{G}\}_{\mathscr{M}}=\{\mathscr{F}:\mathscr{G}\}_{\tilde{\mathscr{M}}}$.

♣

3.1.16 Definition. In the sequel, the probabilistic systems which we shall consider will often be made of some 'elementary' variables, say $(X_i)_{i\in I}$. In this case, the so-called *natural σ-metalgebra* of the system will mean the σ-metalgebra spanned by the X_i. ♢

3.2 Simple tensorization

Now we turn to tensorization. First let us deal with 'simple' tensorization, by which I mean that tensorization is performed on only one variable. The main result of this section will be the 'N against 1' theorem (Theorem 3.2.2).

The problem considered is the following: Let I be a set and $(X_i)_{i\in I}, Y$ be random variables; call $\mathscr{M}$ the natural σ-metalgebra of this system, that is, the σ-metalgebra spanned by the X_i and Y (cf. Definition 3.1.16). Suppose we have bounds $\{X_i:Y\}_{\mathscr{M}}\leqslant\varepsilon_i$ for all i; the question is, can we deduce from them a bound on $\{\vec{X}_I:Y\}$? We shall prove that the answer is "yes", and moreover the bound (FM) we will give is optimal in some way (see § 3.5).

For pedagogical purpose, let us first state and prove a weaker but easier proposition:

3.2.1 Proposition. *With the notation above,*

$$\{\vec{X}_I:Y\}\leqslant\sqrt{\sum_{i\in I}\varepsilon_i^2}. \tag{FD}$$

♣

Proof. By Proposition 1.1.12 we may assume $I=\{1,\dots,N\}$. Let f and g be centered L^2 $\vec{X}$-measurable, resp. Y-measurable, functions; our goal is to bound $|\mathbf{E}[fg]|$.

For all $i\in\{0,\dots,N\}$, denote

$$\mathscr{F}_i=\sigma(X_1,\dots,X_i), \tag{FE}$$

and for all $i\in\{1,\dots,N\}$,

$$f_i=f^{\mathscr{F}_i}-\mathbf{E}[f|\mathscr{F}_{i-1}]. \tag{FF}$$

[†]. More precisely it is a *true* supremum (over K') of *essential* suprema (over $\vec{z}_{K'}$).

Then $f = \sum_i f_i$, where each f_i is $\mathscr{F}_i$-measurable and centered w.r.t. $\mathscr{F}_{i-1}$ (i.e., $\mathbf{E}[f_i|\mathscr{F}_{i-1}] \equiv 0$). Consequently, for all $i_0 < i_1$ one has $\mathbf{E}[f_{i_0} f_{i_1}] = 0$ (since f_{i_0} is $\mathscr{F}_{i_0}$-measurable while f_{i_1} is centered w.r.t. $\mathscr{F}_{i_1-1} \supset \mathscr{F}_{i_0}$) and thus when one expands $\mathrm{Var}(f) = \mathbf{E}\big[(\sum_i f_i)^2\big]$ all the non-diagonal terms vanish, yielding:

$$\mathrm{Var}(f) = \sum_{i=1}^{N} \mathrm{Var}(f_i). \tag{FG}$$

Now, the decomposition "$f = \sum_i f_i$" yields

$$\mathbf{E}[fg] = \sum_{i=1}^{N} \mathbf{E}[f_i g], \tag{FH}$$

so let us bound the $|\mathbf{E}[f_i g]|$. The law of total expectation gives:

$$\mathbf{E}[f_i g] = \int \mathbf{E}\big[f_i g | X_1 = x_1, \dots, X_{i-1} = x_{i-1}\big] d\mathbf{P}[x_1, \dots, x_{i-1}]. \tag{FI}$$

But under $d\mathbf{P}[\cdot|x_1, \dots, x_{i-1}]$, f_i is X_i-measurable and centered while g is Y-measurable, moreover under this law $\{X_i : Y\} \leqslant \{X_i : Y\}_{\mathscr{F}_{i-1}} \leqslant \{X_i : Y\}_{\mathscr{M}} \leqslant \varepsilon_i$, so:

$$|\mathbf{E}\big[f_i g | Y_1 = y_1, \dots, Y_{i-1} = y_{i-1}\big]| \leqslant \varepsilon_i \,\mathrm{Sd}\big(f_i | x_1, \dots, x_{i-1}\big) \mathrm{Sd}\big(g | x_1, \dots, x_{i-1}\big). \tag{FJ}$$

Using the bound $\mathrm{Sd}(h) \leqslant \sqrt{\mathbf{E}[h^2]}$, it follows that:

$$\begin{aligned} |\mathbf{E}[f_i g]| &\leqslant \varepsilon_i \int \sqrt{\mathbf{E}[f_i^2 | x_1, \dots, x_{i-1}]} \sqrt{\mathbf{E}[g^2 | x_1, \dots, x_{i-1}]} d\mathbf{P}[x_1, \dots, x_{i-1}] \\ &\underset{\mathrm{CS}}{\leqslant} \varepsilon_i \sqrt{\int \mathbf{E}[f_i^2 | x_1, \dots, x_{i-1}] d\mathbf{P}[x_1, \dots, x_{i-1}]} \sqrt{\textit{the same for } g} = \varepsilon_i \,\mathrm{Sd}(f) \mathrm{Sd}(g_i). \end{aligned} \tag{FK}$$

So, summing (FK) for all i:

$$|\mathbf{E}[fg]| \leqslant \sum_{i=1}^{N} \varepsilon_i \,\mathrm{Sd}(f_i) \mathrm{Sd}(g) \underset{\mathrm{CS}}{\leqslant} \sqrt{\sum_i \varepsilon_i^2} \sqrt{\sum_i \mathrm{Var}(f_i)} \,\mathrm{Sd}(g) = \sqrt{\sum_i \varepsilon_i^2} \,\mathrm{Sd}(f) \mathrm{Sd}(g). \tag{FL}$$

Since (FL) is true for all f, g, (FD) is proved. ♠

It is striking in Proposition 3.2.1 that the right-hand side of (FD) may be greater than 1, which is never the case for a correlation level. Actually there is some 'loss of optimality' in the proof of the proposition when we bound above $\mathrm{Var}(f_i|\mathscr{F}_{i-1})$ by $\mathbf{E}[f_i^2|\mathscr{F}_{i-1}]$, since $\mathbf{E}[f_i^2|\mathscr{F}_{i-1}] - \mathrm{Var}(f_i|\mathscr{F}_{i-1}) = \mathbf{E}[f_i|\mathscr{F}_{i-1}]^2$ may be different to 0. We will use a technique for 'recycling' that loss to get the following result, which § 3.5 shall prove to be optimal:

3.2.2 Theorem ('N against 1' theorem)**.** *Take the same hypotheses as in Proposition 3.2.1: $\forall i \in I$ $\{X_i : Y\}_{\mathscr{M}} \leqslant \varepsilon_i$, where $\mathscr{M}$ is the natural σ-metalgebra of the system. Then:*

$$\{\vec{X}_I : Y\} \leqslant \sqrt{1 - \prod_{i \in I} (1 - \varepsilon_i^2)}. \tag{FM}$$

♣

3.2.3 Remark. The right-hand side of (FM) is the $\overline{\varepsilon} \in [0,1]$ characterized by $1 - \overline{\varepsilon}^2 = \prod_i (1 - \varepsilon_i)^2$. ♡

3.2.4 Remark. The right-hand side of (FM) is bounded above by $\sqrt{\sum_i \varepsilon_i^2}$, so Theorem 3.2.2 gives back Proposition 3.2.1 as a corollary. ♡

Proof. As in the proof of Proposition 3.2.1, let f and g be centered L^2 $\vec{X}$-measurable, resp. Y-measurable, functions. Assume $I = \{1,\dots,N\}$; denote $\mathscr{F}_i := \sigma(X_1,\dots,X_i)$ and $f_i := f^{\mathscr{F}_i} - \mathbf{E}[f|\mathscr{F}_{i-1}]$. Also denote, for $i \in \{0,\dots,N\}$,

$$g^i := g - \mathbf{E}[g|\mathscr{F}_i]. \tag{FN}$$

As before, one has $\mathrm{Var}(f) = \sum_i \mathrm{Var}(f_i)$ and $\mathbf{E}[fg] = \sum_{i=1}^N \mathbf{E}[f_i g]$. But f_i is centered w.r.t. $\mathscr{F}_{i-1}$ while $(g - g^{i-1})$ is $\mathscr{F}_{i-1}$-measurable, so $\mathbf{E}[f_i g] = \mathbf{E}[f_i g^{i-1}]$. Since, conditionally to $\mathscr{F}_{i-1}$, f_i and g^{i-1} are both centered and resp. X_i- and Y-measurable, the fact that $\{X_i : Y\}_{\mathscr{M}} \leqslant \varepsilon_i$ implies, by the same argument as in the previous proof, that

$$|\mathbf{E}[f_i g^{i-1}]| \leqslant \varepsilon_i \,\mathrm{Sd}(f_i)\,\mathrm{Sd}(g^{i-1}). \tag{FO}$$

Now, for $i \in \{1,\dots,N\}$, denote

$$\overline{g}^i = \mathbf{E}[g^{i-1}|\mathscr{F}_i]. \tag{FP}$$

Since $g^{i-1} = \overline{g}^i + g^i$, where $\overline{g}^i$ is $\mathscr{F}_i$-measurable while g^i is centered w.r.t. $\mathscr{F}_i$, one has:

$$\mathrm{Var}(g^i) = \mathrm{Var}(g^{i-1}) - \mathrm{Var}(\overline{g}^i). \tag{FQ}$$

Then, the point consists in making the following observation: for $\mathrm{Var}(g^i)$ to be large (that is, close to $\mathrm{Var}(g^{i-1})$), $\mathrm{Var}(\overline{g}^i)$ has to be small. But in that case $|\mathbf{E}[f_i g]|$ shall be small: one has indeed, since f_i is $\mathscr{F}_i$-measurable,

$$|\mathbf{E}[f_i g]| = |\mathbf{E}[f_i g^{i-1}]| = |\mathbf{E}[f_i (g^{i-1})^{\mathscr{F}_i}]| = |\mathbf{E}[f_i \overline{g}^i]| \underset{\mathrm{CS}}{\leqslant} \mathrm{Sd}(f_i)\,\mathrm{Sd}(\overline{g}^i). \tag{FR}$$

Let us sum up the relations obtained. One has, for all $i \in \{1,\dots,N\}$:

$$|\mathbf{E}[f_i g]| \leqslant \varepsilon_i \,\mathrm{Sd}(f_i)\,\mathrm{Sd}(g^{i-1}); \tag{FS}$$

$$\mathrm{Sd}(g^i) = \sqrt{\mathrm{Sd}(g^{i-1})^2 - \mathrm{Sd}(\overline{g}^i)^2}; \tag{FT}$$

$$|\mathbf{E}[f_i g]| \leqslant \mathrm{Sd}(f_i)\,\mathrm{Sd}(\overline{g}^i). \tag{FU}$$

Now define $\hat{\varepsilon}_i := |\mathbf{E}[f_i g]|/\mathrm{Sd}(f_i)\,\mathrm{Sd}(g^{i-1})$, or $\hat{\varepsilon}_i = 0$ if the right-hand side is $0/0$. Then (FS) ensures that $\hat{\varepsilon}_i \leqslant \varepsilon_i$, and (FU) means that $\mathrm{Sd}(\overline{g}^i) \geqslant \hat{\varepsilon}_i \,\mathrm{Sd}(g^{i-1})$, so that (FT) yields $\mathrm{Sd}(g^i) \leqslant \sqrt{1-\hat{\varepsilon}_i^2} \times \mathrm{Sd}(g^{i-1})$. Since $g^0 = g$, one has therefore by induction $\mathrm{Sd}(g^i) \leqslant \prod_{i'=1}^{i-1} \sqrt{1-\hat{\varepsilon}_{i'}^2}\,\mathrm{Sd}(g)$, so that the decomposition "$\mathbf{E}[fg] = \sum_i \mathbf{E}[f_i g]$" gives:

$$|\mathbf{E}[fg]| \leqslant \sum_{i=1}^N \Big(\hat{\varepsilon}_i \prod_{i'=1}^{i-1} \sqrt{1-\hat{\varepsilon}_{i'}^2}\Big)\mathrm{Sd}(f_i)\,\mathrm{Sd}(g). \tag{FV}$$

By the Cauchy–Schwarz inequality, it follows that:

$$|\mathbf{E}[fg]| \leqslant \sqrt{\sum_{i=1}^N \hat{\varepsilon}_i^2 \prod_{i'=1}^{i-1} (1-\hat{\varepsilon}_{i'}^2)}\,\mathrm{Sd}(f)\,\mathrm{Sd}(g) = \sqrt{1 - \prod_{i=1}^N (1-\hat{\varepsilon}_i^2)}\,\mathrm{Sd}(f)\,\mathrm{Sd}(g). \tag{FW}$$

Obviously the maximal value for the right-hand side of (FW) is when $\hat{\varepsilon}_i = \varepsilon_i$ for all i, then yielding (FM). ♠

There is an alternative proof, which is less intuitive but whose reasoning shall be used again in the proof of Theorem 3.3.1:

Alternative proof of Theorem 3.2.2. We use the same notation as in the previous proof. As f is $\mathscr{F}$-measurable, $\mathbf{E}[fg] = \mathbf{E}[fg^{\mathscr{F}}]$, so by the Cauchy–Schwarz inequality:

$$|\mathbf{E}[fg]| \leqslant \mathrm{Sd}(f)\mathrm{Sd}(g^{\mathscr{F}}). \tag{FX}$$

Now, by associativity of variance $\mathrm{Sd}(g^{\mathscr{F}}) = \sqrt{\mathrm{Var}(g) - \mathrm{Var}(g - g^{\mathscr{F}})}$, so by (FX) it suffices to prove that

$$\mathrm{Var}(g - g^{\mathscr{F}}) \geqslant \prod_{i=1}^{N} (1 - \varepsilon_i^2)\mathrm{Var}(g). \tag{FY}$$

With our notation, $g - g^{\mathscr{F}} = g^N$ and $g = g^0$; we will prove that for all $i \in \{1, \dots, N\}$,

$$\mathrm{Var}(g^i) \geqslant (1 - \varepsilon_i^2)\mathrm{Var}(g^{i-1}). \tag{FZ}$$

Since g^{i-1} and g^i are centered w.r.t. $\mathscr{F}_{i-1}$, one has

$$\mathrm{Var}(g^{i-1}) = \int \mathrm{Var}(g^{i-1}|x_1, \dots, x_{i-1})\, d\mathbf{P}[x_1, \dots, x_{i-1}], \tag{GA}$$

with a similar decomposition for $\mathrm{Var}(g^i)$, so that it suffices to prove (FZ) conditionally to $\mathscr{F}_{i-1}$.

Conditionally to $\mathscr{F}_{i-1}$, g^{i-1} is centered and Y-measurable. Moreover, $g^i = g^{i-1} - (g^{i-1})^{\sigma(X_i)}$, so by associativity of variance $\mathrm{Var}(g^i) = \mathrm{Var}(g^{i-1}) - \mathrm{Var}\big((g^{i-1})^{\sigma(X_i)}\big)$, and therefore (FZ) is equivalent to

$$\mathrm{Var}\big((g^{i-1})^{\sigma(X_i)}\big) \geqslant \varepsilon_i^2 \mathrm{Var}(g^{i-1}), \tag{GB}$$

which follows directly from the assumption "$\{X_i : Y\}_{\mathscr{F}_{i-1}} \leqslant \varepsilon_i$". ♠

3.3 Double tensorization

Simple tensorization as itself is already interesting since it gives an L^2-type bound for the correlation between X and $\vec{Y}$, which is better than the L^1-type bounds typically obtained by total variation methods. Yet it does not exhaust the full potential of Hilbertian correlations concerning tensorization, since obviously it does not contain results like independent tensorization (cf. § 1.1.e).

The aim of this section is to get sharp tensorization results where we perform tensorizing on *both* sides, without having to assume complete independence like in Theorem 1.1.19. The price to pay is that the techniques involved, though similar in their spirit, will be much more tricky, moreover the bounds obtained will not be completely optimal (see § 3.5).

3.3.a 'N against M' tensorization

The following theorem may be considered as the main result of this part of the thesis. As will be explained in § 3.5.b, it 'contains' qualitatively all the other tensorization theorems (i.e. Theorems 1.1.19, 3.2.2 and 3.3.10).

3.3.1 Theorem ('N against M' theorem)**.** *Let I and J be sets, and let $(X_i)_{i\in I}$ and $(Y_j)_{j\in J}$ be random variables, the σ-metalgebra they generate being denoted by $\mathscr{M}$. Suppose for any i, j, $\{X_i : Y_j\}_{\mathscr{M}} \leqslant \varepsilon_{ij}$ for some $\varepsilon_{ij} \geqslant 0$, and define the operator*

$$\begin{array}{rccl} \boldsymbol{\varepsilon}: & L^2(J) & \to & L^2(I) \\ & (a_j)_{j\in J} & \mapsto & (\sum_{j\in J} \varepsilon_{ij} a_j)_{i\in I}, \end{array} \tag{GC}$$

then:

$$\{\vec{X}_I : \vec{Y}_J\} \leqslant |||\boldsymbol{\varepsilon}||| \wedge 1. \tag{GD}$$

♣

3.3.2 Remark. On $(\mathbb{R}_+)^{I\times J}$, $|||\boldsymbol{\varepsilon}|||$ is a nondecreasing function of each ε_{ij}. ♡

☛ *As the proof of Theorem 3.3.1 is rather technical, I found it useful to write down how it goes on a concrete example. This is performed in Appendix 3.7, which I suggest the reader to look at in parallel with the proof as a complement.*

To prove Theorem 3.3.1, we will need the following

3.3.3 Lemma. *Let $X_1, X_2, \dots, X_N$ and Y be random variables, call $\mathscr{M}$ their natural σ-metalgebra, and assume that for all $i \in \{1,\dots,N\}$,*

$$\{X_i : Y\}_{\mathscr{M}} \leqslant \varepsilon_i. \tag{GE}$$

Let f be an $\bar{L}^2(\vec{X})$ function. For all $0 \leqslant i \leqslant N$, denote $\mathscr{F}_i := \sigma(X_1,\dots,X_i)$, resp. $\mathscr{F}_i^* := \sigma(X_1,\dots,X_i,Y)$, and for all $1 \leqslant i \leqslant N$, define

$$f_i := f^{\mathscr{F}_i} - \mathbf{E}[f|\mathscr{F}_{i-1}], \tag{GF}$$

$$\textit{resp.} \quad f_i^* := f^{\mathscr{F}_i^*} - \mathbf{E}[f|\mathscr{F}_{i-1}^*], \tag{GG}$$

and denote by V_i and V_i^* their respective variances. Then, for all $1 \leqslant i \leqslant N$,

$$V_i^* \geqslant (1-\varepsilon_i^2)V_i - 2\varepsilon_i\sqrt{V_i}\Big(\sum_{i'>i}\varepsilon_{i'}\sqrt{V_{i'}}\Big). \tag{GH}$$

♣

Proof. For $0 \leqslant i \leqslant N$, define

$$\tilde{f}_i := f - \mathbf{E}[f|\mathscr{F}_i], \tag{GI}$$

$$\text{resp.} \quad \tilde{f}_i^* := f - \mathbf{E}[f|\mathscr{F}_i^*], \tag{GJ}$$

and call $\tilde{V}_i$ and $\tilde{V}_i^*$ their respective variances. One has $\tilde{f}_i = \sum_{i'>i} f_{i'}$, resp. $\tilde{f}_i^* = \sum_{i'>i} f_{i'}^*$. Moreover, by the same argument as in the proof of Proposition 3.2.1, all the f_i are orthogonal (that is, $i_0 \neq i_1 \;\Rightarrow\; \mathbf{E}[f_{i_0}f_{i_1}] = 0$), thus

$$\tilde{V}_i = \sum_{i'>i} V_{i'}; \tag{GK}$$

similarly,

$$\tilde{V}_i^* = \sum_{i'>i} V_{i'}^*. \tag{GL}$$

In a first step, we observe that for all i, $\tilde{f}_i - \tilde{f}_i^* = (f - f^{\mathscr{F}_i}) - (f - f^{\mathscr{F}_i^*}) = f^{\mathscr{F}_i^*} - f^{\mathscr{F}_i} = f^{\mathscr{F}_i^*} - (f^{\mathscr{F}_i})^{\mathscr{F}_i^*} = (f - f^{\mathscr{F}_i})^{\mathscr{F}_i^*} = (\tilde{f}_i)^{\mathscr{F}_i^*}$, which by associativity of variance yields the following

3.3.4 Claim.

$$\tilde{V}_i - \tilde{V}_i^* = \mathrm{Var}\big((\tilde{f}_i)^{\mathscr{F}_i^*}\big). \tag{GM}$$

♣

Now, the following claim will be the main tool for proving the lemma:

3.3.5 Claim. *For all* $1 \leqslant i \leqslant N$,

$$\tilde{V}_{i-1} - \tilde{V}^*_{i-1} \leqslant \big(\varepsilon_i \sqrt{V_i} + \sqrt{\tilde{V}_i - \tilde{V}^*_i}\big)^2 \text{ [‡]}. \tag{GN}$$

♣

Admit temporarily Claim 3.3.5. Since $\tilde{V}_N = \tilde{V}^*_N = 0$, (GN) applied with $i = N$ gives $\tilde{V}_{N-1} - \tilde{V}^*_{N-1} \leqslant \varepsilon_N^2 V_N$, which in turn we can use in (GN) with $i = N-1$, and so on, to finally prove by finite (decreasing) induction that, for all i,

$$\tilde{V}_i - \tilde{V}^*_i \leqslant \big(\sum_{i'>i} \varepsilon_{i'} \sqrt{V_{i'}}\big)^2. \tag{GO}$$

Now to get (GH), we note that $V_i = \tilde{V}_{i-1} - \tilde{V}_i$, resp. $V^*_i = \tilde{V}^*_{i-1} - \tilde{V}^*_i$, so, using successively the inequalities (GN) and (GO),

$$\begin{aligned} V_i - V^*_i &= (\tilde{V}_{i-1} - \tilde{V}^*_{i-1}) - (\tilde{V}_i - \tilde{V}^*_i) \\ &\leqslant \big(\varepsilon_i \sqrt{V_i} + \sqrt{\tilde{V}_i - \tilde{V}^*_i}\big)^2 - (\tilde{V}_i - \tilde{V}^*_i) = \varepsilon_i^2 V_i + 2\varepsilon_i \sqrt{V_i}\sqrt{\tilde{V}_i - \tilde{V}^*_i} \\ &\leqslant \varepsilon_i^2 V_i + 2\varepsilon_i \sqrt{V_i}\big(\sum_{i'>i} \varepsilon_{i'} \sqrt{V_{i'}}\big), \end{aligned} \tag{GP}$$

which is equivalent to (GH). ♠

Proof of Claim 3.3.5. Thanks to Claim 3.3.4, what we have to prove is:

$$\mathrm{Var}\big((\tilde{f}_{i-1})^{\mathscr{F}^*_{i-1}}\big) \leqslant \big(\varepsilon_i \sqrt{V_i} + \sqrt{\tilde{V}_i - \tilde{V}^*_i}\big)^2. \tag{GQ}$$

By the definition of conditional expectation and the equality case in the Cauchy–Schwarz inequality, (GQ) is equivalent to saying that for all $\bar{L}^2(\mathscr{F}^*_{i-1})$ function g,

$$|\mathbf{E}[\tilde{f}_{i-1} g]| \leqslant \big(\varepsilon_i \sqrt{V_i} + \sqrt{\tilde{V}_i - \tilde{V}^*_i}\big)\mathrm{Sd}(g). \tag{GR}$$

So let g be a centered L^2 $\mathscr{F}^*_{i-1}$-measurable real function. Since $\tilde{f}_{i-1} = f_i + \tilde{f}_i$, $\mathbf{E}[\tilde{f}_{i-1} g] = \mathbf{E}[f_i g] + \mathbf{E}[\tilde{f}_i g]$, which two terms we shall bound separately.

For the first term, under $\mathbf{P}[\cdot|\mathscr{F}_{i-1}]$, f_i is centered and only depends on X_i, and g only depends on Y. Since $\{X_i : Y\}_{\mathscr{F}_{i-1}} \leqslant \{X_i : Y\}_{\mathscr{M}} \leqslant \varepsilon_i$, it follows that

$$|\mathbf{E}[f_i g|\mathscr{F}_{i-1}]| \leqslant \varepsilon_i \,\mathrm{Sd}(f_i|\mathscr{F}_{i-1})\,\mathrm{Sd}(g|\mathscr{F}_{i-1}), \tag{GS}$$

which yields upon integrating:

$$\begin{aligned} |\mathbf{E}[f_i g]| &\leqslant \varepsilon_i \int \mathrm{Sd}(f_i|\mathscr{F}_{i-1})\mathrm{Sd}(g|\mathscr{F}_{i-1})\,d\mathbf{P} \\ &\underset{\mathrm{CS}}{\leqslant} \varepsilon_i \sqrt{\int \mathrm{Var}(f_i|\mathscr{F}_{i-1})\,d\mathbf{P}}\sqrt{\int \mathrm{Var}(g|\mathscr{F}_{i-1})\,d\mathbf{P}} \\ &= \varepsilon_i \sqrt{V_i}\sqrt{\mathrm{Var}(g) - \mathrm{Var}(g^{\mathscr{F}_{i-1}})} \leqslant \varepsilon_i \sqrt{V_i}\,\mathrm{Sd}(g). \end{aligned} \tag{GT}$$

[‡]. Taking the square root of $(\tilde{V}_i - \tilde{V}^*_i)$ is allowed, since that quantity is nonnegative by Claim 3.3.4.

For the second term, under $\mathbf{P}[\cdot|\mathscr{F}_i]$, g only depends on Y, and $\mathbf{E}[\tilde{f}_i|Y] \equiv \tilde{f}_i - \tilde{f}_i^*$ as we noticed just before Claim 3.3.4, so $\mathbf{E}[\tilde{f}_i g|\mathscr{F}_i] = \mathbf{E}[(\tilde{f}_i - \tilde{f}_i^*)g|\mathscr{F}_i]$, which yields upon integrating:

$$|\mathbf{E}[\tilde{f}_i g]| = |\mathbf{E}[(\tilde{f}_i - \tilde{f}_i^*)g]| \underset{\mathrm{CS}}{\leqslant} \mathrm{Sd}(\tilde{f}_i - \tilde{f}_i^*)\mathrm{Sd}(g) = \sqrt{\tilde{V}_i - \tilde{V}_i^*}\,\mathrm{Sd}(g), \tag{GU}$$

the last equality coming from Claim 3.3.4. Then it just remains to combine (GT) and (GU) to get (GR).

♠

Proof of Theorem 3.3.1. First, thanks to a by now classical approximation argument we may assume that $I = \{1,\dots,N\}$ and $J = \{1,\dots,M\}$. Denote $\mathscr{F} := \sigma(\vec{X}_I)$, resp. $\mathscr{G} := \sigma(\vec{Y}_J)$; our goal is to prove that for all $f \in \bar{L}^2(\mathscr{F})$, all $g \in \bar{L}^2(\mathscr{G})$, one has $|\mathbf{E}[fg]| \leqslant (|||\boldsymbol{\varepsilon}||| \wedge 1)\mathrm{Sd}(f)\mathrm{Sd}(g)$. We will use the same trick as in our alternative proof of Theorem 3.2.2: by the definition of conditional expectation and the Cauchy–Schwarz inequality, proving the inequality above is equivalent to showing that for all $f \in \bar{L}^2(\mathscr{F})$,

$$\mathrm{Var}(f^{\mathscr{G}}) \leqslant (|||\boldsymbol{\varepsilon}|||^2 \wedge 1)\mathrm{Var}(f), \tag{GV}$$

which, by associativity of variance, is in turn equivalent to:

$$\mathrm{Var}(f - f^{\mathscr{G}}) \geqslant (1 - |||\boldsymbol{\varepsilon}|||^2)_+ \mathrm{Var}(f). \tag{GW}$$

For $0 \leqslant i \leqslant N$, resp. $0 \leqslant j \leqslant M$, define $\mathscr{F}_i = \sigma(X_1,\dots,X_i)$, resp. $\mathscr{G}_j = \sigma(Y_1,\dots,Y_j)$. For all $0 \leqslant j \leqslant M$, define

$$f^j := f - \mathbf{E}[f|\mathscr{G}_j], \tag{GX}$$

and for all $1 \leqslant i \leqslant N$, define moreover

$$f_i^j := f^{\mathscr{G}_j \vee \mathscr{F}_i} - \mathbf{E}[f|\mathscr{G}_j \vee \mathscr{F}_{i-1}]. \tag{GY}$$

Denote $V^j := \mathrm{Var}(f^j)$, resp. $V_i^j := \mathrm{Var}(f_i^j)$. For fixed j, the f_i^j are pairwise orthogonal (again by the argument in the proof of Proposition 3.2.1) and their sum is equal to f^j, so:

$$V^j = \sum_{i=1}^N V_i^j. \tag{GZ}$$

Thus, with this notation our goal (GW) becomes:

$$\sum_{i=1}^N V_i^M \geqslant (1 - |||\boldsymbol{\varepsilon}|||^2)_+ \sum_{i=1}^N V_i^0. \tag{HA}$$

The main tool to prove (HA) will be Lemma 3.3.3. Actually the rough formula (GH) is quite impratical, so we introduce a *linearized* version of it: for each $1 \leqslant i \leqslant N$ take some $\alpha_i > 0$ (which for the time being is arbitrary), then by the Cauchy–Schwarz inequality, (GH) implies that:

$$V_i^* \geqslant (1 - \varepsilon_i^2)V_i - \frac{\varepsilon_i V_i}{\alpha_i} \sum_{i'>i} \varepsilon_{i'}\alpha_{i'} - \varepsilon_i \alpha_i \sum_{i'>i} \frac{\varepsilon_{i'} V_{i'}}{\alpha_{i'}}. \tag{HB}$$

3.3.6 Remark. (HB) is devised so that its right-hand side is exactly the same as in (GH) if $V_i \propto \alpha_i^2$ $\forall i$. ♡

Let us reason conditionally to $\mathscr{G}_{j-1}$ for a few lines. Under this conditioning, call $\dot{\mathscr{F}}_i := \sigma(X_1,\dots,X_i)$, resp. $\dot{\mathscr{F}}_i^* := \sigma(X_1,\dots,X_i,Y_j)$, and $\dot{f} := f^{j-1}$. Then $\dot{f}$ is an $\bar{L}^2(\vec{X})$ function, so we are in situation of applying Lemma 3.3.3 to the functions

$$\dot{f}_i := \dot{f}^{\dot{\mathscr{F}}_i} - \mathbf{E}[\dot{f}|\dot{\mathscr{F}}_{i-1}] \tag{HC}$$

$$\text{and} \qquad \dot{f}_i^* := \dot{f}^{\dot{\mathscr{F}}_i^*} - \mathbf{E}[\dot{f}|\dot{\mathscr{F}}_{i-1}^*]. \tag{HD}$$

But in fact we already know these functions: namely, $\dot{f}_i = f_i^{j-1}$ and $\dot{f}_i^* = f_i^j$. Then, applying the linearized version (HB) of Lemma 3.3.3

$$\mathrm{Var}(f_i^j|\mathscr{G}_{j-1}) \geqslant \Big(1-\varepsilon_{ij}^2 - \frac{\varepsilon_{ij}}{\alpha_i}\sum_{i'>i}\varepsilon_{i'j}\alpha_{i'}\Big)\mathrm{Var}(f_i^{j-1}|\mathscr{G}_{j-1}) - \varepsilon_{ij}\alpha_i\sum_{i'>i}\frac{\varepsilon_{i'j}\mathrm{Var}(f_{i'}^{j-1}|\mathscr{G}_{j-1})}{\alpha_{i'}}, \tag{HE}$$

whence upon integrating:

$$V_i^j \geqslant (1-\varepsilon_{ij}^2)V_i^{j-1} - \Big(\sum_{i'>i}\varepsilon_{i'j}\alpha_{i'}\Big)\frac{\varepsilon_{ij}V_i^{j-1}}{\alpha_i} - \varepsilon_{ij}\alpha_i\sum_{i'>i}\frac{\varepsilon_{i'j}V_{i'}^{j-1}}{\alpha_{i'}}. \tag{HF}$$

By Equation (HF), we have transformed our initial problem into a purely abstract operator problem, *posed in an L^1 setting*. To handle it, we need a little notation. Call $L^1(I)$ the set of real functions on I, endowed with the L^1 norm

$$\|(v_i)_{i\in I}\|_1 := \sum_{i\in I}|v_i|. \tag{HG}$$

The dual space of $L^1(I)$ is made of the linear forms $l:\ (v_i)_{i\in I} \mapsto \sum l_i v_i$, equipped with the L^∞ norm

$$\|l\|_\infty := \sup_{i\in I}|l_i|. \tag{HH}$$

We shall write "$L^1(I) \ni v \geqslant 0$" to mean that all the entries of v are nonnegative, and "$(L^1(I))' \ni l \geqslant 0$" to mean that $(v \geqslant 0) \Rightarrow (lv \geqslant 0)$, which is equivalent to say that all the l_i are nonnegative. Now I claim the following lemma, whose proof is postponed:

3.3.7 Lemma. *Suppose given some nonnegative numbers V_i^j for $(i,j) \in \{1,\dots,N\}\times\{0,\dots,M\}$, such that Equation (HF) is satisfied for all i,j. Call $\mathscr{L}$ the nonnegative linear form on $L^1(I)$ defined by*

$$\mathscr{L}v = \sum_{\substack{j\in J\\ i,i'\in I}}\frac{\alpha_{i'}}{\alpha_i}\varepsilon_{ij}\varepsilon_{i'j}v_i, \tag{HI}$$

and assume $\|\mathscr{L}\|_\infty \leqslant 1$, then:

$$\sum_{i=1}^N V_i^M \geqslant \sum_{i=1}^N V_i^0 - \mathscr{L}((V_i^0)_{i\in I}). \tag{HJ}$$

♣

Lemma 3.3.7 has the following immediate

3.3.8 Corollary. *Suppose given some nonnegative numbers V_i^j for $(i,j) \in \{1,\dots,N\}\times\{0,\dots,M\}$, such that Equation (HF) is satisfied for all i,j, then:*

$$\sum_{i=1}^N V_i^M \geqslant \Big(1-\sup_{i\in I}\sum_{\substack{j\in J\\ i'\in I}}\frac{\alpha_{i'}}{\alpha_i}\varepsilon_{ij}\varepsilon_{i'j}\Big)_+\sum_{i=1}^N V_i^0. \tag{HK}$$

♣

Now we finish the proof of Theorem 3.3.1: thanks to Corollary 3.3.8 we have proved that (HK) stands true in our situation for *any* choice of positive $(\alpha_i)_{i\in I}$. The last step then consists in optimizing that choice. Denote "$\alpha > 0$" to mean that all the α_i are positive. One has:

$$\inf_{\alpha>0}\sup_{i\in I}\sum_{\substack{j\in J\\ i'\in I}}\frac{\alpha_{i'}}{\alpha_i}\varepsilon_{ij}\varepsilon_{i'j} = \inf\{\lambda\geqslant 0:\ (\exists\alpha>0)(\forall i)(\sum_{\substack{j\in J\\ i'\in I}}\varepsilon_{ij}\varepsilon_{i'j}\alpha_{i'}\leqslant\lambda\alpha_i)\}$$
$$= \inf\{\lambda\geqslant 0:\ (\exists\alpha>0)(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^*\alpha\leqslant\lambda\alpha)\}. \quad \text{(HL)}$$

But $\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^*$ is a nonnegative operator on $L^2(I)$ (I mean, when seen as a matrix all its entries are nonnegative), so by Lemma 3.8.1 in appendix:

$$\inf\{\lambda\geqslant 0:\ (\exists\alpha>0)(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^*\alpha\leqslant\lambda\alpha)\} = \rho(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^*) = |||\boldsymbol{\varepsilon}|||^2. \quad \text{(HM)}$$

This ends the proof of Theorem 3.3.1. ♠

Proof of Lemma 3.3.7. We prove Lemma 3.3.7 by induction on M. The case $M = 0$ is trivial. Suppose $M \geqslant 1$ and assume the result is true for $(M-1)$. We generalize the notation $\mathscr{L}$ by defining, for $\bullet \in \{\sqcup, 1, *\}$,

$$\mathscr{L}^\bullet v = \sum_{\substack{j\in J^\bullet\\ i,i'\in I}}\frac{\alpha_{i'}}{\alpha_i}\varepsilon_{ij}\varepsilon_{i'j}v_i, \quad \text{(HN)}$$

with $J^1 = \{1\}$, resp. $J^* = \{2,\dots,M\}$, so that $\mathscr{L} = \mathscr{L}^1 + \mathscr{L}^*$. Notice that $\|\mathscr{L}^*\|_\infty \leqslant 1$ since $\|\mathscr{L}\|_\infty \leqslant 1$. For all $i \in I$, define

$$\check{V}_i^1 = (1-\varepsilon_{i1}^2)V_i^0 - \frac{\varepsilon_{i1}V_i^0}{\alpha_i}\sum_{i'>i}\varepsilon_{i'1}\alpha_{i'} - \varepsilon_{i1}\alpha_i\sum_{i'>i}\frac{\varepsilon_{i'1}V_{i'}^0}{\alpha_{i'}}, \quad \text{(HO)}$$

which is the value that V_i^1 would take if there were equality in (HF) for $j = 1$. With that notation, (HF) writes

$$(V_i^1 - \check{V}_i^1)_{i\in I} \geqslant 0, \quad \text{(HP)}$$

and by induction hypothesis we have:

$$\sum_{i=1}^N V_i^M \geqslant \sum_{i=1}^N V_i^1 - \mathscr{L}^*((V_i^1)_{i\in I}). \quad \text{(HQ)}$$

Introducing the $\check{V}_i^1$, we have therefore the following chain of inequalities:

$$\begin{aligned}
\sum_{i=1}^N V_i^M &\underset{\text{(HQ)}}{\geqslant} \sum_{i=1}^N V_i^1 - \mathscr{L}^*((V_i^1)_{i\in I})\\
&\underset{\text{(HP)}}{=} \sum_{i=1}^N \check{V}_i^1 + \|(V_i^1 - \check{V}_i^1)_{i\in I}\|_1 - \mathscr{L}^*((\check{V}_i^1)_{i\in I}) - \mathscr{L}^*((V_i^1 - \check{V}_i^1)_{i\in I})\\
&\underset{\|\mathscr{L}^*\|_\infty\leqslant 1}{\geqslant} \sum_{i=1}^N \check{V}_i^1 - \mathscr{L}^*((\check{V}_i^1)_{i\in I}) = \sum_{i=1}^N V_i^0 - \mathscr{L}^1((V_i^0)_{i\in I}) - \mathscr{L}^*((\check{V}_i^1)_{i\in I})\\
&\underset{\substack{\check{V}^1\leqslant V^0\\ \mathscr{L}^*\geqslant 0}}{\geqslant} \sum_{i=1}^N V_i^0 - \mathscr{L}^1((V_i^0)_{i\in I}) - \mathscr{L}^*((V_i^0)_{i\in I}) = \sum_{i=1}^N V_i^0 - \mathscr{L}((V_i^0)_{i\in I}), \quad \text{(HR)}
\end{aligned}$$

so (HJ) is true for M, whence the lemma by induction. ♠

3.3.9 Remark. Our proof of Theorem 3.3.1 handled the X_i and the Y_j in a fully nonsymmetric way, since we began with putting orders on I and J, which orders played a crucial role in the decomposition of f. Yet the bound (GD) obtained *is* obviously symmetric by re-labelling the basic variables—and this is not due to having proceeded to any 're-symmetrization' step... To date I have no simple explanation for this 'coincidence'. ♡

3.3.b '$\mathbb{Z}$ against $\mathbb{Z}$' tensorization

The proof of the 'N against M' theorem was quite more technical than that of the 'N against 1' theorem; because of that, in order to get tractable computations we had to use suboptimal inequalities at two places:

- Claim 3.3.5 is suboptimal: it has indeed the same shortcoming as Proposition 3.2.1 exhibited compared to Theorem 3.2.2, namely, it does not 'recycle the losses' occurring when one makes g covariate with both f_i and $\tilde{f}_i$ (cf. the discussion on page 120, just after the proof of Proposition 3.2.1).
- Our linearization technique is suboptimal in general, even after optimizing the α_i. In fact, as we said before, Inequality (HB) is optimal if and only if one has $V_i \propto \alpha_i$; thus, for (HF) to be always optimal, one has to have $V_i^j \propto \alpha_i$ for *all* j, with the *same* values for the α_i. This would imply that all the sequences $(V_i^j)_{0\leqslant i<n}$ are proportional, which is not true in general.

So, Theorem 3.3.1 is certainly not optimal [§]—this is confirmed by the example of § 3.7. Nonetheless, there is one particular case in which an alternative reasoning yields an optimal bound [¶]. This case is when some symmetries in the decorrelation hypotheses allow us to transform the original two-parameter problem (indexed by $I\times J$) into a one-parameter problem (indexed by $\mathbb{Z}$). Let us state and prove the corresponding result:

3.3.10 Theorem ('$\mathbb{Z}$ against $\mathbb{Z}$' theorem)**.** *Let I and J be sets isomorphic to $\mathbb{Z}$, and let $(X_i)_{i\in I}$ and $(Y_j)_{j\in J}$ be random variables such that, $\mathscr{M}$ denoting the σ-metalgebra they generate, one has for all $i,j\in\mathbb{Z}$*

$$\{X_i : Y_j\}_{\mathscr{M}} \leqslant \varepsilon(j-i) \tag{HS}$$

for some function $\varepsilon\colon\ \mathbb{Z}\to[0,1]$.

Then

$$\{\vec{X}_I : \vec{Y}_J\} \leqslant \overline{\varepsilon}, \tag{HT}$$

where $\overline{\varepsilon}\in[0,1]$ is characterized by:

$$\operatorname{Arcsin}\overline{\varepsilon} = \Big(\sum_{z\in\mathbb{Z}} \operatorname{Arcsin}\varepsilon(z)\Big)\wedge\frac{\pi}{2}. \tag{HU}$$

♣

3.3.11 Remark. If we apply Theorem 3.3.1 to the situation above, we find $\{\vec{X}_I : \vec{Y}_J\} \leqslant \big(\sum_{z\in\mathbb{Z}}\varepsilon(z)\big)\wedge 1$ (cf. § 3.6.b). The latter expression is always $\geqslant\overline{\varepsilon}$ because of the concavity of the function $\sin(\cdot\wedge\frac{\pi}{2})$ on $\mathbb{R}_+$, and even $>\overline{\varepsilon}$ if $\overline{\varepsilon}\neq 0,1$; so, when it is applicable, Theorem 3.3.10 is strictly stronger than Theorem 3.3.1. ♡

[§]. Though, as we will see in § 3.5.b, it is 'asymptotically optimal'.

[¶]. The bound's being optimal shall be proved by Theorem 3.5.3.

Proof. Let f and g be resp. $\vec{X}_I$- and $\vec{Y}_J$-measurable $\bar{L}^2$ functions. Denote $\mathscr{F} := \sigma(\vec{X})$, resp. $\mathscr{G} := \sigma(\vec{Y})$, and for $i \in \mathbb{Z}$, resp. $j \in \mathbb{Z}$, denote $\mathscr{F}_i := \bigvee_{i' \leqslant i} \sigma(X_{i'})$, resp. $\mathscr{G}_j := \bigvee_{j' \leqslant j} \sigma(Y_{j'})$. For $(i,j) \in \mathbb{Z} \times \mathbb{Z}$, define

$$f_i^j := f^{\mathscr{G}_j \vee \mathscr{F}_i} - \mathbf{E}[f | \mathscr{G}_j \vee \mathscr{F}_{i-1}] \tag{HV}$$

and [‖]

$$g_j^i := g^{\mathscr{G}_j} - \mathbf{E}[g^{\mathscr{G}_j} | \mathscr{G}_{j-1} \vee \mathscr{F}_i]. \tag{HW}$$

Denote $V := \mathrm{Var}(f)$, $W := \mathrm{Var}(g)$, $V_i^j := \mathrm{Var}(f_i^j)$, $W_j^i := \mathrm{Var}(g_j^i)$; also denote

$$S_{ij} := \mathbf{E}[f_i^{j-1} g_j^{i-1}]. \tag{HX}$$

Our auxiliary functions were devised so that

3.3.12 Claim. *Provided the sum in the right-hand side is absolutely convergent,*

$$\mathbf{E}[fg] = \sum_{i,j} S_{ij}. \tag{HY}$$

♣

Proof of Claim 3.3.12. First define $\bar{f} := f^{\mathscr{G}}$, so that $\bar{f}$ is $\mathscr{G}$-measurable and $\mathbf{E}[fg] = \mathbf{E}[\bar{f}g]$. For $j \in \mathbb{Z}$, define $g_j := g^{\mathscr{G}_j} - \mathbf{E}[g | \mathscr{G}_{j-1}]$, resp. $\bar{f}_j := \bar{f}^{\mathscr{G}_j} - \mathbf{E}[\bar{f} | \mathscr{G}_{j-1}]$: we have $g = \sum_j g_j$ and $\bar{f} = \sum_j \bar{f}_j$, which are the respective decompositions of g and $\bar{f}$ on the same basis of orthogonal subspaces of $\bar{L}^2(\mathscr{G})$, so $\mathbf{E}[fg] = \sum_j \mathbf{E}[\bar{f}_j g_j]$. The terms of the right-hand side of that formula are unchanged upon replacing $\bar{f}_j$ by $f^{j-1} := f - \mathbf{E}[f | \mathscr{G}_{j-1}]$, since $\mathbf{E}[(f^{j-1} - \bar{f}_j) g_j]$ is zero—the function $(f^{j-1} - \bar{f}_j)$ is indeed equal to $(f - \mathbf{E}[f | \mathscr{G}_j])$, which is centered conditionally to $\mathscr{G}_j$, while g_j is $\mathscr{G}_j$-measurable. In the end we have:

$$\mathbf{E}[fg] = \sum_j \mathbf{E}[f^{j-1} g_j]. \tag{HZ}$$

So in a first step we have decomposed $\mathbf{E}[fg]$ into a sum indexed by j. Now we decompose each term of that sum into a sum indexed by i. Let us reason conditionally to $\mathscr{G}_{j-1}$. Then f^{j-1} is an $\bar{L}^2(\mathscr{F})$ function and g_j is in $\bar{L}^2(Y_j)$. We compute $\mathbf{E}[f^{j-1} g_j]$ as in the first step of this proof: first we replace g_j by $\bar{g}_j := (g_j)^{\mathscr{F}}$; then we decompose $f^{j-1} = \sum_i f_i^{j-1}$ and $\bar{g}_j = \sum_i \bar{g}_{ji}$, with $f_i^{j-1} := (f^{j-1})^{\mathscr{F}_i} - \mathbf{E}[f^{j-1} | \mathscr{F}_{i-1}]$ [∗], resp. $\bar{g}_{ji} := \bar{g}_j^{\mathscr{F}_i} - \mathbf{E}[\bar{g}_j | \mathscr{F}_{i-1}]$, and by orthogonal decomposition we get $\mathbf{E}[f^{j-1} g_j] = \sum_i \mathbf{E}[f_i^{j-1} \bar{g}_{ji}]$; then we conclude by saying that $\mathbf{E}[f_i^{j-1} \bar{g}_{ji}]$ is actually equal to $\mathbf{E}[f_i^{j-1} g_j^{i-1}]$, since $(g_j^{i-1} - \bar{g}_{ji})$ is centered conditionally to $\mathscr{F}_i$ while f_i^{j-1} is $\mathscr{F}_i$-measurable. In the end we have obtained

$$\mathbf{E}[f^{j-1} g_j] = \sum_i \mathbf{E}[f_i^{j-1} g_j^{i-1}], \tag{IA}$$

which combined with (HZ) yields (HY). ♠

[‖]. Beware: the definition of g_j^i is *not* analogous to the definition of f_i^j !

[∗]. Notation is consistent: this f_i^{j-1} is indeed the same as the f_i^{j-1} defined by (HV), since we are reasoning conditionally to $\mathscr{G}_{j-1}$.

So we have expressed $\mathbf{E}[fg]$ as a function of the S_{ij}. It is also possible to 'read' the values of V and W from the V_i^j, resp. from the W_j^i, *via* the formulas:

$$V = \lim_{j\to-\infty}{}^{\nearrow}\Big(\sum_i V_i^j\Big); \tag{IB}$$

$$W = \sum_j\Big(\lim_{i\to-\infty}{}^{\nearrow} W_j^i\Big). \tag{IC}$$

Now we are looking for relations between the V_i^j, the W_j^i and the S_{ij}. The first relation comes from the decorrelation hypothesis: conditionally to $\mathscr{G}_{j-1}\vee\mathscr{F}_{i-1}$, f_i^{j-1} is in $\bar{L}^2(X_i)$, resp. g_j^{i-1} is in $\bar{L}^2(Y_j)$, and $\{X_i : Y_j\}\leqslant\varepsilon(j-i)$, so:

$$|S_{ij}|\leqslant\varepsilon(j-i)\sqrt{V_i^{j-1}W_j^{i-1}}. \tag{ID}$$

The second relation means that a large value of $|S_{ij}|$ forces W_j^i to diminish. To state it, we observe that, since f_i^{j-1} is $(\mathscr{G}_{j-1}\vee\mathscr{F}_i)$-measurable, $S_{ij}=\mathbf{E}[f_i^{j-1}(g_j^{i-1})^{\mathscr{G}_{j-1}\vee\mathscr{F}_i}]$, so by the Cauchy–Schwarz inequality $|S_{ij}|\leqslant\mathrm{Sd}(f_i^{j-1})\mathrm{Sd}\big((g_j^{i-1})^{\mathscr{G}_{j-1}\vee\mathscr{F}_i}\big)$. Moreover, since $g_j^{i-1}-(g_j^{i-1})^{\mathscr{G}_{j-1}\vee\mathscr{F}_i}=g_j^i$, one has by orthogonality $\mathrm{Var}\big((g_j^{i-1})^{\mathscr{G}_{j-1}\vee\mathscr{F}_i}\big)=\mathrm{Var}(g_j^{i-1})-\mathrm{Var}(g_j^i)$, so our inequality becomes

$$|S_{ij}|\leqslant\sqrt{V_i^{j-1}}\sqrt{W_j^{i-1}-W_j^i} \tag{IE}$$

(where it is understood that $W_j^i\leqslant W_j^{i-1}$), or more eloquently

$$W_j^i\leqslant W_j^{i-1}-(S_{ij})^2/V_i^{j-1} \tag{IF}$$

provided $V_i^{j-1}>0$.

The third and last relation means, on the other hand, that a large value of $|\sum_{i'>i}S_{i'j}|$ forces $\sum_{i'>i}V_i^j$ to diminish. To state it, we denote

$$\tilde{f}_i^j := f-f^{\mathscr{G}_j\vee\mathscr{F}_i}=\sum_{i'>i}f_{i'}^j, \tag{IG}$$

whose variance is $\mathrm{Var}(\tilde{f}_i^j)=\sum_{i'>i}\mathrm{Var}(f_{i'}^j)$ since the $f_{i'}^j$ are pairwise orthogonal. One has

$$\sum_{i'>i}S_{i'j}=\sum_{i'>i}\mathbf{E}[f_{i'}^{j-1}g_j^{i'-1}]=\sum_{i'>i}\mathbf{E}[f_{i'}^{j-1}g_j^i]=\mathbf{E}[\tilde{f}_i^{j-1}g_j^i]=\mathbf{E}[(\tilde{f}_i^{j-1})^{\mathscr{G}_j\vee\mathscr{F}_i}g_j^i], \tag{IH}$$

so by the Cauchy–Schwarz inequality,

$$\Big|\sum_{i'>i}S_{i'j}\Big|\leqslant\mathrm{Sd}\big((\tilde{f}_i^{j-1})^{\mathscr{G}_j\vee\mathscr{F}_i}\big)\mathrm{Sd}(g_j^i). \tag{II}$$

Since $\tilde{f}_i^{j-1}-(\tilde{f}_i^{j-1})^{\mathscr{G}_j\vee\mathscr{F}_i}=\tilde{f}_i^j$, one has by orthogonality

$$\mathrm{Var}\big((\tilde{f}_i^{j-1})^{\mathscr{G}_j\vee\mathscr{F}_i}\big)=\mathrm{Var}(\tilde{f}_i^{j-1})-\mathrm{Var}(\tilde{f}_i^j), \tag{IJ}$$

so our inequality becomes

$$\Big|\sum_{i'>i}S_{i'j}\Big|\leqslant\sqrt{\sum_{i'>i}V_{i'}^{j-1}-\sum_{i'>i}V_{i'}^j}\sqrt{W_j^i}, \tag{IK}$$

or more eloquently:

$$\sum_{i'>i} V_{i'}^{j} \leqslant \sum_{i'>i} V_{i'}^{j-1} - \Big(\sum_{i'>i} S_{i'j}\Big)^2 / W_j^i. \tag{IL}$$

So, we have transformed our initial probabilistic problem into the following analytic one: let $\mathbf{A}$ be an array indexed by $\mathbb{Z}\times\mathbb{Z}$, each entry (i,j) of which contains three numbers $V_i^j \geqslant 0$, $W_j^i \geqslant 0$ and S_{ij}, satisfying (ID), (IE) and (IK)—we will say such an array is *correct*. We define V by (IB) and W by (IC), and we set $S = \sum_{i,j} S_{ij}$ (provided it makes sense); our goal is to get a bound of the form "$|S| \leqslant \overline{\varepsilon}\sqrt{VW}$", with $\overline{\varepsilon}$ only depending on $\varepsilon(\cdot)$.

Note that *A priori* some problems of summability can arise from $\mathbf{A}$'s being infinite, for instance to check (IL) or to define S. However, in the situations which are of interest to us, we can restrict to cases in which $\mathbf{A}$ is of nice particular form. To do this, we first approximate f in $\overline{L}^2(\vec{X})$, resp. g in $\overline{L}^2(\vec{Y})$, by a function depending only on a finite number of X_i, resp. of Y_j —say, we assume f is $\vec{X}_{\dot{I}}$-measurable and g is $\vec{Y}_{\dot{J}}$-measurable for finite $\dot{I}\subset I, \dot{J}\subset J$. Then, we define a new model $(\tilde{X}_i)_{i\in\mathbb{Z}}, (\tilde{Y}_j)_{j\in\mathbb{Z}}$ by $\tilde{X}_i = X_i$ for $i\in\dot{I}$, resp. $\tilde{Y}_j = Y_j$ for $j\in\dot{J}$, and $\tilde{X}_i, \tilde{Y}_j = \partial$ for $i\notin\dot{I}, j\notin\dot{J}$, ∂ being some cemetery point. This new model still gives a correct array, for which $S/\sqrt{VW}$ is arbitrarily close to the initial value of $\mathbf{E}[fg]/\mathrm{Sd}(f)\mathrm{Sd}(g)$; and the new array is of the following form, which we will call *compact*, for which all the quantities of interest are well defined:

- V_i^j is zero as soon as $i\notin\dot{I}$, and it does not depend on j for $j<\min\dot{J}$, nor for $j\geqslant\max\dot{J}$;
- Similarly, W_j^i is zero as soon as $j\notin\dot{J}$, and it does not depend on i for $i<\min\dot{I}$, nor for $i\geqslant\max\dot{I}$;
- S_{ij} is zero as soon as $(i,j)\notin\dot{I}\times\dot{J}$. (This condition automatically follows from the first two if the array is correct).

We define the following operations on arrays:

3.3.13 Definition.

- For $z\in\mathbb{Z}$, we define the *translation operator* τ^z on arrays such that, if the entries of $\mathbf{A}$ at (i,j) are V_i^j, W_j^i, S_{ij}, the entries of $\tau^z\mathbf{A}$ at (i,j) are $V_{i+z}^{j+z}, W_{j+z}^{i+z}, S_{(i+z)(j+z)}$.
- For $\grave{\mathbf{A}}$ and $\acute{\mathbf{A}}$ two arrays with entries $\grave{V}_i^j, \grave{W}_j^i, \grave{S}_{ij}$, resp. $\acute{V}_i^j$, etc., for α,β two real numbers, we define the linear combination $\alpha\grave{\mathbf{A}}+\beta\acute{\mathbf{A}}$ as the array with entries $\alpha\grave{V}_i^j+\beta\acute{V}_i^j, \alpha\grave{W}_j^i+\beta\acute{W}_j^i$, etc..

◇

3.3.14 Lemma. *Correct arrays are stable by translations and by nonnegative linear combinations, i.e., if* $\mathbf{A}$ *and* $\mathbf{B}$ *are correct arrays, then for all* $z\in\mathbb{Z}$ *and* $\alpha,\beta\geqslant 0$, $\tau^z\mathbf{A}$ *and* $\alpha\mathbf{A}+\beta\mathbf{B}$ *are correct too.* ♣

Proof of Lemma 3.3.14. Recall that being correct means satisfying (ID), (IE) and (IK). These conditions are trivially stable by multiplication by a nonnegative constant and by translation [†]. It remains to see that they are stable by addition. The technique being the same for all three inequalities, we just treat the case of (IE). Stability of this condition by addition is a consequence of the following inequality (which is in fact a particuliar case of the *Brunn–Minkowski inequality*, see [32]):

3.3.15 Lemma. *For all* $a_1,b_1,a_2,b_2\geqslant 0$,

$$\sqrt{(a_1+a_2)(b_1+b_2)} \geqslant \sqrt{a_1b_1}+\sqrt{a_2b_2}. \tag{IM}$$

♣

[†]. Getting stability of Condition (ID) by translation is actually the only place where the symmetries of the problem are used.

Proof of Lemma 3.3.15. Take squares on both sides of (IM) and notice that $(a_1+a_2)(b_1+b_2)-(\sqrt{a_1b_1}+\sqrt{a_2b_2})^2 = a_1b_2+a_2b_1-2\sqrt{a_1b_1a_2b_2} = (\sqrt{a_1b_2}-\sqrt{a_2b_1})^2 \geqslant 0$. ♠

For $\grave{\mathbf{A}}$ and $\acute{\mathbf{A}}$ two correct arrays satisfying (IE), applying (IM) with $a_1 = \acute{V}_i^{j-1}, a_2 = \grave{V}_i^{j-1}, b_1 = \acute{W}_j^{i-1} - \acute{W}_j^i, b_2 = \grave{W}_j^{i-1} - \grave{W}_j^i$, we get:

$$\begin{aligned}|\grave{S}_{ij}+\acute{S}_{ij}| \leqslant |\grave{S}_{ij}|+|\acute{S}_{ij}| &\leqslant \sqrt{\grave{V}_i^{j-1}}\sqrt{\grave{W}_j^{i-1}-\grave{W}_j^i}+\sqrt{\acute{V}_i^{j-1}}\sqrt{\acute{W}_j^{i-1}-\acute{W}_j^i} \\ &\leqslant \sqrt{\grave{V}_i^{j-1}+\acute{V}_i^{j-1}}\sqrt{(\grave{W}_j^{i-1}+\acute{W}_j^{i-1})-(\grave{W}_j^i+\acute{W}_j^i)}, \qquad \text{(IN)}\end{aligned}$$

so (IE) is still valid for $(\grave{\mathbf{A}}+\acute{\mathbf{A}})$. ♠

Now, thanks to Lemma 3.3.14 we will reduce our problem on $(\mathbb{Z}\times\mathbb{Z})$-arrays into a problem on $\mathbb{Z}$-arrays. Suppose $\mathbf{A}$ is a correct array with certain values of V, W and S. Then, for $k \geqslant 0$, the array

$$\mathbf{A}_k = \frac{1}{2k+1}\sum_{z=-k}^{k}\tau^z\mathbf{A} \qquad \text{(IO)}$$

is correct too, with the same values of V, W and S as $\mathbf{A}$. Now when $k\to\infty$, $\mathbf{A}_k$ 'looks more and more like a Toeplitz array', that is, an array whose entries at (i,j) only depend on $(j-i)$. To state it rigorously, we need some definitions:

3.3.16 Definition.

- Here, a *Toeplitz array* will mean a $\mathbb{Z}\times\mathbb{Z}$ array whose entries at (i,j) only depend on $(j-i)$. For such an array, for $z\in\mathbb{Z}$ we denote by $V_{(z)}, W_{(z)}, S_{(z)}$ the quantities characterized by $V_i^j = V_{(j-i)}$, etc..
- Actually we can always assume our Toeplitz array is *Toeplitz compact*, which means that there exists some $z^- \leqslant z^+$ such that:
  - $V_{(z)}$ does not depend on z for $z<z^-$, nor for $z\geqslant z^+$;
  - $W_{(z)}$ does not depend on z for $z\leqslant z^-$, nor for $z>z^+$;
  - $S_{(z)}$ is zero as soon as $z<z^-$ or $z>z^+$.
- For a compact Toeplitz array, we define v,w,s as 'renormalized versions' of V,W,S:

$$\begin{aligned} v &:= V_{(z<z^-)}; && \text{(IP)}\\ w &:= W_{(z>z^+)}; && \text{(IQ)}\\ s &:= \sum_{z\in\mathbb{Z}} S_{(z)}. && \text{(IR)}\end{aligned}$$

- A Toeplitz array is said to be *correct* if it is correct when seen as an ordinary array. For a Toeplitz array, Equations (ID), (IF) and (IL) become respectively [‡]:

$$\begin{aligned} |S_{(z)}| &\leqslant \varepsilon(z)\sqrt{V_{(z-1)}W_{(z+1)}}; && \text{(IS)}\\ W_{(z)} &\leqslant W_{(z+1)} - S_{(z)}^2/V_{(z-1)}; && \text{(IT)}\\ V_{(z-1)} &\leqslant v - \Big(\sum_{z'<z} S_{(z')}\Big)^2/W_{(z)}. && \text{(IU)}\end{aligned}$$

◇

[‡]. Note that the way (IU) follows from (IL) is rather tricky, because it appears a difference between two infinite quantities, which has to be 'renormalized' in the convenient way.

With that vocabulary, our informal statement can be made precise: let $\mathbf{A}$ be a compact correct array with entries V_i^j, W_j^i, S_{ij}, and associated quantities V, W, S, and define the arrays $\mathbf{A}_k$ by (IO). Then when $k \to \infty$ one has $(2k+1)\mathbf{A}_k \to \bar{\mathbf{A}}$ (in the sense that each entry of $(2k+1)\mathbf{A}_k$ converges to the corresponding entry of $\bar{\mathbf{A}}$), where $\bar{\mathbf{A}}$ is the Toeplitz array with entries $\bar{V}_i^j, \bar{W}_j^i, \bar{S}_{ij}$ defined by:

$$\bar{V}_{(z)} = \sum_{j-i=z} V_i^j; \tag{IV}$$

$$\bar{W}_{(z)} = \sum_{j-i=z} W_j^i; \tag{IW}$$

$$\bar{S}_{(z)} = \sum_{j-i=z} S_{ij}. \tag{IX}$$

This array $\bar{\mathbf{A}}$ is Toeplitz compact with $z^- = \min \dot{J} - \max \dot{I}$, resp. $z^+ = \max \dot{J} - \min \dot{I}$, and the quantities (IP)–(IR) for $\bar{\mathbf{A}}$ are:

$$\bar{v} = V; \tag{IY}$$

$$\bar{w} = W; \tag{IZ}$$

$$\bar{s} = S. \tag{JA}$$

Moreover $\bar{\mathbf{A}}$ is correct, because all the $(2k+1)\mathbf{A}_k$ are, and being correct is clearly conserved by array convergence.

The consequence of this statement is the following claim, which achieves the reduction to a '$\mathbb{Z}$-indexed' problem I alluded to a few lines above:

3.3.17 Claim. *The supremum of $|S|/\sqrt{VW}$ for correct arrays is not greater than the supremum of $|s|/\sqrt{vw}$ for correct Toeplitz arrays.* ♣

So we have to study (compact) correct Toeplitz arrays. Consider such an array. Denote $\theta(z) := \operatorname{Arcsin} \varepsilon(z)$; then (IS) can be rewritten:

$$\exists \hat{\theta}(z) \in [\pm\theta(z)] \qquad S_{(z)} = \sin\hat{\theta}(z) \cdot \sqrt{V_{(z-1)} W_{(z+1)}}. \tag{JB}$$

Now, notice that for fixed values of the $V_{(z)}$, the $S_{(z)}$ and w, if we have values $W_{(z)}$ such that (IS)–(IU) are satisfied, we can modify those $W_{(z)}$ so that (IT) becomes an equality for all z, an operation which keeps (IS) and (IU) true since it can only make the $W_{(z)}$ increase. So we can suppose that (IT) actually is an equality, i.e. that for all $z \in \mathbb{Z}$,

$$W_{(z)} = w \prod_{z' \geqslant z} \cos^2 \hat{\theta}(z'). \tag{JC}$$

Then it remains to integrate (IU). For $z \in \mathbb{Z}$, denote

$$\Gamma(z) := \sum_{z'<z} \Big(\sin\hat{\theta}(z') \cdot \prod_{z'<z''<z} \cos\hat{\theta}(z'') \cdot \sqrt{V_{(z'-1)}} \Big), \tag{JD}$$

so that (IU) becomes:

$$V_{(z-1)} \leqslant v - \Gamma(z)^2. \tag{JE}$$

$\Gamma(\cdot)$ satisfies the recursion equation

$$\Gamma(z+1) = \sin\hat{\theta}(z)\sqrt{V_{(z-1)}} + \cos\hat{\theta}(z)\Gamma(z), \tag{JF}$$

so by (JE):

$$|\Gamma(z+1)| \leqslant \sin|\hat{\theta}(z)|\sqrt{v - \Gamma(z)^2} + \cos\hat{\theta}(z)|\Gamma(z)|. \tag{JG}$$

From (JG), we will now prove that for all $z \in \mathbb{Z}$:

$$|\Gamma(z)| \leqslant \sin\Big(\frac{\pi}{2} \wedge \sum_{z'<z} \theta(z')\Big)\sqrt{v}. \tag{JH}$$

Indeed, (JH) is equivalent to saying that there exists some $\eta(z) \in [0, \sum_{z'<z} \theta(z')]$ such that $|\Gamma(z)| = \sin\eta(z)\sqrt{v}$, which we prove by induction. First, since our Toeplitz array was supposed compact, $\forall z < z^- \ \ \hat{\theta}(z) = 0$, so the formula is true for $z \leqslant z^-$ with $\eta(z) = 0$. Next, if the formula is true for z, then (JG) yields

$$|\Gamma(z)| \leqslant \big(\sin|\hat{\theta}(z)|\cos\eta(z) + \cos|\hat{\theta}(z)|\sin\eta(z)\big)\sqrt{v} = \sin\big(\eta(z) + |\hat{\theta}(z)|\big)\sqrt{v}, \tag{JI}$$

where $\eta(z) + |\hat{\theta}(z)| \leqslant \sum_{z'<z} \theta(z') + \theta(z) = \sum_{z'<z+1} \theta(z')$, so the formula is true for $(z+1)$, which ends the induction.

To conclude, we write that $s = \sum_z S_{(z)} = \Gamma(z > z^+)\sqrt{w}$. But by (JH), $|\Gamma(z > z^+)| \leqslant \sin\overline{\varepsilon} \cdot \sqrt{v}$, so in the end:

$$|s| \leqslant \sin\overline{\varepsilon} \cdot \sqrt{vw}, \tag{JJ}$$

quod erat demonstrandum. ♠

3.3.18 Corollary ('$\mathbb{Z}^n$ against $\mathbb{Z}^n$' theorem)**.** *Let $n \geqslant 1$; let $(X_x)_{x\in\mathbb{Z}^n}$ and $(Y_y)_{y\in\mathbb{Z}^n}$ be random variables, and assume there exists a function $\varepsilon\colon \mathbb{Z}^n \to [0,1]$ such that for all $x, y \in \mathbb{Z}^n$,*

$$\{X_x : Y_y\}_{\mathscr{M}} \leqslant \varepsilon(y - x), \tag{JK}$$

$\mathscr{M}$ being the natural σ-metalgebra of the system. Then $\{\vec{X} : \vec{Y}\} \leqslant \overline{\varepsilon}$, where $\overline{\varepsilon}$ the number in $[0,1]$ such that

$$\operatorname{Arcsin}(\overline{\varepsilon}) = \Big(\sum_{v\in\mathbb{Z}^n} \operatorname{Arcsin}\varepsilon(v)\Big) \wedge \frac{\pi}{2}. \tag{JL}$$

♣

Proof. To alleviate notation, we define the 'arcsin-sum' as the binary operation $\tilde{+}\colon [0,1]^2 \to [0,1]$ defined by:

$$a \tilde{+} b = \sin\Big((\operatorname{Arcsin} a + \operatorname{Arcsin} b) \wedge \frac{\pi}{2}\Big). \tag{JM}$$

$\tilde{+}$ is associative, commutative and nondecreasing, so it can be extended into an ∞-ary operator $\tilde{\sum}$; with this notation, (JL) merely writes $\overline{\varepsilon} = \tilde{\sum}_{v\in\mathbb{Z}^n} \varepsilon(v)$.

Let $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ be a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. For $1 \leqslant r \leqslant n$, we identify $\mathbb{Z}^r$ with $\mathbb{Z}\mathbf{e}_1 \oplus \mathbb{Z}\mathbf{e}_2 \oplus \cdots \oplus \mathbb{Z}\mathbf{e}_r$; we also denote $\mathbb{Z}_r^\perp := \mathbb{Z}\mathbf{e}_{r+1} \oplus \cdots \oplus \mathbb{Z}\mathbf{e}_n$. What we will prove is actually the following

3.3.19 Claim. *For all $r \in \{1, \dots, n\}$, all $x, y \in \mathbb{Z}_r^\perp$,*

$$\{\vec{X}_{x+\mathbb{Z}^r} : \vec{Y}_{y+\mathbb{Z}^r}\}_{\mathscr{M}} \leqslant \tilde{\sum}_{v\in\mathbb{Z}^r} \varepsilon(y - x + v). \tag{JN}$$

♣

The statement of the lemma then corresponds to the claim for $r = n$.

We prove Claim 3.3.19 by induction on r. The case $r = 1$ is merely Theorem 3.3.10 [§]. Now let us show how to go from the case $r - 1$ to the case r for $r > 1$:

[§]. More precisely, it is the subjective version of that theorem, cf. § 3.4.b.

Take $x, y \in \mathbb{Z}_r^\perp$. We notice that

$$\vec{X}_{x+\mathbb{Z}^r} = (\vec{X}_{x+i\mathbf{e}_r+\mathbb{Z}^{r-1}})_{i\in\mathbb{Z}}, \tag{JO}$$

which we shorthand into $\vec{X}_{x+\mathbb{Z}^r} = (\mathbf{X}_i)_{i\in\mathbb{Z}}$; similarly we write, with obvious notation, $\vec{Y}_{y+\mathbb{Z}^r} =: (\mathbf{Y}_j)_{j\in\mathbb{Z}}$. By induction hypothesis one has for all $i, j \in \mathbb{Z}$:

$$\{\mathbf{X}_i : \mathbf{Y}_j\}_{\mathscr{M}} \leqslant \tilde{\sum}_{v\in\mathbb{Z}^{r-1}} \varepsilon(y - x + (j-i)\mathbf{e}_r + v). \tag{JP}$$

Since the right-hand side of (JP) only depends on $(j-i)$, we can apply Theorem 3.3.10 to the $\mathbf{X}_i$ and the $\mathbf{Y}_j$, which yields

$$\{\vec{X}_{x+\mathbb{Z}^r} : \vec{Y}_{y+\mathbb{Z}^r}\}_{\mathscr{M}} \leqslant \tilde{\sum}_{z\in\mathbb{Z}} \big(\tilde{\sum}_{v\in\mathbb{Z}^{r-1}} \varepsilon(y - x + z\mathbf{e}_r + v)\big) = \tilde{\sum}_{v\in\mathbb{Z}^r} \varepsilon(y - x + v), \tag{JQ}$$

i.e. (JN). ♠

3.4 Generalizations of the tensorization results

3.4.a Minimal Hypotheses

When reading the proofs of the tensorization theorems, you may have noticed that taking the decorrelation hypotheses w.r.t. the whole σ-metalgebra of the system was a needlessly strong assumption. Actually each decorrelation hypothesis can be stated relatively to only *one* σ-algebra, in the following way:

- For Theorem 3.2.2, one needs only assume that for all $i \in I$, X_i and Y are ε_i-decorrelated when seen from $\sigma((X_{i'})_{i'<i})$;
- For Theorems 3.3.1 and 3.3.10, one needs only assume that X_i and Y_j are ε_{ij}-decorrelated (or $\varepsilon(j-i)$-decorrelated) when seen from $\sigma((X_{i'})_{i'<i}, (Y_{j'})_{j'<j})$.

In practice it is rare that one can bound above $\{X_i : Y\}_{\vec{X}_{\{i'<i\}}}$ or $\{X_i : Y_j\}_{(\vec{X}_{\{i'<i\}}, \vec{Y}_{\{j'<j\}})}$ more sharply than $\{X : Y_i\}_{\mathscr{M}}$, resp. $\{X_i : Y_j\}_{\mathscr{M}}$; yet it is worth remembering that the 'genuine' decorrelation hypotheses are weaker than those we wrote, especially when one gets interested in optimality issues (cf. § 3.5).

3.4.1 Remark. In our tensorization proofs we took I and J finite; yet those proofs, and therefore everything in this subsection, remain valid if we take for I or J any (countable) well-ordered set, in particular if I or J is $\mathbb{N}$. ♡

3.4.b Subjective versions of the theorems

In the tensorization theorems I stated, the decorrelation hypotheses were given with regard to the natural σ-metalgebra $\mathscr{M}$ of the system, while the results were given in terms of 'objective' (I mean, not subjective) decorrelations. Yet actually it can be shown that our results are still valid w.r.t. $\mathscr{M}$ —or even w.r.t. any sharper σ-metalgebra $\mathscr{N} \supset \mathscr{M}$, provided decorrelation hypotheses are stated w.r.t. $\mathscr{N}$. As an example, let us state and prove the subjective result corresponding to Theorem 3.2.2:

3.4.2 Corollary. *Let X, $(Y_i)_{i\in I}$ and $(Z_\theta)_{\theta\in\Theta}$ be random variables, and call $\mathscr{N}$ the σ-metalgebra they span. Suppose we have bounds $\{X : Y_i\}_{\mathscr{N}} \leqslant \varepsilon_i$ for all $i \in I$; then:*

$$\{X : \vec{Y}_I\}_{\mathscr{N}} \leqslant \sqrt{1 - \prod_{i\in I}(1 - \varepsilon_i^2)}. \tag{JR}$$

♣

Proof. Up to making up copies of I and Θ, we can assume that $\{0\}$, I and Θ are disjoint, which allows us to denote $Z_0 := X$ and $Z_i := Y_i$ for $i \in I$, so that $\mathcal{N}$ is the σ-metalgebra spanned by the Z_θ for $\theta \in \overline{\Theta} := \{0\} \uplus I \uplus \Theta$. Then (JR) means that for all $\Xi \subset \overline{\Theta}$, for (almost-)all $\vec{z}_\Xi$, one must have:

$$\{X : \vec{Y}_I\} \leqslant \sqrt{1 - \prod_{i \in I}(1 - \varepsilon_i^2)} \quad \text{under the law } \mathbf{P}[\cdot | \vec{Z}_\Xi = \vec{z}_\Xi]. \tag{JS}$$

So, Corollary 3.4.2 will ensue from Theorem 3.2.2 provided we can prove that, denoting by $\mathcal{M}$ the σ-metalgebra spanned by X and the Y_i, one has for all $i \in I$:

$$\{X : Y_i\}_{\mathcal{M}} \leqslant \varepsilon_i \quad \text{under the law } \mathbf{P}[\cdot | \vec{Z}_\Xi = \vec{z}_\Xi]. \tag{JT}$$

But under a law P, saying that $\{X : Y_i\}_{\mathcal{M}} \leqslant \varepsilon_i$ means that for all $\Upsilon \subset \{0\} \uplus I$, for (almost-)all $\vec{z}'_\Upsilon$, one has $\{X : Y_i\} \leqslant \varepsilon_i$ under the law $P[\cdot | \vec{Z}_\Upsilon = \vec{z}'_\Upsilon]$. So, for $P = \mathbf{P}[\cdot | \vec{Z}_\Xi = \vec{z}_\Xi]$, (JT) means that, for all $\vec{z}'_\Upsilon$:

$$\{X : Y_i\} \leqslant \varepsilon_i \quad \text{under the law } \mathbf{P}[\cdot | \vec{Z}_\Xi = \vec{z}_\Xi \text{ and } \vec{Z}_\Upsilon = \vec{z}'_\Upsilon]. \tag{JU}$$

In Formula (JU) we can assume that z_θ and z'_θ coincide for all $\theta \in \Xi \cap \Upsilon$, since otherwise the event "$\vec{Z}_\Xi = \vec{z}_\Xi$ and $\vec{Z}_\Upsilon = \vec{z}'_\Upsilon$" would be empty and there would be nothing to say. Then "$\vec{Z}_\Xi = \vec{z}_\Xi$ and $\vec{Z}_\Upsilon = \vec{z}'_\Upsilon$" is of the form "$\vec{Z}_{\Xi \cup \Upsilon} = \vec{z}_{\Xi \cup \Upsilon}$", where $\Xi \cup \Upsilon \subset \overline{\Theta}$, so that (JU) follows directly from the hypothesis $\{X : Y_I\}_{\mathcal{N}} \leqslant \varepsilon_i$. ♠

3.5 Optimality

3.5.a Exact Optimality

With the minimal hypotheses stated in § 3.4.a, Theorems 3.2.2 and 3.3.10 are optimal:

3.5.1 Theorem. *The bound (FM) in Theorem 3.2.2 is optimal, in the following sense: for any integer N, for all $(\varepsilon_i)_{1 \leqslant i \leqslant N}$ in $[0,1]^N$, one can find random variables $X_1, \ldots, X_N, Y$ such that for all $i \in \{1, \ldots, N\}$,*

$$\{X_i : Y\}_{\vec{X}_{\{i' < i\}}} = \varepsilon_i \tag{JV}$$

and

$$\{\vec{X} : Y\} = \sqrt{1 - \prod_i (1 - \varepsilon_i^2)}. \tag{JW}$$

♣

3.5.2 Theorem. *The bound (HU) in Theorem 3.3.10 is optimal, in the following sense: for any integer N, for all $(\varepsilon(z))_{-N \leqslant z \leqslant N} \in [0,1]^{\{-N, \ldots, N\}}$, one can find random variables $(X_i)_{i \in \mathbb{Z}}$ and $(Y_j)_{j \in \mathbb{Z}}$ such that for all $i, j \in \mathbb{Z}$,*

$$\{X_i : Y_j\}_{(\vec{X}_{\{i' < i\}}, \vec{Y}_{\{j' < j\}})} = \begin{cases} \varepsilon(j - i) & \text{if } |j - i| \leqslant N; \\ 0 & \text{if } |j - i| > N \end{cases} \tag{JX}$$

and $\{\vec{X} : \vec{Y}\} = \overline{\varepsilon}$, with $\overline{\varepsilon}$ defined by:

$$\operatorname{Arcsin} \overline{\varepsilon} = \sum_{z=-N}^{N} \operatorname{Arcsin} \varepsilon(z) \wedge \frac{\pi}{2}. \tag{JY}$$

♣

Actually, as proving Theorem 3.5.2 for *all* the $(\varepsilon(z))_{-N\leqslant z\leqslant N}$ involves some heavy technicalities [64], I will only prove the slightly weaker following

3.5.3 Theorem. *For any integer N, the exists a neighbourhood U of $\vec{0}$ in $[0,1]^{\{-N,\dots,N\}}$ such that, for all $(\varepsilon(z))_{-N\leqslant z\leqslant N}\in U$, one can find random variables $(X_i)_{i\in\mathbb{Z}}$ and $(Y_j)_{j\in\mathbb{Z}}$ satisfying (JX) and (JY)* [¶]. ♣

3.5.4 Remark. On the other hand, Theorem 3.3.1 is obviously not optimal since, as we pointed out, its bound is strictly weaker than that of Theorem 3.3.10. ♡

The proof of Theorem 3.5.1 relies on the following important result:

3.5.5 Lemma. *Let $(X_1,\dots,X_N,Y)$ be an $(N+1)$-dimensional Gaussian vector. For all $1\leqslant i\leqslant N$, define*

$$e_i := \{X_i : Y\}_{\vec{X}_{\{i'<i\}}}, \tag{JZ}$$

then one has exactly:

$$\{\vec{X} : Y\} = \sqrt{1-\prod_i(1-e_i^2)}. \tag{KA}$$

♣

3.5.6 Remark. Maximal correlation, as I told in § 1, is fundamentally a Hilbertian concept. When one deals with Gaussian vectors, the Hilbert spaces involved actually have finite dimensions, so that Lemma 3.5.5 about decorrelations can also be seen as a result about Euclidian spaces. In Appendix 3.9, I will present an unexpected corollary of this lemma, stating a geometric property of the 3-dimensional Euclidian space. ♡

Proof of Lemma 3.5.5. To alleviate notation, we denote $\mathscr{F}_{i-1} := \sigma(\vec{X}_{\{i'<i\}})$. Since $(\vec{X},Y)$ is Gaussian, the law of (X_i,Y) under $\mathbf{P}[\cdot|x_1,\dots,x_{i-1}]$ is Gaussian and only depends on $(x_1,\dots,x_{i-1})$ through an additive constant; consequently, we can speak of "*the* Hilbertian correlation between X_i and Y conditionally to $\mathscr{F}_{i-1}$", which is e_i, and also of "*the* conditional variance of X_i w.r.t. $\mathscr{F}_{i-1}$", resp. "*the* conditional variance of Y", resp. "*the* conditional covariance of (X_i,Y)", which we denote resp. $\mathrm{Var}(X_i|\mathscr{F}_{i-1})$, $\mathrm{Var}(Y|\mathscr{F}_{i-1})$, $\mathrm{Cov}(X_i,Y|\mathscr{F}_{i-1})$. By Theorem 1.2.6, one has:

$$\mathrm{Cov}(X_i,Y|\mathscr{F}_{i-1}) = \pm e_i\,\mathrm{Sd}(X_i|\mathscr{F}_{i-1})\,\mathrm{Sd}(Y|\mathscr{F}_{i-1}). \tag{KB}$$

Now take $g(Y)=Y$ and $f(X)=\sum_{i=1}^N\beta_iX_i$, for some $\beta_i\in\mathbb{R}$ to be chosen later. Then g^{i-1} is equal to $Y-\mathbf{E}[Y|\mathscr{F}_{i-1}]$ and f_i is proportional to $X_i-\mathbf{E}[X_i|\mathscr{F}_{i-1}]$, thus, by (KB) and our model's being Gaussian, all the inequalities until (FU) in the proof of Theorem 3.2.2 actually are equalities for $\varepsilon_i=e_i$. If moreover $\mathrm{Cov}(f_i,g^{i-1}|\mathscr{F}_{i-1})\geqslant 0$ for all i, then we can drop the absolute values in their left-hand sides, and thus (FV) will also be an equality. Then, to get an equality in (FW), it just remains to ensure that the final Cauchy–Schwarz equality is an equality, i.e. to ensure that one has, for all i:

$$\mathrm{Var}(f_i)\propto e_i^2\prod_{i'=1}^{i-1}(1-e_{i'}^2). \tag{KC}$$

If all of that is satisfied, then one will have exactly $\mathbf{E}[fg]=\sqrt{1-\prod_i(1-e_i^2)}\,\mathrm{Sd}(f)\,\mathrm{Sd}(g)$, so that $\{\vec{X}:Y\}\geqslant\sqrt{1-\prod_i(1-e_i^2)}$. The converse inequality being obviously true by (the minimal version of) Theorem 3.2.2, the result will follow.

[¶]. Notice that in the neighbourhood of $\vec{0}$, one can drop the "$\wedge\frac{\pi}{2}$" in the right-hand side of (JY).

So, we have to check that the choice of the β_i can be performed so that (KC) is satisfied, with $\mathrm{Cov}(f_i, g^{i-1}|\mathscr{F}_{i-1})$ of the good sign. To do this, we will choose successively relevant values for β_N, $\beta_{N-1}, \dots, \beta_1$.

We observe that, if $\beta_N, \dots, \beta_{i+1}$ have already been fixed, then $\beta_i \mapsto \mathrm{Cov}(f_i, g^{i-1}|\mathscr{F}_{i-1})$ is an affine function with slope

$$\pm e_i \,\mathrm{Sd}(Y|\mathscr{F}_{i-1}) \frac{\mathrm{Sd}(X_i|\mathscr{F}_{i-1})}{\mathrm{Sd}(X_i)}. \tag{KD}$$

Moreover, $\mathrm{Var}(f_i) = \mathrm{Var}(f_i|\mathscr{F}_{i-1})$ as f_i is centered w.r.t. $\mathscr{F}_{i-1}$; so, since $f_i \propto X_i - \mathbf{E}[X_i|\mathscr{F}_{i-1}]$, (KB) implies:

$$\mathrm{Var}(f_i) = \frac{\mathrm{Cov}(f_i, g^{i-1}|\mathscr{F}_{i-1})^2}{e_i^2 \,\mathrm{Var}(Y|\mathscr{F}_{i-1})}. \tag{KE}$$

So, provided all the three quantities e_i, $\mathrm{Sd}(Y|\mathscr{F}_{i-1})$ and $\mathrm{Sd}(X_i|\mathscr{F}_{i-1})$ are nonzero, there exists a (unique) β_i satisfying (KC).

Now if $\mathrm{Sd}(Y|\mathscr{F}_{i-1})$ is zero, this means that Y is $\mathscr{F}_{i-1}$-measurable; then one of the $e_{i'}$ has to be 1 and thus the result is trivial. Next if $\mathrm{Sd}(X_i|\mathscr{F}_{i-1})$ is zero, this means that X_i is $\mathscr{F}_{i-1}$-measurable; then $e_i = 0$ and $f_i \equiv 0$, so that (KC) is automatically satisfied. Finally if $e_i = 0$ and $\mathrm{Var}(X_i|\mathscr{F}_{i-1}) > 0$, then there exists a (unique) β_i such that $f_i \equiv 0$, for which (KC) is satisfied. So all those particular cases actually work fine too. ♠

Proof of Theorem 3.5.1. For technical reasons, we begin with noticing that the theorem is immediate if some e_i is equal to 1, so that we can assume that all the e_i are < 1. Thanks to Lemma 3.5.5, it suffices to prove that for any sequence of $\varepsilon_i \in [0,1)$ it is possible to build a Gaussian vector $(X, \vec{Y})$ for which $e_i = \varepsilon_i \ \forall i$. To do this, let $\xi, \zeta_1, \dots, \zeta_N$ be i.i.d. $\mathscr{N}(1)$ variables, and take $Y = \xi$ and $X_i = \sqrt{1-\alpha_i}\zeta_i + \sqrt{\alpha_i}\xi$ for some parameters $\alpha_i \in [0,1)$. We want to choose the α_i such that $\vec{e}(\vec{\alpha}) = \vec{\varepsilon}$; this is always possible, by the following method:

- First we compute α_1: By Theorem 1.2.6, one can write down the equation linking α_1 and e_1. It is clear without knowing the precise form of that equation (actually, $e_1 = \sqrt{\alpha_1}$) that e_1 is a continuous increasing function of α_1 with $e_1 = 0$ for $\alpha_1 = 0$ and $e_1 = 1$ for $\alpha_1 = 1$. Therefore there is a unique α_1 such that $e_1 = \varepsilon_1$.
- Then we compute α_2: As we already know the value of α_1, we can treat it as a constant and look for the equation linking α_2 and e_2, which we compute by Theorem 1.2.6 again. That equation, though more complicated than in the previous case (actually, $e_2 = \sqrt{\alpha_2}\sqrt{1-\alpha_1}/\sqrt{1-\alpha_1\alpha_2}$), exhibits the same behaviour: e_2 is a continuous increasing function of α_2 with $e_2(\alpha_2 = 0) = 0$ and $e_2(\alpha_2 = 1) = 1$. Therefore there is a unique α_2 such that $e_2 = \varepsilon_2$.
- We carry on this process until having determined all the α_i.

♠

Proof of Theorem 3.5.3. Again, the principle of the proof will consist in showing how the optimal bound can be attained for relevant Gaussian vectors and linear functions of them.

We consider independent $\mathscr{N}(1)$ variables $(\xi_j)_{j\in\mathbb{Z}}$ and $(\omega_{ij})_{(i,j)\in\mathbb{Z}\times\mathbb{Z}}$. For all i we set:

$$X_i = \sum_{z=-N}^{N} \omega_{i(i+z)}, \tag{KF}$$

resp. for all j:

$$Y_j = \xi_j + \sum_{z=-N}^{N} \alpha_z \omega_{(j-z)j} \tag{KG}$$

for some real parameters $(\alpha_z)_{-N\leqslant z\leqslant N}$ to be fixed later. This model is obviously invariant by translation of the indexes. For $z\in\mathbb{Z}$, define

$$\dot{e}_z := \frac{\mathrm{Cov}(X_i, Y_{i+z}|\mathscr{F}_{i-1}\vee\mathscr{G}_{i+z-1})}{\mathrm{Sd}(X_i|\mathscr{F}_{i-1}\vee\mathscr{G}_{i+z-1})\,\mathrm{Sd}(Y_{i+z}|\mathscr{F}_{i-1}\vee\mathscr{G}_{i+z-1})}, \tag{KH}$$

where the choice of i does not matter. Since our model is Gaussian, by Theorem 1.2.6,

$$\{X_i : Y_{i+z}\}_{(\vec{X}_{\{i'<i\}},\vec{Y}_{\{j'<i+z\}})} = |\dot{e}_z|. \tag{KI}$$

By the properties of Gaussian vectors, it is possible to write down explicitly the equations linking the $\dot{e}_z$ to the α_z. Though these equations may be quite horrendous, some of their properties can be easily established:

3.5.7 Claim.

(i) For $|z|>N$, $\dot{e}_z=0$ (for any choice of the α_z);

(ii) The map $(\alpha_{-N},\dots,\alpha_N)\mapsto(\dot{e}_{-N},\dots,\dot{e}_N)$ is of class $\mathscr{C}^1$ on the neighbourhood of $(0,\dots,0)$, with:

$$\left(\frac{\partial\dot{e}_z}{\partial\alpha_y}\right)(\vec{0}) = \frac{\mathbf{1}_{y=z}}{\sqrt{2N+1}}. \tag{KJ}$$

♣

By the inverse function theorem, one can therefore find neighbourhoods V and U of $\vec{0}$ in $\mathbb{R}^{\{-N,\dots,N\}}$ such that the map $\vec{\alpha}\mapsto\vec{\dot{e}}$ is a $\mathscr{C}^1$-diffeomorphism from V onto U. In particular, for $\vec{\varepsilon}$ in such an U we can always fix the α_z of our model such that $\forall z\ \ \dot{e}_z=\mathbf{1}_{|z|\leqslant N}\varepsilon(z)$, so that (JX) is satisfied.

Now we have to choose f and g. Morally [‖] we have to take the functions f and g having maximal Pearson correlation. Since the model is Gaussian, these functions will be linear, and since the model is invariant by translation, they will likely be invariant by translation too. So we would like to take, formally, $f(\vec{X})=\sum_{i\in\mathbb{Z}}X_i$ and $g(\vec{Y})=\sum_{j\in\mathbb{Z}}Y_j$. As such functions are not properly defined, we will rather consider $f[k](\vec{X})=\sum_{i=-k}^{k}X_i$, resp. $g[k](\vec{Y})=\sum_{j=-k}^{k}Y_j$, and then we will let k tend to infinity.

For these $f[k]$ and $g[k]$, define the $V[k]_i^j$, the $W[k]_j^i$ and the $S[k]_{ij}$ as in the proof of Theorem 3.3.10, which are gathered into the array $\mathbf{A}[k]$. The following properties of the $\mathbf{A}[k]$ follow easily from the structure of our model:

3.5.8 Claim.

(i) All the $V[k]_i^j, W[k]_j^i, S[k]_{ij}$ are bounded uniformly in i,j,k.

(ii)
- *$V[k]_i^j$ is zero as soon as $i\notin\{-k-2N,\dots,k\}$;*
- *$W[k]_j^i$ is zero as soon as $j\notin\{-k,\dots,k\}$.*

(iii) $S[k]_{ij}$ is zero as soon as $|j-i|>N$.

(iv)
- *For $-k\leqslant i\leqslant k-2N$, $V[k]_i^j$ only depends on $(j-i)$, even when k varies. We denote its value by $V_{(j-i)}$.*
- *For $-k\leqslant j\leqslant k$, $W[k]_j^i$ only depends on $(j-i)$, even when k varies. We denote its value by $W_{(j-i)}$.*
- *For $-k\leqslant i\leqslant k-2N$ and $-k\leqslant j\leqslant k$, $S[k]_{ij}$ only depends on $(j-i)$, even when k varies. We denote its value by $S_{(j-i)}$.*

(v)
- *$V_{(z)}$ has some constant value v for $z<-N$;*

[‖]. I say "morally" because nothing ensures that the supremum (AG) would actually be a maximum here.

- *$W_{(z)}$ has some constant value w for $z > N$.*

♣

By Claim 3.5.8, $\mathbf{A}[k]$ converges pointwise to some compact Toeplitz array $\mathbf{A}$, whose entries are the $V_{(z)}, W_{(z)}, S_{(z)}$ introduced at Item (iv) of the claim, whose values v and w are those introduced at Item (v), and whose value s is $\sum_{z=-N}^{N} S_{(z)}$. All the arrays $\mathbf{A}[k]$ are obviously correct since they correspond to true functions, so by passing to the limit $\mathbf{A}$ is correct too.

Since our model is Gaussian, all the inequalities (ID), (IE) and (IK) are actually equalities for the arrays $\mathbf{A}[k]$; moreover, since the $\dot{\varepsilon}_z$ are nonnegative, the $S[k]_{ij}$ are nonnegative. By letting k tend to infinity, it follows that all the inequalities (IS)–(IU) are actually equalities for the array $\mathbf{A}$, with the $S_{(z)}$ nonnegative. Consequently in (JB) one has $\hat{\theta}(z) = \theta(z)$, and all the further inequalities are actually equalities, so that in the end (JJ) becomes:

$$\frac{s}{\sqrt{vw}} = \overline{\varepsilon}. \tag{KK}$$

Now, defining $V[k]$, $W[k]$ and $S[k]$ by resp. (IB), (IC) and (HY) for the arrays $\mathbf{A}[k]$, Claim 3.5.8 shows that, when $k \to \infty$, $V[k] \sim 2kv$, resp. $W[k] \sim 2kw$, resp. $S[k] \sim 2ks$, so (KK) implies that $S[k]/\sqrt{V[k]W[k]} \to \overline{\varepsilon}$. But recall that $V[k]$, $W[k]$ and $S[k]$ are the respective variances and covariance of the functions $f[k] \in \overline{L}^2(\vec{X})$ and $g[k] \in \overline{L}^2(\vec{Y})$, so by the very definition (AG) of Hilbertian correlations,

$$\{\vec{X} : \vec{Y}\} \geqslant \frac{S[k]}{\sqrt{V[k]W[k]}}. \tag{KL}$$

Making $k \to \infty$, it follows that $\{\vec{X} : \vec{Y}\} \geqslant \overline{\varepsilon}$; the converse inequality being obviously true by (the minimal version of) Theorem 3.3.10, this proves Theorem 3.5.3. ♠

3.5.9 Example. In this example we will carry out explicit computations for a Gaussian model close to the model presented in the proof above. We take independent $\mathcal{N}(1)$ variables $\ldots, \zeta_{-1}, \zeta_0, \zeta_1, \ldots$, $\ldots, \xi_{-1/2}, \xi_{1/2}, \xi_{3/2} \ldots$, $\ldots, \omega_{-1/4}, \omega_{1/4}, \omega_{3/4}, \ldots$, and we set

$$X_i = \zeta_i + \sqrt{\alpha}(\omega_{i-1/4} + \omega_{i+1/4}), \tag{KM}$$

$$\text{resp.} \quad Y_j = \xi_j + \sqrt{\alpha}(\omega_{j-1/4} + \omega_{j+1/4}) \tag{KN}$$

for all integer i, resp. all half-integer j, where α is some arbitrary nonnegative parameter. We are going to show that for this system (HU) is actually an equality, in accordance with the proof of Theorem 3.5.3.

For half-integer z denote

$$e_z := \{X_i : Y_{i+z}\}_{(\vec{X}_{\{i'<i\}}, \vec{Y}_{\{j'<i+z\}})}, \tag{KO}$$

where the choice of i does not matter by translation invariance. Clearly $e_{-z} = e_z$ for all z and $e_z = 0$ for $|z| > 1/2$, so to know all the e_z the only nontrivial computation is computing $e_{1/2}$. Let us perform it.

Since everything is Gaussian, by Theorem 1.2.6, $e_{1/2}$ is the value, under the law $\mathbf{P}[\cdot | \vec{X}_{\{i<0\}}, \vec{Y}_{\{j<1/2\}} \equiv 0]$, of

$$|\mathbf{E}[X_0 Y_{1/2}]| / \mathrm{Sd}(X_0)\mathrm{Sd}(Y_{1/2}). \tag{KP}$$

Under the law $\mathbf{P}[\cdot | \vec{X}_{\{i<0\}}, \vec{Y}_{\{j<1/2\}} \equiv 0]$, it is clear that $\zeta_0, \omega_{1/4}, \xi_{1/2}, \omega_{3/4}, \ldots$ have exactly the same (joint) law as under $\mathbf{P}$, and that $\omega_{-1/4}$ is still independent of these (joint) variables, though its

variance shall have diminished. So we need only compute

$$v := \mathrm{Var}(\omega_{-1/4} | \vec{X}_{\{i<0\}}, \vec{Y}_{\{j<1/2\}} \equiv 0). \tag{KQ}$$

Denote $\vec{L}_{\mathrm{r}} := (\dots, X_{-2}, Y_{-3/2}, X_{-1}, Y_{-1/2})$, resp. $\vec{L}_{\mathrm{l}} := (\dots, X_{-2}, Y_{-3/2}, X_{-1})$. We write that (formally)

$$d\mathbf{P}[\vec{L}_{\mathrm{r}} \equiv 0 \text{ and } \omega_{-1/4} = x] \propto e^{-x^2/2v} dx, \tag{KR}$$

and also $d\mathbf{P}[\vec{L}_{\mathrm{l}} \equiv 0 \text{ and } \omega_{-3/4} = y] \propto e^{-y^2/2v} dy$ by translation invariance. But under $\mathbf{P}[\cdot | \vec{L}_{\mathrm{l}} \equiv 0$ and $\omega_{-3/4} = y]$, the law of $(\xi_{-1/2}, \omega_{1/4})$ is the same as under $\mathbf{P}$, so one has:

$$\begin{aligned}
e^{-x^2/2v} &\propto d\mathbf{P}[\vec{L}_{\mathrm{r}} \equiv 0 \text{ and } \omega_{-1/4} = x] \\
&= \int_y dy\, d\mathbf{P}[\vec{L}_{\mathrm{l}} \equiv 0 \text{ and } \omega_{-3/4} = y] d\mathbf{P}[Y_{-1/2} = 0 \text{ and } \omega_{-1/4} = x | \vec{L}_{\mathrm{l}} \equiv 0 \text{ and } \omega_{-3/4} = y] \\
&\propto \int_y d\mathbf{P}[Y_{-1/2} = 0 \text{ and } \omega_{-1/4} = x | \vec{L}_{\mathrm{l}} \equiv 0 \text{ and } \omega_{-3/4} = y] e^{-y^2/2v}\, dy \\
&= \int_y d\mathbf{P}[\xi_{-1/2} = -\sqrt{\alpha}(x+y) \text{ and } \omega_{-1/4} = x | \vec{L}_{\mathrm{l}} \equiv 0 \text{ and } \omega_{-3/4} = y] e^{-y^2/2v}\, dy \\
&\propto \int_y e^{-\alpha(x+y)^2/2} e^{-x^2/2} e^{-y^2/2v}\, dy \propto \exp\Big\{\Big(1 + \alpha - \frac{\alpha^2}{\alpha + 1/v}\Big)\frac{x^2}{2}\Big\},
\end{aligned} \tag{KS}$$

so that v must satisfy:

$$1 + \alpha - \frac{\alpha^2}{\alpha + 1/v} = \frac{1}{v}, \tag{KT}$$

whose only nonnegative solution is

$$v = \frac{\sqrt{1+4\alpha} - 1}{2\alpha}. \tag{KU}$$

So one has $\mathrm{Sd}(X_0 | \vec{L}_{\mathrm{l}} \equiv 0) = \sqrt{1 + \alpha v + \alpha} = (\sqrt{1+4\alpha} + 1)/2$, $\mathrm{Sd}(Y_{1/2} | \vec{L}_{\mathrm{l}} \equiv 0) = \sqrt{1+2\alpha}$ and $\mathbf{E}[X_0 Y_{1/2} | \vec{L}_{\mathrm{l}} \equiv 0] = \alpha$, so that in the end (KP) yields:

$$e_{1/2} = \frac{\sqrt{1+4\alpha} - 1}{2\sqrt{1+2\alpha}}. \tag{KV}$$

With this value, Theorem 3.3.10 states that one has necessarily

$$\{\vec{X} : \vec{Y}\} \leqslant \sin(2 \operatorname{Arcsin} e_{1/2})^{[*]} = 2 e_{1/2} \sqrt{1 - e_{1/2}^2} = \frac{2\alpha}{1+2\alpha}. \tag{KW}$$

We show that (KW) is actually an equality: take indeed $f[k](\vec{X}) := \sum_{i=1}^{k} X_k$, resp. $g[k](\vec{Y}) := \sum_{j=1/2}^{k-1/2} Y_k$, then $\mathrm{Var}(f[k]) = \mathrm{Var}(g[k]) = k(1+2\alpha)$ and $\mathbf{E}[fg] = (2k-1)\alpha$, so that

$$\{\vec{X} : \vec{Y}\} \geqslant \frac{(2k-1)\alpha}{k(1+2\alpha)} \overset{k\to\infty}{\to} \frac{2\alpha}{1+2\alpha}. \tag{KX}$$

♡

[*]. As here one always has $e_{1/2} \leqslant 1/\sqrt{2}$, we can drop the "$\wedge \frac{\pi}{2}$" of Formula (HU).

3.5.10 Remark. One can formally set $\alpha = +\infty$ in the previous example, which actually means that one takes $X_i = \omega_{i-1/4} + \omega_{i+1/4}$, resp. $Y_j = \omega_{j-1/4} + \omega_{j+1/4}$. In this case, both Formulas (KV) and (KX) 'pass to the limit', yielding $e_{1/2} = 1/\sqrt{2}$ and $\{\vec{X} : \vec{Y}\} = 1$. This shows that it is possible indeed that the e_z have 'mild' values and that yet $\vec{X}$ and $\vec{Y}$ are fully correlated. In other words, the "$\wedge\frac{\pi}{2}$" in (HU) is not an 'artifact' of the proof of Theorem 3.3.10 [†], but the expression of a real 'phase transition' phenomenon [‡]. Such a phase transition did not occur for the simple tensorization formula (FM), which shows that double tensorization in intrinsically more complicated than simple tensorization. ♡

3.5.b Asymptotic optimality

In the previous subsection we saw that (the minimal versions of) Theorems 3.2.2 and 3.3.10 were optimal, while Theorem 3.3.1 was not. However it turns out that that result is nevertheless 'asymptotically optimal', in the sense that the bound it gives is equivalent to the optimal bound when the correlations between the variables become weak. Here is a precise statement:

3.5.11 Theorem. *Let $I = \{1,\dots,N\}$ and $J = \{1,\dots,M\}$ be finite sets, and define the function $Opt : [0,1]^{I\times J} \to [0,1]$ by*

$$Opt(\vec{\varepsilon}_{I\times J}) := \sup\big\{\{\vec{X}_I : \vec{Y}_J\}\ ;\ (\forall (i,j)\in I\times J)(\{X_i : Y_j\}_{(\vec{X}_{\{i'<i\}},\vec{Y}_{\{j'<j\}})} \leqslant \varepsilon_{ij})\big\}; \tag{KY}$$

then, when $\vec{\varepsilon}_{I\times J} \to \vec{0}$, one has:

$$Opt(\vec{\varepsilon}) \sim |||\boldsymbol{\varepsilon}|||. \tag{KZ}$$

♣

3.5.12 Remark. In the same way, the simple bound (FD) of Proposition 3.2.1 is asymptotically equivalent to the optimal bound (FM) of Theorem 3.2.2. ♡

Proof. Take $(M+NM)$ i.i.d. $\mathcal{N}(1)$ variables $\xi_1,\dots,\xi_M,\omega_{11},\dots,\omega_{NM}$. For $((\alpha_{ij}))_{i,j} \in \mathbb{R}^{N\times M}$, set

$$\left\{\begin{array}{lcl} X_i &=& \sum_j \omega_{ij}; \\ Y_j &=& \xi_j + \sum_i \alpha_{ij}\omega_{ij}. \end{array}\right. \tag{LA}$$

Denote

$$e_{ij} := \{X_i : Y_j\}_{(\vec{X}_{\{i'<i\}},\vec{Y}_{\{j'<j\}})}, \tag{LB}$$

and define $\dot{e}_{ij}$ as the Pearson correlation coefficient of X_i and Y_j under the law $\mathbf{P}[\cdot|\vec{X}_{\{i'<i\}}, \vec{Y}_{\{j'<j\}} \equiv 0]$. Then, as in the proof of Theorem 3.5.3, one has $e_{ij} = |\dot{e}_{ij}|$, and the function $\vec{\alpha} \mapsto \vec{\dot{e}}$ is $\mathscr{C}^1$ around $\vec{0}$, with

$$\vec{\dot{e}} = \frac{1}{\sqrt{M}}\,\vec{\alpha} + O(\|\vec{\alpha}\|^2) \qquad \text{when } \vec{\alpha} \to \vec{0}. \tag{LC}$$

By the inverse function theorem, $\vec{\alpha} \mapsto \vec{\dot{e}}$ is therefore a diffeomorphism from some neighbourhood V of $\vec{0}$ onto some neighbourhood U of $\vec{0}$, whose inverse function is such that

$$\vec{\alpha} = \sqrt{M}\,\vec{\dot{e}} + O(\|\vec{\dot{e}}\|^2) \qquad \text{when } \vec{\dot{e}} \to \vec{0}. \tag{LD}$$

[†]. On the other hand, it is possible that the "$\wedge 1$" in (GD) was such an artifact, since Theorem 3.3.1 is not optimal.

[‡]. There exist indeed situations going 'beyond the phase transition', i.e. for which $\sum_{z\in\mathbb{Z}} \mathrm{Arcsin}(e_z) > \pi/2$, though this is not the case for Example 3.5.9.

Now let $\vec{\varepsilon} \in (\mathbb{R}_+)^{N\times M} \cap U$. Take $\vec{\alpha} \in V$ such that $\vec{e}(\vec{\alpha}) = \vec{\varepsilon}$, so that the condition of (KY) is satisfied. For $\varphi \in \mathbb{R}^N, \psi \in \mathbb{R}^M$ with $\|\varphi\|, \|\psi\| = 1$, set

$$\begin{cases} f(\vec{X}) & := & \sum_i \varphi_i X_i; \\ g(\vec{Y}) & := & \sum_j \psi_j Y_j. \end{cases} \tag{LE}$$

One has

$$\mathrm{Var}(f) = M, \tag{LF}$$

$$\mathrm{Var}(g) = 1 + O(\|\vec{\alpha}\|^2) = 1 + O(\|\vec{\varepsilon}\|^2) \tag{LG}$$

and

$$\mathbf{E}[fg] = \sum_{i,j} \alpha_{ij}\varphi_i\psi_j = \langle \varphi, \boldsymbol{\varepsilon}\psi\rangle + O(\|\vec{\varepsilon}\|^2), \tag{LH}$$

where the constants implicit in the "$O(\|\vec{\varepsilon}\|^2)$" are uniform in (φ, ψ). So one has

$$Opt(\vec{\varepsilon}) \geqslant \{\vec{X} : \vec{Y}\} \geqslant \frac{|\mathbf{E}[fg]|}{\mathrm{Sd}(f)\mathrm{Sd}(g)} = |\langle \varphi, \boldsymbol{\varepsilon}\psi\rangle| + O(\|\vec{\varepsilon}\|^2), \tag{LI}$$

whence after taking supremum over (φ, ψ):

$$Opt(\vec{\varepsilon}) \geqslant |||\boldsymbol{\varepsilon}||| + O(\|\vec{\varepsilon}\|^2) \overset{\vec{\varepsilon}\to\vec{0}}{\sim} |||\boldsymbol{\varepsilon}|||. \tag{LJ}$$

Since on the other hand $Opt(\vec{\varepsilon}) \leqslant |||\boldsymbol{\varepsilon}|||$ by Theorem 3.3.1, the proposition follows. ♠

3.5.13 Remark. If we state decorrelation hypotheses w.r.t. the *whole* σ-metalgebra of the system (denoted by $*$), no quantity analogous to $\dot{e}_{ij}$ shall exist any more; then one can only write, denoting $e'_{ij} := \{X_i : Y_j\}_*$:

$$e'_{ij}(\vec{\alpha}) = \frac{|\alpha_{ij}|}{\sqrt{M}} + O(\|\vec{\alpha}\|^2). \tag{LK}$$

So, to see how the correlations depend on the parameters, we have to study the map $\vec{\alpha} \mapsto \vec{e}'$, which is approximated by a homothety *only on the cone* $\mathbb{R}_+^{N\times M}$ —and which moreover is no better than continuous here. So we shall replace the inverse function theorem by an alternative technique, which will yield the slightly weaker theorem stated just below. ♡

3.5.14 Theorem. *Define*

$$Opt'((\varepsilon_{ij})_{(i,j)\in I\times J}) := \sup\big\{\{\vec{X}_I : \vec{Y}_J\}\ ;\ (\forall (i,j) \in I \times J)(\{X_i : Y_j\}_* \leqslant \varepsilon_{ij})\big\}; \tag{LL}$$

then for any closed cone C of $\mathbb{R}^{N\times M}$ contained in $(\mathbb{R}_+^*)^{N\times M} \cup \{0\}$, on C, one has

$$Opt'(\vec{\varepsilon}) \overset{\vec{\varepsilon}\to\vec{0}}{\sim} |||\boldsymbol{\varepsilon}|||. \tag{LM}$$

♣

3.6 Machinery for using the tensorization theorems

Up to now we stated the tensorization theorems in a rather 'theoretical' form. To apply these results to 'concrete' situations, some additional techniques may be needed. This section gives such techniques, which we will use later for the applications of Chapter 5.

☛ *In this section, all the probability systems considered will be endowed with their natural σ-metalgebras, cf. Definition 3.1.16. To alleviate notation, I will give no names to these σ-metalgebras, but will plainly denote $\{X : Y\}_*$ to mean "the subjective decorrelation between X and Y seen from the natural σ-metalgebra of the underlying system".*

3.6.a The 'doubling-up' technique

3.6.1 Definition. For I a set and $\mathscr{R}$ a binary relation on I, $J_1, J_2 \subset I$, we will say that "J_2 is $\mathscr{R}$-disjoint to J_1" if $(i,j) \in J_1 \times J_2 \Rightarrow i \not\mathscr{R} j$. ♢

3.6.2 Lemma ('Doubling-up' lemma)**.** *Let I be a (countable) set and let $(X_i)_{i\in I}$ be random variables such that for all $i,j \in I$, $\{X_i : X_j\}_* \leqslant \varepsilon_{ij}$ for a certain family of $\varepsilon_{ij} \in [0,1]$.*

Let $\mathscr{R}$ be a binary relation on I; for $i,j \in I$, denote $\varepsilon^{\mathscr{R}}_{ij} := \mathbf{1}_{i\not\mathscr{R}j}\varepsilon_{ij}$.

Define $\mathbf{I} = I_1 \uplus I_2$ to be a disjoint union of two copies of I; denote by $(i_1)_{i\in I}$, resp. $(j_2)_{j\in I}$, the elements of I_1, resp. I_2. Assume that the following holds for a certain $\varepsilon \in [0,1]$: "if $(Y_{i_\kappa})_{i_\kappa\in\mathbf{I}}$ are random variables such that $\forall i,j \in I \;\; \{Y_{i_1} : Y_{j_2}\}_* \leqslant \varepsilon^{\mathscr{R}}_{ij}$, then $\{\vec{Y}_{I_1} : \vec{Y}_{I_2}\} \leqslant \varepsilon$".

Then for all $J_1, J_2 \subset I$ such that J_2 is $\mathscr{R}$-disjoint to J_1, $\{\vec{X}_{J_1} : \vec{X}_{J_2}\}_* \leqslant \varepsilon$. ♣

3.6.3 Remark. The interest of Lemma 3.6.2 is that, by proving *one* tensorization result on $\{\vec{Y}_{I_1} : \vec{Y}_{I_2}\}$, one gets tensorization results on *all* the $\{\vec{X}_{J_1} : \vec{X}_{J_2}\}$ for J_2 $\mathscr{R}$-disjoint to J_1. ♡

3.6.4 Example.

1. If you take for $\mathscr{R}$ the equality relation, then Lemma 3.6.2 gives a decorrelation result for all disjoint J_1 and J_2.
2. If I is equipped with a distance *dist* and if you take $(i\mathscr{R}j) \Leftrightarrow (dist(i,j) < d_1)$, then you get a decorrelation result for all J_1 and J_2 such that $dist(J_1, J_2) \geqslant d_1$.

♡

Proof. Assume that the hypotheses of the lemma hold and let $J_1, J_2 \subset I$ with J_2 $\mathscr{R}$-disjoint to J_1. For $i_\kappa \in \mathbf{I}$, define

$$Y_{i_\kappa} = \begin{cases} X_i & \text{if } (\kappa = 1 \text{ and } i \in J_1) \text{ or } (\kappa = 2 \text{ and } i \in J_2); \\ \partial & \text{otherwise,} \end{cases} \tag{LN}$$

for ∂ some cemetery point in the range of none of the X_i. Since a constant variable is always independent of any variable, the hypothesis "$\{X_i : X_j\}_* \leqslant \varepsilon_{ij}$" for all $i,j \in I$ implies that $\{Y_{i_1} : Y_{j_2}\}_* \leqslant \varepsilon^{\mathscr{R}}_{ij}$, so, by the assumption of the lemma, $\{\vec{Y}_{I_1} : \vec{Y}_{I_2}\} \leqslant \varepsilon$. But $\vec{X}_{J_1}$ is $\vec{Y}_{I_1}$-measurable, resp. $\vec{X}_{J_2}$ is $\vec{Y}_{I_2}$-measurable, hence $\{\vec{X}_{J_1} : \vec{X}_{J_2}\} \leqslant \varepsilon$.

Getting the subjective result w.r.t. $*$ is just a variant of that reasoning, cf. § 3.4.b. ♠

3.6.b A practical result on $\mathbb{Z}^n$

In Chapter 5, the situations we will handle shall always be of the following form:

3.6.5 Assumption. For some $n \in \mathbb{N}^*$, the system is made of random variables X_i, $i \in \mathbb{Z}^n$, which satisfy the condition

$$\forall i,j \in \mathbb{Z}^n \qquad \{X_i : X_j\}_* \leqslant \varepsilon(j-i) \tag{LO}$$

for some symmetric function $\varepsilon\colon \mathbb{Z}^n \to [0,1]$. ♢

For systems satisfying Assumption 3.6.5, one has the following practical synthetic result:

3.6.6 Lemma. *Consider a norm $|\cdot|$ on $\mathbb{R}^n$, the associated distance on the affine $\mathbb{R}^n$ being denoted by dist. Then for a system satisfying Assumption 3.6.5, for all $J_1, J_2 \subset I$:*

$$\{\vec{X}_{J_1} : \vec{X}_{J_2}\} \leqslant \Big(\sum_{\substack{z\in\mathbb{Z}^n \\ |z|\geqslant dist(J_1,J_2)}} \varepsilon(z)\Big) \wedge 1. \tag{LP}$$

♣

Proof. To alleviate notation, denote $d := dist(J_1, J_2)$. Applying Lemma 3.6.2, taking for "$\mathscr{R}$" the relation "be at distance $< d$" (cf. Example 3.6.4-2), our goal becomes the following: supposing $(Y_{i_\kappa})_{i_\kappa\in\mathbb{Z}_1^n\uplus\mathbb{Z}_2^n}$ are random variables such that $\{Y_{i_1} : Y_{j_2}\}_* \leqslant \mathbf{1}_{|j-i|\geqslant d}\varepsilon(j-i)$, we want to bound above $\{\vec{Y}_{\mathbb{Z}_1^n} : \vec{Y}_{\mathbb{Z}_2^n}\}$.

To do this we apply Theorem 3.3.1, and we get that $\{\vec{Y}_{\mathbb{Z}_1^n} : \vec{Y}_{\mathbb{Z}_2^n}\}$ is bounded by $|||\boldsymbol{\varepsilon}||| \wedge 1$, where $\boldsymbol{\varepsilon}$ is the following operator:

$$\begin{array}{rccl} \boldsymbol{\varepsilon}: & L^2(\mathbb{Z}) & \circlearrowleft & \\ & (g(j))_{j\in\mathbb{Z}} & \mapsto & (\sum_{j\in\mathbb{Z}} \mathbf{1}_{|j-i|\geqslant d}\varepsilon(j-i)g(j))_{i\in\mathbb{Z}}. \end{array} \tag{LQ}$$

To compute $|||\boldsymbol{\varepsilon}|||$, we split $\boldsymbol{\varepsilon}$ as $\sum_{z\in\mathbb{Z}} \mathbf{1}_{|z|\geqslant d}\varepsilon(z)M_z$, where M_z is the operator

$$\begin{array}{rccl} M_z: & L^2(\mathbb{Z}) & \circlearrowleft & \\ & (g(j))_{j\in\mathbb{Z}} & \mapsto & (g(i+z))_{i\in\mathbb{Z}}. \end{array} \tag{LR}$$

Obviously $|||M_z||| = 1$, thus $|||\boldsymbol{\varepsilon}||| \leqslant \sum_{|z|\geqslant d}\varepsilon(z)$ —actually there is even equality—, which ends the proof of Lemma 3.6.6. ♠

3.6.7 Remark. Instead of Theorem 3.3.1, here we could have used Theorem 3.3.18, which would yield a better result; yet that would be very specific to $\mathbb{Z}^n$ (cf. § 3.6.d), and the result would actually be almost equivalent to (LP) (cf. § 3.5.b). ♡

3.6.c Avoiding the artificial phase transition

Let us look again at Formula (LP): the "$\wedge 1$" in it is not really relevant since a correlation level is *always* bounded by 1. In fact the situation is dichotomic: denoting $d := dist(I, J)$, either $\sum_{|z|\geqslant d}\varepsilon(z)$ is < 1 and then (LP) is a true decorrelation result, or it is $\geqslant 1$ and then (LP) tells us actually nothing. In other words, our result has a 'phase transition' depending on the relative values of $\sum_{|z|\geqslant d}\varepsilon(z)$ and 1, similar to the phenomenon we discussed in Remark 3.5.10.

However, as I pointed out in Footnote [†] on page 142, it is not clear whether the phase transition we are dealing with is a real phenomenon: maybe it is rather an artifact due to Theorem 3.3.1's bound's being non-optimal, which could be avoided by a cleverer reasoning. We are strengthened in that thought by observing that, if $\sum_{z\in\mathbb{Z}^n}\varepsilon(z) < \infty$, then for d large enough one has $\sum_{|z|\geqslant d}\varepsilon(d) < 1$, so that there is no phase transition for long distances; why would a transition appear all of a sudden for short distances?

This subsection will show that, indeed, phase transitions can be avoided in the situations we deal with.

3.6.8 Lemma. *For a system satisfying Assumption 3.6.5 with $\varepsilon(z) < 1$ as soon as $z \neq 0$ and $\sum_{z\neq 0}\varepsilon(z) < \infty$, there exists a constant $k < 1$ such that, for all disjoint $J_1, J_2 \subset I$, one has $\{\vec{X}_{J_1} : \vec{X}_{J_2}\} \leqslant k$.* ♣

Proof. As before, using Lemma 3.6.2 we have to bound above $\{\vec{Y}_{\mathbb{Z}_1^n} : \vec{Y}_{\mathbb{Z}_2^n}\}$ in the relevant doubled-up model. Our plan to avoid the phase transition will consist in reducing to the 'long distance' case.

For some $l \in \mathbb{N}^*$, we split $\mathbb{Z}_1^n$, resp. $\mathbb{Z}_2^n$, into a partition of $l^n =: N$ sublattices $Z_1^{(1)}, \dots, Z_1^{(N)}$, resp. $Z_2^{(1)}, \dots, Z_2^{(N)}$, each lattice $Z_\kappa^{(u)}$ being of the form $l\mathbb{Z}^n + z_u$ for some $z_u \in \mathbb{Z}^n / l\mathbb{Z}^n$. I claim two fundamental properties of these sublattices:

3.6.9 Claim. *For all $u, v \in \{1, \dots, N\}$,*

$$\{\vec{Y}_{\mathbb{Z}_1^{(u)}} : \vec{Y}_{\mathbb{Z}_2^{(v)}}\}_* \leqslant \sum_{z \equiv z_v - z_u} \mathbf{1}_{z \neq 0}\, \varepsilon(z). \tag{LS}$$

♣

Proof. It is analogous to the proof of Lemma 3.6.6. ♠

3.6.10 Claim. *Provided l is large enough, the right-hand side of (LS) is (strictly) less than 1 for all the possible values of u, v.* ♣

Proof. Denote $\zeta := \sup_{z \neq 0} \varepsilon(z)$; notice that our assumptions imply that $\zeta < 1$. Since $\sum_{z \in \mathbb{Z}^n} \varepsilon(z)$ converges, there exists some $d_1 < \infty$ such that $\sum_{|z| > d_1} \varepsilon(z) < 1 - \zeta$. Now, denoting $d_0 := \min\{|z| : z \in \mathbb{Z}^n \smallsetminus \{0\}\}$, for $l > 2d_1/d_0$, for all u, v there is at most one z congruent to $z_v - z_u$ [mod. l] such that $|z| \leqslant d_1$, whence the following uniform bound for the right-hand side of (LS):

$$\sum_{z \equiv z_v - z_u} \mathbf{1}_{z \neq 0}\, \varepsilon(z) \leqslant \sum_{|z| > d_1} \varepsilon(z) + \underbrace{\sum_{\substack{|z| \leqslant d_1 \\ z \equiv z_v - z_u \\ z \neq 0}} \varepsilon(z)}_{\substack{\leqslant \zeta \text{ because} \\ \text{the sum has at most} \\ \text{one term, being } \leqslant \zeta}} \leqslant \underbrace{\sum_{|z| > d_1} \varepsilon(z)}_{< 1 - \zeta} + \zeta < 1. \tag{LT}$$

♠

Now, suppose l large enough so that Claim 3.6.10 works. We apply *simple* tensorization (Theorem 3.2.2) to the $\vec{Y}_{\mathbb{Z}_2^{(v)}}$: writing that $\vec{Y}_{\mathbb{Z}_2^n} = (\vec{Y}_{\mathbb{Z}_2^{(1)}}, \dots, \vec{Y}_{\mathbb{Z}_2^{(N)}})$, we get that, for any $u \in \{1, \dots, N\}$,

$$\{\vec{Y}_{\mathbb{Z}_1^{(u)}} : \vec{Y}_{\mathbb{Z}_2^n}\}_* \leqslant \sqrt{1 - \prod_{v=1}^{N} \left(1 - \{\vec{Y}_{\mathbb{Z}_1^{(u)}} : \vec{Y}_{\mathbb{Z}_2^{(v)}}\}_*^2\right)} < 1. \tag{LU}$$

Now we write $\vec{Y}_{\mathbb{Z}_1^n} = (\vec{Y}_{\mathbb{Z}_1^{(1)}}, \dots, \vec{Y}_{\mathbb{Z}_1^{(N)}})$ and we apply simple tensorization again—this time to the $\vec{Y}_{\mathbb{Z}_1^{(u)}}$ —to get:

$$\{\vec{Y}_{\mathbb{Z}_1^n} : \vec{Y}_{\mathbb{Z}_2^n}\} \leqslant \sqrt{1 - \prod_{u=1}^{N} \left(1 - \{\vec{Y}_{\mathbb{Z}_1^{(u)}} : \vec{Y}_{\mathbb{Z}_2^n}\}_*^2\right)} < 1. \tag{LV}$$

Bound (LV) achieves our goal. ♠

3.6.11 Remark. With that proof, the way k depends on $\varepsilon(\cdot)$ is rather complicated; in particular, you cannot express k as a function of only $\sum_{z \neq 0} \varepsilon(z)$ and $\sup_{z \neq 0} \varepsilon(z)$. ♡

3.6.12 Remark. In the case $n = 1$, at first sight Lemma 3.6.8 seems to contradict Theorem 3.5.3, in which we told that Theorem 3.3.10, which does have a phase transition, was optimal. The explanation for this paradox stands in the slight difference between the assumptions of Lemma 3.6.8 and Theorem 3.3.10: while in Lemma 3.6.8 we really imposed that $\{X_i : X_j\}_* \leqslant \varepsilon(\mathfrak{d}(i,j))$, with "$*$" denoting the *full* natural σ-metalgebra of the system, in Theorem 3.3.10—more precisely, in the version of Theorem 3.3.10 Theorem 3.5.3 proved to be optimal, which was the *minimal* version of this theorem (cf. § 3.4.a)—the conditions on subjective decorrelations were a bit looser. That difference makes all the trick when one performs the steps of simple tensorization in the proof of Lemma 3.6.8, because these steps require subjective decorrelations w.r.t. the $\vec{Y}_{I_\kappa^{(u)}}$, which the sole assumptions of Theorem 3.5.3 do not provide. ♡

3.6.d Non-flat geometries

It is natural to ask what we one can do when the basic variables X_i are not indexed by $\mathbb{Z}^n$, but by the vertices of a more general graph, for instance a tree or a finitely generated group. This shall occur indeed if the physical space one works in exhibits some curvature—though Chapter 5 will not handle such situations.

Actually for general graphs there are results analogous to those of §§ 3.6.b and 3.6.c, with similar (though more technical) proofs [64]. Here I will only give the statements of these results.

In this subsection the situation will be the following:

3.6.13 Assumption. The system is made of random variables (X_i) indexed by a (countable) set I. There is a group G acting transitively on I, and I is endowed with a symmetric map $\mathfrak{d}\colon\ I \times I \to \mathfrak{D}$, called the 'abstract distance', which is preserved by the action of G. We assume that one has

$$\forall i,j \in I \quad \{X_i : X_j\}_* \leqslant \varepsilon(\mathfrak{d}(i,j)) \tag{LW}$$

for some function $\varepsilon\colon\ \mathfrak{D} \to [0,1]$. ♢

3.6.14 Definition. For $d \in \mathfrak{D}$, we define $\mathrm{val}(d) := \#\{i \in I :\ \mathfrak{d}(o,i) = d\}$, where the choice of $o \in I$ does not matter. ♢

Then the analogous to Lemma 3.6.6 is the

3.6.15 Lemma. *For $\mathfrak{D}' \subset \mathfrak{D}$, for all $J_1, J_2 \subset I$ such that $(i \in J_1, j \in J_2) \Rightarrow \mathfrak{d}(i,j) \in \mathfrak{D}'$,*

$$\{\vec{X}_{J_1} : \vec{X}_{J_2}\} \leqslant \Big(\sum_{d \in \mathfrak{D}'} \mathrm{val}(d)\varepsilon(d)\Big) \wedge 1. \tag{LX}$$

♣

The analogous of Lemma 3.6.8 is the

3.6.16 Lemma. *Assume that Assumption 3.6.13 is satisfied; denoting by* 0 *the (common) value of the* $\mathfrak{d}(i,i)$*, also assume that* $\mathrm{val}(0) = 1$ *and that* $\varepsilon(d) < 1$ *as soon as* $d \neq 0$*. Assume that* $\sum_{d \in \mathfrak{D}} \mathrm{val}(d)\varepsilon(d) < \infty$*.*

Moreover, assume that the action of G on I is profinite *(cf. [41, Definition 1.1]), i.e. that there is a subset $\mathcal{N} \subset \mathbb{N}^*$ such that for each $N \in \mathcal{N}$, there is a subgroup $G_N \leqslant G$ such that:*

(i) The action of G_N splits I into exactly N orbits $I^{(1)}, \dots, I^{(N)}$;

(ii) G_N *is normal, so that the partition of* I *into the* $I^{(u)}$ *is stable by the action of* G*;*
(iii) Any two distinct points of I *are ultimately separated by the partitions induced by the* G_N*, i.e.:*

$$\overline{\lim_{\substack{N\in\mathcal{N}\\ N\to\infty}}}\,(G_N\cdot o)=\{o\}. \tag{LY}$$

Then there exists a constant $k<1$ *such that, for all disjoint* $J_1,J_2\subset I$*, one has* $\{\vec{X}_{J_1}:\vec{X}_{J_2}\}\leqslant k$*.* ♣

3.6.17 Example. For $I=\mathbb{Z}^n$ on which $G=\mathbb{Z}^n$ acts by translation, equipped with the abstract distance $\mathfrak{d}(x,y)=\{\pm(y-x)\}$, the assumptions of Lemmas 3.6.15 and 3.6.16 are checked, and these lemmas re-give resp. Lemmas 3.6.6 and 3.6.8. ♡

3.6.18 Example. For I the modular group $PSL_2(\mathbb{Z})$ acting by left multiplication on itself, equipped with its natural abstract distance (i.e., $\mathfrak{d}(i,j)=\{i^{-1}j,j^{-1}i\}$), the assumptions of Lemmas 3.6.15 and 3.6.16 are also checked—to see that the action of G on I is profinite, take for the $G_{N(l)}$ the principal congruence subgroups $\Gamma(l)$ of the modular group [68]. Notice that $PSL_2(\mathbb{Z})$ is an example of graph having negative curvature [33]. ♡

3.7 Appendix: Illustration of the proof of Theorem 3.3.1

☛ *This subsection is devised for the readers who would like to understand better the proof of Theorem 3.3.1 by seeing how it works on a concrete example. It only contains pedagogical material, and thus can be skipped safely.*

3.7.a A Gaussian system of variables

In this illustration we take $N=2, M=1$ —since $M=1$, Y_1 will merely be denoted by Y —, and we take (X_1,X_2,Y) Gaussian (and centered), whose law is described through a 3×3 matrix *via* writing that, for some standard Gaussian vector $(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3$,

$$\begin{pmatrix}X_1\\X_2\\Y\end{pmatrix}=\begin{pmatrix}\alpha_1&\alpha_2&\alpha_3\\\beta_1&\beta_2&\beta_3\\\omega_1&\omega_2&\omega_3\end{pmatrix}\begin{pmatrix}\xi_1\\\xi_2\\\xi_3\end{pmatrix}. \tag{LZ}$$

We denote the matrix appearing in (LZ) by $\mathbf{M}$. The rows of $\mathbf{M}$ will be denoted by $\alpha,\beta,\gamma\in\mathbb{R}^3$, and (ξ_1,ξ_2,ξ_3) will be denoted by $\xi\in\mathbb{R}^3$. On $\mathbb{R}^3$ we will use the Euclidian scalar product "$\cdot$" and the associated norm "$\|\cdot\|$".

The advantage of this model is that, by of the general properties of Gaussian vectors (in particular Theorem 1.2.6), all the quantities of interest are computable exactly.

First we compute the correlation levels: by Theorem 1.2.6,

$$\{X_1:Y\}=\frac{|\alpha\cdot\omega|}{\|\alpha\|\,\|\omega\|}, \tag{MA}$$

similarly $\{X_2:Y\}=|\beta\cdot\omega|/\|\beta\|\|\omega\|$; and

$$\{\vec{X}:Y\}=\sqrt{1-\frac{|\omega\cdot(\alpha\vec{\times}\beta)|^2}{\|\omega\|^2\,\|\alpha\vec{\times}\beta\|^2}}, \tag{MB}$$

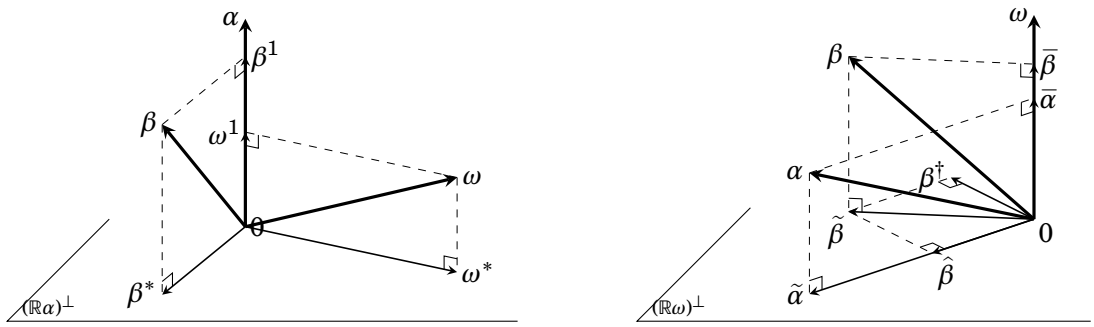

Figure 9: Visual definitions of the vectors derived from α, β and ω: the left drawing shows how to build β^1, $\omega^1, \beta^*, \omega^*$; the right drawing (with different values for α, β, ω) explains the construction of $\overline{\alpha}, \overline{\beta}, \widetilde{\alpha}, \widetilde{\beta}, \widehat{\beta}, \beta^\dagger$.

where "$\vec{\times}$" denotes the cross product on $\mathbb{R}^3$. Concerning the conditional quantities, denote by β^1, resp. ω^1, the (orthogonal) projections of β, resp. ω, on $\mathbb{R}\alpha$, and β^*, resp. ω^*, the projections of the same vectors on $(\mathbb{R}\alpha)^\perp$, i.e. (assuming that $\alpha \neq 0$):

$$\beta^1 := (\beta \cdot \alpha / \|\alpha\|^2)\,\alpha, \quad \omega^1 := (\omega \cdot \alpha / \|\alpha\|^2)\,\alpha; \tag{MC}$$

$$\beta^* := \beta - \beta^1, \quad \omega^* := \omega - \omega^1 \tag{MD}$$

(see Figure 9). Then one has $\mathbf{E}[X_2|X_1] = \beta_1^1\xi_1 + \beta_2^1\xi_2 + \beta_3^1\xi_3 = \beta^1 \cdot \xi$, resp. $\mathbf{E}[Y|X_1] = \omega^1 \cdot \xi$, thus $X_2 - \mathbf{E}[X_2|X_1] = \beta^* \cdot \xi$, resp. $Y - \mathbf{E}[Y|X_1] = \omega^* \cdot \xi$. As (X_1, X_2, Y) is Gaussian, the law of $(X_2 - \mathbf{E}[X_2|X_1], Y - \mathbf{E}[Y|X_1])$ under $\mathbf{P}[\cdot|X_1 = x]$ does not depend on the value of x; therefore we know all the conditional laws of (X_2, Y) under the $\mathbf{P}[\cdot|X_1 = x]$, and for all these laws $\{X_2 : Y\}$ is equal by Theorem 1.2.6 to $|\beta^* \cdot \omega^*| / \|\beta^*\| \|\omega^*\|$, so in the end:

$$\{X_2 : Y\}_{X_1} = \frac{|\beta^* \cdot \omega^*|}{\|\beta^*\|\,\|\omega^*\|}. \tag{ME}$$

$\{X_1 : Y\}_{X_2}$ can be computed by a similar formula.

Now let us 'dissect' the proof of Theorem 3.3.1 on our example. We take f linear, namely

$$f(X_1, X_2) := X_1 + X_2, \tag{MF}$$

so that all the computations shall again be tractable exactly.

Let us start with computing the quantities linked to f^0: one has

$$f^0 = f \quad = \quad (\alpha + \beta) \cdot \xi; \tag{MG}$$

$$f_1^0 = f^{\sigma(X_1)} \quad = \quad X_1 + (X_2)^{\sigma(X_1)} = (\alpha + \beta^1) \cdot \xi; \tag{MH}$$

$$f_2^0 = f - f^{\sigma(X_1)} \quad = \quad \beta^* \cdot \xi, \tag{MI}$$

whence respectively

$$V = V^0 \quad = \quad \|\alpha + \beta\|^2 = \|\alpha\|^2 + \|\beta\|^2 + 2\alpha \cdot \beta; \tag{MJ}$$

$$V_1^0 \quad = \quad \|\alpha + \beta^1\|^2 = \|\alpha\|^2 + 2\alpha \cdot \beta + \frac{(\alpha \cdot \beta)^2}{\|\alpha\|^2}; \tag{MK}$$

$$V_2^0 \quad = \quad \|\beta^*\|^2 = \|\beta\|^2 - \frac{(\alpha \cdot \beta)^2}{\|\alpha\|^2}. \tag{ML}$$

By the way we check that, as claimed by Formula (GZ), $V^0 = V_1^0 + V_2^0$.

Now we turn to the quantities linked to f^1. First we have to compute the conditional laws of (X_1, X_2) under the events "$Y = y$". The technique is the same as for computing $\{X_2 : Y\}_{X_1}$ a few lines above: denoting by $\overline{\alpha}$, resp. by $\overline{\beta}$, the projections of α, resp. β, on $\mathbb{R}\omega$, and $\widetilde{\alpha}$, resp. $\widetilde{\beta}$, the projections of the same vectors on $(\mathbb{R}\omega^\perp)$, i.e. (see Figure 9)

$$\overline{\alpha} := (\alpha \cdot \omega / \|\omega\|^2)\omega, \quad \overline{\beta} := (\beta \cdot \omega / \|\omega\|^2)\omega; \tag{MM}$$

$$\widetilde{\alpha} := \alpha - \overline{\alpha}, \quad \widetilde{\beta} := \beta - \overline{\beta}, \tag{MN}$$

one has $\mathbf{E}[X_1|Y] = \overline{\alpha} \cdot \xi$, resp. $\mathbf{E}[X_2|Y] = \overline{\beta} \cdot \xi$, thus $X_1 - \mathbf{E}[X_1|Y] = \widetilde{\alpha} \cdot \xi$, resp. $X_2 - \mathbf{E}[X_2|Y] = \widetilde{\beta} \cdot \xi$; and $(X_1 - \mathbf{E}[X_1|Y], X_2 - \mathbf{E}[X_2|Y])$ has the same law under all the $\mathbf{P}[\cdot|Y = y]$. So we can compute the quantities linked to f^1 in the same way as we computed those linked to f^0: denoting

$$\widehat{\beta} := \frac{\widetilde{\beta} \cdot \widetilde{\alpha}}{\|\widetilde{\alpha}\|^2} \widetilde{\alpha}; \tag{MO}$$

$$\beta^\dagger := \widetilde{\beta} - \widehat{\beta} \tag{MP}$$

(see Figure 9), one finds

$$f^1 = (\widetilde{\alpha} + \widetilde{\beta}) \cdot \xi; \tag{MQ}$$

$$f_1^1 = (\widetilde{\alpha} + \widehat{\beta}) \cdot \xi; \tag{MR}$$

$$f_2^1 = \beta^\dagger \cdot \xi, \tag{MS}$$

whence respectively:

$$V^1 = \|\widetilde{\alpha} + \widetilde{\beta}\|^2; \tag{MT}$$

$$V_1^1 = \|\widetilde{\alpha} + \widehat{\beta}\|^2; \tag{MU}$$

$$V_2^1 = \|\beta^\dagger\|^2. \tag{MV}$$

As for f^0, we check that $V^1 = V_1^1 + V_2^1$, since $\widetilde{\alpha} + \widetilde{\beta}$ is the orthogonal sum of $\widetilde{\alpha} + \widehat{\beta}$ and $\beta^\dagger$. Moreover one always has $V^1 \leqslant V^0$, resp. $V_2^1 \leqslant V_2^0$: the first inequality follows indeed from $(\widetilde{\alpha} + \widetilde{\beta})$'s being the projection of $(\alpha + \beta)$ on $(\mathbb{R}\omega)^\perp$, and the second one from $\beta^\dagger$'s being the projection of β^* on $(\mathbb{R}\omega + \mathbb{R}\alpha)^\perp$. These inequalities are consistent with the following corollary of Claim 3.3.4, obtained by applying the claim conditionally to $\mathscr{G}_{j-1}$ with the role of "f" played by f^{j-1} and the role of "Y" played by Y_j:

3.7.1 Proposition. *For all* $1 \leqslant j \leqslant M$, *all* $0 \leqslant i \leqslant N$,

$$\sum_{i'>i} V_{i'}^j \leqslant \sum_{i'>i} V_{i'}^{j-1}. \tag{MW}$$

♣

3.7.b Numerical computations

Now let us see a numerical example. Our parameters will be chosen so that the function f defined by (MF) is optimal in the supremum (AG) defining the Hilbertian correlation coefficient $\{\vec{X} : Y\}$; other than that, the behaviour of our example will be generic:

$$\mathbf{M} = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix}. \tag{MX}$$

For that $\mathbf{M}$ the calculations of the previous subsection give:

χ	α	β	γ	$\alpha\vec{\times}\beta$	β^1	ω^1	β^*	ω^*	$\overline{\alpha}$	$\overline{\beta}$	$\widetilde{\alpha}$	$\widetilde{\beta}$	$\widehat{\beta}$	$\beta^\dagger$
χ_1	4	1	1	−3	2	2	−1	−1	1/2	1/2	7/2	1/2	7/6	−2/3
χ_2	1	4	1	−3	1/2	1/2	7/2	1/2	1/2	1/2	1/2	7/2	1/6	10/3
χ_3	1	1	4	15	1/2	1/2	1/2	7/2	2	2	−1	−1	−1/3	−2/3

whence $\{X_1 : Y\} = 1/2$ and $\{X_1 : Y\}_{X_2} = 1/3$, thus $\{X_1 : Y\}_{\mathscr{M}} = 1/2$; and similarly $\{X_2 : Y\}_{\mathscr{M}} = 1/2$.

Then Theorem 3.2.2 yields:

$$\{\vec{X} : Y\} \leqslant 1/\sqrt{2} = 0.707\ldots, \tag{MY}$$

and even, according to the refinements of § 3.4.a:

$$\{\vec{X} : Y\} \leqslant \sqrt{13}/6 = 0.600\ldots; \tag{MZ}$$

on the other hand, the true result is:

$$\{\vec{X} : Y\} = 1/\sqrt{3} = 0.577\ldots. \tag{NA}$$

So here the bound (GD) is (fortunately!) correct, and even rather sharp.

Now, as the proof of Theorem 3.3.1 consists in studying the relations between the V_i^j, let us see what these quantities look like here. One computes:

$$\begin{pmatrix} V_1^0 & V_2^0 \\ V_1^1 & V_2^1 \end{pmatrix} = \begin{pmatrix} 40\frac{1}{2} & 13\frac{1}{2} \\ 24 & 12 \end{pmatrix}. \tag{NB}$$

As a first consequence, we can check the conclusions of Proposition 3.7.1: $V_2^1 = 12 \leqslant V_2^0 = 13\frac{1}{2}$, resp. $V_1^1 + V_2^1 = 36 \leqslant V_1^0 + V_2^0 = 54$. Going further, we check the conclusions of Claim 3.3.5, which forbids the differences $V_2^0 - V_2^1$ and $(V_1^0 + V_2^0) - (V_1^1 + V_2^1)$ to be too large: for the first difference, one has $V_2^0 - V_2^1 = 1\frac{1}{2} \leqslant \varepsilon_2^2 V_2^0 = 3\frac{3}{8}$ [§], and for the second one, $(V_1^0 + V_2^0) - (V_1^1 + V_2^1) = 18 \leqslant (\varepsilon_1 V_1^0 + \sqrt{V_2^0 - V_2^1})^2 = 19.419\ldots$.

3.7.c Some traps to avoid

To finish with this appendix, I would like to comment on what is true or not about the V_i^j in general situations. Proposition 3.7.1 pointed out that for all $\hat{\imath} \in \{0, \ldots, N\}$, $\sum_{i > \hat{\imath}} V_i^j$ is a nonincreasing function of j; in particular, when one looks at the table of the V_i^j, the last term ($\hat{\imath} = N - 1$), resp. the total ($\hat{\imath} = 0$) of line j can only decrease. Moreover, if in some line j all the V_i^j are zero from some position $\hat{\imath} + 1$, then this property remains true in all the lower lines $j' > j$. That can be explained very simply, since saying that all the V_i^j are zero from position $\hat{\imath} + 1$ means indeed that f is $(\mathscr{G}_j \vee \mathscr{F}_{\hat{\imath}})$-measurable, hence *a fortiori* $(\mathscr{G}_{j'} \vee \mathscr{F}_{\hat{\imath}})$-measurable. The following example, in which f turns out to be $2X_1$, illustrates this phenomenon:

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad \Rightarrow \qquad \begin{pmatrix} V_1^0 & V_2^0 \\ V_1^1 & V_2^1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 2 & 0 \end{pmatrix}. \tag{NC}$$

[§]. According to § 3.4.a, one can replace $\varepsilon_2 = 1/2$ by $\{X_2 : Y\}_{X_1} = 1/3$ in this inequality. Then the inequality even becomes an equality: this is linked to the optimality of certain tensorization results for Gaussian variables, cf. § 3.5.

However, keep careful: almost anything else you would like to say about the table of the V_i^j would be false! In particular, for $i < N$, V_i^j is not a nonincreasing function of j in general; it is not even true that $V_i^j = 0 \Rightarrow V_i^{j'>j} = 0$, as shown by the following example:

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \qquad \Rightarrow \qquad \begin{pmatrix} V_1^0 & V_2^0 \\ V_1^1 & V_2^1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1/6 & 1/2 \end{pmatrix}. \tag{ND}$$

It is not true either that, if V^j remains unchanged from one line to another (that is, the total of the V_i^j remains unchanged), then all the V_i^j are unchanged. In fact, that V^{j+1} is equal to V^j means that, conditionally to $\mathscr{G}_j$, f^j is centered w.r.t. Y_{j+1}, and then $f^{j+1} = f^j$. However, the way f^{j+1} decomposes into a sum of f_i^{j+1} may be different to the way f^j decomposed into a sum of f_i^j, because conditioning w.r.t. Y_{j+1} may make the law of the X_i change! That is what happens in the following example:

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \qquad \Rightarrow \qquad \begin{pmatrix} V_1^0 & V_2^0 \\ V_1^1 & V_2^1 \end{pmatrix} = \begin{pmatrix} 1/2 & 3/2 \\ 1 & 1 \end{pmatrix}. \tag{NE}$$

3.8 Appendix: A corollary of the Perron–Frobenius theorem

In this appendix I handle a lemma used in the proof of Theorem 3.3.1. We are working on the vector space $\mathbb{R}^N$ for some $N > 0$; a vector or a matrix is said to be > 0 if all its entries are positive, resp. $\geqslant 0$ if all its entries are nonnegative. Then the *Perron–Frobenius theorem* [39, Theorem 8.3.1] states that if a square matrix A is $\geqslant 0$, then A has some $\geqslant 0$ eigenvector for the eigenvalue $\rho(A)$. Our goal here is prove the following corollary:

3.8.1 Lemma. *Let $A \geqslant 0$ be a square matrix, then:*

$$\inf\{\lambda \geqslant 0 \colon\ (\exists u > 0)(Au \leqslant \lambda u)\} = \rho(A). \tag{NF}$$

♣

Proof. We prove separately each way of the equality. Let us begin with way "$\leqslant$". Let $v \geqslant 0$ be some eigenvector of A for the eigenvalue $\rho(A)$. If $v > 0$, then the value $\lambda = \rho(A)$ checks the condition in the infimum and we are done. Otherwise if $v \not> 0$, up to a permutation of indices it has the form $(0, \dots, 0, v'_{n+1}, \dots, v'_N)$ with $0 < n < N$ and all the v'_i positive. Reasoning by induction, assume that we have proved the way "$\leqslant$" of the lemma for all $n < N$. Then the form of the eigenvector v forces A to write blockwise

$$A = \begin{pmatrix} \tilde{A} & 0 \\ * & * \end{pmatrix} \tag{NG}$$

with $\mathbb{R}^{n \times n} \ni \tilde{A} \geqslant 0$. I claim that $\rho(\tilde{A}) \leqslant \rho(A)$, since if $\tilde{v}$ is an eigenvector of $\tilde{A}$ for the eigenvalue $\rho(\tilde{A})$, then for $t \geqslant 0$

$$A^t(\tilde{v}, 0, \dots, 0) = (\rho(\tilde{A})^t \tilde{v}, *, \dots, *), \tag{NH}$$

so

$$\varlimsup_{t \to \infty} \rho(\tilde{A})^{-t} |A^t(\tilde{v}, 0, \dots, 0)| > 0^{[\P]} \tag{NI}$$

and consequently $\rho(A) \geqslant \rho(\widetilde{A})$. Now let $\varepsilon > 0$. By induction hypothesis there exists some $\mathbb{R}^n \ni w > 0$ such that $\widetilde{A}w \leqslant (\rho(\widetilde{A})+\varepsilon)w$. Thus for $\eta > 0$, $\mathbb{R}^N \ni (\eta w, v') > 0$ and

$$A(\eta w, v') = (\eta \widetilde{A} w, \rho(A)v' + O(\eta)) \leqslant (\eta(\rho(\widetilde{A})+\varepsilon)w, \rho(A)v' + O(\eta)) \overset{\eta \searrow 0}{\leqslant} (\rho(A)+\varepsilon)(\eta w, v'). \quad (\mathtt{NJ})$$

So $(\rho(A)+\varepsilon)$ checks the condition in the right-hand side of the infimum, which ends the proof of the way "$\leqslant$" of (NF).

For the way "$\geqslant$", consider any $\mathbb{R}^N \ni u > 0$ and let again $v \geqslant 0$ be some eigenvector of A for the eigenvalue $\rho(A)$. Then there exists a (unique) $\beta \geqslant 0$ such that $u - \beta v \geqslant 0$ but $u - \beta v \not> 0$. For this β, one of the entries of βv and u is the same, say $\beta v_{i_0} = u_{i_0}$. So if $\lambda < \rho(A)$,

$$\lambda u_{i_0} < \rho(A) u_{i_0} = \rho(A)\beta v_{i_0} = (A(\beta v))_{i_0} \leqslant (A(\beta v))_{i_0} + (A(u-\beta v))_{i_0} = (Au)_{i_0}, \quad (\mathtt{NK})$$

thus $Au \nleqslant \lambda u$. That relation being true for any $u > 0$, λ does not check the condition in the infimum, which proves the way "$\geqslant$" of (NF). ♠

3.9 Appendix: A geometric consequence of results on correlations

As I pointed out in Remark 3.5.6, for Gaussian vectors Hilbertian correlations can be interpreted in terms of Euclidian spaces. In this appendix I will present a funny corollary of Lemma 3.5.5 following from this interpretation. Actually that result itself is more or less a pretext: the real goal of this appendix is in fact to show in an eloquent way the geometric meaning of maximal correlations and the Hilbertian frame that underlies them.

First we need some vocabulary about Euclidian spaces:

3.9.1 Definition.

1. For L_1, L_2 two vector lines in the Euclidian space $\mathbb{R}^2$, or more generally in any Hilbert space, we call *geometric angle* between L_1 and L_2, denoted by $\widehat{L_1 L_2}$, their "angle" in the elementary sense: for arbitrary $\vec{a} \in L_1 \smallsetminus \{0\}, \vec{b} \in L_2 \smallsetminus \{0\}$,

$$\widehat{L_1 L_2} = \operatorname{Arccos} \frac{|\vec{a} \cdot \vec{b}|}{\|\vec{a}\| \, \|\vec{b}\|} \in [0, \pi/2]. \quad (\mathtt{NL})$$

2. For L_1, L_2 and $L_3 \neq L_1, L_2$ three vector lines in the Euclidian space $\mathbb{R}^3$ (or any Hilbert space), we call *apparent angle between L_1 and L_2 seen from L_3* the geometric angle that an observer located somewhere on $L_3 \smallsetminus \{0\}$ would have the impression, due to perspective, that L_1 and L_2 make (see Figure 10): technically, it is the geometric angle $\widehat{L_1' L_2'}$, where L_1' and L_2' are the respective orthogonal projections of L_1 and L_2 onto the plane $(L_3)^\perp$.

◇

Then one has the following corollary of Lemma 3.5.5:

3.9.2 Theorem. *Let L_1, L_2, L_3 be three distinct vector lines of $\mathbb{R}^3$. Denote $\widehat{A} := \widehat{L_2 L_3}, \widehat{B} := \widehat{L_3 L_1}, \widehat{\Omega} := \widehat{L_1 L_2}$, and denote by $\widehat{A'}$ the apparent angle between L_2 and L_3 seen from L_1, resp. $\widehat{B'}$ the apparent angle between L_3 and L_1 seen from L_2, etc.. Then the relative order of $\widehat{A}$ and $\widehat{A'}$ is the same as the relative order of $\widehat{B}$ and $\widehat{B'}$ and as the relative order of $\widehat{\Omega}$ and $\widehat{\Omega'}$, i.e., "$\widehat{A'} < \widehat{A}$" (resp. "$\widehat{A'} = \widehat{A}$", resp. "$\widehat{A'} > \widehat{A}$") is equivalent to "$\widehat{B'} < \widehat{B}$" (resp. "$\widehat{B'} = \widehat{B}$", resp. "$\widehat{B'} > \widehat{B}$"), etc..*

♣

[¶]. Equation (NI) is meaningless if $\rho(\widetilde{A}) = 0$, but in this case there is nothing to prove.

3.9.3 Remark. I found Theorem 3.9.2 by chance, one day that I was looking for a situation where one would have $\widehat{B}' > \widehat{B}$ but $\widehat{A'} < \widehat{A}$, in order to build a 'nice' example for § 3.7.b. I thought that such a situation would be generic, but after having looked for it without success, I realized that it was actually impossible, and that the explanation had a simple interpretation in terms of correlations. ♡

Proof. Fix three arbitrary nonzero vectors α, β, ω of resp. L_1, L_2, L_3; and consider the Gaussian system (LZ) of § 3.7 for these vectors. Then the correlation coefficients between X_1, X_2 and Y can be interpreted as angles between L_1, L_2 and L_3; more precisely, one has the following correspondance:

3.9.4 Proposition.

(i) $\{X_1 : X_2\}$ *is the cosine of the geometric angle between* L_1 *and* L_2*;*
(ii) $\{X_1 : X_2\}_Y$ *is the cosine of the apparent angle between* L_1 *and* L_2 *seen from* L_3*.*

♣

Proof. (i) is nothing but the Euclidian interpretation of Theorem 1.2.6. (ii) follows from the fact that, in the vector space spanned by jointly Gaussian real random variables, conditional expectation corresponds to orthogonal projection and independence corresponds to orthogonality. ♠

By Proprosition 3.9.4, in our situation Lemma 3.5.5 gives:

$$\{(X_1, X_2) : Y\} = \sqrt{1 - \sin^2 \widehat{B} \sin^2 \widehat{A'}}. \tag{NM}$$

Obviously the roles of X_1 and X_2 can be interchanged in the above argument, yielding:

$$\{(X_2, X_1) : Y\} = \sqrt{1 - \sin^2 \widehat{A} \sin^2 \widehat{B}'}. \tag{NN}$$

But (X_1, X_2) and (X_2, X_1) generate the same σ-algebra, so $\{(X_1, X_2) : Y\} = \{(X_2, X_1) : Y\}$, and thus, comparing (NM) and (NN):

$$\frac{\sin \widehat{A'}}{\sin \widehat{A}} = \frac{\sin \widehat{B'}}{\sin \widehat{B}}. \tag{NO}$$

This implies in particular that $\sin \widehat{A}, \sin \widehat{A'}$ and $\sin \widehat{B}, \sin \widehat{B}'$ have the same relative order, so also do $\widehat{A}, \widehat{A'}$ and $\widehat{B}, \widehat{B}'$. A cyclic permutation of L_1, L_2 and L_3 shows that the result is still valid for $\widehat{\Omega}, \widehat{\Omega}'$.

♠

3.9.5 Example. See Figure 10 on page 155. ♡

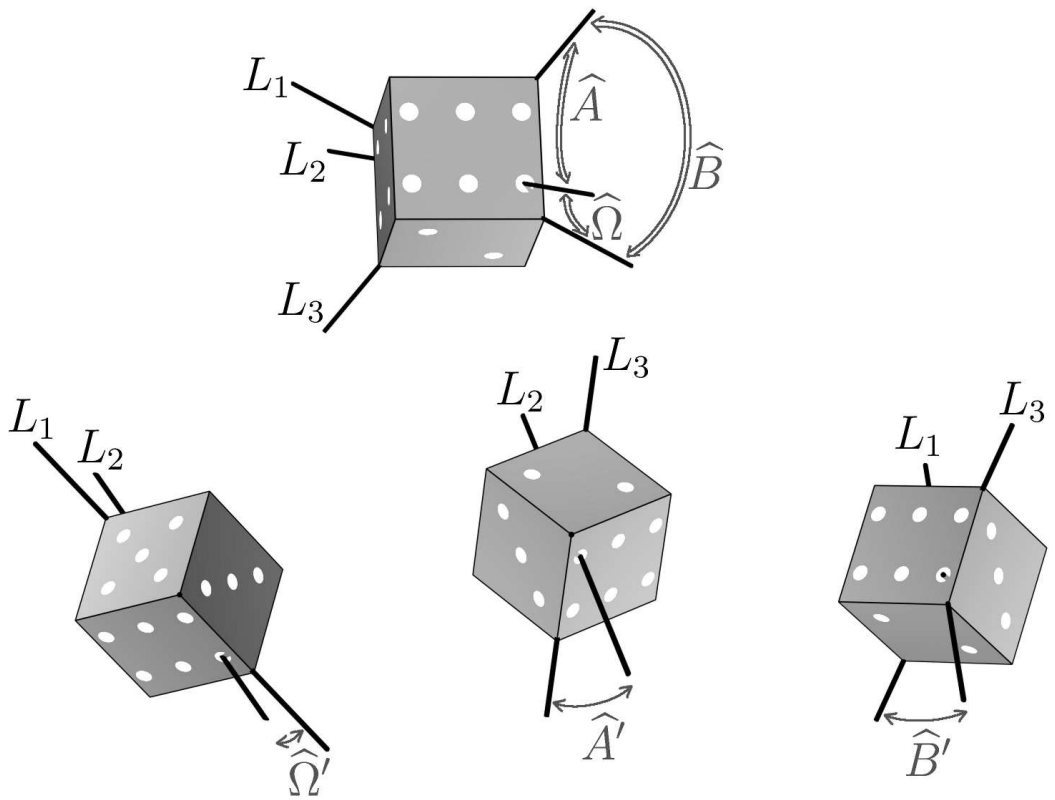

Figure 10: This figure shows four different views of the same 3-dimensional object. What interests us actually is only the three concurrent lines L_1,L_2,L_3, but we added a die centered at their point of concurrency to see depth better on the pictures [Recall that on a die, the total number of points on two opposite faces is always 7.]. On the top picture, the die is shown in generic position. We represent the angles $\widehat{A}$, $\widehat{B}$ and $\widehat{\Omega}$; these angles are 3-dimensional angles, which we underline by drawing them with double strokes. On each of the bottom pictures, the die is viewed from the direction of one of the lines (from the left to the right, L_3, L_1 and L_2), so that this line appears completely foreshortened. We represent the angles $\widehat{\Omega}'$, $\widehat{A}'$ and $\widehat{B}'$ made by the two other lines *as they appear on the drawing*; we underline that these angles are 2-dimensional by drawing them with simple strokes [Note that in the case of $\widehat{B}'$, the angular sector representing $\widehat{B}'$ is not the projection of the angular sector representing $\widehat{B}$, but its supplementary—otherwise $\widehat{B}'$ would be greater than $\pi/2$, which would contradict our 'geometric' definition of angles.]. On this example one has $\widehat{A} \simeq 58°$,$\widehat{B} \simeq 71°$,$\widehat{\Omega} \simeq 15°$ and $\widehat{A}' \simeq 30°$,$\widehat{B}' \simeq 34°$, $\widehat{\Omega}' \simeq 9°$; so, perspective makes angles appear smaller than they are really for all three pairs of lines, which is in accordance with Theorem 3.9.2.

Chapter 4

Other applications of tensorization techniques

In the previous chapter we have been seeing how Hilbertian decorrelation hypotheses between pairs of variables could yield 'global' results on an arbitrary number of variables, by splitting functions of several variables into relevant telescopic sums. I used the word "tensorization" to qualify these results, as the conclusions were of the same nature as the hypotheses.

But the techniques of § 3 can also be applied to get other types of results. In this chapter I am going to show how, from Hilbertian decorrelation hypotheses, one can get results on some classical features of particle systems which are not linked with Hilbertian correlations *a priori*.

I will deal with two such features. First, I will look at the implications of ρ-mixing on the existence of a central limit theorem—more precisely, of a *spatial* central limit theorem, since I am more interested in random fields than in sequences (variables indexed by $\mathbb{Z}^n$ rather than by $\mathbb{Z}$). Very sharp results concerning this issue are already known; however, I find interesting to show how it goes with my 'tensorization-like' approach: this approach takes indeed a quite different way to do the job, which may be neater by certain sides. Moreover, the results are stated with a slighlty different vocabulary—namely, *subjective* Hilbertian correlations.

Next, I will look at the question of spectral gap for Glauber dynamics. Though this point has already been thouroughly studied in a β-mixing paradigm, this work, to the best of my knowledge, is the first to show how ρ-mixing can be used to tackle this issue.

My main goal here is just to show *how* the techniques of this work may be applied to the problems of spatial central limit theorem and convergence of the Glauber dynamics. Accordingly, I favoured the simplicity on proofs against the refinement of the results.

4.1 Spatial central limit theorem

4.1.a Introduction

A fundamental result in probability theory is the central limit theorem (CLT), which, in its standard statement, requires an assumption of complete independence. It is natural to wonder whether that assumption can be relaxed into an hypothesis of 'near independence'. Hilbertian

decorrelations are a natural frame for such a generalization, since the CLT already takes place in an L^2 setting.

Our point of view is motivated by statistical physics. Let $\mathbb{Z}^n$ be a lattice, on each vertex i of which there is a random 'spin' X_i ranged in some space $\mathscr{X}$ not depending on i. We assume that the law of the system is translation invariant, i.e. that for all $z \in \mathbb{Z}^n$, $(X_{i+z})_{i\in\mathbb{Z}^n}$ has the same law as $\vec{X}_{\mathbb{Z}^n}$. Then, for all $z \in \mathbb{Z}^n$, we denote

$$\varepsilon_z = \{X_i : X_{i+z}\}_*. \tag{NP}$$

We are interested in situations where the ε_z are sufficiently 'rapidly decreasing' as $|z| \to \infty$ so that $\sum_{z\in\mathbb{Z}^n} \varepsilon_z < \infty$.

Let $f : \mathscr{X} \to \mathbb{R}$ be a function such that $f(X_0)$ is square-integrable and centered. The question is, does one get a CLT when summing $f(X_i)$ for i in a large subset of $\mathbb{Z}^n$, i.e., does the sum grow as the square root of the number of its terms and have asymptotically normal distribution? For instance, we would like the law of the variable

$$\frac{1}{\sqrt{l^n}} \sum_{\substack{i\in\mathbb{Z}^n \\ 0\leqslant i_1,\dots,i_n<l}} f(X_i) \tag{NQ}$$

to weakly converge, when $l \to \infty$, to some Gaussian distribution.

4.1.1 Remark. Note that the limit distribution, if it exists, will have to be centered, but its variance will not be equal to $\mathrm{Var}(f(X_0))$ in general. ♡

In the case $n = 1$, extremely sharp results for this topic have been known from long; let us cite, among many others, [71, 40, 62, 10]. For $n \geqslant 2$, similar results also exist; see e.g. [10, Theorem 5] for such a result, and [12, § 29] for a survey of the topic. All these proofs relie on some 'coupling' between (bunches of) the spins and other convenient variables which are close to them, but which are *actually* independent, so as to deduce the CLT for the former from the CLT for the latter. On the other hand, my proof will mimick Lévy's proof of the CLT, hence needing no coupling argument.

A priori the results presented here do not improve the state of the art; however, when turning to quantitative versions of these results, it is likely that the difference between the usual method and mine would yield a difference in the corresponding non-asymptotic bounds obtained.

4.1.b Product of weakly coupled variables

My results relie on the following

4.1.2 Lemma. *Let $N \geqslant 1$ and let $\dot{\mathscr{F}}_1,\dots,\dot{\mathscr{F}}_N$ be σ-algebras with $\{\dot{\mathscr{F}}_i : \dot{\mathscr{F}}_j\}_* \leqslant \varepsilon_{ij}$ [*], and denote*

$$\overline{\varepsilon} = \sup_i \sum_{j\neq i} \varepsilon_{ij}. \tag{NR}$$

Let $\Phi_1,\dots,\Phi_N$ be complex-valued random variables with $|\Phi_i| \leqslant 1$ a.s., such that Φ_i is $\dot{\mathscr{F}}_i$-measurable for all i, with all the Φ_i having the same distribution. Then, denoting by φ the common value of the $\mathbf{E}[\Phi_i]$,

$$\Big|\mathbf{E}\Big[\prod_i \Phi_i\Big] - \varphi^N\Big| \leqslant N\overline{\varepsilon}(1+\overline{\varepsilon})(1-|\varphi|^2). \tag{NS}$$

♣

[*]. As in § 3.6, "$*$" stands for "the natural σ-metalgebra of the system". Moreover, in the same way as in § 3.4.a, it is actually possible in the statement of the lemma to replace that σ-metalgebra by smaller ones.

Proof. Denote $\delta := \mathrm{Sd}(\Phi)$. Since $\mathbf{E}[|\Phi^2|] \leqslant 1$, the definition of (complex) variance ensures that $\delta \leqslant \sqrt{1-|\varphi|^2}$.

For all $i \in \{0,\dots,N\}$, denote $\mathscr{F}_i := \bigvee_{i' \leqslant i} \dot{\mathscr{F}}_i$; denote $\Psi^{(i)} := \prod_{i' \leqslant i} \Phi_{i'}$; define

$$\Psi_j^{(i)} := (\Psi^{(i)})^{\mathscr{F}_j} - \mathbf{E}[\Psi^{(i)}|\mathscr{F}_{j-1}] \tag{NT}$$

and denote $\Delta_j^{(i)} := \mathrm{Sd}(\Psi_j^{(i)})$. Also denote

$$\Phi_{i,j} := (\Phi_i)^{\mathscr{F}_j} - \mathbf{E}[\Phi_i|\mathscr{F}_{j-1}]. \tag{NU}$$

Usual manipulation on conditioning shows that, for $i \geqslant 1$,

$$\Psi_j^{(i)} = \Psi_j^{(i-1)}(\Phi_i)^{\mathscr{F}_j} + \Psi_{j-1}^{(i-1)}\Phi_{i,j}. \tag{NV}$$

Since $\|\Phi_i\|_{L^\infty} \leqslant 1$, one has also $\|(\Phi_i)^{\mathscr{F}_j}\|_{L^\infty} \leqslant 1$, hence

$$\mathrm{Sd}\big(\Psi_j^{(i-1)}(\Phi_i)^{\mathscr{F}_j}\big) \leqslant \mathrm{Sd}(\Psi_j^{(i-1)}) = \Delta_j^{(i-1)}. \tag{NW}$$

Similarly, it is obvious that $\|\Psi^{(i-1)}\|_{L^\infty} \leqslant 1$, whence

$$\mathrm{Sd}\big(\Psi_{j-1}^{(i-1)}\Phi_{i,j}\big) \leqslant \mathrm{Sd}(\Phi_{i,j}). \tag{NX}$$

Now, I claim that

4.1.3 Claim.

$$\mathrm{Sd}(\Phi_{i,j}) \leqslant \varepsilon_{ij}\delta. \tag{NY}$$

♣

Proof of Claim 4.1.3. Conditionally to $\mathscr{F}_{j-1}$, $\Phi_{i,j}$ is indeed the projection on $\dot{\mathscr{F}}_j$ of the centered $\dot{\mathscr{F}}_i$-measurable function $(\Phi_i - \mathbf{E}[\Phi_i|\mathscr{F}_{j-1}])$, whose standard deviation is less than $\mathrm{Sd}(\Phi_i) = \delta$ by associativity of the variance, so that $\mathrm{Sd}(\Phi_{i,j}) \leqslant \{\dot{\mathscr{F}}_i : \dot{\mathscr{F}}_j\}_{\mathscr{F}_{j-1}}\delta \leqslant \varepsilon_{ij}\delta$. ♠

In the end, we got that

$$\Delta_j^i \leqslant \Delta_j^{(i-1)} + \varepsilon_{ij}\delta. \tag{NZ}$$

Since $\Delta_j^0 = 0$, one has therefore:

$$\forall i,j \qquad \Delta_j^i \leqslant (1+\overline{\varepsilon})\delta. \tag{OA}$$

Now, denoting $\psi^{(i)} := \mathbf{E}[\Psi^{(i)}]$, one has

$$|\psi^{(i)} - \varphi\psi^{(i-1)}| = \Big|\sum_{j<i} \mathbf{E}\big[\Psi_j^{(i-1)}\Phi_{i,j}\big]\Big| \leqslant \sum_{j<i} \Delta_j^{(i-1)}\,\mathrm{Sd}(\Phi_{i,j}) \leqslant \overline{\varepsilon}(1+\overline{\varepsilon})\delta^2, \tag{OB}$$

and finally

$$|\psi^{(N)} - \varphi^N| = \sum_{i=1}^N |\varphi|^{N-i}|\psi^{(i)} - \varphi\psi^{(i-1)}| \leqslant \sum_{i=1}^N |\psi^{(i)} - \varphi\psi^{(i-1)}| \leqslant N\overline{\varepsilon}(1+\overline{\varepsilon})\delta^2, \tag{OC}$$

which is (NS) if you recall that $\delta^2 \leqslant 1-|\varphi|^2$. ♠

4.1.c A spatial CLT

First I state and prove a CLT on cubes:

4.1.4 Theorem. *Consider a translation-invariant spin model on a lattice $\mathbb{Z}^n$ and define ε_z by (NP). Assume that $\sum_{z\in\mathbb{Z}^n}\varepsilon_z<\infty$. Then for any centered square-summable function $f\colon\mathscr{X}\to\mathbb{R}$, there exists a constant $\sigma<\infty$ such that*

$$\frac{1}{\sqrt{l^n}}\sum_{\substack{i\in\mathbb{Z}^n\\0\leqslant i_1,\dots,i_n<l}} f(X_i)\overset{l\to\infty}{\rightharpoonup}\mathscr{N}(\sigma^2), \tag{OD}$$

where "$\rightharpoonup$" denotes convergence in law. ♣

Proof. Denote by $F(l)$ —or merely F —the left-hand side of (OD).

What will be the value of σ? Clearly we must have

$$\sigma^2=\lim_{l\to\infty}\operatorname{Var}(F(l)), \tag{OE}$$

which yields

$$\sigma=\sqrt{\sum_{z\in\mathbb{Z}}\mathbf{E}[f(X_0)f(X_z)]}, \tag{OF}$$

where the expression under the root sign, which is necessarily nonnegative, is finite because $|\mathbf{E}[f(X_0)f(X_z)]|\leqslant\varepsilon_z\operatorname{Sd}(f(X_0))\operatorname{Sd}(f(X_z))=\varepsilon_z\operatorname{Var}(f)$. By the way, we will denote

$$\sigma_*^2:=\Big(\sum_z\varepsilon_z\Big)\|f\|_{L^2}^2. \tag{OG}$$

Fix some arbitrary $\eta>0$. The assumption that $\sum\varepsilon_z<\infty$ implies the existence of an $l_0<\infty$ such that

$$\sum_{|z|_\infty>l_0}\varepsilon_z\leqslant\eta, \tag{OH}$$

where $|z|_\infty$ denotes $\max(|z_1|,\dots,|z_n|)$. By (OE), we can also fix an $l_1<\infty$ such that

$$|\operatorname{Var}(F(l))-\sigma^2|\leqslant\eta. \tag{OI}$$

Now we will 'tile' the cube of size l into a 'patchwork' made of cubes of size l_1 which I call "tiles", each tile being at distance at least l_0 from the others, plus some "scrap". I denote by $\widetilde{F}$ the part of F due to the tiles and by F^* the part of F due to the scrap.

Index the tiles by $\{1,\dots,N\}$, with $N:=\lfloor(l+l_0)/(l_1+l_0)\rfloor^n$. We write, with obvious notation, $\widetilde{F}=:F_1+\dots+F_N$. For $\lambda\in\mathbb{R}$, denote

$$\Psi(\lambda,l):=\exp(i\lambda\widetilde{F}),\qquad \psi(\lambda,l):=\mathbf{E}[\Psi(\lambda,l)]; \tag{OJ}$$

$$\Phi_j(\lambda,l):=\exp(i\lambda F_j),\qquad \varphi(\lambda,l):=\mathbf{E}[\Phi_j(\lambda,l)]. \tag{OK}$$

Then we are exactly in situation of applying Lemma 4.1.2, which yields:

$$|\psi(\lambda,l)-\varphi(\lambda,l)^N|\leqslant N\eta(1+\eta)\big(1-|\varphi(\lambda,l)|^2\big). \tag{OL}$$

Let us look at the asymptotics of Formula (OL) when $l\to\infty$. We observe that, denoting

$$F_{\mathrm{t}}:=\frac{1}{\sqrt{l_1^n}}\sum_{i\in\textit{ fixed tile}}f(X_i), \tag{OM}$$

one has

$$F_j \overset{\text{law}}{=} \frac{\sqrt{l_1^n}}{\sqrt{l^n}} F_{\mathrm{t}}. \tag{ON}$$

Since F_{t} is centered, its Fourier transform satisfies $\hat{F}_{\mathrm{t}}(0) = 1$, $\hat{F}_{\mathrm{t}}'(0) = 0$ and $\hat{F}_{\mathrm{t}}'' = \mathrm{Var}(F_{\mathrm{t}})$, so that

$$\lim_{l\to\infty} l^n(1-\varphi(\lambda,l)) = \frac{\lambda^2}{2} l_1^n \mathrm{Var}(F_{\mathrm{t}}), \tag{OO}$$

where, denoting $\sigma_{l_1}^2 := \mathrm{Var}(F_{\mathrm{t}})$, we recall that l_1 has been taken sufficiently large so that $|\sigma_{l_1}^2 - \sigma^2| \leqslant \eta$. Then, since $N \overset{l\to\infty}{\sim} l^n/(l_1+l_0)^n$, one has the following asymptotics for (OL):

$$\varphi(\lambda,l)^N \overset{l\to\infty}{\to} \exp\left[-\sigma_{l_1}^2 \left(\frac{l_1}{l_1+l_0}\right)^n \frac{\lambda^2}{2}\right]; \tag{OP}$$

$$N(1-|\varphi(\lambda,l)|^2) \overset{l\to\infty}{\to} \sigma_{l_1}^2 \left(\frac{l_1}{l_1+l_0}\right)^n \lambda^2. \tag{OQ}$$

It remains to control the contribution of F^*.

4.1.5 Claim. *There are at most*

$$\left[1-\left(\frac{l_1}{l_1+l_0}\right)^n\right] l^n + n l_1 l^{n-1} \tag{OR}$$

scrap spins. ♣

By Claim 4.1.5,

$$\|F^*\|_{L^1} \leqslant \mathrm{Var}(F^*) \leqslant \left[1-\left(\frac{l_1}{l_1+l_0}\right)^n + n\frac{l_1}{l}\right]\sigma_*^2; \tag{OS}$$

then the contribution of F^* is controlled using the following immediate

4.1.6 Lemma. *Let X and H be real random variables with $\|H\|_{L^1} < \infty$. Then, for $\lambda \in \mathbb{R}$,*

$$|\mathbf{E}[e^{i\lambda(X+H)}] - \mathbf{E}[e^{i\lambda X}]| \leqslant |\lambda| \|H\|_{L^1}. \tag{OT}$$

♣

In the end, putting everything together we get:

$$\overline{\lim_{l\to\infty}} |\psi(\lambda,l) - e^{-\sigma^2\lambda^2/2}| \leqslant$$
$$\left|e^{-(\sigma^2-\eta)[l_1/(l_1+l_0)]^n\lambda^2/2} - e^{-\sigma^2\lambda^2/2}\right| + \eta(1+\eta)\left(\frac{l_1}{l_1+l_0}\right)^n(\sigma^2+\eta)\lambda^2 + \sqrt{1-\left(\frac{l_1}{l_1+l_0}\right)^n}\,\sigma_*. \tag{OU}$$

Since there were no upper restriction on the value of l_1, we can assume that we have taken it such that $[l_1/(l_1+l_0)]^n \geqslant 1-\eta$. Then (OU) becomes:

$$\overline{\lim_{l\to\infty}} |\psi(\lambda,l) - e^{-\sigma^2\lambda^2/2}| \leqslant \left|e^{-(\sigma^2-\eta)(1-\eta)\lambda^2/2} - e^{-\sigma^2\lambda^2/2}\right| + \eta(1+\eta)(1-\eta)(\sigma^2+\eta)\lambda^2 + \sqrt{\eta}\sigma_*. \tag{OV}$$

The right-hand side of (OV) can be made arbitrarily close to 0 by taking η small enough, so we have proved that

$$\forall \lambda \in \mathbb{R} \quad \mathbf{E}[e^{i\lambda F(l)}] \overset{l\to\infty}{\to} e^{-\sigma^2\lambda^2/2}. \tag{OW}$$

By Lévy's theorem on characteristic functions, this is tantamount to saying that $F(l)$ converges in law to $\mathcal{N}(\sigma^2)$. ♠

The CLT should remain valid for other shapes than a cube, since morally the random field $f(X_i)$ should look like a Gaussian white noise at large scales. Indeed, the same proof as above yields a CLT for general shapes, where moreover convergence is uniform in the shape considered in some way. Let us give a precise statement:

4.1.7 Definition. An open subset $U \subset \mathbb{R}^n$ (not necessarily connected) is said to be $\mathscr{C}^2$ if its boundary M is a $\mathscr{C}^2$ submanifold of $\mathbb{R}^n$ (of codimension 1). We define the *roughness* of U, denoted by $\kappa(U)$, as

$$\kappa(U) := \sup_{x \in M} |||\, \mathrm{II}(x) |||, \tag{OX}$$

where $\mathrm{II}(\cdot)$ denotes the shape tensor of M [38, Chapter 10], which measures the local deviation of M from being flat. Also, the Lebesgue measure of U will be denoted by $\mathrm{vol}(U)$. ♢

4.1.8 Theorem. *Consider a translation-invariant spin model on a lattice $\mathbb{Z}^n$ and define ε_z by (NP). Assume that $\sum_{z \in \mathbb{Z}^n} \varepsilon_z < \infty$. Then for any centered square-summable function $f\colon \mathscr{X} \to \mathbb{R}$, if $(U_k)_{k \in \mathbb{N}}$ is a sequence of $\mathscr{C}^2$ bounded subsets of $\mathbb{R}^n$ with $\sup_k \kappa(U_k) < \infty$ and $(l_k)_{k \in \mathbb{N}}$ is a sequence of positive numbers tending to infinity,*

$$\frac{1}{\sqrt{l_k^n \,\mathrm{vol}(U_k)}} \sum_{i \in l_k U_k \cap \mathbb{Z}^n} f(X_i) \overset{k \to \infty}{\rightharpoonup} \mathscr{N}(\sigma^2), \tag{OY}$$

where σ^2 is the same as in Theorem 4.1.4. ♣

Proof. Just copy the proof of Theorem 4.1.4. The only difference lies in proving the analoguous of Claim 4.1.5, which is where one needs the $\kappa(U_k)$ to be bounded. Observe that we use the non-asymptotic form of our intermediate bounds to get a result independent of the precise shape of the U_k. ♠

4.1.9 Remark. Another generalization of the CLT, still based on the idea that the field $f(X_i)$ looks like a Gaussian white noise at large scales, is the statement that for φ a continuous function with compact support,

$$\frac{1}{\sqrt{l^n}} \sum_{i \in \mathbb{Z}^n} \varphi(X_i / l) f(X_i) \overset{l \to \infty}{\rightharpoonup} \mathscr{N}(\sigma^2 \textstyle\int_{\mathbb{R}^n} \varphi(x)^2 \, dx). \tag{OZ}$$

This can be proved with the same methods as before. ♡

4.2 Spectral gap for the Glauber dynamics

4.2.a Introduction

In this section we are looking at a probabilistic system made of a large number of 'elementary' random variables $(X_i)_{i \in I}$ — I may be seen as lattice and X_i as the state of the particle being at site i. As is customary by now, theorems will only be stated in the case where I is finite, the infinite case being got by passing to the limit.

4.2.1 Definition. Denoting by Ω the states space of $\vec{X}_I$, let $\mathbf{P}$ be a probability measure on Ω. The *Glauber dynamics* [35, 26] associated to $\mathbf{P}$ is the Markov process on Ω having the following law: on each $i \in I$ there is an alarm clock, all the clocks being independent and ringing with law *Poisson*(1). When a clock rings, the state of spin X_i —and only it—is flipped so that the state of X_i immediately after the flip follows the law $\mathbf{P}(X_i | \vec{X}_{I \setminus \{i\}})$.

In formal terms, the Glauber dynamics is the Markov process whose generator $\mathscr{L}$ on $L^\infty(\Omega)$ is defined by:

$$(\mathscr{L}f)(\vec{x}_I) = \sum_{i\in I} \mathbf{E}[f(\vec{X}_I) - f(\vec{x}_I)|\vec{X}_{I\smallsetminus\{i\}} = \vec{x}_{I\smallsetminus\{i\}}]. \tag{PA}$$

$\diamondsuit$

Let us recall some basic facts on the Glauber dynamics (see [47, Chapter IV] for more details). By construction $\mathbf{P}$ is a reversible equilibrium measure for the dynamics, so $\mathscr{L}$ is self-adjoint on $L^2(\mathbf{P})$. Since obviously $\mathscr{L}1 \equiv 0$, one can also define $\mathscr{L}$ on $\bar{L}^2(\mathbf{P})$, on which it is self-adjoint too. This leads to the following definition:

4.2.2 Definition. The *energy* of $f \in \bar{L}^2(\mathbf{P})$ is

$$\mathscr{E}(f,f) = \langle Lf, f\rangle \tag{PB}$$

$\diamondsuit$

The following immediate identity shows that $\mathscr{E}$ is always a nonnegative bilinear form:

4.2.3 Proposition.

$$\mathscr{E}(f,f) = \int_\Omega d\mathbf{P}[\vec{x}_I] \sum_i \mathrm{Var}(f|\vec{X}_{I\smallsetminus\{i\}} = \vec{x}_{I\smallsetminus\{i\}}). \tag{PC}$$

$\clubsuit$

4.2.4 Definition. For $\lambda > 0$, the Glauber dynamics is said to have *spectral gap* $\geqslant \lambda$ if, for all $f \in \bar{L}^2(\mathbf{P})$,

$$\mathscr{E}(f,f) \geqslant \lambda \mathrm{Var}(f). \tag{PD}$$

$\diamondsuit$

What makes spectral gap interesting is that its positiveness is equivalent to exponential convergence to 0 of the semigroup $(e^{-t\mathscr{L}})_{t\geqslant 0}$ on $\bar{L}^2(\mathbf{P})$, the rate of convergence being equal to the width of the spectral gap. As the Glauber dynamics is one of the easiest ways to simulate the law $\mathbf{P}$ for complicated models, the stake of having exponential convergence for it is evident.

Many works have been done on the spectral gap of the Glauber dynamics, see for instance Martinelli's St-Flour course [52]. Several results state that, the less spins are correlated, the larger the spectral gap is. Yet the researchers who work on this topic generally express the decorrelation between the spins in terms of β-mixing (cf. Definition 0.1.4), while it seems be more natural to look at them in terms of Hilbertian decorrelations, since the formula (PD) stating the spectral gap problem takes place in a Hilbertian frame itself. Thus my goal here will be to find a control on the spectral gap expressed in terms of ρ-mixing conditions. Since Hilbertian correlations look to be the minimal frame to study the spectral gap for the Glauber dynamics, hopefully the bounds yielded by this method will be sharp.

Another noticeable feature of my approach is that it remains at a quite abstract level: no symmetry property of I or $\mathbf{P}$ need be assumed, all the work essentially consisting in manipulating relevant quadratic forms.

4.2.b A lower bound for the spectral gap

The central theorem of this section is the following:

4.2.5 Theorem. *Take $I = \{1,\dots,N\}$. Suppose that for all distinct $i,j \in I$ one has $\{X_i : X_j\}_* \leqslant \varepsilon_{ij} < 1$ —we will make the costless assumption that $\varepsilon_{ji} = \varepsilon_{ij}$. For $i \in I$, denote*

$$\tilde{1}_i := \frac{1}{\prod_{i<j\leqslant N}(1-\varepsilon_{ij}^2)} = \frac{\tilde{\varepsilon}_{iN}}{\varepsilon_{iN}}, \tag{PE}$$

and for $i < j$, denote

$$\tilde{\varepsilon}_{ij} := \frac{\varepsilon_{ij}}{\prod_{i<j'\leqslant j}(1-\varepsilon_{ij'}^2)}. \tag{PF}$$

Then the Glauber dynamics has spectral gap at least $|||M|||^{-2}$, where M is the $(N\times N)$ matrix defined by

$$M = \begin{pmatrix} 1 & -\tilde{\varepsilon}_{12} & \cdots & -\tilde{\varepsilon}_{1N} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\tilde{\varepsilon}_{(N-1)N} \\ 0 & \cdots & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \tilde{1}_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \tilde{1}_N \end{pmatrix}. \tag{PG}$$

♣

4.2.6 Remark. The form of the first matrix in the right-hand side of (PG) ensures that it is invertible. Since moreover all the ε_{ij} were supposed < 1, all the $\tilde{\varepsilon}_{ij}$ and the $\tilde{1}_i$ are finite; thus, the lower bound $|||M|||^{-2}$ is strictly positive. ♡

Proof. Let f be a centered square-integrable function on $(\Omega,\mathbf{P})$. For $I' \subset I$, denote $\mathscr{F}_{I'} := \sigma(\vec{X}_{I'})$. For $i \in I$, $I' \subset I \smallsetminus \{i\}$, denote

$$f_i^{I'} := f^{\mathscr{F}_{I'\uplus\{i\}}} - \mathbf{E}[f|\mathscr{F}_{I'}]; \tag{PH}$$

define moreover

$$f_i^{\neq} := f_i^{I\smallsetminus\{i\}}; \tag{PI}$$

$$f_i^{<} := f_i^{\{1,\dots,i-1\}}. \tag{PJ}$$

Then by Proposition 4.2.3, one has

$$\mathscr{E}(f,f) = \sum_i \mathrm{Var}(f_i^{\neq}), \tag{PK}$$

while the usual telescopic argument shows that

$$\mathrm{Var}(f) = \sum_i \mathrm{Var}(f_i^{<}). \tag{PL}$$

So to prove the theorem, we have to establish links between the different values $\mathrm{Var}(f_i^{I'})$. It will be convenient to introduce the shorthands $\Delta_i^{I'} = \mathrm{Sd}(f_i^{I'})$. One has the following

4.2.7 Claim. *For $I' \subset I$ and $i,j \in I \smallsetminus I'$ with $j \neq i$,*

$$\Delta_i^{I'} \leqslant \Delta_i^{I'\uplus\{j\}} + \varepsilon_{ij}\Delta_j^{I'}. \tag{PM}$$

♣

Proof. Assume in a first time that $I' = \emptyset$, and denote $f_i := f_i^{\emptyset}$, $f_j := f_j^{\emptyset}$, $f_i^j := f_i^{\{j\}}$ and $\mathscr{F}_i := \mathscr{F}_{\{i\}}$. Projecting the decomposition "$f_i = f_i^j + (f_i - f_i^j)$" on $L^2(\mathscr{F}_i)$, one has $f_i = (f_i^j)^{\mathscr{F}_i} + (f_j)^{\mathscr{F}_i}$, whence by the Cauchy–Shwarz inequality:

$$\mathrm{Sd}(f_i) \leqslant \mathrm{Sd}\big((f_i^j)^{\mathscr{F}_i}\big) + \mathrm{Sd}\big((f_j)^{\mathscr{F}_i}\big). \tag{PN}$$

One has trivially $\mathrm{Sd}\big((f_i^j)^{\mathscr{F}_i}\big) \leqslant \mathrm{Sd}(f_i^j)$; on the other hand, f_j is X_j-measurable, so $\mathrm{Sd}\big((f_j)^{\mathscr{F}_i}\big) \leqslant \varepsilon_{ij}\,\mathrm{Sd}(f_j)$. In the end, (PN) becomes

$$\mathrm{Sd}(f_i) \leqslant \mathrm{Sd}(f_i^j) + \varepsilon_{ij}\,\mathrm{Sd}(f_j), \tag{PO}$$

which is (PM) for $I' = \emptyset$.

In the case $I' \neq \emptyset$, the same reasoning can be performed, except that one have to work conditionally to $\mathscr{F}_{I'}$. Then, taking $f_i = f_i^{I'}$, $f_j = f_j^{I'}$, $f_i^j = f_i^{I' \uplus \{j\}}$, $\mathscr{F}_i = \mathscr{F}_{I' \uplus \{i\}}$, one gets

$$\mathrm{Sd}(f_i | \mathscr{F}_{I'}) \leqslant \mathrm{Sd}(f_i^j | \mathscr{F}_{I'}) + \varepsilon_{ij}\,\mathrm{Sd}(f_j | \mathscr{F}_{I'}). \tag{PP}$$

Now

$$\mathrm{Sd}(f_i) = \sqrt{\int \mathrm{Sd}\big(f_i | \vec{X}_{I'} = \vec{x}_{I'}\big)^2 d\mathbf{P}[\vec{x}_{I'}]}, \tag{PQ}$$

with similar formulas for f_j and f_i^j, since all these functions are centered w.r.t. $\mathscr{F}_{I'}$. Therefore, integrating (PP) and applying Minkowski's inequality yields:

$$\mathrm{Sd}(f_i) \leqslant \mathrm{Sd}(f_i^j) + \varepsilon_{ij}\,\mathrm{Sd}(f_j), \tag{PR}$$

i.e. (PM). ♠

For $i \leqslant j$, let us denote

$$\Delta_i^{[j]} = \Delta_i^{\{1,\dots,j\} \smallsetminus \{i\}}. \tag{PS}$$

Claim 4.2.7 will be used through the following corollary:

4.2.8 Claim. *For all $i < j$,*

$$\Delta_i^{[j-1]} \leqslant \frac{1}{1 - \varepsilon_{ij}^2}\big(\Delta_i^{[j]} + \varepsilon_{ij}\Delta_j^{<}\big). \tag{PT}$$

♣

Proof. We have to bound $\Delta_i^{[j-1]}$, which here we rather denote $\Delta_a^{[b-1]}$ to avoid confusion with the notation of Claim 4.2.7. Applying Claim 4.2.7 with $I' = \{1, \dots, b-1\} \smallsetminus \{a\}$, $i = a$ and $j = b$, one has

$$\Delta_a^{[b-1]} = \Delta_a^{\{1,\dots,b-1\} \smallsetminus \{a\}} \leqslant \Delta_a^{\{1,\dots,b\} \smallsetminus \{a\}} + \varepsilon_{ab}\Delta_b^{\{1,\dots,b-1\} \smallsetminus \{a\}} = \Delta_a^{[b]} + \varepsilon_{ab}\Delta_b^{\{1,\dots,b-1\} \smallsetminus \{a\}}. \tag{PU}$$

But applying again Claim 4.2.7, this time with $I' = \{1, \dots, b-1\} \smallsetminus \{a\}$, $i = b$ and $j = a$, one has

$$\Delta_b^{\{1,\dots,b-1\} \smallsetminus \{a\}} \leqslant \Delta_b^{\{1,\dots,b-1\}} + \varepsilon_{ab}\Delta_a^{\{1,\dots,b-1\} \smallsetminus \{a\}} = \Delta_b^{<} + \varepsilon_{ab}\Delta_a^{[b-1]}. \tag{PV}$$

Combining (PU) and (PV) then yields (PT). ♠

Now let us show how Claim 4.2.8 implies the theorem. To avoid heavy formalism, I will detail the computations for $I = \{1,2,3,4\}$ (rather denoted by $I = \{a,b,c,d\}$ here to avoid confusions with "1" and "2" taken as numbers), hoping that generalizing is obvious then.

First, note that

$$\Delta_d^< = \Delta_d^{\neq}. \tag{PW}$$

Now, by a direct use of Claim 4.2.8,

$$\Delta_c^< = \Delta_c^{[c]} \leqslant \frac{1}{1-\varepsilon_{cd}^2}(\Delta_c^{[d]} + \varepsilon_{cd}\Delta_d^<) = \tilde{1}_c\Delta_c^{\neq} + \tilde{\varepsilon}_{cd}\Delta_d^{\neq}. \tag{PX}$$

To bound $\Delta_b^<$, we have to iterate Claim 4.2.8 twice:

$$\begin{aligned}\Delta_b^< = \Delta_b^{[b]} &\leqslant \frac{1}{1-\varepsilon_{bc}^2}\Delta_b^{[c]} + \tilde{\varepsilon}_{bc}\Delta_c^< \\ &\leqslant \frac{1}{(1-\varepsilon_{bc}^2)(1-\varepsilon_{bd}^2)}(\Delta_b^{[d]} + \varepsilon_{cd}\Delta_d^<) + \tilde{\varepsilon}_{bc}\Delta_c^< = \tilde{1}_b\Delta_b^{\neq} + \tilde{\varepsilon}_{bd}\Delta_d^{\neq} + \tilde{\varepsilon}_{bc}\Delta_c^< \\ &\underset{(\mathtt{PX})}{\leqslant} \tilde{1}_b\Delta_b^{\neq} + \tilde{1}_c\tilde{\varepsilon}_{bc}\Delta_c^{\neq} + (\tilde{\varepsilon}_{bd} + \tilde{\varepsilon}_{bc}\tilde{\varepsilon}_{cd})\Delta_d^{\neq}.\end{aligned} \tag{PY}$$

Last, bounding $\Delta_a^<$ requires iterating Claim 4.2.8 three times:

$$\begin{aligned}\Delta_a^< = \Delta_a^{[a]} &\leqslant \frac{1}{1-\varepsilon_{ab}^2}\Delta_a^{[b]} + \tilde{\varepsilon}_{ab}\Delta_b^< \\ &\leqslant \frac{1}{(1-\varepsilon_{ab}^2)(1-\varepsilon_{ac}^2)}\Delta_a^{[c]} + \tilde{\varepsilon}_{ac}\Delta_c^< + \tilde{\varepsilon}_{ab}\Delta_b^< \leqslant \tilde{1}_a\Delta^{\neq a} + \tilde{\varepsilon}_{ad}\Delta_d^< + \tilde{\varepsilon}_{ac}\Delta_c^< + \tilde{\varepsilon}_{ab}\Delta_b^< \\ &\underset{(\mathtt{PX},\mathtt{PY})}{\leqslant} \tilde{1}_a\Delta^{\neq a} + \tilde{1}_b\tilde{\varepsilon}_{ab}\Delta_b^{\neq} + \tilde{1}_c(\tilde{\varepsilon}_{ac} + \tilde{\varepsilon}_{ab}\tilde{\varepsilon}_{bc})\Delta_c^{\neq} + (\tilde{\varepsilon}_{ad} + \tilde{\varepsilon}_{ac}\tilde{\varepsilon}_{cd} + \tilde{\varepsilon}_{ab}\tilde{\varepsilon}_{bd} + \tilde{\varepsilon}_{ab}\tilde{\varepsilon}_{bc}\tilde{\varepsilon}_{cd})\Delta_d^{\neq}.\end{aligned} \tag{PZ}$$

One can sum up Equations (PW)–(PZ) into the matricial expression

$$\begin{pmatrix}\Delta_a^< \\ \Delta_b^< \\ \Delta_c^< \\ \Delta_d^<\end{pmatrix} \leqslant \begin{pmatrix}1 & \tilde{\varepsilon}_{ab} & \tilde{\varepsilon}_{ac} + \tilde{\varepsilon}_{ab}\tilde{\varepsilon}_{bc} & \tilde{\varepsilon}_{ad} + \tilde{\varepsilon}_{ac}\tilde{\varepsilon}_{cd} + \tilde{\varepsilon}_{ab}\tilde{\varepsilon}_{bd} + \tilde{\varepsilon}_{ab}\tilde{\varepsilon}_{bc}\tilde{\varepsilon}_{cd} \\ 0 & 1 & \tilde{\varepsilon}_{bc} & \tilde{\varepsilon}_{bd} + \tilde{\varepsilon}_{bc}\tilde{\varepsilon}_{cd} \\ 0 & 0 & 1 & \tilde{\varepsilon}_{cd} \\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\tilde{1}_a\Delta_a^{\neq} \\ \tilde{1}_b\Delta_b^{\neq} \\ \tilde{1}_c\Delta_c^{\neq} \\ \Delta_d^{\neq}\end{pmatrix}. \tag{QA}$$

If we look back at how the square matrix in (QA) has been constructed, we find that

$$\begin{pmatrix}\textit{square} \\ \textit{matrix} \\ \textit{in} \\ \textit{(QA)}\end{pmatrix} = \sum_{k=0}^{\infty}\begin{pmatrix}0 & \tilde{\varepsilon}_{ab} & \tilde{\varepsilon}_{ac} & \tilde{\varepsilon}_{ad} \\ 0 & 0 & \tilde{\varepsilon}_{bc} & \tilde{\varepsilon}_{bd} \\ 0 & 0 & 0 & \tilde{\varepsilon}_{cd} \\ 0 & 0 & 0 & 0\end{pmatrix}^k = \begin{pmatrix}1 & -\tilde{\varepsilon}_{ab} & -\tilde{\varepsilon}_{ac} & -\tilde{\varepsilon}_{ad} \\ 0 & 1 & -\tilde{\varepsilon}_{bc} & -\tilde{\varepsilon}_{bd} \\ 0 & 0 & 1 & -\tilde{\varepsilon}_{cd} \\ 0 & 0 & 0 & 1\end{pmatrix}^{-1}, \tag{QB}$$

so in the end we obtain that

$$\begin{pmatrix}\Delta_a^< \\ \Delta_b^< \\ \Delta_c^< \\ \Delta_d^<\end{pmatrix} \leqslant M\begin{pmatrix}\Delta_a^{\neq} \\ \Delta_b^{\neq} \\ \Delta_c^{\neq} \\ \Delta_d^{\neq}\end{pmatrix}, \tag{QC}$$

where M is given by (PG). Then it is immediate that $\mathrm{Var}(f) = \sum_i(\Delta_i^<)^2 \leqslant |||M|||^2\sum_i(\Delta_i^{\neq})^2 = |||M|||^2\mathscr{E}(f,f)$, QED. ♠

The bound we have obtained for the spectral gap is not symmetric by permutation of the indexes in I. It can however can be bounded by a simpler expression, which is nearly as good as the original one in concrete situations:

4.2.9 Corollary. *In Theorem 4.2.5, M can be replaced by the matrix*

$$M' = \begin{pmatrix} 1 & -\varepsilon_{12} & \cdots & -\varepsilon_{1N} \\ -\varepsilon_{12} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\varepsilon_{(N-1)N} \\ -\varepsilon_{1N} & \cdots & -\varepsilon_{(N-1)N} & 1 \end{pmatrix}^{-1}, \tag{QD}$$

provided $\rho(\mathbf{I}_N - M') < 1$. ♣

Proof. Each entry of M is actually bounded by the corresponding entry of M'. To see it, we 'expand' the entries of M, resp. M'. First, notice that $1/(1-\varepsilon_{ij}^2)$ can be expanded into $1+\varepsilon_{ij}\varepsilon_{ji}+\varepsilon_{ij}\varepsilon_{ji}\varepsilon_{ij}\varepsilon_{ji}+\cdots$, so that one has the expansions

$$\tilde{1}_i = \sum_{i<j_1\leqslant\cdots\leqslant j_k} \prod_{l=1}^{k} \varepsilon_{ij_l}\varepsilon_{j_l i} \tag{QE}$$

and

$$\tilde{\varepsilon}_{ij} = \sum_{i<j_1\leqslant\cdots\leqslant j_k\leqslant j} \Big(\prod_{l=1}^{k} \varepsilon_{ij_l}\varepsilon_{j_l i}\Big)\varepsilon_{ij}. \tag{QF}$$

Then, using the inversion formula $(\mathrm{I}-A)^{-1} = \sum_{k=0}^{\infty} A^k$ for triangular arrays, one obtains that

$$M_{ij} = \sum_{\substack{(i_0,i_1,\ldots,i_k)\\ \textit{first condition}}} \prod_{l=0}^{k-1} \varepsilon_{i_l i_{l+1}}, \tag{QG}$$

where the meaning of "*first condition*" is given by the following

4.2.10 Definition. A sequence $(i_0,\ldots,i_k)$ is said to satisfy the *first condition* if:

(i) $i_0 = i$ and $i_k = j$;
(ii) $i_l \neq i_{l+1}$ for all l;
(iii) $i_{l+1} < i_l$ only if $l \geqslant 1$ and $i_{l+1} = i_{l-1}$;
(iv) If $i_{l+1} < i_l$ and $l \leqslant k-2$, then $i_{l+2} \geqslant i_l$.

◇

One has a similar formula for M':

$$M'_{ij} = \sum_{\substack{(i_0,i_1,\ldots,i_k)\\ \textit{second condition}}} \prod_{l=0}^{k-1} \varepsilon_{i_l i_{l+1}}, \tag{QH}$$

where

4.2.11 Definition. A sequence $(i_0,\ldots,i_k)$ is said to satisfy the *second condition* if it satisfies Conditions (i) and (ii) of Definition 4.2.10. ◇

Since the *second condition* is obviously weaker than the *first condition*, one has $M_{ij} \leqslant M'_{ij}$.

♠

There is a still weaker but even simpler formula:

4.2.12 Corollary. *Defining*

$$\begin{array}{rccc} \boldsymbol{\varepsilon}: & L^2(I) & \to & L^2(I) \\ & (a_i)_{i\in I} & \mapsto & (\sum_{j\neq i}\varepsilon_{ij}a_j)_{i\in I}, \end{array} \tag{QI}$$

the spectral gap of the Glauber dynamics is at least

$$(1-|||\boldsymbol{\varepsilon}|||)_+^2. \tag{QJ}$$

♣

Proof. One has $M' = (\mathrm{I}-\boldsymbol{\varepsilon})^{-1}$, so, provided $|||A|||<1$,

$$|||M'||| = |||(\mathrm{I}-A)^{-1}||| = \Big|\Big|\Big|\sum_{k=0}^{\infty} A^k\Big|\Big|\Big| \leqslant \sum_{k=0}^{\infty} |||A|||^k = (1-|||A|||)^{-1}. \tag{QK}$$

In the case $|||A|||\geqslant 1$, (QJ) is trivial. ♠

4.2.c Avoiding the articial phase transition

A common situation in which we would like to apply the previous results is when $I=\mathbb{Z}^n$ and ε_{ij} is of the form $\varepsilon(j-i)$ for some symmetric function $\varepsilon\colon\ \mathbb{Z}^n\to[0,1]$. Then Corollary 4.2.12 tells that the Glauber dynamics has a (strictly) positive spectral gap as soon as $\sum_{z\neq 0}\varepsilon(z)<1$. But like in § 3.6.c, we are going to prove that that bound is somehow 'artificial' and that it can be relaxed into the neater condition "$\sum_{z\neq 0}\varepsilon(z)<\infty$":

4.2.13 Theorem. *Suppose that $I=\mathbb{Z}^n$ and that for all $i,j\in\mathbb{Z}^n$ one has $\{X_i:X_j\}_*\leqslant\varepsilon(j-i)$ for some symmetric function $\varepsilon\colon\ \mathbb{Z}^n\to[0,1]$ such that $\varepsilon(z)<1$ as soon as $z\neq 0$. Then if $\sum_{z\in\mathbb{Z}^n}\varepsilon(z)<\infty$, the spectral gap of the Glauber dynamics is positive.* ♣

Proof. The assumption on $\sum_z\varepsilon(z)$ allows us to take $l<\infty$ large enough so that

$$\sum_{z\in l\mathbb{Z}^n\smallsetminus\{0\}}\varepsilon(z)<1. \tag{QL}$$

We split $\mathbb{Z}^n$ into a partition of $l^n=:N$ sublattices $Z_1,\dots,Z_N$, each lattice Z_u being of the form $l\mathbb{Z}^n+z_u$ for some $z_u\in\mathbb{Z}^n/l\mathbb{Z}^n$. Then we define an auxiliary dynamics:

4.2.14 Definition. The *sublattice Glauber dynamics* is the Glauber dynamics for $\vec{X}_{\mathbb{Z}^n}$ *considered as the finite-dimensional vector* $(\vec{X}_{Z_1},\dots,\vec{X}_{Z_N})$. In other words, on each $u\in\{1,\dots,N\}$ there is an independent *Poisson*(1) alarm clock, and when clock u rings, the state of the whole $\vec{X}_{Z_u}$ is flipped in one shot according to $\mathbf{P}(X_{Z_u}|\vec{X}_{\mathbb{Z}^n\smallsetminus Z_u})$. ♢

Now let $f\in\bar{L}^2(\Omega)$. In addition to the notation of the proof of Theorem 4.2.5, we introduce the following definition:

4.2.15 Definition. For $u\in\{1,\dots,N\}$, we define

$$f_{(u)}^{\neq} := f-\mathbf{E}[f|\vec{X}_{\mathbb{Z}^n\smallsetminus Z_u}]. \tag{QM}$$

♢

4.2.16 Remark. The $f^{\neq}_{(u)}$ are the equivalent of the $f^{\neq}_{i}$ for the sublattice Glauber dynamics. ♡

Fixing some 'boundary condition' $\vec{x}_{\mathbb{Z}^n \smallsetminus Z_u}$ on $\mathbb{Z}^n \smallsetminus Z_u$, we can apply Corollary 4.2.12 to the Glauber dynamics for $\vec{X}_{Z_u}$ under the law $\mathbf{P}[\cdot|\vec{X}_{\mathbb{Z}^n \smallsetminus Z_u} = \vec{x}_{\mathbb{Z}^n \smallsetminus Z_u}]$. After integrating, one gets that

$$\mathrm{Var}(f^{\neq}_{(u)}) \leqslant (1 - |||\boldsymbol{\zeta}|||)^{-2} \sum_{i \in Z_u} \mathrm{Var}(f^{\neq}_{i}), \tag{QN}$$

where $\boldsymbol{\zeta}$ is the operator on $L^2(l\mathbb{Z}^n)$ defined by

$$(\boldsymbol{\zeta} g)(i) = \sum_{z \in l\mathbb{Z}^n \smallsetminus \{0\}} \varepsilon(z) g(i+z), \tag{QO}$$

whose norm is obviously bounded by $\sum_{z \in l\mathbb{Z}^n \smallsetminus \{0\}} =: \zeta < 1$. Then, summing (QN) for all u:

$$\sum_{u=1}^{N} \mathrm{Var}(f^{\neq}_{(u)}) \leqslant (1-\zeta)^{-2} \mathscr{E}(f,f). \tag{QP}$$

Now, let us apply Theorem 4.2.5 to the sublattice Glauber dynamics [Definition 4.2.14]. It yields that

$$\mathrm{Var}(f) \leqslant |||M|||^2 \sum_{u=1}^{N} \mathrm{Var}(f^{\neq}_{(u)}), \tag{QQ}$$

where M is some $(N \times N)$ matrix depending on the $\{\vec{X}_{Z_u} : \vec{X}_{Z_v}\}_*$. But by Theorem 3.6.8, $\{\vec{X}_{Z_u} : \vec{X}_{Z_v}\}_* < 1$ for all $u \neq v$, thus $|||M||| < \infty$ by Remark 4.2.6. Combining (QP) and (QQ), we finally get that the spectral gap of the Glauber dynamics for $\vec{X}_{\mathbb{Z}^n}$ is bounded below by $|||M|||^{-2} \times (1-\zeta)^2 > 0$. ♠

4.2.17 Remark. Like Theorem 3.6.8, Theorem 4.2.13 could actually be stated in the general case of 'abstract' metric spaces on which some group acts profinitely. ♡

Chapter 5

Concrete examples

It is now time to see what the results of Chapters 3 and 4 yield for concrete models of statistical physics. I will try to give rather different types of examples, so as to illustrate the advantages of working with Hilbertian correlations: this frame is indeed quite general, as it requires little structure on the models considered.

In § 5.1 we will look back at Ising's model, seeing how tensorization of Hilbertian decorrelations improves the results of § 0.1, and what other results are given by the theorems of § 4. We will also consider two kinds of generalizations, namely when the range of interactions becomes infinite and when the strength of the interactions is random (*spin glasses*). In the two next sections we will look at models with continuous states spaces: first a quite general class of linear models [§ 5.2], then a family of nonlinear models [§ 5.3]. Finally in § 5.4 we will see how one can consider time as a supplementary dimension of the system to get contractivity results for non-reversible Markov chains (*hypocoercivity*) on an infinite system of particles.

☛ *In this chapter, all the probability systems considered will be endowed with their natural σ-metalgebras, cf. Definition 3.1.16. To alleviate notation, I will give no names to these σ-metalgebras, but will plainly write "$\{X:Y\}_*$" to mean "the subjective decorrelation between X and Y seen from the natural σ-metalgebra of the underlying system".*

5.1 Back to Ising's model

5.1.a Standard Ising's model

In all this section, we work on the lattice $\mathbb{Z}^n$ equipped with its natural distance $dist$; accordingly $|\cdot|$ will denote the l^1 norm on $\mathbb{R}^n$. Recall the definition of Ising's model and the related notation that we introduced in § 0.1, and Theorem 0.1.7 on the existence of a completely analytical regime.

The following theroem states that Ising's model in completely analytical regime is ρ-mixing, i.e. that two distant bunches of spins are little correlated in the sense of maximal correlation:

5.1.1 Theorem. *For Ising's model on $\mathbb{Z}^n$ in the completely analytical regime,*

(i) There exists some $\psi' > 0$ (the same as in Theorem 0.1.7) such that for all disjoint $I, J \subset \mathbb{Z}^n$, one has when $dist(I,J) \to \infty$ that

$$\{\vec{\omega}_I : \vec{\omega}_J\} \leqslant \exp\big[-(\psi' + o(1))dist(I,J)\big], \tag{QR}$$

where the "$o(1)$" can be easily computed as an explicit function of $\mathrm{dist}(I,J)$, n, T, ψ' and the C' appearing in Theorem 0.1.7.

(ii) There exists some $k<1$ such that for all disjoint $I,J\subset\mathbb{Z}^n$,

$$\{\vec{\omega}_I : \vec{\omega}_J\} \leqslant k. \tag{QS}$$

(iii) Points (i) and (ii) remain valid uniformly under any law of the form $\mathbf{P}[\cdot|\vec{\omega}_K=\vec{\hat{\omega}}_K]$, for $K\subset\mathbb{Z}^n$ and $\vec{\hat{\omega}}_K\in\{\pm1\}^K$ a 'boundary condition' on K.

♣

5.1.2 Remark. Let us compare Theorem 5.1.1 with Theorem 0.1.7. Both theorems state decorrelation between distant bunches of spins above temperature T_c'; the difference relies in using ρ-mixing rather than β-mixing to quantify dependence between the bunches in Theorem 5.1.1.

Both results give an exponential decay of correlations, with the same exponential constant ψ', but Theorem 5.1.1 is more powerful in the sense that the bound (QR) is uniform in the size of I and J while (E) was not. Moreover, thanks to Point (ii) we get a non-trivial result for any choice of disjoint I and J, which was not the case beforehand. Recall that the drawbacks of Theorem 0.1.7 were inherent to β-mixing, as Theorem 0.1.8 shew.

Both result remain valid under conditioning. However, if one takes a *random* boundary condition—that is, if one works under the law $\mathbf{P}[\cdot|\vec{\omega}_K\in C]$ for some non-singleton $C\subset\{\pm1\}^K$ —, then Point (iii) of Theorem 5.1.1 fails (cf. Remark 3.1.4), while (E) is still valid by convexity of the total variation norm. ♡

5.1.3 Remark. Let us compare Theorem 5.1.1 with Theorem 0.1.9. The result of Theorem 0.1.9 can be rewritten:

$$\{\vec{\omega}_{\{0\}\times\mathbb{Z}^{n-1}} : \vec{\omega}_{\{x\}\times\mathbb{Z}^{n-1}}\} \leqslant e^{-\psi x}. \tag{QT}$$

Theorem 5.1.1 can be seen as a generalization of that result to the case where I and J have arbitrary shapes. [*] Moreover, Point (iii) also gives the existence of a conditional version, which we did not have before.

There is however a price to pay for this greater generality, since we had to require complete analyticity rather than just weak mixing, which can be really more restrictive in some cases (cf. Footnote [*] on page 84). ♡

5.1.4 Remark. Continuing the previous remark, a natural open question is whether one can tensorize maximal decorrelation under assumptions of weak mixing type. In the case of Ising's model at least, I expect ρ-mixing to remain true—even for arbitrary shapes—as soon as $T>T_c$, because on the one hand Theorem 0.1.9 proves ρ-mixing between parallel hyperplanes, while on the other hand ρ-mixing seems to hold also in the 'opposite extreme case' when I and J make a check pattern.

By the way, it is likely that the natural condition should not be weak mixing itself but rather something like *strong mixing for cubes* (often called merely *strong mixing* [†], which means that when a boundary condition is fixed outside a cube of arbitrary edge, changing one spin on the

[*]. Note that in the case I and J are hyperplanes, we shew on page 98 that (QR) could be improved into

$$\{\vec{\omega}_{\{0\}\times\mathbb{Z}^{n-1}} : \vec{\omega}_{\{x\}\times\mathbb{Z}^{n-1}}\} \leqslant e^{-\psi' x}. \tag{QU}$$

[†]. Strong mixing *stricto sensu* is actually the same as complete analyticity, so that mathematicians have got used to undermeaning "for cubes"—but strong mixing for cubes is strictly weaker than complete analyticity! [53, § 2].

boundary has an effect in total variation which decreases exponentially with the distance to the spin changed. In fact it has been proved [54] that in dimension 2, weak mixing is equivalent to strong mixing. ♡

Proof of Theorem 5.1.1. Theorem 5.1.1 will be a direct consequence of the work of Chapter 3 as soon as we show that, denoting by $*$ the natural σ-metalgebra of the system (i.e. the σ-metalgebra generated by the ω_i), for all distinct $i,j \in \mathbb{Z}^n$, one has

$$\{\omega_i : \omega_j\}_* \leqslant c_0 C' e^{-\psi' dist(i,j)} \wedge k_0 \tag{QV}$$

for some explicit $c_0 < \infty$ and $k_0 < 1$ only depending on n and T. Then indeed, Proposition 3.6.6 yields

$$\begin{aligned}\{\vec{\omega}_I : \vec{\omega}_J\} &\leqslant \sum_{\substack{\delta\in\mathbb{Z}^n\\ |\delta|\geqslant dist(I,J)}} c_0 C' e^{-\psi'|\delta|} = c_0 C' \sum_{d=dist(I,J)}^{\infty} \#\{\delta\in\mathbb{Z}^n : \ |\delta| = d\} e^{-\psi' d}\\ &\overset{dist(I,J)\to\infty}{\sim} c_0 C' \sum_{d=dist(I,J)}^{\infty} \frac{2^n d^{n-1}}{(n-1)!} e^{-\psi' d} \sim \frac{c_0 C' 2^n}{(n-1)!} dist(I,J)^{n-1} e^{-\psi' dist(I,J)}\\ &= e^{-(\psi'+o(1))dist(I,J)},\end{aligned} \tag{QW}$$

whence Point (i). Moreover, since

$$\sum_{\substack{\delta\in\mathbb{Z}^n\smallsetminus\{0\}\\ |\delta|\leqslant dist(I,J)}} c_0 C' e^{-\psi' dist(i,j)} < \infty, \tag{QX}$$

Point (ii) follows from Lemma 3.6.8, and finally (iii) is a consequence of § 3.4.b about subjective results.

So, we have to prove (QV). Let $\vec{\hat{\omega}}_K \in \{\pm 1\}^K$, $K \subset \mathbb{Z}^n$, be some arbitrary boundary condition, and denote by $\mathbf{P}_{\text{con}}$ the associated law, that is, $\mathbf{P}_{\text{con}} = \mathbf{P}[\cdot|\vec{\omega}_K = \vec{\hat{\omega}}_K]$; our goal is to show that under $\mathbf{P}_{\text{con}}$, for all distinct $i,j \in \mathbb{Z}^n$, one has $\{\omega_i : \omega_j\} \leqslant c_0 C' e^{-\psi' dist(i,j)} \wedge k_0$.

The result is immediate if $i \in K$, resp. $j \in K$ (since then ω_i, resp. ω_j, is constant and thus independent of everything), so we assume $i,j \notin K$. We begin with observing that if K is the set $N(i)$ of all the neighbours of i, equilibrium at i implies that, whatever the boundary condition may be:

$$\mathbf{P}_{\text{con}}[\omega_i = -1], \mathbf{P}_{\text{con}}[\omega_i = +1] \geqslant (e^{4n/T} + 1)^{-1} \tag{QY}$$

—the extremal cases being when $\vec{\hat{\omega}}_{N(i)} \equiv +1$, resp. $\vec{\hat{\omega}}_{N(i)} \equiv -1$. Now in the general case $K \subset \mathbb{Z}^n \smallsetminus \{i\}$, $Law_{\text{con}}[\omega_i]$ is an average of laws of the form $Law(\omega_i|\vec{\omega}_{N(i)} = \vec{\hat{\omega}}_{N(i)})$, so that (QY) remains valid. Similarly, equilibrium on $\{i,j\}$ gives that for all $a,b \in \{\pm 1\}$,

$$\mathbf{P}_{\text{con}}[\omega_i = a \text{ and } \omega_j = b] \geqslant (e^{8n/T} + 2e^{(4n+2)/T} + 1)^{-1}. \tag{QZ}$$

Now, recall that the correlation level between two two-ranged variables can be computed by Formula (BD), where $|p_a^b - p_a p^b|$ is also $\beta(X,Y)/2$. Thus the bound "$\{\omega_i : \omega_j\} \leqslant C_0 e^{-\psi' dist(i,j)}$" is a direct consequence of Theorem 0.1.7, with

$$c_0 = \frac{1/2}{(e^{4n/T}+1)^{-1}(1-(e^{4n/T}+1)^{-1})} = \tanh(4n/T) + 1. \tag{RA}$$

It remains to prove the bound "$\{\omega_i : \omega_j\} \leqslant k_0$". We will use the following corollary of (BD):

5.1.5 Lemma. *With the notation of Remark 1.2.2, there exists a, b in the respective ranges of X, Y such that*

$$\{X : Y\} \leqslant 1 - 4p_a^b. \tag{RB}$$

♣

Proof of Lemma 5.1.5. The difference $p_a^b - p_a p^b$ gets its sign changed whenever a, resp. b, changes, so there are some a and b for which this value is nonpositive; moreover, denoting by $\{a, a'\}$ and $\{b, b'\}$ the respective ranges of X and Y, $p_{a'}^{b'} - p_{a'} p^{b'}$ is also nonpositive. Now one has

$$\frac{p_a p^b}{\sqrt{p_a p_{a'} p^b p^{b'}}} \times \frac{p_{a'} p^{b'}}{\sqrt{p_a p_{a'} p^b p^{b'}}} = 1, \tag{RC}$$

so that either $p_a p^b$ or $p_{a'} p^{b'}$ is $\leqslant \sqrt{p_a p_{a'} p^b p^{b'}}$. Up to changing notation we can assume that it is $p_a p^b$, and then

$$\{X : Y\} = \frac{|p_a^b - p_a p^b|}{\sqrt{p_a p_{a'} p^b p^{b'}}} = \frac{p_a p^b - p_a^b}{\sqrt{p_a p_{a'} p^b p^{b'}}} \leqslant 1 - \frac{p_a^b}{\sqrt{p_a p_{a'} p^b p^{b'}}} \leqslant 1 - 4p_a^b. \tag{RD}$$

♠

Combining Lemma 5.1.5 with (QZ), we then get the desired bound, with

$$k_0 = 1 - 4(e^{8n/T} + 2e^{(4n+2)/T} + 1)^{-1} < 1. \tag{RE}$$

♠

Formula (QV) is also what we need to apply the results of Chapter 4. Indeed, denoting $\varepsilon(z) := \{X_i : X_{i+z}\}_*$, it gives that $\sum_{z \in \mathbb{Z}^n} \varepsilon(z) < \infty$ with $\varepsilon(z) < 1$ as soon as $z \neq 0$, so that Theorems 4.1.8 and 4.2.13 yield respectively:

5.1.6 Theorem. *In completely analytical regime, the spins Ising's model satisfies the central limit theorem, in the sense that the conclusions of Theorem 4.1.8 hold for them.* ♣

5.1.7 Theorem. *In completely analytical regime, the Glauber dynamics for Ising's model has a (strictly) positive spectral gap, and this remains valid uniformly if one fixes a 'boundary condition' on the spins of some $K \subset \mathbb{Z}^n$.* ♣

5.1.8 Remark. As I told in Chapter 4, results of these kinds have already been studied by other methods (see e.g. [10, 22] for the CLT and [52] for the spectral gap). For the standard Ising model in completely analytical regime, which is "very nice", these previous works apply well, so the two theorems above are not new. They are interesting however because of the new method used to prove them, which is quite direct and likely to apply to a broader class of models. Such models will be presented in the sequel of this chapter. ♡

5.1.b Generalizations of Ising's model

The previous results can be adapted to several kinds of generalizations of Ising's model. Let us expose some of them.

Long-range Ising models

A physically important case is the *long-range Ising models* on $\mathbb{Z}^n$. In these models, the states space is unchanged, but the Hamiltonian H becomes

$$H(\vec{\omega}) = -\frac{1}{2}\sum_{i\neq j} J(j-i)\omega_i\omega_j, \tag{RF}$$

where $J\colon\ \mathbb{Z}^n\smallsetminus\{0\}\to\mathbb{R}$ is some symmetric function with non-compact support such that $J(z)\overset{|z|\to\infty}{=} O(|z|^{-(n+\alpha)})$ for some $\alpha>0$.

Let us state a decorrelation result for this class of models. The frame of the proof of the following proposition will work as well for the other generalizations of Ising's model.

5.1.9 Proposition. *There exists an temperature $T_1<\infty$ such that, provided $T\geqslant T_1$:*

(i) Equilibrium for the long-range Ising model is unique;
(ii) Uniformly in i,j, $\{\omega_i:\omega_j\}_*\overset{|j-i|\to\infty}{=} O(|j-i|^{-(n-\alpha)})$;
(iii) There exists some $k_0<1$ such that for all $i\neq j$, $\{\omega_i:\omega_j\}_*\leqslant k_0$.

♣

Proof. The principle of the proof consists in coupling two Glauber dynamics with different initial conditions. Recall that the Glauber dynamics is defined as follows: each spin has an independent clock ringing with rate 1, and when the clock of a spin rings, this spin is flipped so that its final state is drawn according to its equilibrium measure conditionnally to the state of all other spins. Namely, if the clock of spin i rings at time t, denoting as usual $\beta=T^{-1}$,

$$\mathbf{P}[\omega_i(t+)=+1] = \frac{\exp\big(\beta\sum_{j\neq i}J(j-i)\omega_j(t)\big)}{2\cosh\big(\beta\sum_{j\neq i}J(j-i)\omega_j(t)\big)} \tag{RG}$$

and $\mathbf{P}[\omega_i(t+)=-1]=1-\mathbf{P}[\omega_i(t+)=+1]$.

To couple the Glauber dynamics, we will assume that, rather than just "ringing" at time t, the clock of i is a Poisson process on $\mathbb{R}_+\times(0,1)$, points of which are denoted by (t,y). Then, if at time t the clock of spin i has a point (t,y), spin i flips to $+1$ if $y<\mathbf{P}[\omega_i(t+)=+1]$, resp. to -1 if $y\geqslant\mathbf{P}[\omega_i(t+)=+1]$.

Now, consider two Glauber dynamics $\vec{\omega}^-$ and $\vec{\omega}^+$ having the same Poisson process, but starting with different initial conditions. It will be convenient [‡] to assume that $\vec{\omega}^-(t=0)\leqslant\vec{\omega}^+(t=0)$ almost-surely: then, as we will see, for the coupled dynamics one has (a.s.) $\vec{\omega}^-(t)\leqslant\vec{\omega}^+(t)\ \forall t$. At time t, denote by $\Theta(t)$ the set of points where $\vec{\omega}^-$ and $\vec{\omega}^+$ differ:

$$\Theta(t)=\{i\in\mathbb{Z}^n:\ \big(\omega_i^-(t),\omega_i^+(t)\big)=(-1,+1)\}. \tag{RH}$$

When the clock at spin i rings at time t, three cases have to be distinguished:

1. If $y<\exp\big(\beta\sum_{j\neq i}J(j-i)\omega_j^-(t)\big)/2\cosh\big(\beta\sum_{j\neq i}J(j-i)\omega_j^-(t)\big)$, then both ω_i^+ and ω_i^- flip into state $+1$;
2. If $y>\exp\big(\beta\sum_{j\neq i}J(j-i)\omega_j^+(t)\big)/2\cosh\big(\beta\sum_{j\neq i}J(j-i)\omega_j^+(t)\big)$, then both ω_i^+ and ω_i^- flip into state -1;

[‡]. In the cases where interactions can be antiferromagnetic ($J<0$), monotonicity does not stand any more; the proof however remains valid with a heavier formalism, replacing "$>$" by "$\neq$" and putting absolute values at the right places.

3. If $\frac{\exp(\beta\sum_{j\neq i}J(j-i)\omega_j^-(t))}{2\cosh(\beta\sum_{j\neq i}J(j-i)\omega_j^-(t))} < y < \frac{\exp(\beta\sum_{j\neq i}J(j-i)\omega_j^+(t))}{2\cosh(\beta\sum_{j\neq i}J(j-i)\omega_j^+(t))}$, then ω_i^+ flips into state $+1$ while ω_i^- flips into state -1.

Denoting

$$\mathscr{J} := \sum_{z\in\mathbb{Z}^n\smallsetminus\{0\}} J(z), \tag{RI}$$

which is always finite by the assumption on J, the probability of each of the two first cases is bounded below by $e^{-\beta\mathscr{J}}/2\cosh(\beta\mathscr{J})$. The probability of the third case is

$$\frac{\sinh\big(2\beta\sum_{j\in\Theta(t)\smallsetminus\{i\}}J(j-i)\big)}{2\cosh\big(\beta\sum_{j\neq i}J(j-i)\omega_j^-(t)\big)\cosh\big(\beta\sum_{j\neq i}J(j-i)\omega_j^+(t)\big)}, \tag{RJ}$$

which is bounded above by $\beta\sum_{j\in\Theta(t)\smallsetminus\{i\}}J(j-i)$ thanks to the following computational

5.1.10 Lemma. *For $a\leqslant b$ two real numbers,*

$$\sinh(b-a)\leqslant(b-a)\cosh a\cosh b. \tag{RK}$$

♣

Proof. Making the change of variables $x=(a+b)/2, t=(b-a)/2$, we have to prove that for $x\in\mathbb{R}$, $t\geqslant 0$, one has:

$$\sinh(2t)\leqslant 2t\cosh(x-t)\cosh(x+t). \tag{RL}$$

If we consider the right-hand side of (RL) as a function of x, it is symmetric (since cosh is symmetric) and its logarithm is convex (since $\log\circ\cosh$ is convex, its derivative being the increasing function tanh), so its minimum is attained for $x=0$; thus it suffices to prove (RL) in that case, i.e. to prove that $\sinh(2t)\leqslant 2t\cosh^2 t$ for all $t\geqslant 0$. But $\sinh(2t)=2\sinh t\cosh t$, so we can simplify both sides by $2\cosh t$, and then it suffices to prove that $\sinh t\leqslant t\cosh t$, which is true since $\tanh t\leqslant t$ for all $t\geqslant 0$. ♠

Thanks to these estimates, we can define a process Markovian $\Theta^*(t)$ on $\mathfrak{P}(\mathbb{Z}^n)$ such that almost-surely, $\Theta^*(t)\supset\Theta(t)\ \forall t$. This process has the following law:

5.1.11 Definition. The law of Θ^* is defined thanks to independent Poissonian clocks indexed by $(\mathbb{Z}^n)^2$. For $i\neq j$ the clock (i,j) has rate $\beta J(j-i)$, while the clock (i,i) has rate $e^{-\beta\mathscr{J}}/\cosh(\beta\mathscr{J})$. At $t=0$ one has $\Theta^*(0)=\Theta(0)$. If at time t the clock (i,j) rings, with $j\neq i$, then:

- Either $i\in\Theta^*(t-)$ and then Θ^* changes so that $\Theta^*(t+)=\Theta^*(t-)\cup\{j\}$ [§];
- Or $i\notin\Theta^*(t-)$ and then Θ^* does not change.

On the other hand, if at time t the clock (i,i) rings, then Θ^* changes so that $\Theta^*(t+)=\Theta^*(t-)\smallsetminus\{i\}$. ♢

Let $\lambda := \beta\mathscr{J}-(e^{-\beta\mathscr{J}}/\cosh(\beta\mathscr{J}))$. If we take $\mathbf{E}[\#\Theta(t=0)]<\infty$ [¶], it is immediate that $\#\Theta^*(t)/e^{\lambda t}$ is a supermartingale. So, provided T is large enough so that $\lambda<0$, i.e.

$$\beta\mathscr{J}<\frac{e^{-\beta\mathscr{J}}}{\cosh(\beta\mathscr{J})}, \tag{RM}$$

[§]. Of course, if $j\in\Theta^*(t-)$ then Θ^* does actually not change.

[¶]. The general case where Θ^* can be infinite can be got from the finite case by passing to the limit, despite some technicalities of little interest.

the two processes $\vec{\omega}^-(t)$ and $\vec{\omega}^+(t)$ tend to be equal when $t \to \infty$; in particular they have the same equilibrium. That proves Point (i) of the Lemma, since any initial condition stands between the 'extreme' conditions $\vec{\omega}^-(t=0) \equiv -1$ and $\vec{\omega}^+(t=0) \equiv +1$.

Observe that the previous reasoning remains entirely valid if one reasons conditionally to some boundary condition of the form "$\vec{\omega}_K = \vec{\hat{\omega}}_K$", with the same condition on T.

Now we are turning to the correlation between two distant spins. Let $i \in \mathbb{Z}^n$ and let $\vec{\hat{\omega}}_K$ be some boundary condition on some $K \subset \mathbb{Z}^n \smallsetminus \{i\}$. Suppose T satisfies (RM); I want to compare the Glauber dynamics corresponding to the boundary condition "$\vec{\omega}_{K \uplus \{i\}} = (\vec{\hat{\omega}}_K, (+1)^{\{i\}})$"—where $(\vec{\hat{\omega}}_K, (+1)^{\{i\}})$ stands for the function on $K \uplus \{i\}$ which is equal to $\hat{\omega}$ on K and to $+1$ at i —with the Glauber dynamics corresponding to the boundary condition "$\vec{\omega}_{K \uplus \{i\}} = (\vec{\hat{\omega}}_K, (-1)^{\{i\}})$". In this frame, one defines the process Θ^* as previously, except that one imposes that $\Theta^*(t) \cap K = 0$ and $i \in \Theta^*(t)$ for all t. This time, it is the equilibrium behaviour of Θ^* which interests us. Denote by $\mathbf{P}_{\mathrm{eq}}$ the equilibrium law of Θ^*; for $j' \in \mathbb{Z}^n \smallsetminus K$, denote $\theta(j) := \mathbf{P}_{\mathrm{eq}}[j \in \Theta^*]$. Then θ satisfies the following discrete subelliptic equation with Dirichlet boundary conditions:

$$\begin{cases} \forall j \notin K \uplus \{i\} \quad \frac{e^{-\beta \mathscr{J}}}{\cosh(\beta \mathscr{J})} \theta(j) \leqslant \beta \sum_{\substack{i' \in \mathbb{Z}^n \smallsetminus K \\ i' \neq j}} J(i'-j)\theta(i'); \\ \forall k \in K \quad \theta(k) = 0; \qquad \theta(i) = 1. \end{cases} \tag{RN}$$

Define the convolution kernel a on $\mathbb{Z}^n$ by

$$\begin{cases} a(0) = 1; \\ \forall z \neq 0 \quad a(z) = -\frac{\cosh(\beta \mathscr{J})}{e^{-\beta \mathscr{J}}} \beta J(z), \end{cases} \tag{RO}$$

so that (RN) writes in the bulk:

$$a * \theta \leqslant 0. \tag{RP}$$

Writing $a =: \delta_0 - \tilde{a}$, Condition (RM) ensures that $\|\tilde{a}\|_{l^1} < 1$. Since $l^1(\mathbb{Z}^n)$ is a Banach algebra for the convolution operator $*$, with neutral element δ_0, it follows that a is invertible with inverse

$$a^{-*} = \delta_0 + \tilde{a} + \tilde{a} * \tilde{a} + \tilde{a} * \tilde{a} * \tilde{a} + \cdots. \tag{RQ}$$

Since $\tilde{a} \geqslant 0$, a^{-*} is nonnegative everywhere with $a^{-*}(0) > 0$. Therefore the function $F := (a^{-*}(0))^{-1} \delta_i * a^{-*}$ satisfies:

$$\begin{cases} \forall j \notin K \uplus \{i\} \quad (a * F)(j) = 0; \\ \forall k \in K \quad F(k) \geqslant 0; \qquad F(i) = 1. \end{cases} \tag{RR}$$

Comparing (RN) with (RR), since (RN) is subelliptic, we can apply a maximum principle to it [‖], which yields that $\theta \leqslant F$ everywhere. But $J(z) = O(|z|^{-(n+\alpha)})$, so by Lemma 5.5.7 in appendix, $F(j) = O(|j-i|^{-(n+\alpha)})$, and therefore

$$\mathbf{P}[\omega_j = +1 | \vec{\omega}_K = \vec{\hat{\omega}}_K, \omega_i = 1] - \mathbf{P}[\omega_j = -1 | \vec{\omega}_K = \vec{\hat{\omega}}_K, \omega_i = -1] = O(|j-i|^{-(n+\alpha)}), \tag{RS}$$

uniformly in $i, j, K, \vec{\hat{\omega}}_K$.

The end of the proof, namely deducing Point (ii) from (RS) and proving Point (iii), is then performed in the same way as to establish (QV) in the proof of Theorem 5.1.1. ♠

[‖]. The maximum principle is generally stated in a PDE context, see for instance [34, § 3.1], but it works exactly the same for discrete equations.

Thanks to Proposition 5.1.9, we can apply the results of Chapters 3 and 4. One gets the following

5.1.12 Theorem. *For the long-range Ising model on $\mathbb{Z}^n$ at $T \geqslant T_1$,*

(i) For all disjoint $I, J \subset \mathbb{Z}^n$, uniformly in I, J, one has an estimate

$$\{\vec{\omega}_I : \vec{\omega}_J\} \leqslant O(dist(I,J)^{-\alpha}), \tag{RT}$$

where the $O(\cdot)$ can be turned into an explicit constant only depending on $\mathscr{J}$ and T. Moreover, there exists some $k < 1$ (still explicit and only depending on $\mathscr{J}$ and T) such that for all disjoint $I, J \subset \mathbb{Z}^n$,

$$\{\vec{\omega}_I : \vec{\omega}_J\} \leqslant k. \tag{RU}$$

(ii) The spins satisfies the central limit theorem, in the sense that the conclusions of Theorem 4.1.8 hold for them.

(iii) The Glauber dynamics has a positive spectral gap.

(iv) Points (i) and (iii) remain valid uniformly under any law of the form $\mathbf{P}[\cdot|\vec{\omega}_K = \vec{\hat{\omega}}_K]$, for $K \subset \mathbb{Z}^n$ and $\vec{\hat{\omega}}_K \in \{\pm 1\}^K$ a 'boundary condition' on K.

♣

Proof. The proof is the same as the work done in the previous subsection. The only difference is to prove (RT), which follows from the following computation: denoting $D := dist(I,J)$, one has that, when $D \to \infty$,

$$\sum_{\substack{z \in \mathbb{Z}^n \\ |z| \geqslant D}} \frac{1}{|z|^{n+\alpha}} \leqslant \sum_{d=D}^{\infty} \frac{\#\{z \in \mathbb{Z}^n : \ |z| = d\}}{d^{n+\alpha}} = \sum_{d=D}^{\infty} \frac{O(d^{n-1})}{|d^{n+\alpha}|} = O\Big(\sum_{d=D}^{\infty} \frac{1}{d^{1+\alpha}}\Big) = O(D^{-\alpha}). \tag{RV}$$

♠

Spin glasses

Spin glasses are another generalization of Ising's model. In these models, the interaction constants are not invariant by translation any longer. The Hamiltonian writes

$$H(\vec{\omega}) = -\frac{1}{2} \sum_{i \neq j} J(i,j)\omega_i \omega_j \tag{RW}$$

(with $J(j,i) = J(i,j)$), where the $J(i,j)$ themselves are random. We make the following assumptions on the interaction constants:

5.1.13 Assumption. For distinct unordered pairs $\{i,j\}$, all the $J(i,j)$ are independent. Moreover, $J(i,j)$ is distributed according to some law $P_J^{(j-i)}$ only depending on $(j-i)$ [*]. We will assume that all the $P_J^{(z)}$ have bounded support, and we denote by $J_\infty(z)$ the smallest number such that $P_J^{(z)}[|J| \leqslant J_\infty(z)] = 1$. ♢

5.1.14 Remark. Here the $J(i,j)$ can be negative, which corresponds to antiferromagnetic interactions. ♡

[*]. Observe that one has necessarily $P_J^{(-z)} = P_J^{(z)}$ for all z; in particular the function $J_\infty : \mathbb{Z}^n \setminus \{0\} \to \mathbb{R}_+$ shall always be symmetric.

☛ *In spin glass models, there are two levels of randomness: first to fix the $J(i,j)$, next to take $\vec{\omega}$ according to the Gibbs measure associated to H. When both levels of randomness are taken into consideration, one speaks of* annealed *law. Here I am only interested in the* quenched *laws, which deal with the second level of randomness for fixed $J(i,j)$. I will write sentences beginning with "for almost-all quenched systems", which mean that what follows is valid for almost-all Gibbs measures when the $J(i,j)$ are taken randomly according to Assumption 5.1.13.*

The machinery exposed above still works for spin glass models. We obtain the

5.1.15 Theorem. *Suppose that when $|z| \to \infty$, $J_\infty(z)$ decreases at least as fast as $O(|z|^{-(n+\alpha)})$ for some $\alpha > 0$. Then there is a $T_1 < \infty$ such that, for the spin glass model on $\mathbb{Z}^n$ at $T \geqslant T_1$, for almost-all quenched systems,*

(i) If $J_\infty(z) = O(|z|^{-(n+\alpha)})$, then for all disjoint $I, J \subset \mathbb{Z}^n$, uniformly in I, J, one has an estimate

$$\{\vec{\omega}_I : \vec{\omega}_J\} \leqslant O(dist(I,J)^{-\alpha}). \tag{RX}$$

If moreover $J_\infty(z)$ has exponential decay (see Definition 5.5.4 in the appendix), then the right-hand side of (RX) can even be replaced by "$\theta(dist(I,J))$" for some function $\theta(\cdot)$ with exponential decay.
In both cases, there exists some $k < 1$ such that for all disjoint $I, J \subset \mathbb{Z}^n$,

$$\{\vec{\omega}_I : \vec{\omega}_J\} \leqslant k. \tag{RY}$$

(ii) Points (ii)–(iv) of Theorem 5.1.12 hold.

♣

Synthetic vocabulary

For all the models considered in this section, the techniques used and the results stated walked along the same lines. First, one establishes a bound $\{X_i : X_j\}_* \leqslant \varepsilon(j-i) \wedge k_0$ for all $i \neq j$, for some sufficiently rapidly decreasing function $\varepsilon\colon\ \mathbb{Z}^n \to [0,1]$ and some $k_0 < 1$. Then, one applies the results of Chapters 3 and 4, which yield maximal decorrelation for distant bunches of spins (which is sometimes called *(interlaced) ρ^*-mixing*) with uniformly non-full correlation between any two disjoint bunches of spins (which is sometimes denoted "$\rho^*(1) < 1$"), central limit theorem, and spectral gap for the Glauber dynamics.

Since this method will be used again in the following sections, it will be convenient to introduce some synthetic vocabulary:

5.1.16 Definition. If a spin model (spins can have arbitrary range) $\vec{X}$ on $\mathbb{Z}^n$ satisfies some bound "$\{X_i : X_j\}_* \leqslant \varepsilon(j-i) \wedge k_0$" for all distinct $i, j \in \mathbb{Z}^n$, with $\sum_{z \in \mathbb{Z}^n \smallsetminus \{0\}} \varepsilon(z) < \infty$ and $k_0 < 1$, we say that this model is *well-ρ-mixing*. According to our results, for such a model one has ρ^*-mixing with $\rho^*(1) < 1$, CLT and spectral gap.

Moreover,

(i) If $\varepsilon(z) = O(|z|^{-(n+\alpha)})$ when $|z| \to \infty$, then we say that the model is *α-polynomially ρ-mixing*. According to our results, in this case ρ^*-mixing is polynomial with rate α, i.e. Formula (RT) holds.

(ii) If $\varepsilon(z)$ has exponential decay (cf. Definition 5.5.4), then we say that the model is *exponentially ρ-mixing*. According to our results, in this case ρ^*-mixing has an exponential speed of decay (but not with the same rate as $\varepsilon(\cdot)$, cf. Remark 5.5.6), i.e. a formula similar to (QR) holds.

◇

5.2 Quadratic models

☛ *In this subsubsection, an arbitrary norm $|\cdot|$ on $\mathbb{Z}^n$ is fixed.*

5.2.1 Definition. In our *quadratic model*, the states space is $\Omega = \mathbb{R}^{\mathbb{Z}^n}$ for some $n \in \mathbb{N}^*$. For $\vec{\omega}_{\mathbb{Z}^n} \in \Omega$, $i \in \mathbb{Z}^n$, the real number ω_i will be called the *polarization of particle* i. Each particle i is submitted to two types of forces:

- A *pinning force*, preventing the particle from having a too large polarization, which derives from the quadratic potential $\omega_i^2/2$;
- *Interaction forces*: each particle $j \neq i$ exerts a force on i which tends to make the polarizations of particles i and j equal; this force derives from a quadratic potential $\gamma_{j-i}(\omega_j - \omega_i)^2/2$.

In other words, the Hamiltonian of the system is formally defined by

$$H(\vec{\omega}) = \frac{1}{2}\sum_{i\in\mathbb{Z}^n} \omega_i^2 + \frac{1}{4}\sum_{i\neq j} \gamma_{j-i}(\omega_j - \omega_i)^2, \tag{RZ}$$

where the γ_z, for $z \in \mathbb{Z}^n \smallsetminus \{0\}$, are nonnegative numbers which we impose to satisfy the symmetry condition $\gamma_z = \gamma_{-z}$ for all z. Moreover we impose the that the sum of the γ_z is convergent, and we denote

$$\Gamma := \sum_{z\in\mathbb{Z}^n\smallsetminus\{0\}} \gamma_z < \infty. \tag{SA}$$

◇

The Hamiltonian H is a quadratic function of $\vec{\omega}$, so at fixed parameter β the (infinite-dimensional) random vector $\vec{\omega}$ will be Gaussian (and centered). Let us compute its covariance: the probability density of $\vec{\omega}$ w.r.t. the 'Lebesgue measure' on Ω is formally defined by

$$\frac{d\mathbf{P}_\beta[\vec{\omega}]}{\prod_{i\in\mathbb{Z}^n} d\omega_i} \propto \exp\big(\tfrac{1}{2}\omega^\top(\beta Q)\omega\big), \tag{SB}$$

where Q is the (infinite-dimensional) symmetric matrix defined by

$$\begin{cases} Q_{ij} := -\gamma_{j-i} & \text{for } i \neq j; \\ Q_{ii} := 1+\Gamma & \text{on the diagonal,} \end{cases} \tag{SC}$$

thus the covariance matrix of $\vec{\omega}$ is $(\beta Q)^{-1}$. So we have to compute Q^{-1}, the inverse matrix of Q. Since Q is a Toeplitz matrix (with n-dimensional indexes)[†], Q^{-1} —if it exists—will be of the same form. Now, knowing that it is a Toeplitz matrix, Q is described by the function $a_Q : \mathbb{Z}^n \to \mathbb{R}$ such that for all i, j, $Q_{ij} = a_Q(j-i)$. With this notation, (SC) rewrites:

$$\forall z \in \mathbb{Z}^n \qquad a_Q(z) = \mathbf{1}_{z=0}(1+\Gamma) - \mathbf{1}_{z\neq 0}\gamma_z. \tag{SD}$$

When coded by functions like a_Q, the multiplication of Toeplitz matrices becomes the convolution product:

$$\forall M, N \text{ Toeplitz} \qquad a_{MN} = a_M * a_N. \tag{SE}$$

[†]. Recall that saying that matrix Q is *Toeplitz* means that its entries Q_{ij} only depend on $(j-i)$.

So, Q^{-1} will be the Toeplitz matrix whose $a_{Q^{-1}}$ is the inverse of a_Q for the convolution product. Thanks to Condition (SA), such an inverse always exists: indeed we can write $a_Q = (1+\Gamma)(\delta_0 - \tilde{a}_Q)$, where $\tilde{a}_Q$ is a nonnegative function with $\|\tilde{a}_Q\|_{l^1} = \Gamma/(1+\Gamma) < 1$, so that a_Q is invertible with

$$a_Q^{-*} = (1+\Gamma)^{-1}(\delta_0 + \tilde{a}_Q + \tilde{a}_Q * \tilde{a}_Q + \tilde{a}_Q * \tilde{a}_Q * \tilde{a}_Q + \cdots). \tag{SF}$$

In the end, at parameter $\beta > 0$ the covariance matrix of $\vec{\omega}$ has entries:

$$\mathrm{Cov}(\omega_i, \omega_j) = \frac{a_{Q^{-1}}(j-i)}{\beta}. \tag{SG}$$

5.2.2 Remark. All the entries of $\mathrm{Cov}(\vec{\omega})$ are nonnegative, which reflects the fact that all the interaction forces are attractive. ♡

5.2.3 Remark. Since $\mathrm{Cov}(\vec{\omega})$ depends on β only through a constant factor, the behaviour of the system is exactly the same, up to a multiplicative constant, for all $\beta > 0$. Hence the study of correlations will not depend on β. ♡

☛ *In the sequel, we fix arbitrarily $\beta = 1$ and we denote $\mathbf{P}$ for $\mathbf{P}_{\beta=1}$.*

Since the model is Gaussian, by (SG) and Theorem 1.2.6 one has for all $i \neq j$:

$$\{\omega_i : \omega_j\} = \frac{a_{Q^{-1}}(j-i)}{a_{Q^{-1}}(0)}. \tag{SH}$$

Now we have the following claim, with an immediate key corollary:

5.2.4 Claim. *For all $i \neq j$, for all $K \subset \mathbb{Z} \smallsetminus \{i,j\}$,*

$$\{\omega_i : \omega_j\}_{\vec{\omega}_K} \leqslant \{\omega_i : \omega_j\}. \tag{SI}$$

♣

5.2.5 Corollary. *Denoting by $*$ the natural σ-metalgebra of the system, for all $i \neq j$,*

$$\{\omega_i : \omega_j\}_* = \{\omega_i : \omega_j\} = \frac{a_{Q^{-1}}(j-i)}{a_{Q^{-1}}(0)}. \tag{SJ}$$

♣

Proof. The proof of Claim 5.2.4 relies on the following claims:

5.2.6 Claim. *Up to an additive constant, $Law(\vec{\omega}_{\mathbb{Z}^n} | \vec{\omega}_K = \vec{\hat{\omega}}_K)$ is the same for all $\vec{\hat{\omega}}_K \in \mathbb{R}^K$, i.e. there exists a vector-valued function $\vec{\hat{\omega}}_K \mapsto offset(\vec{\hat{\omega}}_K) \in \mathbb{R}^{\mathbb{Z}^n}$ such that the law of $\vec{\omega}_{\mathbb{Z}^n}$ under $\mathbf{P}[\cdot | \vec{\omega}_K = \vec{\hat{\omega}}_K]$ is the same as the law of $\vec{\omega}_{\mathbb{Z}^n} + offset(\vec{\hat{\omega}}_K)$ under $\mathbf{P}[\cdot | \vec{\omega}_K \equiv 0]$.* ♣

5.2.7 Lemma. *For (X,Y) a two-dimensional centered Gaussian vector with X and Y non-degenerate,*

$$\{X : Y\} = \sqrt{\frac{\mathbf{E}[X^2]}{\mathbf{E}[Y^2]}}\, |\mathbf{E}[Y | X = 1]|. \tag{SK}$$

♣

5.2.8 Claim. *For $K \subset \mathbb{Z}^n$, the function offset defined in Claim 5.2.6 is nondecreasing, in the sense that each of the entries of $offset(\vec{\hat{\omega}}_K)$ is a nondecreasing function of each $\hat{\omega}_k$ for $k \in K$.* ♣

5.2.9 Claim. *For $i \in \mathbb{Z}^n$, $K \subset \mathbb{Z}^n \smallsetminus \{i\}$:*

$$\mathit{offset}(1^{\{i\}}, 0^K) \leqslant \mathit{offset}(1^{\{i\}}), \tag{SL}$$

where $(1^{\{i\}}, 0^K)$ stands for the function on $K \uplus \{i\}$ which is equal to 1 at i and to 0 on K, resp. $1^{\{i\}}$ stands for the function on $\{i\}$ mapping i to 1. ♣

Admit temporarily the claims. Let i, j be distinct points of $\mathbb{Z}^n$, let $K \subset \mathbb{Z}^n \smallsetminus \{i, j\}$ and let $\vec{\hat{\omega}}_K \in \mathbb{R}^K$; our goal is to compute $\{\omega_i : \omega_j\}$ under $\mathbf{P}[\cdot | \vec{\omega}_K = \vec{\hat{\omega}}_K]$. First, by Claim 5.2.6 we can suppose that $\vec{\hat{\omega}}_K \equiv 0$. Now under $\mathbf{P}[\cdot | \vec{\omega}_K \equiv 0]$, (ω_i, ω_j) is still Gaussian by the properties of Gaussian vectors, and it is centered by symmetry, therefore by Lemma 5.2.7, $\{\omega_i : \omega_j\}$ is equal to

$$\sqrt{\frac{\mathbf{E}[\omega_i^2 | \vec{\omega}_K \equiv 0]}{\mathbf{E}[\omega_j^2 | \vec{\omega}_K \equiv 0]}} \, |\mathbf{E}[\omega_j | \vec{\omega}_K \equiv 0 \text{ and } \omega_i = 1]| = \sqrt{\frac{\mathbf{E}[\omega_i^2 | \vec{\omega}_K \equiv 0]}{\mathbf{E}[\omega_j^2 | \vec{\omega}_K \equiv 0]}} (\mathit{offset}(1^{\{i\}}, 0^K) \cdot j) \tag{SM}$$

—one has indeed $\mathit{offset}(1^{\{i\}}, 0^K) \cdot j \geqslant 0$, since by Claim 5.2.8, $\mathit{offset}(1^{\{i\}}, 0^K) \geqslant \mathit{offset}(0^{\{i\} \uplus K}) \equiv 0$.

Now, taking $K = \emptyset$ in (SM), we find that under the law $\mathbf{P}$:

$$\{\omega_i : \omega_j\} = \sqrt{\frac{\mathbf{E}[\omega_i^2]}{\mathbf{E}[\omega_j^2]}} (\mathit{offset}(1^{\{i\}}) \cdot j), \tag{SN}$$

which is $\geqslant \sqrt{\mathbf{E}[\omega_i^2]/\mathbf{E}[\omega_j^2]} (\mathit{offset}(1^{\{i\}}, 0^K) \cdot j)$ by Claim 5.2.9. But up to switching the roles of i and j, we can assume that $\mathbf{E}[\omega_i^2]/\mathbf{E}[\omega_j^2] \geqslant \mathbf{E}[\omega_i^2 | \vec{\omega}_K \equiv 0]/\mathbf{E}[\omega_j^2 | \vec{\omega}_K \equiv 0]$, thus getting the desired result:

$$\{\omega_i : \omega_j\} \geqslant \sqrt{\frac{\mathbf{E}[\omega_i^2 | \vec{\omega}_K \equiv 0]}{\mathbf{E}[\omega_j^2 | \vec{\omega}_K \equiv 0]}} (\mathit{offset}(1^{\{i\}}, 0^K) \cdot j) = \{\omega_i : \omega_j\}_{\vec{\omega}_K}. \tag{SO}$$

♠

Proof of the claims.

Claim 5.2.6 – It is a well-known property of Gaussian vectors, which here is stated in an infinite-dimensional setting.

Claim 5.2.7 – Since (X, Y) is centered Gaussian, $Y^{\sigma(X)}$ is the orthogonal projection of the L^2 variable Y on $\mathbb{R}X$, so $\mathbf{E}[Y | X = x] \propto x$. Thus one has:

$$\mathbf{E}[XY] = \int x \, \mathbf{E}[Y | X = x] \, d\mathbf{P}[X = x] = \int x^2 \, \mathbf{E}[Y | X = 1] \, d\mathbf{P}[X = x] = \mathbf{E}[Y | X = 1] \, \mathbf{E}[X^2]. \tag{SP}$$

But for such a Gaussian vector, Theorem 1.2.6 gives that

$$\{X : Y\} = \frac{|\mathrm{Cov}(X, Y)|}{\mathrm{Sd}(x)\mathrm{Sd}(y)} = \frac{|\mathbf{E}[XY]|}{\sqrt{\mathbf{E}[X^2]\mathbf{E}[Y^2]}}, \tag{SQ}$$

which combined with (SP) gives (SK).

Claim 5.2.8 – First, notice that $\mathbf{E}[\vec{\omega}_{\mathbb{Z}^n} | \vec{\omega}_K \equiv 0] = \vec{0}$, so that

$$\mathit{offset}(\vec{\hat{\omega}}_K) = \mathbf{E}[\vec{\omega} | \vec{\omega}_K = \vec{\hat{\omega}}_K]. \tag{SR}$$

Now, allowing temporarily β to vary again, by the properties of Gaussian vectors, the vector-valued variable $\mathbf{E}_\beta[\vec{\omega}_{\mathbb{Z}^n} | \vec{\omega}_K = \vec{\hat{\omega}}_K]$ is Gaussian with constant expectation and covariance matrix

proportional to β. Therefore, the common expectation of all these laws is equal to the constant value of $\vec{\omega}_{\mathbb{Z}^n}$ for $\beta = 0$, which is the $\vec{\omega}$ minimising H under the constraint "$\vec{\omega}_K = \vec{\hat{\omega}}_K$":

$$\mathit{offset}(\vec{\hat{\omega}}_K) = \operatorname*{argmin}_{\vec{\omega}_K = \vec{\hat{\omega}}_K} H(\vec{\omega}). \tag{SS}$$

Since it minimizes energy, the state $\mathit{offset}(\vec{\omega}_K)$ is at equilibrium outside K. In other words, it is the solution of the following subelliptic system:

$$\begin{cases} \forall i \in \mathbb{Z}^n \smallsetminus K \quad -\omega_i + \sum_{j \neq i} \gamma_{j-i}(\omega_j - \omega_i) = 0; \\ \forall i \in K \quad \omega_i = \hat{\omega}_i. \end{cases} \tag{ST}$$

(That system is clearly subelliptic because the pinning and interaction forces all are attractive). By the maximum principle, the solution of (ST) is an increasing function of the boundary condition, which was our claim.

Claim 5.2.9 – Denote $\vec{\omega}^1_{\mathbb{Z}^n} = \mathit{offset}(1^{\{i\}})$. Since $\mathit{offset}(0^{\{i\}}) = 0^{\mathbb{Z}^n}$, by Claim 5.2.8 one has $\vec{\omega}^1_{\mathbb{Z}^n} \geqslant 0^{\mathbb{Z}^n}$. Since obviously $\vec{\omega}^1_i = 1$, one has even $\vec{\omega}^1_{\mathbb{Z}^n} \geqslant (1^{\{i\}}, 0^{\mathbb{Z}^n \smallsetminus \{i\}})$. In particular, $\vec{\omega}^1_{\{i\} \uplus K} \geqslant (1^{\{i\}}, 0^K)$; therefore, using again Claim 5.2.8,

$$\mathit{offset}(\vec{\omega}^1_{\{i\} \uplus K}) \geqslant \mathit{offset}(1^{\{i\}}, 0^K). \tag{SU}$$

Now, we defined $\vec{\omega}^1_{\mathbb{Z}^n}$ as $\mathit{offset}(1^{\{i\}})$, so by Formula (ST) it satisfies

$$-\omega^1_{i'} + \sum_{j \neq i'} \gamma_{j-i'}(\omega_j - \omega'_i) = 0 \tag{SV}$$

for all $i' \in \mathbb{Z}^n \smallsetminus \{i\}$, hence *a fortiori* for all $i' \in \mathbb{Z}^n \smallsetminus (\{i\} \uplus K)$. Since moreover $\vec{\omega}^1$ obviously coincides with $\vec{\omega}^1_{\{i\} \uplus K}$ on $\{i\} \uplus K$, this implies, by Formula (ST) again, that

$$\mathit{offset}(\vec{\omega}^1_{\{i\} \uplus K}) = \vec{\omega}^1_{\mathbb{Z}^n} = \mathit{offset}(1^{\{i\}}). \tag{SW}$$

So, (SU) becomes "$\mathit{offset}(1^{\{i\}}) \geqslant \mathit{offset}(1^{\{i\}}, 0^K)$", what we wanted. ♠

Thanks to Corollary 5.2.5 our tensorization theorems give decorrelation results for the quadratic model:

5.2.10 Theorem. *Provided Condition (SA) holds:*

(i) The quadratic model is well-ρ-mixing, cf. Definition 5.1.16. If $\Gamma < 1$, one can be more specific about the property "$\rho^*(1) < 1$": for all disjoint $I, J \subset \mathbb{Z}^n$, $\{\vec{\omega}_I : \vec{\omega}_J\} \leqslant \Gamma$.

(ii) Moreover, if there is polynomial decay of interactions $\gamma_z = O(1/|z|^{n+\alpha})$, then the model is α-polynomially ρ-mixing, and if γ_z has exponential decay, then the model is exponentially ρ-mixing (but not with the same rate as γ_z in general).

♣

Proof. To prove Point (i), we have to show that

$$\sum_{z \in \mathbb{Z}^n \smallsetminus \{0\}} \frac{a_{Q^{-1}}(z)}{a_{Q^{-1}}(0)} \leqslant \Gamma \tag{SX}$$

—recall that we assumed $\Gamma < \infty$. We write:

$$\sum_{z \in \mathbb{Z}^n \smallsetminus \{0\}} \frac{a_{Q^{-1}}(z)}{a_{Q^{-1}}(0)} = \frac{\sum_{z \in \mathbb{Z}^n} a_{Q^{-1}}(z)}{a_{Q^{-1}}(0)} - 1. \tag{SY}$$

There, $\sum_{z\in\mathbb{Z}^n} a_{Q^{-1}}(z)$ is equal to 1: indeed, $a_{Q^{-1}}$ is the convolution inverse of a_Q, so by Fubini's theorem:

$$\sum_{z\in\mathbb{Z}^n} a_{Q^{-1}}(z) = \Big(\sum_{z\in\mathbb{Z}^n} a_Q(z)\Big)^{-1}, \tag{SZ}$$

where

$$\sum_{z\in\mathbb{Z}^n} a_Q(z) = \sum_{z\in\mathbb{Z}^n\smallsetminus\{0\}} \big(-\gamma(z)\big) + (1+\Gamma) = -\Gamma + 1 + \Gamma = 1. \tag{TA}$$

Now, by (SF), $a_{Q^{-1}}(0)$ is obviously bounded below by $(1+\Gamma)^{-1}$, so in the end:

$$\sum_{z\in\mathbb{Z}^n\smallsetminus\{0\}} \frac{a_{Q^{-1}}(z)}{a_{Q^{-1}}(0)} \leqslant \frac{1}{(1+\Gamma)^{-1}} - 1 = \Gamma. \tag{TB}$$

To prove Point (ii), we have to show that polynomial decay of γ_z implies polynomial decay of $a_{Q^{-1}}$ with the same exponent, resp. that exponential decay of γ_z implies exponential decay of $a_{Q^{-1}}$. This is achieved resp. by Lemmas 5.5.5 and 5.5.7 in the appendix. ♠

5.3 Nonlinear lattice of particles

In this section we will consider a model with continuous spins, but where interactions are non-linear, so that we cannot use the properties of Gaussian variables. One has a lattice of particles indexed by $\mathbb{Z}^n$ (equipped with its l^1 graph structure), each particle i being described by its "polarization" $\omega_i \in \mathbb{R}$. Each particle is submitted to a pinning force deriving from a potential V, and to interaction forces with its neighbours, the interactions deriving from a potential W. In other words, the Hamiltonian is formally

$$H(\vec{\omega}) = \sum_{i\in\mathbb{Z}} V(\omega_i) + \frac{1}{2}\sum_{i\sim j} W(\omega_j - \omega_i). \tag{TC}$$

We make the following assumptions:

5.3.1 Assumption. Both V and W are convex; moreover V is uniformly strictly convex and the Hessian of W is bounded, i.e. there exist constants $v_* > 0$ and $w_* < \infty$ such that for all $x \in \mathbb{R}$, $v_* \leqslant V''(x)$ and $W''(x) \leqslant w^*$. ♢

We are interested in the equilibrium state of the system at some inverse temperature $0 < \beta < \infty$. (In the sequel we suppose that β is fixed).

Let $i \neq j \in \mathbb{Z}$, $K \subset \mathbb{Z}\smallsetminus\{i,j\}$ and $\vec{\hat{\omega}}_K \in \mathbb{R}^K$; we want to study the law of (ω_i,ω_j) under the law $\mathbf{P}[\cdot|\vec{\omega}_K = \vec{\hat{\omega}}_K]$. Then, the probability distribution of the system is formally described by

$$d\mathbf{P}(\omega_i,\omega_j,\vec{\omega}_{K^c\smallsetminus\{i,j\}}) \propto \exp\big(-\beta H(\omega_i,\omega_j,\vec{\omega}_{K^c\smallsetminus\{i,j\}},\vec{\hat{\omega}}_K)\big). \tag{TD}$$

Our assumptions ensure that the function $H(\cdot,\cdot,\cdot,\vec{\hat{\omega}}_K)$ is uniformly convex, so that the equilibrium exists and is unique.

For the sequel, we need to recall the definition of the W_∞ Wasserstein distance:

5.3.2 Definition (see also [19])**.** For μ_1,μ_2 two measures on some metric space (X,d), "$W_\infty(\mu_1,\mu_2) \leqslant \varepsilon$" means that there exists a probability measure γ on E^2 such that the two respective marginals of γ are μ_1 and μ_2 and such that $d(x_1,x_2) \leqslant \varepsilon \quad \gamma$-a.s.. This defines a (possibly infinite) distance on the probability measures on E. ♢

The fundamental lemma of this subsection is the following

5.3.3 Claim. *For $\hat{\omega}_j \in \mathbb{R}$, denote by $\mu(\hat{\omega}_j)$ the law of ω_i under $\mathbf{P}[\cdot|\vec{\omega}_{K\uplus\{j\}} = (\vec{\hat{\omega}}_K, \hat{\omega}_j)]$. There exists a function $\varepsilon\colon \mathbb{Z} \to [0,1]$ with $\varepsilon(d) < 1$ as soon as $d > 0$ and $\varepsilon(d) \overset{d\to\infty}{\leqslant} Ce^{-\psi d}$ for some $\psi > 0$ and $C < \infty$, such that*

$$\forall \hat{\omega}_j^1, \hat{\omega}_j^2 \in \mathbb{R} \qquad W_\infty(\mu(\hat{\omega}_j^1), \mu(\hat{\omega}_j^2)) \leqslant \varepsilon(|j-i|)|\hat{\omega}_j^2 - \hat{\omega}_j^1|. \tag{TE}$$

♣

Proof. The proof relies on the 'explicit' construction of a coupling measure γ between $\mu(\hat{\omega}_j^1)$ and $\mu(\hat{\omega}_j^2)$. To do that, we will construct $\mathbf{P}[\cdot|\vec{\omega}_{K\uplus\{j\}} = (\vec{\hat{\omega}}_K, \hat{\omega}_j)]$ thanks to a reversible Fokker–Planck dynamics, and then couple the dynamics for $\hat{\omega}_j^1$ and $\hat{\omega}_j^2$.

We define the Fokker–Planck dynamics thanks to independent white noises $(dB_t^{i'})_{t\in\mathbb{R}}$ for $i' \in \mathbb{Z}^n \smallsetminus (K \uplus \{j\})$. The motion of point i' is defined by:

$$d\omega_{i'} = -\beta(V'(\omega_{i'}) + \sum_{i''\sim i'} W'(\omega_{i'} - \omega_{i''})) + \sqrt{2}\, dB_t^{i'}, \tag{TF}$$

with the boundary condition $\vec{\omega}_{K\uplus\{j\}} = \vec{\hat{\omega}}_{K\uplus\{j\}}$ for all times. Coupling then consists in taking the same noise for the two processes. The initial condition is not very important since it is asymptotically forgotten, so we will suppose that the two systems have been coupled for an infinite time, so that at any time both systems follow their equilibrium law. We denote by $\vec{\omega}^1(t)$ the system correponding to the boundary condition "$\vec{\omega}_{K\uplus\{j\}} = (\vec{\hat{\omega}}_K, \hat{\omega}_j^1)$", resp. by $\vec{\omega}^2(t)$ the system correponding to the other boundary condition. We denote $\Delta_{i'}(t) := \omega_{i'}^2(t) - \omega_{i'}^1(t)$. Then when the dynamics are coupled, $\vec{\Delta}$ evolves according to the following equation:

$$d(\Delta_{i'}) = -\beta\big[(V'(\omega_{i'}^2) - V'(\omega_{i'}^1)) + \sum_{i''\sim i'} (W'(\omega_{i'}^2 - \omega_{i''}^2) - W'(\omega_{i'}^1 - \omega_{i''}^1))\big]. \tag{TG}$$

Obviously the right-hand side is not a deterministic function of $\vec{\Delta}(t)$, but it can nonetheless be written as

$$-\beta[v(i',t)\Delta_{i'}(t) + w(i',i'',t)(\Delta_{i'}(t) - \Delta_{i''}(t))], \tag{TH}$$

for some $v(i',t)$ and $w(i',i",t)$ satisfying

$$v(i',t) \geqslant v_* \quad \text{and} \tag{TI}$$

$$0 \leqslant w(i',i",t) \leqslant w^* \tag{TJ}$$

by Assumption 5.3.1. Moreover, one has the boundary conditions:

$$\forall t \qquad \begin{cases} \vec{\Delta}_K \equiv 0; \\ \Delta_j = \hat{\omega}_j^2 - \hat{\omega}_j^1. \end{cases} \tag{TK}$$

So, $\vec{\Delta}$ is the solution of some discrete 'damped heat equation', whose coefficients can vary along time though having to satisfy bounds (TI) and (TJ). Such an equation has no stationary solution *stricto sensu*; however there exists some $\vec{\Delta}^+$ such that

$$\vec{\Delta}(t) \leqslant \vec{\Delta}^+ \qquad \Rightarrow \qquad \forall t' \geqslant t \quad \vec{\Delta}(t') \leqslant \vec{\Delta}^+; \tag{TL}$$

namely, this $\vec{\Delta}^+$ is defined as the solution of the following system of equations: $\vec{\Delta}^+_K \equiv 0$, $\Delta^+_j = \hat{\omega}^2_j - \hat{\omega}^1_j$, and for all $i' \notin K \uplus \{j\}$,

$$0 = -v_* \Delta^+_{i'} + \sum_{i'' \sim i'} \mathbf{1}_{\Delta^+_{i''} \geqslant \Delta^+_{i'}} w^* (\Delta^+_{i''} - \Delta^+_{i'}). \tag{TM}$$

One has similarly that

$$\vec{\Delta}(t) \geqslant \vec{0} \qquad \Rightarrow \qquad \forall t' \geqslant t \quad \vec{\Delta}(t') \geqslant \vec{0}. \tag{TN}$$

Consequently, I claim that for all t one has

$$\vec{0} \leqslant \vec{\Delta}(t) \leqslant \vec{\Delta}^+ : \tag{TO}$$

indeed if the initial condition of the system satisfies (TO), then that property remains valid for all subsequent times; now, as I told, initial conditions are asymptomatically forgotten, so in fact (TO) is always satisfied.

One has the following control on $\vec{\Delta}^+$:

5.3.4 Claim. *There exists a function $\varepsilon\colon \mathbb{Z} \to [0,1]$ with $\varepsilon(d) < 1$ as soon as $d > 0$ and $\varepsilon(d) \overset{d\to\infty}{\leqslant} Ce^{-\psi d}$ for some $\psi > 0$ and $C < \infty$, such that*

$$\forall i \in \mathbb{Z}^n \qquad \Delta^+_i \leqslant \varepsilon(|j-i|). \tag{TP}$$

Moreover, the function ε does not depend on K nor on j. ♣

Combining (TO) with Claim 5.3.4 ends the proof of Claim 5.3.3. ♠

Proof of Claim 5.3.4. First, notice that Equation (TM) satisfies a maximum principle, so we know in advance that Δ^+ is uniquely defined with $0 \leqslant \Delta^+ \leqslant 1$ everywhere.

For $i' \sim i''$, denote $w_{i'}(i'') := \mathbf{1}_{\Delta^+_{i''} \geqslant \Delta^+_{i'}} w^*$. Then (TM) can be rewritten into:

$$\Delta_{i'} = \sum_{i'' \sim i'} \frac{w_{i'}(i'')}{v_* + \sum_{i'' \sim i'} w_{i'}(i'')} \times \Delta_{i''} + \frac{v_*}{v_* + \sum_{i'' \sim i'} w_{i'}(i'')} \times 0. \tag{TQ}$$

Now I define the following Markov chain on $\mathbb{Z}^n \uplus \{\partial\}$, ∂ denoting a cemetery point:

5.3.5 Definition.

- If at some time the particle is on some point i' of $\mathbb{Z}^n \smallsetminus (K \uplus \{j\})$, at next time it jumps onto the neighbour i'' of i' with probability $w_{i'}(i'')/(v_* + \sum_{i'' \sim i'} w_{i'}(i''))$, and it jumps onto ∂ with probability $v_*/(v_* + \sum_{i'' \sim i'} w_{i'}(i''))$;
- If the particle is somewhere in $K \uplus \{\partial, j\}$ at some time, then it does not move any more.

♢

Call $(X_t)_{t \in \mathbb{N}}$ such a Markov chain and denote by $\mathscr{L}$ its generator. It is clear that with probability one, X_t eventually remains at some point of $K \uplus \{\partial, i\}$. Extend Δ^+ to $\mathbb{Z}^n \uplus \{\partial\}$ by setting $\Delta^+_\partial = 0$; then, (TQ) merely means that Δ^+ is $\mathscr{L}$-harmonic, and it follows that

$$\Delta^+_i = \mathbf{E}[f(X_\infty) | X_0 = i]. \tag{TR}$$

Thus, to bound above Δ_i^+ I write that

$$\begin{aligned}\mathbf{E}[f(X_\infty)|X_0=i] &= \sum_{\substack{i=i_0\sim\cdots\sim i_t=j\\ i_1,\dots,i_{t-1}\notin K\uplus\{j\}}} \prod_{u=0}^{t-1} \frac{w_{i_u}(i_{u+1})}{v_*+\sum_{i''\sim i_u} w_{i'}(i'')} \\ &\leqslant \sum_{\substack{i=i_0\sim\cdots\sim i_t=j\\ i_1,\dots,i_{t-1}\neq j}} \prod_{u=0}^{t-1} \frac{w_{i_u}(i_{u+1})}{\sum_{i''\sim i_u} w_{i'}(i'')}\Big(\frac{2dw^*}{2dw^*+v_*}\Big)^t \\ &\leqslant \Big(\frac{2dw^*}{2dw^*+v_*}\Big)^{|j-i|} \underbrace{\sum_{\substack{i=i_0\sim\cdots\sim i_t=j\\ i_1,\dots,i_{t-1}\neq j}} \prod_{u=0}^{t-1} \frac{w_{i_u}(i_{u+1})}{\sum_{i''\sim i_u} w_{i'}(i'')}}_{\leqslant 1} \leqslant \Big(\frac{2dw^*}{2dw^*+v_*}\Big)^{|j-i|}. \end{aligned} \tag{TS}$$

♠

From Claim 5.3.3, we take the following

5.3.6 Corollary. *For a Lipschitzian function $f\colon \mathbb{R}\to\mathbb{R}$, denote by $\|f\|_{Lip}$ the optimal Lipschitz constant for f. On $\overline{L}^2(\omega_i)$, define the (possibly infinite) norm $\|\cdot\|_{Lip}$ such that $\|f(\omega_i)\|_{Lip}=\|f\|_{Lip}$* [‡]*; denote by $\overline{Lip}(\omega_i)$ the corresponding Banach space.*

Then under the law $\mathbf{P}[\cdot|\vec{\omega}_K=\vec{\hat{\omega}}_K]$, the map $\pi_{\omega_j\omega_i}$ defined by (AJ) is $\varepsilon(|j-i|)$-contracting when seen as an application from $\overline{Lip}(\omega_i)$ into $\overline{Lip}(\omega_j)$. ♣

Consequently, the map $\pi_{\omega_i\omega_j\omega_i}\colon \overline{Lip}(\omega_i)\to\overline{Lip}(\omega_i)$ is $\varepsilon(|j-i|)^2$-contracting. But the canonical embedding $\overline{Lip}(\omega_i)\mapsto\overline{L}^2(\omega_i)$ is continuous as our hypotheses ensure that $Law(\omega_i)$ is uniformly log-concave, therefore for all $f\in\overline{Lip}(\omega_i)$ one has

$$\overline{\lim_{k\to\infty}}\,|\langle\pi^k_{\omega_i\omega_j\omega_i}f,f\rangle_{\overline{L}^2(\omega_i)}|^{1/k}\leqslant\varepsilon(|j-i|)^2. \tag{TT}$$

Since $\pi_{\omega_i\omega_j\omega_i}$ is self-adjoint in $\overline{L}^2(\omega_i)$ and $\overline{Lip}(\omega_i)$ is a dense subset of $\overline{L}^2(\omega_i)$, it follows by Lemma 0.3.1 that $\pi_{\omega_i\omega_j\omega_i}$ is $\varepsilon(|j-i|)^2$-contracting also in $\overline{L}^2(\omega_i)$. This, by Remark 1.1.10, is equivalent to saying that

$$\{\omega_i:\omega_j\}^{\rightarrow}_{\vec{\omega}_K}\leqslant\varepsilon(|j-i|). \tag{TU}$$

(TU) is what we need to apply Lemma 3.6.8; in the end, we get the

5.3.7 Theorem. *The model (TC) is exponentially ρ-mixing.* ♣

5.4 A hypocoercive system of interacting particles

For the time being we have only been dealing with *spatial* decorrelations. Yet I have had the idea that the ability of Hilbertian decorrelations to get tensorized for infinite sets could be well adapted to the study of *temporal* relaxation of an infinite stochastic system: one can consider indeed time as an extra dimension for the particle system, which leads to a situation analogous to the parallel hyperplanes of § 0.1.c. In the reversible case, we saw that spectral techniques make

[‡]. This definition can be ambiguous if the support of ω_i is not the whole $\mathbb{R}$; in this case, just add an infimum in the definition.

it possible to get L^2 results from L^1 results, cf. Theorem 0.1.9. Here I will show how Hilbertian decorrelations can be used for a *non-reversible* particle stochastic system.

The system which we will study here as an example is governed by a *kinetic Fokker-Planck equation*. This equation, which arises naturally in physics, corresponds to a Hamiltonian evolution perturbed by some noise *acting on speeds*. The study of such systems is made complicated by the fact that diffusion is only performed along certain directions of the states space, so that the non-reversibility of the evolution is essential to ensure convergence to equilibrium. In [83], Villani proves L^2 convergence for such systems in situations where the state of the system lives in a finite-dimensional manifold. Here we will use tensorization of Hilbertian decorrelations in a fundamental way to get a result valid in an infinite-dimensional setting. Moreover, we will get non-trivial bounds for arbitrary small times, which is a new feature compared to [83].

5.4.1 Definition. For real parameters $m, \omega, c, T, \lambda > 0$ [§], we consider a system of particles i indexed by $\mathbb{Z}$, each particle being described by its momentum $p_i \in \mathbb{R}$ and its position $q_i \in \mathbb{R}$. We consider the Hamiltonian

$$H(\vec{p}, \vec{q}) = m^{-1} \sum_{i \in \mathbb{Z}} \frac{p_i^2}{2} + m\omega^2 \sum_{i \in \mathbb{Z}} \frac{q_i^2}{2} + mc^2 \sum_{i \in \mathbb{Z}} \frac{(q_{i+1} - q_i)^2}{2}. \tag{TV}$$

Then the system $(\vec{p}(u), \vec{q}(u))$ evolves according to the Hamiltonian H, plus a white noise independent on each p_i, plus a friction force $F_i = -\lambda p_i$ on each i which dissipates the energy brought by the white noise, friction being adjusted to the noise so that their association constitutes a (volumic) thermal bath at temperature T. One computes that this means that the quadratic variation on p_i is given by $d[p_i] = 2T\lambda m\, du$.

In other words, if $(W_i(u))_{i \in \mathbb{Z}}$ denotes a family of independent brownian motions, the evolution of the system is given by

$$\begin{cases} dp_i &= (-m\omega^2 q_i + mc^2(q_{i-1} + q_{i+1} - 2q_i) - \lambda p_i)\, du + \sqrt{2T\lambda m}\, dW_i \\ dq_i &= m^{-1} p_i\, du. \end{cases} \tag{TW}$$

♢

5.4.2 Remark. The system of Definition 5.4.1 is to be thought as a toy model for a large class of similar systems obtained by generalizing it in several ways. A first example, which would change almost nothing but complicating the formalism, is to replace the states space $\mathbb{R} \times \mathbb{R}$ of each particle by $\mathbb{R}^n \times \mathbb{R}^n$, or to replace the lattice $\mathbb{Z}$ by $\mathbb{Z}^n$. A trickier generalization is to consider the case of non-harmonic interactions: then I expect the results stated below to remain qualitatively true, but proving them might be far more difficult since one cannot use the properties of Gaussian vectors any more. Also, if one allows for infinite-ranged interactions, which speed of decay is required to get temporal decorrelations?

All these questions look quite worthwhile to me, though answering them is out of the scope of this work. Here I will only show how Hilbertian correlations make everything work fine for the toy model, hoping that it shall be useful for the general situation. ♡

Let us consider the equilibrium dynamics of our system. We fix an arbitrary time $0 < t < \infty$. Denote by (p_i, q_i) the state of particle i at time $u = 0$, resp. by (p'_i, q'_i) the state of particle i at time $u = t$. We have to prove the

[§]. m is the mass of each particle, ω is the frequency corresponding to the pinning potential, c is more or less the speed of sound, expressed in inter-atomic distances by unit of time, T is the temperature and λ is the relaxation constant of the friction. Physical homogeneity of these constants are resp. $[\mathsf{M}], [\mathsf{T}^{-1}], [\mathsf{T}^{-1}], [\mathsf{M}\mathsf{L}^2\mathsf{T}^{-2}], [\mathsf{T}^{-1}]$.

5.4.3 Claim. *Provided t is small enough, for all $i,j \in \mathbb{Z}$ (possibly identical), one has $\{p_i : p'_j\}_*$, $\{p_i, q'_j\}_*, \{q_i : p'_j\}_*, \{q_i, q'_j\}_* < 1$, uniformly in i,j. Moreover, still uniformly in i,j, these quantities are bounded by $O(e^{-\gamma|j-i|})$ for some $\gamma > 0$.* ♣

Proof. We denote by η (resp. η') $\in \mathbb{R}^{\mathbb{Z}\times\{p,q\}}$ the global state $(p_i, q_i)_{i\in\mathbb{Z}}$ (resp. $(p'_i, q'_i)_{i\in\mathbb{Z}}$) at time 0 (resp. t). We also denote by $(\varphi^u)_{u\geqslant 0}$ the semigroup of operators on $\mathbb{R}^{\mathbb{Z}\times\{p,q\}}$ corresponding to the evolution of the system in absence of noise, but with the friction remaining. Since the system is linear, the φ^u are linear operators.

By the work of § 5.2, we know that η is distributed according to the centered Gaussian law with covariance matrix $T^{-1}\check{C}$, where $\check{C}$ is defined as $\check{Q}^{-1}$, the matrix $\check{Q}$ being in turn defined by:

$$\check{Q}_{p_ip_i} := m^{-1}; \tag{TX}$$

$$\check{Q}_{q_iq_i} := m(\omega^2 + 2c^2); \tag{TY}$$

$$\check{Q}_{q_iq_{i\pm1}} := -mc^2, \tag{TZ}$$

the other entries of $\check{Q}$ being zero. Observe that, as the matrix of a quadratic form, $\check{Q}$ is bounded (this is obvious from (TX)–(TZ)); moreover, $\check{Q}^{-1}$ (actually exists and) is also bounded: that follows from $\check{Q}$'s being bounded below by the matrix having the same expression with c replaced by 0, which we denote by $\check{Q}^\circ$, which is a strictly positive 'scalar' matrix (modulo some homogeneity constant).

Because of the linear nature of the system, we have moreover that, conditionally to η, the law of η' is some Gaussian vector of the form $\varphi^t\eta + \theta$, where θ is a centered Gaussian vector whose law does not depend on η. Let us denote by $\hat{C}$ the covariance matrix of θ, and $\hat{Q} = \hat{C}^{-1}$ —though for the time being it is not clear that $\hat{Q}$ exists.

Then, we can formally write the covariance matrix $\bar{C}$ of (η,η') as $\bar{C} = \bar{Q}^{-1}$, with:

$$\bar{Q}(\eta,\eta') = \check{Q}(\eta) + \hat{Q}(\eta' - \varphi^t\eta). \tag{UA}$$

(Note that $\bar{Q}$ is a quadratic form on $\mathbb{R}^{\mathbb{Z}\times\{p,q,p',q'\}}$, while $\check{Q}$ and $\hat{Q}$ were defined on $\mathbb{R}^{\mathbb{Z}\times\{p,q\}}$).

5.4.4 Notation. In the sequel, we shorthand "$\mathbb{Z}\times\{p,q\}$" into "$\mathbb{Z}^{\uplus 2}$", resp. "$\mathbb{Z}\times\{p,q,p',q'\}$" into "$\mathbb{Z}^{\uplus 4}$". ♢

Now I claim that there exists constants $0 < r \leqslant R < \infty$ such that $r\mathbf{I} \leqslant \bar{Q} \leqslant R\mathbf{I}$. Well, this is meaningless *stricto sensu*, because all the entries of $\bar{Q}$ do not have the same physical homogeneity, so we have to 'convert' momenta into positions by dividing them by some homogeneity parameter χ, say $\chi = m\omega$ —but other choices may be more relevant.

First, I claim that $\bar{Q} \geqslant \frac{1}{2}(\chi^2 m^{-1} \wedge m\omega^2)\mathbf{I}$. Let indeed $(\eta,\eta') = (\vec{p}_{\mathbb{Z}}, \vec{q}_{\mathbb{Z}}, \vec{p'}_{\mathbb{Z}}, \vec{q'}_{\mathbb{Z}}) \in \mathbb{R}^{\mathbb{Z}^{\uplus 4}}$ with finite support. We observe that

$$\|(\eta,\eta')\|^2 = \sum_{i\in\mathbb{Z}} (\chi^{-2}p_i^2 + q_i^2 + \chi^{-2}p'^2_i + q'^2_i) = \|\eta\|^2 + \|\eta'\|^2, \tag{UB}$$

so that either $\|\eta\|^2 \geqslant \frac{1}{2}\|(\eta,\eta')\|^2$ or $\|\eta'\|^2 \geqslant \frac{1}{2}\|(\eta,\eta')\|^2$. Now, recalling the definition of $\check{Q}^\circ$ a few lines above, $\check{Q}(\eta) \geqslant \check{Q}^\circ(\eta) \geqslant (\chi^2 m^{-1} \wedge m\omega^2)\|\eta\|^2$, so by (UA), $\bar{Q}(\eta,\eta') \geqslant (\chi^2 m^{-1} \wedge m\omega^2)\|\eta\|^2$. Since reversing the direction of time yields the same system with the sign of speeds reversed, which does not change the norms of η and η', one has similarly $\bar{Q}(\eta,\eta') \geqslant (\chi^2 m^{-1} \wedge m\omega^2)\|\eta'\|^2$. The claim follows.

The second point consists in proving that $\bar{Q}$ is bounded above. On the one hand, by (TX)–(TZ),

$$\check{Q}(\eta) \leqslant (m^{-1}\chi^2 \vee m(\omega^2+4c^2))\|\eta\|^2 \leqslant (m^{-1}\chi^2 \vee m(\omega^2+4c^2))\|(\eta,\eta')\|^2. \tag{UC}$$

Next, the difficult point is to prove that $\hat{Q}(\eta' - \varphi^t\eta)$ (exists and) can be bounded above by a multiple of $\|(\eta,\eta')\|^2$. We begin with transforming the original problem of bounding a quadratic form on $\mathbb{R}^{\mathbb{Z}^{\uplus 4}}$ into a problem on $\mathbb{R}^{\mathbb{Z}^{\uplus 2}}$. Indeed, $\|\varphi^t\eta\|$ is bounded by a multiple of $\|\eta\|$, since the operator φ^t dissipates the energy $H(\eta)$, energy which the previous work on $\check{Q}$ proved to be controlled below and above by $\|\eta\|^2$; therefore, it suffices to prove that the quadratic form $\hat{Q}(\eta)$ on $\mathbb{R}^{\mathbb{Z}^{\uplus 2}}$ is bounded by a multiple of $\|\eta\|^2$ to achieve our goal.

The natural quantity to be computed for θ (recall that θ denotes the total effect of noise between times 0 and t) is its covariance matrix $\hat{C}$. Its expression is the following (the notation is explained just below):

$$\hat{C} = 2T\lambda m \int_0^t \varphi^{t-u}\mathbf{I}_p(\varphi^{t-u})^{\mathsf{T}}\,du, \tag{UD}$$

where $\mathbf{I}_p$ is the diagonal matrix being 1 on diagonal entries indexed by some p_i and 0 on diagonal entries indexed by some q_i, and $(\varphi^{t-u})^{\mathsf{T}}$ is the transpose of the linear operator φ^{t-u} seen as a square matrix indexed by $\mathbb{Z}^{\uplus 2}$. This decomposition means that we are summing the contributions of all the elementary noises occuring at times $u \in [0,t]$, using that these elementary noises are independent.

Now we need an approximate expression for φ^u, $u \in [0,t]$. Here for the sake of legibility I will remain at a formal level, giving only limited expansions; it is essential nevertheless to keep in mind that all the "$O(*)$" can be made explicit by using Gronwall's lemma, and that these explicit values ensure that the $O(*)$ behave well provided t is small enough. One finds that

$$\varphi^u\delta_{p_i}\cdot p_j = c^{2|j-i|}\frac{u^{2|j-i|}}{(2|j-i|)!} + O(u^{2|j-i|+2}); \tag{UE}$$

$$\varphi^u\delta_{p_i}\cdot q_j = m^{-1}c^{2|j-i|}\frac{u^{2|j-i|+1}}{(2|j-i|+1)!} + O(u^{2|j-i|+3}); \tag{UF}$$

$$\varphi^u\delta_{q_i}\cdot p_i = m\omega^2 u + O(u^3); \tag{UG}$$

$$\varphi^u\delta_{q_i}\cdot p_{j\neq i} = mc^{2|j-i|}\frac{u^{2|j-i|-1}}{(2|j-i|-1)!} + O(u^{2|j-i|+1}); \tag{UH}$$

$$\varphi^u\delta_{q_i}\cdot q_j = c^{2|j-i|}\frac{u^{2|j-i|}}{(2|j-i|)!} + O(u^{2|j-i|+2}). \tag{UI}$$

Injecting Equations (UE)–(UI) into (UD), one finds that: [¶]

$$\hat{C}_{p_ip_i} = 2T\lambda mt + O(t^3); \tag{UJ}$$

$$\hat{C}_{p_iq_i} = T\lambda t^2 + O(t^4); \tag{UK}$$

$$\hat{C}_{q_iq_i} = \tfrac{2}{3}T\lambda m^{-1}t^3 + O(t^5); \tag{UL}$$

$$\hat{C}_{p_ip_{j\neq i}} = O(t^{2|j-i|+1}); \tag{UM}$$

$$\hat{C}_{p_iq_{j\neq i}} = O(t^{2|j-i|+2}); \tag{UN}$$

$$\hat{C}_{q_iq_{j\neq i}} = O(t^{2|j-i|+3}). \tag{UO}$$

Consequently, the covariance matrix $\hat{C}$ can be seen as a perturbation of the matrix $\hat{C}^\circ$ which is defined by Equations (UJ)–(UO), but with the "$O(*)$" terms replaced by 0. Since $\hat{C}^\circ$ is invertible,

[¶]. Recall that $\hat{C}$, as a covariance matrix, is symmetric.

with an explicitly computable inverse, one finds that $\hat{C}$ is invertible too with:

$$\begin{aligned}
\hat{Q}_{p_ip_i} &= 2T^{-1}\lambda^{-1}m^{-1}t^{-1}+O(t); && \text{(UP)}\\
\hat{Q}_{p_iq_i} &= -3T^{-1}\lambda^{-1}t^{-2}+O(1); && \text{(UQ)}\\
\hat{Q}_{q_iq_i} &= 6T^{-1}\lambda^{-1}mt^{-3}+O(t^{-1}); && \text{(UR)}\\
\hat{Q}_{p_ip_{j\neq i}} &= O(t^{2|j-i|-1}); && \text{(US)}\\
\hat{Q}_{p_iq_{j\neq i}} &= O(t^{2|j-i|-2}); && \text{(UT)}\\
\hat{Q}_{q_iq_{j\neq i}} &= O(t^{2|j-i|-3}). && \text{(UU)}
\end{aligned}$$

In the end, provided that t is small enough, we have proved that $\hat{Q}(\eta)/\|\eta\|^2 \leqslant 6T^{-1}\lambda m t^{-3}+O(t^{-1}) < \infty$.

Actually we have proved more than that: not only we have a bound on the operator norm of $\bar{Q}$, but we have bounded it entry-wise. More precisely, expanding the $O(*)$, we find that provided t is small enough, there exists constants $A<\infty$ and $\gamma>0$ such that for all $i,j\in\mathbb{Z}$,

$$\underbrace{\hat{Q}_{p_ip_j},\hat{Q}_{p_iq_j},\ldots,\hat{Q}_{q'_iq'_j}}_{\text{all 16 possibilities}} \leqslant Ae^{-\gamma|j-i|}. \qquad \text{(UV)}$$

5.4.5 Notation. From now on we denote the basic variables p_i,q_i,p'_i,q'_i of our system by X_i, $i\in\mathbb{Z}^{\uplus 4}$. ♢

Now the question is: for $i\neq j\in\mathbb{Z}^{\uplus 4}$, $K\subset\mathbb{Z}^{\uplus 4}\smallsetminus\{i,j\}$, what is the value of $\{X_i:X_j\}_{\vec{X}_K}$? By the properties of Gaussian variables [Theorem 1.2.6], the answer is the following. Let $\bar{Q}_{|_{\mathbb{Z}^{\uplus 4}\smallsetminus K}}$ be the restriction of $\bar{Q}$ to indexes in $(\mathbb{Z}^{\uplus 4}\smallsetminus K)$. Since $r\mathbf{I}\leqslant\bar{Q}\leqslant R\mathbf{I}$, the same holds for $\bar{Q}_{|_{\mathbb{Z}^{\uplus 4}\smallsetminus K}}$, so this matrix is invertible; denote by $\bar{C}^{|^{\mathbb{Z}^{\uplus 4}\smallsetminus K}}$ its inverse. This matrix is the covariance matrix of (the centered version of) $\vec{X}_{\mathbb{Z}^{\uplus 4}\smallsetminus K}$ under some fixed value for $\vec{X}_K$; thus:

$$\{X_i:X_j\}_{\vec{X}_K}=\frac{|\bar{C}^{|^{\mathbb{Z}^{\uplus 4}\smallsetminus K}}_{ij}|}{\sqrt{\bar{C}^{|^{\mathbb{Z}^{\uplus 4}\smallsetminus K}}_{ii}\bar{C}^{|^{\mathbb{Z}^{\uplus 4}\smallsetminus K}}_{jj}}}. \qquad \text{(UW)}$$

It remains to control the entries of $\bar{C}^{|^{\mathbb{Z}^{\uplus 4}\smallsetminus K}}$, uniformly in K. We need two types of control: first an exponential control when i is far away from j, then a non-trivial control for the values of i and j corresponding to close (or even identical) atoms.

Let us start with the first one. $\bar{C}^{|^{\mathbb{Z}^{\uplus 4}\smallsetminus K}}_{ii}$ and $\bar{C}^{|^{\mathbb{Z}^{\uplus 4}\smallsetminus K}}_{jj}$ are bounded below by R^{-1}, so we just have to bound above $\bar{C}^{|^{\mathbb{Z}^{\uplus 4}\smallsetminus K}}_{ij}$. This is achieved by a direct use of Lemma 5.5.1 in appendix.

Concerning the uniform non-trivial control, since $r\mathbf{I}\leqslant\bar{Q}\leqslant R\mathbf{I}$ one has $r\mathbf{I}\leqslant\bar{Q}_{|_{\mathbb{Z}^{\uplus 4}\smallsetminus K}}\leqslant R\mathbf{I}$, hence $R^{-1}\mathbf{I}\leqslant\bar{C}^{|^{\mathbb{Z}^{\uplus 4}\smallsetminus K}}\leqslant r^{-1}\mathbf{I}$, hence $R^{-1}\mathbf{I}\leqslant(\bar{C}^{|^{\mathbb{Z}^{\uplus 4}\smallsetminus K}})_{|_{\{i,j\}^2}}\leqslant r^{-1}\mathbf{I}$; from this and (UW),

$$\forall i\neq j\in I \qquad \{X_i:X_j\}_*\leqslant\frac{R-r}{R+r}<1. \qquad \text{(UX)}$$

♠

From Claim 5.4.3, we get the main result of this subsection:

5.4.6 Theorem. *For the model of Definition 5.4.1, for all $t > 0$, $\{\eta, \eta'\} < 1$.* ♣

Proof. First, if t is small enough so that Claim 5.4.3 holds, direct application of Lemma 3.6.8 proves the result, as the $\{X_i, X_j\}_*$ are summable (since they decrease exponentially) and they all are < 1.

Now for larger t, fix some $0 < t_1 < t$ so that Claim 5.4.3 holds for t_1. Then we notice that $\eta \to \eta(t_1) \to \eta'$ is a Markov chain (with "$\eta(t_1)$" standing for "$(\vec{p}(t_1), \vec{q}(t_1))$"), so by Proposition 1.1.13, $\{\eta, \eta'\} \leqslant \{\eta, \eta(t_1)\} < 1$. ♠

5.5 Appendix: Inverses of 'nearly diagonal' matrices

The goal of this appendix is to state and prove a few lemmas sharing the same spirit: "*if a matrix is 'nearly diagonal', then it shall be invertible and its inverse shall also be 'nearly diagonal' with the same type of decay*".

5.5.a Matrices with exponential decay

The goal of this subsection is to prove the following

5.5.1 Lemma. *Let $I \subset \mathbb{Z}$ and let $((M_{ij}))_{(i,j)\in I^2}$ be a matrix. Assume that, when seen as a quadratic form on $L^2(I)$, one has $r\mathbf{I} \leqslant M \leqslant R\mathbf{I}$ for $0 < r \leqslant R < \infty$ —in particular, M is invertible. Assume moreover that there exists constants $A < \infty$ and $\gamma > 0$ such that for all $i, j \in I$, $|M_{ij}| \leqslant Ae^{-\gamma|j-i|}$.*

Then there exist constants $A' < \infty$ and $\gamma' > 0$ which are explicit functions of r, R, γ, A (so they do not depend on I*), such that one has the following control on the entries of M^{-1}:*

$$\forall i, j \in I \qquad (M^{-1})_{ij} \leqslant A' e^{-\gamma'|j-i|}. \tag{UY}$$

♣

Proof. Up to multiplying by a scalar, one can assume that $R = 1$. Then M writes $M = \mathbf{I} - H$, where $0 \leqslant H \leqslant (1-r)\mathbf{I}$; since H is symmetric, that inequality means that $|||H||| \leqslant 1 - r < 1$. Therefore, for all $k \in \mathbb{N}$ one has $|||H^k||| \leqslant (1-r)^k$, which allows us to write M^{-1} as a series expansion:

$$M^{-1} = \sum_{k=0}^{\infty} H^k. \tag{UZ}$$

Up to replacing A by $A + 1$, we have the same entry-wise control on H as on M. Then one sees by induction that for all $k \in \mathbb{N}$,

$$\forall i, j \in I \qquad |(H^k)_{ij}| \leqslant A_1^k e^{-\gamma_1|j-i|}, \tag{VA}$$

where γ_1 is an arbitrary parameter in $(0, \gamma)$ and

$$A_1 := \sum_{z\in\mathbb{Z}} Ae^{-\gamma|z|+\gamma_1 z} = \frac{(1-e^{-2\gamma})A}{(1-e^{-(\gamma-\gamma_1)})(1-e^{-(\gamma+\gamma_1)})} \tag{VB}$$

—observe that I does not appear in the expression of A_1. Since A_1 is greater than 1, (VA) is not enough to get an entry-wise control on M^{-1}. But now observe that the bound $|||H^k||| \leqslant (1-r)^k$ implies that all the $(H^k)_{ij}$ are bounded by $(1-r)^k$ in absolute value; thus:

$$|(M^{-1})_{ij}| \leqslant \sum_{k=0}^{\infty} (e^{-\gamma_1|j-i|}A_1^k \wedge (1-r)^k) \leqslant \Big(\frac{A_1}{A_1-1}+\frac{1}{r}\Big)\exp\Big(-\frac{|\log(1-r)|\gamma_1}{|\log(1-r)|+\log A_1}|j-i|\Big), \quad \text{(VC)}$$

from which you read suitable values for A' and γ'. ♠

5.5.b Convolution inverses of rapidly decreasing functions

☛ *In all this subsection, we work on $\mathbb{Z}^n$ for some $n \in \mathbb{N}^*$; $\mathbb{R}^n$ is endowed with some fixed norm $|\cdot|$.*

5.5.2 Remark. Here I will deal with fonctions on $\mathbb{Z}^n$, but the results of this subsection could also be tranposed for functions on $\mathbb{R}^n$. ♡

5.5.3 Definition. If $a : \mathbb{Z}^n \to \mathbb{R}$ is some integrable function with $\|a\|_{l^1} < 1$, we define

$$B[a] = a + a * a + a * a * a + \cdots, \quad \text{(VD)}$$

which is the sum of a convergent series in $l^1(\mathbb{Z}^n)$. $B[a]$ is the function $b \in l^1(\mathbb{Z}^n)$ characterized by:

$$(\delta_0 - a) * (\delta_0 + b) = \delta_0. \quad \text{(VE)}$$

♢

5.5.4 Definition. A function $a : \mathbb{Z}^n \to \mathbb{R}$ is said to have *exponential decay* if there exists some $\beta > 0$ such that, for all $\beta' < \beta$, $a(z) = O(e^{-\beta'|z|})$ when $|z| \to \infty$. The minimal β satisfying that property is called the *(exponential) rate of decay* of a. ♢

5.5.5 Lemma. *Let $a \in l^1(\mathbb{Z}^n)$ with $\|a\|_{l^1} < 1$. If a has exponential decay, then so does $B[a]$.* ♣

Proof. Denoting by $|a|$ the function defined by $|a|(z) = |a(z)|$, it is clear by (VD) that

$$\forall z \in \mathbb{Z}^n \qquad |B[a](z)| \leqslant B[|a|](z), \quad \text{(VF)}$$

therefore it suffices to prove the case where a is nonnegative. In that case, $B[a]$ will also be nonnegative.

Let $(\mathbb{R}^n)^*$ denote the dual space of $\mathbb{R}^n$, endowed with the dual norm

$$\forall \lambda \in (\mathbb{R}^n)^* \qquad |\lambda|_* = \sup_{\substack{z\in\mathbb{R}^n \\ |z|=1}} |\langle \lambda, z\rangle|. \quad \text{(VG)}$$

For a nonnegative function a, we define its Laplace transform $\mathscr{L}\{a\}\colon (\mathbb{R}^n)^* \to \mathbb{R}_+ \cup \{+\infty\}$ by

$$\mathscr{L}\{a\}(\lambda) = \sum_{z\in\mathbb{Z}^n} e^{\langle\lambda,z\rangle} a(z). \quad \text{(VH)}$$

Then, saying that a has exponential decay with rate γ is equivalent to saying that, for all $\lambda \in (\mathbb{R}^n)^*$ with $|\lambda|_* < \gamma$, $\mathscr{L}\{a\}(\lambda)$ is finite.

Since Laplace transform is linear and turns convolution into ordinary product, (VD) yields, for all $\lambda \in (\mathbb{R}^n)^*$:

$$\mathscr{L}\{B[a]\}(\lambda) = \mathscr{L}\{a\}(\lambda) + \mathscr{L}\{a\}(\lambda)^2 + \mathscr{L}\{a\}(\lambda)^3 + \cdots, \quad \text{(VI)}$$

which converges if and only if $\mathscr{L}\{a\}(\lambda) < 1$.

Now, since a is nonnegative, by (VH) the function $\mathscr{L}\{a\}$ is convex, so it is continuous on the interior of the domain where it is finite. By the exponential decay hypothesis, that domain contains a neighbourhood of 0, so $\mathscr{L}\{a\}$ is continuous at 0. And since $\mathscr{L}\{a\}(0) = \sum_{z\in\mathbb{Z}^n} a(z) = \|a\|_{l^1} < 1$, there is a neighbourhood of 0 on which $\mathscr{L}\{a\} < 1$ and thus $\mathscr{L}\{B[a]\} < \infty$. This implies that $B[a]$ has exponential decay. ♠

5.5.6 Remark. This proof also shows that (for nonnegative a) the rate of decay of $B[a]$ will never be greater than the rate of decay of a. In general, it is even strictly smaller, since all the values of λ for which $1 \leqslant \mathscr{L}\{a\}(\lambda) < \infty$ yield a finite Laplace tranform for a but an infinite one for $B[a]$. For example, take $n = 1$ and $a = e^{-1}\delta_1$, which has exponential decay with infinite rate since it is compactly supported; then the k-th convolution power of a is $a^{*k} = e^{-k}\delta_k$, so that $B[a]$ is the function

$$B[a](z) = \mathbf{1}_{z>0}e^{-z}, \tag{VJ}$$

which also has exponential decay, but with rate 1 only. ♡

5.5.7 Lemma. *If $\|a\|_{l^1(\mathbb{Z}^n)} < 1$ and $a(z) = O(1/|z|^\alpha)$ when $|z| \to \infty$ for some $\alpha > n$, then $B[a](z) = O(1/|z|^\alpha)$ when $|z| \to \infty$.* ♣

Proof. Let a satisfy the assumptions of the lemma for some α. Like in the proof of Lemma 5.5.7, we can assume that a is nonnegative. For $d > 0$, we define the function $\varphi_d : \mathbb{Z}^n \to \mathbb{R}$ by:

$$\varphi_d(z) := 1/(|z| \wedge d)^\alpha, \tag{VK}$$

which is in $l^1(\mathbb{Z}^n)$ since $\alpha > n$. Then the key claim is the following sub-lemma, whose proof is postponed:

5.5.8 Lemma. *Under the assumptions of Lemma 5.5.7, there exists some $\rho < 1$ and some $d \in (0,\infty)$ such that, pointwise,*

$$\varphi_d * a \leqslant \rho\varphi_d. \tag{VL}$$

♣

Admitting Lemma 5.5.8, take ρ and d such that (VL) is satisfied. The assumption on a implies that there exists some $C < \infty$ such that $a \leqslant C\varphi_d$; therefore by (VL) one also has $a * a \leqslant C\varphi_d * a \leqslant \rho C\varphi_d$, whence by (VL) again $a * a * a \leqslant \rho C\varphi_d * a \leqslant \rho^2 C\varphi_d$, etc.. In the end,

$$B[a] \leqslant C\varphi_d + \rho C\varphi_d + \rho^2 C\varphi_d + \cdots \leqslant \frac{C}{1-\rho}\varphi_d, \tag{VM}$$

which implies that $B[a](z) = O(1/|z|^\alpha)$. ♠

Proof of Lemma 5.5.8. Denote $S := \|a\|_{l^1}$, which by hypothesis is < 1, and fix $\varepsilon \in (0, 1/2)$ such that $(1-\varepsilon)^\alpha > S$. Let $d \in (0,\infty)$, devised to be quite large; our goal is to bound above $(\varphi_d * a)(z)$ for all $z \in \mathbb{Z}^n$. Since φ_d is bounded above by $d^{-\alpha}$, one has obviously for all $z \in \mathbb{Z}^n$:

$$(\varphi_d * a)(z) \leqslant d^{-\alpha} \sum_{z\in\mathbb{Z}^n} a(z) = Sd^{-\alpha}, \tag{VN}$$

whence $(\varphi_d * a)(z) \leqslant S\varphi_d(z)$ for all z with $|z| \leqslant d$. Since $S < 1$, the claim is therefore okay for $|z| \leqslant d$.

Now, let $z \in \mathbb{Z}^n$ with $|z| > d$. We have to bound above

$$(\varphi_d * a)(z) = \sum_{\substack{x,y\in\mathbb{Z}^n \\ x+y=z}} \varphi_d(x)a(y). \tag{VO}$$

We decompose this sum into three pieces:

$$(\varphi_d * a)(z) = \sum_{|y|\leqslant\varepsilon|z|} \varphi_d(z-y)a(y) + \sum_{\substack{|x|>\varepsilon|z| \\ |z-x|>\varepsilon|z|}} \varphi_d(x)a(z-x) + \sum_{|x|\leqslant\varepsilon|z|} \varphi_d(x)a(z-x), \tag{VP}$$

which we shorthand into "① + ② + ③".

We bound these three terms separately. For ①, we observe that for $|y| \leqslant \varepsilon|z|$, $|z-y| \geqslant (1-\varepsilon)|z|$ by the triangle inequality, thus $\varphi_d(z-y) \leqslant \big((1-\varepsilon)|z|\big)^{-\alpha} = (1-\varepsilon)^{-\alpha}\varphi_d(z)$, whence by summing:

$$① \leqslant (1-\varepsilon)^{-\alpha}\varphi_d(z) \sum_{|y|\leqslant\varepsilon|z|} a(y) \leqslant (1-\varepsilon)^{-\alpha} S\varphi_d(z). \tag{VQ}$$

Similarly, for $|x| \leqslant \varepsilon|z|$, C denoting a constant such that $a \leqslant C\varphi_d$, one has $a(z-x) \leqslant C\big((1-\varepsilon)|z|\big)^{-\alpha}$, thus:

$$③ \leqslant (1-\varepsilon)^{-\alpha} C\|\varphi_d\|_{l^1}\varphi_d(z). \tag{VR}$$

Of course, $\|\varphi_d\|_{l^1}$ depends on d; the important point is that, by dominated convergence, $\|\varphi_d\|_{l^1} \to 0$ when $d \to \infty$.

Finally, provided d is large enough, Term ② will be well approximated by an integral:

$$② \leqslant \sum_{\substack{x\in\mathbb{Z}^n \\ |x|,|z-x|>\varepsilon|z|}} \frac{1}{|x|^\alpha} \times \frac{C}{|z-x|^\alpha} \simeq \int_{\substack{x\in\mathbb{R}^n \\ |x|,|z-x|>\varepsilon|z|}} \frac{C}{(|x|\,|z-x|)^\alpha}\,dx, \tag{VS}$$

where "$\simeq$" means that the ratio between the quantites at each side of that symbol can be made arbitrarily close to 1 when $d \to \infty$, uniformly in z. Indeed, the difference between the sum and the integral is due to two causes: first, approximating the integral on a unit square of $\mathbb{R}^n$ by the value of the integrand at the center of this square, second, summing (or not summing) terms of the discrete sum corresponding to squares that are not entirely in the domain of the integral. For the first cause, on the domain of the integral, $C/(|x||z-x|)^\alpha$ varies of at most $O(1/|z|)$ in relative value on all the unit squares. For the second cause, the border of the domain of the integral is made of two $(n-1)$-dimensional spheres of radius $\varepsilon|z|$, so it crosses $O(|z|^{n-1})$ unit squares. Since $C/(|x||z-x|)^\alpha$ is bounded by $C(\varepsilon(1-\varepsilon))^{-\alpha}|z|^{-2\alpha}$ on the domain of the integral, the (absolute) error due to boundary squares is at most $O(|z|^{n-1-2\alpha})$. As the integral itself is proportional to $|z|^{n-2\alpha}$ (cf. the change of variables below), the relative error due to boundary squares is at most $O(1/|z|)$ too, and $O(1/|z|) = o(1)$ since $|z| > \varepsilon d$.

Making the change of variables $x = |z|x'$, (VS) becomes:

$$② \lesssim C|z|^{n-2\alpha} \int_{|x'|,|1-x'|>\varepsilon} \frac{1}{|x'|^\alpha|1-x'|^\alpha}\,dx', \tag{VT}$$

which I shorthand into "$② \lesssim \mathscr{I}C|z|^{n-2\alpha}$". Since $|z| > d$ and $\alpha > n$, this bound implies:

$$② \lesssim \frac{\mathscr{I}C}{d^{\alpha-n}}\varphi_d(z). \tag{VU}$$

Combining (VQ), (VR) and (VU), one finally gets that when $d \to \infty$, for all $|z| \geqslant d$,

$$(\varphi_d * a)(z) \leqslant \rho(d)\varphi_d(z), \tag{VV}$$

with

$$\rho(d) = (1-\varepsilon)^{-\alpha}(S + C\|a\|_{l^1}) + (1+o(1))\mathscr{I}C/d^{\alpha-n}. \tag{VW}$$

$\rho(d)$ tends to $(1-\varepsilon)^{-\alpha}S < 1$ when $d \to \infty$, so it is < 1 provided d is large enough, which is what we wanted. ♠

Quatrième partie

Flambage de Mc Kean – Vlasov

Résumé

On considère une assemblée de particules soumises à un potentiel attractif à courte portée non singulier, en présence de bruit. L'assemblée est supposée suffisamment dense pour que le système puisse être décrit par une équation aux dérivées partielles (*équation de Vlasov*), avec des frottements suffisamment forts pour que l'évolution soit du premier ordre (*équation de Mc Kean*). On sait alors que la dynamique du système est équivalente à la descente du gradient de l'énergie libre dans la « variété riemannienne de dimension infinie » associée à la métrique de Wasserstein W_2. On s'intéresse au cas où la condition initiale est une densité uniforme sur $\mathbb{R}^d$. Cet état est toujours un équilibre, mais la stabilité de l'équilibre dépend de la température. Notre objectif est de déterminer à quelle température survient la transition de phase et de minorer l'énergie d'activation dans le régime stable, ce qui requiert de prendre en compte les non-linéarités du système.

Pour minorer la fonctionnelle d'entropie, on fait agir sur la mesure de répartition des particules un noyau markovien qui rend sa densité bornée. Cet argument passe par l'établissement d'un résultat nouveau sur la continuité du plongement de la « variété riemannienne » de Wasserstein dans un espace linéaire classique. Mes résultats principaux sont présentés au § 5 : sous certaines hypothèses de régularité sur le potentiel d'interaction, je parviens ainsi à déterminer rigoureusement la température de transition de phase, en minorant l'énergie d'activation avec un exposant critique non-trivial. Quelques améliorations de ces résultats, notamment sur l'affaiblissement des hypothèses, sont en cours d'étude : j'expose brièvement mes projets au § 6.

Conventions et notations

Homogénéité physique

Dans toute cette partie de la thèse, nous travaillerons sur des grandeurs physiques *dimensionnées* ; on notera x [X] pour dire que la grandeur x a pour homogénéité la dimension X. Nous utiliserons les quatre dimensions de base suivantes :

- Quantité de matière N ;
- Longueur L ;
- Énergie E ;
- Temps T.

On notera $\mathcal{N}$ [N^{-1}] le nombre d'Avogadro.

Divers

- Dans toute cette partie de la thèse, « l'espace physique » désignera l'espace affine $\mathbb{R}^d$, d'homogénéité physique L, la dimension d étant un entier fixé. Cet espace est muni de sa structure euclidienne : pour deux vecteurs $v, w \in \mathbb{R}^d$, la norme de v est notée $|v|$ [L] et le produit scalaire de v et w est noté $v \cdot w$ [L^2]. La mesure de Lebesgue sur $\mathbb{R}^d$ est notée dx [L^d].
- Dans ce travail, le caractère π sera utilisé pour noter des mesures de Radon ; quand nous aurons besoin d'invoquer la constante d'Archimède, nous noterons celle-ci $\mathring{\pi}$ pour faire le distinguo.
- Toutes nos fonctions seront sous-entendues réelles, sauf transformation de Fourier où elles pourront être complexes.
- Quand nous dirons d'une fonction f sur $\mathbb{R}^d$ qu'elle est *gentille*, cela signifiera par exemple qu'elle est dans l'espace de Schwartz, *i.e.* qu'elle est infiniment différentiable et que toutes ses dérivées sont intégrables.
- Le crochet de dualité $\langle f, \mu \rangle$, pour f [X] une fonction gentille [‖] et μ [N] une mesure de Radon, désigne $\int f(x)\, d\mu(x)$ [X.N].
- Pour deux fonctions gentilles f [X] et g [Y] sur $\mathbb{R}^d$, on note

$$\langle f, g \rangle_{L^2} := \int_{\mathbb{R}^d} f(x) g(x)\, dx \qquad [\mathsf{X.Y.L}^d]. \tag{A}$$

- La transformée de Fourier d'une fonction gentille f [X] sur $\mathbb{R}^d$ est notée $\hat{f}$ [$\mathsf{X.L}^d$]. Nous suivons la convention usuelle en mathématiques :

$$\hat{f}(\xi \ [\mathsf{L}^{-1}]) := \int_{\mathbb{R}^d} e^{-i\xi \cdot x} f(x)\, dx. \tag{B}$$

- La différentielle d'une fonction gentille f [X] est notée Df [$\mathsf{X.L}^{-1}$] (à valeurs vectorielles).
- Pour μ [X] une mesure sur un espace mesurable X, Y un autre espace mesurable et $f : X \to Y$ une fonction mesurable, on désigne par $f \# \mu$ [X] la mesure-image de μ par f, *i.e.* la mesure sur Y caractérisée par $(f \# \mu)(B) = \mu(f^{-1}(B))$.

Espaces fonctionnels

☛ *Attention, certaines conventions utilisées ici ne sont pas standardes !*

[‖]. On étendra implicitement la notation au cas de fonctions moins régulières dans la mesure où cela fera sens.

0.9 Définition. Pour $s \in \mathbb{N}, p \in [1,+\infty)$, f [X] une fonction gentille sur $\mathbb{R}^d$, on définit la norme (homogène)

$$\|f\|_{s,p} := \Big(\int_{\mathbb{R}^d} |D^s f(x)|^p dx\Big)^{1/p} \qquad [\mathsf{X.L}^{d/p-s}]. \tag{C}$$

Pour $p = \infty$, on définit de même $\|f\|_{s,\infty} := \sup_{x\in\mathbb{R}^d} |D^s f(x)|$ [X.L^{-s}].

Dans le cas où $s \in [0,+\infty) \smallsetminus \mathbb{N}$, on étend la définition en posant formellement

$$|D^s f(x)|^p := \int_{\mathbb{R}^d} \frac{|D^{\lfloor s\rfloor} f(y) - D^{\lfloor s\rfloor} f(x)|^p}{|y-x|^{d+p(s-\lfloor s\rfloor)}}\, dy. \tag{D}$$

L'espace obtenu par complétion de la norme $\|\cdot\|_{s,p}$ est noté $\dot{W}^{s,p}$ et appelé *espace de Sobolev homogène* d'indices (s,p). Pour $s = 0$, $\dot{W}^{0,p}$ sera simplement noté L^p, et $\|f\|_{0,p}$ sera simplement noté $\|f\|_p$. ♢

0.10 Définition. Toujours pour f gentille, pour $(s_1,p_1),(s_2,p_2) \in [0,+\infty)\times[1,+\infty]$, on définit la norme (*inhomogène*)

$$\|f\|_{\substack{s_1,p_1\\ s_2,p_2}} := \|f\|_{s_1,p_1} \vee \|f\|_{s_2,p_2}, \tag{E}$$

et on note $W^{\substack{n_1,p_1\\ n_2,p_2}}$ l'espace obtenu par complétion de celle-ci. Cet espace sera appelé *espace de Sobolev inhomogène* d'indices (s_1,p_1) et (s_2,p_2). ♢

☛ *On veillera à ce que $\|f\|_{s,p}$ notera ici une norme homogène, alors que d'habitude cette notation est utilisée pour ce que nous noterions ici $\|f\|_{\substack{s,p\\ 0,p}}$.*

0.11 Notation. Le dual d'un espace de Banach W sera noté W' ; on le munira de sa topologie forte. ♢

1 Objet de l'étude

Introduction : Flambage

En 1952, un travail original d'A. Turing sur la morphogénèse animale [79] expliqua comment les taches d'un léopard (par exemple) pouvaient se former sans avoir à supposer aucun plan d'organisation supérieur : des espèces chimiques de distribution initiale uniforme, dont les concentrations évoluent sous des équations de réaction-diffusion, font en effet apparaître spontanément, dans certaines conditions, des motifs prononcés. À une autre échelle, on sait également que la formation des grandes structures cosmologiques (galaxies etc.) s'est faite à partir d'un univers primordial essentiellement homogène et isotrope, cette fois-ci *via*-ci un mécanisme d'effondrement gravitationnel régi par une équation de Vlasov – Poisson [60, §§ 4.1 – 5]. On parle de *brisure spontanée de symétrie* pour désigner cette apparition d'hétérogénéité *ex nihilo*. Le principe est que, bien que des raisons de symétrie fassent que la situation homogène est un équilibre, cet équilibre est *instable* et évoluera donc vers des équilibres stables qui, eux, ne seront pas symétriques.

Dans le cas du *flambage* d'une poutre [29], qui relève de cette famille de phénomènes, la stabilité ou non de l'équilibre symétrique dépend d'un paramètre du système — en l'occurrence, de la force exercée sur la poutre. Il se produit ainsi un phénomène de *transition de phase* au moment où cette force dépasse une certaine valeur critique, caractérisé par le retournement de la convexité

de la fonctionnelle d'énergie au point symétrique. En dimension finie, ce genre de transition de phase est bien décrit par la *théorie des catastrophes* [23].

Dans cette partie de la thèse, nous allons étudier mathématiquement un exemple (relativement simple) de modèle *infini-dimensionnel* où une structure spatiale hétérogène apparaît par brisure spontanée de symétrie après une transition de phase. Notre approche visera une rigueur mathématique complète ; en particulier, nous considèrerons nos objets d'étude *non localement* (par quoi j'entends « au-delà de leur développement limité »).

1.a Le modèle

Dans l'espace physique $\mathbb{R}^d$ [L], on considère un ensemble de particules ponctuelles soumises à trois types de forces :

- D'une part, un potentiel d'interaction v [E.N^{-2}] ;
- D'autre part, un bruit blanc gaussien dû à l'agitation thermique, à température T [E.N^{-1}] ;
- Enfin, une force de friction linéaire en la vitesse, dont le coefficient sera noté $\mathcal{N}^{-1}J$, avec J [E.N^{-1}.L^{-2}.T].

Dans la mesure où cela ne change pas le système physique, nous supposerons toujours v symétrique. Le cas qui nous intéresse est celui où le potentiel d'interaction est *non singulier*, *à courte portée* et *attractif*. Pour se fixer les idées, le lecteur pourra se représenter v comme $\mathscr{C}^\infty$, à support compact et négatif ; en fait, pour définir notre modèle, nous aurons simplement besoin de supposer v de classe $\mathscr{C}^1$ et intégrable. On notera informellement L_0 la *portée* de v, c.-à-d. l'échelle typique sur laquelle se font sentir les forces — dans le cas où v est à support compact, ce pourra être par exemple le diamètre du support de v. Nous définissons

1.1 Notation.

$$V := \sup_{\xi\in\mathbb{R}^d}\big(-\hat{v}(\xi)\big) \qquad [\mathsf{E.N}^{-2}.\mathsf{L}^d]. \tag{F}$$

♢

1.2 Hypothèse. On supposera qu'on a $V > 0$ — c'est ainsi qu'on comprendra l'hypothèse selon laquelle « le potentiel d'interaction est attractif »). ♢

1.3 Remarque. Dans la plupart des situations physiques rencontrées, ce seront les phénomènes à grande échelle qui détermineront la transition de phase, c.-à-d. qu'on aura $V = -\hat{v}(0)$, *i.e.* :

$$V = \int_{\mathbb{R}^d}\big(-v(x)\big)\,dx. \tag{G}$$

Ce sera en particulier le cas si v est négatif, ce que nous supposerons la plupart du temps. ♡

Nous supposons que la répartition des particules est suffisamment dense (à l'échelle de L_0) pour qu'on puisse la décrire par une représentation continue : notons ainsi μ [N] la mesure de répartition des particules, et $m(x) := d\mu(x)/dx$ [N.L^{-d}] sa densité au point x. Nous supposons également que les frottements auxquels sont soumis les particules sont suffisamment importants pour qu'on puisse décrire la dynamique du système par une équation du premier ordre [*]. La densité m évolue alors selon l'*équation de Mc Kean – Vlasov* suivante :

$$\partial_t m = J^{-1}\nabla\cdot\big(T\nabla m + m\nabla(v * m)\big). \tag{H}$$

[*]. Pour donner un sens précis à cette affirmation, il faudrait également tenir compte de la densité typique des particules : si celle-ci est R [N.L^{-d}], pour que la dynamique puisse être décrite au premier ordre, il faut avoir, notant $Masse$ [E.L^{-2}.T^2.N^{-1}] la masse molaire des particules, $J^2 \gg Masse\cdot L_0^{-2}VR$.

1.4 Remarque. Ce genre d'équation se rencontre dans le modèle de Keller – Segel quasi-stationnaire [42], avec dans ce cas un potentiel v singulier et une mesure initiale μ_0 de masse finie — alors qu'ici nous considérons des mesures de masse infinie avec un potentiel régulier — ; la question est alors de savoir si la diffusion va l'emporter ou si, au contraire, les particules vont se regrouper en des points de mesure non nulle. Dolbeaut & Perthame [27] ont montré que les deux régimes pouvaient exister : en deçà d'une certaine température de transition, les particules forment des singularités en temps fini, tandis qu'au-delà le comportement du système devient diffusif. En outre, la température de transition de phase dépend uniquement de la masse totale de μ_0. ♡

1.b Descente de gradient

Otto [59] a montré que la dynamique de (H) pouvait s'interpréter comme une descente de gradient dans une « variété riemannienne ». La fonctionnelle de Lyapounov correspondant à cette descente de gradient est l'*énergie libre* du système :

$$\mathscr{F} := \mathscr{U} + T\mathscr{S} \qquad [\mathsf{E}], \tag{I}$$

où $\mathscr{U}$ est l'*énergie interne* :

$$\mathscr{U} := \frac{1}{2}\int_{(\mathbb{R}^d)^2} v(x-y)\,d\mu(x)d\mu(y) \qquad [\mathsf{E}], \tag{J}$$

et $\mathscr{S}$ est l'*entropie* [†] (définie à constante additive près) :

$$\mathscr{S} := \int_{\mathbb{R}^d} \log m(x)\,d\mu(x) \qquad [\mathsf{N}]. \tag{K}$$

1.c Espace de Wasserstein

L'espace fonctionnel associé à la descente de gradient d'Otto est, comme nous l'avons dit, une pseudo variété riemannienne (de dimension infinie, avec des singularités) ; seule sa structure métrique nous importera ici. Cette métrique est celle associé à la distance de Wasserstein W_2, dont nous rappelons la définition ci-dessous.

1.5 Définition ([82, définition 7.1.1])**.** Soient μ, ν [N] deux mesures positives σ-finies sur $\mathbb{R}^d$. Un *couplage* entre μ et ν est une mesure γ sur $\mathbb{R}^d \times \mathbb{R}^d$ dont les deux marginales sont respectivement μ et ν. Le *coût quadratique* de ce couplage est

$$I[\gamma] := \int_{(\mathbb{R}^d)^2} |y-x|^2\,d\gamma(x,y) \qquad [\mathsf{N.L}^2]. \tag{L}$$

Notant $\Gamma(\mu,\nu)$ l'ensemble des couplages entre μ et ν [‡], la *distance de Wasserstein* $W_2(\mu,\nu)$ entre μ et ν est alors définie comme

$$W_2(\mu,\nu) := \inf_{\gamma\in\Gamma(\mu,\nu)} I[\gamma]^{1/2} \qquad [\mathsf{N}^{1/2}.\mathsf{L}]. \tag{M}$$

(On montre facilement qu'il s'agit effectivement d'une distance (à valeurs dans $[0,+\infty]$) sur l'ensemble des mesures sur $\mathbb{R}^d$). ♢

[†]. Qu'un physicien appellerait « néguentropie ».
[‡]. Éventuellement vide si μ et ν n'ont pas la même masse totale.

Notre objectif étant de regarder ce qui se passe au voisinage d'une distribution uniforme, il est alors naturel de se placer dans un espace que j'appellerai dans la suite *espace de Wasserstein*, défini ci-dessous :

1.6 Notation. Dans toute la suite de ce travail, on se fixe une densité non triviale $0 < R < \infty$ $[\mathsf{N.L}^{-d}]$. Nous appellerons « mesure uniforme », notée λ, la mesure de densité uniforme R par rapport à la mesure de Lebesgue :

$$d\lambda(x) := R\,dx \qquad [\mathsf{N}]. \tag{N}$$

Lorsqu'on considèrera une mesure de distribution de particules μ $[\mathsf{N}]$, on lui associera implicitement la mesure signée (de même homogénéité)

$$\pi := \mu - \lambda; \tag{O}$$

de même, on associera à la densité m $[\mathsf{N.L}^{-d}]$ de la mesure μ la densité p (de même homogénéité) de la mesure π :

$$p(x) := m(x) - R. \tag{P}$$

♢

1.7 Définition (espace de Wasserstein)**.** L'*espace de Wassertein*, noté $\mathbf{F}$, est l'ensemble des mesures μ telles que $W_2(\lambda,\mu) < \infty$, muni de la distance de Wasserstein W_2. ♢

1.8 Remarque. L'espace de Wasserstein est foncièrement *non linéaire*. Ainsi, si deux mesures μ_1, μ_2 de $\mathbf{F}$ sont associées repectivement à π_1, π_2, on prendra garde que cela n'a même pas de sens de parler de « $W_2(\pi_1,\pi_2)$ », vu que les mesures π_1 et π_2 ne sont pas positives. ♡

$\mathbf{F}$ est en fait l'espace naturellement adapté non seulement à l'étude de la dynamique continue du système, mais aussi à celle des *fluctuations* de la dynamique réelle du système autour de cette équation — car il faut garder à l'esprit qu'en réalité, le modèle est constitué d'un nombre très grand mais fini de particules. Une façon d'exprimer cela est la proposition immédiate suivante :

1.9 Proposition. *Soit* $f : \mathbb{R}^d \to \mathbb{R}$ $[\mathsf{X.N}^{-1}]$ *une fonction de classe* $\mathscr{C}^2$ *; pour* μ $[\mathsf{N}]$ *une mesure sur* $\mathbb{R}^d$*, notons*

$$F(\mu) := \langle f, \mu \rangle \qquad [\mathsf{X}]. \tag{Q}$$

Considérons N *particules* $X_1, \ldots, X_N \in \mathbb{R}^d$*, chacune étant soumise à une agitation brownienne de variation quadratique par unité de temps (dans chaque direction)* TJ^{-1} $[\mathsf{L}^2.\mathsf{T}^{-1}]$*, et notons*

$$\hat{\mu} := \mathscr{N}^{-1} \sum_{i=1}^{N} \boldsymbol{\delta}_{X_i} \qquad [\mathsf{N}] \tag{R}$$

leur distribution empirique. Alors la fonctionnelle $F(\mu)$ *est soumise à des fluctuations dont la variation quadratique par unité de temps est égale à* $\mathscr{N}^{-1}TJ^{-1}|\nabla_{\mathbf{F}}F(\mu)|^2$ $[\mathsf{X}^2.\mathsf{T}^{-1}]$*, où le gradient de la fonctionnelle* F *dans* $\mathbf{F}$ *est défini formellement par :*

$$|\nabla_{\mathbf{F}}F(\mu)| := \overline{\lim_{\varepsilon \searrow 0}} \Big\{ \frac{|F(\nu) - F(\mu)|}{W_2(\mu,\nu)} : \ 0 < W_2(\mu,\nu) \leqslant \varepsilon \Big\} \qquad [\mathsf{X.N}^{-1/2}.\mathsf{L}^{-1}]^{[\S]}. \tag{S}$$

En d'autres termes, les fluctuations de $\hat{\mu}$ *sont analogues à celles d'un « mouvement brownien » (de dimension infinie) sur la « variété riemannienne »* $\mathbf{F}$*, dont la variation quadratique par unité de temps est* $\mathscr{N}^{-1}J^{-1}T$ $[\mathsf{N.L}^2.\mathsf{T}^{-1}]$. ♣

[§]. On calcule que

$$|\nabla_{\mathbf{F}}F(\mu)| = \Big(\int_{\mathbb{R}^d} |\nabla f(x)|^2 \, d\mu(x) \Big)^{1/2}. \tag{T}$$

Cela dit, compte tenu de l'interprétation en descente de gradient, nous oublierons en général les aspects dynamiques du modèle pour nous concentrer sur la structure statique de $\mathscr{F}$ dans $\mathbf{F}$.

1.d Stabilité

Notre but est d'étudier la stabilité du système autour de l'équilibre λ. Commençons par observer que, pour μ une fonction de $\mathbf{F}$, les expressions (J) et (K) sont infinies *stricto sensu* ; il faut donc les *renormaliser* (en leur enlevant formellement une constante) afin de leur donner une expression convergente. Dans la suite de ce travail, on prendra donc :

$$\mathscr{U}(\mu) := \frac{1}{2}\int_{(\mathbb{R}^d)^2} v(y-x)\big(m(x)m(y)-R^2\big)\,dx\,dy; \tag{U}$$

$$\mathscr{S}(\mu) := \int_{\mathbb{R}^d} \log\big(R^{-1}m(x)\big)\,d\mu(x). \tag{V}$$

Cette renormalisation, faite de sorte que $\mathscr{U}(\lambda), \mathscr{S}(\lambda) = 0$, donne également des expressions sympathiques en fonction de π : posant, pour p [N.L^{-d}],

$$\Phi(p) := (R+p)\log\big(1+R^{-1}p\big) - p \qquad [\mathrm{N.L}^{-d}], \tag{W}$$

on a [¶] :

$$\mathscr{U} = \frac{1}{2}\int_{\mathbb{R}^d} (v * p)(x)\,d\pi(x); \tag{X}$$

$$\mathscr{S} = \int_{\mathbb{R}^d} \Phi\big(p(x)\big)\,dx. \tag{Y}$$

1.10 Remarque. On a $\Phi(p) \geqslant 0$ pour tout p, et $\Phi(p) \overset{p\to 0}{\sim} \frac{1}{2}R^{-1}p^2$. ♡

On définit rigoureusement la stabilité de la façon suivante :

1.11 Définition (stabilité)**.** Pour une température T donnée, nous dirons que l'équilibre homogène est *stable* quand la fonctionnelle $\mathscr{F}$ sur $\mathbf{F}$ atteint un minimum local en λ, *i.e.* quand il existe un voisinage de λ dans $\mathbf{F}$ sur lequel on a $\mathscr{F}(\mu) \geqslant \mathscr{F}(\lambda)$. ♢

1.12 Remarque. À mes yeux, la « bonne » définition de la stabilité [‖] est plutôt la suivante : un équilibre est stable quand tout chemin Lipschitzien $\gamma\colon \mathbb{R}_+ \to \mathbf{F}$ issu de λ vérifie $\mathscr{F}(\gamma(t)) \geqslant \mathscr{F}(\lambda)$ au voisinage de 0 — ce qui est un peu moins contraignant que la définition 1.11.

Le même genre de remarque vaudra pour la définition 1.13. ♡

Quand l'équilibre est stable, la définition suivante nous permet de *quantifier* la stabilité de l'équilibre :

1.13 Définition (énergie d'activation)**.** Quand l'équilibre homogène est stable, son *énergie d'activation* E_{a} [E] est définie comme le supremum des valeurs E vérifiant la propriété suivante : sur la composante connexe de λ au sein du sous-ensemble de $\mathbf{F}$ constitué par les fonctions $\{\mu : \mathscr{F}(\mu) \leqslant \mathscr{F}(\lambda) + E\}$, $\mathscr{F}$ atteint un minimum global en λ. ♢

[¶]. Pour obtenir (X) et (Y), on utilise que pour $\mu \in \mathbf{F}$ on a formellement $\int_{\mathbb{R}^d} d\pi(x) = 0$ — je précise « formellement », car $\int \pi$ n'est pas définie proprement sur $\mathbf{F}$ dès que $d > 2$.

[‖]. Établie d'après des considérations sur des situations similaires en dimension finie.

1.14 Remarque. L'énergie d'activation indique l'énergie minimale qu'il faut fournir au système pour passer de l'état λ à un autre état plus stable. Pour $E_{\mathrm{a}} = 0$, il vaudrait d'ailleurs mieux qualifier l'équilibre λ de *métastable*, puisque la moindre « pichenette » suffit alors à briser cet équilibre.

On peut aussi interpréter l'énergie d'activation en termes de probabilité de flambage spontané : pour un système initialement homogène, en l'absence d'intervention extérieure il faudra attendre un temps (très long !) de l'ordre de $\exp(\mathcal{N}E_{\mathrm{a}}/T)$ fois le temps typique de l'évolution (macroscopique) du système [*] avant d'observer une brisure spontanée de symétrie du seul effet des fluctuations. ♡

1.15 Remarque. Pour peu que la formule (G) s'applique, l'état homogène ne correspond jamais à un minimum absolu de la fonctionnelle $\mathscr{F}$. Considérons en effet la mesure $\mu^\dagger$ obtenue à partir de la mesure λ ainsi : sur une grande boule (de rayon $\gg L_0$) de volume noté B $[\mathsf{L}^d]$, on porte la densité de la mesure à AR pour un certain $A \gg 1$ [1], la masse nécessaire à cette opération ayant été prélevée sur une boule encore plus grande de volume $(A-1)B$. Alors, au premier ordre en B, l'énergie interne (renormalisée) de cette mesure est $-\frac{1}{2}(A-1)AR^2VB$ et son entropie est $-(A\log A)RB$, d'où

$$\mathscr{F}(\mu^\dagger) \sim AR\big(-\tfrac{1}{2}(A-1)RV + (\log A)T\big)B, \tag{Z}$$

qui est strictement négatif pour peu que A soit choisi suffisamment grand.

À partir de cet exemple, on peut même montrer la propriété un peu plus forte selon laquelle l'énergie d'activation de l'équilibre homogène est toujours finie. ♡

Ainsi, nous avons maintenant un objectif mathématique précis : déterminer pour quelles valeurs de T l'équilibre homogène est stable, et minorer l'énergie d'activation dans les situations de stabilité.

2 Minoration de l'énergie libre

☛ *Dans cette section et les suivantes, nous supposons le potentiel v négatif sur tout $\mathbb{R}^d$. (Le cas général sera étudié au § 5.c).*

Pour montrer que l'énergie libre $\mathscr{F}$ atteint un minimum en λ, il nous faut *minorer* judicieusement cette quantité au voisinage de 0. Nous allons donc chercher des minorations de $\mathscr{S}$ et de $\mathscr{U}$.

2.a Entropie

Rappelons que l'entropie du système est $\mathscr{S} = \int_{\mathbb{R}^d} \Phi\big(p(x)\big)\,dx$, où $\Phi(p)$ est défini par (W) ; pour minorer $\mathscr{S}$, il semble donc naturel de minorer $\Phi(\cdot)$. On a vu que $\Phi(p)$ était équivalent à $\frac{1}{2}R^{-1}p^2$ au voisinage de 0, mais il n'y a pas de minoration quadratique globale. On introduit donc un paramètre $\eta \in (0,+\infty)$ $[\mathsf{N}.\mathsf{L}^{-d}]$, et on décompose p en $p_2 + p_1$, où

$$\begin{cases} p_2 := \mathbf{1}_{|p| \leqslant \eta} p; \\ p_1 := \mathbf{1}_{|p| > \eta} p. \end{cases} \tag{AA}$$

L'étude de la fonction Φ donne alors que pour tout p,

$$\Phi(p) \geqslant \frac{\Phi(\eta)}{\eta^2} p_2^2 + \frac{\Phi(\eta)}{\eta}|p_1|. \tag{AB}$$

[*]. NdA : Ce temps typique est $L_0^2 R^{-1} J V^{-1}$ [T].

Intégrant (AB), on obtient une première minoration de $\mathscr{S}$:

$$\mathscr{S} \geqslant \frac{\Phi(\eta)}{\eta^2}\|p_2\|_2^2 + \frac{\Phi(\eta)}{\eta}\|p_1\|_1. \tag{AC}$$

Remarquez que quand $\eta \searrow 0$, on a $\Phi(\eta)/\eta^2 \searrow \frac{1}{2}R^{-1}$ et $\Phi(\eta)/\eta \searrow 0$.

Nous allons maintenant établir une seconde minoration de $\mathscr{S}$. L'idée est d'utiliser un résultat classique sur les chaînes de Markov :

2.1 Théorème ([20, § 4.4.2])**.** *Si P est le noyau d'une chaîne de Markov admettant une mesure invariante μ sur un espace Ω, alors pour toute mesure ν sur Ω, l'entropie relative [20, § 2.3] de ν par rapport à μ est décroissante sous l'action de P :*

$$D_{\mathrm{KL}}(\nu P\|\mu) \leqslant D_{\mathrm{KL}}(\nu\|\mu). \tag{AD}$$

♣

Nous allons appliquer le théorème 2.1 dans le cadre suivant : la chaîne de Markov que nous considérons est la marche aléatoire sur $\mathbb{R}^d$ dont les pas sont distribués suivant la mesure de probabilité K définie par

$$dK(x) := \frac{-v(x)}{V}dx^{[\dagger]}, \tag{AE}$$

et la mesure invariante que nous considérons pour cette chaîne est la mesure uniforme λ. Dans ce cas, sous l'action du noyau de la chaîne, la densité m est transformée en $V^{-1}v * m$, de sorte que p est transformée en $V^{-1}v * p$. (AD) donne alors :

$$\mathscr{S}(p) \geqslant \mathscr{S}(V^{-1}v * p). \tag{AF}$$

Bien que cette borne n'ait pas l'air très intéressante, il se trouvera que $v * p$ peut être contrôlé bien plus facilement que p. Pour l'instant, contentons-nous d'appliquer (AC) à (AF) sous la forme du

2.2 Lemme. *Si $\|v * p\|_\infty \leqslant V\eta$, alors*

$$\mathscr{S} \geqslant \frac{\Phi(\eta)}{\eta^2}V^{-2}\|v * p\|_2^2. \tag{AG}$$

♣

2.b Énergie interne

Nous voulons maintenant minorer $\mathscr{U}$. D'après (X), on a $\mathscr{U} = \frac{1}{2}\langle p, v * p\rangle_{L^2}$; en décomposant p en $p_2 + p_1$ dans le facteur de gauche, on en déduit que

$$-\mathscr{U} \leqslant \frac{1}{2}(\|v * p\|_\infty\|p_1\|_1 + \|p_2\|_2\|v * p\|_2), \tag{AH}$$

d'où par application de l'inégalité de Young :

$$-\mathscr{U} \leqslant \frac{\|v * p\|_\infty}{2}\|p_1\|_1 + \frac{V}{4}\|p_2\|_2^2 + \frac{1}{4V}\|v * p\|_2^2. \tag{AI}$$

[†]. K est bien une mesure de probabilité car nous avons supposé v négatif, ce qui fait que d'une part la mesure K est positive, d'autre part qu'elle est d'intégrale 1 au vu de la remarque 1.3.

Dans cette formule, sous réserve que $\|v * p\|_\infty \leqslant V\eta$, on a par le lemme 2.2 :

$$\frac{1}{4V}\|v * p_2\|_2^2 \leqslant \frac{\eta^2}{\Phi(\eta)}\frac{V}{4}\mathscr{S}, \tag{AJ}$$

et sous réserve que $\|v * p\|_\infty \leqslant V\eta/2$ (condition impliquant la précédente), par (AC) :

$$\frac{\|v * p\|_\infty}{2}\|p_1\|_1 + \frac{V}{4}\|p_2\|_2^2 \leqslant \frac{\eta^2}{\Phi(\eta)}\frac{V}{4}\mathscr{S}. \tag{AK}$$

Au final,

$$\|v * p\|_\infty \leqslant \frac{V\eta}{2}\ \ [\mathsf{E.N}^{-1}] \quad \Rightarrow \quad -\mathscr{U} \leqslant \frac{\eta^2}{\Phi(\eta)}\frac{V}{2}\mathscr{S}, \tag{AL}$$

ce qui donne la minoration suivante sur $\mathscr{F}$:

$$\|v * p\|_\infty \leqslant \frac{V\eta}{2}\ \ [\mathsf{E.N}^{-1}] \quad \Rightarrow \quad \mathscr{F} \geqslant \Big(T - \frac{\eta^2}{\Phi(\eta)}\frac{V}{2}\Big)\mathscr{S}, \tag{AM}$$

où nous rappelons que $\mathscr{S}$ est toujours positive.

3 Plongement de l'espace de Wasserstein dans un espace de Sobolev

Travailler dans l'espace $\mathbf{F}$ est délicat, car celui-ci a uniquement une structure métrique et pas linéaire. Dans cette section, nous allons montrer qu'on peut voir $\mathbf{F}$ comme une partie d'un espace de Banach classique ; plus précisément, nous allons chercher des espaces de mesures $\mathbf{E}$ pour lesquels le plongement canonique [‡] de $\mathbf{F}$ dans $\mathbf{E}$ est continu.

3.1 Notation. Pour $\alpha \in [0,1]$, nous notons $\mathbf{W}^{2-\alpha,2/\alpha} =: \mathbf{W}_\alpha$. ♢

3.a Un théorème simple

3.2 Théorème. *Le plongement canonique de* $\mathbf{F}$ *dans* $\mathbf{W}_0'$ *— plus précisément, l'application* $\mu \mapsto \pi$ *— est continu en* λ. ♣

Démonstration. Soit f [X] une fonction de $\mathbf{W}_0$ [§] et soit T un plan de transport de λ vers $\mu \in \mathbf{F}$, c'est-à-dire que T est une application de $\mathbb{R}^d$ (affine) dans $\mathbb{R}^d$ (vectoriel) et qu'on considère la mesure de couplage entre λ et $\mu := (\mathrm{Id} + T)\#\lambda$ portée par le graphe de $(\mathrm{Id} + T)$. On introduit le raccourci $u(x) := T(x)/|T(x)|$ [1], un vecteur unitaire qui indique la direction dans laquelle le point x se déplace au cours de l'opération de transport.

On écrit :

$$|\langle f, \pi\rangle| = \Big|\int_{\mathbb{R}^d}(f(x+T(x)) - f(x))\,d\lambda(x)\Big| \leqslant R\int_{x\in\mathbb{R}^d}\int_{r=0}^{|T(x)|}|Df(x+ru(x))|\,dr\,dx \qquad [\mathsf{X.N}], \tag{AN}$$

à comparer au coût de transport qui est

$$I[T] = \int_{\mathbb{R}^d}|T(x)|^2\,d\lambda(x) = 2R\int_{x\in\mathbb{R}^d}\int_{r=0}^{|T(x)|} r\,dr\,dx \qquad [\mathsf{N.L}^2]. \tag{AO}$$

[‡]. $\mathbf{F}$ et $\mathbf{E}$ étant tous les deux des espaces de distributions tempérées, le plongement canonique de $\mathbf{F}$ dans $\mathbf{E}$ est l'application qui correspond à l'injection canonique quand on la voit dans l'espace des distributions.

[§]. Rappelons que $\mathbf{W}_0 = W^{2,\infty}$, c.-à-d. que les éléments de $\mathbf{W}_0$ sont les fonctions dont la dérivée est lipschitzienne et de carré intégrable.

En comparant les deux expressions, on s'aperçoit qu'il est intéressant de considérer la valeur de $|Df(x+ru(x))|/2r$, quantité que nous noterons $e(x,r)$ $[\mathsf{X.L}^{-2}]$. Notant $\Omega := \{(x \in \mathbb{R}^d, r \ [L]) : 0 \leqslant r \leqslant |T(x)|\}$ et $d\gamma(x,r) := 2Rr\,dxdr$ $[\mathsf{N.L}^2]$, on peut alors résumer les lignes précédentes en :

$$I[T] = \int_{(x,r)\in\Omega} d\gamma(x,r); \tag{AP}$$

$$|\langle f,\pi\rangle| \leqslant \int_{(x,r)\in\Omega} e(x,r)\,d\gamma(x,r). \tag{AQ}$$

On va maintenant utiliser un lemme de couplage :

3.3 Lemme. *Soit γ $[\mathsf{X}]$ une mesure (positive) σ-finie sur un espace mesurable Ω, et $e : \Omega \to \mathbb{R}_+$ $[\mathsf{Y.X}^{-1}]$ une fonction mesurable. On suppose qu'il existe une fonction $Y_* : \mathbb{R}_+$ $[\mathsf{Y.X}^{-1}] \to \mathbb{R}_+$ $[\mathsf{Y}]$, de classe $\mathscr{C}^1$ et décroissante avec $Y_*(e) \overset{e\to\infty}{\to} 0$, telle que pour tout $\theta \in \mathbb{R}_+$,*

$$\int_{\{e(\omega)\geqslant\theta\}} e(\omega)\,d\gamma(\omega) \leqslant Y_*(\theta). \tag{AR}$$

Définissons à partir de Y_* la fonction $\mathscr{C}^1$ décroissante $X_* : \mathbb{R}_+$ $[\mathsf{Y.X}^{-1}] \to \mathbb{R}_+$ $[\mathsf{X}]$ par

$$X_*(\theta) := \int_\theta^\infty \frac{-Y_*'(\tau)}{\tau}\,d\tau; \tag{AS}$$

on a alors :

$$\int_\Omega e(\omega)\,d\gamma(\omega) \leqslant (Y_* \circ X_*^{-1})\Big(\int_\Omega d\gamma(\omega)\Big). \tag{AT}$$

♣

Démonstration. Pour tout $\theta \in \mathbb{R}_+$ $[\mathsf{Y.X}^{-1}]$, posons $\Omega_\theta := \{\omega \in \Omega : \ e(\omega) \geqslant \theta\}$, et définissons

$$\begin{cases} X(\theta) := \int_{\Omega_\theta} d\gamma(\omega) & [\mathsf{X}]; \\ Y(\theta) := \int_{\Omega_\theta} e(\omega)\,d\gamma(\omega) & [\mathsf{Y}]. \end{cases} \tag{AU}$$

La condition (AR) se réécrit alors "$\forall\theta \ \ Y(\theta) \leqslant Y_*(\theta)$". Maintenant, comme X et Y sont des fonctions monotones de θ, on peut également exprimer θ et Y en fonction de X, ou θ et X en fonction de Y. Comme les fonctions $\theta \mapsto Y$ et Y_* sont décroissantes, en inversant ces fonctions, (AR) devient équivalente à "$\forall y \ \ \theta(y) \leqslant Y_*^{-1}(y)$" ; le lemme consiste alors à montrer que, notant $x_0 := X(\theta = 0)$, on a $Y(x_0) \leqslant (Y_* \circ X_*^{-1})(x_0)$. En fait, nous allons carrément prouver cette propriété pour tout x. En inversant les fonctions (qui, cette fois-ci, sont croissantes), la propriété est équivalente à montrer que pour tout y, on a $X(y) \geqslant (X_* \circ Y_*^{-1})(y)$.

Dérivant (AU), on a

$$dY = \theta\,dX, \tag{AV}$$

d'où

$$X(y) = \int_y^\infty \frac{1}{\theta(\acute{y})}\,d\acute{y} \geqslant \int_y^\infty \frac{1}{Y_*^{-1}(\acute{y})}\,d\acute{y} = \int_y^\infty -d(X_* \circ Y_*^{-1}) = (X_* \circ Y_*^{-1})(y), \tag{AW}$$

ce qui est le résultat annoncé. ♠

Grâce au lemme 3.3, nous devons maintenant majorer $\int_{(x,r)\in\Omega_\theta} |Df(x+ru(x))|\,dxdr$ (où Ω_θ est défini comme dans le lemme) pour θ $[\mathsf{X.L}^{-2}]$ arbitraire — on supposera juste $\theta > \frac{1}{2}\|f\|_{2,\infty}$.

Or "$(x,r) \in \Omega_\theta$" signifie que $|Df(x+ru(x))| \geqslant 2\theta r$, d'où $|Df(x)| \geqslant (2\theta - \|f\|_{2,\infty})r$, ou encore $r \leqslant (2\theta - \|f\|_{2,\infty})^{-1}|Df(x)|$. On a en outre $|Df(x+ru(x))| \leqslant |Df(x)| + r\|f\|_{2,\infty}$, d'où la majoration :

$$\begin{aligned}\int_{(x,r)\in\Omega_\theta} |Df(x+ru(x))|\,dx &\leqslant \int_{r\leqslant \frac{|Df(x)|}{2\theta-\|f\|_{2,\infty}}} \big(|Df(x)| + r\|f\|_{2,\infty}\big)\,dxdr \\ &= \int_{\mathbb{R}^d} \Big(\frac{|Df(x)|^2}{2\theta - \|f\|_{2,\infty}} + \frac{\|f\|_{2,\infty}|Df(x)|^2}{2(2\theta-\|f\|_{2,\infty})^2}\Big)\,dx \\ &= \Big(\frac{1}{2\theta-\|f\|_{2,\infty}} + \frac{\|f\|_{2,\infty}}{2(2\theta-\|f\|_{2,\infty})^2}\Big)\|f\|_{1,2}^2. \qquad \text{(AX)}\end{aligned}$$

On est alors en mesure d'appliquer le lemme 3.3 avec $Y_*(\theta) = R\|f\|_{1,2}^2((2\theta - \|f\|_{2,\infty})^{-1} + \frac{1}{2}\|f\|_{2,\infty}(2\theta - \|f\|_{2,\infty})^{-2})$, et on trouve $X_*(\theta) = R\|f\|_{1,2}^2/(2\theta - \|f\|_{2,\infty})^2$, d'où *in fine* :

$$|\langle f,\pi\rangle| \leqslant \sqrt{R}\|f\|_{1,2} I[T]^{1/2} + \frac{1}{2}\|f\|_{2,\infty} I[T], \qquad \text{(AY)}$$

et donc en prenant l'infimum sur les plans de transport de λ à μ :

$$|\langle f,\pi\rangle| \leqslant \sqrt{R}\|f\|_{1,2} W_2(\lambda,\mu) + \frac{1}{2}\|f\|_{2,\infty} W_2(\lambda,\mu)^2. \qquad \text{(AZ)}$$

On a donc, pour $W_2(\lambda,\mu) \leqslant \varepsilon$, $|\langle f,\pi\rangle| \leqslant (\sqrt{R} + \varepsilon/2)W_2(\lambda,\mu)\|f\|_{\mathbf{W}_0}$ [*inhomogène*]. Dans la mesure où cela est vrai pour toute f, cela signifie que $\|\pi\|_{\mathbf{W}_0'} \leqslant (\sqrt{R} + \varepsilon/2)W_2(\lambda,\mu)$, ce qui montre bien que le plongement canonique de $\mathbf{F}$ dans $\mathbf{W}_0'$ est continu (et même lipschitzien) en λ. ♠

3.4 Remarque. L'étude linéarisée de l'espace de Wasserstein [82, § 7.6] montre qu'on a en fait $W_2(\lambda,\mu) \overset{\mu\to\lambda}{\sim} \sqrt{R}\|p\|_{-1,2}$, où $\|\cdot\|_{-1,2}$ est la norme de Sobolev homogène hilbertienne d'ordre -1. La formule (AZ) est donc optimale au premier ordre, au sens où le terme dominant $\sqrt{R}\|f\|_{1,2}W_2(\mu,\nu)$ ne peut être amélioré. ♡

3.b Un théorème plus fin

Nous allons maintenant donner un théorème qui améliore le théorème 3.2 sur les points suivants :

- On montre que le plongement de $\mathbf{F}$ est continu partout, et plus seulement en λ ;
- L'espace d'arrivée $\mathbf{E}$ est strictement plus fin que dans le cas précédent ; en outre, il est cette fois-ci réflexif (pour $\alpha > 0$).

3.5 Notation. On pose

$$\overline{\alpha} := \frac{2}{d+2}. \qquad \text{(BA)}$$

◇

3.6 Théorème. *Pour $\alpha < \overline{\alpha}$, le plongement canonique de $\mathbf{F}$ dans $\mathbf{W}_\alpha'$ est continu (partout).* ♣

3.7 Remarque. On retrouve le théorème 3.2 comme corollaire pour $\alpha = 0$. Pour $\alpha \leqslant \alpha'$, la théorie de l'interpolation [3, Chap. 6] énonce par ailleurs qu'on a un plongement continu $\mathbf{W}_{\alpha'}' \to \mathbf{W}_\alpha'$, de sorte que le théorème 3.6 est d'autant plus fort que α est grand. ♡

Démonstration. Considérons un plan de transport en deux étapes (T_1, T_2), c'est-à-dire qu'un point situé en x est envoyé d'abord en $x + T_1(x)$, puis en $x + T_1(x) + T_2(x)$. Notons μ_1, μ_2 les images

successives de λ par ce plan de transport, *i.e.* $\mu_1 := (\mathrm{Id}+T_1)\#\lambda$, $\mu_2 := (\mathrm{Id}+T_1+T_2)\#\lambda$; notons I_1 le coût de la première partie du plan de transport, et I_2 le coût de la seconde partie, *i.e.* $I_1 := \int_{\mathbb{R}^d} |T_1(x)|^2\, d\lambda(x)$, $I_2 := \int_{\mathbb{R}^d} |T_2(x)|^2\, d\lambda(x)$ [N.L^2]. Notre objectif sera, pour une fonction gentille f [X], de majorer $|\langle f, \mu_2-\mu_1\rangle|$ par une expression de la forme $G(I_1,I_2)\|f\|_{\mathbf{W}_\alpha}$, où G est croissante en chacune de ses deux variables avec $G(a,b) \overset{b\searrow 0}{\to} 0$ pour tout a fixé. Une telle majoration entraînera le théorème de la même façon que dans la preuve du théorème 3.2, et montrera même que le plongement de $\mathbf{F}$ dans $\mathbf{W}'_\alpha$ est uniformément continu sur tout borné.

On a

$$\langle f, \mu_2-\mu_1\rangle = R\int_{\mathbb{R}^d} \big(f(x+T_1(x)+T_2(x)) - f(x+T_1(x))\big)\, dx \qquad [\mathrm{X.N}]; \tag{BB}$$

dans la suite, nous poserons pour alléger les notations

$$f(x+T_1(x)+T_2(x)) - f(x+T_1(x)) =: \Delta_2 f(x) \qquad [\mathrm{X}]. \tag{BC}$$

Pour $0<\beta\leqslant 1$ [1] un paramètre que nous ne fixons pas encore, notons

$$q(x) := \beta|T_1(x)|^2 + |T_2(x)|^2 \qquad [\mathrm{L}^2], \tag{BD}$$

et sur $\Omega := \mathbb{R}^d$, définissons la mesure

$$d\gamma(x) := q(x)\, d\lambda(x) \qquad [\mathrm{N.L}^2], \tag{BE}$$

de sorte que $\int_{\mathbb{R}^d} d\gamma(x) = \beta I_1 + I_2$. On définit maintenant

$$e(x) := \frac{|\Delta_2 f(x)|}{q(x)} \qquad [\mathrm{X.L}^{-2}], \tag{BF}$$

de sorte que

$$|\langle f, \mu_2-\mu_1\rangle| \leqslant \int_{\mathbb{R}^d} e(x)\, d\gamma(x). \tag{BG}$$

L'idée est alors, comme dans la preuve précédente, d'appliquer le lemme 3.3 — même si l'espace d'intégration Ω sera différent.

Pour la suite de la preuve, nous avons besoin de « mailler » $\mathbb{R}^d$:

3.8 Définition. Dans la suite du texte, l'espace physique est muni d'un repère orthonormé arbitraire, et on fixe une longueur de référence arbitraire $0<L_0<\infty$ [L].

Un *cube* de $\mathbb{R}^d$ désignera un ensemble s'écrivant en coordonnées sous la forme $Q = \prod_{i=1}^d [x_i, x_i+a]$ pour un $a\in(0,\infty)$ [L] qu'on appellera le *côté* du cube, noté $a(Q)$.

Un cube Q sera dit *dyadique* quand $a(Q)$ est de la forme $2^v L_0$ pour un $v\in\mathbb{Z}$ et que les $x_i/(2^v L_0)$ sont tous entiers ou demi-entiers (*i.e.* tous dans $\frac{1}{2}\mathbb{Z}$). ♢

On a le lemme suivant :

3.9 Lemme. *Toute boule de $\mathbb{R}^d$ de rayon $r\in(0,\infty)$ [L] est contenue dans un cube dyadique Q tel que $(2r\leqslant)a(Q)\leqslant 8r$.* ♣

Démonstration. Si $r=L_0/4$, la boule est contenue dans le cube dyadique de côté L_0 dont le centre est le point du réseau $\frac{1}{2}L_0\mathbb{Z}^d$ le plus proche du centre de la boule. Par homothétie, si $r=2^v L_0$ pour v entier, la boule est contenue dans un cube dyadique de côté $4r$. Enfin dans le cas où r est quelconque, il existe un $r'\in[r,2r]$ de la forme $2^v L_0$, et pour cet r' la boule de rayon r est contenue dans une boule de rayon r', elle-même contenue dans un cube dyadique de côté $4r'\leqslant 8r$. ♠

Soit $\theta > 0$ [X.L^{-2}]. Pour $x \in \Omega_\theta$, notant $y_1 := x + T_1(x)$ et $y_2 := y_1 + T_2(x)$, on a $|f(y_2) - f(y_1)| \geqslant \theta(\beta|y_1 - x|^2 + |y_2 - y_1|^2)$. Or notant $r := |y_1 - x| \vee \beta^{-1/2}|y_2 - y_1|$, on a $q(x) \geqslant \beta r^2$, de sorte que x, y_1 et y_2 sont dans la boule centrée sur y_1 de rayon r (rappelons qu'on suppose $\beta \leqslant 1$) et vérifient $|y_2 - y_1| \leqslant \beta^{1/2} r$ et $|f(y_2) - f(y_1)| \geqslant \beta\theta r^2$. Par le lemme 3.9, il existe donc un cube dyadique $Q \ni x, y_1, y_2$ tel que

$$\begin{cases} |y_2 - y_1| \leqslant \beta^{1/2} a(Q)/2; \\ |f(y_2) - f(y_1)| \geqslant \beta\theta a(Q)^2/64. \end{cases} \tag{BH}$$

3.10 Définition. Pour $0 < \beta \leqslant 1$ [1], $\theta > 0$ [X.L^{-2}], un cube dyadique Q sera dit (β,θ)-*correct* quand il contient deux points x_1, x_2 à distance $\leqslant \beta^{1/2} a(Q)$ tels que $|f(x_2) - f(x_1)| \geqslant \beta\theta a(Q)^2$. Dans la suite, on sous-entendra toujours que les cubes (β,θ)-corrects sont dyadiques. ♢

Avec ce vocabulaire *ad hoc*, la formule (BH) établit l'

3.11 Avis. *Pour tout $x \in \Omega_\theta$, il existe un cube (dyadique) $(\beta/4, \theta/16)$-correct de côté $\geqslant 2\beta^{-1/2}|T_2(x)|$ qui contient x et ses deux images successives par le plan de transport.* ♣

3.12 Définition. Un cube (dyadique) Q est dit (β,θ)-*maximal* quand il est (β,θ)-correct et qu'il n'est contenu dans aucun autre cube (β,θ)-correct. ♢

3.13 Remarque. Comme f est supposée gentille, donc bornée, la taille des cubes (β,θ)-corrects est bornée, de sorte que tout cube (β,θ)-correct est contenu dans au moins un cube (β,θ)-maximal. ♡

3.14 Lemme. *Si Q est un cube dyadique (β,θ)-maximal et $x_1, x_2 \in Q$ avec $|x_2 - x_1| \leqslant \beta^{1/2} a(Q)$, alors $|f(x_2) - f(x_1)| < 4\beta\theta a(Q)^2$.* ♣

Démonstration. Tout cube dyadique est contenu dans un cube dyadique de côté double ; soit donc $Q' \supset Q$ avec $r(Q') = 2a(Q)$. L'hypothèse sur (x_1, x_2) nous assure *a fortiori* que $x_1, x_2 \in Q'$ avec $|x_2 - x_1| \leqslant \beta^{1/2} r(Q')$; par conséquent, si par l'absurde on avait $|f(x_2) - f(x_1)| \geqslant 4\beta\theta a(Q)^2 = \beta\theta a(Q')^2$, Q' serait (β,θ)-correct, ce qui contredirait l'hypothèse de maximalité. ♠

Dans l'avis 3.11, comme l'hypothèse sur le côté du cube reste trivialement valide quand on passe à un cube plus grand, on peut appliquer le lemme 3.14 pour obtenir l'

3.15 Avis. *Tout point $x \in \Omega_\theta$ est contenu dans un cube $(\beta/4, \theta/16)$-maximal Q tel que*

$$|\Delta_2 f(x)| \leqslant \frac{\beta}{16}\theta a(Q)^2. \tag{BI}$$

♣

3.16 Corollaire. *Notant $\mathcal{Q}$ l'ensemble des cubes $(\beta/4, \theta/16)$-maximaux,*

$$\int_{\Omega_\theta} e(x)\, d\gamma(x) \leqslant \frac{\beta}{16} R\theta \sum_{Q \in \mathcal{Q}} a(Q)^{d+2}. \tag{BJ}$$

♣

3.17 Lemme. *Il existe des constantes (C_0, C_1) [(1,1)] finies ne dépendant que de (d et) α telles que, si Q est un cube (β,θ)-correct,*

$$\int_Q (C_0 \beta^{-1}\theta^{-2}|Df(x)|^2 + C_1 \beta^{-1/\alpha}\theta^{-2/\alpha}|D^{2-\alpha} f(x)|^{2/\alpha})\, dx \geqslant a(Q)^{d+2}. \tag{BK}$$

♣

Démonstration. Dans un premier temps, supposons que Q est le cube centré en 0 et de côté L_0. Comme nous avons supposé $\alpha < \overline{\alpha}$, l'inégalité de Sobolev [1, théorème 4.12] appliquée à Df sur Q [¶] énonce qu'il existe une constante $c < \infty$ [*inhomogène*] telle que $\|f\|_{1,\infty} \leqslant c\|f\|_{\mathbf{W}_\alpha}$, soit en termes homogènes qu'il existe des constantes (c_0, c_1) $[(\mathsf{L}^{-d/2}, \mathsf{L}^{1-(d/2+1)\alpha})]$ finies telles que

$$\sup_Q |Df| \leqslant c_0\Big(\int_Q |Df(x)|^2\,dx\Big)^{1/2} \vee c_1\Big(\int_Q |D^{2-\alpha}f(x)|^{2/\alpha}\,dx\Big)^{\alpha/2}. \tag{BL}$$

Comme le cube est (β,θ)-correct, il existe $x \in Q$ pour lequel $|Df(x)| \geqslant \beta^{1/2}\theta L_0$, de sorte que (BL) se traduit par

$$\int_Q |Df(x)|^2\,dx \geqslant c_0^{-2}\beta\theta^2 L_0^2 \qquad \text{ou} \qquad \int_Q |D^{2-\alpha}f(x)|^{2/\alpha}\,dx \geqslant c_1^{-2/\alpha}\beta^{1/\alpha}\theta^{2/\alpha}L_0^{2/\alpha}; \tag{BM}$$

dans un cas comme dans l'autre, (BK) est vérifiée avec $C_0 := L_0^d c_0^2$ et $C_1 := L_0^{d+2-2/\alpha} c_1^{2/\alpha}$.

Dans le cas général, si Q est centré en 0 et de côté r $[\mathsf{L}]$, on transforme la fonction f sur Q en une fonction $\check{f}$ sur le cube $\check{Q}$ du cas précédent en posant

$$\check{f}(x) := \Big(\frac{L_0}{r}\Big)^2 f\Big(\frac{r}{L_0}x\Big). \tag{BN}$$

Cette transformation conserve la propriété d'être (β,θ)-correct (au sens où $\check{Q}$ est (β,θ)-correct pour $\check{f}$), ainsi que l'inégalité (BK) dont elle multiplie chacun des membres par $(L_0/r)^{d+2}$, de sorte que la validité du lemme sur $\check{Q}$ entraîne sa validité sur Q. Enfin si Q n'est pas centré en 0, on s'y ramène par une translation, ce qui conserve trivialement la (β,θ)-correction et (BK). ♠

3.18 Lemme. *Tout point de $\mathbb{R}^d$ (de coordonnées toutes non dyadiques) est contenu dans au plus 2^d carrés (β,θ)-maximaux différents.* ♣

Démonstration. Pour $x = (x_i)_{1\leqslant i\leqslant d} \in \mathbb{R}^d$ (non dyadique) et Q un cube dyadique centré en c, j'appelle *quartier de x dans Q*, noté $\mathrm{sgn}_Q(x)$, le d-uplet $(\mathrm{sgn}(x_i - c_i))_{1\leqslant i\leqslant d} \in \{\pm 1\}^d$. Or pour tout $s \in \{\pm 1\}^d$, il existe un unique cube dyadique (β,θ)-maximal Q tel que $\mathrm{sgn}_Q(x) = s$: en effet, pour tout r de la forme $2^v L_0$, il existe un unique cube dyadique $Q_v(x,s)$ de côté r contenant x avec $\mathrm{sgn}_{Q_v(x,s)} = s$, et la suite des $(Q_v(x,s))_{v\in\mathbb{Z}}$ est (strictement) croissante pour l'inclusion, de sorte qu'un seul des ces cubes peut être (β,θ)-maximal. ♠

Grâce aux lemmes 3.17 et 3.18, l'avis 3.16 donne :

$$\int_{\Omega_\theta} e(x)\,d\gamma(x) \leqslant \frac{2^d R}{16}\big(2^{10}C_0\theta^{-1}\|f\|_{1,2}^2 + 2^{10\alpha}C_1\beta^{1-1/\alpha}\theta^{1-2/\alpha}\|f\|_{2-\alpha,2/\alpha}^{2/\alpha}\big). \tag{BO}$$

On peut maintenant appliquer le lemme 3.3. Dans les calculs qui suivent, la définition des constantes C_0 et C_1 est susceptible de changer à chaque ligne, mais ce seront toujours des constantes adimensionnées ne dépendant que de α. On trouve :

$$\begin{aligned} X_*(\theta\ [\mathsf{X.L}^{-2}]) &= R\int_\theta^\infty \big(C_0\|f\|_{1,2}^2\theta^{-3} + C_1\beta^{1-1/\alpha}\|f\|_{2-\alpha,2/\alpha}^{2/\alpha}\theta^{-2/\alpha-1}\big)\,d\theta \\ &\geqslant (C_0 R\|f\|_{1,2}^2\theta^{-2}) \vee (C_1\beta^{1-1/\alpha}R\|f\|_{2-\alpha,2/\alpha}^{2/\alpha}\theta^{-2/\alpha}) \qquad [\mathsf{N.L}^2]; \end{aligned} \tag{BP}$$

$$X_*^{-1}(x\ [\mathsf{N.L}^2]) \geqslant C_0 R^{1/2}\|f\|_{1,2}x^{-1/2} \vee C_1\beta^{(\alpha-1)/2}R^{\alpha/2}\|f\|_{2-\alpha,2/\alpha}x^{-\alpha/2} \qquad [\mathsf{X.L}^{-2}]; \tag{BQ}$$

[¶]. Le cube vérifie la condition de cône, de sorte qu'on peut bien y appliquer l'inégalité de Sobolev.

$$(Y_* \circ X_*^{-1})(x \ \ [\mathsf{N.L}^2]) \leqslant \Big(C_0 R^{1/2} \|f\|_{1,2} x^{1/2} \wedge C_0' \beta^{(1-\alpha)/2} R^{1-\alpha/2} \frac{\|f\|_{1,2}}{\|f\|_{2-\alpha,2/\alpha}} \|f\|_{1,2} x^{\alpha/2} \Big) +$$
$$\Big(C_1' \beta^{1-1/\alpha} R^{3/2-1/\alpha} \Big(\frac{\|f\|_{2-\alpha,2/\alpha}}{\|f\|_{1,2}} \Big)^{2/\alpha} \|f\|_{1,2} x^{1/\alpha-1/2} \wedge C_1 \beta^{(\alpha-1)/2} R^{\alpha/2} \|f\|_{2-\alpha,2/\alpha} x^{1-\alpha/2} \Big)$$
$$\leqslant C_0 R^{1/2} \|f\|_{1,2} x^{1/2} + C_1 \beta^{(\alpha-1)/2} R^{\alpha/2} \|f\|_{2-\alpha,2/\alpha} x^{1-\alpha/2}. \quad \text{(BR)}$$

Fixant $\beta = (I_2/I_1) \wedge 1$, on trouve enfin :

$$|\langle f, \mu_2 - \mu_1 \rangle| \leqslant C_0 R^{1/2} \|f\|_{1,2} (I_1 \wedge I_2)^{1/2} + C_1 R^{\alpha/2} \|f\|_{2-\alpha,2/\alpha} I_1^{(1-\alpha)/2} (I_1 \wedge I_2)^{1/2}, \quad \text{(BS)}$$

qui est bien une expression de la forme désirée — cela montre même que le plongement de $\mathbf{F}$ dans $\mathbf{W}'_\alpha$ est lipschitzien sur tout borné. ♠

4 Minoration d'une norme L^2 par la norme L^∞

Pour minorer l'énergie d'activation, nous aurons besoin de résultats de minoration de $\|v * p\|_2$ en fonction de $\|v * p\|_\infty$. *A priori* cela paraît impossible, mais en fait p n'est pas arbitraire : on sait en effet que la mesure μ est positive, ce qui va nous permettre de nous en sortir.

4.1 Lemme. *Supposons que v est (négatif et) gentil ; en particulier, qu'on a $\|D^2 v\|_\infty < \infty$ et $|D^2 v(x)| \overset{|x|\to\infty}{=} O(|x|^{-\kappa})$ pour un $\kappa > d$. Alors il existe une constante $C < \infty$ $[\mathsf{L}^{d/2}]$ ne dépendant que de v telle que pour toute mesure* positive *μ,*

$$\|v * p\|_2 \geqslant C \|v * m\|_\infty^{-d/4} \|v * p\|_\infty^{1+d/4}. \quad \text{(BT)}$$

♣

4.2 Remarque. Noter l'homogénéité de C, qui ne dépend que de la portée de v et pas de son intensité. ♡

Démonstration. Commençons par définir la « variance L^p » d'une fonction :

4.3 Définition. Pour $f\colon \mathbb{R}^d \to \mathbb{R}$ $[\mathsf{X}]$ une fonction mesurable, $1 \leqslant p \leqslant \infty$, on définit la semi-norme

$$\|f\|_{\overline{p}} := \inf_{a \in \mathbb{R}} \|f - a\|_p \qquad [\mathsf{X.L}^{d/p}]. \quad \text{(BU)}$$

♢

Avec cette définition, on a $\|v * p\|_2 = \|v * m\|_{\overline{2}}$ et $\|v * p\|_\infty = \|v * m\|_{\overline{\infty}}$, de sorte que (BT) se réécrit :

$$\forall f \geqslant 0 \qquad \|f * v\|_{\overline{2}} \geqslant C \left(\frac{\|f * v\|_{\overline{\infty}}}{\|f * v\|_\infty} \right)^{d/4} \|f * v\|_{\overline{\infty}}. \quad \text{(BV)}$$

Pour établir (BV), nous allons montrer l'existence d'une constante $c < \infty$ $[\mathsf{L}^{-2}]$ telle que $\|D^2(f * v)\|_\infty \leqslant c \|f * v\|_\infty$; la conclusion s'ensuivra, comme nous le montrons ci-dessous, avec

$$C = \Big(\int_{\substack{x \in \mathbb{R}^d \\ |x| \leqslant \sqrt{2}}} (1 - |x|^2/2)^2 dx \Big)^{1/2} c^{-d/4\,[\|]}. \quad \text{(BW)}$$

Notant $f * v =: g$ pour alléger les notations, soit en effet a qui minimise $\|g-a\|_2$, de sorte que $\|g\|_{\overline{2}} = \|g-a\|_2$. Par définition de $\|g\|_{\overline{\infty}}$, ou l'argument minimum ou l'argument maximum de g [*] vérifie $|f(x)-a| \geqslant \|f\|_{\overline{\infty}}$ — mettons ici qu'il s'agit de l'argument maximum, noté x_0. On a $Dg(x_0)=0$, d'où par l'inégalité de Taylor–Lagrange, puisqu'on suppose démontré $\|D^2 g\|_\infty \leqslant c\|g\|_\infty$, $g(x)-a \geqslant g(x_0) - \frac{1}{2}\|g\|_\infty |x-x_0|^2 - a \geqslant \|g\|_{\overline{\infty}} - \frac{1}{2}c\|g\|_\infty |x-x_0|^2 \quad \forall x \in \mathbb{R}^d$. En intégrant,

$$\begin{aligned}\|g\|_{\overline{2}}^2 = \|g-a\|_2^2 &\geqslant \int_{\mathbb{R}^d} (g(x)-a)_+^2\,dx \\ &\geqslant \int_{\mathbb{R}^d} (\|g\|_{\overline{\infty}} - \tfrac{1}{2}c\|g\|_\infty |x-x_0|^2)_+^2\,dx \\ &= \|g\|_{\overline{\infty}}^2 c^{-d/2} \left(\frac{\|g\|_{\overline{\infty}}}{\|g\|_\infty}\right)^{d/2} \int_{\substack{x\in\mathbb{R}^d \ [1] \\ |x|\leqslant\sqrt{2}}} (1-|x|^2/2)^2\,dx. \qquad \text{(BX)}\end{aligned}$$

Prouvons donc l'existence de la constante c. On se donne un noyau régularisant K de la forme

$$K(x \ [L]) = (L_0^{-2}|x|^2+1)^{-\kappa'/2} \qquad [1], \qquad \text{(BY)}$$

pour un paramètre arbitraire $\kappa' \in (d,\kappa]$, de sorte que K soit intégrable mais avec des queues polynomiales suffisamment lourdes.

D'après les hypothèses faites sur v, il existe alors une constante $c_1(v) < \infty$ $[\mathsf{E.N}^{-2}.\mathsf{L}^{-2}]$ telle que $|D^2 v(x)| \leqslant c_1 K(x) \quad \forall x \in \mathbb{R}^d$. D'autre part, vu que $(-v)$ est positive intégrale et globalement non nulle, il existe une constante $c_2(v) < \infty$ $[\mathsf{E}^{-1}.\mathsf{N}^2.\mathsf{L}^{-d}]$ telle que $(K*(-v))(x) \geqslant c_2^{-1} K(x) \quad \forall x \in \mathbb{R}^d$. On peut donc écrire :

$$\begin{aligned}\|D^2(f*v)\|_\infty = \|f*D^2 v\|_\infty &\underset{f\geqslant 0}{\leqslant} \|f*|D^2 v|\|_\infty \leqslant c_1\|f*K\|_\infty \\ &\leqslant c_1 c_2 \|f*(K*(-v))\|_\infty = c_1 c_2 \|K*(f*v)\|_\infty \leqslant c_1 c_2 \|K\|_1 \|f*v\|_\infty, \qquad \text{(BZ)}\end{aligned}$$

de sorte que $c = c_1 c_2 \|K\|_1$ convient. ♠

5 Résultats principaux

☛ *Dans cette section, v n'est plus supposé négatif* a priori*, sauf mention explicite du contraire.*

5.a Température de transition

Commençons par énoncer une minoration de la température de transition qui découle simplement de l'étude du système linéarisé :

5.1 Proposition. *Pour $T < RV$, l'équilibre uniforme est instable* [†]. ♣

[‖]. On pourrait calculer explicitement le préfacteur de $c^{-d/4}$, mais cela n'aurait aucun intérêt ; retenons simplement qu'il s'agit d'une constante absolue (ne dépendant que de d).

[*]. Les extrema de g sont en effet atteints, ou alors on a $\|g\|_{\overline{2}} = \infty$ et (BV) est triviale.

[†]. L'instabilité est même valable au sens de la remarque 1.12.

Démonstration. Cette preuve repose sur la linéarisation du système au voisinage de λ ; aussi allons-nous nous placer non pas dans $\mathbf{F}$, mais dans un espace vectoriel topologique qui se plonge continûment dans cet espace (au voisinage de λ) — mettons, dans l'espace des fonctions p gentilles telles que $\int_{\mathbb{R}^d} p(x)\,dx = 0$.

Rappelons que l'expression (X) présente l'énergie libre sous forme d'une forme quadratique de p :

$$\mathscr{U}(\mu) = \frac{1}{2}\langle v * p, p\rangle_{L^2}. \tag{CA}$$

Or l'expression (Y) de l'entropie se bilinéarise formellement au voisinage de λ en

$$\mathscr{S}(\mu) \overset{p\to 0}{\sim} \frac{1}{2R}\langle p, p\rangle_{L^2}, \tag{CB}$$

de sorte qu'au final $\mathscr{F}$ se bilinéarise en la forme quadratique

$$\mathscr{F}(\mu) \overset{p\to 0}{=} \frac{1}{2}\langle (v + R^{-1}T\boldsymbol{\delta}_0) * p, p\rangle_{L^2} + o(p^2), \tag{CC}$$

où le "$o(p^2)$" est à prendre au sens de Gâteaux, *i.e.* le long de chaque droite issue de 0. À cause de la différentiation au sens de Gâteaux, même si la forme quadratique qui apparaît dans (CC) est (strictement) positive, cela ne suffit pas à assurer la stabilité de l'état λ au sens de la définition (1.11) ; en revanche, il suffit de montrer l'existence d'une fonction p telle que $\langle (v + R^{-1}T\boldsymbol{\delta}_0) * p, p\rangle_{L^2} < 0$ pour démontrer que l'équilibre est instable.

Pour étudier cette forme quadratique, on passe dans le domaine de Fourier, où on a

$$\frac{1}{2}\langle (v + R^{-1}T\boldsymbol{\delta}_0) * p, p\rangle_{L^2} = \frac{1}{4\mathring{\pi}}\langle (\hat{v} + R^{-1}T)\overline{\hat{p}}, \hat{p}\rangle_{L^2}, \tag{CD}$$

de sorte que la forme quadratique sera positive si et seulement si $\hat{v}(\xi) + R^{-1}T \geqslant 0 \;\; \forall \xi$, *i.e.* si $-V + R^{-1}T \geqslant 0$, d'où la proposition. ♠

5.2 Théorème. *Si v est négatif avec $v \in \mathbf{W}_\alpha$ pour un $\alpha < \overline{\alpha}$, alors l'équilibre homogène est stable dès que $T > RV$.* ♣

Démonstration. On commence par le

5.3 Lemme. *Sous les hypothèses du théorème 5.2, la fonctionnelle $\mu \mapsto \|v * p\|_\infty$ de $\mathbf{F}$ dans $\mathbb{R}_+$ est continue en λ.* ♣

Démonstration. D'après le théorème 3.2 (ou 3.6 si $\alpha > 0$), il suffit de démontrer que la fonctionnelle $\pi \mapsto \|v * p\|_\infty$ est continue en 0 *vue comme une fonction sur* $\mathbf{W}'_\alpha$. Notant τ^x l'opérateur de translation par $x \in \mathbb{R}^d$ (tel que $\tau^x v(y) = v(y - x)$), on a $(v * p)(x) = \langle \tau^x v, \pi\rangle$, d'où $\|v * p\|_\infty = \sup_{x\in\mathbb{R}} |\langle \tau^x v, \pi\rangle|$; or τ^x est clairement une isométrie de $\mathbf{W}_\alpha$, donc $|\langle \tau^x v, \pi\rangle| \leqslant \|\tau^x v\|_{\mathbf{W}_\alpha}\|\pi\|_{\mathbf{W}'_\alpha} = \|v\|_{\mathbf{W}_\alpha}\|\pi\|_{\mathbf{W}'_\alpha}$, d'où

$$\|v * p\|_\infty \leqslant \|v\|_{\mathbf{W}_\alpha}\|\pi\|_{\mathbf{W}'_\alpha}, \tag{CE}$$

ce qui montre la continuité de la fonctionnelle en 0. ♠

D'après le lemme 5.3, pour tout $\eta > 0$ $[\mathrm{N.L}^{-d}]$ on a donc au voisinage de λ que $\|v * p\|_\infty \leqslant V\eta/2$, de sorte qu'on peut appliquer la formule (AM), qui nous donne que l'équilibre uniforme est stable dès que $T > \eta^2 V/2\Phi(\eta)$. En faisant tendre η vers 0, $\eta^2/\Phi(\eta)$ tend vers $2R$, de sorte qu'en prenant η suffisamment petit on a bien la stabilité de l'équilibre dès que $T > VR$. ♠

5.4 Corollaire. *Si v est négatif avec $v \in \mathbf{W}_\alpha$ pour $\alpha < \overline{\alpha}$, alors la transition de phase du système survient à la température RV.* ♣

5.b Énergie d'activation

5.5 Théorème. *Si v est négatif et dans $\mathbf{W}_0$, avec en outre $|D^2v(x)| \overset{|x|\to\infty}{=} O(|x|^{-\kappa})$ pour un $\kappa > d$, alors pour $T > RV$ l'équilibre homogène a une énergie d'activation non nulle, qui est minorée au voisinage de la température critique par $C(T-RV)^{3+d/2}$ pour un $C > 0$* $[\mathsf{N}^{3+d/2}.\mathsf{E}^{-(2+d/2)}]$. ♣

Démonstration. Prenons η suffisamment petit pour que $T > \eta^2 V/2\Phi(\eta)$, et notons ici $V\eta/2 =: h$ $[\mathsf{E}.\mathsf{N}^{-1}]$. De la même façon que pour le lemme 5.3, mais en remplaçant cette fois le théorème 3.2 par le théorème 3.6, nous avons que la fonctionnelle $\mu \mapsto \|v * \pi\|_\infty$ est continue sur $\mathbf{F}$ tout entier ; par conséquent, $\{\|v * p\|_\infty < h\}$ et $\{\|v * p\|_\infty > h\}$ sont deux ouverts disjoints, qui avec l'ensemble $\{\|v * p\|_\infty = h\}$ partitionnent $\mathbf{F}$.

Supposons que pour un E_{a} $[\mathsf{E}]$, nous sachions démontrer que sur $\{\|v * p\|_\infty = h\}$ on ait $\mathscr{F} \geqslant E_{\mathrm{a}}$. Alors, pour $E < E_{\mathrm{a}}$, les ensembles $\{\mathscr{F} \leqslant E$ et $\|v * p\|_\infty < h\}$ et $\{\mathscr{F} \leqslant E$ et $\|v * p\|_\infty > h\}$ sont deux ouverts de $\{\mathscr{F} \leqslant E\}$ le partitionnant ; par conséquent la composante connexe de λ dans cet ensemble est contenue dans $\{\|v * p\|_\infty < h\}$. Il s'ensuit que $\mathscr{F}$ est positive sur cette composante connexe, donc y atteint son miminum global en λ, ce qui prouve que E_{a} est un minorant de l'énergie d'activation du système. Nous voulons donc trouver E_{a} aussi grand que possible tel que, pour un η bien choisi tel que $\eta^2 V/2\Phi(\eta) < T$,

$$\|v * p\|_\infty = h \;\Rightarrow\; \mathscr{F}(\mu) \geqslant E_{\mathrm{a}}. \tag{CF}$$

Or pour $\|v * p\|_\infty = h$, on sait en combinant (AM) et (AG) que

$$\mathscr{F}(\mu) \geqslant \Big(\frac{\Phi(\eta)T}{\eta^2 V^2} - \frac{1}{2V}\Big)\|v * p\|_2^2. \tag{CG}$$

En outre, on peut appliquer le lemme 4.1 en observant que $\|v * m\|_\infty \geqslant \|v * R\|_\infty = RV$ (attendu que m est une perturbation de la fonction constante égale à R), d'où :

$$\|v * p\|_2 \geqslant \frac{C' \;[\mathsf{L}^{d/2}]}{(RV)^{d/4}} h^{1+d/4}. \tag{CH}$$

En combinant (CG) et (CH), on obtient alors qu'on peut choisir

$$E_{\mathrm{a}} = \frac{(C')^2 V^2}{4\cdot(2R)^{d/2}}\left(\frac{\Phi(\eta)T}{\eta^2 V^2} - \frac{1}{2V}\right)\eta^{2+d/2}. \tag{CI}$$

Développant $T = RV + \varepsilon$, il nous reste à optimiser η en fonction de ε. Un développement à l'ordre 3 de la fonction Φ donne

$$\frac{\Phi(\eta)}{\eta^2} \overset{\eta\to 0}{=} \frac{1}{2R} - \frac{\eta}{6R^2} + o(\eta), \tag{CJ}$$

de sorte qu'en prenant $\eta = \frac{3d+12}{d+6} V^{-1}\varepsilon$, on trouve bien le résultat annoncé, avec C arbitrairement proche de $\frac{R(C')^2}{d+6}\left(\frac{3d+12}{(2d+12)RV}\right)^{2+d/2}$. ♠

5.c Cas où v n'est pas négatif

Les calculs du § 2 recquièrent que v soit négatif afin que le produit de convolution par v puisse être vu comme l'action du noyau d'une chaîne de Markov. Cependant on voudrait également pouvoir minorer $\mathscr{F}$ quand v est positif par endroits — *a priori* cela devrait même être encore plus facile ! Une idée naturelle serait de décomposer v en ses parties négative et positive, mais on aurait des ennuis du fait que v_- et v_+ n'ont pas la même régularité que v.

Il s'avère plus fructeux de plutôt chercher un potentiel w négatif qui « minore » v au sens où l'énergie libre associée à w minore celle associée à v, et pour lequel on ait (presque) la même température de transition que pour v.

5.6 Proposition. *Soient v et w deux potentiels d'interaction sur $\mathbb{R}^d$, et notons $\mathscr{F}_v$ et $\mathscr{F}_w$ les fonctionnelles d'energie libre respectives associées. Pour toute température, on a $\mathscr{F}_v \geqslant \mathscr{F}_w$ sur tout $\mathbf{F}$ si (et seulement si) $-\hat{v}(\xi) \leqslant -\hat{w}(\xi)$ sur tout $\mathbb{R}^d$ $[\mathsf{L}^{-1}]$.* ♣

Démonstration. Dans la différence $(\mathscr{F}_v - \mathscr{F}_w)$, les termes d'entropie s'annulent, de sorte que $\mathscr{F}_v(\mu) - \mathscr{F}_w(\mu) = \mathscr{U}_v(\mu) - \mathscr{U}_w(\mu) = \frac{1}{2}\langle (v-w) * p, p\rangle_{L^2} = \frac{1}{4\pi}\langle (\hat{v}-\hat{w})\bar{\hat{p}}, \hat{p}\rangle$, dont l'étude du signe est immédiate. ♠

5.7 Corollaire. *Si v est dans l'espace de Schwartz, alors la transition de phase du système survient à la température RV, et l'énergie d'activation est strictement positive dès qu'on est au-delà de cette température.* ♣

Démonstration. Pour démontrer le corollaire 5.7, il suffit de prouver que pour tout $\varepsilon > 0$, il existe un potentiel w vérifiant les hypothèses du théorème 5.5 (à savoir, w est négatif et dans $\mathbf{W}_0$ avec $|D^2 w(x)| \overset{|x|\to\infty}{=} O(|x|^{-\kappa'})$ pour un $\kappa' > d$) tel que pour tout $\xi \in \mathbb{R}^d$ $[\mathsf{L}^{-1}]$, on ait $-\hat{v}(\xi) \leqslant -\hat{w}(\xi) \leqslant V + \varepsilon$. En effet, de la première inégalité on tirera par la proposition 5.6 que $\mathscr{F}_v \geqslant \mathscr{F}_w$, et donc qu'à toute température l'énergie d'activation pour v est plus grande que celle pour w ; et de la seconde inégalité on tirera que le "V" du potentiel w est égal à $V + \varepsilon$, de sorte que $\mathscr{F}_w$ et *a fortiori* $\mathscr{F}_v$ présente une énergie d'activation en λ non nulle dès que $T > R(V+\varepsilon)$.

Pour un paramètre $\kappa > d+2$ arbitraire, définissons successivement les fonctions g_1 et $f_1 : \mathbb{R}^d$ $[1] \to \mathbb{R}$ par

$$g_1(\xi) := \frac{(1+|\xi|^2)^{-\kappa/2}}{\left(\int_{\mathbb{R}^d}(1+|\xi|^2)^{-\kappa}\,d\xi\right)^{1/2}}; \tag{CK}$$

$$f_1 := g_1 * g_1. \tag{CL}$$

Les propriétés suivantes de f_1 sont immédiates :

5.8 Proposition.

(i) f_1 est positive, équivalente quand $|\xi| \to \infty$ à $c|\xi|^{-\kappa}$ (pour une constante $c > 0$ dont la valeur exacte nous importe peu), et atteint sa valeur maximale 1 au point 0.

(ii) La transformée de Fourier $\hat{f}_1$ est réelle positive, appartient à $\mathbf{W}_0$, et sa dérivée seconde décroît plus vite que tout polynôme quand $|x| \to \infty$.

♣

Maintenant, la fonction $-\hat{v}$ est bornée supérieurement par V (par définition de V) et décroît plus vite que tout polynôme (puisque v est supposé dans l'espace de Schwartz) ; par conséquent d'après la proposition 5.8-(i), pour $L < \infty$ $[\mathsf{L}^{-1}]$ suffisamment grand, définissant

$$f(\xi\ [\mathsf{L}^{-1}]) := (V+\varepsilon) f_1(\xi/L), \tag{CM}$$

on a $-\hat{v}(\xi) \leqslant f(\xi) \leqslant V + \varepsilon$ $\forall \xi \in \mathbb{R}^d$. Introduisons alors w comme la transformée de Fourier inverse de $-f$. Comme f se déduit de f_1 par des homothéties de rapports positifs sur les espaces de départ et d'arrivée, elle hérite de f_1 les propriétés de la proposition 5.8-(ii), donc le potentiel w est négatif et dans $\mathbf{W}_0$ avec une décroissance plus rapide que tout polynôme, et satisfait ainsi les hypothèses du théorème 5.5, ce que nous voulions. ♠

6 Travaux en cours

Les travaux que présente cette partie de la thèse sont encore en cours, et j'escompte y apporter plusieurs améliorations substantielles dans les mois à venir. Je voudrais évoquer ces pistes de recherche ci-dessous : dans la mesure où une thèse est une forme de période d'essai au travail de chercheur, il me semble en effet naturel que ce manuscrit s'arrête sur un instantané du travail en question, avec ses perspectives et ses questions ouvertes. Certaines des pistes présentées ne demanderont si tout va bien qu'un approfondissement sur certains résultats et techniques d'analyse ; d'autres en revanche sont plus spéculatives.

6.a Sur le plongement optimal de F

Je pense que la condition "$\alpha < \overline{\alpha}$" du théorème 3.6 n'est pas optimale. Plus précisément, il me semble qu'en utilisant des inégalités de type Sobolev plus poussées, le théorème 3.6 devrait pouvoir être adapté pour tout $\alpha < 2\overline{\alpha}$.

En effet, dans notre démonstration du lemme 3.17, nous avons utilisé la condition de (β,θ)-correction seulement pour dire que $\|f\|_{1,\infty} \geqslant \beta^{1/2}\theta L_0$. Or la contrainte de (β,θ)-correction est au départ une condition de type L^∞ non pas sur Df, mais sur f elle-même, et on perd très vraisemblablement à omettre cela.

On devrait ainsi avoir le résultat suivant :

6.1 Conjecture. *Supposons $\alpha < 2\overline{\alpha}$. Alors pour un cube Q fixé, il existe une constante $c < \infty$ [inhomogène] telle que toute fonction f régulière sur Q vérifie*

$$\sup f - \inf f \leqslant c\|f\|_{\mathbf{W}_\alpha}. \tag{CN}$$

♣

6.2 Corollaire. *Sous la même hypothèse, il existe des constantes C_0 et C_1 telles que pour tout cube (β,θ)-correct,*

$$\int_Q (C_0\beta^{-2}\theta^{-2}|Df(x)|^2 + C_1\beta^{-2/\alpha}\theta^{-2/\alpha}|D^{2-\alpha}f(x)|^{2/\alpha})\,dx \geqslant a(Q)^{d+2}. \tag{CO}$$

♣

La validité de la conjecture 6.1 entraînerait la continuité (lipschitzienne) en λ du plongement de $\mathbf{F}$ dans $\mathbf{W}'_\alpha$ pour tout $\alpha < 2\overline{\alpha}$. Par contre, cela ne suffit pas à avoir la continuité globale.

En poussant plus loin, on pourrait chercher à tirer de l'hypothèse de (β,θ)-correction une contrainte de type « L^∞ sur la dérivée fractionnaire $D^\eta f$ » pour un $\eta \in (0,1]$. On devrait alors obtenir le lemme suivant :

6.3 Conjecture. *Pour $\eta \in (0,1]$, $\alpha < (2-\eta)\overline{\alpha}$, il existe des constantes C_0 et C_1 telles que pour tout cube (β,θ)-correct,*

$$\int_Q (C_0\beta^{-1}\theta^{-2}|Df(x)|^2 + C_1\beta^{(\eta-2)/\alpha}\theta^{-2/\alpha}|D^{2-\alpha}f(x)|^{2/\alpha})\,dx \geqslant a(Q)^{d+2}. \tag{CP}$$

♣

La conjecture 6.3 entraîne la continuité du plongement dans $\mathbf{W}'_\alpha$ dès que $\alpha < 2\overline{\alpha}$, mais cette fois-ci il ne s'agira que d'une continuité $((2-\alpha/\overline{\alpha})\wedge 1)$-hölderienne [‡]. Comme nous le verrons, cette restriction n'est pas un artefact dû à la preuve mais bien un phénomène incontournable.

6.4 Remarque. En dimension $d = 1$ on a $2\overline{\alpha} = 4/3 > 1$, de sorte qu'on peut alors prendre $\alpha = 1$ pour obtenir un plongement (continu en λ et $\frac{1}{2}$-hölderien ailleurs) de $\mathbf{F}$ dans l'espace *homogène* $\mathbf{W}_1 = \dot{W}^{1,2}$. ♡

La limite $2\overline{\alpha}$ est optimale, car $\mathbf{F}$ ne peut pas être plongé dans $\mathbf{W}_{2\overline{\alpha}}$. En effet, on peut obtenir pour un coût de transport arbitrairement faible à partir de la mesure uniforme une mesure π de la forme $n\boldsymbol{\delta}_{x_0} - f(x)dx$ (où $x_0 \in \mathbb{R}^d$, $n > 0$ [N]) pour f [N.L^{-d}] une fonction gentille, et une telle mesure n'est pas dans $\mathbf{W}'_{2\overline{\alpha}}$ [1, exemple 4.43].

Par ailleurs, une fois qu'on a créé un atome pour μ, on peut déplacer (dans l'espace physique) cet atome d'une distance r au prix d'un déplacement W_2 de la mesure μ qui va en $O(r)$ quand $r \to 0$. Comme la différence entre deux diracs de même masse distants de r a quant à elle une norme dans $\mathbf{W}'_\alpha$ qui va en $r^{(2-\alpha/\overline{\alpha})\wedge 1}$ quand $r \to 0$, il s'ensuit que la continuité du plongement de $\mathbf{F}$ dans $\mathbf{W}'_\alpha$ en une mesure ayant un atome est au mieux $((2-\alpha/\overline{\alpha})\wedge 1)$-hölderienne. Ainsi les résultats de plongement obtenus si nous parvenons à mener notre programme à bien seront essentiellement optimaux.

Cela dit, ce n'est pas parce que le théorème 3.2 cesse d'être valide pour $\alpha \geqslant 2\overline{\alpha}$ qu'il en est forcément de même pour le théorème 5.2 sur la stabilité de l'équilibre homogène. Il serait donc intéressant de chercher des contre-exemples à ce théorème pour voir à quel point les hypothèses de régularité sur v y sont nécessaires.

6.b Sur l'énergie d'activation

Retour sur le § 4 Pour minorer l'énergie d'activation, j'ai utilisé au § 4 une technique consistant à minorer $\|v * p\|_2$ en fonction de $\|v * p\|_\infty$. Malheureusement, le lemme 4.1 sur lequel repose cette méthode recquiert une hypothèse supplémentaire forte sur v. Cette hypothèse correspond à une contrainte d'intégrabilité sur $D^2 v$; plus précisément, on a en fait besoin qu'il existe une fonction A intégrable sur $\mathbb{R}^d$ et un rayon $r > 0$ tels que $|y-x| \leqslant r \;\Rightarrow\; |D^2 f(x)| \leqslant A(y)$. (Cette contrainte est un peu plus faible que celle du lemme, mais je l'avais éludée car elle n'est guère éclairante et conduit à une démonstration plus technique). On peut en tout cas trouver des contre-exemples montrant qu'une hypothèse de ce type est indispensable pour que le lemme 4.1 s'applique. Il est à noter que cette contrainte apparemment surprenante se retrouve également pour démontrer la continuité de la fonctionnelle $\mathscr{U}$ sur $\mathbf{F}$, de sorte que son sens est sans doute plus profond qu'il n'y paraît.

Cela dit, peut-être peut-on contourner le lemme 4.1 et minorer quand même l'énergie d'activation pour des potentiels v ne satisfaisant pas l'hypothèse d'intégrabilité sur $D^2 v$. Une idée serait d'essayer d'appliquer à cette situation la technique du § 5.c, à savoir de trouver un potentiel w tel que $-\hat{w}$ majore $-\hat{v}$, ayant la même température de transition, et vérifiant pour sa part l'hypothèse supplémentaire. Dans quelles conditions est-ce possible ?

Je signalerai enfin que le lemme 4.1, qui a été écrit à partir d'hypothèses sur $D^2 v$, devrait pouvoir s'adapter sous des hypothèses du même genre sur $D^\eta v$ pour un $\eta \in (0,2]$. Ainsi, si v est dans $\mathbf{W}'_\alpha$ et vérifie une hypothèse d'intégrabilité supplémentaire appropriée, on devrait pouvoir obtenir une formule du genre de (BT) avec l'exposant $d/4$ remplacé par $\frac{2\overline{\alpha}-\alpha}{2\overline{\alpha}} d/4$, et donc au final

[‡]. Plus exactement, pour tout $\varepsilon > 0$ le plongement sera $((2-\alpha/\overline{\alpha}-\varepsilon)\wedge 1)$-hölderien (sur les bornés).

un exposant de $3 + \frac{2\overline{\alpha}-\alpha}{2\overline{\alpha}} d/2$ dans la version correspondante du théorème 5.5. Toutefois, compte tenu des remarques précédente et suivante il est probable que cela n'ait que peu d'intérêt.

Exposant critique pour l'énergie d'activation D'autre part, j'ai établi au théorème 5.5 que l'énergie d'activation apparaissait avec un exposant de $(3+d/2)$ au plus, mais j'ignore quelle est la valeur optimale de cet exposant. Il s'agit là selon moi d'une question particulièrement importante. Dans le cas d'une brisure spontanée de symétrie en dimension finie en tout cas, la théorie des catastrophes montre que l'exposant critique générique est 3, ce qui suggère une amélioration possible. Il serait intéressant de déjà mener des calculs et simulations numériques pour se faire une idée de la véritable valeur de l'exposant critique, puis si possible de démontrer le résultat ainsi conjecturé.

6.c Sur les potentiels d'interaction non négatifs

J'ai donné au § 5.c une technique pour établir la température de transition de phase dans le cas général où v n'est pas négatif. Nous avons ainsi vu qu'on récupérait la formule du théorème 5.2 pour v suffisamment gentille. Peut-on améliorer ce résultat ?

Voici d'abord un point qui pourrait s'avérer intéressant pour les questions d'énergie d'activation. La fonction f_1 que j'ai donnée étant strictement inférieure à 1 dès que $\xi \neq 0$, nous sommes obligés d'augmenter sans cesse la valeur de L à mesure que ε tend vers 0, ce qui entraîne une augmentation ennuyeuse de l'exposant du théorème 5.5 (bien que je n'aie pas abordé cette question tout à l'heure). Pour supprimer cet écueil, il suffirait de remplacer la fonction f_1 par une fonction ayant un plateau. J'aimerais donc que la réponse à la question suivante soit positive :

6.5 Question. *Existe-t-il un fonction f_1 qui satisfasse les propriétés de la proposition 5.8 et qui en outre soit constante au voisinage de* 0 *?* ♣

Un défi plus ambitieux est d'affaiblir les hypothèses de régularité sur v dans le théorème 5.7 (du moins pour l'aspect « température de transition » de ce théorème). L'idéal serait d'étendre ce résultat pour tout $v \in \mathbf{W}_\alpha$, ce qui soulève la question suivante :

6.6 Question. *Soit $\alpha \in [0,1]$. Pour toute fonction $v \in \mathbf{W}_\alpha$, existe-t-il une fonction $w \in \mathbf{W}_\alpha$ négative telle que $-\hat{w}(\xi) \geqslant -\hat{v}(\xi)$ $\forall \xi$ et que $\int(-w)$ soit arbitrairement proche* [§] *de V ?* ♣

[§]. Voire égal, ce qui répondrait du même coup au premier point de cette sous-section.

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Quelques problèmes d'inspiration physique en théorie des probabilités

Cette thèse présente quatre travaux de recherche mêlant probabilités et analyse, ayant en commun de s'appuyer sur l'intuition physique, tant dans la position des problèmes que dans leur résolution :

1. On borne les probabilités de transition des chaînes de Markov réversibles discrètes, améliorant la borne de Carne grâce à une démonstration alternative.
2. On démontre la convergence vers la limite de champ moyen dans une approche uniforme et non asymptotique pour un modèle de Boltzmann spatialement homogène.
3. On étudie le coefficient de ρ-mélange entre deux tribus, montrant en particulier comment cette quantité peut être tensorisée dans un cadre général, ce qui implique des résultats de décorrélation entre groupes infinis de spins en physique statistique.
4. On s'intéresse, pour une équation de McKean – Vlasov, à la stabilité de l'équilibre homogène en fonction de la température, minorant notamment l'énergie d'activation.

Some questions in probability theory viewed with a physical twist

This thesis presents four research themes on probability and analysis, which have in common to state and solve problems in physically intuitive contexts:

1. I give an upper bound for the transition probabilities of discrete reversible Markov chains, improving Carne's bound by an alternative proof.
2. I prove convergence to the mean field limit in a uniform and non-asymptotic context for a spatially homogeneous Boltzmann model.
3. I study the ρ-mixing coefficient between two σ-algebras, showing in particular how this quantity can be tensorized in a general framework. This implies decorrelation results between infinite bunches of spins in statistical mechanics.
4. I study stability of the homogeneous equilibrium for a McKean–Vlasov equation as temperature varies; among other things, I give a lower bound for the activation energy.