Temps local et diffusion en environnement aléatoire

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TEMPS LOCAL ET DIFFUSION EN ENVIRONNEMENT ALÉATOIRE

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Introduction

1.1 Marche aléatoire en milieu aléatoire

Pour que le modèle d'un système physique réel soit utilisable en pratique, il doit être suffisamment simple pour pouvoir être étudié mathématiquement. Cela nécessite de négliger l'effet des irrégularités de l'environnement dans lequel le système évolue. Cependant les simplifications effectuées peuvent entraîner de grandes différences entre l'évolution du système modélisé et l'évolution du système réel. Une manière de remédier à cela et de prendre en compte cette irrégularité est d'introduire la notion d'environnement non homogène.

Une manière classique de représenter un environnement non homogène est de le considérer comme le résultat d'une expérience aléatoire. On peut alors étudier l'évolution d'une particule dans de tels milieux. L'aléatoire apparaît donc sous deux formes dans le modèle : premièrement, dans la construction du milieu, puis, pour un environnement donné, l'évolution de la particule dans ce milieu est une seconde source d'aléa.

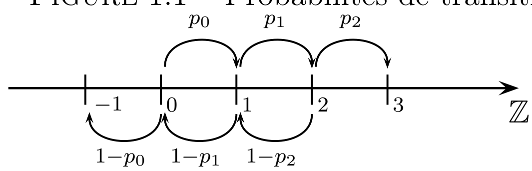
Historiquement, les premiers modèles mathématiques de milieux aléatoires à avoir été introduits sont des modèles discrets car plus simples à définir. Étant donnée une suite $p = (p_i)_{i\in\mathbb{Z}} \in]0,1[^{\mathbb{Z}}$, on appelle *marche aléatoire en milieu* p, la marche au plus proche voisin $(S_n)_{n\in\mathbb{N}}$ sur $\mathbb{Z}$ vérifiant $S_0 = 0$ et

$$P_p(S_{n+1} = i+1 | S_n = i) = 1 - P_p(S_{n+1} = i-1 | S_n = i) = p_i.$$

On peut rendre les p_i aléatoires en munissant l'ensemble $]0,1[^{\mathbb{Z}}$ d'une loi de probabilité $\mathcal{P}$. On appelle alors *marche aléatoire en milieu aléatoire* de loi $\mathcal{P}$ tout processus $(S_n,\ n \in \mathbb{N})$ de loi $\mathbb{P} = \int \mathcal{P}(\mathrm{d}p) P_p$.

Ce modèle fut introduit par Chernov [21] en 1967 pour modéliser la réplication de l'ADN puis repris par Temkin [90] en 1972 pour des modèles en

FIGURE 1.1 – Probabilités de transition



génétique et en métallurgie. Ce dernier domaine a influencé la terminologie des milieux aléatoires : la loi $\mathbb{P}$ est ainsi appelée loi *annealed*, «recuite», et la loi à milieu fixé P_p, loi *quenched*, «trempée».

Pour étudier le comportement des marches en milieu aléatoire, il est intéressant d'introduire la notion de potentiel : pour tout $i \in \mathbb{Z}$, on pose $\omega_i = \log(\frac{1-p_i}{p_i})$. On définit alors le potentiel $V = (V_i, i \in \mathbb{Z})$ à valeurs dans $\mathbb{R}$ par

$$\begin{cases} V_0 = 0, \\ V_i = \displaystyle\sum_{j=1}^{i} \omega_j & \text{si} \quad i \geq 1, \\ V_i = \displaystyle\sum_{j=i+1}^{0} -\omega_j & \text{si} \quad i \leq -1. \end{cases}$$

La marche peut alors être vue comme une particule se déplaçant dans le potentiel V,

$$P_p(S_{n+1} = i+1 | S_n = i) = \frac{e^{V_{i-1}}}{e^{V_{i-1}} + e^{V_i}} = \frac{e^{-\omega_i/2}}{e^{-\omega_i/2} + e^{\omega_i/2}}$$

et de nombreux résultats sur le comportement asymptotique de la marche peuvent s'exprimer simplement en fonction du potentiel et des ω_i. Remarquons que l'on peut retrouver les variables p_i à partir de V. Ainsi il est équivalent pour le cas unidimensionnel de se donner un potentiel ou la suite des probabilités ; ceci n'est plus vrai lorsque la dimension est plus grande que 1.

On suppose à partir de maintenant que sous $\mathcal{P}$ les variables p_i sont indépendantes, identiquement distribuées et de même loi. C'est ce type de milieu qui a été le plus étudié, on peut néanmoins citer Alili [3], Zeitouni [94] ou Mayer-Wolf, Roitershtein, Zeitouni [57] pour des résultats dans des milieux plus généraux. Dans le cas i.i.d., Solomon [77] montre en 1974 que la marche $(S_n,\ n \in \mathbb{N})$ est récurrente si et seulement si $\mathbb{E}[\omega_0] = 0$. Dans le même article, il étudie la vitesse limite dans le cas transient : il existe un $v \in [-1, 1]$

dépendant uniquement de la loi de l'environnement tel que $\mathbb{P}$-p.s.

$$\lim_{n\to\infty} \frac{S_n}{n} = v.$$

Contrairement aux marches aléatoires usuelles, les marches aléatoires en milieu aléatoire peuvent être transientes de vitesse nulle. Kesten, Kozlov et Spitzer [53] donnent un théorème de type central limite qui montre qu'il existe des régimes différents fonction de l'unique solution κ de l'équation

$$\mathbb{E}\left(\frac{1-p_0}{p_0}\right)^{\kappa} = 1. \tag{1.1}$$

Dans le cas récurrent, Sinaï montre en 1982 que sous des hypothèses relativement générales sur le milieu, la marche en milieu aléatoire est considérablement plus lente que la marche aléatoire simple sur $\mathbb{Z}$ et qu'elle se localise au voisinage de certains points. Plus précisément, si les trois conditions suivantes soient vérifiées,

$$\mathbb{E}[\omega_0] = 0, \tag{1.2}$$

$$\sigma^2 := \mathbb{E}\left[\omega_0^2\right] > 0, \tag{1.3}$$

$$\exists \delta > 0,\, \mathbb{P}(\delta < p_0 < 1-\delta) = 1 \tag{1.4}$$

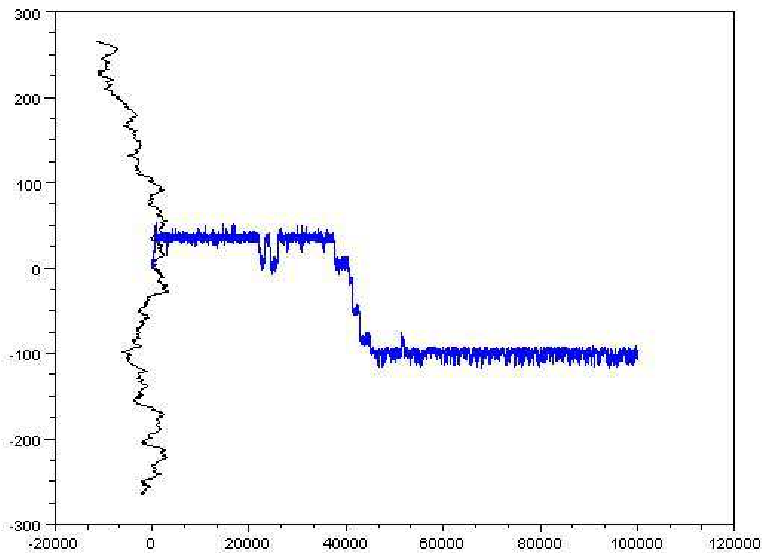
alors il existe une variable aléatoire non dégénérée m_∞ telle que

$$\frac{\sigma^2 S_n}{(\log n)^2} \xrightarrow[n\to\infty]{\mathcal{L}} m_\infty. \tag{1.5}$$

L'hypothèse (1.4), appelée condition d'ellipticité, est une hypothèse technique qui peut être relaxée (voir par exemple Zeitouni [94] ou Andreoletti [4]), l'hypothèse (1.3) évite le cas de la marche aléatoire simple. On appelle *marche de Sinaï* une marche aléatoire en milieu aléatoire vérifiant ces trois hypothèses. La loi de la variable m_∞ a été explicitée indépendamment par Golosov [36] et Kesten [52]. La lenteur relative des marches de Sinaï par rapport à la marche simple sur $\mathbb{Z}$ s'explique par la présence de puits dans le potentiel. En effet, la marche ayant tendance à se déplacer vers la gauche quand la pente du potentiel est positive et vers la droite lorsqu'elle est négative, la particule se retrouve piégée au fond des vallées.

Il y a également une abondante littérature sur les modèles de marches en milieu aléatoire défini sur des domaines plus complexes que l'axe $\mathbb{Z}$. Nous ne les décrirons pas en détail ici, citons néanmoins pour l'étude de milieu défini sur $\mathbb{Z}^d,\ d \geq 2$ Kalikow [47], Zerner [95], Zerner et Merkl [96] Sznitman

FIGURE 1.2 – Trajectoire d'une marche de Sinaï (en bleu, le temps est en horizontal) et d'un potentiel (en vertical en noir) avec $p_0 \stackrel{\mathcal{L}}{=} \mathcal{U}([0.1, 0.9])$



[81, 78, 79, 80, 82], Rassoul-Agha [63, 62, 64], Comets et Zeitouni [23, 24] ou encore Enriquez et Sabot [32]. Pour les modèles où l'environnement est défini sur un arbre, environnement plus riche qu'un potentiel sur $\mathbb{Z}$ tout en restant plus simple que les modèles multi-dimensionnels car il y a toujours un unique plus court chemin reliant deux points de l'arbre, on peut citer Lyons et Pemantle [54] qui ont donné un critère de récurrence/transience et Hu et Shi [43, 44] qui ont étudié le comportement asymptotique de la marche en milieu récurrent.

1.2 Diffusion en milieu aléatoire

Le potentiel décrit à la section précédente est, dans le cas de la marche de Sinaï, une marche aléatoire simple centrée de variance finie. Le théorème de Donsker nous dit donc que, correctement renormalisé, il converge vers un mouvement brownien. Il est donc naturel de vouloir construire un analogue continu de la marche de Sinaï comme une diffusion dans un environnement brownien. L'idée est donc de prendre maintenant pour environnement un mouvement brownien réel $(W(x)\ ,\ x \in \mathbb{R})$ et de définir la *diffusion en milieu*

W comme un mouvement brownien drifté par la pente de W, ie comme la solution de l'équation différentielle stochastique suivante

$$\begin{cases} \mathrm{d}X(t) = \mathrm{d}\beta(t) - \frac{1}{2}W'(X(t))\mathrm{d}t, \\ X(0) = 0. \end{cases}$$

où β est un mouvement brownien indépendant de W. Comme W est presque sûrement non dérivable, cette équation est purement formelle. On doit donc pour être correct (mais malheureusement moins explicite !) définir X comme un processus qui, conditionnellement à l'environnement W, a pour générateur infinitésimal,

$$A_W := \frac{1}{2} e^{W(x)} \frac{\mathrm{d}}{\mathrm{d}x} e^{-W(x)} \frac{\mathrm{d}}{\mathrm{d}x}.$$

Il est alors facile de généraliser cette définition en prenant pour le milieu non plus un mouvement brownien mais par exemple un processus càd-làg (i.e. continu à droite et admettant une limite à gauche en tout point) localement borné quelconque que l'on choisit nul en zéro par convention. On note, dans l'introduction, P_W pour la loi de la diffusion à milieu W fixé et $\mathbb{P}$ la mesure de probabilité totale.

Il existe un lien étroit entre diffusions et marches en milieu aléatoire, ainsi Schumacher [69] a construit une marche de Sinaï à partir d'une diffusion dans un environnement bien choisi. Pour cela, on prolonge sur $\mathbb{R}$ le potentiel V associée à une marche en milieu aléatoire, en posant pour tout réel x, $V(x) := V_{[x]}$ où $[x]$ désigne la partie entière de x. On note $\mathbf{X}$ la diffusion dans ce potentiel et on définit les temps d'arrêt suivants : $\mu_0 := 0$ et

$$\forall n \in \mathbb{N},\ \mu_{n+1} := \inf\{t \geq \mu_n,\ |\mathbf{X}(t) - \mathbf{X}(\mu_n)| = 1\},$$

alors

$$P_V\big(\mathbf{X}(\mu_{n+1}) = i+1 | \mathbf{X}(\mu_n) = i\big) = p_i$$

où P_V est la mesure de probabilité conditionnellement au milieu V. Ce qui signifie que $\big(\mathbf{X}(\mu_n),\ n \in \mathbb{N}\big)$ est une marche en milieu p. De plus, asymptotiquement $\mu_n \sim n$; cette construction permet ainsi de ramener l'étude des marches en environnement aléatoire à un problème de calcul stochastique.

Revenons maintenant à la diffusion en milieu brownien standard, le comportement de ce processus présente de nombreuses similitudes avec celui de la marche de Sinaï. Ainsi, il est récurrent et, de plus, Brox [17] a montré que la diffusion a le même comportement asymptotique que la marche :

$$\frac{X(t)}{(\log t)^2} \xrightarrow[t\to\infty]{\mathcal{L}} m_\infty$$

où m_∞ est la même variable que dans l'équation (1.5).

Hu et Shi [40] ont montré quant à eux que les comportements asymptotiques extrêmes de la diffusion et de la marche sont les mêmes : presque sûrement,

$$\limsup_{t\to\infty} \frac{X(t)}{(\log t)^2 \log_3 t} = \limsup_{n\to\infty} \frac{\sigma^2 S_n}{(\log n)^2 \log_3 n} = \frac{8}{\pi^2},$$
$$\liminf_{t\to\infty} \frac{\log_3 t}{(\log t)^2} \max_{0\le s\le t} |X(s)| = \liminf_{n\to\infty} \frac{\log_3 n}{(\log n)^2} \max_{0\le k\le n} \sigma^2 S_k = 1$$

où $\log_1 = \log$ et $\log_{i+1} = \log_i \circ \log$ pour tout entier $i \geq 1$.

On peut alors se demander si il existe un équivalent du théorème de Donsker pour les milieux aléatoire : est-ce qu'une marche de Sinaï correctement renormalisée en temps et en espace converge vers un processus de Brox au sens de Skorohod ? On pourrait penser que le processus

$$\left(\frac{\sigma^2 S_{[nt]}}{(\log n)^2} \ , \ t \geq 0 \right)$$

convient ; malheureusement (1.5) implique que pour tout $t > 0$,

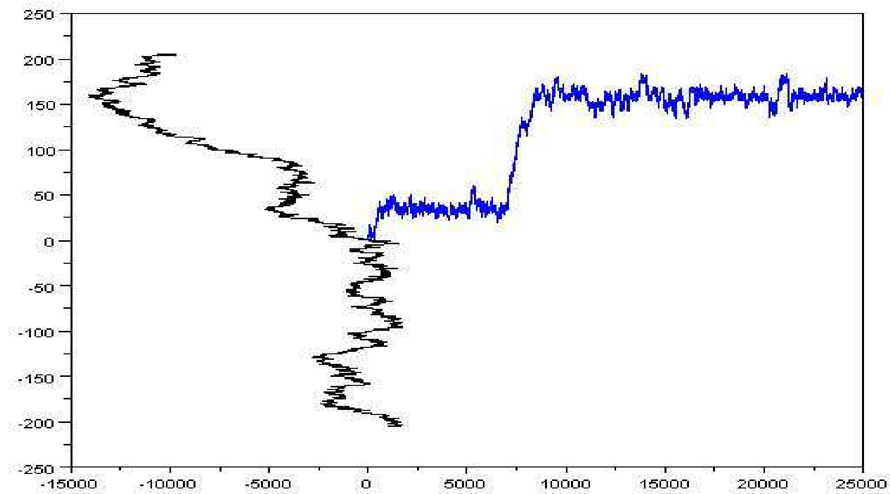
$$\frac{\sigma^2 S_{[nt]}}{(\log n)^2} \xrightarrow[n\to\infty]{\mathcal{L}} m_\infty.$$

Néanmoins, Seignourel [70] a montré que si on s'autorise à modifier l'environnement à chaque itération, il peut y avoir convergence. Précisément, il existe un famille de processus $\{S_n^m \ , \ n \in \mathbb{N}, \ m \in \mathbb{N}^*\}$ telle que pour tout $m \in \mathbb{N}^*$, fixé, $\{S_n^m \ , \ n \in \mathbb{N}\}$ soit une marche de Sinaï et $\left(\sigma^2 S^n_{[n^2 t]}/n \ , \ t \geq 0\right)$ converge en loi pour la topologie de Skorohod vers le processus de Brox.

Nous avons remarqué que les puits du potentiel ralentissaient fortement les processus en milieu aléatoire. Il est ainsi particulièrement intéressant notamment dans le cas où le potentiel est un mouvement brownien standard d'étudier la structure de ces puits. Pour cela nous allons définir la notion de vallée introduite par Brox [17]. Soit h un réel strictement positif. On appelle *h-maximum* de W un réel x tel qu'il existe $a < x < b$ vérifiant $W(a) \vee W(b) \leq W(x) - h$ et pour tout $y \in [a, b]$, $W(y) \leq W(x)$. De manière analogue, on dit qu'un réel x est un *h-minimum* de W si x est un h-maximum de $-W$. Si W est un mouvement brownien, presque sûrement, les h-maxima, les h-minima alternent, l'ensemble $\mathcal{E}_h$ des h-extrema n'admet pas de point d'accumulation et 0 n'est pas un extremum local. Ainsi, presque sûrement, il existe un unique triplet (p_h, m_h, q_h) de h-extremas successifs tels que

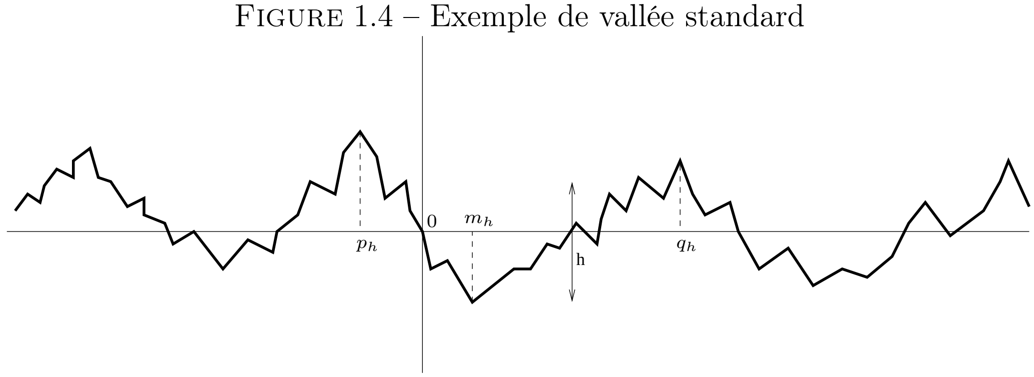
– p_h et q_h soient des h-maxima,
– m_h soit un h-minimum,

Figure 1.3 – Trajectoire (en bleu) d'une diffusion dans un milieu brownien standard (en noir)



– 0 appartient à $]p_h, q_h[$.
On appelle ce triplet, la *vallée standard de hauteur* h du milieu W.

Figure 1.4 – Exemple de vallée standard



Brox [17] a montré que la diffusion met un temps de l'ordre de t pour sortir de la vallée de hauteur $\log t$. Plus précisément, à l'instant t, le processus se trouve près du fond de cette vallée,

$$\frac{1}{(\log t)^2}|X(t) - m_{\log t}| \xrightarrow[t\to\infty]{\mathbb{P}} 0. \qquad (1.6)$$

Par la suite, Tanaka [87] et Hu [39] ont affiné la localisation de la diffusion. La convergence précédente est particulièrement importante car elle montre clairement qu'asymptotiquement, on peut ramener l'étude des propriétés asymptotique de la diffusion à l'étude de quantités dépendant uniquement du milieu. Ici, par exemple, comme $m_{\log t}/(\log t)^2$ a même loi que m_1, on voit immédiatement que $X(t)/(\log t)^2$ converge en loi vers m_1.

Le cas où le milieu est un mouvement brownien drifté :

$$W_\kappa(x) := W(x) - \frac{\kappa}{2}x$$

a également été beaucoup étudié. Citons les travaux de Kawazu et Tanaka [50, 51] puis de Hu, Shi et Yor [45] montrant qu'il existe trois régimes différents selon la valeur de κ : si $\kappa \in]0,1[$, la vitesse est nulle, si $\kappa > 1$, la vitesse limite est non nulle et pour le cas critique $\kappa = 1$, le temps d'atteinte de r par X est asymptotiquement de l'ordre de $r \log r$. On a là encore l'analogie avec le cas des marches en milieu aléatoire. Les potentiels plus généraux ont été moins étudiés, il y a malgré tout des résultats lorsque le potentiel est un processus de Lévy, voir Carmona [18], Cheliotis [19] et Singh [75, 76].

Une autre généralisation intéressante est d'étudier les diffusions dans des milieux aléatoires en dimension strictement plus grande que 1. Le milieu étant alors un champ aléatoire. Cela reste pour l'instant assez peu étudié. Tanaka [86] a malgré tout montré que la diffusion dans un milieu brownien multidimensionnel est récurrente et Mathieu [55, 56] que la variation typique de la diffusion restait de l'ordre de $(\log t)^2$.

1.3 Processus des temps locaux

Nous avons vu que les processus en milieu aléatoire présentaient d'importants phénomènes de localisation que ce soit dans le cas continu ou dans le cas discret. Le processus qui intervient naturellement dans l'étude de ce phénomène est le temps local qui «mesure» le temps passé en un point par le processus. Dans le cas de la marche, il se définit simplement : pour tout entier $n \in \mathbb{N}$ et tout entier $x \in \mathbb{Z}$, on pose

$$\xi(n,x) := \sum_{0\le k\le n} \mathbf{1}_{S_k=x} = \mathrm{card}\{0 \le k \le n\ ,\ S_k = x\}.$$

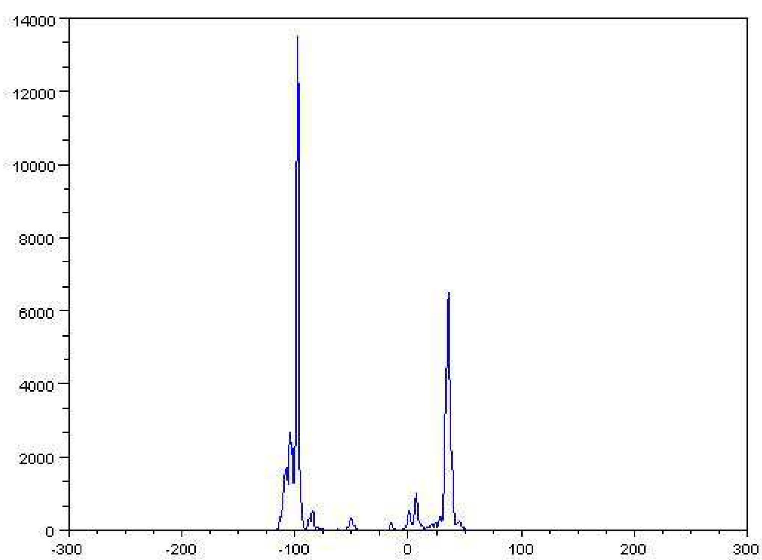
Ainsi, $\xi(n, x)$ représente le nombre de fois où la marche est passé au point x avant l'instant n. Pour les diffusions en milieu aléatoire, le temps local est défini comme la densité de la mesure d'occupation. La mesure d'occupation représente le temps passé par la diffusion dans un ensemble : pour tout $t > 0$ et tout borélien A,

$$\nu_t(A) := \int_0^t \mathbf{1}_A(X(s))\mathrm{d}s. \tag{1.7}$$

Le temps local associée à la diffusion est la version presque sûrement càd-làg (continue si le milieu est continu) de la densité de ν_t par rapport à la mesure de Lebesgue (on montrera par la suite qu'une telle quantité est bien définie) :

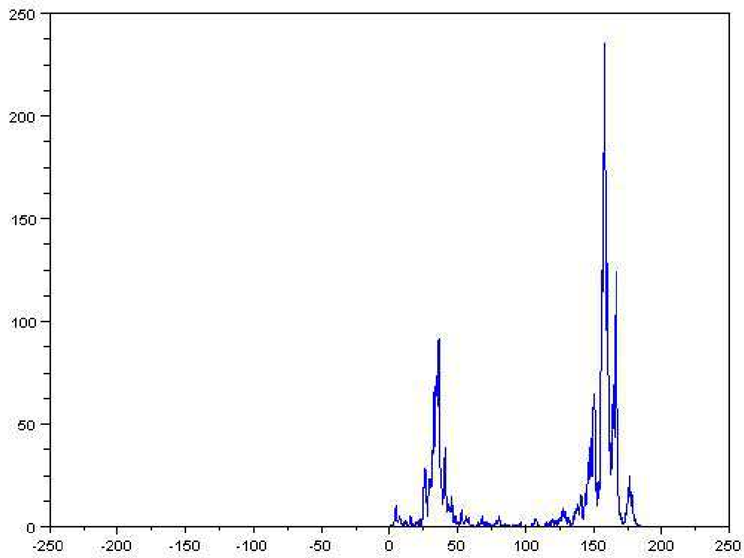
$$\nu_t(A) = \int_{-\infty}^{\infty} \mathbf{1}_A(x) L_X(x, t)\mathrm{d}x.$$

FIGURE 1.5 – Temps local en $n = 100000$ associé à la marche de Sinaï de la figure 1.2



Étudier ces processus permet de mieux comprendre où la marche et la diffusion passent la plus grande partie de leur temps. Dans le cas récurrent : la marche de Sinaï et la diffusion en milieu brownien non drifté, Révész [65, 66] puis Hu et Shi [41] ont montré que le temps local en un point x fixé

FIGURE 1.6 – Temps local en $t = 25000$ associé à la diffusion en milieu brownien de la figure 1.3



peut prendre des valeurs beaucoup plus grandes mais également beaucoup plus faibles que dans le cas de la marche simple ou du mouvement brownien standard. Cela met une fois encore en évidence le phénomène de localisation évoqué précédemment. Comme l'évolution du temps local dépend grandement de l'endroit considéré, le processus passant beaucoup de temps au fond des vallées et très peu aux sommets, on s'intéresse non pas au temps local en un point fixé, mais au maximum du temps local à un instant donné :

$$\forall t \geq 0,\ L_X^*(t) := \max_{x\in\mathbb{R}} L_X(t,x) \text{ et } \forall n \in \mathbb{N},\ \xi^*(n) := \max_{x\in\mathbb{Z}} \xi(n,x).$$

L'étude de ces processus met en avant plusieurs faits surprenants. D'abord, la concentration extrêmement forte autour des points le plus visités, Révész [65] puis Shi [71] ont montré qu'il existe une constante $c \in]0;\infty[$ telle que presque sûrement

$$\limsup_{n\to\infty} \frac{\xi^*(n)}{n} = c. \tag{1.8}$$

La valeur de la constante a par la suite été calculé dans le cas d'un milieu sur $\mathbb{N}$ (avec une frontière réfléchissante en zéro : $p_0 = 1$) par Gantert, Peres et

Shi [34]. Ainsi, infiniment souvent la marche passe une fraction strictement positive de son temps en un seul point. Cette limite décrit le temps maximal que peut passer la marche au point favori. Il est également intéressant de regarder le temps minimal passé en ce point. Dembo, Gantert, Peres et Shi [25] ont montré que dans le cas où le milieu est pris uniquement sur $\mathbb{N}$, on a le comportement asymptotique suivant : il existe une constante $c' \in]0, \infty[$ telle que

$$\liminf_{n\to\infty} \frac{\log_3 n}{n} \xi^*(n) = c'. \tag{1.9}$$

C'est-à-dire que même dans le cas le moins favorable, la marche de Sinaï passe beaucoup plus de temps au point favori que la marche simple. Quant au cas continu, Shi [71] a montré que le maximum du temps local peut aller encore plus vite : presque sûrement,

$$\limsup_{t\to\infty} \frac{L_X^*(t)}{t \log_3 t} > \frac{1}{32}. \tag{1.10}$$

Cette relation montre, outre le phénomène de localisation que les temps locaux dans le cas continu et discret n'ont pas exactement le même comportement. Un résultat directement lié concerne la localisation du point favori. Il semble naturel comme le processus se retrouve pris au piège dans la vallée standard de hauteur $\log t$ que le point le plus visité

$$m^*(t) := \min\{x \in \mathbb{R},\ L_X(t,x) = L_X^*(t)\}$$

se situe au voisinage du fond de cette vallée. Cheliotis [20] a montré que c'est en effet le cas,

$$m^*(e^t) - m_t \xrightarrow[t\to\infty]{\mathbb{P}} 0. \tag{1.11}$$

Ce résultat a son pendant presque sûr : pour tout $c > 6$, il existe une fonction strictement croissante λ de $\mathbb{R}^+$ dans $\mathbb{R}^+$ telle que presque sûrement

$$\lim_{t\to\infty} \frac{\lambda(t)}{t} = 1 \text{ et } \limsup_{t\to\infty} \frac{|m^*(e^{\lambda(t)}) - m_t|}{(\log t)^c} = 0.$$

Le supremum du temps local a également été étudié dans le cas transient : Gantert et Shi [35] ont montré que pour une marche transiente à vitesse nulle la convergence (1.8) reste vraie. Par contre, pour une marche à vitesse strictement positive

$$\limsup_{n\to\infty} \frac{\xi^*(n)}{n} = 0.$$

et ils caractérisent dans ce cas les classes de Lévy au sens de la lim sup : pour toute suite strictement positive croissante (a_n),

$$\sum_{n=1}^{\infty} \frac{1}{na_n} \begin{cases} < +\infty \\ = +\infty \end{cases} \Longleftrightarrow \limsup_{n\to\infty} \frac{\xi^*(n)}{(na_n)^{1/\kappa}} = \begin{cases} 0 \\ +\infty \end{cases} \quad \mathbb{P}\text{-a.s.}$$

où κ est l'unique solution de l'équation (1.1).

Le temps local de la diffusion en milieu brownien drifté W_κ a été étudié par Devulder [27]. Si $0 < \kappa < 1$, comme dans le cas récurrent, on observe un comportement différent de celui des modèles discrets :

$$\limsup_{t\to\infty} \frac{L_X^*(t)}{t} = +\infty \quad \mathbb{P}\text{-p.s.}$$

Par contre si la dérive est suffisamment forte, les processus, discrets et continus se comporte de la même manière : si $\kappa > 1$, pour toute fonction strictement positive croissante $a(\cdot)$,

$$\sum_{n=1}^{\infty} \frac{1}{na(n)} \begin{cases} < +\infty \\ = +\infty \end{cases} \Longleftrightarrow \limsup_{t\to\infty} \frac{L_X^*(t)}{(ta(t))^{1/\kappa}} = \begin{cases} 0 \\ +\infty \end{cases} \quad \mathbb{P}\text{-a.s.}$$

1.4 Construction du modèle en temps continu

La définition de la diffusion en milieu aléatoire à partir de son générateur est difficilement manipulable. C'est pourquoi il est utile d'en chercher une expression plus simple. Si V est un processus càd-làg localement borné, on peut en effet représenter la diffusion $\mathbf{X}$ en milieu V comme un mouvement brownien changé en temps et en espace (voir [17]). Plus précisement, il existe un mouvement brownien B indépendant de V tel que en notant

$$S_V(x) := \int_0^x e^{V(y)} \mathrm{d}y \ , \ x \in \mathbb{R} \tag{1.12}$$

et

$$T_{V,B}(t) := \int_0^t e^{-2V(S_V^{-1}(B(s)))} \mathrm{d}s \ , \ t > 0 \tag{1.13}$$

le processus $\mathbf{X}$ puisse se réécrire

$$\mathbf{X} = S_V^{-1} \circ B \circ T_{V,B}^{-1}. \tag{1.14}$$

Pour simplifier les notations, on écrira S et T pour respectivement S_V et $T_{V,B}$. Remarquons que cette expression n'est valable que dans le cas d'une diffusion unidimensionnelle, c'est une des raisons pour laquelle l'étude des diffusions en milieu aléatoire multidimensionnelle est plus complexe.

L'équation (1.14) permet de montrer que le processus des temps locaux de $\mathbf{X}$ est bien défini et donne une expression en fonction du processus des temps locaux L du mouvement brownien B. En effet, en utilisant successivement le changement de variable $u = T^{-1}(s)$, la formule des temps d'occupation pour le mouvement brownien et le changement de variable $x = S(y)$, on obtient pour tout $t > 0$ et tout ensemble borélien A,

$$\begin{aligned}\int_0^t \mathbf{1}_A(\mathbf{X}(s))\mathrm{d}s &= \int_0^t \mathbf{1}_A(S^{-1} \circ B \circ T^{-1}(s))\mathrm{d}s \\ &= \int_0^{T^{-1}(t)} \mathbf{1}_A(S^{-1} \circ B(s))e^{-2V(S^{-1}(B(u)))}\mathrm{d}u \\ &= \int_{-\infty}^{\infty} \mathbf{1}_A(S^{-1}(x))e^{-2V(S^{-1}(x))}L(T^{-1}(t), x)\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \mathbf{1}_A(y)e^{-2V(y)}L(T^{-1}(t), S(y))\mathrm{d}y.\end{aligned}$$

Ainsi le temps local de la diffusion $\mathbf{X}$ vérifie l'égalité suivante :

$$L_X(t, x) = e^{-V(x)}L(T^{-1}(t), S(x)) \ , \ x \in \mathbb{R}, \ t \geq 0. \tag{1.15}$$

1.5 Description des résultats obtenus

Le but principal de la thèse est d'étudier le comportement asymptotique du temps local d'une diffusion en milieu aléatoire récurrent.

Le chapitre 2 est un travail réalisé en collaboration avec Pierre Andreoletti accepté pour publication dans la revue *Journal of Theoretical Probability*. On y étudie la loi limite du temps local. Comme pour l'équation (1.6), on commence par se ramener à une quantité ne dépendant que du milieu. Pour présenter le résultat plus simplement, commençons par une définition : on dit que le processus $(Y(t, x) \ , \ t \geq 0, x \in \mathbb{R})$ converge uniformément sur tout compact en probabilité vers le processus $(Y_\infty(x) \ , \ x \in \mathbb{R})$ si pour tout $K > 0$, $\sup_{x\in[-K,K]} |Y(t, x)-Y_\infty(x)|$ converge vers 0 en probabilité quand t tends vers l'infini. On note alors

$$Y(t, \cdot) \xrightarrow[t\to\infty]{\text{ucp}} Y_\infty.$$

Si $Y(t, x)$ est presque sûrement strictement positif, on dit que $\tilde{Y}(t, \cdot)$ est ucp-équivalent à $Y(t, \cdot)$ et on note $\tilde{Y}(t, \cdot) \underset{t}{\overset{\text{ucp}}{\sim}} Y(t, \cdot)$ si

$$\frac{\tilde{Y}(t, \cdot)}{Y(t, \cdot)} \xrightarrow[t\to\infty]{\text{ucp}} 1.$$

Dans le chapitre 2, on considère la diffusion $\mathbf{X}$ en milieu brownien standard W. Nous montrons que le temps local de $\mathbf{X}$ recentré au fond de la vallée de hauteur $\log t$ est ucp-équivalent à une variable dépendant uniquement du milieu. Plus précisement, on définit le milieu recentré au fond de la vallée par

$$W_{m_{\log t}}(\cdot) := V(\cdot + m_{\log t}) - W(m_{\log t})$$

et on note alors pour $r \in]0,1[$ fixé,

$$a_{rt} := \sup\left\{x \leq 0,\ W_{m_{\log t}}(x) \geq r\log t\right\} \text{ et}$$
$$b_{rt} := \inf\left\{x \geq 0,\ W_{m_{\log t}}(x) \geq r\log t\right\}.$$

Le théorème 2.1.3 du chapitre 2 nous dit alors que

$$\left(\frac{L_X(t, m_{\log t} + x)}{t}\ ,\ x \in \mathbb{R}\right) \overset{\text{ucp}}{\underset{t}{\sim}} \frac{e^{-W_{m_{\log t}}(x)}}{\int_{a_{rt}}^{b_{rt}} e^{-W_{m_{\log t}}}}. \tag{1.16}$$

L'étude de la loi du mouvement brownien au voisinage du fond de la vallée faite par Tanaka [85] et le résultat précédent nous permettent d'obtenir la loi limite du processus des temps locaux quand t tend vers l'infini. On pose pour $x \in \mathbb{R}$, $R(x) := R_1(x)\mathbf{1}_{\{x\geq 0\}} + R_2(-x)\mathbf{1}_{\{x<0\}}$, où R_1 et R_2 sont deux processus de Bessel de dimension 3 issus de 0 indépendants, alors

$$\left(\frac{L_X(t, m_{\log t} + x)}{t}\ ,\ x \in \mathbb{R}\right) \xrightarrow[t\to\infty]{\mathcal{L}} \left(\frac{e^{-R(x)}}{\int_{-\infty}^{+\infty} e^{-R}}\ ,\ x \in \mathbb{R}\right). \tag{1.17}$$

Le résultat (1.16) est encore vrai pour des milieux plus généraux, en fait il suffit que le milieu soit un processus càd-làg localement borné V vérifiant presque sûrement :

1. V est stable : il existe $\alpha > 0$ tel que pour tout $c > 0$, les processus V et $c^{-1}V(c^\alpha \cdot)$ aient même loi.
2. $\liminf_{x\to\pm\infty} V(x) = -\infty$ et $\limsup_{x\to\pm\infty} V(x) = +\infty$
3. V est continu en ses extremas locaux et les valeurs prises par V en ces points sont deux à deux distinctes.

C'est par exemple le cas lorsque le milieu est un processus de Lévy stable d'indice $\alpha \in [1,2]$ qui n'est pas un drift pur. L'étude du temps local de la diffusion dans ce type de milieu est l'objet du chapitre 3. C'est un travail réalisé en collaboration avec Guillaume Voisin accepté pour publication dans la revue *Stochastics : An International Journal of Probability and Stochastic Processes*. Encore une fois, on peut utiliser la relation (1.16) pour obtenir la loi asymptotique du temps local et généraliser le résultat du chapitre précédent. En effet, un processus de Bessel de dimension 3 peut être vu comme un mouvement brownien conditionné à rester positif (voir Doob [30] et McKean [58]). Et cette notion s'étend aux processus de Lévy : à tout processus de Lévy V, on peut associer un processus $V^\uparrow$ appelé processus de Lévy conditionné à rester positif (voir par exemple Doney [28]). On considère alors un processus de Lévy V défini sur $\mathbb{R}$, stable d'indice $\alpha \in [1,2]$ qui n'est pas un drift pur et on se donne les deux processus indépendants $V^\uparrow$ et $(-V)^\uparrow$, $V^\uparrow$ ayant même loi que $(V(x),\ x \in \mathbb{R})$ conditionné à rester positif et $(-V)^\uparrow$ ayant même loi que $(-V(x),\ x \in \mathbb{R})$ conditionné à rester positif. On définit le processus

$$\widetilde{V}(x) := V^\uparrow(x)\mathbf{1}_{x\geq 0} + (-V)^\uparrow(-x)\mathbf{1}_{x<0}.$$

Nous montrons dans le chapitre 3 la convergence faible pour la topologie de Skorohod :

$$\left(\frac{L_X(t, m_{\log t} + x)}{t},\ x \in \mathbb{R}\right) \xrightarrow[n\to\infty]{\mathcal{L}} \left(\frac{e^{-\widetilde{V}(x)}}{\int_{-\infty}^{\infty} e^{-\widetilde{V}(y)}\mathrm{d}y},\ x \in \mathbb{R}\right). \tag{1.18}$$

Cette convergence ainsi que (1.17) montre qu'asymptotiquement la diffusion se comporte comme si elle se trouvait piégée dans une vallée de hauteur infinie du milieu. Dans ce même chapitre, nous étendons par ailleurs le résultat de Cheliotis (1.11) sur le positionnement de point favori au cas d'un milieu Lévy stable d'indice $\alpha \in [1,2]$ qui n'est pas un drift pur. Ces deux résultats nous permettent d'obtenir la convergence en loi de $L_X^*(t)$:

$$\frac{L_X^*(t)}{t} \xrightarrow[t\to\infty]{\mathcal{L}} \frac{1}{\int_{-\infty}^{+\infty} e^{-\widetilde{V}}}.$$

Finalement, dans le chapitre 4, on se restreint de nouveau au cas où le milieu est un mouvement brownien standard. Il s'agit là encore d'un article soumis pour publication. Nous nous intéressons maintenant au comportement presque sûr du temps local. Cela nécessite d'effectuer des estimées plus fines que dans le chapitre 2.

Nous voulons ici vérifier à quel point la concentration du temps local peut être forte, c'est-à-dire que nous cherchons à savoir si $L_X^*(t)$ peut croître plus vite que $t\log_3 t$ (voir équation (1.10)). La réponse est non : il existe une constante $c > 0$ telle que presque sûrement,

$$\limsup_{t\to\infty} \frac{L_X^*(t)}{t\log_3(t)} \leq c.$$

La question associée qui arrive alors naturellement est de savoir si la localisation peut être beaucoup plus faible, pour cela on s'intéresse au comportement de la lim inf. Nous montrons, au chapitre 4 également, que dans ce cas, la renormalisation correcte est la même que pour la marche discrète : presque sûrement,

$$0 < \liminf_{t\to\infty} \frac{\log_3(t)}{t} L_X^*(t) < \infty.$$

Les différences entre le cas continu et le cas discret (cf (1.8) et (1.9)) peuvent s'interpréter de la façon suivante : dans les deux cas, $L_X^*(t)/t$ se comporte approximativement au mieux, comme l'inverse de l'intégrale de l'exponentielle de l'environnement «le plus pentu» et au pire, comme l'inverse de l'intégrale de l'exponentielle de l'environnement «le plus plat». Néanmoins si dans le cas de la marche, il y a réellement un environnement le plus pentu car la loi de p_0 est à support compact dans $]0,1[$, dans le cas continu, l'environnement peut être aussi pentu qu'on le souhaite. Ceci explique la différence pour la lim sup. Par contre, dans les deux cas, l'environnement peut-être aussi plat que nécessaire. Pour calculer explicitement ces taux, il est nécessaire d'étudier à quel point les vallées peuvent être encaissées et surtout on doit s'intéresser à l'endroit où la diffusion passe la plus grande partie de son temps. En effet, comme nous l'avons déjà fait remarquer, pour des temps t suffisamment larges, le processus $\mathbf{X}$ se retrouve au voisinage du fond $m_{\log t}$ de la vallée standard de hauteur $\log t$. Cet évènement se produit avec une probabilité qui tend vers 1 lorsque t tend vers l'infini, malheureusement, elle ne croît pas assez vite pour qu'on puisse en déduire des résultats presque sûrs. Infiniment souvent, le potentiel possède plusieurs vallées de hauteur proche et la particule peut se retrouver piégée au voisinage d'un autre point que $m_{\log t}$; ce phénomène a déjà été mis en évidence par Shi et Zindy [73] dans le cas de la marche. Nous relaxons donc l'hypothèse de localisation en étudiant le voisinage non plus d'un unique point mais de quatre. En effet, la particule peut partir du mauvais côté, c'est-à-dire partir à gauche de 0 alors que le fond de la vallée est à droite ou inversement. On considére donc les fonds des vallées à droite

$$\tau_r^d := \inf\{x \geq 0 \ , \ W(x) - \min_{[0,x]} W = r\},$$
$$m_r^d := \inf\{x \geq 0 \ , \ W(x) = \min_{[0,\tau_r^d]} W\}$$

et à gauche

$$\tau_r^g := \sup\{x \leq 0 \ , \ W(x) - \min_{[x,0]} W = r\},$$
$$m_r^g := \sup\{x \leq 0 \ , \ W(x) = \min_{[\tau_r^g,0]} W\}.$$

Mais même ainsi, on ne peut être sûr que pour des temps t grands, la diffusion aura atteint l'un des deux fonds de la vallée de hauteur $\log t$, ni qu'elle ne sera pas sortie de cette vallée. On doit donc considérer les fonds des vallées de hauteur un peu plus petite que $\log t$:

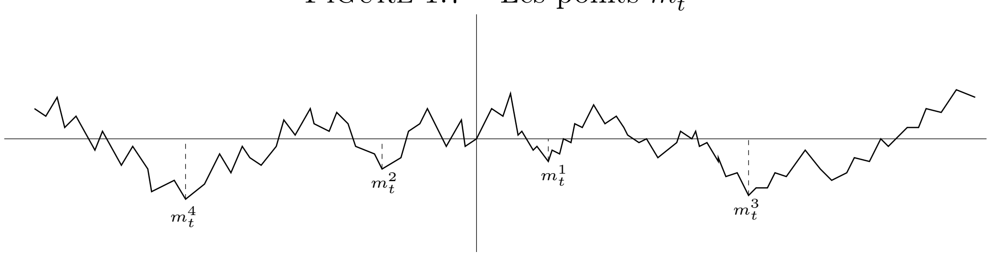
$$m_t^1 := m^d_{\log t - c_1 \log_2 t} \text{ et } m_t^2 := m^g_{\log t - c_1 \log_2 t}, \quad c_1 > 0 \text{ fixé}$$

pour que la diffusion ait atteint l'un ou l'autre de ces points et les fonds des vallées de hauteur un peu plus grande que $\log t$:

$$m_t^3 := m^d_{\log t + c_2 \log_2 t} \text{ et } m_t^4 := m^g_{\log t + c_2 \log_2 t}, \quad c_2 > 0 \text{ fixé}$$

pour que le processus n'ait pas quitté la vallée.

FIGURE 1.7 – Les points m_t^i



Dans le chapitre 4, on montre alors que pour $\epsilon > 0$ fixé et en définissant correctement c_1 et c_2 à partir d'une constante $c_0 > 0$ quelconque, si l'on note

$$I_t := \bigcup_{i=1}^{4} [m_t^i - (\log_2 t)^{4+\epsilon},\ m_t^i + (\log_2 t)^{4+\epsilon}],$$

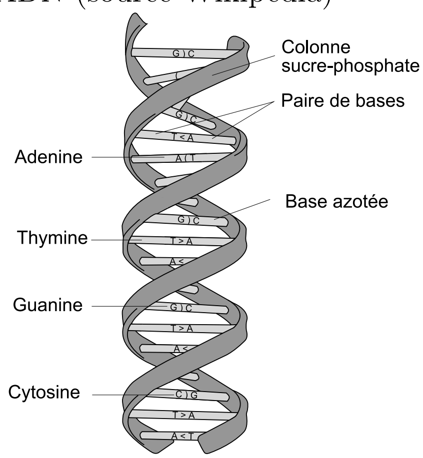
alors presque sûrement

$$\lim_{t\to\infty} \frac{(\log t)^{c_0}}{t} \nu_t \left(\overline{I_t}\right) = 0$$

où ν_t est la mesure d'occupation définie en (1.7). C'est-à-dire que presque sûrement, pour des temps suffisamment grands, la diffusion a passé la plus grande partie de son existence au voisinage d'au plus quatre points. Des résultats sur la concentration de la mesure d'occupation ont déjà été obtenu dans le cas de la marche de Sinaï par Andreoletti [5].

Le dernier chapitre de la thèse est consacré à une application des milieux aléatoires dans un cas pratique. Le but est d'identifier une séquence d'ADN. Commençons par quelques rappels de biologie. L'acide désoxyribonucléique (ADN) est une molécule, présente dans toutes les cellules vivantes, qui renferme l'ensemble des informations nécessaires au développement et au fonctionnement d'un organisme. Elle est composée de séquences de nucléotides ; on parle de polymères de nucléotides ou encore de polynucléotides. Chaque nucléotide est constitué de trois éléments liés entre eux : un groupe phosphate lié à un sucre, le désoxyribose, lui-même lié à une base azotée. Il existe quatre bases azotées différentes : l'adénine (notée A), la thymine (notée T), la cytosine (notée C) et la guanine (notée G). Une molécule d'ADN est composé de deux brins se faisant face, et formant une double hélice. Ceci est possible car les nucléotides trouvés dans un brin possèdent des nucléotides complémentaires avec lesquels ils peuvent interagir par des liaisons hydrogènes (liaisons faibles). Il y a deux liaisons hydrogènes entre A et T et trois entre C et G. En face d'une adénine, il y a toujours une thymine ; en face d'une cytosine, il y a toujours une guanine.

FIGURE 1.8 – Structure d'une molécule d'ADN (source Wikipedia)



On cherche à identifier la séquence de nucléotides composant un brin (la séquence du second brin s'en déduit), c'est-à-dire qu'on cherche à obtenir une suite d'éléments de l'alphabet {A,T,C,G} correspondant aux quatre nucléotides formant l'enchaînement de l'ADN. Pour cela, on exerce une force mécanique f constante sur chacun des brins qui entraîne le dégraphage progressif des deux brins en cassant les liaisons hydrogènes. Comme les liaisons A-T sont plus faibles que les C-G, le dégraphage des secondes liaisons met plus de temps et si la force f n'est pas trop forte, certaines liaisons peuvent même se reformer. L'idée de Baldazzi, Cocco, Marinari et Monasson ([8], [9]) et Cocco et Monasson [22] est de modéliser ce phénomène par un processus en milieu aléatoire. On note $b = (b_i,\ 1 \leq i \leq M)$ la suite des acides aminés, i.e. chaque b_i est un élément de l'alphabet $\{A, T, C, G\}$. On note $Y = (Y_t,\ t \geq 0)$ le nombre de liaisons dégraphées au temps t. On modélise le dégraphage de la manière suivante : conditionnellement à b, Y est un processus de Markov à valeurs dans $\{1, \cdots, M\}$. Lorsqu'il est en i saute en $i+1$ au taux $e^{-\beta g(b_i, b_{i+1})}$ et en $i-1$ au taux $e^{-\beta g_1(f)}$ si $i \in \{2, \ldots, M-1\}$, si Y est en 1, il saute en 2 au taux $e^{-\beta g(b_1, b_2)}$ et si Y est en M, il saute en $M-1$ au taux $e^{-\beta g_1(f)}$.

Dans le dernier chapitre, on explique comment estimer la séquence d'ADN à partir d'une (ou plusieurs) trajectoire(s) de Y et on étudie le comportement de l'estimateur de maximum de vraisemblance.

Chapitre 2

Limite en loi du temps local en milieu brownien

2.1 Introduction

2.1.1 The model

Let $(W(x), x \in \mathbb{R})$ be a càdlàg real-valued stochastic process with $W(0) = 0$. A *diffusion process in the environment* W is a process $(X(t), t \in \mathbb{R}^+)$ whose conditional generator, given W, is

$$\frac{1}{2}e^{W(x)}\frac{d}{dx}\left(e^{-W(x)}\frac{d}{dx}\right).$$

Notice that for almost surely differentiable W, $(X(t), t \in \mathbb{R}^+)$ is the solution of the following stochastic differential equation

$$\begin{cases} \mathrm{d}X(t) = \mathrm{d}\beta(t) - \frac{1}{2}W'(X(t))\mathrm{d}t, \\ X(0) = 0. \end{cases}$$

where β is a standard one-dimensional Brownian motion independent of W. Of course when W is not differentiable, the previous equation has no rigorous sense.
The study of such a process starts with a choice for W, a classic one, originally introduced by S. Schumacher [69] and T. Brox [17], is to take for W a Lévy process. In fact only a few papers deal with the discontinuous case, see for example P. Carmona [18] or A. Singh [76], and most of the results concern continuous W, *i.e.* $(W(x) := B_x - \kappa x/2, x \in \mathbb{R})$, with $\kappa \in \mathbb{R}^+$ and B a two sided Brownian motion independent of β.
The case $\kappa > 0$ was first studied by K. Kawazu and H. Tanaka [50], then

by H. Tanaka [87] and Y. Hu, Z. Shi and M. Yor [45], more recently by for example M. Taleb [83], and A. Devulder [27, 26]. The universal characteristic of this X is the transience, however a wide range of limit behavior appears, depending on the value of κ see [45].
In this paper we choose $\kappa = 0$, X is then recurrent and [17] shows that it is sub-diffusive with asymptotic behavior in $(\log t)^2$, moreover X has the property, for a given instant t, to be localized in the neighborhood of a random point $m_{\log t}$ depending only on t and W. The limit law of $m_{\log t}/(\log t)^2$ and therefore of $X(t)/(\log t)^2$ were made explicit independently by H. Kesten [52] and A. O. Golosov [37].
In fact, the aim of H. Kesten and A. O. Golosov was to determine the limit law of the discrete time and space analogous of Brox's model introduced by F. Solomon [77] and then studied by Ya. G. Sinai [74]. This random walk in random environment, usually called Sinai's walk, $(S_n, n \in \mathbb{N})$ has actually the same limit distribution as Brox's one.
Turning back to Brox's diffusion, notice that H. Tanaka [87, 85] obtained a deeper localization and later Y. Hu and Z. Shi [40] get the almost sure rates of convergence. It appears that these rates of convergence are exactly the same as the rate of convergence for Sinai's walk. The question of an invariance principle, that could exist between these two processes rises and remains open (see Z. Shi [72] for a survey). In fact a first attempt to link these two processes appears for the first time in the articles of S. Schumacher [69] and K. Kawazu, Y. Tamura and H. Tanaka [48].

This work is devoted to the limit distribution of the local time of X. Indeed to the diffusion X corresponds a local time process $(L_X(t,x))_{t\geq 0, x\in\mathbb{R}}$ defined by the occupation time formula : L_X is the unique $\mathbb{P}$-a.s. jointly continuous process such that for any bounded Borel function f and for any $t \geq 0$,

$$\int_0^t f(X(s))\mathrm{d}s = \int_{\mathbb{R}} f(x) L_X(t,x)\mathrm{d}x.$$

The first results on the behavior of L_X can be found in [71] and [41]. In particular in [41] it is proven that, for any $x \in \mathbb{R}$

$$\frac{\log(L_X(t,x))}{\log t} \xrightarrow{\mathcal{L}} U \wedge \hat{U},\ t \to +\infty \tag{2.1}$$

where U and $\hat{U}$ are independent variables uniformly distributed in $(0,1)$, $\xrightarrow{\mathcal{L}}$ is the convergence in law and $x \wedge y = \min(x,y)$. Notice that in the same paper Y. Hu and Z. Shi also prove that this behavior is the same for Sinai's walk: if we denote by $\xi(n,x) := \sum_{i=1}^n \mathbf{1}_{S_i = x}$ the local time of S in $x \in \mathbb{Z}$ at

time n then
$$\frac{\log(\xi(n,x))}{\log n} \xrightarrow{\mathcal{L}} U \wedge \hat{U},\ n \to +\infty.$$
For previous works on the local time for Sinai's diffusion we refer to the book of P. Révèsz [66].

In this article we show that the normalized local time process $(L(t, x + m_{\log t})/t,\ x \in \mathbb{R})$ behaves asymptotically as a process which only depends on the environment W. We also make explicit the limit law of this process when t goes to infinity, it involves some 3-dimensional Bessel processes. The supremum of the local time process of X is given by
$$\forall t \geq 0, \quad L_X^*(t) := \sup_{x \in \mathbb{R}} L_X(t, x),$$
as a consequence of our results we show that $L_X^*(t)/t$ converges weakly and determine its limit law. We also find interesting to compare the discrete Theorems of [34] with ours, pointing out the analogies and the differences.

2.1.2 Preliminary definitions and results

First let us describe the probability space where X is defined. It is composed of two Wiener's spaces, one for the environment and the other one for the diffusion itself. Let $\mathcal{W}$ be the space of continuous functions $W : \mathbb{R} \to \mathbb{R}$ satisfying $W(0) = 0$ and $\mathcal{A}$ the σ-field generated by the topology of uniform convergence on compact sets on $\mathcal{W}$. We equip $(\mathcal{W}, \mathcal{A})$ with Wiener measure $\mathcal{P}$ i.e the coordinate process is a "two-sided" Brownian motion. We call *environment* an element of $\mathcal{W}$.
We also define the set $\Omega := C([0; +\infty), \mathbb{R})$, the σ-field $\mathcal{F}$ on Ω generated by the topology of uniform convergence on compact sets and the probability measure P such that the coordinate process on Ω is a standard Brownian motion.
We denote by $\mathbb{P}$ the probability product $\mathcal{P} \otimes P$ on $\mathcal{W} \times \Omega$. We indifferently denote by W an undetermined element of $\mathcal{W}$ and the first coordinate process on $\mathcal{W} \times \Omega$ (i.e a “two-sided“ Brownian motion under $\mathbb{P}$ or P) similarly B is indifferently an element of Ω and the second coordinate process on $\mathcal{W} \times \Omega$.
Finally $\stackrel{\mathcal{L}_W}{=}$ means "equality in law" under P that is under a fixed environment W, and $\stackrel{\mathcal{L}}{=}$ (respectively $\xrightarrow{\mathcal{L}}$) for an equality (resp. a convergence) in law under $\mathbb{P}$. We can now state our first result:

Theorem 2.1.1. *We have*
$$\frac{L_X^*(t)}{t} \xrightarrow{\mathcal{L}} \frac{1}{\int_{-\infty}^{\infty} e^{-R(y)}\, dy} \tag{2.2}$$

where for any $x \in \mathbb{R}$, $R(x) := R_1(x)\mathbf{1}_{\{x \geq 0\}} + R_2(-x)\mathbf{1}_{\{x<0\}}$, R_1 and R_2 are two independent 3-dimensional Bessel processes starting at 0.

First notice that $\int_{-\infty}^{\infty} e^{-R(y)} \mathrm{d}y < +\infty$ *a.s.*. Now we would like to state the equivalent of this theorem for Sinai's walk recently found by [34],

$$\frac{\xi^*(n)}{n} \xrightarrow{\mathcal{L}} \sup_{x \in \mathbb{Z}} \pi(x) \tag{2.3}$$

where

$$\pi(x) = \frac{\exp(-Z_x) + \exp(-Z_{x-1})}{2 \sum_{y \in \mathbb{Z}} \exp(-Z_y)}, \ x \in \mathbb{Z}, \tag{2.4}$$

and Z is a sum of i.i.d random variables (with mean zero, strictly positive variance and bounded) null at zero and conditioned to stay positive (see below Theorem 1.1 and Section 4 of [34] for the exact definition).
The analogy between the local time for X and the local time for S takes place in the fact that both R and S can be obtained from classical diffusion conditioned to stay positive, R_1 and R_2 are Brownian motions conditioned to stay positive (see [92]) and Z a simple symmetric random walk conditioned to stay positive (see [36] and [11]). Note also that A. O. Golosov also proved that $\sum_{y \in \mathbb{Z}} \exp(-Z_y) < +\infty$.
However Z and R have not the same nature, one is discrete the other one continuous, notice also that the increments of Z are supposed to be bounded (see hypothesis 1.2 in [34]), and it is not the case for R. Finally the numerator of $\pi(x)$ is not as simple as the numerator of our result, but this comes essentially from the difference of nature of the two processes discrete for one and continuous for the other.

Theorem 2.1.1 is an easy consequence of an interesting intermediate result (Theorem 2.1.2 below). Before introducing that result we need some extra definitions on the environment, these basic notions have been introduced by [17] see also [60]. Let $h > 0$, we say that $W \in \mathcal{W}$ admits a *h-minimum* at x_0 if there exists ξ and ζ such that $\xi < x_0 < \zeta$ and for any $x \in [\xi, \zeta]$,

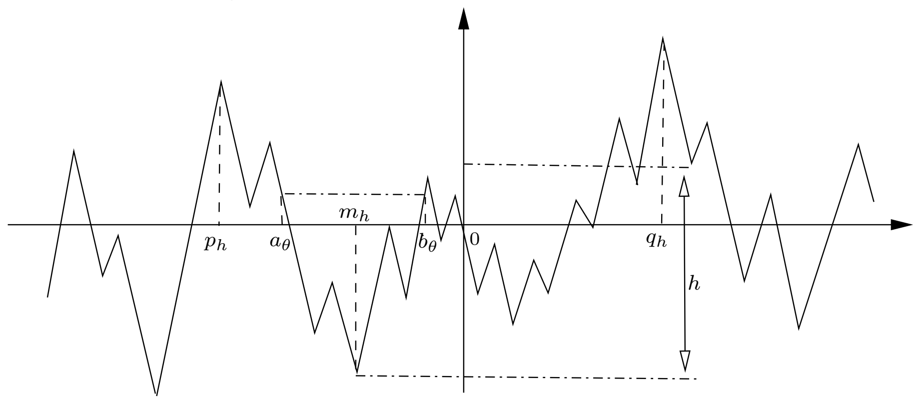
- $W(x) \geq W(x_0)$,
- $W(\xi) \geq W(x_0) + h$,
- $W(\zeta) \geq W(x_0) + h$.

Similarly we say that W admits a *h-maximum* at x_0 if $-W$ admits a h-minimum at x_0. We denote by $M_h(W)$ the set of *h-extrema* of W. It is easy to establish that $\mathcal{P}$-a.s. M_h has no accumulation point and that the points of h-maximum and of h-minimum alternate. Hence there exists exactly one triplet $\Delta_h = (p_h, m_h, q_h)$ of successive elements in M_h such that

– m_h and 0 lay in $[p_h, q_h]$,
– p_h and q_h are h-maxima,
– m_h is a h-minimum.

We call this triplet the standard h-valley of W (see Figure 2.1). We can now

Figure 2.1: Example of standard valley



state our second result:

Theorem 2.1.2.

$$\left(\frac{L_X(t, m_{\log t} + x)}{t}\right)_{x\in\mathbb{R}} \xrightarrow{\mathcal{L}} (\mathfrak{R}(x))_{x\in\mathbb{R}}, \ \textit{with}$$

$$\mathfrak{R}(x) := \frac{e^{-R(x)}}{\int_{-\infty}^{\infty} e^{-R(y)}\, dy}.$$

and R is the same as in Theorem 2.1.1.

This result is the analogue of Theorem 1.2 of [34], we recall their result:

$$\left(\frac{\xi(n, b_n + x)}{n}, x \in \mathbb{Z}\right) \xrightarrow{\mathcal{L}} (\pi(x))_{x\in\mathbb{Z}}, \tag{2.5}$$

where $\pi(x)$ is given by (2.4) and b_n plays the same role for S as $m_{\log t}$ plays for X. Notice that Theorem 2.1.1 can be deduced directly from Theorem 2.1.2, in the same way that (2.3) can be deduced from (2.5).

There is a second remark we can make here, H. Tanaka [88] shows that the limit when t goes to infinity of the distribution of the process $(X(t) - m_{\log t})$ converges to a distribution with a density (with respect to Lebesgue measure) given by $\mathfrak{R}$. There is a natural explanation to understand, at least intuitively, why $\mathfrak{R}$ appears in both limits. For that we need first to present another result, indeed Theorems 2.1.1 and 2.1.2 are consequences of a result, in probability, on the asymptotic behavior of the local time in a neighborhood of $m_{\log t}$ (Theorem 2.1.3 below) together with results on the random environment (see Sections 3.2 and 3.3). Before stating our third result, we need a new notation, let $(W_x, x \in \mathbb{R})$ be the *shifted difference of potential*,

$$\forall x \in \mathbb{R}, W_x(\cdot) := W(x + \cdot) - W(x). \tag{2.6}$$

Theorem 2.1.3.
Let $r \in (0,1)$, then for all $\delta > 0$,

$$\lim_{\alpha \to +\infty} \mathbb{P}\left(\sup_{a_{\alpha r} \le x \le b_{\alpha r}} \left| \frac{L_X(e^\alpha, m_\alpha + x)}{e^\alpha} \frac{\int_{a_{\alpha r}}^{b_{\alpha r}} e^{-W_{m_\alpha}(y)} dy}{e^{-W_{m_\alpha}(x)}} - 1 \right| \le \delta \right) = 1$$

where for any $\theta > 0$,

$$a_\theta = a_\theta(W_{m_\alpha}) := \sup \{x \le 0 / W_{m_\alpha}(x) \ge \theta\} \text{ and}$$
$$b_\theta = b_\theta(W_{m_\alpha}) := \inf \{x \ge 0 / W_{m_\alpha}(x) \ge \theta\},$$

see also Figure 2.1.

We are ready to give an idea of the reason why $\mathfrak{R}$ appears in the paper of H. Tanaka [88] and the present one. The important term in the last result, which appears when we study the inverse of the local time in m_α (see section 2.2.2), is the following

$$\bar{\mathfrak{R}}(\alpha, x) := \frac{e^{-W_{m_\alpha}(x)}}{\int_{a_{\alpha r}}^{b_{\alpha r}} e^{-W_{m_\alpha}(y)} \mathrm{d}y}.$$

First, we will show in Section 2.3.2 that the process $(\bar{\mathfrak{R}}(\alpha, x), x \in \mathbb{R})$ converges weakly when α goes to infinity to $(\mathfrak{R}(x), x \in \mathbb{R})$, so Theorem 2.1.3 leads to Theorem 2.1.2. We therefore focus on $\bar{\mathfrak{R}}(\alpha, x)$.
Note that $(\bar{\mathfrak{R}}(\alpha, x), x \in (a_{\alpha r}, b_{\alpha r}))$ can be seen as a local (in time) invariant probability measure for the process X. Indeed until the instant e^α, $(X(s) \equiv X(W, s),\ s \le e^\alpha)$ spends, with a high probability, most of its time between the two points $(a_{\alpha r}, b_{\alpha r})$. So X can be approached, in some sense, by a simpler process $(\tilde{X}(s) \equiv \tilde{X}(\alpha W, s),\ s \le e^\alpha)$ with the same generator as X but reflected at fixed barriers, $\tilde{a}$ and $\tilde{b}$ (see [17] or [88] page 159). This new process obviously possesses an invariant probability measure given by $\tilde{\mu}_\alpha(dx) := \tilde{\mathfrak{R}}(\alpha, x) dx = e^{-\alpha W_{m_1}(x)} \mathrm{d}x / \int_{\tilde{a}}^{\tilde{b}} e^{-\alpha W_{m_1}}$. And $\tilde{\mathfrak{R}}(\alpha, x)$ is naturally involved in the limit behavior of the normalized local time $L_{\tilde{X}}(e^\alpha, m_1 + \cdot)/e^\alpha$ and also in the limit distribution of $\tilde{X}(e^\alpha) - m_1$. To turn back to $\bar{\mathfrak{R}}$ a self

similarity argument of the environment can be used. To finish with this discussion, it is interesting to notice that for the discrete time model the result in law plays an important role to get the almost sure asymptotic of the limit sup of ξ^*. Indeed (2.3) (together with Lemmata 3.1 and 3.2 in [34]) leads to

$$\limsup_n \frac{\xi^*(n)}{n} = \text{const} \in]0, +\infty[, \mathbb{P}.a.s.$$

and the const is known explicitly. For Brox's model, $L_X^*(t)$ is possibly larger than t and, in fact, if we use a similar argument than [34], then the limit in law (Theorem 2.1.2) only implies

$$\limsup_t \frac{L_X^*(t)}{t} = +\infty, \ \mathbb{P}.a.s.$$

So the limit in law implies a weaker result for the limsup than the already known result of Z. Shi ([71]) who proved $\limsup_t \frac{L_X^*(t)}{t \log\log\log t} \geq \text{const}, \ \mathbb{P}.a.s.$.

2.1.3 Basic facts for diffusion with potential

In this section we recall basic definitions and tools traditionally used to study diffusion in random environment. For all $W \in \mathcal{W}$, define

$$\forall x \in \mathbb{R}, S_W(x) := \int_0^x e^{W(y)} \mathrm{d}y \tag{2.7}$$

and

$$\forall t \geq 0, T_W(t) := \int_0^t e^{-2W(S_W^{-1}(B(s)))} \mathrm{d}s. \tag{2.8}$$

As Brox points out in [17], the standard diffusion theory implies that the process

$$\begin{array}{rcl} X : \mathcal{W} \times \Omega & \longrightarrow & \Omega \\ (W, B) & \longmapsto & S_W^{-1} \circ B \circ T_W^{-1} \end{array} \tag{2.9}$$

is under $\mathbb{P}$ a diffusion in Brownian environment. To simplify notations, we write when there is no possible mistake S and T for respectively S_W and T_W.

Using Formula (2.9), we easily obtain that for any $x \in \mathbb{R}$ and $t \geq 0$,

$$L_X(t, x) = e^{-W(x)} L_B(T^{-1}(t), S(x)) \tag{2.10}$$

where L_B is the local time process of the Brownian motion B.

T. Brox ([17]) noticed also that it is more convenient to study the asymptotic behavior of the process

$$X_\alpha(W, \cdot) := X(\alpha W, \cdot)$$

and the one of its local time process L_{X_α} instead of X and L_X.

For all $x \in \mathbb{R}$, we denote

$$W^\alpha(x) := \frac{1}{\alpha} W(\alpha^2 x).$$

For each $\alpha > 0$, we have a link between X_α and X given by:

Lemma 2.1.4. *For each $W \in \mathcal{W}$ and $\alpha > 0$,*

$$(X_\alpha(W^\alpha, t))_{t\geq 0} := (X(\alpha W^\alpha, t))_{t\geq 0} \stackrel{\mathcal{L}_W}{=} \left(\frac{1}{\alpha^2} X(W, \alpha^4 t)\right)_{t\geq 0},$$

$$\left(L_{X_\alpha(W^\alpha,\cdot)}(t,x)\right)_{t\geq 0, x\in\mathbb{R}} \stackrel{\mathcal{L}_W}{=} \left(\frac{1}{\alpha^2} L_{X(W,\cdot)}(\alpha^4 t, \alpha^2 x)\right)_{t\geq 0, x\in\mathbb{R}}.$$

We do not give any detail of the proof of this Lemma, the first relation can be found in Brox (see [17], Lemma 1.3) and the second is a straightforward consequence of the first one.

Formulas (2.7), (2.8) and (2.9) for X_α are given by

$$\forall t \geq 0, X_\alpha(t) = S_\alpha^{-1}(B(T_\alpha^{-1}(t))) \tag{2.11}$$

where

$$\forall x \in \mathbb{R}, S_\alpha(x) := S_{\alpha W}(x) = \int_0^x e^{\alpha W(y)} \mathrm{d}y \tag{2.12}$$

and

$$\forall t \geq 0, T_\alpha(t) := T_{\alpha W}(t) = \int_0^t e^{-2\alpha W(S_\alpha^{-1}(B(s)))} \mathrm{d}s, \tag{2.13}$$

also for the local time we have,

$$\forall t \geq 0, \forall x \in \mathbb{R}, L_{X_\alpha}(t,x) = e^{-\alpha W(x)} L_B(T_\alpha^{-1}(t), S_\alpha(x)). \tag{2.14}$$

The rest of the paper is organized as follows: in the first part of Section 2.2 we get the asymptotic of the local time within a random amount of time, which is the inverse of the local time at $m_{\log t}$, in Section 2.2.2 the asymptotic of the inverse of the local time itself is studied. Note that Sections 2.2.1 and 2.2.2 can be read independently.

Propositions 2.2.1 and 2.2.5 of Sections 2.2 are the key results to get Theorem 2.1.3 proved at the beginning of Section 2.3. In the second and third subsection of Section 2.3, Theorems 2.1.1 and 2.1.2 are proved. Note that these Theorems come from Theorem 2.1.3 together with the study of a functional of the random environment.

2.2 Asymptotics for the local time L_{X_α} and its inverse σ_{X_α}

We begin with some definitions that will be used all along the paper. For any process M we define the following stopping times with the usual convention $\inf \emptyset = +\infty$,

$$\forall x \in \mathbb{R}, \tau_M(x) := \inf\{t \geq 0 / M(t) = x\}, \tag{2.15}$$

$$\forall x \in \mathbb{R}, \forall r \geq 0, \sigma_M(r, x) := \inf\{t \geq 0 / L_M(t, x) \geq r\}. \tag{2.16}$$

We define for any $W \in \mathcal{W}$ and for all $x, y \in \mathbb{R}$,

$$\overline{W}(x,y) := \begin{cases} \sup_{[x,y]} W & \text{if } y \geq x, \\ \sup_{[y,x]} W & \text{if } y < x \end{cases}$$

and

$$\underline{W}(x,y) := \begin{cases} \inf_{[x,y]} W & \text{if } y \geq x, \\ \inf_{[y,x]} W & \text{if } y < x, \end{cases}$$

they represent respectively the maximum and the minimum of W between x and y. Finally we introduce the process starting in $x \in \mathbb{R}$,

$$\left(X_\alpha^x(W,t)\right)_{t\geq 0} := \left(x + X_\alpha(W_x, t)\right)_{t \geq 0} = \left(x + X(\alpha W_x, t)\right)_{t\geq 0}$$

where W_x is the *shifted difference of potential* (see (2.6)). Notice that we have the equivalent of (2.11):

$$\forall t \geq 0, X_\alpha^x(t) = x + (S_\alpha^x)^{-1}(B((T_\alpha^x)^{-1}(t))) \tag{2.17}$$

where $S_\alpha^x := S_{\alpha W_x}$ and $T_\alpha^x := T_{\alpha W_x}$ and it is easy to establish that for a fixed $W \in \mathcal{W}$, X_α^x is a strong Markov process.

2.2.1 Asymptotic behavior of L_{X_α} at time $\sigma_{X_\alpha}(e^{\alpha h(\alpha)}, m)$

In this first sub-section we study the asymptotic behavior of the local time at the inverse of the local time in $m := m_1$, recall that m_1 is the coordinate of the bottom of the basic valley defined page 5.

Proposition 2.2.1.
Let $r \in (0,1)$, $W \in \mathcal{W}$. Let h be a function such that $\lim_{\alpha \to +\infty} h(\alpha) = 1$. Then, for all $\delta > 0$,

$$\lim_{\alpha\to+\infty} P\left(\sup_{a_r \leq x \leq b_r}\left|\frac{L_{X_\alpha}(\sigma_{X_\alpha}(e^{\alpha h(\alpha)}, m), m+x)}{e^{\alpha h(\alpha) - \alpha W_m(x)}} - 1\right| \leq \delta\right) = 1.$$

where a_r and b_r are defined in Theorem 2.1.3 page 29..

Proof. For simplicity, we assume that $m = m_1(W) \geq 0$. The proof is based on the decomposition of the local time into two terms, the first one is the contribution of the local time in x_α before $\tau_{X_\alpha}(m)$ (the first time X_α hits m) and the second one is the contribution of the local time between $\tau_{X_\alpha}(m)$ and $\sigma_{X_\alpha}(e^{\alpha h(\alpha)}, m)$ (the inverse of the local time in m):

$$L_{X_\alpha}(\sigma_{X_\alpha}(e^{\alpha h(\alpha)}, m), m+x) = L_{X_\alpha}(\tau_{X_\alpha}(m), m+x) \\ + \left(L_{X_\alpha}(\sigma_{X_\alpha}(e^{\alpha h(\alpha)}, m), m+x) - L_{X_\alpha}(\tau_{X_\alpha}(m), m+x)\right).$$

We treat these two terms in the Lemmata 2.2.2 and 2.2.3 below. Lemma 2.2.2 states that, asymptotically, the local time in a point x until the process reaches m is negligible compared to $e^{\alpha h(\alpha) - \alpha W_m(x)}$. Thanks to the strong Markov property for X_α, it remains to study the asymptotic behavior of

$$\left(L_{X_\alpha^m}(\sigma_{X_\alpha^m}(e^{\alpha h(\alpha)}, m), m+x)\right)_{a_r \leq x \leq b_r}$$

when α goes to infinity, this is what is done in Lemma 2.2.3 which says that the local time in x of X_α^m within the interval of time $[0, \sigma_{X_\alpha}(e^{\alpha h(\alpha)}, m)]$ is of the order of $e^{\alpha h(\alpha) - \alpha W_m(x)}$. □

Let us state and prove

Lemma 2.2.2. *Let $W \in \mathcal{W}$. For any $\delta > 0$,*

$$\lim_{\alpha \to +\infty} P\left(\sup_{a_r \leq x \leq b_r} \frac{L_{X_\alpha}(\tau_{X_\alpha}(m), m+x)}{e^{\alpha h(\alpha) - \alpha W_m(x)}} \leq \delta\right) = 1.$$

Proof. First, as we have assumed that $m \geq 0$, for all $x > 0$, $L_{X_\alpha}(\tau_{X_\alpha}(m), m + x) = 0$ so we only have to consider non positive x. Moreover, denote $\bar{a} = (m + a_r) \vee 0$, for all $x \in [a_r, 0]$, $L_{X_\alpha}(\tau_{X_\alpha}(\bar{a}), m + x) = 0$, therefore thanks to the strong Markov property for X_α, we only need to prove that

$$\lim_{\alpha \to +\infty} P\left(\sup_{a_r \leq x \leq 0} \frac{L_{X_\alpha^{\bar{a}}}(\tau_{X_\alpha^{\bar{a}}}(m), m+x)}{e^{\alpha h(\alpha) - \alpha W_m(x)}} \leq \delta\right) = 1. \tag{2.18}$$

It follows from (2.17) with $x = \bar{a}$ that

$$\tau_{X_\alpha^{\bar{a}}}(m) = \tau_{X_\alpha(W_{\bar{a}}, \cdot)}(|a_r| \wedge m) = T_\alpha^{\bar{a}}(\tau_B(S_\alpha^{\bar{a}}(|a_r| \wedge m))),$$

so according to (2.14), we have for all $x \in \mathbb{R}$,

$$L_{X_\alpha^{\bar{a}}}(\tau_{X_\alpha^{\bar{a}}}(m), m+x) = e^{-\alpha W_{\bar{a}}(|a_r| \wedge m + x)} L_B(\tau_B(S_\alpha^{\bar{a}}(|a_r| \wedge m)), S_\alpha^{\bar{a}}(|a_r| \wedge m + x)).$$

The classic scaling property of the local time of the Brownian motion given by:

$$\forall \lambda > 0, \forall y_1 > 0, \quad (\lambda L_B(\tau_B(y_1), y))_{y\in\mathbb{R}} \overset{\mathcal{L}_W}{=} (L_B(\tau_B(\lambda y_1), \lambda y))_{y\in\mathbb{R}}\,, \qquad (2.19)$$

yields that the processes $\left(L_{X^{\bar{a}}_\alpha}\left(\tau_{X^{\bar{a}}_\alpha}(m), m+x\right)\right)_{x\in\mathbb{R}}$ and

$$\left(S^{\bar{a}}_\alpha(|a_r| \wedge m) e^{-\alpha W_{\bar{a}}(|a_r|\wedge m + x)} L_B\left(\tau_B(1), s_\alpha(|a_r| \wedge m + x)\right)\right)_{x\in\mathbb{R}}$$

are equal in law, where $s_\alpha(z) := S^{\bar{a}}_\alpha(z)/S^{\bar{a}}_\alpha(|a_r| \wedge m)$.
We claim that for all $x \in [a_r, 0]$

$$\begin{aligned} &S^{\bar{a}}_\alpha(|a_r| \wedge m) e^{-\alpha W_{\bar{a}}(|a_r|\wedge m + x)} L_B(\tau_B(1), s_\alpha(|a_r| \wedge m + x)) \\ &\le m e^{\alpha(\overline{W}_m(-|a_r|\wedge m, 0) - W_m(x))} \sup_{y\le 1} L_B(\tau_B(1), y). \end{aligned}$$

Indeed

$$S^{\bar{a}}_\alpha(|a_r| \wedge m) = \int_0^{|a_r|\wedge m} e^{\alpha W_{\bar{a}}(y)} \mathrm{d}y \le m e^{\alpha \overline{W}_{\bar{a}}(0, |a_r|\wedge m)},$$

for all $x \in [a_r, 0]$

$$\overline{W}_{\bar{a}}(0, |a_r| \wedge m) - W_{\bar{a}}(|a_r| \wedge m + x) = \overline{W}_m(-|a_r| \wedge m, 0) - W_m(x)$$

and $s_\alpha(|a_r| \wedge m + x) \le 1$.

Assembling the above estimates, we get for any $\delta > 0$,

$$\begin{aligned} &P\left(\sup_{a_r\le x\le 0} \frac{L_{X^{\bar{a}}_\alpha}(\tau_{X^{\bar{a}}_\alpha}(m), m+x)}{e^{\alpha h(\alpha) - \alpha W_m(x)}} \le \delta\right) \\ &\ge P\left(m e^{\alpha \overline{W}_m(-|a_r|\wedge m, 0) - \alpha h(\alpha)} \sup_{y\le 1} L_B(\tau_B(1), y) \le \delta\right). \end{aligned}$$

As $\overline{W}_m(-|a_r| \wedge m, 0) \le r$, $\lim_{\alpha\to+\infty} h(\alpha) = 1$ and $\sup_{y\le 1} L_B(\tau_B(1), y) < \infty$ P-a.s., the right hand side of the last inequality tends to 1 as α goes to infinity. (2.18) is proved together with the Lemma. $\square$

We continue with the proof of the second Lemma

Lemma 2.2.3. *Let $W \in \mathcal{W}$. For any $\delta > 0$,*

$$\lim_{\alpha\to+\infty} P\left(\sup_{a_r\le x\le b_r} \left|\frac{L_{X^m_\alpha}(\sigma_{X^m_\alpha}(e^{\alpha h(\alpha)}, m), m+x)}{e^{\alpha h(\alpha) - \alpha W_m(x)}} - 1\right| \le \delta\right) = 1.$$

Proof. For simplicity we denote for any process M, $\sigma_M(r) := \sigma_M(r,0)$, we will also assume without loss of generality that $m = 0$, Lemma 2.2.3 can therefore be rewritten in the following way:

$$\lim_{\alpha\to+\infty} P\left(\sup_{a_r\le x\le b_r}\left|\frac{L_{X_\alpha}(\sigma_{X_\alpha}(e^{\alpha h(\alpha)}),x)}{e^{\alpha h(\alpha)-\alpha W(x)}}-1\right|\le\delta\right)=1. \tag{2.20}$$

Like for τ in Lemma 2.2.2, we easily get that $\sigma_{X_\alpha}(t) = T_\alpha(\sigma_B(t))$, thus formula (2.14), together with the scale invariance for the local time of the Brownian motion

$$\forall r>0, \forall\lambda>0, (\lambda L_B(\sigma_B(r),y))_{y\in\mathbb{R}} \stackrel{\mathcal{L}_W}{=} (L_B(\sigma_B(\lambda r),\lambda y))_{y\in\mathbb{R}} \tag{2.21}$$

yields

$$\left(L_{X_\alpha}(\sigma_{X_\alpha}(e^{\alpha h(\alpha)}),x)\right)_{x\in\mathbb{R}} \stackrel{\mathcal{L}_W}{=} \left(e^{\alpha h(\alpha)-\alpha W(x)}L_B(\sigma_B(1),\widetilde{s}_\alpha(x))\right)_{x\in\mathbb{R}},$$

where $\widetilde{s}_\alpha(x) := S_\alpha(x)e^{-\alpha h(\alpha)}$. Therefore for any $\delta>0$,

$$\begin{aligned} &P\left(\sup_{a_r\le x\le b_r}\left|\frac{L_{X_\alpha}(\sigma_{X_\alpha}(e^{\alpha h(\alpha)}),x)}{e^{\alpha h(\alpha)-\alpha W(x)}}-1\right|\le\delta\right)\\ &=P\left(\sup_{\widetilde{s}_\alpha(a_r)\le y\le\widetilde{s}_\alpha(b_r)}|L_B(\sigma_B(1),y)-1|\le\delta\right). \end{aligned}$$

Moreover,

$$|\widetilde{s}_\alpha(a_r)|\vee\widetilde{s}_\alpha(b_r)\le(|a_r|\vee b_r)e^{-\alpha h(\alpha)+\alpha r},$$

thus $\lim_{\alpha\to+\infty}\widetilde{s}_\alpha(a_r) = \lim_{\alpha\to+\infty}\widetilde{s}_\alpha(b_r) = 0$. As $y\to L_B(\sigma_B(1),y)$ is continuous at 0, (2.20) and therefore, the lemma, are proved. □

2.2.2 Asymptotic behavior of $\sigma_{X_\alpha}(e^{\alpha h(\alpha)}, m)$

Before stating the main Proposition of this section, we need a preliminary result on the random environment which gives precisions on the standard h-valley $\Delta_h(W) = (p_h, m_h, q_h)$ defined Section 2.1.2. We denote

$$W^{\#}(x,y) := \max_{[x\wedge y, x\vee y]}(W(z)-\underline{W}(x,z))$$

where $x\vee y = \max(x,y)$. Notice that the function $W^{\#}$ represents the largest barrier of potential we have to cross in the path from x to y. We call *depth of the valley* $\Delta_h(W)$ the quantity

$$D(\Delta_h(W)) := (W(p_h)-W(m_h))\wedge(W(q_h)-W(m_h))$$

and *inner directed ascent* the quantity

$$A(\Delta_h(W)) := W^{\#}(p_h, m_h) \vee W^{\#}(q_h, m_h).$$

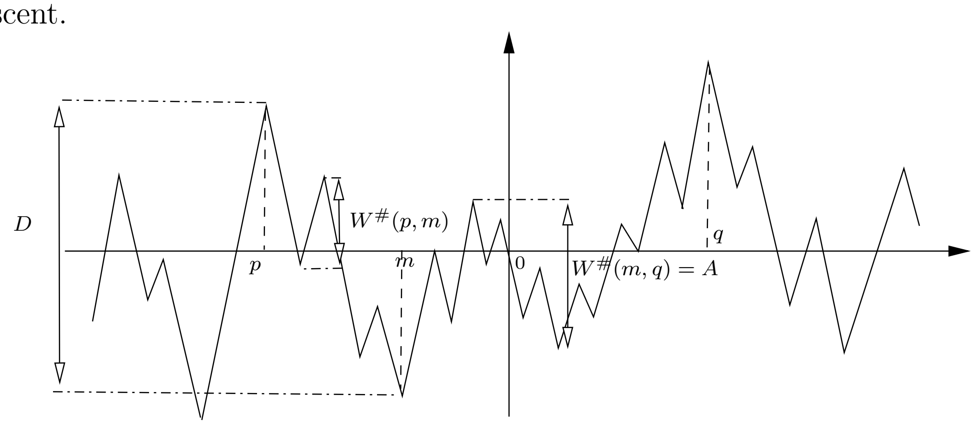
Note that the above notions have already been introduced by Sinai [74], Brox [17], and Tanaka [87]. According to Brox, we have the following lemma.

Lemma 2.2.4. *There exists a subset $\widetilde{\mathcal{W}}$ of $\mathcal{W}$ of $\mathcal{P}$-measure 1 such that for any $W \in \widetilde{\mathcal{W}}$, the standard 1-valley $\Delta_1(W) := (p_1, m_1, q_1)$ satisfies*

$$A(\Delta_1(W)) < 1 < D(\Delta_1(W)).$$

Throughout this Section we write p, m, q, D and A for respectively $m_1(W)$, $p_1(W)$, $q_1(W)$, $D(\Delta_1(W))$ and $A(\Delta_1(W))$.

Figure 2.2: Example of 1-standard valley with its depth and its inner directed ascent.



We can now state the main result of this section:

Proposition 2.2.5.
Let $W \in \widetilde{\mathcal{W}}$, $r \in (0,1)$ and h be a function such that $\lim_{\alpha\to+\infty} h(\alpha) = 1$, then for all $\delta > 0$,

$$\lim_{\alpha\to+\infty} P\left(\left|\frac{\sigma_{X_\alpha}(e^{\alpha h(\alpha)}, m)}{e^{\alpha h(\alpha)} \int_{a_r}^{b_r} e^{-\alpha W_m(x)}\, dx} - 1\right| \le \delta\right) = 1$$

where a_r and b_r are defined in Theorem 2.1.3 page 29.

To lighten notations, in the rest of the paper we denote,

$$g(\alpha) := \int_{a_r}^{b_r} e^{-\alpha W_m(y)} \mathrm{d}y,\ \alpha > 0.$$

Proof. We assume that $m > 0$, we get the other case by reflection, note that we work at fixed W which belongs to $\widetilde{\mathcal{W}}$. We follow the same steps of the proof of Proposition 2.2.1: we decompose $\sigma_{X_\alpha}(e^{\alpha h(\alpha)}, m)$ into two terms,

$$\sigma_{X_\alpha}(e^{\alpha h(\alpha)}, m) = \tau_{X_\alpha}(m) + \left(\sigma_{X_\alpha}(e^{\alpha h(\alpha)}, m) - \tau_{X_\alpha}(m)\right).$$

The first one $\tau_{X_\alpha}(m)$ is treated in Lemma 2.2.6, we show that its contribution is negligible comparing to $g(\alpha)e^{\alpha(\alpha)}$. Then thanks to the strong Markov property, it is enough to prove that $\sigma_{X_\alpha}(e^{\alpha h(\alpha)}, m)/g(\alpha)e^{\alpha h(\alpha)}$ converge to 1 in P probability, this is what Lemma 2.2.7 tells. □

Let us state and prove the first Lemma

Lemma 2.2.6. *Let $W \in \widetilde{\mathcal{W}}$. For any $\delta > 0$,*

$$\lim_{\alpha \to +\infty} P\left(\frac{\tau_{X_\alpha}(m)}{g(\alpha)e^{\alpha h(\alpha)}} \leq \delta\right) = 1.$$

Proof. This proof has the same outline of the proof of point (i) of Lemma 3.1 in [17], however because of some slight differences and for completeness we give some details.

By definition of the local time together with (2.14) and (2.19), the hitting time of m can be written

$$\begin{aligned}
\tau_{X_\alpha}(m) &= \int_{-\infty}^{m} L_{X_\alpha}(\tau_{X_\alpha}(m), z)\mathrm{d}z, \\
&= \int_{-\infty}^{m} e^{-\alpha W(z)} L_B(\tau_B(S_\alpha(m)), S_\alpha(z))\mathrm{d}z, \\
&\overset{\mathcal{L}_W}{=} S_\alpha(m) \int_{-\infty}^{m} e^{-\alpha W(z)} L_B(\tau_B(1), \hat{s}_\alpha(z))\mathrm{d}z
\end{aligned} \tag{2.22}$$

where $\hat{s}_\alpha(z) := S_\alpha(z)/S_\alpha(m)$. Let $n := \operatorname*{argmax}_{[0,m]} W$ we denote

$$\begin{aligned}
I_{\alpha,1} &:= S_\alpha(m) \int_{-\infty}^{n} e^{-\alpha W(z)} L_B(\tau_B(1), \hat{s}_\alpha(z))\mathrm{d}z, \\
I_{\alpha,2} &:= S_\alpha(m) \int_{n}^{m} e^{-\alpha W(z)} L_B(\tau_B(1), \hat{s}_\alpha(z))\mathrm{d}z,
\end{aligned}$$

and formula (2.22) can be rewritten

$$\tau_{X_\alpha}(m) \overset{\mathcal{L}_W}{=} I_{\alpha,1} + I_{\alpha,2}. \tag{2.23}$$

The rest of the proof consists essentially in finding an upper bound for $I_{\alpha,1}$ and $I_{\alpha,2}$.

We begin with $I_{\alpha,1}$, first we prove that, with a probability which tends to 1 when α goes to infinity, the process X_α does not visit coordinates smaller than p, where p is the left vertex of the standard 1-valley defined page 27. Thanks to this, the lower bound in the integral of $I_{\alpha,1}$ will be p and not $-\infty$, the upper bound for $I_{\alpha,1}$ follows almost immediately.

Let us define

$$l := \inf\left\{x \leq 0 / L_B(\tau_B(1), x) > 0\right\}, \tag{2.24}$$

we claim that,

$$P.a.s, \ \exists \alpha_0 \text{ such that } \forall \alpha > \alpha_0, \hat{s}_\alpha^{-1}(l) > p. \tag{2.25}$$

Indeed

$$\hat{s}_\alpha^{-1}(l) \geq p \Longleftrightarrow l \geq \hat{s}_\alpha(p) = -\frac{\int_p^0 e^{\alpha W(x)} \mathrm{d}x}{\int_0^m e^{\alpha W(x)} \mathrm{d}x},$$

moreover Laplace's method gives

$$\lim_{\alpha \to +\infty} \frac{1}{\alpha} \log \int_p^0 e^{\alpha W(x)} \mathrm{d}x = \overline{W}(p, 0) = W(p), \text{ and}$$
$$\lim_{\alpha \to +\infty} \frac{1}{\alpha} \log \int_0^m e^{\alpha W(x)} \mathrm{d}x = \overline{W}(0, m) = W(n),$$

so

$$\hat{s}_\alpha(p) = -\exp\left(\alpha(W(p) - W(n)) + o(\alpha)\right),$$

finally according to the definition of the standard valley $W(p) > W(n)$, therefore

$$\lim_{\alpha \to +\infty} \hat{s}_\alpha(p) = -\infty$$

and (2.25) is true.

On the event $\{\hat{s}_\alpha^{-1}(l) > p\}$, we have

$$\begin{aligned} I_{\alpha,1} &= S_\alpha(m) \int_p^n e^{-\alpha W(z)} L_B(\tau_B(1), \hat{s}_\alpha(z)) \mathrm{d}z, \\ &\leq S_\alpha(m)(n-p) e^{-\alpha \underline{W}(p,n)} \max_{x \leq 1} L_B(\tau_B(1), x), \end{aligned}$$

moreover

$$S_\alpha(m) \leq m e^{\alpha \overline{W}(0,m)} \leq (q-p) e^{\alpha W(n)},$$

and we get the upper bound

$$I_{\alpha,1} \leq (q-p)^2 e^{\alpha A} \max_{x \leq 1} L_B(\tau_B(1), x) \tag{2.26}$$

where A is the inner direct ascent of the valley defined at the beginning of this section.

We continue with $I_{\alpha,2}$, the main ingredient to get an upper bound in this case is to use the First Ray-Knight theorem which leads to the study of an integral involving a two-dimensional squared Bessel process: first we rewrite $I_{\alpha,2}$ in the following way

$$I_{\alpha,2} = S_\alpha(m)\int_n^m e^{-\alpha W(z)} L(\tau_B(1), 1-\bar{s}_\alpha(z))\mathrm{d}z \tag{2.27}$$

where

$$\bar{s}_\alpha(z) := 1 - \hat{s}_\alpha(z) = \frac{1}{S_\alpha(m)}\int_z^m e^{\alpha W(x)}\mathrm{d}x.$$

Let R be a two-dimensional squared Bessel process starting from the origin, according to the First Ray-Knight theorem

$$\left(L_B(\tau_B(1), 1-\bar{s}_\alpha(z))\right)_{z\in[0,m]} \overset{\mathcal{L}_W}{=} \left(R(\bar{s}_\alpha(z))\right)_{z\in[0,m]},$$

together with the scale invariance $(t^2 R_{\frac{1}{t}})_{t\in\mathbb{R}_+} \overset{\mathcal{L}_W}{=} (R_t)_{t\in\mathbb{R}_+}$ we get

$$\int_n^m e^{-\alpha W(z)} R(\bar{s}_\alpha(z))\mathrm{d}z \overset{\mathcal{L}_W}{=} \int_n^m \left\{e^{-\alpha W(z)}\bar{s}_\alpha(z)\right\}\bar{s}_\alpha(z) R(\frac{1}{\bar{s}_\alpha(z)})\mathrm{d}z. \tag{2.28}$$

We are now able to get a preliminary upper bound for $I_{\alpha,2}$:

$$\begin{aligned}
&S_\alpha(m)\int_n^m \left\{e^{-\alpha W(z)}\bar{s}_\alpha(z)\right\}\bar{s}_\alpha(z) R(\frac{1}{\bar{s}_\alpha(z)})\mathrm{d}z,\\
&\leq \max_{n\leq z\leq m}\left[e^{-\alpha W(z)}\int_z^m e^{\alpha W(x)}\mathrm{d}x\right]\int_n^m \bar{s}_\alpha(z) R(\frac{1}{\bar{s}_\alpha(z)})\mathrm{d}z,\\
&\leq (q-p)\exp\left\{\alpha\max_{n\leq z\leq m}\left[-W(z)+\overline{W}(z,m)\right]\right\}\int_n^m \bar{s}_\alpha(z) R(\frac{1}{\bar{s}_\alpha(z)})\mathrm{d}z,\\
&\leq (q-p)^2 e^{\alpha A}\underbrace{\frac{1}{m-n}\int_n^m \bar{s}_\alpha(z) R(\frac{1}{\bar{s}_\alpha(z)})\mathrm{d}z}_{J_\alpha}
\end{aligned} \tag{2.29}$$

and the last inequality comes from the relation

$$\max_{n\leq z\leq m}\left[-W(z)+\overline{W}(z,m)\right] = W^{\#}(n,m) \leq A.$$

According to Jensen's inequality and Fubini's theorem the expectation of $(J_\alpha)^2$ satisfies

$$\begin{aligned}
E\left[(J_\alpha)^2\right] &\leq \frac{1}{m-n}\int_n^m E[\bar{s}^2_\alpha(z)(R)^2(\frac{1}{\bar{s}_\alpha(z)})]\mathrm{d}z,\\
&= \frac{1}{m-n}\int_n^m\int_0^{+\infty}\frac{\bar{s}_\alpha(z)^3 y^2}{2}e^{-\frac{\bar{s}_\alpha(z)y}{2}}\mathrm{d}y\mathrm{d}z = 8.
\end{aligned} \tag{2.30}$$

End of the proof of the Lemma: using (2.23), we obtain for all $\alpha > 0$

$$\begin{aligned} P\left(\frac{\tau_{X_\alpha}(m)}{g(\alpha)e^{\alpha h(\alpha)}} \leq \delta\right) &= P\left(\frac{I_{\alpha,1}+I_{\alpha,2}}{g(\alpha)e^{\alpha h(\alpha)}} \leq \delta\right), \\ \geq P\left(\frac{I_{\alpha,1}}{g(\alpha)e^{\alpha h(\alpha)}} \leq \frac{\delta}{2} \;;\; \hat{s}_\alpha^{-1}(l) \geq p\right) &+ P\left(\frac{I_{\alpha,2}}{g(\alpha)e^{\alpha h(\alpha)}} \leq \frac{\delta}{2}\right) - 1. \end{aligned}$$

For the first term in the above expression, (2.26) yields

$$\begin{aligned} & P\left(\frac{I_{\alpha,1}}{g(\alpha)e^{\alpha h(\alpha)}} \leq \frac{\delta}{2} \;;\; \hat{s}_\alpha^{-1}(l) \geq p\right) \geq \\ & P\left(\max_{x \leq 1} L_B(\tau_B(1), x) \leq G(\alpha) \;;\; \hat{s}_\alpha^{-1}(l) \geq p\right) \end{aligned} \tag{2.31}$$

where

$$G(\alpha) := \frac{\delta g(\alpha)}{2(q-p)^2} e^{\alpha(h(\alpha)-A)}.$$

By Laplace's method we know that $\lim_{\alpha \to +\infty} \frac{\log g(\alpha)}{\alpha} = 0$, so $g(\alpha) = e^{\circ(\alpha)}$, therefore as $\lim_{\alpha \to +\infty} h(\alpha) = 1 > A$, $G(\alpha)$ tends to infinity when α does. Using that $y \to L_B(\tau_B(1), y)$ is P-a.s. finite and (2.25) we get that (2.31) tends to 1 when α goes to infinity.

For the second term we collect (2.27), (2.28) and (2.29), we get

$$P\left(\frac{I_{\alpha,2}}{g(\alpha)e^{\alpha h(\alpha)}} > \frac{\delta}{2}\right) \leq P\left(J_\alpha > G(\alpha)\right),$$

then by Tchebytchev's inequality and (2.30)

$$P\left(J_\alpha > G(\alpha)\right) \leq \frac{E\left[(J_\alpha)^2\right]}{(G(\alpha))^2} \leq \frac{8}{(G(\alpha))^2},$$

by using once again that $G(\alpha)$ tends to infinity we get the Lemma. $\square$

Next step is to prove

Lemma 2.2.7. *Let $W \in \widetilde{\mathcal{W}}$. For any $\delta > 0$,*

$$\lim_{\alpha \to +\infty} P\left(\left|\frac{\sigma_{X_\alpha^m}(e^{\alpha h(\alpha)}, m)}{g(\alpha)e^{\alpha h(\alpha)}} - 1\right| \leq \delta\right) = 1.$$

Proof. Just like for the proof of Lemma 2.2.3 we assume without loss of generality that $m = 0$, as a consequence $\sigma_{X_\alpha^m}(e^{\alpha h(\alpha)}, m) = \sigma_{X_\alpha}(e^{\alpha h(\alpha)})$ and we simply have to establish that :

$$\lim_{\alpha \to +\infty} P\left(\left|\frac{\sigma_{X_\alpha}(e^{\alpha h(\alpha)})}{g(\alpha)e^{\alpha h(\alpha)}} - 1\right| \leq \delta\right) = 1. \tag{2.32}$$

In the same way we get (2.22), one can prove that

$$\sigma_{X_\alpha}(e^{\alpha h(\alpha)}) \stackrel{\mathcal{L}_W}{=} e^{\alpha h(\alpha)} \int_{-\infty}^{+\infty} e^{-\alpha W(x)} L_B(\sigma_B(1), \widetilde{s}_\alpha(x)) \mathrm{d}x, \tag{2.33}$$

recall that $\widetilde{s}_\alpha(y) = S_\alpha(y)e^{-\alpha h(\alpha)}$. The rest of the proof is devoted to estimate the integral and the main difficulty is to get the upper bound.
We begin with the lower bound, we easily get that

$$\begin{aligned}\int_{-\infty}^{+\infty} e^{-\alpha W(x)} L_B(\sigma_B(1), \widetilde{s}_\alpha(x)) \mathrm{d}x &\geq \int_{a_r}^{b_r} e^{-\alpha W(x)} L_B(\sigma_B(1), \widetilde{s}_\alpha(x)) \mathrm{d}x, \\ &\geq \inf_{y \in [\widetilde{s}_\alpha(a_r), \widetilde{s}_\alpha(b_r)]} L_B(\sigma_B(1), y) g(\alpha),\end{aligned}$$

where a_r and b_r are defined at the end of Theorem 2.1.3, therefore

$$\begin{aligned}&P\left(\frac{\sigma_{X_\alpha}(e^{\alpha h(\alpha)})}{g(\alpha)e^{\alpha h(\alpha)}} \geq 1 - \delta\right) \geq \\ &P\left(\inf_{y \in [\widetilde{s}_\alpha(a_r), \widetilde{s}_\alpha(b_r)]} L_B(\sigma_B(1), y) \geq 1 - \delta\right).\end{aligned} \tag{2.34}$$

Also we recall that $r \in (0, 1)$ therefore we can prove easily by using the Laplace method that $\lim_{\alpha \to +\infty} \widetilde{s}_\alpha(a_r) = \lim_{\alpha \to +\infty} \widetilde{s}_\alpha(b_r) = 0$. We conclude by noticing that $P-a.s.$, $\lim\limits_{\alpha \to +\infty} \inf_{y \in [\widetilde{s}_\alpha(a_r), \widetilde{s}_\alpha(b_r)]} L_B(\sigma_B(1), y) = 1$, thanks to the continuity of the function $y \to L_B(\sigma_B(1), y)$ at 0.
We continue with the upper bound. First we use the same idea of the proof of Lemma 2.2.6 when we had to deal with $I_{\alpha,1}$: we establish that with a probability which tends to 1 as α goes to infinity, X_α does not exit from the standard valley (p, m, q). Define

$$L := \inf\left\{x \leq 0 / L_B(\sigma_B(1), x) > 0\right\}, U := \sup\left\{x \geq 0 / L_B(\sigma_B(1), x) > 0\right\},$$

we claim that,

$$P-a.s.\ \exists \alpha_0,\ \forall \alpha > \alpha_0, p < \widetilde{s}_\alpha^{-1}(L) < 0 < \widetilde{s}_\alpha^{-1}(U) < q. \tag{2.35}$$

Indeed, we have

$$\widetilde{s}_\alpha^{-1}(L) \geq p \Longleftrightarrow L \geq \widetilde{s}_\alpha(p) = -e^{-\alpha h(\alpha)} \int_p^0 e^{\alpha W(x)} \mathrm{d}x,$$

and by Laplace's method we get $\widetilde{s}_\alpha(p) = -e^{\alpha(W(p)-h(\alpha))+o(\alpha)}$. It follows from the fact that $W \in \widetilde{\mathcal{W}}$ and $\lim\limits_{\alpha\to+\infty} h(\alpha) = 1 < D \leq W(p)$ that $\lim_{\alpha\to+\infty} \widetilde{s}_\alpha(p) = -\infty$, P-a.s.. In a similar way we obtain $\lim_{\alpha\to+\infty} \widetilde{s}_\alpha(q) = +\infty$, P-a.s. and (2.35) is satisfied.

On the event $\{p < \widetilde{s}_\alpha^{-1}(L) < 0 < \widetilde{s}_\alpha^{-1}(U) < q\}$, we can write

$$\begin{aligned}
\int_{-\infty}^{+\infty} e^{-\alpha W(x)} L_B(\sigma_B(1), \widetilde{s}_\alpha(x)) \mathrm{d}x = &\int_p^{a_r} e^{-\alpha W(x)} L_B(\sigma_B(1), \widetilde{s}_\alpha(x)) \mathrm{d}x \\
&+ \int_{a_r}^{b_r} e^{-\alpha W(x)} L_B(\sigma_B(1), \widetilde{s}_\alpha(x)) \mathrm{d}x \\
&+ \int_{b_r}^{q} e^{-\alpha W(x)} L_B(\sigma_B(1), \widetilde{s}_\alpha(x)) \mathrm{d}x.
\end{aligned}$$

We only have to found an upper bound for these integrals, first we have

$$\begin{aligned}
&\int_p^{a_r} e^{-\alpha W(x)} L_B(\sigma_B(1), \widetilde{s}_\alpha(x)) \mathrm{d}x + \int_{b_r}^{q} e^{-\alpha W(x)} L_B(\sigma_B(1), \widetilde{s}_\alpha(x)) \mathrm{d}x \\
&\leq (q-p) \exp(-\alpha \min_{x\in I_r} W(x)) \sup_{y\in\mathbb{R}} L_B(\sigma_B(1), y)
\end{aligned}$$

where $I_r := [p, a_r] \cup [b_r, q]$, and moreover

$$\begin{aligned}
\int_{a_r}^{b_r} e^{-\alpha W(x)} L_B(\sigma_B(1), \widetilde{s}_\alpha(x)) \mathrm{d}x &\leq \sup_{y\in[\widetilde{s}_\alpha(a_r), \widetilde{s}_\alpha(b_r)]} L_B(\sigma_B(1), y) \int_{a_r}^{b_r} e^{-\alpha W(x)} \mathrm{d}x, \\
&= g(\alpha) \sup_{y\in[\widetilde{s}_\alpha(a_r), \widetilde{s}_\alpha(b_r)]} L_B(\sigma_B(1), y).
\end{aligned}$$

Therefore, assembling the last two inequalities and the equality in law (2.33)

$$\begin{aligned}
&P\left(\frac{\sigma_{X_\alpha}(e^{\alpha h(\alpha)})}{g(\alpha)e^{\alpha h(\alpha)}} - 1 \leq \delta\right) \geq \\
&P\Bigg(\sup_{y\in[\widetilde{s}_\alpha(a_r), \widetilde{s}_\alpha(b_r)]} L_B(\sigma_B(1), y) - 1 + \frac{(q-p)}{g(\alpha)} e^{-\alpha \min_{I_r} W} \sup_{y\in\mathbb{R}} L_B(\sigma_B(1), y) \leq \delta; \\
&p < \widetilde{s}_\alpha^{-1}(L) < 0 < \widetilde{s}_\alpha^{-1}(U) < q\Bigg). \qquad (2.36)
\end{aligned}$$

By hypothesis $r < 1$, so $\lim_{\alpha\to+\infty} \widetilde{s}_\alpha(a_r) = \lim_{\alpha\to+\infty} \widetilde{s}_\alpha(b_r) = 0$, moreover $y \to L_B(\sigma_B(1), y)$ is P-a.s. continuous at 0, it follows

$$\lim_{\alpha\to+\infty} \sup_{y\in[\widetilde{s}_\alpha(a_r),\widetilde{s}_\alpha(b_r)]} L_B(\sigma_B(1), y) = 1 \quad P\text{-a.s.}.$$

We also know that $\lim_{\alpha\to+\infty} \frac{\log g(\alpha)}{\alpha} = 0$, moreover according to the definition of $\widetilde{\mathcal{W}}$, $\min_{x\in I_r} W(x) > 0$, and finally $\sup_{y\in\mathbb{R}} L_B(\sigma_B(1), y)$ is P-a.s. finite, so

$$\lim_{\alpha\to+\infty} \frac{(q-p)}{g(\alpha)} e^{-\alpha \min_{I_r} W} \sup_{y\in\mathbb{R}} L_B(\sigma_B(1), y) = 0 \quad P\text{-a.s.}.$$

Putting the last two assertions together with (2.36) we get the upper bound and finally the Lemma. □

2.3 Proof of the main results

One of the key result of this paper is Theorem 2.1.3, the other results can be deduced from that theorem together with estimates on the random environment, so we naturally start with the

2.3.1 Proof of Theorem 2.1.3

We begin with the asymptotic behavior of L_{X_α} within a deterministic interval of time :

Proposition 2.3.1.
Let $r \in (0,1)$, $W \in \widetilde{\mathcal{W}}$ and h a real function such that $\lim_{\alpha\to\infty} h(\alpha) = 1$. For all $\delta > 0$, we have

$$\lim_{\alpha\to+\infty} P\left(\sup_{a_r\le x\le b_r} \left| \frac{L_{X_\alpha}(e^{\alpha h(\alpha)}, m+x)}{e^{\alpha h(\alpha)}} \frac{\int_{a_r}^{b_r} e^{-\alpha W_m(y)} dy}{e^{-\alpha W_m(x)}} - 1 \right| \le \delta \right) = 1.$$

Proof. Let $\delta > 0$, and $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{\alpha\to+\infty} f(\alpha) = 1$, define

$$A_{\alpha,f} := \left\{ \sup_{a_r\le x\le b_r} \left| \frac{L_{X_\alpha}(\sigma_{X_\alpha}(e^{\alpha f(\alpha)}, m), m+x)}{e^{\alpha f(\alpha)-\alpha W_m(x)}} - 1 \right| \le \delta \right\}$$

and

$$B_{\alpha,f} := \left\{ \left| \frac{\sigma_{X_\alpha}(e^{\alpha f(\alpha)}, m)}{g(\alpha) e^{\alpha f(\alpha)}} - 1 \right| \le \delta \right\}$$

where, as in the previous section, $g(\alpha) = \int_{a_r}^{b_r} e^{-\alpha W_m(y)}\mathrm{d}y$. We also define two functions

$$h^+(\alpha) := h(\alpha) - \alpha^{-1}\log\left(g(\alpha)(1-\delta)\right),$$
$$h^-(\alpha) := h(\alpha) - \alpha^{-1}\log\left(g(\alpha)(1+\delta)\right).$$

On B_{α,h^+} the following inequality holds:

$$\begin{aligned}\sigma_{X_\alpha}(e^{\alpha h^+(\alpha)}, m) &\geq (1-\delta)g(\alpha)e^{\alpha h^+(\alpha)},\\ &\geq e^{\alpha h(\alpha)},\end{aligned}$$

moreover in its first coordinate the local time is an increasing function, therefore on $A_{\alpha,h^+} \cap B_{\alpha,h^+}$, $\forall x \in [a_r, b_r]$,

$$\begin{aligned}L_{X_\alpha}(e^{\alpha h(\alpha)}, m+x) &\leq L_{X_\alpha}(\sigma_{X_\alpha}(e^{\alpha h^+(\alpha)}, m), m+x),\\ &\leq e^{\alpha h^+(\alpha)-\alpha W_m(x)}(1+\delta),\\ &\leq \frac{e^{\alpha h(\alpha)-\alpha W_m(x)}}{g(\alpha)}\frac{1+\delta}{1-\delta}.\end{aligned}$$

In the same way, on $A_{\alpha,h^-} \cap B_{\alpha,h^-}$ we obtain

$$L_{X_\alpha}(e^{\alpha h(\alpha)}, m+x) \geq \frac{e^{\alpha h(\alpha)-\alpha W_m(x)}}{g(\alpha)}\frac{1-\delta}{1+\delta}.$$

By Laplace's method, $\lim\limits_{\alpha\to+\infty} \frac{\log g(\alpha)}{\alpha} = 0$, so h^+ and h^- tend to 1 when α goes to infinity and we can apply Propositions 2.2.1 and 2.2.5, finally

$$\lim_{\alpha\to+\infty} P\left(A_{\alpha,h^+} \cap B_{\alpha,h^+} \cap A_{\alpha,h^-} \cap B_{\alpha,h^-}\right) = 1$$

and the proposition is proved. □

We turn back to the proof of the Theorem, notice that the difference between Theorem 2.1.3 and Proposition 2.3.1 above is the process itself: one deals with X whereas the other deals with X_α. To finish the proof we need to show that thanks to Lemma 2.1.4 we can get the theorem from the proposition.

Let $\alpha > 0$, recall that $W^\alpha(.) := \alpha^{-1}W(\alpha^2\cdot)$. First, remark that for all $W \in \mathcal{W}$,

$$\begin{aligned}m_1(W^\alpha) &= \alpha^{-2}m_\alpha(W),\\ a_r(W^\alpha_{m_1(W^\alpha)}) &= \alpha^{-2}a_{\alpha r}(W_{m_\alpha(W)}),\\ b_r(W^\alpha_{m_1(W^\alpha)}) &= \alpha^{-2}b_{\alpha r}(W_{m_\alpha(W)}),\end{aligned}$$

and for any $x \in \mathbb{R}$,

$$W^{\alpha}_{m_1(W^\alpha)}(x) = \frac{1}{\alpha} W_{m_\alpha(W)}(\alpha^2 x).$$

Now replacing t by $\alpha^{-4}e^{\alpha}$ in the second part of Lemma 2.1.4 page 31, we obtain for all $W \in \mathcal{W}$,

$$\left(L_{X(\alpha W^\alpha,\cdot)}(\alpha^{-4}e^{\alpha}, m_1(W^\alpha) + \alpha^{-2}x)\right)_{x\in\mathbb{R}} \overset{\mathcal{L}_W}{=} \left(\frac{1}{\alpha^2} L_{X(W,\cdot)}(e^{\alpha}, m_\alpha(W) + x)\right)_{x\in\mathbb{R}}.$$

Therefore, for any $\alpha > 0$, $\delta > 0$ and $W \in \mathcal{W}$,

$$P\left(\sup_{a_{\alpha r}\le x\le b_{\alpha r}} \left| \frac{L_{X(W,\cdot)}(e^{\alpha}, m_\alpha(W)+x)}{e^{\alpha}} \frac{\int_{a_{\alpha r}}^{b_{\alpha r}} e^{-W_{m_\alpha}(y)}\mathrm{d}y}{e^{-W_{m_\alpha}(x)}} - 1\right| < \delta\right) =$$
$$P\left(\sup_{a_r\le x\le b_r} \left| \frac{L_{X(\alpha W^\alpha,\cdot)}(\alpha^{-4}e^{\alpha}, m_1(W^\alpha)+x)}{\alpha^{-2}e^{\alpha}} \frac{\int_{\alpha^2 a_r}^{\alpha^2 b_r} e^{-\alpha W^{\alpha}_{m_1}(\alpha^{-2}y)}\mathrm{d}y}{e^{-\alpha W^{\alpha}_{m_1}(x)}} - 1\right| < \delta\right).$$

Moreover, for all $\alpha > 0$, $\mathcal{P}$ is invariant under the transformation $W \mapsto W^{\alpha}$, we get that

$$\mathbb{P}\left(\sup_{a_{\alpha r}\le x\le b_{\alpha r}} \left| \frac{L_{X(W,\cdot)}(e^{\alpha}, m_\alpha(W)+x)}{e^{\alpha}} \frac{\int_{a_{\alpha r}}^{b_{\alpha r}} e^{-W_{m_\alpha}(y)}\mathrm{d}y}{e^{-W_{m_\alpha}(x)}} - 1\right| < \delta\right) =$$
$$\mathbb{P}\left(\sup_{a_r\le x\le b_r} \left| \frac{L_{X(\alpha W,\cdot)}(\alpha^{-4}e^{\alpha}, m_1(W)+x)}{\alpha^{-4}e^{\alpha}} \frac{\int_{a_r}^{b_r} e^{-\alpha W_{m_1}(y)}\mathrm{d}y}{e^{-\alpha W_{m_1}(x)}} - 1\right| < \delta\right)$$

and we recall that $\mathbb{P} = \mathcal{P}\otimes P$. To finish the proof we notice that $\mathcal{P}(\widetilde{\mathcal{W}}) = 1$, $\alpha^{-4}e^{\alpha} = e^{\alpha(1-\frac{4}{\alpha}\log\alpha)}$ and $\lim_{\alpha\to+\infty}(1-\frac{4}{\alpha}\log\alpha) = 1$, so applying Proposition 2.3.1 we get Theorem 2.1.3. $\square$

As we said at the beginning of the section, once Theorem 2.1.3 is proved, we get the other results by studying in details some properties of the random environment.

2.3.2 Proof of Theorems 2.1.1 and 2.1.2

Denote $m^*(t) := \min\{x \in \mathbb{R},\ L_X(x,t) = L^*_X(t)\}$ the (first) favorite point of the diffusion at time t. D. Cheliotis has shown in [20] that

$$m^*(e^{\alpha}) - m_\alpha \xrightarrow[t\to\infty]{\mathbb{P}} 0.$$

Then, for any $r \in (0,1)$ as $\lim_{\alpha\to+\infty} m_\alpha - a_{\alpha r} = \lim_{\alpha\to+\infty} b_{\alpha r} - m_\alpha = \infty$, the processes $L_X^*(e^\alpha)/e^\alpha$ and $\sup_{[a_{\alpha r}, b_{\alpha r}]} L_X(e^\alpha, \cdot)/e^\alpha$ have the same limit in law and Theorem 2.1.1 is a straight forward consequence of Theorem 2.1.2. So we are left to prove Theorem 2.1.2. The main ingredients to get this theorem are Theorem 2.1.3 and Lemmata 2.3.2 and 2.3.3 below. We begin with the proof of the first Lemma:

Lemma 2.3.2. *For all $r \in (0,1)$, $\mathcal{P}$-almost surely for all $a_r \le a < 0 < b \le b_r$, we get*

$$\int_{a_r}^{b_r} e^{-\alpha W_{m_1}(y)} dy \underset{\alpha\to+\infty}{\sim} \int_a^b e^{-\alpha W_{m_1}(y)} dy.$$

Proof. For any $W \in \mathcal{W}$,

$$\begin{aligned}\int_{a_r}^{b_r} e^{-\alpha W_{m_1}(y)} dy - \int_a^b e^{-\alpha W_{m_1}(y)} dy &= \int_{a_r}^{a} e^{-\alpha W_{m_1}(y)} dy + \int_b^{b_r} e^{-\alpha W_{m_1}(y)} dy \\ &\le (b_r - a_r) e^{-\alpha \min_I W_{m_1}}\end{aligned}$$

where $I := [a_r, a] \cup [b, b_r]$. Moreover, by Laplace's method,

$$\lim_{\alpha\to+\infty} \frac{1}{\alpha} \log \int_{a_r}^{b_r} e^{-\alpha W_{m_1}(y)} dy = -\min_{[a_r,b_r]}(W_{m_1}).$$

As $a_r \le 0 \le b_r$, this minimum is equal to 0, thus $\int_{a_r}^{b_r} e^{-\alpha W_{m_1}(y)} dy = e^{o(\alpha)}$ and so,

$$\frac{\int_{a_r}^{b_r} e^{-\alpha W_{m_1}(y)} dy - \int_a^b e^{-\alpha W_{m_1}(y)} dy}{\int_{a_r}^{b_r} e^{-\alpha W_{m_1}(y)} dy} \le (b_r - a_r) e^{-\alpha \min_I W_{m_1} + o(\alpha)}.$$

And for any $W \in \widetilde{\mathcal{W}}$, $\min_I W_{m_1} > 0$, then letting α go to infinity we obtain the equivalence for any $W \in \widetilde{\mathcal{W}}$. As $\mathcal{P}(\widetilde{\mathcal{W}}) = 1$, this implies the result of the lemma. $\square$

Now, the proof of the theorem will be finished once we will have shown

Lemma 2.3.3. *For any positive constant K and for any bounded continuous functional F on $C(\mathbb{R},\mathbb{R})$ such that $F(f)$ depends only on the values of the function f on $[-K,K]$,*

$$\lim_{\alpha\to+\infty} \mathbb{E}\left[F\left(\frac{e^{-W_{m_\alpha}}}{\int_{a_{\alpha r}}^{b_{\alpha r}} e^{-W_{m_\alpha}(y)}\, dy}\right)\right] = \mathbb{E}\left[F\left(\frac{e^{-R}}{\int_{-\infty}^{\infty} e^{-R(y)}\, dy}\right)\right]. \qquad (2.37)$$

We recall that

$$\forall x \in \mathbb{R}, R(x) := R_1(x)\mathbf{1}_{\{x\ge 0\}} + R_2(-x)\mathbf{1}_{\{x<0\}},$$

R_1 and R_2 being two independent 3-dimensional Bessel processes.

Proof. According to the scaling property of Brownian Motion,

$$\mathbb{E}\left[F\left(\frac{e^{-W_{m_\alpha}(\cdot)}}{\int_{a_{\alpha r}}^{b_{\alpha r}} e^{-W_{m_\alpha}(y)}\mathrm{d}y}\right)\right] = \mathbb{E}\left[F\left(\frac{e^{-\alpha W_{m_1}(\alpha^{-2}\cdot)}}{\alpha^2\int_{a_r}^{b_r} e^{-\alpha W_{m_1}(y)}\mathrm{d}y}\right)\right].$$

So, thanks to Lemma 2.3.2, denoting

$$\tilde{a}_r := \begin{cases} a_r & \text{if } m_1 < 0 \\ a_r \vee 0 & \text{if } m_1 \geq 0 \end{cases} \text{ and } \tilde{b}_r := \begin{cases} b_r & \text{if } m_1 \geq 0 \\ b_r \wedge 0 & \text{if } m_1 < 0 \end{cases},$$

it is enough to prove that

$$\lim_{\alpha\to+\infty} \mathbb{E}\left[F\left(\frac{e^{-\alpha W_{m_1}(\alpha^{-2}\cdot)}}{\alpha^2\int_{\tilde{a}_r}^{\tilde{b}_r} e^{-\alpha W_{m_1}(y)}\mathrm{d}y}\right)\right] = \mathbb{E}\left[F\left(\frac{e^{-R}}{\int_{-\infty}^{\infty} e^{-R(y)}\mathrm{d}y}\right)\right].$$

This can be done using exactly the same arguments as in the proof of Lemma 3.2 in Tanaka [87]. Therefore we will not present a proof here. $\square$

Chapitre 3

Temps local en milieu Lévy stable

3.1 Introduction.

Let $(V(x), x \in \mathbb{R})$ be a real-valued càd-làg stochastic process such that $V(0) = 0$. A *diffusion in the environment* V is a process $(X(V,t), t \geq 0)$ (or $(X(t), t \geq 0)$ if there is no ambiguity on the environment) whose generator given V is

$$\frac{1}{2}e^{V(x)}\frac{d}{dx}\left(e^{-V(x)}\frac{d}{dx}\right).$$

Remark that if V is almost surely differentiable and V' is bounded, then $(X(t), t \geq 0)$ is the solution of the following SDE:

$$\begin{cases} \mathrm{d}X(t) = \mathrm{d}\beta(t) - \frac{1}{2}V'(X(t))\mathrm{d}t, \\ X(0) = 0 \end{cases}$$

where β is a standard Brownian motion independent of V. Historically, this model has been introduced by Schumacher [69]. It was mainly studied when the environment V is a Brownian motion (Brox [17] and Kawazu and Tanaka [49]), a drifted Brownian motion (Kawazu and Tanaka [50], Hu, Shi and Yor [45], Devulder [26] and Talet [84]) or a process which has the same asymptotic behavior as a Brownian motion(Hu and Shi [41, 40]), this last case allows to study the discrete analogue of the diffusion in a Brownian environment, the so-called Sinai's random walk. More recently, general Lévy environments have been studied by Carmona [18], Tanaka [89], Cheliotis [19] and Singh [76].

The local time process $(L_X(t,x), t \geq 0, x \in \mathbb{R})$ is defined by the occupation time formula, i.e., it is the unique process, almost surely continuous in the variable t, such that for all bounded Borel function f and for all $t \geq 0$,

$$\int_0^t f(X_s)\mathrm{d}s = \int_{\mathbb{R}} f(x)L_X(t,x)\mathrm{d}x. \tag{3.1}$$

The existence of such a process can be easily proved, see e.g. Shi [71]. In the Brownian case, the local time process was largely studied see e.g. Hu and Shi [41, 42], Shi [71], Devulder [27] and Andreoletti and Diel [7]. In this paper, we generalize the result of [7] to the Lévy environment case. However several difficulties arise: contrary to the Brownian motion, the law of a Lévy process is not stable by time inversion; moreover the processes $(V(x) - \inf_{0\le y\le x} V(y), x \geq 0)$, $(\sup_{0\le y\le x} V(y) - V(x), x \geq 0)$ and $|V|$ do not have the same law.

In what follows, we focus on the local time of the diffusion when the environment V is an α-stable Lévy process on $\mathbb{R}$. We say that V is α-stable if there exists $\alpha > 0$, such that for all $c > 0$

$$(c^{-1}V(c^\alpha x), x \in \mathbb{R}) \overset{\mathcal{L}}{=} (V(x), x \in \mathbb{R})$$

where $\overset{\mathcal{L}}{=}$ is the equality in law. And V is an α-stable Lévy process if on top of that it is a Lévy process on $\mathbb{R}$.

One of the main results of this paper is the convergence in law of the supremum of the local time

$$\forall t \geq 0, \quad L^*_X(t) = \sup_{x\in\mathbb{R}} L_X(t, x)$$

which measures the time passed by the diffusion in the most visited point.

We next recall the notion of *valley* introduced in the Brownian case by Brox [17] (see also [60]). Denote by $\mathcal{V}$ the space of càd-làg functions $\omega : \mathbb{R} \to \mathbb{R}$, endowed with the Skorohod topology (the space $\mathcal{V}$ is then Polish). Let $c > 0$, $\omega \in \mathcal{V}$ has a *c-minimum* in the point x_0 if there exists ξ and ζ such that $\xi < x_0 < \zeta$ and

- $\omega(\xi) \geq (\omega(x_0) \wedge \omega(x_0-)) + c$,
- $\omega(\zeta-) \geq (\omega(x_0) \wedge \omega(x_0-)) + c$ and
- for all $x \in (\xi, \zeta)$, $x \neq x_0$, $\omega(x) \wedge \omega(x-) > \omega(x_0) \wedge \omega(x_0-)$.

Analogously, ω has a *c-maximum* in x_0 if $-\omega$ has a c-minimum in x_0. Denote by $M_c(\omega)$ the set of *c-extrema* of ω. If ω takes pairwise distinct values in its local extrema and is continuous at these points, then we can check that $M_c(\omega)$ has no accumulation points and that the c-maxima and the c-minima alternate . These assumptions are verified for α-stable Lévy processes with $\alpha \in [1, 2]$. Thus there exists a triplet $\Delta_c = \Delta_c(\omega) = (\mathfrak{p}_c, \mathfrak{m}_c, \mathfrak{q}_c)$ of successive elements of M_c such that

- $\mathfrak{m}_c$ and 0 belongs to $[\mathfrak{p}_c, \mathfrak{q}_c]$,
- $\mathfrak{p}_c$ and $\mathfrak{q}_c$ are c-maxima,
- $\mathfrak{m}_c$ is a c-minimum.

This triplet is called the *standard valley* with height c of ω. The stability of the environment V allows us to reduce the study of a valley with height c to a valley with height 1.

In the following, we denote

$$\big(\omega^+(x), x \geq 0\big) = (\omega(x), x \geq 0) \ , \ \big(\omega^-(x), x \geq 0\big) = (\omega((-x)-), x \geq 0) \,. \tag{3.2}$$

Our main result gives the limit law of the local time process recentered on the minimum of the standard valley with height $\log t$.

Theorem 3.1.1. *Let V be an α-stable Lévy process with $\alpha \in [1, 2]$ which is not a pure drift.*
As $t \to \infty$, the process $\left(\dfrac{L_X(t, \mathfrak{m}_{\log t} + x)}{t}, x \in \mathbb{R}\right)$ converges weakly in the Skorohod topology to the process $\left(\dfrac{e^{-\widetilde{V}(x)}}{\int_{-\infty}^{\infty} e^{-\widetilde{V}(y)}\, dy}, x \in \mathbb{R}\right)$. The law of $\tilde{V}$ is $\tilde{\mathbb{P}}$ and under $\tilde{\mathbb{P}}(d\omega)$, ω^+ and ω^- are independent, and distributed respectively as $\mathbb{P}^{\uparrow}$ and $\hat{\mathbb{P}}^{\uparrow}$, where $\mathbb{P}^{\uparrow}$ is the law of the Lévy process V^+ conditioned to stay positive, and $\hat{\mathbb{P}}^{\uparrow}$ is the law of the dual process V^- conditioned to stay positive.

Moreover, we obtain the localization of the favorite point $m^*(t)$ of the diffusion X.

Theorem 3.1.2. *With the same assumptions as in previous theorem, for any $\delta \geq 0$,*

$$\lim_{t \to +\infty} \mathcal{P}\left(|m^*(t) - \mathfrak{m}_{\log t}| \leq \delta\right) = 1$$

where $m^*(t) = \inf\big\{x \in \mathbb{R},\ L_X(t, x) = \sup\{L_X(t, y), y \in \mathbb{R}\}\big\}$.

The previous theorems imply the following corollary.

Corollary 3.1.3. *With the same assumptions as before, we get*

$$\frac{L_X^*(t)}{t} \xrightarrow[t\to\infty]{\mathcal{L}} \frac{1}{\int_{-\infty}^{\infty} e^{-\widetilde{V}(y)}\, dy}$$

where $\widetilde{V}$ is the same process as in the previous theorem and $\xrightarrow{\mathcal{L}}$ denotes the convergence in law.

Proof. Recall that a Lévy process is continuous in probability. Thus, for any $x > 0$,

$$\mathcal{P}\big(V(x) = V(x-) \text{ and } V(-x) = V((-x)-)\big) = 1.$$

Fix $\delta > 0$. Theorem 3.1.2 shows that it is sufficient to study the supremum of the process $x \mapsto L_X(t, \mathfrak{m}_{\log t} + x)$ on the compact set $[-\delta, \delta]$. Proposition VI.2.4 of [46] tells us that the function $F \mapsto \sup_{[-\delta,\delta]} F$ defined on $\mathcal{V}$ is continuous for the Skorohod topology for F such that $F(\pm\delta) = F((\pm\delta)-)$. From Theorem 3.1.1 the result follows. $\square$

The proof of Theorems 3.1.1 and 3.1.2 is based on the study of the law of the environment. Indeed, for large times the diffusion is located in a valley whose right slope looks like the Lévy process V^+ conditioned to stay positive and the left slope like V^- conditioned to stay positive (see Proposition 3.2.2 for a rigorous statement). And, for large t its local time process, normalized and centered at $\mathfrak{m}_{\log t}$, behaves like $e^{-V_{\mathfrak{m}_{\log t}}} / \int e^{-V_{\mathfrak{m}_{\log t}}}$ where $V_{\mathfrak{m}_{\log t}}$ is the environment recentered at $\mathfrak{m}_{\log t}$ (see Theorem 3.3.6 for more details).

The law of the valley has already been obtained by Tanaka [85] in the special case of a Brownian environment; in this case, the right and left slopes are both 3-dimensional Bessel processes. In [89], Tanaka proved similar results in the Lévy environment case.

This paper is organized as follows. In Section 3.2, we define the environment V and we give the asymptotic law of the standard valley V on $\mathbb{R}_+$ (Proposition 3.2.7) and in Section 3.3, we prove the convergence of the local time: Theorem 2.1.2 is a consequence of Theorem 2.1.3 and Proposition 3.3.8 and we also show the localization of the favorite point: Theorem 3.1.2 is the last statement of Theorem 2.1.3.

3.2 Standard valley of a Lévy process.

3.2.1 Lévy process on $\mathbb{R}$.

A $\mathcal{V}$-valued random process V is a Lévy process on $\mathbb{R}$ if its increments are independent and stationary (see [18]) i.e.

1. $V(0) = 0$,
2. if $x_0 < x_1 < \cdots < x_n$ are real numbers, then the random variables $(V(x_{i+1}) - V(x_i), 0 \le i \le n-1)$ are independent,
3. for all x and y, the law of $V(x+y) - V(x)$ only depends on y.

Denote by $\mathbb{P}$ the law of the Lévy process V.

Remark 1. *An equivalent definition of a Lévy process indexed by $\mathbb{R}$ is given by Cheliotis in [19]: let $(V^+(t), t \ge 0)$ be an $\mathbb{R}$-valued Lévy process starting from 0 and $(V^-(t), t \ge 0)$ a process independent of V^+ and with the same* *law as $-V^+$, we define $(V(t), t \geq 0)$ by*

$$\forall t \in \mathbb{R},\ V(t) = \mathbf{1}_{t \geq 0} V^+(t) + \mathbf{1}_{t \leq 0} V^-((-t)-). \tag{3.3}$$

This process has càd-làg paths and satisfies the three assumptions of the previous definition. Conversely, if the three assumptions hold for a process, then it can be decomposed in two Lévy processes as in Formula (3.3).

Define $\underline{V}^+$ the infimum process of V^+ and $\overline{V}^+$ the supremum process: for any $t \geq 0$, $\underline{V}^+(t) = \inf_{0 \leq s \leq t} V^+(s)$ and $\overline{V}^+(t) = \sup_{0 \leq s \leq t} V^+(s)$. Let $\mathcal{N}$ be the excursion measure of the process $V^+ - \underline{V}^+$ away from 0. We denote by L a local time of $V^+ - \underline{V}^+$ in 0 and $L^{-1}(t) = \inf\{s \geq 0; L(s) > t\}$ its right continuous inverse.

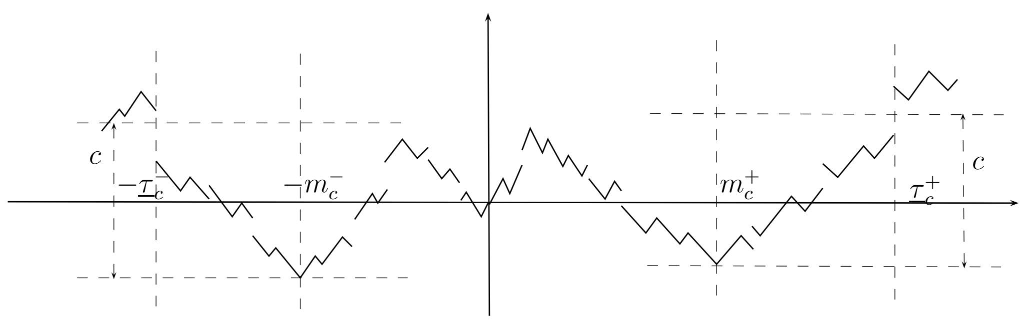
Denote by $\mathcal{V}^+$ the set of càd-làg functions $\omega : \mathbb{R}_+ \to \mathbb{R}$ endowed with the Skorohod topology. For $c > 0$ fixed, we define the first passage time above a level c of ω and the last previous time in which ω reaches 0: for all $\omega \in \mathcal{V}^+$, $\tau_c(\omega) = \inf\{t \geq 0; \omega(t) \geq c\}$ and $n_c(\omega) = \sup\{t \leq \tau_c(\omega); \omega(t) = 0 \text{ or } \omega(t-) = 0\}$. We then denote

$$\underline{\tau}_c^+ = \tau_c\left(V^+ - \underline{V}^+\right)\ ,\ m_c^+ = n_c\left(V^+ - \underline{V}^+\right) \text{ and}$$

$$J_c^+ = \left(V^+(m_c^+) + c\right) \vee \overline{V}^+(m_c^+).$$

Similarly, we define analogue variables $\underline{\tau}_c^-$, m_c^- and J_c^- using the process V^-.

Figure 3.1: Standard valley with height c



Remark 2. *We show that the minimum of the standard valley with height c defined in the introduction satisfies $\mathfrak{m}_c = m_c^+$ if $J_c^+ < J_c^-$ and $\mathfrak{m}_c = -m_c^-$ otherwise (see [52] for a proof in the Brownian case which is also true in our case). Thus to study the process V in a neighborhood of $\mathfrak{m}_c$, it is sufficient to study V^+ in a neighborhood of m_c^+ and V^- in a neighborhood of m_c^-.*

Therefore, in this section we study the Lévy process V^+ defined on $\mathbb{R}_+$. However, we still denote $\mathbb{P}$ its law and until the end of this section, all the probability measures are measures on $\mathcal{V}^+$. In order to simplify notations, we write V, m_c and $\underline{\tau}_c$ for V^+, m_c^+ and $\underline{\tau}_c^+$.

3.2.2 Valley of the Lévy process on $\mathbb{R}_+$.

We say that $(V(t), 0 \leq t \leq \underline{\tau}_c)$ is the standard valley with height c of the Lévy process on $\mathbb{R}_+$.

Furthermore, we say that 0 is regular for $(0, +\infty)$ (resp. for $(-\infty, 0)$) if $\mathbb{P}(\inf\{t > 0; V(t) > 0\} = 0) = 1$ (resp. $\mathbb{P}(\inf\{t > 0; V(t) < 0\} = 0) = 1$). We will only work with Lévy process for which 0 is regular for $(-\infty, 0)$ and $(0, +\infty)$. Thereby we recall some properties of such processes stated in [19], which give us the existence of the triplet $(\mathfrak{p}_c, \mathfrak{m}_c, \mathfrak{q}_c)$ defined in the introduction.

For $\omega \in \mathcal{V}^+$, the point x_0 is called a left local maximum (resp. minimum) if there exists $\varepsilon > 0$ such that $\omega(x) \leq \omega(x_0-)$ (resp. $\omega(x) \geq \omega(x_0-)$) for all $x \in (x_0 - \varepsilon, x_0)$. Similarly, we define a right local maximum and minimum.

Lemma 3.2.1. *([19], Lemma 3)*
Let V be a Lévy process such that 0 is regular for $(-\infty, 0)$ and $(0, +\infty)$, then $\mathbb{P}$-a.s.:

1. *V is continuous at its left local extrema and at its right local extrema,*
2. *In no two local extrema V has the same value.*

The point (2) of the previous Lemma gives us the uniqueness of the minimum m_c such that $V(m_c) = \inf_{[0,\underline{\tau}_c]} V$. Moreover, m_c is a local minimum, thus according to point (1) the process V is continuous in m_c, that means $V(m_c) = V(m_c-)$.

Remark 3. *If $\omega \in \mathcal{V}^+$ is càd-làg on a bounded interval I, then there exists $x, y \in \bar{I}$ such that $\sup_I \omega = \omega(x) \vee \omega(x-)$ and $\inf_I \omega = \omega(y) \wedge \omega(y-)$. So if ω is continuous at its local extrema, we get:*

$$\exists x \in \bar{I},\ s.t.\ \sup_I \omega = \omega(x) \text{ and } \exists y \in \bar{I},\ s.t.\ \inf_I \omega = \omega(y).$$

The pre-infimum and post-infimum processes.

We now define the pre-infimum and post-infimum processes $(\underset{\leftarrow}{V}_{m_c}(t), 0 \leq t \leq m_c)$ and $(\underset{\rightarrow}{V}_{m_c}(t), 0 \leq t \leq \underline{\tau}_c - m_c)$ by

$$\underset{\leftarrow}{V}_{m_c}(t) = V((m_c - t)-) - V(m_c) \quad \text{and} \quad \underset{\rightarrow}{V}_{m_c}(t) = V(m_c + t) - V(m_c).$$

When there is no possible mistake, we write $\underset{\leftarrow}{V}$ instead of $\underset{\leftarrow}{V}_{m_c}$ and $\underset{\rightarrow}{V}$ instead of $\underset{\rightarrow}{V}_{m_c}$. We also denote respectively $\underset{\leftarrow}{\mathbb{P}}^c$ and $\underset{\rightarrow}{\mathbb{P}}^c$ the laws of the processes $\underset{\leftarrow}{V}$ stopped at time m_c and $\underset{\rightarrow}{V}$ stopped at time $\tau_c(\underset{\rightarrow}{V})$.

We now study the law of these two processes. Therefore, we need the notion of Lévy process conditioned to stay positive.

Lévy process conditioned to stay positive.

We present here Bertoin's construction by concatenation of the excursions of the Lévy process above 0 (see Section 3, [12]).

The process V is a semi-martingale, its local continuous martingale part is null or proportional to a standard Brownian motion. Denote by ℓ the local time in a semi-martingale sense of V at 0, see for example [61] and consider the time passed by V in $(0,\infty)$ and its right continuous inverse,

$$A_t^+ = \int_0^t ds \mathbf{1}_{V(s)>0} \text{ and } \alpha_t^+ = \inf\{s \geq 0; A_s^+ > t\}.$$

We define the process $V^\uparrow$ by

$$\begin{cases} V(\alpha_t^+) + \frac{1}{2}\ell_{\alpha_t^+} + \sum_{0<s\leq\alpha_t^+}(0 \vee V(s-))\mathbf{1}_{V(s)\leq 0} - (0 \wedge V(s-))\mathbf{1}_{V(s)>0} & \text{if } t < A_\infty^+, \\ +\infty & \text{otherwise.} \end{cases}$$

We denote by $\mathbb{P}^\uparrow$ the law of the process $V^\uparrow$ and $\hat{\mathbb{P}}^\uparrow$ the law of the dual process $\hat{V} = -V$ conditioned to stay positive.

Remark 4. *We will mainly use this expression to prove that when the process V is stable, $V^\uparrow$ is stable too.*

We state some properties of the Lévy process conditioned to stay positive (see [12]):

– $\mathbb{P}$-a.s. for all $t > 0$, $V^\uparrow(t) > 0$,
– in the case where V has no Brownian part, $\ell = 0$,
– the process $V^\uparrow$ tends to $+\infty$ as t goes to $+\infty$.

For every process Y which is lower bounded, we define the future infimum: for all $t \geq 0$, $\underline{\underline{Y}}(t) = \inf_{s\geq t} Y(s)$. For $c > 0$, we introduce the time

$$\hat{m}_c^\uparrow = n_c(\hat{V}^\uparrow - \underline{\underline{\hat{V}}}^\uparrow).$$

We write $\mathbb{P}^{\uparrow,c}$ (resp. $\hat{\mathbb{P}}^{\uparrow,c}$) for the law of the process $V^\uparrow$ (resp. $\hat{V}^\uparrow$) stopped at time $\tau_c(V^\uparrow)$ (resp. $\hat{m}_c^\uparrow$).

Valley with height c.

Unlike the Brownian case where the law of the post-infimum process is equal to the law of a 3 dimensional Bessel process, in the Lévy case, this law is only absolutely continuous with respect to the law of a Lévy process conditioned to stay positive. In order to have the law of the standard valley, we use Proposition 4.7 of Duquesne [31] and therefore we have to introduce some notations.

For $t \geq 0$, define $U(t) = -V(L^{-1}(t))$ if $L(\infty) > t$ and $U(t) = +\infty$ otherwise (recall that L is the local time of $V - \underline{V}$ in 0). The process (L^{-1}, U) is called the ladder process, it is a subordinator. We then define the potential measure $\mathcal{U}$ associated with U: for all measurable positive function F,

$$\int_{\mathbb{R}} \mathcal{U}(dx) F(x) = \mathbb{E}\left[\int_0^{L(\infty)} dv F(U(v))\right].$$

For the excursion measure $\mathcal{N}$ and for all measurable non-negative function f on $\mathcal{V}$, we write

$$\mathcal{N}(f) = \int f(v) \mathcal{N}(dv).$$

Proposition 3.2.2. *We consider a Lévy process V, which does not go to $+\infty$ neither to $-\infty$ and such that 0 is regular for $(0,+\infty)$ and for $(-\infty,0)$.*

Then the pre-infimum and post-infimum processes $(\underset{\leftarrow}{V}(t), 0 \leq t \leq m_c)$ and $(\underset{\rightarrow}{V}(t), 0 \leq t \leq \tau_c(\underset{\rightarrow}{V}))$ are independent. Moreover

1. *The law $\underset{\leftarrow}{\mathbb{P}}^c$ of the pre-infimum process $(\underset{\leftarrow}{V}(t), 0 \leq t \leq m_c)$ is equal to the law $\hat{\mathbb{P}}^{\uparrow,c}$ of the process $(\hat{V}^{\uparrow}(t), 0 \leq t \leq \hat{m}_c^{\uparrow})$.*
2. *The law $\underset{\rightarrow}{\mathbb{P}}^c$ of the post-infimum process $(\underset{\rightarrow}{V}(t), 0 \leq t \leq \tau_c(\underset{\rightarrow}{V}))$ is absolutely continuous with respect to the law $\mathbb{P}^{\uparrow,c}$ of the process $(V^{\uparrow}(t), 0 \leq t \leq \tau_c(V^{\uparrow}))$.*
   More precisely, $\underset{\rightarrow}{\mathbb{P}}^c(d\omega) = f_c\left(\omega(\tau_c(\omega))\right) \mathbb{P}^{\uparrow,c}(d\omega)$
   where for all $x \in \mathbb{R}_+$, $\frac{1}{f_c(x)} = \mathcal{U}\left([0,x)\right) \mathcal{N}(\tau_c(v) < \infty)$.

Remark 5.

1. *When V is a Lévy process with no positive jumps, $\omega(\tau_c(\omega)) = c$, $\mathbb{P}^{\uparrow,c}-$a.s. Thus, $\omega \mapsto f_c(\omega(\tau_c(\omega)))$ is constant and equal to 1 and the post-infimum process is equal in law to the process $V^{\uparrow}$ conditioned to stay positive.*

2. *$\mathbb{P}$-a.s., $\underset{\leftarrow}{V}$ is continuous in its local extrema (see Lemma 3.2.1), then if c tends to $+\infty$, using point (1) of Proposition 3.2.2, $\hat{V}^{\uparrow}$ and $V^{\uparrow}$ are continuous at their local extrema too.*

Proof of the Proposition 3.2.2. (1) We begin with the proof of the first part of Proposition 3.2.2 and the independence of the two processes for $c = 1$. To simplify the notations, we will not write the index 1.

We want to express the pre-infimum process $\underset{\leftarrow}{V}$ in terms of the excursions of V above its minimum and of the associated jump of $\underline{V}$. We consider the couple of processes $(V - \underline{V}, -\underline{V})$ and we give an expression using excursions. We use the proof of Lemma 4 of [10], remark that the result is still true for every Lévy process that does not converge to $-\infty$, see Theorem 27 of [28].

In order to use excursion theory, we denote by $\bar{\mathcal{V}}$ the set of the càd-làg paths $v : [0, \infty) \to \mathbb{R}_+$ such that

(i) $v(t) > 0$ for $0 < t < \sigma(v) = \inf\{s > 0, v(s) = 0\}$,

(ii) $v(0) = v(t) = 0$ for $t \geq \sigma(v)$.

We consider the excursion process of $V - \underline{V}$ above 0 defined by:

$$p_1(t) = \begin{cases} r \mapsto (V - \underline{V})(L^{-1}(t-) + r)\mathbf{1}_{[0, L^{-1}(t) - L^{-1}(t-)]}(r) & \text{if } L^{-1}(t-) < L^{-1}(t) \\ 0 & \text{if } L^{-1}(t-) = L^{-1}(t) \end{cases}$$

and the jumps process of $\underline{V}$ defined by:

$$p_2(t) = \underline{V}(L^{-1}(t-)) - \underline{V}(L^{-1}(t)).$$

Thus the process $p(\cdot) = (p_1(\cdot), p_2(\cdot))$ is a $\bar{\mathcal{V}} \times \mathbb{R}_+$-valued Poisson point process (see for example [38]). Remark that $p_1\left(L(\underline{\tau})\right)$ is the excursion starting from m, that is to say that $L(\underline{\tau})$ is the first time t where the excursion $p_1(t)$ has a height bigger than 1. Thus, the processes $(p(t); 0 \leq t < L(\underline{\tau}))$ and $(p(t); t \geq L(\underline{\tau}))$ are independent. The pre-infimum process depends only on the first and the post-infimum process depends only on the second, then these two processes are independent too.

In order to obtain the process $\underset{\leftarrow}{V}$, we first invert the excursions of $V - \underline{V}$ then we invert the time in each of these excursions.

Denote by $\tilde{p}_i = \big(p_i(L(\underline{\tau}) - t)\mathbf{1}_{t<L(\underline{\tau})} + p_i(t)\mathbf{1}_{t \geq L(\underline{\tau})}\, ; t \geq 0\big)$ the process p_i where the order of the excursions before $L(\underline{\tau})$ is changed for $i = 1$ and $i = 2$. The law of the Poisson point process p is stable by inversion of the order of its excursions before $L(\underline{\tau})$, we deduce the equality in law of the two processes $(\tilde{p}_1(t), \tilde{p}_2(t))$ and $(p_1(t), p_2(t))$.

Consider now the time inversion in the excursions defined for $v \in \bar{\mathcal{V}}$ by

$$[v](s) = \begin{cases} v((\sigma(v) - s)-) & \text{for } 0 \leq s \leq \sigma(v) \\ 0 & \text{for } s > \sigma(v). \end{cases}$$

Using this transformation in the previous law, we show that the processes $([\tilde{p}_1(t)], \tilde{p}_2(t)\,; t \geq 0)$ and $([p_1(t)], p_2(t)\,; t \geq 0)$ have same law. With the first one we get the process $\underset{\leftarrow}{V}$. Indeed, $([\tilde{p}_1(t)]\,; t \geq 0)$ is the excursion process of $\underset{\leftarrow}{V}$ above its future infimum and $(\tilde{p}_2(t)\,; t \geq 0)$ represents the jumps of the future infimum of $\underset{\leftarrow}{V}$. We now use the couple $([p_1(t)], p_2(t)\,; t \geq 0)$ to get a Lévy process conditioned to stay positive.

We recall the notations of [10], for $t \geq 0$, we use $g(t) = \sup\{s < t; V(s) = \underline{V}(s)\}$ and $d(t) = \inf\{s > t; V(s) = \underline{V}(s)\}$, the left and right extremities of the excursion interval of $V - \underline{V}$ away from 0 that contains t and

$$\mathcal{R}_{V-\underline{V}}(t) = \begin{cases} (V - \underline{V})((d(t) + g(t) - t)-) & \text{if } g(t) < d(t) \\ 0 & \text{otherwise.} \end{cases}$$

See Figure 3.2 and 3.3.

The process L is also a local time in 0 for the process $\mathcal{R}_{V-\underline{V}}$. So the excursion process of $\mathcal{R}_{V-\underline{V}}$ above 0, parametrized by L, is the Poisson point process $([p_1(t)], t \geq 0)$ and, similarly, the excursion process of

$$\left((V - \underline{V})((m-t)-)\mathbf{1}_{0 \leq t < m} + \mathcal{R}_{V-\underline{V}}(t)\mathbf{1}_{t \geq m},\ t \geq 0\right)$$

above 0 is the Poisson point process $([\tilde{p}_1(t)], t \geq 0)$. We have an analogue equality for the jumps associated with every excursion. Then we get the equality in law of the processes

$$\begin{aligned}((V - \underline{V})((m-t)-)\mathbf{1}_{0 \leq t < m} + \mathcal{R}_{V-\underline{V}}(t)\mathbf{1}_{t \geq m},& \\ (\underline{V}((m-t)-) - \underline{V}(m))\mathbf{1}_{0 \leq t < m} - \underline{V}(d(t))\mathbf{1}_{t \geq m}\ ;\ t \geq 0)&\end{aligned}$$

and $(\mathcal{R}_{V-\underline{V}}(t), -\underline{V}(d(t); t \geq 0))$. Adding the two parts for $t < m$, we get:

$$\left(\underset{\leftarrow}{V}(t); 0 \leq t < m\right) \stackrel{\mathcal{L}}{=} \left(\mathcal{R}_{V-\underline{V}}(t) - \underline{V}(d(t)); 0 \leq t < m\right),$$

See Figure 3.3 and 3.4.

Figure 3.2: The Lévy process $(V(t); t \geq 0)$

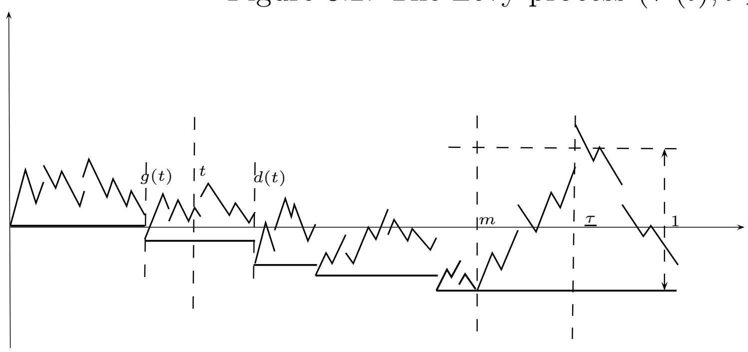
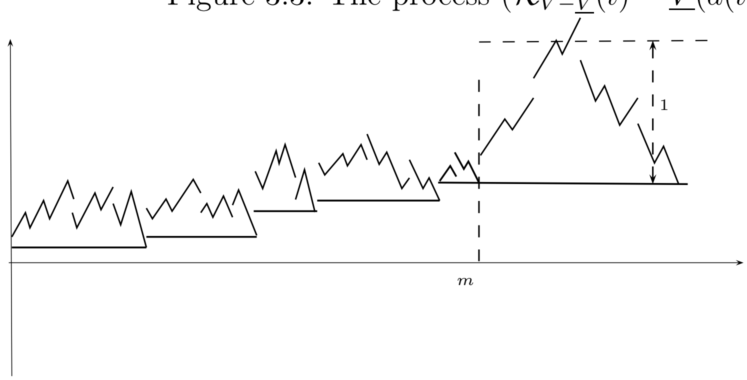


Figure 3.3: The process $(\mathcal{R}_{V-\underline{V}}(t) - \underline{V}(d(t)); t \geq 0)$



We now use Theorem 28 of [28] : the law of $(\mathcal{R}_{V-\underline{V}}(t) - \underline{V}(d(t)), t \geq 0)$ under $\mathbb{P}$, is $\hat{\mathbb{P}}^{\uparrow}$.

Now we express the final time m in terms of $(\mathcal{R}_{V-\underline{V}}(t) - \underline{V}(d(t)), t \geq 0)$.

Recall that $\underline{\tau} = \inf\{t \geq 0; V(t) - \underline{V}(t) \geq 1\}$ and that $m = \sup\{t \leq \underline{\tau}; V(t) - \underline{V}(t) = 0\}$. We define the variable $\underline{\tau}' = \tau(\mathcal{R}_{V-\underline{V}}) = \inf\{t \geq 0; \mathcal{R}_{V-\underline{V}} \geq 1\}$. Notice that $\underline{\tau}$ and $\underline{\tau}'$ belong to the same excursion interval of $V - \underline{V}$. The variable m remains stable by time inversion in every excursions of $V - \underline{V}$ and by the inversion of the excursions before m, then $m = \sup\{t \leq \underline{\tau}'; \mathcal{R}_{V-\underline{V}}(t) = 0\}$. We write

$$Y(t) := \mathcal{R}_{V-\underline{V}}(t) - \underline{V}(d(t)).$$

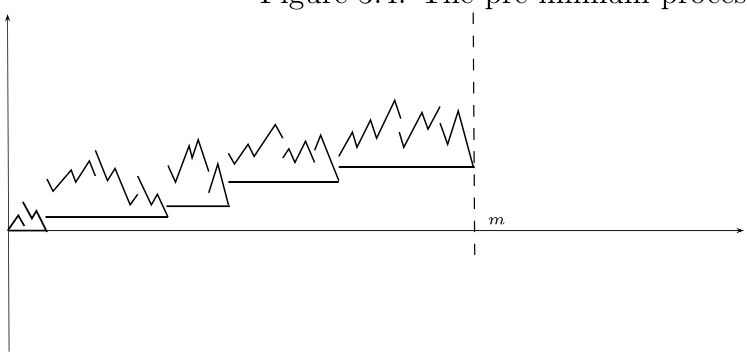
We prove that

$$\mathcal{R}_{V-\underline{V}}(t) = Y(t) - \underline{\underline{Y}}(t),$$

it is enough to prove that, for a fixed $t \geq 0$, the post-infimum process $\underline{\underline{Y}}(t) = \inf_{s \geq t} Y(s)$ is equal to $-\underline{V}(d(t))$.

On the interval $[t, d(t))$, the process $(\underline{V}(d(s)), t \leq s < d(t))$ is constant

Figure 3.4: The pre-infimum process



and

$$\inf_{t \le s < d(t)} \mathcal{R}_{V-\underline{V}}(s) = \mathcal{R}_{V-\underline{V}}(d(t)-) = 0$$

then $\inf_{t \le s < d(t)} Y(s) = -\underline{V}(d(t))$. The process $\mathcal{R}_{V-\underline{V}}$ is non-negative and vanishes at times $\{d(t)-, t \ge 0\}$ (see Corollary 1, [68]), and the process $-\underline{V}(d(.))$ increases by jumps at times $\{d(t), t \ge 0\}$, then $\underline{\underline{Y}}(t) = \inf_{t \le s < d(t)} Y(s)$ and so $\underline{\underline{Y}}(t) = -\underline{V}(d(t))$. We deduce that

$$m = \sup\{t \le \underline{\tau}'; Y(t) - \underline{\underline{Y}}(t) = 0\}.$$

Finally, under $\mathbb{P}$, the law of the process $(\mathcal{R}_{V-\underline{V}}(t) - \underline{V}(d(t)))_{0 \le t \le m}$ is the law of the process $\hat{V}^{\uparrow}$ stopped at time $\hat{m}^{\uparrow}$ and the point (1) is proved.

(2) Now we consider the post-infimum process. Let F be a bounded measurable function on $\mathcal{V}^+$. We can write

$$\underset{\rightarrow}{\mathbb{E}}^c \left[F(\omega)\right] = \mathcal{N}\left(F\left(v(\cdot \wedge \tau_c)\right) | \tau_c(v) < \infty\right).$$

We use Proposition 4.7 of [31]: for all bounded measurable function G on $\mathcal{V}^+$,

$$\begin{aligned} \mathbb{E}^{\uparrow,c}\left[G(\omega)\right] &= \mathcal{N}\left(G\left(v(\cdot \wedge \tau_c)\right)\mathcal{U}\left([0, v(\tau_c))\right) | \tau_c < \infty\right)\mathcal{N}\left(\tau_c(v) < \infty\right) \\ &= \mathcal{N}\left(\frac{G(v(\cdot \wedge \tau_c))}{f_c(v(\tau_c))}\Big| \tau_c(v) < \infty\right) \end{aligned}$$

where we recall that for all $x \in \mathbb{R}_+$, $\frac{1}{f_c(x)} = \mathcal{U}\left([0,x)\right)\mathcal{N}(\tau_c(v) < \infty)$. Using the function $G(v(\cdot \wedge \tau_c)) = F\left(v(\cdot \wedge \tau_c)\right) f_c(v(\tau_c))$, we obtain the second point of the proposition.

$\square$

3.2.3 Stable Lévy process

Definition 3.2.3. *([13], Section VIII)*
The Lévy process V is stable with index $\alpha \in (0,2]$ if, for all $c > 0$

$$\left(c^{-1}V(c^{\alpha}t), t \geq 0\right) \overset{\mathcal{L}}{=} (V(t), t \geq 0).$$

For $\alpha \in (0,1) \cup (1,2)$, the Fourier transform of an α-stable Lévy process V is : for all $t \geq 0$ and for all $\lambda \in \mathbb{R}$, $\mathbb{E}\left[e^{i\lambda V(t)}\right] = e^{-t\psi(\lambda)}$ with

$$\psi(\lambda) = k|\lambda|^{\alpha}\left(1 - i\beta \ sgn(\lambda)\tan(\frac{\pi\alpha}{2})\right)$$

where $k > 0$ and $\beta \in [-1,1]$.
The Lévy measure of the process V is absolutely continuous with respect to the Lebesgue measure, it can be written as follows:

$$\pi(dx) = c^{+}x^{-\alpha-1}\mathbf{1}_{x>0}dx + c^{-}x^{-\alpha-1}\mathbf{1}_{x<0}dx$$

where c^{+} and c^{-} are positive numbers non equal to 0 together and such that $\beta = \frac{c^{+}-c^{-}}{c^{+}+c^{-}}$.

In the quadratic case ($\alpha = 2$), V is a Brownian motion and its characteristic exponent can be written $\psi(\lambda) = k\lambda^2$ where k is a positive number. The Lévy measure is then trivial.

In the case $\alpha = 1$, V is a Cauchy process and $\psi(\lambda) = k|\lambda| + id\lambda$ with $k > 0$ and $d \in \mathbb{R}$. The Lévy measure is then proportional to $x^{-2}dx$.

Remark 6. *If the process V is α-stable, then the pre-infimum processes $\left(c^{-1}\underset{\leftarrow}{V}_{m_c}(c^{\alpha}t), t \geq 0\right)$ and $\underset{\leftarrow}{V}_{m_1}$ (stopped at correct times) have the same law. This also holds for the post-infimum processes $\left(c^{-1}\underset{\rightarrow}{V}_{m_c}(c^{\alpha}t), t \geq 0\right)$ and $\underset{\rightarrow}{V}_{m_1}$.*

Lemma 3.2.4. *Let V be an α-stable Lévy process which is not a pure drift. Then V is recurrent if and only if $\alpha \geq 1$.*

Proof. We use a condition of recurrence given in [13], Theorem I.17: V is recurrent if and only if $\int_{-r}^{r} \Re\left[\frac{1}{\psi(\lambda)}\right] d\lambda = +\infty$ where $\Re[z]$ is the real part of the complex number z.

For $\alpha = 1$, $\int_{-r}^{r} \frac{k|\lambda|}{(k^2+d^2)\lambda^2} d\lambda = +\infty$ because $k \neq 0$.

For $\alpha \neq 1$, $\int_{-r}^{r} \frac{1}{k|\lambda|^{\alpha}(1+\beta^2\tan^2(\frac{\pi\alpha}{2}))} d\lambda < +\infty$ if and only if $\alpha < 1$. $\square$

Then we use a stable Lévy process with index $\alpha \in [1,2]$ and which is not a pure drift. We need others properties for V, mainly to use the results of Section 3.2.2.

Lemma 3.2.5. *Let V be a stable Lévy process with index $\alpha \in [1,2]$ and which is not a pure drift. Then,*

– *0 is regular for $(0,+\infty)$ and for $(-\infty,0)$,*
– *$\mathbb{P}$-a.s., $\limsup_{t\to+\infty} V(t) = +\infty$ and $\liminf_{t\to+\infty} V(t) = -\infty$.*

Proof.

• Rogozin criterion (see Prop. VI.11 of [13]) gives a necessary and sufficient condition to have the regularity in 0:

0 is regular for $(0,+\infty)$ (resp. for $(-\infty,0)$) if and only if $\int_0^1 t^{-1}\mathbb{P}(V(t) \geq 0)dt = +\infty$ (resp. $\int_0^1 t^{-1}\mathbb{P}\left(V(t) \leq 0\right)dt = +\infty$).

Remark, using the stability property of V, that $\mathbb{P}(V(t) \geq 0)$ does not depend on t. Moreover this probability is positive if V is not a pure drift, see [13], VIII.1.

With the previous criterion, 0 is regular for $(0,+\infty)$ and for $(-\infty,0)$.

• We have $\int_1^\infty t^{-1}\mathbb{P}(V(t) \geq 0)dt = \int_1^\infty t^{-1}\mathbb{P}(V(t) \leq 0)dt = +\infty$, this implies that $\mathbb{P}$-a.s., $\limsup_{t\to+\infty} V(t) = +\infty$ and $\liminf_{t\to+\infty} V(t) = -\infty$, see [13], Theo. VI.12.

□

Lemma 3.2.6. *We consider a stable Lévy process V with index $\alpha \in (0,2]$. Then the process $V^\uparrow$ is an α-stable process.*

Proof. In the case where $\alpha = 2$, the stable Lévy process V is a standard Brownian motion, $V^\uparrow$ is a 3-dimensional Bessel process (see [59]) and then is stable with index 2.

If $\alpha \in (0,2)$, the stable Lévy process has no Brownian part. Recall that α_t^+ is the right continuous inverse of $A_t^+ = \int_0^t \mathbf{1}_{V(s)>0}ds$. These two processes are stable with index 1. We use the definition given in Section 3.2.2:

$$V^\uparrow(t) = V(\alpha_t^+) + \sum_{0<s\leq\alpha_t^+} \left(0 \vee V(s-)\right)\mathbf{1}_{V(s)\leq 0} - \left(0 \wedge V(s-)\right)\mathbf{1}_{V(s)>0},$$

We deduce that the process $V^\uparrow$ is stable with index α.

□

Proposition 3.2.7. *We consider a stable Lévy process with index $\alpha \in [1,2]$ and which is not a pure drift. When c converges to $+\infty$:*

1. *the law $\underset{\leftarrow}{\mathbb{P}}^c$ converges weakly (in the Skorohod sense) to the law $\hat{\mathbb{P}}^\uparrow$.*
2. *the law $\underset{\rightarrow}{\mathbb{P}}^c$ converges weakly (in the Skorohod sense) to the law $\mathbb{P}^\uparrow$.*

To prove the second point, we need the following lemma.

Lemma 3.2.8. *We fix $\varepsilon > 0$ and we define*

$$\sigma_\varepsilon = \sigma_\varepsilon(V^\uparrow) = \inf\left\{t > 0 / V^\uparrow(t) - \underline{\underline{V}}^\uparrow(t) = 0 \text{ and } \exists s < t, V^\uparrow(s) - \underline{\underline{V}}^\uparrow(s) > \varepsilon\right\}.$$

Then the process $\left(V^\uparrow_{\sigma_\varepsilon}(t), t \geq 0\right) = \left(V^\uparrow(\sigma_\varepsilon + t) - V^\uparrow(\sigma_\varepsilon), t \geq 0\right)$ is independent of the process $\left(V^\uparrow(t), 0 \leq t \leq \sigma_\varepsilon\right)$ and its law is given by $\mathbb{P}^\uparrow$.

Proof. Thanks to Theorem 28 of [28], the law of the process $\mathcal{R}_{\overline{V}-V} + V(d(\cdot))$ is $\mathbb{P}^\uparrow$ (where $\mathcal{R}$ and d are defined like page 57 but using the process $\overline{V} - V$). It is enough to prove the result in the case where $V^\uparrow = \mathcal{R}_{\overline{V}-V} + V(d(\cdot))$. But in this case, σ_ε can be written as

$$\sigma_\varepsilon = \inf\left\{t > 0/\overline{V}(t) - V(t) = 0 \text{ and } \exists s < t, \overline{V}(s) - V(s) > \varepsilon\right\}$$

then σ_ε is a stopping time for V.

Thus the process $(V_{\sigma_\varepsilon}(t), t \geq 0) = (V(\sigma_\varepsilon + t) - V(\sigma_\varepsilon), t \geq 0)$ is a Lévy process with law $\mathbb{P}$ independent of $(V(t), 0 \leq t \leq \sigma_\varepsilon)$. We can check that $V^\uparrow_{\sigma_\varepsilon}(t) = \mathcal{R}_{\overline{V}_{\sigma_\varepsilon} - V_{\sigma_\varepsilon}}(t) + V_{\sigma_\varepsilon}(d(t))$, this proves the Lemma. □

Proof of Proposition 3.2.7. (1) Thanks to Proposition 3.2.2, the laws of $\underset{\leftarrow}{\mathbb{P}}^c$ and $\hat{\mathbb{P}}^{\uparrow,c}$ are equal. When $c \to \infty$, $\hat{\mathbb{P}}^\uparrow$-a.s., $\hat{m}^\uparrow_c$ tends to infinity, then $\underset{\leftarrow}{\mathbb{P}}^c = \hat{\mathbb{P}}^{\uparrow,c}$ converges to $\hat{\mathbb{P}}^\uparrow$.

(2) Now we prove that for a continuous bounded function F in the Skorohod topology such that $F(\omega) = F(\omega \mathbf{1}_{[-K,K]})$, we have

$$\lim_{c\to\infty} \underset{\rightarrow}{\mathbb{E}}^c\left[F(\omega)\right] = \mathbb{E}^\uparrow\left[F(\omega)\right].$$

As $\mathbb{P}$-a.s., $\lim_{c\to\infty} \tau_c(\underset{\rightarrow}{V}_{m_c}) = \infty$, so $\lim_{c\to\infty} \underset{\rightarrow}{\mathbb{P}}^c(K \leq \tau_c(\omega)) = 1$ and thus

$$\lim_{c\to\infty} \underset{\rightarrow}{\mathbb{E}}^c\left[F(\omega)\right] = \lim_{c\to\infty} \underset{\rightarrow}{\mathbb{E}}^c\left[F(\omega), K \leq \tau_c(\omega)\right].$$

We first use the stability of V. Thanks to Remark 6,

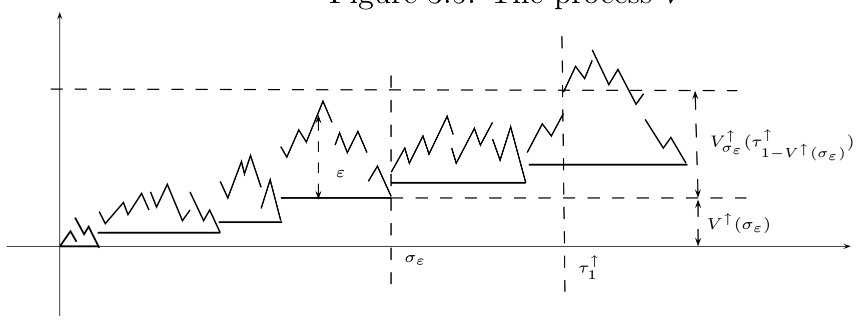
$$\begin{aligned}\underset{\rightarrow}{\mathbb{E}}^c\left[F(\omega), K \leq \tau_c(\omega)\right] &= \underset{\rightarrow}{\mathbb{E}}^1\left[F\left(c\,\omega(c^{-\alpha}\cdot)\right), K \leq c^\alpha \tau_1(\omega)\right], \\ &= \mathbb{E}^\uparrow\left[F\left(c\,\omega(c^{-\alpha}\cdot)\right) f_1\left(\omega(\tau_1)\right), K \leq c^\alpha \tau_1(\omega)\right]\end{aligned}$$

where we use the result (2) of Proposition 3.2.2, for the second equality.

We now prove the asymptotic independence of $F\left(c\,\omega(c^{-\alpha}\cdot)\right)$ and $f_1\left(\omega(\tau_1)\right)$ under $\mathbb{P}^\uparrow$. Then we express these two terms with respect to the processes $(\omega(t), 0 \leq t \leq \sigma_\varepsilon)$ and $(\omega(t), \sigma_\varepsilon \leq t)$ where, for $\varepsilon > 0$, we recall that:

$$\sigma_\varepsilon = \sigma_\varepsilon(\omega) = \inf\left\{t > 0/\omega(t) - \underline{\underline{\omega}}(t) = 0 \text{ and } \exists s < t, \omega(s) - \underline{\underline{\omega}}(s) > \varepsilon\right\}.$$

Figure 3.5: The process $V^\uparrow$



Fix $\varepsilon > 0$, remark that

$$\lim_{c\to+\infty} \mathbb{P}^\uparrow (K \le c^\alpha \tau_1) = \lim_{c\to+\infty} \mathbb{P}^\uparrow (K \le c^\alpha \sigma_\varepsilon) = 1.$$

Then $K \le c^\alpha \tau_1$ can be replaced by $K \le c^\alpha \sigma_\varepsilon$.

Moreover, $\mathbb{P}^\uparrow$-a.s., $\lim_{\varepsilon\to 0} \sigma_\varepsilon = 0$. Indeed, if there exists $\delta > 0$ such that for all $\varepsilon > 0$, $\sigma_\varepsilon > \delta$, then the excursion of $\omega - \underline{\underline{\omega}}$ containing time δ is the first one. And this is $\mathbb{P}^\uparrow$-a.s. impossible because 0 is regular for the process V. Thus $\lim_{\varepsilon\to 0} \mathbb{P}^\uparrow (\sigma_\varepsilon < \tau_1) = 1$. By conditioning with respect to $(\omega(t), 0 \le t \le \sigma_\varepsilon)$ and using that $\{\sigma_\varepsilon < \tau_1^\uparrow\} = \{\forall t \le \sigma_\varepsilon, V_t^\uparrow < 1\}$ is measurable with respect to $(\omega(t), 0 \le t \le \sigma_\varepsilon)$, we get:

$$\begin{aligned} &\mathbb{E}^\uparrow \left[F\left(c\,\omega(c^{-\alpha}\cdot)\right) f_1\left(\omega(\tau_1)\right), c^{-\alpha}K \le \sigma_\varepsilon < \tau_1\right] \qquad (3.4)\\ &= \mathbb{E}^\uparrow \left[F\left(c\,\omega(c^{-\alpha}\cdot)\right) \mathbf{1}_{c^{-\alpha}K\le\sigma_\varepsilon<\tau_1} \mathbb{E}^\uparrow \left[f_1\left(\omega(\tau_1)\right) \Big| (\omega(t), 0 \le t \le \sigma_\varepsilon)\right]\right]. \end{aligned}$$

We compute the limit of the conditional expectation when ε converges to 0. When $\sigma_\varepsilon < \tau_1$, we can write $\omega(\tau_1) = \omega(\sigma_\varepsilon) + \omega_{\sigma_\varepsilon}(\tau_{1-\omega(\sigma_\varepsilon)})$ where $\omega_{\sigma_\varepsilon}$ is the path starting from σ_ε : $(\omega_{\sigma_\varepsilon}(t), t \ge 0) = (\omega(\sigma_\varepsilon + t) - \omega(\sigma_\varepsilon), t \ge 0)$, see Figure 3.5. Thanks to Lemma 3.2.8,

$$\mathbb{E}^\uparrow \left[f_1\left(\omega(\tau_1)\right) \Big| (\omega(t), 0 \le t \le \sigma_\varepsilon)\right] = \chi(\omega(\sigma_\varepsilon))$$

where $\chi(x) = \mathbb{E}^\uparrow \left[f_1\left(x + \omega(\tau_{1-x})\right)\right]$. Remark that $\mathbb{P}^\uparrow$-a.s, $\lim_{\varepsilon\to 0} \omega(\sigma_\varepsilon) = 0$. We now prove that $\mathbb{P}^\uparrow$-a.s., $\lim_{y\to 1-} \omega(\tau_y) = \omega(\tau_1)$: $\mathbb{P}^\uparrow$-a.s., ω is continuous at its local extrema (see (2) of Remark 5). Thus, thanks to Remark 3, $\mathbb{P}^\uparrow$-a.s., $\sup_{[0,\tau_1)} \omega$ is attained in a point $x_0 \in [0, \tau_1]$ and $\omega(x_0) \le 1$.

If $x_0 = \tau_1$, then $\omega(\tau_1) = 1$ and $\lim_{y\to 1-} 1 - \omega(\tau_y) \le \lim_{y\to 1-} 1 - y = 0$. And if $x_0 < \tau_1$, then $\omega(x_0) < 1$ and then, for $1 - y$ small enough, $\tau_1 = \tau_y$, thereby $\lim_{y\to 1-} \omega(\tau_y) = \omega(\tau_1)$.

Moreover, in the case where V is a stable process with index $\alpha \in [1,2)$, an explicit expression of $\mathcal{U}$ is given in Example 7 of [29]:

$$\mathcal{U}([0,x)) = \frac{x^{\alpha\rho}}{\Gamma(\alpha\rho+1)} \text{ where } \rho = \mathbb{P}(V(t) \geq 0) \text{ does not depend on } t.$$

Thus using the expression of f_c given in Proposition 3.2.2, we get:

$$f_1(x) = \frac{\Gamma(\alpha\rho+1)}{x^{\alpha\rho}\mathcal{N}(\tau_1(\omega) < \infty)}, \tag{3.5}$$

then f_1 is continuous on $\mathbb{R}_+^*$ and

$$\begin{aligned} \lim_{\varepsilon\to 0} \mathbb{E}^{\uparrow}\left[f_1(\omega(\tau_1)) \,\middle|\, (\omega(t), 0 \leq t \leq \sigma_\varepsilon)\right] = \lim_{\varepsilon\to 0} \chi(\omega(\sigma_\varepsilon)) = \mathbb{E}^{\uparrow}\left[f_1(\omega(\tau_1))\right], \\ = 1,\ \mathbb{P}^{\uparrow}\text{-a.s.}, \end{aligned} \tag{3.6}$$

because $f_1(\omega(\tau_1))$ is the density of $\underset{\rightarrow}{\mathbb{P}}^1$ with respect to $\mathbb{P}^{\uparrow,1}$.

Using the stability of the process $V^{\uparrow}$ given in Lemma 3.2.6, we get that for all $c \geq 0$,

$$\mathbb{E}^{\uparrow}\left[F\left(c\,\omega(c^{-\alpha}\cdot)\right)\right] = \mathbb{E}^{\uparrow}\left[F(\omega)\right] = \mathbb{E}\left[F\left(V^{\uparrow}\right)\right].$$

So to prove the proposition, it is enough to show that

$$\lim_{\varepsilon\to 0}\lim_{c\to\infty}\left|\mathbb{E}^{\uparrow}\left[F\left(c\,\omega(c^{-\alpha}\cdot)\right)\mathbf{1}_{c^{-\alpha}K\leq\sigma_\varepsilon}\chi\left(\omega(\sigma_\varepsilon)\right)\mathbf{1}_{\sigma_\varepsilon<\tau_1}\right] - \mathbb{E}^{\uparrow}\left[F\left(c\,\omega(c^{-\alpha}\cdot)\right)\right]\right| = 0.$$

We introduce $\mathbb{E}^{\uparrow}\left[F\left(c\,\omega(c^{-\alpha}\cdot)\right)\mathbf{1}_{c^{-\alpha}K\leq\sigma_\varepsilon}\right]$, and we get the following upper bound :

$$\begin{aligned} &\left|\mathbb{E}^{\uparrow}\left[F\left(c\,\omega(c^{-\alpha}\cdot)\right)\mathbf{1}_{c^{-\alpha}K\leq\sigma_\varepsilon}\chi\left(\omega(\sigma_\varepsilon)\right)\mathbf{1}_{\sigma_\varepsilon<\tau_1}\right] - \mathbb{E}^{\uparrow}\left[F\left(c\,\omega(c^{-\alpha}\cdot)\right)\right]\right| \leq \\ &\qquad \mathbb{E}^{\uparrow}\left[\left|F\left(c\,\omega(c^{-\alpha}\cdot)\right)\mathbf{1}_{c^{-\alpha}K\leq\sigma_\varepsilon}\left(\chi\left(\omega(\sigma_\varepsilon)\right)\mathbf{1}_{\sigma_\varepsilon<\tau_1} - 1\right)\right|\right] + \\ &\qquad \mathbb{E}^{\uparrow}\left[\left|F\left(c\,\omega(c^{-\alpha}\cdot)\right)\left(\mathbf{1}_{c^{-\alpha}K\leq\sigma_\varepsilon} - 1\right)\right|\right]. \end{aligned}$$

For ε fixed, we have $\lim_{c\to+\infty}\mathbb{P}^{\uparrow}\left(c^{-\alpha}K \leq \sigma_\varepsilon\right) = 1$. Then for the second part, we get :

$$\lim_{c\to\infty}\mathbb{E}^{\uparrow}\left[\left|F\left(c\,\omega(c^{-\alpha}\cdot)\right)\left(\mathbf{1}_{c^{-\alpha}K\leq\sigma_\varepsilon} - 1\right)\right|\right] \leq \|F\|_\infty \lim_{c\to\infty}\mathbb{E}^{\uparrow}\left[\left|\mathbf{1}_{c^{-\alpha}K\leq\sigma_\varepsilon} - 1\right|\right] = 0$$

and for the first part, we have :

$$\begin{aligned} &\overline{\lim_{c\to\infty}}\,\mathbb{E}^{\uparrow}\left[\left|F\left(c\,\omega(c^{-\alpha}\cdot)\right)\mathbf{1}_{c^{-\alpha}K\leq\sigma_\varepsilon}\left(1 - \chi\left(\omega(\sigma_\varepsilon)\right)\mathbf{1}_{\sigma_\varepsilon<\tau_1}\right)\right|\right] \\ &\qquad \leq \|F\|_\infty\,\mathbb{E}^{\uparrow}\left[\left|1 - \chi\left(\omega(\sigma_\varepsilon)\right)\mathbf{1}_{\sigma_\varepsilon<\tau_1}\right|\right]. \end{aligned}$$

Let ε tends to 0, we have $\mathbb{P}^{\uparrow}$-a.s, $\lim_{\varepsilon\to 0}\sigma_\varepsilon = 0$ and we use (3.6) to get the convergence :

$$\lim_{\varepsilon\to 0}\lim_{c\to\infty}\mathbb{E}^{\uparrow}\left[\left|F\left(c\,\omega(c^{-\alpha}\cdot)\right)\mathbf{1}_{c^{-\alpha}K\leq\sigma_\varepsilon}\left(1 - \chi\left(\omega(\sigma_\varepsilon)\right)\mathbf{1}_{\sigma_\varepsilon<\tau_1}\right)\right|\right] = 0.$$

This completes the proof. $\square$

3.3 Diffusion in a random environment

3.3.1 Asymptotic behavior of a diffusion in a stable environment

We suppose that the diffusion X in the random environment V is defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and we write, as previously, $\mathbb{P}$ the law of V. But we take a general case for the environment: *in all this section, we only suppose that V is an α-stable càd-làg process on $\mathbb{R}$ ($\alpha > 0$) and not necessary a Lévy process*. We will prove that the asymptotic behavior of L_X, the local time of the diffusion, depends only on the environment.

We first give some properties of X. We can prove (see for example [72]) that there exists a Brownian motion B independent of V such that $\mathcal{P}$-a.s.,

$$\forall t > 0, \quad X(V,t) = S_V^{-1}\left(B\left(T_V^{-1}(t)\right)\right) \tag{3.7}$$

where

$$\forall x \in \mathbb{R}, \quad S_V(x) := \int_0^x e^{V(y)}\mathrm{d}y$$

and

$$\forall t \geq 0, \quad T_V(t) := \int_0^t e^{-2V(S_V^{-1}(B(s)))}\mathrm{d}s.$$

In order to simplify the notations, we write S and T respectively for S_V and T_V, when there is no confusion. From Formula (3.7) and the occupation time Formula (3.1), we deduce an expression of the local time process of X using the local time process L_B of the Brownian motion B. For all $x \in \mathbb{R}$ and all $t \geq 0$,

$$L_X(t,x) = e^{-V(x)}L_B(T^{-1}(t), S(x)). \tag{3.8}$$

To study the asymptotic behavior of the diffusion, we first work in a quenched environment.

As Brox in the Brownian environment case (see [17]), we introduce the process $X_c(V,t) := X(cV,t)$. For all $x \in \mathbb{R}$, we write $V^c(x) := c^{-1}V(c^\alpha x)$. There is a direct link between the law of X_c and the law of X:

Lemma 3.3.1. *For every fixed c , conditionally on the environment V, the processes $\left(c^{-\alpha}X(V,c^{2\alpha}t); t \geq 0\right)$ and $(X_c(V^c,t); t \geq 0)$ have the same law.*

Proof. We can easily check the result with Formula (3.7). $\square$

With the definition of the local time (Formula (3.1)), we get an analogous relation for the local time processes:

Corollary 3.3.2. *For every fixed c, conditionally on V, the local time processes* $\left(L_{X_c(V^c,\cdot)}(t,x); t \geq 0, x \in \mathbb{R}\right)$ *and* $\left(c^{-\alpha} L_{X(V,\cdot)}(c^{2\alpha}t, c^{\alpha}x); t \geq 0, x \in \mathbb{R}\right)$ *have the same law.*

We will first give a convergence result for the process L_{X_c} and then we will deduce a convergence result for the process L_X. The asymptotic behavior of L_{X_c} with a quenched environment can be only described with respect to the environment if this one is a "good" environment in the sense of the following definition:

Definition 3.3.3. *We call a good environment every path $\omega \in \mathcal{V}$ that checks*

1. $\liminf\limits_{x \to \pm\infty} \omega(x) = -\infty$ *and* $\limsup\limits_{x \to \pm\infty} \omega(x) = +\infty$,
2. *ω is continuous at its local extrema and the values of ω in these points are pairwise distinct.*

Denote by $\widetilde{\mathcal{V}}$ the set of good environments.

Remark that if the paths of V are in $\widetilde{\mathcal{V}}$, then the minimum $\mathfrak{m}_c(V)$ of the standard valley with height c (see page 49) is well defined.

Then we can define, for every $c > 0$ and every $r > 0$,

$$a_{cr} = a_{cr}(V_{\mathfrak{m}_c}) := \sup\{x \leq 0 / V_{\mathfrak{m}_c}(x) > cr\} \text{ and}$$
$$b_{cr} = b_{cr}(V_{\mathfrak{m}_c}) := \inf\{x \geq 0 / V_{\mathfrak{m}_c}(x) > cr\}$$

where for $x \in \mathbb{R}$, $(V_x(y), y \in \mathbb{R})$ is the recentered environment in x: for all $y \in \mathbb{R}$,

$$V_x(y) := V(x+y) - V(x).$$

Now we can give the result in a quenched environment. We denote by $\mathcal{P}_V$ the probability measure $\mathcal{P}(\cdot|V)$.

Proposition 3.3.4.
We fix a real number $r \in (0,1)$ and we choose an $\mathbb{R}$-valued function h such that $\lim\limits_{c \to \infty} h(c) = 1$. If V is a good environment, then for all $\delta > 0$,

$$\lim_{c \to +\infty} \mathcal{P}_V\left(\sup_{a_r \leq x \leq b_r} \left| \frac{L_{X_c}(e^{ch(c)}, \mathfrak{m}_1 + x)}{e^{ch(c)}} \frac{\int_{a_r}^{b_r} e^{-cV_{\mathfrak{m}_1}(y)}\,dy}{e^{-cV_{\mathfrak{m}_1}(x)}} - 1 \right| \leq \delta \right) = 1$$

and

$$\lim_{c \to +\infty} \mathcal{P}_V\left(\sup_{x \in \mathbb{R}\setminus[a_r,b_r]} L_{X_c}(e^{ch(c)}, \mathfrak{m}_1 + x) \leq \frac{\delta e^{ch(c)}}{\int_{a_r}^{b_r} e^{-cV_{\mathfrak{m}_1}(y)}\,dy}\right) = 1.$$

Proof. Ideas of the proof are essentially the same as in Sections 2.2.1 and 2.2.2 of the previous chapter. For the first one, we proceed as in Proposition 2.2.1. The proof of the second formula is the same as the one of Proposition 2.2.5 but instead of computing with the integral of the local time, we use the supremum on $\mathbb{R}\setminus[a_r,b_r]$ of the same local time.

In both cases, same arguments as the ones of Proposition 2.3.1 allow to write the formula with a deterministic time instead of a random time. $\square$

Denote by $m_c^*(t)$ the favorite point of X_c,

$$m_c^*(t)=\inf\left\{x\in\mathbb{R},\ L_{X_c}(t,x)=\sup_{y\in\mathbb{R}}L_{X_c}(t,y)\right\}.$$

The previous proposition implies that this point is localized in the neighborhood of $\mathfrak{m}_1$:

Proposition 3.3.5. *If V is a good environment $\mathbb{P}$-a.s. then for any $\delta>0$ such that*

$$\lim_{\varepsilon\to 0}\limsup_{c\to\infty}\mathbb{P}\left(\inf_{[a_r,-\delta/c^\alpha]\cup[\delta/c^\alpha,b_r]}cV_{\mathfrak{m}_1}\le\varepsilon\right)=0 \tag{3.9}$$

we have

$$\lim_{c\to+\infty}\mathcal{P}\left(\left|m_c^*(e^{ch(c)})-\mathfrak{m}_1\right|\le\delta/c^\alpha\right)=1.$$

Proof. Let $\varepsilon\in(0,1/2)$, for any $c>1$, we define two events:

$$A_c:=\left\{\sup_{a_r\le x\le b_r}\left|\frac{L_{X_c}(e^{ch(c)},\mathfrak{m}_1+x)}{e^{ch(c)}}\frac{\int_{a_r}^{b_r}e^{-cV_{\mathfrak{m}_1}(y)}\mathrm{d}y}{e^{-cV_{\mathfrak{m}_1}(x)}}-1\right|\le\varepsilon\right\}$$

and

$$B_c:=\left\{\sup_{x\in\mathbb{R}\setminus[a_r,b_r]}L_{X_c}(e^{ch(c)},\mathfrak{m}_1+x)\le\frac{\varepsilon e^{ch(c)}}{\int_{a_r}^{b_r}e^{-cV_{\mathfrak{m}_1}(y)}\mathrm{d}y}\right\}.$$

On $A_c\cap B_c$, we have the following inequality

$$L_{X_c}(e^{ch(c)},m_c^*(e^{ch(c)}))\ge L_{X_c}(e^{ch(c)},\mathfrak{m}_1)\ge\frac{(1-\varepsilon)e^{ch(c)}}{\int_{a_r}^{b_r}e^{-cV_{\mathfrak{m}_1}(y)}\mathrm{d}y}.$$

Thereby $m_c^*(e^{ch(c)})\in(a_r,b_r)$ and so

$$L_{X_c}(e^{ch(c)},m_c^*(e^{ch(c)}))\le\frac{(1+\varepsilon)e^{ch(c)}}{\int_{a_r}^{b_r}e^{-cV_{\mathfrak{m}_1}(y)}\mathrm{d}y}e^{-cV_{\mathfrak{m}_1}(m_c^*(e^{ch(c)}))}.$$

Necessarily,

$$cV_{\mathfrak{m}_1}(m_c^*(e^{ch(c)}) \leq \log \frac{1+\varepsilon}{1-\varepsilon}.$$

Denote

$$\mathcal{E}_c^\varepsilon = \left\{ V \in \widetilde{\mathcal{V}}\ ,\ \inf_{[a_r,-\delta/c^\alpha]\cup[\delta/c^\alpha,b_r]} cV_{\mathfrak{m}_1} > \log \frac{1+\varepsilon}{1-\varepsilon} \right\}.$$

If $V \in \mathcal{E}_c^\varepsilon$, then on $A_c \cap B_c$,

$$\left| m_c^*(e^{ch(c)}) - \mathfrak{m}_1 \right| \leq \delta/c^\alpha.$$

Thus,

$$\mathcal{P}\left(\left| m_c^*(e^{ch(c)}) - \mathfrak{m}_1 \right| > \delta/c^\alpha \right) \leq \mathcal{P}(\overline{A_c}) + \mathcal{P}(\overline{B_c}) + \mathcal{P}(\overline{\mathcal{E}_c^\varepsilon})$$

and therefore

$$\limsup_{c\to\infty} \mathcal{P}\left(\left| m_c^*(e^{ch(c)}) - \mathfrak{m}_1 \right| > \delta/c^\alpha \right) \leq \lim_{\varepsilon\to 0} \limsup_{c\to\infty} \mathcal{P}(\overline{\mathcal{E}_c^\varepsilon}) = 0$$

according to Hypothesis (3.9). □

Now we can come back to the process L_X and prove the following theorem, main result of this section:

Theorem 3.3.6. *Fix a real number $r \in (0,1)$. If the environment V is a good environment $\mathbb{P}$-a.s. and there exists $\alpha > 0$ such that V is α-stable then for all $\delta > 0$,*

$$\lim_{c\to+\infty} \mathcal{P}\left(\sup_{a_{cr}\leq x\leq b_{cr}} \left| \frac{L_X(e^c, \mathfrak{m}_c + x)}{e^c} \frac{\int_{a_{cr}}^{b_{cr}} e^{-V_{\mathfrak{m}_c}(y)}\, dy}{e^{-V_{\mathfrak{m}_c}(x)}} - 1 \right| \leq \delta \right) = 1.$$

and

$$\lim_{c\to+\infty} \mathcal{P}\left(\sup_{x\in\mathbb{R}\setminus[a_{cr},b_{cr}]} L_X(e^c, \mathfrak{m}_c + x) \leq \frac{\delta e^c}{\int_{a_{cr}}^{b_{cr}} e^{-V_{\mathfrak{m}_c}(y)}\, dy} \right) = 1.$$

Recall the definition of m^* of Theorem 3.1.2:

$$m^*(t) = \inf\left\{ x \in \mathbb{R},\ L_X(t,x) = \sup_{y\in\mathbb{R}} L_X(t,y) \right\}.$$

If for any $\delta > 0$, we have

$$\lim_{\varepsilon\to 0} \limsup_{c\to\infty} \mathbb{P}\left(\inf_{[a_{cr},-\delta]\cup[\delta,b_{cr}]} V_{\mathfrak{m}_c} \leq \varepsilon \right) = 0 \tag{3.10}$$

then

$$\lim_{c\to+\infty} \mathcal{P}\left(|m^*(e^c) - \mathfrak{m}_c| \leq \delta \right) = 1.$$

Remark 7. *We deal in this article with stable Lévy processes and we will see later that they verify Hypothesis* (3.10)*. Moreover, the previous theorem also applies when the environment V is a fractional Brownian motion.*

Proof. We fix $c > 0$ and we recall that $V^c(\cdot) := c^{-1}V(c^\alpha \cdot)$. By the stability of V, the processes V and V^c have same law. We first remark that, as the paths of V are in $\widetilde{\mathcal{V}}$, $\mathcal{P}$-a.s., the standard valley is well defined and therefore

$$\begin{aligned} \mathfrak{m}_1(V^c) &= c^{-\alpha}\mathfrak{m}_c(V), \\ a_r(V^c_{\mathfrak{m}_1(V^c)}) &= c^{-\alpha}a_{cr}(V_{\mathfrak{m}_c(V)}), \\ b_r(V^c_{\mathfrak{m}_1(V^c)}) &= c^{-\alpha}b_{cr}(V_{\mathfrak{m}_c(V)}), \end{aligned}$$

and for all $x \in \mathbb{R}$,

$$V^c_{\mathfrak{m}_1(V^c)}(x) = \frac{1}{c}V_{\mathfrak{m}_c(V)}(c^\alpha x). \tag{3.11}$$

Thus, using the substitution $z = c^\alpha y$, we get that

$$\int_{a_r}^{b_r} e^{-cV_{\mathfrak{m}_1}(y)}dy = c^{-\alpha}\int_{a_{cr}}^{b_{cr}} e^{V_{\mathfrak{m}_c}(z)}dz.$$

Now we replace t by $c^{-2\alpha}e^c$ and x by $\mathfrak{m}_1(V^c)+c^{-\alpha}x$ in Corollary 3.3.2, then we get that, conditionally on V, $\left(L_{X_c(V^c,\cdot)}(c^{-2\alpha}e^c, \mathfrak{m}_1(V^c) + c^{-\alpha}x), x \in \mathbb{R}\right)$ and $\left(c^{-\alpha}L_{X(V,\cdot)}(e^c, \mathfrak{m}_c(V) + x), x \in \mathbb{R}\right)$ are equal in law. Moreover the processes V and V^c having the same law, $\left(L_{X_c(V,\cdot)}(c^{-2\alpha}e^c, \mathfrak{m}_1(V) + c^{-\alpha}x), x \in \mathbb{R}\right)$ and $\left(c^{-\alpha}L_{X(V,\cdot)}(e^c, \mathfrak{m}_c(V) + x), x \in \mathbb{R}\right)$ have the same law without conditioning. Thus for all $c > 0$, $K > 0$, $r \in (0, 1)$ and $\delta > 0$,

$$\mathcal{P}\left(\sup_{a_{cr}\le x\le b_{cr}}\left|\frac{L_{X(V,\cdot)}(e^c, \mathfrak{m}_c(V) + x)}{e^c}\frac{\int_{a_{cr}}^{b_{cr}} e^{-V_{\mathfrak{m}_c}(y)}\mathrm{d}y}{e^{-V_{\mathfrak{m}_c}(x)}} - 1\right| < \delta\right) =$$

$$\mathcal{P}\left(\sup_{a_r\le x\le b_r}\left|\frac{L_{X_c(V,\cdot)}(c^{-2\alpha}e^c, \mathfrak{m}_1(V) + x)}{c^{-2\alpha}e^c}\frac{\int_{a_r}^{b_r} e^{-cV_{\mathfrak{m}_1}(y)}\mathrm{d}y}{e^{-cV_{\mathfrak{m}_1}(x)}} - 1\right| < \delta\right).$$

To finish, it is enough to remark that $c^{-2\alpha}e^c = e^{c(1-\frac{2\alpha}{c}\log c)}$ and $\lim_{c\to+\infty}(1 - \frac{2\alpha}{c}\log c) = 1$. Then, using Proposition 3.3.4, we obtain the first convergence of Theorem 3.3.6. The other ones are proved in the same way. ☐

3.3.2 Limit law of the renormalized local time

Now we suppose again that our environment is an α-stable Lévy process with $\alpha \in [1, 2]$ and is not a pure drift.

Lemma 3.3.7. *Hypothesis* (3.10) *is true for such a Lévy process:*

$$\lim_{\varepsilon\to 0}\limsup_{c\to\infty}\mathbb{P}\left(\inf_{[a_{cr},-\delta]\cup[\delta,b_{cr}]} V_{\mathfrak{m}_c}\leq\varepsilon\right)=0.$$

Then, the second part of Theorem 2.1.3 gives the convergence in probability of $m^*(e^c)-\mathfrak{m}_c$ *to* 0.

Proof. Recall that $\mathfrak{m}_c = m_c^+$ or $\mathfrak{m}_c = m_c^-$. As the proof is similar in the both cases, we only study the first one: (if $\beta<\alpha$ then we use the convention $[\alpha,\beta]=\emptyset$ and $\inf_{[\alpha,\beta]}V_{\mathfrak{m}_c}=+\infty$)

$$\mathbb{P}\left(\inf_{[a_{cr},-\mathfrak{m}_c]\cup[-\mathfrak{m}_c,-\delta]\cup[\delta,b_{cr}]} V_{\mathfrak{m}_c}\leq\varepsilon,\mathfrak{m}_c=m_c^+\right)\leq$$
$$\mathbb{P}\left(\inf_{[a_{cr},-\mathfrak{m}_c]} V_{\mathfrak{m}_c}\leq\varepsilon,\mathfrak{m}_c=m_c^+\right)+\mathbb{P}\left(\inf_{[-m_c^+,-\delta]} V_{m_c^+}\leq\varepsilon\right)+\mathbb{P}\left(\inf_{[\delta,b_{cr}]} V_{m_c^+}\leq\varepsilon\right).$$

Using the stability of V, $\mathfrak{m}_c$, a_{cr} and b_{cr} (see (3.11) above), we get for any $\varepsilon>0$:

$$\begin{aligned}\mathbb{P}\left(\inf_{[a_{cr},-\mathfrak{m}_c]} V_{\mathfrak{m}_c}\leq\varepsilon,\mathfrak{m}_c=m_c^+\right)&=\mathbb{P}\left(\inf_{[a_r,-\mathfrak{m}_1]} V_{\mathfrak{m}_1}\leq\varepsilon c^{-1},\mathfrak{m}_1=m_1^+\right)\\&\xrightarrow[c\to\infty]{}\mathbb{P}\left(\inf_{[a_r,-\mathfrak{m}_1]} V_{\mathfrak{m}_1}\leq\ 0,\mathfrak{m}_1=m_1^+\right).\end{aligned}$$

By uniqueness of the minima (Lemma 3.2.1), this last probability is equal to 0. Moreover, $\mathbb{P}$-a.s. $\inf_{[\delta,+\infty)}\hat{V}^{\uparrow}>0$, then Point (1) of Proposition 3.2.2 gives

$$\lim_{\varepsilon\to 0}\limsup_{c\to\infty}\mathbb{P}\left(\inf_{[-m_c^+,-\delta]} V_{m_c^+}\leq\varepsilon\right)=0.$$

Using now the stability of V and then Point (2) of Proposition 3.2.2, we obtain:

$$\begin{aligned}\mathbb{P}\left(\inf_{[\delta,b_{cr}]} V_{m_c^+}\leq\varepsilon\right)&=\mathcal{P}\left(\inf_{[c^{-\alpha}\delta,b_r]} V_{m_1^+}\leq\varepsilon c^{-1}\right)\\&=\int\mathbf{1}_{\left\{\inf_{[c^{-\alpha}\delta,\tau_r(\omega)]}\omega\leq\varepsilon c^{-1}\right\}}\underset{\rightarrow}{\mathbb{P}}^1(d\omega)\\&=\int\mathbf{1}_{\left\{\inf_{[c^{-\alpha}\delta,\tau_r(\omega)]}\omega\leq\varepsilon c^{-1}\right\}}f_1\left(\omega(\tau_1(\omega))\right)\mathbb{P}^{\uparrow,1}(d\omega)\\&\leq\frac{\Gamma(\alpha\rho+1)}{\mathcal{N}(\tau_1<\infty)}\mathbb{P}\left(\inf_{[\delta,\tau_{cr}(V^{\uparrow})]} V^{\uparrow}\leq\varepsilon\right)\end{aligned}$$

where the last inequality comes from the expression of f_1 given by (3.5). Finally,

$$\lim_{\varepsilon\to 0}\limsup_{c\to\infty}\mathbb{P}\left(\inf_{[\delta,b_{cr}]} V_{m_c^+}\le\varepsilon\right)\le\frac{\Gamma(\alpha\rho+1)}{\mathcal{N}(\tau_1<\infty)}\mathbb{P}\left(\inf_{[\delta,\infty)} V^{\uparrow}=0\right)=0.$$

This concludes the proof. □

According to Theorem 3.3.6, to obtain the weak convergence of the local time process, we only have to study the convergence of $e^{-V_{\mathfrak{m}_c}}/\int_{a_{rc}}^{b_{rc}} e^{-V_{\mathfrak{m}_c}}$, what we do now.

For $\omega\in\mathcal{V}$, recall the definition of ω^+ and ω^- given in (3.2) and notations of the first part:

$$\begin{aligned}\underline{\tau}_c^+ &= \inf\{t\ge 0; V^+(t)-\underline{V}^+(t)\ge c\},\\ m_c^+ &= \sup\{t\le\underline{\tau}_c^+; V^+(t)-\underline{V}^+(t)=0\}\text{ and}\\ J_c^+ &= \big(V^+(m_c^+)+c\big)\vee\overline{V}^+(m_c^+)\end{aligned}$$

and similar variables defined using V^-, $\underline{\tau}_c^-$, m_c^- and J_c^-. As said before, we can prove that the minimum of the standard valley $\mathfrak{m}_c$ is equal to m_c^+ if $J_c^+<J_c^-$ and $-m_c^-$ otherwise.

Proposition 3.3.8. *Let V be a stable Lévy process with index $\alpha\in[1,2]$ which is not a pure drift. We fix $r\in(0,1)$.*

When $c\to\infty$, the process $\dfrac{e^{-V_{\mathfrak{m}_c}}}{\int_{a_{rc}}^{b_{rc}} e^{-V_{\mathfrak{m}_c}(y)}dy}$ converges weakly (in the Skorohod sense) to $\dfrac{e^{-\tilde V}}{\int_{-\infty}^{+\infty} e^{-\tilde V(y)}dy}$ where the law of $\tilde V$ is $\tilde{\mathbb{P}}$ and, under $\tilde{\mathbb{P}}(d\omega)$, ω^+ and ω^- are independent, and distributed respectively as $\mathbb{P}^{\uparrow}$, the law of the Lévy process V conditioned to stay positive, and $\hat{\mathbb{P}}^{\uparrow}$ the law of the dual process V^- conditioned to stay positive.

In order to prove Proposition 3.3.8, we need the following Lemmas. In the first one, we prove that the integral $\int_{-\infty}^{+\infty} e^{-\tilde V(y)}dy$ is finite.

Lemma 3.3.9.

$$\text{We have } \tilde{\mathbb{P}}\text{-a.s. } \int_{-\infty}^{+\infty} e^{-\tilde V(y)}dy<\infty.$$

Proof. It is enough to prove the integrability on $\mathbb{R}_+$ for the process $\hat V^{\uparrow}$ whose law is $\hat{\mathbb{P}}^{\uparrow}$. Fix $t\ge 0$ and recall that $U(t)=-V(L^{-1}(t))$. We prove the integrability of e^{-U_t}. The process U is a subordinator, then it is characterized by its Laplace exponent ψ_U (see Section III.1, [13]): for all $\lambda \geq 0$,

$$\mathbb{E}\left[e^{-\lambda U(t)}\right] = e^{-t\psi_U(\lambda)}.$$

For all $t > 0$, $U(t) > 0$ then $\psi_U(1) > 0$, we obtain that $\mathbb{E}\left[e^{-U(t)}\right] = e^{-t\psi_U(1)}$ is integrable on $\mathbb{R}_+$. We recall the notation of the Proposition 3.2.2, we get

$$U(t) = -\underline{V}(d(t)) \leq \mathcal{R}_{V-\underline{V}}(t) - \underline{V}(d(t)).$$

Thanks to this Proposition, the process $(\mathcal{R}_{V-\underline{V}}(t) - \underline{V}(d(t)); t \geq 0)$ is equal in law to the process $\hat{V}^{\uparrow}$. Thus, $\mathbb{E}\left[e^{-\hat{V}^{\uparrow}(t)}\right]$ is also integrable on $\mathbb{R}_+$, we deduce that $e^{-\hat{V}^{\uparrow}}$ is $\hat{\mathbb{P}}^{\uparrow}$-a.s. integrable.

□

Lemma 3.3.10. *Let ω be a non-negative càd-làg function with $\omega(0) = 0$ which is continuous in its extrema and such that 0 is the unique point of minimum of ω. Then for all $a \leq \alpha < 0 < \beta \leq b$, we get*

$$\int_a^b e^{-c\omega(y)} dy \underset{c \to +\infty}{\sim} \int_\alpha^\beta e^{-c\omega(y)} dy.$$

Proof.

$$\begin{aligned}\int_a^b e^{-c\omega(y)} dy - \int_\alpha^\beta e^{-c\omega(y)} dy &= \int_a^\alpha e^{-c\omega(y)} dy + \int_\beta^b e^{-c\omega(y)} dy \\ &\leq |\alpha - a| e^{-c \inf_{y \in (a,\alpha)} \omega(y)} + |b - \beta| e^{-c \inf_{y \in (\beta,b)} \omega(y)}.\end{aligned}$$

As ω is positive on $\mathbb{R} \setminus \{0\}$ and continuous at its extrema, for all $x \neq 0$, $\inf_{(a,\alpha)} \omega > 0$ and $\inf_{(\beta,b)} \omega > 0$ (see Remark 3), then $\lim_{c \to +\infty} \int_a^b e^{-c\omega(y)} dy - \int_\alpha^\beta e^{-c\omega(y)} dy = 0$.

By the Laplace method,

$$\lim_{c \to +\infty} \frac{1}{c} \log \int_a^b e^{-c\omega(y)} dy = \sup_{(a,b)} (-\omega).$$

As $a \leq 0 \leq b$ then $\sup_{(a,b)}(-\omega) = 0$, thus $\int_a^b e^{-c\omega(y)} dy = e^{o(c)}$.

$$\lim_{c \to +\infty} \frac{\int_a^b e^{-c\omega(y)} dy - \int_\alpha^\beta e^{-c\omega(y)} dy}{\int_a^b e^{-c\omega(y)} dy} \leq \lim_{c \to +\infty} e^{-c \inf_{(a,\alpha)} \omega - c \inf_{(\beta,b)} \omega - o(c)} = 0.$$

This implies the equivalence. □

Proof of Proposition 3.3.8. We follow the same idea as in the proof of Proposition 3.2.7.

It is enough to prove that for a bounded continuous function F in the Skorohod topology and such that $F(\omega) = F(\omega \mathbf{1}_{[-K,K]})$,

$$\lim_{c\to+\infty} \mathbb{E}\left[F\left(\frac{e^{-V_{\mathfrak{m}_c}}}{\int_{a_{rc}}^{b_{rc}} e^{-V_{\mathfrak{m}_c}(y)}dy}\right)\right] = \mathbb{E}\left[F\left(\frac{e^{-\tilde{V}}}{\int_{-\infty}^{+\infty} e^{-\tilde{V}(y)}dy}\right)\right].$$

Using Formula (3.11) and the fact that $\mathfrak{m}_1 \in \{-m_1^-, m_1^+\}$,

$$\mathbb{E}\left[F\left(\frac{e^{-V_{\mathfrak{m}_c}}}{\int_{a_{rc}}^{b_{rc}} e^{-V_{\mathfrak{m}_c}(y)}dy}\right)\right] = \mathbb{E}\left[F\left(\frac{e^{-cV_{\mathfrak{m}_1}(c^{-\alpha}\cdot)}}{c^\alpha \int_{a_r}^{b_r} e^{-cV_{\mathfrak{m}_1}(y)}dy}\right), \mathfrak{m}_1 = m_1^+\right] +$$
$$\mathbb{E}\left[F\left(\frac{e^{-cV_{\mathfrak{m}_1}(c^{-\alpha}\cdot)}}{c^\alpha \int_{a_r}^{b_r} e^{-cV_{\mathfrak{m}_1}(y)}dy}\right), \mathfrak{m}_1 = -m_1^-\right].$$

We will compute the two terms of the sum by the same method, thus we only study the first one. Thanks to Lemma 3.3.10, it suffices to take the integral from $-\mathfrak{m}_1 \vee a_r$ to b_r. By definition, the law of the process $\left(\tilde{V}(t), t\geq 0\right)$ is $\mathbb{P}^\uparrow$ and the law of the process $\left(\tilde{V}((-t)-), t\geq 0\right)$ is $\hat{\mathbb{P}}^\uparrow$, moreover they are independent. Then we write them respectively $V^\uparrow$ and $\hat{V}^\uparrow$. Using the fact that $\mathfrak{m}_1 = m_1^+$ if and only if $J_1^+ < J_1^-$ and the result of Proposition 3.2.2, we get that

$$\mathbb{E}\left[F\left(\frac{e^{-cV_{\mathfrak{m}_1}(c^{-\alpha}\cdot)}}{c^\alpha \int_{-\mathfrak{m}_1\vee a_r}^{b_r} e^{-cV_{\mathfrak{m}_1}(y)}dy}\right), \mathfrak{m}_1 = m_1^+, \frac{K}{c^\alpha} \leq \mathfrak{m}_1 \wedge \tau_1(\underset{\rightarrow}{V})\right] =$$

$$\mathbb{E}\left[F\left(\frac{e^{-c\tilde{V}(c^{-\alpha}\cdot)}}{c^\alpha \int_{-\tilde{a}_r}^{\tilde{b}_r} e^{-c\tilde{V}(y)}dy}\right) f_1\left(V^\uparrow(\tau_1)\right), \tilde{J}_1^+ \leq \tilde{J}_1^-, \frac{K}{c^\alpha} \leq \hat{m}_1^\uparrow \wedge \tau_1(V^\uparrow)\right] \tag{3.12}$$

where $\tilde{a}_r$, $\tilde{b}_r$, $\tilde{J}_1^+$ and $\tilde{J}_1^-$ are defined similarly as a_r, b_r, J_1^+ and J_1^- but using the process $\tilde{V}$:

$$
\begin{aligned}
\tilde{a}_r &:= \hat{m}_1^\uparrow \vee \inf \left\{ x \geq 0 / \hat{V}^\uparrow(x) > r \right\}, \\
\tilde{b}_r &:= \inf \left\{ x \geq 0 / V^\uparrow(x) > r \right\}, \\
\tilde{J}_1^+ &:= \left(1 - \hat{V}^\uparrow(\hat{m}_1^\uparrow)\right) \vee \left(\sup_{(0,\hat{m}_1^\uparrow)} \hat{V}^\uparrow - \hat{V}^\uparrow(\hat{m}_1^\uparrow) \right), \\
\text{and } \tilde{J}_1^- &:= \left(1 - W^\uparrow(m_1^\uparrow)\right) \vee \left(\sup_{(0,m_1^\uparrow)} W^\uparrow - W^\uparrow(m_1^\uparrow) \right)
\end{aligned}
$$

where $W^\uparrow$ is a process with law $\mathbb{P}^\uparrow$ independent of $\tilde{V}$. To simplify the notation, we write $F_{\tilde{a}_r,\tilde{b}_r}$ for $F\left(\frac{e^{-c\tilde{V}(c^{-\alpha}\cdot)}}{c^\alpha \int_{-\tilde{a}_r}^{\tilde{b}_r} e^{-c\tilde{V}(y)} dy} \right)$.

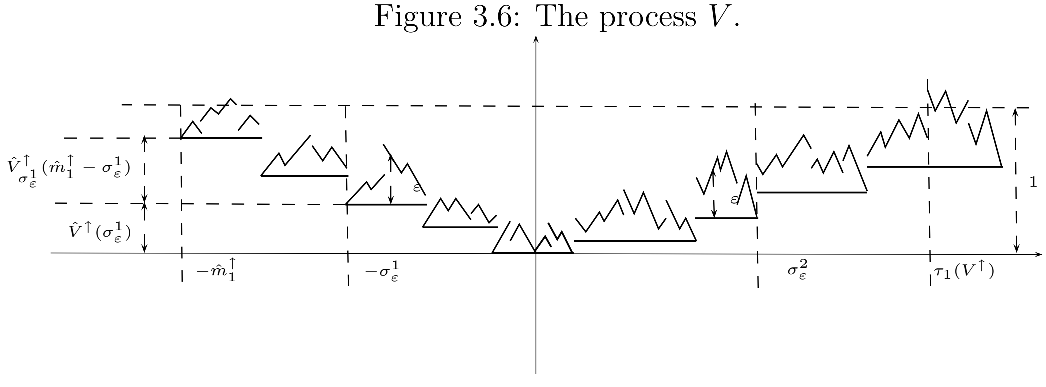
As in the proof of Proposition 3.2.7, we fix $\varepsilon > 0$ and we separate the several terms of the expectation with respect to $(\tilde{V}(t), t \leq -\sigma_\varepsilon^1)$, $(\tilde{V}(t), -\sigma_\varepsilon^1 \leq t \leq \sigma_\varepsilon^2)$ and $(\tilde{V}(t), \sigma_\varepsilon^2 \leq t)$, where:

$$
\sigma_\varepsilon^1 = \sigma_\varepsilon(\hat{V}^\uparrow) = \inf \left\{ t > 0 / \hat{V}^\uparrow(t) - \underline{\underline{\hat{V}^\uparrow}}(t) = 0 \text{ and } \exists s < t, \hat{V}^\uparrow(s) - \underline{\underline{\hat{V}^\uparrow}}(s) > \varepsilon \right\},
$$

$$
\sigma_\varepsilon^2 = \sigma_\varepsilon(V^\uparrow) = \inf \left\{ t > 0 / V^\uparrow(t) - \underline{\underline{V^\uparrow}}(t) = 0 \text{ and } \exists s < t, V^\uparrow(s) - \underline{\underline{V^\uparrow}}(s) > \varepsilon \right\},
$$

to obtain the asymptotic independence of $F_{\tilde{a}_r,\tilde{b}_r}$, of $V^\uparrow(\tau_1)$ and of $\tilde{J}_1^+$.

Figure 3.6: The process $\tilde{V}$.



We need now to study the set $\{\tilde{J}_1^+ \leq \tilde{J}_1^-\}$. Denote

$$
M_+ = \sup_{(0,\hat{m}_1^\uparrow)} \hat{V}^\uparrow - \hat{V}^\uparrow(\hat{m}_1^\uparrow) \text{ and } M_\varepsilon = \sup_{(0,\sigma_\varepsilon^1)} \hat{V}^\uparrow - \hat{V}^\uparrow(\hat{m}_1^\uparrow).
$$

For ε small enough, $M_\varepsilon < M_+$, in this case: $\sigma_\varepsilon^1 < \hat{m}_1^\uparrow$, $\sup_{(0,\hat{m}_1^\uparrow)} \hat{V}^\uparrow = \sup_{(\sigma_\varepsilon^1,\hat{m}_1^\uparrow)} \hat{V}^\uparrow$ and $\hat{V}^\uparrow(\hat{m}_1^\uparrow) = \hat{V}^\uparrow(\sigma_\varepsilon^1) + \hat{V}^\uparrow_{\sigma_\varepsilon^1}(\hat{m}_1^\uparrow - \sigma_\varepsilon^1)$. Moreover, for ε small enough, $\sigma_\varepsilon^2 < \tau_1(V^\uparrow)$, thus $\tilde{J}_1^+ = \phi_\varepsilon\left(\hat{V}^\uparrow(\sigma_\varepsilon^1)\right)$ with

$$\phi_\varepsilon(x) = \left(1 - x - \hat{V}^\uparrow_{\sigma_\varepsilon^1}(\hat{m}_1^\uparrow - \sigma_\varepsilon^1)\right) \vee \left(\sup_{(\sigma_\varepsilon^1,\hat{m}_1^\uparrow)} \hat{V}^\uparrow_{\sigma_\varepsilon^1} + \hat{V}^\uparrow_{\sigma_\varepsilon^1}(\hat{m}_1^\uparrow - \sigma_\varepsilon^1)\right).$$

We follow the idea of the proof of Proposition 3.2.7 with Lemma 3.2.8. We use Lemma 3.3.10 for the integral in the function F and we remark that

$$\lim_{c\to+\infty} \mathbb{P}\left(\frac{K}{c^\alpha} \le \hat{m}_1^\uparrow \wedge \tau_1(V^\uparrow)\right) = \lim_{c\to+\infty} \mathbb{P}\left(\frac{K}{c^\alpha} \le \sigma_\varepsilon^1 \wedge \sigma_\varepsilon^2\right) = 1.$$

Recall (3.12), we obtain

$$\lim_{c\to+\infty} \mathbb{E}\left[F_{\tilde{a}_r,\tilde{b}_r}\, f_1\left(V^\uparrow(\tau_1)\right), \tilde{J}_1^+ \le \tilde{J}_1^-, \frac{K}{c^\alpha} \le \hat{m}_1^\uparrow \wedge \tau_1(V^\uparrow)\right] =$$

$$\lim_{\varepsilon\to0}\lim_{c\to+\infty} \mathbb{E}\left[F_{\sigma_\varepsilon^1,\sigma_\varepsilon^2}\, \chi_2\left(V^\uparrow(\sigma_\varepsilon^2)\right)\, \chi_1\left(\hat{V}^\uparrow(\sigma_\varepsilon^1)\right), \frac{K}{c^\alpha} \le \sigma_\varepsilon^1 \wedge \sigma_\varepsilon^2, \sigma_\varepsilon^1 \le \hat{m}_1^\uparrow, \sigma_\varepsilon^2 \le \tau_1(V^\uparrow)\right]$$

where

$$\chi_1(x) = \mathbb{P}\left(\left(1 - x - \hat{V}^\uparrow(\hat{m}_1^\uparrow)\right) \vee M_+ < \tilde{J}_1^-\right) \text{ and}$$
$$\chi_2(x) = \mathbb{E}\left[f_1\left(x + V^\uparrow(\tau_{1-x})\right)\right].$$

Moreover, $\lim_{\varepsilon\to0} \chi_1\left(\hat{V}^\uparrow(\sigma_\varepsilon^1)\right) = \mathbb{P}(\tilde{J}_1^+ \le \tilde{J}_1^-) = \mathbb{P}(\mathfrak{m}_1 > 0)$ and thanks to (3.6), $\lim_{\varepsilon\to0} \chi_2\left(V^\uparrow(\sigma_\varepsilon^2)\right) = 1$. Then we obtain:

$$\lim_{\varepsilon\to0}\lim_{c\to+\infty} \left|\mathbb{E}\left[F_{\tilde{a}_r,\tilde{b}_r}\left(\mathbb{P}(\mathfrak{m}_1 > 0) - \chi_1\left(\hat{V}^\uparrow(\sigma_\varepsilon^1)\right)\, \chi_2\left(V^\uparrow(\sigma_\varepsilon^2)\right)\right)\right]\right| = 0.$$

We use the stability of $V^\uparrow$ and of $\hat{V}^\uparrow$ (see Lemma 3.2.6). Moreover, for all $r \in (0,1)$, $\lim\limits_{c\to+\infty} \tilde{a}_{cr} = -\infty$ and $\lim\limits_{c\to+\infty} \tilde{b}_{cr} = +\infty$ a.s., then $e^{-\tilde{V}} / \int_{a_{rc}}^{b_{rc}} e^{-\tilde{V}}$ converges to $e^{-\tilde{V}} / \int_{-\infty}^{+\infty} e^{-\tilde{V}}$ in the Skorohod topology on $[-K,K]$. Thus

$$\lim_{c\to+\infty} \mathbb{E}\left[F\left(\frac{e^{-cV_{\mathfrak{m}_1}(c^{-\alpha}\cdot)}}{c^\alpha \int_{a_r}^{b_r} e^{-cV_{\mathfrak{m}_1}}}\right), \mathfrak{m}_1 = m_1^+\right] = \mathbb{E}\left[F\left(\frac{e^{-\tilde{V}}}{\int_{-\infty}^{+\infty} e^{-\tilde{V}}}\right)\right] \mathbb{P}(\mathfrak{m}_1 > 0).$$

The same study for the case where $\mathfrak{m}_1 = -m_1^-$ gives

$$\lim_{c\to+\infty} \mathbb{E}\left[F\left(\frac{e^{-cV_{\mathfrak{m}_1}(c^{-\alpha}\cdot)}}{c^\alpha \int_{a_r}^{b_r} e^{-cV_{\mathfrak{m}_1}}}\right), \mathfrak{m}_1 = -m_1^-\right] = \mathbb{E}\left[F\left(\frac{e^{-\tilde{V}}}{\int_{-\infty}^{+\infty} e^{-\tilde{V}}}\right)\right]\mathbb{P}(\mathfrak{m}_1 < 0).$$

And this concludes the proof. $\square$

Using Theorem 3.3.6 and Proposition 3.3.8, we deduce Theorem 3.1.1 given in the introduction.

Chapitre 4

Comportement asymptotique presque sûre du temps local en milieu brownien

4.1 Introduction

Let $(W(x)\ ,\ x \in \mathbb{R})$ be a two-sided one-dimensional Brownian motion on $\mathbf{R}$ with $W(0) = 0$. We call *diffusion process in the environment* W a process $(\mathbf{X}(t), t \in \mathbb{R}^+)$ whose infinitesimal generator given W is

$$\frac{1}{2} e^{W(x)} \frac{d}{dx} \left(e^{-W(x)} \frac{d}{dx} \right).$$

Notice that if we take W differentiable, $(\mathbf{X}(t), t \in \mathbb{R}^+)$ would be solution of the following stochastic differential equation

$$\begin{cases} \mathrm{d}\mathbf{X}(t) = \mathrm{d}\beta(t) - \frac{1}{2} W'(\mathbf{X}(t))\mathrm{d}t, \\ \mathbf{X}(0) = 0 \end{cases}$$

in which β is a standard one-dimensional Brownian motion independent of W. Of course as we choose for W a Brownian motion, the previous equation does not have any rigorous sense but it explains the denomination *environment* for W.

This process was first introduced by Schumacher [69] and Brox [17]. The diffusion $\mathbf{X}$ is recurrent and sub-diffusive with asymptotic behavior in $(\log t)^2$. Moreover Brox showed in [17] that $\mathbf{X}$ has the property to be localized in the neighborhood of a point $m_{\log t}$ only depending on t and W. Later, Tanaka [87, 85] and Hu [39] obtained deeper localization results. The limit law of

$m_{\log t}/(\log t)^2$ and therefore of $\mathbf{X}(t)/(\log t)^2$ is independently made explicit by Kesten [52] and Golosov [37].

Kesten and Golosov's aim was actually to determine the limit law of a random walk in random environment, introduced by Solomon [77], which is often considered as the discrete analogue of Brox's model. Sinai [74] proved that this random walk $(S_n, n \in \mathbb{N})$, now called Sinai's walk, has the same limit distribution as Brox's. Hu and Shi [40] got the almost sure rates of convergence of the lim sup and lim inf of $\mathbf{X}$ and S. It appears that these rates are the same.

In the present paper, we are interested in the local time of the diffusion $\mathbf{X}$. This process, denoted $(\mathbf{L}(t,x), t \geq 0, x \in \mathbb{R})$, is the density of the occupation measure of $\mathbf{X}$: $\mathbf{L}$ is the unique a.s. jointly continuous process such that for each Borel set A and for any $t \geq 0$,

$$\nu_t(A) := \int_0^t \mathbf{1}_A(\mathbf{X}_s)\mathrm{d}s = \int_{\mathbb{R}} \mathbf{1}_A(x)\mathbf{L}(t,x)\mathrm{d}x. \tag{4.1}$$

We shall see later that such a process exists. The first results on the behavior of $\mathbf{L}$ can be found in [71] and [41]. In particular Hu and Shi proved in [41] that for any $x \in \mathbb{R}$,

$$\frac{\log(\mathbf{L}(t,x))}{\log t} \xrightarrow{\mathcal{L}} U \wedge \hat{U},\ t \to +\infty$$

where U and $\hat{U}$ are independent variables uniformly distributed in $(0,1)$ and $\xrightarrow{\mathcal{L}}$ is the convergence in law. Note that in the same paper it is also proved that Sinai's walk local time ξ has the same behavior. The process ξ is the time spent by S at x before time n for $n \in \mathbf{N}$ and $x \in \mathbb{Z}$: $\xi(n,x) := \sum_{k=0}^n \mathbf{1}_{S_k = x}$. This result shows that the local time in a fixed point can vary a lot. So to study localization, a good quantity to look at is the maximum of the local time of $\mathbf{X}$,

$$\mathbf{L}^*(t) := \sup_{x \in \mathbb{R}} \mathbf{L}(t,x).$$

Shi was the first one to be interested in this process; in [71] he gave a lower bound on the lim sup behavior,

$$\limsup_{t \to \infty} \frac{\mathbf{L}^*(t)}{t \log_3(t)} \geq \frac{1}{32}$$

where for any $i \in \mathbf{N}^*$, $\log_{i+1} = \log \circ \log_i$ and $\log_1 = \log$. In the same paper, he computed the similar rate in the discrete case: define for $n \in \mathbf{N}$, $\xi^*(n) := \sup_{x \in \mathbb{Z}} \xi(n,x)$, there is constant $c \in (0,\infty)$ such that

$$\limsup_{n \to \infty} \frac{\xi^*(n)}{n} = c.$$

Thereby he highlighted a different behavior for the discrete model and for the continuous one. The limit law of $\mathbf{L}^*(t)$ was then determined in [7]:

$$\frac{\mathbf{L}^*(t)}{t} \xrightarrow{\mathcal{L}} \frac{1}{\int_{-\infty}^{\infty} e^{-\widetilde{W}(x)}\mathrm{d}x}, \quad t \to +\infty$$

where $\widetilde{W}$ has the same law as the environment conditioned to stay positive. But unlike the discrete case (see [34]), this result does not allow to obtain an upper bound on the almost sure behavior.

Here we prove that the quantity $\log_3(t)$ is the correct renormalization for the lim sup and an analog result for the lim inf:

Theorem 4.1.1. *Almost surely,*

$$\frac{1}{32} \leq \limsup_{t\to\infty} \frac{\mathbf{L}^*(t)}{t\log_3(t)} \leq e^2/2 \text{ and}$$
$$\frac{j_0^2}{64} \leq \liminf_{t\to\infty} \frac{\log_3(t)\mathbf{L}^*(t)}{t} \leq \frac{e^2\pi^2}{4}.$$

where j_0 is the smallest strictly positive root of Bessel function J_0.

We can compare these results to the ones in the discrete case, see [71] (and [34] for the value of the constant) for the lim sup and [25] for the lim inf: there exists two constants $c, c' \in (0,\infty)$ such that almost surely,

$$\limsup_{n\to\infty} \frac{\xi^*(n)}{n} = c \quad \text{and} \quad \liminf_{n\to\infty} \frac{\log_3(n)\xi^*(n)}{n} = c'.$$

As Shi noted it, the lim sup in the continuous case and in the discrete case have a different normalization, but we see with Theorem 4.1.1 that the normalization is the same for the lim inf.

There is the following heuristic interpretation: in the discrete case like in the continuous one, $\mathbf{L}^*(t)/t$ behaves approximately in the best case like the inverse of the integral of the exponential of the "steepest" environment and in the worst case like the inverse of the integral of the exponential of the "flattest" one. However if in discrete case there is a steepest environment, in the continuous case it is possible to be as steep as we want, and in the two cases, the environment can be as flat as we want. This explains the difference for the lim sup. To compute these rates, we need to study carefully both the behavior of the "steepest" and "flattest" environment and the place where the diffusion spent the most of its time. Indeed, as we said, for large t the process $\mathbf{X}$ is in the neighborhood of a point $m_{\log t}$. This event happens with a probability which tends to 1 when t goes to infinity, however it does not grow fast enough with t to derive almost sure results. So here we relax the localization of the particle to the neighborhood of four points instead of one. Then we obtain the following theorem:

Theorem 4.1.2. *Fix $\epsilon > 0$ and $c_0 > 0$. There are four processes m_t^1, m_t^2, m_t^3 and m_t^4 only depending on the environment W such that if we denote*

$$I_t := \bigcup_{i=1}^{4} [m_t^i - (\log_2 t)^{4+\epsilon},\ m_t^i + (\log_2 t)^{4+\epsilon}],$$

then, almost surely,

$$\lim_{t \to \infty} \frac{(\log t)^{c_0}}{t} \nu_t \left(\mathbb{R} \setminus I_t \right) = 0.$$

That is to say, for large t the diffusion has spent much of its time in the neighborhood of at most four points.

The rest of the paper is organized as follows: in Section 4.2, we present some technical tools used in the article, in Section 4.3, we show that environment W has "good" properties with a large enough probability, Section 4.4 is centered on the study of the local time of the process $\mathbf{X}$ at properly chosen random times, then in Section 4.5, we look at the behavior of these random times to deduce results in deterministic time and finally in Section 4.6, precise almost sure asymptotic result on the environment are given to finish the proofs of Theorems 4.1.1 and 4.1.2.

4.2 Three useful theorems and some technical estimates

We begin with a useful representation of $\mathbf{X}$ we use throughout the article. The martingale representation theorem says that, given the environment W, $\mathbf{X}$ is a Brownian motion rescaled in time and space. Precisely, there is a Brownian motion B started at 0, independent of W such that if we define the scale function

$$\forall x \in \mathbb{R}, S_W(x) := \int_0^x e^{W(y)} \mathrm{d}y$$

and the random time change

$$\forall t \geq 0, T_{W,B}(t) := \int_0^t e^{-2W(S_W^{-1}(B(s)))} \mathrm{d}s,$$

then

$$\mathbf{X} = S_W^{-1} \circ B \circ T_{W,B}^{-1}. \tag{4.2}$$

To simplify notations, we write S and T for respectively S_W and $T_{W,B}$. Using (4.2), we easily obtain that a continuous function $\mathbf{L}$ verifies (4.1) only if

$$\forall x \in \mathbb{R},\ \forall t \geq 0,\ \mathbf{L}(t,x) = e^{-W(x)} L(T^{-1}(t), S(x)) \tag{4.3}$$

where L is the local time process of the Brownian motion B. And so the local time of $\mathbf{X}$ is defined correctly. Here are presented some results used throughout the article. We begin the section with some notations :

$$\begin{gathered}
\forall x \geq 0,\ \boldsymbol{\tau}(x) := \inf\{t \geq 0 / \mathbf{X}(t) = x\}, \\
\forall x \geq 0,\ \tau(x) := \inf\{t \geq 0 / B(t) = x\}, \\
\forall x \geq 0, \forall r \geq 0,\ \sigma_{X_\alpha}(r,x) := \inf\{t \geq 0 / \mathbf{L}(t,x) \geq r\} \text{ and} \\
\forall x \geq 0, \forall r \geq 0,\ \sigma(r,x) := \inf\{t \geq 0 / L(t,x) \geq r\}.
\end{gathered}$$

More generally, the quantities related to $\mathbf{X}$ are written in bold font and the ones related to B in normal font. Furthermore, in the rest of the paper, letter K stands for a universal constant whose value can change from one line to another and for any process M, $\tau_M(x)$ will denote the hitting time of height x by M. We also denote by $\mathbb{P}$ the total probability and by P^W the probability given the environment W.

As said before, the study of the process $\mathbf{X}$ is reduced by a change in time and space to the study of a Brownian motion. We therefore use in our proofs the following two theorems (see [67]) that describe the law of the local time of a Brownian motion stopped at some properly chosen random times:

Theorem 4.2.1 (Ray-Knight)**.** *Let a be a positive real number. The process $(L(\tau(a), a-y),\ y \in [0,a])$ is a square of a 2-dimensional Bessel process started at 0. And conditionally on $L(\tau(a),0)$, $(L(\tau(a),-y),\ y \geq 0)$ is a square of a 0-dimensional Bessel process started at $L(\tau(a),0)$ independent of $(L(\tau(a),y),\ y \geq 0)$.*

Theorem 4.2.2 (Ray-Knight)**.** *Let r be a positive real number. The processes $(L(\sigma(r,0),y),\ y \in \mathbb{R}^+)$ and $(L(\sigma(r,0),-y),\ y \in \mathbb{R}^+)$ are two independent squares of 0-dimensional Bessel processes started at r.*

In the rest of the article, we denote by Z a square of a 0-dimensional Bessel process started at 1 and by Q a square of a 2-dimensional Bessel process started at 0. To use the previous theorems, we have to estimate the behavior of these two processes.

Lemma 4.2.3. *For all $v, \delta, M > 0$, we have* *(i)* $\mathbb{P}\left(\sup_{0\le t\le v}|Z(t)-1|\ge\delta\right)\le 4\frac{\sqrt{(1+\delta)v}}{\delta}\exp\left(-\frac{\delta^2}{8(1+\delta)v}\right)$,

(ii) $\mathbb{P}\left(\sup_{t\ge 0}Z(t)\ge M\right)=\frac{1}{M}$,

(iii) $\mathbb{P}\left(\sup_{0\le t\le v}Q(t)\ge M\right)\le 4e^{-\frac{M}{2v}}$,

(iv) *There is a constant* $K>0$ *such that for any* $0<a<b$,

$$\mathbb{P}\left(\sup_{a\le t\le b}\frac{1}{t}Q(t)\ge M\right)\le Ke^{-\frac{M}{2\log(8b/a)}}.$$

Proof. (i) and (ii) are proved in [84] Lemma 3.1 (the results are stated for a Bessel process but they are actually true for a *squared* Bessel process).

(iii) Denote by B and $\tilde{B}$ two independent Brownian motions. The processes Q and $B^2+\tilde{B}^2$ have the same law, so

$$\begin{aligned}\mathbb{P}\left(\sup_{0\le t\le v}Q(t)\ge M\right)&\le\mathbb{P}\left(\left(\sup_{0\le t\le v}|B|(t)\right)^2+\left(\sup_{0\le t\le v}|\tilde{B}|(t)\right)^2\ge M\right)\\&\le 4\mathbb{P}\left(\left(\sup_{0\le t\le v}B(t)\right)^2+\left(\sup_{0\le t\le v}\tilde{B}(t)\right)^2\ge M\right)\end{aligned}$$

where the second inequality comes from the reflection principle. For a fixed v, $\sup_{0\le s\le v}B(s)\overset{\mathcal{L}}{=}|B(v)|$, then

$$\mathbb{P}\left(\sup_{0\le t\le v}Q(t)\ge M\right)\le 4\mathbb{P}\left(Q(v)\ge M\right)=4e^{-\frac{M}{2v}}.$$

as $Q(v)$ has exponential distribution of mean $2v$.

(iv) Thanks to the scaling property of Q,

$$\begin{aligned}\mathbb{P}\left(\sup_{a\le t\le b}\frac{1}{t}Q(t)\ge M\right)&=\mathbb{P}\left(\sup_{a/b\le t\le 1}\frac{1}{t}Q(t)\ge M\right)\\&\le\mathbb{P}\left(\sup_{0\le t\le 1}\sqrt{\frac{Q(t)}{t\log(8/t)}}\ge\sqrt{\frac{M}{\log(8b/a)}}\right).\end{aligned}$$

We then conclude with Lemma 6.1 in [40] which says that there exist a constant $K>0$ such that for all $x>0$,

$$\mathbb{P}\left(\sup_{0\le t\le 1}\sqrt{\frac{Q(t)}{t\log(8/t)}}\ge x\right)\le Ke^{-\frac{x^2}{2}}.$$

$\square$

We also need to study the behavior of the environment W. We start with a notation for the minimum of W on an interval

$$\underline{W}(x,y) := \begin{cases} \min_{z\in[x,y]} W(z) & \text{if } x \le y \\ +\infty & \text{if not} \end{cases},$$

another one for the maximum

$$\overline{W}(x,y) := \begin{cases} \max_{z\in[x,y]} W(z) & \text{if } x \le y \\ -\infty & \text{if not} \end{cases}$$

and a last one for the environment reversed in time

$$\left(\widehat{W}(x),\ x \in \mathbb{R}\right) := (W(-x),\ x \in \mathbb{R}).$$

Define now

$$\begin{aligned} H_v &:= \inf\{x \ge 0\ ;\ W(x) - \underline{W}(0,x) \ge v\}, \\ m_v &:= \inf\{x \ge 0\ ;\ W(x) = \underline{W}(0,H_v)\} \end{aligned}$$

and $\widehat{H}_v$ and $\widehat{m}_v$ the corresponding points for $\widehat{W}$. Brox showed in [17] that at time e^v, with high probability, the process $\mathbf{X}$ has spent much of its time in the neighborhood of m_v or of $\widehat{m}_v$. For our study, we need to know the law of the environment in the neighborhood of these points; it is given by the following theorem due to Tanaka (Lemma 3.1 in [87], see also the proposition page 164 in [85]).

Theorem 4.2.4. *Let R be a Bessel process of dimension 3 started at 0 and define*

$$\begin{aligned} \tau_R(v) &:= \inf\{x \ge 0/R(x) \ge v\}, \\ \zeta_R(v) &:= \inf\{x \ge 0/R(x) - \inf_{y\ge x} R(y) \ge v\} \text{ and} \\ \rho_R(v) &:= \sup\{x \le \zeta_R(v)/R(x) - \inf_{y \ge x} R(y) = 0\}. \end{aligned}$$

Under $\mathbb{P}$, the process $(W(-x+m_v) - W(m_v),\ x \in [0,m_v])$ and the process $(W(x+m_v) - W(m_v),\ x \in [0,H_v - m_v])$ are independent and the following equalities in law hold:

$$\big(W(-x+m_v) - W(m_v),\ x \in [0,m_v]\big) \overset{\mathcal{L}}{=} \big(R(x),\ x \in [0,\rho_R(v)]\big)$$

and

$$\big(W(x+m_v) - W(m_v),\ x \in [0,H_v - m_v]\big) \overset{\mathcal{L}}{=} \big(R(x),\ x \in [0,\tau_R(v)]\big).$$

Therefore, to use this theorem it is necessary to have informations on the behavior of Bessel processes of dimension 3.

Lemma 4.2.5. *Let R be a 3-dimensional Bessel process started in 0. There is a positive real number K such that for every $a, x > 0$,*

(i) $\displaystyle \frac{a}{\sqrt{x}} e^{-a^2/2x} \leq \mathbb{P}\left(\sup_{[0,x]} R > a\right) \leq K\left(\frac{a}{\sqrt{x}} + \frac{\sqrt{x}}{a}\right) e^{-a^2/2x}.$

(ii) $\displaystyle \mathbb{P}\left(\sup_{[0,x]} R < a\right) \geq \frac{1}{K} e^{-\pi^2 x/(2a^2)}$

(iii) $\displaystyle \mathbb{P}\left(\int_0^\infty e^{-R(x)}\,dx > a\right) \leq K e^{-j_0^2 a/8}$ *where j_0 is the smallest strictly positive root of Bessel function J_0.*

Proof. *(i)* According to the reflection principle for Brownian motion, one can find a $K > 0$ such that

$$\mathbb{P}\left(\sup_{[0,x]} R > a\right) \leq K\mathbb{P}\left(R(x) > a\right).$$

Moreover, $\mathbb{P}\left(\sup_{[0,x]} R > a\right) \geq \mathbb{P}\left(R(x) > a\right)$. So Item *(i)* of the lemma is a consequence of usual estimates for 3-dimensional Bessel processes.

(ii) Recall the Bessel function of the first kind (see [1] chapter 9)

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x,$$

its smallest positive root is π. Then, according to Theorem 2 of [93], there is a positive number K such that

$$\mathbb{P}\left(T_R(1) \geq x\right) \sim \frac{1}{K} e^{-\pi^2 x/2}.$$

Then, (the value of K can change) $\mathbb{P}\left(T_R(1) \geq x\right) \geq K^{-1} e^{-\pi^2 x/2}$ and

$$\mathbb{P}\left(\sup_{[0,x]} R > a\right) = \mathbb{P}\left(T_R(a) \geq x\right) = \mathbb{P}\left(T_R(1) \geq x/a^2\right) \geq \frac{1}{K} e^{-\pi^2 x/(2a^2)}.$$

(iii) Le Gall's Ray-Knight theorem (Proposition 1.1 of [33]) shows that $1/4 \int_0^\infty e^{-R(x)}\mathrm{d}x$ has the same law as $T_Q(1)$, the hitting time of height 1 by a squared Bessel process of dimension 2 started at 0. Then according to Theorem 2 of [93], as in the proof of the previous item,

$$\mathbb{P}\left(\int_0^\infty e^{-R(x)}\mathrm{d}x > a\right) = \mathbb{P}\left(T_Q(1) > \frac{a}{4}\right) \leq K e^{-j_0^2 a/8}.$$

□

4.3 Estimates on the environment

The process $\widehat{W}$ has the same law as W, this allows to restrict the study to W on $\mathbb{R}^+$ and to get similar results on $\mathbb{R}^-$ by symmetry.

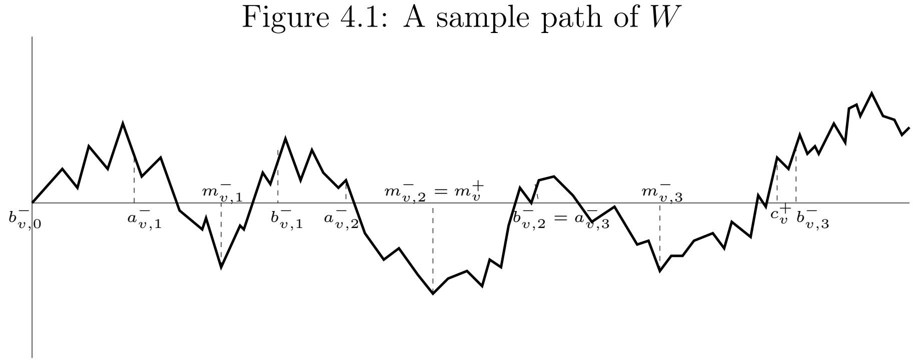
We study the environment on $[0, H_v]$, the *valley* of height v and particularly in the neighborhood of m_v as it is the place where the diffusion spends most of its time. Unfortunately, the probability that at time e^v, the process has reached the bottom m_v and has not left the valley is not growing fast enough to derive almost sure results. So we rather study the valley of height $v - c_1 \log v$, where c_1 is a positive real number whose value will be determined later, so that, with high probability, at time e^v, the process has reached the bottom of this valley and the valley of height $v + c_3 \log v$ so that the process is still inside at time e^v. We therefore fix three constants $c_1, c_2, c_3 > 0$ with $c_1 \geq c_2$ and define recursively for any $v > 1$, $b^-_{v,0} := 0$ and

$$\begin{aligned} \forall i \in \mathbb{N},\ b^-_{v,i+1} &:= \inf\{x \geq b^-_{v,i}\ ;\ W(x) - \underline{W}(b^-_{v,i}, x) \geq v - c_1 \log v\}, \\ m^-_{v,i+1} &:= \inf\{x \geq b^-_{v,i}\ ;\ W(x) = \underline{W}(b^-_{v,i}, b^-_{v,i+1})\}. \end{aligned}$$

Denote also for any $i \in \mathbb{N}^*$,

$$\begin{aligned} a^-_{v,i} &:= \sup\{x \leq m^-_{v,i}\ ;\ W(x) - W(m^-_{v,i}) \geq v - c_2 \log v\} \vee b^-_{v,i-1}, \\ c^+_v &:= \inf\{x \geq 0\ ;\ W(x) - \underline{W}(0, x) \geq v + c_3 \log v\}, \\ m^+_v &:= \inf\{x \geq 0\ ;\ W(x) = \underline{W}(0, c^+_v)\}, \\ b^+_v &:= \inf\{x \geq m^+_v\ ;\ W(x) - W(m^+_v) \geq v - c_1 \log v\} \text{ and} \\ a^+_v &:= \sup\{x \leq m^+_v\ ;\ W(x) - W(m^+_v) \geq v - c_2 \log v\} \vee 0. \end{aligned}$$

Figure 4.1: A sample path of W



Obviously, there is a $i \in \mathbb{N}^*$ such that $m^+_v = m^-_{v,i}$. We want to prove that, with a probability large enough, $m^+_v \in \{m^-_{v,1}\ ;\ m^-_{v,2}\}$ and moreover that, in the valley $[0, c_v^+]$, the points after b_v^+ are higher than $W(m_v^+)+(c_1+c_3)\log v$. This can be expressed formally as follows:

$$\Gamma_v^1 := \left\{c_v^+ \leq b_{v,3}^- \ ; \ \underline{W}(b_v^+, c_v^+) - W(m_v^+) \geq (c_1+c_3)\log v\right\}.$$

We also need many more technical conditions to ensure that the environment does not stray too far from its average behavior:

$$\begin{aligned}\Gamma_v^2 := &\left\{b_{v,2}^- \leq v^6 \ ; \ W(m_{v,1}^-) \geq -v^2 \ ; \ W(m_{v,2}^-) - W(b_{v,1}^-) \geq -v^2\right\} \\ &\bigcap \left\{m_{v,1}^- - a_{v,1}^- \geq \frac{1}{v^2} \ ; \ m_{v,2}^- - a_{v,2}^- \geq \frac{1}{v^2}\right\} \\ &\bigcap \left\{c_v^+ - m_v^+ \geq v \ ; \ W(c_v^+) - \underline{W}((c_v^+ - \log v) \vee m_v^+, c_v^+) \leq 2\log v\right\} \\ &\bigcap \left\{\overline{W}((m_{v,1}^- - \log v) \vee a_{v,1}^-, m_{v,1}^-) - W(m_{v,1}^-) \leq 2\log v\right\} \\ &\bigcap \left\{\overline{W}((m_{v,2}^- - \log v) \vee a_{v,2}^-, m_{v,2}^-) - W(m_{v,2}^-) \leq 2\log v\right\}.\end{aligned}$$

We would also wish sometimes that $m_v^+ = m_{v,1}^-$ (which of course is obtained with a probability lower than the previous one). For that we use the event:

$$\Gamma_v^3 := \left\{c_v^+ \leq b_{v,2}^- \ ; \ \underline{W}(b_v^+, c_v^+) - W(m_v^+) \geq (c_1+c_3)\log v\right\}.$$

Define similarly $\widehat{\Gamma}_v^1$, $\widehat{\Gamma}_v^2$ and $\widehat{\Gamma}_v^3$ from $\widehat{W}$.

We will therefore work on

$$\Gamma_v = \Gamma_v^1 \cap \Gamma_v^2 \cap \widehat{\Gamma}_v^1 \cap \widehat{\Gamma}_v^2 \tag{4.4}$$

or on

$$\Gamma'_v = \Gamma_v^3 \cap \Gamma_v^2 \cap \widehat{\Gamma}_v^3 \cap \widehat{\Gamma}_v^2. \tag{4.5}$$

The first step, as stated before, is to show that these events occur with a high enough probability. We denote for every event A, $\overline{A} := \Omega \setminus A$.

Proposition 4.3.1. *There exists a constant $K > 0$ such that for v large enough,*

$$\mathbb{P}(\overline{\Gamma}_v) \leq K\left(\frac{\log v}{v - c_1\log v}\right)^2 \ \textit{and} \ \mathbb{P}(\overline{\Gamma'}_v) \leq \frac{K\log v}{v - c_1\log v}.$$

Proof. Start with the upper bound for $\mathbb{P}(\overline{\Gamma}_v^1)$. Define

$$\begin{aligned}W_1 &:= (W(b_{v,1}^- + x) - W(b_{v,1}^-))_{x\in[0, b_{v,2}^- - b_{v,1}^-]} \text{ and} \\ W_2 &:= (W(b_{v,2}^- + x) - W(b_{v,2}^-))_{x\in[0, b_{v,3}^- - b_{v,2}^-]}.\end{aligned}$$

The event $\left\{c_v^+ > b_{v,3}^-\right\}$ is included in

$$\left\{\sup_{[0,b_{v,2}^- - b_{v,1}^-]} W_1 \leq (c_1+c_3)\log v \ ; \ \sup_{[0,b_{v,3}^- - b_{v,2}^-]} W_2 \leq (c_1+c_3)\log v\right\}.$$

The processes W_1 et W_2 are independent and are distributed like

$$\left(W(x)\ ,\ x \in [0, b_{v,1}^-]\right).$$

Therefore,

$$\begin{aligned}
\mathbb{P}\left(c_v^+ > b_{v,3}^-\right) &\leq \left(\mathbb{P}\left(\sup_{[0,b_{v,1}^-]} W \leq (c_1+c_3)\log v\right)\right)^2 \\
&= \left(\mathbb{P}\left(W \text{ hits } -v + (2c_1+c_3)\log v \text{ before } (c_1+c_3)\log v\right)\right)^2 \\
&= \left(\frac{(c_1+c_3)\log v}{v - c_1\log v}\right)^2.
\end{aligned}$$

For the second part of $\overline{\Gamma}_v^1$, according to Theorem 4.2.4, if we denote by R a Bessel process of dimension 3 started at $v - c_1\log v$, then

$$\begin{aligned}
\mathbb{P}\Big(\underline{W}(b_v^+, c_v^+) - W(m_v^+) < (c_1{+}c_3)\log v\Big)& \\
&= \mathbb{P}\left(\underline{R}(0, \tau_R(v + c_3\log v)) < (c_1+c_3)\log v\right) \\
&= \left(\frac{(c_1+c_3)\log v}{v - c_1\log v}\right)^2.
\end{aligned}$$

For the last equality, see for example Property 2.2.2 of part II, chap 5 in [16]. We obtain in the same way

$$\mathbb{P}(\overline{\Gamma}_v^3) \leq \frac{(c_1+c_3)\log v}{v - c_1\log v}.$$

Continue with an upper bound for $\mathbb{P}(\overline{\Gamma}_v^2)$. As $W - \underline{W}$ has the same law as $|W|$,

$$\begin{aligned}
\mathbb{P}(b_{v,2}^- > v^6) &\leq \mathbb{P}(b_{v,2}^- - b_{v,1}^- > \frac{v^6}{2}) + \mathbb{P}(b_{v,1}^- > \frac{v^6}{2}) \\
&= 2\mathbb{P}(\tau_{|W|}(v - c_1\log v) > \frac{v^6}{2}) \\
&\leq 2\mathbb{P}(\tau_W(v) > \frac{v^6}{2}) \leq \frac{K}{v^2}.
\end{aligned}$$

Moreover, $-W(m_{v,1}^-)$ and $W(b_{v,1}^-) - W(m_{v,2}^-)$ are exponentially distributed with mean $v - c_1 \log v$ (see for example the first lemma of [60]) and are independent. Thus

$$\begin{aligned}
\mathbb{P}\left(W(b_{v,1}^-) - W(m_{v,2}^-) < -v^2\right) &= \mathbb{P}\left(W(m_{v,1}^-) < -v^2\right) \\
&\le \mathbb{P}\left(W(m_{v,1}^-) < -(v - c_1 \log v)^2\right) \\
&= v^{c_1} e^{-v}.
\end{aligned}$$

Thanks to Theorem 4.2.4, still denoting by R a Bessel process of dimension 3 but now started at 0, as $a_{v,i}^- \ge b_{v,i-1}^-$,

$$\begin{aligned}
\mathbb{P}\left(m_{v,i}^- - a_{v,i}^- < \frac{1}{v^2}\right) \le & \mathbb{P}\left(m_{v,i}^- - a_{v,i}^- < \frac{1}{(v - c_1 \log v)^2}\right) \\
\le & \mathbb{P}\left(\tau_R(v - c_1 \log v) < \frac{1}{(v - c_1 \log v)^2}\right) \\
& + \mathbb{P}\left(m_{v,i}^- - b_{v,i-1}^- < \frac{1}{(v - c_1 \log v)^2}\right).
\end{aligned}$$

Yet, according to the scaling property of Brownian motion and a lemma proved by Cheliotis in [20] (claim at the end of the proof of Lemma 13), there is a constant $K > 0$, such that

$$\begin{aligned}
\mathbb{P}\left(m_{v,i}^- - b_{v,i-1}^- < \frac{1}{(v - c_1 \log v)^2}\right) &= \mathbb{P}\left(m_1 < \frac{1}{(v - c_1 \log v)^4}\right) \\
&\le \frac{K}{(v - c_1 \log v)^2}.
\end{aligned}$$

Moreover, Item (i) of Lemma 4.2.5 gives

$$\mathbb{P}\left(\tau_R(v - c_1 \log v) < \frac{1}{(v - c_1 \log v)^2}\right) \le K(v - c_1 \log v)^2 e^{-(v - c_1 \log v)^4/2}.$$

We also obtain the following upper bound:

$$\begin{aligned}
\mathbb{P}\left(c_v^+ - m_v^+ < v\right) &= \mathbb{P}\left(\tau_R(v + c_3 \log v) < v\right) \\
&\le \mathbb{P}\left(\tau_R(v) < v\right) \\
&\le K\sqrt{v} e^{-v/2}.
\end{aligned}$$

It remains to control $\mathbb{P}(W(c_v^+) - \underline{W}((c_v^+ - \log v) \vee m_v^+, c_v^+) > 2 \log v)$. Let $\beta = v + c_3 \log v$. Using one more time Theorem 4.2.4, we see that

$$\begin{aligned}
& W(c_v^+) - \underline{W}((c_v^+ - \log v) \vee m_v^+, c_v^+) \\
&\quad = W(c_v^+) - W(m_v^+) - \left(\min_{t \in [0, \log v \wedge (c_v^+ - m_v^+)]} W((c_v^+ - t) - W(m_v^+)\right)
\end{aligned}$$

has the same law as

$$\beta - \min_{t\in[0,\log v\wedge\tau_R(\beta)]} R(\tau_R(\beta)-t) = \max_{t\in[0,\log v\wedge\tau_R(\beta)]}(\beta - R(\tau_R(\beta)-t)).$$

And according to Proposition 4.8, Chapter VII of [67], the processes

$$(\beta - R(\tau_R(\beta)-t),\ t\in[0,\tau_R(\beta)]) \text{ and } (R(t),\ t\in[0,\tau_R(\beta)])$$

have the same law. Therefore,

$$\mathbb{P}\Big(W(c_v^+) - \underline{W}((c_v^+ - \log v)\vee m_v^+, c_v^+) > 2\log v\Big) \\ = \mathbb{P}\left(\max_{t\in[0,\log v\wedge\tau_R(\beta)]} R(t) > 2\log v\right)$$

and Item (i) of Lemma 4.2.5 implies

$$\mathbb{P}\left(\max_{t\in[0,\log v]} R(t) > 2\log v\right) \le K\frac{\sqrt{\log v}}{v^2}.$$

Finally, we just have to obtain an upper bound for

$$\mathbb{P}(\overline{W}((m_{v,1}^- - \log v)\vee a_{v,1}^-, m_{v,1}^-) - W(m_{v,1}^-) > 2\log v)$$

to prove the proposition. It can be obtained as the previous one. □

We now come back to the local time of the diffusion $\mathbf{X}$.

4.4 Asymptotic behavior of L at particular times

Let r be a positive real number. As for the numbers c_i, its value will be fixed later. Define $\boldsymbol{\sigma}_{v,1}^- := \boldsymbol{\sigma}(re^v, m_{v,1}^-)$ the inverse of local time in $m_{v,1}^-$ and in the same way $\boldsymbol{\sigma}_{v,2}^-$, $\widehat{\boldsymbol{\sigma}}_{v,1}^-$, $\widehat{\boldsymbol{\sigma}}_{v,2}^-$ and $\boldsymbol{\sigma}_v^+$. At these times, it is possible to estimate the local time of $\mathbf{X}$ in the neighborhood of the corresponding point. We first give an estimate at a fixed environment in Proposition 4.4.1, then in Proposition 4.4.4, the estimate is independent of the environment provided that this one belongs to Γ_v.

Proposition 4.4.1. *Define for* $i\in\{1,2\}$,

$$\mathcal{A}_v^i := \left\{\forall x\in[a_{v,i}^-, b_{v,i}^-], \left|\frac{\mathbf{L}(\boldsymbol{\sigma}_{v,i}^-, x)}{re^{v-W(x)+W(m_{v,i}^-)}} - 1\right| \le \delta\right\},$$
$$\mathcal{B}_v^i := \left\{\forall x\in[b_{v,i-1}^-, a_{v,i}^-), \mathbf{L}(\boldsymbol{\sigma}_{v,i}^-, x) \le \delta re^v\right\},$$
$$\mathcal{C}_v := \left\{\forall x\in[b_v^+, c_v^+], \mathbf{L}(\boldsymbol{\sigma}_v^+, x) \le \delta re^v\right\} \text{ and}$$
$$\mathcal{D}_v := \left\{\forall x > c_v^+, \mathbf{L}(\boldsymbol{\sigma}_v^+, x) = 0\right\}$$

and in the same way $\widehat{\mathcal{A}_v^i}$, $\widehat{\mathcal{B}_v^i}$, $\widehat{\mathcal{C}}_v$ *and* $\widehat{\mathcal{D}_v}$ *from* $\widehat{W}$. *There is a constant* $K > 0$ *such that for any* $v \geq 2$, *for any* $0 \leq \delta \leq 1$ *and any* $r > 0$,

$$\begin{aligned}
P^W\left(\overline{\mathcal{A}_v^i}\right) &\leq \frac{K}{\delta}\sqrt{\frac{m_{v,i}^-}{rv^{c_2}}}\exp\left(-\frac{\delta^2 rv^{c_2}}{Km_{v,i}^-}\right),\\
P^W\left(\overline{\mathcal{B}_v^i}\right) &\leq K\exp\left(-\frac{\delta rv^{c_1}}{4(m_{v,i}^- - b_{v,i-1}^-)\log\left(8\frac{S(m_{v,i}^-)-S(b_{v,i-1}^-)}{S(m_{v,i}^-)-S(a_{v,i}^-)}\right)}\right) + \frac{2}{\delta v^{c_1-c_2}},\\
P^W\left(\overline{\mathcal{C}_v}\right) &\leq \frac{1}{\delta}e^{-\underline{W}(b_v^+,c_v^+)+W(m_v^+)} \text{ and}\\
P^W\left(\overline{\mathcal{D}_v}\right) &\leq \frac{re^{v+W(m_v^+)}}{2(S(c_v^+)-S(m_v^+))}.
\end{aligned}$$

Similar estimates hold for $P^W(\widehat{\mathcal{A}_v^i})$, $P^W(\widehat{\mathcal{B}_v^i})$, $P^W(\widehat{\mathcal{C}}_v)$ *and* $P^W(\widehat{\mathcal{D}_v})$.

Proof. We estimate the probabilities of the events relative to W, the ones relative to $\widehat{W}$ follow by symmetry. To simplify notations, all along the proof, we shall not mark the index v for variables and events. Begin with the events $\mathcal{A}^1$ and $\mathcal{B}^1$ ($\mathcal{A}^2$ and $\mathcal{B}^2$ can be studied in the same way).

The local time can be decomposed in two terms. The first one represents the contribution of the local time before $\boldsymbol{\tau}(m_1^-)$ (the first time where $\mathbf{X}$ reaches m_1^-) and is negligible compared to the second one which represents the contribution of the local time between $\boldsymbol{\tau}(m_1^-)$ and $\boldsymbol{\sigma}_1^- = \boldsymbol{\sigma}(re^v, m_1^-)$:

$$\mathbf{L}(\boldsymbol{\sigma}_1^-, x) = \mathbf{L}(\boldsymbol{\tau}(m_1^-), x) + \big(\mathbf{L}(\boldsymbol{\sigma}_1^-, x) - \mathbf{L}(\boldsymbol{\tau}(m_1^-), x)\big). \tag{4.6}$$

The following lemma describes the behavior of the first term.

Lemma 4.4.2. *For any* $v, \delta, r > 0$,

$$\begin{aligned}
P^W_{\mathcal{A}^1,1} &:= P^W\left(\sup_{x\in[a_1^-,m_1^-]}\frac{\mathbf{L}(\boldsymbol{\tau}(m_1^-),x)}{re^{v-(W(x)-W(m_1^-))}} > \delta\right)\\
&\leq K\exp\left(-\frac{\delta rv^{c_2}}{2m_1^-}\right),\\
P^W_{\mathcal{B}^1,1} &:= P^W\left(\sup_{x\in[b_0^-,a_1^-)}\mathbf{L}(\boldsymbol{\tau}(m_1^-),x) > \delta re^v\right)\\
&\leq K\exp\left(-\frac{\delta rv^{c_1}}{2(m_1^- - b_0^-)\log\left(8\frac{S(m_1^-)-S(b_0^-)}{S(m_1^-)-S(a_1^-)}\right)}\right).
\end{aligned}$$

To highlight the fact that the computations are identical for $\mathcal{B}^1$ and for $\mathcal{B}^2$ we make the quantity b_0^- appears, although it is zero.

Proof. Thanks to (4.2), it is easy to verify that $\boldsymbol{\tau}(m_1^-) = T(\tau(S(m_1^-))$. So using (4.3), for any $x \geq 0$,

$$\mathbf{L}(\boldsymbol{\tau}(m_1^-), x) = e^{-W(x)} L(\tau(S(m_1^-)), S(x)).$$

Then,

$$P^W_{\mathcal{A}^1,1} = P^W \left(\sup_{x \in [a_1^-, m_1^-]} L(\tau(S(m_1^-)), S(x)) > \delta r e^{v + W(m_1^-)} \right).$$

According to the first Ray-Knight theorem, Theorem 4.2.1,

$$\big(L(\tau(S(m_1^-)), S(m_1^-) - y),\ y \in [0, S(m_1^-)]\big)$$

is distributed as a square of a Bessel process of dimension 2 started at 0. Therefore, with Item *(iii)* of Lemma 4.2.3,

$$P^W_{\mathcal{A}^1,1} \leq K \exp\left(- \frac{\delta r e^{v + W(m_1^-)}}{2(S(m_1^-) - S(a_1^-))} \right)$$

and by definition of a_1^-,

$$\begin{aligned} S(m_1^-) - S(a_1^-) &= \int_{a_1^-}^{m_1^-} e^{W(x)} \mathrm{d}x \leq m_1^- e^{\overline{W}(a_1^-, m_1^-)} \\ &\leq m_1^- e^{v - c_2 \log v + W(m_1^-)}. \end{aligned}$$

Hence the first upper bound of the lemma is obtained.

Continue with the second one: using a similar argument and denoting by Q a squared Bessel process of dimension 2 started at 0, we get

$$\begin{aligned} P^W_{\mathcal{B}^1,1} &= P^W \left(\sup_{x \in [b_0^-, a_1^-)} e^{-W(x)} Q(S(m_1^-) - S(x)) > \delta r e^{v} \right) \\ &= P^W \left(\sup_{x \in [b_0^-, a_1^-)} \frac{e^{-W(x)}(S(m_1^-) - S(x))}{S(m_1^-) - S(x)} Q(S(m_1^-) - S(x)) > \delta r e^{v} \right). \end{aligned}$$

By definition of m_1^-, for any $x \in [b_0^-, a_1^-)$,

$$e^{-W(x)}(S(m_1^-) - S(x)) \leq (m_1^- - b_0^-) e^{-W(x) + \overline{W}(x, m_1^-)}.$$

As b_1^- is the first positive number x such that $W(x) - \underline{W}(b_0^-, x) \geq v - c_1 \log v$,

$$(m_1^- - b_0^-)e^{-W(x)+\overline{W}(x,m_1^-)} \leq (m_1^- - b_0^-)e^{v-c_1 \log v}.$$

Thus, coming back to the probability $P^W_{\mathcal{B}^1,1}$, we obtain

$$P^W_{\mathcal{B}^1,1} \leq P^W \left(\sup_{u \in [S(m_1^-)-S(a_1^-), S(m_1^-)-S(b_0^-))} \frac{1}{u} Q(u) > \frac{\delta r v^{c_1}}{m_1^- - b_0^-} \right).$$

According to Item (iv) of Lemma 4.2.3, we finally have

$$P^W_{\mathcal{B}^1,1} \leq K \exp\left(-\frac{\delta r v^{c_1}}{2(m_1^- - b_0^-) \log\left(8 \frac{S(m_1^-)-S(b_0^-)}{S(m_1^-)-S(a_1^-)}\right)} \right).$$

This conclude the proof of the lemma. $\square$

Now, we study the second term of (4.6).

Lemma 4.4.3. *For any $v, \delta, r > 0$,*

$$\begin{aligned}
P^W_{\mathcal{A}^1,2} &:= P^W \left(\sup_{x \in [a_1^-, b_1^-]} \left| \frac{\mathbf{L}(\boldsymbol{\sigma}_1^-, x) - \mathbf{L}(\boldsymbol{\tau}(m_1^-), x)}{r e^{v-(W(x)-W(m_1^-))}} - 1 \right| > \delta \right) \\
&\leq \frac{8}{\delta} \sqrt{\frac{(1+\delta) m_1^-}{r v^{c_2}}} \exp\left(-\frac{\delta^2 r v^{c_2}}{8(1+\delta) m_1^-} \right), \\
P^W_{\mathcal{B}^1,2} &:= P^W \left(\sup_{x \in [b_0^-, a_1^-)} \mathbf{L}(\boldsymbol{\sigma}_1^-, x) - \mathbf{L}(\boldsymbol{\tau}(m_1^-), x) > \delta r e^v \right) \\
&\leq \frac{1}{\delta v^{c_1 - c_2}}.
\end{aligned}$$

Proof. It is easy to verify that the inverse of local time $\boldsymbol{\sigma}$ satisfies the following equality for every $r > 0$ and $y \in \mathbb{R}$,

$$\boldsymbol{\sigma}(r, y) = T(\sigma(r e^{W(y)}, S(y))). \tag{4.7}$$

Thus, thanks to (4.3), for any $r > 0$ and $y \in \mathbb{R}$,

$$\mathbf{L}(\boldsymbol{\sigma}_1^-, x) = e^{-W(x)} L(\sigma(r e^{W(m_1^-)+v}, S(m_1^-)), S(x)).$$

And so the following expression for the local time holds,

$$\begin{aligned}
&\mathbf{L}(\boldsymbol{\sigma}_1^-, x) - \mathbf{L}(\boldsymbol{\tau}(m_1^-), x) \\
&\qquad = e^{-W(x)} \left(L(\sigma(r e^{W(m_1^-)+v}, S(m_1^-)), S(x)) - L(\tau(S(m_1^-)), S(x)) \right) \\
&\qquad \stackrel{\mathcal{L}}{=} e^{-W(x)} r e^{W(m_1^-)+v} L(\sigma(1, 0), s(x))
\end{aligned}$$

where

$$s(x) := (S(x) - S(m_1^-))\frac{e^{-W(m_1^-)-v}}{r}.$$

Denote by Z the square of a Bessel process of dimension 0 started at 1. According to the second Ray-Knight theorem (Theorem 4.2.2), we have

$$\begin{aligned} P^W_{\mathcal{A}^1,2} \leq P^W &\left(\sup_{0 \leq y \leq |s(a_1^-)|} |Z(y) - 1| > \delta \right) \\ &+ P^W \left(\sup_{0 \leq y \leq s(b_1^-)} |Z(y) - 1| > \delta \right). \end{aligned}$$

Therefore, using Item (i) of Lemma 4.2.3,

$$\begin{aligned} P^W_{\mathcal{A}^1,2} \leq &\frac{4}{\delta}\sqrt{(1+\delta)|s(a_1^-)|}\exp\left(-\frac{\delta^2}{8(1+\delta)|s(a_1^-)|} \right) \\ &+ \frac{4}{\delta}\sqrt{(1+\delta)s(b_1^-)}\exp\left(-\frac{\delta^2}{8(1+\delta)s(b_1^-)} \right). \end{aligned}$$

Moreover, the definition of a_1^- implies that

$$|s(a_1^-)| = \frac{e^{-W(m_1^-)-v}}{r}\int_{a_1^-}^{m_1^-} e^{W(x)}\mathrm{d}x \leq \frac{m_1^-}{r}e^{W(a_1^-)-W(m_1^-)-v} \leq \frac{m_1^-}{rv^{c_2}}.$$

We get likewise $|s(b_1^-)| \leq b_1^-/(rv^{c_1})$. As $c_1 \geq c_2$, these last two inequalities lead to the first point of the lemma. We now prove the second inequality of the lemma. If $b_0^- = a_1^-$, we have obviously $P^W_{\mathcal{B}^1,2} = 0$ else, reasoning in the same way as before, we obtain

$$P^W_{\mathcal{B}^1,2} = P^W \left(\sup_{x \in [b_0^-, a_1^-)} e^{-W(x)+W(m_1^-)} L(\sigma(1,0), s(x)) > \delta \right).$$

One more time, thanks to the second Ray-Knight theorem and denoting by Z a squared Bessel process of dimension 0 started at 1, we obtain

$$\begin{aligned} P^W_{\mathcal{B}^1,2} &\leq P^W \left(e^{-\underline{W}(b_0^-, a_1^-)+W(m_1^-)} \sup_{u \geq 0} Z(u) > \delta \right) \\ &= \frac{1}{\delta} e^{-\underline{W}(b_0^-, a_1^-)+W(m_1^-)}. \end{aligned}$$

The second line is a consequence of Item (ii) of Lemma 4.2.3. Denote by n^1 the unique real number in $[b_0^-, a_1^-]$ such that $W(n^1) = \underline{W}(b_0^-, a_1^-)$ then

$$\begin{aligned} W(m_1^-) - W(n^1) &= \overline{W}(n^1, a_1^-) - W(n^1) - (\overline{W}(n^1, a_1^-) - W(m_1^-)) \\ &\leq v - c_1 \log v - (v - c_2 \log v) = (c_2 - c_1)\log v. \end{aligned}$$

Therefore,

$$P^W_{\mathcal{B}^1,2} \leq \frac{1}{\delta v^{c_1-c_2}}$$

and the lemma is proved. $\square$

Combining the results of Lemmas 4.4.2 and 4.4.3 yields to the upper bounds of $P^W(\overline{\mathcal{A}^1})$ and $P^W(\overline{\mathcal{B}^1})$ of Proposition 4.4.1.

We continue with the estimate of $P^W(\mathcal{C})$. Reducing once again the local time of $\mathbf{X}$ to the local time of a Brownian motion by a time and space change, we obtain

$$P^W(\overline{\mathcal{C}}) = P^W\left(\sup_{x\in[b^+,c^+]} e^{-W(x)+W(m^+)} L(\sigma(1,0), s(x)) > \delta\right)$$

where $s(x)$ is the same as before but m_1^- is replaced by m^+. As before, Z denotes a squared 0-dimensional Bessel process starting at 1 and the second Ray-Knight theorem gives:

$$P^W(\overline{\mathcal{C}}) \leq P^W\left(e^{-\underline{W}(b^+,c^+)+W(m^+)} \sup_{u\geq 0} Z(u) > \delta\right).$$

So Item (ii) of Lemma 4.2.3 yields to the upper bound of Proposition 4.4.1.

Finally, we show that with a high probability diffusion $\mathbf{X}$ does not hit c^+ before time $\boldsymbol{\sigma}_1^-$. I.e. we find an upper bound for $P^W(\overline{\mathcal{D}})$. The scale change in time and space of $\mathbf{X}$ and the usual properties of Brownian motion give (see e.g. [16] Formula 4.1.2 page 185)

$$\begin{aligned}
P^W\left(\overline{\mathcal{D}}\right) &= P^W\Big(\boldsymbol{\tau}(c^+) < \boldsymbol{\sigma}\left(re^v, m^+\right)\Big) \\
&= P^W\left(\tau(S(c^+)) < \sigma\left(re^{v+W(m^+)}, S(m^+)\right)\right) \\
&= 1 - \exp\left(-\frac{re^{v+W(m^+)}}{2(S(c^+)-S(m^+))}\right) \leq \frac{re^{v+W(m^+)}}{2(S(c^+)-S(m^+))}.
\end{aligned}$$

This completes the proof of the proposition. $\square$

We now give upper bounds independent of the environment provided that it is in the set Γ_v defined in (4.4).

Proposition 4.4.4. *We use the same notations as in the previous proposition. There is a constant $K > 0$ such that for any $v \geq 2$, any $0 \leq \delta \leq 1$,* *and any $r>0$, if $W\in\Gamma_v$, for $i\in\{1,2\}$,*

$$\begin{aligned}
P^W\left(\overline{\mathcal{A}^i_v}\right) \le & \frac{K}{\delta\sqrt{rv^{c_2-6}}}\exp\left(-\frac{\delta^2 r v^{c_2-6}}{K}\right),\\
P^W\left(\overline{\mathcal{B}^i_v}\right) \le & K\exp\left(-\frac{\delta r v^{c_1-8}}{K}\right)+\frac{2}{\delta v^{c_1-c_2}},\\
P^W\left(\overline{\mathcal{C}_v}\right) \le & \frac{1}{\delta v^{c_1+c_3}} \text{ and}\\
P^W\left(\overline{\mathcal{D}_v}\right) \le & \frac{r}{v^{c_3-2}\log v}.
\end{aligned}$$

Once again similar estimates hold for $P^W(\widehat{\mathcal{A}^i_v})$, $P^W(\widehat{\mathcal{B}^i_v})$, $P^W(\widehat{\mathcal{C}_v})$, $P^W(\widehat{\mathcal{D}_v})$.

Proof. We only have to control the values of the upper bounds of Proposition 4.4.1 when $W\in\Gamma_v$. For $P^W\left(\overline{\mathcal{A}^i_v}\right)$, it is enough to notice that $m^-_{v,i}\le b^-_{v,2}$ and that, on Γ_v, the variable $b^-_{v,2}$ is smaller than v^6. Then, we also obtain the upper bound $m^-_{v,i}-b^-_{v,i-1}\le b^-_{v,2}\le v^6$. To estimate $P^W\left(\overline{\mathcal{B}^i_v}\right)$, it remains to study

$$\frac{S(m^-_{v,i})-S(b^-_{v,i-1})}{S(m^-_{v,i})-S(a^-_{v,i})}\le\frac{(m^-_{v,i}-b^-_{v,i-1})e^{\overline{W}(b^-_{v,i-1},m^-_{v,i})}}{(m^-_{v,i}-a^-_{v,i})e^{W(m^-_{v,i})}}.$$

First remark that $\overline{W}(b^-_{v,i-1},m^-_{v,i})-W(b^-_{v,i-1})\le v-c_1\log v\le v$. Then it is easy to see that, on Γ_v, the following inequality holds:

$$\frac{S(m^-_{v,i})-S(b^-_{v,i-1})}{S(m^-_{v,i})-S(a^-_{v,i})}\le v^8e^{v+v^2}.$$

This implies the second upper bound of the proposition. As on Γ_v, we have $\underline{W}(b^+_v,c^+_v)-W(m^+_v)\ge(c_1+c_3)\log v$, the third estimate is obtained immediately. It remains the upper bound of $P^W\left(\overline{\mathcal{D}_v}\right)$. Remark that

$$\begin{aligned}
S(c^+_v)-S(m^+_v) &\ge \int_{(c^+_v-\log v)\vee m^+_v}^{c^+_v} e^{W(x)}\mathrm{d}x\\
&\ge \left(\log v\wedge(c^+_v-m^+_v)\right)e^{\underline{W}((c^+_v-\log v)\vee m^+_v,c^+_v)}.
\end{aligned}$$

And on Γ_v we have $c^+_v-m^+_v\ge v$ and $\underline{W}((c^+_v-\log v)\vee m^+,c^+_v)\ge W(c^+_v)-2\log v$, then

$$P^W\left(\overline{\mathcal{D}_v}\right)\le\frac{rv^2}{\log v}e^{v+W(m^+_v)-W(c^+_v)}.$$

As $W(c^+_v)-W(m^+_v)=v+c_3\log v$, this concludes the proof. □

4.5 Asymptotics of local time in deterministic time

We fix now the constants c_i: take a real number $c > 0$, then c_1, c_2 and c_3 are chosen as follows $c_1 := 2c + 8$, $c_2 := c + 6$ and $c_3 := c + 2$. Thanks to Proposition 4.4.4, we can now study the process $\mathbf{L}$ at the time

$$\boldsymbol{\sigma}_v := \widehat{\boldsymbol{\sigma}}^-_{v,1} \wedge \widehat{\boldsymbol{\sigma}}^-_{v,2} \wedge \boldsymbol{\sigma}^-_{v,1} \wedge \boldsymbol{\sigma}^-_{v,2}.$$

Define for v large enough and $i \in \{1, 2\}$,

$$I_{v,i} := \int_{a^-_{v,i}}^{b^-_{v,i}} e^{-W(x)+W(m^-_{v,i})} \mathrm{d}x$$

and define similarly $\widehat{I}_{v,i}$ from $\widehat{W}$. Consider finally

$$j_v := \widehat{I}_{v,1} \wedge \widehat{I}_{v,2} \wedge I_{v,1} \wedge I_{v,2} \text{ et } J_v := \widehat{I}_{v,1} + \widehat{I}_{v,2} + I_{v,1} + I_{v,2}.$$

Roughly speaking, Proposition 4.5.1 shows that the process $\boldsymbol{\sigma}_v/re^v$ stays between j_v and J_v. Moreover, the occupation measure is concentrated in the neighborhood of $m^-_{v,1}$, $m^-_{v,2}$, $\widehat{m}^-_{v,1}$ and $\widehat{m}^-_{v,2}$. More precisely, for v large enough and $i \in \{1, 2\}$, define the last time the environment is less than $W(m^-_{v,i}) + \log 1/\delta$ between $m^-_{v,i}$ and $b^-_{v,i}$,

$$d^-_{v,i} := \sup\{m^-_{v,i} \leq x \leq b^-_{v,i} \ , \ W(x) - W(m^-_{v,i}) \leq \log 1/\delta\}$$

and the first time the environment is less than $\log 1/\delta$ between $a^-_{v,i}$ and $m^-_{v,i}$,

$$e^-_{v,i} := \inf\{a^-_{v,i} \leq x \leq m^-_{v,i} \ , \ W(x) - W(m^-_{v,i}) \leq \log 1/\delta\}.$$

Consider then the interval $U^-_{v,i} := [e^-_{v,i}, d^-_{v,i}]$. Define the analogous variables for $\widehat{W}$. The next proposition shows that at time $\boldsymbol{\sigma}_v$, the diffusion has spent much of its time in the set

$$A_v := U^-_{v,1} \cup U^-_{v,2} \cup \widehat{U}^-_{v,1} \cup \widehat{U}^-_{v,2}.$$

and $\mathbf{L}^*$ is approximately re^v.

Proposition 4.5.1. *Define the event*

$$\mathcal{E}_v := \Big\{ \nu_{\boldsymbol{\sigma}_v}(\overline{A}_v) \leq 4rv^6e^v\delta \ ; \ re^v \leq \mathbf{L}^*(\boldsymbol{\sigma}_v) \leq re^v(1+\delta) \ ;$$
$$j_v(1-\delta) \leq \frac{\boldsymbol{\sigma}_v}{re^v} \leq J_v + 2v^6\delta \Big\}.$$

There is a constant $K > 0$ such that for any $0 < \delta < 1$ and any $r > 0$, if $W \in \Gamma_v$,

$$P^W\left(\overline{\mathcal{E}}_v\right) \leq K\left(\frac{1}{\delta\sqrt{rv^c}}\exp\left(-\frac{\delta^2 rv^c}{K}\right) + \exp\left(-\frac{\delta rv^c}{K}\right) + \frac{1}{\delta v^c} + \frac{r}{v^c}\right).$$

Proof. We prove that on the intersection of all the events of Proposition 4.4.4, the event

$$\left\{re^v \leq \mathbf{L}^*(\boldsymbol{\sigma}_v) \leq re^v(1+\delta)\ ;\ j_v(1-\delta) \leq \frac{\boldsymbol{\sigma}_v}{re^v} \leq J_v + 2v^6\delta\right\}$$

is realized. As $\boldsymbol{\sigma}_v$ is the first time the diffusion has "spent a time" re^v in one of the four points $m_{v,i}^-$ or $\widehat{m}_{v,i}^-$, we already have $re^v \leq \mathbf{L}^*(\boldsymbol{\sigma}_v)$. Moreover, the local time process is non-decreasing, so for every $x \in \mathbb{R}$,

$$\mathbf{L}(\boldsymbol{\sigma}_v, x) = \mathbf{L}(\widehat{\boldsymbol{\sigma}}_{v,1}^-, x) \wedge \mathbf{L}(\widehat{\boldsymbol{\sigma}}_{v,2}^-, x) \wedge \mathbf{L}(\boldsymbol{\sigma}_{v,1}^-, x) \wedge \mathbf{L}(\boldsymbol{\sigma}_{v,2}^-, x).$$

On Γ_v, the time $\boldsymbol{\sigma}_v^+$ is equal to $\boldsymbol{\sigma}_{v,1}^-$ or to $\boldsymbol{\sigma}_{v,2}^-$ and the time $\widehat{\sigma_{X_\alpha v}}$ to $\widehat{\boldsymbol{\sigma}}_{v,1}^-$ or to $\widehat{\boldsymbol{\sigma}}_{v,2}^-$, then Proposition 4.4.4 gives $\mathbf{L}^*(\boldsymbol{\sigma}_v) \leq re^v(1+\delta)$.

Continue with the estimate of $\boldsymbol{\sigma}_v$: by definition of the local time,

$$\boldsymbol{\sigma}_v = \int_{-\infty}^{+\infty} \mathbf{L}(\boldsymbol{\sigma}_v, x)\mathrm{d}x,\ \mathbb{P}\text{-a.s.}$$

Using Proposition 4.4.4 one more time, we easily obtain:

$$\begin{aligned}\int_0^{+\infty} \mathbf{L}(\boldsymbol{\sigma}_v, x)\mathrm{d}x \leq& \int_0^{a_{v,1}^-} \mathbf{L}(\boldsymbol{\sigma}_{v,1}^-, x)\mathrm{d}x + \int_{a_{v,1}^-}^{b_{v,1}^-} \mathbf{L}(\boldsymbol{\sigma}_{v,1}^-, x)\mathrm{d}x \\ &+ \int_{b_{v,1}^-}^{a_{v,2}^-} \mathbf{L}(\boldsymbol{\sigma}_{v,2}^-, x)\mathrm{d}x + \int_{a_{v,2}^-}^{b_{v,2}^-} \mathbf{L}(\boldsymbol{\sigma}_{v,2}^-, x)\mathrm{d}x + \int_{b_v^+}^{c_v^+} \mathbf{L}(\boldsymbol{\sigma}_v^+, x)\mathrm{d}x \\ &\leq (I_{v,1} + I_{v,2} + \delta c_v^+)re^v.\end{aligned}$$

The integral $\int_{-\infty}^0 \mathbf{L}(\boldsymbol{\sigma}_v, x)\mathrm{d}x$ has a similar upper bound. As $c_v^+ + \widehat{c}_v^+ \leq 2v^6$ on Γ_v, the upper bound of the proposition follows immediately.

If $\boldsymbol{\sigma}_v = \boldsymbol{\sigma}_{v,1}^-$, we have

$$\boldsymbol{\sigma}_{v,1}^- \geq \int_{a_{v,1}^-}^{b_{v,1}^-} \mathbf{L}(\boldsymbol{\sigma}_{v,1}^-, x)\mathrm{d}x \geq I_{v,1}(1-\delta)re^v.$$

The same computation when $\boldsymbol{\sigma}_v$ takes one of the three other possible values yields the lower bound stated in the proposition.

Finally, as

$$\nu_{\boldsymbol{\sigma}_v}(\overline{A}_v) = \int_{-\infty}^{\infty} \mathbf{1}_{\overline{A}_v}\mathbf{L}(\boldsymbol{\sigma}_v, x)\mathrm{d}x,$$

we can obtain the bound proceeding in the same way as before. $\square$

As the behavior of $\boldsymbol{\sigma}_v$ is controlled, same kind of results in deterministic time can be obtained.

Proposition 4.5.2. *For any* $0 < \delta \leq 1/2$, *for* v *large enough, if* $W \in \Gamma_v$,

$$\begin{aligned}
&P^W\left(\frac{e^v}{J_v + 2v^6\delta} \leq \mathbf{L}^*(e^v) \leq \frac{e^v(1+\delta)}{j_v(1-\delta)}\right) \\
&\qquad \geq 1 - K\left(\frac{1}{\delta\sqrt{v^{c-6}}}\exp\left(-\frac{\delta^2 v^{c-6}}{K}\right) + \exp\left(-\frac{\delta v^{c-6}}{K}\right) + \frac{1}{\delta v^c} + \frac{1}{v^{c-4}}\right).
\end{aligned}$$

Proof. We use the real number r which appears in all propositions since the beginning. Define

$$\rho(v) := \frac{1}{J_v + 2v^6\delta} \text{ and } r(v) := \frac{1}{j_v(1-\delta)}.$$

We write $\boldsymbol{\sigma}_v^\rho$ for the time $\boldsymbol{\sigma}_v$ associated with ρ and $\boldsymbol{\sigma}_v^r$ for the one associated with r. Consider the events

$$\begin{aligned}
\Omega^\rho &:= \left\{\frac{\boldsymbol{\sigma}_v^\rho}{\rho(v)e^v} \leq J_v + 2v^6\delta\right\} = \{\boldsymbol{\sigma}_v^\rho \leq e^v\}, \\
\Omega^r &:= \left\{\mathbf{L}^*(\boldsymbol{\sigma}_v^r) \leq r(v)e^v(1+\delta) \; ; \; j_v(1-\delta) \leq \frac{\boldsymbol{\sigma}_v^r}{r(v)e^v}\right\} \\
&= \{\mathbf{L}^*(\boldsymbol{\sigma}_v^r) \leq r(v)e^v(1+\delta) \; ; \; e^v \leq \boldsymbol{\sigma}_v^r\}.
\end{aligned}$$

The maximum of local time is a non decreasing function, then on Ω^r,

$$\mathbf{L}^*(e^v) \leq \mathbf{L}^*(\boldsymbol{\sigma}_v^r) \leq r(v)e^v(1+\delta)$$

and on Ω^ρ,

$$\rho(v)e^v \leq \mathbf{L}^*(\boldsymbol{\sigma}_v^\rho) \leq \mathbf{L}^*(e^v).$$

Therefore it is enough to find a lower bound for $\mathbb{P}(\Omega^r \cap \Omega^\rho)$. According to the previous proposition, we only have to estimate r and ρ. First, on Γ_v, the following inequality holds

$$j_v \leq J_v \leq c_v^+ + \widehat{c}_v^+ \leq 2v^6.$$

Moreover $m_{v,1}^- - a_{v,1}^- \geq v^{-2}$ and

$$\overline{W}((m_{v,1}^- - \log v) \vee a_{v,1}^-, m_{v,1}^-) - W(m_{v,1}^-) \leq 2\log v,$$

thus

$$I_{v,1} \geq \int_{(m_{v,1}^- - \log v)\vee a_{v,1}^-}^{m_{v,1}^-} e^{-W(x)+W(m_{v,1}^-)}\mathrm{d}x \geq \frac{1}{v^4}.$$

The lower bounds for $I_{v,2}$, $\widehat{I}_{v,1}$ and $\widehat{I}_{v,2}$ are found in the same way. Finally,

$$\frac{K}{v^6} \le \rho(v) \le r(v) \le 2v^4$$

and the estimate of the proposition follows easily. $\square$

Using similar arguments, we can also obtain a result in deterministic time for the occupation measure.

Proposition 4.5.3. *For any $0 < \delta \le 1/2$, for v large enough, if $W \in \Gamma_v$,*

$$\begin{aligned} &P^W\left(\nu_{e^v}(\overline{A}_v) \le 8v^{10}e^v\delta\right) \\ &\quad \ge 1 - K\left(\frac{1}{\delta\sqrt{v^{c-6}}}\exp\left(-\frac{\delta^2 v^{c-6}}{K}\right) + \exp\left(-\frac{\delta v^{c-6}}{K}\right) + \frac{1}{\delta v^c} + \frac{1}{v^{c-4}}\right). \end{aligned}$$

Fix now $c_0 > 10$ and recall that $c_1 = 2c + 8$. Proposition 4.5.3 used with $\delta = v^{-c_0}/8$ and $c > 6 + 2c_0$ and the upper bound for $\mathbb{P}(\Gamma_v)$ of Proposition 4.3.1 yield

$$\mathbb{P}\left(\nu_{e^v}(\overline{A}_v) \le e^v/v^{c_0-10}\right) \ge 1 - K\left(\frac{\log v}{v - c_1\log v}\right)^2 \quad \text{for } v \text{ large enough.} \tag{4.8}$$

Proposition 4.5.2 with $\delta = v^{-7}$ and $c > 20$ and the upper bound for $\mathbb{P}(\Gamma_v)$ of Proposition 4.3.1 yield for v large enough to

$$\mathbb{P}\left(\frac{e^v}{J_v + 2v^{-1}} \le \mathbf{L}^*(e^v) \le \frac{e^v(1+v^{-7})}{j_v(1-v^{-7})}\right) \ge 1 - K\left(\frac{\log v}{v - c_1\log v}\right)^2. \tag{4.9}$$

and if we use the event Γ'_v instead of Γ_v, we obtain

$$\mathbb{P}\left(\frac{e^v}{I_{v,1} + \widehat{I}_{v,1} + 2v^{-1}} \le \mathbf{L}^*(e^v) \le \frac{e^v(1+v^{-7})}{I_{v,1} \wedge \widehat{I}_{v,1}(1-v^{-7})}\right) \ge 1 - \frac{K\log v}{v - c_1\log v}. \tag{4.10}$$

4.6 Proof of Theorems 4.1.1 and 4.1.2

As shown by Proposition 4.5.2, the asymptotic behavior of $\mathbf{L}^*$ has a direct link with the ones of j_v, J_v. Therefore, the proof of Theorem 4.1.1 requires to study the integrals $I_{v,i}$ and $\widehat{I}_{v,i}$.

4.6.1 Maximum and minimum speed

Begin with a lower bound for the maximum speed:

Lemma 4.6.1. *Let* $v_n = \exp(n)$. $\mathbb{P}$-*a.s.,*

$$\limsup_{n\to\infty} \frac{I_{v_n,1} \wedge \widehat{I}_{v_n,1}}{\log_2 v_n} \geq \frac{4}{e^2\pi^2}.$$

Proof. First, define for n large enough, the sequence of events

$$E_n := \left\{ m^-_{v_n,1} > m^-_{v_{n-1},1} \ ; \ \int_{m^-_{v_n,1}}^{b^-_{v_n,1}} e^{-W(x)+W(m^-_{v_n,1})}\mathrm{d}x \geq \frac{4\log n}{e^2\pi^2} \right\}.$$

Denote by $(\mathcal{G}_n)$ the filtration generated by $(W(x),\ 0 \leq x \leq b^-_{v_n,1})$. The process $W_n := (W(x + b^-_{v_{n-1},1}) - W(b^-_{v_{n-1},1}),\ x \geq 0)$ is a Brownian motion independent of $\mathcal{G}_{n-1}$. The event E_n can be expressed in term of W_n: $E_n = E_{n,1} \cap E_{n,2}$ where

$$E_{n,1} = \{W_n \text{ hits } -v_{n-1} + c_1 \log v_{n-1} \text{ before } v_n - v_{n-1} - c_1\} \text{ and}$$
$$E_{n,2} = \left\{ \int_{m^-_{v_n,1}(W_n)}^{b^-_{v_n,1}(W_n)} e^{-W_n(x)+W_n(m^-_{v_n,1})}\mathrm{d}x \geq \frac{4\log n}{e^2\pi^2} \right\}$$

Therefore, E_n is independent of $\mathcal{G}_{n-1}$ and $\mathcal{G}_n$-measurable. Moreover, thanks to Theorem 4.2.4, $E_{n,1}$ and $E_{n,2}$ are also independent from each other and

$$\begin{aligned}&\mathbb{P}(W_n \text{ hits } -v_{n-1} + c_1 \log v_{n-1} \text{ before } v_n - v_{n-1} - c_1) \\ &\quad = \frac{v_n - v_{n-1} - c_1}{v_n - c_1 \log v_n} \geq (1 - e^{-1} - c_1 e^{-n})\end{aligned}$$

and

$$\begin{aligned}&\mathbb{P}\left(\int_{m^-_{v_n,1}(W_n)}^{b^-_{v_n,1}(W_n)} e^{-W_n(x)+W_n(m^-_{v_n,1})}\mathrm{d}x \geq \frac{4\log n}{e^2\pi^2} \right) \\ \geq &\mathbb{P}\left(\int_0^{T_R(2)} e^{-R(x)}\mathrm{d}x \geq \frac{4\log n}{e^2\pi^2} \right) \geq \mathbb{P}\left(T_R(2) \geq \frac{4\log n}{\pi^2} \right).\end{aligned}$$

Then, according to Item (ii) of Lemma 4.2.5,

$$\mathbb{P}(E_n) \geq \frac{1 - e^{-1} - c_1 e^{-n}}{K\sqrt{n}}.$$

We now define the similar event for $\widehat{W}$:

$$\widehat{E}_n := \left\{ \widehat{m}^-_{v_n,1} > \widehat{m}^-_{v_{n-1},1} \ ; \ \int_{m^-_{v_n,1}}^{b^-_{v_n,1}} e^{-W(x)+W(m^-_{v_n,1})} \mathrm{d}x \geq \frac{4\log n}{e^2\pi^2} \right\}.$$

The events E_n, $\widehat{E}_n$ are independent, thus

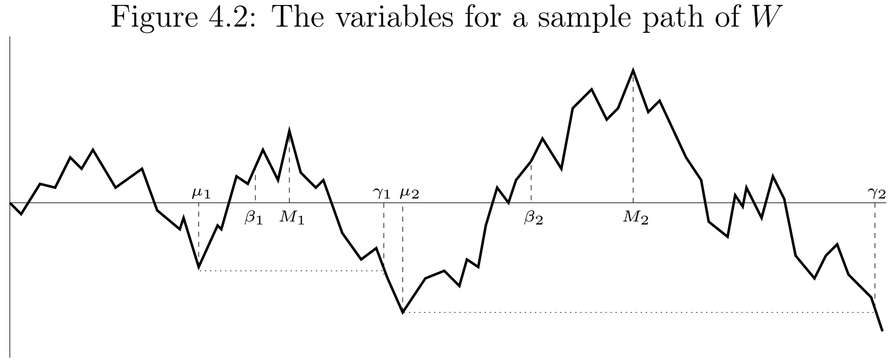
$$\mathbb{P}(E_n \cap \widehat{E}_n) = \mathbb{P}(E_n)\mathbb{P}(\widehat{E}_n) \geq \frac{(1-e^{-1}-c_1e^{-n})^2}{K^2 n}.$$

The second Borel-Cantelli lemma yields the conclusion. □

We are not interested in an upper bound of the minimum speed because this would lead, except for the value of the constant, to the result obtained by Shi in [71]. We now look for almost sure bounds of the integral. To this end, we study the successive values μ_n the process $(m_v,\ v \geq 2)$ can take. These are precisely defined as follows: define $\gamma_0 = 0$, $h_0 = 2$ and recursively for any $n \in \mathbb{N}$,

$$\begin{aligned}
\beta_{n+1} :=& \inf\{x \geq \gamma_n \ ; \ W(x) - \underline{W}(\gamma_n, x) = h_n\}, \\
\mu_{n+1} :=& \inf\{x \geq \gamma_n \ ; \ W(x) = \underline{W}(\gamma_n, \beta_{n+1})\}, \\
\gamma_{n+1} :=& \inf\{x \geq \beta_{n+1} \ ; \ W(x) = W(\mu_{n+1})\}, \\
M_{n+1} :=& \inf\{x \geq \beta_{n+1} \ ; \ W(x) = \overline{W}(\beta_{n+1}, \gamma_{n+1})\}, \\
h_{n+1} :=& W(M_{n+1}) - W(\mu_{n+1}) \text{ and} \\
\mathcal{F}_{n+1} :=& \sigma\left(W(x), 0 \leq x \leq \gamma_{n+1}\right).
\end{aligned}$$

Figure 4.2: The variables for a sample path of W



Lemma 4.6.2. *There is a positive number K such that for any $n \in \mathbb{N}^*$ and any $\lambda > 0$,*

$$\mathbb{P}\left(\int_{\gamma_{n-1}}^{M_n} e^{-W(x)+W(\mu_n)} dx \geq \lambda\right) \leq K e^{-j_0^2 \frac{\lambda}{16}} \text{ and}$$
$$\mathbb{P}\left(\int_{\mu_n}^{\beta_n} e^{-W(x)+W(\mu_n)} dx \leq \lambda\right) \leq K \left(2/(e\sqrt{\lambda}) + e\sqrt{\lambda}/2\right) e^{-2/(e^2\lambda)}.$$

where j_0 is the smallest strictly positive root of the Bessel function J_0.

Proof. The process $(W(\gamma_{n-1}+x) - W(\gamma_{n-1}),\ x \geq 0)$ is a Brownian motion independent of $\mathcal{F}_{n-1}$. Therefore given $h_{n-1} = h$, Theorem 4.2.4 gives the law of the process

$$\left(W(\mu_n + x) - W(\mu_n), -\mu_n + \gamma_{n-1} \leq x \leq \beta_n - \mu_n\right).$$

Moreover, according to Proposition 3.13 Chapter 6 of [67], given $h_{n-1} = h$ and $W(M_n) = M$,

$$(W(x+\beta_n) - W(\beta_n) + h_{n-1}),\ 0 \leq x \leq M_n - \beta_n)$$

is a 3-dimensional Bessel process started at h and killed when it hits $M + h$, thus $(W(\mu_n + x) - W(\mu_n), 0 \leq x \leq M_n - \mu_n)$. is a 3-dimensional Bessel process started at 0 and killed when it hits $M+h$. So if we denote by R and $\widetilde{R}$ two independent Bessel processes of dimension 3 started at 0, then

$$\mathbb{P}\left(\int_{\gamma_{n-1}}^{M_n} e^{-W(x)+W(\mu_n)} \mathrm{d}x \geq \lambda\right) \leq \mathbb{P}\left(\int_0^\infty e^{-R(x)} \mathrm{d}x + \int_0^\infty e^{-\widetilde{R}(x)} \mathrm{d}x \geq \lambda\right).$$

Le Gall's Ray-Knight Theorem (Proposition 1.1 [33]) shows that $\int_0^\infty e^{-R(x)}\mathrm{d}x$ has the same law as $4 \cdot T_Q(1)$ where $T_Q(1)$ is the hitting time of height 1 by a square of a 2-dimensional Bessel process started at 0. Thus we denote by $\widetilde{T}_Q(1)$ an independent copy of $T_Q(1)$ and the previous inequality gives

$$\begin{aligned}\mathbb{P}\left(\int_{\gamma_{n-1}}^{M_n} e^{-W(x)+W(\mu_n)} \mathrm{d}x \geq \lambda\right) &\leq \mathbb{P}\left(T_Q(1) + \tilde{T}_Q(1) \geq \frac{\lambda}{4}\right)\\ &\leq 2\mathbb{P}\left(T_Q(1) \geq \frac{\lambda}{8}\right).\end{aligned}$$

Moreover according to Theorem 2 of [93], as in the proof of the previous lemma,

$$\begin{aligned}\mathbb{P}\left(\int_{\gamma_{n-1}}^{M_n} e^{-W(x)+W(\mu_n)} \mathrm{d}x \geq \lambda\right) &\leq \mathbb{P}\left(T_Q(1) + \tilde{T}_Q(1) \geq \frac{\lambda}{4}\right)\\ &\leq K e^{-j_0^2 \frac{\lambda}{16}}.\end{aligned}$$

For the second bound, we denote $T_R(2) := \inf\{x \geq 0, R(x) \geq 2\}$ and, thanks to Theorem 4.2.4,

$$\begin{aligned}\mathbb{P}\left(\int_{\mu_n}^{\beta_n} e^{-W(x)+W(\mu_n)}\mathrm{d}x \leq \lambda\right) \leq & \mathbb{P}\left(\int_0^{T_R(2)} e^{-R(x)}\mathrm{d}x \leq \lambda\right) \\ \leq & \mathbb{P}\left(T_R(2) \leq e^2\lambda\right) \\ \leq & K\left(2/(e\sqrt{\lambda}) + e\sqrt{\lambda}/2\right) e^{-2/(e^2\lambda)}.\end{aligned}$$

This concludes the proof. $\square$

We also need the following lemma:

Lemma 4.6.3. *Let $a < \exp(1) < b$. $\mathbb{P}$-a.s., for n large enough,*

$$a^n < h_n < b^n,\ \ a^n < W(\mu_n) - W(\mu_{n+1}) < b^n \text{ and } a^{2n} < \gamma_n < b^{2n}.$$

Proof. Begin with the law of the sequence (h_n). For any $h \geq 1$, any $n \in \mathbb{N}$ and any $x \geq 2$,

$$\begin{aligned}\mathbb{P}\left(\frac{h_{n+1}}{h_n} \leq h | h_n = x\right) =& \mathbb{P}\left(\frac{h_{n+1} - h_n}{h_n} \leq h - 1 | h_n = x\right) \\ =& \mathbb{P}\left(\tau_W((h-1)x) \geq \tau_W(-x)\right) = 1 - \frac{1}{h}.\end{aligned}$$

Thus the variables $r_n := h_{n+1}/h_n$ are independent and $\log r_n$ is exponentially distributed with mean 1. Therefore, $\log h_n - \log h_0 = \sum \log r_k$ has the gamma distribution $\Gamma(n, 1)$: for any $1 < a < \exp(1)$ and n large enough,

$$\mathbb{P}\left(h_n \leq a^n\right) \leq \int_0^{n\log a} \frac{x^{n-1}}{(n-1)!}e^{-x}\mathrm{d}x \leq \frac{(n\log a)^n}{a^n(n-1)!}$$

as the function $x \to x^{n-1}e^{-x}$ is non decreasing on $[0, n-1]$ and so on $[0, n\log a]$ if n is larger than $(1 - \log a)^{-1}$. The Stirling Formula $n! \sim (\frac{n}{e})^n\sqrt{2\pi n}$ give

$$\frac{(n\log a)^n}{a^n(n-1)!} \sim \sqrt{\frac{n}{2\pi}}\left(\frac{e\log a}{a}\right)^n.$$

As for any $a \in]1, e[$, $0 < \frac{e\log a}{a} < 1$, the series $\sum \mathbb{P}\left(h_n \leq a^n\right)$ converges. Then the first lower bound is a direct consequence of the Borel-Cantelli lemma. The upper bound is proved in the same way.

For the second result, note that, given $h_{n-1} = x$, $W(\mu_{n-1}) - W(\mu_n)$ has the same law as $W(m_x)$. Therefore,

$$\mathbb{P}\left(W(\mu_{n-1}) - W(\mu_n) < h | h_{n-1} = x\right) = 1 - e^{-h/x} \leq \frac{h}{x}.$$

Take $1 < d < a < \exp(1)$,

$$\begin{aligned}
&\mathbb{P}\big(W(\mu_{n-1}) - W(\mu_n) < d^{n-1}\big) \\
&\quad \leq \mathbb{P}\left(W(\mu_{n-1}) - W(\mu_n) < d^{n-1} \ ; \ h_{n-1} > a^{n-1}\right) + \mathbb{P}\left(h_{n-1} \leq a^{n-1}\right) \\
&\quad \leq \left(\frac{d}{a}\right)^{n-1} + \mathbb{P}\left(h_{n-1} \leq a^{n-1}\right).
\end{aligned}$$

The previous proof implies that the sum of

$$\mathbb{P}\left(W(\mu_{n-1}) - W(\mu_n) < d^{n-1}\right)$$

converges and the Borel-Cantelli lemma shows that for large n,

$$W(\mu_{n-1}) - W(\mu_n) \geq d^{n-1}.$$

The other bound can be obtained in the same way.

The last inequality with γ_n uses same kind of arguments: as before, we can show that for any $d > 0$,

$$\mathbb{P}\left(\frac{\gamma_n - \beta_n}{h_n^2} > d\right) \leq \mathbb{P}\left(\frac{\gamma_n - \beta_n}{W(\mu_n)^2} > d\right) = \mathbb{P}\left(\tau_W(1) \geq d\right)$$

and

$$\mathbb{P}\left(\frac{\beta_n - \gamma_{n-1}}{h_n^2} > d\right) \leq \mathbb{P}\left(\frac{\beta_n - \gamma_{n-1}}{h_{n-1}^2} > d\right) \leq \mathbb{P}\left(\tau_W(1) \geq d\right).$$

Then,

$$\mathbb{P}\left(\frac{\gamma_n - \gamma_{n-1}}{h_n^2} > d\right) \leq 2\mathbb{P}\left(\tau_W(1) \geq d/2\right) \leq \frac{K}{\sqrt{d}}.$$

Now, let $\epsilon > 0$ and $\gamma_0 = 0$, we have for any $n \geq 1$,

$$\begin{aligned}
\mathbb{P}\left(\frac{\gamma_n}{h_n^2} > (1+\epsilon)^{2n}\right) &\leq \mathbb{P}\left(\sum_{k=1}^{n} \frac{\gamma_k - \gamma_{k-1}}{h_k^2} > (1+\epsilon)^{2n}\right) \\
&\leq \sum_{k=1}^{n} \mathbb{P}\left(\frac{\gamma_k - \gamma_{k-1}}{h_k^2} > \frac{(1+\epsilon)^{2n}}{n}\right) \leq \frac{Kn^{3/2}}{(1+\epsilon)^n}.
\end{aligned}$$

Therefore Borel-Cantelli lemma implies that $\mathbb{P}$-a.s., for n large enough, $\gamma_n \leq (1+\epsilon)^{2n} h_n^2$. It is then easy to deduce the upper bound for γ_n. And the lower bound can be obtain easily using same techniques. □

It is now possible to control the asymptotic behavior of the integral.

Proposition 4.6.4. $\mathbb{P}$-*almost surely,*

$$\liminf_{v\to\infty} I_{v,1}\log_2 v \geq 2/e^2 \text{ and}$$
$$\limsup_{v\to\infty} I_{v,1}(\log_2 v)^{-1} \leq 16/j_0^2.$$

Proof. Begin with the lower bound. Fix $d > e^2/2$. According to Lemma 4.6.2, for any $n \in \mathbb{N}^*$,

$$\mathbb{P}\left(\int_{\mu_n}^{\beta_n} e^{-W(x)+W(\mu_n)}\mathrm{d}x \leq \frac{1}{d\log(n-1)}\right) \leq K\frac{\sqrt{\log(n-1)}}{(n-1)^{2d/e^2}}.$$

So the first Borel-Cantelli lemma implies that $\mathbb{P}$-a.s. for n large enough,

$$\int_{\mu_n}^{\beta_n} e^{-W(x)+W(\mu_n)}\mathrm{d}x > \frac{1}{d\log(n-1)}.$$

For v large enough, $\mathbb{P}$-a.s. there is a unique $n \in \mathbb{N}^*$ such that $h_{n-1} < v - c_1\log v \leq h_n$. Thus $m_{v,1}^- = \mu_n$ and $b_{v,1}^- \geq \beta_n$ and

$$\int_{m_{v,1}^-}^{b_{v,1}^-} e^{-W(x)+W(m_{v,1}^-)}\mathrm{d}x \geq \int_{\mu_n}^{\beta_n} e^{-W(x)+W(\mu_n)}\mathrm{d}x > \frac{1}{d\log(n-1)}.$$

According to Lemma 4.6.3, if n is large enough, $h_{n-1} > 2^{n-1}$. Thereby $\mathbb{P}$-a.s., for v large enough,

$$\int_{m_{v,1}^-}^{b_{v,1}^-} e^{-W(x)+W(m_{v,1}^-)}\mathrm{d}x \geq \frac{1}{d(\log_2 v - \log_2 2)}.$$

When d tends to $e^2/2$, we obtain the result of the proposition.

For the upper bound, fix $d > 16/j_0^2$. One more time, thanks to Lemma 4.6.2 and Borel-Cantelli lemma, $\mathbb{P}$-a.s. for n large enough,

$$\int_{\gamma_{n-1}}^{M_n} e^{-W(x)+W(\mu_n)}\mathrm{d}x < d\log(n-1).$$

For v large enough, $\mathbb{P}$-a.s., there is a unique $n \in \mathbb{N}^*$ such that $h_{n-1} < v - c_1\log v \leq h_n$. Therefore $m_{v,1}^- = \mu_n$ and $b_{v,1}^- \leq M_n$ and so

$$\begin{aligned}\int_{a_{v,1}^-}^{b_{v,1}^-} e^{-W(x)+W(m_{v,1}^-)}\mathrm{d}x \leq& \int_0^{M_n} e^{-W(x)+W(\mu_n)}\mathrm{d}x \\ \leq& \gamma_{n-1}e^{-W(\mu_{n-1})+W(\mu_n)} + \int_{\gamma_{n-1}}^{M_n} e^{-W(x)+W(\mu_n)}\mathrm{d}x\end{aligned}$$

As $2 < \exp(1) < 3$, if n is large enough, according to Lemma 4.6.3,

$$\gamma_{n-1} e^{-W(\mu_{n-1})+W(\mu_n)} \leq 3^n e^{-2^n} \text{ and}$$

$$\int_{\gamma_{n-1}}^{M_n} e^{-W(x)+W(\mu_n)} \mathrm{d}x \leq d \log(n-1) \leq d(\log_2 v - \log_2 2).$$

Thus,

$$\limsup \frac{1}{\log_2 v} \int_{a_{v,1}^-}^{b_{v,1}^-} e^{-W(x)+W(m_{v,1}^-)} \mathrm{d}x \leq d.$$

When d tends to $16/j_0^2$, this gives the upper bound. $\square$

We can now come back to the local time process.

4.6.2 End of the proof of Theorem 4.1.1

The previous results allow to know the asymptotic behavior of $\mathbf{L}^*$. Using (4.9) with $v_n = n^{2/3}$ and Borel-Cantelli lemma, we obtain that, $\mathbb{P}$-almost surely for n large enough,

$$\frac{e^{v_n}}{J_{v_n} + 2/v_n} \leq \mathbf{L}^*(e^{v_n}) \leq \frac{e^{v_n}(1 + v_n^{-7})}{j_{v_n}(1 - v_n^{-7})}.$$

Thereby Proposition 4.6.4 gives the following inequalities,

$$\limsup_{n\to\infty} \frac{\mathbf{L}^*(e^{v_n})}{e^{v_n} \log_2 v_n} \leq \frac{1}{\liminf j_{v_n} \log_2 v_n} \leq e^2/2 \text{ and}$$

$$\liminf_{n\to\infty} \frac{\log_2 v_n}{e^{v_n}} \mathbf{L}^*(e^{v_n}) \geq \liminf_{n\to\infty} \frac{\log_2 v_n}{J_{v_n}} \geq \frac{j_0^2}{64}.$$

Denote by $[x]$ the integer part of x, as $\mathbf{L}^*$ is non decreasing, we get

$$\frac{j_0^2}{64} \leq \liminf_{v\to\infty} \frac{\log_2 [v^{3/2}]^{2/3}}{e^{[v^{3/2}]^{2/3}}} \mathbf{L}^*(e^{[v^{3/2}]^{2/3}}) \leq \liminf_{v\to\infty} \frac{\log_2 v}{e^v} \mathbf{L}^*(e^v)$$

and similarly $\limsup_{v\to\infty} \frac{\mathbf{L}^*(e^v)}{e^v \log_2 v} \leq e^2/2$.

For the last inequality of Theorem 4.1.1, (4.10) with $v_n = e^n$ and Borel-Cantelli lemma imply that, $\mathbb{P}$-almost surely for n large enough,

$$\frac{e^{v_n}}{I_{v_n,1} + \widehat{I}_{v_n,1} + 2e^{-n}} \leq \mathbf{L}^*(e^{v_n}) \leq \frac{e^{v_n}(1 + e^{-7n})}{I_{v_n,1} \wedge \widehat{I}_{v_n,1}(1 - e^{-7n})}.$$

Then Lemma 4.6.1 yields directly

$$\liminf_{v\to\infty} \frac{\log_2 v}{e^v} \mathbf{L}^*(e^v) \leq \frac{e^2 \pi^2}{4}.$$

And the proof of the theorem is completed.

4.6.3 End of the proof of Theorem 4.1.2

Fix $c_0 > 10$. According to (4.8) used with $v_n = n$ and Borel-Cantelli lemma, $\mathbb{P}$-almost surely for n large enough,

$$\nu_{e^n}(\overline{A}_n) \leq e^n / n^{c_0 - 10}.$$

As $t \to \nu_t(A)$ is a nondecreasing function for every Borel set A, $\mathbb{P}$-almost surely for t large enough,

$$\nu_t(\overline{A}_{[\log t]+1}) \leq e \frac{t}{(\log t)^{c_0 - 10}}$$

To obtain the theorem, we need to find a bound for the width of $A_{[\log t]+1}$. Therefore, we only have to estimate the behavior of the processes $d_{v,i}^-$ and $e_{v,i}^-$. Introduce the following sequences

$$\begin{aligned} \forall n \geq 1,\ \delta_n &:= \sup\{\mu_n \leq x \leq \beta_n,\ W(x) - W(\mu_n) \leq nc_0 \log 3 + \log 8\}, \\ \epsilon_n &:= \inf\{\gamma_{n-1} \leq x \leq \mu_n,\ W(x) - W(\mu_n) \leq nc_0 \log 3 + \log 8\}. \end{aligned}$$

Lemma 4.6.5. *Let $\epsilon > 0$. Then $\mathbb{P}$-a.s., for n large enough,*

$$\delta_n - \mu_n \leq ((n-1)\log 2)^{4+\epsilon} \text{ and } \mu_n - \epsilon_n \leq ((n-1)\log 2)^{4+\epsilon}.$$

Proof. Notice that for any $u > 0$, $n \in \mathbb{N}^*$,

$$\begin{aligned} &\mathbb{P}(\delta_n - \mu_n > u | h_{n-1} = x) \\ =&\mathbb{P}\left(\underline{W}(\mu_n + u, \beta_n) - W(\mu_n) < nc_0 \log 3 + \log 8 | h_{n-1} = x\right). \end{aligned}$$

As the law of the environment near μ_n is the one of a Bessel process R of dimension 3 started at 0 (Theorem 4.2.4), we have

$$\begin{aligned} \mathbb{P}(\delta_n - \mu_n > u | h_{n-1} = x) &= \mathbb{P}\left(\min_{u \leq y \leq \tau_R(x)} R(y) < nc_0 \log 3 + \log 8 \right) \\ &\leq \mathbb{P}\left(\min_{u \leq y < \infty} R(y) < nc_0 \log 3 + \log 8 \right). \end{aligned}$$

The left member of the inequality does not depend on x, so it is also an upper bound for $\mathbb{P}(\delta_n - \mu_n < u)$. According to Proposition 3.5, Chap VI in [67], $\min_{u \leq y < \infty} R(y)$ has the same law as the supremum of a Brownian motion, therefore with $u = ((n-1)\log 2)^{4+\epsilon}$,

$$\mathbb{P}(\delta_n - \mu_n > ((n-1)\log 2)^{4+\epsilon}) \leq \frac{K}{n^{1+\epsilon/2}}.$$

So Borel-Cantelli lemma gives the first result and the second one can be obtained in a similar way. ∎

For v large enough, there is a unique integer $n \geq 1$ such that $h_{n-1} < v - c_1 \log v \leq h_n$ and so $\mu_n = m^-_{v,1}$. Moreover, according to Lemma 4.6.3, $2^{n-1} \leq v \leq 3^n$, then $c_0 \log v \leq c_0 n \log 3$ and

$$d^-_{v,1} - m^-_{v,1} \leq \delta_n - \mu_n \leq (\log 2^{n-1})^{4+\epsilon} \leq (\log v)^{4+\epsilon}.$$

We have the same upper bound for $m^-_{v,1} - e^-_{v,1}$ and thereby Theorem 4.1.2 is proven.

Chapitre 5

Une application au décodage de l'ADN

5.1 Introduction

5.1.1 The physical approach

In this introduction we first summarize some ideas and results of the works of V. Baldazzi, S. Cocco, E. Marinari and R. Monasson ([8], [9]), and S. Cocco and R. Monasson [22] who are interested in a method for the sequencing of DNA molecules. They study a mechanical way, described below, instead of traditional bio-chemical or gel electrophoresis technics. These experiments for mechanical unzipping were first realized by Bockelmann, Helsot and coworkers [15] and [14]. The principle is based on the fact that the force which links the two bases of a given pair depends on whether it is a $C-G$ or a $A=T$ (see Figure 5.1). Indeed the links $A=T$ is weaker for biochemical reasons than the link between $C-G$. Moreover there is also some stacking effects between adjacent bases, that is to say, the force needed to break, for example, the link $C-G$ is different if the C is following by a A, or a T. This last factor is not negligible (see the table below) and therefore must be taken into account if we want the model to be as sharp as possible.

We now give a brief description of the experiment (for more details see [8]), the extremities of the DNA molecule are stretched apart under a force f. The force f is chosen in such way that it is large enough so the molecule can be totally unzipped. However f is also not too strong so that naturally the molecule rebuild itself. Though there is back and force movement of the number of open pair bases, this back and force movement generates a signal which can be measured by biologists. This signal can be modelised by a birth and death process with random probabilities of transition.

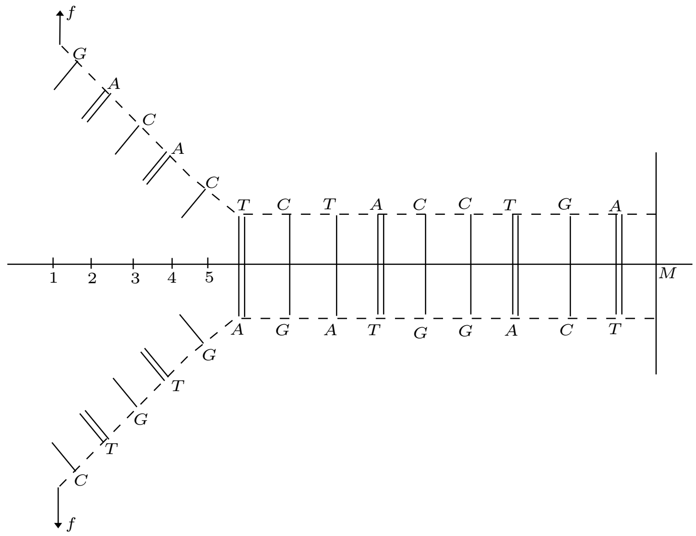


Figure 5.1: $X_{\cdot} = 5$, $b_1 : C - G$, $b_2 : T = A$

g_0	A	T	C	G
A	1.78	1.55	2.52	2.22
T	1.06	1.78	2.28	2.54
C	2.54	2.22	3.14	3.85
G	2.28	2.52	3.90	3.14

Figure 5.2: Binding free energies (units of $k_B T$)

5.1.2 The model

We denote by M the length of the DNA chain and by $(b_1, b_2, \cdots, b_M)$ the sequence of bases of one of the strand of the molecule. So b_i is the i^{th} base which can be either a A, a T, a C or a G and the corresponding base of the other strand can be deduced. We will consider both a discrete and continuous time-sequence of the number of open base pairs, the first one is denoted X, the second one Y. We now make the link between X (and Y) and b. For this, we need some physical notions: the free energy excess g when the first x base pairs of the molecule are open is

$$g(x) := \sum_{i=1}^{x} g_0(b_i, b_{i+1}) - x g_1(f).$$

There are two different parts: first, $g_0(b_i, b_{i+1})$ is the binding energy of the pair i. Notice that stacking effects are taken into account: g_0 depends on the base content b_i and on the next pair b_{i+1}. The second contribution $g_1(f)$ is the work to stretch under a force f the open part of the two strands when one more base pair is opened, especially g_1 increases when f does. Note that g_1 is known, whereas $\sum_{i=1}^{x} g_0(b_i, b_{i+1})$ is random as we are looking for the b_i's. A typical trajectory of g is given in [9] page 7, it looks like Figure 5.3.

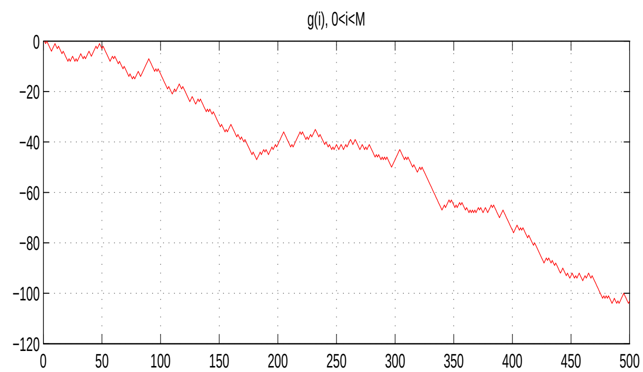


Figure 5.3: A typical trajectory of g, $M = 500$

The number of opened base pairs evolves randomly with a probability directly connected to the difference of free energy g between two consecutive base pairs. Therefore it can be represented by a random walk in random environment:

The discrete case is defined as follows, assume that the random sequence $g_0 := (g_0(b_x, b_{x+1}), 1 \leq x \leq M-1)$ is fixed, then the probabilities of transition of the number of open pairs, are given by, for all $2 \leq x \leq M-1$

$$p_x = \mathbb{P}(X_{.+1} = x+1 | X_. = x, g_0) := \frac{1}{1 + \exp\left(\beta(g(x) - g(x-1))\right)}, \qquad (5.1)$$

where β is a constant parameter which is proportional to the inverse of the temperature, also we assume $p_1 = 1$. Note that this definition is such that the larger is f the greater is the probability to open a new pair. We easily get a simple expression for this probability which is

$$\mathbb{P}(X_{.+1} = x+1 | X_. = x,\ g_0) = \frac{1}{1 + \exp(\beta \Delta g(b_x, b_{x+1}))}, \qquad (5.2)$$

where we denote

$$\Delta g(b_x, b_{x+1}) := g_0(b_x, b_{x+1}) - g_1(f). \qquad (5.3)$$

Formula (5.2) shows that we only need to have local information on the sequence b to get the probability of transition at site $x+1$. Though in the discrete case $X_{.}$ can only move forward with probability p_x or backward with probability $1-p_x$. We will discuss about some results on this well known model in the next section. A typical trajectory of X looks like Figure 5.4.

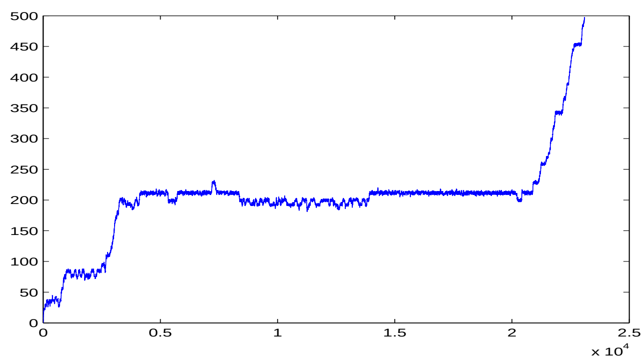


Figure 5.4: A typical trajectory of the number of unzipping pairs, $M = 500$.

For *the continuous time model*, the physicists also take into account the time it takes X to go from a site to another. Thus we introduce a second time continuous model Y. Given the g_0, when Y is at the site x, it jumps in $x+1$ with rate $re^{-\beta g_0(b_x,b_{x+1})}$ and in $x-1$ with rate $re^{-\beta g_1(f)}$ where r is a constant which value depends on biological parameters. That is, given the DNA sequence b, Y is a Markov process with finite state space $\{1,\dots,M\}$ killed when it hits M whose transition rates are for $x>1$,

$$p(x,y) = \begin{cases} re^{-\beta g_0(b_x,b_{y+1})} & \text{if } y = x+1, \\ re^{-\beta g_1(f)} & \text{if } y = x-1, \\ -r\left(e^{-\beta g_0(b_x,b_{x+1})} + e^{\beta g_1(f)}\right) & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

and for $x=1$,

$$p(1,y) = \begin{cases} re^{-\beta g_0(b_1,b_2)} & \text{if } y = 2, \\ -re^{-\beta g_0(b_1,b_2)} & \text{if } y = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The process Y can be represented as the couple (X,T) where X is the sequence of the discrete jumps and has the same law as in (5.1) and T is the sequence of the successive times spent in each site between two jumps.

We describe now briefly same results obtained by the physicists with the continuous time case

5.1.3 Some results obtained by the physicists

In their paper [8], [9], [22], they assume first that the model is without stacking effect, considering that g_0 is only a function of b_x and that $(g_0(b_x), x)$ is a sequence of independent and identically distributed random variables. In this case they compute the maximum likelihood estimator for b_x. For a better accuracy they consider several total unzipping instead of a single one, that is to say they look at a sequence of R independent trajectories $(Y_{\cdot}^{(l)}, l \leq R)$. In a second step they study the decreasing of the probability that this estimator gives a wrong sequence, and they show that this probability decreases exponentially:

$$\forall i \leq M, \ \mathbb{P}(b_i \neq \hat{b}_i) \leq \exp(-R/R_c(i)).$$

The constant R_c is also estimated numerically. For the general case (with stacking effects) they use Viterbi algorithm [91] to compute the maximum of likelihood. Then they estimate the probability to be wrong with both analytics and numerical methods, they get a similar result than for the independent case.

After some discussions with S. Cocco and R. Monasson some questions rise. Is it possible to get a general and rigorous method which can be applied to all these cases ? How the choice of the force can be used in order to improve the results ? What is the difference between the discrete and continuous time model ? We study all those questions in the present paper.

5.1.4 A mathematical point of view

First we would like to recall some basic facts for the discrete time model. If we forget, for the moment, that the state space is finite, $(X_k, \ k \in \mathbb{N})$ is a random walk on a random environment on $\mathbb{Z}$ as Solomon defined it in [77]. We know, for example that if the $g_0(b_x, b_{x+1}) = g_0(b_x)$ are i.i.d with mean zero and $g_1 = 0$, then X is almost surely recurrent and transient on the other case. For the recurrent case, X is a Sinai's walk [74], for the transient one, the first study is due to H. Kesten, M.V. Kozlov, F. Spitzer [53]. Here we are interested on what a trajectory of the walk can say about the environment, this aspect has not been studied a lot, there is a paper of O. Adelman, N. Enriquez [2] and for the special case of Sinai's walk a paper of P. Andreoletti [6]. More precisely [6] shows that the $g(x)$ can be estimated from a single trajectory of the walk by studying the asymptotic (in time) of the local time which is the amount of time the walk spends in one site. However this approach can not be used to give informations on a particular site, typically on $g_0(x)$ for a given x.
To move from Solomon walks to the problem asked by the physicists we have

to make a sacrifice, more especially we are no longer interested in asymptotic in time. Indeed if the time goes to infinity that means that either we have to wait a very long time to reach the end of the molecule, or that if it is reached it can move back to the beginning. This last case is not possible because when the end of the molecule is reached then the two separate strands will not be able to reform the molecule properly. In compensation, we only have to study the process X or Y until it reaches M, that is until time

$$\tau_M = \inf\{k > 0, X_k = M\}. \tag{5.4}$$

So we are interested in the discrete time process $(X_k,\ k \le \tau_M)$ and the continuous one $Y = (X_k, T_k,\ k \le \tau_M)$. Note also that M is the length of the DNA molecule, in term of the number of pairs, which can be big but finite. The other good news is the fact that the DNA molecule can be unzipped a large number of times, we have called, this number R, and we will be looking at asymptotic in this variable. So we are looking at R independent trajectories denoted $(Z^{(l)}_{t_l},\ 1 \le l \le R,\ 0 \le t_l \le \tau^{(l)}_M)$ of random walks on a same unknown environment b with $\tau^{(l)}_M$ the first time the walk l hits M (Z is either X or $Y = (X, T)$). We state most of our results without any assumptions on the distribution of the sequence b. However to simplify the expressions we assume sometimes that all molecules are equiprobable.
The method is based on the fact that, given the trajectory of a random walk (or R random walks) on an environment b, the probability that a given estimator $\hat{b}$ gives a good sequence (typically $\mathbb{P}(b = \hat{b})$) depends only on elementary functions of the trajectory of this random walk.
For the *discrete time*, the important quantities are the number of times X goes from x to $x+1$ or to $x-1$:

$$L^{+,(l)}_x := \sum_{k=0}^{\tau^{(l)}_M - 1} \mathbf{1}_{X^{(l)}_k = x; X^{(l)}_{k+1} = x+1}, \quad L^{-,(l)}_x := \sum_{k=0}^{\tau^{(l)}_M - 1} \mathbf{1}_{X^{(l)}_k = x; X^{(l)}_{k+1} = x-1},$$

$$L^{+,R}_x := \sum_{l=1}^{R} L^{+,(l)}_x \quad \text{and} \quad L^{-,R}_x := \sum_{l=1}^{R} L^{-,(l)}_x.$$

For the *continuous time*, we have also to consider the total time spent in each site until the instant $\tau^{(l)}_M$ (which is as in the discrete case the hitting time of M for the processes $X^{(l)}$): for any $x \in [1, M]$,

$$S^{(l)}_x = \sum_{i=0}^{\tau^{(l)}_M} T^{(l)}_x \mathbf{1}_{X^{(l)}_i = x} \quad \text{and} \quad S^R_x = \sum_{l=1}^{R} S^{(l)}_x.$$

We will denote by X^R (Y^R in the continuous case) the σ-field generated by the trajectories of the R independent random walks killed when they hit the coordinate M. $\mathbb{P}$ denotes the probability distribution of the whole system, whereas P^α is the probability distribution for a given sequence of nucleotides α. Also E^α (resp. Var^α for the variance) is the expectation associated to P^α.

In Section 5.2, we start by the estimation base by base, we define the *information* at site x for both cases and show that the expression of the probability to get a given base at a site x conditionally on the trajectories are a simple function of the information. Then we study the asymptotic (in R) of the probability that the maximum likelihood estimator gives a wrong base, we define and study a typical number of unzipping R_c which measures the quality of our prediction. In a second time we are interested in the estimation of the whole molecule, we start with a general expression of the probability to get a specific sequence given the trajectories of R random walks. We show that the maximum likelihood estimator converges, then we study the probability to make at least h mistakes by considering this estimator, and study the decreasing of the probability to be wrong. We focus on the continuous case, and just quote the differences with the discrete case.

In Section 5.3, we focus on some possible improvements. The first one consists on modifying locally the force in order to trap the system in a specific region. It has a direct effect on the time spent in this region and therefore on the quality of the prediction. The second one consists also in modifying the force, but this time it is function of the binding energies.

5.2 Bayes estimator, asymptotics in R and typical number of needed unzipping R_c

In this section we will always assume that f is constant. We start with the estimation site by site:

5.2.1 Prediction site by site

We start with a general proposition true both for continuous and discrete time cases, then we discuss the differences between the two cases. First we define the following function i_x, it is called local information at site x of the system, it differs for the two cases. Let $x \in \{2, \cdots, M-1\}$ and $\alpha_{x-1}, \alpha_x, \alpha_{x+1} \in \{A, T, C, G\}^3$.

For the discrete case, the information is defined by

$$\begin{aligned} i_x(\alpha_{x-1},\alpha_x,\alpha_{x+1}) :=& \\ L_x^{+,R}\log(1+e^{\beta\Delta g(\alpha_x,\alpha_{x+1})}) &+ L_x^{-,R}\log(1+e^{-\beta\Delta g(\alpha_x,\alpha_{x+1})}) \\ +L_{x-1}^{+,R}\log(1+e^{\beta\Delta g(\alpha_{x-1},\alpha_x)}) &+ L_{x-1}^{-,R}\log(1+e^{-\beta\Delta g(\alpha_{x-1},\alpha_x)}). \end{aligned}$$

and *for the continuous case,*

$$\begin{aligned} i_x(\alpha_{x-1},\alpha_x,\alpha_{x+1}) :=& \beta g_0(\alpha_x,\alpha_{x+1})L_x^{+,R} + S_x^R r e^{-\beta g_0(\alpha_x,\alpha_{x+1})} \\ &+ \beta g_0(\alpha_{x-1},\alpha_x)L_{x-1}^{+,R} + S_{x-1}^R r e^{-\beta g_0(\alpha_{x-1},\alpha_x)}. \end{aligned}$$

We are now ready to state the

Proposition 5.2.1. *For all $x \in \{2,\cdots,M-1\}$, and for $\alpha_x \in \{A,T,C,G\}$, denoting $b^x = (b_1,b_2,\cdots,b_{x-1},b_{x,+1},\cdots,b_{M-1})$, we have*

$$\mathbb{P}\left(b_x = \alpha_x | Z^R,\ b^x\right) = \frac{\exp(-I_x(\alpha_x,b))}{\sum_{\bar{\alpha}_x}\exp(-I_x(\bar{\alpha}_x,b))} \tag{5.5}$$

where

$$I_x(u,b) = I_x(u,b)(Z^R) := i_x(b_{x-1},u,b_{x+1}) - \log\mathbb{P}(b_x = u|b^x),$$

and Z^R is either X^R for the discrete case or Y^R for the continuous one. The maximum likelihood estimator $\hat{b}_x$ for b_x, is given by:

$$\hat{b}_x = \sum_{\alpha_x\in\{A,T,C,G\}} \alpha_x \mathbf{1}_{\{I_x(\alpha_x,b)=\min_{\bar{\alpha}} I_x(\bar{\alpha},b)\}}. \tag{5.6}$$

and we have for all R large enough

$$\mathbb{P}\left(\hat{b}_x \neq b_x | Z^R, b^x\right) = \exp\left(-R/R_c(x) + \epsilon_x(R)\right), \tag{5.7}$$

where $R_c(x)$ is called the typical number of random walks at site x, *it is defined by*

$$1/R_c(x) := \lim_{R\to\infty} \frac{-\log\mathbb{P}\left(\hat{b}_x \neq b_x | Z^R, b^x\right)}{R}. \tag{5.8}$$

Moreover, $\epsilon_x(R) \lesssim (2R\log\log R)^{1/2}((Var^b L_x^{+,(1)})^{1/2} + (Var^b L_x^{-,(1)})^{1/2})$ in the discrete case and $\epsilon_x(R) \lesssim (2R\log\log R)^{1/2}((Var^b L_x^{+,(1)})^{1/2} + (Var^b S_x^{(1)})^{1/2})$ in the continuous one. We denote $a \lesssim b$ if there exists a strictly positive constant r such that $a \leq rb$.

We will give the proof of a more general result in Section 5.2.2 so we do not give any details here.

Notice that $1/R_c(x)$ is no more and no less the rate function in the large deviation theory. The above proposition is general and gives only few informations on the decreasing of the probability to be wrong, so we now separate the two cases, and discuss about R_c.

The discrete case. First define the function $G_a : \mathbb{R} \to \mathbb{R}_+$,

$$G_a(u) := \log\left(\frac{1+e^{\beta u}}{1+e^{\beta a}}\right) + e^{\beta a}\log\left(\frac{1+e^{-\beta u}}{1+e^{-\beta a}}\right).$$

Recall that Δg is defined in (5.3). Thanks to the law of large numbers,

$$1/R_c(x) \approx \frac{1}{\bar{p}_{x-1}}\Delta G^-(b_{x-1}) + \frac{1}{\bar{p}_x}\Delta G^+(b_x),$$

with:

$$\begin{aligned}
\Delta G^-(b_{x-1}) &:= \min_{\alpha_x \neq \hat{b}_x}(G_{\Delta g(b_{x-1},\hat{b}_x)}(\Delta g(b_{x-1},\alpha_x)),\\
\Delta G^+(b_{x+1}) &:= \min_{\alpha_x \neq \hat{b}_x}(G_{\Delta g(\hat{b}_x,b_{x+1})}(\Delta g(\alpha_x,b_{x+1})).
\end{aligned}$$

$$\frac{1}{\bar{p}_x} = \frac{1}{\bar{p}_x(b)} := \sum_{k=x+1}^{M-1}\exp\left(\beta\left(g(k)-g(x)\right)\right)+1 = \frac{1}{P^b_{x+1}(\tau_x \leq \tau_M)}. \tag{5.9}$$

Note that we want $\Delta G^-(b_{x-1})$ and $\Delta G^+(b_{x+1})$ as large as possible, both of them measure the difference between the correct information which involves $\hat{b}_x$, and the others informations. Unfortunately they are possibly very small:

$$G_a(u) = \frac{\beta^2(u-a)^2}{2(e^{-\beta a}+1)} + o(u-a)^2, \tag{5.10}$$

thus,

$$\Delta G^-(b_{x-1}) \approx \frac{\beta^2}{2(e^{-\beta a}+1)}\min_{\alpha_x \neq \hat{b}_x}\left(\Delta g(b_{x-1},\alpha_x) - \Delta g(b_{x-1},\hat{b}_x)\right)^2$$

when this difference is small. Moreover even if this difference is not small, ΔG^+ and ΔG^- can however be small: indeed, assume $u<0$ and $a<0$, then for large β,

$$G_a(u) \approx \exp(\beta a)(\exp(\beta(u-a)) - 1 - \beta(u-a)). \tag{5.11}$$

This situation may appear when f is large and the binding energy at site x of the molecule is weak. We will see in Section 5.3 a method to avoid this kind of situation. We also have the following inequality:

$$\frac{1}{\bar{p}_x} \geq \exp(\beta(\max_{x \leq l \leq M}(g(l) - g(x)))) = \exp(\beta M_x), \tag{5.12}$$

with $M_x := \max_{x \leq l \leq M} \left\{ \sum_{k=x+1}^{l} g_0(b_k, b_{k+1}) - (l-x) g_1(f) \right\}$. As expected, the convergence is better if there are obstacles in the path from x to M. Finally, we have

$$1/R_c(x) \geq \exp(\beta M_{x-1}) \Delta G^- + \exp(\beta M_x) \Delta G^+, \tag{5.13}$$

with

$$\Delta G^- := \min\{\Delta G^-(\gamma), \gamma \in \{A, T, C, G\}\}$$

and

$$\Delta G^+ := \min\{\Delta G^+(\gamma), \gamma \in \{A, T, C, G\}\}.$$

A few words about $\epsilon_x(R)$ which comes from the iterated logarithm law: typically $(Var^b(L_x^+))^{1/2}$ as well as $(Var^b(L_x^-))^{1/2}$ behaves like $1/\bar{p}_x$ (see Lemma 5.2.2). Then $\epsilon_x(R)$ is always negligible comparing to the main term.

Formula useful for the estimation. As we have seen above, $R_c(x)$ characterizes locally the environment. However, what is really important to control the quality of the estimation at a point x is not the number of walks R but the total number of passages at this point, L_x^R. That is why we define the *typical number of visits at site* x, $L_c(x)$ by

$$1/L_c(x) := \lim_{R \to +\infty} \frac{-\log \mathbb{P}\left(\hat{b}_x \neq b_x | Z^R, b^x\right)}{L_x^R}. \tag{5.14}$$

One more time, we have $1/L_c(x) \geq \Delta G^+ \wedge \Delta G^-$.

Total amount of time to reach M. An other important factor is the time required to unzip totally R times the DNA molecule. It should not be too large. This time is given by:

$$\tau_M^R = \sum_{l=1}^{R} \tau_M^{(l)} = \sum_{l=1}^{R} \sum_{x=1}^{M-2} (L_{x-1}^{+,(l)} + L_x^{+,(l)} - 1).$$

so we have

$$
\begin{aligned}
R\exp(\beta \max_x M_x) \lesssim E^b\left[\tau_M^R\right] =& R\sum_{x=1}^{M-2}\left(\frac{1}{\bar{p}_{x-1}}+\frac{1}{\bar{p}_x}-1\right)\\
\lesssim& RM\exp(\beta \max_x M_x).
\end{aligned}
$$

Here is a problem with β: indeed as we have seen in the previous paragraph (see (5.12)), large β can lead to a better prediction, however it slows down the system. Of course it is worse if there is an obstacle between x and M because in this case M_x is large too.

The continuous time case.
Like for the discrete case we first define a function $F:\mathbb{R}\to\mathbb{R}_+$ by

$$
\begin{aligned}
&F(u) = e^{\beta u} - 1 - \beta u \text{ and}\\
&\Delta F^- = \min\left(F(g_0(\alpha,u)-g_0(\alpha,v)), \alpha, u, v \in \{S,W\}, u\neq v\right),\\
&\Delta F^+ = \min\left(F(g_0(u,\alpha)-g_0(v,\alpha)), \alpha, u, v \in \{S,W\}, u\neq v\right),
\end{aligned}
$$

then a similar analysis than for the discrete case leads to

$$
1/R_c(x) \geq \frac{\Delta F^+}{\bar{p}_x} + \frac{\Delta F^-}{\bar{p}_{x-1}} \text{ and } 1/L_c(x) \geq \Delta F^+ \wedge \Delta F^-.
$$

Note that the bad case observed for the discrete case (see equation (5.11)) does not appear here, however when $u-a$ is small, $F(u-a)$ is as $G_a(u)$ of the order of $a-u$ but the constant is better.

In the next section we look at the entire molecule, we define global information and study the decreasing of the probability to make a mistake by using the maximum likelihood estimator.

5.2.2 Inferring the whole molecule

First we present the joint distribution of $L_x^{+,(1)}, L_x^{-,(1)} = L_{x-1}^{+,(1)} - 1$, in fact it is not more difficult to get the joint distribution of $(L_x^{+,(1)}, 1\leq x\leq M-1)$ and as we have not found it in the literature, we first prove the following lemma for one random walk and a constant force f:

Lemma 5.2.2. *If we denote $k = (k_i\ ,\ i \in \{1, \dots, M-2\})$, then the distribution of $L^+ := L^{+,(1)}$ is*

$$
\begin{aligned}
P^b\left(L^+ = k\right) =& p_{M-1}(1-p_{M-1})^{k_{M-2}-1} \prod_{i=2}^{M-2} \binom{k_i + k_{i-1} - 2}{k_i - 1} p_i^{k_i}(1-p_i)^{k_{i-1}-1},\\
=& \prod_{i=2}^{M-1} \binom{k_i + k_{i-1} - 2}{k_i - 1} p_i^{k_i}(1-p_i)^{k_{i-1}-1}
\end{aligned}
$$

with $k_{M-1} = 1$. In particular, for $x \in \{2, \cdots, M-1\}$,

$$
\begin{aligned}
&P^b\left(L_x^+ = k_x, L_x^- = k_{x-1}\right) = P^b\left(L_x^+ = k_x, L_{x-1}^+ = k_{x-1}+1\right)\\
=&\binom{k_x + k_{x-1} - 1}{k_x - 1}(1-p_x)^{k_{x-1}}(p_x(1-\bar{p}_x))^{k_x-1}(p_x\bar{p}_x),
\end{aligned}
\tag{5.15}
$$

where $\bar{p}_x$ is given by (5.9). Moreover

$$
\begin{aligned}
E^b\left(L_x^+\right) = \frac{1}{\bar{p}_x},\ E^b\left(L_x^-\right) = \frac{e^{\beta \Delta g(b_x, b_{x+1})}}{\bar{p}_x} \text{ and } E^b\left(S_x^{(1)}\right) = \frac{e^{\beta g_0(b_x, b_{x+1})}}{r\bar{p}_x},\\
Var^b\left(L_x^+\right) = \frac{1}{\bar{p}_x}\left(\frac{1}{\bar{p}_x} - 1\right) \text{ and } Var^b\left(S_x^{(1)}\right) = \frac{e^{2\beta g_0(b_x, b_{x+1})}p_x}{r^2\bar{p}_x}.
\end{aligned}
$$

Proof. The equality (5.15) of the lemma can easily be obtained by using the Markov property of X given b, the mean and the variance of L_x^+ and $S_x^{(1)}$ are direct consequences. Therefore we just prove the expression of the joint distribution of L^+. Define now for $n \geq 1$, $A_n := \bigcap_{j=n}^{M-1}\{L_j^+ = k_j\}$ where $k_{M-1} = 1$ (there is always only one jump from $M-1$ to M),

$$
\begin{aligned}
P^b\left(L^+ = k\right) &= P^b\left(A_1\right) = P^b\left(L_1^+ = k_1 \middle| A_2\right) P^b\left(A_2\right)\\
&= P^b\left(L_1^+ = k_1 \middle| L_2^+ = k_2\right) P^b\left(A_2\right)
\end{aligned}
$$

where the second equality comes from the Markov property of the walk X given b. Equation (5.15) implies for any $x \in \{2, \cdots, M-1\}$,

$$
P^b\left(L_{x-1}^+ = k_{x-1} \middle| L_x^+ = k_x\right) = \binom{k_x + k_{x-1} - 2}{k_x - 1}(1-p_x)^{k_{x-1}-1}p_x^{k_x}
$$

Thus,

$$
P^b\left(L^+ = k\right) = \binom{k_2 + k_1 - 2}{k_2 - 1} p_2^{k_2}(1-p_2)^{k_1-1} P^b\left(A_2\right)
$$

and we get the result of Lemma 5.2.2 recursively. $\square$

We now define the *global information* I of the whole molecule. Let $\alpha \in \{A,T,C,G\}^M$.
For the discrete case X^R,

$$I(\alpha) := -\log \mathbb{P}(b=\alpha) + \sum_{x=1}^{M-1} L_x^{+,R} \log(1+e^{\beta \Delta g(\alpha_x,\alpha_{x+1})}) + L_x^{-,R} \log(1+e^{-\beta \Delta g(\alpha_x,\alpha_{x+1})}).$$

and *for the continuous case* $Y^R = (X^R, T^R)$,

$$I(\alpha) = -\log \mathbb{P}(b=\alpha) + \sum_{x=1}^{M-1} \beta g_0(\alpha_x,\alpha_{x+1}) L_x^{+,R} + r e^{-\beta g_0(\alpha_x,\alpha_{x+1})} S_x^R \quad (5.16)$$

We now give a general result and its proof:

Theorem 5.2.3. *For any* $\alpha \in \{A,T,C,G\}^M$*, we have:*

$$\mathbb{P}\left(b=\alpha \middle| Z^R\right) = \frac{e^{-I(\alpha)}}{\sum_{\bar{\alpha}} e^{-I(\bar{\alpha})}}. \quad (5.17)$$

The maximum likelihood estimator is the element $\hat{b}$ *of* $\{A,T,C,G\}^M$ *which minimizes the function* I*. If the function*

$$G_0 : \ \alpha \to (g_0(\alpha_x,\alpha_{x+1}),\ x \in \{1,\cdots,M-1\})$$

is injective, then the maximum likelihood estimator converges almost surely to the DNA chain b*.*

Proof. We only give the proof for the continuous case, the discrete one is simpler and uses the same ideas. For a realization of $Y^R = \left(X^{(1)},\cdots,\ X^{(R)},\ T^{(1)},\cdots,\ T^{(R)}\right)$, Bayes Lemma gives :

$$\mathbb{P}\left(b=\alpha|Y^R=y\right) = \mathbb{P}\left(b=\alpha \middle| \bigcap_{l=1}^{R} \left\{X^{(l)}=x^{(l)},\ T^{(l)}=t^{(l)}\right\}\right)$$
$$= \frac{\mathbb{P}\left(\bigcap_{l=1}^{R} \left\{X^{(l)}=x^{(l)},\ T^{(l)}=t^{(l)}\right\} \middle| b=\alpha\right) \mathbb{P}(b=\alpha)}{\mathbb{P}\left(\bigcap_{l=1}^{R} \left\{X^{(l)}=x^{(l)},\ T^{(l)}=t^{(l)}\right\}\right)}$$

(We still use $\mathbb{P}$ to denote a probability density.) When $X_i^{(l)} = x_i^{(l)}$, and the environment α are given, $T_i^{(l)}$ is an exponential variable independent from the other $X^{(k)},\ T^{(k)}$, of parameter $r\left(e^{-\beta g_0(\alpha_{x_i},\alpha_{x_i+1})} + e^{-\beta g_1(f)}\right)$ if $x_i^{(l)} \in \{2,\cdots,M-1\}$ or $re^{-\beta g_0(\alpha_1,\alpha_2)}$ if $x_i^{(l)} = 1$. Thus,

$$\begin{aligned}
&\mathbb{P}\left(\bigcap_{l=1}^{R}\left\{X^{(l)} = x^{(l)},\ T^{(l)} = t^{(l)}\right\}\middle| b=\alpha\right)\\
=&\mathbb{P}\left(\bigcap_{l=1}^{R} X^{(l)} = x^{(l)}\middle| b=\alpha\right)\prod_{l=1}^{R}\prod_{i=1}^{\tau_M^{(l)}-1}\mathbb{P}(T_i^{(l)} = t_i^{(l)}|b=\alpha,\ X_i^{(l)} = x_i^{(l)})\\
=&\mathbb{P}\left(\bigcap_{l=1}^{R} X^{(l)} = x^{(l)}\middle| b=\alpha\right)(re^{-\beta g_0(\alpha_1,\alpha_2)})^{l_1^R}e^{-s_1^R re^{-\beta g_0(\alpha_x,\alpha_{x+1})}}\\
&\times\prod_{x=2}^{M-1}(r(e^{-\beta g_0(\alpha_x,\alpha_{x+1})} + e^{-\beta g_1(f)}))^{l_x^R}e^{-s_x^R r(e^{-\beta g_0(\alpha_x,\alpha_{x+1})}+e^{-\beta g_1(f)})}
\end{aligned}$$

$$\text{where } l_i^R = \sum_{l=1}^{R}\sum_{k=1}^{\tau_M^{(l)}-1}\mathbf{1}_{x_k^{(l)}=i} \quad\text{and}\quad s_i^R = \sum_{l=1}^{R}\sum_{k=1}^{\tau_M^{(l)}-1}\tau_{x^{(l)}}^{(l)}\mathbf{1}_{x_k^{(l)}=i}.$$

Moreover

$$\begin{aligned}
&\mathbb{P}\left(\bigcap_{l=1}^{R} X^{(l)} = x^{(l)}\middle| b=\alpha\right)\\
=&\prod_{x=2}^{M-1}\left(\frac{e^{-\beta g_0(\alpha_x,\alpha_{x+1})}}{e^{-\beta g_0(\alpha_x,\alpha_{x+1})} + e^{-\beta g_1(f)}}\right)^{l_x^{+,R}}\left(\frac{e^{-\beta g_1(f)}}{e^{-\beta g_0(\alpha_x,\alpha_{x+1})} + e^{-\beta g_1(f)}}\right)^{l_{x-1}^{+,R}-1}\\
=&\prod_{x=2}^{M-1}\frac{e^{-l_x^{+,R}\beta g_0(\alpha_x,\alpha_{x+1})-(l_{x-1}^{+,R}-1)\beta g_1(f)}}{\left(e^{-\beta g_0(\alpha_x,\alpha_{x+1})} + e^{-\beta g_1(f)}\right)^{l_x^R}}
\end{aligned}$$

where

$$l_i^{+,R} = \sum_{l=1}^{R}\sum_{k=1}^{\tau_M^{(l)}}\mathbf{1}_{x_k^{(l)}=i,\ x_{k+1}^{(l)}=i+1}.$$

Then we have the following equality

$$\begin{aligned}
&\mathbb{P}\left(\bigcap_{l=1}^{R}\left\{X^{(l)} = x^{(l)},\ T^{(l)} = t^{(l)}\right\}\middle| b=\alpha\right)\\
=&\prod_{x=2}^{M-1} r^{l_x^R}e^{-s_x^R r(e^{-\beta g_0(\alpha_x,\alpha_{x+1})}+e^{-\beta g_1(f)})-l_x^{+,R}\beta g_0(\alpha_x,\alpha_{x+1})-(l_{x-1}^{+,R}-1)\beta g_1(f)}.\\
&\times r^{l_1^R}e^{-s_1^R re^{-\beta g_0(\alpha_1,\alpha_2)}-l_1^{+,R}\beta g_0(\alpha_1,\alpha_2)}.
\end{aligned}$$

It is now easy to obtain the expression of $\mathbb{P}(b = \alpha | Y^R)$ of the theorem.

We now prove the convergence of the maximum likelihood estimator. According to Lemma 5.2.2, for any $x \in \{1, \cdots, M-1\}$, the strong law of large number and the central limit theorem give

$$L_x^{+,R} = \frac{R}{\bar{p}_x(b)} + \epsilon_x(R) \ \text{ and } S_x^R = \frac{Re^{\beta g_0(b_x, b_{x+1})}}{r\bar{p}_x(b)} + \epsilon_x(R)$$

Then for any $\alpha \in \{A, T, C, G\}^M$, $\mathbb{P}$-a.s. the information $I(\alpha)$ is equivalent to

$$I(\alpha) = R \sum_{x=1}^{M-1} (\beta g_0(\alpha_x, \alpha_{x+1}) + e^{-\beta(g_0(\alpha_x,\alpha_{x+1}) - g_0(b_x, b_{x+1}))}) \frac{1}{\bar{p}_x(b)} + o(R).$$

This quantity is minimal if and only if for each $x \in \{1, \cdots, M-1\}$,

$$g_0(\alpha_x, \alpha_{x+1}) = g_0(b_x, b_{x+1}).$$

So if the function G_0 is injective, then, almost surely, for R large enough, $I(\alpha)$ is minimal iff $\alpha = b$. $\square$

5.2.3 Control of the quality of the estimation for the continuous time case

In this part, we suppose that the function G_0 defined in the previous theorem is injective. Fix an $x \in \{1, \cdots, M-1\}$. The probability to make a mistake at site x when you know the environment around the point x is

$$\mathbb{P}(b_x = \hat{b}_x | Y^R,\ b^x) = \frac{e^{-i_x(b_{x-1}, \hat{b}_x, b_{x+1})}}{\sum_{\alpha_x} e^{-i_x(b_{x-1}, \alpha_x, b_{x+1})}}$$

As $\hat{b}_{x-1}$ and $\hat{b}_{x+1}$ converges to b_{x-1} and b_{x+1} according to Theorem 5.2.3, it is possible to estimate this probability by substituting the couple of variables (b_{x-1}, b_{x+1}) by $(\hat{b}_{x-1}, \hat{b}_{x+1})$. However this estimate is good only if $\hat{b}_{x-1}$ and $\hat{b}_{x+1}$ are themselves good estimates of b_{x-1} and b_{x+1}. It is also possible to obtain simpler asymptotic estimates which does not depend on b. As said before, for any $x \in \{1, \cdots, M-1\}$,

$$\begin{aligned} L_x^{+,R} = \frac{R}{\bar{p}_x} + \epsilon_x(R) \ \text{ and } S_x^R &= \frac{R}{\bar{p}_x r e^{g_0(b_x, b_{x+1})}} + \epsilon_x(R) \\ &= \frac{L_x^{+,R}}{r e^{g_0(b_x, b_{x+1})}} + \epsilon_x(R) \end{aligned}$$

where

$$\begin{aligned}\epsilon_x(R) &\lesssim (2R\log\log R)^{1/2}((Var^b L_x^{+,(1)})^{1/2} + (Var^b S_x^{(1)})^{1/2})\\ &= o((R\log\log R)^{1/2}).\end{aligned}$$

Then for any $\alpha_x \in \{A, T, C, G\}$, $i_x(b_{x-1}, \alpha_x, b_{x+1})$ is asymptotically equivalent to

$$\begin{aligned}i_x(b_{x-1}, \alpha_x, b_{x+1}) = (\beta g_0(\alpha_x, b_{x+1}) + e^{-\beta(g_0(\alpha_x, b_{x+1}) - g_0(b_x, b_{x+1}))})L_x^{+,R}\\ + (\beta g_0(b_{x-1}, \alpha_x) + e^{-\beta(g_0(b_{x-1},\alpha_x) - g_0(b_{x-1}, b_x))})L_{x-1}^{+,R} + \epsilon_x(R).\end{aligned}$$

This quantity is minimal if and only if $\alpha_x = b_x$ then

$$\sum_\alpha e^{-i_x(b_{x-1},\alpha_x,b_{x+1})} = e^{-(\beta g_0(b_x,b_{x+1})+1)L_x^{+,R} + (\beta g_0(b_{x-1},b_x)+1)L_{x-1}^{+,R} + \epsilon_x(R)}.$$

As $\mathbb{P}$-almost surely, for R large enough,

$$\hat{b}_{x-1} = b_{x-1},\ \hat{b}_x = b_x \text{ and } \hat{b}_{x+1} = b_{x+1},$$

we obtain the following asymptotic estimate for the probability to make a wrong prediction at site x:

$$\begin{aligned}&-\log\mathbb{P}(b_x \neq \hat{b}_x | Y^R,\ b^x) \geq\\ &F\big(g_0(\hat{b}_x, \hat{b}_{x+1}) - g_0(u_x^+, \hat{b}_{x+1})\big)L_x^{+,R} +\\ &F\big(g_0(\hat{b}_{x-1}, \hat{b}_x) - g_0(\hat{b}_{x-1}, u_x^-)\big)L_{x-1}^{+,R} + \epsilon_x(R)\end{aligned}$$

where

$$\begin{aligned}u_x^+ &= \operatorname{argmin}\{u \neq \hat{b}_x,\ F\big(g_0(\hat{b}_x, \hat{b}_{x+1}) - g_0(u, \hat{b}_{x+1})\big)\},\\ u_x^- &= \operatorname{argmin}\{u \neq \hat{b}_x,\ F\big(g_0(\hat{b}_{x-1}, \hat{b}_x) - g_0(\hat{b}_{x-1}, u)\big)\}.\end{aligned}$$

Therefore this estimate is still true when b^x is not given:

$$\begin{aligned}-\log\mathbb{P}(b_x \neq \hat{b}_x | Y^R) \geq &F\big(g_0(\hat{b}_x, \hat{b}_{x+1}) - g_0(u_x^+, \hat{b}_{x+1})\big)L_x^{+,R} +\\ &F\big(g_0(\hat{b}_{x-1}, \hat{b}_x) - g_0(\hat{b}_{x-1}, u_x^-)\big)L_{x-1}^{+,R} + \epsilon_x(R)\end{aligned}$$

Same kind of arguments give the probability to be wrong on a chain of length $h+1$ starting at x.

Corollary 5.2.4. *Let n_e be the number of errors done by estimating the DNA chain. Almost surely for R large enough,*

$$\begin{aligned}-\log \mathbb{P}\left(n_e \geq h\ |Y^R\right) &\geq KM_h^R + o(R)\\ &\geq KhR + o(R).\end{aligned}$$

where

$$M_h^R = \min\left(\sum_{x\in I_h} L_x^{+,R}\ /\ I_h \subset \{1,\dots,M-1\}\ ,\ |I_h| = h\right).$$

Proof. We begin with the definition of the information for the sites x to $x+h$: for $\alpha_{x-1},\cdots,\alpha_{x+h+1} \in \{A,T,C,G\}^{h+3}$,

$$\begin{aligned}i_{x,x+h}(\alpha_{x-1},\cdots,\alpha_{x+h+1}) &= -\log \mathbb{P}(b_x = \alpha_x,\dots,b_{x+h} = \alpha_{x+h})\\ &\quad + \sum_{y=x-1}^{x+h} \beta g_0(\alpha_y,\alpha_{y+1})L_y^{+,R} + re^{-\beta g_0(\alpha_y,\alpha_{y+1})}S_y^R.\end{aligned}$$

We can now express the probability we are looking for. Denote by $b^{x,x+h}$ the vector $(b_1,\cdots,b_{x-1},b_{x+h+1},\cdots,b_M)$, then

$$\begin{aligned}&\mathbb{P}\left(b_x \neq \hat{b}_x,\dots,\ b_{x+h} \neq \hat{b}_{x+h}|Y^R,\ b^{x,\dots,x+h}\right)\\ &= \sum_{\alpha_x\neq\hat{b}_x,\dots,\alpha_{x+h}\neq\hat{b}_{x+h}} \frac{e^{-i_{x,x+h}(b_{x-1},\alpha_x,\cdots,\alpha_{x+h},b_{x+h+1})}}{\sum_{\bar{\alpha}_x,\dots,\bar{\alpha}_{x+h}} e^{-i_{x,x+h}(b_{x-1},\bar{\alpha}_x,\cdots,\bar{\alpha}_{x+h},b_{x+h+1})}}\end{aligned}$$

Reasoning in the same way as in the previous proof, we obtain

$$\sum_{\bar{\alpha}_x,\dots,\bar{\alpha}_{x+h}} e^{-i_{x,x+h}(b_{x-1},\bar{\alpha}_x,\cdots,\bar{\alpha}_{x+h},b_{x+h+1})} = e^{-\sum_{y=x-1}^{x+h}(\beta g_0(b_y,b_{y+1})+1)L_y^{+,R}+o(R)}.$$

and then

$$\begin{aligned}&-\log \mathbb{P}\left(b_x \neq \hat{b}_x,\dots,\ b_{x+h} \neq \hat{b}_{x+h}|Y^R\right) \geq\\ &F\big(g_0(\hat{b}_{x-1},\hat{b}_x) - g_0(\hat{b}_{x-1},v_{x-1})\big)L_{x-1}^{+,R}+\\ &\sum_{y=x}^{x+h-1} F\big(g_0(\hat{b}_y,\hat{b}_{y+1}) - g_0(u_y,v_y)\big)L_y^{+,R}\\ &+ F\big(g_0(\hat{b}_{x+h},\hat{b}_{x+h+1}) - g_0(u_{x+h},\hat{b}_{x+h+1})\big)L_{x+h}^{+,R} + o(R)\end{aligned}$$

where

$$(u_y, v_y) = \operatorname{argmin}\{u \neq \hat{b}_y,\ v \neq \hat{b}_{y+1},\ F\big(\beta g_0(\hat{b}_y,\ \hat{b}_{y+1}) - \beta g_0(u, v)\big)\}.$$

We can have as before a lower bound which does not depend from the estimator:

$$\begin{aligned} -\log \mathbb{P}\left(b_x \neq \hat{b}_x, \ldots,\ b_{x+h} \neq \hat{b}_{x+h} | Y^R\right) &\geq K \sum_{y=x-1}^{x+h} L_y^{+,R} + o(R) \\ &\geq K(h+2)R + o(R). \end{aligned}$$

which now leads easily to the corollary. □

Corollary 5.2.4 shows in particular that the probability to make at least h mistakes decreases exponentially with h.

The discrete time case leads to very similar results, in fact the main difference is that ΔF^+ and ΔF^- are replaced by ΔG^+ and ΔG^-.

5.3 Possible improvements of the method

In this paragraph, we use the results of the previous sections to propose two simple extensions which can be used to improve the predictions. The idea of both of them is based on the fact that the force f can be modified and adapted to the context.

5.3.1 Forces depending on the site we are interested in

In this section, we will discuss about $\frac{1}{\bar{p}_x}$, which appears in the lower-bounds of $1/R_c(x)$. Like we have seen before, $\frac{1}{\bar{p}_x}$ can be large, depending on the sequence g_0 and the force at site x, we recall that

$$\frac{1}{\bar{p}_x} = \sum_{l=x+1}^{M-1} \exp\left(\beta \left\{ \sum_{k=x+1}^{l} g_0(b_k, b_{k+1}) - (l-x) g_1(f) \right\}\right)$$

For example when the force f is not too large, typically when $g_1(f) << \max(g_0(\alpha_x, \alpha_{x+1}))$, then valleys, that is to say portions of the sequences b such that $\sum_{k=x+1}^{l} g_0(b_k, b_{k+1}) - (l-x) g_1(f) >> 1$, can appear. So the quality of our prediction is good (the decreasing of the probability to be wrong is in $e^{-\mathrm{const} R e^{\beta M_x}}$) only in some specific regions of the molecule.
When the above conditions do not appear, the force can be modified in

order to slow down locally the system [1]. Assume that we are interested in a specific region centered on the point y, $[y-A, y+A]$, $A>0$ where the $(L^+_{y+x}+L^-_{y+x}, x\in[-A,A])$ or the $(S_{y+x},\ x\in[-A,A])$ are small. Then we can take for $x\in[-A,A]$,

$$g_1(f_{y+x}) = C(A-x), \tag{5.18}$$

and we get:

$$\frac{1}{\bar{p}_{y+x}} \geq \exp\left(\beta\left\{\sum_{k=y+x+1}^{A+y} g_0(b_k,b_{k+1}) - \frac{C}{2}(A-x)(A-x-1)\right\}\right)$$

especially for $x=0$,

$$\frac{1}{\bar{p}_{y}} \geq \exp\left(\beta\left\{\sum_{k=y+1}^{A+y} g_0(b_k,b_{k+1}) - \frac{C}{2}A(A-1)\right\}\right)$$

Then if $\mathbb{E}(g_0(b_k,b_{k+1})) \geq \frac{C}{2}(A-1)$ and A is large enough, $\frac{1}{\bar{p}_y}$ will be quite large too. Once again this will work if the region we are looking at is quite far from the end of the molecule, that is to say, A is large. On the other case what could be a good idea is to unzip the molecule from the other end.

We now move to another possible improvement.

5.3.2 The energy point of view: forces depending on the values of the environment

In this paragraph we do not try to find directly the sequence of bases but the associated binding energies. We denote $g_0(x)$ for $g_0(b_x,b_{x+1})$and we assume that they are K distinct values for $g_0(x)$, typically for the DNA they are given by Table 5.2. The random variables $(g_0(x),x)$ are not independent, for the DNA molecule the dependence is also given by Table 5.2. For example, the energy 1.06 can only be followed by 1.78, 1.55, 2.52 or 2.22. We also assume that the whole sequences g_0 are equiprobable.

First let us introduce some new notations. We will denote $\mu_1,\mu_2,\cdots,\mu_K$, the possible values of $g_0(.)$, they are ordered in such a way that $\mu_i>\mu_{i+1}$ for all i. We also assume that the force f can take $K+1$ different values $\{f_1,f_2,\cdots,f_{K-1},f_K,f_{K+1}=0\}$, and then, that $g_1(.)$ takes equally $K+1$ distinct values denoted $\{r_1,r_2,\cdots,r_{K-1},r_K,r_{K+1}=0\}$. We also suppose

1. According to the physicists, it is possible.

that $r_1 > r_2 > \cdots > r_{K-1}$, and

$$\begin{aligned}
&\mu_1 - r_1 < 0,\ \mu_1 - r_2 > 0,\ \forall i > 1\ \mu_i - r_2 < 0,\\
&\mu_2 - r_2 < 0,\ \mu_2 - r_3 > 0,\ \forall i > 2\ \mu_i - r_3 < 0,\\
&\cdots\\
&\mu_K - r_K < 0,\ \mu_K - r_{K+1} = \mu_K > 0,\ .
\end{aligned}$$

Let us define

$$q_m^i := (1 + e^{\beta(\mu_m - r_i)})^{-1}, \tag{5.19}$$

which is the probability to go on the right if the force f_i is applied and if the value of the environment is equal to μ_m. Notice that if f_1 is applied then for all $x \leq M$,

$$p_x := (1 + \exp(\beta(g_0(x) - r_1)))^{-1} \geq q_1^1 > 1/2,$$

and we denote $\Gamma_1 := \{x \leq M, p_x = q_1^1\}$. Then if f_2 is applied, for all $x \leq M,\ x \notin \Gamma_1$,

$$p_x \geq q_2^2, \tag{5.20}$$

in the same way as before we denote Γ_2 the sites such the equality in (5.20) is satisfied. We get a partition $\{\Gamma_1, \Gamma_2, \cdots, \Gamma_K\}$ of $\{1, \cdots, M\}$. Finally we can look at a certain number of random walks for each values taken by the force. We denote by R_j the number of random walks we consider for the force with value f_j.

From now on we will only focus on the discrete time case because it is the one where the gain is the most important, however what we suggest can be applied to the continuous time model as well. We introduce the information at site x, if the force f_j is applied then

$$i_x^j(m) := L_x^{+,R_j} \log q_m^j + L_x^{-,R_j} \log(1 - q_m^j), \tag{5.21}$$

and the relative information at site x

$$i_x^j(m,l) := i_x^j(m) - i_x^j(l). \tag{5.22}$$

We define the function $H_a : \mathbb{R} \to \mathbb{R}_+$,

$$H_a(u) := \log(1 + \exp(\beta u)) + \exp(\beta a) \log(1 + \exp(-\beta u)).$$

Proposition 5.3.1. *Assume that the forces f_k and then f_{k+1} are applied (everywhere) then, for all x, all sequence g_0^x and all estimator $\hat{g}_0(x)$,*

$$\begin{aligned}
&\mathbb{P}\left(\hat{g}_0(x)=\mu_k,\ g_0(x)=\mu_k|X^{R_k},X^{R_{k+1}},\ g_0^x\right)\\
&=\left(1+\sum_{m=1,m\neq k}^{K}\exp\left(-i_x^k(m,k)-i_x^{k+1}(m,k)\right)\right)^{-1}\mathbf{1}_{\hat{g}_0(x)=\mu_k}.
\end{aligned}$$

Let us define the following estimator:

$$\hat{g}_0(x)=\inf\left\{k>0,\ \frac{L_x^{-,R_k}}{L_x^{+,R_k}}<1,\ \frac{L_x^{-,R_{k+1}}}{L_x^{+,R_{k+1}}}>1\right\} \tag{5.23}$$

then

$$\begin{aligned}
\frac{1}{R_c^k(x)}&:=-\lim_{R_k=R_{k+1}=R\to\infty}\log\frac{1}{R}\left(\mathbb{P}\left(\hat{g}_0(x)=\mu_k,\ g_0(x)=\mu_k|X,\ g_0^x\right)-1\right)\\
&\geq\frac{H^{(k)}}{\bar{p}_x^k}+\frac{H^{(k+1)}}{\bar{p}_x^{k+1}},
\end{aligned}$$

where

$$\begin{aligned}
H^{(k)}&:=\min_{l\in\{k-1,k+1\}}H_{\mu_l-r_k}(\mu_l-r_k)-H_{\mu_k-r_k}(\mu_k-r_k)\\
H^{(k+1)}&:=\min_{l\in\{k-1,k+1\}}H_{\mu_l-r_k}(\mu_l-r_{k+1})-H_{\mu_k-r_{k+1}}(\mu_k-r_{k+1}),\ \textit{with}
\end{aligned}$$

$$\frac{1}{\bar{p}_x^l}=\frac{1}{\bar{p}_x^l(g_0^x)}:=\sum_{z=x+1}^{M-1}\exp\left(\sum_{y=x+1}^{z}g_0(y)-r_l\right)+1. \tag{5.24}$$

g_0^x is (like for b^x) the sequence $(g_0(1),g_0(2),\cdots,g_0(x-1),g_0(x+1),\cdots,g_0(M))$.

Here we avoid a bad situation seen in the first section (see (5.11)), indeed, the worst case under the force f_{k+1}, leads to: for $a=\mu_k-r_{k+1}>0, u=\mu_{k+1}-r_k+1>0$,

$$H_a(u)-H_a(a)\gtrsim(\exp(\beta(a-u))-1-\beta(a-u)), \tag{5.25}$$

which increases with β. However we have to be careful with this method, indeed in order to catch the small values of the energies, f_k should be small and slows down the system, indeed recall that

$$\begin{aligned}
E^b\left[\tau_M\right]=&R\sum_{x=1}^{M-2}\left(\frac{1}{\bar{p}_x^k}+\frac{1}{\bar{p}_{x-1}^k}-1\right)+R\sum_{x=1}^{M-2}\left(\frac{1}{\bar{p}_x^{k-1}}+\frac{1}{\bar{p}_{x-1}^{k-1}}-1\right)\\
\gtrsim&R\exp(\beta\max_x M_x^k),
\end{aligned}$$

with $M_x^k := \max_{x\le l\le M}\left\{\sum_{l=x+1}^{l} g_0(b_l,b_{l+1})-r_k\right\}$ large if k is large.
An alternative approach is first to apply a large force f_1, from 0 to $x-1$ in order to reach rapidly the region we are interested in, then apply all the forces in x and then, after $x+1$ apply a small force (for example f_K) in order to slow down the system and stay focus on x. For each value of the force $(f_i, i\le K)$ at site x, R random walks are used. More precisely f is, as before, function of the energy and now of the sites:

$$f_i(z)=f_1\mathbf{1}_{1\le z\le x-1}+f_i\mathbf{1}_{z=x}+f_K\mathbf{1}_{z\ge x+1} \tag{5.26}$$

we get

$$\begin{aligned}\frac{1}{R_c(x)} &:= -\lim_{R\to\infty}\frac{1}{R}\log\left(\mathbb{P}\left(\hat g_0(x)\neq g_0(x)|X^R,\ g_0^x, f_i(.), i\le K\right)\right)\\ &\ge \frac{1}{\bar p_x^K}(H^{\rightarrow}+H^{\leftarrow})\\ H^{\rightarrow} &:= \max_{k\le K-1}\min_{l\in\{k-1,k+1\}}(H_{\mu_l-r_k}(\mu_l-r_k)-H_{\mu_k-r_k}(\mu_k-r_k))\\ H^{\leftarrow} &:= \max_{k\le K-1}\min_{l\in\{k-1,k+1\}}(H_{\mu_l-r_k}(\mu_l-r_{k+1})-H_{\mu_k-r_{k+1}}(\mu_k-r_{k+1})).\end{aligned}$$

The main interest in the above result comparing to the preceding one is the fact that $\frac{1}{\bar p_x^K}$ is large but the time to reach x is small. Of course this also increases the amount of time to reach the end of the molecule, but it can be stopped. The proof to get these results are very close to the one of Section 5.2 so we do not give any details.

In the above proposition we assume that g_0^x is known which can be restrictive, indeed we do not know anything on the molecule, we can remove this assumption and get the following result. First we need to introduce another function of information $\bar i$: for fixed x, m and j

$$\bar i_x^j(\mu_m^x) := L_x^{+,R_j}\log(1-\bar p_x^j(\mu_m^x))+R(\log(\bar p_x^j(\mu_m^x))-\log(1-\bar p_x^j(\mu_m^x))).$$

recall that $\bar p_x^j$ is given by (5.24), it is a function of (compatible) energies $g_0(.)$ form $x+1$ to M, moreover μ_m^x is the sequence (from $x+1$ to M) of (compatible) energies compatible with the energy μ_m at site x.

Proposition 5.3.2. *For all x, all estimator $\hat g_0(x)$, assume that each of the forces $(f_i(.), i\le K)$ are applied to respectively $(R_i, i\le K)$ random walks, then*

$$\begin{aligned}&\mathbb{P}\left(\hat g_0(x)=g_0(x)|\,L_x^{+,R_i}, L_{x,}^{-,R_i}, f_i(.),\ i\le K\right)\\ &=\left(1+\sum_{m=1,\mu_m\neq\hat g_0(x)}^{K} e^{\sum_{j=1}^K-(i_x^j(m)-i_x^j(0))}\frac{\sum_{\mu_m^x} e^{\bar i_x^K(\mu_m^x)}}{\sum_{\mu_x^0} e^{\bar i_x^K(\mu_0^x)}}\right)^{-1}\end{aligned}$$

where the sum over the μ_m^x is the sum over all the compatible energies (from $x+1$ to the M) with μ_m of site x. μ_0^x is the same, but the compatibility is with $\hat{g}_0(x)$ instead of μ_m. $i_x^l(m)$ is given by (5.21), and for $i_x^l(0)$ we replace μ_m in $i_x^l(m)$ by $\hat{g}_0(x)$.
Moreover if we take for the estimator at site x defined in (5.23) *and we assume that all the R_i are equal we get as before*

$$\begin{aligned}\frac{1}{R_c(x)} &:= -\lim_{R\to\infty}\log\frac{1}{R}\left(\mathbb{P}\left(\hat{g}_0(x)\neq g_0(x)|L_x^{+,R_i},L_{x,}^{-,R_i},i\leq K\right)\right)\\ &\geq \frac{1}{\bar{p}_x^K}(H^{\rightarrow}+H^{\leftarrow}).\end{aligned}$$

Proof. Thanks to Lemma 5.2.2, we easily get the first part of the proposition,

$$\lim_{R\to+\infty}\frac{1}{R}\bar{i}_x^j(\mu_m^x)=\left(\frac{1}{\bar{p}_x^j(\mu_0^x)}-1\right)\log(1-\bar{p}_x^j(\mu_m^x))+\log(\bar{p}_x^j(\mu_m^x)).$$

Also we have that $H(x):=(1/a-1)\log(1-x)+\log x$ reaches its maximum in a, we deduce from that $\frac{\sum_{\mu_m^x} e^{\bar{i}_x^K(\mu_m^x)}}{\sum_{\mu_x^0} e^{\bar{i}_x^K(\mu_x^0)}}\leq 1$, so we get the lower bound for $1/R_c(x)$. $\square$

We finish with a discussion about the link between the energies and the sequences of bases. First let us recall the table of the binding free energies for DNA at room temperature:

g_0	A	T	C	G
A	1.78	1.55	2.52	2.22
T	1.06	1.78	2.28	2.54
C	2.54	2.22	3.14	3.85
G	2.28	2.52	3.90	3.14

We notice that the largest free energies which correspond to the most stable links are on the bottom right end corner of the table, in fact the largest binding energy is obtained when a G is followed by a C. Notice also that $g_0(G,G)=g_0(C,C)$ so we can not distinguish this two different links by looking only at the free energy. In the same way the lowest free energy are made with bases with a T and A followed by the same letters, again $g_0(A,A)=g_0(T,T)$. For the rest of the table we have the equality $g_0(W,S)=g_0(\bar{S},\bar{W})$, where S is either a C or a G and W a A or a T, $\bar{S}$ the complementary of S, and the same for $\bar{W}$.

Moreover it is possible to reconstruct the DNA molecule from the compatible binding energies only if there is only one sequence of base pairs which corresponds to the sequence of energies (see Theorem 5.2.3). This is not always the case, for example when the molecule repeats the same scheme which is undetermined: the energy of $C - C - \cdots - C$ is equal to the energy of $G - G - \cdots - G$, in the same way $A - C - A - C - \cdots - A - C$ has the same energy than $G - T - G - T - \cdots - G - T$. Notice that if this kind of sequence is broken only once in the molecule then we turn to a determined case.

Bibliographie

[1] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions.* Dover, New York, 1965.

[2] O. Adelman and N. Enriquez. Random walks in random environment : What a single trajectory tells. *Israel J. Math.*, 142 :205–220, 2004.

[3] S. Alili. Asymptotic behaviour for random walks in random environments. *J. Appl. Probab.*, 36(2) : 334–349, 1999.

[4] P. Andreoletti. Alternative proof for the localisation of Sinai's walk. *Journal of Statistical Physics*, 118 :883–933, 2005.

[5] P. Andreoletti. On the concentration of Sinai's walk. *Stoch. Proc. Appl.*, 116 :1377–1408, 2006.

[6] P. Andreoletti. On the estimation of the potential of Sinai's rwre. *To appear in Braz. Journal of Prob. and Stat.*, 2010.

[7] P. Andreoletti and R. Diel. Limit law of the local time for Brox's diffusion. *Journal of Theoretical Probability*, A paraître.

[8] V. Baldazzi, S. Cocco, E. Marinari, and R. Monasson. Infering dna sequences from mechanical unzipping : an ideal-case study. *Physical Review Letters E*, **96** : 128102–1–4, 2006.

[9] V. Baldazzi, S. Cocco, E. Marinari, and R. Monasson. Infering dna sequences from mechanical unzipping data : the large-bandwith case. *Physical Review Letters E*, **75** : 011904–1–33, 2007.

[10] J. Bertoin. Sur la décomposition de la trajectoire d'un processus de lévy spectrallement positif en son infimum. *Ann. Inst. H. Poincaré*, 27 : 537–547, 1991.

[11] J. Bertoin. Spliting at the infimum and excursions in half-lines for random walks and levy processes. *Stoch. Proc. Appl.*, 47 : 17–35, 1993.

[12] J. Bertoin. Splitting at the infimum and excursions in half-lines for random walks and lévy processes. *Stoch. Proc. App.*, 47(1) : 17–35, 1993.

[13] J. Bertoin. *Lévy processes*. Cambridge University Press, 1996.

[14] U. Bockelmann, B. Essevaz-Roulet, and F. Heslot. Molecular stick-slip motion revealed by opening dna with piconewton forces. *Phys. Rev. Letters*, **79**(22) : 4489–4492, 1997.

[15] U. Bockelmann, B. Essevaz-Roulet, and F. Heslot. Dna strand separation studied by single molecule force measurements. *Phys. Rev. E*, **58**(2) : 2386–2394, 1998.

[16] A. Borodin and P. Salminen. *Handbook of Brownian Motion-Facts and Formulae*. Birkhäuser, 1996.

[17] T. Brox. A one-dimensional diffusion process in a Wiener medium. *Ann. Probab.*, 14(4) : 1206–1218, 1986.

[18] P. Carmona. The mean velocity of a Brownian motion in a random lévy potential. *Ann. Probab.*, 25(4) : 1774–1788, 1997.

[19] D. Cheliotis. One-dimensional diffusion in an asymmetric environment. *Ann. Inst. H. Poinc., Probab. and Stat.*, 42(6) : 715–726, 2006.

[20] D. Cheliotis. Localization of favorite points for diffusion in random environment. *Stochastic Process. Appl.*, 118(7) : 1159–1189, 2008.

[21] A. A. Chernov. Replication of a multicomponent chain, by the "lightning mechanism". *Biophysics*, 12 : 336–341, (1962).

[22] S. Cocco and R. Monasson. Reconstructing a random potential from its random walks. *Europhysics Letters*, **81** : 1–6, 2008.

[23] F. Comets and O. Zeitouni. A law of large numbers for random walk in random environments. *Ann. Probab.*, 32(1B) :880–914, 2004.

[24] F. Comets and O. Zeitouni. Gaussian fluctuations for random walks in random mixing environments. *Israel J. Math.*, 148 :87–113, 2005.

[25] A. Dembo, N. Gantert, Y. Peres, and Z. Shi. Valleys and the maximal local time for random walk in random environment. *Probability Theory and Related Fields*, 137 : 443–473, 2007.

[26] A. Devulder. Almost sure asymptotics for a diffusion process in a drifted Brownian potential. *Preprint*.

[27] A. Devulder. The maximum of the local time of a diffusion in a drifted Brownian potential. *Preprint*.

[28] R. A. Doney. *Fluctuation theory for Lévy processes, Ecole d'été de Probabilités de St-Flour XXXV*. Springer, 2005.

[29] R. A. Doney and A. E. Kyprianou. Overshoots and undershoots of lévy processes. *Ann. Appl. Prob.*, 16 : 91–106, 2006.

[30] J. L. Doob. Conditional Brownian motion and the boundary limits of harmonic functions. *Bull. Soc. Math. France*, 85 : 431–458, 1957.

[31] T. Duquesne. Path decompositions for real levy processes. *Ann. Inst. H. Poincaré*, 39(2) : 339–370, 2003.

[32] N. Enriquez and C. Sabot. Random walks in a dirichlet environment. *Electron. J. Probab.*, 11(31) : 802–817, 2006.

[33] J.F. Le Gall. Sur la mesure de Hausdorff de la courbe brownienne. *Sém. Probability. XIX. Lect. Notes Math.*, 1123 : 297–313, 1985.

[34] N. Gantert, Y. Peres, and Z. Shi. The infinite valley for a recurrent random walk in random environment. *Ann.Inst. H. Poincaré Prob. et Stat.*, 46(2) :525–536, 2010.

[35] N. Gantert and Z. Shi. Many visits to a single site by a transient random walk in random environment. *Stoch. Proc. Appl.*, 99 : 159–176, 2002.

[36] A. O. Golosov. Localization of random walks in one-dimensional random environments. *Communications in Mathematical Physics*, 92 : 491–506, 1984.

[37] A. O. Golosov. On limiting distribution for a random walk in a critical one-dimensional random environment. *Com. of the Mosc. Math. Soc.*, pages 199–200, 1986.

[38] P. Greenwood and J. Pitman. Fluctuation identities for lévy processes and splitting at the maximum. *Adv. Appl. Prob.*, 12 : 893–902, 1980.

[39] Y. Hu. Tightness of localization and return time in random environment. *Stochastic Process. Appl.*, 86(1) : 81–101, 2000.

[40] Y. Hu and Z. Shi. The limits of Sinai's simple random walk in random environment. *Ann. Probab.*, 26(4) : 1477–1521, 1998.

[41] Y. Hu and Z. Shi. The local time of simple random walk in random environment. *J. of Theoret. Probab.*, 11(3), 1998.

[42] Y. Hu and Z. Shi. The problem of the most visited site in random environment. *Probab. Theory Relat. Fields*, 116(2) : 273–302, 2000.

[43] Y. Hu and Z. Shi. Slow movement of random walk in random environment on a regular tree. *Ann. Prob.*, 35(5) : 1978–1997, 2007.

[44] Y. Hu and Z. Shi. A subdiffusive behaviour of recurrent random walk in random environment on a regular tree. *Probab. Theory Related Fields*, 138(3-4) : 521–549, 2007.

[45] Y. Hu, Z. Shi, and M. Yor. Rates of convergence for one-dimensional diffusion in a drifted Brownian potential. *Trans. Amer, Math. Soc*, 2000.

[46] A.N. Shiryaev J. Jacod. *Limit Theorems for Stochastic Processes.* Springer Verlag, 1987.

[47] S. A. Kalikow. Generalised random walk in random environment. *The Ann. of Prob.*, 9(5) : 753–768, (1981).

[48] K. Kawazu, Y. Tamura, and H. Tanaka. Limit theorems for one-dimensional diffusions and random walks in random environments. *Probab. Theory Related Fields*, 80 : 501–541, 1989.

[49] K. Kawazu and H. Tanaka. On the maximum of a diffusion process in a drifted brownian environment. *Séminaire de Probabilités XXVII, Lectures Notes in Math.*, 1557 : 78–85, 1993.

[50] K. Kawazu and H. Tanaka. A diffusion process in a Brownian environment with drift. *J. Math. Soc. Japan*, 49 : 189–211, 1997.

[51] K. Kawazu and H. Tanaka. Invariance principle for a Brownian motion with large drift in a white noise environment. *Hiroshima math J.*, 28 : 129–137, 1998.

[52] H. Kesten. The limit distribution of Sinai's random walk in random environment. *Physica*, 138A : 299–309, 1986.

[53] H. Kesten, M.V. Kozlov, and F. Spitzer. A limit law for random walk in a random environment. *Comp. Math.*, 30 : 145–168, 1975.

[54] R. Lyons and R. Pemantle. Random walk in a random environment and first-passage percolation on trees. *Ann. Prob.*, 20(1) : 125–136, 1992.

[55] P. Mathieu. Zero white noise limit through dirichlet forms, with application to diffusions in a random media. *Proba. Theory Relat. Fields*, 99(4) : 549–480, 1994.

[56] P. Mathieu. Limit theorems for diffusions with a random potential. *Stochastic Process. Appl.*, 60 : 103–111, 1995.

[57] E. Mayer-Wolf, A. Roitershtein, and O. Zeitouni. Limit theorems for one-dimensional transient random walks in markov environments. *Ann. Inst. H. Poincaré Probab. Statist.*, 40(5) : 635–659, 2004.

[58] H. P. McKean. Excursions of a non-singular diffusion. *Z. Wahrsch. Verw. Gebiete*, 1 : 230–239, 1963.

[59] H. P. McKean. Excursions of non-singular diffusion. *Z. Wahrsch.*, 1 : 230–239, 1963.

[60] J. Neveu and J. Pitman. Renewal property of the extrema and tree property of the excursion of a one-dimensional Brownian motion. séminaire de probabilitées xxiii. *Lecture Notes Math.*, 1372 :239–247, 1989.

[61] P. Protter. *Stochastic integration and stochastic differential equations : a new approach.* Springer, 1990.

[62] F. Rassoul-Agha. The point of view of the particule on the law of large numbers for random walks in a mixing random environment. *Ann. Probab.*, 31(3) : 1441–1463, 2003.

[63] F. Rassoul-Agha. Large deviations for random walks in a mixing random environment, and other (non-markov) random walk. *Comm. Pure Appl. Math.*, 57(9) : 1178–1196, 2004.

[64] F. Rassoul-Agha. On the zero-one law and the law of large numbers for random walk in mixing random environment. *Electron. Comm. Probab.*, 10 : 36–44, 2005.

[65] P. Révész. In random environment the local time can can be very big. *Astérisque*, pages 157–158,321–339, 1988.

[66] P. Révész. *Random walk in random and non-random environments.* World Scientific, 1989.

[67] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion.* Springer, 1999.

[68] L. C. G. Rogers. A new identity for real lévy processes. *Ann. Inst. H. Poincaré*, 20 : 21–34, 1984.

[69] S. Schumacher. Diffusions with random coefficients. *Contemp. Math.*, 41 : 351–356, 1985.

[70] P. Seignourel. Discrete schemes for processes in random media. *Prob. Th. Rel. Fields*, 118 :293–322, 2000.

[71] Z. Shi. A local time curiosity in random environment. *Stoch. Proc. Appl.*, 76(2) : 231–250, 1998.

[72] Z. Shi. Sinai's walk via stochastic calculus. *Panoramas et Synthèses*, 12 : 53–74, 2001.

[73] Z. Shi and O. Zindy. A weakness in strong localization for sinaï's walk. *Ann. Probab.*, 35(3) :1118–1140, 2007.

[74] Y. G. Sinai. The limit behaviour of a one-dimensional random walk in a random medium. *Theory Probab. Appl.*, 27(2) : 256–268, 1982.

[75] A. Singh. Limiting behavior of a diffusion in an asymptotically stable environment. *Ann. Inst. H. Poincaré Prob. Stat.*, 43(1) :101–138, 2007.

[76] A. Singh. Rates of convergence of a transient diffusion in a spectrally negative lévy potential. *Ann. Probab.*, 36(3) : 279–318, 2008.

[77] F. Solomon. Random walks in random environment. *Ann. Probab.*, 3(1) : 1–31, 1975.

[78] A. S. Sznitman. Slowdown and neutral pockets for a random walk in random environment. *Probability Theory and Related Fields*, 115(3) : 287–323, 1999.

[79] A. S. Sznitman. Slowdown estimates and central limit theorem for random walks in random environment. *J. Eur. Math. Soc.*, 2(2) : 93–143, 2000.

[80] A. S. Sznitman. On a class of transient random walks in random environment. *Ann. Probab.*, 29(2) : 723–764, 2001.

[81] A. S. Sznitman. An effective criterion for ballistic behavior of random walk in random environement. *Probability Theory and Related Fields*, 122(4) : 509–544, 2002.

[82] A. S. Sznitman. On new examples of ballistic random walks in random environment. *Ann. Probab.*, 31(1) : 285–322, 2003.

[83] M. Taleb. Large deviations for a Brownian motion in a drifted Brownian potential. *Ann. Probab.*, 29(3) : 1173–1204, 2001.

[84] M. Talet. Annealed tail estimates for a Brownian motion in a drifted Brownian potential. *Ann. Probab.*, 35(1) : 32–67, 2007.

[85] H. Tanaka. Limit theorem for one-dimensional diffusion process in Brownian environment. *Lecture Notes in Mathematics*, 1322, 1988.

[86] H. Tanaka. Recurrence of a diffusion process in a multidimensional brownian environment. *Proc. Japan Acad. Ser. A Math. Sci.*, 69 : 377–381, 1993.

[87] H. Tanaka. Localization of a diffusion process in a one-dimensional Brownian environmement. *Comm. Pure Appl. Math.*, 17 : 755–766, 1994.

[88] H. Tanaka. Limit theorem for a Brownian motion with drift in a white noise environment. *Chaos Solitons Fractals*, 11 : 1807–1816, 1997.

[89] H. Tanaka. Lévy processes conditioned to stay positive and diffusions in random environments. *Advanced Studies in Pure Math.*, 39 : 355–376, 2004.

[90] D. E. Temkin. One dimensional random walk in two component chain. *Soviet Math. Dokl.*, 13(5), (1972).

[91] A. J. Viterbi. Error bounds for convolutional codes and an asymptotically optimum decoding algorithm. *IEEE Trans. Inf. Theory*, **13**(2) : 260–269, 1967.

[92] D. Williams. Path decomposition and continuity of local time for one dimensional diffusions. *Proc London Math. Soc.*, 28 : 738–768, 1974.

[93] S.J. Taylor Z. Ciesielski. First passage times and sojourn times for Brownian motion in space and the exact Hausdorff measure of the sample path. *Trans. Amer. Math. Soc.*, 103 : 434–450, 1962.

[94] O. Zeitouni. Random walks in random environment. *Lecture Notes in Math.*, 1837 :189–312, 2004.

[95] M. P. W. Zerner. Lyapounov exponents and quenched large deviations for multidimentional random walk in random environment. *The Ann. of Prob.*, 26(4) : 1446–1476, 1998.

[96] M. P. W. Zerner and F. Merkl. A zero-one law for planar random walks in random environment. *The Ann. of Prob.*, 29(4) : 1716–1732, 2001.

Roland DIEL

Temps local d'une diffusion en environnement aléatoire. Applications

On appelle diffusion en milieu aléatoire la solution de l'équation différentielle stochastique suivante :

$$\mathrm{d}X(t) = \mathrm{d}B(t) - \frac{1}{2}W'(X(t))\mathrm{d}t$$

où B est un mouvement brownien standard et W, le milieu, est un processus càd-làg qui n'est pas nécessairement dérivable (l'EDS précédente n'a alors qu'un sens formel). Schumacher [69] et Brox [17] ont montré que dans le cas où W est un mouvement brownien, la diffusion X a un comportement sous-diffusif et se localise au voisinage de certains points du milieu.
Cette thèse est principalement consacrée à l'étude du comportement asymptotique du processus des temps locaux de X. Ce processus $L_X(t,x)$ représente le temps passé par X au point x avant le temps t. C'est donc un outil bien adapté pour étudier la localisation de la diffusion. On décrit ici la loi limite du temps local lorsque le milieu est un mouvement brownien standard ou plus généralement un processus de Lévy stable. On s'intéresse également au temps passé par la diffusion au voisinage des points les plus visités et au comportement asymptotique presque sûr du maximum du temps local.
Dans la dernière partie de la thèse, on utilise le temps local d'une version discrète du modèle, pour obtenir des informations sur le milieu. Le but étant d'appliquer ce modèle au séquençage de l'ADN.

Mots clés : Processus en milieu aléatoire, temps local, mouvement brownien, processus de Lévy.

Local time of a diffusion in random environment. Applications

A diffusion in random environment is the solution of the following stochastic differential equation:

$$\mathrm{d}X(t) = \mathrm{d}B(t) - \frac{1}{2}W'(X(t))\mathrm{d}t$$

where B is a standard Brownian motion and W a càd-làg process which is not necessarily differentiable (the previous SDE has then only a formal sense). Schumacher [69] and Brox [17] have shown that the diffusion X has a sub-diffusive behavior when W is also a standard Brownian motion. Moreover they point out a localization phenomena for X.
This thesis is principally devoted to the description of the asymptotic behavior of the local time process of X. The local time $L_X(t,x)$ represents the time spent by X before t at point x. This is thereby a useful tool to study the localization of the diffusion. Here is described the limit law of the local time when the environment is a Brownian motion or more generally a stable Lévy process. We are also interested in the time spent by X in the neighborhood of the most visited points and in the almost sure asymptotic behavior of the maximum of the local time.
In the last chapter of the thesis the notion of local time is used in a discrete version of the model to obtain informations on the environment. The goal is to apply this model to DNA sequencing.

Keywords: Processes in random environment, local time, Brownian motion, Lévy process.

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