The works of KŐNIG Dénes (1884–1944) in the domain of mathematical recreations and his treatment of recreational problems in his works of graph theory.

Mitsuko Wate Mizuno

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DOCTORAT

Histoire des mathématiques

AUTEUR : WATE MIZUNO Mitsuko

TITRE : The works of KŐNIG Dénes (1884–1944) in the domain of mathematical recreations and his treatment of recreational problems in his works of graph theory.

TITRE EN FRANÇAIS : Les oeuvres de KŐNIG Dénes (1884–1944) dans le domaine des récréations mathématiques et son traitement de problèmes récréatifs dans ses travaux de théorie des graphes.

Thèse dirigée par Karine CHEMLA

Soutenue le 3 décembre 2010

JURY

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Chapter 1

Introduction

This thesis deals with a chapter of the history of graph theory. Its main focus is on the historical transition from mathematical recreations to graph theory, as it can be studied from the various writings by KŐNIG Dénes (1884–1944).

In this chapter, I will describe a motivation for this research, the historical background of this research, and the significance of this research.

1.1 Motivation for this research

KŐNIG Dénes (1884–1944) is a Hungarian mathematician born in Budapest. He is now recognised as the "father of graph theory" with his treatise *Theorie der endlichen und unendlichen Graphen* written in 1936 [121].

In examining publications on KŐNIG's works, I found that he published two books that are apparently not related to graph theory. These two books compose a series entitled *Mathematikai mulatságok (Mathematical entertainments/amusements)*, and published successively in 1902 and 1905 [86, 87], when he was still a student. According to GALLAI Tibor (1912-1992) [224], a mathematician who was a student of KŐNIG, these books were very successful[1].

Fortunately in 2007, I found the original version of the second book of 1905, and the recomposed version of the first book of 1902 at the Library of Eötvös Loránd University in Budapest. In the next year, thanks to my Hungarian friend TÓTH Imre, deceased in 2010, I also obtained a duplicate of the original version of the first book of 1902.

[1]These books were recomposed with TypoTEX, and reprinted in 1991 and 1992. The digital version is sold on the Internet. Therefore, it is not very difficult to get the recomposed versions. However in the recomposed versions, some parts are modified from the original version. It is more serious that the pages of references at the end of the second book of 1905 are totally missing in the digitalized version.

In examining these books, I built a hypothesis that mathematical recreations and graph theory are not completely separated in KŐNIG's mind. I suppose that the works of the young KŐNIG on mathematical recreations are historically connected to his later works on graph theory.

In this thesis, I will try to verify this hypothesis by examining historical documents related to mathematical recreations and those related to graph theory.

1.2 Historical background of this research

1.2.1 Graph theory

A historical approach to graph theory was made by BIGGS, LLOYD and WILSON in their book *Graph Theory, 1736–1936* in 1976 [215]. In this book, important texts of many mathematicians related to graph theory are collected and translated into English. Moreover, this book provides a historical approach to graph theory with mathematical complementary commentaries. This aspect is very useful for people studying graph theory.

This book dealt with KŐNIG's treatise of 1936 as one of two landmarks of the history of graph theory —the other landmark was Euler's article on the problem of seven bridges of Königsberg published in 1736 [51]. The authors consider KŐNIG's book of 1936 as "the first full-length book on the subject" (Preface, [215]) and "the first comprehensive treatise on graph theory" (p.219, [215]).

1.2.2 KŐNIG's works

As historical approaches to KŐNIG's works, the works of GALLAI Tibor and those of LIBOR Józsefné are remarkable.

GALLAI Tibor gave biographical notes in Hungarian in 1965 [224]. GALLAI listed all the works of KŐNIG. He classified KŐNIG's mathematical works into two categories: ones belonging to set theory, geometry or combinatorial topology, and the others belonging to graph theory. He described the outline of each mathematical work of KŐNIG. GALLAI gave also a short German version of the biographical notes, which was contributed to the republished version of 1986 of KŐNIG's treatise *Theorie der endlichen und unendlichen Graphen* (1936) [121]. This German version was translated into English, and inserted into the English version of KŐNIG's treatise published in 1990 [123].

LIBOR Józsefné published an article "Megemlékezés Kőnig Dénesről (Recollection of Dénes Kőnig)" in 2006. She examined not only the works of

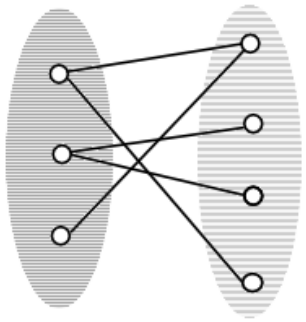

Figure 1.1: A bipartite graph

KŐNIG, but also the role of KŐNIG's works in the later development of graph theory. She focused on "König's theorem" published in 1931 [117]. This theorem concerns the separation of vertices of bipartite graphs. A bipartite graph is defined as follows: A "bipartite graph" is a graph the vertices of which can be divided into two disjoint sets such that every edge connects a vertex in one set to a vertex in the other set (Figure 1.1).

"König's theorem" is the following theorem: in a bipartite graph the minimal number of vertices which exhaust the edges equals the maximal number of edges no two of which have a common vertex. This theorem can be expressed in the matrix-theoretic formulation: in every matrix the minimal number of lines (rows, columns) in whose union contain all the nonzero elements is equal to the maximal number of nonzero elements which pairwise lie in distinct lines. This theorem was also proved by KŐNIG.

She mentioned also EGERVÁRY Jenő's simple new proof of the matrix form of König's theorem together with an interesting generalization published in the same year as König's theorem [49].

According to LIBOR, König's theorem and EGERVÁRY's generalization played an important role in the development of mathematical economics. Harold William KUHN, an American mathematician, inspired from EGERVÁRY's generalization, using König's theorem, constructed a solution algorithm to the so-called assignment problem in mathematical economics (1955) [132]. KUHN named this process the "Hungarian method". According to LIBOR, the Hungarian method has been applied to other problems in economics.

I will give the outline of KŐNIG's works in Chapter 2. In particular, we shall see that his first two publications, written in Hungarian, were devoted to mathematical recreations.

KŐNIG's works on mathematical recreations have not much been examined yet, but they contain many important features from the point of view of

the history of graph theory. I will examine in Chapter 6 the role of mathematical recreations in the formation of graph theory by KŐNIG. For this purpose, I will translate KŐNIG's books on mathematical recreations in Hungarian of 1902 and 1905 [86, 87] into English in Chapters 4 and 5.

Although no problem in the book of 1902 was treated again in KŐNIG's treatise of 1936 [121], many problems in the book of 1905 were treated again in the treatise of 1936; conversely, many problems in the treatise of 1936 are already treated in the book of 1905 [87], but the way of treatment is different from each other.

Both in the treatise of 1936 and in the book of 1905, KŐNIG used many diagrams to discuss the problems, but the features of diagrams are different from each other.

I will analyze the historical transition of the features —the form and the treatment— of diagrams of graph theory (Chapter 7).

1.2.3 Mathematical recreations

As we will see in Chapter 3, collecting various problems as mathematical recreations was began in 17th century in Europe. These collections themselves have an aspect of historical research on recreational problems, but these collections were also examined from a historical point of view by many historians and mathematicians.

Anne-Marie DÉCAILLOT [217, 218, 219] examined all the mathematical activities of Édouard LUCAS (1842–1891), who published four volumes of books *Récréations mathématiques, I–IV* [148, 149, 151, 152].

David SINGMASTER [240] precisely examined all the different editions of the book *Mathematical Recreations and problems of past and present times* (1892) [12] (the title was changed to *Mathematical Recreations and essays* on the fourth edition in 1905 and later)[2] written by Walter William Rouse BALL (1850–1925).

Albrecht HEEFFER [226] examined the first book that was entitled with the words "Recreation mathematicqve (mathematical recreations)".

1.3 Significance of this research

The history of graph theory is already well examined by some historians and mathematicians. The history of mathematical recreations is much examined by the other historians and mathematicians. However, the relation between graph theory and mathematical recreations is not yet much examined.

[2]The first edition was published in 1892, and augmented repeatedly in the later editions.

Biggs, Lloyd and Wilson [215] mention also the works on mathematical recreations related to graph theory, referring to *Récréations mathématiques, I–IV* (1882–1894) by Lucas [148, 149, 151, 152], *Mathematical Recreations and problems of past and present times* (1892) by Ball [12] and *Mathematische Unterhaltungen und Spiele* (1901) by Wilhelm Ahrens[3]. However, because the book of Biggs, Lloyd and Wilson is dedicated to the history of graph theory, these books on mathematical recreations are not examined in their book. Moreover, they put side by side the publications on mathematical recreations and Kőnig's treatise without raising the question of the historical relationship between the two. However, Kőnig's writings clearly show such a relationship. On the one hand, Kőnig's treatise is full of problems of mathematical recreations. On the other hand, in his youth, Kőnig wrote two small booklets of mathematical recreations. My thesis aims at understanding the relationship between the two. Moreover, I try to understand how Kőnig depended on previous publications in mathematical recreations to which he referred. This is how I intend to tackle the question of the relationship between mathematical recreations and the emergence of graph theory. This is also the reason why I did extensive research on the various 19th century writings on mathematical recreations in Chapter 3.

As we will see in this thesis, mathematical recreations played an important role on the formation of graph theory. Research on the relation between graph theory and mathematical recreations is therefore significant from the point of view of the history of mathematics.

[3] *Mathematische Unterhaltungen und Spiele* (1901) was later augmented and divided into two volumes [3, 4].

Chapter 2

Kőnig's works

2.1 Introduction

KŐNIG Dénes was born in Budapest in 1884 to KŐNIG Gyula (1849–1913) and OPPENHEIM Eliz (1863–1916). His father Gyula, also a mathematician, is famous for his works in the domain of set theory. Most of his works were published in German with his German name "Julius König". KŐNIG Gyula studied medicine in Vienna from 1866, but he wanted to study natural science, and moved to Heidelberg where famous scientists were in the teaching staff. In 1869, KÖNIGSBERGER was appointed to be chair of mathematics at Heidelberg. In those days, KÖNIGSBERGER was interested in elliptic functions, which interested also KŐNIG Gyula. KŐNIG Gyula obtained a doctorate with his dissertation *Zur Theorie der Modulargleichungen der elliptischen Functionen (On the theory of modular equations of the elliptic functions)* under KÖNIGSBERGER's supervision in 1870. In 1871, KŐNIG Gyula was appointed to be instructor at the University of Budapest. In 1873, he was appointed to be professor at Teacher's College of Budapest, and in 1874 to be professor at Budapest Polytechnic, and continued working there. Even after his retirement of 1905, he continued teaching on some topics. Besides educations, he worked on research in algebra, number theory, geometry, set theory, and analysis.

KŐNIG Dénes cited later an article of his father. In the treatise of KŐNIG Dénes in 1936 on graph theory that we will discuss soon, Gyula's article on set theory of 1906 [124] was cited at two places in relation to graph theory: one in Section 3 "Der verschärfte Äquivalenzsatz (The sharpened equivalence theorem)" of Chapter 6 "Spezielle Untersuchungen über unendliche Graphen (Special examinations on infinite graphs)" and the other in Section 5 "Mengentheoretische Formulierungen (Set theoretical formulations)" of Chapter 13

"Faktorenzerlegung regulärer unendlicher Graphen (Factorization of regular infinite graphs)".

In 1872, KŐNIG Gyula published an article "Ueber eine reale Abbildung der s. g. Nicht-Euklidischen Geometrie (On a real expression of so-called non-Euclidean geometry)", in which he treated intuitive ways to prove the consistency of non-Euclidean geometries. In 1903, he published a treatise *Einleitung in die allgemeine Theorie der algebraischen Gröszen*[1] *(Introduction to the general theory of algebraic quantities)* [126].

In the preface, KŐNIG Gyula declared that, based on the long history until Leopold KRONECKER (1823–1891) forming the general theory of algebraic quantities, he adopted the systematic representation of the theory. It is also interesting to us that the name of J. KÜRSCHÁK is listed among people he thanks to: later on, KŐNIG Dénes worked on his dissertation under the supervision of KÜRSCHÁK.

KŐNIG Gyula worked especially on set theory in his last years. In 1904, he talked "Zum Kontinuum-Problem (on continuum hypothesis)" in the third International Congress of Mathematicians in Heidelberg [128]. He disproved here the continuum hypothesis that there is no set whose cardinality lies between that of the Natural numbers and that of the Real numbers. However, his proof contains an error. Later on, in 1940, Kurt GÖDEL (1906–1978) showed that the continuum hypothesis cannot be disproved on the axiomatic system of Zermelo–Fraenkel set theory, even if the axiom of choice is adopted; moreover, in 1963, Paul Joseph COHEN (1934–2007) showed that neither the continuum hypothesis nor the axiom of choice can be proved on the axiomatic system of Zermelo–Fraenkel set theory.

KŐNIG Gyula worked on his own approach to set theory, logic and arithmetic: his final book *Neue Grundlagen der Logik, Arithmetik und Mengenlehre (New foundations of logic, arithmetic and set theory)* was published after his death in 1914.

Besides KŐNIG Gyula, one of Dénes' teachers of mathematics was BEKE Manó (1862–1946) at the gymnasium. BEKE, before becoming a teacher, studied in Göttingen in the school year 1892/1893. At this opportunity, BEKE became known to Felix Christian KLEIN (1849–1925)[2], who was a

[1] In some library catalogues, the orthography "Grössen" or "Größen" appears instead of "Gröszen".

[2] KLEIN had many important achievements in the domains of geometry and function theory. In 1872, he published an article "Vergleichende Betrachtungen über neuere geometrische Forschungen (Comparative studies of recent geometric research)" on the occasion of his inauguration as a professor at Erlangen University [82]. In this article, he suggested a research program on geometries. For this purpose, he classified many kind of geometries according to properties under groups of transformation. This classification

professor at the University of Göttingen in 1886–1913, and who was working actively to reform the teaching methods of mathematics.

KLEIN tried to strengthen the relation between mathematics and its application to science and technology. KLEIN also deplored the discontinuity between school mathematics and university mathematics. According to KLEIN, the students of mathematics had to forget their knowledge of mathematics twice: first when one began one's university studies, one had to forget school mathematics; secondly when one became a teacher, one had to forget university mathematics and return to school mathematics.

KLEIN was trying to get rid of the discontinuity by improving educations to the mathematical teachers. He promoted instruction with practical subjects and development of spatial intuition, and published the lectures of elementary mathematics from an advanced viewpoint [81, 79, 80] and so on [234, 235, 236].

I suppose that KLEIN's policy to consider the importance of applied mathematics and spatial intuition supported introduction of mathematical recreations to mathematical educations. In fact, as we will see in 2.2 and in Chapters 4 and 5, KŐNIG published two books on mathematical recreations, and they can be considered as a part of the activities of reforming mathematical teaching in Hungary, taking over KLEIN's activities.

BEKE, after coming back to Hungary from Göttingen, "worked actively to reform mathematics teaching in Hungary, applying Klein's conceptions." [237]. In 1895, BEKE became a gymnasium teacher in Budapest, and taught young KŐNIG.

In 1902, KŐNIG won the first prize in the "Eötvös Loránd matematikai tanulóverseny (Eötvös Loránd mathematical students' competition)". He continued to study mathematics at the Budapesti Műegyetem (Budapest Polytechnic)[3].

In the school year of 1904/1905, KŐNIG studied in Göttingen. On this

gives a synthetic view to various geometries studied by mathematicians at that time. In the studies on geometry, properties invariant under a given group of transformations are to be studied. This program is called "Erlanger Programm (Erlangen Program)".

In 1886, he began to work at the University of Göttingen. In 1895, he succeeded to call David HILBERT (1862–1943) from Königsberg as the chair of mathematics at the University of Göttingen. KLEIN and HILBERT made an effort to make Göttingen be the world centre of mathematics. After KLEIN's death, in 1929, the Mathematical Institute was built in Göttingen thanks to the Rockefeller Foundation.

KLEIN worked also for modernizing mathematical teaching. He tried to change the institution for teaching mathematics, and published in Göttingen many books for teachers of mathematics on teaching methods of mathematics.

[3]Budapesti Műegyetem later on became Budapesti Műszaki és Gazdaságtudományi Egyetem (BME, Budapest University of Technology and Economics).

occasion, he attended Hermann MINKOWSKI's lectures on the problem of the four-colour map, which is later considered as one of the important problems of graph theory, as we can see, for example, in Chapter 9 of the book of BIGGS, LLOYD and WILSON [215]. As we will see in the next section, KŐNIG treated the problem of the four-colour map in two different genres: mathematical recreations and graph theory.

2.2 KŐNIG's works

In 1899, when KŐNIG was still a gymnasium student, his first publication "Két maximum-minimum probléma elemi tárgyalása (Elementary discussion of two maximum-minimum problems)" appeared [85].

In 1902, his first book *Mathematikai mulatságok, első sorozat (Mathematical recreations, first series)*[4] was published [86], which was a collection of various problems of mathematical recreations. The preface of this book was written by BEKE Manó. In the preface, BEKE insisted on the following points:

- most high-school students very actively dealt with problems of mathematical recreations,
- this book of KŐNIG includes not only elementary problems but also something beyond them,
- these problems complement the school curriculum from many points of view.

Depending on these descriptions of BEKE, we can regard the publication of this book of 1902 and its sequel book of 1905 as one of the activities of reforming mathematical educations.That is, KŐNIG was one of the first generations who shared the benefit of the reformed educations of mathematics in Hungary.

Around this period, many Hungarian mathematicians were interested in mathematical recreations. In fact, among the references in KŐNIG's books on mathematical recreations, many articles are cited from a Hungarian magazine on physics and mathematics (Mathematikai és Physikai Lapok: Mathematical and physical reviews), or that on mathematics for high-school students (Középiskolai Mathematikai Lapok: High-school mathematical reviews). It is remarkable that this magazine intended for high-school students include

[4] In modern Hungarian, 'mathematical' is written without 'h' as "matematikai", but in the beginning of 20th century, the spelling with 'h' was still used.

many articles on mathematical recreations. This fact suggests that mathematical recreations were often used for mathematical educations in Hungary already when KŐNIG was a student.

KŐNIG studied in Göttingen in the school year of 1904/1905, and attended Hermann MINKOWSKI's lectures on "Analysis Situs" [159], where the problem of the four-colour map was treated. This problem was later discussed in KŐNIG's treatise in 1936. In 1905, KŐNIG published an article on this topic "A térképszínezésről (On the map-colouring)" [88]. He proved in this article the following theorem:

> If only one connected boundary borders a map of country drawn on a plane, and if every prefecture of the country touches the national border along a certain part of line, then the prefectures can be coloured with three colours in the way that the prefectures with a common border should always get different colours.

Although no term of graph theory was used in this article, it is KŐNIG's first article related to graph theory. This fact was asserted by GALLAI Tibor, a student of KŐNIG Dénes, in his article on the biography of KŐNIG published in 1965 [224].

In the same year, KŐNIG published the second book *Mathematikai mulatságok, második sorozat (Mathematical recreations, second series)* [87]. It is interesting that Chapter III of this book is exactly "The four-colour map". In this book, differently from the article on this problem published in the same year [88], he devoted those pages mainly to introduce examples related to the problem. This difference reflects the difference of genre between these publications: the article was written for mathematicians, while the book was written for a wider readership.

As we will see in Chapter 6, among 9 chapters of this book, Chapters III to VIII already contain problems that will be treated again in his later book on graph theory in 1936.

Based on these facts, we can conclude that MINKOWSKI's lecture played an important role to bring KŐNIG to questions and publications which were later related to graph theory.

In 1905, KŐNIG came back to Budapest, and continued his mathematical studies under the supervision of KÜRSCHÁK József (1864–1933)[5]. KŐNIG

[5]KÜRSCHÁK studied mathematics and physics at the Budapesti Műegyetem (Budapest Polytechnic) from 1881 to 1886. After graduation, he taught in secondary schools for two years, and returned to Polytechnic to study mathematics, and obtained his doctorate in 1890. He was appointed to be tutor, and then associate professor in the same year. He

received his doctorate in 1907 with the dissertation "A többméretű tér forgásainak és véges forgáscsoportjainak analytikus tárgyalása (The analytic discussion on the rotations of the space of larger dimension and on the finite rotation groups of it)" [89]. Although KŐNIG's dissertation treated geometrical problems, the supervisor KÜRSCHÁK was interested also in something related to mathematical recreations and to graph theory: as we will see in 5.10.1 and 5.10.2, KÜRSCHÁK's works are cited at two places of KŐNIG's books on mathematical recreations published in 1902 and 1905; moreover, in KŐNIG's treatise of 1936 on graph theory, KÜRSCHÁK's works [135, 136] are cited at two places.

In the same year, he began to work at the Budapesti Műegyetem (Budapest Polytechnic), and continued to be attached to it until his suicide in 1944, when the Nazis occupied Hungary. His suicide was probably due to Nazi persecution.

GALLAI examined old records at the Budapesti Műegyetem, and made clear KŐNIG's teaching programs in the years 1907–1944 when KŐNIG was working at the Polytechnic as a Polytechnic private tutor and an entrusted lecturer [224]. According to GALLAI, KŐNIG's lectures were titled as follows: Nomogram, Analysis situs, Set theory, Real numbers, Real numbers and functions, Graph theory. GALLAI described also as follows:

> **GALLAI (1965) [224], p. 278.**
> Gráfelmélet címen először csak az 1927/28-as tanévben hirdet előadást, ez a tárgykör azonban már 1911-óta szerepel előadásaiban, mert Analysis Situs című előadása gráfelméleti fejezeteket is tartalmazott.
> **GALLAI (1965) [224], p. 278. My translation from Hungarian.**
> As title of his lecture, the word "Graph theory" appeared barely in the school year of 1927/28, but he taught graph theory since 1911, because the title of his lecture "Analysis situs" implies also the subjects of graph theory.

This remark of GALLAI suggests that graph theory was established as a domain of mathematics between 1911 and 1927. Let's observe the interplay between analysis situs and what became graph theory in his publications of the succeeding years.

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continued teaching at Polytechnic throughout his life. He worked much in the domain of geometry, as well as in algebra. He founded the theory of valuations. He was interested also in mathematical recreations, and published articles in this domain. His articles on knight's move on chessboard [135, 136] were cited in KŐNIG's treatise on graph theory in 1936 [121].

In 1911, KŐNIG published 2 articles: "Vonalrendszerek Kétoldalú felületeken (Line systems on two-sided [= orientable] surfaces)" and "A vonalrendszerek nemszámáról (On the genus number of line systems)" [94, 95]. In these publications, the problem of the four-colour map was not mentioned, but later in the book of 1936, these articles were cited in a section concerning four-colour map. On this point, these articles can be related to MINKOWSKI's lectures on analysis situs, which he heard in Göttingen. In these articles, he did not use the term "graph". However, he used the term "vonalrendszer (line system)", the signified of which can be considered as one of the representations of a graph from the point of view of KŐNIG's later book in 1936. If we pay attention to these terms, we will notice that KŐNIG still continued to use the term "vonalrendszer (line system)" in later publications on analysis situs: his article in 1915 ("Vonalrendszerek és determinánsok (Line systems and determinants)" [102]); his text book in 1918 (*Az analysis situs elemei (The elements of analysis situs)*) [105]. At least in the years 1916–1918, KŐNIG used both terms "vonalrendszer (line system)" and "graph": at the latest in 1916, he used the term "graph" in his article "Graphok[6] és alkalmazásuk a determinánsok és halmazok elméletére (Graphs and their application to the theory of determinants and sets)" [103]. The German translation was published in the same year [104]. Later in 1976, the English translation was inserted in Chapter 10 "The factorization of graphs" of the book *Graph Theory, 1736–1936* of BIGGS, LLOYD and WILSON [215].

In this article, KŐNIG gave the proofs to the following theorems[7]:

A) Minden páros körüljárású reguláris graphnak van elsőfokú faktora. (Every bipartite regular graph possesses a factor of the first degree.)

B) Minden páros körüljárású reguláris k-adfokú graph k-számú elsőfokú faktor szorzatára bomlik. (Every bipartite regular graph of the kth degree splits into k factors of the first degree.

C) Ha egy páros körüljárású graph bármelyik csúcsába legfeljebb k-számú él fut, akkor minden éléhez oly módon lehet k-számú index valamelyikét hozzárendelni, hogy ugyanabba a csúcsba futó két élhez mindenkor két különböző index legyen rendelve. (Supposing that each point of a bipartite graph is incident with at most k edges, then one can assign one of k labels to each edge of the graph in such a way that two edges incident with the same point must have different labels.)

[6] In modern Hungarian, graph is written as "gráf", but KŐNIG still used the foreign spelling.

[7] The English translations are cited from the book of BIGGS, LLOYD and WILSON [215].

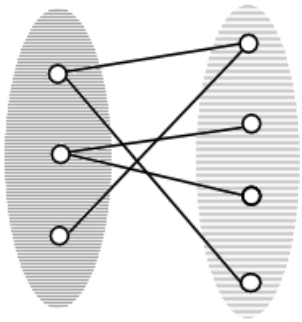


Figure 2.1: A bipartite graph

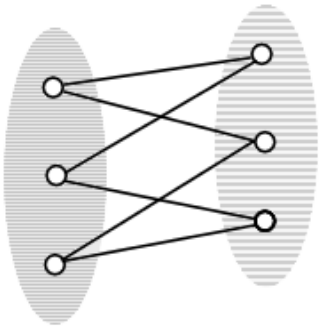


Figure 2.2: A bipartite regular graph

Although I mentioned the definition of "bipartite graph" in Chapter 1, I write it here again with additional information about the definitions of some other terms appeared in the theorems above.

A "bipartite graph" is a graph the vertices of which can be divided into two disjoint sets such that every edge connects a vertex in one set to a vertex in the other set(Figure 2.1).

A graph is said to be "regular" when the same number of edges leave each vertex, and this number is called the "degree" of the graph (Figure 2.2).

These theorems were later inserted in Chapter 11 "Faktorenzerlegung regulärer endlicher Graphen (Factorization of finite graphs)" of KŐNIG's treatise *Theorie der endlichen und unendlichen Graphen (Theory of finite and infinite graphs)* in 1936 [121].

Already in 1914, he mentioned the theorems in his lecture in the *Congrès de philosophie mathématique* in Paris.

Although the contents of the lecture belong to graph theory from our

point of view, it is not clear if he used the term "graph" already in the lecture, because the text of the lecture was published much later, in 1923, under the title "Sur un problème de la théorie générale des ensembles et la théorie des graphes (on a problem of the general theory of sets and the theory of graphs)" [110]. There was a long interval between the lecture in Paris and the publication of the text.

In 1918, he published a treatise, *Az analysis situs elemei (The elements of analysis situs)* [105], in which topology and "vonalrendszer (line system)" were treated. Perhaps this was based on the notes he used for his lectures on "Analysis situs". This book can be an evidence supporting GALLAI's remark that subjects of graph theory were taught in his lectures on "Analysis situs".

It seems, indeed, that he published other books in relation to the lectures he gave. For instance, in 1920, he published a textbook of mathematics *Mathematika: Műegyetemi előadás építész- és vegyészhallgatók számára (Mathematics: polytechnic lecture for architect and chemist students)* [106], which seems to be used for his lectures on "Real numbers and functions". This supports the idea that the previous book was also the notes on the basis of which he taught Analysis situs.

In 1931, he published an article "Graphok és matrixok (Graphs and matrices)" [117], in which the theorem on the separation of vertices of bipartite graphs was proved.

All these previous works paved the way to the publication in which later commentators recognized the birth of graph theory: in 1936, KŐNIG Dénes published the book *Theorie der endlichen und unendlichen Graphen (Theory of finite and infinite graphs)* [121]. In this book, he discussed graph theory based on set theory and combinatorics. He treated also various problems related to graph theory, among which recreational problems were found.

This fact makes us remember that KŐNIG published two books on mathematical recreations in 1902 and in 1905. As we will discuss in Chapter 6, the book of 1905 and the treatise of 1936 contain many same problems. However, the ways of treatment of these problems in the treatise of 1936 are different from those in the books of 1905. This difference reflects that the properties of these books are different from each other: KŐNIG's books of 1902 and 1905 are collections of mathematical recreations for a wide readership; on the other hand, the book of 1936 is a treatise of graph theory, which is "the first full-length book on the subject" according to BIGGS, LLOYD and WILSON, as mentioned in the preface of their book[215].

Chapter 3

Mathematical recreations before KŐNIG

KŐNIG worked on mathematical recreations when he was a student, and published two books in this domain in 1902 [86] and 1905 [87]. He gave some references at the end of the first book[86], and also some detailed information on the referred problems on mathematical recreations at the end of the second book[87].

In Chapter 6, I will analyze the relation between KŐNIG's works and the works of others on mathematical recreations before KŐNIG. As an introduction to that analysis, I will give in this chapter a brief history of mathematical recreations.

3.1 Mathematical recreations before BACHET's compilations

Some of the problems treated as mathematical recreations in modern books were already known in ancient times. According to Louis BECQ DE FOUQUIÈRES (1831–1887) a litterateur, Puzzles of calculation of numbers, construction of mosaics[1] etc. appeared already in ancient Greece[22]. BECQ DE FOUQUIÈRES gave also the purpose of treating such puzzles in referring to Plato's dialogue *Laws (VII)*.

[1] According to LUCAS, the puzzles of construction of mosaics are known by children under the name of "casse-tête chinois (Chinese puzzle)" ([149], p. 123), but he did not mention the relation between China and this kind of puzzles.

> **Becq de Fouquières [22], p. 71, l. 16.**
> Dans les *Lois* de Platon (VII) on peut lire un passage remarquable, relatif à la gymnastique intellectuelle à laquelle il faut soumettre l'intelligence des enfants. Il y est dit qu'il faut les exercer à une foule de petits calculs à leur portée, comme par exemple de partager, en un nombre plus ou moins grand de leurs camarades, un certain nombre de pommes ou de couronnes, en sorte qu'ils soient forcés tout en s'amusant de recourir à la science de nombres. Il est évident en effet que les enfants dans leurs jeux développent la faculté qu'ils ont de compter, de comparer, d'ajouter, de diviser et qu'ainsi ils arrivent à se familiariser avec les nombres.

According to Becq de Fouquières, in ancient Greece, the mathematical recreations were not only enjoyed but also given the educational purposes, and that both were inherited by people in France in 19th century.

Some of the problems that are considered as mathematical recreations in modern times were already known in ancient times. However, some of the problems were not considered as recreations.

For example, magic squares were known already in ancient times in China as well as in India, but the purpose of them were not always recreations.

For example in China, a magic square of third order (3 rows and 3 columns) was found on a divination board excavated in 1977 from a tomb of the Han Dynasty (206 B. C.–220 A. D.) in Fuyang, Anhui[2]. This board was created not for recreation but for divination.

Mathematical approaches to magic squares can be found much later. One of them was in Persia in the 10th century. Another one was in the Southern Song in the 13th century.

Abū'l-Wafā' al-Būzjānī (940–998)[3] and 'Alīb. Aḥmad al-Anṭākī gave methods to construct magic squares. [190, 239].

Yang Hui (around 1238–1298)[4] gave methods different from Abū'l-Wafā' to construct magic squares in the 13th century [209].

Abū'l-Wafā' al-Būzjānī worked also on the geometrical mosaic puzzles. These works in Persian were analyzed in French by Franz Woepcke (1826–1864)[5] in his article "Analyse et extrait d'un recueil de constructions géométriques par Aboûl Wafâ" in 1855 [243]. Woepcke's article consists of 2 parts:

[2]Taiyi Jiugong divination board, in possession of Youli Zhouyi Museum. (太乙九宫占盘, 羑里周易博物馆.)

[3]Mathematician and astronomer in Persia.

[4]Mathematician of the Southern Song.

[5]Woepcke is a German mathematician, and historian on Arabic and Persian mathematics.

one part is WOEPCKE's analysis on the selected extracts of Abū'l-Wafā' al-Būzjānī, and the other part is his French translation of the extracts of the works of Abū'l-Wafā' al-Būzjānī. WOEPCKE analyzed that the manuscripts consist of the works of Abū'l-Wafā' al-Būzjānī described by the students of Abū'l-Wafā' al-Būzjānī. Abū'l-Wafā' al-Būzjānī treated many problems of division of a square into some squares, and of composition of a square from some squares. The problem 7 in the second section of WOEPCKE's article includes an example similar to the problem 5 of the chapter on "Decomposition and recomposition of plane figures" in KŐNIG's book on mathematical recreations published in 1902 [86], which we will see later in 4.6, though KŐNIG did not refer to Abū'l-Wafā' al-Būzjānī's works. The problem 7 of Abū'l-Wafā' al-Būzjānī treated composition of a square from some equal squares. On the other hand, the problem 5 of KŐNIG treated decomposition of a square into three congruent squares. In WOEPCKE's extracts, the problem of decomposition of a square is not found.

3.2 Collections as mathematical recreations

Around 17th century, some problems were collected as mathematical recreations, and books in this genre began to be published.

The first collection of various problems as "mathematical" and yet "pleasing and delectable" was published in Europe. In 1612, Claude-Gaspar BACHET DE MÉZIRIAC (1581–1638) published *Problèmes plaisans et délectables qui se font par les nombres* [7]. This book was revised with augmentation in 1624 [8], and republished many times even until today. BACHET described in the preface the purpose of the book: he wanted to let the treatments of games come to light. It is to prove that the knowledge of games and real recreations practiced with joy can sometimes be useful. Some problems in this book of BACHET was treated later in KŐNIG's two books of mathematical recreations: guessing numbers (see 4.3), magic squares (see 4.4), weight measurements applied with binary and ternary system (see 5.2), selection from groups ("Turks and Christians", "Josephus and 40 refugees": see 5.7), ferrying a wolf, a goat and cabbages (see 5.8).

The concept of "mathematical recreation" maybe first appeared as the title of a book in 1624. According to Albrecht HEEFFER [226], an octavo entitled *Recreation mathematicqve, composee de plusieurs problemes plaisants et facetievx, En faict d'Arithmeticque, Geometrie, Mechanicque, Opticque, et autres parties de ces belles sciences* was published from the university of Pont-à-Mousson in France in 1624 [50]. HEEFFER notes also that the frontispiece mentions no name of an author but the dedication is signed "H. van ETTEN",

or "Henry van ETTEN" in the first English edition of 1633, who was a student at the local university. Although most bibliographical references and library catalogs mention "Jean LEURECHON" as the author, HEEFFER challenges this authorship [226]. According to HEEFFER, the source of most of the arithmetical and combinatorial problems treated in this book of 1624 is in the book of BACHET of 1612 [7].

In 1694, Jacques OZANAM (1640–1718) also published a collection of mathematical recreations in two volumes [165]. He referred to BACHET's book, and he added into his collection more ancient exemples. OZANAM's collection includes Large numbers (see 4.1), guessing numbers (4.3), ferrying a wolf, a goat and cabbages (5.8).

The books of BACHET and OZANAM were cited in almost all later books on mathematical recreations.

3.3 Bloom of mathematical recreations in the second half of 19th century

Édouard LUCAS (1842–1891) gave remarkable works on Mathematical recreations with four volumes of books published between 1882–1894[6] [148, 149, 151, 152].

Many results on recreational problems after OZANAM were added into LUCAS' books. Even the most recent problem —TARRY's talk on "geometry of situation" in the conference in 1886— was included in the fourth volume of LUCAS in 1894 [152], as the chapter "Les réseaux et les dominos".

In 1892 in the United Kingdom, Walter William Rouse BALL (1850–1925) published *Mathematical Recreations and problems of past and present times* [12]. The title was renamed *Mathematical Recreations and essays* for the fourth and later editions, and became a long seller.

David SINGMASTER made a precise examination on all the editions of BALL's book [240].

This book of BALL was translated from English to French by FITZ-PATRICK as shown in Table 3.1[7].

[6]The final volume was published three years after LUCAS's death. This means that LUCAS could not provide the errata to the book. In fact, in the chapter of "La Géométrie des réseaux et le problème des dominos", a diagram of heptagon lacks one diagonal.

[7]I suppose that KŐNIG cites Ball's book via French translation by FITZ-PATRICK, because he refers always to the page numbers of the French version, which do not correspond to the same pages of the English version. This means that KŐNIG read the French version enriched by FITZ-PATRICK.

Table 3.1: French versions of BALL's book (The numbers between brackets [ ] are the numbers listed in the bibliography of the present thesis.)

BALL	FITZ-PATRICK
1st/2nd (1892 [12])	
3rd (1896 [13])	French translation (1898 [14])
4th (1905 [15])	French translation enriched with many additions by FITZ-PATRICK (3 vols, 1907–1909 [16, 17, 18]); Reprinted in 1926–1927.
5th (1911 [19])	
...	
10th (1922 [21])	
...	

The 2nd edition is a simple reprint of the 1st edition. The 3rd edition includes augmentation of the contents, and the 4th edition includes more augmentation.

The 2nd French translation dependents on BALL's 4th edition. It includes much augmentation by FITZ-PATRICK, and consists of 3 volumes (1907–1909; reprinted 1926–1927). One of his additions can be found in the chapter of "Problèmes des tracés continus", which corresponds to "unicursal problems" of the original version, in the second volume of French version (1908/1926) [17]. FITZ-PATRICK added here "Tarry's method", which fully depends on TARRY's talk in 1886 on the geometry of situation [201], just as same as the fact that LUCAS included in his fourth volume of mathematical recreations [152].

This augmentation of the French version influenced BALL's later editions of the original English version.

In Germany, Wilhelm AHRENS (1872–1927) published *Mathematische Unterhaltungen und Spiele* in 1901 [1]. KŐNIG's books of 1902/1905 took many diagrams from this AHRENS' book.

3.4 Mathematical approach to recreational problems

We take a look here at some examples of mathematical approaches to recreational problems.

3.4.1 Seven bridges of Königsberg

The first mathematical approach to the problem of seven bridges of Königsberg can be found in the article by Leonhard EULER (1707–1783) published in 1736 [51].

I will give the outline of EULER's article. Moreover, I will compare two French translations of the article: one by Émile COUPY in 1851, and the other by Édouard LUCAS in 1882. The following points will be clear by the comparison: Both COUPY and LUCAS completed a table of EULER; Both applied the result of EULER to the Seine in Paris; The translation of COUPY is more exact than LUCAS; LUCAS added and changed phrases freely depending on his interpretation, which made the text more intelligible to readers; The translation of "Situs" by COUPY is inconsistent; LUCAS' one has consistency; COUPY did not mention the lack of proof on 2 of 3 theorems of EULER; LUCAS paid attention to it, mentioned it in his note, and described the proof.

It will also be clear that some mathematicians in the times of COUPY were interested in mathematical recreations, and that the proofs described by LUCAS were quite similar to those of HIERHOLZER (1873), but LUCAS did not mention it.

EULER's article on seven bridges of Königsberg

Leonhard EULER published an article titled "Solutio problematis ad geometriam situs pertinentis" in 1736 [51], in which he solved the problem known as "the problem of seven bridges of Königsberg". I will introduce this article briefly in this section.

The article of EULER consists of 21 sections, each of which has no title.

§ 1: Aim of this article. This article gives a specimen of *Geometriam Situs.*

§ 2: Introduction of the problem. 7 bridges connect 4 land areas as shown in the FIG. 3.1. Is there a route to cross every bridge once and only once? More generally, is there such a route for any other forms of rivers and bridges?

§ 3: Choice of the way to solve the problem. One can solve the problem by checking over every possible course, but EULER chooses a simpler method, with which one can find if such a route exists or not.

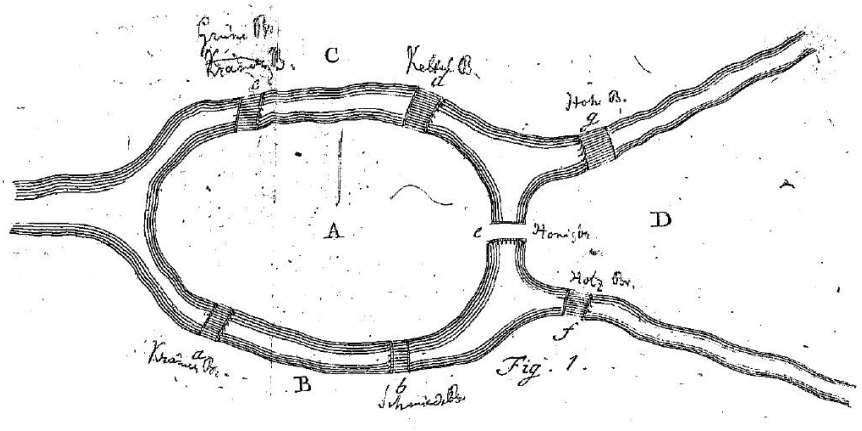


Figure 3.1: *Figura 1* in *Tabula VIII* inserted between p. 128 and p. 129 of the article of EULER [51].

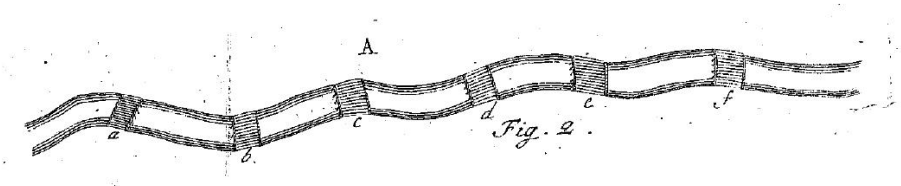


Figure 3.2: *Figura 2* in *Tabula VIII* inserted between p. 128 and p. 129 of the article of EULER [51].

§§ 4–5: Symbolization of the objects of the problem. EULER gives symbols A, B, C, D to the land areas and $a, b, \ldots, g$ to the bridges. Each route is described by a sequence of symbols of land areas in the order of passage. For example, ABD is a route depart from A via B to D, no matter which bridges are crossed. To describe a route to cross 7 bridges, 8 symbols are necessary.

§§ 6–9: Solution of the problem of seven bridges of Königsberg. Because there are 2 bridges between A and B, the sequence of symbols of the route demanded should include 2 sets of adjacent A and B. The same consideration is applied to the other bridges. EULER tries to find a law to judge if such a sequence of 8 symbols exists or not. Suppose that a traveler crosses the bridge a of the FIG. 3.2. In the sequence of the route, A appears once no matter if the traveler departs from A or arrives at A. Similarly, if A has 3 bridges and the traveler crosses them, A appears twice in the sequence of the route. Generally, if A has any odd number of $2n + 1$ of bridges, A

appears $n+1$ times in the sequence of the route. In the case of Königsberg, A has 5 bridges, and each of B, C, D has 3 bridges. Therefore A appears 3 times, and each of B, C, D appears 2 times in the sequence of route. But such a sequence cannot be realized with only 8 symbols. This means that there is no route to cross every bridge of Königsberg once and only once.

§§ 10–13: Generalization of the problem. EULER generalizes the problem to all the forms of bridges and land areas. EULER says that, if the sum of numbers of symbols to appear in the sequence is larger than "the number of bridges + 1", it is impossible to find such a route to cross every bridge once and only once. EULER says also that it is possible if the sum of numbers of symbols to appear in the sequence is equal to "the number of bridges+ 1", but it is not proved.

In the case where A has any even number of bridges, we should consider if the traveler starts from A or not. If the traveler does not start from A which has $2n$ bridges, A appears n times in the sequence of symbols of the route. If the traveler starts from A which has $2n$ bridges, A appears $n+1$ times in the sequence of symbols of the route.

We put off considering the starting point. Then, a symbol of a land area appears $n+1$ times if the land area has any odd number $2n+1$ of bridges, and n times if the land area has any even number $2n$ of bridges in the sequence of symbols of the route. If the sum of the symbols to appear in the sequence is equal to "the number of bridges + 1", there is a route to cross every bridge once and only once, where the starting point cannot be any land area which has any even number of bridges. If the sum of the symbols to appear in the sequence is equal to "the number of bridges", there is a route to cross every bridge once and only once, where the starting point should be a land area which has any even number of bridges, so as to increase by 1 the number of symbols to appear in the sequence. But EULER gives no proof for the case where such a route exists.

§ 14: Invention of an algorithm depending on §§ 10–13. EULER gives an algorithm to know if one can cross every bridge once and only once in any form of rivers and bridges. But actually, it is rather an algorithm to know if such a route is impossible or not,

because EULER gives no proof for the case where such a route exists.

1. Label the land area with symbols $A, B, C, \ldots$.

2. Write down "the number of bridges + 1".

3. Make a table with a column which consists of $A, B, C, \ldots$, and with the next column which consists of the number of bridges connected to each land area.

4. Asterisk the symbols of land areas which have any even number of bridges.

5. Make another column which consists of:
   n if the land area has any even number $2n$ of bridges,
   $n+1$ if the land area has any odd number $2n+1$ of bridges.

6. Sum up the numbers of the last column. If the sum is equal to the number written in the step 2, or if the sum is less by 1 than it, a route to cross every bridge once and only once is possible, where in the former case, the starting point should be one of the land areas without asterisk; in the latter case, the starting point should be one of the land areas with asterisk.

EULER makes such a table for the problem of Königsberg.

§ 15: Example different from seven bridges of Königsberg. EULER gives an example where a route to cross every bridge once and only once exists, applies the above-mentioned algorithm to it, and gives such a route. But we should still note that he gives no proof for the case where such a route exists.

§§ 16–17: Proof of the handshaking lemma To obtain a simpler way to judge if a route to cross every bridge once and only once exists or not, EULER proves thatthe number of land areas which have any odd number of bridges is an even number. This is called the handshaking lemma in our times. Proof: Count bridges which each land area has. The sum of these numbers is just twice as many as the number of all the bridges, because every bridge connects just 2 land areas, and it is counted double. Therefore the sum of the numbers of bridges which each land area has is an even number. If the number of land areas which have any odd number of bridges is an odd number, the sum of the numbers of bridges which each land area has cannot be an even number. So the number of land areas which have any odd number of bridges is an even number.

§§ 18–19: Simplification of the algorithm of § 14. Because the sum of the numbers of bridges which each land area has is twice as many as the number of all the bridges, $\frac{\text{(The sum of the numbers of bridges which each land area has)}+2}{2}$ is

just the number written in the step 2 of § 14. If every land area has anyeven number of bridges, the sum in the step 6 of § 14 is less by 1 than the number written in the step 2, with which EULER means that a route to cross every bridge once and only once is possible. Because every land area has any even number of bridges, any of them can be the starting point.
If only 2 land areas have any odd number of bridges and the other areas have any even number of bridges, the sum in the step 6 of § 14 is just as same as the number written in the step 2, with which EULER means that a route to cross every bridge once and only once is possible. In this case, one of the land areas which have any even number of bridges should be the starting point.
But we should still note that he gives no proof for it.
If the land areas which have anyodd number of bridges are 4, 6, 8 or more, the sum in the step 6 of § 14 is larger by 1, 2, 3 or more than the number written in the step 2. Then there is no route to cross every bridge once and only once.

§ 20: Summary of §§ 18–19. If more than 2 land areas have any odd number of bridges, it is impossible to cross every bridge once and only once.
If just 2 land areas have any odd number of bridges, and if the traveler choose one of such land areas as the starting point, it is possible to cross every bridge once and only once.
If every land area has any even number of bridges, no matter which land area is chosen as the starting point, it is possible to cross every bridge once and only once.
But we should still note that EULER gives no proof for the 2 latter propositions.

§ 21: Method to simplify the way to find the route. Remove pairs of bridges which connect 2 common land areas, and it will be easier to find a route to cross every bridge once and only once. After finding the route, put back the removed bridges as they were, and it will be easy to modify the route so as to include them.

Portrait of COUPY

Perhaps Émile COUPY is the first person who published a French translation [41] of EULER's article of seven bridges of Königsberg [51]. I introduce here the portrait of COUPY, who is not much known in our times.

TAB. 3.2 consists of the references concerning COUPY.

I regret that I could not find any publication of COUPY in the years between 1855 and 1868.

Table 3.2: COUPY in publications

N° Ref.	Year	Role	Name, Title	Contents
[38]	1844	Author	M. ÉMILE COUPY, bachelier ès sciences mathématiques.	Solution of a question by Fodot about the playing cards.
[39]	1847	Author	M. ÉMILE COUPY, Bachelier ès sciences et professeur de mathématiques à Orléans.	Solution of a problem of algebra about the mixtures.
[40]	1849	Author	M. J. COUPY, Professeur à l'École militaire de la Flèche.	Note on a theorem described by Louis Guillard about the cylinder and the cone circumscribed to a sphere.
[41]	1851	Translator	M. E. COUPY, Professeur au collège militaire de la Flèche.	Translation of EULER's article on seven bridges of Königsberg.
[42]	1851	Author	M. Emile Coupy	An account of his travels to London.
[43]	1853	Author	M. E. COUPY, Professeur au Collège militaire.	Solution of a question by Huet on the formula to give all the years in which February has five Sundays.
[58]	1853	Mentioned by the author	M. Coupy, professeur au Prytanée de la Flèche.	Coupy pointed out an error in a proof by H. Faure answering a question on the linear recursive sequence made by Richelot.
[207]	1855	Mentioned by the editor	M. Coupy, professeur au Prytanée de la Flèche.	Coupy sent to the editor a proof of a property and its various numerical applications concerning the algebraic equations and an arithmetic progression.
[45]	1855	Author	M. COUPY, Professeur au Prytanée de la Flèche.	Coupy pointed out an error in an article by W. Loof about the recurring decimals.
[46]	1855	Questioner	E. COUPY.	Question about the construction of a triangle under a sufficient condition.
[44]	1855	Author of a letter cited	ÉMILE COUPY, Professeur.	Biography of Mathurin Jousse of 17th century who published three books: one about the carpentry, one about the lock-work and one about the geometry concerning the architecture.
[47]	1868	Author	Y..	Biography of Marie Dorval, actress.

Depending on this table, COUPY seems to have been interested also in domains different from mathematics. In 1851, the year of the publication of his translation of EULER's article, he published also an account of his travels to London [42]. In 1868, he published a biography of an actress under the pen name of "Y..". How can we know that "Y.." is COUPY? It is written in the *Dictionnaire des pseudonymes* by Georges d'Heilly, the name of the author under the book title, or by Georges d'Heylli as pseudonym under his preface of the book. According to the catalogue of the National Library of France, the name "Georges d'Heilly" is also a pseudonym, and his real name is Edmond Antoine POINSOT, who was born in Nogent-sur-Seine, Aube on August 16 of 1833, and dead in Paris in 1902, and who was a litterateur, a section manager at the *Légion d'honneur* and the founder and the director of the *Gazette anecdotique, littéraire, artistique et bibliographique* (1876-1902) by profession.

The reason why COUPY used a pseudonym for his book [47] is explained by himself in the book of d'Heilly:

> **d'Heilly [227], p. 367, l. 16.**
> [...] M. Coupy est un lettré, un bibliophile, et aussi un journaliste; il a collaboré aux feuilles locales de ses diverses résidences, et surtout à Orléans, à La Flèche, etc... Il y a donné des articles de littérature et de théâtre. C'est un homme modeste, d'une vie douce et retirée que charme le culte constant des lettres. Il n'a pas voulu signer son livre: «Un professeur de mathématiques, nous disait-il, signant un livre sur une actrice! Ou le professeur passera pour bien léger, ou le livre pour bien mauvais.» Je connais l'homme et j'ai lu le livre, et je n'ai pas craint de nuire à son succès en dévoilant le nom, la qualité et aussi les qualités de son érudit auteur. [...]

According to d'Heilly, the formal name of COUPY is Philippe-Émile COUPY. The author of the article [40] is entered as "M. J. COUPY", so it might be a different person from Émile COUPY, or a misprint.

COUPY passed his boyhood in Blois according to the book *Une excursion à Londres* written by himself in 1851 [42], but I did not find his birth year. I see his profession in the titles of his articles.

He was already a Bachelor of Mathematical Science in 1844. He became a professor of mathematics at Orléans before 1847. He was transferred to the military school of la Flèche before 1851 (or before 1849 if "J. Coupy" is the same person as "Émile Coupy"), and to the Imperial Military Prytaneion of la Flèche before 1853.

Depending on the contents of his articles, I suppose that he was interested in the mathematical recreations: Some of his articles [38, 39, 41, 43] seem to be mathematical recreations, that is, recreative problems which require mathematical methods; from a certain point of view, some other articles [40, 58, 45, 46] can be regarded as recreative, though we cannot distinguish clearly recreative mathematics and non-recreative ones.

I suppose that some other mathematicians were also interested in the mathematical recreations, because the persons who made questions of the mathematical recreations solved by COUPY should have been interested in it.

COUPY describes [41] the reason for the translation of EULER's article of seven bridges of Königsberg [51] that it was mentioned in a memoir by Louis POINSOT (1777–1859) [175] and in a textbook of algebra by LHUILIER [142] [8]. I suppose that it is also a factor of the translation that COUPY was generally interested in the recreations.

Translations of EULER's article

As well as COUPY, Édouard LUCAS published a French translation of EULER's article on seven bridges of Königsberg [51]. LUCAS' works and his biography were already much examined by Anne-Marie DÉCAILLOT-LAULAGNET. Her article on this subject was published in 1998 [217]. Her doctoral thesis in 1999 also dealt with the subject [218].

LUCAS introduced the problem of bridges as the second recreation in his book *Récréations mathématiques I* published in 1882 [148], 31 years after COUPY's translation [41]. He inserted a French translation of EULER's article, and made some comments on it.

LUCAS did not mention the translation of COUPY explicitly, but I suppose that he knew it, because he described the following:

> **LUCAS [148], p. 21, l. 7.**
> [...] nous donnons, d'après les *Nouvelles Annales de Mathématiques,* un commentaire de cet opuscule, qui a paru en latin dans les *Mémoires de l'Académie des sciences de Berlin* pour l'année 1759, et qui a pour titre: *Solutio problematis ad Geometriam situs pertinentis.* [...]

The *Nouvelles Annales de Mathématiques* is the periodical which contains the translation of EULER's article by COUPY. But LUCAS did not enter COUPY's translation in his references, and every translated sentence

[8] COUPY spelt the name as "Lhuillier", but it is spelt as "Lhuilier" in the publications of LHUILIER himself.

in LUCAS' book includes different words and phrases from COUPY's translation. Therefore I suppose that LUCAS translated EULER's article almost independently of COUPY's translation.

I will pick up here some parts of translations of COUPY and LUCAS, and examine the difference of them.

Translation of § 1 This is the beginning of § 1 of EULER's article:

> **EULER [51], p. 128, l. 9.**
> Praeter illam Geometriae partem,
> quae circa quantitates versatur, et omni tempore summo studio est exculta, alterius partis etiamnum admodum ignotae primus mentionem fecit *Leibnitzius*, quam Geometriam situs vocauit. [...]

> **Translation by COUPY [41], p. 107, l. 1.**
> Outre cette partie de la géométrie qui traite des grandeurs et qui a été de tout temps cultivée avec beaucoup de zèle, il en est une autre, jusqu'à nos jours complètement inconnue, dont Leibnitz a fait le premier mention et qu'il appela *géométrie de position*. [...]

> **Translation by LUCAS [148], p. 21, l. 13.**
> Outre cette partie de la Géométrie qui s'occupe de la grandeur et de la mesure, et qui a été cultivé dès les temps les plus reculés, avec une grande application, Leibniz a fait mention, pour la première fois, d'une autre partie encore très inconnue actuellement, qu'il a appelée Geometria situs. [...]

COUPY changed the initial letter of "geometrie (Geometriae)" from capital to small, and LUCAS kept it as it was. COUPY translated "versatur" as "s'occupe", and LUCAS as "traite". COUPY translated "quantitates" as "grandeurs" in the plural as it was. LUCAS changed it to the singular "grandeur", and added a new word "mesure" which is not written in EULER's original text.

I suppose that COUPY tried to translate it as exactly as possible, and LUCAS rather tried to make the text intelligible to readers.

COUPY translated the important phrase "Geometriam situs" as "géometrie de position", but LUCAS kept here as the Latin phrase, where he changed only the accusative case "-am" to the nominative case "-a". It seems a proper choice because it represents the phrase called by Leibniz. As for the translation of "situs", COUPY has an inconsistency: he translated it as "situation" in

the title, but as "position" in the text. On the other hand, LUCAS has a consistency in translating it as "situation" in every case without this sentence.

Concerning the translation for "geometria situs", Louis POINSOT already described his point of view. According to POINSOT, the signification of the "géométrie de situation" of LEIBNITZ and that of the "géométrie de position" of Lazare Nicolas Marguérite CARNOT are completely different. POINSOT mentioned the «géométrie de situation» in the footnote on p. 17 of his memoir in 1810 [175] citing the words of LEIBNITZ:

> **POINSOT [175], p. 17.**
> [...] La géométrie de situation, comme je l'ai dit, a pour objet l'ordre et les lieux dans l'espace, sans aucune considération de la grandeur ni de la continuité des figures; de sorte que la partie de l'analyse mathématique qui pourrait naturellement s'y appliquer, est la science des propriétés des nombres ou l'*analyse indéterminée,* comme l'analyse ordinaire s'applique naturellement aux problèmes déterminés de la géométrie, et le calcul différentiel à la théorie des courbes, où les affections changent par des nuances insensibles. Je n'ai pas pu trouver l'endroit des Actes de Leipsick, où *Leibnitz* a dit un mot de la géométrie de situation; mais il me semble qu'il en avait une idée conforme à celle que nous en donnons ici, et c'est ce qu'on voit assez clairement dans ce passage d'une de ses lettres sur les jeux mathématiques: «Après les jeux qui dépendent uniquement des nombres, dit-il, viennent les jeux où entre encore la situation, comme dans le Trictrac, dans les Dames, et sur-tout dans les Échecs. Le jeu nommé *Solitaire* m'a plu assez. [...]». (Lettre VIII, à M. de Montmort. *Leibn. Opera philologica*).

Moreover, POINSOT mentioned the «géométrie de position» of CARNOT citing his memoir of 1806 [26]:

> **Poinsot [175], pp. 17–18.**
> Quant à la *Géométrie de position* de M. *Carnot,* elle n'a point du tout le même objet. L'auteur a eu principalement en vue d'établir, par la corrélation des figures, la véritable théorie des quantités négatives. C'est ce dont on peut voir un précis rapide à la fin d'un excellent Mémoire qu'il a donné depuis, *sur la relation qui existe entre les distances mutuelles de cinq points pris dans l'espace;* Mémoire qui contient encore sur la pyramide triangulaire, une suite d'élégans théorèmes exprimés par les données mêmes de la figure; et cette ingénieuse théorie des *transversales,* dont les principes simples et féconds mériteraient bien d'être admis au nombre des élémens de la géométrie.

We can see in these descriptions of Poinsot that "géometrie de situation" and "géometrie de position" already had different meanings in French language around the beginning of the 19th century. According to this terminology, the translation of the expression "Geometria situs" should be "géometrie de situation".

Footnote to § 3

> **Euler [51], pp. 129–130, l. 28.**
> [...] Quamobrem missa hac methodo, in aliam inquisiui, quae plus non largiatur, quam ostendat, vtrum talis cursus institui queat, an secus; talem enim methodum multo simpliciorem fore sum suspicatus.

> **Translation by Coupy [41], p. 108, l. 19.**
> [...] Ayant donc laissé de côté cette méthode, j'en ai cherché une autre qui me donne non pas toutes les manières de passer, mais me montre seulement celle qui satisfait à la question; et je regarde une pareille méthode comme de beaucoup plus simple que la précédente.

> **Translation by Lucas [148], p. 23, l. 20.**
> [...] Donc, en laissant de côté ces considerations, j'ai recherché s'il n'était pas préférable d'imaginer une méthode qui permît de juger, au premier abord, de la possibilité ou de l'impossibilité du problème; je pensais, en effet, qu'une telle méthode devait être beaucoup plus simple. [Footnote by Lucas:] «Cette remarque d'Euler comporte un très grand caractère de généralité qu'elle ne paraît pas avoir tout d'abord. J'ai observé que, dans un grand nombre de problèmes de la Géométrie de situation, il y a souvent une différence considérable dans la manière de traiter la possibilité et l'impossibilité; en général, l'impossibilité se manifeste plus facilement que la possibilité, [...] .»

Coupy translated this part exactly as Euler described. On the other hand, Lucas changes the expression of the latter half. Lucas inserted the words "possibilité" and "impossibilité" so as to emphasize that Euler treated the problem of bridges as a problem of possibility and impossibility. Moreover, he added a footnote to it, and stated clearly that such a treatment is often appears in problems of "géometrie de situation", that the methods to treat the possibility and the impossibility are quite different, and that the treatment of impossibility is generally easier than that of possibility. This warning is quite useful for readers to understand what Euler proved and what he did not proved.

Possibility and impossibility in § 13 In § 13, Euler described 2 cases where a route to cross every bridge once and only once exists. But they are not proved by Euler. Coupy translated this part exactly as Euler describes. On the other hand, Lucas completely changed the contents of this part.

> **Euler [51], p. 134, l. 27.**
> Deinde si numerus omnium vicium adaequet numerum pontium vnitate auctum, tum transitus desideratus succedit, at initium ex regione, in quam impar pontium numerus ducit, capi debet. Sin autem numerus omnium vicium fuerit vnitate minor, quam pontium numerus vnitate auctus, tum transitus succedet incipiendo ex regione, in quam par pontium numerus ducit, quia hoc modo vicium numerus vnitate est augendus.

> **Translation by COUPY [41], p. 113, l. 17.**
> Ensuite, si le nombre de fois que toutes les lettres doivent s'écrire est égal au nombre des pont augmenté de 1, alors le passage désiré a lieu, mais on doit commencer à marcher d'une région à laquelle conduisent un nombre impair de ponts; mais si ce nombre de fois est inférieur de 1 au nombre des ponts augmenté de 1, alors le passage a lieu en commençant par une région à laquelle conduise un nombre pair de ponts, parce que par ce moyen le nombre des fois qu'on doit écrire les lettres est augmenté de 1.

> **Translation by LUCAS [148], p. 28, l. 7.**
> Nous aurons alors deux cas à considérer, suivant que le départ s'effectue d'une régio impaire ou d'une région paire.
> Dans le premier cas, le problème sera impossible si le nombre total des répétitions des lettres ne surpasse pas d'une unité le nombre total des ponts. Dans le cas de départ d'une région paire, le problème sera impossible, si le nombre total des répétitions des lettres n'égale pas le nombre des ponts; car, en commençant par une région paire, on devra augmenter d'une unité pour cette région, et pour celle-là seulement, le nombre des répétitions de la lettre correspondante.

LUCAS changed the propositions to the inverse of them, so the meaning is different from that of EULER. I suppose that LUCAS noticed that EULER did not prove these propositions, and he wanted to make them coherent. As a result, the contents of § 13 became more reasonable than the original text.

Table in § 14 There was a lack in a table of EULER. He forgot to write the sum of the last column in § 14 (TAB. 3.3), though he did not forget it for the other table in § 15. Both COUPY and LUCAS complemented the sum "9" to the table in § 14, and used the table in § 15 as it was.

Incidentally, Norman L. Biggs and E. Keith Lloyd and Robin J. Wilson, who translated the same article into English in their book published in 1976 [215], did not complete the table, and kept it as it was.

The following description might be the reason for it:

> **Biggs, Lloyd and Wilson [215], p. 2, l. 12.**
> However, the reader may prefer to stop after Paragraph 9, and go on to our commentary at the end of the article, since EULER's main results will be proved more succinctly later in the chapter.

Numerus pontium 7, habetur ergo 8

Pontes	
A, 5	3
B, 3	2
C, 3	2
D, 3	2

(a)

Nombre des ponts 7; j'ai donc 8.

Ponts.		
A	5	3
B	3	2
C	3	2
D	3	2
		9

(b)

Table 3.3: (a) Table of § 14 of Euler [51], p. 136.
(b) Table of § 14 of Coupy [41], p. 114.

Biggs, Lloyd and Wilson newly described the part after § 9 in their own way. I suppose that Biggs, Lloyd and Wilson did not attach importance to Euler's procedure after § 9, and that the table in § 14 is not important for the readers of their book.

Comments on Euler's article

Proof of possibility As we viewed in 3.4.1, Euler did not prove the following 2 propositions of § 20:

> If just 2 land areas have any odd number of bridges, and if the traveler choose one of such land areas as the starting point, it is possible to cross every bridge once and only once.
> If every land area has any even number of bridges, no matter which land area is chosen as the starting point, it is possible to cross every bridge once and only once.

But Coupy did not mention this lack. I suppose that he was not aware of it depending on the following comment by Coupy:

> **Coupy [41], p. 119, l. 5.**
> [...] en appelant D la rive droite, G la rive gauche, A et B les îles de la Cité et Saint-Louis, on reconnaît que 11 ponts conduisent en A, 8 en B, 14 en G, 15 en D; donc le problème est possible, d'après la règle du n° 20, pourvu qu'on parte de la Cité ou de la rive droite, et il est très facile de trouver effectivement la marche à suivre. [...]

2 of 3 propositions of the "règle du n° 20" were not yet proved, but he used one of them without proof, and concluded that it was possible to cross every bridge in Paris once and only once.

On the other hand, LUCAS mentioned that EULER did not prove the possible cases, and he described the proof of them in the Note II at the end of his book.

Note II consists of the proofs of 2 theorems. The first theorem is called the handshaking lemma in our times:

> **LUCAS [148], p. 222, l. 28.**
> THÉORÈME I — *Dans tout réseau géométrique formé de lignes droites ou courbes, le nombre des points impairs est toujours zéro ou un nombre pair.*

It was already proved by EULER in §§ 16–17. LUCAS mentioned it, and showed another proof for this theorem.

I describe here again the proof of EULER which I mentioned in 3.4.1:

> Count bridges which each land area has. The sum of these numbers is just twice as many as the number of all the bridges, because every bridge connects just 2 land areas, and it is counted double. Therefore the sum of the numbers of bridges which each land area has is an even number. If the number of land areas which have any odd number of bridges is an odd number, the sum of the numbers of bridges which each land area has cannot be an even number. So the number of land areas which have any odd number of bridges is an even number.

In the proof of LUCAS, the land areas are called as *les diverses stations du réseau, les divers points d'embranchement, les têtes de ligne*, and bridges are called as *chemins*. He removes the *chemin* one by one. Every time he removes it, the *parité* (the property even or odd) of the number of *chemins* connected to each *station* at each end of the *chemin* is changed. That is, if each of 2 *stations* has any odd number of *chemins*, each will lose 1 *chemin*, and will have an even number of *chemins* after the removal of the *chemin*; if each of them has any even number of *chemins*, each will have an odd number of *chemins*; if one of them has any odd number of *chemins* and the other has any even number of *chemins*, these 2 *stations* will exchange their *parités*. Therefore, the removal of *chemins* will not change the *parités* of the number of *stations* which have any odd number of *chemins*. After the removal of all the *chemins*, the number of *stations* which have any odd number of *chemins* is 0. Therefore, at the initial state before the removal of *chemins*, the number of *stations* which have any odd number of *chemins* should be 0 or an even number. Q. E. D.

This proof of LUCAS consists of the same idea as HIERHOLZER who proved also the handshaking lemma

> **Hierholzer [70], p. 32, l. 17.**
> Ein Linienzug kann nur eine gerade Anzahl ungerader Knotenpunkte besitzen.

in the final part of his article in 1873 [70].

The second theorem of Note II of Lucas is the theorem concerning the possibility of crossing every bridge once and only once:

> **Lucas [148], p. 223, l. 21.**
> Théorème II — *Tout réseau géométrique qui contient* $2n$ *points impairs peut être décrit par un nombre minimum de* n *traits sans répétition. Tout réseau géometrique, qui ne contient que des points pairs, peut être décrit par un seul trait sans répétition.*

One of three propositions written by Euler in § 20 deals only with the case of $n = 1$ of this theorem. In other words, this theorem is more general than the proposition of Euler. The article of J. B. Listing published in 1847, which is entered in the references by Lucas, contains the same theorem, though Listing did not prove it.

Lucas' proof of the second theorem include again the same idea as Hierholzer [70]. Only two differences exist between them:

- Hierholzer, as well as Euler, deals with connected line systems; Lucas deals also with separated line systems.
- Hierholzer deals only with the case of $n = 1$ of Lucas' theorem.

For the proof, Lucas as well as Hierholzer suppose *le réseau géométrique continu* (Lucas), that is, *ein zusammenhängender Linienzug* (Hierholzer), and try to trace all the lines continuously without repeating any line. Although Lucas' theorem includes also separated line systems, he deals only with connected line systems at the first step of the proof. And after proving the case of connected systems, he will apply the proved result to each connected part of the separated systems.

In the case where every *station* of a connected system has any even number of *chemins*: He starts from any *station* M, and traces the *chemins* freely without repeating any *chemin*. Because every *station* has any even number of *chemins*, no matter which *station* except M he arrives, there should be a *chemin* never traced. So he will not be blocked at any *station* except M, and will come back some time to the starting point M. If there is still any *chemin* never traced, he starts again from any *station*, and will come back again to the starting point. By repeating this procedure, all the

chemins will be traced, and all the routes can be unified so as to be traced once and only once. Therefore, in the case where every *station* of a connected system have any even number of *chemins*, there is a route to trace all the *chemins* continuously without repeating any of them.

In the case where $2n$ *stations* of a connected system have any odd number of *chemins*: He starts from any *station* A which has any odd number of *chemins*, and traces the *chemins* freely without repeating any *chemin*. Every time he arrives at a *station* which has an even number of *chemins*, there should be a *chemin* never traced. It means that the *station* B where he will be blocked should have an odd number of *chemins*. After arriving at B, he removes the *chemins* already traced. Then, because every *parité* of the visited *stations* does not change, and because the *parités* of A and B will be changed from odd to even, the number of *stations* which have any odd number of *chemins* will be reduced to $2n - 2$. After repeating it n times, the number of *stations* which have any odd number of *chemins* will be 0. In the proof of HIERHOLZER, he deals only with the case of $n = 1$, so there is no need to repeat this procedure. Then only the *stations* which have any even number of *chemins* are left, and it is the case already proved that there is a route to trace all the *chemins* continuously without repeating any of them. Choosing the final B as the starting point M, he can trace all the *chemins* with n strokes without repeating any of them. Q. E. D.

For both theorems I and II, the ideas of the proofs of LUCAS are the same as HIERHOLZER, but LUCAS did not enter the article of HIERHOLZER in his references. Therefore it is unclear if LUCAS knew the article of HIERHOLZER. Possibly LUCAS did not read the article of HIERHOLZER directly but learnt its contents from others.

The Seine Both COUPY and LUCAS mentioned the bridges over the Seine in Paris, to which they applied the result of EULER.

COUPY considered the area from Iéna bridge to Austerlitz bridge, so the Isle of the Swans, which was constructed in 1827, was not included. On the other hand, LUCAS included it, so he considered 3 islands. This difference comes from the enlargement of the territory of Paris in 1860: COUPY's article (1851) was before the enlargement, and LUCAS' book (1882) was after the enlargement.

COUPY counted all the bridges over the Seine in Paris. On the other hand, LUCAS considered only the bridges connected to the islands: 9 bridges connect the islands and the right bank, and 8 bridges connect the islands and the left bank; then, no matter how many bridges connect directly the

right bank and the left bank, it is clear that one bank has an odd number of bridges, and the other has an even number of bridges, so there is no need to count them.

COUPY and LUCAS applied the theorem which EULER did not prove, and which LUCAS completed in his Note II, that is:

> If just 2 land areas have any odd number of bridges, and if the traveler choose one of such land areas as the starting point, it is possible to cross every bridge once and only once.

According to COUPY, the Isle of the Cité had 11 bridges, the Saint-Louis Isle had 8, the left bank had 14, and the right bank had 15. From this information, it becomes clear that the bridges to the Isle of the Cité were more by 1, the bridges to the Saint-Louis Isle were more by 2, the bridges to the left bank were less by 4, and the bridges to the right bank were less by 3 than our times. From the description of LUCAS, we obtain also the information that there was a bridge called the *Estacade* between the Saint-Louis Isle and the right bank.

Conclusion

The same points on the treatments of EULER's article (1736) by COUPY (1851) and by LUCAS (1882) are:

- that they complemented a term to the table of § 14 of EULER, and
- that they applied the result of EULER to the Seine in Paris.

The difference between them are the followings:

- The translation of COUPY is more exact than LUCAS. LUCAS added and changed phrases freely depending on his interpretation, which made the text more intelligible to readers.
- The translation of "Situs" by COUPY is inconsistent: It is translated as "situation" in the title, but as "position" in the text. LUCAS' translation of it has consistency: the part called by Leibnitz is kept in Latin as it was, and all the other parts were translated as "situation".
- COUPY did not mention the lack of proof on 2 of 3 theorems. LUCAS paid attention to it early in § 3, mentioned it in his note, and described the proof in his Note II at the end of the book.

In addition, it became clear:

- that some mathematicians in the times of COUPY, around the year 1851, were interested in mathematical recreations;
- that the proofs in LUCAS' Note II (1882) were quite similar to those of HIERHOLZER (1873), but LUCAS did not mention it.

3.4.2 Labyrinths

Some collectors of labyrinths showed in their publications that labyrinths, or figures seem like a labyrinth, can be found already in ancient times worldwide [155, 23, 76].

Mathematical approach to labyrinths can be found in the following ways:

1. Algorithms to find a way from a point to another point in a labyrinth without any help of a map of the labyrinth;
2. Geometrical approach to the forms of paths and walls of labyrinths, including algorithms to create the forms.

Solving problems of labyrinths

Algorithm of Trémaux Édouard Lucas, in the book *Récréations mathématiques I* published in 1882, presented an algorithm to find a goal point of a labyrinth without any help of a map of the labyrinth [148].

According to Lucas, the algorithm was found by Trémaux, who was an old student of the Polytechnic School, and an engineer of telegraph.

The aim is to find a goal by walking in a labyrinth, of which a starting point and a goal are fixed.

The algorithm of Trémaux consists of the following steps:

Rule 1: Leave the starting point of a labyrinth. At the first junction, choose an arbitrary pathway, enter it and go ahead until you arrive at a dead end or a junction:

- If you arrive at a dead end, go back to the last junction, and mark the pathway not to enter it.
- If you arrive at a junction, and if it is the first time to be here, make a directional marker on the last pathway, choose an arbitrary pathway, make a directional marker also on it and enter it.

Rule 2: If you arrive at a junction already passed through, and if it is the first time you passed through the last pathway, make on this pathway two directional markers of coming and going, and go back to the last junction.

Rule 3: If you arrive at a junction already passed through, and if it is the second time you passed through the last pathway, mark this pathway again, and:

- Choose a pathway not yet passed through if any, make a directional marker on it, and enter it.
- If all the pathways are already passed through, choose a pathway through which you passed only once, make a directional mark on the pathway, and enter it.

This algorithm is called the "Depth-First Search algorithm" today. If the extent of a labyrinth is finite, this algorithm brings you to the goal point without fail, though the path is not necessarily the shortest.

Successors of Trémaux Øystein ORE gave an algorithm to find a goal point of a labyrinth in 1959, which is different from that of TRÉMAUX [163]. This algorithm is called the "Breadth-First Search algorithm" today, and consists of the following rules:

Rule 1: Leave the starting point of a labyrinth. At the first junction, choose a pathway, enter it and go ahead until you arrive at a dead end or a junction:

- If you arrive at a junction already passed through or a dead end, go back through the last pathway, and mark it not to enter.
- If you arrive at a junction, and if it is the first time to be here, mark the last pathway, and go back.

At the first junction, choose another pathway with no marking, and proceed the same thing as above.

Rule 2: After passing through all the pathways from the first junction, choose a pathway with no marking, enter it, go ahead until you arrive at a dead end or a next junction, and proceed the same thing as Rule 1. After passing through all the pathway from this junction, go back to the first junction, choose another pathway with no marking, and redo the same thing as above.

With this algorithm, you will find the shortest path. Moreover, even if the extent of labyrinth is infinite, the goal point at a finite distance can be found.

However, the number of pathways to be passed through increases according to an exponential function of the distance (number of pathways) from the starting point. Therefore, if there are many junctions, even if the goal point is near the starting point, you should pass through many pathways. In other words, when a computer searches the goal point of a labyrinth, this algorithm requires more junctions to be put in the queue of its memory than the algorithm of TRÉMAUX.

Richard E. KORF gave another algorithm *Depth-First Iterative Deepening Search* in 1985 [130]. This algorithm is similar to the algorithm of TRÉMAUX. The difference is to limit first the "depth of search", that is, the number of pathways to be passed through from the starting point. And then obey the rules of the algorithm of TRÉMAUX. If the goal point is not found within this limit, quit once this search, and extend the limit, and restart the search. Repeat this procedure, and you will find the shortest path to the goal point. This algorithm has two advantages: differently from the algorithm of TRÉMAUX, it will not search distant points in vain; differently from the algorithm of ORE, it will not occupy much part of your memory.

This article of KORF was published in the journal *Artificial intelligence*. Many new algorithms to find a goal point in a labyrinth are published in the domain of artificial intelligence.

Geometrical approach

It is possible that algorithms to create some kinds of labyrinths, or figures seems like a labyrinth, were found already in ancient times, because of the similarity of figures found in the same area: figures shown on ancient coins of Crete, labyrinths on the floors of Gothic cathedrals and so on[9].

Manuscripts of C. F. Gauss Among the mathematical works, the manuscripts of Carl Friedrich Gauss between 1823 and 1844 seem to be related to the forms of labyrinths. These manuscripts were inserted in the book *Werke* published after his death [66, 67][10] .

In these manuscripts, Gauss examined not the labyrinths but the knots, but the notions in his procedure can be related to those of labyrinth: he treated an entangled loop as a path starting from a point on the loop, following points on the loop and returning to the starting point.

This manuscript [67] is an important result of knot theory. Gauss named the points of intersection of a loop $a, b, c, ...$ Starting from a point, he followed all the points on the loop: the order of passage of the points is represented with a sequence of letters. Using this notation, Gauss indicated that some orders of passage of the points are not possible. For example, when there are 2 points of intersection on a loop, the orders *aabb* and *abba* are possible, but *abab* is impossible.

Such a sequence of letters is called "Gauss code" today.

Pierre Rosenstiehl's notation of labyrinth in his article "Les mots du labyrinthe" in 1973 [180] is similar to "Gauss code". He described a labyrinth as a graph: the pathways are edges of a graph. The directions of traveler going through an edge A is denoted with a and a'. A path of a traveler is represented with a sequence of letters indicating directions. He called this notation "les mots de labyrinthe". His method is similar to the method of "Gauss code" of knot theory. For example, a "mot de labyrinthe" aa' (he called it "nœud") of Rosenstiehl means that a traveler goes and back through a pathway in a labyrinth. This journey returns the same result as staying at the first point, therefore aa' can be eliminated from the whole sequence of letters.

[9] The figures of those labyrinths are shown in the collections of labyrinths [155, 23, 76].

[10] According to the remark of Paul Stäckel on the manuscripts of Gauss related to the knots [66, 67], the periods of writing them are considered as follows: "I. Zur geometria Situs" [66] consist of 9 parts, which form one suite in a notebook, but the periods of writing should be different. The parts 1 and 2 are maybe written between 1823 and 1827, and the part 9 is maybe written after 1840; "II. zur Geometrie der Lage für zwei Raumdimensionen" [67] is maybe the foundation of the note dated December 30 of 1844 written on the page of the part I.2.

Such a property is similar to "Gauss code" aa, which can be eliminated with a Reidemeister move.

ROSENSTIEHL made another topological approach to the form of labyrinths [184]. He remarked that the cathedral labyrinths can be classified with the number of layers of path $2l$ and the number of "semi-axes", at which the path is folded, a. Some labyrinths of cathedrals have 12 layers of pathways and 3 "semi-axes". In this article, he showed the homeomorphism of a form of a cathedral labyrinth and a figure of fishes and birds by continuous transformations.

3.5 Citation by KŐNIG

The following collections were cited by KŐNIG Dénes in his books on mathematical recreations published in 1902 and 1905 [86, 87]: *Problèmes plaisans et délectables qui se font par les nombres* by Claude-Gaspar BACHET DE MÉZIRIAC [7, 8, 9, 10] in the early 17th century; *Récreations mathématiques et physiques* by Jacques OZANAM [165, 172] in the 17th century; *Récreations mathématiques* (4 volumes) and *L'Arithmétique amusante* by Édouard LUCAS [148, 149, 151, 152, 153] and *Mathématiques et mathématiciens: pensées et curiosités* by Alphonse REBIÈRE [177, 178] in the 19th century; *Mathematical Recreations and problems of past and present times* (*Mathematical Recreations and essays* after the 4th ed.) by Walter William Rouse BALL [12, 13, 14, 15, 16, 17, 19] (and many later editions), *Mathematische Mußestunden* by Hermann SCHUBERT [187, 188, 189] and *Récréations arithmétiques* and *Curiosités géométriques* by Émile FOURREY [65, 60] around 1900; *Mathematische Unterhaltungen und Spiele* and *Mathematische Spiele* by Wilhelm AHRENS [1, 3, 4, 2] in the early 20th century. Also in the period close to the publication of KŐNIG's treatise of 1936, a book in this genre was published in Belgium: *La mathématique des jeux ou récréations mathématiques* by Maurice KRAITCHIK [131], which might suggest interest of mathematicians at that time in mathematical recreations.

In the UK and the USA of the 19th and the 20th century, amateur mathematicians or scientific writers —Samuel LOYD, Henry Ernst DUDENEY, Martin GARDNER and so on— also published in this domain. Some problems in their collections were taken from the collections listed above. This fact is interesting from the point of view of the popularization of mathematics. However, we cannot find any relation of them to KŐNIG's works.

Chapter 4

Mathematikai Mulatsagok 1: Mathematical recreations 1

4.1 Convention of translation

Here is the original text with my translation of *Mathematikai Mulatsagok, első sorozat* (Mathematical recreations, first series) by Dénes KŐNIG, 1902. The second series will be treated in the next chapter.

As GALLAI Tibor mentioned in his article in 1965 [224], these books were reprinted many times. However, I have no information on the versions after 1905 before 1991.

The editions of 1991 and 1992 are re-typeset versions with TypoTEX of the books of 1902 and 1905 respectively. The largest difference from the first editions is that the part "Az első és második sorozat problemáinak eredetéről és irodalmáról (About sources and bibliography of the problems of the first and second series)" at the end of the book of 1905 was totally omitted in the new versions. Apart from this, there are not many modifications from the books of 1902 and 1905, but small changes related to the orthography can be found. I took here the original orthography of 1902 and 1905 as it were.

In the editions of 1991 and 1992, some notes of the anonymous editor were added, but I don't translate them here in order to avoid the copyright infringement. This restriction does not make much problem to our purpose, because the notes of the editor did not modify the original text but add only small supplementary informations.

Each page number between brackets [ ] in the original text indicates the page number of the original texts of 1902 and 1905.

4.0 Előszó: Preface

Note of the translator:
This preface was written by Beke Manó (1862–1946, Emanuel Beke in German), who was one of the mathematical teachers of Kőnig Dénes during his teenage years at the gynmasium. See Chapter 2 for his role related to Kőnig's mathematical activities.

[p. 3]
Nagy örömömre szolgált, hogy fiatal barátom, legkedvesebb tanítványaim egyike vállalkozott arra, hogy „Mathematikai mulatságok“ czímen oly könyvecskét szerkeszszen, mely hivatva van némileg arra, hogy a mathematikai problemák iránti érdeklődést az iskolában és ezen kívül is fölkeltse, s az olvasót némely általánosan elterjedt, és sokszor a középiskolai tanításanyagba egyáltalában bele nem illeszthető mathematikai kérdéssel foglalkoztassa.

Ilyen irányú könyv már igen sok van; a külföldi irodalom legjelesebb ilyen gyűjteményeit, melyekből a szerző anyagát válogatta, a könyvecske végén felsorolva találhatja az olvasó; még magyar nyelvű gyűjtemény is létezik; de az ezen munkácskánál sokkal elemibb, inkább csak a közkeletű számtani és geometriai rejtvényeket tartalmazza.

It was my great joy that my young friend who is one of my dearest students undertook to edit a booklet entitled “mathematical entertainments”. He has somewhat a talent for this work. He undertook also to arouse an interest in the mathematical problems in the school as well as out of school and an interest of readers in general public, letting them deal with many mathematical questions exceeding the subjects to be taught in high-school.

There are already a great many books in such an orientation; publications from abroad have the most remarkable collections in this genre. The author chose his subjects from those collections, and readers can find the sources listed at the end of the booklet. Even if a collection in Hungarian language exists, it is much more elementary than the work here, and includes rather only commonplace arithmetic and geometry puzzles.

Ez a könyvecske nemcsak ilyen, általánosan elterjedt, rendszerint elsőfokú egyenletek által megfejthető talányokat, mondhatnók, népszerű találós számtani feladatokat tartalmaz, hanem az említett külföldi jelesebb művek nyomán betekintést nyujt a számok csudás világába:

This booklet includes not only such things, but also something beyond the elementary problems. It includes not only popular puzzling arithmetic problems including riddles which can be solved usually with linear equations, but also the more remarkable works which were printed in foreign countries as mentioned above, and which lead us to the wonderful world of the numbers.

adataival kissé közelebb hozza a nagy számokat felfogásunk határaihoz, bemutatja nehány szám és számsorozat meglepő szabályosságait, újabb, nem egészen közismeretű [p. 4] módokat közöl a számok kitalálásának annyira elterjedt szórakoztató játékához, megismerteti a mathematika egyik ősrégi, nem csak játékszerű, hanem tudományos szempontból is érdekes problémáját:

Using his data, he brings the big numbers somewhat nearer to the borders of our understanding. He introduces the surprising regularities of some numbers and number series. He shows methods for guessing numbers as an amusing game very widely played. Some methods are new and not totally common knowledge. He tells problems of ancient mathematics which are not only toylike, but also interesting from a scientific viewpoint.

a képeslapokban és ifjúsági iratokban is sokszor szereplő bűvös négyzeteket, közöl néhány mathematikai játékot, melyek között egyesek, mint pl. a 15 török és 15 keresztény, a farkas, a kecske és a káposzta átszállítása, Josephus és a barátjának megmenekülése stb. a mathematikai irodalom legrégibb problémái közé tartoznak és még ma is bizonynyal sokaknak igen kedves fejtörő gyakorlatúl fognak szolgálni.

He shows magic squares appearing often also in the postcards and in the publications for young people. He shows some mathematical games, for example: 15 Turks and 15 Christians; the transport of wolf, goat and cabbage with a ferry; rescue of Josephus and his friend etc. These games belong to the oldest problems in the mathematical documents. And even today, very nice puzzling practices will be certainly useful for many people.

Még a mathematikai iskolai tananyag szempontjából is fontosak azok a fejtegetések, a melyek mathematikai hamisságok címen egyes, még tanulók között is igen elterjedt hibás bizonyításokra vonatkoznak, megjelölve mindenütt a bizonyítás hibáját.

Nem kevésbbé fontosak a szorosan vett iskolai anyag kibővítése szempontjából azok a feladatok, melyek a síkidomok szétszedését és összerakását tárgyalják, s a melyek a mellett, hogy kedves játékul szolgálhatnak, egyuttal az egyenlő területű idomok összefüggésébe is mélyebb betekintést nyújtanak.

A művecske tartalmának eme rövid jellemzése után fölvethető az a kérdés, hogy szükséges-e és hasznose ilyen könyvecske?

Véleményünk szerint szükséges, hogy a szorosan vett iskolai mathematikai tanításanyagon kívül is foglalkozzék a mathematika iránt némi érdeklődést tanúsító olvasó ilyen számtani és geometriai vonatkozású kérdésekkel.

De nem is mesterségesen teremtett szükségletről van szó.

The following discussions are important even to the mathematical school curriculum: the discussions in the chapter entitled mathematical errors concerning wrong proofs very widely found even among students. In the discussions, all the mistakes of proof are pointed out.

The problems on the decomposition and the composition of plane figures are no less important for increasing teaching materials required in the school curriculum. They can be used as a nice game; besides, they provide also deeper insight into the relation between equal plane figures.

After this short characterisation of the content of the opuscule, it is possible to raise the following question: is such a booklet necessary and useful?

In our opinion, besides the teaching materials of mathematics required in the school curriculum, it is also necessary to use reading materials containing problems concerning arithmetic and geometry that draw students' interests in mathematics.

This is not a baseless opinion.

Az ilyen irányú problémák iránti érdeklődés a művelt körökben, sőt, miként a számtani vonatkozású népies találós mesék nagy száma és különösen a magyar találós számtani feladatok [p. 5] leleményessége mutatja, még a népben is megvan.

Indeed, educators are interested in the following problems: how to show a great many popular riddles concerning arithmetic; and especially how to show the inventiveness of the Hungarian puzzling arithmetic problems found even among the common people.

Hiszen alig akadunk olyan társaságra, a hol ilyen mathematikai rejtvények ne szerepelnének.

In fact, we hardly find any society where such mathematical puzzles would not appear.

Egyes szellemes találós kérdések bámulatos gyorsasággal járják be az egész országot, sőt az egész művelt világot.

Each witty riddle goes around all over the country with surprizing velocity, especially among educated people.

Középiskolai működésem alatt többször iparkodtam a mulatságos kérdések gyűjteményét a tanulók gyűjtése révén növelni és mindannyiszor meggyőződtem arról, hogy az osztály nagy része igen élénken foglalkozik ilyen mathematikai mulatságos kérdésekkel.

In my high-school activity, I strove repeatedly to increase the collection of the recreative questions from students, and every time I am sure that the most part of the class very actively deals with such mathematical recreative questions.

Megvan tehát a természetes érdeklődés ilyen problémák iránt és ennek kielégítését czélozó művecskét szükségesnek kell mondanunk.

People are therefore naturally interested in such problems, and we need a booklet to satisfy their interest.

A második kérdésre még könnyebben válaszolhatunk.

We can answer the second question [“is such a booklet useful?”] even more easily.

Hiszen minden természetes szellemi szükséglet kielégítésére szolgáló könyvet hasznosnak kell tekintenünk.

In fact, we should notice that a book used for satisfying any of the natural intellectual needs is useful.

De mi az ilyen könyvtől, mely a középiskolai tananyag körén kívül eső, azt sok tekintetben kiegészítő, érdekes kérdésekkel foglalkozik, melyek között sok mély jelentőségű, még tudományos szempontból is nevezetes valamely mathematikai lángész éles előrelátásából eredő szabályosságok foglaltatnak:

But such a book deals with interesting questions. These questions complement from many points of view something falling out of high-school curriculum. Some of these questions have very deep significance. These questions include regularities remarkable even from a scientific viewpoint, which originate in the sharp foresight of certain mathematical geniuses.

azt is várhatjuk, hogy talán akad olyan olvasó, a kit ellenállhatlan erővel vonz a számok varázsa a mathematika bűvös világába.

we can also expect that some of readers will possibly be attracted by the magic of the numbers into the magical world of the mathematics with irresistible strength.

Ezen érdekes tüneményeknek a mathematika történetének tanúsága szerint már eddig is megvolt az üdvös hatásuk a mathematikai tudományok fejlődésére.

According to the evidence of the history of the mathematics, the significant effect of these interesting phenomena was already found on the development of mathematical sciences.

Reméljük, hogy e könyvecske ha csak parányi mértékben is, hozzájárul ahhoz, hogy a mathematikai tudományokkal nálunk többen és szívesebben foglalkozzanak; már pedig: „artem geometriae discere atque exercere publice interest“ (Cod. Justin. IX. k. 18. ez. 2.)*

We hope that, even if only in a quite small measure, this booklet contributes toward letting readers deal with mathematical sciences more largely and more favourably to us; already though: “artem geometriae discere atque exercere publice interest (To learn and apply the science of geometry is to the public interest)” (Cod. Justin. IX. title 18. section 2.)*

Dr. Beke Manó.

*Note of the translator:
It is a citation from *Codex Justinianus (Code of Justinian)*. The sentence is followed by: "ars autem mathematica damnabilis interdicta est omnino (However, the damnable magician's art is forbidden)."
The word "mathematica" here means astrology, which belongs rather to magic than to science.

4.1 Nagy számok: Large numbers

[p. 7]

Az az írásmód, melylyel a legnagyobb számokat is igen röviden leírhatjuk,

The writing method, with which we can write down the largest numbers very briefly,

hozzászoktatta a mathematikával nem foglalkozó közönséget is a nagy számokhoz.

accustomed also public readers unfamiliar with mathematics to the large numbers.

De minthogy ez az írásmód nem tesz nagy különbséget a számok közt

However, this writing style does not make any large difference between the numbers

(hiszen egy 0 hozzáírásával megtízszerezhetjük,

(since we can write down ten times a number by adding a digit 0,

kettővel megszázszorozhatjuk a számot),

hundred times the number by adding two 0 s),

azért, bár könnyen leírhatjuk őket, bármily nagyok is,

therefore, we can easily write them down including any large numbers, but

megkülönböztetni és elképzelni őket nem tudjuk.

we don't know how to distinguish and imagine them.

A laikus talán már a milliót és billiót sem különbözteti meg képzeletében.

Perhaps people who are not expert in mathematics don't distinguish even one million and one billion in their imagination*.

*Note of the translator:
Kőnig uses long scale: million is 10^6, and billion is 10^{12} here.
The short scale is used by many English-speaking people, and the long scale is used mainly by Europeans. In the United Kingdom, people used the long scale traditionally, but they have been using the short scale officially since 1974, when Harold Wilson, the prime minister at that time, announced to the House of Commons that the meaning of "billion" in papers concerning Government statistics would thenceforth be 10^9, in conformity with the United States usage.

Pedig, hogy mekkora e számok közt a különbség, azt eléggé mutatja,	Now, the difference between million and billion can be sufficiently shown as follows:
hogy egy millió másodpercz eltelésé-hez 12 napra sincs szükség,	one million seconds is shorter than 12 days,
míg egy billió másodpercz 33 333 *év* alatt telik el.	while one billion seconds correspond 33 333 years*.

*Note of the translator:
33333 years $\approx 1,05$ billion seconds. More precisely, one billion seconds $\approx$ 31710 years.

Azt sem igen hinné el az ember, hogy Krisztus születése óta tavaly múlt el az első ezermillió (milliárd) percz.	Maybe no one believes that one thousand-million (milliard) minutes have passed since the first Christmas.*

*Note of the translator:
According to Gallai Tibor [224], this book was published in 1902.
999 691 200 minutes = 1902 years.
Kőnig uses the long scale, because
thousand millions = one milliard in the long scale,
while
thousand millions = one billion in the short scale.

A nagy számokban való tévedéseknek oka az, hogy az ipar, kereskedelem, stb. nagyon ritkán használ 8-jegyűnél nagyobb számokat és csak a mathematikai, physikai és különösen az astronómiai tudomány szorul ennél nagyobb számokra.

The reason for the unfamiliarity with the large numbers is that the industry, the commerce etc. very rarely use numbers larger than 8 digits, while mathematics, physics and especially astronomical science need numbers larger than 8 digits.

[p. 8] De hogy a mindennapi életben is gyakran juthatunk elképzelhetetlen nagy számokhoz, azt igazolni fogják a következő példák.

But the following examples will justify that we can often encounter unimaginable large numbers also in our daily life.

32 kártya 3 személy között 2 753 294 408 204 640-félekép osztható szét, úgy t. i.,

The number of ways to distribute 32 cards to 3 persons is 2 753 294 408 204 640, i.e.*,

*Note of the translator:
$\binom{32}{10}\binom{22}{10}\binom{12}{10} = \frac{32!}{10!\,22!}\frac{22!}{10!\,12!}\frac{12!}{10!\,2!}$

hogy a 3 személy 10–10 kártyát kap,

each of 3 persons receives 10 cards,

kettőt pedig külön teszünk.

and 2 cards are put aside.

Az 52 lapos (whist-) kártya 4 személy közt már

The number of ways to distribute 52 whist cards to 4 persons is

53 644 737 765 488 792 839 237 440 000*

félekép osztható szét.

*Note of the translator:
$\binom{52}{13}\binom{39}{13}\binom{26}{13} = \frac{52!}{13!\,39!}\frac{39!}{13!\,26!}\frac{26!}{13!\,13!}$

Némiképp meggyőződhetünk e szám nagyságáról,

We can somehow understand the largeness of this number

ha megjegyezzük,

if we remark the following:

hogy ha a föld egész felületén sűrűn egymás mellett egy négyzetméter felületű asztaloknál whistet játszanának,

5 perczenkint egy játékot,

akkor is több mint ezermillió évre volna szükség, hogy az összes lehető módon szétoszthassák az 52 kártyát.

if whist would be played at tables with a surface of a square metre put side by side densely on the surface of the whole earth,

and if each game takes 5 minutes,

then they will need more than one thousand million years to realize all ways to distribute 52 cards*.

*Note of the translator:
If the surface of the earth is $5.1 \times 10^{14}\,\mathrm{m}^2$, it requires

$$\frac{53\ 644\ 737\ 765\ 488\ 792\ 839\ 237\ 440\ 000}{5.1 \times 10^{14}} \times 5\,\text{minutes} \approx 1\ 000\ 625\ 574\,\text{years}.$$

Más ismeretes példa a következő:

Shehram, a sakkjáték állítólagos feltalálója jutalmúl annyi búzaszemet kívánt királyától,

a mennyi összekerül a sakktáblán,

ha ennek első mezejére 1 szemet tesz és minden következőre kétszer annyit, mint az előbbire.

We have another example as follows:

Shehram, the alleged inventor of chess, was rewarded by the king with wheat grains as much as he wanted,

and the amount was fixed with the chessboard:

at the first time, one grain was put in a square on the chessboard, and in the next square, 2 times as much as the previous square, and the same way is continued.

*Note of the translator:
A chessboard consists of 8×8 squares, and one should fill all of them in multiplying the number of grains in this way.

Az így összekerülő magok száma $(2^{64} - 1)$ nagyobb 18 trilliónál és ennyi maggal a föld egész felületét csaknem 1 cm magas réteggel lehetne bevonni.

Then the number of grains $(2^{64} - 1)$ will be more than 18 trillion grains, and this amount of grains could cover the whole earth with about 1 cm of thickness*.

*Note of the translator:
If the earth surface is $5.1 \times 10^{18}\,\mathrm{cm}^2$, the number of grains per cm^2 is

$$\frac{2^{64} - 1}{5.1 \times 10^{18}} \approx 3.6\,(\mathrm{grains/cm}^2).$$

Only 3.6 grains can occupy 1 cm^3, or it is possible that KŐNIG used a somewhat smaller number for the area of the earth surface.

Ugyancsak egy geometriai haladvány összegezésére vezet a következő feladat.

Similarly, the following problem leads to the sum of a geometric series.

Reggel 9 órakor gyilkosságot követnek el három szemtanú előtt.

A murder happened at 9 o'clock in the morning, and three persons witnessed it.

Mind a három tanú a következő negyedórában 3-3 embert tudósít a gyilkosságról.

Each of the three persons informs 3 persons of the murder in a quarter hour.

Ez a 9 a következő negyedórában megint hármat-hármat, stb.

Each of the 9 persons informs again three persons of the murder in a quarter hour, and so on.

A kérdés most már az,

Now the question is:

hogy ha ily módon a föld összes lakói tudomást szerezhetnének a gyilkosságról,

If all inhabitants on the earth can hear the news of the murder in this way,

mennyi idő múlva történnék ez meg?

how much time will it take?

— Feleletűl azt a meglepő eredményt nyerjük,

— We get a surprising result as the answer:

hogy [p. 9] már d. u. fél 2 órakor az összes földi lakókhoz eljutott a gyilkosság híre.	At 1:30 p.m., all the inhabitants on the earth are already informed of the news of the murder.
18 negyedóra alatt ugyanis	That is to say, in 18 quarter hours,

$$3 + 3^2 + 3^3 + ... + 3^{19}$$

ember értesül a gyilkosságról.	persons are informed of the murder.
Ez az összeg pedig	this amount is calculated as

$$3\,\frac{3^{19}-1}{2} = 1\ 743\ 392\ 199,$$

már nagyobb a földön élő emberek számánál.	it is already more than the number of living people on the earth*.

*Note of the translator:
According to the recent research [83], the world population in the year 1900 is estimated as 1 633 848 213, though it may be somewhat different from the data used by KŐNIG.

Igen nagy számra vezet egy most élő személy (A) ősei számának meghatározása.	A calculation of the number of ancestors of a living person (A) leads to a very large number.
Mindenkinek van két szüleje,	everyone has two parents,
négy nagyszüleje,	four grandparents,
nyolc dédszüleje, ...,	eight great-grandparents, ...,
2^n n-ed rangú őse.	2^n of n-th ancestors.
Tegyük fel, hogy egy századra három emberöltő jut és hogy így	Supposing that one century contains three generations,
(A)-nak 100 év előtt 8,	(A) has 8 ancestors 100 years ago,

200 év előtt 64,	64 ancestors 200 years ago,
... 1900 év előtt (időszámításunk kezdetén) 8^{19} őse élt.	... 8^{19} ancestors were living 1900 years ago (at the beginning of our calendar).
Ez körülbelül 144 000 billió ember;	This is 144 000 billion persons*;

*Note of the translator:
$144115188075855872 \approx 144000 \times 10^{12}$

hogy ennyi ember elférhessen a földön,	to find room on the earth for so many people,
minden négyzet-deciméterre 2–3 embernek kellene jutni.	2–3 persons should be fit in one square-decimetre.

*Note of the translator:
If the earth surface is $5.1 \times 10^{16}\,\mathrm{dm}^2$,

$$\frac{1.44 \times 10^{17}}{5.1 \times 10^{16}} \approx 2.82\,(\mathrm{people/dm}^2).$$

Eredményünkben tehát valami hibának kell lenni, s ezt megtaláljuk, ha a rokonok közti házasságra gondolunk.	Accordingly, our result should have some mistake. If we consider the marriage between relatives, We will find what is the mistake.
Unokatestvérek házasságából származó gyermekeknek pld. csak 6 dédszülejük lehet stb.	For example, a child from a couple of cousins have only 6 great-grandparents, and so on*.

*Note of the translator:
If the parents of a child are cousins, a parent of the father and a parent of the mother are born from common parents. It means that only one pair of great-grand parents should be counted for 2 grand parents.

Az ősök nagy számából mindenesetre azt következtethetjük,	Then we can somehow conclude depending on the large number of ancestors that

hogy egy ember ősei közt sok rokonnak kellett házasságot kötni.

many marriage relations should exist among the ancestors of one person.

A kamatoskamat-számításnál is meglepő nagy számokhoz juthatunk.

We can reach surprisingly large numbers also from *the calculation of compound interests.*

Így kiszámították,

According to the calculation,

hogy egy Jézus korában 4%-ra elhelyezett fillér 1875. év végére 800 000 *quadrillio* koronára növekedett volna.

if one invests a little amount of money at the first year of our calendar under the condition of 4% of compound interest, it would grow to a fortune of 800 000 *quadrillion* times of money at the end of the year 1875*.

*Note of the translator:
According to the long scale, 1 quadrillion = 10^{24}, but

$$1.04^{1875} = 865986626476236508270156786660 24 \approx 865987 \times 10^{26}$$

This difference of digits may come from a simple mistake, or it is possible that the scale used by KŐNIG was different from ours. The year 1875 is chosen maybe for simplifying the calculation:

$$1.04^{1875} = 1.04^{3\times 5^4} = 1.04^{3^{5^{5^5}}}$$

Végül megjegyezzük még,

Finally we remark that

hogy a mathematikai jelölések mily kényelmesek ily nagy számokkal való számolásra.

the mathematical notations are convenient for counting such large numbers.

A csakis három jegygyel leírt

For example, the notation with only three numbers

$$9^{9^9}$$

[p. 10] szám pld. sokszorta nagyobb a mindeddig említett nagy számoknál.	is many times as large as the numbers mentioned above.
Mert	Since

$$9^9 = 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9$$

is már mintegy 387 millió, és 9-et most még ennyiszer kell önmagával megszorozni,	is already about 387 millions, and then 9 should be multiplied by itself about 387 million times,
hogy az említett óriási számot nyerjük.	so we get an enormous number as mentioned above.
Az így keletkező szám jegyeinek száma 369 millió és leírásához több,	To write down the calculated number, we need to write down more than 369 million digits,
mint 18 ezer kilométerre volna szükség, ha egy deciméterre húsz számjegy férne is el;	and 18 thousand kilometres will be necessary for it if twenty digits can be fit in one decimetre*;

*Note of the translator:
The number 9^{9^9} has $\log 9^{9^9} = 369693099,631570359$ digits, and it requires

$$\frac{369,693099631570359 \times 10^6}{20} \approx 18 \times 10^6 \,\text{decimetres} = 1800 \,\text{kilometres}.$$

There seems to be a lack of a comma: it should be $1,8$ thousand kilometres.

pontos meghatározásához pedig egy emberélet sem volna elegendő.	however, such a number cannot be written down during a human lifetime.

4.2 Érdekes számok és eredmények: Interesting numbers and results

[p. 11]
A mathematikában gyakran fordulnak elő olyan érdekes jelenségek, melyek a laikusnak valóban csodásnak tűnhetnek fel, míg a mathematikus, ki ezeket meg tudja magyarázni, természetesnek találja őket.

There are interesting phenomena in mathematics: people who are not expert in mathematics may find such phenomena really wonderful, while the mathematician who can explain them finds them natural.

1. Meglepő tulajdonsága van pld. az 142 857 számnak.

1. For example the number 142 857 has a surprising property.

Szorozzuk meg 2-vel, 3-mal, 4-gyel, 5-tel, 6-tal ezt a számot, akkor szorzatúl a

We multiply this number by 2, 3, 4, 5, 6, then we get the product numbers

$$285714, \quad 428571, \quad 571428, \quad 814285, \quad 857142$$

számokat nyerjük.

Ezek a számok nemcsak ugyanazokat a számjegyeket tartalmazzák, mint az eredeti szám, hanem e jegyek még ugyanabban a sorrendben is következnek egymásután, ha t. i. az első számjegyet (1) az utolsó (7) után következőnek tekintjük.

These numbers not only involve the same digits as the original number, but also these digits come in the same order in the next turn; more precisely, we regard the digits from the first (1) to the last (7) as a series*.

*Note of the translator:
Such a number is called a cyclic number, as well as the number 0588235294117647 mentioned in the next paragraph.

Ha 7-szeresét veszszük a 142 857 számnak, akkor csupa 9-esből álló számot nyerünk:

If we consider 7 times the number 142 857, then we get a number which consists of only the digit 9:

$$142857 \times 7 = 999999$$

Ha 7-nél nagyobb számmal szorozzuk meg a 142 857 számot, akkor 7-, vagy több jegyű számot [p. 12] nyerünk.

If we multiply the number 142 857 by a number bigger than 7, then we get a number in 7 or more digits.

Vágjuk el ebből az utolsó hat jegyet és adjuk össze az így keletkező két csoportot, mindkettőt egy számnak olvasva, akkor visszanyerjük az eredeti számjegyeket s az eredeti sorrendet.

We cut off the last six digits of this number, and, reading each of the two parts as a separate number, we add together the two numbers obtained in this way, then we get the original digits and the original order again*.

*Note of the translator:
Except the case of multiplying 142 857 by one of the multiples of 7 which will be shown later.

Így például

For example as follows:

$$142857 \times 24 = 3428568$$

és

and

$$3 + 428568 = 428571$$

Ha 7 valamely többszörösével szorozzuk meg a 142 857 számot, akkor a most említett módon nem az eredeti számjegyeket nyerjük, hanem ismét a hat 9-esből álló számot, pld. :

If we multiply the number 142 857 by one of the multiples of 7, then we do not get the original digits in the way mentioned here, but again the number which consists of six 9s, for example:

$$142857 \times 42 = 5999994$$

és

and

$$5 + 9.999,994 = 999.999$$

Nem akarjuk az olvasót ezen jelenség magyarázásával untatni, csak annyit jegyzünk meg, hogy ha 1/7-et tizedes törtté alakítjuk, akkor a keletkező szakaszos tizedes tört szakasza épen az 142,857 szám lesz.

We do not want to bore the reader by explaining this phenomenon. We notice only the following: if we turn 1/7 into a decimal fraction, then the repeating block of the recurring decimal fraction generated will be exactly the number 142,857*.

*Note of the translator:

$$
\begin{aligned}
1/7 &= 0.\overline{142857} \\
2/7 &= 0.\overline{285714} \\
3/7 &= 0.\overline{428571} \\
4/7 &= 0.\overline{571428} \\
5/7 &= 0.\overline{714285} \\
6/7 &= 0.\overline{857142} \\
7/7 &= 1 = 0.\overline{9} \\
8/7 &= 1.\overline{142857} \\
9/7 &= 1.\overline{285714} \\
&\ldots
\end{aligned}
$$

where a series of digits with an overline means a recurring decimal. When $0.\overline{142857}$ is multiplied by one of the multiples of 7, it will be an integer, which can be represented by a decimal with 9 repeating.
When $0.\overline{142857}$ is multiplied by a number larger than 7, the digit over the 6 repeating digits will be added to the next larger block of 6 repeating digits. For example 24/7 can be calculated as follows:

$$
\begin{aligned}
0.\overline{142857} \times 24 = {} & 3.428568 \\
& + 0.000003428568 \\
& + 0.000000000003428568 \\
& + \ldots \\
& = 3.\overline{428571}
\end{aligned}
$$

These additions correspond to the calculation $3 + 428568$ above mentioned by Kőnig.
Suppose that a prime number p is in a numeral system in base b (p does not divide b); when the number $\frac{b^{p-1}-1}{p}$ gives a repeating block which is a cyclic number, this p is called a long prime or a full reptend prime in base b. The first few full reptend primes in base 10 (decimal) are:

$$p = 7, 17, 19, 23, 29, 47, 59, 61, 97, 109, \ldots$$

2. Ép így az 1/17-del egyenlő szakaszos tizedes tört periódusa, a 16-jegyű

2. Just like above, the repeating block of recurring decimal fraction equal to 1/17, that is, the 16-digits number

0588235294117647

szám, ugyanoly tulajdonságú, mint a 142 857-es szám. 4-szerese például:	has the same kind of property as the number 142 857. For example, 4 times the number is:

2352941176470588

44-szerese pedig	And 44 times the number is

25882352941176468;

a két csoport összege:	the sum of two parts (the last 16 digits and the rest) is:

$$2 + 5882352941176468 = 5882352941176470$$

megint az eredeti számjegyekből áll.	which consists of the original digits again.
[p. 13] 3. Érdekes tulajdonsága van a 37-es számnak is; szorozzuk meg vele a	3. The number 37 also has an interesting property; we multiply it by any of the numbers

3, 6, 9, 12, 15, 18, 21, 24, 27

számtani sor kilencz tagjának bármelyikét; akkor mindig három egyenlő jegyből álló számot kapunk, pld.	which are nine terms of an arithmetic progression; then we get always a number which consists of three identical digits, for example

$$37 \times 18 = 666.$$

Hogy ezt a jelenséget megmagyarázzuk, elég megjegyezni, hogy

In order to explain this phenomenon, the following remark is enough:

$$3 \times 37 = 111$$

és, hogy így

and

$$6 \times 37 = 2 \times 111 = 222,$$
$$9 \times 37 = 3 \times 111 = 333,$$

stb. stb.

etc. etc.

4. Épígy ha az

4. Just like above, if we multiply any of the nine terms of the arithmetic progression

$$15873, \quad 31746, \quad 47619, \quad 63492, \quad 79365, \quad 95238, \quad 111111, \quad 126984, \quad 142857$$

számtani sor kilencz tagjának bármelyikét 7-tel megszorozzuk, hat egyenlő jegygyel leírt számot nyerünk.

by 7, we get a number written with six identical digits.

Ez pedig az

And this comes from the equality

$$15873 \times 7 = 111111$$

egyenlőségből tűnik ki.

5. Érdekes tulajdonságuk van az u. n. *tökéletes szám*-oknak is.

Igy hívják az oly számot, mely osztói összegével egyenlő.

(Megjegyzendő, hogy az egység az osztók közé számítandó, míg maga a szám nem.)

A legkisebb tökéletes szám*): 6, osztói: 1, 2, 3 és valóban $1 + 2 + 3 = 6$.

*) 1-et nem szokták a tökéletes számok közé számítani.

5. The numbers called *perfect numbers* also have an interesting property.

If the sum of divisors of a number is equal to the number itself, this number is called a perfect number.

(Note that only the proper divisors are counted, the number itself is not counted.)

The smallest perfect number*): 6, divisors: 1, 2, 3 and in fact 1+2+3 = 6.

*) 1 is not counted in the perfect numbers.

Páratlan tökéletes [p. 14] számot nem ismernek, a párosok pedig mind meg vannak adva az

No odd perfect number is known, but all the even perfect numbers are recognized with the formula

$$N = 2^{\alpha-1}(2^\alpha - 1)$$

képlettel, hol α helyébe csak oly értékek teendők, melyekre nézve $2^\alpha - 1$ törzsszám (α maga tehát szintén törzsszám).

where we have to give to α only values for which $2^\alpha - 1$ is a prime number (so α itself is also a prime number)*.

*Note of the translator:
If $2^\alpha - 1$ is a prime number, α is a prime number. It is proven as follows:
if α is not a prime number,

$$\begin{aligned} 2^\alpha - 1 &= 2^{pq} - 1 \\ &= (2^p)^q - 1 \\ &= (2^p - 1)\left((2^p)^{(q-1)} + (2^p)^{(q-2)} + ... + 2^p + 1\right) \end{aligned}$$

It is a product of integers, and contradicts with the condition that $2^\alpha - 1$ is a prime number. ■
Marin Mersenne (1588–1648) described some prime numbers $2^n - 1$ when he treated *numeri perfecti* (perfect numbers) in the chapter of *Præfatio generalis* (general preface) of his book *Cogitata Physico-Mathematica* in 1644 [158]. His description includes some errors, but the prime numbers represented with $2^n - 1$ are named Mersenne primes. Finding larger Mersenne primes is a problem examined by many scientists even now.

Ha	If

$$\alpha = 2, \quad 3, \quad 5, \quad 7, \quad 13, \quad 17, \quad 19, \quad 31,$$

akkor rendre nyerjük az első nyolcz tökéletes számot:	then we get successively the first eight perfect numbers:

$$\begin{aligned} N = 6, \quad 28, \quad 496, \quad 8,128, \quad 33,550,336, \\ 8,589,869,056, \quad 137,438,691,328, \\ 2,305,843,008,139,952,128. \end{aligned}$$

6. Hasonló tulajdonsága van a *barátságos számpárnak*.	6. The *friendly number pair* has similar properties.
Így neveznek két oly számot, melyek mindegyike a másik osztóinak összegével egyenlő.	If each of two numbers is equal to the sum of the divisors of the other, this pair of numbers is called a friendly number pair.

Igy 220 és 284 barátságos számpár, mert 220 osztóinak összege

220 and 284 are such a friendly number pair, because the sum of divisors of 220 is

$$1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 = 284,$$

284 osztóinak összege pedig

and the sum of divisors of 284 is

$$1 + 2 + 4 + 71 + 142 = 220.$$

Barátságos számpár pld. a következő kettő is:

For example, the following pairs are also friendly numbers:

$$\begin{array}{lll} 10,744 & \text{és (and)} & 10,856, \\ 63,020 & \text{és (and)} & 76,084. \end{array}$$

7. Érdekes eredményre vezetnek, mint itt látható, a következő műveletek is.

7. The following operations, as shown here, also lead us to an interesting result.

[p. 15]

$$\begin{aligned} 1^2 &= 1 \\ 11^2 &= 121 \\ 111^2 &= 12321 \\ 1111^2 &= 1234321 \\ 11111^2 &= 123454321 \\ 111111^2 &= 12345654321 \\ 1111111^2 &= 1234567654321 \\ 11111111^2 &= 123456787654321 \\ 111111111^2 &= 12345678987654321 \end{aligned}$$

$$
\begin{aligned}
11 &= 11 \\
11^2 &= 121 \\
11^3 &= 1331 \\
11^4 &= 14641 \\
&\cdots\cdots
\end{aligned}
$$

$$
\begin{aligned}
1 \times 9 + 2 &= 11 \\
12 \times 9 + 3 &= 111 \\
123 \times 9 + 4 &= 1111 \\
1234 \times 9 + 5 &= 11111 \\
12345 \times 9 + 6 &= 111111 \\
123456 \times 9 + 7 &= 1111111 \\
1234567 \times 9 + 8 &= 11111111 \\
12345678 \times 9 + 9 &= 111111111 \\
123456789 \times 9 + 9 &= 1111111111
\end{aligned}
$$

$$
\begin{aligned}
9 \times 9 + 7 &= 88 \\
98 \times 9 + 6 &= 888 \\
987 \times 9 + 5 &= 8888 \\
9876 \times 9 + 4 &= 88888 \\
98765 \times 9 + 3 &= 888888 \\
987654 \times 9 + 2 &= 8888888 \\
9876543 \times 9 + 1 &= 88888888 \\
98765432 \times 9 + 0 &= 888888888 \\
987654321 \times 9 - 1 &= 8888888888
\end{aligned}
$$

[p. 16]

$$
\begin{aligned}
8 \times 1 &+ 1 = 9 \\
8 \times 12 &+ 2 = 98 \\
8 \times 123 &+ 3 = 987 \\
8 \times 1234 &+ 4 = 9876 \\
8 \times 12345 &+ 5 = 98765 \\
8 \times 123456 &+ 6 = 987654 \\
8 \times 1234567 &+ 7 = 9876543 \\
8 \times 12345678 &+ 8 = 98765432 \\
8 \times 123456789 &+ 9 = 987654321
\end{aligned}
$$

$$
\begin{aligned}
2 \times 99999 &= 199998 \\
3 \times 99999 &= 299997 \\
4 \times 99999 &= 399996 \\
5 \times 99999 &= 499995 \\
6 \times 99999 &= 599994 \\
7 \times 99999 &= 699963 \\
8 \times 99999 &= 799992 \\
9 \times 99999 &= 899991 \\
10 \times 99999 &= 999990
\end{aligned}
$$

$$
\begin{aligned}
9 \times 6 &= 54 \\
99 \times 66 &= 6534 \\
999 \times 666 &= 665334 \\
9999 \times 6666 &= 66653334 \\
&\cdots\cdots\cdots\cdots
\end{aligned}
$$

$$
\begin{aligned}
4^2 &= 16\\
34^2 &= 1156\\
334^2 &= 111556\\
3334^2 &= 11115556\\
33334^2 &= 1111155556\\
&\cdots\cdots\cdots
\end{aligned}
$$

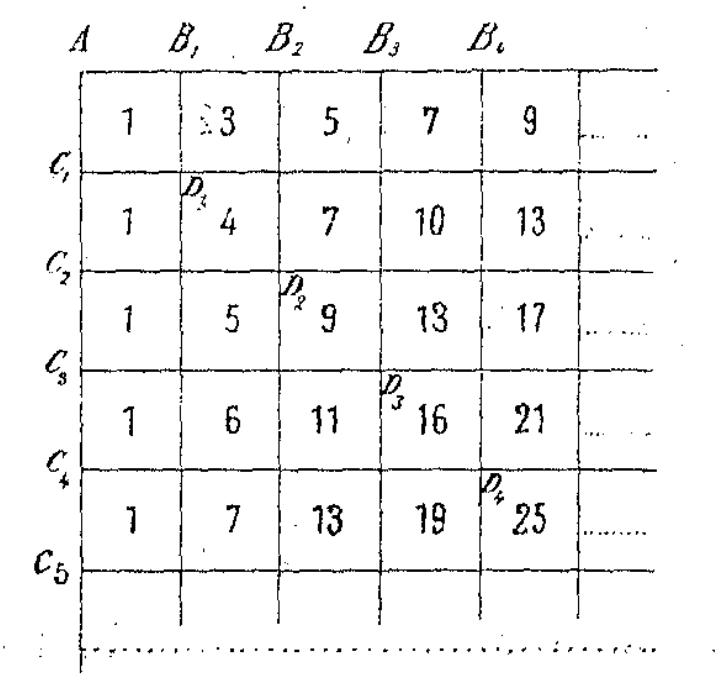
$$
\begin{aligned}
7^2 &= 49\\
67^2 &= 4489\\
667^2 &= 444889\\
6667^2 &= 44448889\\
66667^2 &= 4444488889\\
&\cdots\cdots\cdots
\end{aligned}
$$

[p. 17]
Mindezen műveletekben nagyon kitűnik az a szabályosság és törvényszerűség, melylyel a mathematikai számításokban mindig találkozunk.

We can see the regularity and the lawfulness in all these operations, as we always find them in mathematical calculations.

8. Írjuk egymás alá azon számtani sorokat, melyeknek első tagja: 1, különbségük pedig 2, 3, 4, ... ; így nyerjük a következő táblázatot:

8. Let's write the arithmetic progressions arranging them one under the other. The first term of them is: 1, and their differences are 2, 3, 4, ...; we get the following table in this way:



1. ábra. (Figure 1.)

Bármeddig folytatjuk is ezt a táblázatot, az $AB_1D_1C_1$, $AB_2D_2C_2, \ldots$ $AB_nD_nC_n$ négyzetben lévő számok összege teljes négyzet lesz.

A $B_1B_2D_2C_2C_1D_1, \ldots,$ $B_nB_{n+1}D_{n+1}C_{n+1}C_nD_n$ idomokban álló számok pedig összegül mindig teljes köböt adnak.

Például:

This table can be enlarged to any size. The sum of numbers in the square $AB_1D_1C_1$, $AB_2D_2C_2, \ldots$ $AB_nD_nC_n$ will be a perfect square.

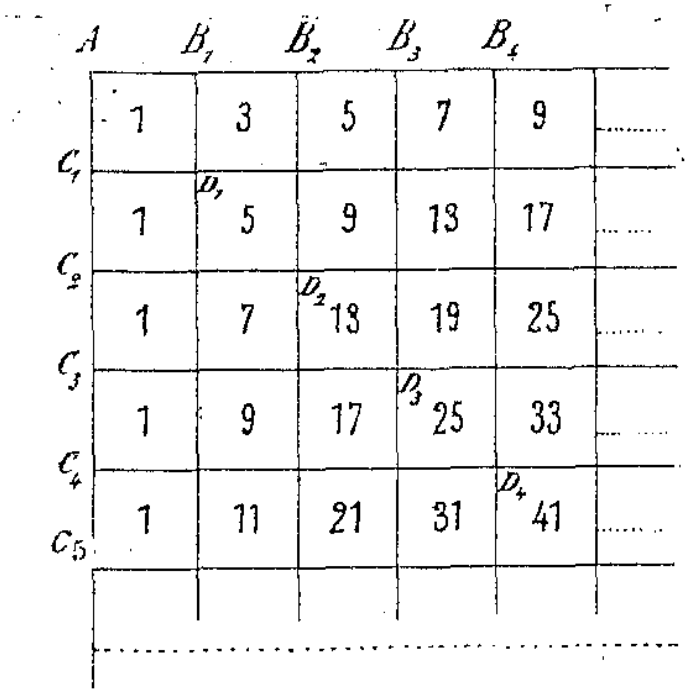
The sum of numbers in the forms $B_1B_2D_2C_2C_1D_1, \ldots,$ $B_nB_{n+1}D_{n+1}C_{n+1}C_nD_n$ gives always a perfect cube.

For example:

$$1 + 1 + 3 + 4 = 9 = 3^2 \quad \text{és (and)} \quad 3 + 4 + 1 = 8 = 2^3.$$

Hasonló tulajdonsága van a 2. ábrában látható táblázatnak, melyben az egyes sorokban álló számtani sorok első tagja ismét: 1, de a különbség rendre: 2, 4, 6, ...

The table shown in Figure 2 has similar properties, where the first term of the arithmetic progressions in the single lines is: 1, but the difference is successively: 2, 4, 6, ...



2. ábra. (Figure 2.)

Ha ugyanis a $B_1B_2D_2C_2C_1D_1, ...,$ $B_nB_{n+1}D_{n+1}C_{n+1}C_nD_n$ idomokban álló számokat összeadjuk, mindig két teljes köb összegét nyerjük, például:

More precisely, if we add together the numbers in the forms $B_1B_2D_2C_2C_1D_1, ...,$ $B_nB_{n+1}D_{n+1}C_{n+1}C_nD_n$, we get always the sum of two perfect cubes, for example:

$$5 + 9 + 13 + 7 + 1 = 35 = 27 + 8 = 3^3 + 2^3$$

[p. 18]
Az első ábrában lévő táblázatnak az említetten kívül még egy érdekes tulajdonsága van.

Ha ugyanis ebben a táblázatban bárhol egy oly tetszőleges nagyságú négyzetet jelölünk ki, (l. 3. ábrát), melynek egyik átlója összeesik az 1, 4, 9 ... (négyzet-) számok vonalával,

The table in Figure 1 has one more interesting property in addition to the property mentioned above.

More precisely, if we pick up a square of arbitrary size anywhere in this table (see Figure 3), so that one of the diagonals of the square corresponds to the line of numbers 1, 4, 9 ... (square numbers),

1	3	5	7	9	11	13	15	17	...
1	4	7	10	13	16	19	22	25	...
1	5	9	13	17	21	25	29	33	...
1	6	11	16	21	26	31	36	41	...
1	7	13	19	25	31	37	43	49	...
1	8	15	22	29	36	43	50	57	...
1	9	17	25	33	41	49	57	65	...
1	10	19	28	37	46	55	64	73	...

3. ábra. (Figure 3.)

akkor azon [p. 19] számok összege, melyeket ez a négyzet tartalmaz, mindig teljes négyzet lesz; például:

then the sum of numbers contained in this square will be always a perfect square; for example:

$$9 + 13 + 17 + 11 + 16 + 21 + 13 + 19 + 25 = 144 = 12^2$$

9. Végül még egy apróságot mutatunk, mely mathematikai szempontból nagyon jelentéktelen ugyan, de elég érdekes ahhoz, hogy e fejezetbe felvegyük.

9. Finally we show one more small topic which is in fact very insignificant from a mathematical point of view, but which is comparatively enough interesting to insert into this chapter.

A 9-es szám hatfélekép felírható két szám hányadosaként úgy, hogy mind a tíz számjegyet egyszer és csak egyszer használjuk fel:

We can write the number 9 as a quotient of two numbers in six ways using all of ten digits once and only once:

$$9 = \frac{97524}{10836} = \frac{95832}{10648} = \frac{95742}{10638} = \frac{75249}{08361} = \\ = \frac{58239}{06471} = \frac{57429}{06381} \quad ^*.$$

*Note of the translator:
An error is found in the original text as $\frac{95823}{10648}$, and fixed here by the translator.
For using all ten digits, some of the denominators have 0 in the high-order end.

Hasonlók 100-nak következő felbontásai:

Similarly, 100 also has the following representations:

$$100 = 91\frac{5,742}{638} = 91\frac{7,524}{836} = 91\frac{5,823}{647} = 94\frac{1,578}{263} = \\ = 96\frac{2,148}{537} = 96\frac{1,428}{357} = 96\frac{1,752}{438}^*,$$

*Note of the translator:
These representations mean the sums of an integer and a fraction, for example $91+\frac{5,742}{638}$.

csakhogy 0 ezekben nem szerepel.

however 0 does not appear in these representations*.

*Note of the translator:
In the representations of 100, only 9 digits are used. We can trivially put 0 in the high-order end of denominators and make them 10 digits.

Ha a törtvonáson kívül a + jelet is felhasználjuk, idevehetjük a következő felbontásokat is:

If we use the + sign in addition to the fraction line, we can get the following representations:

$$100 = 97 + \frac{5+3}{8} + \frac{6}{4} + \frac{1}{2} = 75 + 24 + \frac{9}{18} + \frac{3}{6}$$

stb. stb.

etc. etc.

4.3 Számok kitalálása: Guessing numbers

[p. 20]
Egy gondolt szám nagyon sokfélekép határozható meg egy oly szám segitségével, melyet az eredeti számból bizonyos sorrendben rajta elvégzett ismert műveletekkel nyerünk.

A number in the other's mind can be determined in a great many ways with the help of a number drawn from the original number, by a certain order of known operations.

Ha egy ily eredményt ismerünk, könnyen kitalálhatjuk a gondolt számot, mert erre egy egyenletet állíthatunk fel.

If we know such a result, we can guess the number in the other's mind easily, because we can establish an equation for the guessing.

Így, ha tudjuk, hogy a szám négyszereséhez 3-at adva 11-hez jutunk, akkor a gondolt x számra a

If we know that a number was multiplied by 4 and added to 3, and we obtained 11, then we get the following equation for the number in mind x:

$$4x + 3 = 11$$

egyenletet nyerjük és innen a gondolt szám:

and hence the number in mind is:

$$x = 2.$$

Egy ily egyenlet megoldásán alapszik a legtöbb módszer, melynek segítségével egy gondolt számot ki lehet találni.

Most of the methods are based on the solution of an equation. With the help of such an equation, we can find out the number in the other's mind.

1) Szólítsuk fel pld. A-t, ki a számot gondolta, hogy szorozza meg a gondolt számot 5-tel, adjon a szorzathoz 15-öt, az így nyert számot oszsza el 5-tel és mondja meg az eredményt (a).

1) For example, let A think of a number, multiply the number in mind with 5, and then add 15 to the multiplied result, divide the result by 5, and tell the [final] result (a).

Ha ezen eredményből 3-at levonunk, a képzelt számot nyerjük, mert

If we subtract 3 from this result, we get the number in A's mind, because

[p. 21]

$$a = \frac{5x + 15}{5} = x + 3$$

és innen valóban

and it is indeed

$$x = a - 3.$$

2) Másféleképen lehet a gondolt számot kitalálni, ha például az

2) It is possible to guess the number in the other's mind in a different way. Suppose, for example, let x undergo the operation in the expression

$$a = (x + 5)\,3 - 7$$

kifejezésben látható műveleteket végeztetjük x-szel;

ebben az esetben a gondolt szám:

In this case, the number in mind is

$$x = \frac{a - 8}{3}.$$

Nehezebbnek tűnik fel már a számok kitalálása, ha magát a gondolt számot többször belevonjuk a számításba:

It seems to be more difficult to guess the numbers if the number in the other's mind is used more than once in the calculation as following examples.

3) Szólítsuk fel pld. A-t, hogy szorozza meg 3-mal a gondolt számot, adjon a szorzathoz 2-t, felezze meg ezt az összeget és az így keletkező számból vonja le a gondolt szám felét.

3) For example, let A perform the following operations: multiply by 3 the number in mind, add 2 to the result, divide this sum into half, and, from the resulting number, deduct the half of the number in mind.

Ha A megmondja az így keletkező eredményt, akkor ebből egyszerűen 1-et levonunk s megkapjuk a gondolt számot, mert

If A tells the result got in this way, then we simply subtract 1 [from the result], and we will get the number in A's mind, because

$$a = \frac{3x + 2}{2} - \frac{x}{2} = x + 1; \quad x = a - 1.$$

4) Végeztessük el A-val a gondolt számon a következő műveleteket:

4) Let A do the next operations on the number in mind:

a) szorozza meg a-val a gondolt számot,

a) multiply the number in mind by a,

b) oszsza el a szorzatot b-vel,

c) szorozza a hányadost c-vel,

d) oszsza el a szorzatot d-vel,

e) és az így keletkező számot oszsza el a gondolt számmal.

[p. 22]
f) adja a hányadoshoz a gondolt számot és mondja meg az eredményt.

Ha ebből az eredményből $\frac{ac}{bd}$-t levonunk, megkapjuk a gondolt számot.

Ha ugyanis x-szel a fenti műveleteket végezzük, rendre a következő számokat nyerjük:

b) divide the product by b,

c) multiply the quotient by c,

d) divide the product by d,

e) and divide the resulting number by the number in mind.

f) add the number in mind to the quotient and tell the result.

If we subtract $\frac{ac}{bd}$ from this result, we get the number in A's mind.

Indeed, if the above operations are carried out with x, we get successively the following numbers:

$$ax, \frac{ax}{b}, \frac{acx}{b}, \frac{acx}{bd}, \frac{ac}{bd}, \frac{ac}{bd} + x;$$

ha ez utóbbi számból $\frac{ac}{bd}$-t levonunk, valóban x-et nyerjük.

Czélszerű az a, b, c, d számokat úgy választani, hogy $\frac{ac}{bd}$ egész szám legyen, s hogy így gyorsan levonhassuk a végső eredményből.

Ha pld.

if the last number is subtracted by $\frac{ac}{bd}$, we get actually x.

The numbers a, b, c, d that we give can be chosen so that $\frac{ac}{bd}$ is an integer, and thus it allows us to subtract the number quickly from the final result.

If, for example,

$$a = 12, b = 4, c = 7, d = 3,$$

akkor

then

$$\frac{ac}{bd} = \frac{12 \cdot 7}{4 \cdot 3} = 7.$$

Tegyük fel, hogy A pld. 5-öt gondolta, akkor a fenti műveletek eredményéül a

Suppose that A thought of 5, then the results of these operations are the following numbers:

$$60, 15, 105, 35, 7, 12$$

számokat nyeri és $12 - 7$ valóban 5-öt, a gondolt számot adja.

and $12 - 7$ is indeed 5, thus we got the number in A's mind.

Az eddig említett módszereknél A bármily számot gondolhatott, míg az itt következőben feltételezzük, hogy A *positiv egész* számot gondolt.

In the methods mentioned above, A can think of any number, while we assume in the following method that A thinks of a *positive integer*.

5) Mondjuk A-nak, hogy szorozza meg 3-mal a gondolt számot és mondja meg, hogy páros-e vagy páratlan az eredmény.

5) We ask A to multiply the number in mind by 3, and to tell us if the result is even or odd.

Ha páros, felezze meg, ha páratlan, akkor adjon előbb 1-et hozzá és csak azután vegye a felét;

If it is an even number, divide it by 2; if it is an odd number, add 1 to the number, and then divide it by 2;

szorozza meg [p. 23] az eredményt 3-mal és vonjon le a keletkező számból annyiszor kilenczet, a hányszor csak lehet és mondja meg, hogy hányszor lehetett.

multiply the result by 3, and subtract nine from the resulting number as many times as possible; and then tell us how many times the number nine can be subtracted.

Tegyük fel, hogy n-szer lehetett; a gondolt szám akkor $2n$ vagy $2n+1$, a szerint, hogy a gondolt szám páros, vagy páratlan volt.

Suppose that it can be subtracted n-times; the number in mind is then $2n$ or $2n + 1$, according to whether the number in mind was even or odd.

(*A* páros illetőleg páratlan számot gondolt, ha a szám 3-szorosa, az első művelet eredménye, páros vagy páratlan.)

(*A* thought of even number if the result of the first operation, 3 times the number, was even; *A* thought of odd number if the result of the first operation was odd).

Ha ugyanis a gondolt szám: $x = 2n$, páros, akkor a fenti műveletekkel a következő számokat nyerjük:

If since the number in mind is even, $x = 2n$, then we get the following numbers with the operations above:

$$6n, 3n, 9n, n;$$

az utolsó szám kétszerese $2n$, valóban a gondolt számot adja.

twice of the last number $2n$ gives indeed the number in *A*'s mind.

Ha pedig a gondolt szám páratlan: $x = 2n + 1$, akkor a műveletsorozat a

And if the number in mind is odd, $x = 2n + 1$, then the following sequence of numbers

$$6n + 3, 6n + 4, 3n + 2, 9n + 6, n$$

számokhoz vezet.

are led.

Az utolsó szám kétszereséhez 1-et adva nyerjük a gondolt $x = 2n + 1$ számot.

Add 1 to the twice of the last number, and we get the number in *A*'s mind $x = 2n + 1$.

Nagyon sok módszert lehetne még egyes számok kitalálására bemutatni, de, minthogy ilyenek az emlitettek mintájára nagyon könnyen készíthetők, áttérünk azon módszerekre, melyek segítségével *több (n)* gondolt számot találhatunk ki.

A great many methods could be introduced for guessing numbers, but we switch to the methods by means of which we can guess *more (n)* numbers in the other's mind, since such methods can very easily prepare the sample described above.

Általában n ismeretlen szám meghatározásához n egyenletre lévén szükség, a gondolt számokon elvégzett n műveletsorozat eredményét kell ismernünk.

Generally, n equations are necessary to define n numbers unknown to me. we have to know the results of n successive operations done on the numbers in the other's mind.

Ha azonban az n gondolt szám bizonyos feltételeknek van alávetve, akkor esetleg n-nél kevesebb adat, sőt egyetlen szám is meghatározhatja őket, mint azt majd a következőkben bemutatjuk.

However, if n numbers in the other's mind satisfy certain conditions, then the data can be less than n. Indeed the numbers can be defined with only one number as shown below.

Előbb azonban felemlítünk két oly módszert, melyekben a gondolt számok semmiféle feltételnek sincsenek alávetve.

First, however, we describe two methods in which the numbers in the other's mind are not subject to any conditions.

[p. 24]
6) Szólítsuk fel A-t, hogy gondoljon két számot és mondja meg összegüket és különbségüket.

6) Let A think of two numbers and tell us the sum and the difference.

A két eredménynek, melyet igy megtudunk, fél összege és fél különbsége adja a két gondolt számot.

From the two results, we learn a half of the sum of them and a half of the difference between them, we thus get two numbers in A's mind.

Ha ugyanis

Indeed, if

$$x + y = s \quad \text{és (and)} \quad x - y = d,$$

akkor összegezés és kivonás által nyerjük, hogy

then we get from addition and subtraction of them:

$$x = \frac{s+d}{2} \quad \text{és (and)} \quad y = \frac{s-d}{2}.$$

7) Ha A három számot gondolt (x, y, z), szólítsuk fel, hogy mondja meg az első és második, első és harmadik, végűl a második és harmadik szám összegét (s_1, s_2, s_3).

Ha az így nyert három eredmény fél összegéből 1) s_1-et, 2) s_2-t, 3) s_3-at levonunk, a három gondolt számot nyerjük, mert

7) Let A think of three numbers (x, y, z), and tell us the sum of the first and the second numbers, that of the first and the third numbers, finally that of the second and the third numbers (s_1, s_2, s_3).

If we take a half of the sum of three results 1) s_1, 2) s_2, 3) s_3, then we get three numbers in A's mind, because

$$\frac{1}{2}(s_1 + s_2 + s_3) = \frac{1}{2}(\{x + y\} + \{x + z\} + \{y + z\}) = x + y + z,$$

s ha e számból az $s_1 = x + y$, $s_2 = x + z$, illetőleg az $s_3 = y + z$ számot levonjuk, valóban a z, y, x, számokat nyerjük.

A következő két példában feltételezzük, hogy a gondolt két, illetőleg három szám 10-nél kisebb (positiv egész), tehát egyjegyű szám.

8) Felszólítjuk A-t, hogy jegyezzen meg magának két 10-nél kisebb számot*), szorozza meg 5-tel az egyiket, adjon 7-et a szorzathoz, szorozza [p. 25] meg 2-vel az eredményt, s az így keletkező számhoz adja hozzá a második gondolt számot.

*) A a két szám helyett pld. egy dominókövet s választhat, mely mindig két egyjegyű számot artalmaz.

and if each of the numbers $s_1 = x+y$, $s_2 = x + z$ and $s_3 = y + z$ is subtracted from this number, then we get indeed the numbers z, y, x.

In the following two examples, we suppose that each of two or three numbers is smaller than 10 (positive integer), that is, one-digit number.

Let A think of two numbers smaller than 10*), multiply one of them by 5, add 7 to the product, multiply the result by 2, and adds the second number in A's mind to the arising number.

*) A can substitute, for example, a domino piece for the two numbers and make a choice. A set of domino pieces consists of pairs of one-digit numbers.

Ha A megmondja a végeredményt, akkor ebből egyszerűen 14-et levonunk s a megmaradó kétjegyű szám két számjegye a két gondolt szám lesz.

Ha ugyanis A az x és y számokat gondolta, akkor a fenti műveletekkel rendre a következő számokat nyeri:

If A tells the final result, then simply subtract 14 from this, and two numbers in A's mind will be the two digits of the remaining two-digit number.

If A thought of the numbers x and y, then the following numbers are got successively with the above mentioned operations:

$$5x,\ 5x+7,\ 2\,(5x+7)=10x+14,\ 10x+y+14;$$

ha az utolsó számból 14-et levonunk, valóban oly kétjegyű számot nyerünk $(10x+y)$, melynek jegyei: x és y.

Három szám kitalálására még a következő módot említjük.

9. * Kérjük fel A-t, hogy szorozza meg 2-vel az első számot, adjon a szorzathoz 3-at, szorozza meg 5-tel az eredményt és adjon hozzá 7-et.

if we subtract 14 from the last number, we get indeed a two-digit number $(10x+y)$, the digits of which are x and y.

Let us discuss the following method for guessing three numbers.

9. * Let A multiply the first number by 2, add 3 to the product, multiply the result by 5, and add 7 to it.

*Note of the translator:
sic. The paragraph numbers after this paragraph are written without parenthesis ")" as well.

Adja hozzá ezen összeghez a második gondolt számot és adjon hozzá 3-at az uj összeg kétszereséhez.

Szorozza meg végül 5-tel az eredményt és adja hozzá a harmadik gondolt számot.

Let A add the second number in A's mind to this sum, and add 3 to the double of the new sum.

Finally, multiply the result by 5, and add to it the third number in A's mind.

Ezen műveleteket a három (x, y és z) számon elvégezve az

These operations on the three numbers (x, y and z) are written as follows:

$$5 \cdot [2 \cdot (5 \cdot \{2x + 3\} + 7 + y) + 3] + z$$

kifejezéshez jutunk, mely egyszerűsítve igy írható:

This expression can be simplified and written as follows:

$$100x + 10y + z + 235.$$

Ebből az utolsó kifejezésből kitűnik, hogy ha ezt az eredményt ismerjük, egyszerűen levonunk belőle 235-öt s ez által oly háromjegyű számot nyerünk melynek három számjegye a három gondolt számmal egyenlő.

This last expression means that, if we know the result of this, we subtract simply 235 from it, we thus get a three-digit number which consists of three digits equal to the three numbers in A's mind.

A számok egyik oszthatósági törvényén alapszik a következő módszer.

The following method is based on one of the divisibility laws of the numbers.

[p. 26]

10. A felír egy tetszőlegesen nagy (pos. egész) számot és egy másikat, mely ugyanazokat a jegyeket tartalmazza, mint az első, de más Sorrendben elhelyezve, A levonja a kisebb számot a nagyobból és a különbséget megszorozza egy tetszőleges (pos. egész) számmal.

10. A writes down an arbitrary large number (positive integer), and then another number consisting of the same digits as the first one, but placed in the different order. A subtracts the smaller number from the larger one, and multiplies the difference by an arbitrary number (positive integer).

Az így keletkező számból egy 0-tól különböző számjegyet kitöröl és megmondja az így maradó számot.

A deletes from the resulting number one non-0 digit, and tells us the remaining number.

Hogy a kitörölt számot megtudjuk, adjuk össze a számnak, melyet *A* velünk tudat, a számjegyeit.

To find out the deleted number, we add together all the digits of the number told by *A*.

A kitörölt szám most az az egyjegyű szám lesz, mely ezt az összeget 9 valamely többszörösére egészíti ki.

The deleted number will be such a one-digit number that complements this sum to one of the multiples of 9.

A felírja például a

A writes down, for example the number

7832864

számot és levonja belőle az ugyanezen számjegyeket tartalmazó

and subtracts from it the number consisting of the same digits

2883674

számot, akkor marad

then the following number remains:

4949190

melyet megszoroz pld. 37-tel s így nyeri a

A multiplies it, for example by 37, then gets the number

183120030

számot; kitörli pld. a 8-as számjegyet és tudatja velünk az

A deletes from this number, for example, the digit 8, informs us of the number

13120030

számot.

Ha ezen szám jegyeit összeadjuk, 10-et kapunk, az ezután következő legkisebb szám mely 9-czel osztható: 18 s erre a számra 10-et valóban a kitörölt 8-as szám egészíti ki*).

If we add together the digits of number, then we get 10. The smallest number larger than 10 divisible by 9 is 18. In this number, the deleted number 8 indeed complements 10*).

*) Hogy ezen eljárás helyességét kimutassuk, elég bebizonyítani, hogy az A által felirt két szám különbsége osztható 9-czel.

Ekkor ugyanis ennek minden többszöröse és minden többszörösben a jegyek összege szintén 9 többszöröse lesz.

*) Rather than making clear the correctness of this procedure, we prove that the difference of two numbers written by A is divisible by 9.

This implies that all the multiples of this number as well as the sum of the digits in all the multiples will be multiples of 9*.

*Note of the translator:
I will give a complement after the proof.

Minden tizes rendszerben felírt szám:

All the decimal numbers are written as follows:

$$N = a_n 10^n + a_{n-1} 10^{n-1} + ... + a_1 10 + a_0$$

igy is irható:

They can be written also as follows:

$$N = a_n \left(10^n - 1\right) + a_{n-1} \left(10^{n-1} - 1\right) + ... + a_1 \left(10 - 1\right) + a_n + a_{n-1} + ... + a_1 + a_0$$

Mivelhogy a zárójelekben álló $(10k - 1)$ alakú számok 9-czel oszthatók, azért N a következőképen irható:

Since the numbers in the form $(10k - 1)$ in the parentheses are divisible by 9, N thus can be written as follows:

$$N = 9r + s,$$

hol s az N számjegyeinek összegét jelenti.	where s means the sum of the digits of N.
Az A által felírt két számra (N_1, N_2) nézve tehát szintén	The two numbers (N_1, N_2) written by A have similarly the following form*:

$$N_1 = 9p + s, \quad N_2 = 9q + s,$$

*Note of the translator:
In the original text, r is used instead of p, but it must be a misprint in comparing with the equation below.

hol s, a számjegyek összege, a feltétel szerint a két számban ugyanaz.	where the sum of the digits s is a number common to the two numbers according to the condition.
A két szám különbsége:	The difference of the two numbers is

$$N_1 - N_2 = 9(p - q)$$

tehát valóban 9-nek többszöröse.	It is indeed a multiple of 9.

*Note of the translator:
As KŐNIG has written in his note, all the multiples of $N_1 - N_2$ as well as the sum of the digits in all the multiples are multiples of 9.
Because $N_1 - N_2$ is a multiple of 9, it is clear that all the multiples of $N_1 - N_2$ are multiples of 9. Therefore, we should prove only that the sum of the digits of $N_1 - N_2$ is a multiple of 9.
The difference $N_1 - N_2$ is also a decimal number, therefore, it also can be written with its digits $a_n, a_{n-1}, ..., a_1, a_0$ as follows:

$$N_1 - N_2 = a_n\left(10^n - 1\right) + a_{n-1}\left(10^{n-1} - 1\right) + ... + a_1\left(10 - 1\right) + a_n + a_{n-1} + ... + a_1 + a_0$$

Because $N_1 - N_2 = 9(p - q)$, the sum of the digits of $N_1 - N_2$ is

$$a_n + a_{n-1} + ... + a_1 + a_0 = 9(p-q) - \left[a_n\left(10^n - 1\right) + a_{n-1}\left(10^{n-1} - 1\right) + ... + a_1\left(10 - 1\right)\right]$$

Every $10^k - 1$ is a multiple of 9, therefore the part between the brackets "[" and "]" can be represented with $9t$, then

$$a_n + a_{n-1} + ... + a_1 + a_0 = 9(p - q - t)$$

The sum of the digits of $N_1 - N_2$ is therefore a multiple of 9.

[p. 27]
Az eddigi példákban mindig bizonyos műveletek eredményét ismerve, kitaláltuk a számot, melyet valaki gondoit.

In the examples above, we always guessed the number in the other's mind from the result of certain operations.

Most még két oly módot mutatunk be, melynek segítségével a gondolt számot nem találhatjuk ugyan ki, de megmondhatjuk az eredményt, melyre egy műveletsorozattal jutunk.

Now we present two methods with which we cannot find the number in the other's mind, but we can tell the result of a series of operations.

Ezen eljárások azon alapszanak, hogy a gondolt szám maga a műveletek által a számításból kiesik és a végső eredmény független lesz a gondolt számtól.

The procedures in it depend on the case that the number in the other's mind itself falls out of the calculation during the operations, and the final result will be independent of the number in the other's mind.

11. Végeztessük A-val a gondolt számon a következő műveleteket.

11. Let A perform the following operations on the number in A's mind.

Szorozza meg 2-vel a gondolt számot, adjon az eredményhez 5-öt, s a keletkező szám 3-szorosából vonja le a gondolt szám 6-szorosát.

Multiply the number in A's mind by 2, add 5 to the result; and then, from 3 times the resulting number, subtract 6 times the number in A's mind.

Az így keletkező szám, bármily számot gondolt is A, 15 lesz, mert x bármely értékénél:

No matter which number is in A's mind, the resulting number will be 15, which is drawn from any x:

$$3(2x + 5) - 6x = 15.$$

[p. 28]
Megmondhatjuk az eredményt akkor is, ha gondolt számon (x) pld. a következő két kifejezésben látható műveleteket végeztetjük el:

We can tell the result also in the cases when we let A perform, for example, on the number in A's mind (x), the operations which can be seen in the following two expressions:

$$\frac{4\,(3x+2)+4}{12}-x,\ \sqrt{4\,(x+3)\,(x+4)+1}-2x,$$

a végeredmény az első esetben mindig 1, az utóbbiban pedig 7 lesz.

the final result in the first case will be 1 for any x, that in the latter one will be 7.

Némikép hasonlít a 10. példához a következő:

The following example is similar to the example of the paragraph 10:

12. A felír egy háromjegyű számot s egy másikat, mely ugyanazon jegyeket, mint az első, fordított sorrendben tartalmazza.

Levonja a kisebbet a nagyobból*) s a keletkező háromjegyű szám első jegyét megmondja nekünk.

12. A writes down a three-digit number, and another number which contains the same digits as the first one, but in reverse order.

A subtracts the smaller one from the larger one*) and tells us the first digit of the resulting three-digit number.

*) Attól az esettől, midőn a két szám egyenlő, eltekintünk (ekkor $a = c$)

*) Ignore the cases that the two numbers are equal (in this case, $a = c$).

Hogyan lehet most az egész különbséget kitalálni?

Then, how is it possible to guess the difference in every case?

Legyen

Let

$$A = 100a + 10b + c$$

*Note of the translator:
This A is a number, while the person in the text is also represented by a letter A.

és	and

$$B = 100c + 10b + a$$

a két szám, melyet A fölirt (a, b és c számjegyet jelent).	be the two numbers that A wrote down (a, b and c means digits).
Vegyük fel, hogy $A > B$.	Suppose that $A > B$.
A két felírt szám különbsége:	The difference of the two written numbers is:

$$A - B = 100\,(a - c) + (c - a)\,,$$

ami így is írható;	It can be written also as follows:

$$100\,(a - c) - 10 + (10 + c - a)$$

vagy így:	or as follows:

$$100\,(a - c - 1) + 90 + (10 + c - a)\,.$$

Minthogy $a > c$ (csak így lehet $A > B$), azért $a - c - 1$ is, $10 + c - a$ is egyjegyű számot (számjegyet) jelent, s így $A - B$ oly háromjegyű szám, [p. 29] melynek első jegye: $a - c - 1$, második jegye: 9, s a harmadik: $10 + c - a$.	Since $a > c$ (only when $A > B$), thus also $a-c-1$, as well as $10+c-a$, represents a one-digit number (digit), and similarly, $A - B$ represents a three-digit number, the first digit of which is $a-c-1$, the second digit is 9, and the third digit is $10+c-a$.

Első és harmadik jegyének összege tehát $(a - c - 1 + 10 + c - a)$ mindig 9-czel egyenlő.

The sum of the first and the third digits is therefore $(a - c - 1 + 10c - a)$, which is always equal to 9.

Ha tehát ismerjük az $A - B$ szám első jegyét, megkapjuk a harmadikat, ha az elsőt 9-ből levonjuk, a második jegy pedig függetlenül a felírt számtól mindig 9.

Therefore, if we know the first digit of the number $A - B$, we get the third digit by subtracting the first digit from 9. The second digit is always 9 independently of the number written down.

Ha tehát megtudjuk, hogy a keletkező különbség első jegye: $1, 2, ..., 9$, akkor az egész $A - B$ különbség:

Therefore, if we find out that the first digit of the resulting difference is $1, 2, ..., 9$, then all the differences $A - B$ are as follows:

198, 297, 396, 495, 594,
693, 792, 891, ill. (and) 990.

E kilencz értéken kívül más értéket $A - B$ nem vehet fel.

$A - B$ cannot take any value other than these nine values.

4.4 Bűvös négyzetek: Magic squares

[p. 30]
Bűvös négyzetnek oly négyzetet nevezünk, melynek n^2 mezejébe úgy van n^2 (rendesen az első n^2) szám beírva, hogy bármely sorban, oszlopban, bármely átló mentén álló számok mindig ugyanazt az adják (4. ábra).

We call the following square a magic square: n^2 numbers (precisely, the first n^2 numbers*) are written in n^2 cells of a square, so that the sum of numbers in any row, in any column, or along any diagonal is always the same (Figure 4).

*Note of the translator:
Integers $1, 2, ..., n^2$.

6	1	8
7	5	3
2	9	4

4. ábra. (Figure 4.)

Minthogy az első n^2 szám összege:

Because the sum of the first n^2 numbers is:

$$1 + 2 + 3 + ... + n^2 = \frac{n^2(n^2+1)}{2}$$

azért az első n^2 számból képzett bűvös négyzet egy sorában stb. álló számok összege $\frac{n(n^2+1)}{2}$ lesz;

therefore the sum of the numbers in any single row etc. of a magic square formed with the first n^2 numbers will be $\frac{n(n^2+1)}{2}$;

így az első 9, 16, 25, 36, 49, 64, 81, 100,... számból képzett bűvös négyzetnél ez az összeg rendre 15, 34, 65, 111, 175, 260, 369, 505,... lesz.

so the sum of a magic square formed from the first 9, 16, 25, 36, 49, 64, 81, 100,... numbers will be successively 15, 34, 65, 111, 175, 260, 369, 505,....

Ebben a fejezetben néhány módszert fogunk bemutatni, melyek segítségével majd könnyen szerkeszthetünk bűvös négyzetet.

In this chapter, we will present some methods, with the help of which we can certainly easily construct magic squares.

Mivel a bűvös négyzetek megszerkesztésében igen nagy szerepe van annak, hogy n páros-e, vagy páratlan, azért külön fogjuk tárgyalni a páros és páratlan mezejű négyzeteket.

In the composition of magic squares, it will play a very important role that n is either an even or an odd number. Therefore, we will treat separately the squares with an even number of cells and those with an odd number of cells.

[p. 31]

A) *Páratlan mezejű bűvös négyzetek.*

A) *Magic squares with an odd number of cells.*

Első (indus) módszer.

First (Indian) method.

Az 1-es számot a legfelső sor közepébe írjuk, a 2-est pedig a középsőtől eggyel jobbra levő oszlop legalsó mezejébe, most felfelé és jobbra haladva írjuk átló irányban a 3-as, 4-es,... számokat.

We write the number 1 into the middle of the top row, and 2 into the lower cell of one column right to the middle column, then we write the numbers 3, 4,... progressing up- and right-wards in a diagonal direction.

Ha elérjük a négyzet jobboldali szélét, akkor a következő számot a következő (fölötte lévő) sor első (baloldali) mezejébe [p. 32] írjuk és átló ($\nearrow$) irányban megint tovább haladunk.

When we reach the right edge of the square, then we write the next number into the first (left-end) cell of the preceeding (the one above) row, and we continue again in a diagonal ($\nearrow$) direction.

Ha pedig a négyzet felső szélét érjük el, a jobbra levő oszlop legalsó mezejébe írjuk a következő számot.

But, if we reach the upper edge of the square, we write the next number into the lower cell of the right column.

Ha már elfoglalt mezőre jutunk, vagy az utolsó sor utolsó mezejére, akkor az utoljára leírt szám alá írjuk a következőt.

If we come to a cell already occupied, or to the last cell of the last row, then we write the next number under the last-written number.

Így keletkezik pld. az 5. ábrán látható 7×7 mezős bűvös négyzet.

For example a magic square of 7×7 cells shown in Figure 5 is made in this way*.

*Note of the translator:
Kőnig gave references for these two books on mathematical recreations at the end of the second book [87], and he wrote there that he cited the book *Du royaume de Siam* (1691) by Simon de La Loubère. See Section 5.10.1 for this part of my translation.
According to Kőnig, it is the first book describing this Indian method in European language. La Loubère devotes one chapter to "le probème des carées magiques selon les Indiens (the problem of the magiques squares according to the Indians)" [139, 140].
This book of La Loubère was published by two publishers: Coignard is "imprimeur et libraire ordinaire du Roy (a printing company and a book shop of the King)"; Wolfgang is not. The chapter we cite here is in pp. 295–359 of the publisher Coignard, or pp. 235–288 of the publisher Wolfgang, the contents are the same; Kőnig cites the book of the publisher Wolfgang). In this chapter, Kőnig cited also Bachet's method, which we will see below.
La Loubère is "envoyé extraordinaire du Roy auprés du Roy de Siam (a special envoy of the King to the King of Siam)" according to the book of La Loubère [139, 140].
Jean-Pierre Niceron (1685–1738) gives more precise information about La Loubère [231]:
Simon de La Loubère (1642–1729) was a poet, and a secretary at the Swiss embassy; the King was interested in the establishment of the religion and of the commerce of the kingdom of Siam, and La Loubère was sent there as a special envoy: he departed from Brest on March 1st of 1687, arrived at Siam at the end of September, and stayed there until January of 1688: in these 3 months, he collected exact fundamental knowledge on the history and the nature of the country, on the origin, the usage, the customs, the industry and the religion of the habitants.

30	39	48	1	10	19	28
38	47	7	9	18	27	29
46	6	8	17	26	35	37
5	14	16	25	34	36	45
13	15	24	33	42	44	4
21	23	32	41	43	3	12
22	31	40	49	2	11	20

5. ábra. (Figure 5.)

Második (Bachet-féle) módszer.	*The second (Bachet-style) method**.

*Note of the translator:
Claude Gaspar Bachet de Méziriac (1581–1638) is a French mathematician, who published *Problèmes plaisants et délectables qui se font par les nombres* (enjoyable and delectable problems which are made with the numbers) first edition in 1612 [7]; second edition in 1624 [8]. After his death, the third (1874 [9]) and the fourth (1879 [10]) edition revised, simplified and augmented by A. Labosne appeared; the fifth edition with a preface by Jean Itard containing the biography of Bachet appeared in 1959 [11]. Kőnig cites the third edition.

Ez a módszer is általában mindenféle páratlan mezejű bűvös négyzet szerkesztésére alkalmas, de könnyebb megérthetőség kedvéért ezt csak egy példára fogjuk alkalmazni.

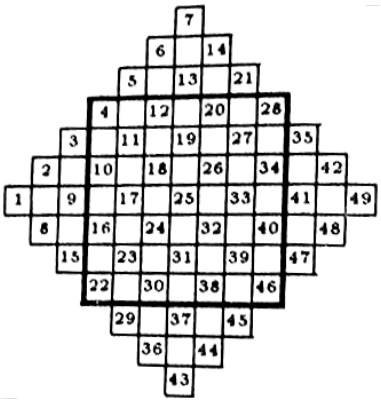
This method is suitable for constructing all kinds of magic squares with an odd number of cells generally, but we will apply it to only one example that is easier to understand.

Szerkeszszünk pld. az első $7 \times 7 = 49$ számból bűvös négyzetet.

We construct a magic square for example with the first $7 \times 7 = 49$ numbers.

Írjuk fel természetes sorrendben az első 49 számot, mint a 6. ábrában látható, átló irányban.

We write down the first 49 numbers in a natural order in a diagonal direction as shown in Figure 6.



6. ábra. (Figure 6.)

Így a négyzet 25 mezejét már betöltöttük.

We filled already 25 cells of the square in this way.

A négyzet 4 oldalán kívül álló 6-6 számot toljuk most egymáshoz való helyzetük megváltoztatása nélkül, a szemközt levő oldalig.

[p. 33]
Ez a 24 szám az üresen levő helyeket tölti meg, s így keletkezik a 7. ábrában látható bűvös négyzet.

Now we push the numbers 6 by 6 which are put outside of the 4 sides of the square to the opposite side without changing the mutual disposition of the numbers.

These 24 numbers fill the vacant places, and the magic square shown in Figure 7 is constructed in this way.

4	29	12	37	20	45	28
35	11	36	19	44	27	3
10	42	18	43	26	2	34
41	17	49	25	1	33	9
16	48	24	7	32	8	40
47	23	6	31	14	39	15
22	5	30	13	38	21	46

7. ábra. (Figure 7.)

Harmadik (Moschopulos-féle vagy lóugrásos) módszer.

The third (Moschopulos-style or knight's move) method.*

*Note of the translator:

Manuel MOSCHOPULUS (1265?-1316?) is a philologist in Byzantine Empire, who writes also about magic squares in Greek in around 1315 [162, 161]. Philippe de LA HIRE (1640–1718), a mathematician in France, finds a manuscript accidentally in the National Library of France, and gives an analysis of it in 1705 [137]. Adam Wilhelm Siegmund GÜNTHER (1848–1923), a teacher of mathematics in Germany, reproduces a manuscript found in Munich in his book without translation in 1876 [68]. Paul TANNERY (1843–1904), a French historian of sciences, mentions GÜNTHER's reproduction with his regret "...d'après un manuscript de Munich malheureusement trop incorrect pour qu'une nouvelle édition ne soit pas désirable (...based on a manuscript of Munich, which is unfortunately too incorrect for a new edition not to be desirable)," and he reproduces the manuscript of the National Library of France with his translation into French in 1886 [200]. John Calvin MCCOY translated TANNERY's translation into English in 1941 [157]. P. G. BROWN (a mathematician, University of New South Wales in Australia) translated a manuscript in Greek directly into English in 2005 [24]. KŐNIG cites GÜNTHER's reproduction, but does not cite TANNERY's (see 5.10.1). It is possible that KŐNIG did not know TANNERY's article at that time.

Az 1-es számot a legalsó sor közepébe írjuk.	We write the number 1 into the middle of the lower row.
(Ha n nem osztható 3-mal, mint a 8. ábrában, akkor bárhová tehetjük az 1-est).	(If n is not divisible by 3 as shown in Figure 8, then we can put the 1 anywhere).

21	39	8	33	2	27	45
5	23	48	17	42	11	29
38	14	32	1	26	44	20
22	47	16	41	10	35	4
13	31	7	25	43	19	37
46	15	40	9	34	3	28
30	6	24	49	18	36	12

8. ábra. (Figure 8.)

A 2-es számot [p. 34] egy oszloppal jobbra és két sorral följebb írjuk (lóugrás), a 3-ast ismét egy oszloppal jobbra és két sorral feljebb, stb.

We write the number 2 in the cell one column to the right and two rows higher (knight's move), the number 3 again in the cell one column to the right and two rows higher, etc.

Ha már elfoglalt mezőre jutnánk, akkor a következő szám nem lóugrásnyira, hanem ugyanazon oszlopba, *4 mezővel feljebb* írandó, mint az utoljára leírt szám.

If we come to a cell already occupied, then the next number is to be written not into the cell of knight's move, but into the same column *4 cells higher* than the number written last.

A bűvös négyzetek ily képzésénél is, mint az első módszernél: az első sor (oszlop) az utolsó sor (oszlop) után következőnek tekintendő.

The construction of magic squares is also like the first method: the first row (column) is considered as succeeding to the last row (column).

A 8. ábra pontos áttekintésével hamar begyakorolhatjuk ezt az eljárást is.

We can practice this procedure soon in examining Figure 8 precisely.

Negyedik (Horner vagy Scheffler-féle) módszer.

*The fourth (Horner or Scheffler-style) method**.

*Note of the translator:
William George Horner (1786–1837) was an English mathematician, who wrote an article "On the algebra of magic squares" in 1871 [71]. Hermann Scheffler (1820–1903), a German mathematician, published *Die magischen Figuren* in 1882 [186], which is cited by Ahrens on p. 231 of *Mathematische Unterhaltungen und Spiele* [1].

Válaszszunk négy oly positiv, vagy negativ egész a, a', b, b' számot, hogy ez a négy szám, valamint a következő öt is:

Let's pick up four positive or negative integers a, a', b, b', so that these four numbers should be relative primes to n and different from 0, as well as the following five numbers:

$$ab' - a'b,\ a + a',\ b + b',\ a - a',\ b - b'$$

n-nel relatív prím és 0-tól különböző legyen.

Ebben az esetben bűvös négyzetet nyerünk, ha az n^2 mezőbe a következőképpen osztunk be n^2 számot*):

In this case, we get a magic square if we divide n^2 numbers among the n^2 cells as follows*):

$1+0+0n$	$1+a+bn$	$1+2a+2bn$	...
$1+a'+b'n$	$1+(a+a')+(b+b')n$	$1+(2a+a')+(2b+b')n$	...
$1+2a'+2b'n$	$1+(a+2a')+(b+2b')n$	$1+(2a+2a')+(2b+2b')n$	...
$1+3a'+3b'n$	$1+(a+3a')+(b+3b')n$	$1+(2a+3a')+(2b+3b')n$	...
....	.....	.....	...
....	.....	.....	...

*) A táblázatban előforduló mennységek mind $1+p+qn$ alakú összegek. Megjegyzendő, hogy ha p vagy q az n számnál nagyobb, akkor helyükbe p-nek, illetve q-nak n-nel való elosztásakor fellépő maradék teendő. Ha $p=n$, vagy $q=n$, akkor p, ill. q helyébe 0 teendő.

*) All the sums appearing in the table are in the form of $1+p+qn$. Note that, if p or q is larger than the number n, then such a p or q should be replaced with its remainder arising from division with n. If $p=n$ or $q=n$, then such a p or q should be replaced with 0.

Ha $n=5; a=2,\ a'=1,\ b=1,\ b'=2$, akkor a 9. ábrabeli bűvös négyzet keletkezik.

If $n=5; a=2,\ a'=1,\ b=1,\ b'=2$, then the magic square of Figure 9 is constructed.

1	8	15	17	24
12	19	21	3	10
23	5	7	14	16
9	11	18	25	2
20	22	4	6	13

9. ábra. (Figure 9.)

[p. 35]

B) *Páros mezejű bűvös négyzetek.*

2×2 *mezejű bűvös négyzet* az első 4 számból, mint arról könnyen meggyőződhetünk, nem képezhető.

4×4 *mezejű bűvös négyzet.* Írjuk be természetes sorrendben az első 16 számot a 16 mezőbe [p. 36] (10. ábra) és hagyjuk változatlanul a négy sarokban (1, 4, 16, 13) és a négy középen (6, 7, 11, 10) lévő számot.

B) *Magic squares with an even number of cells.*

Magic square of 2×2 *cells* cannot be formed from the first 4 numbers as we can easily verify it.

Magic square of 4×4 *cells.* We write the first 16 numbers in the 16 cells in a natural order (Figure 10), and we leave four numbers in the angles (1, 4, 16, 13) and four numbers in the middle (6, 7, 11, 10) invariable.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

10. ábra. (Figure 10.)

A megmaradó nyolcz szám helyébe pedig írjuk azon számot, mely őket 17-re $(4 \times 4 + 1)$ kiegészíti, tehát a

But we replace the remaining eight numbers with their complements to 17 $(4 \times 4 + 1)^*$, therefore we replace the numbers

$$2, 3, 8, 12, 15, 14, 9, 5$$

számok helyébe rendre a

successively with the numbers

$$15, 14, 9, 5, 2, 3, 8, 12$$

számokat írjuk. Az így keletkező négyzet (11. ábra) bűvös négyzet lesz.

The square constructed in this way (Figure 11) will be a magic square.

---

*Note of the translator:

The sum of each row/column/diagonal of $n \times n$ square is a multiple of $n^2 + 1$ as shown at the beginning of this chapter. Both of the sums of diagonal lines of Figure 10 are 34. This number is already equal to the number $\frac{4\cdot(4^2+1)}{2}$ which composes a 4×4 magic square, therefore we keep the numbers $(1, 6, 11, 16)$ and $(4, 7, 10, 13)$ as they are.

We compare the sums of the first row and the fourth row: $1+2+3+4 = 10 = 34-24$ and $13+14+15+16 = 58 = 34+24$, therefore we will get the sum 34 if we replace the pairs $(2, 3)$ and $(14, 15)$, of which the difference of sums is $(14+15)-(2+3) = 24$. Similarly, we compare the sums of the second column and the third column $2+6+10+14 = 32 = 34-2$ and $3+7+11+15 = 36 = 34+2$, and we will know that the replacement of the pairs $(2, 14)$ and $(3, 15)$ will give the sum 34. To realise these two replacements, we have to replace 2 and 15, as well as 3 and 14, which are complements to 17 each other.

By doing the similar procedure for the first and the fourth columns and the second and the third rows, we will get the magic square in Figure 11.

Ahrens [1], Lucas [148, 152] and Schubert [189] describe this method, which Kőnig cites (see 5.10.1).

Kőnig cites also the French translation [14] of Ball's book [13], which contains the description of a similar method to construct magic squares with any even number of cells.

Ball, in the chapter entitled "Magic squares" in his book *Mathematical Recreations and problems of past and present times* of the third edition (1896) [13] and later, cites his article "Even magic squares" of the *Messenger of Mathematics*, Cambridge, September, 1893, vol. 23, pp. 65–69, as the first printed description of this method.

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

11. ábra. (Figure 11.)

Nagyobb számú, páros mezejű bűvös négyzetek szerkesztésére csak bonyodalmasabb módszerek vannak, mint a páratlan mezejűek képzésére.

Ezért oly módszert, mely minden próbálgatást, találgatást kizár, itt nem mutathatunk be.

6×6 mezejű bűvös négyzet készítése végett induljunk ki ismét abból a négyzetből, mely az első 36 számot természetes sorrendben elhelyezve tartalmazza (12. ábra).

For constructing *magic squares with a larger even number of cells,* there are only methods more complicated than the construction of magic squares with an odd number of cells.

Therefore such a method cannot be introduced here without experimenting and guessing.

For making a magical square with 6×6 cells, let us start again from that square which contains the first 36 numbers placed in a natural order (Figure 12).

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

12. ábra. (Figure 12.)

Az átlók mentén levő számokat [p. 37] változatlanul hagyva, a többiek helyébe írjuk azokat a számokat, melyek 37-re $(6\times 6+1)$ egészítik ki őket.

we let the numbers along the diagonals invariable, and we replace the other numbers with their complements to 37 $(6 \times 6 + 1)$.

Az így keletkező négyzetből (13. ábra) 6 felcseréléssel bűvös négyzetet nyerhetünk (14. ábra).

From the square constructed in this way (Figure 13), we can get a magic square by 6 interchanges (Figure 14)*.

*Note of the translator:

Interchange 3 and 33; 7 and 25; 14 and 20; 13 and 18; 9 and 10; 2 and 5.

Figure 13 and Figure 14 are misarranged in the original text, and it is fixed in this translation: the figure in Figure 14 of the original text is placed in Figure 13 here, and vice versa.

1	35	34	33	32	6
30	8	28	27	11	25
24	23	15	16	20	19
18	17	21	22	14	13
12	26	10	9	29	7
31	5	4	3	2	36

13. ábra. (Figure 13.)

1	35	34	3	32	6
30	8	23	27	11	7
24	23	15	16	14	19
13	17	21	22	20	18
12	26	9	10	29	25
31	2	4	33	5	36

14. ábra. (Figure 14.)

$n \times n$ *mezejű négyzet* készítésére általánosságban is használhatjuk ezt a módszert: a két [p. 38] átlóra eső számokat kivéve, az összes szám helyébe az őket $(n^2 + 1)$-re kiegészítő számokat írjuk s az így keletkező négyzetből néhány symmetrikusan fekvő pont felcserélésével bűvös négyzetet nyerhetünk.

We can use this method for making *a square with* $n \times n$ *cells* for general n: leaving the numbers put in the two diagonals as they are, we replace all the other numbers with their complements to $(n^2 + 1)$, and we interchange some pairs of numbers placed symmetrically in such a square, then we can get a magic square.

Evvel a módszerrel természetesen nem képezhetjük az összes páros mezejű bűvös [p. 39] négyzetet.

Not all the magical squares with an even number of cells can be naturally constructed with this method.

Ilyen különben nagyon sok van.

There are a great many others.

Kiszámították pld., hogy az első 16 számból 880, az első 36-ból pedig már sok milllió bűvös négyzet képezhető.

For example, it was counted that 880 magic squares can be formed with the first 16 numbers*, and millions of magic squares can be formed with only the first 36 numbers.

*Note of the translator:
Before Euler's works on magic squares published in 1782 and 1849 [55, 56] (the latter article was delivered in 1776 to the St. Petersburg Academy, but originally published much later), Bernard Frénicle de Bessy (1605–1675), amateur mathematician and counselor at the "Cour des monnaies" in Paris, determined and counted all the 880 magic squares of 4×4 cells before 1675, and it was published in 1693 and reprinted in 1729 [61, 62, 63].

Bűvös négyzetek bűvös részekkel.

Magic squares with magic parts.

Minden külön megjegyzés nélkül ide iktatunk még két oly bűvös négyzetet, melyeknek vastag vonalakkal bekerített részei magukban is bűvös négyzetet alkotnak (l. 15. és 16. ábrát).

We insert two magic squares here without any particular comment. In these magic squares, the parts in themselves bordered with thick lines also form magic squares (see Figures 15 and 16).

40	1	2	3	42	41	46
38	31	13	14	32	35	12
39	30	26	21	28	20	11
43	33	27	25	23	17	7
6	16	22	29	24	34	44
5	15	37	36	18	19	45
4	49	48	47	8	9	10

15. ábra. (Figure 15.)

11	18	13	74	81	76	29	36	31
16	14	12	79	77	75	34	32	30
15	10	17	78	73	80	33	28	35
56	63	58	38	45	40	20	27	22
61	59	57	43	41	39	25	23	21
60	55	62	42	37	44	24	19	26
47	54	49	2	9	4	65	72	67
52	50	48	7	5	9	70	68	66
51	46	53	6	1	8	69	64	71

16. ábra. (Figure 16.)

4.5 Mathematikai hamisságok: Mathematical errors

[p. 40]
Ebben a fejezetben néhány oly arithmetikai és geometriai bizonyítást fogunk bemutatni, mely hibás eredményre vezet.

In this chapter, we will present some arithmetic and geometric proofs leading to a wrong result.

Az ily *arithmetikai* bizonyítások legnagyobb része azon hibás tételen alapszik, hogy valamely egyenlőség helyes marad akkor is, ha a két oldalról ugyanazt a tényezőt elhagyjuk. Az

The largest part of such *arithmetic* demonstrations is based on the wrong proposition that every equation is kept true even if the same terms are deleted from the two sides of an equation. The following equation:

$$ax = bx$$

egyenlőségből ugyanis nem következik, hogy $a = b$, mert, ha $x = 0$, akkor ezen egyenlőség fennáll, bármily értéket vesz is fel a és b.

does not imply that $a = b$ because, if $x = 0$, this equality is kept true for any values of a and b.

1. Jelölje a és b ugyanazt a számot, akkor*

1. Let a and b be the same numbers, then*

$$ab = a^2,$$

*Note of the translator:
The equal sign was complemented by the translator.

tehát — therefore

$$ab - b^2 = a^2 - b^2;$$

vagyis — that is,

$$b(a - b) = (a + b)(a - b),$$

innen — from this

$$b = a + b, \quad b = 2b, \quad \text{tehát} \quad 1 = 2.$$

Adjunk mindkét oldalhoz 1-et, 2-t, ..., akkor nyerjük, hogy

Add to both sides 1,2, ..., and we get:

$$2 = 3, 3 = 4, ...;$$

vagyis

that is,

$$1 = 2 = 3 = ...$$

s így *az összes számok egyenlők.*

Thus *all the numbers are equal to each other.*

[p. 41]
Ugyancsak hibás eredményre juthatunk, mint a következő három bizonyítás mutatja, ha nem veszszük tekintetbe a négyzetgyök kétértékűségét.

If we do not take into account the two values of the square root, we can get to a wrong result as shown by three proofs given below.

Ez is különben a 0-val való osztásra vezethető vissza. Mert abból, hogy

Otherwise the division with 0 can bring a wrong result as follows: from the equation

$$x^2 = y^2, \quad \text{vagyis (that is)} \quad (x + y)(x - y) = 0,$$

csak úgy következik, hogy

only the following result is derived:

$$x = y \quad \text{és (and)} \quad x - y = 0,$$

ha az

if the two sides of the equation

$$(x+y)(x-y)=0$$

egyenlőség két oldalát $(x+y)$-nal osztjuk. Ez a művelet pedig, ha

are divided by $(x+y)$. However, this operation brings the wrong result if

$$x+y=0, \qquad x=-y$$

hibás eredményre vezet.

2. Legyen két egymástól *különböző* a és b szám számtani közepese c, vagyis:

2. Let c be the mean value of two *different* numbers a and b, that is:

$$a+b=2c,$$

innen

From this,

$$(a+b)(a-b)=2c(a-b);$$

és a kijelölt műveleteket elvégezve:

and expand it to

$$a^2-b^2=2ac-2bc,$$

vagy

or

$$a^2 - 2ac = b^2 - 2bc;$$

adjunk mindkét oldalhoz c^2-et, akkor: | add c^2 to both sides, then

$$(a-c)^2 = (b-c)^2,$$

tehát | therefore

$$a = b,$$

ami pedig feltételünkkel ellenkezik. | which contradicts our condition.

Kissé nehezebb már a hiba megtalálása a következő példákban. | It will be somewhat more difficult to find the mistake in the following examples.

[p. 42]
3. A | 3. The identity

$$\sqrt{-1} = \sqrt{-1}$$

identitás a következő alakban írható: | can be written in the following form:

$$\sqrt{\frac{-1}{1}} = \sqrt{\frac{1}{-1}}$$

vagy | or

$$\frac{\sqrt{-1}}{\sqrt{1}} = \frac{\sqrt{1}}{\sqrt{-1}};^*$$

*Note of the translator:
In the original text, the left side was $\frac{\sqrt{-1}}{\sqrt{-1}}$, which was corrected as above by the translator. This transformation is anyhow incorrect, and brings the wrong result as shown below.

szorozzuk meg mindkét oldalt $\sqrt{1}$, $\sqrt{-1}$-gyel, akkor nyerjük, hogy

multiply both sides with $\sqrt{1} \cdot \sqrt{-1}$, then we get the following equation:

$$\left(\sqrt{-1}\right)^2 = \left(\sqrt{1}\right)^2;$$

de $\left(\sqrt{-1}\right)^2 = -1$ és $\left(\sqrt{1}\right)^2 = 1$, tehát

but $\left(\sqrt{-1}\right)^2 = -1$ and $\left(\sqrt{1}\right)^2 = 1$, therefore

$$-1 = 1.$$

4. A

The identity

$$\sqrt{x-y} = i\sqrt{y-x}$$

identitás (hol i a képzetes egységet jelenti) x és y bármely értékénél helyes marad.

(where i means the imaginary unit*) is always true for any values x and y.

*Note of the translator:
This i is either $\sqrt{-1}$ or $-\sqrt{-1}$. The following deduction confuses these two cases.

Legyen tehát

Therefore, let

1. $x = a$, $y = b$ akkor (then) $\sqrt{a-b} = i\sqrt{b-a}$ (1)
2. $x = b$, $y = a$ akkor (then) $\sqrt{b-a} = i\sqrt{a-b}$ (2)

(1)-et és (2)-t összeszorozva | Multiply (1) by (2), then

$$\sqrt{a-b}\sqrt{b-a} = i^2\sqrt{a-b}\sqrt{b-a}$$

és innen ismét | and from this again

$$1 = i^2 = -1$$

5. Legyen x oly szám, mely az | 5. Let x be a number satisfying the following equation:

$$a^x = -1$$

egyenletnek eleget tesz. Emeljük négyzetre a két oldalt, akkor | Square both sides, then

$$a^{2x} = 1$$

tehát | Therefore

$$2x = 0, \qquad x = 0 \qquad ^*$$

*Note of the translator:
This deduction is wrong, because the logarithm of an imaginary number is not 0 according to Johann Bernoulli discussed in his correspondence with Gottgried Wilhelm Leibniz between March 1712 and June 1713 [216, 141]:

$$\frac{\log\sqrt{-1}}{\sqrt{-1}} = \frac{\pi}{2}$$

[p. 43]
ezt helyettesítve eredeti egyenletünkbe:

Substitute this to our original equation, then

$$a^0 = -1$$

s így

and then

$$1 = -1$$

6. Legyen a, b, c és d négy oly szám, hogy

6. Let a, b, c and d be four numbers such as:

$$ad = bc, \text{ vagyis (that is) } \frac{a}{b} = \frac{c}{d}$$

Ha $a < b$, akkor $\frac{a}{b}$ s a vele egyenlő $\frac{c}{d}$ is valódi tört, tehát $c < d$. Ha most

If $a < b$, then $\frac{a}{b}$ and also its equal value $\frac{c}{d}$ are proper fractions, therefore $c < d$*. Now, if

*Note of the translator:
It is not true for negative numbers.

$$a = d = -1 \text{ és (and) } b = c = +1,$$

akkor az a, b, c, d számok közt fennáll a kivánt összefüggés és $a < b$, tehát $c < d$ vagyis $+1 < -1$, ami pedig nem igaz.

Mint az utóbbi két példában látható, akkor is hibás eredményre juthatunk, ha oly tételeket, melyek csak a számok egy bizonyos csoportjára (az 5. példában a valós számokra, a 6.-ban a positiv számokra) érvényesek, úgy tekintünk, mintha az öszszes számokra is érvényesek volnának.

Az „arithmetikai hamisságok“ negyedik csoportja abból a hibás feltevésből indul ki, hogy végtelen sok szám összege mindig végtelen nagy.

7. Ezen alapszik Zeno ismeretes sophismája, mely szerint Achilles a teknősbékát soha utól nem éri.

Vegyük fel, hogy a teknősbéka egy stádiummal halad Achilles előtt s hogy sebessége Achilles sebességének $\frac{1}{10}$-ével egyenlő; vagyis, hogy egyenlő idők alatt Achilles 10-szerte nagyobb utat jár be a teknősbékáénál.

then among the numbers a, b, c, d, there is the relation required above, and $a < b$, therefore $c < d$, that is $+1 < -1$, but it is not true.

As shown in the latter of the two examples, we can be brought to a wrong result if the theorems are valid only for a certain group of the numbers (in the example 5, the theorem is valid only for the real numbers; in the example 6, only for the positive numbers), nevertheless we consider as if they would be valid for all the numbers.

The fourth group of „arithmetic errors“ begins with the wrong hypothesis that the sum of infinitely many numbers is after all infinitely large.

Zeno's known sophism that Achilles never catches up with the tortoise on the road is based on this.

Suppose that the tortoise goes forward in front of Achilles in a race, and that the velocity of tortoise is equal to $\frac{1}{10}$ of Achilles' velocity; that is, Achilles runs 10 times long road than that of the tortoise at the same time.

[p. 44]
Zeno most így okoskodik. Míg Achilles az első stádium végéhez ér, a teknősbéka $\frac{1}{10}$ stádiummal haladt tovább, míg Achilles ezt az utat is bevégzi, a teknősbéka mát megint előbbre jutott $\frac{1}{100}$ stádiummal.

Majd $\frac{1}{1{,}000}$, $\frac{1}{10{,}000}$, stb. stádiummal halad Achilles előtt.

Achilles tehát folyton közelebb ér a teknősbékához, teljesen azonban sohasem éri el.

A hiba itt abban rejlik, hogy Zeno csak az útnak egy véges darabjára bizonyítja be sophismáját, melynek hosszát az

Zeno considers here as follows: until Achilles reaches the end of the first part of the race [that is, the starting point of the tortoise], the tortoise goes forward by $\frac{1}{10}$ length of Achilles goes; until Achilles finishes this length, the tortoise now goes ahead again by $\frac{1}{100}$ length.

Then the tortoise progresses by $\frac{1}{1{,}000}$, $\frac{1}{10{,}000}$, and so on in front of Achilles.

Achilles thus constantly approaches the tortoise, but never reaches it at all.

The mistake here resides in the following thing: Zeno only proves the sophism on a finite piece of the road, the length of which is expressed with the sum

$$1 + \frac{1}{10} + \frac{1}{100} + ...\text{in inf.}$$

összeg fejezi ki stádiumokban. E sor *végtelen* ugyan, de összege a *véges* $1^1/_9$ számmal egyenlő.

Geometriai bizonyításoknál még könnyebben juthatunk hibás eredményhez, amit legtöbbnyire a hibás rajz okoz, és bár az ily bizonyításokban a hiba sokkal burkoltabban jelenhetik meg, mint az arithmetikaiaknál, mégis pontos rajz készítésével rögtön felfedezhetjük a hibát.

in the races. This addition continues *infinitely,* but the sum is equal to the *finite* number $1^1/_9$.

We can be brought to a wrong result even more easily in *geometric* proofs. The wrong result is mostly caused by a wrong drawing, and, although such a mistake in proofs appears less apparently than the arithmetic one. we can somehow discover the mistake immediately with an accurate drawing.

1. Egyike a legérdekesebb ilyen bizonyításoknak a következő, mely bebizonyítja, hogy *valamennyi háromszög egyenlőszárú.*

Legyen ugyanis ABC tetszőleges háromszög BC oldalának felezőpontja D, s emeljünk e pontban BC-re merőlegest, rajzoljuk meg továbbá a BAC szög felezőjét.

Ha 1.) e szögfelező a D-ben emelt merőlegest *nem* metszi, akkor párhuzamos vele s így merőleges BC-re; a háromszög tehát egyenlőszárú ($AB = AC$).

Ha 2.) metszik egymást ezen egyenesek, akkor a metszőpont lehet *a)* az ABC háromszögön belül és [p. 45] *b)* az ABC háromszögön kivül.

Mindkét esetben bebizonyítjuk, hogy $AB = AC$.

a) Legyen az O metszőpont a háromszögön belül (17. ábra).

1. Here is one of the most interesting proofs: a proof that *every triangle is isoscele.*

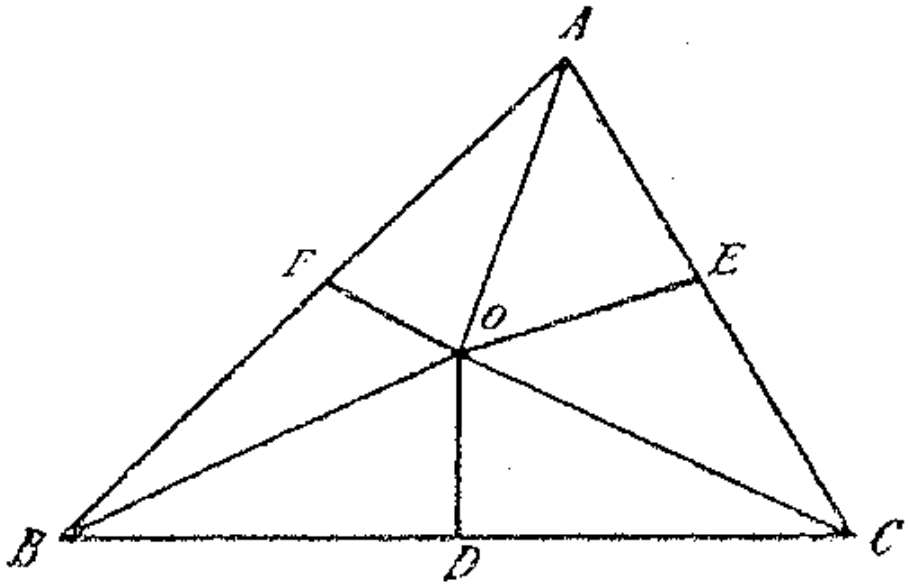
Let ABC be an arbitrary triangle, D be the midpoint of the edge BC; draw a line including this point perpendicular to BC; furthermore draw the bisector of the angle BAC.

If 1.) this angle bisector does *not* cut the line perpendicular to D, then the bisector is parallel with the perpendicular line, and it is perpendicular to BC, the triangle is thus isosceles ($AB = AC$).

If 2.) these straight lines cut each other, then the intersection can be *a)* inside the triangle ABC, or *b)* outside the triangle ABC.

Let us prove in both cases that $AB = AC$.

a) Let O be an intersection inside the triangle (17. figure).



17. ábra.

Bocsássuk le O-ból AB-re és AC-re az OF illetőleg OE merőjegeseket.	Let OF, OE be the perpendicular lines from O to AB and to AC.
Minthogy O a szögfelezőn van, azért	Since O is on the angle bisector, therefore

$$OF = OE \text{ és (and) } AF = AE,$$

s mivel O az OD merőlegesen van, azért	and since O is on the perpendicular line OD, therefore

$$OB = OC.$$

A BOF és COE derékszögű háromszögek congruensek, s így	The right triangles BOF and COE are congruent with each other, and

$$FB = EC;$$

de | but

$$AF = AE,$$

tehát | thus

$$AF + FB = AE + EC,$$

vagyis | that is,

$$AB = AC.$$

b) Ha O a háromszögön kivül van (18. ábra), akkor ugyanazon jelölést használva, mint előbb, ugyanúgy nyerjük, hogy

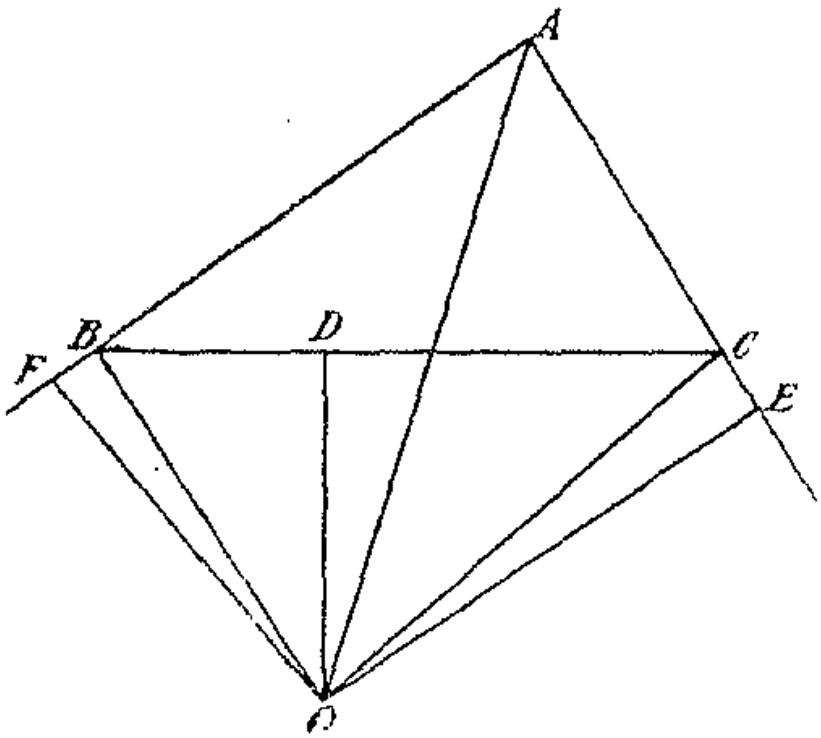
b) If O is out of the triangle (Figure 18.), then using the same notations as before, we get similarly that

$$AF = AE \qquad (1)$$

és, hogy a BOF és COE derékszögű háromszögek congruensek, tehát:

and that the right triangles BOF and COE are congruent, therefore

$$BF = CE. \qquad (2)$$



18. ábra.

[p. 46] (1)-ből és (2)-ből ismét nyerjük, hogy	From (1) and (2), we get

$$AF - BF = AB - CE \text{ vagyis (that is) } AB = AC.$$

Az olvasóra bízzuk, hogy ebben a nagyon pontosnak látszó bizonyításban a hibát megkeresse.	It is left to the reader to find the error in this proof which is apparently very accurate.

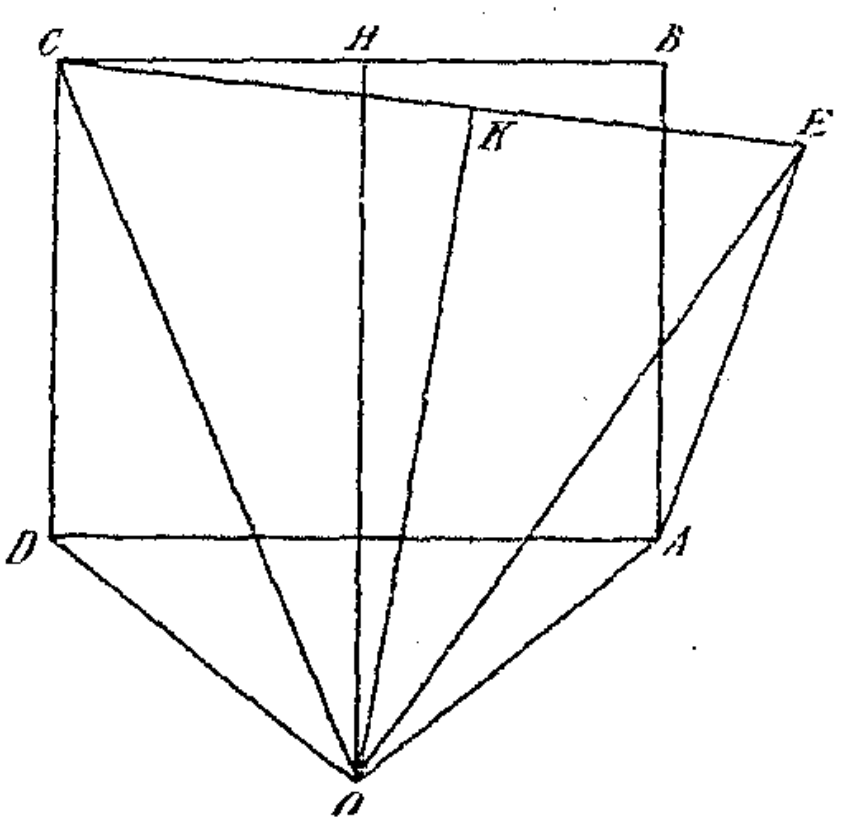
*Note of the translator:
The intersection O cannot be inside the triangle, therefore the condition of a) cannot be satisfied. In the proof of b), if one of the points E and F is inside of the triangle, the other should be on the edge of the triangle. For example, if E is outside of the triangle, F is on the edge AB, therefore the equation $AF - BF = AB - CE$ does not imply $AB = AC$.

2. Bebizonyítjuk, hogy a *tompaszög egyenlő a derékszöggel.*

Rajzoljunk $ABCD$ derékszögű négyszög (l. 19. ábrát) A csúcsán keresztül egy oly $AE = AB$ hosszúsága egyenest, mely mint a rajzon látható [p. 47] AB-vel hegyes szöget zár be.

2. We prove that the *obtuse angle is equal to the right angle.*

Draw a rectangle $ABCD$ (see Figure 19), and a line $AE = AB$ through the vertex A which, as indicated in the drawing, forms with AB an acute angle.



19. ábra.

CB egyenes H középpontjában emelt merőlegesek mindenesetre metszik egymást O pontban. A szerkesztés alapján

Through the middle point H of the straight line CB, a perpendicular line cutting CB extends to the point O. Based on the conditions,

$$OD = OA \text{ és (and) } OC = OE,$$

továbbá	furthermore

$$AE = AB = DC,$$

s így	and

$$OAE\triangle \cong ODC\triangle;$$

tehát	thus

$$ODC\sphericalangle = OAE\sphericalangle;$$

de	but

$$ODA\sphericalangle = OAD\sphericalangle,$$

mert OAD háromszög egyenlőszárú s így végre	because the triangle OAD triangle is isosceles, and finally

$$DAE\sphericalangle = CDA\sphericalangle = 90^\circ$$

pedig a szerkesztés alapján DAE tompa szög.	though, based on the conditions, DAE is an obtuse angle.

A hiba ezen bizonyításba ott kerül be, midőn feltételezzük, hogy az O pont az A-n túl meghosszabbított EA egyenesen túl van.

3. *Ha egy négyszög két szemközt fekvő oldala egyenlő, akkor a másik két oldal párhuzamos.*

Legyen ugyanis $ABCD$ négyszögben $AB = CD$, továbbá AD középpontja: M és BC-é: N.

Ha az M-ben és N-ben AD-re, ill. BC-re emelt merőlegesek nem metszik egymást, akkor AD és BC párhuzamos.

Ha pedig metszik egymást, akkor a metszőpont lehet a négyszögön belül, vagy kívül.

a) Legyen az O metsző-pont [p. 48] az $ABCD$ négyszögön belül (20. ábra).

The mistake is brought into this proof at the time when we presuppose that the line extended from the point O to the point A is beyond the straight line EA.

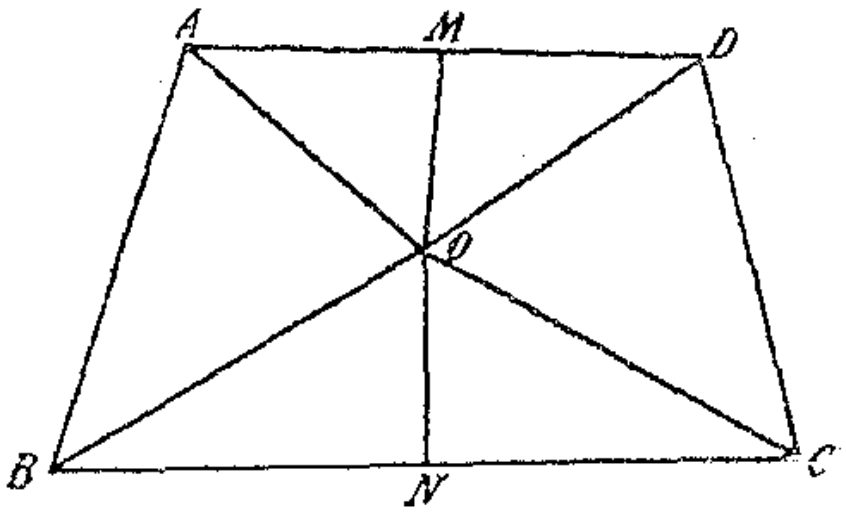
3. *If the two opposite edges of a quadrangle are equal, then the other two edges are parallel.*

In a quadrangle $ABCD$, let $AB = CD$, furthermore M is the middle point of AD, and N is the middle point of BC.

If a line through M perpendicular to AD and a line through N perpendicular to BC do not cut each other, then AD and BC are parallel.

If they cut each other, the intersection can be inside or outside the quadrangle.

a) Let O be a cutting point inside the quadrangle $ABCD$ (Figure 20).



20. ábra.

A szerkesztés alapján:	Based on the conditions:

$$OA = OD \text{ és (and) } OB = OC,$$

s így az OAB és ODC háromszögek congruensek, mert oldalaik rendre egyenlők, tehát	and the triangles OAB and ODC are congruent, because their edges are equal successively, thus

$$AOB\sphericalangle = DOC\sphericalangle.$$

De	And

$$AOM\sphericalangle = MOD\sphericalangle, \qquad BON\sphericalangle = NOC\sphericalangle;$$

ha tagonkint összeadjuk ezt a három egyenletet, akkor nyerjük, hogy	if we add these three equations together, then we get

$$AOB\sphericalangle + AOM\sphericalangle + BON\sphericalangle = DOC\sphericalangle + MOD\sphericalangle + CON\sphericalangle,$$

és innen

and here

$$MOA\sphericalangle + AOB\sphericalangle + BON\sphericalangle = 180^\circ;$$

M, O és N tehát egy egyenesbe esik s így $AD \parallel BC$

b) Ha O pont $AECD$-n kívül van (21. ábra), akkor ismét

M, O and N thus falls into one straight line, and $AD \parallel BC$

b) If the point O is outside $AECD$ (Figure 21), then again

$$AOB\sphericalangle = DOC\sphericalangle \text{ és (and) } BON\sphericalangle = NOC\sphericalangle,$$

tehát

thus

$$AOB\sphericalangle + BON\sphericalangle = DOC\sphericalangle + NOC\sphericalangle,$$

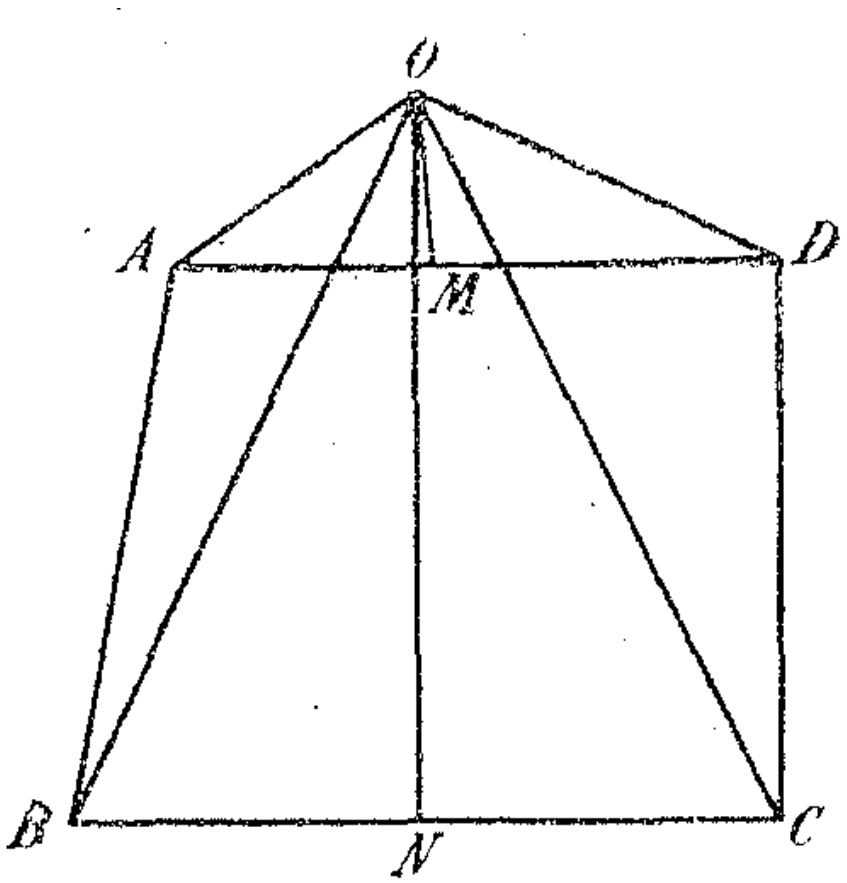
vagy

or

$$AON\sphericalangle = NOD\sphericalangle;$$

ON tehát AOD szög felezője s így összeesik OM-mel. AD és BC tehát ismét párhuzamos.

ON is therefore the bisector of the angle AOD, and crosses OM. AD and BC are thus parallel again.



21. ábra.

Ezen bizonyításban elkövetett hibára is rögtön rájövünk, ha meggondoljuk, hogy az O pont [p. 49] nem fog az AB, BC és CD egyenesek által meghatározott háromszögbe esni.

We recognize immediately the mistake committed in this proof if we consider that the point O will not fall into a particular triangle composed with the straight lines AB, BC and CD.

*Note of the translator:

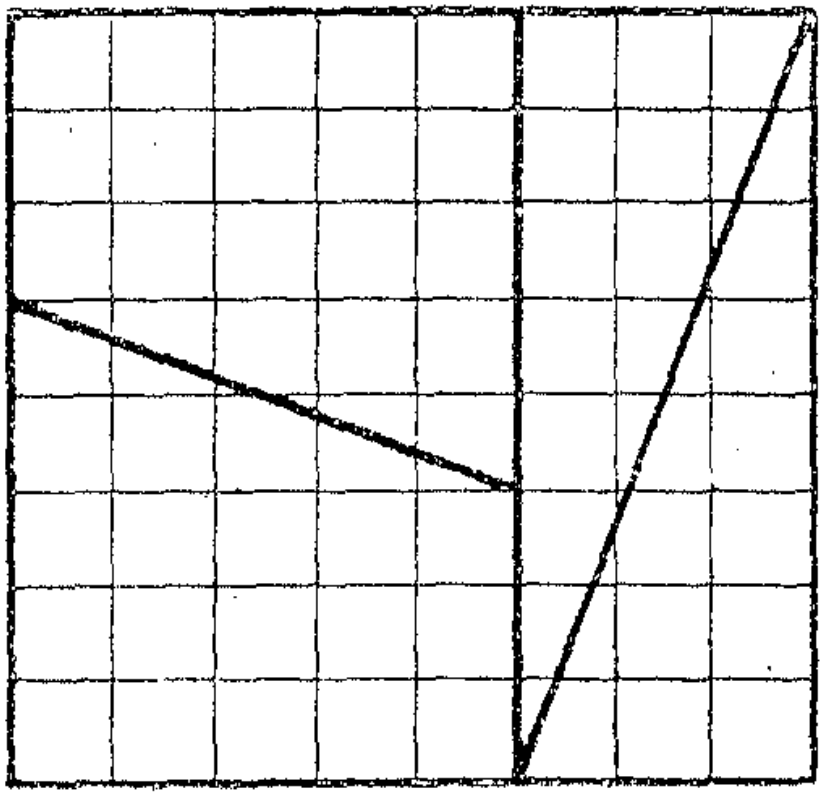
When the point O is outside of the large triangle composed with the lines AB, BC and CD, one of the triangles AOB and DOC is outside of the quadrangle $ABCD$. Therefore the equations above do not imply that the line ON is the bisector of the angle AOD.

4. Ebben az utolsó példában bebizonyítjuk még, hogy a sakktábla módjára 64 *mezőre osztott négyzetalakú papírlap felbontható négy oly részre, melyeket ismét egymás mellé helyezve, egy 65 ugyanakkora mezőt tartalmazó idomot nyerünk.*

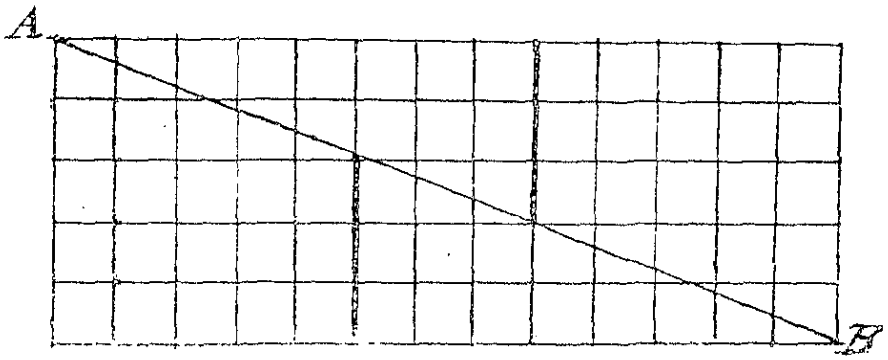
[p. 50]
Erre az eredményre jutunk, ha az eredeti 64 mezős négyzetet a 22. ábra vastagabb vonalai mentén négy darabra vágjuk és ezt a négy darabot a 23. ábrában látható módon ismét összerakjuk.

4. In this last example, we prove that *a sheet of paper divided into 64 square fields like a chessboard can be cut into four parts, with which we get a figure containing 65 the same size of field by rearranging the parts beside each other.*

We are brought to this result if we cut the original square with 64 field in Figure 22 into four pieces along the thicker lines, and we can see these four pieces put together again in the manner of Figure 23.



22. ábra.



23. ábra.

Az első ábrabeli idom területe $8 \times 8 = 64$ kis négyzetből áll, míg a 23. ábrában keletkező derékszögű négyszög területe $5 \times 13 = 65$ alapnégyzet, s így $64 = 65$.

The area of the form in the first figure consists of $8 \times 8 = 64$ small squares, while the area of the rectangular quadrangle occurring in Figure 23 consists of $5 \times 13 = 65$ squares, thus $64 = 65$.

Könnyen rájöhetünk, hogy a hiba úgy keletkezik, hogy az a négy csúcspont, mely a 23. ábrában látszólag az AB átlón fekszik, valóságban nem fekszik egy egyenesen, hanem ez a négy pont egy oly idomot határoz meg, melynek területe épen egy kis négyzet területével egyenlő.

We can recognize easily that the mistake comes from the following thing: in the form consisting of those four parts in Figure 23, AB apparently lies on a diagonal, but actually it does not lie on a straight line; these four points [A, B and two vertices of the triangles between A and B] compose a form, the area of which is exactly equal to the area of one small square.

Mindezen példákból csak az a tanulság, hogy pontosan kell rajzolni!

From all these examples, only the following moral instruction arises: it is necessary to draw precisely!

4.6 Síkidomok szétszedése és összeállítása: Decomposition and recomposition of plane figures

*Note of the translator:
As Kőnig mentioned in the part of references at the end of the second book of 1905 (see p. 271), most of the topics are taken from the books of Lucas (*Récréations mathématiques II*, 1883 [149]) and Schubert (*Mathematische Mussestunden III*, 1900 [189]). We discussed on the similar problems on p. 26.

[p. 51]
A geometriában két terület egyenlőségének bizonyításánál gyakran használjuk azt az eljárást, hogy a két idomot oly részekre bontjuk, melyeknek egyenlőségét már könnyebben bebizonyíthatjuk.

In the geometry, for the proof of the equality of two areas, we often use the following procedure: we split the two forms into parts for which we can prove the equality more easily.

Egyenlő lesz továbbá két terület akkor is, ha egyenlő területű idomok hozzátételével egyenlő területű idomokat nyerünk.

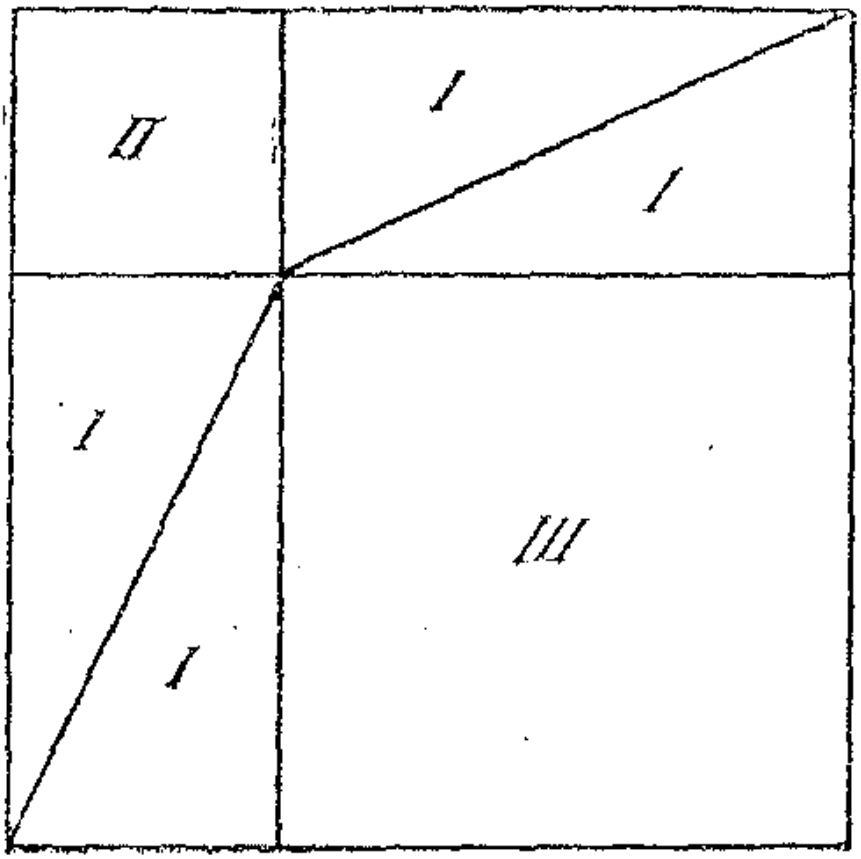
Furthermore, two areas will be equal also if we get the forms with combination of forms with equal area.

1. Ezen tétel segítségével bebizonyíthatjuk a geometria egyik alaptételét, a *Pythagorastételt* is.

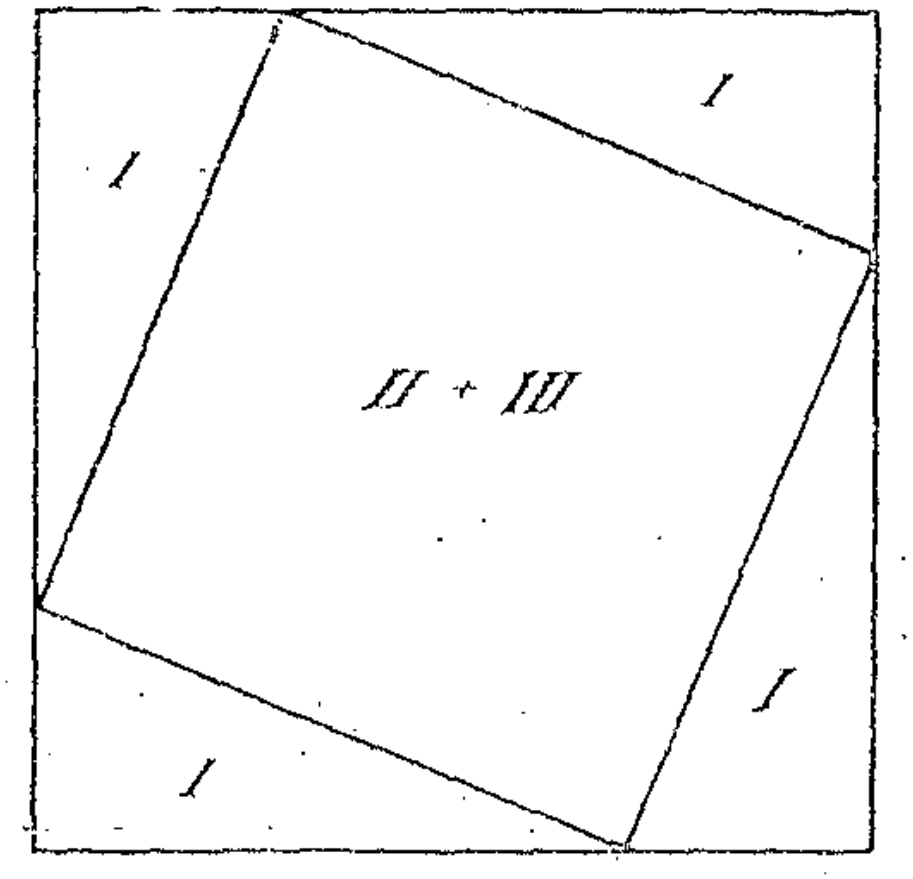
1. With the help of this theorem, we can prove also one of the fundamental theorems of the geometry, the *Pythagoras' theorem.*

Bontsuk fel ugyanis két congruens négyzet egyikét a 24. ábrában látható módon s a másikat úgy mint azt a 25. ábra mutatja.

Let us divide one of two congruent squares in the manner shown in Figure 24, and the other one in the manner shown in Figure 25.



24. ábra.



25. ábra.

Ha most a két idomból a négy 1-gyel jelzett háromszöget elveszszük, akkor egyenlő területű részek maradnak.

A 24. ábrában pedig marad az I-es derékszögű háromszög befogói fölé rajzolt két négyzet; a 25.-ben pedig ugyanezen derékszögű háromszög átfogója fölé rajzolt negyzet.

If we remove the marked four equal triangles from the two forms, then the parts with an equal area are left.

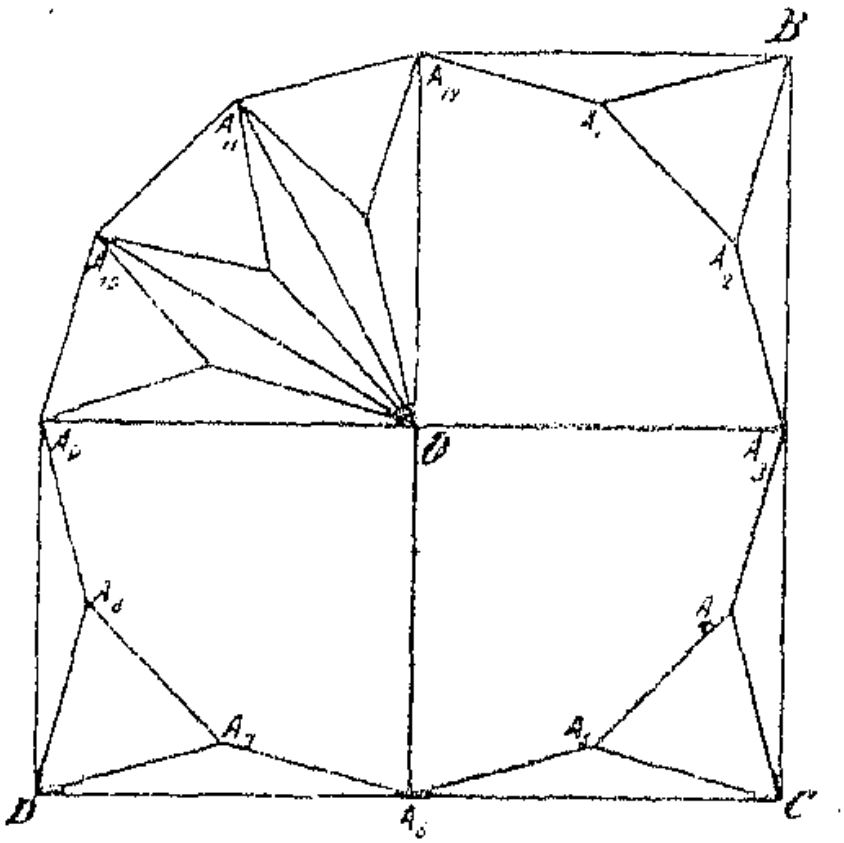
In Figure 24, two squares drawn adjacent to the legs of the right triangles I are left; on the other hand in Figure 25, a square drawn adjacent to the hypotenuse of right triangles are left.

Ez utóbbi tehát egyenlő az első kettő összegével.

2. Síkidomok szétszedésével és összeállításával bizonyítható azon ismeretes tétel is, mely szerint *az r-sugarú körbe irt szabályos 12-szög területe* $3r^2$ (26. ábra).

The area of the latter square is thus equal to the sum of the areas of the first two squares.

2. With decomposition and recomposition of plane figures, we can prove the known theorem, according to which *the area of regular dodecagon inscribed in the circle of radius r is* $3r^2$ (Figure 26).



26. ábra.

[p. 52]

Az $OA_{12}A_{11}$ és $BA_{12}A_1A_2A_3$ idomok ugyanis mint az az ábrában lévő felbontásból kitűnik, egyenlő területűek.

Mind a kettőt tehát 3-szor véve, ismét egyenlő területekhez jutunk, s így $OA_9A_{10}A_{11}A_{12}$ idom területe egyenlő a $BA_{12}A_1A_2A_3$, $CA_3A_4A_5A_6$ és $DA_6A_7A_8A_9$ ötszögek területének összegével s innen végre a szabályos 12-szög egyenlő a körülírt kör sugara fölé rajzolt három négyzet összegével.

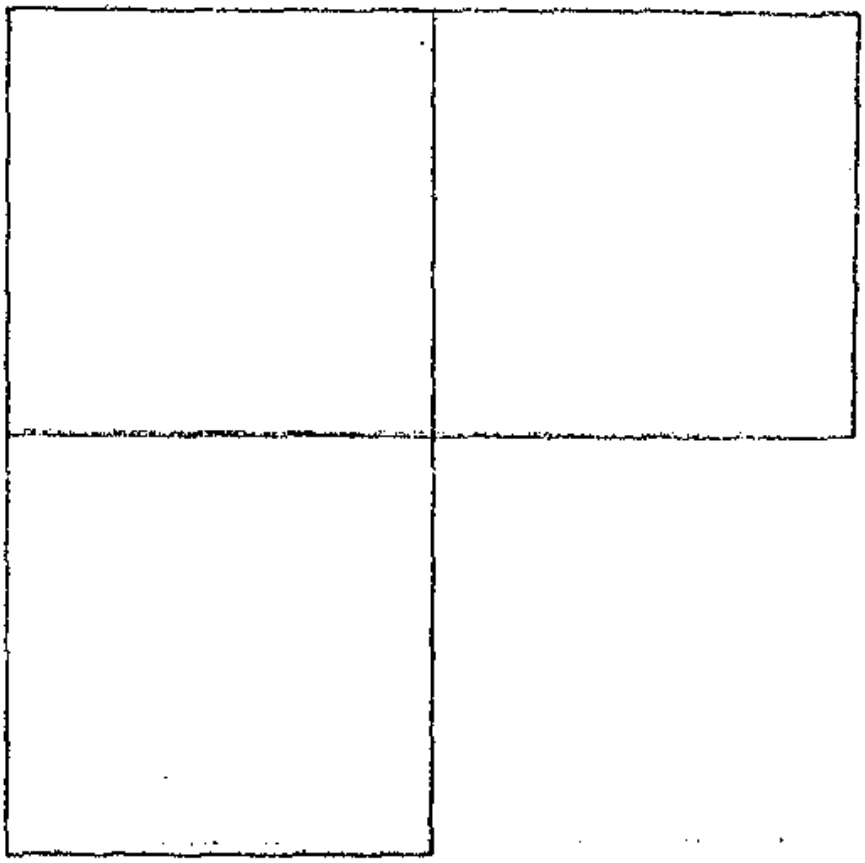
[p. 53]

3. *Hogyan lehet a három négyzet összetevéséből keletkező 27. ábrát négy congruens részre bontani?*

2. The forms $OA_{12}A_{11}$ and $BA_{12}A_1A_2A_3$ have equal areas as clarified by the decomposition in a figure.

We thus have both forms 3 times, then we get equal areas again: the area of the form $OA_9A_{10}A_{11}A_{12}$ is equal to the sum of the areas of the pentagons $BA_{12}A_1A_2A_3$, $CA_3A_4A_5A_6$ and $DA_6A_7A_8A_9$. Therefore finally the regular dodecagon inscribed in the circle is equal to the sum of three squares.

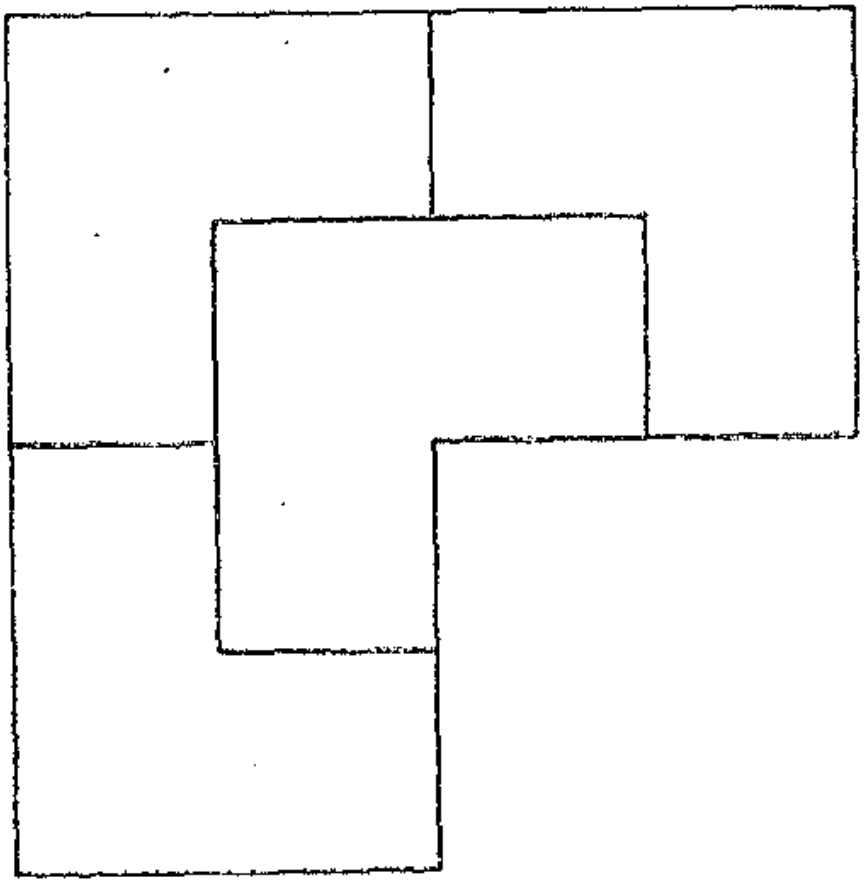
3. *How can the three squares composed as shown in Figure 27 be decomposed into four congruent parts?*



27. ábra.

A megoldást mutatja a 28. ábra.

Figure 28 shows the solution.



28. ábra.

4. *Vágjunk szét egy négyzetalakú papírlapot, húsz congruens háromszögre.*

A négyzet egyik csúcsát összekötjük a szembenfekvő oldal középpontjával s ezen egyenessel a szembenfekvő csúcson át [p. 54] párhuzamost vonunk.

4. *Let us cut a square-shaped sheet of paper into twenty congruent triangles.*

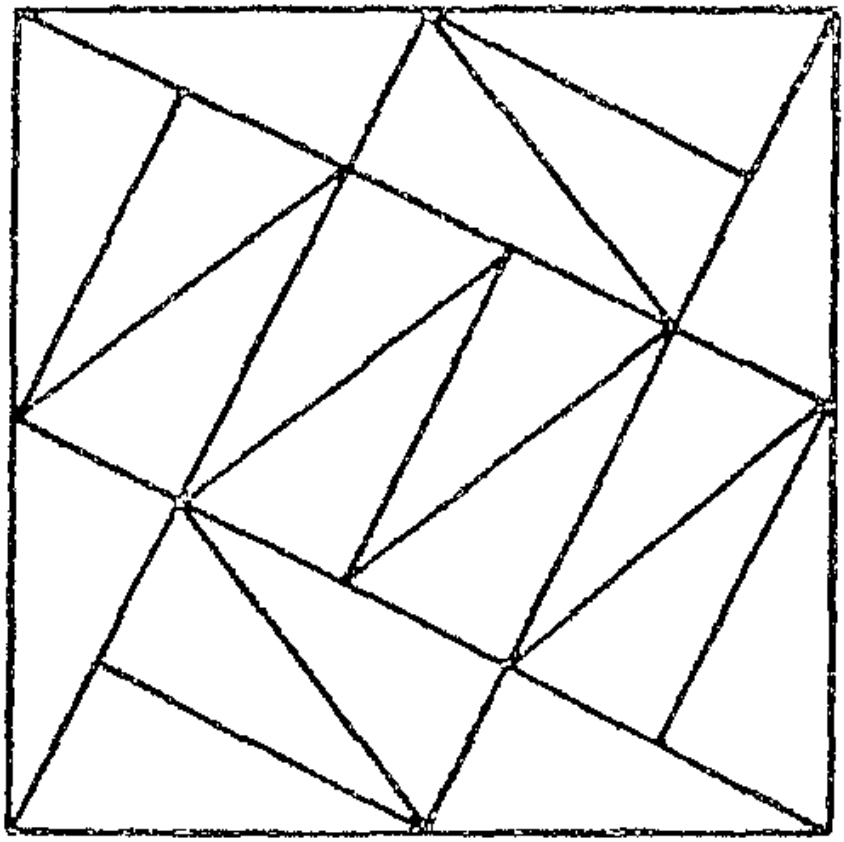
We draw a straight line from one of the vertices of the square to the middle point of the opposite edge, and then draw a line parallel to this straight line through the opposite vertex.

A másik két csúcsból pedig e két egyenesre merőlegest bocsátunk.

From the other two vertices, draw two straight lines perpendicular to the first two lines.

A további szerkesztést már világosan mutatja a 29. ábra.

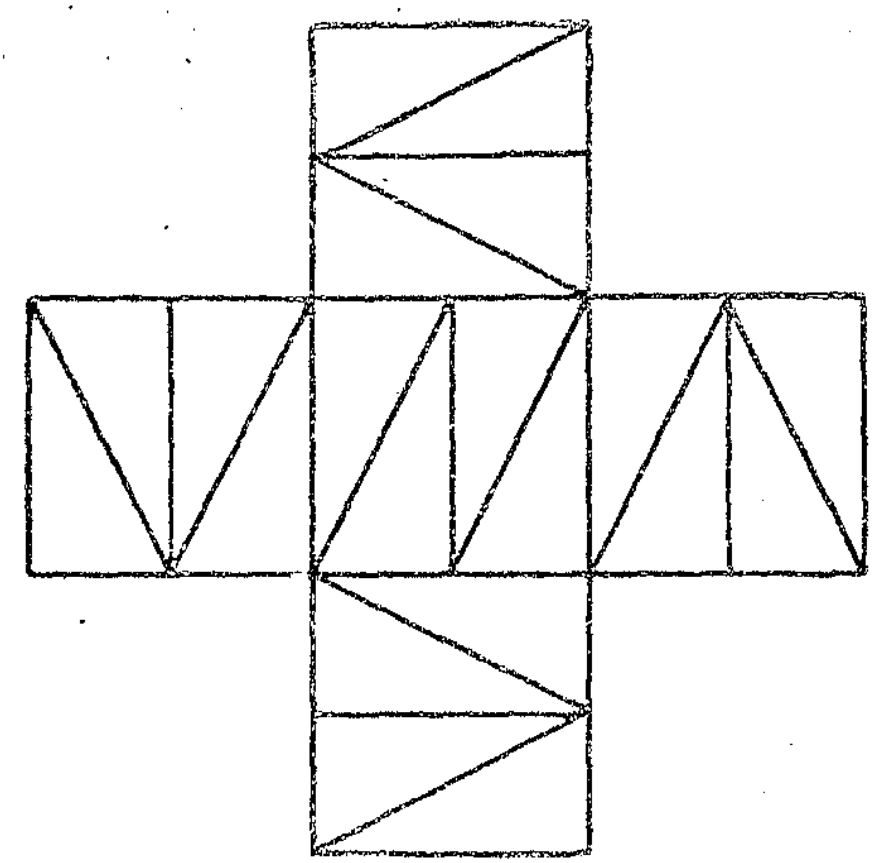
The additional construction is already shown clearly in Figure 29.



29. ábra.

Érdekes, hogy a húsz háromszögből egy keresztalakú idomot rakhatunk össze, mint azt a 30. ábra mutatja.

Interestingly, from the twenty triangles can be put together in a cross-shaped form as shown in Figure 30.



30. ábra.

Sokkal nehezebb már a következő feladatok megoldása:

[p. 55]
5. *Bontsunk fel egy négyzetet hét oly részre, hogy ezen részekből három congruens négyzetet alkothassunk.*

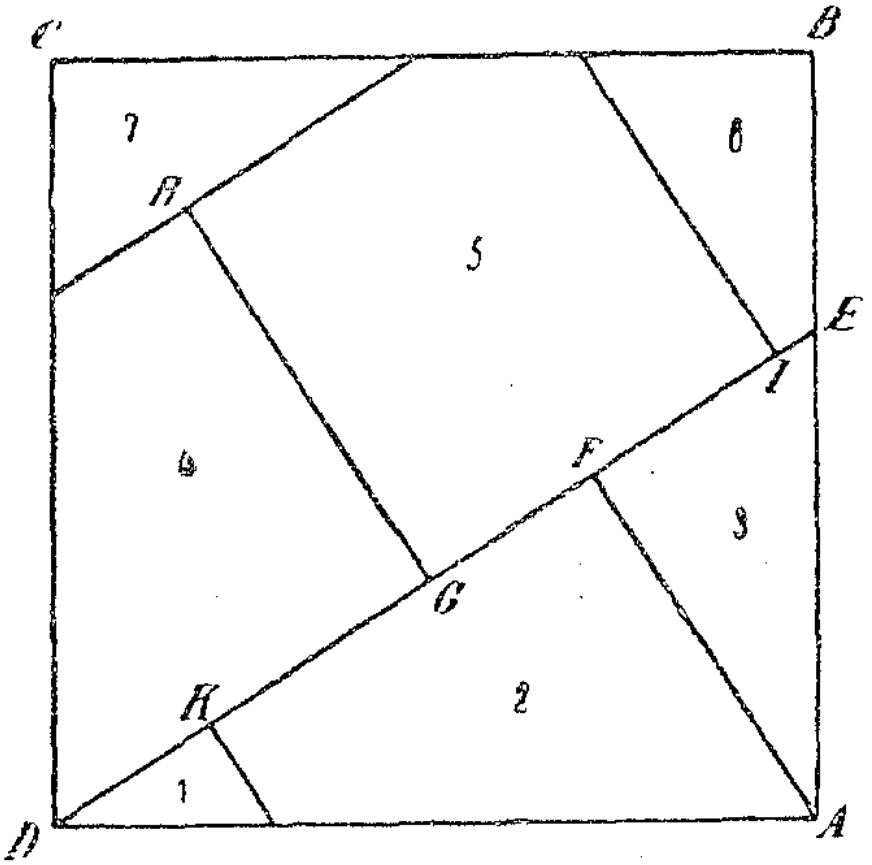
The solutions to the following problems are much more difficult:

5. *Let us decompose a square into seven parts so that we can form three congruent squares from these parts**.

*Note of the translator:
This problem and its solution are the same as those of LUCAS: *Récréations mathématiques* vol. 2 in 1883 [149], Chapter 5 "Les jeux de casse-tête (jigsaw puzzles)", the section of "Transformations d'un carré (Transformations of a square)", p. 145. LUCAS cited COATPONT's article "Sur un problème de M. Busschop" (1877) [34], in which the same problem and the same solution as LUCAS' were treated. COATPONT mentioned a BUSSCHOP's problem in the book of Eugène Charles CATALAN (1814–1894) *Théorèmes et problèmes de géométrie élémentaire* (1872) [29], in which just the same problem and the same solution as COATPONT's are presented on p. 194. We cannot follow more ancient appearance of this problem, but a similar problem was treated already by Abū'l-Wafā' al-Būzjānī in the 10th century (see my discussion on p. 26.)

[p. 56]
Az $ABCD$ négyzet (31. ábra) AB oldalára felmérjük a négyzet átlójának felével egyenlő AE darabot és DE-re az AF és CG merőlegeseket bocsátjuk, a H, I és K pontokat pedig megkapjuk, ha a GH, GI és FK egyeneseket AF-fel egyenlőknek veszszük fel.

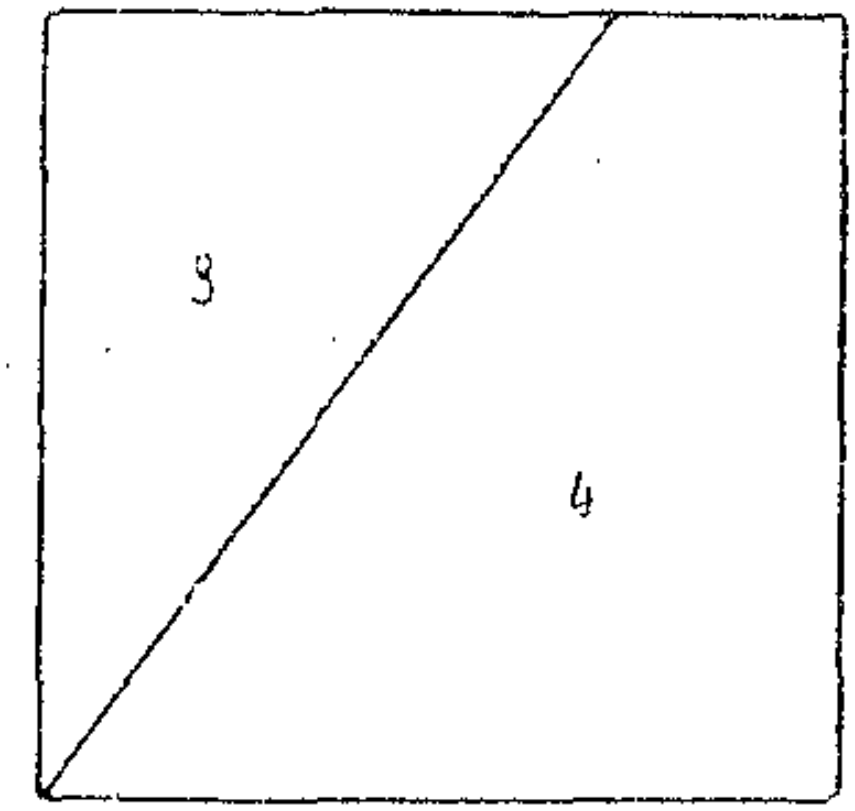
On the edge AB of the square $ABCD$ (Figure 31), we measure the line segment AE equal to the half of diagonal of the square, draw AF and CG perpendicular to DE, and we fix the points H, I and K so as [the length of] straight lines GH, GI and FK to be equal to AF.



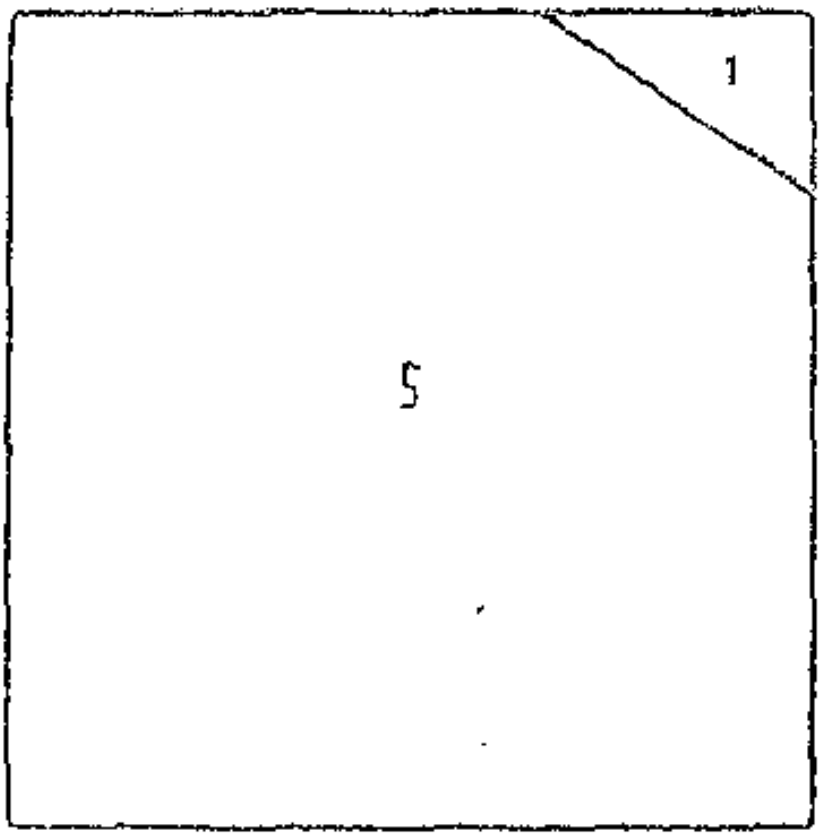
31. ábra.

Végül a K-n, G-n és I-n át AF-fel párhuzamosakat, H-n át pedig rája merőleges egyenest rajzolunk. Az így keletkező hét részből a 32., 33., és 34. ábrában látható módon három congruens négyzetet állíthatunk össze.

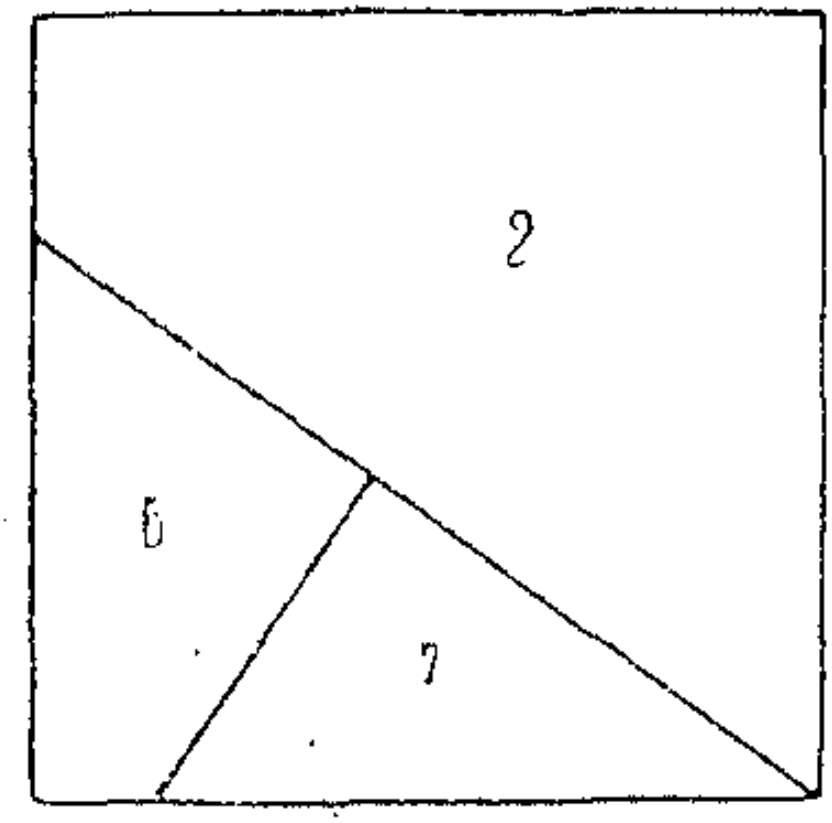
Finally we draw straight lines parallel to AF crossing the K, G and I, and a straight line perpendicular to it crossing H. Then we can construct three congruent squares from the resulting seven parts in the manner shown on Figures 32, 33, and 34.



32. ábra.



33. ábra.



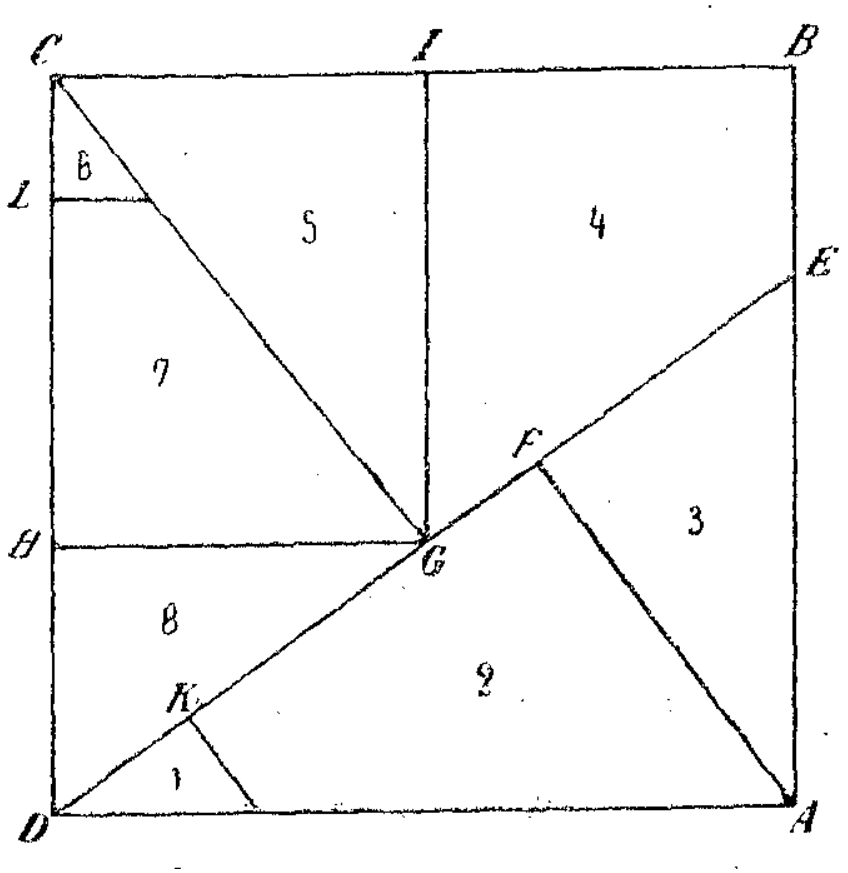
34. ábra.

6. *Bontsunk fel egy négyzetet úgy nyolcz részre, hogy e részekből két oly négyzetet állíthassunk össze, hogy a nagyobb a kisebbik kétszeresével legyen egyenlő.*

6. *Divide a square into eight parts. From these parts, let us compose two squares so that the larger square is equal to twice as large as the smaller one.*

[p. 57]
Az E, F, G és K pontokat (35. ábra) az előbbi szerkesztés segítségével határozzuk meg, majd G-n át megrajzoljuk a négyzet oldalaival párhuzamos GH, GI egyeneseket és H-tól C felé felmérjük a $HL = HG$ távolságot.

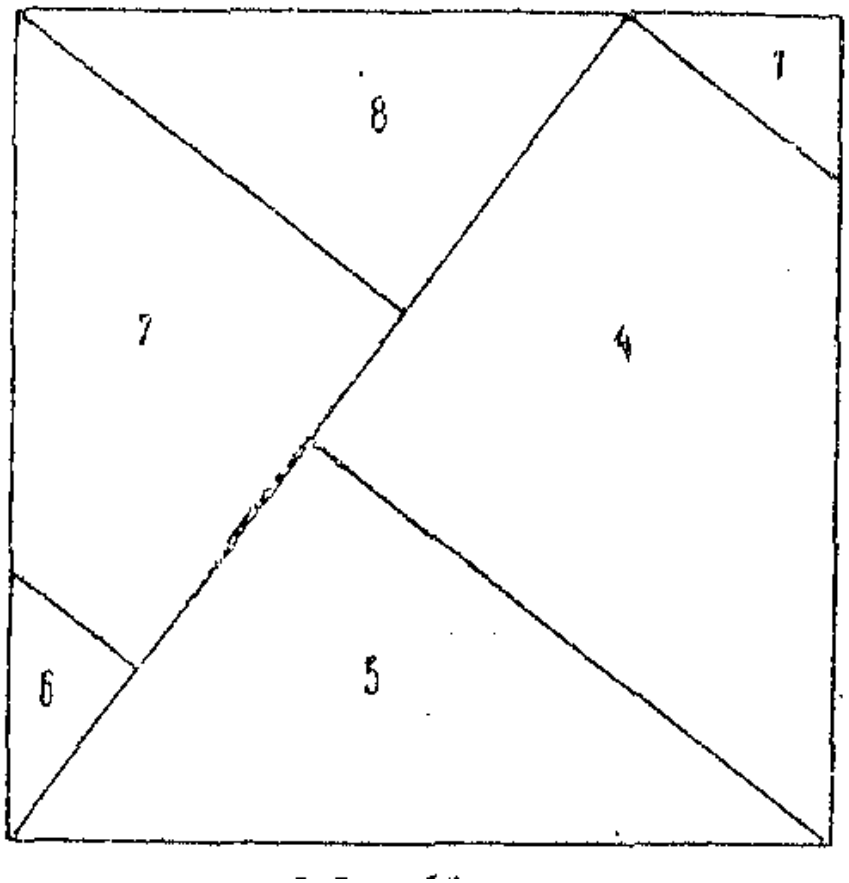
We define the points E, F, G and K (Figure 35) with the help of the first construction, then crossing G we draw straight lines GH and GI which are parallel to the edges of the square, and we measure the distance $HL = HG$ from H towards C.



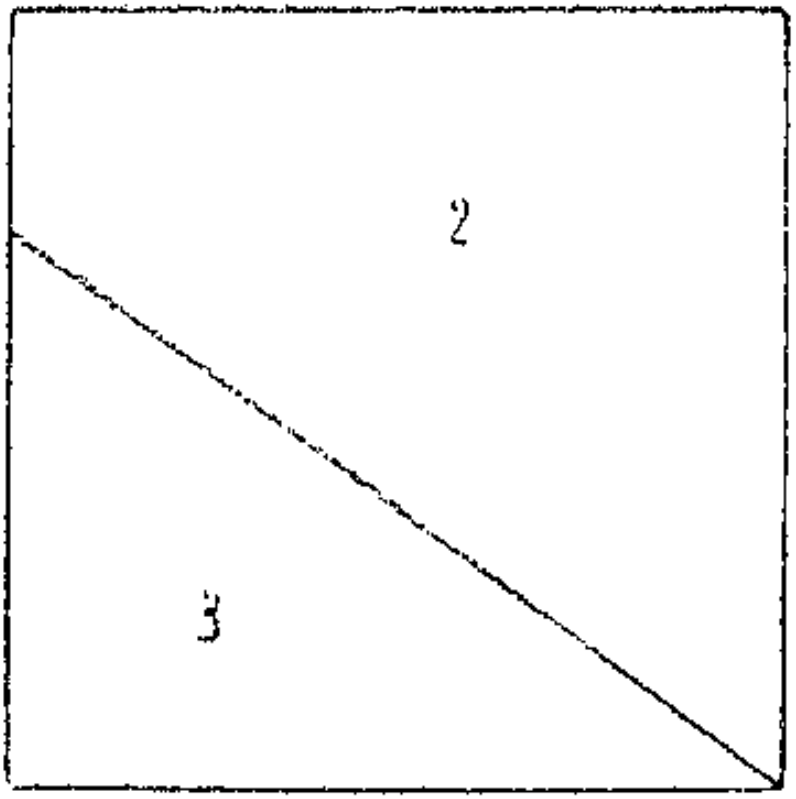
35. ábra.

[p. 58]
A keletkező nyolcz részt a 36. és 37. ábrában látható módon elhelyezve, valóban két oly négyzetet nyerünk, hogy az egyik a másiknak a kétszerese.

The resulting arrangement of the eight parts is shown in Figures 36 and 37. We get indeed two squares, one of which is twice as large as the other one.



36. ábra.



37. ábra.

Végül még két ilyen feladatot említünk, de ezekben a szerkesztés olyan bonyolódott és hoszszas, hogy a szerkesztés leírása nélkül, csakis a felbontást és összeállítást mutató ábrákat mutatjuk be.

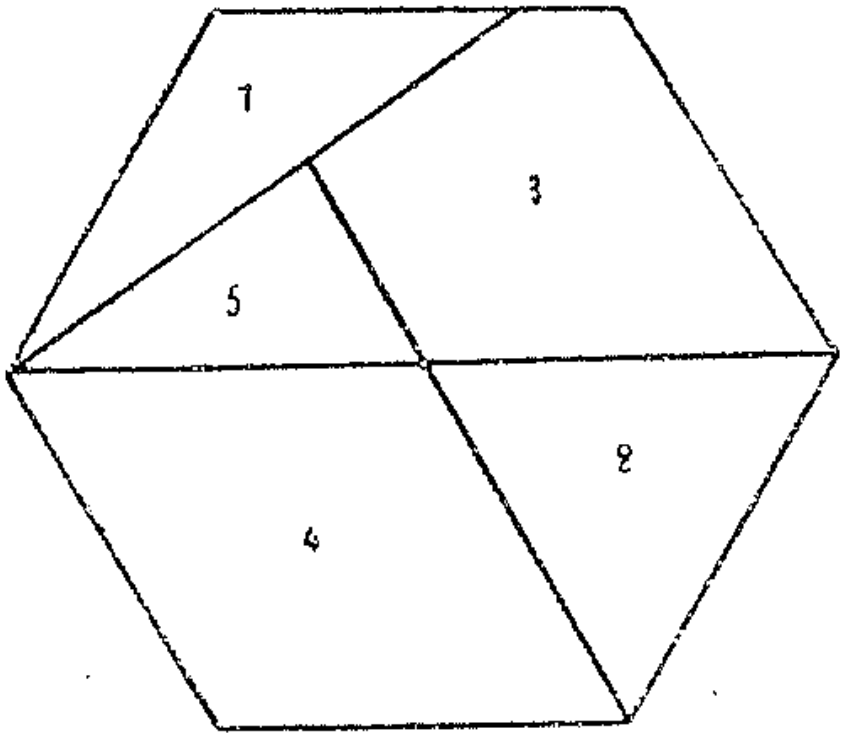
[p. 59]
Ez a két feladat a következő:

We finally mention two problems, but these construction was getting so complicated and lengthy that we present only figures indicating the decomposition and recomposition without description on the construction.

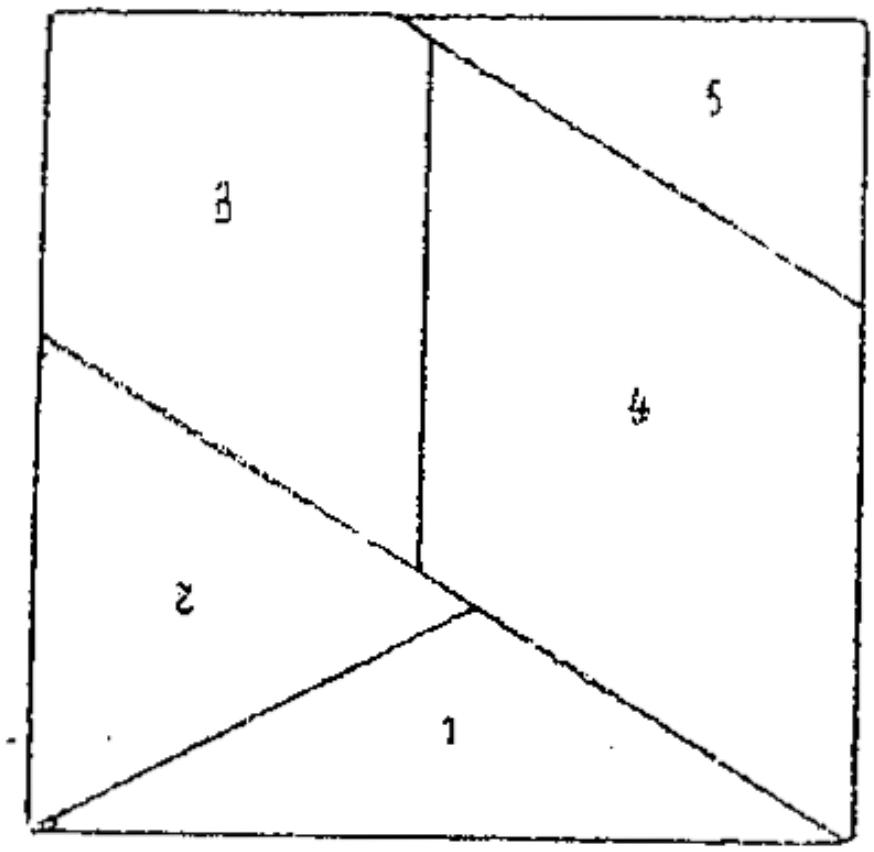
These two problems are as follows:

7. *Bontsunk fel egy szabályos hatszöget öt oly részre, hogy ezen részekből egy négyzetet rakhassunk össze.* A megoldást mutatja a 38. és 39. ábra.

7. *Decompose a regular hexagon into five parts so that a square can be composed together from these parts.* Shows the solution in Figures 38 and 39.



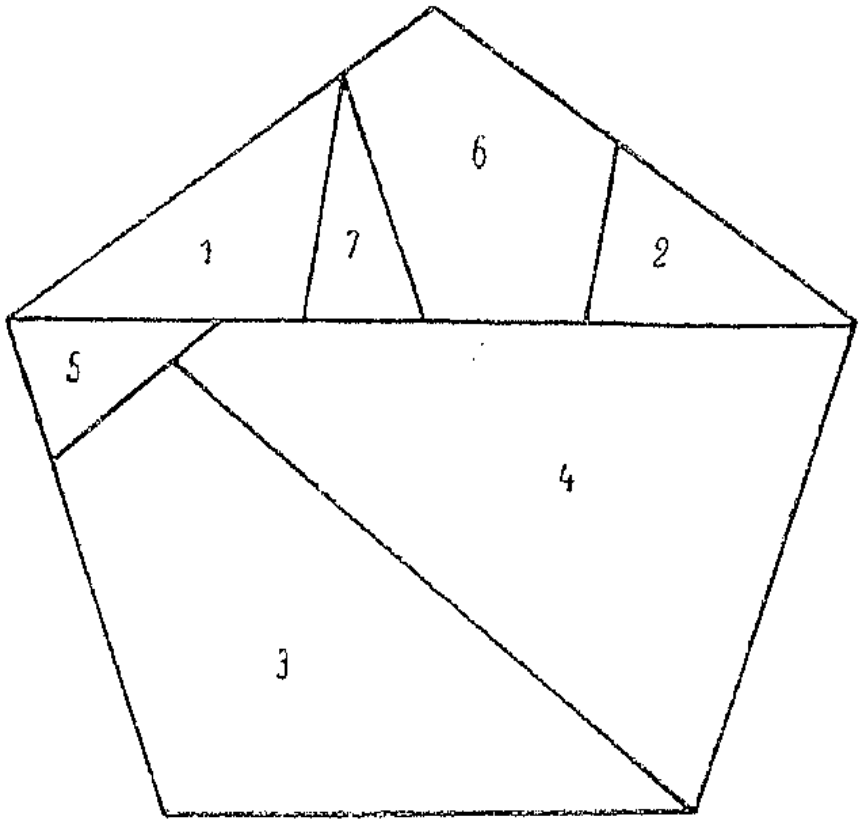
38. ábra.



39. ábra.

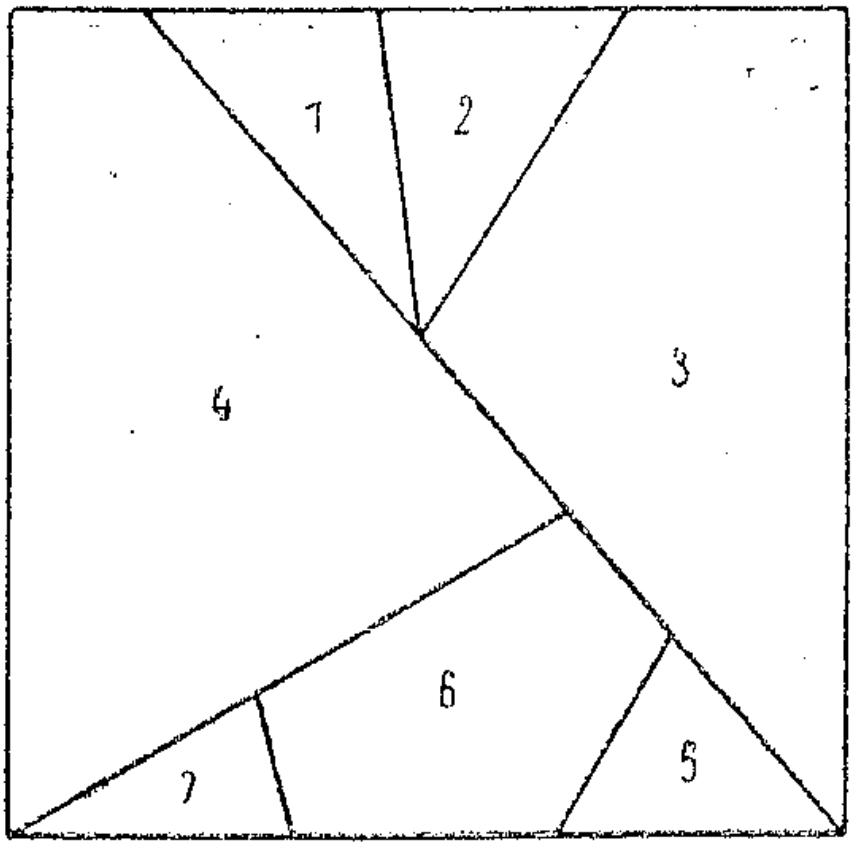
8. *Bontsunk fel ugyanezen feltétellel egy szabályos ötszöget hét részre (40. és 41. ábra).*

8. *Decompose a regular pentagon into seven parts under the same conditions as above (Figures 40 and 41).*



40. ábra.

[p. 60]



41. ábra.

Vége az első sorozatnak. | The end of the first series.

4.7 Felhasznált munkák: List of works used

[p. 61]
Ahrens: Mathematische Unterhaltungen und Spiele. (Leipzig, Teubner, 1901.)
Bachet: Problèmes plaisants et délectables, qui se font par les nombres. (Paris, Gauthier-Villars, 1874.)
Ball: Récréations et problèmes mathématiques. (Paris, Hermann, 1898.)
Fourrey: Récréations arithmétiques. (Paris, Nony, 1901.)
Lucas: Récréations mathématiques, I–IV. (Paris, Gauthier-Villars, 1895.)
Lucas: L'arithmétique amusante. (Paris, Gauthier-Villars, 1895.)
(*Ozanam*:)* Récréations mathématiques. (Rouen, Osmont, 1629.)
Rebière: Mathématiques et mathématiciens. (Paris, Nony, 1898.)
Schubert: Mathematische Mussestunden, I–III. (Leipzig, Göschen, 1900.)

* Névtelenűl jelent meg.

* Anonymously appeared.

Továbbá a „Középiskolai Mathematikai Lapok“, „Mathesis“ (Paris, Gauthier-Villars, 1901.) című folyóiratok egyes kisebb közleményei.

Furthermore, individual smaller reports in journals called *Középiskolai Mathematikai Lapok (High-school mathematical reviews)*, *Mathesis* (Paris, Gauthier-Villars, 1901.)

*Note of the translator:
Ahrens' book of 1901 [1] was later augmented and divided into two volumes [3, 4]. Kőnig later referred to these augmented editions in his treatise of 1936 [121].
Bachet's first edition was published already in 1612 [7]. Kőnig here referred to the third edition revised, simplified and augmented by A. Labosne [9]. See my note on p. 106.
Ball's first edition in English was published in 1892 [12]. Kőnig referred to the French translations by Fitz-Patrick in his books of mathematical recreations [86, 87] as well as in his treatise of 1936 [121]. See also p. 28 for this topic.
Fourrey's first edition was published in 1899 [65]. Kőnig referred to the second edition.
Lucas' *Récréations mathématiques* was first published in 1882 (vol. 1) [148], 1883 (vol. 2) [149], 1893 (vol. 3) [151] and 1894 (vol. 4) [152]. Kőnig did not refer to another important book of Lucas *Théorie des nombres*, which was published in 1891 [150], although this book includes many recreational problems. I suppose that, when Kőnig published his books on mathematical recreations in 1902 and 1905, he did not know this book of Lucas. Kőnig later referred to this book of Lucas in his treatise *Theorie der endlichen und unendlichen Graphen* (1936 [121]) for a recreational problem on the munimum number of strokes to trace a graph.
The anonymously published *Récréations mathématiques : Composées, de plusieurs problemes, plaisans & facetieux, d'arithmetique, geometrie, astrologie, optique, perspective, mechanique, chymie, & d'autres rares & curieux secrets... Premiere et seconde partie. La troisiesme partie, contient un recueil de plusieurs gentilles & recreatives inventions de feux d'artifice : la maniere de faire toutes sortes de fuzées, simples & composées : le tout représenté par figures* [5] in 1629 was not written by Ozanam (1640–1718) because of the year. However, Ozanam published *Récreations mathématiques et physiques* in 1694 [165]. On p. 289 and p. 297 of the present dissertation, I suppose that Kőnig referred to Ozanam's book of 1964, not to the book of 1629.
Rebière's book was first published in 1889 [177]. Kőnig referred to the third edition [178].
Schubert's book was first published in 1897 [187]. In the edition of 1900, it was much augmented and divided into three volumes [189]. According to Singmaster [240], it appeared also in an abbreviated one-volume form in 1904.

Chapter 5

Mathematikai Mulatsagok 2: Mathematical recreations 2

Here is the original text with my translation of *Mathematikai Mulatsagok, második sorozat* (Mathematical recreations, second series) by Dénes KŐNIG, 1905.

5.1 A mathematikai valószínűségről: About the mathematical probability

[p. 3]

Valamely esemény bekövetkezésének *mathematikai valószínűségén* az illető esemény bekövetkezésére kedvező esetek és az összes lehetséges esetek számának a viszonyát értik. E szerint pld. annak a valószínűsége, hogy az 1–9 számokkal megjelölt teljesen egyforma kilenc golyó közül épen a 4-eset válasszuk, $\frac{1}{9}$, mert az összes lehetséges kilenc eset közül a kérdéses esemény bekövetkezésére csak egy kedvező.

The relation between the number of favourable cases and the number of all the possible cases is understood as *mathematical probability* of the occurrence of one event among the related events. According to this, for example, from nine bullets which are totally alike and marked with the numbers 1–9, the probability of choosing the bullet 4 is just $\frac{1}{9}$, because only one case is favourable among all nine possible cases of occurrences of related events.

A mathematikai valószínűség értelmezéséből következik, hogy ha két, ugyanazon eseménycsoport alá tartozó és egymástól független esemény egyikének v_1, másikának v_2 a valószínűsége, akkor annak a valószínűsége, hogy a kettő egyike bekövetkezzék: $v_1 + v_2$, míg annak a valószínűsége, hogy mindkettő (egyidőben, illetve egymásután) bekövetkezzék: $v_1 v_2$.

The interpretation of the mathematical probability brings the following. Suppose that two events are in one same group of events, and they are independent from each other. If one of the events has v_1 of probability, and the other one has v_2 of probability, then the probability that one of them occurs is $v_1 + v_2$, while the probability that both of them (simultaneously, or precisely successively) occur is $v_1 v_2$.

Hogy pld., a fenti példára térve vissza, vagy a 4-es, vagy 6-os golyót húzzuk ki egy húzásra, annak valószínűségét $\frac{2}{9}$ fejezi ki, míg annak valószínűsége, hogy először 4-est és azután ezt visszatéve, a 6-ost húzzuk ki, csak $\frac{1}{81}$.

For example, returning to the example above, the probability that we draw the bullet 4 or 6 in one draw is $\frac{2}{9}$. On the other hand, the probability that we draw the bullet 4 first and put this back, and afterwards we draw the bullet 6 is only $\frac{1}{81}$.

[p. 4]
A valószínűségszámítás különösen hazárdjátékok tétjeinek megállapításánál alkalmazható. Ha annak valószínűsége, hogy A nyer v_1, míg azé, hogy B nyer v_2, akkor a játék az esetben lesz csak igazságos, azaz csak az esetben nem lesz egyiknek sem fölénye a másik fölött, ha A tétje (t_1), úgy aránylik B-éhez (t_2), mint $v_1 : v_2$. Tegyük fel ugyanis, hogy az összes n eshetőség közül A csak a, B pedig b esetben nyer, úgy, hogy

The calculation of probability is particularly applicable to the statement of stakes of gamblings. Suppose that the probability of winning of A is v_1 and the probability of winning of B is v_2. Then the game will be fair, that is, one of them will not have superiority to the other nor vice versa, only if the proportion of the stake of A (t_1) to the stake of B (t_2) is $v_1 : v_2$. Suppose that, among all n possibilities, A wins only a cases, and B wins only b cases as follows:

$$v_1 = \frac{a}{n}, \quad v_2 = \frac{b}{n}; \tag{1}$$

n játék után tehát a legvalószínűbb, hogy A elnyerte B-nek a-számú tétjét, azaz nyert at_2-t, míg B nyert A-tól bt_1-et. A játék kiegyenlítődik tehát, ha

Then after n games, it is the most probable that A won a times stakes, that is, at_2 from B, while B won bt_1 from A. Then the game is a tie if

$$at_2 = bt_1, \quad \text{azaz (that is)} t_1 : t_2 = a : b$$

(1) alapján e feltétel, mint előre kimondottuk, így is írható:

Using the equation (1) mentioned above, this condition can be written also as follows:

$$t_1 : t_2 = v_1 : v_2$$

Ha a játék hirtelenül abbamarad, akkor is a valószínűségszámítás szabja meg a betétnek a játszók között való igazságos elosztását.

If the game is interrupted, then a calculation of probability defines the fair distribution of stake between the players.

A betét t. i. nem mindig egyenletesen osztandó ki a játszók között, mert természetes, hogy az, aki a nyeréshez már közelebb jutott, nagyobb összeget követelhet, mint az, ki az eddigi játék alatt csak kevesebbel jutott előre.

In fact, the stake is not always equally distributed between the players, because it is natural that, during the game is played, the person who is already near the winning prize can require a larger amount than the person who is still far from the winning prize.

A és B pld. megállapodik, hogy az nyeri mindkettőjük betétjét, aki előbb nyer öt játékot. Nyerési esélyeik egyformák lévén, tétjük is ugyanaz. Miután A négy, B pedig három játékot nyert, a játék abbamarad.

For example, A and B agree that if any of them wins five games first, the one wins the stake. Their winning chances are equal, so their stakes are equal. After that, A won four games and B won three games, then the game finished.

Kérdés, hogy betétjük milyen arányban osztandó fel közöttük. Hogy 4 : 3, A és B nyert játékainak a viszonya, általában nem a helyes arányt adja, kitűnik onnan is, hogy ez nem lehet független a betét elnyeréséhez szükséges nyerendő játékok számától.

Here is a question: in which proportion should their stakes be shared between them? Their games have the relation of 4 : 3 between the games A won and the games B won. Generally, it does not give a correct proportion*. It becomes clear also from here that this proportion can not be independent from the number of winning games which is necessary for the gain of stake.

*Note of the translator:
The "correct proportion" signifies the proportion reasonable from their equal winning chances in the fifth game.

Világos ugyanis, hogy sokkal kisebb volna az A és B nyerési [p. 5] esélye közötti különbség, ha nem 5, hanem például 25 nyert játék volna szükséges a tétek elnyeréséhez. Hogy a helyes arányhoz jussunk, azt kell tekinteni, hogy miképpen folytatódhatott volna a játék.

In other words, it is clear that the difference between the winning chance of A and that of B would be much smaller if, for example, winning not 5 but 25 games were necessary for the gain of stakes. To ensure the correct proportion, we have to consider how the game could have been continued.

Két játék mindenesetre eldönti a tétek sorsát. Ha u. i. csak egyet a kettő közül A nyer, már megvan az öt nyert játéka; ha mindkettőt elveszíti, akkor B nyerte el a betétet. Tegyük fel egyöntetűség kedvéért, hogy mind a két játékot okvetlenül kijátsszák, még ha A nyeri is az elsőt, és így a második játék már nem gyakorolhat befolyást. Négy eset lehetséges, melyek közül egyik sem valószínűbb, mint a másik:

Two games determine whatever the fate of the stakes. In other words, if A wins only one of two games, then A won five games; if A loses both of two games, then B won the stake. Let us assume for uniformity that all two games are surely finished, and that the second game can not be influenced by the first game even if A wins the first game. Four events are possible, and none of them is more probable than the others:

1. Az első játékot megnyeri A, a másodikat A
2. Az első játékot megnyeri A, a másodikat B
3. Az első játékot megnyeri B, a másodikat A
4. Az első játékot megnyeri B, a másodikat B

1. A wins the first game, A wins the second game.
2. A wins the first game, B wins the second game.
3. B wins the first game, A wins the second game.
4. B wins the first game, B wins the second game.

Az 1., 2., 3., esetben A a győztes és csak a 4.-ben húzza ki B a tételeket. Annak a valószínűsége tehát, hogy A, illetve B győzzön $\frac{3}{4}$ és $\frac{1}{4}$. A betét is tehát 3 : 1 arányban osztandó ki, A kapván a nagyobb részt.

A is the winner in the cases 1, 2, 3, and B draws the stakes only in the case 4. Then the probability of winning of A is $\frac{3}{4}$, and that of B is $\frac{1}{4}$. Therefore the stake is distributed in the proportion of 3 : 1, and A receives the larger part.

Hasonlóképpen nyerhető az az eredmény is, hogy három játszó esetében, ha 4, 3, illetve 2 nyert játékaiknak a száma

Similarly, in the case of three players, the result can be as follows: if one of the players won 4 games, another won 3 games, and the other won 2 games, then the probability of winning 5 games of the first, the second or the third player is:

$$\frac{19}{27}, \frac{6}{27}, \frac{2}{27}$$

fejezi ki annak valószínűségét, hogy az első, második vagy harmadik játékos jut előbb 5 nyert játékhoz. A játék abbahagyása esetén tehát 19 : 6 : 2 arányban osztandó el az együttes betét, ha megállapodásuk szerint öt nyert játék lett volna szükséges annak elnyeréséhez.

for each. In the case of interrupting the game, the common stake is to be divided in the proportion 19 : 6 : 2 if winning of five games would have been necessary for their getting of stakes according to their agreement.

Nem kell elfelejteni, hogy a nyert eredmények csak akkor igazságosak, ha valamennyi játékos ugyanazon valószínűséggel nyeri az egyes játszmákat, azaz ha csak a sors és nem az esetleges tehetség vagy [p. 6] gyakorlottság dönti el azokat.

We should not forget that the won results are correct only if every player wins each game in an equal probability, that is, only if neither the potential talent nor the practice but only the luck decides them.

Ellenkező esetben is végezhetők ezek a meggondolások, csak a játékosok nyerési valószínűségének viszonyát kell számokkal megadnunk. Ez esetben a számítások sokkal bonyolultabbak lesznek, s így ezekkel nem is foglalkozunk, csak egy eredményt említünk.

These considerations can be made also in the opposite case. We only have to consider the numbers related to the probability of winning of players. In this case, the calculations will be much more complicated, so we do not deal with them, and we mention only one result.

A kétszer olyan jól játszik, mint B, azaz annak valószínűsége, hogy A nyer egy játékot: $2/3$, míg B csak $1/3$ valószínűséggel nyer. Megállapodnak, hogy az a nyertes, aki előbb nyer meg hat játékot. Az első hat játék közül A 4-et, B pedig 2-t nyer. Ha a játék a hatodik játszma után abbamarad, a betét $232 : 11$ arányban osztandó fel köztük, természetesen A-nak jutván a nagyobbik rész.

A plays two times better than B, that is, the probability of winning of A is $\frac{2}{3}$, while B wins with only $\frac{1}{3}$ probability. The final winner is the one who wins six games first. Among the first six games, A wins 4 games, and B wins 2 games. If the game is interrupted after the sixth game, the stake is to be shared bewteen them in the proportion $232 : 11^*$, of which the bigger part is naturally left for A.

*Note of the translator:
If they continue the game, the cases that A wins and the cases that B wins are as follows:

final winner \ game	7th	8th	9th	10th	11th
A	A	A	-	-	-
	A	B	A	-	-
	A	B	B	A	-
	A	B	B	B	A
	B	A	A	-	-
	B	A	B	A	-
	B	A	B	B	A
	B	B	A	A	-
	B	B	A	B	A
	B	B	B	A	A
B	A	B	B	B	B
	B	A	B	B	B
	B	B	A	B	B
	B	B	B	A	B
	B	B	B	B	-

Considering the winning pribabilities $\frac{2}{3}$ of A and $\frac{1}{3}$ of B, the proportion of their probabilities is:

$$\begin{aligned} v_A : v_B &= (2\cdot 2\cdot 3\cdot 3\cdot 3 + \cdots + 1\cdot 1\cdot 1\cdot 2\cdot 2) : (2\cdot 1\cdot 1\cdot 1\cdot 1 + \cdots + 1\cdot 1\cdot 1\cdot 1\cdot 3) \\ &= 232 : 11 \end{aligned}$$

A mathematika egy ágában sem olyan nehéz talán a tévedések kikerülése, mint a valószínűségszámításban. Meggyőződhetünk erről a következő, különben igen egyszerűnek látszó probléma tárgyalásánál.

Mi annak a valószínűsége, hogy feldobván három pénzdarabot, mind a három ugyanazon (fej- vagy írás-) oldalra essék?

Among the branches of mathematics, It may be the most difficult to avoid mistakes in the calculation of probability. We can make sure of this below, where the description of problem seems very simple.

In tossing three coins, what is the probability that all the three coins fall showing the same (obverse or reverse) side?

Minthogy csak fej- és írásoldal van, a három darab közül kettő mindig ugyanazon oldalra esik. Annak a valószínűsége, hogy a harmadik egy bizonyos oldalra essék: $\frac{1}{2}$, s így ez a valószínűsége annak is, hogy a harmadik pénzdarab ugyanazon oldalra essék, amelyre az első kettő esett.

There are only one obverse side and one reverse side, therefore two of three coins fall showing the same side without fail. The probability that the third coin shows a certain side is $\frac{1}{2}$. Then it is also the probability that the third coin falls showing the same side as the first two coins.

Tehát $\frac{1}{2}$ volna a kérdéses valószínűség. Másrészt annak a valószínűsége, hogy az első pénzdarab fej oldalra essék, $\frac{1}{2}$. Ugyancsak ez a valószínűsége annak is, hogy a második, valamint annak is, hogy a harmadik fej oldalra essék.

Therefore $\frac{1}{2}$ would be the probability asked for. On the other hand, the probability that the first coin falls showing the obverse side is $\frac{1}{2}$. It is the same as the probability that the second coin falls showing the obverse side, as well as the probability that the third coin falls showing the obverse side.

Hogy tehát mind a három fej oldalra essék, annak $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$ a valószínűsége. Természetesen annak is $\frac{1}{8}$ a valószínűsége, hogy mind a három írásoldalra essék. Így tehát annak valószínűsége, hogy mind a három ugyanazon oldalra essék: $\frac{1}{8} + \frac{1}{8} = \frac{1}{4}$.

Therefore the probability that all three coins fall showing the same side is $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$. Naturally, $\frac{1}{8}$ is the probability that all three coins show the reverse side. So the probability that all three coins show the same side is $\frac{1}{8} + \frac{1}{8} = \frac{1}{4}$.

[p. 7]
Melyik most már a helyes eredmény: $\frac{1}{2}$ vagy $\frac{1}{4}$? Tényleges kísérletezéssel is rájuthatunk, hogy az utóbbi. A hiba az első bizonyításba ott kerül be, midőn ott a „harmadik“ pénzdarabról, mint egy teljesen meghatározottról beszélünk, pedig az esetenként más és más; valamint az a két pénzdarab is, mely ugyanazon oldalra esik, nem lesz mindig ugyanaz a két pénzdarab.

Now which one is the correct result: $\frac{1}{2}$ or $\frac{1}{4}$? The actual experimentation can make it clear that the latter one is correct. The mistake gets into the first demonstration when we talked about the “third” coin as a totally fixed one. But there are some cases that the third one is another one; and the two coins which fall showing the same side will not always be the same two coins.

A következő valószínűségszámítási kérdés is könnyen okozhat nehézségeket. *Egy edényben* n-*számú krajcár van; bizonyos számú krajcárt (legalább egyet) kiveszünk belőle. Mi a valószínűsége annak, hogy a kivett krajcárok száma páros, és mi azé, hogy páratlan?*

Ha n páros szám, akkor első pillanatban úgy látszik, hogy e két valószínűség egyenlő, azaz mind a kettő $\frac{1}{2}$, hiszen 1-től n-ig ez esetben ugyanannyi a páros, mint a páratlan szám, t. i. $\frac{n}{2}$. Mégis kitűnik, hogy páros n esetében sem lesz $\frac{1}{2}$ a keresett valószínűségek értéke, ha meggondoljuk, hogy n például 6 lévén, annak a valószínűsége, hogy 2 krajcárt veszünk ki, nem ugyanaz, mint azé, hogy 3-at veszünk ki.

Valóban 6 krajcárból 1, 3 és 5 krajcár

$$6, \quad \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} = 20, \quad \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 6$$

félekép vehető ki, páratlan számú tehát összesen 32-féleképen.

The next question on calculation of probability may easily cause difficulties. *There are* n-*kreuzers in a pot; we take a certain number of kreuzers (at least one) out of it. What is the probability that the number of the taken kreuzers is an even number, and what is that of an odd number?*

If n is an even number, then it seems at first sight that these two probabilities are equal, that is, each of them is $\frac{1}{2}$, because in this case, there are as many even numbers as odd numbers from 1 to n, that is, $\frac{n}{2}$. However, it will be clear that in the case of even n the value of the probabilities asked for will not be $\frac{1}{2}$. If we consider the case of $n = 6$ for example, then the probability that we pick up 2 kreuzers is not the same as the probability that we pick up 3 kreuzers.

In fact, the combinations that we can take 1,3 and 5 kreuzer(s) out of 6 kreuzers are:

Therefore, there are 32 combinations in total.

Páros számúra hasonlóképpen a 31-es számot nyerjük. Így tehát $n = 6$ esetében annak a valószínűsége, hogy páros, illetve páratlan számú krajcárt vegyünk ki, $\frac{3}{6} \cdot \frac{1}{3}$ és $\frac{3}{6} \cdot \frac{2}{3}$. Látjuk tehát, hogy e valószínűségek, bár igen közel vannak hozzá, mégsem $\frac{1}{2}$-del egyenlők.

We get similarly 31 combinations for the even number. Therefore in the case of $n = 6$, the probability that we take an even number of kreuzers is $\frac{31}{63}$, and that for an odd number is $\frac{32}{63}$. We see therefore that these probabilities are very near to $\frac{1}{2}$.

Általánosságban minden (páros és páratlan) n-re kimutatható, hogy *páratlan számú krajcár kivétele a valószínűbb eset.* Ismeretes ugyanis, hogy n elemből $1, 2, ..., n$ elem kiválasztása

Generally, it can be proven for any n (even and odd) that *the case of taking an odd number of kreuzers out is more likely to happen.* Therefore it is known that, for the selection of $1, 2, ..., n$ element(s) from n elements, there are

$$n, \binom{n}{2}, \binom{n}{3}, ..., 1$$

[p. 8]
félekép történhetik. E számok összege ismeretes képlet szerint[1]):

combinations. According to a known formula, the sum of these numbers is[1]):

[p. 9]

$$\binom{n}{1} + \binom{n}{2} + \binom{n}{3} + ... + \binom{n}{n-1} + \binom{n}{n} = 2^n - 1. \qquad \text{(I)}$$

Hasonlóképpen páros számú krajcár

Similarly, for an even number of kreuzers, there are

$$\binom{n}{2} + \binom{n}{4} + ... = 2^{n-1} - 1 \qquad \text{(II)}$$

félekép, páratlan számú pedig

combinations. And for an odd number of kreuzers, there are

$$\binom{n}{1} + \binom{n}{3} + ... = 2^{n-1} \qquad \text{(III)}$$

félekép választható ki. Annak valószínűsége tehát, hogy páros, illetve páratlan számú krajcárt vegyünk ki az edényből:

combinations to be taken out. Therefore the probability that we take an even or an odd number of kreuzers out of the pot is:

$$v_2 = \frac{2^{n-1}-1}{2^n-1} \text{ és (and) } v_1 = \frac{2^{n-1}}{2^n-1}.$$

Valóban tehát mindig $v_1 > v_2$ és 2^{n-1} korona tehető $2^{n-1}-1$ korona ellenébe, hogy az n-számú krajcárból kimarkolt krajcárok száma páratlan.

Therefore indeed always $v_1 > v_2$, and, an odd number of kreuzers can be grasped out of the n-number of kreuzers in 2^{n-1} games, while an even number of kreuzers can be grasped in $2^{n-1}-1$ games.

*Note of the translator:
"Korona (krone)" is a money unit at that time.
In 1892, the money unit of Hungary was changed as follows:
100 krajcár (kreuzers) = 1 forint (1858-1892);
100 fillér = 1 korona (krone) (1892–1918).
But in this context, "korona" is translated as one game of grasping kreuzers out of a pot. This chapter concerns combinatorics, and not explicitely the *geometria situs* nor the graph theory. However, KŐNIG treated in 1936 graph theory as a branch of combinatorics (see 7.2 on p. 306). Moreover, considering also the fact that, in all the other chapters of his second book in 1905, he treats something concerning *geometria situs*, which he treats again in 1936 as examples of graph theory, and the fact that the problems treated in his first book in 1902 do not appear in his book of the graph theory in 1936, it is possible that KŐNIG has already established a connection between combinatorics and graph theory as early as 1905.

[1]) Az (I), (II), (III), képletek körül elegendő az (I) és (II) bizonyítása, mert (III) ezekből egyszerű kivonással kiadódik. Ami (I)-et illeti, azt egyszerűen úgy nyerjük, hogy az általános binomiális képletben, mely a következő:

[1]) Among the equations (I), (II) and (III), it is enough to prove (I) and (II) because (III) is easily extracted from them. Concerning (I), we get it in a simple way with the following general binomial formula:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + ... + \binom{n}{n-1}ab^{n-1} + b^n,$$

mind a, mind b helyébe 1-et írunk. (II) az (I)-ből vezethető le. Az

in which we substitute every a and every b with 1. (II) can be deduced from (I). With the help of the basic formula

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

alapképlet segítségével ugyanis (I)-ben az első és második, harmadik és negyedik stb. tag egy-egy taggá vonható össze. (I) tehát, ha n páros, így írható:

We can write the first and the second terms in (I) together in one term, the third and the fourth terms as well, etc. Therefore if n of (I) is an even number, it can be written as follows:

$$\binom{n+1}{2} + \binom{n+1}{4} + ... + \binom{n+1}{n} = 2^n - 1$$

és $n = m - 1$-et téve (m páratlan):

and substituting with $n = m - 1$ (m is an odd number):

$$\binom{m}{2} + \binom{m}{4} + ... + \binom{m}{m-1} = 2^{m-1} - 1 \qquad (1)$$

Ha n páratlan, (I) baloldalán páratlan számú tag van és így az utolsó tag, melynek értéke 1, a párosításnál fennmarad. Ekkor tehát (I) így alakul;

If n is an odd number, the number of terms on the left side of (I) is an odd number. The value of the last term is 1, and an even number of terms are remaining. In this case (I) is developed as follows:

$$\binom{n+1}{2} + \binom{n+1}{4} + ... + \binom{n+1}{n-1} + 1 = 2^n - 1$$

és ismét $n = m - 1$-et helyettesítve (m páros):

and we substitute again with $n = m - 1$ (m is an even number):

$$\binom{m}{2} + \binom{m}{4} + ... + \binom{m}{m-2} + 1 = 2^{m-1} - 1^* \tag{2}$$

(II) a most levezetett (1) és (2) képleteket *egyiittesen* fejezi ki.

Épen ez teszi lehetettlenné az utolsó tag kiírását mind (II), mind (III) baloldalán; ez azonban nem okozhat félreértést, minthogy az $\binom{n}{n+1}, \binom{n}{n+2}, ...$ jelek 0-t jelentenek.

(II) is expressed *by the unification* of the formulae (1) and (2) just derived.

It is totally impossible to write the last term of the left side of every (II) and every (III). However, this cannot bring a misunderstanding because the signs $\binom{n}{n+1}, \binom{n}{n+2}, ...$ mean 0.

*Note of the translator:
The last term of the left side of the equation (2) is 1, which can be written as $\binom{m}{m}$.
This equation is corrected by the translator because the original text has a misprint as follows:

$$\binom{m}{2} + \binom{m}{4} + ... + \binom{m}{m-2} + {}^{2m-1}.$$

5.2 A kettes számrendszerről: About the binary numeral system

[p. 10]
Több gyakorlatilag is érdekes eredmény vonható le a (közönséges egész) számoknak azon ismeretes tulajdonságáról, hogy bármely számrendszerben (és mindegyikben csak egyféleképp) felírhatók. Ez úgy is kifejezhető, hogy minden szám

More practically interesting results can be drawn from a known property of numbers (general integer) that the integers can be written down in any numeral systems (and with only one method in each system). In other words, all the numbers can be written in the following form:

$$a_1x^n + a_2x^{n-1} + ... + a_1x^{n-i} + ... + a_{n-1}x + a_n$$

alakban írható, hol x az egységnél nagyobb egész számot jelent, az a-k pedig a $0, 1, 2, ..., x-2, x-1$ értékeket vehetik fel. A számok mindennapi (10-es rendszerben való) kiírása is tulajdonképpen ily alakban történik, csak + jeleket és 10-nek hatványait hagyjuk el rövidség kedvéért, minthogy a O használat ával minden kétértelműség ki van zárva. Természetesen x nem csupán 10-et jelenthet. Némi tekintetben legegyszerűb lesz a számok jelölése, ha x-nek a 2-t választjuk, azaz a kettes számrendszerben írjuk a számokat. Ekkor ugyanis az a-k (a számjegyek) csak 0-t vagy 1-et jelenthetnek. Innen következik, [p. 11] hogy a kettes rendszerben való felírás bármely számot, mint 2 különböző hatványainak összegét adja és így minden szám felírható ily összeg alakjában.

where x represents the integer larger by one than the unit*, and the a-s can be one of the values $0, 1, 2, ..., x-2, x-1$. The usual written form of numbers (base 10 system) is also essentially in this present form, but + signs and the powers of 10 are omitted for abbreviation, because the use of 0 excludes any ambiguity. Of course, x represents not necessarily 10. The representation of the numbers would be simplest from a point of view. If we select 2 as x, then the numbers we write are in the binary system. In this case, a-s (the digits) can be only 0 or 1. Therefore, in the binary system, all the numbers are written uniquely in the form of the sum of any different powers of 2.

*Note of the translator:
That is, x is the *base* of the numeral system.

Ezen alapszik a számok kitalálásának következő módja. Készítsünk k számú számtáblát; az első, második, ... , k-adik tábla első száma legyen rendre, 1, 2, $2^2 = 4$, $2^3 = 8$, ..., 2^{k-2}, 2^{k-1}. Az első $2^k - 1$ szám mindegyikét most már úgy írjuk ezen k táblára, hogy ha az N szám 2 különböző hatványainak összegeként írva a következő:

Based on this [numeral system], the numbers are written in the following way. Prepare k tables; let the first number of the first, second, ..., k-th table be the sequence $1, 2, 2^2 = 4, 2^3 = 8, ..., 2^{k-2}, 2^{k-1}$. In these k tables, we write down all the first $2^k - 1$ numbers in the following way: if the number N is written as a sum of different powers of 2 as follows:

$$N = 2^{\alpha_1} + 2^{\alpha_2} + ... + 2^{\alpha_r}$$

akkor N-et az $\alpha_1 + 1$-edik, $\alpha_2 + 1$-edik, ... és $\alpha_r + 1$-edik táblára írjuk. Ha most már valaki egy 2^k-nál kisebb számot gondol és ideadja mindazon táblákat, melyeken az illető szám előfordul, akkor a gondolt számot megkapjuk, ha ezen táblák első számait összeadjuk. Ez az összeg lesz ugyanis az egyetlen szám, mely ezeken a táblákon, és csak is ezeken, rajta van. Bármily és bárhány táblát választunk is ki a k tábla közül, mindig lesz egy és csak egy szám, mely ezeken és csak ezeken rajta van, t. i. e táblák első számainak összege.*)

then write N in the α_1+1-th, α_2+1-th, ... and $\alpha_r + 1$-th table. If someone has in his mind a number smaller than 2^k, and gives me all the tables in which the respective numbers occur, then we obtain the number in his mind by means of adding together the first numbers in these tables. This sum will be one and only one number on these tables. No matter which and how many tables are selected from k tables, it will always be one and only one number, which is in these and only these tables, precisely, the sum of the first numbers of the table.*)

*) Hogy ily módon bizonyos táblák kiválasztásával minden a táblákon előforduló szám kiadódik, a következőképpen látható be. A k táblából $1, 2, ..., k$ tábla $\binom{k}{1}, \binom{k}{2}, ..., \binom{k}{k}$ féleképp választható ki, s így az ily módon meghatározható számok száma:

*) With the selection of certain tables in this way, it is recognized that all the numbers appear on the tables as follows. There are $\binom{k}{1}, \binom{k}{2}, ..., \binom{k}{k}$ different ways to select $1, 2, ..., k$ table(s) from k tables, so the number of the numbers that can be determined in this way is:

$$\binom{k}{1} + \binom{k}{2} + ... + \binom{k}{k} = (1+1)^k - 1 = 2^k - 1,$$

ami valóban megegyezik a táblákon lévő számok számával.

which is just the same as the number of all the numbers on the tables.

Ha pl. akkor a táblákon a következő módon kell a számokat elhelyezni:

If, for example, $k = 5$, then the numbers should be placed in the following ways on the tables*:

[p. 12]

I. tábla:
1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31.
II. tábla:
2, 3, 6, 7, 10, 11, 14, 15, 18, 19, 22, 23, 26, 27, 30, 31.
III. tábla:
4, 5, 6, 7, 12, 13, 14, 15, 20, 21, 22, 23, 28, 29, 30, 31.
IV. tábla:
8, 9, 10, 11, 12, 13, 14, 15, 24, 25, 26, 27, 28, 29, 30, 31.
V. tábla:
16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31.

*Note of the translator:
Each table represents a digit: "I. tábla (Table I)" represents a digit 2^0; "II. tábla (Table II)" represents a digit 2^1; and so on. Using this table, a decimal number can be written in the form of a sum of powers of 2. For example, the number 13 appears in the tables I, III and IV, therefore this number is written as "$2^0 + 2^2 + 2^3$.

A k más értékeinél is hasonlóan megalkothatók a táblák, melyek a 0-t is beszámítva (melynek a szabály szerint egyiken sem szabad rajta lenni), az első 2^k szám bármelyikének kitalálására alkalmasak. Mindig azt az érdekes eredményt fogjuk látni, hogy a k tábla mindegyikére ugyanannyi szám jut*); $k = 5$ [p. 13] esetében pld. minden tábla 16 számból áll.

Also for the other values of k, including 0, the tables can be similarly constructed (in this case, nothing allowed to be on the table according to the rule), and any of the first 2^k numbers can be composed with the tables. We will see in every case the interesting results that the same number of numbers is written in all the k tables*); for example in the case of $k = 5$, each table consists of 16 numbers.

*) Ez a következőképpen bizonyítható. Minden a táblákon előforduló szám

*) This is proved as follows. The number appearing in all the tables is in the following form:

$$N = a_1 2^{k-1} + a_2 2^{k-2} + ... + a_{k-i+1} 2^{i-1} + ... + a_k$$

alakú, hol minden a 1-et vagy 2-t jelent. Valamely N szám akkor fog az i-edik táblában (melynek első száma 2^{i-1}) előfordulni, ha 2^{i-1} együtthatója, a_{k-i+1} nem 0, tehát 1. Emellett a többi a-k (melyek $k-1$ számban vannak) akár 0-sal, akár 1-gyel lehetnek egyenlők. Az i-edik táblában előforduló számok száma tehát annyi, mint ahányféleképp a $k-1$ számú $a_1, ..., a_{k-i}, a_{k-i+1}, ..., a_k$ jelek helyébe 0 vagy 1 tehető, vagyis: 2^{k-1}. Minthogy e kifejezés független i-től, azért valóban minden táblába ugyanannyi, 2^{k-1} szám jut.

where each a represents 1 or 2. Then a certain number N will occur in the the i-th table (the first number of which is 2^{i-1}) if a_{k-i+1}, the coefficient of 2^{i-1}, is not 0 but 1. In addition, the remaining a-s (the number of which is $k-1$) can be equal to 0 or 1. The number of the numbers occurring in the i-th table is the same as the number of $a_1, ..., a_{k-i}, a_{k-i+1}, ..., a_k$ of $k-1$, which can be replaced by 0 or 1, that is, 2^{k-1}. Since this term is independent of i, it is just the same for all the tables, and we will get the number 2^{k-1} as a result.

Mennél több táblát készítünk, annál meglepőbb lesz a számok ilyen kitalálása, mert k növekedésével a kitalálható számok száma, $n = 2^k - 1$ rohamosan nő, mint az n és k itt látható összetartozó értékei mutatják:

The more tables are prepared, the more surprising is the fabrication of numbers, because with the increase of k, the number of numbers $n = 2^k - 1$ increases rapidly, then the values of n and k are shown as follows:

$$k = 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, ...$$
$$n = 7, 15, 31, 63, 127, 255, 511, 1023, 2047, 4095, ...$$

A 2-es számrendszerben való felírásukon alapszik a számok kitalálásának következő módja is. Ketten megállapodnak, hogy egy pénzdarab egyik oldala (pld. a „korona“) 1-et s a másik („írás“) 0-t jelentsen. Egyikük felszólít egy harmadikat, hogy egy 32-nél (általában 2^k-nál) kisebb számot gondoljon és mondja meg neki a gondolt számot. Erre ő úgy tud öt (általában k számú) pénzdarabot egymás mellé elhelyezni, hogy a másik, ki a számot nem ismeri, ezt ki tudja találni. Az eljárás az említettek után nagyon egyszerű. A gondolt szám u. i. $k = 5$ esetére szorítkozva

Based on the writing of numbers in the binary system, the numbers can be fabricated also in the following ways. Two people agree that one side of a coin (for example the „head“) means 1, and the other side („tail“) means 0. One of them calls for a third person, thinks of a number less than 32 (generally at 2^k), and tells the third person the number in his mind. According to this, the third person can place five (generally k) coins side by side, and the other, who does not know the number, can find this. The procedure is very simple as follows. The number in his mind, reduced to $k = 5$ in this case, is

$$a_1 \cdot 2^4 + a_2 \cdot 2^3 + a_3 \cdot 2^2 + a_4 \cdot 2 + a_5$$

alakban írható, hol az a-k ismét 1-et vagy 0-t jelentenek. Az első, második, harmadik, negyedik és ötödik pénzdarabot most már „írás“-oldalára, vagy „korona“ oldalára kell fektetni a szerint, hogy a_1, a_2, a_3, a_4 és a_5 0-t vagy 1-et jelent.

It can be written in a form in which the a-s represent again 1 or 0. The first, second, third, fourth and fifth coins should be placed now showing „tail“ or „head“ of a coin, so that a_1, a_2, a_3, a_4 and a_5 mean 0 or 1.

Ily módon a szám egyértelműleg meghatározható; például a pénzdarabok következő elhelyezése,

Thus, the number is uniquely determined, for example, as the following placement of the coins,

KIKKI

holl I az írásoldalt, K a koronaoldalt jelenti, a 2-es rendszer 10110 számát jelenti, ami a tízes rendszerben:

where I means the head [írás], K means the tail [korona]. This placement means the number 10110 of the binary system, which is the following number in the decimal system:

$$1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2 + 0 = 22.$$

[p. 14]
Ez esetben is, mint előbb, annál meglepőbb lesz a mutatvány, mennél nagyobbra választjuk a k-t, mert a k pénzdarabbal így meghatározható számok száma (hiszen ez ismét $2^k - 1$) k rohamosan nő.

In this case also, if we choose larger k, this amount will be more surprising than before, because that is the number of numbers k which can be defined with coins (since this is again $2^k - 1$) which increases rapidly.

A 2-es számrendszerről most elmondottak érdekesen alkalmazhatók a súlymérésre. Felmerülhet ugyanis a kérdés, hogy miképen kell a súlyokat legcélszerűbben úgy megválasztani, hogy 1-től n kilóig minden egész számú kiló megmérhető legyen. Célszerű mindenesetre akkor lesz a választás, ha lehetőleg kevés súlyra lesz szükség. Ha az n szám 2-nek k-adik és $k+1$-edik hatványa közt van, akkor az $1, 2, ..., n$ kilós súlyok lemérésére k súly legalább is szükséges (ennek bizonyítását itt nem tárgyaljuk). Hogy k súly elegendő, azt mutatja a következő sorozat:

The binary system is now interestingly applicable to the counterweight measurement. Indeed, we can ask the following question: "how the counterweights should be the most efficiently chosen so that all the number of kilograms from 1 to n kilograms can be measured?" In any case the choice will be favorable if we need counterweights as little as possible. If number n is between k-th power of 2 and $k+1$-th power of 2, then for measuring the $1, 2, ..., n$ kilogram of weights, at least k counterweights are necessary. (we don't discuss the proof here). The following sequence shows that k counterweights are sufficient:

$$1, 2, 4, ...2^{k-1} \quad \text{kiló},$$

melyekkel nemcsak n-ig, hanem egészen $2^k - 1$-ig minden egész számú kiló lemérhető, mert hiszen minden 2^k-nál kisebb szám e sorozat bizonyos tagjainak összegeként írható. — Nézzük most már, hogy micsoda súlyokat kell választanunk, ha a mérleg azon csészéjébe is rakhatunk súlyokat, melybe a megmérendő tárgyat tettük. Ez esetben 3 hatványai fogják a legcélszerűbben megválasztott súlyok mérőszámát adni, mert *minden N szám mint 3 különböző hatványainak algebrai*) összege írható.*

*) Oly értelemben használva e szót, hogy a tagok negatív előjellel is vehetők.

with which, not only up to n, but every kilo up to $2^k - 1$ is measurable, because all the numbers smaller than 2^k can be written as the sum of certain terms of a sequence. — Let's see now what kind of counterweights should be chosen if we can put counterweights into the cup of the balance for measuring the objects. In this case, it will be the most expedient that the power of 3 will be chosen to give the index number of counterweights, because *every N can be written as an algebraic*) sum of different powers of 3.*

*) Using this word in a sense that that the terms with a negative sign also can be taken.

Ha ugyanis N a 3-as rendszerben felírva:

Indeed, if N is written in the ternary system:

$$N = a_n 3^n + a_{n-1} 3^{n-1} + ... + a_i 3^i + ... + a_1 3 + a_0$$

akkor azon jobbról számított első tag helyébe, melyben $a_i = 2$, ez írható:

then, on the right side, the first term with $a_i = 2$ can be replaced as follows:

$$a_i 3^i = 2 \cdot 3^i = 3 \cdot 3^i - 3^i = 3^{i+1} - 3^i;$$

[p. 15]
ha ezután is, egy taggá vonva össze 3-nak ugyanazon hatványát tartalmazó tagokat, marad még 3-nak oly hatványa, melynek együtthatója 2, erre nézve végezzük el ezt az átalakítást, ismét jobbról balra haladva, mindaddig míg N valóban 3 különböző hatványainak algebrai összegeként adódik ki, még pedig úgy, hogy ha 3^n a 3-nak legnagyobb N-nél kisebb hatványa, akkor 3-nak legfeljebb $n+1$-edik hatványa fog szerepelni. Hogy mérhetünk meg ezek szerint pld. egy 154 kilós testet?

and then, if we carry out this conversion on such terms with a power of 3 with a coefficient 2, and again move it from right to left, as long as N is actually written as an algebraic sum of different powers of 3, and if 3^n is the largest power of 3 smaller than N, then at least $n+1$-th power of 3 will be included. According to these procedure, how, for example, a mass of 154 kilograms can be measured?

$$N = 154 = 3^4 + 2\cdot 3^3 + 2\cdot 3^2 + 1 = 3^4 + 2\cdot 3^3 + 3^3 - 3^2 + 1$$
$$= 3^4 + 3\cdot 3^3 - 3^2 + 1 = 2\cdot 3^4 - 3^2 + 1 = 3^5 - 3^4 - 3^2 + 1$$

Valóban, ha azon csészébe, melyben a 154 kilós test van még egy $3^4 = 81$ és $3^2 = 9$ kilós súlyt teszünk s a másikba egy $3^5 = 243$ és egy 1 kilósat, akkor a mérleg egyensúlyba jön.

Indeed, if the body of 154 kilograms is in the plate plate of balance, and if we put in the same plate a counterweight of $3^4 = 81$ kilograms and a counterweight of $3^2 = 9$ kilograms, and in the other plate a counterweight of $3^5 = 243$ and a counterweight of 1 kilogram, then it comes to balance.

Ezen fejezet címe alá foglalható még némi tekintetben az a probléma, mely valamely verseny résztvevőinek a versenyben való elrendezésére vonatkozik. Oly versenyekre gondolunk itt, melyben tetszőleges számú egyén vehet részt és mindenkor kettő-kettő küzd egymással; a kettő közül az egyik mindig győz, a másik veszít („remis“ nincsen).

Even the following problem is included in the title of this chapter from a certain point of view. The problem concerns arranging the participants in a competition. We think of competitions in which an arbitrary number of individuals can participate, and every time two people fight against each other; one of them always wins, and the other loses ("remis" ["redo" in French] if a game ended in a tie).

Vesztes a további játékban nem vesz többé részt s a győztes az, aki végül egyedül marad veretlen.

The loser will not take further game, and only one winner will be finally left undefeated.

Mindezen szabályok betartásánál is a verseny igen sokféleképp folyhat le. Legegyszerűbb a beosztás akkor, ha a résztvevők száma 2-nek valamely hatványa. Ha pld. négyen vesznek részt — jelöljük őket 1, 2, 3, 4-gyel, — akkor a verseny a következő séma szerint folyhat le:

Even if all these rules are obeyed, the competition can be carried out much variously. The simplest case is that the number of participants is some power of 2. If, for example, four people participate — we call them with numbers 1, 2, 3, 4 —, then the competition can be carried out according to the following schedule:

$$\left.\begin{array}{l}\left.\begin{array}{l}1\\2\end{array}\right\}(1,2)\\ \left.\begin{array}{l}3\\4\end{array}\right\}(3,4)\end{array}\right\}\text{győztes (winner)}$$

[p. 16]
Szavakban: 1 játszik 2-vel, a győztest jelöljük $(1,2)$-vel; 3 játszik 4-gyel, és ezek közt legyen $(3,4)$ a győztes. Végül $(1,2)$ és $(3,4)$ játszik együtt s a ki köztük győz, az az egész verseny nyertese. — Egészen hasonló a verseny lefolyása nyolc résztvevő esetében. Hiszen az első forduló után csak négy játékos „marad fenn“ és ezek közül a most részletezett módon kerül ki a verseny győztese stb. Ha általában 2^k résztvevő van, akkor az első, második, ..., k-adik forduló után még $2^{k-1}, 2^{k-2}, ..., 1$ játékos marad veretlen s így a győztest a k-adik forduló adja ki. Minthogy az első, második, ... k-adik forduló $2^{k-1}, 2^{k-2}, ..., 1$ játszmából áll, azért az egész verseny játszmáinak a száma:

It means: 1 plays with 2, and a winner is selected from $(1,2)$; 3 plays with 4, and a winner is selected from $(3,4)$. Finally, the winner from $(1,2)$ and the winner from $(3,4)$ play with each other, and the winner of this game is the winner of the whole contest. — It is quite similar to the competition of eight participants. Since only four players „remain“ after the first round, and among them the winner of the competition is selected, etc. If there are generally 2^k participants, then $2^{k-1}, 2^{k-2}, ..., 1$ players remain undefeated after the first, second, ..., k-th round. In this way, the final winner turns out after the k-th round. The first, second, ... k-th round consists of $2^{k-1}, 2^{k-2}, ..., 1$ games, therefore the number of games in the whole competition is:

$$2^{k-1} + 2^{k-2} + ... + 1 = 2^k - 1$$

1-gyel kisebb a résztvevők számánál. Ezek után áttérhetünk azon esetre, midőn a résztvevők száma, n nem hatványa 2-nek, hanem pld. 2^{k-1} és 2^k közt van. Ekkor is mindig elérhető, hogy már a második fordulóba kerülő játékosok száma 2 valamely hatványa és pedig 2^{k-1} legyen. Ez esetben azonban az első fordulóban nem vehet valamennyi játékos részt, hanem egynéhány a második fordulóba jut a nélkül, hogy az elsőben győzött, sőt játszott volna. Ezek az u. n. „bye“-ek.

It is smaller by 1 than the number of participants. After this, we can switch to the case that the number of participants n is not a power of 2, but, for example, between 2^{k-1} and 2^k. At this time also we can get the result that the number of players getting into the second round is already one of the powers of 2, and it should be 2^{k-1}. In this case, however, not all the players can participate in the first round, but some of them can get into the second round with no winning nor playing in the first round. These are so-called „bye“-s.

Legyen ezeknek szabadon választható száma b, míg a többieké: $2a$ úgy, hogy

Let the number of these people be an arbitrarily chosen number b, while the others be $2a$, so that

$$b + 2a = n.$$

Míg a b-számú játékos mind bejut a második fordulóba, addig a $2a$-számú játékos közül, minthogy ezek az első fordulóban is játszanak, csak a fele, t. i. a. A második fordulóban tehát $a + b$ játszó játszik. Most már b választásával el akarjuk érni, hogy ez az $a+b$ szám 2 valamely hatványa legyen. Ez egyszerűen elérhető úgy, hogy b-nek $2^k - n$-et vesszük. Ekkor t. i. nyerjük, hogy

All the b-number of players get into the second round, while only a half of $2a$-number of players, i. e. a-number of players, get into the second round, because they play in the first round. Therefore, $a + b$ players play in the second round. Now, we want to select b so that $a+b$ is one of the powers of 2. We can select b as $2^k - n$ for simplicity. Then we get:

[p. 17]

$$a = \frac{n-b}{2} = \frac{n-2^k+n}{2} = n - 2^{k-1}$$

és valóban | and indeed

$$a + b = n - 2^{k-1} + 2^k - n = 2^k - 2^{k-1} = 2^{k-1}$$

A második fordulótól kezdve tehát a verseny épp úgy folyhatik le, mint a mikor 2^{k-1} a résztvevők száma és így a fordulók száma, $k-1$ lesz. Az első forduló játszmáinak a száma

Therefore, starting from the second round of the competition, the number of participants is 2^{k-1}, the number of rounds will be $k-1$*. The number of games in the first round is

*Note of the translator:
In the original text, the numbers are written as "2^k and k, but replaced as above by the translator according to the equation above.

$$a = \frac{n-b}{2} = n - 2^{k-1}$$

A második, ..., k-adik fordulóban pedig $2^{k-2}, ..., 1$ a játszmák száma. Az összes játszmák száma tehát

In the second, ..., k-th round, the number of games is $2^{k-2}, ..., 1$. Then the total number of games is

*Note of the translator:
In the original text, the number of games of the second round is written as "2^{k-1}, but replaced as above by the translator. Because the number of participants in the second round is 2^{k-1}, the number of games in the second round should be 2^{k-2}.

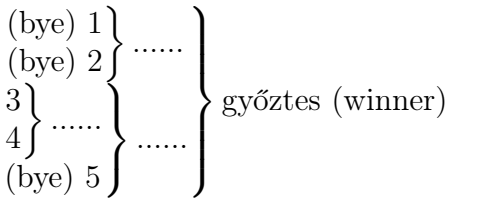
$$n - 2^{k-1} + \left(2^{k-2} + ... + 1\right) = n - 2^{k-1} + 2^{k-1} - 1 = n - 1,$$

ami ugyanaz az eredmény, mint amelyet 2^k játékos esetében nyertünk. Különben is világos, hogy n játékos esetében mindig $n-1$ játszmából áll a verseny, mert a győztest kivéve mind a többi $n-1$ játékos kibukik. Minthogy vesztes a további versenyben nem vesz már részt, azért ehhez éppen $n-1$ játszma szükséges és elegendő.

which is the same result as the result we have gotten in the case of 2^k players. Moreover, in the case of n players, it is clear that the competition consists of $n-1$ games, and the $n-1$ players other than the winner lose. Since they need not lose any more game in the competition, precisely $n-1$ games are necessary and sufficient.

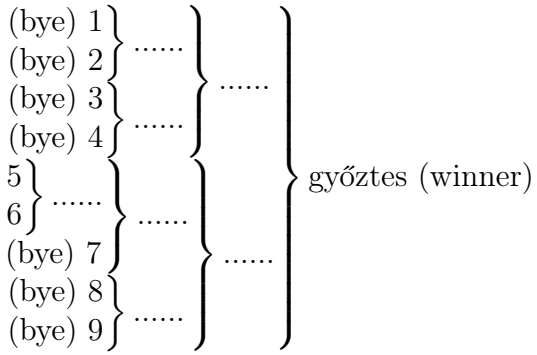
A versenyzők ily beosztása különösen teniszversenyeken van általánosan elfogadva és pld. 5 versenyző esetében, midőn a bye-ek száma $2^3 - 5 = 3$ a verseny így folyik le.

Such an arrangement of competitors is particularly widely accepted for tennis tournaments. For example, in the case of 5 competitors, the competition is held so that the number of bye-games is $2^3 - 5 = 3$.



[p. 18]
és pld. 9 versenyző esetében, midőn $2^4 - 9 = 7$ bye van, pld. ily módon:

And, for example, in the case of 9 competitors, there are $2^4 - 9 = 7$ bye-games. [The whole competition is] for example as follows:



5.3 A négyszínű térkép: The four colour map

[p. 19]
E fejezetben a következő tapasztalati tétellel akarunk foglalkozni.

Egy megyékre osztott ország térképének színezésére legfeljebb négy szin szükséges, ha a színezést úgy akarjuk végrehajtani, hogy érintkező megyék különböző színűek legyenek.

Érintkezőnek nevezünk itt két megyét akkor, ha nem csak egy pontjuk közös, hanem egy egész vonal mentén erintkezik.

Hiszen, ha két megye csak egy pontban érintkeznek, akkor a rajz az esetben is feltünteti határukat, ha színűek megegyezik.

Tételünkből következik, hogy bármikép legyen is az ország megyékre osztva, öt megye közül mindig kiválasztható kettő úgy, hogy ezek ne legyenek érintkező megyék[1]).

[1]) Az alább közölt bizonyítás csak ezt a tételt bizonyítja, mely csak közvetlen folyománya a főtételnek, de avval épenséggel nem azonos.

In this chapter, we would like to describe the following empirical proposition.

For colouring a map of a country divided into prefectures, at most four colours are required, if we want to realize the colouring so that the adjacent prefectures can be distinguished.

We say here that two prefectures are adjacent if they share not only one point but a whole line.

It is because, if two prefectures share only one point, then the boundary is clear in the drawing even if a common colour is used on these prefectures.

It is derived as a result from our proposition that, in every drawing of the country divided into prefectures, we can always select two prefectures among five prefectures, so that the two prefectures are not adjacent[1]).

[1]) The proof shown below proves only this solution of the Four-Color-Map Problem proposition which is only a direct result of the main proposition, though it is not identical at all.

Ha ugyanis az öt megye közül bármely kettő érintkeznék, akkor nem lehetne kettőt közülök ugyanazon szinre festeni és az egész ország térképének szinezésére öt szin volna szükséges.

More precisely, if any two prefectures among five were adjacent to each other, it would be impossible to colour two of them with a common colour, and five colours would be required for colouring the map of whole country.

Gyakran egy terület úgy van felosztva, hogy már négy megyét sem lehet [p. 20] úgy kiválasztani, hogy közülök bármely kettő érintkezzék.

An area is sometimes divided in such a way that one cannot select four prefectures among which every pair of prefectures is adjacent.

Magyarországon pld. Bereg-, Ugocsa-, Maramaros- és Szatmármegyén (l. 1. ábrát) kívül más oly négy megye nem található, hogy bármelyikből el lehessen jutni bármely másba anélkül, hogy egy harmadikon kelljen áthaladni.

For example in Hungary, except Bereg, Ugocsa, Maramaros and Szatmár prefectures (see Figure 1), we cannot find such four prefectures that one can leave from an arbitrary prefecture and get into any other prefecture without passing through a third prefecture.

1. ábra.

Amennyire nehéz a tétel bizonyítása (általánosságban máig sincsen bebizonyítva), annyira egyszerű rendesen a térképnek az említett módon való színezését végrehajtani.

The proof of the proposition is difficult (not yet fully proved until today), though it is very simple to execute the proper colouring of the map in the way mentioned in the proposition*.

*Note of the translator:
The proposition mentioned at p. 189 was proven as four-colour-theorem later in 1976 and published in 1977 by Kenneth Appel and Wolfgang Haken [6]. They dealt with planar graphs, and they considered colouring the vertices of them so that adjacent vertices do not have the same colour. That is, each coloured area corresponds to a vertex of a graph, and each border between two areas corresponds to an edge joining two vertices that correspond to the both areas divided with the border. The proposition mentioned at p. 189 is equivalent to the proposition "4 colours are enough for colouring all the vertices of every planar graph in such a way." They deal with a set S of graphs, where every planar graph has at least one subgraph $\in S$. S is called an unavoidable set. We can select an element of S so that it has the following property: if a graph from which a subgraph $\in S$ is removed can be coloured with 4 colours in the way mentioned here, then the graph before removal of the subgraph also can be coloured with 4 colours in the way mentioned here. Such an element of S is called a reducible configuration. If there is an unavoidable set which consists only of reducible configurations, the given proposition is proven using mathematical induction. Appel and Haken found an algorithm to test this reducibility, and created an unavoidable set which consists of 1834 reducible configurations thanks to the assistance of computer.

Egy tetszőleges megyénél kezdve a többieket csak akkor látjuk el más, még eddig fel nem használt színnel, ha valamennyi felhasznált színűvel érintkezik.

Ha új színre nincs szükség, akkor vagy magától kiadódik az alkalmazandó szín, vagy, ha nem adódnék ki, akkor egy ily (három különböző színnel ellátott megyét érintő) megyére térünk át előbb.

At an arbitrary prefecture when we start colouring the other prefectures, we use a colour not yet used if the prefecture to be coloured is adjacent to a prefecture already coloured.

If there is no need of any new colour*, the applicable colour is automatically fixed, or if the colour is not fixed, we rather switch first to another prefecture (which is adjacent to three different colours).

*Note of the translator:
That is the case that any areas coloured with one of the used colours are not adjacent to the area to be coloured.

Ezen eljárással *többnyire* minden próbálgatás nélkül eljuthatunk a térképnek a feltételnek megfelelő színezéséhez.

A tétel különben, így kimondva csak egyszerűen összefüggő[1]) (sík-, gömb-, ellipsoid- stb.) felületre [p. 21] lehet érvényes.

[1]) Így hívják az oly felületet, melyet minden zárt görbe két oly részre oszt, hogy egyikből a másikba csak a görbe metszésével lehet eljutni.

In this way, *almost* without any recolouring, we can get colouring satisfying the condition of map.

Otherwise the proposition can be valid only on simply connected[1]),* (plane, sphere, ellipsoid etc.) surfaces.

[1]) The surface is called like this when every closed curve divides the surface into two parts so that we can go out of one part and get to the other only by cutting of the curve.

*Note of the translator:
It is remarkable that Kőnig used a notion of topology even in the text of mathematical recreation. This may help readers being interested in higher mathematics. This fact supports the idea that this book was published in the context of reforming mathematical education that we discussed in Chapter 2.

Gyűrűs felületen pld., mely pld. úgy keletkezik, hogy egy kör egy a síkjában lévő őt nem metsző egyenes körül forog, a négy szín helyébe *hét* szín kerül. *)

*) Érdekes körülmény, hogy ellentétben az egyszerűbbnek látszó síkra vonatkozó tétellel, ennek a tételnek a bizonyítása alig nyújt nehézségeket.

For example, on a surface of torus, which is formed for example by rotating a ring on a plane around a straight line in a direction not parallel to the plane. *seven* colours will be required instead of four colours*).

) It is interesting that the proof of this proposition is less difficult in contrast with the proof of the proposition on a plane which is apparently simpler.

*Note of the translator:
HEAWOOD proved in 1890 that at most 7 colours are required for colouring every map on a torus [69]. The year of publication of this article was a short time ago from the publication of this book of KŐNIG. This work should be a part of the most advanced mathematics at that time. It is interesting that KŐNIG mentioned this kind of higher mathematics in a book of mathematical recreations. However, this is not the first citation of this article in the books of mathematical recreations. This article was cited already by AHRENS in his book *Mathematische Unterhaltungen Und Spiele* in 1901 [1].

Ugyancsak nem elegendő a négy szín, ha az ország egy síkban terül ugyan el, de a megyéknek egymástól különálló, egymással nem érintkező részei vannak s ha ugyanazon megyének különálló részeit is ugyanazon színnel akarjuk ellátni.

Similarly, four colours are not sufficient if the country extends in one plane but the prefectures have some parts disconnected and non-adjacent to each other, and if we want to colour the detached parts of such a prefecture with a common colour*.

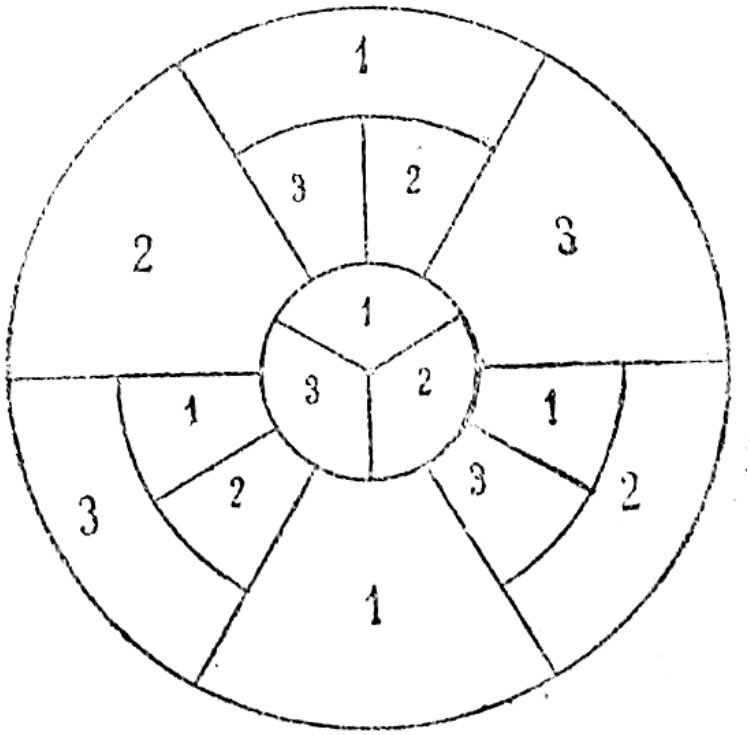
*Note of the translator:
For example, if, in each region, there are disconnected parts of all the other regions, then the required number of colours is equal to the number of regions.

Az ily ország térképének színezésére szükséges színek száma természetesen attól függ, hogy egy-egy megyének hány egymástól különálló része van.

The number of colours required for colouring a map of such a country depends naturally on the number of detached parts of each prefecture.

Vannak azonban esetek, midőn négynél kevesebb szín is elegendő.

But there are some cases where less than four colours are sufficient.



2. ábra.

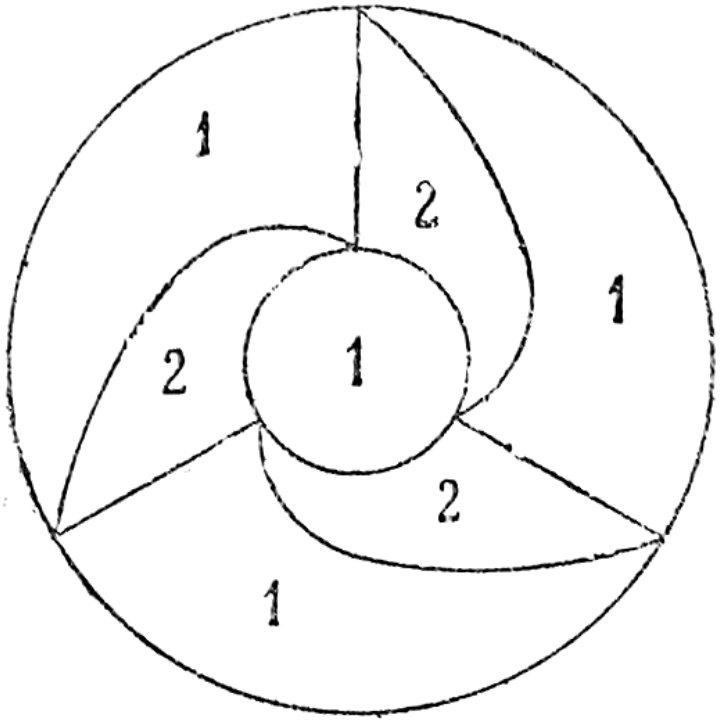
Így például, ha egy pontban legfeljebb három határvonal fut össze és minden megye páros számú megyét érint, akkor csak három szín szükséges (l. 2. ábrát); valamint akkor is, ha [p. 22] minden megye a külső határ mentén fekszik.

For example, if at most three boundary lines gather at a point, and if each prefecture is adjacent to an even number of prefectures, then only three colours are required (see Figure 2); the same number of colours required if every prefecture is along the outer boundary.

*Note of the translator:
That is the case that a part of the boundary of every prefecture is a part of the national border.

Ha pedig két megye minden közös pontjába páros számú határvonal fut össze, akkor már két szín elegendő (l. 3. ábrát).

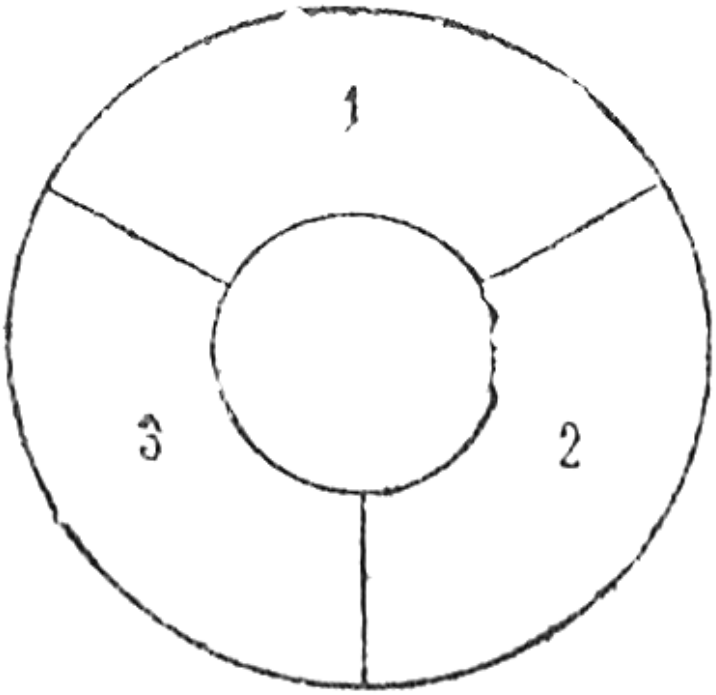
Moreover, if an even number of boundary lines gather at a common point of two prefectures, then two colours are already sufficient (see Figure 3).



3. ábra.

A kérdéses tétel, mint már említettük, máig sincs bebizonyítva — sőt újabban helyességét is kétségbe vonták — de a következő meggondolás valószínűvé teheti az olvasó előtt a tétel helyességét.

As we have already described above, this proposition is not yet proved until today — indeed on the contrary, the correctness is doubted more recently — but the consideration below will make readers believe the correctness of the proposition.



4. ábra.

[p. 23]
Három megye, melyek közül bármely kettő érintkezik s a melyek tehát három különböző színre festendők, egész általánosságban egy három részre osztott körgyűrű-alakban vehető fel, mint a 4. ábra mutatja.

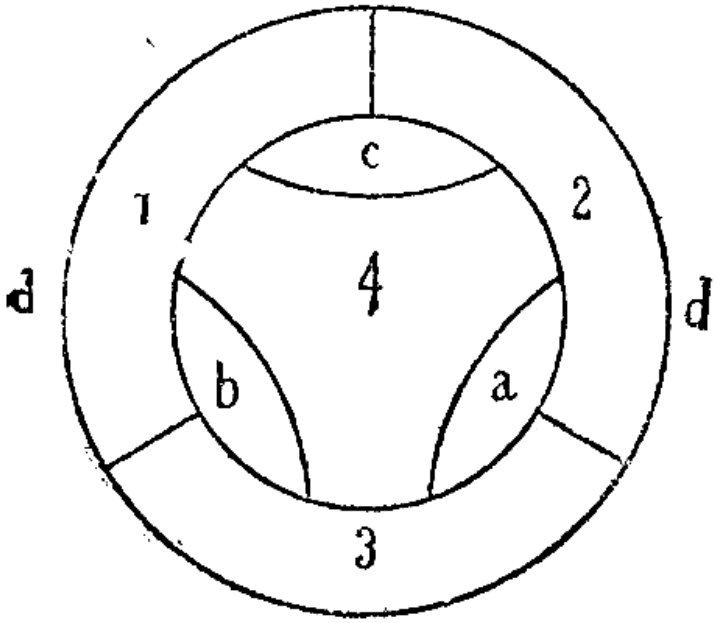
(A körülfogott belső terület el is tűnhetik.)

Three prefectures, among which any two prefectures are adjacent to each other, and therefore which are coloured with three different colours, can be treated usually in a ring divided into three parts as shown in the figure 4.

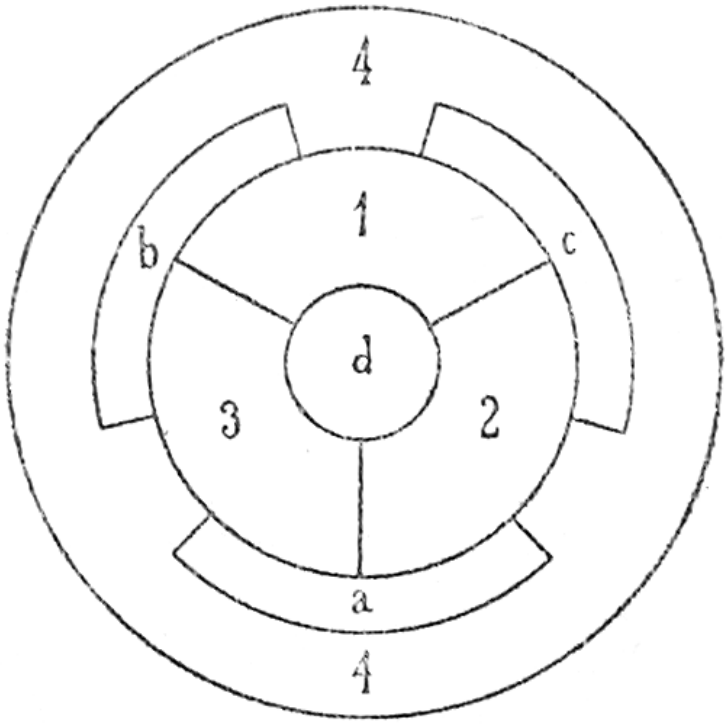
(The encircled inner area can be deleted.)

Egy negyedik e hármat érintő s így külön színnel ellátantó megye (az ábrákon e nagyedik megye 4-essel van jelezve) most már vagy egészen belül (5. ábra) vagy egészen kívül (6. ábra) lesz, mert hiszen feltesszük, hogy egy megyének két különálló része nem lehet.

The fourth prefecture adjacent to these three prefectures is to be coloured with a different colour (this fourth prefecture is indicated with the numeral 4 in the figures), and it is entirely inside (the figure 5) or entirely outside (the figure 6), because we suppose that there cannot be two detached parts of one prefecture.



5. ábra.



6. ábra.

Akár a belsőt, akár a külsőt vesszük az első háromhoz (az 5. és 6. ábrán egész általánosságban van a 4-es megye megrajzolva) mindig meggyőződhetünk, hogy oly ötödik megye, mely e négyet érinti, nem létezik, mert ennek az ötödik megyének azzal a tulajdonsággal kellene bírnia, hogy belőle úgy lehessen az első négy bármelyikébe eljutni, hogy egy harmadikon ne kelljen áthaladni.

No matter if we take the inside or the outside of the first three (the prefecture with numeral 4 is drawn usually in the figures 5 and 6), we can always be sure that the fifth prefecture adjacent to these four can not exist, because the fifth prefecture should have a property that it is possible to go from this prefecture to each of the first four prefectures without passing through a third prefecture.

Már pedig, mint az 5. és 6. ábrán látható, nincs a síknak olyan, a négy megye által le nem foglalt része, melyből a négy megye mindegyikébe közvetlenül át lehetne jutni.

But as already shown in the figures 5 and 6, on the plane, there is no part which is occupied by none of the four prefectures, and from which one can go directly to each of the four prefectures.

E le nem foglalt terület ugyanis mindkét ábrán [p. 24] négy a, b, c, d betűkkel jelölt részre oszlik és a-ból az 1-sel, b-ből a 2-sel, c-ből a 3-sal, d-ből a 4-sel jelölt megyébe közvetlenül nem lehet átjutni.

This area, which is not occupied by any of the four prefectures, is divided into parts indicated with four letters a, b, c, d in both of the figures, and it is impossible to go directly from a to the prefecture indicated with the numeral 1, from b to 2, from c to 3, from d to 4.

(A 6. ábrában a külső körön kívül levő terület a-val is, b-vel is, c-vel is jelölendő, mert a 4-es terület e területet az 1-es, 2-es és 3-as területtől egyaránt elválasztja.)

(In the figure 6, the area outside of the largest circle can be indicated with a and b and c, because the area indicated with the numeral 4 separates this area and the areas of the numeral 1, the numeral 2 and the numeral 3.)

5.4 A königsbergi hídak: The bridges of Königsberg

[p. 25]

Königsbergben van egy Kneiphof nevezetű sziget;

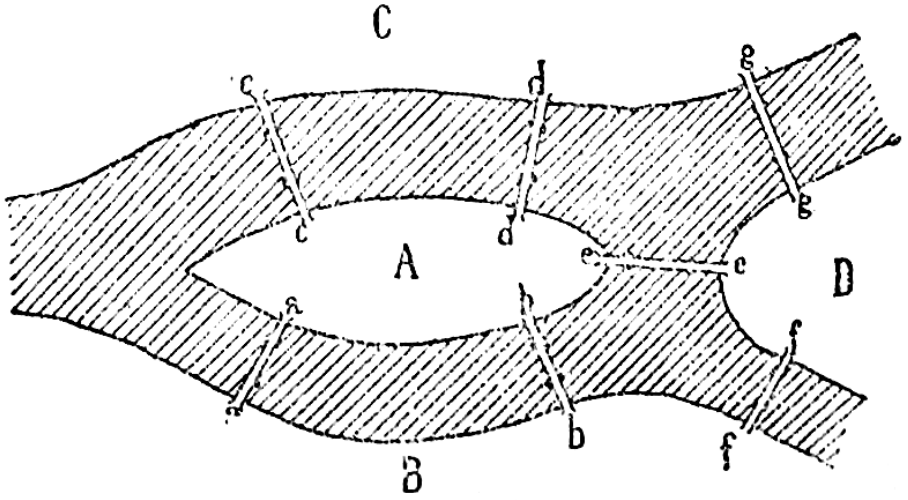
In Königsberg, there is an island called Kneiphof;

a Pregel folyó, mely a szigetet alkotja, a sziget után ismét két ágra szakad.

the Pregel river, which forms the island, is torn again in two branches after the island.

A folyón 7 híd vezet át, melyek közül 5 magából a szigetből indul ki (l. 7. abrát).

7 bridges are built over the river, and 5 of them start from the island (see Figure 7).



7. ábra.

Kérdés, hogy be lehet-e úgy járni az összes königsbergi hídakat, hogy minden hídon csak egyszer haladjon át az ember?

The problem is as follows: can one pass all the bridges of Königsberg by crossing every bridge just once?

Jelöljük a négy területet, melyet a folyó ágai meghatároznak, a nagy A, B, C és D betűkkel és a hídakat, mint az ábrán látható, a kis $a, b, ...f$, betűkkel.

We mark the four land areas, which form branches of the river, with four letters A, B, C and D, and the bridges with lowercase letters $a, b, ...f$, as shown on the figure.

[p. 26]
Ha az A területből átmegyünk B-be, akkor jelöljük a megtett utat AB-vel akár az a, akár a b hídon mentünk át.

If we cross from the area A to B, we represent the passed path by AB, no matter which bridge was chosen between a and b.

Ha B-ből továbbmegyünk az f hídon át D-be, akkor a második utat BD-vel jelölük, s a két utat együtt ABD-vel.

If we advance from the area B to D via bridge f, we represent the second path by BD, and represent both together by ABD.

Ha D-től a g hídon továbbmegyünk C-be, akkor az egész utat A-tól C-ig $ABDC$-vel jelöljük.

If we advance from the area D to C via bridge g, we represent the whole path from A to C by $ABDC$.

E jel most már azt jelenti, hogy A-ból kiindulva, áthaladva előbb B-n, aztán D-n, C-be jutottunk.

This sequence of symbols tells that we started from A, crossed to B, and then to D, and arrived at C.

Ezen út alatt közben hídon kellett áthaladnunk, épigy minden utat, mely négy hídon vezet keresztül, öt betűvel jelezhetünk.

We have to cross bridges in this route, and every route crossing four bridges can be indicated with just five letters.

Általában n hídon átvezető utnak a jele $n+1$ betűből áll.

In general, a sequence of symbols of a journey crossing n bridges consists of $n+1$ letters.

A königsbergi hét hídon átvezető minden út jele tehát *nyolc betűből fog állani.*

Therefore, any sequence of symbols for a path crossing the seven bridges of Königsberg *will consist of eight letters.*

Ha egy bizonyos X-területből csak *egy* híd indul ki, akkor a bejárt út „jel"-ében az X betű *egyser* fog csak előfordulni, akár X-ből kiindulva mentünk át ezen az egy hídon, akár ellenkező irányban.

If just *one* bridge is connected to a certain region X, then the letter X will appear just *once* in the sequence of "symbols" of the path to follow, no matter if we started from X crossing the bridge or finished in X reversely.

Ha X-et a többi „terület"-tel *három* híd köti össze, akkor könnyen belátható, hogy az X-betű *kétszer* fog a bejárat út jelében szerepelni.

If X is connected to the other "areas" with *three* bridges, then it is easily guessed that the letter X will appear *two times* in the sequence of symbols of the path to pass through.

Ha általában $2n+1$ híd indul X-ből, akkor az út jelében X $\frac{2n+2}{2} = n+1$-szer fog előfordulni.

In general, if $2n+1$ bridges are connected to X, X will appear $\frac{2n+2}{2} = n+1$ times in the sequence of symbols of the path.

A königsbergi problémánál most már A-ból: 5; B-ből, C-ből és D-ből pedig 3 híd indul ki.

In the problem of Königsberg, 5 bridges are connected to A, while 3 bridges are connected to each of B, C and D.

Így annak az útnak a jele, mely mind a hét hídon egyszer és csak egyszer vezet át.

This idea is applied to the sequence of symbols of a path crossing all seven bridges once and only once.

3-szor tartalmazná A-t és kétszer-kétszer B-t, C-t és D-t.

The sequence should contain three A-s, two B-s, two C-s and two D-s.

Az egész jelnek tehát $3+2+2+2 = 9$ betűből kell állni, nem pedig 8-ból, mint azt más meggondolás segítségével fentebb bebizonyítottuk.

Then, the whole sequence of symbols should contain $3 + 2 + 2 + 2 = 9$ letters. On the other hand, it should contain only 8 letters according to the result of another consideration mentioned formerly.

A königsbergi hídak tehát nem járhatók be úgy, hogy legalább az egyiken közülök kétszer ne kelljen áthaladni.

Therefore, if we should not cross any of the bridges twice, we cannot walk through the bridges of Königsberg so that such a sequence of symbols would exist.

Megoldhatóvá lesz ellenben a feladat, ha egy nyolcadik híd közvetlenül a B és C területet köti össze.

On the other hand, if an eighth bridge connected areas B and C directly, the problem could be solved.

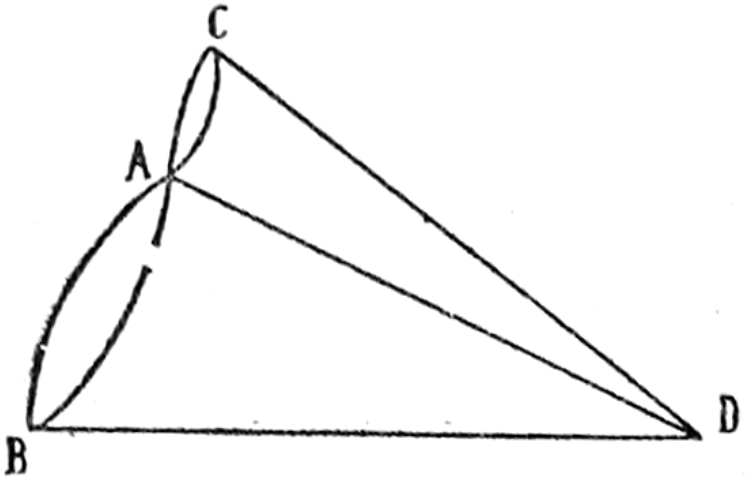
A königsbergi hidak problémája sokkal egyszerűbb lesz, ha az ú. n. területeket pontokkal és a [p. 27] hidakat vonalakkal helyettesítjük.

The problem of Königsberg will be much simpler if we substitute points for something called areas, and lines for bridges.

Így keletkezik a 8. ábra, melyre vonatkozólag a kérdést így tehetjük fel: *megrajzolható-e a 8. ábra egy folytonos vonallal?*

Figure 8 is drawn in this way, and we can reformulate the problem based on this figure: *is it possible to draw Figure 8 with one continuous line?**

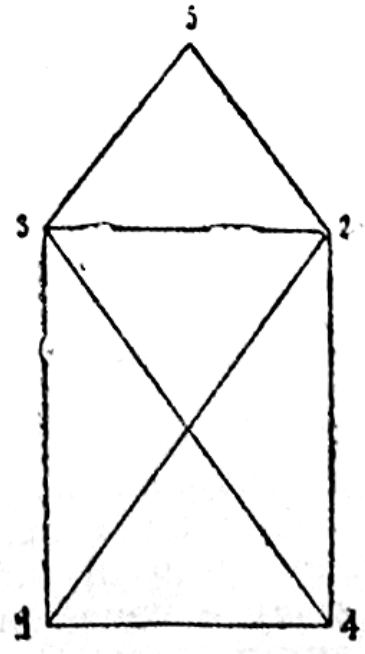
*Note of the translator:
This reformulation is the most important in this problem. This is the key idea which connects mathematical recreations and graph theory. We will discuss this point in Chapters 6 and 7.



8. ábra.

Az ehhez hasonló feladatok közt a legismertebb az, mely a 9. ábrának egy vonalban való megrajzolását kívánja.

Among problems similar to it, the best known is to draw the figure 9 with one stroke.



9. ábra.

Az ily feladatok megoldásának általános módját itt nem tárgyaljuk és csak a megoldás lehetőségére vonatkozólag teszünk egy észrevételt, melynek [p. 28] tekintetbe vétele azonban magában is gyakran már eredményre vezet.

We do not deal here with the general approach to this kind of problems*, and we make only one remark related to the possibility of solution. However, this remark will often naturally lead us to the result.

*Note of the translator:
Euler deals with the general approach to this kind of problems, but he proves his theorems only partially in 1736 [51]. Hierholzer proves them completely in 1873 [70]. See 3.4.1 (p. 30–) for details.

Ha egy vonalrendszer kettőnél több oly pontot tartalmaz, melyben páratlan számú vonal fut össze, akkor e vonalrendszer egy vonalban nem rajzolható meg.

If a line system contains more than two points at which an odd number of lines gather, then this line system* can not be drawn with one stroke.

*Note of the translator:
The term "vonalrendszer (line system)" is a keyword related to graph theory. In Kőnig's later articles and books, this term was often used. For the detailed discussion on this term, see Chapter 2.

Egy folytonos vonal ugyanis *csak kezdő és végpontjában összefutó páratlan számú vonalat foglalhat magában* és minden más pontjából párosszámú vonal indul ki, mert ahányszor a folytonos vonal egy ily pontba eljut, annyszor távozik is belőle.

It is *one continuous line* particularly when *an odd number of lines are gathered only to the starting point and to the end point* and an even number of lines are gathered to all the other points, because each time the continuous line reaches one of such points, it also depart from the point.

Ezen tétel segítségével is könnyen átláthatjuk, hogy a königsbergi probléma megoldhatatlan.

Thanks to this proposition, we can also easily see that the problem of Königsberg is insoluble.

A 8-ik ábrában ugyanis A-ba: 5; B-be, C-be és D-be: 3 vonal fut össze; négy pontból indul ki tehát páratlan számú vonal s így e vonalrendszer megrajzolásához legalább félannyi, tehát két folytonos vonalra van szükség.

In the figure 8, 5 lines gather at A, 3 at each of B, C and D; in other words, an odd number of lines are gathered to 4 points, therefore, at least a half number of continuous lines, namely, two continuous lines are necessary for drawing this line system.

A 9. ábrában ellenben csak a két alsó pontba fut össze páratlan számú (3) vonal; ez a vonalrendszer tehát esetleg megrajzolható egy oly folytonos [p. 29] vonallal, mely e pontok egyikéből indul ki s a másikban végződik.

Contrarily, in the figure 9, an odd number (3) of lines gather at just two points at the bottom; therefore, this line system can be drawn with one continuous line on condition that one starts from one of the two points and finishes at the other.

Valoban az

In fact, the path

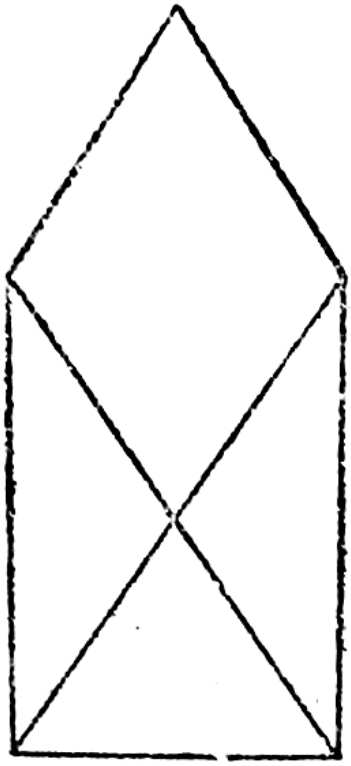
1 2 3 4 2 5 3 1

út megoldást szolgáltat.

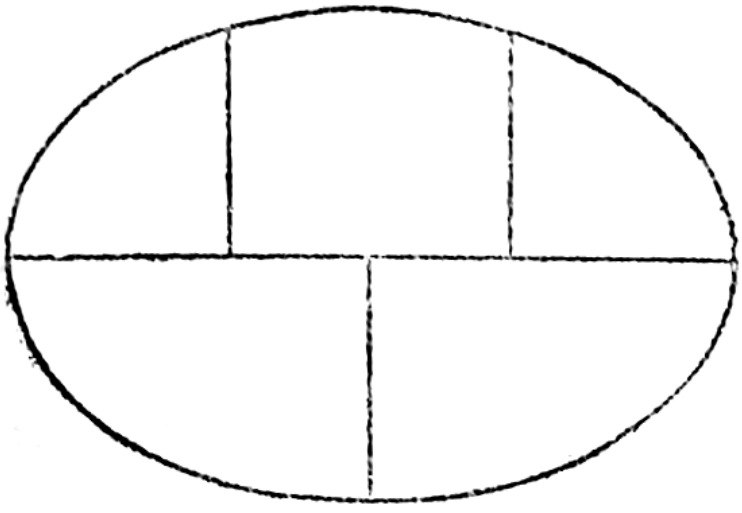
gives a solution.

Ezek után könnyen bebizonyítható még, hogy pld a 10. ábra megrajzolásához legalább két és a 11. ábra megrajzolásához legalább négy folytonos vonal szükséges.

According to these, it can be easily proved that, for example, the drawing of the figure 10 requires at least 2 continuous lines, and the drawing of the figure 11 requires at least 4 continuous lines.



10. ábra.



11. ábra.

5.5 Az iskoláslányok sétái: Daily walk of schoolgirls

[p. 30]
Tizenöt iskoláslány naponkint egyszer sétálni megy, hármasával egy sorban.

Hogyan kell őket naponként az 5 sorban úgy elosztani, hogy egy hét alatt minden lány minden másikkal egyszer és csak egyszer kerüljön egy sorba?

Minden lány minden nap két leánnyal kerül egy sorba, s hogy mind a többi 14-gyel egyszer összekerüljön, valóban $\frac{14}{2} = 7$ napra, vagyis egy hétre van szükség.

Kiszámították, hogy e feladatnak 15 567 552 000 különböző megoldása van, de minthogy általában a 15 lány nagyon sokféleképp $\left(\frac{15!}{5!(3!)^5}\right)$ osztható el az 5 sorban, azért annak a valószínűsége, hogy egy tetszőleges elosztás egy hétre feladatunknak megfeleljen, igen kicsi.

Fifteen schoolgirls go for a walk once a day, divided [into groups consisting of] three by three for each time.

How they should be divided into 5 groups, so that each girl will be with each of the other girls together in a group once and only once within a week?

Everyday, every girl is with two girls together in a group, and with all of the other 14 once and only once, then $\frac{14}{2} = 7$ days, that is, one week is necessary.

It is calculated that this problem has 15 567 552 000 different solutions, and usually the 15 girls can be divided into 5 groups very variously in $\left(\frac{15!}{5!(3!)^5}\right)$ ways*. Therefore the probability of the distribution to answer to our problem in one week is very small.

*Note of the translator:
The number 15 567 552 000 was derived from a very complicated calculation. The detail of the calculation can be found in J. POWER's article "On the problem of the fifteen school girls" (1867) [176]. The outline of POWER's procedure is as follows.
A group of 3 girls is called a "triad". There are 5 triads in a day, therefore 35 triads in a week. this is one set of triads.
POWER counted the number of different sets of triads, which is represented by t:

$$t = \frac{8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 \cdot 14 \cdot 15}{4}$$

He counted also the number of solutions belonging to one set of triads, and got the result 240. Then he got the whole number of solutions:

$$240t = (8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 \cdot 14 \cdot 15) \cdot 60 = 15\ 567\ 552\ 000$$

On the other hand, KŐNIG gave the number of ways to divide 15 girls into 5 groups: the number of combinations to divide 15 girls into 5 groups A, B, C, D, E is $\frac{15!}{(3!)^5}$; in the case of the problem of schoolgirls, the 5 groups are not distinguished, therefore the number of combinations should be divided by 5!, that is, $\left(\frac{15!}{5!(3!)^5}\right)$.

Be fogjuk mutatni a feladat megoldását két különböző módszer segítségével.	We will show two different methods to solve the problem.
Első (Peirce-féle) módszer.	First (Peirce's) method*.

*Note of the translator:
This method originated in PEIRCE's article "Cyclic solutions of the school-girl puzzle" (1861) [173]. LUCAS (1883) [149] and AHRENS [1] introduced briefly this method.

Jelöljük a 15 lány egyikét p-vel s a többi 14-et az	We call one of the 15 girls p, and the other 14 as follows:

$$a_1, a_2, a_3, a_4, a_5, a_6, a_7$$

és	and

$$b_1, b_2, b_3, b_4, b_5, b_6, b_7$$

betűkkel.

[p. 31]
A 15 elem egyik 5 hármassorba való elhelyezése a következő:

in letters.

Place the 15 elements in 5 lines of three-rows as follows:

$$\begin{array}{ccc} p & a & b \\ b & a & a \\ b & a & a \\ b & a & a \\ b & b & b \end{array}$$

Itt mind a hét a, mind a hét b, valamint p is szerepel; hogy ez egy meghatározott elrendezés legyen, még csak az indexeket kell elhelyezni.

Here, all the seven a-s, all the seven b-s and p appear; for making this be a particular arrangement, we should only arrange the indices.

Látni fogjuk, hogy tisztán ezzel, az indexek elhelyezésével nyerhető lesz mind a hét, feltételünknek megfelelő elrendezés.

We will see here clearly that the placement of the indices allows us to obtain all the seven arrangements satisfying our condition.

Az első, hétfői elrendezésben legyen az indexek elrendezése ez:

In the first arrangement on Monday, the indices are arranged as follows:

$$\begin{array}{ccc} p & a_1 & b_1 \\ b_4 & a_5 & a_7 \\ b_6 & a_3 & a_4 \\ b_7 & a_2 & a_6 \\ b_2 & b_3 & b_5 \end{array}$$

A keddi elrendezést innen úgy kapjuk, hogy p-t változatlanul hagyva, az indexeket 1-gyel megnagyobbítjuk; ahova azonban 8 jutna, oda 1-et írunk.

From this, we get the arrangement on Tuesday as follows: let p invariable, and we increase the indices by 1; however, we write 1 instead of 8.

Hatszor ismételve ezt, ezen megoldáshoz jutunk.

We repeat it six times, and we get the following solution.

Hétfő (Mon.)	Kedd (Tue.)	Szerda (Wed.)	Csütört. (Thu.)	Péntek (Fri.)	Szombat (Sat.)	Vasárnap (Sun.)
pa_1b_1	pa_2b_2	pa_3b_3	pa_4b_4	pa_5b_5	pa_6b_6	pa_7b_7
$b_4a_5a_7$	$b_5a_6a_1$	$b_6a_7a_2$	$b_7a_1a_3$	$b_1a_2a_4$	$b_2a_3a_5$	$b_3a_4a_6$
$b_6a_3a_4$	$b_7a_4a_5$	$b_1a_5a_6$	$b_2a_6a_7$	$b_3a_7a_1$	$b_4a_1a_2$	$b_5a_2a_3$
$b_7a_2a_6$	$b_1a_3a_7$	$b_2a_4a_1$	$b_3a_5a_2$	$b_4a_6a_3$	$b_5a_7a_4$	$b_6a_1a_5$
$b_2b_3b_5$	$b_3b_4b_6$	$b_4b_5b_7$	$b_5b_6b_1$	$b_6b_7b_2$	$b_7b_1b_3$	$b_1b_2b_4$

Hogy ez a módszer valóban mindig a feladat megoldására vezet, az a következőkből előre is kitűnik.

This approach always leads to a solution of the problem. It is shown as follows.

Az egy sorban levő a betűk mutatójának különbsége [p. 32] a hétfői elrendezésben mind az öt sorban más.

The indices of letters in a row are different in all the five rows of the arrangement on Monday.

Ezért a_3 és a_7 például csak egyszer kerülhet egy sorba.

Therefore, for example, a_3 and a_7 can be together in a row only once.

Az a_3 és a_7 ugyanis abban a sorban fog egymás mellé kerülni, hol már a_2 és a_6, a_1 és a_5 stb. is egy sorban volt, vagyis hol az a betűk mutatójának a különbsége 4, de ilyen sor csak egy van (a negyedik, amelyben kedden valóban összekerül a_3 és a_7), s így a_3 és a_7 *csak egyszer* kerülhet össze, de hogy bármely két *a egyszer mindig* összekerül, az onnan következik, hogy az a-k mutatói (az első hét szám) közt létező minden különbség az eredeti elhelyezésünk egyik sorában előfordul.*)

a_3 and a_7 will be next to each other in a row where a_2 and a_6, a_1 and a_5 etc. were in a row. That is, the indices of the letter a have 4 variations*, but their arrangement is only one (a_3 and a_7 are actually together in the fourth row on Tuesday). In this way, a_3 and a_7 are together *only once,* and *every* double combination is *once* in a row. Therefore, all the existing variations among the indices of a (the first seven numbers) appear in one of the rows of our original placement.*)

*Note of the translator:
When three of $a_1, a_2, ..., a_7$ are fixed in different rows, the rest four a-s can be in a row with one of the fixed a-s, or can be apart from the other a-s, therefore there are 4 variations. But the combination with p and b-s should be considered, therefore only one arrangement of a-s is possible.

*) Tulajdonképpen csak ± 2, ± 1 és ± 4 fordul elő, mint két a indexének a különbsége. Ámde, minthogy 8 helyett is 1-et írunk, azért oly kombinációból, melyben $\pm d$ az indexek különbsége, az indexek 1-gyel, 2-vel stb. való nagyításával oly kombináció is keletkezhetik, melyben nem $\pm d$, hanem $\pm(7-d)$ az indexek különbsége. Ily értelemben ± 5, ± 6 és ± 3 is előfordul, mint a hétfői elrendezés egy-egy sorában lévő a-k indexeinek különbsége.

*) In fact, only ± 2, ± 1 and ± 4 are found as variations of the indices of two letters, though we write 1 instead of 8. Therefore, deriving from the combination in which the variation of the indices 1, 2 etc. $\pm d$, we can form also the following combination: not $\pm d$ but $\pm(7-d)$ is the variation of the indices. In this sense, ± 5, ± 6 and ± 3 are also found as variation of indices of a-s in each row of the arrangement on Monday.

Amit most az a-kra elmondottunk, a b-kre is érvényes.

These things related to the a-s are valid also to the b-s.

Az egy sorban lévő a-k, b-k mutatójának a különbsége is mind az öt sorban más és minden lehető különbség előfordul, az a-k és b-k is tehát egyszer és csak egyszer kerülnek egy sorba.

The variations of the indices a-s and b-s in the single row are in all the five rows, and all the possible variations occur. The a-s and b-s are together in a row once and only once.

Hogy végül p minden elemmel egyszer és csak egyszer kerül össze, az könnyen belátható.

We can easily see that p is finally together [in a row] with every element once and only once.

Peirce módszere tehát evvel helyesnek bizonyult.

Peirce' method is thus considered to be correct.

Minthogy a hétfői elrendezés említett tulajdonsága mellett is sokféleképp állítható össze, azért e módszer több megoldás találására is alkalmas.

Besides the above mentioned way of the arrangement for Monday, we can set up various arrangements, and this method is suitable for finding more solutions.

Második (Horner-féle) módszer.	Second (Horner's) method.

*Note of the translator:
This method of Joseph HORNER was published by his friend J. POWER in the article "On the problem of the fifteen school girls" (1867) [176].

Osszuk a 15 elemet egy 7-es és egy 8-as csoportba.	Divide the 15 elements into a group of 7 [elements] and a group of 8 [elements].
Legyenek az első tagjai $a_1, a_2, a_3, b_1, b_2, b_3, b_4$ és a másodiké $c_1, c_2, c_3, c_4, d_1, d_2, d_3, d_4$.	Let the first members be $a_1, a_2, a_3, b_1, b_2, b_3, b_4$ and the second be $c_1, c_2, c_3, c_4, d_1, d_2, d_3, d_4$.
Képezzük a második csoport 8 elemének összes (28) kettős kombinációit és írjunk négyet-négyet a 28 kombináció közül úgy egymás alá, hogy a keletkező hét szakasz mindegyike mind a nyolc elemet tartalmazza.	Let the 8 elements of the second group form all the (28) possible two by two combinations. Write the two by two combinations among the 28 combinations four by four, one under another. Each of the arising seven sections contains all the eight elements.
Így keletkezik pld. a következő táblázat:	As a result we get, for example, the following table:

[p. 33]

I.	II.	III.	IV.	V.	VI.	VII.
c_1c_2	c_1c_3	c_1c_4	c_1d_1	c_1d_2	c_1d_3	c_1d_4
c_3c_4	c_2c_4	c_2c_3	c_2d_2	c_2d_1	c_2d_4	c_2d_3
d_1d_2	d_1d_3	d_1d_4	c_3d_3	c_3d_4	c_3d_1	c_3d_2
d_3d_4	d_2d_4	d_2d_3	c_4d_4	c_4d_3	c_4d_2	c_4d_1

Írjuk most a 15 elem második (7-es) csoportjának minden tagját e táblázat egyes szakaszainak mind a négy sora elé (a_1-et az I. sorai elé, a_2-t a II. sorai elé stb.).

We write now the second (7 elements) group of the 15 elements. We write all the 7 elements on the left of each of the four rows in each section of this table (a_1 on the left of the row I, a_2 on the left of the row II etc.).

Igy a 15 elem 28 hármas kombinációját nyerjük.

Thus, we get 28 combinations consist of 3 among the 15 elements.

Ezek közt most már nemcsak a c-k és a d-k vannak minden lehető módon egyszer és csak egyszer összehozva, hanem egyszersmind az a-k a c-kkel és a d-kkel, továbbá a b-k a c-kkel és d-kkel.

Among them, not only the c-s and the d-s are once and only once together with each other in all the possible ways, but at the same time, the a-s are also with the c-s and the d-s, furthermore the b-s are also with the c-s and d-s.

Csak az a és b betűk nem fordulnak még elő együtt.

Only the letters a and b are not together with each other.

Az $a_1, a_2, a_3, b_1, b_2, b_3, b_4$ elemek hármas kombinációi, melyekben bármely két elem egyszer és csak egyszer kerül össze, pld. a következők.

the three by three combinations of the $a_1, a_2, a_3, b_1, b_2, b_3, b_4$ elements in which any two elements are together with each other once and only once are for example as follows.

$$a_1a_2a_3, a_1b_1b_2, a_1b_3b_4, a_2b_1b_3, a_2b_2b_4, a_3b_1b_4, a_3b_2b_3$$

Ezekkel együtt a következő (28+7 =)35 hármas kombinációt nyerjük.

Together with them, we get the following (28+7 =)35 three by three combinations.

I.	II.	III.	IV.	V.	VI.	VII.
$a_1a_2a_3$						
$a_1b_1b_2$	$a_2b_1b_3$	$a_3b_1b_4$				
$a_1b_3b_4$	$a_2b_2b_4$	$a_3b_2b_3$				
$a_1c_1c_2$	$a_2c_1c_3$	$a_3c_1c_4$	$b_1c_1d_1$	$b_2c_1d_2$	$b_3c_1d_3$	$b_4c_1d_4$
$a_1c_2c_4$	$a_2c_2c_4$	$a_3c_2c_3$	$b_1c_2d_2$	$b_2c_2d_1$	$b_3c_2d_4$	$b_4c_2d_3$
$a_1d_1d_2$	$a_2d_1d_3$	$a_3d_1d_4$	$b_1c_3d_3$	$b_2c_3d_4$	$b_3c_3d_1$	$b_4c_3d_2$
$a_1d_3d_4$	$a_2d_2d_4$	$a_3d_2d_3$	$b_1c_4d_4$	$b_2c_4d_3$	$b_3c_4d_2$	$b_4c_4d_1$

Ez a 35 hármas kombináció a 15 elem minden kettős kombinációját egyszer és csak egyszer tartalmazza.

Feladatunk tehát meg van oldva, ha e 35 kombinációt oly 7 ötös csoportba tudjuk osztani, hogy minden csoport mind a 15 elemet tartalmazza.

Ha tekintetbe vesszük, hogy az egy csoportba [p. 34] kerülő hármas kombinációk az utolsó táblázat más-más szakaszába kell, hogy tartozzanak (egy szakasz bármely két sora tartalmaz ugyanis egy közös elemet), könnyen nyerjük a 15 iskoláslány problémájának következő megoldását:

These 35 three by three combinations contain all the double combinations of the 15 elements once and only once.

Our problem is therefore solved if these 35 combinations can be divided into 7 groups of five combinations so that each group contains all 15 elements.

If we consider that the three by three combination being together in a single group should belong to different groups of the last table (any two rows of a section contain a common element), we get easily the following solution to the problem of the 15 schoolgirls:

I.	II.	III.	IV.	V.	VI.	VII.
$a_1a_2a_3$	$a_1b_1b_2$	$a_1b_3b_4$	$a_1d_3d_4$	$a_1d_1d_2$	$a_1c_2c_4$	$a_1c_1c_2$
$b_1c_1d_1$	$a_2c_1c_3$	$a_2c_2c_4$	$a_2b_1b_3$	$a_2b_3b_4$	$a_3d_1d_3$	$a_2d_2d_4$
$b_2c_3d_4$	$a_3d_2d_3$	$a_3d_1d_4$	$a_3c_1c_4$	$a_3c_2c_3$	$a_2b_2b_3$	$a_3b_1b_4$
$b_3c_4d_2$	$b_2c_2d_4$	$b_1c_3d_3$	$b_2c_2d_1$	$b_1c_4d_4$	$b_1c_2d_2$	$b_2c_4d_3$
$b_4c_2d_3$	$b_4c_4d_1$	$b_2c_1d_2$	$b_4c_3d_2$	$b_3c_1d_3$	$b_4c_1d_4$	$b_3c_2d_1$

Ismervén a 15 iskoláslány problémájának megoldását, elég egyszerűen át lehet térni 45 lány esetére, kik szintén hármasával egy sorban végzik naponkint sétáikat.

After knowing a solution of the problem of the 15 schoolgirls, it is possible to change the subject quite simply to the case of 45 girls who take a walk three by three in a row everyday.

Tekintsük a 45 iskoláslányt három különböző intézet növendékeinek és pedig legyenek az I. II. és III. intézet növendékei.

Let us regard the 45 schoolgirls as the students of three different schools, and let us call them the students of school I, II or III.

$$\begin{array}{rl} \text{I.} & a_1, b_1, c_1, d_1, e_1, f_1, g_1, h_1, i_1, j_1, k_1, l_1, m_1, n_1, o_1 \\ \text{II.} & a_2, b_2, c_2, d_2, e_2, f_2, g_2, h_2, i_2, j_2, k_2, l_2, m_2, n_2, o_2 \\ \text{III.} & a_3, b_3, c_3, d_3, e_3, f_3, g_3, h_3, i_3, j_3, k_3, l_3, m_3, n_3, o_3 \end{array}$$

Minden lánynak 44 társa lévén, 22 napra és elhelyezkedésre van szükség, hogy bármely kettő egyszer egy sorba kerüljön.

Every girl has 44 partners. 22 days and 22 arrangements are required. Any two of the girls should be once together in a single row.

E 22 elrendezés közül 7 úgy nyerhető, hogy a lányok intézetenként végzik az első héten sétájukat, mint azt az előbbiekben elmondottuk.

We can get 7 of these 22 arrangements as follows: the girls grouped in each school take a walk. In the first week, these arrangements are the same as those in the problem above.

A többi 15 elrendezésében már természetesen három más-más intézethez tartozó lány fog minden sorba kerülni, mert hiszen hét nap alatt ugyanazon intézet bármely két növendéke okvetlenül összejutott már egy sorba.

The girls belonging to the three different schools are naturally together in a group in the rest 15 arrangements. Every row will thus contain the girls from different schools. It is because, after seven days, any two students from the same school have surely been together in a row.

E 15 elhelyezés elsőjét úgy nyerjük, hogy a fenti táblázatban ugyanazon oszlopban levő betűket helyezzük egy-egy sorba.

We get these 15 placements as follows: in the table above, take the letters in a same column, and put them in each row.

Az így keletkező elrendezés a következő:

The resulting arrangement is as follows:

[p. 35]

$$\begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ \ldots & \ldots & \ldots \\ o_1 & o_2 & o_3 \end{array}$$

A második elrendezést hasonlóképpen egy másik táblázat szolgáltatja, mely az elsőből úgy keletkezik, hogy a második sort egy, a harmadikat pedig két betűvel balra toljuk és az első betű elé kerülő betűket sorrendjük megtartásával a sor végére helyezzük.

As the second arrangement, another table is generated from the first arrangement as follows: we move one letter of the second row to the left, two letters of the third row to the left, and we place the first letters to the end of the row with keeping their order.

Így keletkezik ez a második táblázat:

Thus, the second table is produced:

$$\begin{array}{rl} \text{I.} & a_1, b_1, c_1, d_1, e_1, f_1, g_1, h_1, i_1, j_1, k_1, l_1, m_1, n_1, o_1; \\ \text{II.} & b_2, c_2, d_2, e_2, f_2, g_2, h_2, i_2, j_2, k_2, l_2, m_2, n_2, o_2, a_2; \\ \text{III.} & c_3, d_3, e_3, f_3, g_3, h_3, i_3, j_3, k_3, l_3, m_3, n_3, o_3, a_3, b_3; \end{array}$$

honnan a lányok következő második, illetve a kilencedik elhelyezkedése adódik ki.

The girls in the second table are grouped as follows, then the ninth arrangement is produced*.

*Note of the translator:

This is the ninth among 22 arrangements.

$$\begin{array}{ccc} a_1 & b_2 & c_3 \\ b_1 & c_2 & d_3 \\ c_1 & d_2 & e_3 \\ \ldots & \ldots & \ldots \\ o_1 & a_2 & b_3 \end{array}$$

Hasonló módon készíthető még 13 táblázat (a 16. már azonos lesz az elsővel) s ezekből az oszlopok sorrá alakításával nyerhető a 45 lány utolsó 13 napi elhelyezkedése.

With the similar method, 13 more tables can be produced (the 16th table will be the same as the first). Thus, by converting the columns into a row, we can get the arrangements of the last 13 days of 45 girls.

Közvetlenül látható, hogy az így nyerhető $7 + 15 = 22$ elrendezés feladatunk megoldását adja.

We can see immediately that we can get $7 + 15 = 22$ arrangements in this way, and it gives the solution to our problem.

Az alkalmazott módszer nemcsak $n = 45$ esetére nyújt megoldást, ha $n = 15$-re a feladat meg van oldva, hanem általában $n = 3k$ esetére, ha $n = k$ esetében egy megoldás ismeretes.

The applied method not only gives a solution in the case of $n = 45$, when the problem for $n = 15$ is solved. This method gives a solution generally in the case of $n = 3k$ if a solution in the case of $n = k$ is known.

Világos, hogy k csak $6r + 3$ alakú lehet, mert egyrészt páratlannak, másrészt 3-mal oszthatónak kell lennie.

It is clear that k can take only the form $6r + 3$. This is an odd number because of the right term. Moreover, it should be a number that can be divided by 3.

Az elhelyezkedések száma ez esetben — jelöljük ezt mindenkorra ν_k-val — a következő:

The number of the arrangements in this case — we always name this number as ν_k — is as follows:

$$\nu_k = \frac{k-1}{2} = 3r + 1$$

[p. 36]
Áttérve $k = 6r + 3$-ról $n = 3k = 18r + 9$ iskoláslány esetére, az elhelyezkedések száma:

Applying $k = 6r + 3$ to the case of schoolgirls $n = 3k = 18r + 9$, the number of the arrangements is:

$$\nu_{3k} = \frac{n-1}{2} = 9r + 4$$

Fennáll tehát a következő összefüggés:

There is therefore the following relation:

$$\nu_{3k} = \nu_k + k$$

Az említett módszerrel is először ν_k (a tárgyalt $k = 15$ esetben 7), azután k (15) számú elhelyezést nyerünk.

Also with the method above, we get first ν_k (7 in the related case of $k = 15$), and afterwards we get the arrangements of the number k (15).

Problémánk most már megoldottnak tekintendő az

Our problem should be regarded as already solved for the following cases:

$$n = 15, 45, 135, ..., 15i, ...$$

esetekben. Hogy még

Even the cases

$$n = 9, 27, 81, ..., 9i...$$

esetekben is megoldhassuk elegendőnek bizonyult $n = 9$ esetben egy megoldás találása.

are also can be regarded as solved if it is already proved that a solution in the case of $n = 9$ is found.

Ez a következő módon könnyen található.

This is easily found with the following method.

Írjuk a 9 iskolás lányt jelölő $a, b, ..., i$ betűt négyzetalakjában, egy-egy sorba hármat, és az első két oszlopot írjuk le még egyszer az utolsó után.

We write the 9 schoolgirls in the square form of the letters $a, b, ..., i$, three girls in each row, and we write again the first two columns after the last column.

Az így keletkező

As a result, we get the following table

$$\begin{array}{ccc|cc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array}$$

tábla közvetlenül szolgáltatja mind $\nu_9 = 4$ számú elhelyezést.

and this table immediately provides all the $\nu_9 = 4$ arrangements.

Az elsőt a sorok, a másodikat az oszlopok, a harmadikat és negyediket a két átlóval párhuzamosan elhelyezkedő 3–3 betű adja.

We put the letters 3 by 3, and get the first square from rows, the second square from columns, the third and fourth squares from two parallel diagonals.

A négy elhelyezkedés tehát a következő:

The four arrangements are as follows:

p. 37]

I.	II.	III.	IV.
abc	adg	aei	ceg
def	beh	bfg	afh
ghi	cfi	cdh	bdi

Utólag igazolható, hogy e négy elhelyezkedés megfelel a követelményeknek.

Then we can verify that these four arrangements meet the requirements.

Nem fölösleges talán megjegyezni, hogy e módszer természetszerűleg csak $\nu_n = 4$ és így $n = 2\nu_n + 1 = 9$ esetében használható.

Egész általánosságban az iskoláslányok problémája így fogalmazható:

n-számú iskoláslány naponként egyszer sétálni megy, p-számú lány jutván egy-egy sorba; hogyan kell őket naponként az $\frac{n}{p}$ sorban úgy elhelyezni, hogy $\frac{n-l}{p-1}$ nap alatt bármely kettő egyszer és csak egyszer egy sorba kerüljön.

(Ehhez valóban $\frac{n-1}{p-1}$ nap szükséges, egy nap ugyanis minden lány $p-1$ számúval jut egy sorba, összesen pedig kívüle $n-1$ lány van.)

Az általános feladatot itt nem oldjuk meg, csak azt vizsgáljuk, hogy micsoda összefüggésnek kell n és p közt lenni, hogy a feladat megoldható legyen.

A szükséges feltétel a következő: 1. n-nek oszthatónak kell lenni p-vel és 2. $n-1$-nek oszthatónak kell lenni $p-1$-gyel.

Ha mindkét feltétel teljesül, akkor világos, hogy $n-p = n-1-(p-1)$ osztható p-vel is, $p-1$-gyel is és így $p(p-1)$-gyel is, tehát megoldhatósághoz szükséges, hogy

Perhaps there is no need to say that this method is used naturally only in the case of $\nu_n = 4$ and $n = 2\nu_n+1 = 9$.

In general, the problem of schoolgirls can be described as follows:

n schoolgirls go for a walk once a day, and p girls walk together in each row; how they should be grouped everyday in the $\frac{n}{p}$ rows, so that after $\frac{n-1}{p-1}$ days any two girls are together in a row once and only once?

(Actually $\frac{n-1}{p-1}$ days are necessary for it, since every girl is together in a row with $p-1$ girls in a day, and there are $n-1$ other girls.)

We do not solve here the general problem. We only examine what should be the relation between n and p so that the problem can be solved.

The necessary conditions are as follows: 1. n should be divisible by p and 2. $n-1$ should be divisible by $p-1$.

If both conditions come true, then it is clear that $n-p = n-1-(p-1)$ is also divisible by p, by $p-1$ and by $p(p-1)$ as well. Therefore, so that the problem can be solved, the following condition is necessary:

$$\frac{n-p}{p(p-1)} = a$$

egész szám legyen, azaz n ily alakban legyen írható:

is an integer. The n in this formula can be written as follows:

$$n = p(p-1)a + p$$

Ez a keresett, n és p közt fennálló összefüggés.

This is the required relation between n and p.

5.6 Tait problémái és hasonló feladatok: Tait's problem and similar problems

*Note of the translator:
This chapter is mainly based on an article of Peter Guthrie TAIT entitled "Listing's Topologie" (1884) [197]. In addition, KŐNIG treated also a problem related to knight's move on a chessboard, which was not treated in the article of TAIT. Although the sources of the problems were different, these problems were later on related to graph theory in KŐNIG's treatise of 1936. For more precise discussion on this fact, see 6.3.

[p. 38]
1. *Négy fehér és négy fekete kavics úgy van egy sorban elhelyezve, hogy mindig felváltva egy fehér, egy fekete következik egymás után. Hogyan lehet a nyolc követ úgy áthelyezni, hogy külön a fehérek és külön a feketék kerüljenek együvé, ha két egymás mellett fekvő követ úgy szabad két egymás mellett levő üres helyre elhelyezni, hogy a két kő sorrendje ne változzék.*

1. *Four white and four black gravels are placed in a row, so that a white one always follows a black one, and vice versa. How can it be moved to eight stones for which the white stones and the black stones stand apart, if two adjacent stones can be moved to two adjacent vacant places, and if the order of the two stones should not be changed?*

Jelöljük a fehér köveket a-val, a feketéket b-vel.

We express the white stones with a, the black ones with b.

A feladat legrövidebben négy lépéssel oldható meg.

The problem can be solved with four steps at the shortest.

Ezen legrövidebb megoldást mutatja a következő táblázat:

The following table shows this shortest solution:

	•	•	a	b	a	b	a	b	a	b
1)	b	a	a	b	a	b	a	•	•	b
2)	b	a	a	b	•	•	a	a	b	b
3)	b	•	•	b	a	a	a	a	b	b
4)	b	b	b	b	a	a	a	a	•	•

Nehezebb az analóg feladat megoldása több kő esetében.

It is more difficult to solve the similar problem in the case of more stones.

Ha a fehér és fekete kövek száma külön-külön:

If the number of the white and black stones is:

$$n = 5, 6, 7, 8, 9, 10, ...$$

akkor a legrövidebb megoldáshoz szükséges lépések száma rendre:

then the number of steps necessary to the shortest solution is respectively:

$$5, 7, 7, 9, 9, 11, ...$$

[p. 39]

2. Ha a játékszabályt úgy változtatjuk, hogy *két kő áthelyezésekor sorrendjük mindenkor megváltoztatandó,* akkor az előbbi feladat ($n = 4$ esetében) csak öt lépéssel oldható meg, például a következőképpen:

2. If the rules of the game are changed so that *two stones always exchange their places when they are moved,* then the former problem (in the case of $n = 4$) can be solved with only five steps, for example as follows:

	•	•	a	b	a	b	a	b	a	b
1)	a	b	a	b	a	b	a	•	•	b
2)	a	b	a	•	•	b	a	a	b	b
3)	a	•	•	a	b	b	a	a	b	b
4)	a	a	a	a	b	b	•	•	b	b
5)	a	a	a	a	b	b	b	b	•	•

3. *Nyolc pénzdarab van az asztalon egy sorban. Hogyan és hány lépéssel lehet elérni, hogy a pénzdarabok mind párosával legyenek az asztalon, ha egy lépés abból áll, hogy egy pénzdarabot (akár jobbra, akár balra) két mellette fekvőn átugratunk és a harmadikra rátesszük?*

Jelöljük az üres helyet 0-val, s a helyet, hol 1, illetőleg 2 pénzdarab van, 1-gyel, illetőleg 2-vel.

A feladat egyik négy lépéses megoldását mutatja a következő táblázat:

3. *Eight coins are put in a row on a table. All the coins are put in pairs [two by two]. A step consists of the following acts: a coin (the right or the left) skips over two coins lying beside it, and is placed on the third coin. How and how many steps can be achieved?*

We express the empty space with 0, and the place of 1 or 2 coin(s) with 1 or 2.

One of the solutions to the problem is shown in four steps in the following table:

	1	1	1	1	1	1	1	1
1)	1	2	1	1	0	1	1	1
2)	1	2	0	1	0	1	2	1
3)	2	2	0	0	0	1	2	1
4)	2	2	0	0	0	0	2	2

Ha a pénzdarabok száma n kisebb 8-nál, a feladat nem oldható meg, ellenben ha $n > 8$ és páros, akkor feladatunknak mindig van megoldása.

If the number of coins n is less than 8, the problem cannot be solved. If, however, $n > 8$ and is an even number, then there is always a solution to the problem.

4. *Két fehér és két fekete kő úgy van egy sorban elhelyezve, hogy az 1. és 2. helyen 1–1 fehér, a 4. és 5. helyen 1–1 fekete kő van; a 3. hely üres.*

Hogyan és hány lépéssel cserélhetnek helyet a fehér és fekete kövek, ha egy lépés abból áll, hogy egy követ egy hellyel jobbra, vagy egy hellyel balra tolunk, vagy pedig egy szomszédos kövön egy üres helyre átugratunk?

Könnyen nyerjük a következő nyolclépéses megoldást (a ismét a fehér; b a fekete köveket jelöli):

4. *Two white stones and two black stones are placed in a row, so that the white stones are put one by one on the places 1 and 2; the black stones are put one by one on the places 4 and 5; the place 3 is empty.*

A step consists of the following acts: a stone moves to the right or to the left, or it skips over an adjacent stone to a blank place. How and with how many steps can the white stones and black stones exchange their place?

We get easily a solution with the following eight steps (a represents again the white stones; b represents the black stones)*:

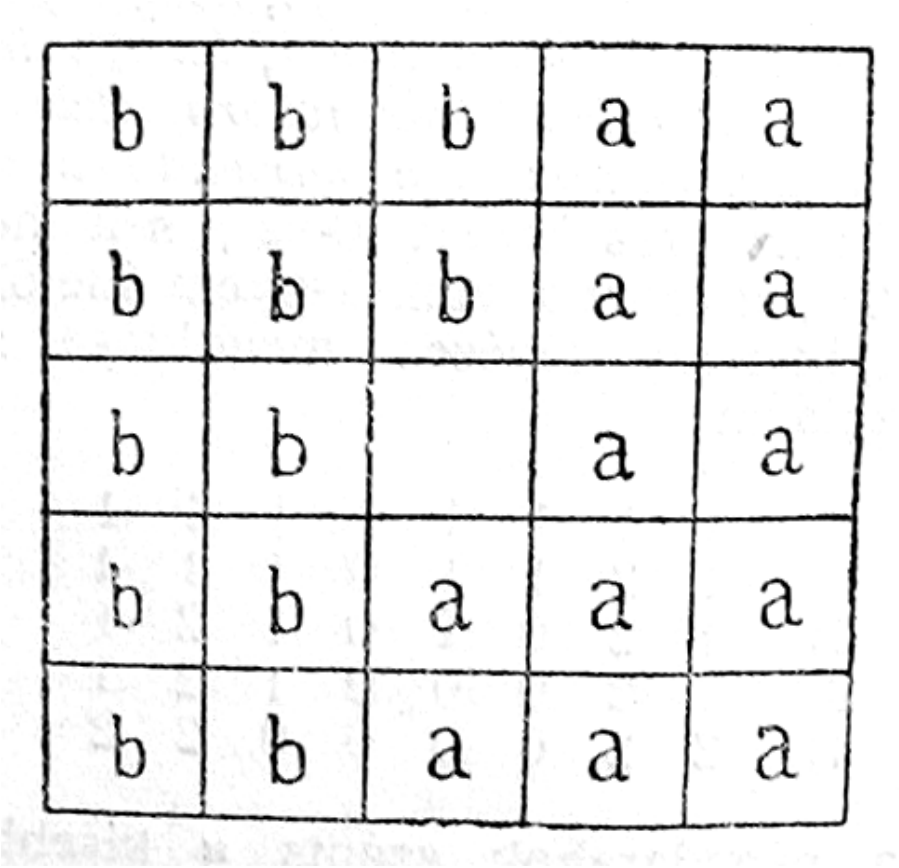
[p. 40]

	a	a	•	b	b
1)	a	•	a	b	b
2)	a	b	a	•	b
3)	a	b	a	b	•
4)	a	b	•	b	a
5)	•	b	a	b	a
6)	b	•	a	b	a
7)	b	b	a	•	a
8)	b	b	•	a	a

*Note of the translator:
The procedure consists of the following steps: 1) a moves to the right; 2) b skips over a; 3) b moves to the left; 4) a skips over b; 5) a skips over b; 6) b moves to the left; 7) b skips over a; 8) a moves to the right.

5. *Hasonló feltételekkel, mint az előbbi feladatban, hogyan cserélhetnek helyet a fehér és fekete kövek a következő táblában?*

5. *Under rules similar to those in the problem above, how can the white stones and black stones exchange their place in the following table?*



12. ábra

Először is a középső vízszintes sorban cseréljük fel a fehér és fekete köveket, az utolsó feladatban megadott módon; azután a középső függélyes sorban hajtjuk ezt végre, de nem egyhuzamban, hanem amint az egyik vízszintes sorban a középső hely megüresedik: abban a sorban cseréljük fel éppúgy, mint előbb, a fehéreket a feketékkel.

First we exchange the white stones and the black stones in the middle horizontal row with the method shown in the last problem; and then we carry it out in the middle vertical row, not [the whole step] at once, so that the middle place of one of the horizontal rows is emptied: in that horizontal row, we exchange the white ones and the blacks ones with the same method as before.

Minthogy pedig a középső függélyes sor tagjainak felcserélésekor minden hely többször is üres lesz, azért feladatunk ily módon igen röviden megoldható.

Because the middle places of horizontal rows will be empty one after another corresponding to the steps of the replacement of the stones in the middle vertical row, our problem can be solved most shortly with such a method.

[p. 41]
Mivelhogy egy sorban vagy oszlopban a fehér és fekete kövek nyolc lépéssel cserélhetnek helyet, azért ezen feladat (legrövidebben) $6 \times 8 = 48$ lépésben oldható meg.

Because the white stones and the black stones in a row or in a column can exchange their places with eight steps, this problem can be solved with $6 \times 8 = 48$ steps (the most shortly).

Ha általánosságban a négyzet egy oldala mentén $2p + 1$ mező van, akkor a feladat $2p(p + 1)(p + 2)$ lépésben oldható meg.

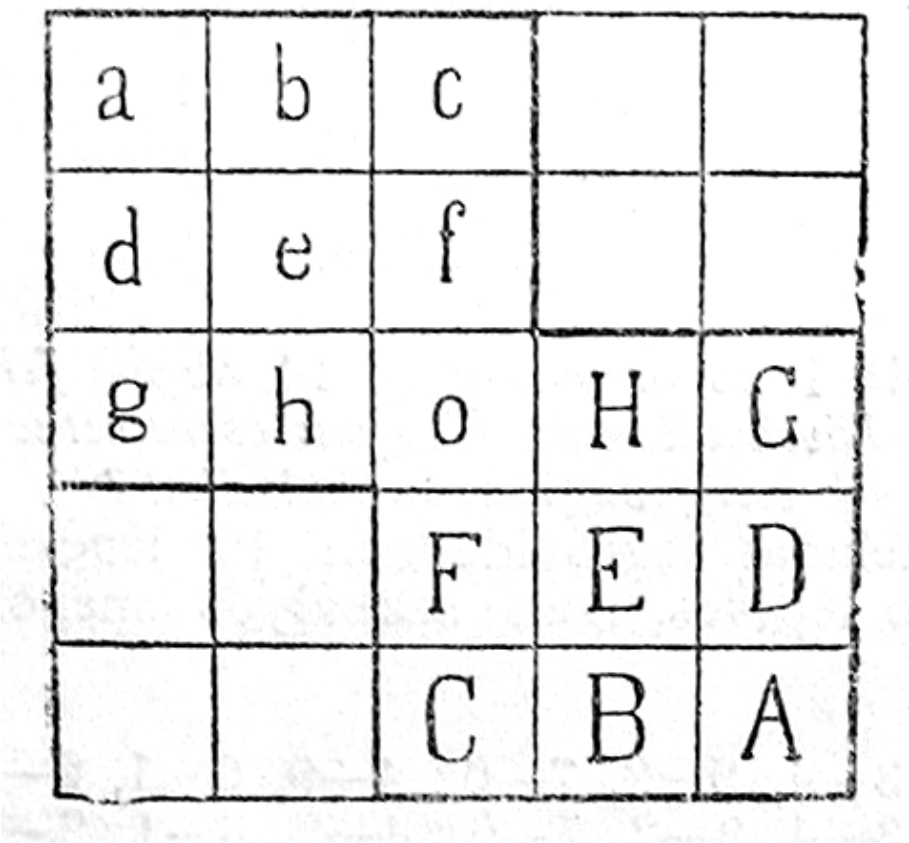
If, in general, an edge of the square has $2p + 1$ fields, then the problem can be solved in $2p(p+1)(p+2)$ steps.

6. *Az előbbi két feladat feltételeivel hogyan cserélhetnek helyet a* 13. *ábra szerint elhelyezett fehér és fekete kövek, ha csak a vastagabb vonallal bekerített területen belül mozoghatnak?*

6. *How can we change the rules of the above two problems to solve the problem in which the white stones and the black stones are placed as shown in the figure* 13, *if the stones cannot go outside of the area framed with the thick lines?*

A kis betűk a fehér, a nagyok a fekete köveket jelentik, a O-val jelzett mező üres.

The lower-case letters indicate the white stones, the upper-case letters indicate the black stones, the field expressed with O is empty.



13. ábra

A feladat sok megoldásának egyikét ábrázolja ez a sorozat.

The following series represents one of many solutions to the problem.

hHOfFEHGOcbhgdfFCOhHBACOcabhHO
cfFDGHBCOghefFOhHO

*Note of the translator:
The letters in the sequence indicate a place that becomes empty, which was initially the place o. According to the sequence of letters, for example at the first step, a white stone h moves to o, and the place h becomes empty; and at the second step, a black stone H skips over the white stone h at the place o, and comes to the place h; and so on.

Az egyes betűk jelentik itt a (13. ábra szerint) sorban annak a kőnek a helyét, melynek rendre az egyetlen (de változó) üres helyet kell elfoglalnia.

A feladat oly könnyen megoldható, hogy inkább az lehet a kérdés, hogy ki oldja meg rövidebben.

[p. 42]
Nehéz lesz azonban a feladat, ha kikötjük, hogy a fehér kövek csak jobbra és lefelé, a feketék pedig csak balra és felfelé mozoghatnak.

A fenti megoldás ezen megszorításnak is eleget tesz.

A következő feladat megoldását sem könnyű megtalálni.

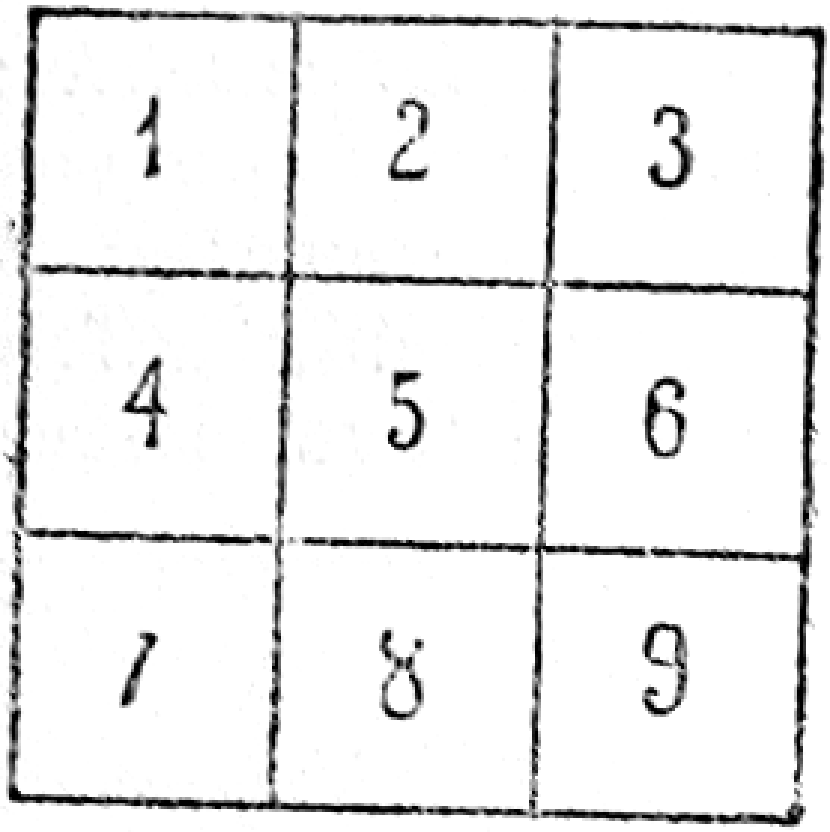
The single letters mean here (according to the figure 13) the place of the stone in the row, where the only (but variable) blank should successively occupy.

The problem can be solved so easily that it can be rather a question to be solved more shortly.

However, the problem is difficult if we insist that the white stones can move only to the right and down, while the black ones only to the left and upwards.

The solution shown above satisfies also these conditions.

It is not easy to find solutions to the following problem.



14. ábra.

7. *Az* 1. *és* 3. *mezőn (l.* 14. *ábrát) fehér, a* 7. *és* 9-*en fekete kövek állanak. Hogyan cserélhetnek ezek helyet ha mindegyik csak lóugrásban haladhat?*

7. *White stones are in the fields* 1 *and* 3 *(see Figure 14), and black stones in the fields* 7 *and* 9. *How can they exchange their places if every stone can move only in the knight's move?**

*Note of the translator:
This problem originated in a manuscript of P. Guarini di Forli (1512). Lucas [152] and Ahrens [1] treated this problem in their books. We will discuss the relation between Tait's problems, knight's moves and graph theory in 6.3.

A feladat legrövidebben 16 lépésben oldható meg.

E 16 lépéses, igen szabályos megoldás a következő:

The problem can be solved the most shortly in 16 steps.

The most regular solution in these 16 steps is as follows:

1–8, 3–4, 9–2, 7–6; 4–9, 6–1, 2–7, 8–3;
1–8, 3–4, 9–2, 7–6; 4–9, 6–1, 2–7, 8–3.

5.7 Elhelyezkedések körben: Positions on a ring

[p. 43]

1. *Hogyan lehet körben álló gyerekeket különböző módokon úgy felállítani, hogy mindegyik mindegyiknek egyszer és csak egyszer legyen szomszédja?*

Minden gyereknek egyszerre két szomszédja van s így azon gyerekek száma, kik már pld. A-nak szomszédjai voltak, mindig páros.

A-t is beszámítva tehát, a gyerekek számának páratlannak kell lenni, hogy feladatunk megoldható legyen.

Ennek és a következő feladatnak megoldását egy-egy geometriai ábra segítségével lehet legkönnyebben nyerni.

Könnyebb érthetőség kedvéért legyen a gyerekek száma $2n + 1 = 11$, bár a következő meggondolás egész általánosan minden n-re használható.

1. *How is it possible to let children stand together on a ring, and let them exchange their positions so that every child will be a neighbor [to each of the other children] once and only once?*

All the children have two neighbors, and thus the number of children who will be, for example, A's neighbors is always even.

Therefore, the number of children including A must be an odd number, so that the problem can be solved.

This problem and the following problem can be solved the most easily with the help of a geometric figure.

To understand more easily, let the children's number be $2n+1 = 11$, although the consideration below can be applied generally to any n.

Osszuk fel egy kör területét (l. a 15. ábrát) $2n = 10$ egyenlő részre s írjuk az osztópontokhoz a gyerekeket jelölő $B, C, D, ..., K$ betűket.

Let us divide a round area into (see Figure 15) $2n = 10$ equal parts [of circumference], and identify the children at the dividing points with letters $B, C, D, ..., K$.

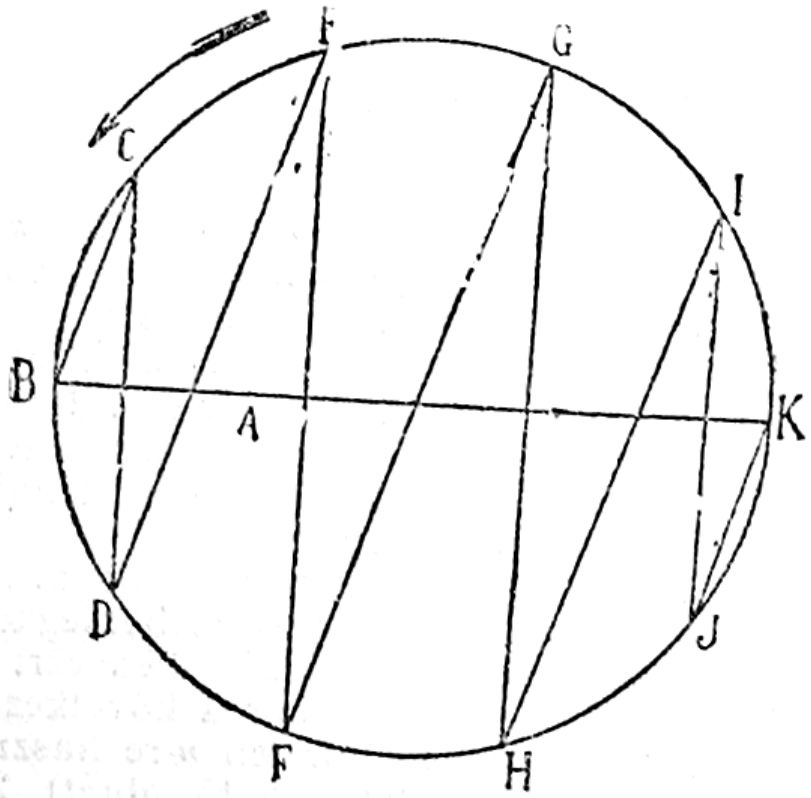
Az A betűt, mint a rajzban látható, az átmérőre írjuk.

The letter A is, as shown in the drawing, written on the diameter.

Legyen a gyerekek első elhelyezése a következő:

Let the first disposition of the children be as follows:

I. *A B C D E F G H I J K*



15. ábra.

mint azt a törtvonal az ábrában mutatja (az első betű az utolsó után következőnek tekintendő).

As shown in the figure, children stand on the points on the ring cut by a straight line (we consider that the first letter succeeds to the last letter)*.

*Note of the translator:
We should notice that not the circle but the straight lines connect neighbors. The solution of the problem is represented by the sequences of letters on the straight lines.

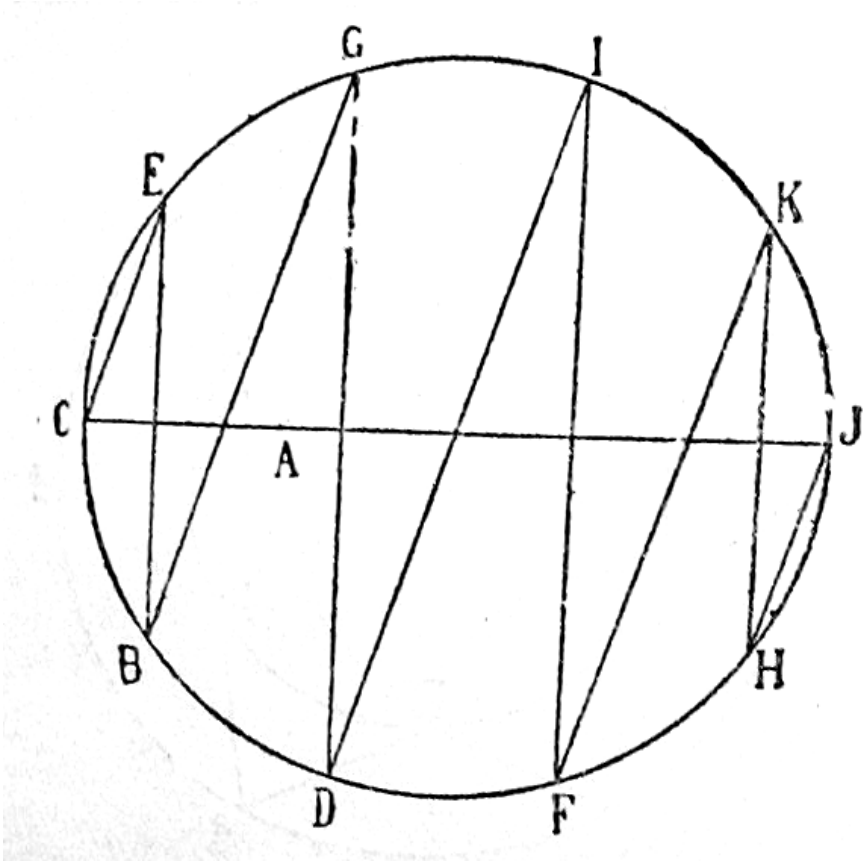
[p. 44]
Hogy a gyerekek második elhelyezését nyerjük, forgassuk el a kört, az annak pontjait jelölő betűkkel együtt a nyíl irányában $\frac{360}{2n} = 36$ fokkal, változatlanul hagyva az ábra egyenes vonalait s az ezen lévő A betűt.

We obtain the second disposition of the children as follows: rotate the circle together with the points indicated with letters in the direction of the arrow by $\frac{360}{2n} = 36$ degrees, changing neither the straight lines nor the letter A on the line of the figure.

Ha a törtvonal mentén most álló számokat sorban leírjuk (l. a 16. ábrát), nyerjük a gyerekek második elhelyezését:

If we describe here the letters along the dividing lines (see Figure 16), we get the second arrangement of the children:

II. *A, C, E, B, G, D, I, F, K, H, J.*



16. ábra.

Forgassuk el ugyanannyival tovább a kört, akkor a következő sorrendben nyerjük a betűket:

We rotate the circle more in the same way, then we get the letters in the order as follows:

III. *A, E, G, C, I, B, K, D, J, F, H*

Hasonlóan nyerjük a gyerekek negyedik és ötödik elhelyezését is:

Similarly, we will get the fourth and fifth arrangements:

[p. 45]

IV. *A, G, I, E, K, C, J, B, H, D, F;*
V. *A, I, K, G, J, E, H, C, F, B, D.*

Így minden gyerek egyszer és csak egyszer volt a többinek szomszédja.

Thus, all children once and only once was a neighbor of the others.

Minthogy egy gyereknek egyszerre csak két szomszédja lehet, azért valóban n különböző elhelyezésre van szükség, hogy a többi $2n$ gyerek egyszer melléje juthasson.

One child can have only two neighbors at once, thus it is necessary that the child stands [successively] at n different locations, and that [each of] the other $2n$ children stands only once beside the child.

2. *Egy asztal körül hölgyek és urak egyenlő számban ülnek, mindig fölváltva egy hölgy, egy úr. Többszörös helyváltoztatással hogyan érheti el a társaság, hogy minden hölgy egyszer és csak egyszer kerüljön ugyanazon szomszédságba és fordítva?*

2. *An equal number of women and men sat around a table, always alternately a lady next to a gentleman and vice versa. With multiple replacements, how can every woman sits besides each man once and only once, and vice versa [every men]?*

Minthogy minden hölgynek egyszerre két szomszédja van, azért a feladat akkor oldható meg, ha az urak és hölgyek száma külön-külön páros, tehát $2n$ alakú.

Each woman has always two neighbors. Therefore, the problem can be solved if each of the number of men and the number of women is an even number, thus in the form $2n$.

[p. 46]
Legyen pld. $2n = 6$.

For example, let $2n = 6$.

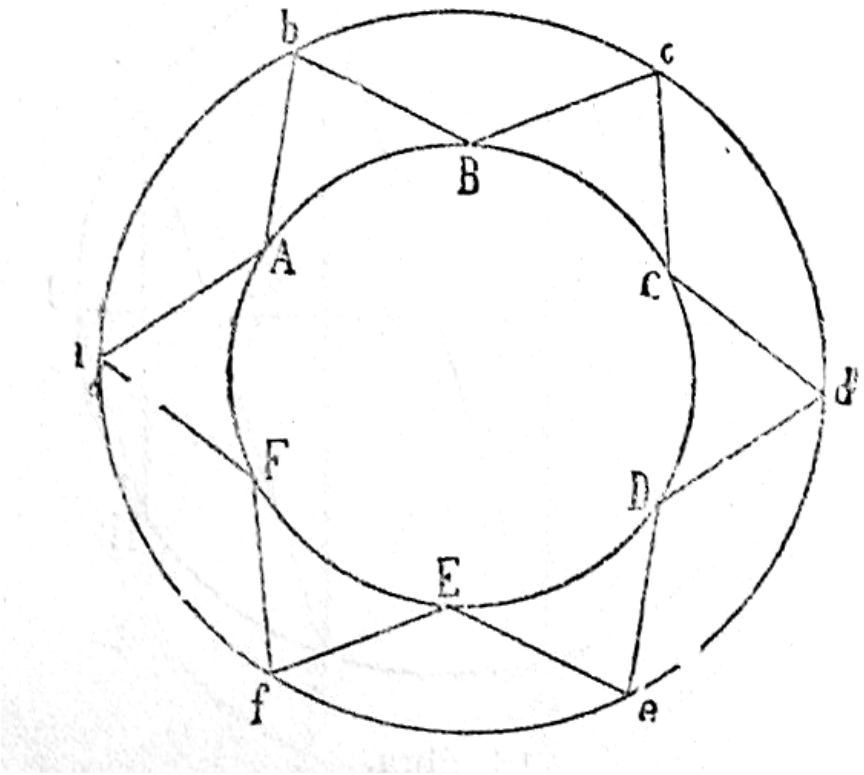
Jelöljük a hölgyeket a külső kör a, b, c, d, e, f pontjaival (l. a 17. ábrát) s az urakat a belső kör A, B, C, D, E, F pontjaival.

We express the women of the outer circle with points a, b, c, d, e, f (see Figure 17) and the men of the inner circle with points A, B, C, D, E, F.

Az ábrából kitűnik a társaság első elhelyezkedése:

The diagram shows the first arrangement of the group:

I. *a, A, b, B, c, C, d, D, e, E, f, F* *



17. ábra.

(az első *a* betű ismét az utolsó *F* után következőnek tekintendő).

(the first letter *a* continues again after the last F)**.

*Note of the translator:
Although the letters in the figure is not very clear, the letters are displaced on the circles clockwise in alphabetical order.

** As same as the problem of children, the straight lines connect neighbors. Because the whole connected straight lines form a loop, a is adjacent to F.

Hogy a társaság második és harmadik elhelyezését nyerjük, forgassuk el a belső kört középpontja körül, míg A a C, majd az E helyébe jut. Így nyerjük a következő elhelyezéseket:

To obtain the second and third arrangements of the group, we rotate the inner circle around the center. A is then replaced by C, and then E. Thus we get the following arrangements:

II. *a, E, b, F, c, A, d, B, e, C, f, D;*
III. *a, C, b, D, c, E, d, F, e, A, f, B.*

Az I., II., III. elhelyezésében valóban minden kis betű egyszer és csak egyszer fordul elő minden nagy betű mellett — és viszont.

In the arrangements I, II and III, every lowercase letter occurs indeed adjacently to every uppercase letter once and only once — And vice versa.

[p. 47]

3. *Egy asztal körül n-számú házaspár ül; úgy akarnak elhelyezkedni, hogy férj és feleség ne kerüljön egymás mellé és mégis hölgynek csak urak, úrnak csak hölgyek legyenek szomszédjai. Lehetséges-e az ilyen elhelyezkedés, és ha igen, hányféle megfelelő elhelyezkedés létezik?*

3. *n couples sit around a table; let them dispose so that a husband and a wife should not be side by side, and yet men are adjacent only to women, women only to men. Is such a position possible? If so, how many proper positions are there?*

A feladat $n = 2$ esetében nem oldható meg, de minden nagyobb n-re megoldható.

The problem cannot be solved in the case of $n = 2$, but can be solved for all the larger n-s.

Jelöljük az urakat a nagy $A, B, C, ...$ betűkkel és nejeiket a megfelelő kis betűkkel.

We express the men with uppercase letters $A, B, C, ...$, and their wives with the suitable lowercase letters.

Világos, hogy $n = 3$ esetében csak egy megoldás létezik, amidőn t. i. minden asszony urával szemben ül:

It is clear that, in the case of $n = 3$, there is only one solution. That is, every woman sits not adjacently to her husband:

$$AbCaBc$$

képviseli ezen elhelyezkedést.

Négy házaspár esetében ez a két megoldás létezik:

represents this arrangement.

In the case of four couples, the following two solutions exist:

$$AcBdCaDb$$

és

and

$$AdBaCbDc$$

Az n növekedésével a megoldások száma rohamosan nő, $n = 5$ esetében például már 13 megoldás van.

Ezek egyike a következő:

If n becomes larger, the number of solutions becomes much larger. For example, in the case of $n = 5$, the number of solutions is already 13.

Here is one of the solutions:

$$AcBaCeDbEd$$

4. *Egy hajóban 15 keresztyén és 15 török ül. Vihar keletkezik és így a hajó legfeljebb 15 utast tud partra szállítani. Megállapodnak, hogy a sorsra bízzák e 15 utas kiválasztását. Ezért körbe állnak és egy bizonyosnál elkezdve minden kilencediket a tengerbe dobják, mindaddig, míg csak 15-en maradnak a hajón. Milyen rendben állott fel a 30 utas, ha épen a 15 török került a tengerbe, anélkül, hogy csak egy keresztyén is elpusztult volna?*

4. *15 Christian and 15 Turks are seated on a boat. Storm occurs, therefore the boat can carry only passengers up to 15 to the coast. They agree that this selection of 15 passengers is left to the fate. Therefore, they do a round starting from a certain one, and the ninth is thrown into the sea until the moment when only 15 people remain on the boat. How were the 30 passengers properly arranged if only the 15 Turks got into the sea without any Christian being thrown?**

*Note of the translator:
This problem is found not in the books *Récréations mathématiques* (1882–1894) but on p. 12 of *Arithmétique amusante* (1895) by Lucas, though Kőnig does not refer to this part of the book. On the other hand, Ahrens' *Mathematische Unterhaltungen und Spiele* (1901), to which Kőnig referred, includes also this problem (p. 287). See Section 5.10.2 for Kőnig's references.

A feladat a következőképpen könnyen megoldható:

Vegyünk fel egy egyenesen vagy egy körön 30 pontot (utóbbi esetben is az első pont az utolsó után [p. 48] következőnek tekintendő) és az elsőnél kezdve a számlálást, jelöljünk meg minden kilencediket, míg csak 15-öt meg nem jelöltünk (a megjelölt pontok később számításon kívül hagyandók).

A megjelölt 15 helyet fogja a 15 török elfoglalni, a többit a 15 keresztyén.

This task is easily solved as follows:

Let 30 points be on a straight line or a circle (in the latter case, the first point to be considered as coming after the last one). Start the count from the first one, let us mark every ninth point, until the moment when only 15 points are left unmarked (the marked points are ignored in the next count).

The marked 15 places will be occupied by 15 Turks, the others by 15 Christians.

Így nyerjük a következő elhelyezést: 4 keresztyén, 5 török, 2 k., 1 t., 3 k., 1 t., 1 k., 2 t., 2 k., 3 t., 1 k., 1 t., 2 k., 1 t., mely könnyen észbentárható a következő latin vers segítségével:

Thus the following arrangement is obtained: 4 Christians, 5 Turks, 2 C. 1 T. 3 C. 1 T. 1 C. 2 T. 2 C. 3 T. 1 C. 1 T. 2 C. 1 T. It can be easily expressed with the help of the following Latin poem:

Populeam virgam mater regina ferebat

hol az egyes magánhangzók az ábécé szerinti sorrendjükben az 1–5 számokat jelölik, *a, e, i, o, u* rendre 1-et, 2-t, 3-at, 4-et, 5-öt jelent.

where the single vowels indicate the 1–5 numbers according to the alphabet, that is, *a, e, i, o, u* means 1,2, 3,4,5 successively.

A magánhangzók sorrendje e versben:

The sequence of vowels in this verse is:

$$o, u, e, a, i, a, a, e, e, i, a, e, e, a$$

mely tehát a számok következő sorához vezet:

which thus leads to the following numbers:

$$4, 5, 2, 1, 3, 1, 1, 2, 2, 3, 1, 2, 2, 1$$

Ilyen számban ülnek felváltva keresztyének és törökök a hajóban.

Following this sequence of numbers, Christians and Turks sit in the boat.

A 15 keresztyén és 15 török feladatához nagyon hasonlít a következő feladat.

The problem of 15 Christians and 15 Turks is very similar to the following problem.

5. *Josephus egy ízben 40 társával egy pincébe menekült. Társai elhatározzák, hogy inkább megölik magukat, semhogy az ellenség elfogja őket. Körbe állnak és megegyeznek, hogy minden harmadikat megölik, az utolsó pedig maga ölje meg magát. Melyik helyre állott Josephus s hova állította legjobb barátját, úgy hogy ő utolsónak, barátja pedig utolsó előttinek maradt?*

Ennek a feladatnak a megoldását is ép úgy nyerhetjük, mint az előbbiét: Josephus a 31., barátja pedig a 16. helyre állott.

Ugyancsak egy bizonyos sor minden n-edik tagjának kiválasztásán alapszik a következő feladat.

6. *Kezünkbe veszünk bizonyos számú kártyát, a legfelsőt leteszszük az* [p. 49] *asztalra, a másodikat pedig a kezünkbe levő kártya-csomó alá, a harmadikat megint az asztalra, az elsőre rá, a negyediket a kártyák alá, stb. egész addig, míg az összes kártyákat le nem raktuk. Mily sorrendben kell a kártyákat a kezünkbe venni, hogy a kártyák az asztalon bizonyos előre megállapított sorrendben következzenek egymásután?*

Vegyünk pld. 4 kártyát és jelöljük ezeket az 1, 2, 3, 4 számokkal.

Ha a következő sorrendben veszszük kezünkbe a kártyákat:

5. *Josephus escaped into a cellar with 40 fellow refugees. His fellows decide that they rather kill themselves than being captured by the enemy. They do a round and count so that every third one is killed, and let the last one kill himself. Which place should Josephus begin with, and where Josephus and his best friend should be placed, so that he is the last one, and his friend left just before the last?*

We can solve also this problem in a similar way to the previous problem: Josephus stands at the 31st place, and his friend stands at the 16th place.

Similarly, The problem below is based on the selection of the n-th member on a line.

6. *Take a certain number of cards in our hand, and put the top card down on the table; put the second card under the bundle of cards in our hand, the third card again on the table on which the first one was put, the fourth card under the cards, etc. until the time when all cards are put on the table. In what order the cards in our hands should be taken in advance, so that the cards on the table will be ordered properly?*

Take, for example, 4 cards, and mark them with numbers 1, 2, 3, 4.

If we take the following sequence of cards in our hands:

1, 2, 3, 4 (I)

akkor az asztalon a	then [after the above-mentioned procedure], the cards on the table will be ordered as follows:

4, 2, 3, 1 (II)

sorrendben fognak feküdni. Ha pld. az	If, for example, we want to obtain the cards on the table in the following order:

1, 2, 3, 4

sorrendet akarjuk elérni, vagyis hogy az első helyre	that is, if we want the order above be changed as follows:

4	helyébe (to)	1
2	„	2
3	„	3
1	„	4

jusson, akkor (I)-ben is így kell a számokat felcserélni; a	then also in (I), the numbers should be replaced; from the order

4, 2, 3, 1

elhelyezésből valóban a kívánt	indeed the desired order

1, 2, 3, 4

sorrendhez jutunk.	is obtained.

Ugyanezen eljárást használhatjuk, ha a kártyák száma nagyobb.

The same procedure can be used even if the number of cards is larger.

7. Más feladat lehet meghatározni, hogy *n számú kártya esetében a fenti eljárást többször ismételve, hányadszorra* (x) *nyerjük vissza az eredeti sorrendet.*

7. We can suggest another problem below. *With n cards, repeat the procedure above several times. After how many tries* (x)*, we get again the original order?*

Hogy ezt mindig vissza kell valamikor nyernünk, az természetes, mert hiszen n elem $n!$-féleképp, tehát véges számú módon helyezhető el egymás után.

It is natural that we must get sometime the original order again. Since there are n elements, there are $n!$ different ways, which can thus be placed after each other, on a finite number of manners.

[p. 50]
Ha pld. $n = 5$ kártyánk van

If, for example, there are $n = 5$ cards

1, 2, 3, 4, 5

sorrendben, akkor rendre a következő elrendezéseket nyerjük:

then, we get the following order:

1) 5, 3, 1, 2, 4
2) 4, 1, 5, 3, 2
3) 2, 5, 4, 1, 3
4) 3, 4, 2, 5, 1
5) 1, 2, 3, 4, 5

tehát $x = 5$-ödszörre nyertük vissza az eredeti sorrendet.

Therefore at $x = 5$ we got again the original order.

Hasonlóan nyerjük, hogy

We get similarly that

$$x = 2, 3, 2, 5, 4, 6, 6, 8, 6, ...$$

ha	if

$$n = 2, 3, 4, 5, 6, 7, 8, 9, 10, ...$$

5.8 Átkelési, átöntési és vasúti feladatok: Problems of traversing, pouring and railway

[p. 51]
E fejezetbe tartozó feladatok között a legismertebb a következő:

1. *Egy révésznek át kell szállítani a folyón egy farkast, egy kecskét és egy káposztafejet. Csónakja azonban olyan kicsiny, hogy rajta kívül legfeljebb a két állat egyike vagy a káposztafej fér még el. Természetesen az átszállításnál vigyáznia kell a révésznek, hogy sem a farkas és a kecske, sem a kecske és a káposzta ne maradjon együtt sem az innenső, sem a túlsó parton, ha ő maga nincs jelen.*

A megoldás igen egyszerű.

Átviszi a kecskét és visszajön a farkasért (vagy a káposztáért), ezt átviszi és a kecskét megint visszahozza.

Most átviszi a káposztafejet (illetőleg a farkast), úgy hogy az innenső parton marad még a kecske s a túlsón van a farkas és a káposzta, végül visszajön a révész és átviszi a kecskét is.

Among the problems in this chapter, the best-known is the following problem:

1. *A ferryman has to transport a wolf, a goat and a cabbage across the river. His boat is so small that he can transport at best two among animals and a cabbage apart from himself. Of course the ferryman has to manage the transfer [on the boat]: neither the wolf and the goat, nor the goat and the cabbage must not be left together either on the hither side or on the opposite side of the river, if the ferryman is not beside them.*

The solution is simple.

Carry over the goat and comes back for the wolf (or for the cabbage), carry this over and bring back the goat again.

Carry over the cabbage now (which joins the wolf), so that the goat is left on the hither side yet, and the wolf and the cabbage are on the opposite one. The ferryman comes back finally and carries over the goat.

2. *Egy szakasz katonaság egy folyóhoz jut, melyen nem vezet át híd. Két kisfiú játszadozik egy csónakban, mely azonban oly kicsiny, hogy legfeljebb egy katonát bír el,* [p. 52] *de a két gyerek egyszerre is elfér benne. Hogyan kelhet át az egész katonaság a folyón?*

Az eljárás a következő:

A két fiú átevez a túlsó oldalra, egyikük kiszáll s a másik visszahozza a csónakot. Egy katona egyedül átkel és a túlsó parton lévő fiú visszahozza a csónakot.

A két fiú megint együtt átevez, az egyik a túlsó parton marad, a másik áthozza a csónakot; egy katona most megint átkel, stb. stb.

Így tehát kétszer annyi ide- és odacsónakázásra van szükség, mint ahány katona van a szakaszban, hogy az egész szakasz a túlsó oldalra jusson.

3. *Három úr három szolgájával egy folyón akar átkelni. A csónak, melyet a parton találnak, legfeljebb két embert bír el. Hogyan fogják az urak az átkelést intézni, ha, félve a szolgáktól, nem akarják, hogy (akár az innenső, akár a túlsó parton) a szolgák többségben legyenek felettük?*

2. *A section army guards a river, across which there is no bridge. Two little boys play in a boat. This boat is too small to hold more than a soldier, but these two children fit well simultaneously. How can all the soldiers get across the river?*

The procedure is as follows:

The two boys row to the other side; one of them gets out and the other one brings back the boat. A soldier gets across alone, and the boy on the opposite side brings back the boat.

The two boys row together again, one of them stays on the opposite side, the other one brings back the boat; a soldier gets across again now, etc. etc.

Thus, it is necessary to row here and there twice as many times as the number of soldiers in the section for the entire section to reach the other side.

3. *Three lords want to cross a river with three servants. The boat, which is found on the riverside, is able to carry at most two men. How will the lords arrange the crossing if, being afraid of the servants, they do not want to let the number of the servants larger than themselves. (both on this side, and on the opposite side)?*

A legrövidebb megoldás a következő:

Az első szolga átviszi először a másodikat, aztán a harmadikat és visszahozza a csónakot.

Most áthajózik két úr és visszatér egy úr egy szolgával, úgy, hogy még két úr és szolga van az innenső parton.

A két úr átkel, s a túlsó oldalon lévő szolga egymásután átviszi két társát.

4. *Három féltékeny férj feleségével egy folyón akar átkelni. Egyetlen hajójukban legfeljebb két személynek van helye. Hogyan fognak e csónak segítségével mind a hatan a túlsó partra jutni, ha a férjek megállapodtak, hogy egyik asszony se lehessen férfitársaságban, ha ura nincs jelen?*

A megoldás egészen a 3-as feladat mintájára történhetik:

Két asszony átevez, az egyik visszajön és átviszi a harmadikat, majd áthozza a csónakot és férjével az innenső parton marad, míg a másik két férj átevez.

Az egyik mindjárt visszajön feleségével, ezt itt hagyja és átviszi a harmadik férjet.

The shortest way is as follows:

The first servant carries over the second servant first, carries the third servant secondly, and then brings back the boat.

Two lords row over now and one lord returns with one servant, then there are still two lords and one servant on the hither side.

The two lords get across, and the servant on the other side carries over his two fellows.

4. *Three jealous husbands and their wives want to cross a river. At most two persons can be carried at once with their boat. With this boat, how will all the six persons manage to get to the opposite side, if the husbands came to an agreement that a woman cannot be with a man if her husband is not present?*

The solution can be entirely based on the sample of the 3rd problem:

Two women row, one of them comes back, and carries over the third woman. She brings over the boat, and she is left on this side with her husband, while the other two husbands row over.

One of them is back with his wife, leave the wife here, and carry forward the third husband.

A túlsó parton lévő asszony aztán átviszi egymásután a másik [p. 53] kettőt, vagy csak az egyiket; a másikért akkor ura jön át.

The woman on the opposite side carries over the other two, or only one of them; for the other woman, her husband comes round.

Ezt az egész megoldást magában foglalja a következő kis latin vers:

The whole of this solution is included in the following little Latin verse:

It duplex mulier, redit una, vehitque manentem
Itque una, utuntur tunc duo puppe viri.
Par vadit, redeunt bini; mulierque sorores
Advehit; ad propriam sive maritus abit.

A feladat igen sok irányban általánosítható (a partok közt szigetek terülhetnek el, az átkelő házaspárok száma nagyobb lehet; sőt vannak olyan tárgyalások is, melyek a bigámiát is megengedik), de ezen feladatokkal itt nem foglalkozunk.

The problem can be generalized in a great many directions (islands can be extended between the riversides, the number of crossing couples can be larger; there are indeed discussions in which the bigamy is allowed), but we do not deal with these problems here.

4. *Két embernek 8 liter bora van egy 8 literes edényben, hogyan felezhetik meg ezt a bort, ha a 8 literes edényen kívül csak egy 5 literes és egy 3 literes edény áll rendelkezésükre?*

5.* *Two men have 8 litres of wine in an 8-litre pot. How this wine can be divided into halves if, apart from the 8-litre pot, only 5-litre pot and 3-litre pot are at their disposal?*

*Note of the translator:
"4" in the original text may be a misprint. The number of the paragraph should be 5.

A megoldást találgatás, próbálgatás segítségével nyerhetjük. A legegyszerűbb megoldás ez:

We can get the solution with the help of guessing and experimenting. The simplest solution is this:

Az 1. (8 literes) edényből telitöltjük a 2.-at (az 5-literest) ebből pedig a 3. (3 lit.) edényt, úgy, hogy az első edényben: 3, a 2.-ban: 2, a 3.-ban: 3 liter bor lesz.

From the 1st (8-litre) pot we charge the 2nd (5-litre), from this we charge the 3rd (3-litre) pot, so that the wine will be in the first pot: 3, in the second pot: 2, in the third pot: 3 litres.

A 3.-ban lévő 3 liter bort az 1.-be, a 2.-ban lévő 2 litert a 3.-ba öntjük s az elsőből telitöltjük a 2.-at; ekkor a három edényben 1, illetőleg 2 liter bor lesz.

We pour 3 litres of wine in the 3rd pot into the 1st pot, 2 litres in the 2nd pot into the 3rd, and from the first pot we charge the 2nd pot; at this time the wine will be in the three pots, 1 or 2 litres*.

*Note of the translator:
1st: 1 litre; 2nd: 5 litres; 3rd: 2 litres.

A 2.-ból most megtöltjük a 3.-at, s az ebben összekerülő 3 litert az elsőbe öntjük.

from the 2nd we fill now the 3rd, and we give the 3 litres in the 3rd into the first.

Ezzel a feladat meg van oldva, mert az 1. és 2. edényben 4–4 liter bor van.

This problem is solved, because 4 litres of wine are in each of the 1st and the 2nd pots

A feladat ezen megoldását mutatja a következő könnyen megérthető kis táblázat:

The following little table shows this solution of the problem, which can be understood easily:

[p. 54]

8	5	3
literes edény		
8	0	0
3	5	0
3	2	3
6	2	0
6	0	2
1	5	2
1	4	3
4	4	0

Más megoldást mutat a következő táblázat:

Another solution is shown in the following table:

8	5	3
literes edény		
8	0	0
5	0	3
5	3	0
2	3	3
2	5	1
7	0	1
7	1	0
4	1	3
4	4	0

[p. 55]

6)* *Három ember 24 liter bort kap egy 24 literes edényben; ezenkívül van még egy 5, egy 11 és 13 literes üres edényük. Hogyan fognak a boron egyenlően megosztozni?*

6. *Three people get 24 litres of wine in a 24-litre pot; there are also a 5-litre, a 11-litre and a 13-litre empty pots. How will they share the wine equally?*

*Note of the translator:
This ")" should be a misprint of ".".

A feladat legegyszerűbb két megoldását mutatja a következő két táblázat:

24	13	11	5
literes edény			
24	0	0	0
0	8	11	5
16	8	0	0
16	0	8	0
3	13	8	0
3	8	8	5
8	8	8	0

the simplest two solutions to the problem are shown with the following two tables:

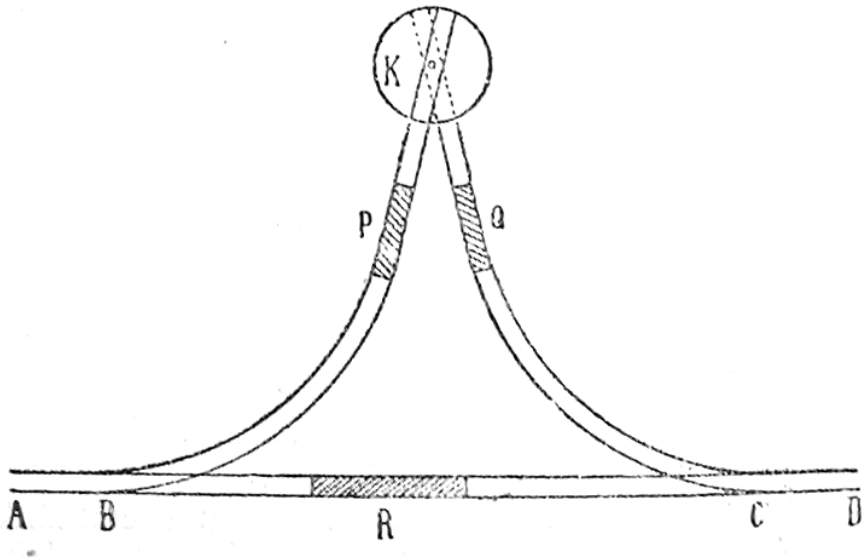
24	13	11	5
literes edény			
24	0	0	0
8	0	11	5
8	11	0	5
8	13	0	3
8	13	3	0
8	8	3	5
8	8	8	0

Végül még három másfajta feladatot említünk.

7. *Az ABCD sínpárt a forgó K koronggal két sínpár, BK és CK köt össze (lásd a 18-dik ábrát).*

Finally, we mention three other problems.

7. *Two tracks BK and CK with a turntable K are connected to the track ABCD (see Figure-18).*



18. ábra.

A K korong arra való, hogy a BK sínpárról KC sínpárra (vagy fordítva) vigye a vasúti kocsikat. A hosszabb lokomotív azonban már nem fér el rajta. Egy P-vel jelölt kocsi a BK sínen, egy másik, Q a CK sínen van, az R lokomotív pedig a B és C közti sínpáron áll. Kérdés, hogy mikép viheti az említett feltételek mellett az R lokomotív a P kocsit Q helyére, Q-t pedig P helyére.

The turntable K change the way from the track BK to the track KC (or vice versa); let it carry the railway carriages. However, the locomotive [R] is longer than K, and there is no room for it on K. A car marked with P is on the rail BK, and a car Q is on the rail CK. The locomotive R is at a standstill on the track between B and C. The question is how the locomotive R can carry the car P to the place where Q is, and the car Q to the place where P is, under the conditions mentioned above.

A feladat a következő öt lépésben oldható meg:

1. Az R lokomotív eltolja a P kocsit a K korongra és

The problem is solved by the following five steps:

1. The locomotive R pushes away the car P onto the turntable K, and

2. B-n és C-n keresztül, kétszer irányt változtatva, CK-ra jut, hol a Q kocsit tovább tolja egészen P-ig, [p. 56] melyet időközben a korongon átfordítottak a CK sínpárra.

3. P-t és Q-t összekapcsolva, R mindkettőt BC-re viszi, P-t otthagyja, Q-t pedig ugyanazon az úton, melyen jött, visszaviszi a K korongra.

4. R visszajön P-ért s azt C-n át Q régi helyére CK-ra viszi és végül.

5. C-n és B-n keresztül, kétszer irányt változtatva BK-n Q-hoz kapcsolódik, melyet időközben átfordítottak a KB sínre, és elviszi Q-t P régi helyére, BK-ra. Ezzel a követelt helycsere megtörtént.

A 18. ábrán látható sínrendszer alkalmat ad a következő kérdésre is.

8. *B és C közt egy lokomotív és egy vasúti kocsi áll. Miképpen cserélhet a kettő helyet?*

A megoldást, mely egészen az előbbi mintájára nyerhető, itt nem részletezzük.

2. R goes through B and C, changes its direction twice, comes to CK, where it pushes the car Q entirely to P, which is meanwhile rotated to the track CK.

3. Connecting P and Q, R carries both to BC, leaves P, carries Q back to the same way as it came, puts it back onto the turntable K.

4. R returns for P, and takes it through C to the place on CK where Q were.

5. R goes through C and B, changes its direction twice, through BK approaches Q, which is meanwhile rotated to the track KB, and takes Q away onto BK where P were before. The required exchange of places thus occurred.

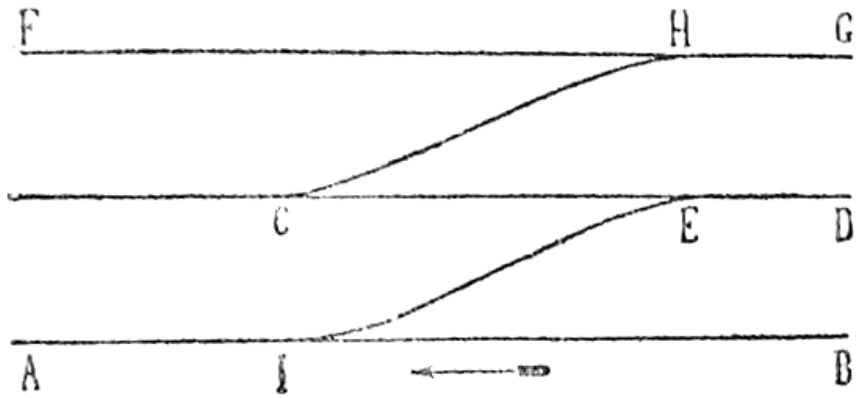
The railway systems shown in Figure 18 will give also the following question.

8. *A locomotive and a rail car are between B and C. How can they exchange their two places?*

We can get the solution entirely from the former example, therefore we do not precisely describe it here.

9. *Valamely vasúti állomás sínrendszerét a 19. ábra mutatja, hol minden sínpár egy vonallal van ábrázolva. Az AB, CD és FG síneket csak az IE és CH sinek kapcsolják össze. Egy vonat érkezik az AB sínpárra BA irányban. Az elöl lévő lokomotívtól számítva a 9-ik és a [p. 57] 12-ik kocsit a vonatnak az állomáson és pedig a GH sínrészen kell hagyni s a többi kocsival folytatni útját. Miként intézhető mindez a legegyszerűbben?*

9. *Figure 19 shows the rail system for railway stations, where all tracks are represented with a line. Only the tracks IE and CH connect the tracks AB, CD and FG. A train comes into the track AB in the direction BA. Counted from the locomotive ahead, the 9th and the 12th cars of the train should stay at the station on the track GH, though let the rest of the cars continue their way. How can all of them be managed the most simply?*



19. ábra.

A megoldás a következő:

A lokomotív 1. az első 11 kocsival előrehalad IA-ra, majd I-n és E-n keresztül ED-re tolja a 10-ik és 11-ik kocsit;

The solution is as follows:

The locomotive 1 with the first 11 cars goes ahead to IA, then pushes the 10th and the 11th cars through I and E to ED;

2. a többi kocsival visszatér IA-ra; itt irányt változtatva B felé indul, míg a csak BI-n hagyott kocsik elsejét, a 12-est el nem éri; ezt magához csatolja és visszatér IA-ra;

2. the other cars go back to IA; here the locomotive changes the direction and goes toward B, while the 12th car is waiting on BI; attaches this car to themselves, and goes back to IA;

3. innen a tíz kocsival I-n és E-n át E és D közé vonul, maga előtt tolván a 10. és 11. kocsit;

3. from here with the ten cars, the locomotive goes through I and E to E and D, pushing ahead the 10th and 11th cars;

4. most már a tíz kocsival EC-n, majd CH-n át egyszerűen HG-re jut, hol az utolsó két kocsit az ott hagyandó 9-est és 12-est ott hagyja, végül pedig

4. The locomotive goes now with the ten cars to EC, then through CH to HG simply, where the train leaves the last two cars, the 9th and the 12th. Finally,

5. azon az úton, melyen jött, elmegy először a DE-n hagyott 10. és 11. kocsiért, majd ezeket a 8 kocsihoz kapcsolva E-n és I-n keresztül BI-re; összekapcsolva a 11. és 13. kocsit folytathatja az egész vonattal útját A felé.

5. on the way they came through, first go to DE where the 10the and the 11th cars were left, connect them to the 8th car, then go through E and I to BI, connect the 11th and 13th cars, the whole train can thus continue its way toward A.

5.9 Apróságok (Örök naptár. Versenyszámolás. Meglepő eredmények): Trivial matters (Perpetual calendar, race-calculation, surprising results)

[p. 58]

Örök naptár. Gyakran előforduló feladat, meghatározni, hogy egy bizonyos dátum a hét melyik napjára esett (illetőleg fog esni).

Perpetual calendar. It is a frequently occurring problem: how to determine on which day of the week a certain date fell (or will fall).

A következő két lapon közölt négy tábla segítségével ez igen könnyen meghatározható.

Adjuk ugyanis össze az évszám százasai, egyesei, továbbá a hónap mellett az 1., 2., 3. illetőleg 4. táblában álló (vastagabban nyomtatott) számokat és vonjunk le ezen összegből annyiszor 7-et, a hányszor csak lehet (vagyis oszszuk el 7-tel és határozzuk meg a maradékot); a maradó számtól a 4. táblázatban jobbra álló nap lesz a keresett nap.

This is definable most easily with the help of four tables shown on the following two pages.

The Tables 1, 2, 3 and 4 are correspond to hundreds and units of year number, furthermore to months. In the Tables 1, 2, 3 and 4, some numbers are written besides (printed more thickly). Add together these numbers in boldface*. Subtract 7 from this sum as many times as possible. (that is, divide it by 7, and we determine the remainder); from the remainder number, we obtain the requested day, which will be the day standing at the right of the table 4.

*Note of the translator:
Add also the number corresponding to the date on Table 4.

Határozzuk meg pld. hogy 1884. szept. 21. mily napra esett.

Determine, for example, what day was September 21 in 1884*.

*Note of the translator:
This date is the birthday of Kőnig Dénes.

Az 1. táblában (In Table 1)	18	mellett találjuk a we find the number at the side	3	számot,
A 2. „ (In Table 2)	84	„	0	„
A 3. „ (In Table 3)	szept. Sep.	„	4	„
A 4. „ (In Table 4)	21	„	0	„
		E számok összege Sum of these numbers	7	

mely szám 7-tel osztva maradékul 0-t ád; 0 mellett pedig a 4. táblázatban vasárnap áll s így ez a keresett nap, melyre 1884. szept. 21. esett.

Ez a naptár 1582. okt. 15-től 3500. február 28-ig minden napra használható.

This remainder divised by 7 is 0; in Table 4, Sunday stands beside 0, which is the requested day on which September 21 in 1884 fell.

This calendar can be used for every day from October 15 in 1582 to February 28 in 3500.

[p. 59]

1. táblázat. (Table 1.)
(The hundreds of year-number:)

Az évszám százasai:					
15	19	23	27	31	**1**
16	20	24	28	32	**0**
17	21	25	29	33	**5**
18	22	26	30	34	**3**

2. táblázat. (Table 2.)
(The units of year-number:)

Az évszám egyesei:				
0	28	56	84	**0**
1	29	57	85	**1**
2	30	58	86	**2**
3	31	59	87	**3**
4	32	60	88	**5**
5	33	61	89	**6**
6	34	62	90	**0**
7	35	63	91	**1**
8	36	64	92	**3**
9	37	65	93	**4**
10	38	66	94	**5**
11	39	67	95	**6**
12	40	68	96	**1**
13	41	69	97	**2**
14	42	70	98	**3**
15	43	71	99	**4**
16	44	72		**6**
17	45	73		**0**
18	46	74		**1**
19	47	75		**2**
20	48	76		**4**
21	49	77		**5**
22	50	78		**6**
23	51	79		**0**
24	52	80		**2**
25	53	81		**3**
26	54	82		**4**
27	55	83		**5**

[p. 60]

3. táblázat. (Table 3.)

A hónapok	
Március	2
Április	5
Május	0
Június	3
Július	5
Augusztus	1
Szeptember	4
Október	6
November	2
December	4
Január*)	0
Február*)	3

*) Ha a keresett nap januárban vagy februárban van, akkor az adott évszámnál eggyel kisebb szám veendő számításba, mintha az év tulajdonkép márc. 1.-én kezdődnék.

Table 3.

The months	
March	2
April	5
May	0
June	3
July	5
August	1
September	4
October	6
November	2
December	4
January*)	0
February*)	3

*) If you are looking for a day in January or February, then the next lower number than the year number is to be counted. As of the year, actually begins on March 1.

4. táblázat. (Table 4.)

A hó hanyadika? (What day of the month?)						Napok (Day)
1	8	15	22	29	**1**	Hétfő (Monday)
2	9	16	23	30	**2**	Kedd (Tuesday)
3	10	17	24	31	**3**	Szerda (Wednesday)
4	11	18	25	...	**4**	Csütörtök (Thursday)
5	12	19	26	...	**5**	Péntek (Friday)
6	13	20	27	...	**6**	Szombat (Saturday)
7	14	21	28	...	**0**	Vasárnap (Sunday)

[p. 61]
Versenyszámlálás.

Ezen nagyon ismeretes játék abból áll, hogy ketten, jelöljük őket A-val és B-vel, felváltva egy 11-nél kisebb számot választanak és a választott számokat mindenkor összeadják.

Győztes a játékban az lesz, aki előbb éri el a 100-as számot.

Tegyük fel, hogy A már elérte a 89-es számot.

Ez esetben A a játékot is megnyerheti, mert bármily 11-nél kisebb számot ad is B a 89-hez, A az összeget egy ugyancsak 11-nél kisebb szám hozzáadásával 100-ra tudja kiegészíteni.

Hogy pedig A 89-hez juthasson, ahhoz — mint hasonló módon belátható — 78-at kell elérnie.

78-at pedig csak akkor tudja mindenesetre elérni, ha előbb 67-et mondott.

Tovább folytatva e meggondolást, kitűnik, hogy A nyeri a játékot, ha egymás után eljut a következő számtani sor tagjaihoz:

Competition counting.

This well known game consists of two people named with A and B; they select alternately a number smaller than 11, and the selected numbers are added together every time.

The winner of the game will be the one who first reached the number 100.

Suppose that A has reached the number 89.

In this case, A is to win the game, because B add to 89 any number less than 11, then A can also add a number less than 11 to the addition, and complete 100.

For that A can reach 89, — as a similar manner can be considered — A should reach 78.

If one reached 67 first, then one can reach 78 in any case.

Continuing this consideration, it is clear that A wins the game if A gets one after another the terms of the following arithmetic sequence:

1, 12, 23, 34, 45, 56, 67, 78, 89.

Az első tehát, 1-et mondva, mindenkor megnyerheti a játékot.

The first should take therefore 1, then always win the game.

Ha általánosságban m a legnagyobb szám, melyet választani szabad és n az elérendő szám, akkor a győztesnek a következő számtani sor tagjait kell elérni:

In general, if m is the largest number of free choice, and n is the the number to be reached, then the winner has to reach the members of the following arithmetic sequence:

$$a, a+(m+1), a+2(m+1), ...a+k(m+1),$$

hol k és a az n-nek $m+1$-gyel való elosztásakor fellépő hányados, illetőleg maradék, úgy hogy e sor utolsó tagja

where k and a are a quotient and a remainder occurring from the division of n with $m+1$,
so that the last member of this sequence is

$$a + k(m+1) = n$$

maga az elérendő szám. Helyes marad akkor is az az eljárás, ha n osztható $m+1$-gyel, tehát $a = O$ (pld. $n = 80$, $m = 7$) csakhogy ebben az esetben nem A, a kezdő nyer, hanem B.

which is the target number itself. That procedure can be performed if n is divisible by $m+1$, so $a = 0$ (for example, $n = 80$, $m = 7$); however in this case, not the first player A but B wins.

Kissé megváltozik a játék, ha azt vesztesnek tekintjük, ki az n számot eléri. Ez esetben az a győztes, ki $n-1$-et éri el s így ez a játék is vissza van vezetve az előbbire.

The game changes slightly if we regard as a loser the one who reaches the number n. In this case, the winner is who reaches $n-1$, and this game is traced back to the former game.

Ezen számolást nem csak elvontan számokon [p. 62] végezhetjük, hanem gyufákon vagy más apróbb tárgyakon, mint a következő játék mutatja.

We can make this kind of counting not only abstractly on numbers, but also on matchsticks or other smaller objects, as shown in the following game.

Az asztalon 15 gyufa van, melyből A és B fölváltva 1, 2, 3, 4 vagy 5 gyufát elvesz. A vesztes az, kinek az utolsó gyufát kell elvennie.

There are 15 matchsticks on the table, from which A and B alternately take away 1, 2, 3, 4 or 5 matchstick(s). The loser is the one who has to take away the last matchstick.

Ebben a játékban is mindig a kezdő nyerhet, ha először 2 gyufát vesz el.

In this game, always the first player can win if that player takes away 2 matchsticks at the first turn.

Ha a másik most a gyufát vesz el, akkor a $6 - a$ gyufát kell elvennie, stb.

If the other now removes the matchsticks, then the one should remove $6 - a$ matchsticks, etc.

Meglepő eredmények.

Surprising results.

1. *A egy bizonyos T összeggel leül B-vel játszani. Megállapodnak, hogy minden játékért, melyet A nyer, az ő (A) meglévő pénzének $\frac{1}{n}$-részét kapja B-től. Ha viszont B nyer, A meglévő pénzének $\frac{1}{n}$-részét B-nek adja. Kérdés, hogy miután A ugyanannyi játékot nyert, mint veszített, végeredményben több vagy kevesebb-e a pénze, mint amennyivel a játékot elkezdte.*

1. *A certain A gambles with B with the amount T. They stop for all games which A wins; this one (A) receives $\frac{1}{n}$ of the money [T] from B. If B wins on the other hand, B gets $\frac{1}{n}$ of A's money [T]. The question is that, after A won more games than he lost, you have more or less money in the final result, than the amount at the beginning of the game.*

Ha A az első játékot megnyeri, akkor T-ből

If A wins the first game, then starting from T,

$$T + \frac{1}{n}T = \frac{n+1}{n}T$$

lett vagyona; ha a második játékot ugyancsak A nyeri, már

became A's property; If A wins also the second game, now the amount of money will be

$$\frac{n+1}{n}T + \frac{1}{n}\cdot\frac{n+1}{n}T = \left(\frac{n+1}{n}\right)^2 T$$

pénze lesz. Ha általában i játékot nyer egymásután, vagyonát

If A wins i games in succession, one's property is expressed as

$$\left(\frac{n+1}{n}\right)^i T$$

fejezi ki. Egész hasonlóan nyerjük, hogy ha A i-szer veszít, akkor T-ből

As a whole we get similarly that if A loses i-times, then starting from T

$$\left(\frac{n-1}{n}\right)^i T$$

-re fogy a vagyona. Ha most már i-szer nyer és i-szer veszít, akkor (a szorzás commutatív volta miatt bármilyen sorrendben történjék is ez)

the property is reduced. If now one already won i times and lost i times, then (the multiplication being commutative, it holds in any kind of order)

[p. 63]

$$T_{2i} = \left(\frac{n+1}{n}\right)^i \left(\frac{n-1}{n}\right)^i T = \left(\frac{n^2-1}{n^i}\right)^i T$$

vagyona marad. Látjuk tehát, hogy A *veszít,* mert (a zárójelben lévő tört valódi lévén), mindenkor

property remains. We see, therefore, that A *loses* and (because of the part between brackets) always

$$T_{2i} < T$$

Ez az eredmény első pillanatra igazságtalannak látszik és A hátrányban látszik lenni B mögött.

Belátható azonban, hogy másrészt viszont A-nak van előnye, mert a játékszabály szerint bármily kis összeggel kezd is játszani, összes pénzét sohasem vesztheti el.

Ellenben B semmiféle határt nem szabhat az általa elveszthető összegnek.

Ha tehát azt az eredményt nyertük volna, hogy i nyert és i vesztett játék után A és B épen a pénzénél van, akkor világos, hogy hasonló játékban mindenki inkább A, mint B szerepét szeretné vinni.

2. *Két alkalmazottnak évi 1000–1000 korona fizetése van. Az első, ki évenkint kapja a fizetési, minden évben 20 korona fizetésjavításban részesül, a másodikat félévenkint fizetik és félévenkint 5 koronával növelik a fizetését. Melyikük kap többet?*

Az első év végén az első alkalmazott 1000, a második $500+505 = 1005$ koronát kap, a második év végén az elsőnek fizetnek 1020 koronát, míg a második addig már $510+515 = 1025$ koronát kapott stb.

A második tehát minden évben 5 koronával többet kap, mint az első.

This result seems unfair at first glance and A appears at a disadvantage to be behind B.

However it is reasonable that A has an advantage on the other hand, because A starts playing with an any little sum according to the rules of the game, A can never lose his money.

Conversely, B cannot impose any limit to keep the amount.

If we got the result of i won games and i lost games, then A and B have just one's original amount. Then it is clear that in a similar game everybody would like to play rather A's role than B's role.

2. *Two employees each receives 1000 crowns as the payment per year. The first one receives the salary annually, and the payment is raised by 20 crowns every year. The payment to the second one is increased by 5 crowns every half year. Which one gets more?*

In the first year, the first employee receives 1000, the second employee receives $500 + 505 = 1005$ crowns. At the end of the second year, the first is paid with 1020 crowns, while the second one he obtained $510+515 = 1025$ crowns, etc.

The second gets 5 crowns more than the first one every year.

3. *Egy pohárban bor, egy másikban ugyanannyi víz van. A borból egy kanálnyit a vizes pohárba öntünk s az ebben keletkező vegyülékből ugyancsak egy kanálnyit beleöntünk a bort tartalmazó pohárba. Kérdés, hogy ezen két művelet után az első pohárból hiányzik-e több bor, vagy a másodikból több víz.*

E kérdésre sokan valószínűleg azt felelik, hogy az elsőből hiányzik több bor, talán azért, mert nem gondolnak arra, hogy a második művelettel bor is kerül vissza [p. 64] az első pohárba.

A kérdéses két mennyiség azonban *egyenlő,* amint az a következő módon könnyen belátható.

Mindkét pohárban a műveletek után ugyanannyi folyadék van, mint azok előtt.

Az első pohárból tehát annyi bor hiányzik, amennyi víz van benne, vagyis amennyi víz hiányzik a másikból.

És éppen ez az, amit bizonyítanunk kellett. A kérdéses két mennyiség különben ki is számítható.

3. *There is wine in a glass; there is the same amount of water in another glass. From the wine glass, we pour a spoonful of wine into the water glass and in the same way, from the resulting mixture, we pour a spoonful into the glass containing the wine. The question: is more wine missing from the first glass after these two operations, or is more water missing from the second glass?*

To this question, many people probably say that more is missing from the first glass of wine, perhaps because they do not think that that the second operation will also let wine be back to the first glass.

The two quantities about which the question is raised are however *equal,* which can be easily understood with the following manner.

In both glasses, there is as much liquid as before after the operations.

Therefore, from the first glass, wine is missing as much as the water in it; that is, from the other glass, water is missing as much as the wine in it.

And it is precisely what we had to prove. The two quantities about which the question is raised can be otherwise calculated.

Legyen e célból a bor és a víz köbtartalma külön-külön a, a kanálba férő folyadéké b.

For this aim, let the volume of the wine and the volume of the water be a for each; the volume of the liquid fitting into the spoon be b.

Az első művelet után tehát a második pohár a vizet és b bort tartalmaz.

Therefore, after the first operation, the second glass contains a water and b wine.

A második művelet alkalmával a kanálba kerülő folyadék, melyet az első pohárba öntünk, természetesen szintén $a : b$ arányban tartalmaz vizet és bort: $\frac{ab}{a+b}$ vizet és $\frac{b^2}{a+b}$ bort, mert

On the occasion of the second operation, a spoon of the liquid is poured into the first glass. of course it contains water and wine in a proportion $a : b$. This spoon contains $\frac{ab}{a+b}$ water and $\frac{b^2}{a+b}$ wine, because

$$\frac{ab}{a+b} + \frac{b^2}{a+b} = b$$

és

and

$$\frac{ab}{a+b} : \frac{b^2}{a+b} = a : b.$$

Az első pohárból hiányzó bormennyiség tehát:

Therefore, the following volume of wine is missing from the first glass:

$$b - \frac{b^2}{a+b} = \frac{ab}{a+b}$$

és láttuk, hogy ugyanennyi víz hiányzik a másodikból.

and we have seen that the same quantity of water is missing from the second glass.

Vége a második sorozatnak.

The end of the second series.

5.10 Az első és második sorozat problemáinak eredetéről és irodalmáról: About sources and bibliography of the problems of the first and second series

5.10.1 ELSŐ SOROZAT: FIRST SERIES

[p. 65]

A nagy számokról szóló fejezet anyaga csaknem kizárólag *Schubert**) munkájából (I. köt., 26. l.) van véve, de már *Ozanam* is foglalkozik az elképzelhetetlen nagy számokkal (I. 176–182. és 203. l.) A sakktáblára kerülő búzaszemek nagy számáról *Ahrens* is ír (28. l.) valamint bizonyos hír terjesztőinek nagy számáról, melyről könyvünkben a 8–9 lapokon van szó. Néhány példát igen nagy számokra *Lucas* is említ (*A. A.* 150. l.)

*) Ahol a szerzők nevét munkájuk megemlítése nélkül említjük, ott az első sorozat végén található jegyzékben lévő munkájuk értendő. A lapszámok is az ott említett kiadásra vonatkoznak. *Lucas* nevénél, ki két munkával szerepel a felhasznált munkák jegyzékében, ha könyvének címe megemlítve nincs, a *Récréations Mathématiques* értendő. Az *Arithmétique Amusante* „A. A.“-val van rövidítve.

The substance of the chapter **about the large numbers** is almost exclusively taken from *Schubert**) work (vol. I, p. 26), but *Ozanam* also already deals with the unimaginable large numbers (I pp. 176–182 and 203). *Ahrens* also writes (p. 28) about the large number of wheat grains put on the chessboard, as well as about the large number of distributors of certain news which is treated in pp. 8–9 of our book. *Lucas* also mentions some examples on very large numbers (*A. A.* p. 150).

*) Where we mention the authors' name without mentioning their works, their works can be found at the end of the first series in the list of works. The page numbers concern the publication mentioned there. For *Lucas*' name who has two works in the list of works used, it is understood as *Récréations Mathématiques* if the book title is not mentioned. The *Arithmétique Amusante* is abbreviated with „A. A.“.

Érdekes számok- és eredményekre is leginkább *Schubert* (I. 15. l.) szolgáltat példákat, bár egynéhány már *Lucas A. A.*-ban (64–68. l.) és a *Recr. Math.* IV. kötetében (232. l.) található. Az *A. A.*-belieket *Ibn Albanna** XIII. századbeli *Talkhys* nevű munkájából meríti. *Fourrey*-nál a 7–16. és 23–25. lapon található néhány érdekes szorzási eredmény, míg a tökéletes számokról [p. 66] és barátságos számpárról, melyeknek eredete *Pythagoras*-ra vezethető vissza, a 93–94. lapokon szól.

For Interesting numbers and results also mostly *Schubert* (I. p. 15) supplies examples, though some of them can be found already in *Lucas*' *A. A.* (pp. 64–68) and the *Recr. Math.* vol. IV. (p. 232). Drawings in *A. A.* are from the work named *Talkhys* by *Ibn Albanna** of the XIII century. A few interesting multiplication results are shown on pp. 7–16 and 23–25 of *Fourrey*, while the origin of the perfect numbers and the friendly number pair, which can be led back to *Pythagoras*, are described on pp. 93–94.

*Note of the translator:
Ibn AL-BANNA (1256–1321) is mathematician and astronomer in Morocco. "Talkhys", in another transliteration "talkhis", means "summary" in Arabic, which indicates one of his works titled "Talkhis amal al-hisab (Summary of arithmetical operations)".

A tökéletes számokra vonatkozólag különben *Euler* bizonyította be először, hogy más páros tökéletes szám, mint a melyet a 14. lapon* lévő már *Euklides* által is ismert képlet szolgáltat, nem létezik.

Additionally, concerning the perfect numbers, *Euler* proved first that there is no even perfect number other than the ones shown on p. 14*. *Euclid* is also known as giving a formula, but it does not exist.

*Note of the translator:
The page number is of the original version of KŐNIG's book, which is p. 76 here.

Hogy N mindenkor tökéletes szám, ha $2^\alpha - 1$ prímszám, sokkal könnyebben bizonyítható. (Ennek bizonyítását követelte az 1903-i tanulóverseny egyik feladata, l. Math. és Phys. Lapok, XII. évf. 344. l.)

It can be much more easily proven that N is always a perfect number if $2^\alpha - 1$ is a prime number. (The proof for this was required in one of the problems of students' competition* in 1903, see Math. és Phys. Lapok, vol. XII, p. 344.)

*Note of the translator:
A Mathematikai és Physikai Társulat tanulóversenye (The student competition of Mathematical and Physical Society) was held on the review every year.

Hogy páratlan tökéletes szám nem létezik, az mindmáig csak egy igen nagy határig van bebizonyítva*.

The proposition that odd perfect number does not exist is proven only within one very large restriction until now*.

*Note of the translator:
Even in 2009, it is not yet proven.

A tökéletes vagy barátságos számokkal *Ball* (52. l.), *Rebière* (444. l.) és *Schubert* (I. 100. l.) is foglalkozik.

Ball (p. 52), *Rebière* (p. 444) and *Schubert* (I. p. 100) also deal with perfect numbers or friendly numbers.

Az 1–3. ábrákra vonatkozó tételek a francia *Mathesis* c. folyóiratból (Paris, 1901. évf.) vannak véve. A 9-es stb. szám különböző alakokban való kiírását *Lucas*-nál (*A. A.* 83. l.) és *Schubert*-nél (I. 190. l.) találhatjuk.

The propositions concerning Figures 1–3 are taken from the French periodical titled *Mathesis* (Paris, volume of 1901). We can find publications concerning the number 9 etc. in different shapes in *Lucas* (*A. A.* p. 83) and *Schubert* (I. p. 190).

Számok kitalálására vonatkozólag *Schubert*-nél (I. 1–14. l.), *Ball*-nál (5–22. l.) és *Fourrey*-nál (5–22. l.) találhatunk sok példát.

Concerning **guessing numbers**, we can find many examples in *Schubert* (I. pp. 1–14), *Ball* (pp. 5–22) and *Fourrey* (pp. 5–22).

De már *Ozanam* is számok kitalálására szolgáló módszerekkel kezdi könyvét és *Bachet* is ily kérdésekkel vezeti be „Problème“-jeit.

But *Ozanam* also already starts his book with methods useful for guessing numbers, and *Bachet* also inserts this kind of „Problème“-s with questions.

A bűvös négyzetek hazájául többnyire Indiát említik.

India is mostly mentioned as the homeland of **magic squares**.

Európában először *Dürer* „Melancolia“ nevezetű rézmetszetén merül fel bűvös négyzet, és pedig az, melyet a 11 ábra mutat.

In Europe a magic square appears first in the copper engraving of *Dürer* called „Melancolia“, and this is shown in Figure 11.

Az első sor két középső száma itt összeolvasva a kép keletkezésének évét (1514) adja.

Reading here two middle numbers of the first row together, the year of production of the picture (1514) is given.

A könyvünkben elsőnek közölt módszert *La Loubère* francia utazó Indiából hozta magával 1688-ban (l. *La Loubère*: „Du royaume de Siam“ Amsterdam, 1691. II. 235. l.)

The method first published in our book was brought from India by a French traveler *La Loubère* in 1688 (see *La Loubère*: „Du royaume de Siam“ Amsterdam, 1691, II p. 235).

Legrégibbnek a könyvünkben is bemutatott *Moschopulos*-féle módszert tekintik.

The oldest method introduced by *Moschopulos* is also considered in our book.

A XIV. századból származó *Moschopulos*-féle görög szöveget *Günther* reprodukálja). „Vermischte Untersuchungen zur Geschichte der mathematischen Wissenschaften“, Leipzig, Teubner, 1876; 195–203. l.)

The Greek text by *Moschopulos* written in the XIV century is reproduced by *Günther* („Vermischte Untersuchungen zur Geschichte der mathematischen Wissenschaften (Mixed examinations to the history of the mathematical sciences)“, Leipzig, Teubner, 1876; pp. 195–203).

Időben legközelebb áll talán hozzá *Bachet* eljárása (könyvének 88. lapján), melyet *Günther* fenti munkájában „Terrassen Methode“-nak nevez a amely a *Moschopulos*-étól csak látszólag különbözik.

Bachet's procedure (page 88 of his book), which is maybe in the nearest time to it, and which was called „Terrassen Methode (terraces method)“ in *Günther*'s above mentioned work, differs only apparently from the *Moschopulos*.

A negyediknek közölt módszer újabb keletű (l./ *Horner*: „On the algebra of magic squares“ [p. 67] Quarterly Journal of Mathematics, XI. köt. 1871.; 57–65, l.).

The fourth method was published in more recent date (see *Horner*: „On the algebra of magic squares“ Quarterly Journal of Mathematics, vol. XI, 1871.; pp. 57–65)*.

*Note of the translator:
The first method is described by La Loubère before 1691 [139, 140], the second method by Bachet before 1624 [8], the third method by Moschopulos before around 1315 [162, 161], and the fourth by Horner in 1871 [71]. Therefore the first/second/third methods have already been described in an European language before 1691, while the fourth method was not published before 1871. The third method was not widely known before the analysis by La Hire in 1705 [137], or before the reproduction by Günther in 1876 [68]. See also my note on p. 108.

Mind a négy módszer megtalálható *Ahrens* könyvében, hol a bűvös négyzetek a 209–247. lapokon vannak tárgyalva. Különben *Lucas* (I. k. XIII. l. és IV. k. 89. l.) *Ball* (180–203. l.), *Schubert* (II. 17–48. l.) és *Fourrey* (197–261. l.) is igen részletesen foglalkozik az immár óriási irodalommal bíró bűvös négyzetek elméletével.

All four methods can be found in *Ahrens*' book, where the magic squares are treated on pp. 209–247. On the other hand, *Lucas* (vol. I, p. XIII and vol. IV, p. 89), *Ball* (pp. 180–203), *Schubert* (II, pp. 17–48) and *Fourrey* (pp. 197–261) also deal much in detail with the theory of magical squares, which are favourably comparable with recent great literatures.

(Az irodalomra vonatkozólag l. *Günther* említett munkája mellett *Cantor*: „Vorl. über Geschichte der Math.“, Leipzig, Teubner, 1880; l. r. 436., 539., stb. lapokat.)

(Concerning these documents, see *Cantor* mentioned by *Günther*: „Vorlesungen über Geschichte der Mathematik“, Leipzig, Teubner, 1880; see the parts on pp. 436, 539 etc.)

A mathematikai hamisságok közt mindenesetre Zeno sophismáját és 64 = 65 bizonyítását kell a legrégibbnek tekintenünk, bár a modern gyűjtemények, pld. *Lucas* (II. 152. l.) és *Fourrey* (193. l.), melyek csak az utóbbit közlik, nem számolnak be ennek eredetéről.

Zeno sophismáját *Aristoteles* őrizte meg.

Mathematikai hamisságok nagyobb számban találhatók *Rebière* (406, 409 l.), *Ball* (36. és 61. l.) és *Schubert* (I. 134. és VII. 159. l.) gyüjteményeiben.

A Középiskolai Math. Lapok is több ízben közölt ilyeneket mind az algebrából, mind a geometriából (IV. évf. 11. l., 109. l.; V. évf. 7. 1.; VI. évf. 28. l.).

A síkidomok szétszedése és összeállítása című fejezet anyagát nagyrészt *Lucas* (II. 125., 129. és 145. l.) és *Schubert* (III. 127. l.) könyveiből vettük.

Among **the mathematical errors** Zeno's paradox and the proof of 64 = 65 should be considered as the oldest ones in any case, though modern collections, for example *Lucas* (II, p. 152) and *Fourrey* (p. 193), which tell only the latter one, do not report this origin.

Aristotle recorded *Zeno*'s paradox.

Mathematical errors in larger numbers can be found in the collections of *Rebière* (pp. 406, 409), *Ball* (pp. 36 and 61) and *Schubert* (I, p. 134 and VII, p. 159).

The Középiskolai Mathematikai Lapok (High school mathematical reviews) also sometimes published such things either from algebra, or from geometry (vol. IV, p. 11, p. 109; vol. V, p. 7; vol. VI, p. 28).

The substance of the chapter titled **decomposition and recomposition of plane figures** are largely took from the books of *Lucas* (II, pp. 125, 129 and 145) and *Schubert* (III, p. 127).

A 26. ábrabeli felbontás *Kürschák*-tól származik (Math. és Phys. Lapok VII. évf. 53 l.; továbbá X. évf. 279. l. és a Középiskolai Mathematikai Lapok X. évf. 119. l., hol a szabályos 12 szög más érdekes felbontásai találhatók.)

The solution of Figure 26 originates in *Kürschák* (Mathematikai és Physikai Lapok (mathematical and physical reviews), vol. VII, p. 53; furthermore, see vol. X, p. 279 and the Középiskolai Mathematikai Lapok (highschool mathematical reviews) vol. X p. 119, where other interesting resolutions of regular dodecagon can be found.)

A 27–28. ábrára vonatkozó feladatot *Rebière* is közli (488. l.).

The problem concerning Figures 27–28 is reported also by *Rebière* (p. 488).

5.10.2 MÁSODIK SOROZAT: SECOND SERIES

[p. 68]
A mathematikai valószínűség nem igen szokott hasonló gyűjteményekben tárgyalva lenni, csak *Schubert* foglalkozik igen részletesen vele (II. 167–247. l.)

The mathematical probability is not described in very ordinary collections. Only *Schubert* deals much in detail with it (II, pp. 167–247).

A három pénzdarab sophismáját *Ball* könyvéből vettük (42. l.), hol azonban a sophisma nincsen megmagyarázva.

We took the paradox of three coins from *Ball*'s book (p. 42), but the paradox is not explained there.

E probléma különben Francis *Galton*-tól származik és először az angol „Nature“ c. folyóíratban jelent meg (XLIX. köt. 365. l.) 1894-ben.

Additionally, this problem originates in Francis *Galton*, and appeared first in the English magazine named *Nature* (vol. XLIX, p. 365) in 1894.

Az abbamaradt kártyajátékokról szóló egyik kérdés pedig *Schubert*-nél található. (II. 206. l.)

But one of the questions concerning ceased card games can be found in *Schubert* (II, p. 206).

Az n krajcárra vonatkozó valószínűségi feladat végeredménye megvan pld. *Bendt:* Katechismus der algebraischen Analysis (Webers illustrierte Katechismen, Leipzig, 1901.) 80. lapján és *Rebière*-nél (443. l.)

The final result of problems of probability concerning the n kreuzers is found for example in *Bendt:* Katechismus der algebraischen Analysis (Introduction to the algebraic analysis) (Webers illustrierte Katechismen, Leipzig, 1901) on p. 80 and in *Rebière* (p. 443).

A kettes számrendszert legelőször *Leibniz* alkalmazta.

The binary numeral system is first applied by *Leibniz*.

A számok számtáblákkal való kitalálásának rendkívül ismert módja valószínűleg *Bachet*-tól származik (198. l.)

A well-known method for guessing the numbers in number tables probably originates in *Bachet* (p. 198).

Magyar gyermeklapok is közölték már és megtalálható *Kőnig* „Algebrá"-jában is (I. kiadás, III. füzet, 17. l.), valamint *Lucas*-nál (I. 154. l. és *A. A.* 169. l.) és *Schubert*-nél is (I. 82. l.)

Hungarian children's papers already reported the method, and it can be found also in *Kőnig*'s „Algebra" (1st edition, vol. III, p. 17)*, in *Lucas* (I, p. 154 and *A. A.* p. 169) and in *Schubert* (I, p. 82)

*Note of the translator:
KŐNIG Gyula, *Algebra: a középtanodák felsőbb osztályai számára az új gymnasiumi tanterv értelmében (Algebra: for the higher levels of high schools in the new curriculum of gymnasium)* vol. III, Eggenberger-féle könyvkereskedés, Budapest, 1881. KŐNIG Gyula is the father of KŐNIG Dénes.

A számok másik, pénzdarabok segítségével való kitalálásának módja tárgyalva van a következő helyeken: *Schubert* (I. 85. l.) és *Ahrens* (29. l.).

Another method for guessing the numbers with the help of coins is treated in the following places: *Schubert* (I, p. 85) and *Ahrens* (p. 29).

A 2. és 3-as rendszernek súlymérésre való alkalmazása [p. 69] megtalálható már *Bachet*-nál (154. l.).

The application of binary and ternary system for weight measurement can be found already in *Bachet* (p. 154).

Tőle átvette: *Lucas* (I. 151. l. és *A. A.* 165. l.), *Ball* (45. l.), *Schubert* (I. 87. l.), *Fourrey* (53. l.) és *Ahrens* (40. l.). A Math. és Phys. Lapokban *Zemplén* Győző tárgyalta a 3-as rendszernek súlymérésre való alkalmazását, bebizonyítva, hogy minden szám 3 különböző hatványainak algebrai összegeként írható. (VIII. évf. 135. l.). A tennisversenynél szokásos elrendezések még nem voltak mathematikailag tárgyalva.

It was taken over by: *Lucas* (I, p. 151 and *A. A.* p. 165), *Ball* (p. 45), *Schubert* (I, p. 87), *Fourrey* (p. 53) and *Ahrens* (p. 40). In the *Math. és Phys. Lapok*, *Zemplén* Győző treated ternary system used for weight measurement. He proved that, for all the various powers of number 3, the algebraic sum of them can be written. (vol. VIII, p. 135). The ordinary arrangements of tennis tournament were not yet mathematically treated.

A négyszínű térkép című fejezetben említett tételt *Ball* szerint *Möbius* vetette fel 1840-i előadásában és *Guthrie* tette először közzé „Note on the colourings of maps“ címen 1880-ban. (proceedings of the Royal Society of Edinbourgh, X. köt., 728. l.).

according to *Ball*, the proposition mentioned in the chapter titled **four-colour map** was presented in a lecture of *Möbius* in 1840, and *Guthrie* first published an article entitled “Note on the colourings of maps” in 1880 (*Proceedings of the Royal Society of Edinbourgh*, vol. X, p. 728).

Legutóbb, a Mathematische Annalen 1903-i kötetében (LVIII. 413. l.) *Wernicke* foglalkozott a négy szín problémájával, de az ő bizonyítása is hibásnak bizonyult.

Recently, in the volume of 1903 of *Mathematische Annalen* (LVIII, p. 413), *Wernicke* dealt with the problem of four colours, but his proof is proved to be wrong.

Térképkészítők különben (ugyancsak *Ball* szerint) már régebben is alkalmazták e tételt.

On the other hand, map manufacturers (according to *Ball*) already applied this proposition earlier.

A 13-24. lapokon* közölt bizonyítás leginkább *Ball*-éra (72–73. l.) támaszkodik, de ennél mindenesetre szigorúbb és szabatosabb.

The proof on pp. 23–24* depends mostly on *Ball* (pp. 72–73), but it is somehow more strict and more correct here.

*Note of the translator:
sic.
It should be pp. 23–24, which corresponds to the part from p. 196 of this translation.

A 21–22. lapokon bizonyítás nélkül közölt három tétel közül az első és harmadik, melyek szerint bizonyos esetekben négy szín sem szükséges, *Kempe*-től származik (American Journal of Mathematics, 1879. II. köt. 193–200 l.) és *Ball* (75–76. l.) is reprodukálja őket.

A második tétel azonban új. Különben *Lucas* (IV. 168. l.) és *Ahrens* (340–350. l.) is tárgyalja a négy szín problémáját. E fejezet lényeges hibával megjelent az „Uránia“ IV. évfolyamában (87. l.).

A königsbergi hidak problémáját, mely az analysis situs történetileg első kérdésének tekinthető, *Euler* vetette fel és oldotta meg először.

Értekezése „Solutio problematis ad geometriam situs pertinentis“ címmel a pétervári Akadémia 1741.-i évkönyvében jelent meg. (Commentationes Academiae Petropolitanae, VIII. évf. 128. l.)

A 25–26. lapokon nagyjában *Euler* bizonyítását közöljük, mely *Lucas* munkájában is megtalálható (I. kötet 21. l.).

According to the first and the third propositions among the three propositions shown without proof on pp. 21–22, four colours are necessary in certain cases. It originates in *Kempe* (*American Journal of Mathematics*, 1879, vol. II, pp. 193–200), and is reproduced by *Ball* (pp. 75–76).

The second proposition is new. In addition, *Lucas* (IV, p. 168) and *Ahrens* (p. 340–350) deal with the problem of four colours. This subject appeared with an essential mistake in the volume IV of „Uránia“ (p. 87).

The problem of **the bridges of Königsberg** can be considered as the first question in the history of analysis situs. *Euler* treated and solved it first.

His treatise titled “Solutio problematis gives geometriam situs pertinentis” appeared in the *Commentaries of Petersburg Academy* in 1741. (Commentationes Academiae Petropolitanae, vol. VIII, p. 128)

We show *Euler*’s proof on the whole on pp. 25–26. It can be found also in *Lucas*’ work (volume I, p. 21).

A probléma egyéb irodalma: *Ball* (204. l.), *Schubert* (III. 58. l.) és *Ahrens* (317. l.). A 9. és 10. ábrára vonatkozó kérdések, bár könyvben tárgyalva nem találhatók, közismert [p. 70] feladványok. A 11. ábrára vonatkozó kérdést *Kürschák* vetette fel.

Az iskoláslányok sétáira vonatkozó problémát az angol *Kirkman* vetette fel 1850-ben a Philosophical Magazine-ben megjelent „On the Triads made with Fifteen Things“ c. értekezésében.

Ugyanitt és ugyanezen évben jelent meg az első megoldás *Cayley*-től. A könyvünkben közölt két módszer elseje, s *Peirce*-féle a The Astronomical Journal-ban (VI. köt. 169. l.) látott napvilágot „Cyclic solutions of the school-girl puzzle“ címen, 1861-ben.

Hat évvel későbbi a Quarterly Journal of Mathematics VIII. évf. 236–251. lapjain „On the problem of the fifteen schoolgirls“ címen megjelent *Horner*-féle megoldás, melyet *Cape* próbálkozásaival együtt *Power* tesz ott közé, ki *Horner* megoldását nem tartotta valódi megoldásnak az előforduló próbálgatások miatt.

The other publications on the problem: *Ball* (p. 204), *Schubert* (III, p. 58) and *Ahrens* (p. 317). The questions concerning Figures 9 and 10 are well-known puzzles, though they cannot be found treated in any book. The question concerning Figure 11 was treated by *Kürschák*.

The problem on **daily walk of schoolgirls** was treated by *Kirkman*, an English man. His treatise titled “On the Triads made with Fifteen Things” appeared in 1850 in *Philosophical Magazine*.

The first solution was published by *Cayley* in the same magazine in the same year. The first of the two methods shown in our book was published by *Peirce* in the Astronomical Journal (vol. VI, p. 169) with the title „Cyclic solutions of the school-girl puzzle“ in 1861.

Six years later, a solution by *Horner* appeared in an article titled „On the problem of the fifteen schoolgirls“ in the Quarterly Journal of Mathematics vol. VIII, pp. 236–251*. *Cape* and *Power* published their attempt applying the solution. As a result of their attempt, they did not consider *Horner*'s solution as a real solution.

*Note of the translator:
This is Power's article. Power introduced the method of his friend Horner in his article.

Francia nyelven először *Lucas* (II. 176. l.) ír ezen nehéz problémáról s az angolok módszereihez sok új érdekes dolgot csatol.

Lucas (II, p. 176) wrote this difficult problem in French language first, and he added a lot of new interesting things to the methods of English people.

A 34–36. lapokon levő módszert pld., mely k-ról $3k$-ra engedi a probléma általánosítását, *Lucas* teszi közzé először, *Walecki*-nek tulajdonítván ennek felfedezését.

Among the methods on pp. 34–36, for example, the method to allow the generalisation of the problem from k to $3k$ was published by *Lucas* first. He ascribed this discovery to *Walecki*.

Ball (152. l.), *Schubert* (II. 49. l.) és *Ahrens* (274. l.) szintén hosszasan foglalkozik a 15 illetve n iskoláslány kérdésével.

Ball (p. 152), *Schubert* (II, p. 49) and *Ahrens* (p. 274) deal verbosely with the schoolgirl's problem for the case of $n = 15$.

A VI. fejezet feladatai közül csak az elsőt közölte *Tait* (Philosophical Magazine, 1884. I. 30–46. l.), ki egy vasúti kocsiban hallotta e feladatot felvetni.

Among the problems of **Chapter 6** [Tait's problem], only the first one was shown by *Tait* (Philosophical Magazine, 1884, I, pp. 30–46), who heard this problem asked in a railway carriage.

A játékszabály 2. feladatbeli megváltoztatása *Lucas*-tól származik (*A. A.*, 97. l.) és *Ahrens* (12–13. l.) is tárgyalja.

The alteration of the rules for the game in the problem 2 originates in *Lucas* (*A. A.*, p. 97). *Ahrens* (pp. 12–13) also deals with it.

A 3-dik feladatot is *Lucas* veti fel (II. 139. l.). *Schubert* a II. kötet 99–114. lapjain foglalkozik az 1–4. feladatokkal.

The 3rd problem is also shown by *Lucas* (II, p. 139). *Schubert* deals with problems 1–4 in volume II, pp. 99–114.

Ball könyvében a 92–101. lapokon találhatók; ez utóbbitól származik a 6. feladat. A 7. feladatot *Lucas* (IV. 207. l.) és *Ahrens* (167. l.) közli *Guarini*-nak egy 1512.-i kéziratából.

The problem 6 can be found on pp. 92–101 of *Ball*'s book. The problem 7 is shown by *Lucas* (IV, p. 207) and *Ahrens* (p. 167). It is also found in a *Guarini*'s manuscript in 1512*.

*Note of the translator:
According to LUCAS, it is found under the No. 42 of a manuscript of P. GUARINI DI FORLI (1512).

A VII. fejezet első három feladata *Ahrensnél* (257. l.), *Lucas*-nál (II. 161. l.) és *Schubert*-nél (II. 115. l.) található.

The first three problems of **Chapter 7** [Positions on a ring] can be found in *Ahrensnél* (p. 257), *Lucas* (II, p. 161) and *Schubert* (II p. 115) .

Sokkal régibb ezeknél a 4. és 5. feladat, mely a 15 keresztyén és 15 török, illetve Josephus problémáját tárgyalja.

Among them, the problems 4 and 5 are much older: the problem concerning 15 Christians and 15 Turks, and the Josephus problem.

Ami az előbbit illeti, az *Cantor:* „Vorl. über Geschichte der Math." II. 332. l. szerint *Chuquet*: „Le [p. 71] Triparty en la science des nombres" 1484-i kéziratában található először.

According to *Cantor*: *Vorlesungen über Geschichte der Mathematik (lectures on history of mathematics)* II, p. 332, the former problem can be found first in a manuscript of *Chuquet*: *Le Triparty en la science des nombres* in 1484.

Később *Cardan* („Practica Arithmeticae Generalis", 1539. IX. 117. l.) és *Bachet* (118. l.) említi.

Cardan (*Practica Arithmeticae Generalis (general practical arithmetic)*, 1539, IX, p. 117) and *Bachet* (p. 118) mention the problem later.

Ez utóbbi helyen található a Josephus-féle probléma első tárgyalása is.

In the latter document, the first description of the Josephus problem can also be found.

Az újabb könyvekből *Schubert* (II. 1. l.) és *Ahrens* (286. l.) tárgyalja egész általános elmélet alapján e problémákat.

Among the newer books, *Schubert* (II, p. 1) and *Ahrens* (p. 286) deal with these problems based on a whole general theory.

A VIII. fejezet a legismertebb feladatokat tartalmazza.

Chapter VIII [Problems of traversing, pouring and railway] includes the best-known problems.

Az első négy részben vagy egészben a mathematikai mulatságok irodalmának minden gyűjteményében megtalálható: *Ozanam* (I. 21. l.), *Bachet* (148. l.), *Lucas* (I. 1. l. és *A. A.* 125. l.), *Ball* (88. l.), *Schubert* (II. 159. l.), *Fourrey* (161. l.), *Ahrens* (1. l.) Eredetük ismeretlen; *Cantor* szerint (l. *Ahrens* jegyzetét az 1. lapon) az első feladat eredete az 1000. évre nyúlik vissza.

Almost all of the first four parts can be found in all the collections of the works of mathematical entertainments: *Ozanam* (I, p. 21), *Bachet* (p. 148), *Lucas* (I, p. 1 and *A. A.* p. 125), *Ball* (p. 88), *Schubert* (II, p. 159), *Fourrey* (p. 161), *Ahrens* (p. 1). Their origin is unknown; according to *Cantor* (see *Ahrens*' note on p. 1), the origin of the first problem goes back to 1000 years ago.

Az átöntési feladatok is már megtalálhatók *Bachet*-nál (138. l.) továbbá *Ahrens*-nál (53. l.), *Fourrey*-nál (167. l.) és *Ball*-nál (28. l.), valamint egy 1816.-i magyar könyvben, *Czövik* Istvan „Magyiás ezermester"-ében, melyben a 15 keresztyén és 15 török problémája is megtalálható. Kevésbbé ismert e fejezet utolsó három feladata, melyek kis változtatással *Fourrey* könyvéből (184. l.) vannak véve.

The problem of pouring already can be found in *Bachet* (p. 138), *Ahrens* (p. 53), *Fourrey* (p. 167) and *Ball* (p. 28). It can be found also in a Hungarian book in 1816, „Magyiás ezermester (magician)" of *Czövik* Istvan*. In this book, the problem of 15 Christians, and 15 Turks can also be found. The last three problems of this chapter are less known. They are taken from *Fourrey*'s book (p. 184) with a little change.

*Note of the translator:
Czövek István (1777–1828) is a lawyer in Hungary. The contents of *Magyiás ezermester* are collected and translated from German books of Karl von Eckartshausen (1752–1803, German philosopher): *Aufschlüsse zur Magie*, 4 vols., München. 1788-1792.

A 9.-et *Ahrens* (2. l.) és *Ball* (86. l.) is felemlíti.

The problem 9* is described by *Ahrens* (p. 2) and *Ball* (p. 86).

*Note of the translator:
The problem 9 is "Valamely vasúti állomás sínrendszer (Rail system for railway stations)".

Az apróságok közt közölt örök naptár táblázataít *Ahrens*-tól vettük át (384. l.; l. még *Lucas* IV. 8. l.). Az ezután közölt versenyszámlálásnak nevezett játék; az irodalomban szerzője után többnyire Bachet-féle játék néven szerepel.

The table of eternal calendar of **the trivial matters** was taken from *Ahrens* (p. 384; see also *Lucas* IV p. 8). The game called the counting competition is mostly published by Bachet who named the game.

Bachet könyvének 115. alapján valóban megtalálható már e gyerekújságokból is jól ismert játék tárgyalása. Egyéb tárgyalások találhatók *Ahrens*-nál (72. l.) *Schubert*-nél és *Ball*-nál (29. l.).

The description of the game can be found already in p. 115 of *Bachet*'s book. It is also well-known in children's newspapers. Other descriptions can be found in *Ahrens* (p. 72), *Schubert* and *Ball* (p. 29).

A meglepő eredmények címen közölt három utolsó feladat *Ball*-tól származik, ki azonban a másodikat kivéve csak a kérdést és feleletet közli.

The three last problems entitled the surprising results originate from *Ball*. He shows however only the problem and the answer apart from the second problem.

Végül még *Mikola* Sándor nem rég megjelent két kis füzetét kell felemlítenünk, melyek több könyvünkben is tárgyalt kérdéssel foglalkoznak.

Finally we have to mention a small notebook of *Mikola* Sándor which recently appeared. problems in the notbook are treated also in our books. are treated also in more books of us.

*Note of the translator:

Mikola Sándor (1781–1945) is a Hungarian physicist and a teacher.

A két füzet „Mathematikai szünórák“ címen, mint a „Stampfel-féle tudományos zsebkönyvtár“ 112. és 114. száma jelent meg (Pozsony-Budapest, [p. 72] 1903).

The two notebooks titled *Mathematical breaks* appeared as the number 112 and 114 of *Stampfel-style scientific pocket library* (Pozsony-Budapest, 1903).

Nem emlékeztünk meg a *Rátz* László szerkesztésében megjelenő Középiskolai Mathematikai Lapokban megjelent minden egyes hasonló tárgyú közleményről sem, mely pedig — különösen „Tréfás feladatok“ című rovatában — már sok idetartozó feladatot tárgyalt.

We forgot to mention *Rátz* László appearing in *Középiskolai Matematikai Lapok (Highschool mathematical reviews)* and each of his writings about objects similar to ours. He treats however — particularly in the column titled „Tréfás feladatok (Joking problems)“ — a lot of problems cited in this book.

Ez utóbbiak összegyűjtve megtalálhatók a szerkesztő „Mathematikai Gyakorlókönyv“-ében is. (Budapest, 1904.; I. köt. 70–85. l.)

The latter person can be found in the editors of „Mathematikai Gyakorlókönyv (Practice book of mathematics)“ (Budapest, 1904.; vol. I, pp. 70–85).

Chapter 6

KŐNIG's works on Mathematical recreations

6.1 Introduction

Our aim is to clarify the relation of KŐNIG's works on mathematical recreations and those on graph theory.

One part of our objects consists of KŐNIG's works on graph theory, which are several articles and the treatise *Theorie der endlichen und unendlichen Graphen* of 1936 [121]. According to GALLAI Tibor [224], the following articles of KŐNIG belong to graph theory: "A térképszínezésről (On the map-colouring)" [88] (1905), "Vonalrendszerek Kétoldalú felületeken (Line systems on two-sided [= orientable] surfaces)" [94] (1911), "A vonalrendszerek nemszámáról (On the genus number of line systems)" [95] (1911), "Sur un problème de la théorie générale des ensembles et la théorie des graphes (On a problem of general set theory and graph theory)" [110] (published in 1923, but the lecture was already given in 1914 in the *Congrès de philosophie mathématique* in Paris), "Vonalrendszerek és determinánsok (Line systems and determinants)" [102] (1915), "Graphok és alkalmazásuk a determinánsok és halmazok elméletére (Graphs and their application to theory of determinants and set theory)" [103] (1916), "Sur les rapports topologiques d'un problème d'analyse combinatoire (On the topological relations of a problem of combinatorial analysis)" [111] (1924), "Halmazok többértelmű leképezéseiről (On ambiguous mappings of sets)" [112] (1925), "Sur les correspondances multivoques des ensembles (On the multivalued correspondences of sets)" [114] (1926), "Über eine Schlussweise aus dem Endlichen ins Unendliche (About an inference from the finite to the infinite)" [116] (1927), "Graphok és matrixok (Graphs and matrices)" [117] (1931), "Egy Végességi tétel és alkalmazásai

(A finiteness theorem and its applications)" [118] (1932), "Über trennende Knotenpunkte in Graphen (About separating nodes in graphs)" [120] (1933).

In the treatise of 1936, most of these articles were cited, and the same subjects were discussed again, but the article on the problem of four-colour map published in 1905 [88] was not mentioned in the book of 1936[1].

In this article of 1905, KŐNIG proved the following theorem: if a map of a country drawn on a plane is bordered with a single continuous borderline, and if all the prefectures of the country are adjacent to the frontier along one of the line segments, then the prefectures can be coloured with three colours in the way that prefectures with a common border are always coloured with different colours.

Although GALLAI considered this article as belonging to graph theory, KŐNIG did not use any term of graph theory in the proof. However, in Chapter XII of his treatise of 1936, the problem of four-colour map was mentioned with terms of graph theory as a theorem not yet proved by anyone. As I commented in 5.3, the theorem was proved later in 1976 and published in 1977 by Kenneth APPEL and Wolfgang HAKEN [6].

In KŐNIG's articles that GALLAI considered as belonging to graph theory, no relation to mathematical recreations is mentioned. However, in the treatise of 1936, in which KŐNIG's works on graph theory are eventually collected, some problems are discussed with mentioning of mathematical recreations. This fact will be useful for comparing KŐNIG's works on graph theory and those on mathematical recreations. I will therefore select this treatise as one of our objects that consists of graph theory.

The other part of our objects consists of KŐNIG's works on mathematical recreations. They are not yet much examined by historians, but in Chapters 4 and 5 I translated KŐNIG Dénes' two books on mathematical recreations, which were published in 1902 and 1905[86, 87].

Depending on them, I will discuss in this chapter the following questions:

- Did his books on mathematical recreations of 1902 and 1905 [86, 87] have any relation to his treatise on graph theory of 1936 [121]?
- How did his books on mathematical recreations of 1902 and 1905 related to his treatise on graph theory of 1936?

To answer these questions, we will examine his books in 1902, 1905 and 1936, and compare them with each other.

[1]The article in 1916 [103] and that in 1925 [112] were not cited in the book of 1936, but their German translations "Über Graphen und ihre Anwendungen auf Determinantentheorie und Mengenlehre (About graphs and their applications to theory of determinants and set theory)" [104] (1916) and "Über mehrdeutige Abbildungen von Mengen (About ambiguous mappings of sets)" [115] (1926) were cited and discussed.

Table 6.1: Were problems in each chapter of the book of 1902 treated again in 1936?

Chapter	Title and Contents	In 1936
I	Nagy számok (Large numbers): The way to distinguishing and understanding large numbers.	No
II	Érdekes számok és eredmények (Interesting numbers and results): Cyclic numbers, numbers with repeated digits, perfect numbers, friendly number pair, Mersenne numbers and so on.	No
III	Számok kitalálása (Guessing numbers): Guessing the initial numbers from the results of operations.	No
IV	Bűvös négyzetek (Magic squares)	No
V	Mathematikai hamisságok (Mathematical errors): Incorrect proofs in arithmetic and geometric problems.	No
VI	Síkidomok szétszedése és összeállítása (Decomposition and recomposition of plane figures): Geometric puzzles.	No

6.2 Recreational problems in the book of 1902

I examined the problems in the book of 1902 [86], chapter by chapter, and I considered whether the problems were treated again in the book of 1936 [121].

As we can see in Table 6.1, no problem in the book of 1902 [86] was treated in the book of 1936 [121].

Chapter I "Large numbers" and Chapter II "Interesting numbers and results" in the book of 1902 belong to number theory.

Chapter III "Guessing numbers" belongs to algebra.

Chapter IV "Magic squares" belongs to number theory as well as group theory: an application of group theory to magic squares was already published by Edmond Maillet in 1894 [154].

Chapter V "Mathematical errors" belongs to proof theory.

Chapter VI "Decomposition and recomposition of plane figures" belongs to geometry.

These recreational problems were not related to graph theory in Kőnig's treatise of 1936. It is remarkable that Kőnig's first book on mathematical recreations was published in 1902, which means that this publication preceded his studying abroad in Göttingen in 1904/1905. In Göttingen, Kőnig attended Hermann Minkowski's lecture titled "Analysis situs". In this lec-

ture, MINKOWSKI lectured on properties of topology of four-dimensional surfaces, among which the problem of four-colour map was treated. Just in the same year, in 1905, KŐNIG published an article on the problem of four-colour map [88], and the second book of mathematical recreations [87].

As we shall see, it is possible that MINKOWSKI's lecture exerted an influence on KŐNIG's selection of recreational problems in the book of 1905 [87]. In the next section, we will examine the problems treated in the book of 1905.

6.3 Recreational problems in the book of 1905

I examined the book of 1905 [87], chapter by chapter, and determined which problems were treated again in the treatise of 1936 [121].

See Table 6.2 for the results.

Chapter I "About mathematical probability" belongs to probability theory.

Chapter II "About the binary numeral system" belongs to number theory.

We can see that the problems in the chapters III, IV, VII and XIII were treated again in the treatise of 1936. As I will describe below, the problems in the chapters V and VI are also strongly related to some problems treated in the treatise of 1936.

As for the problems in the chapters V, VI, VIII, IX of Table 6.2, We can make the following remarks.

***1 of Table 6.2:** The problem of "Daily walk of schoolgirls" is not treated in the book of 1936, but a similar and simpler problem is mentioned in §1 of Chapter 11 of the book of 1936[2].

[2]"Daily walk of schoolgirls" is a problem of combinatorics. This problem can be treated with the method of "block designs". The method of block designs was developed by William John YOUDEN (1900–1971) [211] and Frank YATES (1902–1994) [210] in 1930's.

A block design is a pair (X, E), where X is a set of v elements, and E is a set of b subsets (not empty) of X, which satisfies the following conditions: an arbitrary element of X is contained in r elements of E; an arbitrary element of E contains k elements of X; an arbitrary pair of different elements of X are contained in λ elements of E. An element of E is called a block. The numbers v, b, r, k, λ are parameters of the block design.

The problem of "Daily walk of schoolgirls" is considered as a problem to determine the block design with parameters $v = 15, b = 35, r = 7, k = 3, \lambda = 1$.

The method of block designs was later related to graph theory by some mathematicians, for example Jacobus Hendricus van LINT (1932–2004) and Peter Jephson CAMERON (1947–), who published the book *Graph theory, coding theory and block designs* in 1975.

Table 6.2: Were problems in each chapter of the book of 1905 treated again in 1936?

Chapter	Title and Contents	In 1936
I	A mathematikai valószínűségről (About mathematical probability)	No
II	A kettes számrendszerről (About the binary numeral system)	No
III	A négyszínű térkép (The four colour map)	Ch. 11; Ch. 12
IV	A königsbergi hídak (The bridges of Königsberg)	Ch. 2
V	Az iskoláslányok sétái (Daily walk of schoolgirls)	*1
VI	Tait problémái és hasonló feladatok (Tait's problem and similar problems): Tait problem deals with re-arranging positions of 2 kinds of coins.	*2
VII	Elhelyezkedések körben (Positions on a ring)	Ch. 11 Note 9
VIII	Átkelési, átöntési és vasúti feladatok (Problems of traversing, pouring and railway)	Ch. 8 §3; *3
IX	Apróságok (Örök naptár. Versenyszámolás. Meglepő eredmények) (Trivial matters (Perpetual calendar, race-calculation, surprising results))	No; *4

*: See notes in the paragraphs in Section 6.3.

***2 of Table 6.2:** The same problem was not treated in the book of 1936. However, in the same chapter of the book of 1905, KŐNIG put also other problems as 'similar' problems to the Tait problem, and one of them dealt with knight's moves on a chessboard, a less simple example of which was treated in Chapter 2 of 1936-book.

Devoting one chapter to Tait's problem is maybe the influence of AHRENS' book *Mathematische Unterhaltungen und Spiele* written in 1901 [1]. In Chapter II "Ein Problem Tait" of AHRENS' book, he treats the problem which was treated by TAIT in Section (12) of his article "Listing's Topologie" in 1884 [197], which introduced Johann Benedict LISTING's treatise "Vorstudien zur Topologie" of 1847 [146], with TAIT's addition of problems. We can see here that the interaction between analysis situs and mathematical recreations started before KŐNIG. However, KŐNIG brought later more problems from mathematical recreations in connection with analysis situs when he shapes graph theory.

Although Tait's problem was described in TAIT's article mentioning the name of LISTING, LISTING [146] did not mention the problem. TAIT wrote just before describing the problem of re-arranging coins as follows [197]: "A few weeks ago, in a railway-train, I saw the following problem proposed." So we can see mathematicians picking up problems around them. The problem is concerning re-arranging positions of 2 kinds of coins. KŐNIG treated, in addition to this problem, some other problems as similar to TAIT's, one of which was concerning knight's move on the chessboard. It is remarkable that AHRENS treated this problem not together with TAIT's problems in Chapter II, but in Chapter XI "Der Rösselsprung (knight's move)", while KŐNIG put both problems together in one chapter. In other words, AHRENS did not focus on the mathematical similarity, but simply classified the problems depending on the kinds of games. On the other hand, in KŐNIG's book of 1905, he classified the problems depending on another standard, possibly from a mathematical point of view.

***3 of Table 6.2:** It is also remarkable that 2 different problems —"ferrying a wolf, a goat and a cabbage" and "pouring wine"— were treated in a common chapter in both books of 1905 and 1936, while they were not always treated together in the books cited by KŐNIG in the book of 1905 (see Table 6.3). Only BACHET (1874) [9] and FOURREY (1901) [65] treated these two kinds of problems in a common part, but we should pay attention to the fact that BACHET's last part was titled "S'ensuivent quelques autres petits subtilitez des nombres, qu'on propose ordinairement (some other small complicated things of numbers ordinarily proposed)", and that FOURREY's Chapter 11

Table 6.3: Ferry problem and wine problem in the citation by KŐNIG (1905)

Book	Ferry	Wine
OZANAM (1694) [165]	Part I, Problem 18, p. 21	Nothing
BACHET (3rd ed. 1874) [9]	Last part, §4, p. 148	Last part, §3, p. 138
LUCAS (1882) [148]	vol. I, Chapter 1, p. 1	Nothing
LUCAS (1895) [153]	p. 125	Nothing
BALL (1898) [14]	Chapter 2, Section 6, p. 88	Chapter 1, Section 1, p. 28
SCHUBERT (1900) [189]	vol. II, §16, p. 159	Nothing
FOURREY (1901) [65]	Chapter 11, p. 161	Chapter 11, p. 167
AHRENS (1901) [1]	Chapter 1, Section 1, p. 1	Chapter 4, Section 1, p. 53

was titled "Problèmes anciens (ancient problems)". It means that their classification was not based on mathematical notion, while KŐNIG puts these problems together depending on a mathematical point of view, at least in the book of 1936, possibly also in the book of 1905.

***4 of Table 6.2:** This chapter is a collection of various problems put at the end of the whole series in 2 books in 1902 and 1905. I suppose that KÖNIG thought that these problems are suitable neither to put in the other chapters, nor to creat an independent chapter for each. Depending on this supposition, it is reasonable that this chapter does not contain any problem related to graph theory.

In conclusion from Table 6.2, the problems in the chapters from III to VIII of the book of 1905 appeared again in some way in the book of 1936. This is all the more interesting that no problem of the 1902 book found its way into the 1936 book.

This difference supports our hypothesis that MINKOWSKI's lecture on the problem of four-colour map, which KŐNIG attended in Göttingen in 1904–1905, just between the publications of 1902 and 1905, played an important role to bring KŐNIG to the works which were later related to graph theory.

6.4 Recreational problems in the book of 1936

I examined here the book of 1936 chapter by chapter, to determine where any recreational problems appeared, and if these recreational problems were already treated in the book of 1905. The English chapter titles of the book of 1936 are taken from Richard McCoart's translation (1990) [123].

In Chapter 1 "Foundations" of the treatise of 1936, definitions of terms of graph theory are given, and some theorems are proved using the terms.

In Chapter 2 "Euler trails and Hamiltonian cycles", Eulerian paths and Hamiltonian paths are discussed. To prove the related theorems, the terms and the theorems in Chapter 1 were used.

In Chapter 3 "The labyrinth problem", the problem of labyrinth is discussed. The problem is to give an algorithm to find a goal point in walking in a labyrinth without a map. The algorithms described in this chapter are discussed using a theorem in Chapter 2.

In Chapter 4 "Acyclic Graphs", the graphs without closed path are called acyclic graphs and discussed. A finite and connected acyclic graph is called a tree.

In Chapter 5 "Center of trees", The center of a tree is defined. If all the end edges of a tree are deleted, the remaining edges also form a tree. By repeating the deletion of the end edges of the remaining tree, all the edge(s) of remaining tree will be finally end edge(s). If there remains only 1 end edge, the edge is called an axis, and the 2 endpoints of the edge are called bicenters (central points). If there remains end edges more than 1, the common vertex of all the remaining end edges is the center. Some theorems on the properties of the center are proved, and applications to forms of molecules are discussed.

In Chapter 6 "Infinite graphs", graphs of finite degree are introduced, and the related theorems on infinite connected graphs of finite degree are proven. And then the Infinity Lemma and the sharpened Equivalence Theorem are proved. The Infinity Lemma is the following lemma:

> Let $\Pi_1, \Pi_2, ...$ be a countably infinite sequence of finite, non-empty, pairwise disjoint sets of points. Let the points contained in these sets form the vertices of a graph. If G has the property that every point of $\Pi_{n+1} (n = 1, 2, ...,$ ad inf.) is joined with a point of Π_n by an edge of G, then G has a singly infinite path $P_1 P_2 ...$, where $P_n (n = 1, 2, ...$ ad inf.) is a point of Π_n.

The sharpened Equivalence Theorem is shown as an example of an application of graph theory to abstract set theory. The theorem is as follows:

> Let the set Π of vertices of the graph G come from two disjoint

Table 6.4: Recreational problems in each chapter of the book of 1936

Chapter	Title and Contents	In 1905
1	Die Grundlagen (Foundations)	No; *a
2	Eulersche und Hamiltonsche Linien (Euler trails and Hamiltonian cycles)	*b
3	Das Labyrinthenproblem (The labyrinth problem)	*c
4	Kreislose Graphen (Acyclic Graphs)	No; *d
5	Zentren der Bäume (Center of trees)	No
6	Spezielle Untersuchungen über unendliche Graphen (Infinite graphs) [If translated word by word, the chapter title is "Special analysis on infinite graphs".]	No
7	Basisprobleme für gerichtete Graphen (Basis problems for directed graphs)	*e
8	Verschiedene Anwendungen der gerichteten Graphen (Logik. — Theorie der Spiele. — Gruppentheorie.) (Various applications of directed graphs)	*f
9	Zyklen und Büschel und die entsprechenden linearen Formen ([Directed] Cycles and stars and the corresponding linear forms)	No
10	Komposition der Kreise und der Büschel (Composition of cycles and stars)	No
11	Faktorenzerlegung regulärer endlicher Graphen (Factorization of regular finite graphs)	*g
12	Faktorenzerlegung regulärer endlicher Graphen dritten Grades (Factorization of regular finite graphs of degree 3)	*h
13	Faktorenzerlegung regulärer unendlicher Graphen (Factorization of regular infinite graphs)	No
14	Trennende Knotenpunkte und Knotenpunktmengen (Separating vertices and sets of vertices)	No

*: See notes in the paragraphs in Section 6.4.

sets Π_1 and Π_2 and let the set K of edges of G come from two disjoint sets K_1 and K_2 in such a way that

1) every edge of G joins a Π_1-vertex with a Π_2-vertex;

2) every Π_1-vertex is the endpoint of one and only one K_1-edge end every Π_2-vertex is the endpoint of at most one K_1-edge;

3) every Π_2-vertex is the endpoint of one and only one K_2-edge and every Π_1-vertex is the endpoint of at most one K_2-edge.

Then G has a factor of first degree.

KŐNIG insists that, if the terminology of graph theory was avoided in the proof of this theorem, the result would be the simplest proof of the Equivalence Theorem given by his father Julius KÖNIG (KŐNIG Gyula) [124]. It is remarkable that most of the part of this Chapter depends on the articles of KŐNIG Dénes already published in 1916, 1926 and 1927 [104, 114, 116].

In Chapter 7 "Basis problems for directed graphs", the vertex basis and the edge basis of a directed graph are defined, and the related theorems are proved.

In Chapter 8 "Various applications of directed graphs", the concepts and results of Chapter 7 are applied to axiomatic theory, game theory and group theory. KŐNIG's article of 1927 [116] is cited and discussed in a part of this chapter.

In Chapter 9 "[Directed] Cycles and stars and the corresponding linear forms", a linear form of a directed graph is defined. Linear forms belonging to directed cycles of graphs are discussed. A directed star is defined as a subgraph of a directed graph which is formed by the edges ending in the same vertex. Linear forms belonging to directed stars of graphs are discussed. The discussions are applied to Gustav Robert KIRCHHOFF's results [77] concerning electric circuits.

In Chapter 10 "Composition of cycles and stars", a composition of a finite number of graphs is defined. A composition of undirected cycles of finite graphs and a composition of undirected stars of graphs are discussed. A theorem in this chapter already appeared in KŐNIG's article of 1916 [104].

In Chapter 11 "Factorization of regular finite graphs", the "product" of graphs $G_1G_2...$ is defined, where graphs $G_1, G_2, ...$ are called "factors". Applications to the problems of combinatorics are discussed. Most of this chapter depends on Julius PETERSEN's works [174], but KŐNIG's article of 1916 [104] and that of 1932 [118] are also discussed. Two theorems are already stated in KŐNIG's lecture in the congress in Paris in 1914 [110].

In Chapter 12 "Factorization of regular finite graphs of degree 3", bridges and leaves of a graph are defined, and related theorems are proved. The new proof of FRINK's theorem [64] is given. The problem of four-colour map is considered as very closely connected with the factorization of regular finite graphs of degree 3, and 7 pages are devoted to the discussion.

In Chapter 13 "Factorization of regular infinite graphs", bipartite graphs are considered, and the related theorems are proved. To one of the theorems, also set theoretical formulations are given. KŐNIG's important articles [110, 104, 114, 115] are discussed in this chapter.

In Chapter 14 "Separating vertices and sets of vertices", cut points and blocks of a graph are defined, and separating sets of vertices of a graph is discussed. This separation is considered also for bipartite graphs. KŐNIG's articles [102, 104, 117, 120] are discussed.

***a of Table 6.4:** No recreational problem appeared. Some technical terms were defined in this chapter, and no application was treated.

***b of Table 6.4:**

- An example of graph which "cannot be traversed in fewer than four trails" was given in Section 1, Chapter 2 of the treatise of 1936 (see Figure 6.1). This diagram originates in Thomas CLAUSEN's article "De linearum tertii ordinis proprietatibus" (1844) [32] as a figure that requires four strokes to trace on. This figure was treated again by LISTING's "Vorstudien zur Topologie" (1847) [146], by LUCAS' *Récréations mathématiques* IV (1894) [152] and so on. A similar example appeared in Chapter IV of the book of 1905 (see Figure 6.2), for which KŐNIG referred to KÜRSCHÁK, who was a teacher of KŐNIG at the Polytechnic.

- The problem of Königsberg bridges was also treated in Section 1. This problem appeared also in Chapter IV of the book of 1905.

- The problem of knight's move was treated in Section 2. Not exactly the same but a simplified problem of knight's move was treated in Section 7 of Chapter VI in the book of 1905 with Figure 14.

***c of Table 6.4:** The problems of mazes were treated in this chapter, while no maze appeared in the book of 1905. Yet they were one of the major topics of mathematical recreations in the 19th century.

Figure 6.1: “Fig. 5” of the book of 1936.

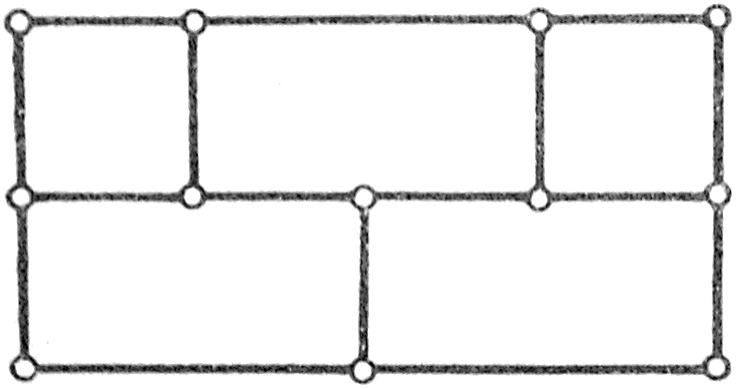
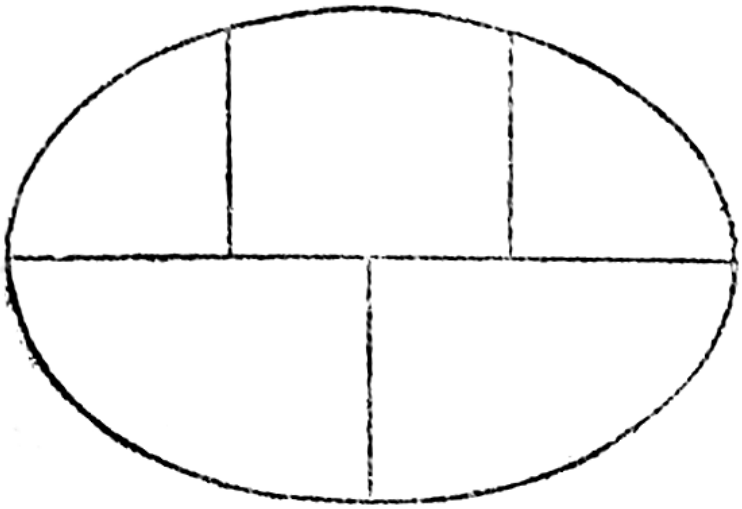


Figure 6.2: “11. ábra.” of the book of 1905.



***d of Table 6.4:** No recreational problem was treated as Acyclic Graphs. Yet, this object appeared in other famous books of mathematical recreations, which KŐNIG read. In these previous books, it is described as something related to scientific disciplines[3].

***e of Table 6.4:** In §1 "Die punktbasis (The vertex basis)" of this chapter, the problem of queens on a chessboard is discussed. The problem is: "How many queens are necessary to attack every unoccupied square of a chessboard?" Though this problem can be considered as one of mathematical recreations, it did not appear in the book of 1905.

***f of Table 6.4:**

- In §3 "Solo-Spiele (Solitaire games)", the problem of pouring wine and the problem of ferrying a wolf, a goat and a cabbage were treated. The problem of pouring wine appeared in §5 of Chapter VIII of 1905. The problem of ferrying a wolf, a goat and a cabbage appeared in §1 of Chapter VIII of 1905.

- §4 "Spiele zu zweit (Games for two people)" consists of discussion on games. It is natural that this discussion did not appear in the book of 1905, because these results depend on the works after 1905: by ZERMELO (1912) [212], KŐNIG (1927) [116], KALMÁR (1928) [75] and EUWE (1929) [57]. Still they connect to mathematical recreations.

***g of Table 6.4:**

- §1 "Faktoren der regulären Graphen (Factors of regular graphs)" contains a diagram from Chapter 18: Das Farben Karten Problem (problem of coloured map) in the version of 1918 of AHRENS' book [3]. This diagram appeared also in the version of 1901 of AHRENS' book [1], which was cited by KŐNIG in the book of 1905. The same diagram did not appear in the book of 1905, but the problem of colouring map was treated in Chapter III of the book of 1905.

[3]BALL's book *Mathematical Recreations and problems of past and present times* (1892) [12] contains a section of "geometrical trees", the notion of which corresponds to "Bäume (trees)" defined in this chapter of the book of 1936. BALL derived trees from blind alleys in mazes, and mentioned the application of the theory of trees to chemical and biological theories, but BALL did not discuss any recreational aspect of trees.

- In §1, KŐNIG mentioned "a problem of bringing together $2n$ participants $2n-1$ times into n pairs in such a way that everyone is paired with everyone else exactly once." He cited here LUCAS (1883) for "Les promenades du pensionnat (promenades of the school dormitory" [149]. Not exactly the same but a similar and less simple problem "daily walk of schoolgirls" was treated in Chapter V of the book of 1905.

- In §2, the problems "Les rondes enfantines (The circles of children)", "Les rondes paires (The pair circles)" and "Les rondes alternées (The alternate circles)" are treated. They are drawn from LUCAS' book (1883) [149]. These problems were already treated in Chapter VII of the book of 1905.

- In §5, three applications were treated, which seem to be recreational problems, but nothing is cited for these problems.

***h of Table 6.4:** The problem of four colour map was treated here. This problem was treated in Chapter III of the book of 1905.

6.4.1 Recreational problems not treated in the book of 1905 but in the book of 1936

We saw that the recreational problems of mazes (Chapter 3), queens on a chessboard (Chapter 7, §1) and games for two people (Chapter 8, §4) treated in the book of 1936 were not treated in 1905.

Among them, the absence of games for two people is natural, because the related results were based on the works after 1905 by ZERMELO (1912) [212], KŐNIG (1927) [116], KALMÁR (1928) [75] and EUWE (1929) [57], as we have already said.

As for mazes (Chapter 3) and queens on a chessboard (Chapter 7), they were treated in the books cited in the book of 1905, therefore, these problems were among alternatives of the problems to be put in the book of 1905. In spite of this fact, why were these problems not treated in the book of 1905?

Mazes KŐNIG related Chapter 3 "Das Labyrinthenproblem (The labyrinth problem)" of the book of 1936 to a theorem in §1 "Die Eulerschen und verwandte Sätze (The Euler theorem and related theorems)" of Chapter 2 "Eulersche und Hamiltonsche Linien (Euler trails and Hamiltonian cycles)", and discussed methods to solve mazes, referring to WIENER's method (1873) [208], TRÉMAUX' method (in LUCAS's book, 1882) and TARRY's method [202].

It is remarkable that one chapter was devoted to mazes in the book of 1936, while some books of mathematical recreations cited by KŐNIG treated mazes in the same chapter as the problem of Königsberg bridges, as shown below:

Ahrens (1901) [1]: Kapitel XVII Brücken und Labyrinthe (Chapter XVII Bridges and mazes),
§1. Das Euler'sche Brückenproblem (The problem of bridges of Euler);
§2. Labyrinthe (Mazes);
§3. Durchwanderung aller Wege eines Labyrinths (Passing through all the paths of a maze).

Bachet [9]: [4] Neither labyrinth nor bridges were mentioned.

Ball [14]: [5] Chapitre VI Les problèmes sur les tracés continus (Chapter VI Unicursal problems),
§Problème d'Euler (Euler's problem);
§Labyrinthes (Mazes);
§Les arbres géométriques (Geometrical trees);
§Le jeu d'Hamilton (The Hamiltonian game);
§Marche du cavalier sur l'échiquier (Knight's path on a chess-board).

Fourrey (1901) [65]: Neither labyrinth nor bridges were mentioned.

Lucas (1882: Récréations mathématiques I) [148]: Deuxième Récréation: Le jeu des ponts et des îles (Chapter 2: The game of bridges and islands);
Troisième récréation: Le jeu des labyrinthes (Chapter 3: The game of mazes).

Lucas (1895: L'arithmétique amusante) [153]: Neither labyrinth nor bridges were mentioned.

Ozanam (1694) [165]: Neither labyrinth nor bridges were mentioned.

Rebière (1898, the third edition) [177]: Maze was not mentioned. The problem of bridges was mentioned as one of the questions in the chapter of "Problèmes frivoles et humoristiques (trivial and humorous problems)".

[4]First edition in 1612 [7]; second edition in 1624 [8]; KŐNIG cited the third edition of 1874 [9]

[5]First edition in 1892 [12]; KŐNIG cited the translation of the third edition to French in 1898 [14]

Schubert (1900) [189]: Maze was not mentioned. The problem of bridges was treated in Zweiter Abschnitt: Anordnungs-probleme (Part 2: problems of position),
§23 Eulersche Wanderungen (Euler trails);
§24 Hamiltonsche Rundreisen (Hamiltonian cycles);
§25 Rösselsprünge (knight's move).

3 of 9 books mentioned in the book of 1905 contain both problems of bridges and mazes. AHRENS and BALL treated bridges and mazes in a common chapter, while LUCAS treated them separately.

Yet in the book of 1905, KŐNIG did not treat mazes in spite of the fact that the three authors whom KŐNIG quoted treated them. We can build up a hypothesis that he might have thought in 1905 that the problem of mazes was something near to the problem of bridges, and that it was not worth devoting one chapter to the mazes.

Queens on a chessboard This problem does not appear in the book of 1905, though AHRENS devotes one chapter to this problem (Kapitel X Königinnen auf dem Schachbret (Queens on the chessboard)) [1]. However, it is remarkable that this chapter of AHRENS' 1901 version [1], which KŐNIG cited in 1905, consists of only 8 pages with no section, while the same chapter of AHRENS' 1921 version [4], which KŐNIG cited in 1936, consists of 6 sections and 1 appendix, 34 pages. I suppose that this difference influenced the difference of the book of 1905 and the book of 1936.

6.4.2 Summary regarding the book of 1936

As shown in Table 6.4, the chapters 2, 3, 7, 8, 11, 12 among 14 chapters contain recreational problems in some way, and recreational problems in the chapters 2, 8, 11, 12 were already treated in the chapters III to VIII of the book of 1905.

The recreational problems of mazes, queens on the chess board and games for two people were treated in the book of 1936, but they did not appear in the book of 1905. We have suggested hypotheses on the reasons for the absence of them. The games for two people were based on the works after 1905, therefore it was impossible to appear in the book of 1905.

As for mazes and queens on the chess board, we considered some possible reasons for the absence of these problems from the book of 1905.

6.5 Difference in treatment of problems between the books of 1905 and 1936

We saw that

- all the problems in the chapters from III to VIII of the book of 1905 appeared again in some way in the book of 1936;
- 6 chapters among 14 chapters in the book of 1936 contain recreational problems, and
  4 chapters among them contain recreational problems which were already treated in the book of 1905.

These books treat many identical problems, but, because of the difference of genres, we can expect that there be differences in the treatment. Then, what is the difference in treatment of recreational problems between the books of 1905 and 1936? In order to consider this problem, it will be helpful to focus on one of the recreational problems, and compare the treatment of it. We will select here the problem of bridges of Königsberg, because it is much discussed in both 1905 and 1936.

6.5.1 Bridges in the book of 1905

Chapter IV of the book of 1905 begins with the problem itself:

> **Kőnig (1905) [87], Chapter IV. My translation from Hungarian.**
> In Königsberg, there is an island called Kneiphof; the Pregel river, which forms the island, is torn again in two branches after the island. 7 bridges are built over the river, and 5 of them start from the island (see Figure 6.3). The problem is as follows: can one pass all the bridges of Königsberg by crossing every bridge just once?

In 1905, Kőnig attached a letter symbol to each land area and to each bridge, and he explained briefly the solution that Euler had given to this problem in 1736 [51]. Euler had proved that one cannot pass all the bridges of Königsberg by crossing every bridge just once. And then Kőnig gave a diagram shown in Figure 6.4 of our numbering[6], maybe taken from Ahrens' book in 1901 [1], to make the problem "sokkal egyszerűbb (much more simple)".

[6] A part of the line between A and B is cut, but it is only an error of printing. It should be continuous for consistency of the text.

Figure 6.3: "7. ábra." of the book of 1905.

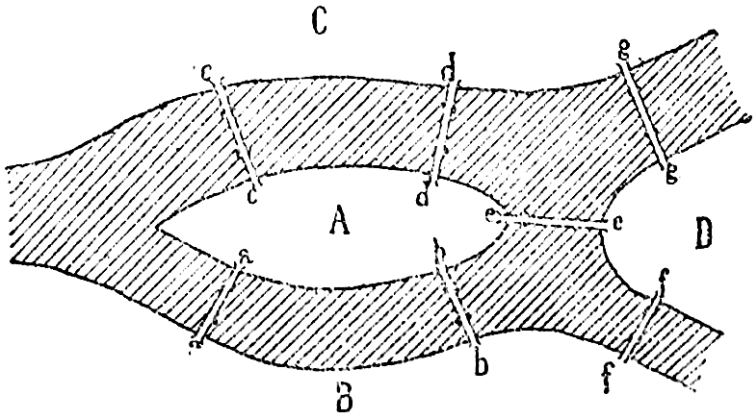
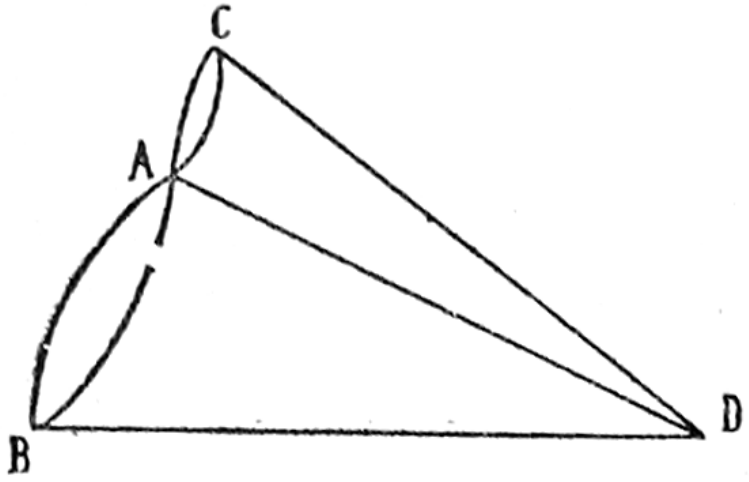


Figure 6.4: "8. ábra." of the book of 1905.



With this diagram, in 1905, KŐNIG modified the problem as follows: "Is it possible to draw the diagram [Figure 6.4 of our numbering] with one continuous line?" Using such a "vonalrendszer (line system)" — this is the concept he used — KÖNIG gave some other examples of the problem, asking if the "line system" can be drawn with one continuous line. Finally KÖNIG raised another question: "how many continuous lines are required for drawing a 'line system'?" And he mentioned the diagram that we discussed in 6.4 (Figure 6.2), which requires at least 4 continuous lines.

6.5.2 Bridges in the book of 1936

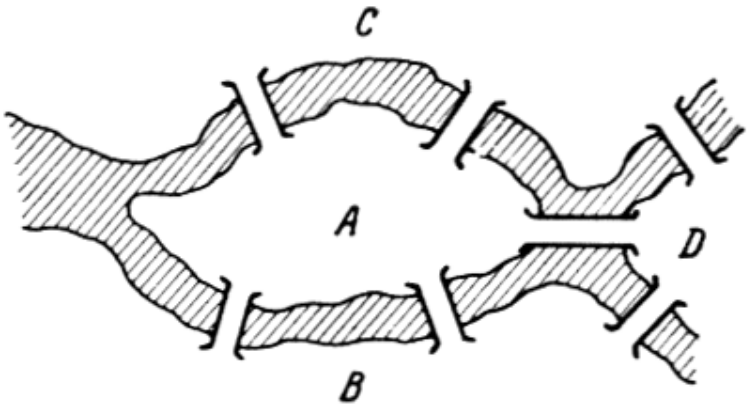
In Chapter 2 of the book of 1936, Euler trails and Hamiltonian cycles were discussed. For this discussion, the definition of basic concepts described in Chapter 1 was required, in contrast with the book of 1905. The basic concepts were defined using the concepts of set theory: "graph", "vertex", "edge" etc.

At the beginning of Chapter 2, §1: Die Eulerschen und verwandte Sätze (The Euler theorem and related theorems), KŐNIG mentioned mathematical games. Although we have the English translation by Richard MCCOART [123], I use here my translation for some reasons:

> **KŐNIG (1936) [121], Chapter 2, §1.**
> Manche Fragestellungen, die sich auf gewisse mathematische Spiele beziehen und denen die Graphtheorie und überhaupt die Analysis Situs ihre ersten Untersuchungen verdankt, führen auf die Frage, wann ein Graph als ein geschlosser Kantenzug aufgefaßt (gezeichnet) werden kann.
> **My translation**
> Some questions, which refer to certain mathematical games, and to which graph theory and generally[7]the analysis situs owe their first examinations, lead to the question of when a graph can become understood (drawn) as a closed path.
> **Translation by MCCOART (1990) [123]**
> Some questions, which relate to certain mathematical games and to which graph theory and especially topology owe their first interest, lead to the question of when a graph can be represented as a closed trail.

[7]KŐNIG considers the graph theory as something related to analysis situs. The adverb "überhaupt" means "above all", which we can interpret as either "generally" or "especially", therefore we can not conclude from the sentence above if he considers that the graph theory is a part of analysis situs, or that analysis situs is a part of the graph theory, or he thinks of any other relation between them. MCCOART chose "especially". However, if

Figure 6.5: "Fig. 6" of the book of 1936.



In other words, according to KŐNIG, the graph theory started from examinations of questions which refer to certain mathematical games.

In spite of this relation between graph theory and certain mathematical games, the approach to the problem of bridges was different from that of the book of 1905. In the book of 1936, KŐNIG gave first some theorems concerning a closed path, and then he proved the theorems using the concepts defined in Chapter 1 based on the concepts of set theory.

It is interesting that the problem "how many continuous lines are required for drawing a 'line system'?" was treated *after* the problem of bridges in the book of 1905 (Figure 6.2), while this problem was treated *before* the problem of bridges in the book of 1936 (Figure 6.1); this problem was treated as an example of one of the theorems, and this theorem was used for solving the problem of bridges.

After the definition of the terms of graph theory and proofs of some theorems, the problem of bridges was introduced with a drawing of the river as an application of graph (see Figure 6.5). To solve this problem, KŐNIG first gave a diagram of graph (see Figure 6.6). And then, he applied the theorems proved beforehand to this problem.

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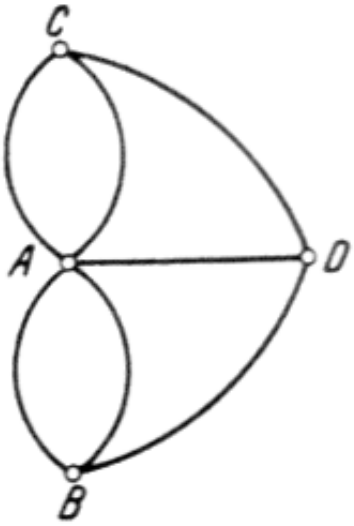
we remember GALLAI's remark that KŐNIG taught also the subjects of graph theory in his lecture titled "Analysis situs" between 1911–1927 [224], it seems suitable to translate it as "generally". Moreover, MCCOART translated "Analysis Situs" as "topology", but this English term seems not to signify something that KŐNIG meant with the term "Analysis Situs", in which a prototype of both topology and graph theory was included. KŐNIG used also the term "Topologie" several times in the book of 1936, and this term indeed fits to the English term "topology".

Figure 6.6: "Fig. 7" of the book of 1936.



6.5.3 Summary of the difference between the books of 1905 and 1936

Common problems were treated in both books of 1905 and 1936, but the ways of treatment were different from each other.

As we have already discussed in 2.2, KŐNIG's books on mathematical recreations can be regarded as one of the activities of reforming mathematical educations.

For this purpose, KŐNIG's books on mathematical recreations were written not for professional use but for wide readership including high school students. Therefore, it was important to draw the readers' interest in the topics and the related mathematics of higher level, and it was not necessary to describe precisely mathematical theorems nor proofs.

We saw in this section that the book of 1905 represented these aspects. In the book of 1905, the problem was given first, and it was solved without using the concepts of graph theory, and then another representation of the problem was given, which is similar to the representation of a graph.

As we discussed in Chapter 2, a main purpose of the publication of these books on mathematical recreations was educational use with the methods of KLEIN and BEKE especially for high school students.

On the other hand in the book of 1936, we saw that the concepts of graph theory were first defined based on set theory, and some theorems were

proved, and then the problem of bridges was treated as an example of the application of the theorems.

It is natural if we take into account that the book of 1936 was a treatise of mathematics for professional use. For this purpose, mathematical description of theorems and precise proofs of them are indispensable.

The examination of different kinds of books in this chapter supports these suggestions.

Chapter 7

Historical transition of the features of diagrams of graph theory

7.1 Introduction

In KŐNIG's treatise in 1936, some problems are selected from mathematical recreations, and they play an important part. This fact corresponds to the fact that KŐNIG, when he was still a student, published two books on mathematical recreations. In fact, KŐNIG's treatise of 1936 and one of his books on mathematical recreations are closely related to each other (see Chapter 6).

In this chapter, I will analyse how, in the context of mathematical recreations, some features of the diagrams of graph theory, as well as some concepts of it, took shape. Moreover, I will establish that KŐNIG inherited the features of the diagrams and the concepts from some publications on mathematical recreations.

In the treatise of 1936, many diagrams are used for representing graphs. They are mostly of a single kind of diagram, consisting of curved or straight lines which represent edges, and small circles which represent vertices. This representation of graphs in diagrams of graph theory continues to be used in the texts of graph theory until today.

This chapter focuses on the question of how this representation of graphs in diagrams of graph theory took shape. To address this issue, it appears that one needs to examine some problems from mathematical recreations which in the treatise of 1936, KŐNIG treated on the basis of the way mathematicians before him had dealt with them. Four problems appear to have played a key part in this process: the problem of Königsberg bridges, a problem of

describing polygons, a problem of configuring dominoes and the problem of circulating in mazes. This selection of problems will allow us to deploy a historical approach to diagrams, and to identify how some of the key features of the diagrams of graph theory took shape in different contexts.

In the treatise of 1936, KŐNIG discussed these different problems using the same concepts attached to the notion of graph. In other terms, the concepts of graph allowed viewing problems that, when they appeared, looked unrelated, as depending on the same concepts attached to a single notion of graph, and thereby as related. In fact, when examining the changing solution given to these problems along the 19th century and early 20th century, one can identify a process of progressive integration of the problems throughout the various publications of mathematical recreations in which they were treated. In these publications, mathematicians introduced concepts that allowed to unify problems progressively, and that entered in the shaping of the concepts attached to the same notion, that is, the notion of graph in KŐNIG's treatise of 1936.

Moreover, in these earlier writings, different types of diagrams were used for different topics. Among these types of diagrams, a certain type became gradually influential. It was on the basis of this type of diagrams that the different problems became gradually understood as concerning the same object. The process of the integration of the features of diagram was not originate in one source of a certain problem. The elements of the features of diagram appeared in various contexts, and they were integrated step by step. Along with the integration of the features of diagram, the concepts corresponding to them were also integrated.

On the basis of these facts, we will be led, in Section 7.3, to the hypothesis that the two historical processes mentioned are related to each other: the process which shaped the diagrams of graph, and the process which shaped the concepts of graph.

To support this hypothesis, in what follows, we shall focus in particular on the following issues: how were different topics of mathematical recreations integrated, and how graph theory was shaped by this process?

7.2 Why did KŐNIG use diagrams in 1936?

The treatise *Theorie der endlichen und unendlichen Graphen* in 1936 [121] makes explicit the reason why KŐNIG Dénes used diagrams in it. Since this gives us information as to how KŐNIG viewed the diagrams he was using, let us first examine what the author has to say about them.

KŐNIG put various problems orderly and discussed them as graph theory

in this treatise.

In the preface, he explained that graph theory can be understood from two standpoints: one standpoint views the topic as the first part of general topology, while the other understands it as a branch of *combinatorics* and abstract *set theory*.

If this treatise had embraced the first standpoint, that of general topology, we could suppose that diagrams would be used for representing topological concepts.

However, KŐNIG took actually the latter standpoint, that of *combinatorics* and abstract *set theory*. He explained the reason for this decision as follows:

> **KŐNIG: *Theorie der endlichen und unendlichen Graphen* in 1936 [121], Preface. (My translation from German.)**
> In this book, we take this second standpoint, mainly because we attribute to the elements of graphs — points and edges — no geometrical content at all: the points (vertices) are arbitrary distinguishable elements, and an edge is nothing else but a unification of its two endpoints. This abstract point of view —which Sylvester (1873) emphasized already[1] — will be strictly kept in our representation, with the exception of some examples and applications.

In spite of KŐNIG's decision to take the standpoint of *combinatorics* and abstract *set theory*, he used in his treatise a geometrical way to represent elements of a graph (points and edges). Moreover, he introduced diagrams that were not presupposing any geometrical point of view or any geometrical axioms.

We can therefore naturally raise a very simple question: why did KŐNIG, in this treatise, use a geometrical way of representing parts in a graph and

[1] KŐNIG made here a reference to the article by SYLVESTER entitled "On recent discoveries in mechanical conversion of motion" in 1873 [191]. This article treated a mode of producing motion in a straight line by a system of pure link-work without the aid of grooves or wheel-work, or any other means of constraint than that due to fixed centres, and joints for attaching or connecting rigid bars. Maybe here KŐNIG had the following part of SYLVESTER's article in mind: "The theory of ramification is one of pure colligation, for it takes no account of magnitude or position; geometrical lines are used, but have no more real bearing on the matter than those employed in genealogical tables have in explaining the laws of procreation. [New paragraph] The sphere within which any theory of colligation works is not spatial but logical—such theory is concerned exclusively with the necessary laws of antecedence and consequence, or in one word of *connection* in the abstract, or in other terms is a development of the doctrine of the compound parenthesis." (*the Collected mathematical papers of James Joseph Sylvester*, vol. 3, pp. 23–24.)

diagrams, although no geometrical content is attributed to the elements of graphs?

KŐNIG answered himself to half of the question. He made clear that he used a geometrical way of representing elements of a graph because it gave him a very comfortable terminology.

The question thus remains: how were the diagrams to be read if they were not geometrical?

We can suppose that KŐNIG used diagrams for representing the geometrical notation used in this treatise, even though neither any "geometrical point of view" nor any "geometrical axiom" was presupposed.

To inquire further into this supposition, I will examine in Section 7.3 some of the diagrams shown in this treatise.

In the book *Graph Theory, 1736–1936* (Norman L. BIGGS, E. Keith LLOYD and Robin J. WILSON, 1976 [215]), gathers most of the source material with which we shall deal later on. However, they read it, using the concepts and diagrams of graph. Therefore, they bypass the question of the emergence of these concepts and diagrams. These are the questions with which I shall reconsider this source material and other documents.

7.3 Diagrams in KŐNIG's treatise of 1936 and their historical background

Let us consider here how KŐNIG used diagrams in *Theorie der endlichen und unendlichen Graphen* in 1936. The point will be here to compare these diagrams with those he himself published before 1936, and those in the publications by other mathematicians. As I wrote in Section 7.1, I will select only the diagrams used in the problems of bridges, polygons, dominoes and mazes, since these problems were treated also in many publications before his treatise of 1936, and we can therefore get enough literatures to compare with KŐNIG's treatise in 1936.

These problems can be found in the books on mathematical recreations written by some mathematicians. In these books, certain mathematical problems were collected under the concept of "recreation", "pleasure", "delectation", "leisure", "amusement", "game" or "curiosity"[2].

As we saw in Chapter 2, KŐNIG Dénes also published 2 books on mathematical recreations [86, 87]. One of them includes the problem of bridges which we will discuss in 7.3.1. This provides us with the publications to

[2]See Chapter 3 for the history of mathematical recreations.

which he has access in his youth and allows us to analyze the evolution of his approach to some problems between 1905 and 1936.

LUCAS' works were the topic of research by some historians (Anne-Marie DÉCAILLOT [217, 218, 219]).

According to DÉCAILLOT, LUCAS was attracted by "geometry of situation", and, from the problems which had been considered as "geometry of situation", he drew recreations, but without any analysis ([218], p. 5). The "geometry of situation" was not yet well structured at that time (PONT: *La topologie algébrique des origines à Poincaré* [232], EPPLE: "Topology, matter, and space I : topological notions in 19th-century natural philosophy" [222])[3].

Through the examination of diagrams used in the above mentioned four problems, we will clarify how a part of the "geometry of situation" was structured, and which diagrams were involved in this process.

7.3.1 Appearance of graph-like diagram for the problem of seven bridges of Königsberg

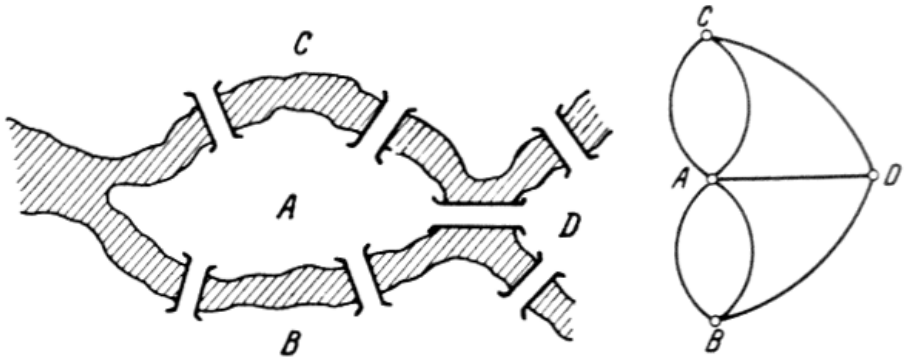
The problem of seven bridges of Königsberg was mathematically considered and published first by Leonhard EULER in 1736 [51]. We examined EULER's article in 3.4.1. In 6.5, we compared KŐNIG's treatment on this problem in the book of 1905 and that in the book of 1936.

In the chapter titled "Eulersche und Hamiltonsche Linien (Eulerian and Hamiltonian lines)" of the book of 1936, KŐNIG treated the problems of the so-called Eulerian circuits and the Hamiltonian circuits. An Eulerian circuit is a closed path which goes through each edge of a graph once and only once; a Hamiltonian circuit is a closed path which goes through each vertex of a graph once and only once.

KŐNIG gave some theorems concerning Eulerian circuit in the first section of this chapter. One of the theorems is as follows: "One can go through all the edges of a graph in a closed path if and only if the graph is a connected Euler graph" (Theorem 2, my translation from German).

In this context, in the second section entitled "Das Brücken- und Dominoproblem (The problem of bridges and the problem of dominoes)", KŐNIG mentioned the problem of seven bridges of Königsberg as an example of application of the theorems.

[3] As DÉCAILLOT says, "among the mathematical games and recreations of EULER, the traces of strings on the chessboard of VANDERMONDE in the 18th century, the 'higher mathematics' of RIEMANN and the Analysis situs of POINCARÉ, the geometry of situation has a fluctuating content which is not structured but progressively during the 19th century" ([218], p. 129, my translation from French).



a. **b**.

Figure 7.1: Taken from Section 2.2 of KŐNIG's book of 1936 [121].

The problem is as follows: in Königsberg, there was a river flowing from the east to the west; across the river, there were 7 bridges; the problem is to find a smart method to know if there is a way to cross every bridge once and only once.

KŐNIG introduced this problem using a simplified map as in Figure 7.1.**a**, and then he gave a diagram as in Figure 7.1.**b**, which consists of small circles representing vertices corresponding to land areas, and straight or curved lines representing edges corresponding to bridges. In this way, the diagram shown in Figure 7.1.**b** represents geometrical elements which can correspond to a graph. I will call such a diagram a "graph-like diagram".

In the diagram in Figure 7.1.**b**, KŐNIG displayed only the elements necessary *for solving* the problem. From this diagram, we can see that each vertex is connected to an odd number of edges. KŐNIG concluded, using the theorem of graph theory mentioned above, that there is not such a way to cross every bridge just once.

This diagram is thus useful *for solving* the problem using geometrical concepts representing a graph.

Originally, how did EULER deal with the problem in his article "Solutio problematis ad geometriam situs pertinentis (solution of a problem relating to the geometry of situation)" in 1736 [51]?

EULER introduced a map illustrating the situation. On the map, EULER displayed symbols for his proof. EULER's proof needed only the symbols of the land areas and the number of bridges connected to each land area. For him, the names of bridges were not necessary. Yet we can see that EULER kept these informations on the map (Figure 7.2). Moreover, the proof does not make any reference to a diagram. In fact, more precisely, EULER did not

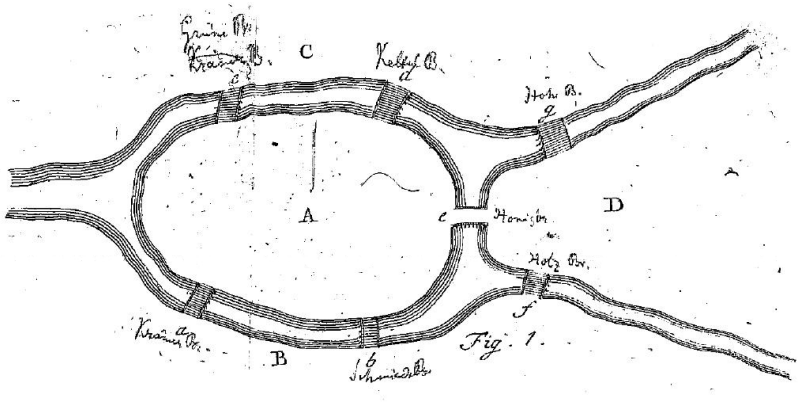


Figure 7.2: Taken from EULER's article in 1736 [51].

give any graph-like diagram for this problem.

The informations that were not necessary for the solution to the problem were to be removed from the diagrams included in the texts of subsequent mathematicians who addressed the problem. Indeed, the problem will be taken up in several publications devoted to mathematical recreations.

Let us consider them since this analysis will put in a situation to determine who first introduced a graph-like diagram in this context and how he influenced KŐNIG for this feature of the diagrams.

In 1851, Émile COUPY translated this article of EULER into French [41]. And in 1882, Édouard LUCAS translated it again into French in the chapter about the problem of bridges in vol. 1 of his series on mathematical recreations [148]. But neither COUPY nor LUCAS made significant modification to EULER's figures.

In 1892, Walter William Rouse BALL dealt with the problem within a more general context, since he mentioned it in the chapter about "unicursal problems" in his book devoted to mathematical recreations [12]. In this new context, BALL gave a new kind of diagram (Figure 7.3) for the problem. In this diagram, BALL represented the bridges with lines —some lines curved and others straight— indicated with lowercase letters, and the land areas with points indicated with uppercase letters. BALL mentioned the correspondence of the notation introduced for bridges to EULER's map, but he did not use it in his consideration of the problem, and indeed it is not necessary for solving the problem. However, it is remarkable that lines in this diagram represent geometrical elements which can correspond to a graph. As a diagram given to the problem of seven bridges of Königsberg, this is maybe the first graph-like diagram.

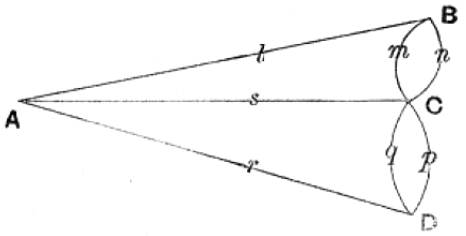


Figure 7.3: Taken from Chapter 6 of Ball's 2nd ed. in 1892 [12].

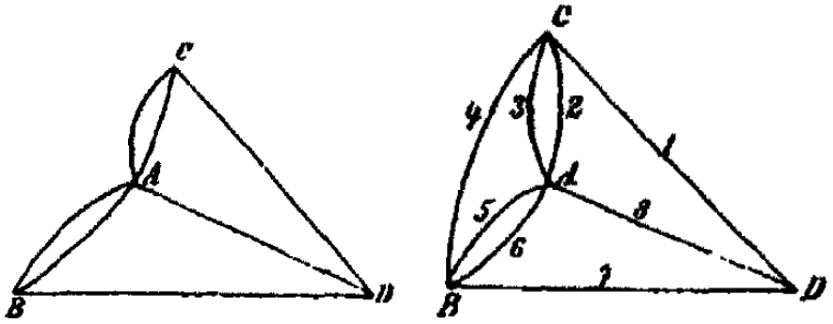


Figure 7.4: Taken from Section 17.1 of Ahrens' book of 1901 [1].

In 1901, Wilhelm Ahrens also, in his book on mathematical recreations [1], treated the problem of bridges. However, for him, the problem fell in the chapter about "Brücken und Labyrinthe (bridges and labyrinths)", that is, withing a classification of problems made on the basis of their topic. Still, he gave here diagrams quite similar to Ball's graph-like diagram (Figure 7.4).

The diagram **a** of Figure 7.4 has no mark for lines representing the bridges, but only associates letters to points representing the land areas. We find here thus only the information necessary for solving the problem. One can thus see that the diagram is drawn for the problem, and not as a representation of a general mathematical object. The diagram **b** of Figure 7.4 represents the case with 8 bridges, where one can pass through all the bridges once and only once. The diagram **b** has digits attached to lines, but their meaning is different from the lowercase letters shown in Ball's diagram: Ahrens let these numbers represent the order of passing through the bridges, therefore these numbers are necessary information for representing a solution to the problem.

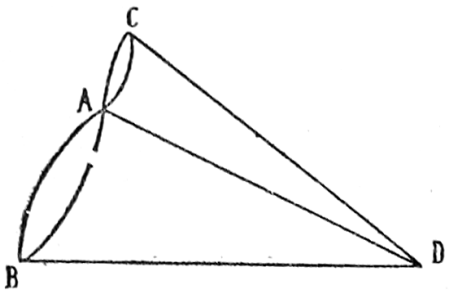


Figure 7.5: Taken from Chapter 4 of KŐNIG's book of 1905 [87].

In 1905, KŐNIG himself also, in one of the books that in his youth he devoted to mathematical recreations in 1905 —long before the publication of his treatise of 1936— treated the problem of bridges in the chapter about "A Königsbergi hidak (the bridges of Königsberg)" [87]. In relation to this problem, the book of 1905 quotes EULER [51], LUCAS [148], BALL [14], SCHUBERT [189], AHRENS [1]. However, as for the diagram given to this problem, he took it from AHRENS' book (Figure 7.5). Figure 7.5 is almost the same as AHRENS' graph-like diagram[4]. We can suppose that KŐNIG's diagram for the problem of bridges in 1905 was influenced by AHRENS' diagram in 1901.

It is interesting that this diagram of KŐNIG in 1905 was still different from his diagram in his treatise in 1936 where, instead of the points indicated with uppercase letters, small circles were used, which represented the vertices of a graph. We will see that the representation of vertices of a graph with small circles and the full notation of the elements of a graph in 1936 betrays an influence different from the problem of bridges.

7.3.2 Polygons, dominoes and the introduction of the flexible strings

In the same section entitled "Das Brücken- und Dominoproblem (The problem of bridges and the problem of dominoes)" in KŐNIG's treatise of 1936, he treated 2 other problems coming from mathematical recreations —one bearing on polygons and another one on dominoes—. He dealt with them as other examples for his theorem on Eulerian circuits. Examining the history of the treatment of these problems and of their relation to each other will

[4]A part of the line between A and B of KŐNIG's diagram is not connected, but it is only an error of printing. It should be continuous for consistency of the text.

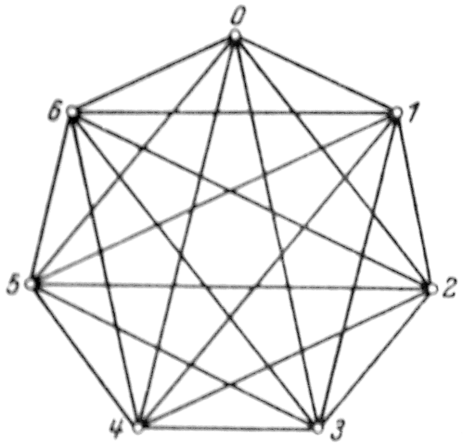


Figure 7.6: Taken from Section 2.2 of KŐNIG's book of 1936 [121].

show how another feature of the diagrams for graphs took shape within this context.

The problem on polygons can be formulated as follows: a polygon being given, can we go along every edge and every diagonal just once with only one stroke?

As for the problem on dominoes, it can be formulated as follows: one set of dominoes consists of 28 pieces; on each piece, a pair of integers from 0 to 6 is shown; we put aside here the double numbered pieces with (0,0), (1,1) etc. because they play no part in the question considered; the question is to arrange all the remaining 21 pieces so that adjacent numbers are equal to each other.

KŐNIG related the three types of problems to each other, which can be done when one can reformulate them in terms of problems related to graphs. Previously, they were not precisely discussed together in the same context. As we will see soon, KŐNIG was not the first one to have perceived their link, but he was the first mathematician to treat them explicitly as related problems, and he did this on the basis of diagrams of graph theory. Moreover, in the context in which the problems were understood as being connected with each other, another feature of the diagram took shape: the nature of the lines representing edges to be "flexible strings". Let us explain what we mean by these statements.

For the case of a heptagon, KŐNIG represented the problem of polygons by a diagram shown in Figure 7.6. We can trace every edge and every diagonal of this diagram just once with only one stroke.

Here, the diagram was used *for solving* the problem.

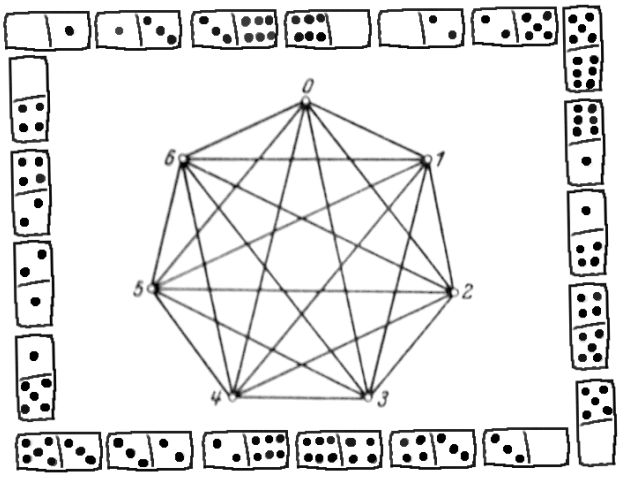


Figure 7.7: Domino pieces which I arranged so as to correspond to KŐNIG's diagram shown in Figure 7.6.

Moreover, it was also by using the same diagram of a heptagon that KŐNIG solved the problem of dominoes, thereby displaying the link between the two. He let each vertex of a heptagon correspond to a number on a domino piece, and each edge of it to one domino piece. By means of this representation, the solution to the problem of dominoes corresponded exactly to the solution to the problem of a heptagon. KŐNIG did not give any specific diagram for dominoes, but we can easily understand the relation of the solutions to these two different problems as I drew in Figure 7.7.

What is important is that the problem of polygons and the problem of dominoes were not treated as being the same in any of the previous mathematical texts in which they both appeared. However, they were progressively shaped as corresponding to the same diagram, to the same question related to this problem and to the same solution. In other words, the diagram played a key part in shaping the identity between the two problems. Let us outline the process of this integration.

In 1809, Louis POINSOT treated the problem of polygons in his lecture about "*les polygones et les polyèdres* (the polygons and the polyhedrons)". This lecture was published as a memoir in 1810 [175]. The problem we consider was described in the section 18 of this memoir.

POINSOT described the problem as follows:

> **Poinsot (1809/1810) [175], Section 18, pp. 28–29, my translation from French.**
> [...] The problem is, between points placed in the space as you like, to lead a same flexible string [*fil flexible*] which unites the points two by two in all the possible ways, so that the two ends of the string come to be rejoined at the end, and that the total length[5] should be equal to the sum of all the mutual distances.

And Poinsot explained why the solution is possible only for an odd number of points. He did so, using the concept of a "flexible string": when the points are in even number, one can still lead a string which connects the points two by two in all the possible ways, but this string should pass twice from any of the points to any other, before the two ends be rejoined and, the string being closed, the total length be equal to the twice of all the mutual distances of the proposed points.

Poinsot treated this problem with points in a space, which means that the points and the flexible string does not necessarily form a polygon. However, in the succeeding sections in his memoir, he discussed this problem in the case that the points are projected onto a plane. By projection of the points onto a plane, we can consider this problem as of polygons. In fact, in the section 23, Poinsot related this problem in the case of 4 points to a quadrilateral with 2 diagonals; and finally in the sections 24 and 25, he applied this problem to arbitrary polygons.

In his publication, Poinsot used no diagram to discuss this problem. It is nevertheless remarkable that he used the concept of a "flexible string" *to describe* the path and *to solve* the problem.

In fact, the idea of flexible strings can be traced back to an article by Alexandre-Théophile Vandermonde (1735–1796), which Poinsot mentioned at the beginning of the memoir. He wrote:

> **Poinsot (1809/1810) [175], pp. 16–17, my translation.**
> [...] *Vandermonde* gave, in the Memoirs of the Academy of science for 1771, a simpler solution[6], which was deduced from a particular notation which he invented for this sort of problems, and which he applied also to the representation of a textile or net formed with the successive knots of several strings. [...]

[5] Despite the fact that Poinsot speaks of a flexible string, he uses the length. This means that one looses the "geometry of situation".

[6] Poinsot mentioned Vandermonde [206] in the context of the problem of knight's move on a chessboard, as one of the problems concerning the "geometry of situation". He meant that Vandermonde's solution was simpler than Euler's [54].

We can therefore suppose that POINSOT got the concept of flexible strings from VANDERMONDE's article "Remarques sur des problèmes de situation (remarks on the problems of situation)" in 1771 [206].

However, VANDERMONDE did not treat polygons in his article of 1771. Therefore POINSOT took his idea of flexible strings, and used it in a completely different context. Moreover, as we saw in the memoir of POINSOT in 1810 (see p. 316), POINSOT did not only use what VANDERMONDE used, but he added measures of the length, that is, "geometry of situation" was lost in POINSOT's memoir.

On the other hand, VANDERMONDE's concern was the notation to be used by the worker who makes a *braid*, a *net*, or *knots*. These workers do not conceive these spatial situations in terms of size, but in terms related to the situation of strings with respect to each other. What the workers see is the order in which the strings are interlaced.

VANDERMONDE attempted to create a system of notation more conform to the process of the worker's mind. This notation was the basis on which he would work out a solution for the problem. For this purpose, he needed a notation which would represent only the idea formed from his work, which could be sufficient for making again a similar thing any time.

The object of VANDERMONDE's article of 1771 was only to make a hint of the possibility of this kind of notation, and its usage in questions related to textiles composed of strings.

For this purpose, VANDERMONDE described each point on a string with its spatial position. To represent the spatial position, one splits a 3-dimensional space into parallelepipeds. each parallelepiped is indicated with a triple of numbers —VANDERMONDE called it "trois nombres assemblés, ainsi c_a^b (three numbers gathered, so that c_a^b)"—, each term of which corresponds to a position of the parallelepiped on each axis of the space. By putting the triples in the order of the parallelepipeds where a string passed through, one gets a sequence of triples, which denotes a form of the string.

From such a sequence of triples, One can reproduce the textile or knots by making a string go through the parallelepipeds indicated by the triples in order.

VANDERMONDE applied this notation to a 2-dimensional space for solving the problem of knight's move on a chessboard:

> **Vandermonde (1771) [206], p. 568, my translation from French.**
> Let the *knight* go all over the squares of a chessboard without visiting twice the same square, as a result determine a certain trace of the *knight* on the chessboard; or else, supposing a pin fixed at the center of each square, determine the course of a string passed one time around each pin, according to a law of which we will search the expression.

Vandermonde let a trace of a knight correspond to a string, each square to a pin.

To a trace of a knight on a chessboard, the above mentioned notation can be applied. Because the trace is on a plane, the sequence corresponding to the trace is a sequence of pairs of numbers, each number of which consists of any of the numbers 1, 2, 3, 4, 5, 6, 7 and 8.

A knight's move on a chessboard can be denoted as $\begin{smallmatrix} b & b\pm1 \\ a & a\pm2 \end{smallmatrix}$ or $\begin{smallmatrix} b & b\pm2 \\ a & a\pm1 \end{smallmatrix}$.

To simplify the solution, we use the symmetry of the knight's trace: if we create a sequence of knight's move, and interchange the numbers of the pairs: 8 to 1, 7 to 2, 6 to 3, 5 to 4 and vice versa. Then we will get a new sequence denoting a trace of knight symmetry to the original one.

Therefore, to obtain a knight's trace visiting all the squares on a chessboard once and only once, we first need to create only a trace within the squares denoted with a sequence of pairs with numbers 1, 2, 3 and 4, and then we get another sequence by exchanging the numbers of one axis, still another sequence by exchanging the numbers of the other axis, and the other sequence by exchanging the numbers of both axis.

We thus obtain 4 sequences denoting 4 separate traces which, as a whole, visiting all the squares on a chessboard once and only once.

These 4 sequences can be connected, without breaking knight's move, by joining two sequences, or by inserting one sequence between two pairs of another sequence.

We should pay attention to the fact that, *for solving* the problem, Vandermonde did not use the concepts of "strings" and "pins", but sequences of numbers which denote strings.

In spite of not using the formulation in terms of a string and of pins *for solving* the problem, Vandermonde gave a diagram corresponding to these concepts *to represent his result* regarding "la forme de la trace du cavalier sur l'échiquier, déterminée par cette suite (the form of the trace of the knight on the chessboard, determined by this sequence)", compare Figure 7.8. We can see in this diagram that Vandermonde used small circles representing pins fixed on a chessboard. These features of his diagram evoke the form of

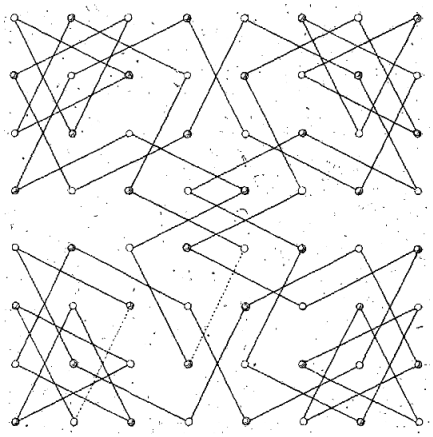


Figure 7.8: Taken from VANDERMONDE's article in 1771 [206].

diagram used in KŐNIG's treatise in 1936, as well as in the texts on graph theory of the present day.

Inspired by VANDERMONDE's article of 1771, which POINSOT mentioned, we can suppose that POINSOT had in his mind not only the concept of "flexible string" but also the concept of "pins". In fact, POINSOT lead a flexible string between the "points placed in the space as you like", just like VANDERMONDE let a string go through each "pin fixed at the center of each square" on a chessboard. In other terms, POINSOT's "points" played the role devoted to "pins" in VANDERMONDE's geometrical representation.

In terms used in KŐNIG's treatise of 1936, POINSOT applied these concepts to a problem related to Eulerian circuits, whereas VANDERMONDE applied them to a problem related to Hamiltonian circuits. In fact, in 1936, KŐNIG considered both problems as examples of more general problems: the problem of polygons was treated as a problem of Eulerian circuits, while the problem of knight's moves was treated as a problem of Hamiltonian circuits.

However, without identifying the kinds of general problems they were, POINSOT had already recognized that one could formulate these two distinct problems by means of a set of common concepts —a flexible string and pins.

7.3.3 Polygons and dominoes again: a single diagram and a single way of using it for two distinct problems

Later on, KŐNIG went further: he not only described them on the basis of the common concepts of graph theory, but he also formulated the general prob-

lems bearing on the general object of graphs for which they were particular cases.

POINSOT's lecture of 1809 mentioned no relation between the problem of polygons and the problem of dominoes. It is interesting, though the commentary of Orly TERQUEM (1782–1862) on POINSOT's works on polygons, published in 1849 [204], alluded to a relation, since at the end of this article on polygons —After the description of POINSOT's problem of polygons, which we have already examined in 7.3.2— TERQUEM mentioned the question of dominoes:

> **TERQUEM (1849) [204], p. 74, my translation from French.**
> [...] The determination of the *number* of possible solutions for an odd number of points is a problem of which the solution is desired. I proposed it to some distinguished geometers, but I got nothing. The domino game presents a question of this type: in how many ways can one place all the dominoes on only one line obeying the law of the game? One can suppose that the *double*-numbered pieces are put aside.

Clearly, TERQUEM gave neither any precise description of the relation, nor any diagram representing this idea. However, this seems to be the first mention of a relation between the problem of polygons and the problem of dominoes. One may assume that the concepts of pins and strings played a key part in the process of shaping the problems as comparable to each other.

Similarly, regarding the relation between the problem of polygons and the problem of dominoes, already in 1883 LUCAS was aware of the relation between the problem of a heptagon and the problem of dominoes, because he mentioned it in his note to the chapter about "Le jeu de dominos (the domino games)" put at the end of vol. 2 of his series of mathematical recreations [149]. But he did not give any precise description about this relation at that time.

At last in 1894, in the chapter about "La Géométrie des réseaux et le problème des dominos (the geometry of nets and the problem of dominoes)" in vol. 4 of his series of mathematical recreations [152], LUCAS declared that the idea of relating a heptagon to the problem of dominoes mentioned in vol. 2 was given by LAISANT[7].

If we now go back to the history of the relationship between the problem of bridges and that of polygons, we can note that in 1882, LUCAS treated the problem of polygons with the diagrams shown in Figure 7.9. Further, he included the problem in vol. 1 of his series on mathematical recreations, in the chapter devoted to the problem of bridges [148]. In this case, it is not

[7]Charles-Ange LAISANT (1841–1920) was a mathematician, and was a director of some reviews of mathematics.

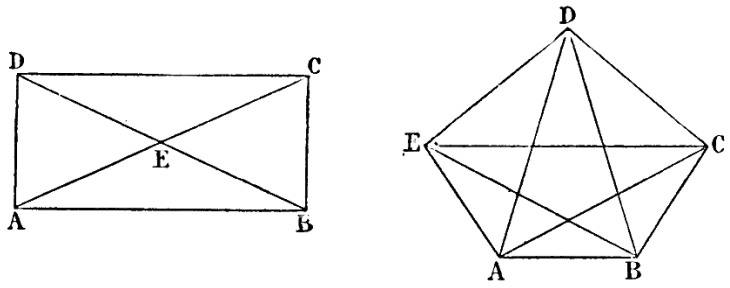


Figure 7.9: Taken from Chapter 2 of LUCAS' vol. 1 in 1882 [148].

by making use of common terms to formulate different problems that LUCAS indicated something common between them. LUCAS did so by classifying them in the same chapter of his book.

This fact indicates that LUCAS recognized that both problems could be treated in the same way, thereby contributing to their integration within a wider chapter devoted to questions having some similarity with each other. We see here a reason why it is meaningful to examine the classification of problems carried out in books of mathematical recreations: when we are interested in a process of integration that led problems to be understood as bearing on similar questions or as related to similar objects, the classification of problems conveys meanings that are important to read since they are not formulated in other ways.

Moreover, LUCAS described in this chapter relationships between a wider set of problems, since he stressed the relations between four different topics of mathematical recreations —bridges, mazes, polygons and dominoes.

However, LUCAS did not discuss these four topics explicitely on the same basis, while KŐNIG did it on the basis of diagrams of graph theory.

The links that LUCAS could establish between these topics depended on ideas that TARRY had presented at a conference in 1886 [201], which we will discuss more precisely in Section 7.4.

Let us simply stress for the moment that again we can identify a mathematical work by means of the impact that can be read in the classification of problems. Let us first consider the history of the treatment of the fourth type of problems that LUCAS linked to the first three considered above.

7.3.4 Mazes of which the junctions became important

KŐNIG's treatise in 1936 also included a chapter entitled "Das Labyrinthenproblem (The problem of mazes)". In it, he treated the following problem:

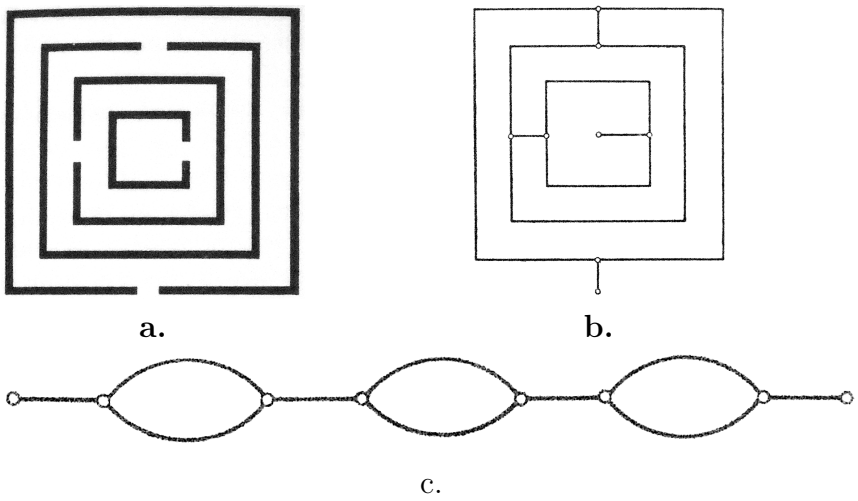


Figure 7.10: Taken from Section 3.1 of KŐNIG's book of 1936 [121].

how can I arrive at a certain place —a branching point or a loop— in walking in a maze without map? He gave 3 different diagrams for the same example: in the diagram **a** in Figure 7.10, the lines are the walls of a maze, I walk therefore the path between the lines; in the diagram **b** in Figure 7.10, the lines and the small circles are paths and junctions of a maze, which is represented by means of the edges and vertices of a graph; the diagram **c** in Figure 7.10 is a transformation of the diagram **b**. With the diagram **c**, KŐNIG showed that the absolute position of vertices and edges are ignored in graph theory, and that only the relation between the vertices and the edges are important for solving the problem.

A solution to this problem had been first published in 1882 by LUCAS in the chapter about "Labyrinthes (mazes)" in vol. 1 of his series on mathematical recreations [148]. LUCAS says about it that this solution had been given by TRÉMAUX, a telegrapher and a former student of the polytechnic school. LUCAS included a proof of the correctness of the solution. However, in KŐNIG's treatise of 1936, he noted that "Dieser Lucassche Beweis ist nicht vollständig (this LUCAS' proof is not complete)".

Let us concentrate on the diagrams used by LUCAS. In the proof to the solution, LUCAS used diagrams representing a part of a maze. One of LUCAS' diagrams is shown in Figure 7.11. In LUCAS' diagrams, the lines represent "chemins (paths)", and points to which lines are gathering represent "carrefours (junctions)".

We can see in this diagram other characteristics: the arrows indicate the directions of the walk; the marks crossing the lines indicate the paths through

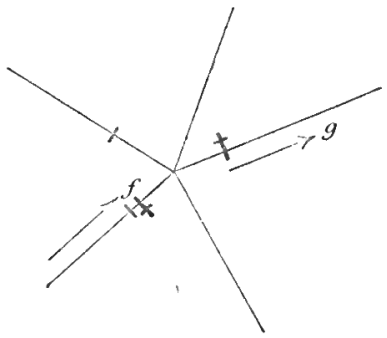


Figure 7.11: Taken from Chapter 3 of LUCAS' vol. 1 in 1882 [148].

which one already walked. In other terms, we have here a graph-like diagram with further marks.

LUCAS used lines representing paths of a maze in 1882, just like KŐNIG did in his treatise of 1936, using for this problem lines which represented edges of a graph. KŐNIG thus used a general type of diagram to represent the maze problem whereas LUCAS drew a diagram specific to the problem considered. However, both diagrams look alike. Moreover, LUCAS did not use small circles in his diagrams for the problem of mazes. This is correlated to the fact that LUCAS does not consider the vertices of the graph as relevant elements for the solution. On the other hand in KŐNIG's diagrams in 1936, small circles represented vertices, corresponding to junctions of a maze, and this representation was used also for all the other problems of bridges, polygons and dominoes. We will see in Section 7.4 how this common form of diagram became used in all the problems of graph theory, and how this detail bears witness to the historical process by means of which KŐNIG adopted these representations.

To sum up our conclusions so far, we saw in this section that KŐNIG, in his treatise of 1936, discussed different problems of mathematical recreations —bridges, polygons, dominoes and mazes— using the same general concepts attached to graph. We saw also the process of the integration of these different problems in the writings of other mathematicians before 1936. Some mathematicians used the same concepts for integrating different problems in various ways: lines for bridges and polygons; points and lines for polygons and dominoes. The concepts of pins and strings of VANDERMONDE, though did not concern directly to the integration, inspired POINSOT to work on polygons in a way different from the usual geometry in his period. These

concepts can be related to the concepts KŐNIG used later on. In earlier writings, different types of diagrams were used for different topics. Among these types of diagrams, a certain type became gradually influential. On the basis of this type of diagrams, the different problems became understood as concerning the same object.

On the basis of these facts, we are led to the hypothesis that the following two historical processes are related each other: one process which shaped the diagrams of graph, and the other process which shaped the concepts of graph.

This remark leads us to focus now in greater detail on the following question: how were different topics of mathematical recreations integrated into graph theory?

7.4 TARRY's roles

I examined texts written by KŐNIG and other mathematicians before 1936, and found that TARRY's talk in a conference in 1886 played two key roles in relation to my two questions —that is, how the representation of graphs in diagrams of graph theory took shape, and how different topics of mathematical recreations were integrated into graph theory.

I shall now establish that the first role played by TARRY's talk related to his way of using diagrams, and its second role relates to the integration of ways of treating different topics —bridges, mazes, polygons and dominoes.

Let us first say a few words on the person. Gaston TARRY (1843–1893) was a public servant working for French financial administration in Alger, and an amateur mathematician. He gave a talk in the 15th session of the *Association française pour l'avancement des sciences* in Nancy in 1886 [201]. Its title was: "Géométrie de situation: nombre de manières distinctes de parcourir en une seule course toutes les allées d'un labyrinthe rentrant, en ne passant qu'une seule fois par chacune des allées (Geometry of situation: number of distinct ways of walking in only one course along all the alleys of a recurring maze, in passing through each of the alley only once)." Here, by "labyrinthe rentrant (recurring maze)" he meant a maze for which the number of alleys leading to each junction is always an even number.

This problem is different from our problem of mazes. The subject of this problem is, in modern terms, the number of all the Eulerian circuits of the maze read as graph. That is, TARRY dealt with something related to the problem of bridges using the concepts attached to mazes, for example "walk", "alleys", "junctions" etc.

The proceedings of this session consist of 2 volumes: Volume 1 presents

the abstracts of talks prepared by the secretariat of the Association, and Volume 2 contains the articles written by the speakers and the diagrams corresponding to the articles put at the end of the volume.

The abstract of TARRY's talk was written by someone else[8], and it reads as follows:

> **Editor: secrétariat de l'Association (1887) [201], p. 81 of Part 1, my translation from French.**
> Mr. TARRY, in Alger.
> *On a problem of the geometry of situation.* — Mr. Tarry proves two theorems on the figures[9] which one can draw with only one continuous stroke, without interruption nor repetition. These two theorems allow one to find the number of solutions[10] in a very large number of cases; he applies his procedure to the problem of Reiss[11], on the game of dominoes, and find the results of Doctor Reiss again in two pages, while the much longer solution of Reiss occupies 60 pages in No. 4 of the *Annali di Matematica.*
> *Discussion.* — The president of the section[12] insists on the extreme elegance and the great simplicity of this new method.

Although dominoes are mentioned in this abstract, there is no mention of dominoes in TARRY's text itself. The details of the problem of dominoes were maybe given only to the audience of his talk.

The question we need to tackle then is to understand by means of which concepts and diagrams that became possible.

[8]It is unclear who the authors of the abstracts were, but someone of the bureau of the section which contained TARRY's talk may have been the author: Président d'honneur: M. le Géneral FROLOW, major général du génie russe (Russian general major of engineer); Président: M. Ed. LUCAS, Prof. de math. spéciale au Lycée Saint-Louis (Professor of higher mathematics at the Saint-Louis High school); Vice-Président: M. Laisant, Député de la Seine, Anc. Él. de l'Éc. Polyt. (Deputy of the Seine, Alumnus of the Polytechnic School); Secrétaire: M. HEITZ, Él. de l'Éc. centr. des Arts et Manufact. (Student of the central school of the Arts and Manufacture).

[9]TARRY talked about figures of mazes according to his article in vol. 2 of the proceedings.

[10]The problem treated by TARRY was therefore different from the problem treated by POINSOT, who described the possibility of tracing all the edges and diagonals of a polygon once and only once with one stroke.

[11]Michel REISS (1805–1869) is a mathematician from Frankfurt who worked mainly on the theory of determinants. He published an article about dominoes "Evaluation du nombre de combinaisons desquelles les 28 dés d'un jeu du domino sont susceptibles d'après la règle de ce jeu" (1871) of 58 pages [179].

[12]That is, Édouard LUCAS.

In the text of the proceedings, TARRY proved 2 theorems and a corollary. He first proved the "Théorème des impasses (Theorem of the dead-ends)". An alley the both ends of which lead to an identical junction is called an "impasse (dead-end)". In the terminology of graph theory of today, the "impasse (dead-end)" corresponds to a loop on a vertex. The theorem is as follows:

> Theorem of the dead-ends: In a recurring maze, if a dead-end is deleted, then the number of distinct courses of the maze is reduced, and then the number of distinct courses of the reduced maze multiplied by the number of the alleys leading to the junction situated on the deleted dead-end is equal to the number of distinct courses of the primitive maze. Each of the other dead-ends on the junction are also counted as two alleys.

Let N be the number of distinct courses of the reduced maze. Let $2n$ be the number of its alleys leading to the junction that was situated on the deleted dead-ends. The theorem can be written with N and $2n$: the number of distinct course of the primitive maze is equal to $N \times 2n$. The proof is as follows: consider any of the N distinct courses of the reduced maze; in this course, you will pass n times through the junction situated on the deleted dead-end; to walk in the primitive maze, in any of these n passages, you can interrupt the walk when you arrive at the junction of the dead-end, walk entirely this dead-end, which can be done in two different directions, and, after coming back to the junction, complete your walk in the maze; as a result, each of the N distinct courses of the reduced maze will supply $2n$ distinct courses of primitive maze; the N distinct courses of the reduced maze will supply therefore $N \times 2n$ courses of the primitive maze; evidently, these $N \times 2n$ courses of the primitive maze are all distinct, and there is no other way to walk the primitive maze in only one course; the theorem is thus proved.

And then TARRY gave the following corollary:

> Corollary: If $2(n+k)$ alleys lead to a junction, and $2k$ of them belong to k dead-ends, then the number of distinct courses of the given maze is equal to the product of $n(n+1)(n+2).....(n+k-1)2^k$ and the number of distinct courses of the reduced maze got after deletion of k dead-ends of the given maze.

In fact, if you add successively each of these k dead-ends to the reduced maze, then this procedure brings the numbers of distinct courses successively multiplied by $2n$, $2(n+1)$, $2(n+2)$, ... $2(n+k-1)$.

For calculating the number of distinct courses, using the theorem of the dead-ends, we can eliminate the dead-ends of the given maze, and simplify the calculation to the case of the maze with no dead-end.

TARRY proved then the following theorem:

> Theorem (to reduce junctions): A recurring maze consisting of k junctions without dead-end is given; N is one of the junctions of the recurring maze; let $2n$ be the number of alleys leading to N; the number of distinct courses of the given maze is equal to the sum of the numbers of distinct courses of $1\times3\times5\times7...(2n-1)$ recurring mazes consisting of not more than $k-1$ junctions. These $1 \times 3 \times 5 \times 7...(2n-1)$ new mazes are obtained by the following procedure:
>
> 1. group the $2n$ alleys leading to the junction N into n pairs of alleys in all the possible ways;
> 2. and then, in each of the groups,
>   (a) replace each pair of alleys with a new alley joining the 2 junctions to which the endpoints of the pair of alleys leads, or,
>   (b) in the case that the 2 alleys of the pair lead to an identical junction, replace the pair of alleys with a dead-end at this junction.

TARRY proved this theorem as follows: group the $2n$ alleys leading to the junction N into pairs of alleys in all the possible ways; we will get $(2n-1)(2n-3)...\times5\times3\times1$ different groups; to each of these groups, we relate all the courses of the given maze; in these courses, in each passage through the junction N, the alley leading to and the alley away from belong to a pair of the group considered; we can see easily that the number of courses to be found will be equal to the sum of the numbers of distinct courses corresponding to each group, in the way just shown above; consider the courses of one of these groups, and examine the n pairs of alleys that compose the group; in each of these n pairs of alleys, the 2 alleys, which are considered as ways out of the junction N, lead to two different junctions A, B or to one identical junction C; in the former case, we can replace the 2 alleys NA, NB with a new alley AB that joins the junctions A and B without changing the number of courses, because this change means replacing the track ANB or BNA with the equivalent tracks AB or BA; in the latter case, the two alleys joining the junctions N and C can be replaced with a dead-end passing through the junction C; the theorem is thus proved.

After proving these theorems, TARRY gave a procedure to calculate the number of distinct courses of any recurring maze:

1. Apply the "theorem of dead-ends" and "the theorem to reduce junctions" to a given recurring maze. According to the theorems, the number of junction of the maze will be reduced, and we will get an equation between the number of distinct courses of the reduced maze and that of the maze before reduced.

2. Repeat the process 1 so that we get finally mazes containing only 2 junctions without dead-end.

3. Count the number of the mazes containing only 2 junctions without dead-end: 2 junctions of such a maze are connected with $2n$ alleys, therefore the number of distinct courses is equal to $2(2n-1)(2n-2)...4 \times 3 \times 2 \times 1$ if each direction of the walk is count.

4. Substitute for the variable of the last equation the number of distinct courses of the last maze, that is, the maze containing only 2 junctions without dead-end. We will thus get the value of the variable of the preceding equation.

5. Repeat the substitutions, and we will get finally the number of distinct courses of the primitive maze.

To show the application of this procedure, TARRY selected such a recurring maze that the alleys form the edges and the diagonals of a heptagon. We recognize here that this application gives the number of possible solutions to the POINSOT's problem of polygons.

TARRY used a new kind of diagram as shown in Figures 7.12 and 7.13.

In these diagrams, he made use of several signs such as circles, equilateral triangles joined to a circle.

TARRY gave the explanatory notes to read his diagrams. The sentences between "[" and "]" bellow are my comments.

Circle: Junction of the maze.
[We recognize here that the elements that were to become the vertices of the graph are explicitly noted.]

Straight line connecting two circles: Alley of the maze connecting two junctions.
[A straight line represents any kind of alley and is used from the viewpoint that it is described, as was the case for the polygons above.]

Equilateral triangle having one corner on a circle: Dead-end (alley, both ends of which lead to the same junction corresponding to the circle).

Letter beside each figure: The letter indicates each figure, and at the same time in the equation, represents the number of courses corresponding to this figure. [See the detail bellow.]

In the first diagrams of Figure 7.12, we can see the equation $X = 15H$. X represents the number of distinct courses of the heptagonal maze, which we want to obtain. H represents the number of distinct courses of the hexagonal maze, which is a reduced maze of the heptagonal maze. We can get the number "15" from the number of alleys leading to one of the junctions of the heptagonal maze "6": when we group the 6 alleys leading to the junction into pairs of alleys in all the possible ways, we get $5 \times 3 \times 1$ different groups, that is, 15. Applying the theorem to reduce junctions, we get the equation $X = 15H$. When we delete the junction of the heptagonal maze X, the 6 alleys are reduced to 3 alleys, which form double lines in the figure of the hexagonal maze H.

In the next figures, the hexagonal maze H is reduced to the pentagonal mazes P_1, P_2, P_3, P_4. These 4 pentagonal mazes are drawn differently because multiple lines are differently connected depending on the ways of grouping of the 6 alleys leading to one of the junctions of the hexagonal maze into 3 pairs of alleys: for P_1, we count the groups bringing 5 double alleys; for P_2, we count the groups bringing 3 double alleys and 1 triple alley; for P_3, we count the groups bringing 4 double alleys; for P_4, we count the groups bringing 2 triple alleys. Moreover, each of P_3 and P_4 has a dead-end, we therefore apply to them the corollary with $k = 1, n = 2$. We thus multiply the count of groups by $(2 + 1 - 1)2^1 = 4$ to obtain the number of distinct courses. Then we get the equation $H = 8P_1 + 4P_2 + 4 \times 2P_3 + 4 \times P_4$.

We continue the similar procedure, and we get the following equations

successively:

$$
\begin{aligned}
X &= 15H \\
H &= 8P_1 + 4P_2 + 8P_3 + 4P_4
\end{aligned}
$$

$$
\begin{aligned}
P_1 &= 6Q_1 + 4Q_2 + 16Q_3 + 16Q_4 \\
P_2 &= 8Q_1 + 16Q_3 + 2Q_5 + 16Q_6 \\
P_3 &= 2Q_1 + Q_2 \\
P_4 &= 2Q_1 + Q_5
\end{aligned}
$$

$$
\begin{aligned}
Q_1 &= 6T_1 + 24T_2 + 48T_3 \\
Q_2 &= 8T_1 + 24T_2 + 64T_4 \\
Q_3 &= 2T_1 + 4T_2 \\
Q_4 &= 2T_2 + 4T_3 \\
Q_5 &= 48T_2 + 24T_5 \\
Q_6 &= 2T_2 + 2T_5
\end{aligned}
$$

$$
\begin{aligned}
T_1 &= 6D_1 + 144D_2 \\
T_2 &= 2D_1 + 16D_2 \\
T_3 &= 12D_2 \\
T_4 &= 2D_2 + 4D_3 \\
T_5 &= D_1
\end{aligned}
$$

$$
\begin{aligned}
D_1 &= 240 \\
D_2 &= 12 \\
D_3 &= 2
\end{aligned}
$$

From these equations, we get finally the number of distinct courses of the heptagonal maze $X = 129976320$.

TARRY did not mention the problem of dominoes in his text of the proceedings, but we can see on the diagram sheets the caption "TARRY — PROBLÈME DES DOMINOS (Tarry — problem of dominoes)". This caption supports the description of the abstract that TARRY applied his theorems to the problem of dominoes. Moreover, we recognize that the diagrams given for the calculation of the number of distinct courses of a heptagonal maze was, in his talk, used for the problem of dominoes. We recognize therefore that TARRY related the problem of polygons with the problem of dominoes.

In TARRY's diagrams, We can find a clear representation of junctions of a maze, which corresponds to vertices of a graph in modern terms, while such a representation was not found in LUCAS's diagram used to solve the problem of maze (Figure 7.11), nor in that of polygons (Figure 7.9). TARRY's representation suggests importance of junctions, which can correspond to vertices of a graph, which is still important in KŐNIG's treatise of 1936.

It is remarkable that TARRY related the problem of bridges to the concepts of mazes, though he, in this talk, did not treat the same problem of mazes as we discussed in 7.3.4. Moreover, he applied his result to the problem of dominoes using diagrams of polygons. However, he did not integrate explicitly all the four problems of bridges, polygons, dominoes and mazes, while KŐNIG did it in his treatise of 1936.

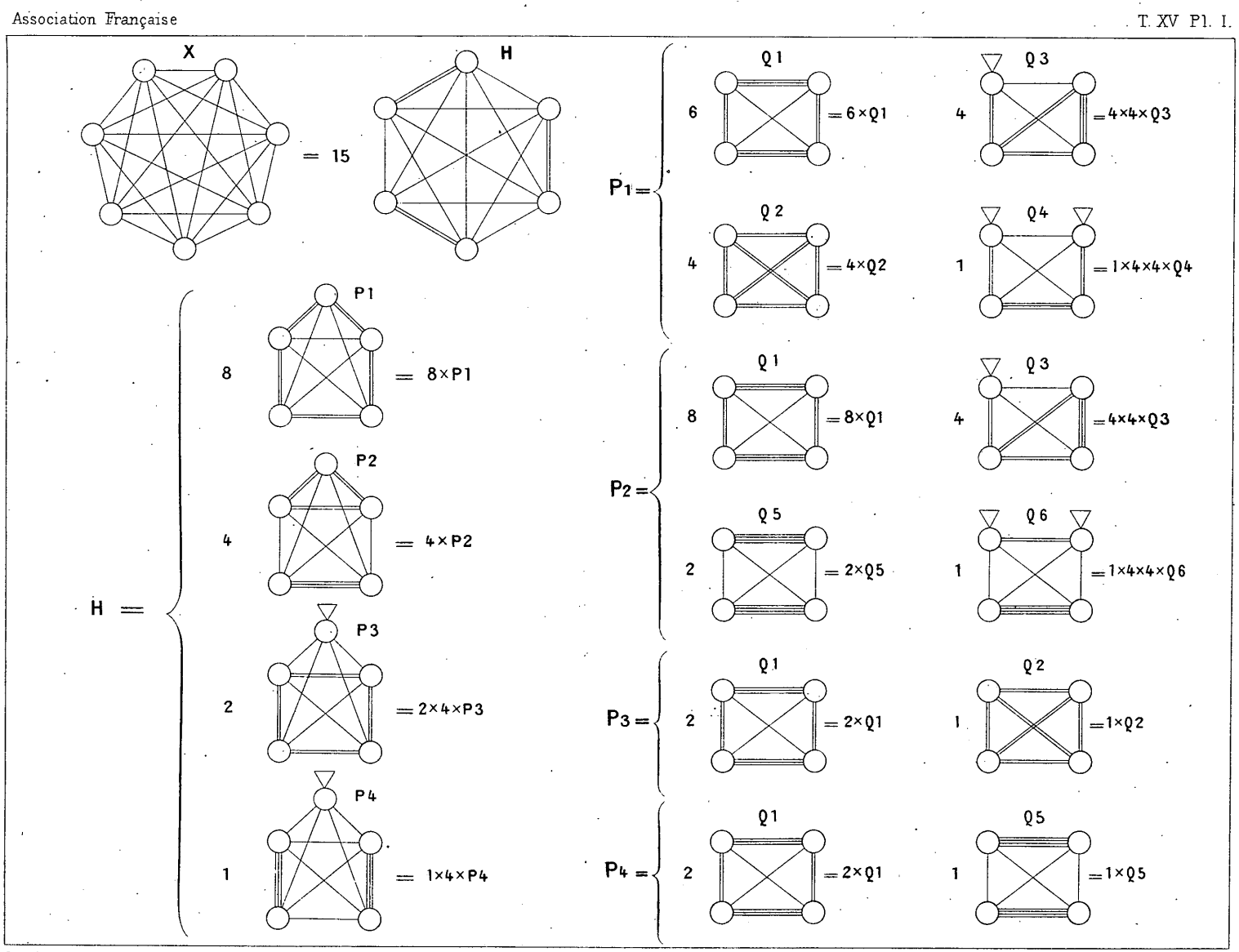


Figure 7.12: TARRY's diagrams [201].

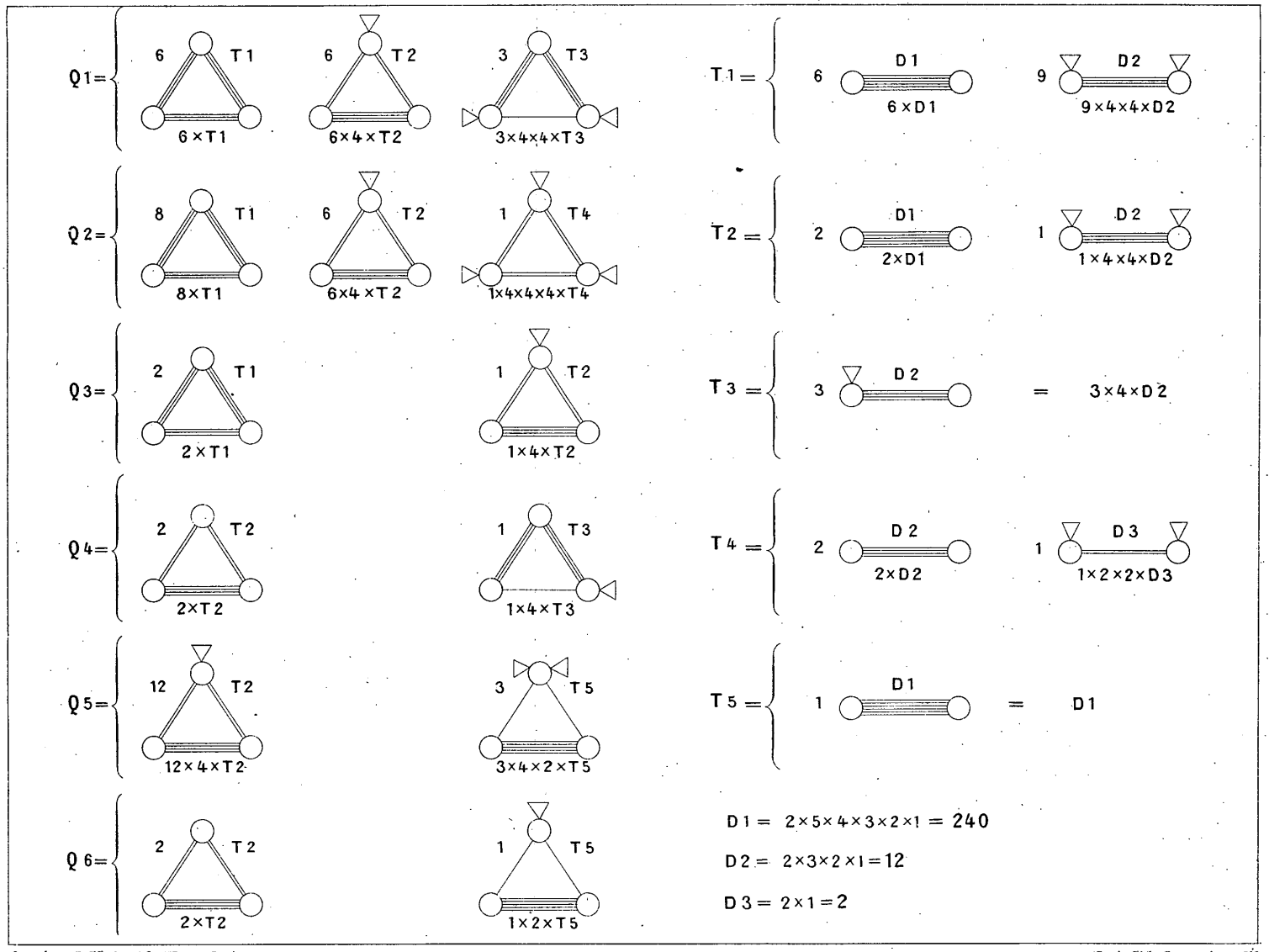


Figure 7.13: TARRY's diagrams (continued) [201].

Chapter 8

Conclusion

Through my examination on documents related to mathematical recreations and those related to graph theory, I considered that mathematical recreations made an important part of the basis on which graph theory was built.

In this chapter, I will give the reasons for this consideration.

8.1 KŐNIG's books on mathematical recreations and that of graph theory

In Chapter 6, we discussed the works of KŐNIG Dénes on mathematical recreations and his works on graph theory.

To consider the relation between his books on mathematical recreations in 1902/1905 and that on graph theory in 1936, we viewed his works before 1936, examined KŐNIG's books in 1902/1905 and 1936, and compared them with each other.

BEKE Manó, one of KŐNIG's teacher in his adolescence, was leading the activities of reforming mathematical educations in Hungary, applying the concepts of Felix KLEIN of Göttingen. KŐNIG published two books on mathematical recreations in 1902 and 1905, which we can regard as one of the activities of reforming mathematical educations led by BEKE.

Between these two publications on mathematical recreations, KŐNIG studied in Göttingen in 1904–1905, where he was interested in the problem of four-colour map lectured by Hermann MINKOWSKI.

In 1905, KŐNIG published an article on this problem, included this problem into the book of 1905. In 1911, he published two articles that were later cited in the book of 1936 for the problem of four-colour map.

Based on these facts, we can suppose that MINKOWSKI's lecture in Göttingen played an important role to bring KŐNIG to the works which were

later related to graph theory.

This hypothesis is supported by our examination on the book of 1902 and that of 1905: The book of 1902 has no relation to the book of 1936; on the other hand, the book of 1905 has remarkable relation to the book of 1936.

We examined also the difference between the books of 1905 and 1936. We focused on one problem treated in both books of 1905 and 1936, and compared the ways of treatment. Although a common problem was treated in both books of 1905 and 1936, the ways of treatment were different from each other.

Because the books on mathematical recreation of 1902 and 1905 were published as one of the activities of reforming mathematical educations of BEKE, KŐNIG wrote these books so as to draw the readers' interest in the topics and the related mathematics of higher level. It was not necessary to describe precisely mathematical theorems nor proofs.

The ways of treatment of a problem in the book of 1905 represented these aspects.

On the other hand, the book of 1936 was a treatise of mathematics for professional use. Therefore, in contrast with the book of 1905, the book of 1936 should consist of theorems and proofs.

8.2 Diagrams used in texts related to graph theory

In Chapter 7, we discussed how the representation of graphs by means of the diagrams of graph theory took shape, and In Chapter 6, we discussed the works of KŐNIG Dénes on mathematical recreations and his works on graph theory.

We examined up some problems KŐNIG treated in his treatise of 1936, and analyzed the diagrams attached to them in the texts before 1936. Earlier, these diagrams did not always have the form of the present day, and the forms were not unique. The features of diagrams in early days were different from each other, depending on the problem in which the diagram was used, while the features of diagrams in the present day are unique in different topics.

This raises two questions: how did these diagrams take shape? Which part did diagrams play in shaping the problems as same and integrating them into a single theoretical body?

In these two respects, we can conclude on the basis of the previous discussion that TARRY's talk in 1886 [201] played two important and related roles: one role in the way of using the diagrams, and the other role in the inte-

gration of the treatments of different topics —bridges, mazes, polygons and dominoes— as bearing on the same kind of diagrammatic object. This thus suggests that the two processes are intimately connected with each other. Let us consider the two roles in turn.

With respect to diagrams, TARRY's key role was that he used a unique type of diagrams in different topics, and that *for solving* the problem of dominoes in his talk, thereby connecting this problem to other problems for which similar diagrams had been introduced and which had been reformulated as problems related to this kind of diagrams.

In fact, we identified another graphical representation and another conception of the object under study used at beginning of 19th century to connect a smaller set of distinct problems now understood as both bearing on graph: a form of diagrams with lines and small circles which already appeared in an article of VANDERMONDE in 1771 [206]. However note that its status was different at that time: VANDERMONDE used the diagram not for stating and then solving a problem, but *for representing his solution* to a problem. The remarkable contribution of VANDERMONDE (1771) to graph theory is that he used the concepts of "épingle (pin)" and "fil (string)" for a problem of knight's move on a chessboard, thereby introducing in particular the line independently from its shape and distinguishing only some points to represent a situation. Moreover, he gave a diagram representing his result using these concepts. In KŐNIG's treatise of 1936, the problem of knight's move on a chessboard was considered, in the context of graph theory, to be a problem of Hamiltonian circuit. But KŐNIG's treatise of 1936 is not the first text in which the same concepts were used as a basis to define problems as referring to questions related to circuits, the circuits being represented by means of the same elements. POINSOT, in his lecture of 1809 [175], applied the concepts of VANDERMONDE to the problem of polygons. In other terms, POINSOT recognized that the same concepts can be used to formulate two apparently different problems, one on a Hamiltonian circuit and another one, which in KŐNIG's treatise in 1936 was mentioned as a problem of Eulerian circuits. It is also remarkable that POINSOT used these concepts *for solving* the problem, and not only representing the solution, though no diagram representing the concepts can be found. It was KŐNIG who, for the first time, explicitly related these two problems in the same chapter, and considered them using the same basis of graph theory.

The second role of TARRY can be perceived through the following facts:

TARRY related the problem of bridges to the concepts of mazes. Moreover, he applied his result to the problem of dominoes using diagrams of polygons.

Before TARRY's talk of 1886, no one related these four kinds of topics.

However, TARRY did not integrate explicitly all the four problems of

bridges, polygons, dominoes and mazes, while KŐNIG did it in his treatise of 1936.

It is possible that TARRY's ideas of 1886 influenced somehow —directly or not— the formation of the representation of graphs in KŐNIG's treatise in 1936, as well as the concepts with which graphs were approached.

8.3 Future works

In examining KŐNIG's works, I understood little by little that systems and methods of mathematical education was changing in Hungary in the years around 1900. I suppose that KŐNIG received benefit of new education of mathematics. In fact, KŐNIG won the first prize of mathematical student competition in 1899. Such a competition might be a part of the movement of reforming mathematical education. His first books on mathematical recreations seem to be written in order to be used for education in high-schools. Among the references in these books, many articles from *Középiskolai Mathematikai Lapok (High-school mathematical reviews)* were found. He also became a teacher of mathematics, and wrote textbooks for his students. Based on this background, it will be interesting to consider the relation between mathematical education and mathematical recreations in examining KŐNIG's textbooks, articles in *Középiskolai Mathematikai Lapok*, publications on mathematical recreations in Hungary, etc.

Examining those documents may give informations about mathematics and educations in Hungary. However, the movement of changing education was not happened in Hungary. In fact, such a movement in Hungary was brought from Göttingen. It will also be interesting to consider the difference of movements related to mathematical education between countries, including acceptance of mathematical recreations into mathematical education.

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