Affine T-varieties: additive group actions and singularities

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$\mathbb{T}$-variétés affines : actions du groupe additif et singularités

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Au vu des rapports de Ivan ARZHANTSEV *et* Hubert FLENNER

To my wife, Ximena

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>
> *Extracto del último discurso*
> SALVADOR ALLENDE

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Contents

Introduction

Let $\mathbf{k}$ be an algebraically closed field of characteristic 0. The algebraic torus $\mathbb{T}_n = \mathbb{T}$ of dimension n is the algebraic variety $(\mathbf{k}^*)^n$ with its natural structure of algebraic group. A $\mathbb{T}$-variety is an algebraic variety endowed with an effective action of the torus $\mathbb{T}$.

This thesis is devoted to the study of two aspects of normal affine $\mathbb{T}$-varieties: the additive group actions and the characterization of singularities.

The introduction is divided in three parts. First, we introduce a combinatorial description of normal affine $\mathbb{T}$-varieties, this corresponds to Chapter 1. We also give a historical overview on the subject. In the second part we present the results concerning the additive group actions on affine $\mathbb{T}$-varieties, these results are developed in Chapters 2 and 3. Finally, we expose the results of Chapter 4 about the classification of singularities on $\mathbb{T}$-varieties. In this introduction all varieties are assumed to be normal.

Normal $\mathbb{T}$-varieties

A character (resp. one-parameter subgroup) of the torus is a morphism $\chi : \mathbb{T} \to \mathbf{k}^*$ (resp. $\lambda : \mathbf{k}^* \to \mathbb{T}$) that is at the same time a group homomorphism. The set of all characters (resp. one-parameter subgroups) form a lattice M (resp. N) of rank n and there is a natural duality given by (see Section 1.3.1)

$$\langle \chi, \lambda \rangle = \ell, \quad \text{if} \quad \chi \circ \lambda(t) = t^{\ell}\,.$$

It is the standard convention to consider M and N as abstract lattices. In this case, the torus $\mathbb{T} = \operatorname{Spec} \mathbf{k}[M]$ and for every $m \in M$ (resp. $p \in N$) we denote by χ^m (resp. λ_p) the corresponding character (resp. one-parameter subgroup) of the torus. We also let $N_{\mathbb{Q}}$ and $M_{\mathbb{Q}}$ be the rational vector spaces $N \otimes \mathbb{Q}$ and $M \otimes \mathbb{Q}$, respectively. The natural duality between M and N extends in an obvious way to a duality between the vector spaces $M_{\mathbb{Q}}$ and $N_{\mathbb{Q}}$.

It is well known that a $\mathbb{T}$-action on an affine variety $X = \operatorname{Spec} A$ gives rise to an M-grading on A, where M is the character lattice of $\mathbb{T}$, see Theorem 1.3.7. Moreover, letting $K^{\mathbb{T}} \subseteq \operatorname{Frac} A$ be the field of $\mathbb{T}$-invariant rational functions on X, without loss of generality, we may assume that

$$A = \bigoplus_{m \in \sigma^\vee \cap M} A_m \chi^m, \quad \text{where} \quad A_m \subseteq K^{\mathbb{T}}\,,$$

and $\sigma^\vee$ is the weight cone of the M-grading i.e., the cone spanned in $M_{\mathbb{Q}}$ by all the lattice vectors m such that $A_m \neq 0$, see Section 1.3.3. In the sequel, for any cone $\sigma^\vee \subseteq M_{\mathbb{Q}}$ we denote the set $\sigma^\vee \cap M$ by $\sigma^\vee_M$.

For an algebraic torus $\mathbb{T}$ acting on an algebraic variety X, the complexity of this action is defined as the codimension of the general orbit. If the $\mathbb{T}$-action is effective, then the complexity is $\dim X - \dim \mathbb{T}$. Moreover, the complexity of the $\mathbb{T}$-action equals the transcendence degree of $K^{\mathbb{T}}$ over $\mathbf{k}$.

In 2006, Altmann and Hausen [AH06] gave a combinatorial description of normal affine $\mathbb{T}$-varieties that generalizes two well established theories: the theory of toric varieties, that corresponds to $\mathbb{T}$-varieties of complexity zero; and the theory of quasihomogeneous varieties that corresponds to $\mathbb{T}_1 = \mathbf{k}^*$-varieties. It also generalizes a combinatorial description in the particular case of complexity one given by Mumford [KKMS73, Chapter 4].

Let us now introduce the above mentioned descriptions of toric varieties, of quasihomogeneous varieties, of $\mathbb{T}$-varieties of complexity one, and finally of $\mathbb{T}$-varieties of arbitrary complexity, in more detail.

Toric varieties. The theory of toric varieties first appeared in 1970 in the influential work of Demazure on the Cremona group [Dem70]. It was later developed independently by Kempf, Knudsen, Mumford and Saint-Donat [KKMS73], Miyake and Oda [MO75], and Satake [Sat73]. See also the surveys by Danilov [Dan78] and Teissier [Tei81].

This theory represents a bridge between convex and algebraic geometry, which in particular allows to treat a large class of algebraic varieties in a combinatorial way. In the present, there are several textbooks covering the basic theory [Oda88; Ful93; CLS]. This is still an active domain of research.

Let $\mathbb{T}$ be an algebraic torus, M be its character lattice, and N be its one-parameter subgroup lattice. A toric variety is a normal $\mathbb{T}$-variety of complexity zero.

A fan Σ in $N_{\mathbb{Q}}$ is a collection of pointed convex polyhedral cones in $N_{\mathbb{Q}}$ such that for all $\sigma \in \Sigma$, each face of σ also belongs to Σ; and for all $\sigma, \sigma' \in \Sigma$, the intersection $\sigma \cap \sigma'$ is a face of each of them. There is a natural way to associate to a fan Σ a toric variety X_Σ, and every toric variety arises in this way, see Section 1.4.

The case of affine toric varieties is particularly simple. These varieties correspond to fans Σ consisting of only one maximal cone σ and all of its faces. In this case, we denote X_Σ by X_σ. Furthermore, the algebra of regular functions of an affine toric variety X_σ is the semigroup algebra

$$\mathbf{k}[X_\sigma] = \mathbf{k}[\sigma_M^\vee] := \bigoplus_{m \in \sigma_M^\vee} \mathbf{k} \cdot \chi^m \,.$$

In this setting, the variety X_σ is completely determined by the pointed cone $\sigma \subseteq N_{\mathbb{Q}}$ or, equivalently, by the weight cone $\sigma^\vee \subseteq M_{\mathbb{Q}}$.

Quasihomogeneous affine varieties. A quasihomogeneous variety is a variety endowed with an effective action of the torus $\mathbb{T}_1 = \operatorname{Spec} \mathbf{k}[\mathbb{Z}] = \mathbf{k}^*$ [1]. A quasihomogeneous affine variety is called unmixed if the corresponding $\mathbb{Z}$-grading is positive i.e., if the weight cone $\sigma^\vee$ is $\mathbb{Q}_{\geq 0}$, and hyperbolic if the weight cone $\sigma^\vee$ is $\mathbb{Q}$.

There is a well known description of a quasihomogeneous affine variety X by means of rational divisors ($\mathbb{Q}$-divisors) on a variety Y of dimension $\dim X - 1$.

[1] This definition differs from the concept of quasihomogeneity in theory of algebraic group actions.

This description first appeared for unmixed $\mathbf{k}^*$-actions. For surfaces it was encountered in the works of Dolgachev [Dol75] and Pinkham [Pin77; Pin78], and latter on was generalized by Demazure [Dem88][2] to arbitrary dimension.

For hyperbolic $\mathbf{k}^*$-surfaces this description was developed by Flenner and Zaidenberg in [FZ03]. Finally, in arbitrary dimension the description follows easily from the results in [FZ03] and [Dem88]. It is also a corollary of [AH06].

A variety Y is called semiprojective if it is projective over an affine variety. Let Y be a normal semiprojective variety and let D be an ample $\mathbb{Q}$-divisor on Y. Letting $\mathcal{O}_Y(D)$ be the sheaf $\mathcal{O}_Y(\lfloor D\rfloor)$, where $\lfloor D\rfloor$ is the integral part of D, we define the algebra

$$A[Y,D]=\bigoplus_{m\in\mathbb{Z}_{\geq 0}} A_m\chi^m, \quad \text{where} \quad A_m=H^0(Y,\mathcal{O}_Y(mD))\,.$$

In this setting, $X=\operatorname{Spec}A[Y,D]$ is a normal affine variety of dimension $\dim Y+1$ endowed with an unmixed $\mathbf{k}^*$-action. Conversely, every unmixed affine $\mathbf{k}^*$-variety arises in this way [Dem88, Theorem 3.5]. The variety Y in this description is, in general, not unique. However Y can be made unique by imposing the condition $Y\simeq\operatorname{Proj}A[Y,D]$.

Let as before Y be a normal semiprojective variety and let D_+, D_- be two ample $\mathbb{Q}$-divisors such that $D_++D_-\leq 0$. We define the algebra

$$A[Y,D_+,D_-]=\bigoplus_{m\in\mathbb{Z}} A_m\chi^m, \quad \text{where} \quad A_m=\begin{cases} H^0(Y,\mathcal{O}_Y(mD_+)) & \text{if } m\geq 0,\\ H^0(Y,\mathcal{O}_Y(-mD_-)) & \text{otherwise.}\end{cases}$$

The condition $D_++D_-\leq 0$ ensures that $A[Y,D_+,D_-]$ is indeed an algebra, see Section 1.5. In this setting $X=\operatorname{Spec}A[Y,D_+,D_-]$ is a normal affine variety of dimension $\dim Y+1$ endowed with a hyperbolic $\mathbf{k}^*$-action. Conversely, every affine hyperbolic $\mathbf{k}^*$-variety arises in this way [FZ03].

$\mathbb{T}$-varieties of complexity one. In Chapter 4 of [KKMS73], Mumford gave a combinatorial description of $\mathbb{T}$-varieties of complexity one admitting a rational quotient that is also a regular morphism, see Definition 1.3.3.

More generally, Timashev [Tim97] gave a combinatorial description of normal varieties endowed with an effective action of a reductive group of complexity one. When specialized to the case of $G=\mathbb{T}$ [Tim08], this description coincides with the one given previously by Mumford. The description due to Timashev is also available in the case where the $\mathbb{T}$-action does not admit a rational quotient which is a morphism. We recall briefly the description of affine $\mathbb{T}$-varieties due to Timashev.

Let C be a smooth projective curve, M and N be mutually dual lattices of rank n, $\mathcal{H}^+=N_\mathbb{Q}\times\mathbb{Q}_{\geq 0}$, and $\mathcal{H}=N_\mathbb{Q}\times\{0\}\subseteq\mathcal{H}^+$. A hypercone Θ on C is a set of pointed polyhedral cones $\mathcal{C}_z\subseteq\mathcal{H}^+$, for all $z\in C$ such that the following conditions hold.

- (i) The cone $\mathcal{C}_z\cap\mathcal{H}=:\sigma$ does not depend on $z\in C$.
- (ii) $\mathcal{C}_z=\sigma\times\mathbb{Q}_{\geq 0}$ for all but finitely many z.
- (iii) Let Δ_z denote the projection onto $N_\mathbb{Q}$ of the polyhedron $\mathcal{C}_z\cap(\mathcal{H}+(\bar{0},1))$ and $\Delta=\sum_{z\in C}\Delta_z$, then the polyhedron Δ is a proper subset of σ.

[2] This paper was officially published in 1988, but it first appeared in the Demazure-Giraud-Teissier seminary in 1979.

(*iv*) Let h_z (resp. h_Θ) be the support function[3] of Δ_z (resp. Δ), and $\Theta_m = \sum_{z\in C} h_z(m)\cdot z$, for all $m\in\sigma_M^\vee$. If $\Delta\neq\emptyset$ then for every $m\in\sigma_M^\vee$ such that $h_\Theta(m)=0$ a multiple of the divisor Θ_m is principal.

We let $C^\circ=\{z\in C\mid \mathcal{C}_z\neq\sigma\}$. For every hypercone Θ on a smooth projective curve C we define the algebra

$$A[C,\Theta]=\bigoplus_{m\in\sigma_M^\vee} A_m\chi^m,\quad \text{where}\quad A_m=H^0(C^\circ,\mathcal{O}_C(\Theta_m))\,.$$

In this setting $X=\operatorname{Spec} A[C,\Theta]$ is a normal affine variety of dimension $\operatorname{rank} M+1$ endowed with an effective $\mathbb{T}$-action. Conversely, every affine $\mathbb{T}$-variety of complexity one arises in this way [Tim08, Theorem 2].

$\mathbb{T}$-varieties of arbitrary complexity. We pass now to the announced combinatorial description of normal affine $\mathbb{T}$-varieties of arbitrary complexity due to Altmann and Hausen [AH06].

Let M and N be mutually dual lattices of rank n, and $\mathbb{T}=\operatorname{Spec}\mathbf{k}[M]$. We let as before $N_\mathbb{Q}=N\otimes\mathbb{Q}$ and $M_\mathbb{Q}=M\otimes\mathbb{Q}$. Let σ be a pointed polyhedral cone in $N_\mathbb{Q}$. A polyhedron Δ is called a σ-polyhedron if can be decomposed as the Minkowski sum of a bounded polyhedron and σ.

A σ-polyhedral divisor on a semiprojective variety Y is a formal sum

$$\mathfrak{D}=\sum_Z \Delta_Z\cdot Z\,,$$

where Z runs over all prime divisors on Y, Δ_Z is a σ-polyhedron, and $\Delta_Z=\sigma$ for all but finitely many prime divisors Z. For $m\in\sigma^\vee$ we can evaluate $\mathfrak{D}$ by letting $\mathfrak{D}(m)$ be the $\mathbb{Q}$-divisor

$$\mathfrak{D}(m)=\sum_Z h_Z(m)\cdot Z\,,$$

where h_Z is the support function of Δ_Z. A σ-polyhedral divisor $\mathfrak{D}$ is called proper if $\mathfrak{D}(m)$ is semiample and $\mathbb{Q}$-Cartier for all $m\in\sigma^\vee$, and $\mathfrak{D}(m)$ is big[4] for all $m\in\operatorname{rel.int}(\sigma^\vee)$.

To any proper σ-polyhedral divisor $\mathfrak{D}$ on a semiprojective variety Y we associate the algebra

$$A[Y,\mathfrak{D}]=\bigoplus_{m\in\sigma_M^\vee} A_m\chi^m,\quad \text{where}\quad A_m=H^0(Y,\mathcal{O}_Y(\mathfrak{D}(m))\subseteq\mathbf{k}(Y)\,.$$

THEOREM (Altmann and Hausen). *Let $\mathfrak{D}$ be a proper σ-polyhedral divisor on a semiprojective variety Y. Then $X[Y,\mathfrak{D}]:=\operatorname{Spec}A[Y,\mathfrak{D}]$ is a normal affine $\mathbb{T}$-variety of dimension $\operatorname{rank}M+\dim Y$. Conversely, every normal affine $\mathbb{T}$-variety is isomorphic to $X[Y,\mathfrak{D}]$ for some semiprojective variety Y and some proper σ-polyhedral divisor $\mathfrak{D}$ on Y.*

In [AHS08], divisorial fans were introduced to extend this combinatorial description to normal not necessarily affine $\mathbb{T}$-varieties. This provides a generalization of the passage from cones to fans in toric geometry.

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[3]See Section 1.1.2 for a definition.

[4]Recall that a $\mathbb{Q}$-Cartier divisor D on Y is called big if there exists a divisor D_0 in the linear system $|rD|$, for some $r>1$, such that $Y\setminus\operatorname{Supp}D_0$ is affine.

In the following, we show how this description restricts to the particular cases of toric varieties, quasihomogeneous varieties, and $\mathbb{T}$-varieties of complexity one.

Affine toric varieties. Affine toric varieties correspond to the case where Y is reduced to a point. Since the only divisor on Y is $\emptyset$, for any proper σ-polyhedral divisor $\mathfrak{D}$ the evaluations $\mathfrak{D}(m)=\emptyset$, for all $m\in\sigma_M^\vee$, and so $H^0(Y,\mathcal{O}_Y(\mathfrak{D}(m)))=\mathbf{k}$. This yields

$$A[Y,\mathfrak{D}]=\bigoplus_{m\in\sigma_M^\vee}\mathbf{k}\chi^m,\quad\text{and so}\quad X[Y,\mathfrak{D}]=X_\sigma\,.$$

Affine quasihomogeneous varieties. Let X be a quasihomogeneous variety. In the case where X is unmixed, we let $X\simeq\operatorname{Spec}A[Y,D]$, for some ample $\mathbb{Q}$-Cartier divisor D on a semiprojective variety Y. Letting $M=\mathbb{Z}$, $\sigma=\mathbb{Q}_{\geq 0}$, and

$$\mathfrak{D}=[1,\infty)\cdot D\quad\text{yields}\quad A[Y,D]=A[Y,\mathfrak{D}]\,.$$

In the case where X is hyperbolic, we let $X\simeq\operatorname{Spec}A[Y,D_+,D_-]$, for some ample $\mathbb{Q}$-Cartier divisors D_+,D_- on a semiprojective variety Y such that $D_++D_-\leq 0$. Letting $M=\mathbb{Z}$, $\sigma=\{0\}$, and

$$\mathfrak{D}=\{1\}\cdot D_++[0,1]\cdot(-D_+-D_-)\quad\text{yields}\quad A[Y,D_+,D_-]=A[Y,\mathfrak{D}]\,.$$

Affine $\mathbb{T}$-varieties of complexity one. Let X be an affine $\mathbb{T}$-variety of complexity one. We can assume that $X=\operatorname{Spec}A[C,\Theta]$ where Θ is a hypercone over a smooth projective curve C. With the notation as in the definition of a hypercone, (i) shows that all the polyhedra Δ_z are σ-polyhedra. By (ii)

$$\mathfrak{D}=\sum_{z\in C^\circ}\Delta_z\cdot z$$

is a σ-polyhedral divisor on C°. Finally, (iii) and (iv) ensure that $\mathfrak{D}$ is proper. With these definitions, it is clear that $A[C,\Theta]=A[C^\circ,\mathfrak{D}]$, see also [Vol07].

Additive group actions

The additive group $\mathbb{G}_a$ over an algebraically closed field $\mathbf{k}$ of characteristic zero is defined as the affine variety $\mathbb{A}^1\simeq\mathbf{k}$ endowed with the natural structure of algebraic group induced by the addition on $\mathbf{k}$.

Let $X=\operatorname{Spec}A$ be an affine variety. A derivation $\partial:A\to A$ is called locally nilpotent (LND for short) if for every $a\in A$ there exists $k\in\mathbb{Z}_{\geq 0}$ such that $\partial^k(a)=0$. A $\mathbb{G}_a$-action on X gives rise to an LND on A and every $\mathbb{G}_a$-action on X arises in this way, see Section 2.1.

The study of $\mathbb{G}_a$-actions goes back to Hilbert who calculated the rings of invariants of certain linear $\mathbb{G}_a$-actions on $\mathbb{A}^n$ up to integral closure. In 1959, Nagata gave a counterexample to the famous Hilbert's fourteenth problem, which uses a linear action of $\mathbb{G}_a^{13}$ on $\mathbb{A}^{32}$ [Nag59].

In 1968, Rentschler classified all the locally nilpotent derivations of the polynomial ring in two variables over a field of characteristic zero, and showed how this gives the equivalent classification of all $\mathbb{G}_a$-actions on $\mathbb{A}^2$ [Ren68].

The modern interest in $\mathbb{G}_a$-actions and LNDs comes from the introduction by Kaliman and Makar-Limanov [ML96; KML97] of the ring absolute constants, now called the Makar-Limanov invariant (ML invariant for short). The ML invariant of an affine variety $X=\operatorname{Spec}A$ is defined as the intersection of the kernels of all the LNDs on A.

Let us consider the Koras-Russell affine cubic threefold $X = \operatorname{Spec} A$, where

$$A = \mathbf{k}[x, y, z, t]/(x + x^2y + z^2 + t^3)\,.$$

The ML invariant was first introduced to distinguish X from $\mathbb{A}^3$. In fact $\mathrm{ML}(X) = \mathbf{k}[x]$ while $\mathrm{ML}(\mathbb{A}^3) = \mathbf{k}$. This was the last step in the proof of the fact that all the $\mathbf{k}^*$-actions on $\mathbb{A}^3$ are linearizable [KKMLR97].

We describe now the results in Chapters 2 and 3, where we investigate $\mathbb{G}_a$-actions on affine $\mathbb{T}$-varieties, or equivalently LNDs on normal affine M-graded domains. These results are contained in the paper [Lie10] and the preprint [Lie09a].

Let as before M and N be mutually dual lattices of rank n, $N_{\mathbb{Q}} = N \otimes \mathbb{Q}$, $M_{\mathbb{Q}} = M \otimes \mathbb{Q}$, and $\mathbb{T} = \operatorname{Spec} \mathbf{k}[M]$. We also let σ be a pointed polyhedral cone in $N_{\mathbb{Q}}$. We consider an integrally closed affine effectively M-graded domain

$$A = \bigoplus_{m\in\sigma_M^\vee} A_m\chi^m \subseteq K^{\mathbb{T}}[M], \quad \text{where} \quad A_m \subseteq K^{\mathbb{T}}\,,$$

and we let $X = \operatorname{Spec} A$ be the corresponding affine $\mathbb{T}$-variety.

A derivation $\partial : A \to A$ is called homogeneous if it sends homogeneous elements into homogeneous elements i.e., if there exists a lattice vector $e = \deg \partial \in M$ such that

$$\partial(A_m\chi^m) \subseteq A_{m+e}\chi^{m+e}, \quad \text{for all} \quad m \in \sigma_M^\vee\,.$$

A $\mathbb{G}_a$-action on X is called compatible with the $\mathbb{T}$-action if the corresponding LND is homogeneous, geometrically this means that the $\mathbb{G}_a$-action is normalized by the torus $\mathbb{T}$.

In Lemma 2.1.7 we show that we can associate to any LND on A a homogeneous one. A homogeneous LND ∂ on A can be extended to a derivation on $K^{\mathbb{T}}[M]$ by the Leibniz rule. We also denote this extension by ∂.

We say a homogeneous LND ∂ on A, or equivalently, a compatible $\mathbb{G}_a$-action on X, is of fiber type if $\partial(K^{\mathbb{T}}) = 0$ and of horizontal type otherwise. In geometric terms, a compatible $\mathbb{G}_a$-action is of fiber type if the general orbits of the $\mathbb{G}_a$-action are contained in the orbit closures of the $\mathbb{T}$-action.

Let $\mathrm{LND}(A)$ be the set of all LNDs on A. The Makar-Limanov invariant of A, or equivalently of X, is defined as

$$\mathrm{ML}(A) = \bigcap_{\partial\in\mathrm{LND}(A)} \ker\partial\,.$$

Similarly, letting $\mathrm{LND_h}(A)$ be the set of all homogeneous LNDs on A, we define the homogeneous Makar-Limanov invariant of A as

$$\mathrm{ML_h}(A) = \bigcap_{\partial\in\mathrm{LND_h}(A)} \ker\partial\,.$$

We say that the ML invariant of A is trivial if $\mathrm{ML}(A) = \mathbf{k}$. Clearly, the triviality of the homogeneous ML invariant implies that of the usual one.

$\mathbb{G}_a$-actions on toric varieties. Letting $\sigma \subseteq N_{\mathbb{Q}}$ be a pointed polyhedral cone, we let $A = \mathbf{k}[\sigma_M^\vee]$, and $X_\sigma = \operatorname{Spec} A$. Fix a ray ρ of σ with primitive vector ρ_0 and dual facet $\tau \subseteq \sigma^\vee$. We define S_ρ as the set

$$S_\rho = \left\{m \in M \mid \langle\rho_0, m\rangle = -1, \text{ and } \langle\rho', m\rangle \geq 0\, \forall\rho' \in \sigma(1) \setminus \rho\right\}\,,$$

where $\sigma(1)$ is the set of all rays of σ. The main result of Section 2.2 is the following classification, which is valid over an arbitrary field of characteristic zero, not necessarily algebraically closed.

THEOREM A. *To any pair (ρ, e), where ρ is a ray of σ and e is a lattice vector in S_ρ, we can associate in a natural way a homogeneous LND $\partial_{\rho,e}$ on $A = \mathbf{k}[\sigma_M^\vee]$ with kernel $\ker \partial_{\rho,e} = \mathbf{k}[\tau_M]$ and $\deg \partial_{\rho,e} = e$.*

Conversely, if $\partial \neq 0$ is a homogeneous LND on A, then $\partial = \lambda\partial_{\rho,e}$ for some ray $\rho \subseteq \sigma$, some lattice vector $e \in S_\rho$, and some $\lambda \in \mathbf{k}^*$.

In [Dem70] an analog result is proven for smooth not necessarily affine toric varieties. In *loc. cit.* the elements in the set $\mathcal{R} = -\bigcup_{\rho\in\sigma(1)} S_\rho$ are called the roots of σ.

As usual, we denote a ray and its primitive vector by the same letter ρ. Let ρ be a ray of σ and $e \in S_\rho$, then the LND $\partial_{\rho,e}$ is given by

$$\partial_{\rho,e}(\chi^m) = \langle m, \rho\rangle \chi^{m+e} .$$

As a first corollary of Theorem A we show that the equivalence classes of homogeneous LNDs on the toric variety X_σ are in one to one correspondence with the rays of σ. Concerning the ML invariant of toric varieties we obtain the following result, see Proposition 3.2.1.

THEOREM B. *Let $\theta \subseteq M_\mathbb{Q}$ be the maximal subspace contained in $\sigma^\vee$. Then*

$$\mathrm{ML}(A) = \mathrm{ML}_\mathrm{h}(A) = \mathbf{k}[\theta_M] .$$

In particular $\mathrm{ML}(A) = \mathbf{k}$ if and only if σ is of full dimension i.e., if and only if X does not have a non-trivial torus factor in X.

$\mathbb{G}_\mathrm{a}$-actions of fiber type on $\mathbb{T}$-varieties of arbitrary complexity. We fix a smooth semiprojective variety Y and a proper σ-polyhedral divisor

$$\mathfrak{D} = \sum_Z \Delta_Z \cdot Z \quad \text{on} \quad Y .$$

Letting $\mathbf{k}(Y)$ be the field of rational functions on Y, we consider the affine variety $X = \operatorname{Spec} A$, where

$$A = A[Y,\mathfrak{D}] = \bigoplus_{m\in\sigma_M^\vee} A_m\chi^m, \quad \text{with} \quad A_m = H^0\left(Y, \mathcal{O}(\mathfrak{D}(m))\right) \subseteq \mathbf{k}(Y) .$$

We also fix a homogeneous LND ∂ of fiber type on A, and we let $\bar{A} = \mathbf{k}(Y)[\sigma_M^\vee]$ be the affine semigroup algebra of $\sigma_M^\vee$ over the field $\mathbf{k}(Y)$. The LND ∂ can be extended to a homogeneous locally nilpotent $\mathbf{k}(Y)$-derivation $\bar{\partial}$ on $\bar{A}$. The derivations on $\bar{A}$ were classified in Theorem A.

In Section 2.4 we apply this remark to classify the LNDs of fiber type on $\mathbb{T}$-varieties of arbitrary complexity. This is done first in the particular case of complexity one in Section 2.3.1.

For any $e \in S_\rho$, we let $\Phi_e^* = H^0(Y, \mathcal{O}_Y(-D_e)) \setminus \{0\}$, where D_e is the $\mathbb{Q}$-divisor on Y defined by

$$D_e := \sum_Z \max_{m\in\sigma_M^\vee\setminus\tau_M} (h_Z(m) - h_Z(m+e)) \cdot Z .$$

For a ray $\rho \subseteq \sigma$ we denote by τ the corresponding dual facet of $\sigma^\vee$. The main result concerning the classification of LNDs of fiber type on $A[Y,\mathfrak{D}]$ is the following theorem.

THEOREM C. *To any triple (ρ, e, φ), where ρ is a ray of σ, $e \in S_\rho$, and $\varphi \in \Phi_e^*$, the derivation $\partial_{\rho,e,\varphi} := \varphi\partial_{\rho,e}$ is a homogeneous LND of fiber type on $A = A[Y,\mathfrak{D}]$ of degree e with kernel*

$$\ker \partial_{\rho,e,\varphi} = \bigoplus_{m\in\tau_M} A_m\chi^m\,.$$

Conversely, every non-trivial homogeneous LND ∂ of fiber type on A is of the form $\partial = \partial_{\rho,e,\varphi}$ for some ray $\rho \subseteq \sigma$, some lattice vector $e \in S_\rho$, and some function $\varphi \in \Phi_e^*$.

The kernel of the LND $\partial_{\rho,e,\varphi}$ depends only on the ray ρ. So the equivalence classes of LNDs of fiber type on A are in one to one correspondence with the rays ρ of σ satisfying that there exists $e \in S_\rho$ such that Φ_e^* is non-empty. The following theorem gives a condition for the latter to happen.

THEOREM D. *Let $\rho \subseteq \sigma$ be the ray dual to a facet $\tau \subseteq \sigma^\vee$. Then there exists $e \in S_\rho$ such that Φ_e^* is non-empty if and only if the divisor $\mathfrak{D}(m)$ is big for all lattice vector $m \in \operatorname{rel.int}(\tau)$.*

In particular, the LNDs of fiber type on $A = A[Y,\mathfrak{D}]$ are in one to one correspondence with the rays ρ of σ such that $\mathfrak{D}(m)$ is big for all lattice vector $m \in \operatorname{rel.int}(\tau)$.

Form Theorem D we obtain the following corollary that gives a condition for the triviality of the ML invariant of A.

COROLLARY E. *Let $A = A[Y,\mathfrak{D}]$. If Y is projective, $\operatorname{rank} M \geq 2$, σ is full dimensional, and $\mathfrak{D}(m)$ is big for all non-zero lattice vector $m \in \sigma^\vee$, then $ML(A) = \mathbf{k}$.*

$\mathbb{G}_a$-actions on $\mathbb{T}$-varieties of complexity one. The case of compatible $\mathbb{G}_a$-actions on affine $\mathbf{k}^*$-surfaces was first studied by Flenner and Zaidenberg in [FZ05a]. This paper was our motivation in the next part of the thesis. In Section 2.3.3 we show how that our results restrict to those in [FZ05a] in the case of affine $\mathbf{k}^*$-surfaces.

In the case of affine $\mathbb{T}$-varieties of complexity one we give in Section 2.3 a classification of all homogeneous LNDs. Let σ be a pointed cone in $N_\mathbb{Q}$. We fix a smooth curve C and a proper σ-polyhedral divisor $\mathfrak{D}$ on C

$$\mathfrak{D} = \sum_{z\in C} \Delta_z \cdot z\,.$$

Letting $\mathbf{k}(C)$ be the field of rational functions of C, we consider the affine variety $X = \operatorname{Spec} A$, where

$$A = A[C,\mathfrak{D}] = \bigoplus_{m\in\sigma_M^\vee} A_m\chi^m, \quad \text{with} \quad A_m = H^0\left(C, \mathcal{O}_C(\mathfrak{D}(m))\right) \subseteq \mathbf{k}(C)\,.$$

We define the degree of $\mathfrak{D}$ as the polyhedron

$$\deg \mathfrak{D} = \sum_{z\in C} \Delta_z\,.$$

The classification of the homogeneous LNDs of fiber type on A is given in Theorem C above. Furthermore, in the case of complexity one we can replace the condition "$\mathfrak{D}(m)$ is big for all lattice vector $m \in \operatorname{rel.int}(\tau)$" in Theorem D by the simpler one "ρ is disjoint from $\deg \mathfrak{D}$".

The classification of LNDs of horizontal type is more involved. First, we prove that the existence of a homogeneous LND of horizontal type on A implies that the base curve C is isomorphic to $\mathbb{A}^1$ or to $\mathbb{P}^1$, see Lemma 2.3.14. In the following we assume that $C = \mathbb{A}^1$ or $C = \mathbb{P}^1$.

The main classification result for homogeneous LNDs of horizontal type on $A = A[C, \mathfrak{D}]$ is Theorem 2.3.26. The statement of this theorem requires too much notation to be included in this introduction. Here, we only state its main corollary in Theorem F.

Letting $h_z : \sigma^\vee \to \mathbb{Q}$ be the support function of Δ_z, we define the normal quasifan $\Lambda(\mathfrak{D})$ of $\mathfrak{D}$ as the coarsest refinement of the quasifan of $\sigma^\vee \subseteq M_\mathbb{Q}$ such that for every $z \in C$, the function h_z is linear in each cone $\eta \in \Lambda(\mathfrak{D})$. We say that a maximal cone $\eta \in \Lambda(\mathfrak{D})$ is good if there exists $z_0 \in C$ such that $h_z|_\eta$ is integral, for all $z \in C \setminus \{z_0\}$.

With these definitions we can state the following classification of the equivalence classes of homogeneous LNDs of horizontal type on $A = A[C, \mathfrak{D}]$, see Corollaries 2.3.27 and 2.3.28.

THEOREM F. *Let $\mathfrak{D}$ be a proper σ-polyhedral divisor on C, and let $A = A[C, \mathfrak{D}]$. The equivalence classes of homogeneous LNDs of horizontal type on A are in one to one correspondence with:*

(*i*) *The good maximal cones η in the normal quasifan $\Lambda(\mathfrak{D})$, in case where $C = \mathbb{A}^1$.*
(*ii*) *The pairs (z_∞, η), where $z_\infty \in C$ and η is a good maximal cone in the normal quasifan of $\Lambda(\mathfrak{D}|_{C_0})$, with $C_0 := C \setminus \{z_\infty\}$, in case where $C = \mathbb{P}^1$.*

In Theorem 3.3.4 we compute the homogeneous ML invariant of A. Again, it requires too much notation to be included in this introduction.

The ML invariant and rationality. As stated before the ML invariant is an important tool for affine geometry. In particular, it allows to distinguish certain varieties from the affine space. Nevertheless, this invariant is far from being optimal. Indeed, the ML invariant of the affine space $\mathbb{A}^n$ is trivial i.e., $\mathrm{ML}(\mathbb{A}^n) = \mathbf{k}$. However, it can also be trivial for a non-rational affine variety.

Recall that a variety is rational if its field of functions is a purely transcendental extension of the base field $\mathbf{k}$. In Section 3.3.1 we apply Corollary E to give, to our best knowledge, the first example of a non-rational affine variety having a trivial ML invariant. This example is generalized in Section 3.4.

We give here a geometrical instance of these examples. Let Y be a projective variety, H be an ample Cartier divisor on Y, and $n \geq 2$. We let $\widetilde{X}$ be the total space of the vector bundle associated to the locally free sheaf $\bigoplus_{i=1}^n \mathcal{O}_Y(H)$, and X be the contraction of the zero section of $\widetilde{X}$ to a point. In Example 3.4.3 we show that $\mathrm{ML}(X) = \mathbf{k}$, while X has the birational type of $Y \times \mathbb{P}^n$.

In Theorem 3.4.1 we apply this example to give the following birational characterization of normal affine varieties with trivial ML invariant.

THEOREM G. *Let X be an affine variety over the field $\mathbf{k}$. If $\mathrm{ML}(X) = \mathbf{k}$ then $X \simeq_{\mathrm{bir}} Y \times \mathbb{P}^2$ for some variety Y. Conversely, in any birational class $Y \times \mathbb{P}^2$ there is an affine variety X with $\mathrm{ML}(X) = \mathbf{k}$.*

To avoid such pathological examples, we introduce in Section 3.5 a field version of the ML invariant, we call it the FML invariant. This invariant is defined as

$$\mathrm{FML}(A) = \bigcap_{\partial\in\mathrm{LND}(A)} \mathrm{Frac}(\ker\partial)\,.$$

For any finitely generated normal domain A there is an inclusion $\mathrm{ML}(A) \subseteq \mathrm{FML}(A)$. Since $\mathrm{FML}(\mathbb{A}^n) = \mathbf{k}$ the FML invariant is stronger than the classical one in the sense that it can distinguish more varieties form the affine space that the classical one.

For an affine variety X, we conjecture that $\mathrm{FML}(X) = \mathbf{k}$ implies that X is rational. In Theorem 3.5.6 we confirm this conjecture for dimensions up to 3.

Finitely generated rings of invariants. The generalized Hilbert's fourteenth problem can be formulated as follows. Let $\mathbf{k} \subseteq L \subseteq K$ be field extensions, and let $A \subseteq K$ be a finitely generated $\mathbf{k}$-algebra. Is it true that the $\mathbf{k}$-algebra $A \cap L$ is also finitely generated?

In the case where $K = \mathrm{Frac}\ A$ and $\mathrm{Spec}\ A$ has a $\mathbb{G}_a$-action, we consider $L = K^{\mathbb{G}_a}$ so that $A\cap L$ is the subring of invariants of the $\mathbb{G}_a$-action. So $A\cap L = \ker\partial$, where ∂ is the associated LND on A. In this case the answer to the question above is known to be negative even for the polynomial ring in $n \geq 5$ variables [DF99]. On the other hand, in Section 2.5 we show the following result.

THEOREM H. *Let A be a normal finitely generated effectively M-graded algebra, where M is a lattice of finite rank, and let ∂ be a homogeneous LND on A. If the complexity of the corresponding $\mathbb{T}$-action on* Spec A *is zero or one, or the LND ∂ is of fiber type, then* $\ker\partial$ *is finitely generated.*

This theorem follows from our classification results. The hard case, where the LND is of horizontal type, follows as well from a result due to Kuroda [Kur03].

Furthermore, in Corollary 2.5.5, we apply Kuroda's result to prove that $\ker\partial$ is also finitely generated in the case where $X = \mathrm{Spec}\,A$ is rational and the $\mathbb{T}$-action is of complexity two.

Normal singularities with torus actions

Let X be a normal variety endowed with an effective torus action. By a classic theorem of Sumihiro (see Theorem 1.3.4) every point $x \in X$ has an affine open neighborhood invariant by the torus action. Hence, local problems can be reduced to the affine case.

We give now the geometrical counterpart of the combinatorial description of normal affine $\mathbb{T}$-varieties due to Altmann and Hausen. Let Y be a normal semiprojective variety and $\mathfrak{D}$ be a proper σ-polyhedral divisor on Y. We define the M-graded $\mathcal{O}_Y$-algebra

$$\widetilde{A} = \widetilde{A}[Y,\mathfrak{D}] := \bigoplus_{m\in\sigma_M^\vee} \mathcal{O}_Y(\mathfrak{D}(m))\,.$$

So that taking the global sections of $\widetilde{A}[Y,\mathfrak{D}]$ yields the M-graded algebra $A[Y,\mathfrak{D}]$ defined before

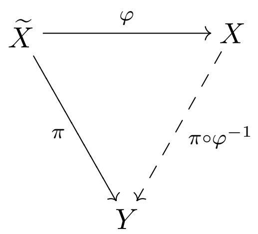
$$A = A[Y,\mathfrak{D}] = H^0(Y,\widetilde{A}[Y,\mathfrak{D}])\,.$$

We also let

$$\widetilde{X} = \widetilde{X}[Y,\mathfrak{D}] := \mathbf{Spec}_Y\,\widetilde{A}[Y,\mathfrak{D}]\,.$$

Here, $\mathbf{Spec}_Y$ stands for the relative spectrum of a $\mathcal{O}_Y$-algebra. See [Har77, Ch. II Ex. 5.17] for a definition.

The $\mathbf{Spec}_Y$ construction provides a $\mathbb{T}$-invariant affine morphism $\pi : \widetilde{X} \to Y$ which is thus a rational quotient for the $\mathbb{T}$-action on $\widetilde{X}$. The global sections functor provides a $\mathbb{T}$-equivariant birational morphism $\varphi : \widetilde{X} \to X = X[Y, \mathfrak{D}]$ and so $\pi \circ \varphi^{-1}$ is again a rational quotient for the $\mathbb{T}$-action on X. We can summarize all this considerations in the following commutative diagram, where all the arrows pointing down are rational quotients.



With these definitions, we have the following theorem.

Theorem (Altmann and Hausen).

(*i*) $\widetilde{X}$ *is a variety.*
(*ii*) *The affine morphism* π *is a good quotient for the* $\mathbb{T}$*-action on* $\widetilde{X}$*.*
(*iii*) *The birational morphism* φ *is proper.*

We describe now the results in Chapter 4 where we investigate singularities on affine $\mathbb{T}$-varieties. These results are contained in the preprint [Lie09b] and are currently being generalized in a joint work with Süß [LS10].

The combinatorial description $(Y, \mathfrak{D})$ of a $\mathbb{T}$-variety X is not unique. Indeed, if we consider a blow up $\psi : \widetilde{Y} \to Y$ of Y at a closed point and the proper σ-polyhedral divisor $\psi^*\mathfrak{D}$, then $X[Y, \mathfrak{D}] \simeq X[\widetilde{Y}, \psi^*\mathfrak{D}]$, see Lemma 4.2.1 for a more general statement.

We define the support of a σ-polyhedral divisor $\mathfrak{D}$ on a semiprojective variety Y as

$$\operatorname{Supp}\mathfrak{D} = \sum_{\Delta_Z \neq \sigma} Z\,.$$

We say that $\mathfrak{D}$ is an SNC σ-polyhedral divisor if Y is smooth, $\mathfrak{D}$ is proper, and $\operatorname{Supp}\mathfrak{D}$ is a simple normal crossing (SNC) divisor. In Corollary 4.2.5 we show that every affine $\mathbb{T}$-variety admits a combinatorial description $(Y, \mathfrak{D})$ such that $\mathfrak{D}$ is an SNC σ-polyhedral divisor.

Recall that a normal variety X is called toroidal if for every $x \in X$ the formal neighborhood of x is isomorphic to the formal neighborhood of a point in a toric variety. With this definitions, in Section 4.2 we prove the following result.

Theorem I. *Let* $\mathfrak{D}$ *be a proper* σ*-polyhedral divisor on a normal semiprojective variety* Y*. If* $\mathfrak{D}$ *is SNC then* $\widetilde{X}[Y, \mathfrak{D}]$ *is a toroidal variety.*

This theorem shows in particular that the proper birational morphism

$$\varphi : \widetilde{X} = \widetilde{X}[Y, \mathfrak{D}] \to X = X[Y, \mathfrak{D}]$$

is a partial desingularization of X having only toric singularities. Moreover, a desingularization of $\widetilde{X}$ can be obtained by toric methods and so also a desingularization of X.

Since toric singularities are well understood (see Section 1.6), in the following we will use the morphism $\varphi : \widetilde{X} \to X$ to study the singularities of X.

Let X be a normal variety and let $\psi : W \to X$ be a (full) desingularization of X. Usually, the classification of singularities involves the higher direct images of the structure sheaf $R^i\psi_*\mathcal{O}_W$. These sheaves are defined via

$$U \longrightarrow H^0(U, R^i\psi_*\mathcal{O}_W) := H^i\left(\psi^{-1}(U), \mathcal{O}_W|_{\psi^{-1}(U)}\right) .$$

The sheaves $R^i\psi_*\mathcal{O}_W$ are independent of the particular choice of a desingularization of X. Furthermore, X is normal if and only if $R^0\psi_*\mathcal{O}_W := \psi_*\mathcal{O}_W = \mathcal{O}_X$. In the following theorem we compute the sheaves $R^i\psi_*\mathcal{O}_W$ of a normal affine $\mathbb{T}$-variety $X[Y, \mathfrak{D}]$ in terms of the combinatorial data, see Theorem 4.3.3.

Theorem J. *Let $X = X[Y, \mathfrak{D}]$, where $\mathfrak{D}$ is an SNC σ-polyhedral divisor on Y. If $\psi : W \to X$ is a desingularization, then for every $i \geq 0$, the higher direct image $R^i\psi_*\mathcal{O}_W$ is the sheaf associated to*

$$\bigoplus_{u\in\sigma^\vee_M} H^i(Y, \mathcal{O}(\mathfrak{D}(m)))$$

A normal variety X is said to have rational singularities if $R^i\psi_*\mathcal{O}_W = 0$ for all $i \geq 1$, see e.g., [Art66; KKMS73; Elk78]. In the following theorem we apply Theorem J to give a criterion for X to have rational singularities.

Theorem K. *Let $X = X[Y, \mathfrak{D}]$, where $\mathfrak{D}$ is an SNC σ-polyhedral divisor on Y. Then X has rational singularities if and only if for every $m \in \sigma^\vee_M$*

$$H^i(Y, \mathcal{O}_Y(\mathfrak{D}(m))) = 0, \quad \forall i \in \{1, \ldots, \dim Y\} .$$

The “only if” part of Theorem K for $m = 0$ gives as a corollary that if $X[Y, \mathfrak{D}]$ has rational singularities, then the structure sheaf $\mathcal{O}_Y$ of Y is acyclic. Furthermore, in the case of complexity one, we have a more explicit result.

Corollary L. *If Y is a smooth curve, then X has rational singularities if and only if*

(*i*) *Y is affine, or*
(*ii*) *$Y = \mathbb{P}^1$ and $\deg\lfloor\mathfrak{D}(m)\rfloor \geq -1$ for all $m \in \sigma^\vee_M$.*

Rational singularities are Cohen-Macaulay. Recall that a local ring is Cohen-Macaulay if its Krull dimension equals to the depth. A variety X is called Cohen-Macaulay if all the local rings $\mathcal{O}_{X,x}$ are Cohen-Macaulay, see Section 1.6.

Let as before $\psi : W \to X$ be a desingularization of X. By a well known theorem due to Kempf (see Lemma 4.3.6), a variety X has rational singularities if and only if X is Cohen-Macaulay and the induced map $\psi_*\omega_W \hookrightarrow \omega_X$ is an isomorphism. We apply Kempf's Theorem to prove the following result.

Theorem M. *Let $X = X[Y, \mathfrak{D}]$, where $\mathfrak{D}$ is an proper σ-polyhedral divisor on Y. Assume that the following hold.*

(*i*) *For every facet $\tau \subseteq \sigma^\vee$, the divisor $\mathfrak{D}(m)$ is big for all $m \in \operatorname{rel.int}(\tau)$.*
(*ii*) *For every prime divisor Z on Y and every vertex p on Δ_Z, the divisor $\mathfrak{D}(m)|_Z$ is big for all $m \in \operatorname{rel.int}(\operatorname{cone}((\Delta_Z - p)^\vee))$.*

Then X is Cohen-Macaulay if and only if X has rational singularities.

In the case of complexity one, condition (*ii*) in Theorem M is always satisfied. From Theorem M we obtain the following corollary characterizing isolated Cohen-Macaulay singularities in complexity one.

Corollary N. *Let $X = X[Y, \mathfrak{D}]$, where Y is a smooth curve. Assume that X has only isolated singularities, then the following hold.*

(*i*) *If $\operatorname{rank} M = 1$, then X is Cohen-Macaulay.*
(*ii*) *If $\operatorname{rank} M \geq 2$, then X is Cohen-Macaulay if and only if X has rational singularities.*

A normal surface singularity (X, x) is called elliptic if $R^1\psi_*\mathcal{O}_W = \mathbf{k}$, see e.g., [Lau77; Wat80; Yau80]. An elliptic singularity is called minimal if it is Gorenstein i.e., is Cohen-Macaulay and the canonical sheaf ω_X is invertible.

In Proposition 4.4.2 we give a criterion for a surface with a $\mathbf{k}^*$-action to be Gorenstein. Its formulation requires too much notation to be included in this introduction. In Theorem 4.4.3 we characterize (minimal) elliptic singularities in term of the combinatorial data, here we only state the elliptic singularities part of the theorem.

Let $\operatorname{rank} M = 1$ and Let $X = X[Y, \mathfrak{D}]$, where Y is a smooth curve and $\mathfrak{D}$ is an SNC σ-polyhedral divisor on Y. If Y is affine, then X has rational singularities, so in the following we assume that Y is projective i.e., that the action is elliptic. In this setting we may assume that $\sigma = \mathbb{Q}_{\geq 0}$ and so $\mathfrak{D}$ is completely determined by $\mathfrak{D}_1 := \mathfrak{D}(1)$. Furthermore, there is a unique attractive fixed point $\bar{0}$.

Theorem O. *Let $X = X[Y, \mathfrak{D}]$ be a normal affine surface with an effective elliptic $\mathbf{k}^*$-action, and let $\bar{0} \in X$ be the unique fixed point. Then $(X, \bar{0})$ is an elliptic singularity if and only if one of the following two conditions holds:*

(*i*) $Y = \mathbb{P}^1$, $\deg\lfloor m\mathfrak{D}_1\rfloor \geq -2$ *and* $\deg\lfloor m\mathfrak{D}_1\rfloor = -2$ *for one and only one* $m \in \mathbb{Z}_{>0}$.
(*ii*) *Y is an elliptic curve, and for every $m \in \mathbb{Z}_{>0}$, the divisor $\lfloor m\mathfrak{D}_1\rfloor$ is not principal and* $\deg\lfloor m\mathfrak{D}_1\rfloor \geq 0$.

Introduction (version française)

Soit $\mathbf{k}$ un corps algébriquement clos de caractéristique nulle. Le tore algébrique $\mathbb{T}_n = \mathbb{T}$ de dimension n est la variété algébrique $(\mathbf{k}^*)^n$ avec sa structure naturelle de groupe algébrique. Une $\mathbb{T}$-variété est une variété algébrique munie d'une action effective du tore $\mathbb{T}$.

Cette thèse est consacrée à l'étude de deux aspects des $\mathbb{T}$-variétés affines normales : les actions du groupe additif et la caractérisation des singularités.

Cette introduction est divisée en trois parties. D'abord, on introduit une description combinatoire des $\mathbb{T}$-variétés affines normales, ceci correspond au Chapitre 1. On fournit aussi un aperçu historique du sujet. Dans la seconde partie, on présente les résultats concernant les actions du groupe additif dans des $\mathbb{T}$-variétés affines, ces résultats sont contenus dans les Chapitres 2 et 3. Enfin, on expose les résultats du Chapitre 4 sur la classification des singularités de $\mathbb{T}$-variétés. Dans cette introduction, toutes les variétés sont supposées être normales.

$\mathbb{T}$-variétés normales

Un caractère (resp. sous-groupe à un paramètre) du tore est un morphisme $\chi : \mathbb{T} \to \mathbf{k}^*$ (resp. $\lambda : \mathbf{k}^* \to \mathbb{T}$) qui est en même temps un homomorphisme de groupes. L'ensemble de tous les caractères (resp. sous-groupes à un paramètre) forme un réseau M (resp. N) de rang n et il y a une dualité naturelle donnée par (voir Section 1.3.1)

$$\langle \chi, \lambda \rangle = \ell, \quad \text{si} \quad \chi \circ \lambda(t) = t^\ell \,.$$

La convention de notation standard veut que l'on considère M et N comme des réseaux abstrait. Dans ce cas, le tore $\mathbb{T} = \operatorname{Spec} \mathbf{k}[M]$ et pour tout $m \in M$ (resp. $p \in N$) on note χ^m (resp. λ_p) le caractère (resp. sous-groupe à un paramètre) du tore correspondant. On note $N_\mathbb{Q}$ et $M_\mathbb{Q}$ les espaces vectoriels rationnels $N \otimes \mathbb{Q}$ et $M \otimes \mathbb{Q}$, respectivement. La dualité naturelle entre M et N s'étend d'une façon unique en une dualité entre les espaces vectoriels $M_\mathbb{Q}$ et $N_\mathbb{Q}$.

Il est bien connu qu'une action de $\mathbb{T}$ dans une variété affine $X = \operatorname{Spec} A$ engendre une graduation de A indexée par M (M-graduation), où M est le réseau des caractères de $\mathbb{T}$, voir le Théorème 1.3.7. De plus, si l'on note $K^\mathbb{T}$ le corps des fonctions rationnelles sur X invariantes par $\mathbb{T}$, sans perte de généralité, on peut supposer que

$$A = \bigoplus_{m \in \sigma^\vee \cap M} A_m \chi^m, \quad \text{où} \quad A_m \subseteq K^\mathbb{T} \,,$$

et $\sigma^\vee$ est le cône des poids de la M-graduation c'est-à-dire, le cône dans $M_\mathbb{Q}$ engendré par tous les éléments du réseau m tels que $A_m \neq 0$, voir Section 1.3.3. Dans la suite, pour tout cône $\sigma^\vee \subseteq M_\mathbb{Q}$ on note $\sigma^\vee_M$ l'ensemble $\sigma^\vee \cap M$.

Pour un tore algébrique $\mathbb{T}$ agissant sur une variété algébrique X, la complexité de cette action est définie comme la codimension d'une orbite générale. Si l'action de $\mathbb{T}$ est effective, la complexité est $\dim X - \dim \mathbb{T}$. De plus, la complexité de l'action est aussi donné par le degré de transcendance de $K^{\mathbb{T}}$ sur $\mathbf{k}$.

En 2006, Altmann et Hausen [AH06] ont donné une description combinatoire des $\mathbb{T}$-variétés affines normales qui généralise deux théories bien établies : la théorie des variétés toriques, qui correspondent aux $\mathbb{T}$-variétés de complexité zéro ; et la théorie des variétés quasi-homogènes, qui correspondent aux $\mathbb{T}_1 = \mathbf{k}^*$-variétés. Cette description généralise également une description combinatoire dans le cas particulier de complexité un donnée par Mumford [KKMS73, Chapter 4].

Introduisons maintenant les descriptions, mentionnées ci-dessus, des variétés toriques, des variétés quasi-homogènes, des $\mathbb{T}$-variétés de complexité un, et enfin plus en détail des $\mathbb{T}$-variétés de complexité arbitraire.

Variétés toriques. La théorie des variétés toriques est apparu en 1970 dans l'influent article de Demazure sur le groupe de Cremona [Dem70]. Elle a ensuite été développée indépendamment par Kempf, Knudsen, Mumford et Saint-Donat [KKMS73], Miyake et Oda [MO75], et Satake [Sat73]. Voir aussi les articles de survol par Danilov [Dan78] et Teissier [Tei81].

Cette théorie représente un pont entre la géométrie convexe et la géométrie algébrique, qui permet, en particulier, de traiter une large classe de variétés algébriques de manière combinatoire. Aujourd'hui, il existe plusieurs livres portant sur la théorie de base [Oda88; Ful93; CLS]. Les variétés toriques sont encore un domaine actif de recherche.

Soit $\mathbb{T}$ un tore, M son réseau de caractères, et N son réseau de sous-groupes à un paramètre. Une variété torique est une $\mathbb{T}$-variété normale de complexité zéro.

Un éventail Σ dans $N_{\mathbb{Q}}$ est une collection de cônes polyédraux fortement convexes dans $N_{\mathbb{Q}}$ telle que pour tout $\sigma \in \Sigma$, chaque face de σ appartient aussi à Σ ; et pour tout $\sigma, \sigma' \in \Sigma$, l'intersection $\sigma \cap \sigma'$ est une face de chacun d'entre eux. Il existe une façon naturelle d'associer à un éventail Σ une variété torique X_Σ, et toute variété torique est obtenue de cette façon, voir la Section 1.4.

Le cas des variétés toriques affines est particulièrement simple. Ces variétés correspondent aux éventails Σ formés d'un seul cône maximal σ et de toutes ses faces. Dans ce cas, on note X_σ la variété X_Σ. De plus, l'algèbre des fonctions régulières sur une variété torique affine X_σ est l'algèbre de semi-groupe

$$\mathbf{k}[X_\sigma] = \mathbf{k}[\sigma_M^\vee] := \bigoplus_{m \in \sigma_M^\vee} \mathbf{k} \cdot \chi^m \,.$$

Dans ce cadre, la variété X_σ est uniquement déterminé par le cône $\sigma \subseteq N_{\mathbb{Q}}$ ou, de manière équivalente, par le cône des poids $\sigma^\vee \subseteq M_{\mathbb{Q}}$.

Variétés affines quasi-homogènes. Une variété quasi-homogène est une variété munie d'une action effective du tore $\mathbb{T}_1 = \operatorname{Spec} \mathbf{k}[\mathbb{Z}] = \mathbf{k}^*$ [5]. Une variété quasi-homogène affine est dit non-hyperbolique si la $\mathbb{Z}$-graduation correspondante est positive c'est-à-dire, si le cône des poids $\sigma^\vee$ est $\mathbb{Q}_{\geq 0}$, et hyperbolique si le cône des poids est $\mathbb{Q}$.

[5]Cette définition diffère du concept du quasi-homogénéité dans la théorie des actions des groupes algébriques.

Il est bien connu qu'une variété quasi-homogène affine X peut être décrit par des diviseurs à coefficients rationnels ($\mathbb{Q}$-diviseurs) sur une variété Y de dimension $\dim X - 1$.

Cette description est apparue d'abord pour des actions de $\mathbf{k}^*$ non-hyperboliques. Pour les surfaces elle se trouve dans les œuvres de Dolgachev [Dol75] et Pinkham [Pin77; Pin78]. Plus tard elle a été généralisée par Demazure [Dem88][6] en dimension quelconque.

Pour des $\mathbf{k}^*$-surfaces hyperboliques cette description à été développée par Flenner et Zaidenberg dans [FZ03]. Enfin, en dimension quelconque la description est une conséquence des résultats dans [FZ03] et [Dem88]. Elle est aussi un corollaire de [AH06].

Une variété Y est dite semi-projective si elle est projective au-dessus d'une variété affine. Soit Y une variété semi-projective normale et soit D un $\mathbb{Q}$-diviseur ample sur Y. On note $\mathcal{O}_Y(D)$ le faisceau $\mathcal{O}_Y(\lfloor D\rfloor)$, où $\lfloor D\rfloor$ est la partie entière de D, et on définit l'algèbre

$$A[Y,D]=\bigoplus_{m\in\mathbb{Z}_{\geq 0}} A_m\chi^m, \quad \text{où} \quad A_m=H^0(Y,\mathcal{O}_Y(mD))\,.$$

Dans ce cadre, $X=\operatorname{Spec} A[Y,D]$ est une variété affine normale de dimension $\dim Y+1$ munie d'une action non-hyperbolique de $\mathbf{k}^*$. Inversement, toute $\mathbf{k}^*$-variété affine non-hyperbolique est obtenue de cette façon [Dem88, Théorème 3.5]. En général, la variété Y dans cette description n'est pas unique. Cependant, le choix de Y peut être rendu unique en imposant la condition $Y\simeq \operatorname{Proj} A[Y,D]$.

Soit comme avant Y une variété semi-projective normale et soit D_+, D_- deux $\mathbb{Q}$-diviseurs amples tels que $D_+ + D_- \leq 0$. On définit l'algèbre

$$A[Y,D_+,D_-]=\bigoplus_{m\in\mathbb{Z}} A_m\chi^m, \quad \text{où} \quad A_m=\begin{cases} H^0(Y,\mathcal{O}_Y(mD_+)) & \text{si } m\geq 0,\\ H^0(Y,\mathcal{O}_Y(-mD_-)) & \text{sinon.}\end{cases}$$

La condition $D_+ + D_- \leq 0$ assure que $A[Y,D_+,D_-]$ est en effet une algèbre, voir la Section 1.5. Dans ce cadre, $X=\operatorname{Spec} A[Y,D_+,D_-]$ est une variété affine normale de dimension $\dim Y+1$ munie d'une action hyperbolique de $\mathbf{k}^*$. Inversement, toute $\mathbf{k}^*$-variété affine hyperbolique est obtenue de cette façon [FZ03].

$\mathbb{T}$-variétés de complexité un. Dans le Chapitre 4 de [KKMS73], Mumford a donné une description combinatoire des $\mathbb{T}$-variétés de complexité un admettant un quotient rationnel qui est aussi un morphisme, voir la Définition 1.3.3.

Plus généralement, Timashev [Tim97] a donné une description combinatoire des variétés normales munies d'une action effective de complexité un d'un groupe réductif. Lorsque l'on spécialise cette description au cas $G=\mathbb{T}$ [Tim08], elle coïncide avec celle donnée auparavant par Mumford. La description de Timashev considère aussi le cas où l'action de $\mathbb{T}$ n'admet pas un quotient rationnel qui est un morphisme. Dans la suite on présente sommairement la description des $\mathbb{T}$-variétés affines due à Timashev.

Soit C une courbe projective lisse, M et N des réseaux mutuellement duaux de rang n, $\mathcal{H}^+=N_{\mathbb{Q}}\times\mathbb{Q}_{\geq 0}$, et $\mathcal{H}=N_{\mathbb{Q}}\times\{0\}\subseteq\mathcal{H}^+$. Un hyper-cône Θ sur C est

[6] Cette article à été officiellement publié en 1988, mais il est apparu en 1979 dans le séminaire Demazure-Giraud-Teissier.

un ensemble des cônes polyédraux fortement convexes $\mathcal{C}_z \subseteq \mathcal{H}^+$, pour tout $z \in C$ satisfaisant les propriétés suivantes.

(*i*) Le cône $\mathcal{C}_z \cap \mathcal{H} =: \sigma$ ne dépend pas de $z \in C$.

(*ii*) $\mathcal{C}_z = \sigma \times \mathbb{Q}_{\geq 0}$ pour tous sauf un nombre fini de z.

(*iii*) On note Δ_z la projection sur $N_\mathbb{Q}$ du polyèdre $\mathcal{C}_z \cap (\mathcal{H} + (\bar{0}, 1))$ et on pose $\Delta = \sum_{z\in C} \Delta_z$. Alors, le polyèdre Δ est un sous-ensemble propre de σ.

(*iv*) Soit h_z (resp. h_Θ) la fonction de support[7] de Δ_z (resp. Δ), et $\Theta_m = \sum_{z\in C} h_z(m)\cdot z$, pour tout $m \in \sigma_M^\vee$. Si $\Delta \neq \emptyset$ alors pour chaque $m \in \sigma_M^\vee$ tel que $h_\Theta(m) = 0$, un multiple du diviseur Θ_m est principal.

On pose $C^\circ = \{z \in C \mid \mathcal{C}_z \neq \sigma\}$. Pour tout hyper-cône Θ sur une courbe projective lisse C on définit l'algèbre

$$A[C,\Theta] = \bigoplus_{m\in\sigma_M^\vee} A_m\chi^m, \quad \text{où} \quad A_m = H^0(C^\circ, \mathcal{O}_C(\Theta_m))\,.$$

Dans ce cadre, $X = \operatorname{Spec} A[C,\Theta]$ est une variété affine normale de dimension $\operatorname{rang} M + 1$ munie d'une action effective de $\mathbb{T}$. Inversement, toute $\mathbb{T}$-variété affine normale de complexité un est obtenue de cette façon [Tim08, Theorem 2].

$\mathbb{T}$-variétés de complexité quelconque. On expose maintenant la description combinatoire des $\mathbb{T}$-variétés affine normales de complexité quelconque due à Altmann et Hausen [AH06].

Soit M et N des réseaux mutuellement duaux de rang n, $\mathbb{T} = \operatorname{Spec} \mathbf{k}[M]$, $N_\mathbb{Q} = N \otimes \mathbb{Q}$, et $M_\mathbb{Q} = M \otimes \mathbb{Q}$. On fixe un cône polyédral fortement convexe σ dans $N_\mathbb{Q}$. On dit qu'un polyèdre Δ est un σ-polyèdre s'il peut être décomposé comme la somme de Minkowski d'un polyèdre borné et de σ.

Un diviseur σ-polyédral sur une variété semi-projective Y est une somme formelle

$$\mathfrak{D} = \sum_Z \Delta_Z \cdot Z\,,$$

où Z parcourt tous les diviseurs premiers sur Y, Δ_Z est un σ-polyèdre, et $\Delta_Z = \sigma$ pour tous sauf un nombre fini de diviseurs premiers Z. Pour chaque $m \in \sigma^\vee$ on définit l'évaluation $\mathfrak{D}(m)$ de $\mathfrak{D}$ en m comme le $\mathbb{Q}$-diviseur :

$$\mathfrak{D}(m) = \sum_Z h_Z(m) \cdot Z\,,$$

où h_Z est la fonction de support de Δ_Z. Un diviseur σ-polyédral $\mathfrak{D}$ est dit propre si $\mathfrak{D}(m)$ est semi-ample et $\mathbb{Q}$-Cartier pour tout $m \in \sigma^\vee$, et $\mathfrak{D}(m)$ est abondant[8] pour tout $m \in \operatorname{rel.int}(\sigma^\vee)$.

A tout diviseur σ-polyédral propre $\mathfrak{D}$ sur une variété semi-projective Y on peut associer l'algèbre

$$A[Y,\mathfrak{D}] = \bigoplus_{m\in\sigma_M^\vee} A_m\chi^m, \quad \text{où} \quad A_m = H^0(Y, \mathcal{O}_Y(\mathfrak{D}(m)) \subseteq \mathbf{k}(Y)\,.$$

[7]Voir la Section 1.1.2 pour la définition de fonction de support.

[8]On dit qu'un diviseur $\mathbb{Q}$-Cartier D sur Y est abondant s'il existe un diviseur D_0 dans le système linéaire $|rD|$, pour quelque $r > 1$, tel que $Y \setminus \operatorname{Supp} D_0$ est affine.

THÉORÈME (Altmann et Hausen). *Soit $\mathfrak{D}$ un diviseur σ-polyédral propre sur une variété semi-projective Y. Alors $X[Y,\mathfrak{D}] := \operatorname{Spec} A[Y,\mathfrak{D}]$ est une $\mathbb{T}$-variété affine normale de dimension* $\operatorname{rang} M + \dim Y$. *Inversement, toute $\mathbb{T}$-variété affine normale est isomorphe à $X[Y,\mathfrak{D}]$ pour une certaine variété semi-projective Y et un certain diviseur σ-polyédral propre $\mathfrak{D}$ sur Y.*

Dans [AHS08], des éventails divisoriaux ont été introduits pour étendre cette description combinatoire aux $\mathbb{T}$-variétés normales, non nécessairement affines. Ceci donne une généralisation du passage des cônes aux éventails dans la géométrie torique.

Dans la suite, on montre la façon dont cette dernière description généralise les cas particuliers des variétés toriques, des variétés quasi-homogènes, et des $\mathbb{T}$-variétés de complexité un.

Variétés toriques affines. Les variétés toriques affines correspondent au cas où Y est réduite à un point. Comme l'unique diviseur sur Y est $\emptyset$, pour tout diviseur σ-polyédral propre $\mathfrak{D}$ les évaluations $\mathfrak{D}(m) = \emptyset$, pour tout $m \in \sigma_M^\vee$, et alors $H^0(Y, \mathcal{O}_Y(\mathfrak{D}(m))) = \mathbf{k}$. Ceci donne

$$A[Y,\mathfrak{D}] = \bigoplus_{m\in\sigma_M^\vee} \mathbf{k}\chi^m, \quad \text{et} \quad X[Y,\mathfrak{D}] = X_\sigma\,.$$

Variétés affines quasi-homogènes. Soit X une variétés affines quasi-homogènes. On suppose d'abord que X est non-hyperbolique. Soit D un diviseur $\mathbb{Q}$-Cartier ample sur une variété semi-projective Y tel que $X \simeq \operatorname{Spec} A[Y,D]$. Si l'on pose $M = \mathbb{Z}$, $\sigma = \mathbb{Q}_{\geq 0}$, et

$$\mathfrak{D} = [1,\infty)\cdot D \quad \text{on a} \quad A[Y,D] = A[Y,\mathfrak{D}]\,.$$

On suppose maintenant que X est hyperbolique. Soit D_+, D_- deux diviseurs $\mathbb{Q}$-Cartier amples sur une variété semi-projective Y tels que $D_+ + D_- \leq 0$ et $X \simeq A[Y, D_+, D_-]$. Si l'on pose $M = \mathbb{Z}$, $\sigma = \{0\}$, et

$$\mathfrak{D} = \{1\}\cdot D_+ + [0,1]\cdot(-D_+ - D_-) \quad \text{on a} \quad A[Y,D_+,D_-] = A[Y,\mathfrak{D}]\,.$$

$\mathbb{T}$-variétés affines de complexité un. Soit X une $\mathbb{T}$-variété affine de complexité un. On peut supposer que $X = \operatorname{Spec} A[C,\Theta]$ où Θ est un hyper-cône sur une courbe projective lisse C. Avec la notation de la définition d'un hyper-cône, (i) montre que tous les polyèdres Δ_z sont des σ-polyèdres. D'après (ii)

$$\mathfrak{D} = \sum_{z\in C^\circ} \Delta_z \cdot z$$

est un diviseur σ-polyédral sur C°. Enfin, (iii) et (iv) assurent que $\mathfrak{D}$ est propre. Avec ces définitions, il est évident que $A[C,\Theta] = A[C^\circ,\mathfrak{D}]$, voir aussi [Vol07].

Actions du groupe additif

Le groupe additif $\mathbb{G}_a$ sur un corps algébriquement clos $\mathbf{k}$ de caractéristique nulle est défini comme la variété affine $\mathbb{A}^1 \simeq \mathbf{k}$ munie de la structure naturelle de groupe algébrique induite par l'addition dans $\mathbf{k}$.

Soit $X = \operatorname{Spec} A$ une variété affine. Une dérivation $\partial : A \to A$ est dite localement nilpotente (DLN en abrégé) si pour tout $a \in A$ il existe $k \in \mathbb{Z}_{\geq 0}$ tel que $\partial^k(a) = 0$. Une action du groupe additif sur X donne lieu à une DLN de A et toute action du groupe additif sur X est obtenue de cette façon, voir la Section 2.1.

L'étude des actions du groupe additif remonte à Hilbert qui a calculé les anneaux d'invariants de certaines actions linéaires de $\mathbb{G}_a$ sur $\mathbb{A}^n$ à clôture intégrale près. En 1959, Nagata a donné un contre-exemple au célèbre quatorzième problème de Hilbert, qui utilise une action linéaire de $\mathbb{G}_a^{13}$ sur $\mathbb{A}^{32}$ [Nag59].

En 1968, Rentschler a classifié toutes les dérivations localement nilpotentes de l'anneau des polynômes à deux variables sur un corps de caractéristique nulle, et a montré que celle-ci donne une classification des actions du groupe additif sur $\mathbb{A}^2$ [Ren68].

L'intérêt moderne dans les actions du groupe additif et dans les DLN provient de l'introduction par Kaliman et Makar-Limanov de l'anneau des constantes absolues, maintenant appelé l'invariant Makar-Limanov (l'invariant de ML en abrégé). L'invariant de ML d'une variété affine $X = \operatorname{Spec} A$ est défini comme l'intersection des noyaux de toutes les DLN sur A.

On considère la variété de Koras-Russell $X = \operatorname{Spec} A$ donné par

$$A = \mathbf{k}[x, y, z, t]/(x + x^2y + z^2 + t^3)\,.$$

L'invariant de ML a été introduit pour distinguer X de $\mathbb{A}^3$. En effet, $\mathrm{ML}(X) = \mathbf{k}[x]$ tandis que $\mathrm{ML}(\mathbb{A}^3) = \mathbf{k}$. Ceci fût le dernier pas dans la preuve du fait que toutes les actions du groupe multiplicatif sur $\mathbb{A}^3$ sont linéarisables [KKMLR97].

Dans la suite on décrit les résultats des Chapitres 2 et 3, où l'on étudie les actions du groupe additif $\mathbb{G}_a$ sur des $\mathbb{T}$-variétés affines, où de manière équivalente, les DLN des algèbres intègres de type fini M-graduées. Ces résultats sont contenus dans l'article [Lie10] et la pré-publication [Lie09a].

Soit M et N des réseaux mutuellement duaux de rang n, et $\mathbb{T} = \operatorname{Spec} \mathbf{k}[M]$. Soit aussi $N_{\mathbb{Q}} = N \otimes \mathbb{Q}$, et $M_{\mathbb{Q}} = M \otimes \mathbb{Q}$. On fixe un cône polyédral fortement convexe σ dans $N_{\mathbb{Q}}$, on considère une algèbre intègre de type fini intégralement close M-graduée

$$A = \bigoplus_{m \in \sigma_M^\vee} A_m \chi^m \subseteq K^{\mathbb{T}}[M], \quad \text{où} \quad A_m \subseteq K^{\mathbb{T}}\,,$$

et on pose $X = \operatorname{Spec} A$ la $\mathbb{T}$-variété affine correspondante.

Une dérivation $\partial : A \to A$ est dite homogène si elle envoie des éléments homogènes sur des éléments homogènes c'est-à-dire, s'il existe un élément du réseau $e = \deg \partial \in M$ tel que

$$\partial(A_m \chi^m) \subseteq A_{m+e} \chi^{m+e}, \quad \text{pout tout} \quad m \in \sigma_M^\vee\,.$$

Une action du groupe additif sur X est dite compatible avec l'action de $\mathbb{T}$ si la DLN correspondante est homogène, en termes géométriques cela signifie que l'action du $\mathbb{G}_a$ est normalisée par le tore $\mathbb{T}$.

Dans le Lemme 2.1.7 on montre qu'à chaque DLN de A on peut associer une DLN homogène. Une DLN homogène ∂ de A peut s'étendre à une dérivation de $K^{\mathbb{T}}[M]$ par la règle de Leibniz. On note aussi cette extension par ∂.

On dit qu'une DLN homogène ∂ de A, où de manière équivalente, une action du groupe additif sur X, est de type fibre si $\partial(K^{\mathbb{T}}) = 0$ et de type horizontal sinon. En termes géométriques, une action de $\mathbb{G}_a$ compatible est de type fibre si les orbites générales de l'action de $\mathbb{G}_a$ sont contenues dans les adhérences des orbites de l'action de $\mathbb{T}$.

Soit $\mathrm{LND}(A)$ l'ensemble de toutes les DLN de A. L'invariant de Makar-Limanov de A (ou de X) est défini comme

$$\mathrm{ML}(A) = \bigcap_{\partial \in \mathrm{LND}(A)} \ker \partial \,.$$

De manière similaire, on note $\mathrm{LND}_{\mathrm{h}}(A)$ l'ensemble de toutes les DLN homogènes de A, on définit l'invariant de Makar-Limanov homogène de A comme

$$\mathrm{ML}_{\mathrm{h}}(A) = \bigcap_{\partial \in \mathrm{LND}_{\mathrm{h}}(A)} \ker \partial \,.$$

On dit que l'invariant de ML de A est trivial si $\mathrm{ML}(A) = \mathbf{k}$. Évidement, la trivialité de l'invariant de ML homogène entraîne celle de l'invariant de ML usuel.

Actions du groupe additif sur des variétés toriques. Soit $\sigma \subseteq N_{\mathbb{Q}}$ un cône polyédral fortement convexe . On pose $A = \mathbf{k}[\sigma_M^\vee]$ et $X_\sigma = \operatorname{Spec} A$. On considère un rayon ρ de σ avec vecteur primitif ρ_0 et son mur dual $\tau \subseteq \sigma^\vee$. On définit S_ρ comme l'ensemble

$$S_\rho = \left\{ m \in M \mid \langle \rho_0, m \rangle = -1, \text{ et } \langle \rho', m \rangle \geq 0 \,\forall \rho' \in \sigma(1) \setminus \rho \right\} \,,$$

où $\sigma(1)$ est l'ensemble de tous le rayons de σ. Le résultat principal de la Section 2.2 est la classification suivante, valable sur un corps de caractéristique nulle qui n'est pas forcement algébriquement clos.

Théorème A. *A toute couple (ρ, e), où ρ est un rayon de σ et e est un élément du réseau dans S_ρ, on peut associer une DLN homogène $\partial_{\rho,e}$ de $A = \mathbf{k}[\sigma_M^\vee]$ avec noyau* $\ker \partial_{\rho,e} = \mathbf{k}[\tau_M]$ *et degré* $\deg \partial_{\rho,e} = e$*.*

Inversement, si $\partial \neq 0$ est une DLN homogène de A, alors $\partial = \lambda \partial_{\rho,e}$ pour un certain rayon $\rho \subseteq \sigma$, un certain élément du réseau $e \in S_\rho$, et un certain $\lambda \in \mathbf{k}^*$.

Dans [Dem70] le résultat analogue est obtenu pour une variété torique lisse qui n'est pas nécessairement affine. Dans *loc. cit.* les éléments de l'ensemble $\mathcal{R} = -\bigcup_{\rho \in \sigma(1)} S_\rho$ sont appelés les racines de σ.

Comme d'habitude, on note un rayon et son vecteur primitif par la même lettre ρ. Soit ρ un rayon de σ et $e \in S_\rho$, alors la DLN $\partial_{\rho,e}$ est donné par

$$\partial_{\rho,e}(\chi^m) = \langle m, \rho \rangle \chi^{m+e} \,.$$

Comme première application du Théorème A on montre que les classes d'équivalence de DLN homogènes de la variété torique X_σ sont en bijection avec les rayons de σ. Pour l'invariant de ML d'une variété torique, on obtient le résultat suivant, voir la Proposition 3.2.1.

Théorème B. *Soit $\theta \subseteq M_{\mathbb{Q}}$ le sous-espace vectoriel maximal contenu dans $\sigma^\vee$. Alors*

$$\mathrm{ML}(A) = \mathrm{ML}_{\mathrm{h}}(A) = \mathbf{k}[\theta_M] \,.$$

En particulier, $\mathrm{ML}(A) = \mathbf{k}$ *si et seulement si σ est de dimension maximale c'est-à-dire, si et seulement si X n'est pas isomorphe à $Y \times \mathbb{T}'$ pour un tore $\mathbb{T}'$ de dimension positive.*

Actions du groupe additif de type fibre sur des $\mathbb{T}$-variétés de complexité quelconque. Soit Y une variété semi-projective et $\mathfrak{D}$ le diviseur σ-polyédral propre

$$\mathfrak{D} = \sum_Z \Delta_Z \cdot Z \quad \text{sur} \quad Y\,.$$

On note $\mathbf{k}(Y)$ le corps des fonctions rationnelles de Y, et on considère la variété affine $X = \operatorname{Spec} A$, où

$$A = A[Y,\mathfrak{D}] = \bigoplus_{m\in\sigma_M^\vee} A_m\chi^m, \quad \text{avec} \quad A_m = H^0\left(Y, \mathcal{O}(\mathfrak{D}(m))\right) \subseteq \mathbf{k}(Y)\,.$$

On choisit aussi une DLN homogène ∂ de type fibre de A, et on considère l'algèbre $\bar{A} = \mathbf{k}(Y)[\sigma_M^\vee]$ du semi-groupe $\sigma_M^\vee$ au-dessus du corps $\mathbf{k}(Y)$. La DLN ∂ peut s'étendre à une $\mathbf{k}(Y)$-dérivation localement nilpotente homogène $\bar{\partial}$ de $\bar{A}$. Les dérivations de $\bar{A}$ ont été classifiées dans le Théorème A.

Dans la Section 2.4 on utilise cette remarque pour classifier les DLN de type fibre des $\mathbb{T}$-variétés de complexité quelconque. Ceci est d'abord fait pour le cas particulier de complexité un dans la Section 2.3.1.

Pour tout $e \in S_\rho$, on pose $\Phi_e^* = H^0(Y, \mathcal{O}_Y(-D_e)) \setminus \{0\}$, où D_e est le $\mathbb{Q}$-diviseur sur Y défini par

$$D_e := \sum_Z \max_{m\in\sigma_M^\vee\setminus\tau_M} (h_Z(m) - h_Z(m+e)) \cdot Z\,.$$

Étant donné un rayon $\rho \subseteq \sigma$ on note τ le mur de $\sigma^\vee$ dual à ρ. Notre résultat principal par rapport aux DLN de type fibre de $A[Y,\mathfrak{D}]$ est le théorème suivant.

Théorème C. *Pour chaque triplet (ρ, e, φ), où ρ est un rayon de σ, $e \in S_\rho$, et $\varphi \in \Phi_e^*$, la dérivation $\partial_{\rho,e,\varphi} := \varphi\partial_{\rho,e}$ est une DLN homogène de type fibre de $A = A[Y,\mathfrak{D}]$ de degré e avec noyau*

$$\ker \partial_{\rho,e,\varphi} = \bigoplus_{m\in\tau_M} A_m\chi^m\,.$$

Inversement, si ∂ est une DLN homogène de type fibre non-trivial de A, alors $\partial = \partial_{\rho,e,\varphi}$ pour un certain rayon $\rho \subseteq \sigma$, un certain élément du réseau $e \in S_\rho$, et une certaine fonction rationnelle $\varphi \in \Phi_e^*$.

Le noyau de la DLN $\partial_{\rho,e,\varphi}$ ne dépend que du rayon ρ. Alors les classes d'équivalence de DLN de type fibre de A sont en bijection avec les rayons ρ de σ satisfaisant la condition suivante : il existe $e \in S_\rho$ avec Φ_e^* non-vide. Le théorème suivant donne une interprétation géometrique de cette condition.

Théorème D. *Soit $\rho \subseteq \sigma$ le rayon dual à un mur $\tau \subseteq \sigma^\vee$. Alors il existe $e \in S_\rho$ tel que Φ_e^* est non-vide si et seulement si le diviseur $\mathfrak{D}(m)$ est abondant pour tout élément du réseau $m \in \operatorname{rel.int}(\tau)$.*

En particulier, les DLN de type fibre de $A = A[Y,\mathfrak{D}]$ sont en correspondance bijective avec les rayons ρ de σ tels que $\mathfrak{D}(m)$ est abondant pour tout élément du réseau $m \in \operatorname{rel.int}(\tau)$.

A partir du Théorème D on obtient le corollaire suivant qui donne une condition pour que l'invariant de ML de A soit trivial.

Corollaire E. *Soit* $A = A[Y, \mathfrak{D}]$. *Si* Y *est projective,* $\operatorname{rang} M \geq 2$*,* σ *est de dimension maximale, et* $\mathfrak{D}(m)$ *est abondant pour tous les éléments du réseau* $m \in \sigma^\vee$ *différents de zéro, alors* $ML(A) = \mathbf{k}$*.*

Actions du groupe additif sur des $\mathbb{T}$-variétés de complexité un. Le cas des actions compatibles du groupe additif sur des $\mathbf{k}^*$-surfaces affines a été étudié d'abord par Flenner et Zaidenberg dans [FZ05a]. Cette article a été la motivation pour développer la partie de cette thèse décrite ci-dessous. Dans la Section 2.3.3 on montre que nos résultats généralisent les résultats de [FZ05a].

Pour le cas des $\mathbb{T}$-variétés affines de complexité un on donne, dans la Section 2.3, une classification complète de toutes les DLN homogènes. Soit σ un cône polyédral fortement convexe dans $N_\mathbb{Q}$. On considère une courbe lisse C et un diviseur σ-polyédral propre $\mathfrak{D}$ sur C

$$\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z\,.$$

On note $\mathbf{k}(C)$ le corps de fonctions rationnelles de C, et on considère la variété affine $X = \operatorname{Spec} A$, où

$$A = A[C, \mathfrak{D}] = \bigoplus_{m \in \sigma_M^\vee} A_m \chi^m, \quad \text{avec} \quad A_m = H^0\left(C, \mathcal{O}_C(\mathfrak{D}(m))\right) \subseteq \mathbf{k}(C)\,.$$

On définit aussi le degré de $\mathfrak{D}$ comme le polyèdre

$$\deg \mathfrak{D} = \sum_{z \in C} \Delta_z\,.$$

La classification des DLN homogènes de type fibre de A a été donnée dans le Théorème C. De plus, dans le cas de complexité un on peut remplacer la condition "$\mathfrak{D}(m)$ est abondant pour tout élément du réseau $m \in \operatorname{rel.int}(\tau)$" dans le Théorème D par la condition suivante "ρ est disjoint de $\deg \mathfrak{D}$" qui est plus simple à vérifier.

La classification des DLN homogènes de type horizontal est plus compliquée. D'abord, on montre que l'existence d'une DLN de type horizontal de A entraîne que la courbe de base C est isomorphe à $\mathbb{A}^1$ où à $\mathbb{P}^1$, voir le Lemme 2.3.14. Dans la suite on suppose que $C = \mathbb{A}^1$ ou $C = \mathbb{P}^1$.

Le résultat principal pour les DLN homogènes de type horizontal de $A = A[C, \mathfrak{D}]$ est le Théorème 2.3.26. L'énoncé de ce théorème est trop technique pour être inclus dans cette introduction. Ici, on donne dans le Théorème F son corollaire le plus important.

On note $h_z : \sigma^\vee \to \mathbb{Q}$ la fonction de support de Δ_z, et on définit le quasi-éventail normal $\Lambda(\mathfrak{D})$ de $\mathfrak{D}$ comme le plus petit raffinement du quasi-éventail de $\sigma^\vee \subseteq M_\mathbb{Q}$ tel que pour tout $z \in C$, la fonction h_z est linéaire dans chaque cône $\eta \in \Lambda(\mathfrak{D})$. On dit qu'un cône maximal $\eta \in \Lambda(\mathfrak{D})$ est convenable s'il existe $z_0 \in C$ tel que $h_z|_\eta$ est entier, pour tout $z \in C \setminus \{z_0\}$.

Avec ces définitions on peut énoncer la classification suivante des classes d'équivalence de DLN homogènes de type fibre de $A = A[C, \mathfrak{D}]$, voir les Corollaires 2.3.27 et 2.3.28.

Théorème F. *Soit* $\mathfrak{D}$ *un diviseur* σ*-polyédral propre sur* C*, et soit* $A = A[C, \mathfrak{D}]$*. Les classes d'équivalence des DLN homogènes de type horizontal de* A *sont en bijection avec :*

(*i*) *Les cônes maximaux convenables* η *du quasi-éventail normal* $\Lambda(\mathfrak{D})$*, dans le cas où* $C = \mathbb{A}^1$.

(*ii*) *Les couples* (z_∞, η)*, où* $z_\infty \in C$ *et* η *est un cône maximal convenable du quasi-éventail normal* $\Lambda(\mathfrak{D}|_{C_0})$*, avec* $C_0 := C \setminus \{z_\infty\}$*, dans le cas où* $C = \mathbb{P}^1$.

Dans le Théorème 3.3.4 on calcule l'invariant de ML homogène de A. À nouveau, l'énoncé de ce résultat est trop technique pour être inclus dans cette introduction.

L'invariant de ML et rationnalité. Comme on l'a dit antérieurement, l'invariant de ML est un outil important en géométrie affine. En particulier, il permet de distinguer certaines variétés de l'espace affine. Cependant, cet invariant est loin d'être optimal. En effet, l'invariant de ML de l'espace affine $\mathbb{A}^n$ est trivial c'est-à-dire, $\mathrm{ML}(\mathbb{A}^n) = \mathbf{k}$. Pourtant, il peut aussi être trivial pour des variétés affines non rationnelles.

Rappelons qu'une variété est rationnelle si son corps de fonctions rationnelles est une extension purement transcendante du corps de base $\mathbf{k}$. Dans la Section 3.3.1 on applique le Corollaire E pour donner, à notre connaissance, le premier exemple d'une variété affine non-rationnelle d'invariant de ML trivial. Cet exemple est généralisé dans la Section 3.4.

On donne ici un cas particulier de ces exemples. Soit Y une variété projective, H un diviseur de Cartier ample sur Y, et $n \geq 2$. On considère l'espace total $\widetilde{X}$ du fibré vectoriel associé au faisceau localement libre $\bigoplus_{i=1}^n \mathcal{O}_Y(H)$, et on note X la contraction de la section nulle de $\widetilde{X}$ en un point. Dans l'Exemple 3.4.3 on montre que $\mathrm{ML}(X) = \mathbf{k}$, tandis que X est birationnelle à $Y \times \mathbb{P}^n$.

Dans le Théorème 3.4.1 on applique cet exemple pour donner la caractérisation birationnelle suivante des variétés affines normales d'invariant de ML trivial.

THÉORÈME G. *Soit* X *une variété affine au-dessus du corps* $\mathbf{k}$. *Si* $\mathrm{ML}(X) = \mathbf{k}$ *alors* $X \simeq_{\mathrm{bir}} Y \times \mathbb{P}^2$ *pour une certaine variété* Y. *Inversement, dans chaque classe d'équivalence birationnelle* $Y \times \mathbb{P}^2$ *il y a une variété affine* X *avec* $\mathrm{ML}(X) = \mathbf{k}$.

Pour éviter ces exemple pathologiques, on introduit dans la Section 3.5 une version de l'invariant ML que l'on calcule dans le corps de fonctions rationnelles, on l'appelle l'invariant FML. Cet invariant est défini comme

$$\mathrm{FML}(A) = \bigcap_{\partial \in \mathrm{LND}(A)} \mathrm{Frac}(\ker \partial)\,.$$

Pour toute algèbre affine intègre et intégralement close A il existe une inclusion $\mathrm{ML}(A) \subseteq \mathrm{FML}(A)$. Comme $\mathrm{FML}(\mathbb{A}^n) = \mathbf{k}$ l'invariant FML est plus puissant que l'invariant classique puisqu'il permet de distinguer plus de variétés de l'espace affine que l'invariant classique.

On conjecture que $\mathrm{FML}(X) = \mathbf{k}$ entraîne la rationnalité de X. Dans le Théorème 3.5.6 on confirme cette conjecture en dimension inférieure ou égale à 3.

Anneaux d'invariants de type fini. Le quatorzième problème de Hilbert généralisé peut être formulé comme suit. Soit $\mathbf{k} \subseteq L \subseteq K$ une extension de corps, et soit $A \subseteq K$ une $\mathbf{k}$-algèbre de type fini. Est-il vrai que la $\mathbf{k}$-algèbre $A \cap L$ est aussi de type fini ?

Dans le cas où $K = \mathrm{Frac}\, A$ et $\mathrm{Spec}\, A$ admet une action du groupe additif, on pose $L = K^{\mathbb{G}_a}$ et donc $A \cap L$ est l'anneau d'invariants de l'action de $\mathbb{G}_a$. On a que

$A \cap L = \ker \partial$, où ∂ est la DLN de A correspondante. Dans ce cas, la réponse à la question au-dessus est négative, même si A est l'algèbre des polynômes de $n \geq 5$ variables [DF99]. De l'autre côté, dans la Section 2.5 on montre le résultat suivant.

THÉORÈME H. *Soit A une algèbre affine intègre intégralement close et M-graduée, où M est un réseau de rang fini, et soit ∂ une DLN homogène de A. Si la complexité de l'action correspondante de $\mathbb{T}$ sur* Spec A *est zéro ou un, ou la DLN ∂ est de type fibre, alors* $\ker \partial$ *est de type fini.*

Ce théorème suit de nos différentes classifications. Le cas difficile, où la DLN est de type horizontal, est aussi corollaire d'un résultat du à Kuroda [Kur03].

De plus, dans le Corollaire 2.5.5, on utilise le résultat de Kuroda pour montrer que $\ker \partial$ est de type fini aussi dans le cas où $X = \operatorname{Spec} A$ est rationnelle et l'action de $\mathbb{T}$ est de complexité deux.

Singularités normaux avec une action du tore

Soit X une variété normale munie d'une action effective d'un tore algébrique. D'après un résultat bien connu de Sumihiro (voir le Théorème 1.3.4) tout point $x \in X$ a un voisinage affine invariant par l'action du tore. Alors les problèmes locaux peuvent être réduits au cas affine.

On donne maintenant le point de vue géométrique de la description combinatoire des $\mathbb{T}$-variétés affines normales due à Altmann et Hausen. Soit Y une variété semi-projective normale et soit $\mathfrak{D}$ un diviseur σ-polyédral propre sur Y. On définit la $\mathcal{O}_Y$-algèbre M-graduée

$$\widetilde{A} = \widetilde{A}[Y, \mathfrak{D}] := \bigoplus_{m \in \sigma_M^\vee} \mathcal{O}_Y(\mathfrak{D}(m)) .$$

Dans ce cas, prendre les sections globales de $\widetilde{A}[Y, \mathfrak{D}]$ donne l'algèbre M-graduée $A[Y, \mathfrak{D}]$ définie auparavant

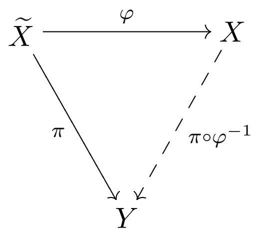
$$A = A[Y, \mathfrak{D}] = H^0(Y, \widetilde{A}[Y, \mathfrak{D}]) .$$

On définit aussi le schéma

$$\widetilde{X} = \widetilde{X}[Y, \mathfrak{D}] := \mathbf{Spec}_Y\, \widetilde{A}[Y, \mathfrak{D}] .$$

Ici, $\mathbf{Spec}_Y$ est le spectre relatif d'une $\mathcal{O}_Y$-algèbre. Voir [Har77, Ch. II Ex. 5.17] pour une définition.

La construction de $\mathbf{Spec}_Y$ donne un morphisme affine $\pi : \widetilde{X} \to Y$ invariant par $\mathbb{T}$ qui est donc un quotient rationnel pour l'action de $\mathbb{T}$ sur $\widetilde{X}$. Le foncteur sections globales donne un morphisme birationnel $\varphi : \widetilde{X} \to X = X[Y, \mathfrak{D}]$ équivariant par rapport à $\mathbb{T}$. On a alors que $\pi \circ \varphi^{-1}$ est un quotient rationnel pour l'action de $\mathbb{T}$ sur X. On peut résumer ces considérations dans le diagramme suivant, où toutes les flèches vers le bas sont des quotients rationnels.



Avec ces définitions, on a le théorème suivant.

THÉORÈME (Altmann et Hausen).

(i) *$\widetilde{X}$ est une variété.*

(ii) *Le morphisme affine π est un bon quotient pour l'action de $\mathbb{T}$ sur $\widetilde{X}$.*

(iii) *Le morphisme birationnel φ est propre.*

On décrit maintenant les résultats au chapitre 4, où on étudie les singularités des $\mathbb{T}$-variétés affines. Ces résultats sont contenus dans la pré-publication [Lie09b]. Une généralisation de ces résultats dans un travail conjoint avec Süß [LS10] est en cours de rédaction.

La description combinatoire $(Y, \mathfrak{D})$ d'une $\mathbb{T}$-variété X n'est pas unique. En effet, si l'on considère l'éclatement $\psi : \widetilde{Y} \to Y$ d'un point fermé de Y et le diviseur σ-polyédral propre $\psi^*\mathfrak{D}$, on a que $X[Y, \mathfrak{D}] \simeq X[\widetilde{Y}, \psi^*\mathfrak{D}]$, voir le Lemme 4.2.1 pour un énoncé plus précis.

On définit le support d'un diviseur σ-polyédral $\mathfrak{D}$ sur une variété semi-projective Y par

$$\operatorname{Supp} \mathfrak{D} = \sum_{\Delta_Z \neq \sigma} Z \,.$$

On dit que $\mathfrak{D}$ est un diviseur σ-polyédral SNC si Y est lisse, $\mathfrak{D}$ est propre, et $\operatorname{Supp} \mathfrak{D}$ est un diviseur à croisements normaux simples. Dans le Corollaire 4.2.5 on montre que toute $\mathbb{T}$-variété admet une description combinatoire $(Y, \mathfrak{D})$ telle que $\mathfrak{D}$ est un diviseur σ-polyédral SNC.

Rappelons qu'une variété normale X est dite toroïdale si pour tout $x \in X$, le voisinage formel de x est isomorphe au voisinage formel d'un point d'une variété torique. Avec ces définitions, dans la Section 4.2 on montre le résultat suivant.

THÉORÈME I. *Soit $\mathfrak{D}$ un diviseur σ-polyédral propre sur une variété normale semi-projective Y. Si $\mathfrak{D}$ est SNC alors $\widetilde{X}[Y, \mathfrak{D}]$ est une variété toroïdale.*

En particulier, ce théorème entraîne que le morphisme birationnel propre

$$\varphi : \widetilde{X} = \widetilde{X}[Y, \mathfrak{D}] \to X = X[Y, \mathfrak{D}]$$

est une désingularisation partielle de X n'ayant que des singularités toriques. De plus, une désingularisation de $\widetilde{X}$, et donc aussi de X, peut être obtenue par des méthodes toriques.

Comme les singularités toriques sont bien comprises (voir Section 1.6), dans la suite on utilisera le morphisme $\varphi : \widetilde{X} \to X$ pour étudier les singularités de X.

Soit X une variété normale et soit $\psi : W \to X$ désingularisation (complète) de X. Habituellement, la classification des singularités utilise les images directes supérieures du faisceau structural $R^i\psi_*\mathcal{O}_W$. Ces faisceaux sont définis par

$$U \longrightarrow H^0(U, R^i\psi_*\mathcal{O}_W) := H^i\left(\psi^{-1}(U), \mathcal{O}_W|_{\psi^{-1}(U)}\right) \,.$$

Les faisceaux $R^i\psi_*\mathcal{O}_W$ sont indépendants de la désingularisation de X choisie. De plus, X est normale si et seulement si $R^0\psi_*\mathcal{O}_W := \psi_*\mathcal{O}_W = \mathcal{O}_X$. Dans le théorème suivant on calcule les faisceaux $R^i\psi_*\mathcal{O}_W$ d'une $\mathbb{T}$-variété affine normale $X[Y, \mathfrak{D}]$ en fonction de la donnée combinatoire, voir le Théorème 4.3.3.

Théorème J. *Soit $X = X[Y, \mathfrak{D}]$, où $\mathfrak{D}$ est un diviseur σ-polyédral SNC sur Y. Si $\psi : W \to X$ est une désingularisation, alors pour tout $i \geq 0$, l'image directe supérieure $R^i\psi_*\mathcal{O}_W$ est le faisceau associé à*

$$\bigoplus_{u \in \sigma_M^\vee} H^i(Y, \mathcal{O}(\mathfrak{D}(m)))$$

Une variété normale X a des singularités rationnelles si $R^i\psi_*\mathcal{O}_W = 0$ pour tout $i \geq 1$, voir e.g., [Art66; KKMS73; Elk78]. Dans le théorème suivant on applique le Théorème J pour donner un critère pour que X ait des singularités rationnelles.

Théorème K. *Soit $X = X[Y, \mathfrak{D}]$, où $\mathfrak{D}$ est un diviseur σ-polyédral SNC sur Y. Alors X a des singularités rationnelles si et seulement si pour tout $m \in \sigma_M^\vee$*

$$H^i(Y, \mathcal{O}_Y(\mathfrak{D}(m))) = 0, \quad \forall i \in \{1, \ldots, \dim Y\}\,.$$

La restriction du Théorème K au cas $m = 0$ montre que si $X[Y, \mathfrak{D}]$ a des singularités rationnelles, alors le faisceau structural $\mathcal{O}_Y$ de Y est acyclique. De plus, dans le cas de complexité un, on a un résultat plus explicite.

Corollaire L. *Si Y est une courbe lisse, alors X a des singularités rationnelles si et seulement si*

(i) *Y est affine, ou*

(ii) *$Y = \mathbb{P}^1$ et $\deg\lfloor\mathfrak{D}(m)\rfloor \geq -1$ pour tout $m \in \sigma_M^\vee$.*

Les singularités rationnelles sont de Cohen-Macaulay. Rappelons qu'un anneau local est de Cohen-Macaulay si sa dimension de Krull est égale à sa profondeur. Une variété X est dite de Cohen-Macaulay si tous les anneaux locaux $\mathcal{O}_{X,x}$ sont de Cohen-Macaulay, voir la Section 1.6.

Soit comme avant $\psi : W \to X$ une désingularisation de X. D'après un théorème bien connu de Kempf (voir le Lemme 4.3.6), une variété X a des singularités rationnelles si et seulement si X est de Cohen-Macaulay et le comorphisme induit $\psi_*\omega_W \hookrightarrow \omega_X$ est un isomorphisme. On applique ce théorème pour démontrer le résultat suivant.

Théorème M. *Soit $X = X[Y, \mathfrak{D}]$, où $\mathfrak{D}$ est un diviseur σ-polyédral propre sur Y. On suppose que les conditions suivantes sont satisfaites.*

(i) *Pour chaque mur $\tau \subseteq \sigma^\vee$, le diviseur $\mathfrak{D}(m)$ est abondant pour tout $m \in \operatorname{rel.int}(\tau)$.*

(ii) *Pour chaque diviseur premier Z sur Y et chaque sommet p de Δ_Z, le diviseur $\mathfrak{D}(m)|_Z$ est abondant pour tout $m \in \operatorname{rel.int}(\operatorname{cone}((\Delta_Z - p)^\vee))$.*

Alors X est de Cohen-Macaulay si et seulement si X a des singularités rationnelles.

Dans le cas de complexité un, la condition (ii) dans le Théorème M est toujours satisfaite. D'après le Théorème M on obtient le corollaire suivant qui caractérise les singularités de Cohen-Macaulay isolées dans le cas de complexité un.

Corollaire N. *Soit $X = X[Y, \mathfrak{D}]$, où Y est une courbe lisse. On suppose que X n'a que des singularités isolées.*

(i) *Si $\operatorname{rang} M = 1$, alors X est de Cohen-Macaulay.*

(ii) *Si $\operatorname{rang} M \geq 2$, alors X est de Cohen-Macaulay si et seulement si X a des singularités rationnelles.*

Une singularité normale (X, x) d'une surface est dite elliptique si $R^1\psi_*\mathcal{O}_W = \mathbf{k}$, voir e.g., [Lau77; Wat80; Yau80]. Une singularité elliptique est minimale si elle est de Gorenstein c'est-à-dire, elle est de Cohen-Macaulay et le faisceau canonique ω_X est inversible.

Dans la Proposition 4.4.2 on donne un critère pour qu'une surface munie d'une action de $\mathbf{k}^*$ soit de Gorenstein. L'énoncé de ce résultat est trop technique pour être inclus dans cette introduction. Dans le Théorème 4.4.3 on caractérise les singularités elliptiques (minimales) en fonction de la donnée combinatoire. Ici, on énonce juste la partie concernant les singularités elliptiques.

Soit $\operatorname{rang} M = 1$ et $X = X[Y, \mathfrak{D}]$, où Y est une courbe lisse et $\mathfrak{D}$ est un diviseur σ-polyédral SNC sur Y. Si Y est affine, alors X a des singularités rationnelles. Dans la suite on suppose que Y est projective c'est-à-dire, que l'action est elliptique. Dans ce cadre on peut supposer que $\sigma = \mathbb{Q}_{\geq 0}$, et donc $\mathfrak{D}$ est complètement déterminé par $\mathfrak{D}_1 := \mathfrak{D}(1)$. De plus, il y a un unique point fixe $\bar{0}$.

Théorème O. *Soit $X = X[Y, \mathfrak{D}]$ une surface affine normale munie d'une action elliptique de $\mathbf{k}^*$, et soit $\bar{0} \in X$ l'unique point fixe. Alors $(X, \bar{0})$ est une singularité elliptique si et seulement si l'une de conditions suivantes est satisfaite.*

(i) $Y = \mathbb{P}^1$, $\deg\lfloor m\mathfrak{D}_1\rfloor \geq -2$ *et* $\deg\lfloor m\mathfrak{D}_1\rfloor = -2$ *pour exactement un* $m \in \mathbb{Z}_{>0}$.

(ii) Y *est une courbe elliptique, et pour tout* $m \in \mathbb{Z}_{>0}$ *le diviseur* $\lfloor m\mathfrak{D}_1\rfloor$ *n'est pas principal et* $\deg\lfloor m\mathfrak{D}_1\rfloor \geq 0$.

CHAPTER 1

Combinatorial description of $\mathbb{T}$-varieties

By a $\mathbb{T}$-variety we mean an algebraic variety endowed with an effective action of an algebraic torus $\mathbb{T}$. In this chapter we recall a combinatorial description of torus actions on normal algebraic varieties with special emphasis in affine varieties. The exposition is divided in two cases: the case of toric varieties, and the case of general $\mathbb{T}$-varieties. Obviously the later contains the former, however, the theory in the toric case is more developed.

Before getting into the announced description of $\mathbb{T}$-varieties, we recall the basic definitions and results from convex geometry and semigroup algebras that will be needed in this thesis. We also give a brief review on the classification of toric singularities.

1.1. Convex geometry

In this section we recall the standard terminology and the basic facts of convex geometry needed in this thesis. The proofs of these facts can be found in any book on convex or toric geometry, such as [Oda88; Ful93; Ewa96; CLS].

Let N be a lattice of rank n and $M = \operatorname{Hom}(N, \mathbb{Z})$ be its dual lattice. We also let $N_\mathbb{Q} = N \otimes \mathbb{Q}$, $M_\mathbb{Q} = M \otimes \mathbb{Q}$ be the corresponding rational vector spaces, and we consider the natural duality $M_\mathbb{Q} \times N_\mathbb{Q} \to \mathbb{Q}$, $(m, p) \mapsto \langle m, p\rangle$.

We define the *Minkowski sum* of two sets Δ and Δ' in $N_\mathbb{Q}$ as

$$\Delta + \Delta' = \left\{p + p' \mid p \in \Delta,\ p' \in \Delta'\right\}$$

For a finite set $S \subseteq N_\mathbb{Q}$ we define the *convex polyhedral cone* σ spanned by S as the positive span of S i.e.,

$$\sigma = \operatorname{cone}(S) = \left\{\sum \alpha_p p \mid p \in S,\ \alpha_p \geq 0\right\},$$

and the *bounded convex polyhedron* Δ of S as the convex hull of S [1] i.e.,

$$\Delta = \operatorname{conv}(S) = \left\{\sum \alpha_p p \mid p \in S,\ \alpha_p \geq 0,\ \sum \alpha_p = 1\right\}.$$

Remark 1.1.1. The previous definitions also make sense for an infinite set S provided that the sums are taken to be finite and there is a finite subset $S_0 \subseteq S$ such that $\operatorname{cone}(S) = \operatorname{cone}(S_0)$ and $\operatorname{conv}(S) = \operatorname{conv}(S_0)$.

A *convex polyhedron* is the Minkowski sum of a convex polyhedral cone and a bounded convex polyhedron. By this definition, convex polyhedral cones and bounded convex polyhedra are also convex polyhedra.

Since we only consider convex polyhedral sets, we usually refer to convex polyhedral cones and convex polyhedra simply as cones and polyhedra, respectively.

[1] This is the usual definition of a convex polytope. In the interest of homogeneity in the notation, we do not use this notation.

1.1.1. Convex polyhedral cones. Let σ be a cone in $N_{\mathbb{Q}}$, σ is called *full dimensional* if the topological dimension of σ coincides with the rank of M. σ is called *pointed* if it contains no subspaces of positive dimension. Furthermore, a cone is called *regular* (resp. *simplicial*) if the set of primary vectors of its rays can be completed into a basis of N (resp. $N_{\mathbb{Q}}$).

Given a cone $\sigma \in N_{\mathbb{Q}}$, its *dual cone* is defined by

$$\sigma^{\vee} = \{m \in M_{\mathbb{Q}} \mid \langle m, \sigma \rangle \geq 0\} .$$

The cone $\sigma^{\vee}$ is also a convex polyhedral cone, and duality is reflexive i.e., $\sigma = (\sigma^{\vee})^{\vee}$. The cone $\sigma^{\vee}$ is full dimensional if and only if σ is pointed.

The *relative interior* $\operatorname{rel.int}(\sigma)$ of a cone σ is the topological interior of σ in the vector space spanned by σ. A *supporting hyperplane* of σ is an hyperplane H such that $H \cap \operatorname{rel.int}(\sigma) = \emptyset$ and a *supporting halfspace* H^+ of σ is the halfspace delimited by a supporting hyperplane that contains σ. Given any $m \in \sigma^{\vee}$ the sets H_m and H_m^+ defined as follows

$$H_m = \{p \in N_{\mathbb{Q}} \mid \langle m, p \rangle = 0\}, \qquad H_m^+ = \{p \in N_{\mathbb{Q}} \mid \langle m, p \rangle \geq 0\}$$

are a supporting hyperplane and a supporting halfspace, respectively. Furthermore, every supporting hyperplane (halfspace) arises in this way.

A *face* of σ is the intersection of σ with a supporting hyperplane. A *facet* of σ is a face of codimension 1. A *ray* of σ is a face of dimension 1. By duality, there is a bijective correspondence between rays $\rho \in \sigma$ and facets $\tau \in \sigma^{\vee}$ given by $\tau = \rho^{\perp} \cap \sigma^{\vee}$, where $\rho^{\perp}$ denotes the subspace of $M_{\mathbb{Q}}$ orthogonal to ρ. A cone σ is pointed if and only if $0 \in N$ is a face of σ.

A *quasifan* Σ in $\mathbb{N}_{\mathbb{Q}}$ is a finite collection of cones such that

(*i*) For all $\sigma \in \Sigma$, each face of σ is also in Σ.
(*ii*) For all $\sigma, \sigma' \in \Sigma$, the intersection $\sigma \cap \sigma'$ is a face of each.

Furthermore, a *fan* is a quasifan satisfying

(*iii*) Every cone $\sigma \in \Sigma$ is pointed.

A quasifan is completely determined by the set of its maximal cones.

1.1.2. Tailed polyhedra. Let σ be a pointed cone in $N_{\mathbb{Q}}$. We say that a polyhedron in $N_{\mathbb{Q}}$ is *σ-tailed* if it can be decomposed as the Minkowski sum of a bounded polyhedron and σ. A σ-polyhedron is called *full dimensional* if its topological dimension coincides with the rank of M. If σ is full dimensional, then any σ-polyhedron is full dimensional.

We define $\operatorname{Pol}_{\sigma}(N_{\mathbb{Q}})$ to be the set of all σ-tailed polyhedra in $N_{\mathbb{Q}}$. The set $\operatorname{Pol}_{\sigma}(N_{\mathbb{Q}})$ equipped with the Minkowski sum forms a commutative semigroup with neutral element σ.

We let also $\operatorname{CPL}_{\mathbb{Q}}(\sigma^{\vee})$ denote the set of all piecewise linear $\mathbb{Q}$-valued functions $h : \sigma^{\vee} \to \mathbb{Q}$ which are concave and positively homogeneous i.e.,

$$h(m + m') \geq h(m) + h(m'), \text{ and } h(\lambda m) = \lambda h(m), \forall m, m' \in \sigma^{\vee}, \ \forall \lambda \in \mathbb{Q}_{\geq 0} .$$

The set $\operatorname{CPL}_{\mathbb{Q}}(\sigma^{\vee})$ with the usual addition forms a commutative semigroup with neutral element 0.

For a σ-tailed polyhedron $\Delta \in \operatorname{Pol}_{\sigma}(N_{\mathbb{Q}})$ we define its support function

$$h_{\Delta} : \sigma^{\vee} \to \mathbb{Q}, \quad m \mapsto \min_{p \in \Delta} \langle m, p \rangle .$$

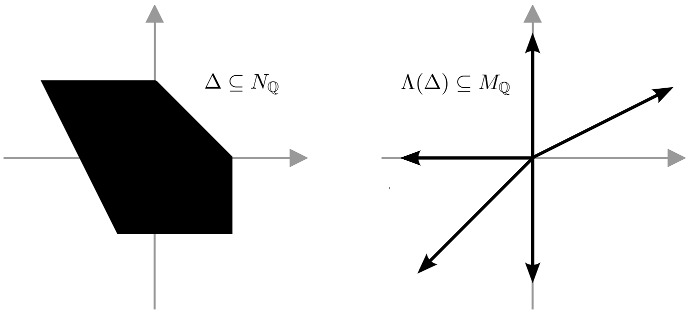


FIGURE 1. The σ-polyhedron Δ and its normal fan.

The support function of any σ-tailed polyhedron is piecewise linear, positively homogeneous and convex. Furthermore, the map $\mathrm{Pol}_\sigma(N_{\mathbb{Q}}) \to \mathrm{CPL}_{\mathbb{Q}}(\sigma^\vee)$ given by $\Delta \mapsto h_\Delta$ is an isomorphism of semigroups.

For a function $h \in \mathrm{CPL}_{\mathbb{Q}}(\sigma^\vee)$ we define its *normal quasifan* $\Lambda(h)$ as the coarsest refinement of the quasifan of $\sigma^\vee$ in $M_{\mathbb{Q}}$ such that h is linear in each cone $\delta \in \Lambda(h)$. For a σ-polyhedron $\Delta \in \mathrm{Pol}_\sigma(N_{\mathbb{Q}})$ we define its normal quasifan $\Lambda(\Delta)$ as the normal quasifan of the support function h_Δ. The normal quasifan of Δ is a fan if and only if Δ is full dimensional.

Alternatively, the normal quasifan of a σ-polyhedron $\Delta \in N_{\mathbb{Q}}$ can be obtained as follows. Given a vertex $v \in \Delta$, we define the cones

$$\sigma_v = \mathrm{cone}(\Delta - v) \subseteq N_{\mathbb{Q}} \quad \text{and} \quad \omega_v = \sigma_v^\vee \in M_{\mathbb{Q}}\,.$$

Now, the set of cones ω_v, for all vertex v is the set of maximal cones of the normal quasifan of Δ in $M_{\mathbb{Q}}$.

EXAMPLE 1.1.2. Letting $N = \mathbb{Z}^2$ and $\sigma = \{(0,0)\}$, in $N_{\mathbb{Q}} = \mathbb{Q}^2$ we consider the polyhedron Δ with vertices $(2,0)$, $(0,2)$, $(-3,2)$, $(-1,-2)$ and $(2,-2)$. The cones σ_v and ω_v are

$$\begin{aligned}
\sigma_{(2,0)} &= \mathrm{cone}((0,-1),(-1,1)), & \omega_{(2,0)} &= \mathrm{cone}((-1,0),(-1,-1)),\\
\sigma_{(0,2)} &= \mathrm{cone}((1,-1),(-1,0)), & \omega_{(0,2)} &= \mathrm{cone}((-1,-1),(0,-1)),\\
\sigma_{(-3,2)} &= \mathrm{cone}((1,0),(1,-2)), & \omega_{(-3,2)} &= \mathrm{cone}((0,-1),(2,1)),\\
\sigma_{(-1,-2)} &= \mathrm{cone}((-1,2),(1,0)), & \omega_{(-1,-2)} &= \mathrm{cone}((2,1),(0,1)),\\
\sigma_{(2,-2)} &= \mathrm{cone}((-1,0),(0,1)), & \omega_{(2,-2)} &= \mathrm{cone}((0,1),(-1,0))\,.
\end{aligned}$$

The σ-polyhedron Δ and its normal quasifan, which is a fan in this case, are shown in Figure 1.

1.2. Semigroup algebras

In this section we gather some basic results about semigroup algebras needed for this thesis. A more detailed exposition can be found in any book on toric geometry, see for instance [Ful93; Oda88; CLS].

Let $(S,+)$ be a commutative semigroup with an identity element[2]. If the binary operation in S is clear from the context we denote $(S,+)$ by S. We define the

[2]Some authors refer to $(S,+)$ as a commutative monoid.

semigroup algebra of $(S,+)$ as the $\mathbf{k}$-algebra

$$\mathbf{k}[S]=\bigoplus_{m\in S}\mathbf{k}\cdot\chi^m$$

where χ^m is a new variable for every $m\in S$, and the multiplicative structure of $\mathbf{k}[S]$ is given by the relations

$$\chi^m\cdot\chi^{m'}=\chi^{m+m'},\quad \text{for all}\quad m,m'\in S\,.$$

REMARK 1.2.1.

(*i*) If S is a group under the considered operation this construction coincides with the group algebra studied in group theory.

(*ii*) Given a semigroup S and an algebra A, we can define an algebra $A[S]$ in the same way.

Let $\beta\subset S$ be a generating set of S i.e.,

$$S=\left\{\sum\alpha_s s \text{ finite}\mid s\in\beta \text{ and } \alpha_s\in\mathbb{Z}_{\geq 0}\right\}\,.$$

It is clear from the definition of a semigroup algebra that $\mathbf{k}[S]$ is generated as a $\mathbf{k}$-algebra by the elements $\chi^\beta:=\{\chi^s\mid s\in\beta\}$. Furthermore, if the generating set β is minimal, then χ^β is a minimal generating set of $\mathbf{k}[S]$ as a $\mathbf{k}$-algebra.

EXAMPLES 1.2.2. We present a list of basic examples to cover a variety of different possibilities illustrating the definitions below.

(*i*) Let $S=(\mathbb{Z}_{\geq 0},+)$. S is generated by the element 1 and so the semigroup algebra $\mathbf{k}[S]$ is generated as an algebra by χ^1. The element χ^1 satisfies no non-trivial polynomial relation, thus $\mathbf{k}[S]$ is the polynomial algebra in the variable χ^1.

(*ii*) Let $S=\{0,2,3,\ldots\}$ with addition as the binary operation. A generating set of $\mathbf{k}[S]$ is $\{2,3\}$ and so the algebra $\mathbf{k}[S]$ is generated by $x=\chi^2$ and $y=\chi^3$. Furthermore, this elements satisfy $x^3-y^2=0$ and so the $\mathbf{k}[S]=\mathbf{k}[x,y]/(x^3-y^2)$.

(*iii*) Let $S=(\mathbb{Z}_r,+)$ be the group of integers modulo r. S is spanned as a semigroup by the element $[1]$ (the class of 1) and so $\mathbf{k}[S]$ is generated by $x=\chi^{[1]}$. The element x satisfies the relation $x^r-1=0$ and so $\mathbf{k}[\mathbb{Z}_r]=\mathbf{k}[x]/(x^r-1)$.

(*iv*) Let $S=\mathbb{Z}_{>0}$ with the multiplication as binary operation. By the unique factorization theorem on $\mathbb{Z}$, the semigroup S is generated by the set β of positive prime numbers, and $\mathbf{k}[S]$ is generated by χ^β. Again by the unique factorization, there are no non-trivial relations between the elements of χ^β and so $\mathbf{k}[S]$ is a polynomial algebra in infinitely many variables indexed by the positive prime numbers.

(*v*) Let $S=(\mathbb{Z},+)$. A generating set of S is $\{1,-1\}$ and so $\mathbf{k}[S]$ is generated by $x=\chi^1$ and $y=\chi^{-1}$. These elements satisfy the relation $xy-1=0$ and so

$$\mathbf{k}[S]=\mathbf{k}[x,y]/(xy-1)=\mathbf{k}[x,x^{-1}]\,.$$

The algebra $\mathbf{k}[S]$ is just the algebra of Laurent polynomials in one variable.

(*vi*) Let M be a lattice of rank n and let $\{\mu_1,\cdots,\mu_n\}$ be a base of M as a free $\mathbb{Z}$-module. The addition as a module induces a structure of semigroup on M, and M is generated as a semigroup by the elements $\beta = \{\pm\mu_1, \cdots, \pm\mu_n\}$. The algebra $\mathbf{k}[M]$ is generated by χ^β. These elements satisfy the relations

$$\chi^{\mu_i}\chi^{-\mu_i} - 1 = 0, \qquad i \in \{1, \cdots, m\}$$

and so

$$\mathbf{k}[M] = \mathbf{k}[\chi^{\pm\mu_1}, \cdots, \chi^{\pm\mu_n}]/(\chi^{\mu_1}\chi^{-\mu_1} - 1, \cdots, \chi^{\mu_n}\chi^{-\mu_n} - 1)\,.$$

Letting $x_i = \chi^{\mu_i}$ for $i \in \{1, \cdots, n\}$, $\mathbf{k}[M]$ can be written as

$$\mathbf{k}[M] = \mathbf{k}[x_1, x_1^{-1}, \cdots, x_n, x_n^{-1}]\,.$$

This algebra is known as the algebra of Laurent polynomials in n variables. There is a splitting of $\mathbf{k}[M]$ as

$$\mathbf{k}[M] = \bigotimes_{i=1}^{n} \mathbf{k}\left[x_i, x_i^{-1}\right]\,.$$

Definition 1.2.3. A commutative semigroup with an identity element is *affine* if it is finitely generated and can be embedded in a lattice M. An affine semigroup S is *saturated* if for all $k \in \mathbb{Z}_{\geq 0}$ and $m \in M$, $k \cdot m \in S$ implies $m \in S$.

The semigroups in Example 1.2.2 (i), (ii), (v), and (vi) are affine semigroups since they are trivially embedded in a lattice and all of these but the one in (ii) are saturated. In contrast, the semigroup in (iii) has torsion and hence it cannot be embedded in a lattice, which is torsion free. Finally, the semigroup in (iv) fails to be finitely generated.

Let S be a semigroup. The properties of being affine or saturated can be translated into well known properties of the algebra $\mathbf{k}[S]$. This is done in the following lemma.

Lemma 1.2.4.

(i) *Let S be a semigroup. If S is affine then $\mathbf{k}[S]$ is an affine domain.*
(ii) *Let S be an affine semigroup. Then S is saturated if and only if $\mathbf{k}[S]$ is an integrally closed domain.*

Proof. See [CLS, Prop. 1.1.14] for the first assertion and [CLS, Th. 1.3.5] for the second one. □

Let as before M be a lattice of rank n and let $M_\mathbb{Q} = M \otimes \mathbb{Q}$ be the associated rational vector space. Let $\omega \subseteq M_\mathbb{Q}$ be a full dimensional convex polyhedral cone. Then the set of lattice points $\omega \cap M$ forms a saturated affine semigroup.

Proposition 1.2.5. *For every saturated affine semigroup S there exists a lattice M and a full dimensional polyhedral cone $\omega \subset M_\mathbb{Q}$ such that $S \simeq \omega \cap M$.*

Proof. For a proof of this proposition, see Theorem 1.3.5 in [CLS]. □

Notation 1.2.6. Let $\omega \subseteq M_\mathbb{Q}$ be a convex polyhedral cone. Throughout the thesis we denote the semigroup $\omega \cap M$ by ω_M.

The class of saturated affine semigroups will be crucial to this thesis. Given a saturated affine semigroup ω_M, the affine variety $\operatorname{Spec}\omega_M$ is called an affine toric variety. Such varieties are the main object of our study in Section 1.4.

1.3. Algebraic torus actions

In this section we fix the notation and recall some general facts about the actions of algebraic tori on normal varieties. For a more general view on torus actions, or more generally, on linear group actions see [CLS; Hum75; Bor91].

1.3.1. Algebraic tori. Let M be a lattice of rank n and $N = \operatorname{Hom}(M, \mathbb{Z})$ be its dual lattice. We fix dual bases $\{\nu_1, \cdots, \nu_n\}$ and $\{\mu_1, \cdots, \mu_n\}$ for N and M, respectively. We also let $N_\mathbb{Q} = N \otimes \mathbb{Q}$, $M_\mathbb{Q} = M \otimes \mathbb{Q}$, and we consider the natural duality $M_\mathbb{Q} \times N_\mathbb{Q} \to \mathbb{Q}$, $(m, p) \mapsto \langle m, p \rangle$.

Definition 1.3.1. The *algebraic torus* associated to M is defined as the affine algebraic variety $\mathbb{T}_M = \operatorname{Spec} \mathbf{k}[M]$.

We usually refer to an algebraic torus simply as a torus and we denote it by $\mathbb{T}$ when the lattice M is obvious from the context. Letting $x_i = \chi^{\mu_i}$, $\forall i \in \{1, \dots, n\}$, by Example 1.2.2 (vi)

$$\mathbf{k}[M] = \bigoplus_{m \in M} \mathbf{k}\chi^m = \bigotimes_{i=1}^{n} \mathbf{k}\left[x_i, x_i^{-1}\right] .$$

Thus, the torus $\mathbb{T}$ is isomorphic to $(\mathbf{k}^*)^n$. Via this isomorphism, the coordinate-wise multiplication on $(\mathbf{k}^*)^n$ induces the structure of a linear algebraic group on $\mathbb{T}$.

A *character*[3] of the torus $\mathbb{T}$ is a morphism $\chi : \mathbb{T} \to \mathbf{k}^*$ that is also a group homomorphism. For every $m \in M$ the regular function χ^m gives a character of the torus. Conversely, every character arises in this way (see [Hum75, §16]). Thus, the character group of $\mathbb{T}$ is naturally isomorphic to the lattice M.

More explicitly, for any $m = m_1\mu_1 + \dots m_n\mu_n \in M$, under the isomorphism $\mathbb{T} \simeq (\mathbf{k}^*)^n$ the character $\chi^m : \mathbb{T} \to \mathbf{k}^*$ is given by $(x_1, \dots, x_n) \mapsto x_1^{m_1} \cdots x_n^{m_n}$.

A *one-parameter subgroup* of the torus $\mathbb{T}$ is a morphism $\lambda : \mathbf{k}^* \to \mathbb{T}$ that is also a group homomorphism. Equivalently, a one-parameter subgroup is given by the comorphism $\lambda^* : \mathbf{k}[M] \to \mathbf{k}[t, t^{-1}]$. For every $p \in N$ the morphism $\lambda_p^* : \mathbf{k}[M] \to \mathbf{k}[t, t^{-1}]$ given by $\chi^m \mapsto t^{\langle m,p \rangle}$ is the comorphism of a one-parameter subgroup. Conversely, every one-parameter subgroup arises in this way (see [Hum75, §16]). Therefore, the group of one-parameter subgroups is naturally isomorphic to the lattice N.

More explicitly, for any $p = p_1\nu_1 + \dots p_n\nu_n \in N$, under the isomorphism $\mathbb{T} \simeq (\mathbf{k}^*)^n$ the one-parameter subgroup $\lambda_p : \mathbf{k}^* \to \mathbb{T}$ is given by $t \mapsto (t^{p_1}, \dots, t^{p_n})$.

Given a character χ^m and a one-parameter subgroup λ_p, the composition $\chi^m \circ \lambda_p : \mathbf{k}^* \to \mathbf{k}^*$ is given by $t \mapsto t^{\langle m,p \rangle}$.

1.3.2. Torus actions. Let G be an algebraic group and let X be an algebraic variety. An (algebraic) action of the group G on X is a group homomorphism $\phi : G \to \operatorname{Aut}(X)$ such that the map $G \times X \to X$, sending (g, x) to $\phi(g)(x)$ is a morphism. A G-action is called *non-trivial* if $\operatorname{coker}(\phi) \neq \operatorname{Aut}(X)$, *effective* if $\ker(\phi) = \{1\}$, and *locally free* if the stabilizer of a general point is trivial.

Let now X, X' be two varieties endowed with a G-action. We say that a morphism $\theta : X \to X'$ is called *G-equivariant* if

$$\theta(\phi(g)(x)) = \phi(g)(\theta(x)), \quad \text{for all} \quad x \in X,\ g \in G .$$

[3]This is a particular case of the more general notion of a character of group.

The morphism $\theta : X \to X'$ is called *G-invariant* if it is G-equivariant for the trivial G-action on X'.

Let $\mathbb{T} = \operatorname{Spec} \mathbf{k}[M]$ be an algebraic torus and let ϕ be a non-trivial $\mathbb{T}$-action on an algebraic variety X. Without loss of generality, we may assume that the action is effective and locally free. Indeed, if the $\mathbb{T}$-action is not effective, we may replace $\mathbb{T}$ by its quotient modulo $\ker(\phi)$. This is again a torus and the new action is effective. Furthermore, any effective $\mathbb{T}$-action is locally free.

DEFINITION 1.3.2. A $\mathbb{T}$-*variety* is an algebraic variety endowed with an effective action of the algebraic torus $\mathbb{T}$.

For a $\mathbb{T}$-action on an algebraic variety X, the *complexity* is defined as the codimension of a general orbit. If the $\mathbb{T}$-action is effective the complexity is $\dim X - \dim \mathbb{T}$. The complexity of a $\mathbb{T}$-action is also given by $\operatorname{tr.deg}_{\mathbf{k}}(\mathbf{k}(X)^{\mathbb{T}})$, where $\mathbf{k}(X)^{\mathbb{T}}$ in the field of $\mathbb{T}$-invariant rational functions.

In particular, a $\mathbb{T}$-variety of complexity zero is a $\mathbb{T}$-variety having an open orbit. Since any effective $\mathbb{T}$-action is locally free, it follows that a $\mathbb{T}$-variety X of complexity zero corresponds to a $\mathbb{T}$-equivariant embedding of the torus $\mathbb{T} \hookrightarrow X$. Here, we regard the torus $\mathbb{T}$ as a $\mathbb{T}$-variety with the action by multiplication.

The notion of a quotient of an algebraic variety by a torus action (or more generally by an algebraic group action) is rather delicate, and will not be developed here, see [MFK94]. Nevertheless, we will need two different definitions of a quotient.

DEFINITION 1.3.3. Let X be a $\mathbb{T}$-variety. A *rational quotient* of the $\mathbb{T}$-action on X is a $\mathbb{T}$-invariant rational morphism $r : X \dashrightarrow W$ such that the comorphism $r^* : \mathbf{k}(W) \hookrightarrow \mathbf{k}(X)$ induces an isomorphism $\mathbf{k}(W) \simeq \mathbf{k}(X)^{\mathbb{T}}$. A *good quotient* is a $\mathbb{T}$-invariant affine morphism $q : X \to W$ such that the natural morphism $\mathcal{O}_W \to q_*(\mathcal{O}_X)^{\mathbb{T}}$ is an isomorphism.

For a $\mathbb{T}$-action, there always exists a rational quotient. In contrast, the existence of a good quotient imposes strong restrictions on a $\mathbb{T}$-action.

In the sequel, we restrict to normal $\mathbb{T}$-varieties. The study of the local behavior of normal $\mathbb{T}$-varieties can be restricted to the study of affine $\mathbb{T}$-varieties due to the following theorem.

THEOREM 1.3.4 (Sumihiro). *Let X be a normal $\mathbb{T}$-variety. Then every point $x \in X$ has a $\mathbb{T}$-invariant affine open neighborhood.*

PROOF. See [Sum74] for a proof, cf. [Sum75]. □

REMARK 1.3.5. The condition that X is normal is essential in Theorem 1.3.4. Indeed, let $C \subseteq \mathbb{P}^2$ be the nodal cubic defined by the equation $y^2 z = x^2(x + z)$. By the Jacobian criterion C has a unique singular point at $P = (0 : 0 : 1)$. The complement of this point is isomorphic to $\mathbf{k}^*$ and the action of $\mathbf{k}^* = \mathbb{T}_1$ on itself given by multiplication extends to an action of $\mathbb{T}_1$ on C with P a unique fixed point. Any $\mathbb{T}_1$-invariant open neighborhood of P contains P and $\mathbb{T}_1$ and hence it is the whole curve C which is not affine.

By Theorem 1.3.4, any normal $\mathbb{T}$-variety has a $\mathbb{T}$-invariant affine open covering. Hence, a description of normal $\mathbb{T}$-varieties can be obtained by addressing the following two problems.

(i) Describe normal affine $\mathbb{T}$-varieties; and
(ii) describe a way to patch them together.

Since we deal with affine varieties, we will mainly address the first problem. The second problem will only be studied in the case of toric varieties. We refer the reader to [AHS08] for a general treatment of the second problem that uses the combinatorial methods which will be explained in Section 1.5, see also [FKZ07].

1.3.3. Torus actions on affine varieties. In this section we show that the algebra of regular functions of an affine $\mathbb{T}$-variety is naturally graded by M. Let us first give a definition of graded algebra adapted to our setting.

Definition 1.3.6. Let S be a semigroup and let A be an algebra. We say that A is an *S-graded algebra* if there exists a direct sum decomposition

$$A = \bigoplus_{s \in S} A_s$$

such that $A_s \cdot A_{s'} \subseteq A_{s+s'}$ for all $s, s' \in S$.

The simplest example of an S-graded algebra is the semigroup algebra $\mathbf{k}[S]$.

Let S be a saturated affine semigroup. By Proposition 1.2.5, there exist a lattice M and a cone $\omega \subseteq M_{\mathbb{Q}}$ such that S is isomorphic to ω_M. In this setting, any ω_M-graded algebra is also an M-graded algebra by setting $A_m = 0$ for all $m \notin \omega_M$. Since all the algebras in this thesis are graded by a saturated affine semigroup ω_M, we will follow the convention of saying that they are graded by the lattice M.

We turn now to affine $\mathbb{T}$-varieties. Recall that the algebra of regular functions of the torus $\mathbb{T}$ is canonically isomorphic to the semigroup algebra of its character lattice M.

Theorem 1.3.7. *Let $X = \operatorname{Spec} A$, where A is an affine domain. Then there is a bijective correspondence between the $\mathbb{T}$-actions on X and the M-gradings on A.*

Proof. If $\mathbb{T} \times X \to X$ is a $\mathbb{T}$-action, then the correspondence is given by pulling back the natural M grading on $A[M]$ by the comorphism $A \to \mathbf{k}[M] \otimes A = A[M]$. See [KR82] for more details. □

Definition 1.3.8. We say that an M-grading on an algebra A is *effective* if the set $\{m \in M : A_m \neq 0\}$ is not contained in a proper sublattice of M. A $\mathbb{T}$-action is effective if an only if the corresponding M-grading is effective.

We consider an effectively M-graded affine domain

$$A = \bigoplus_{m \in M} \tilde{A}_m \,,$$

and we let $K = \operatorname{Frac} A$. For any $m \in M$ we define

$$K_m = \left\{ f/g \in K \mid f \in \tilde{A}_{m+e},\, g \in \tilde{A}_e \right\} .$$

If $f/g \in K_0 \setminus \{0\}$ then the same hold for g/f and so K_0 is a field. Clearly, K_0 corresponds to the field of $\mathbb{T}$-invariant rational functions, so we will denote K_0 by $K^{\mathbb{T}}$. There is a tower of field extensions $\mathbf{k} \subseteq K^{\mathbb{T}} \subseteq K$.

Recall that $\{\mu_1, \ldots, \mu_n\}$ is a basis of M, we fix for every $i \in \{1, \ldots, n\}$ an element $\chi^{\mu_i} \in K_{\mu_i}$ and we let

$$\chi^m = \prod_i (\chi^{\mu_i})^{m_i}, \quad \text{where } m = \sum_i m_i \mu_i \in M \,.$$

By the definition of K_m we have $K_m = \chi^m K^{\mathbb{T}}$, and since $\tilde{A}_m \subseteq K_m$ we can write $\tilde{A}_m = A_m\chi^m$, where $A_m \subseteq K^{\mathbb{T}}$. Thus, without loss of generality, we assume in the sequel that

$$A = \bigoplus_{m\in M} A_m\chi^m, \quad \text{where} \quad A_m \subseteq K^{\mathbb{T}}\,.$$

Recall that the complexity of the $\mathbb{T}$-action equals the transcendence degree of $K^{\mathbb{T}}$ over $\mathbf{k}$. In particular, for a $\mathbb{T}$-variety X of complexity zero $K^{\mathbb{T}} = \mathbf{k}$ and so $A \subseteq \mathbf{k}[M]$. Since the torus $\mathbb{T}$ is an open subset in X, χ^m can be chosen to be a character of $\mathbb{T}$ regarded as a rational function on X. More generally, in arbitrary complexity the algebra A is contained in the semigroup $K^{\mathbb{T}}$-algebra $K^{\mathbb{T}}[M]$, and $\operatorname{Frac} A = \operatorname{Frac} K^{\mathbb{T}}[M]$. Hence χ^m can be chosen to be a character of the $K^{\mathbb{T}}$-torus $\operatorname{Spec} K^{\mathbb{T}}[M]$ regarded as a rational function on X.

Definition 1.3.9. The *weight cone* $\omega \subseteq M_{\mathbb{Q}}$ of A is the cone in $M_{\mathbb{Q}}$ spanned by the set $\{m \in M \mid A_m \neq 0\}$.

Recall that for a cone $\omega \subseteq M_{\mathbb{Q}}$, ω_M stands for the semigroup of lattice points in ω. The algebra A is also graded by the semigroup ω_M, and so we have

$$A = \bigoplus_{m\in \omega_M} A_m\chi^m, \quad \text{where} \quad A_m \subseteq K^{\mathbb{T}}\,. \tag{1}$$

Since A is finitely generated, the cone ω is a convex polyhedral cone and since the M-grading is effective, ω is of full dimension.

Remark 1.3.10.

(*i*) Throughout this thesis, we assume that all the M-graded algebras are in the standard form of the algebra A in (1).

(*ii*) We will sometimes represent $K^{\mathbb{T}}$ by the field of fractions $\mathbf{k}(Y)$, where $X = \operatorname{Spec} A \dashrightarrow Y$ is a rational quotient of the corresponding $\mathbb{T}$-action, see Section 1.5.

(*iii*) With this notation, the field of fractions of A if given by $\operatorname{Frac} A = K^{\mathbb{T}}(M)$, where $K^{\mathbb{T}}(M)$ denotes the field of fractions of the semigroup $K^{\mathbb{T}}$-algebra $K^{\mathbb{T}}[M]$.

1.4. Toric varieties

In the rest of this chapter, we will show several combinatorial descriptions of normal $\mathbb{T}$-varieties. We begin in this section with the simplest case i.e., the case of toric varieties.

A *toric variety* is a normal $\mathbb{T}$-variety of complexity zero. There is a well established theory of toric varieties See e.g., [Dem70], Chapter 1 in [KKMS73], [Dan78], [Oda88], [Ful93], and [CLS]. In this section we review the definitions and results needed for this thesis.

The following proposition gives a combinatorial description of affine toric varieties in terms of convex polyhedral cones.

Proposition 1.4.1. *Let X be a normal affine variety with A as its ring of regular functions. Then X admits the structure of a toric variety if and only if A is isomorphic to the algebra of a saturated affine semigroup.*

Proof. Let ω_M be a saturated affine semigroup and let $A = \mathbf{k}[\omega_M]$. By Theorem 1.3.7 X admits a $\mathbb{T}$-action. In this case, the field $K^{\mathbb{T}} = \mathbf{k}$ and so the complexity of the $\mathbb{T}$-action is $\operatorname{tr.deg}_{\mathbf{k}} K^{\mathbb{T}} = 0$. Finally, X is normal by Lemma 1.2.4.

Conversely, assume that X admits a $\mathbb{T}$-action of complexity zero, then for this action $K^{\mathbb{T}} = \mathbf{k}$ and so by (1)

$$A = \bigoplus_{m\in\omega_M} A_m\chi^m, \quad \text{where} \quad A_m \in \{0, \mathbf{k}\}\,.$$

Furthermore, the normality of A implies that $A_m = \mathbf{k}$, for all $m \in \omega_M$. Thus

$$A = \bigoplus_{m\in\omega_M} \mathbf{k}\chi^m = \mathbf{k}[\omega_M]\,.$$ □

In the particular case of a toric variety the weight cone $\omega \subseteq M_{\mathbb{Q}}$, which completely determines X, corresponds to the cone spanned by all lattice vectors $m \in M$ such that the character $\chi^m : \mathbb{T} \to \mathbf{k}^*$ extends to a regular function on X.

Remark 1.4.2. The usual description of an affine toric variety $X = \operatorname{Spec} \mathbf{k}[\omega_M]$ is by means of the cone $\sigma \subseteq N_{\mathbb{Q}}$ dual to the weight cone $\omega \subseteq M_{\mathbb{Q}}$, this is denoted by $X = X_\sigma$. Of course, these two descriptions are equivalent by duality. The reason for this choice is that general toric varieties are better described in $N_{\mathbb{Q}}$ than in $M_{\mathbb{Q}}$, see the description below.

In the following, we sketch the description of general toric varieties by means of polyhedral fans. For a detailed treatment see any of the references at the beginning of this section.

Let $\sigma \subseteq N_{\mathbb{Q}}$ be a convex polyhedral cone and let $\tau \subseteq \sigma$ be a face of σ. By duality $\omega := \sigma^\vee \subseteq \tau^\vee$ and so $\mathbf{k}[\omega_M] \subseteq \mathbf{k}[\tau^\vee_M]$. This yields an open immersion of toric varieties $X_\tau \hookrightarrow X_\sigma$.

Let now $\Sigma \subseteq N_{\mathbb{Q}}$ be a fan, σ_1 and σ_2 be any two cones in Σ, and $\tau = \sigma_1 \cap \sigma_2$. By the previous analysis, X_τ can be seen as an open set sitting inside X_{σ_1} and X_{σ_2}. We define a scheme X_Σ by gluing the affine varieties X_σ, for all $\sigma \in \Sigma$ along the open sets X_τ defined above.

Theorem 1.4.3. *Let Σ be a fan in $N_{\mathbb{Q}}$. The scheme X_Σ is a normal separated toric variety.*

Proof. See Theorem 3.1.5 in [CLS]. □

The main result that we will need in this thesis about general toric varieties is the following theorem known as the orbit-cone correspondence.

Theorem 1.4.4. *Let N be a lattice of rank n and let X_Σ be the toric variety of a fan Σ in $N_{\mathbb{Q}}$, then there is a bijective correspondence between the cones σ of dimension ℓ in Σ and the $\mathbb{T}$-orbits $\operatorname{orb}(\sigma)$ of dimension $n - \ell$. Moreover, for any cone σ in Σ, the open affine variety X_σ is the union of the orbits*

$$X_\sigma = \bigcup_{\tau \text{ is a face of } \sigma} \operatorname{orb}(\tau)\,.$$

Proof. See Theorem 3.2.6 in [CLS] for a proof. □

1.5. Normal affine $\mathbb{T}$-varieties

In [AH06], Altmann and Hausen gave a combinatorial description of normal affine $\mathbb{T}$-varieties of arbitrary complexity similar to the description of toric varieties by means on convex polyhedral cones. In [AHS08] this description was expanded to describe all normal $\mathbb{T}$-varieties. Their theory generalizes the description of toric varieties given in Section 1.4, as well as many other descriptions of $\mathbb{T}$-varieties given previously under different restrictions.

In [Tim97], a combinatorial description of reductive group actions of complexity one is given and in [Tim08] it is specialized for torus actions. For torus actions of complexity one, the descriptions in [AH06] and [Tim97] are equivalent and agree with the one given earlier in a slightly more restrictive setting by Mumford [KKMS73, Chapters 2 and 4], see [Tim08] and [Vol07] for more details.

Furthermore, the case of $\mathbb{T}_1 = \mathbf{k}^*$-actions was studied in [Dem88] and [Wat81] and the particular case of $\mathbf{k}^*$-surfaces has been treated by several authors, see [Dol75; Pin77; Pin78; FZ03]. This has led to an almost full understanding of $\mathbf{k}^*$-actions on normal surfaces [FKZ09]. In particular, the description of Altmann and Hausen specializes to the ones given previously in [Dem88; FZ03].

In the following we recall the main features of the description of normal affine $\mathbb{T}$-varieties due to Altmann and Hausen.

Let $\mathbf{k}$ be an algebraically closed field of characteristic zero. A variety Y is called *semiprojective* if its algebra of global regular functions $\mathbf{k}[Y]$ is finitely generated and Y is projective over $\operatorname{Spec}\mathbf{k}[Y]$.

NOTATION 1.5.1. For every $a \in \mathbb{Q}$, we denote the integral part of a by $\lfloor a \rfloor$ and the fractional part by $\{a\}$. Similarly, for a $\mathbb{Q}$-divisor $D = \sum_Z a_Z \cdot Z$ on Y, we define the integral part and the fractional part, respectively, by

$$\lfloor D \rfloor = \sum_Z \lfloor a_Z \rfloor \cdot Z, \quad \text{and} \quad \{D\} = \sum_Z \{a_Z\} \cdot Z$$

For any Weil divisor D on Y, the *sheaf of sections* $\mathcal{O}_Y(D)$ is classically defined via

$$U \longrightarrow H^0(U, \mathcal{O}_Y(D)) := \{f \in \mathbf{k}(Y) \mid \operatorname{div}(f|_U) + D|_U \geq 0\}\,.$$

For a $\mathbb{Q}$-divisor D, we can define the sheaf of sections $\mathcal{O}_Y(D)$ in the same way. Obviously, with this definition we have

$$\mathcal{O}_Y(D) = \mathcal{O}_Y(\lfloor D \rfloor)\,.$$

Let as before, N be a lattice of rank n and $M = \operatorname{Hom}(N, \mathbb{Z})$ be its dual lattice. We also let $N_{\mathbb{Q}} = N \otimes \mathbb{Q}$, $M_{\mathbb{Q}} = M \otimes \mathbb{Q}$, and we consider the natural duality $M_{\mathbb{Q}} \times N_{\mathbb{Q}} \to \mathbb{Q}$, $(m,p) \mapsto \langle m, p \rangle$. Let Y be a normal semiprojective variety and σ be a cone in $N_{\mathbb{Q}}$ with dual cone $\omega \in M_{\mathbb{Q}}$.

DEFINITION 1.5.2. A *σ-polyhedral divisor* on Y is a formal sum

$$\mathfrak{D} = \sum_Z \Delta_Z \cdot Z, \quad \text{where} \quad \Delta_Z \in \operatorname{Pol}_\sigma(N_{\mathbb{Q}})\,,$$

and $\Delta_Z = \sigma$ for all but finitely prime divisors Z. Here the sum runs over all prime divisors $Z \subseteq Y$.

Let $\mathfrak{D}$ be a σ-polyhedral divisor. For a prime divisor Z on Y we denote the support function of Δ_Z by $h_Z := h_{\Delta_Z}$. For every $m \in \omega$ we can evaluate $\mathfrak{D}$ in m by letting $\mathfrak{D}(m)$ be the $\mathbb{Q}$-divisor on Y

$$\mathfrak{D}(m) = \sum_Z h_Z(m) \cdot Z\,.$$

Definition 1.5.3. For a σ-polyhedral divisor $\mathfrak{D}$ on Y, we define its *normal quasifan* $\Lambda(\mathfrak{D})$ as the coarsest common refinement of the quasifans $\Lambda(h_Z)$, for all prime divisors $Z \subseteq Y$.

Recall that a $\mathbb{Q}$-Cartier divisor D on Y is called *semiample* if there exists $r > 1$ such that the linear system $|rD|$ is base point free, and *big* if there exists a divisor $D_0 \in |rD|$, for some $r > 1$, such that $Y \setminus \operatorname{Supp} D_0$ is affine.

Definition 1.5.4. A σ-polyhedral divisor is called *proper* if the following hold.

(i) The divisor $\mathfrak{D}(m)$ is $\mathbb{Q}$-Cartier and semiample for all $m \in \omega$, and

(ii) $\mathfrak{D}(m)$ is big for all $m \in \operatorname{rel.int}(\omega)$.

Let $\mathfrak{D} = \sum_Z \Delta_Z \cdot Z$ be a proper σ-polyhedral divisor on Y. The concavity of the support functions h_Z ensures that

$$\mathfrak{D}(m+m') \geq \mathfrak{D}(m) + \mathfrak{D}(m'), \quad \text{for all} \quad m, m' \in \omega_M\,.$$

And so, there exist multiplication maps

$$\mathcal{O}_Y(\mathfrak{D}(m)) \otimes \mathcal{O}_Y(\mathfrak{D}(m')) \to \mathcal{O}_Y(\mathfrak{D}(m+m')), \quad \text{for all} \quad m, m' \in \omega_M\,.$$

This ensures that the sheaves $\mathcal{O}_Y(\mathfrak{D}(m))$, where $m \in \omega_M$ can be put together into a M-graded $\mathcal{O}_Y$-algebra

$$\widetilde{A} = \widetilde{A}[Y, \mathfrak{D}] := \bigoplus_{m \in \omega_M} \mathcal{O}_Y(\mathfrak{D}(m))\,.$$

Moreover, taking the global sections of $\widetilde{A}$ yields an M-graded algebra

$$A = A[Y, \mathfrak{D}] := H^0(Y, \widetilde{A}[Y, \mathfrak{D}])\,.$$

Since the global section functor commutes with direct sums, we can give a nicer description of A as

$$A = A[Y, \mathfrak{D}] = \bigoplus_{m \in \omega_M} A_m \chi^m, \quad \text{where} \quad A_m = H^0(Y, \mathcal{O}_Y(\mathfrak{D}(m))) \subseteq \mathbf{k}(Y)\,.$$

The following theorem gives a combinatorial description of $\mathbb{T}$-varieties of arbitrary complexity analogous to the classical combinatorial description of toric varieties.

Theorem 1.5.5. *For any proper σ-polyhedral divisor $\mathfrak{D}$ on a semiprojective variety Y, the M-graded algebra $A[Y, \mathfrak{D}]$ is a normal finitely generated effectively M-graded domain of dimension* $\operatorname{rank} M + \dim Y$.

Conversely, if $\mathbf{k}$ is algebraically closed then any normal finitely generated effectively M-graded domain is isomorphic to $A[Y, \mathfrak{D}]$ for some semiprojective variety Y and some proper σ-polyhedral divisor $\mathfrak{D}$ on Y.

Proof. The proof of Theorem 1.5.5 is the main subject of the paper [AH06] and due to its length it is out of the scope of this thesis. It applies some strong geometrical results. In particular, the converse statement is based on results from geometric invariant theory. □

DEFINITION 1.5.6. Let $X = \operatorname{Spec} A$ be a normal affine $\mathbb{T}$-variety. A *combinatorial description* of X is a pair $(Y, \mathfrak{D})$ such that $A \simeq A[Y, \mathfrak{D}]$. The semiprojective variety Y is called the *base variety*.

The combinatorial description $(Y, \mathfrak{D})$ of a $\mathbb{T}$-variety X is not unique. Indeed, if we consider a blow up $\psi : \widetilde{Y} \to Y$ of Y at a closed point and the proper σ-polyhedral divisor $\psi^*\mathfrak{D}$, then $X \simeq X[\widetilde{Y}, \psi^*\mathfrak{D}]$. See Lemma 4.2.1 for a more precise statement.

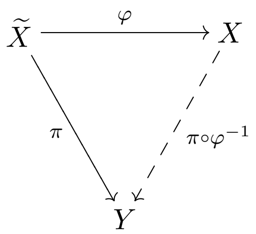
Even if it is not obvious from Theorem 1.5.5, given a normal affine $\mathbb{T}$-variety X a combinatorial description can be obtained in an explicit way. In Section 11 of [AH06] a recipe is given to obtain such a description.

To give the geometrical counterpart of this classification, we define

$$X = X[Y, \mathfrak{D}] := \operatorname{Spec} A[Y, \mathfrak{D}], \quad \text{and} \quad \widetilde{X} = \widetilde{X}[Y, \mathfrak{D}] := \mathbf{Spec}_Y \, \widetilde{A}[Y, \mathfrak{D}]\,.$$

Here, $\mathbf{Spec}_Y$ stands for the relative spectrum of a $\mathcal{O}_Y$-algebra. See [Har77, Ch. II Ex. 5.17] for a definition. A priori, X and $\widetilde{X}$ are only schemes, but Theorem 1.5.5 implies that X is a normal affine variety. Moreover, the gradings on A and $\widetilde{A}$ endow X and $\widetilde{X}$ with $\mathbb{T}$-actions.

The $\mathbf{Spec}_Y$ construction provides a $\mathbb{T}$-invariant affine morphism $\pi : \widetilde{X} \to Y$ which is thus a rational quotient for the $\mathbb{T}$-action on $\widetilde{X}$. The global section functor provides a $\mathbb{T}$-equivariant birational morphism $\varphi : \widetilde{X} \to X$ and so $\pi \circ \varphi^{-1}$ is again a rational quotient for the $\mathbb{T}$-action on X. We can summarize all these considerations in the following commutative diagram, where all the arrows pointing down are rational quotients.



With these definitions, we have the following Theorem.

THEOREM 1.5.7.
(*i*) *$\widetilde{X}$ is a variety.*
(*ii*) *The affine morphism π is a good quotient for the $\mathbb{T}$-action on $\widetilde{X}$.*
(*iii*) *The birational morphism φ is proper.*

REMARK 1.5.8. In the particular case where Y is affine the morphism φ is always an isomorphism since any quasi-coherent sheaf on an affine variety is generated by its global sections.

Recall that a variety X is called *toroidal* if every closed point $x \in X$ has a formal neighborhood isomorphic to a formal neighborhood of a point in a toric variety. We will show in Chapter 4 that every $\mathbb{T}$-variety X has a combinatorial description $(Y, \mathfrak{D})$ such that $\widetilde{X}$ is toroidal.

Since toric singularities are well understood (see Section 1.6), in Chapter 4 we will use the morphism $\varphi : \widetilde{X} \to X$ to study the singularities of X.

REMARK 1.5.9. It is not true, in general, that the singularities of $\widetilde{X}$ are milder that those of X. In Example 2.5.1 X is the affine space, hence smooth and $\widetilde{X}$ has some singularities.

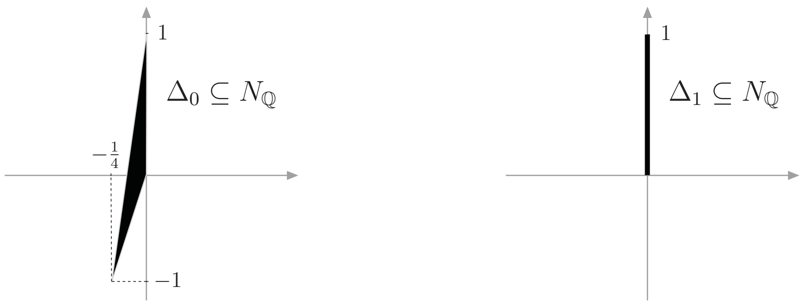


FIGURE 2. The polyhedra Δ_0 and Δ_1 in Example 1.5.11.

The following remark will be useful in Chapter 3.

REMARK 1.5.10. Since every graded piece $H^0(Y, \mathcal{O}_Y(\mathfrak{D}(m)))$ of A is contained in $\mathbf{k}(Y)$, there is a natural embedding $A \hookrightarrow \mathbf{k}(Y)[M]$. Moreover, the field of fractions of both algebras coincide, thus $X = \operatorname{Spec} A$ is birationally equivalent to $Y \times \mathbb{P}^n$ where $n = \operatorname{rank} M$, and to the scheme $\operatorname{Spec} \mathbf{k}(Y)[M]$. The former scheme, which is a $\mathbf{k}(Y)$-variety, is the algebraic torus over the field $\mathbf{k}(Y)$ associated to the lattice M.

The description in Theorem 1.5.5 for an affine $\mathbf{k}^*$-surface is particularly simple. Let $X = X[Y, \mathfrak{D}]$ be a normal affine $\mathbf{k}^*$-surface, then Y is a smooth curve, $M \simeq \mathbb{Z}$, and σ is isomorphic to one of the pointed cones $\{0\}$ and $\mathbb{Q}_{\geq 0}$ in $M_\mathbb{Q} \simeq \mathbb{Q}$.

In [FK91] (see also [FZ03]) all $\mathbf{k}^*$-surfaces are divided into three types: elliptic, parabolic and hyperbolic. These correspond to the cases Y projective, Y affine and $\sigma = \mathbb{Q}_{\geq 0}$, and Y affine and $\sigma = \{0\}$, respectively.

In the general case, we will use the following terminology. An M-graded domain $A[Y, \mathfrak{D}]$ (or, equivalently, a $\mathbb{T}$-variety X) will be called *elliptic* if Y is projective. A non-elliptic $\mathbb{T}$-variety will be called *parabolic* if σ is of full dimension and *hyperbolic* if $\sigma = \{0\}$. If $\dim X \geq 3$, this does not exhaust all the possibilities.

EXAMPLE 1.5.11. Letting $N = \mathbb{Z}^2$ and $\sigma = \{(0,0)\}$, in $N_\mathbb{Q} = \mathbb{Q}^2$ we consider the triangle Δ_0 with vertices $(0,0)$,$(0,1)$ and $(-1/4,-1)$ and the segment $\Delta_1 = \{0\} \times [0,1]$, see Figure 2.

Let $Y = \operatorname{Spec} \mathbf{k}[t]$ and $\mathfrak{D} = \Delta_0 \cdot [0] + \Delta_1 \cdot [1]$. In Figure 3, for the normal quasifans $\Lambda(h_{\Delta_0})$, $\Lambda(h_{\Delta_1})$ and $\Lambda(\mathfrak{D})$ in $M_\mathbb{Q} = \mathbb{Q}^2$, for $i = 0,1$ we show the values of $h_i = h_{\Delta_i}$ on each maximal cone.

We let $A = A[Y, \mathfrak{D}]$ as in Theorem 1.5.5 and $X = \operatorname{Spec} A$. The torus $\mathbb{T} = (\mathbf{k}^*)^2$ acts on X. Since Y is affine and $\sigma = \{(0,0)\}$, X is hyperbolic as $\mathbb{T}$-variety. By Theorem 1.5.5 we have

$$A_{(4,0)} = t\mathbf{k}[t], \quad A_{(-1,0)} = \mathbf{k}[t], \quad A_{(-4,1)} = \mathbf{k}[t], \quad \text{and} \quad A_{(8,-1)} = t(t-1)\mathbf{k}[t]\,.$$

An easy calculation shows that the elements

$$u_1 = -t\chi^{(4,0)}, \quad u_2 = \chi^{(-1,0)}, \quad u_3 = -\chi^{(-4,1)}, \quad \text{and} \quad u_4 = t(t-1)\chi^{(8,-1)}$$

generate A as an algebra. Furthermore, they satisfy the irreducible relation $u_1 + u_1^2u_2^4 + u_3u_4 = 0$, and so

$$A \simeq \mathbf{k}[x_1, x_2, x_3, x_4]/(x_1 + x_1^2x_2^4 + x_3x_4)\,. \tag{2}$$

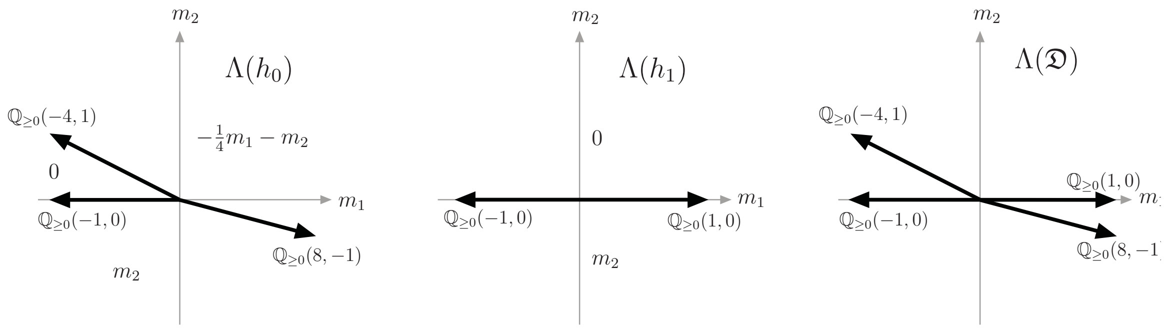


FIGURE 3. The normal quasifans $\Lambda(h_{\Delta_0})$, $\Lambda(h_{\Delta_1})$ and $\Lambda(\mathfrak{D})$ in Example 1.5.11.

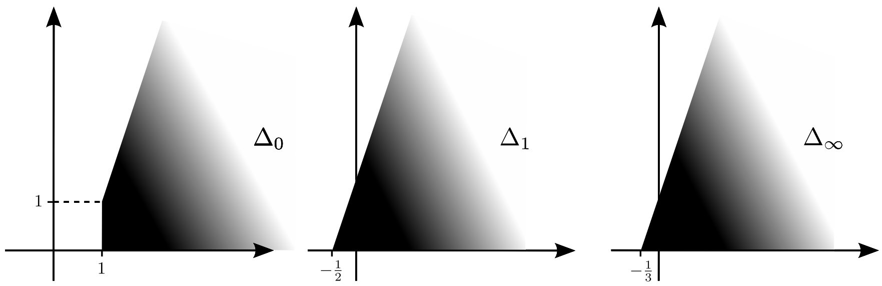


FIGURE 4. The polyhedra Δ_0, Δ_1 and Δ_∞ in Example 1.5.12.

The $\mathbb{Z}^2$-grading on A is given by $\deg x_1 = (4,0)$, $\deg x_2 = (-1,0)$, $\deg x_3 = (-4,1)$, and $\deg x_4 = (8,-1)$. The curve Y and the proper polyhedral divisor $\mathfrak{D}$ can be recovered from this description of A following the recipe in [AH06, Section 11].

EXAMPLE 1.5.12. Letting $N = \mathbb{Z}^2$ and $\sigma = \operatorname{cone}((1,0),(1,6))$, in $N_\mathbb{Q} = \mathbb{Q}^2$ we consider the σ-polyhedra $\Delta_0 = \operatorname{conv}((1,0),(1,1)) + \sigma$, $\Delta_1 = (-1/2, 0) + \sigma$, and $\Delta_\infty = (-1/3, 0) + \sigma$, see Figure 4.

Let $Y = \mathbb{P}^1$ so that $\mathbf{k}(Y) = \mathbf{k}(t)$, where t is a local coordinate at zero. We consider the polyhedral divisor $\mathfrak{D} = \Delta_0 \cdot [0] + \Delta_1 \cdot [1] + \Delta_\infty \cdot [\infty]$, and we let $A = A[Y, \mathfrak{D}]$ and $X = \operatorname{Spec} A$. An easy calculation shows that the elements

$$u_1 = \chi^{(0,1)}, \quad u_2 = \frac{t-1}{t^2}\chi^{(2,0)}, \quad u_3 = \frac{(t-1)^2}{t^3}\chi^{(3,0)}, \quad \text{and} \quad u_4 = \frac{(t-1)^3}{t^5}\chi^{(6,-1)}$$

generate A as an algebra. Furthermore, they satisfy the irreducible relation $u_2^3 - u_3^2 + u_1u_4 = 0$, and so

$$A \simeq \mathbf{k}[x_1, x_2, x_3, x_4]/(x_2^3 - x_3^2 + x_1x_4)\,.$$

1.5.1. Complexity one case. In this section we review the classification of $\mathbb{T}$-varieties restricted to this case.

Let $X = X[Y, \mathfrak{D}]$ be a $\mathbb{T}$-variety of complexity one. Since Y is a rational quotient, its dimension equals the complexity of the $\mathbb{T}$-action. Thus, Y is a curve. For this reason, the semiprojective variety Y will be denoted by C in this case.

Every normal curve is smooth, and every smooth curve is either affine or projective. Furthermore, prime divisors on C are simply closed points. This make things rather explicit in complexity one. We let $\mathfrak{D}$ be the σ-polyhedral divisor on C

$$\mathfrak{D} = \sum_{z\in C} \Delta_z \cdot z\,.$$

Definition 1.5.13. We define the *degree* of $\mathfrak{D}$ as the σ-polyhedron

$$\deg \mathfrak{D} = \sum_{z\in C} \Delta_z\,.$$

The degree of the evaluation $\mathfrak{D}(m)$ can be expressed in terms of support function of $\deg \mathfrak{D}$ i.e.,

$$\deg \mathfrak{D}(m) = h_{\deg \mathfrak{D}}(m), \quad \text{for all} \quad m \in \omega_M\,.$$

Moreover, the normal quasifan $\Lambda(\mathfrak{D})$ of $\mathfrak{D}$ equals the normal quasifan $\Lambda(\deg \mathfrak{D})$ of the σ polyhedron $\deg \mathfrak{D}$.

The condition that a σ-polyhedral divisor is proper can be stated in terms of the σ-polyhedron $\deg \mathfrak{D}$.

Lemma 1.5.14. *A σ-polyhedral divisor $\mathfrak{D}$ on a smooth curve C is proper if and only if either C is affine or C is projective and the following two conditions hold.*

(*i*) *The polyhedron* $\deg \mathfrak{D}$ *is a proper subset of the cone σ.*

(*ii*) *If $h_{\deg \mathfrak{D}}(m) = 0$ then m is contained in the boundary of $\omega = \sigma^\vee$ and a multiple of $\mathfrak{D}(m)$ is principal.*

Proof. Since C is smooth, any $\mathbb{Q}$-divisor is Cartier. If C is affine then every divisor is ample, hence big and semiample. If C is projective then a divisor D is big if and only if $\deg D > 0$ and semiample if and only if it is principal or $\deg D > 0$. With these considerations, (*i*) corresponds to Definition 1.5.4 (*ii*) and (*ii*) to Definition 1.5.4 (*i*). □

Since two smooth projective curves are birationally equivalent if and only if they are isomorphic, the condition for two normal affine $\mathbb{T}$-varieties of complexity one to be equivariantly isomorphic is particularly simple.

Theorem 1.5.15. *The M-graded domains $A[C,\mathfrak{D}]$ and $A[C',\mathfrak{D}']$ are isomorphic if and only if $C \simeq C'$, and under this isomorphism, $\mathfrak{D}(m) - \mathfrak{D}'(m)$ is linear on m, and principal for all $m \in \omega_M$.*

Proof. See Theorem 8.8 in [AH06]. □

In the case where $A = A[C,\mathfrak{D}]$ is non-elliptic we have that $C = \operatorname{Spec} A_0$ is affine. This allows us to prove the following useful property.

Lemma 1.5.16. *Let A be an M-graded algebra of complexity one. If A is non-elliptic then A_m is a locally free A_0-module of rank 1 for every $m \in \omega_M$.*

Proof. This is the algebraic counterpart of the well known fact that the sheaf $\mathcal{O}_C(D)$ is invertible, for all Cartier divisor D on C. □

Following [FZ03, Proposition 4.12], in the next lemma we show the way in which our combinatorial description is affected when passing to a certain cyclic covering. This rather technical lemma will be needed in the proof of Lemma 2.3.24.

Lemma 1.5.17. *Let $A = A[C,\mathfrak{D}]$, where C is a smooth curve and $\mathfrak{D}$ is a proper σ-polyhedral divisor on C. Consider the normalization A' of the cyclic ring extension $A[s\chi^e]$, where $e \in M$, $s^d = f \in A_{de} \subseteq \mathbf{k}(C)$ and $d > 0$. Then $A' = A[C',\mathfrak{D}']$, where C' and $\mathfrak{D}'$ are defined as follows:*

(*i*) *If A is elliptic, then A' is also elliptic and C' is the smooth projective curve with function field $\mathbf{k}(C)[s]$.*
(*ii*) *If A is non-elliptic, then A' is also non-elliptic and $C = \operatorname{Spec} A'_0$, where A'_0 is the normalization of A_0 in $\mathbf{k}(C)[s]$.*
(*iii*) *In both cases, $\mathfrak{D}' = \sum_{z\in C} \Delta_z \cdot p^*(z)$, where $p : C' \to C$ is the projection.*

PROOF. The normalization A' admits a natural M-grading. The latter is defined by the M-grading on A and by letting $\deg s\chi^e = e$. Let $K = \operatorname{Frac} A$. Since $(s\chi^e)^d - f\chi^{de} = 0$, A' is the normalization of A in the function field $K' := K[s\chi^e]$. But $\chi^{-e} \in K$, so $K' = K[s]$. Moreover $K[s] = \mathbf{k}(C)[s] \otimes \operatorname{Frac} \mathbf{k}[M]$, so the function field of C' is $\mathbf{k}(C)[s]$, and A'_0 is the normalization of A_0 in the field $\mathbf{k}(C)[s]$. This proves (*i*) and (*ii*).

For every $m \in N$ we have $\mathfrak{D}'(m) = \sum_{z\in C} h_z(m)p^*(z) = p^*(\mathfrak{D}(m))$. Therefore for every $f \in \mathbf{k}(C)$ there are equivalences:

$$\operatorname{div}_C(f) + \mathfrak{D}(m) \geq 0 \Leftrightarrow \operatorname{div}_{C'}(p^*f) + p^*(\mathfrak{D}(m)) \geq 0 \Leftrightarrow \operatorname{div}_{C'}(f) + \mathfrak{D}'(m) \geq 0\,.$$

Let $m \in \omega_M$ and let $r > 0$ be such that $\mathfrak{D}(rd \cdot m)$ is integral. Then

$$\begin{aligned} g \in A'_m \Leftrightarrow g^{rd} \in A_{rdm} &\Leftrightarrow \operatorname{div}_C(g^{rd}) + \mathfrak{D}(rd\cdot m) \geq 0 \\ &\Leftrightarrow \operatorname{div}_{C'}(g^{rd}) + \mathfrak{D}'(rd\cdot m) \geq 0 \Leftrightarrow \operatorname{div}_{C'}(g) + \mathfrak{D}'(m) \geq 0\,, \end{aligned}$$

which proves (*iii*). □

1.6. Singularities and toric varieties

In Chapter 4 we will study singularities of normal $\mathbb{T}$-varieties of arbitrary complexity. In this section we briefly recall the classification of singularities needed for Chapter 4. We also gather several results about singularities of toric varieties that will be useful in the sequel.

Some of these results are rather new and so the proofs cannot be found in the books cited in Section 1.4. Nevertheless, all of them can be found in the survey [Dai02] by Dais.

1.6.1. Different types of singularities. In all this section we let X be a variety (not necessarily affine) and we denote the local ring of X at a point x by $\mathcal{O}_{X,x}$.

Let R be a Noetherian local ring and let $\mathfrak{m}$ be the unique maximal ideal. The ring R is called *regular* if

$$\dim R = \dim(\mathfrak{m}/\mathfrak{m}^2)\,,$$

and *normal* if it is a domain and integrally closed in its field of fractions.

A finite sequence $a_1, \ldots, a_\ell$ of elements in R is defined to be a *regular sequence* if a_1 is not a zero-divisor of R and for all $i = 2, \ldots, \ell$, a_i is not a zero divisor of $R/(a_1, \ldots, a_{i-1})$. The *depth* of R is the maximum of the lengths of all regular sequences contained in the maximal ideal $\mathfrak{m}$. The local ring R is called *Cohen-Macaulay* if

$$\operatorname{depth} R = \dim R\,.$$

A local Cohen-Macaulay ring R is called *Gorenstein* if there is a maximal regular sequence contained in the maximal ideal generating an irreducible ideal.

DEFINITION 1.6.1. A variety X is *regular*, *normal*, *Cohen-Macaulay*, or *Gorenstein* if all the local rings $\mathcal{O}_{X,x}$ are of this type. An affine variety $X = \operatorname{Spec} A$ is *factorial*[4] if A is a unique factorization domain.

Let now $\psi : Z \to X$ be a desingularization of X i.e., Z is smooth and ψ is a proper birational morphism that is an isomorphism outside the singular locus of X

$$\psi|_{\psi^{-1}(X^{\mathrm{reg}})} : \psi^{-1}(X^{\mathrm{reg}}) \xrightarrow{\sim} X^{\mathrm{reg}} .$$

We also assume that the singular locus is a divisor $\sum_i E_i$ with only simple normal crossings (SNC). We define the i-th direct image sheaf $R^i\psi_*\mathcal{O}_Z$ via

$$U \longrightarrow H^0(U, R^i\psi_*\mathcal{O}_Z) := H^i\left(\psi^{-1}(U), \mathcal{O}_Z|_{\psi^{-1}(U)}\right) .$$

DEFINITION 1.6.2. We say that a variety X has *rational singularities* if it is normal and

$$R^i\psi_*\mathcal{O}_Z = 0, \quad \text{for all} \quad i > 0 .$$

Similarly, we say that X has *elliptic singularities* if it is normal and

$$R^i\psi_*\mathcal{O}_Z = 0 \text{ for all } i \neq 0, \dim X, \quad \text{and} \quad R^{\dim X}\psi_*\mathcal{O}_Z = \mathbf{k} .$$

Assume now that X is normal. For the regular part X^{reg} of X, we define the *canonical sheaf* $\omega_{X^{\mathrm{reg}}}$ as the top exterior product of the sheaf of differentials

$$\omega_{X^{\mathrm{reg}}} = \bigwedge_{i=1}^{\dim X} \Omega_{X^{\mathrm{reg}}} .$$

The canonical sheaf of X^{reg} is invertible and so there exists a Cartier divisor $K_{X^{\mathrm{reg}}}$ on X^{reg} such that

$$\omega_{X^{\mathrm{reg}}} = \mathcal{O}_{X^{\mathrm{reg}}}(K_{X^{\mathrm{reg}}}) .$$

The Zariski closure K_X of $K_{X^{\mathrm{reg}}}$ is a Weil divisor on X. We call K_X the *canonical divisor* of X.

DEFINITION 1.6.3. A normal variety X is called $\mathbb{Q}$-*factorial* if every Weil divisor is $\mathbb{Q}$-Cartier. X is called $\mathbb{Q}$-*Gorenstein* if the canonical divisor K_X is $\mathbb{Q}$-Cartier. If X is $\mathbb{Q}$-Gorenstein, then the *Gorenstein index* of X is the smallest integer ℓ such that ℓK_X is Cartier.

REMARK 1.6.4. Contrary to what the notation suggests, a normal $\mathbb{Q}$-Gorenstein variety of index 1 is not necessarily Gorenstein. Nevertheless, a normal variety X is Gorenstein if and only if it is Cohen-Macaulay and $\mathbb{Q}$-Gorenstein of index 1. This is usually taken as the definition of the Gorenstein property for normal varieties.

Assume now that X is $\mathbb{Q}$-Gorenstein of index ℓ and recall that $\psi : Z \to X$ is a desingularization of X. We can pull back the canonical divisor by setting

$$\psi^* K_X = \tfrac{1}{\ell}\psi^*(\ell K_X) .$$

Let E_i be the exceptional prime divisors of the morphism ψ. We define the *discrepancy divisor* of ψ as

$$K_Z - \psi^* K_X = \sum_i a_i E_i ,$$

and the *discrepancies* of ψ as the coefficients a_i of the discrepancy divisor.

[4]This definition is stronger than the usual definition of factoriality in projective geometry that asks for all the local rings of X to be factorial.

DEFINITION 1.6.5. We say that X has *terminal, canonical, log-terminal or log-canonical singularities* if all the discrepancies are > 0, ≥ 0, > -1 or ≥ -1, respectively. This definition is independent of the particular choice of the desingularization.

This last definition is inspired by the classes of singularities needed to run the minimal model program.

1.6.2. Toric singularities. By Theorem 1.3.4, without loss of generality, we can restrict the analysis to the case of affine toric varieties. Indeed, given a fan $\Sigma \in N_{\mathbb{Q}}$, the variety X_Σ belongs to any of the classes of singularities defined in the previous section if and only if X_σ belong to it, for all cone $\sigma \in \Sigma$.

THEOREM 1.6.6. *Let $\sigma \subseteq N_{\mathbb{Q}}$ be a convex polyhedral cone and let X_σ be the corresponding toric variety, then the following hold.*

(*i*) *X_σ is smooth if and only if σ is regular.*
(*ii*) *If X_σ is factorial if and only if X_σ is smooth.*
(*iii*) *X_σ is $\mathbb{Q}$-factorial if and only if σ is simplicial.*
(*iv*) *X_σ is Cohen-Macaulay.*
(*v*) *X_σ has rational singularities.*
(*vi*) *X_σ is $\mathbb{Q}$-Gorenstein if and only if there exists $m_G \in M_{\mathbb{Q}}$ such that $\langle m_G, \rho\rangle = 1$, for every ray $\rho \subseteq \sigma$. In this case, the Gorenstein index is the smallest integer ℓ such that $\ell \cdot m_G \in M$.*
(*vii*) *If X_σ is $\mathbb{Q}$-Gorenstein then X has log-terminal singularities.*
(*viii*) *If X_σ is $\mathbb{Q}$-Gorenstein then X has canonical singularities if and only if*

$$\sigma \cap \{p \in N_{\mathbb{Q}} \mid \langle m_G, p\rangle < 1\} = \{0\}\,.$$

PROOF. See the survey [Dai02] for proofs or references to proofs. □

CHAPTER 2

$\mathbb{G}_a$-action on $\mathbb{T}$-varieties

The group $\mathbb{G}_a$ is the additive group of an algebraically closed field $\mathbf{k}$. In this chapter we give some classification results about compatible $\mathbb{G}_a$-actions on affine $\mathbb{T}$-varieties. More precisely, we give a full classification of compatible $\mathbb{G}_a$-actions in two cases: for toric varieties, and for $\mathbb{T}$-varieties of complexity 1. In general complexity, we give a classification of compatible $\mathbb{G}_a$-actions whose general orbits are contained in the closures of the general orbits of the $\mathbb{T}$-action. Finally, we show that in all these three cases the ring of invariants of a $\mathbb{G}_a$-action is finitely generated.

2.1. Locally nilpotent derivations and $\mathbb{G}_a$-actions

Any affine $\mathbf{k}$-algebra A can be regarded as a vector space over the base field $\mathbf{k}$. A *derivation* $\partial : A \to A$ is a linear morphism satisfying the Leibniz rule

$$\partial(aa') = a\partial(a') + a'\partial(a), \quad \text{for all} \quad a, a' \in A\,.$$

Definition 2.1.1. A derivation on A is called *locally nilpotent* (LND for short) if for every $a \in A$ there exists $n \in \mathbb{Z}_{\geq 0}$ such that $\partial^n(a) = 0$. The *additive group* $\mathbb{G}_a$ is defined as the algebraic variety $\mathbb{A}^1 \simeq \mathbf{k}$ endowed with the group structure induced by the addition on $\mathbf{k}$.

Given an LND ∂ on A, the map

$$\phi_\partial : \mathbb{G}_a \times A \to A, \qquad (t,a) \mapsto e^{t\partial}a := \sum_{i=0}^{\infty} \frac{t^i \partial^i a}{i!}$$

defines a $\mathbb{G}_a$-action on $X = \operatorname{Spec} A$. Conversely, given a $\mathbb{G}_a$-action ϕ on X, the map

$$\partial_\phi : A \to A, \qquad a \mapsto \frac{d}{dt}(\phi^*(a))\bigg|_{t=0}$$

defines an LND on A. The following well known lemma shows that these maps are mutually inverse.

Lemma 2.1.2. *The maps defined above are mutually inverse and so there is a bijective correspondence between LNDs on A and $\mathbb{G}_a$-actions on $X = \operatorname{Spec} A$.*

Proof. See [Fre06, Section 1.5]. □

Remark 2.1.3. Under the above correspondence, the kernel $\ker \partial$ corresponds to the ring of invariants $\mathbf{k}[X]^{\mathbb{G}_a}$ of the corresponding $\mathbb{G}_a$-action.

In the following lemma we collect some well known facts about LNDs over a field of characteristic 0 not necessarily algebraically closed, needed for later purposes, see e.g., [ML; Fre06].

Lemma 2.1.4. *Let A be a finitely generated normal domain over a field of characteristic 0. If ∂ and ∂' are two LNDs on A, then the following hold:*

(i) $\ker\partial$ *is a normal subdomain of codimension 1.*
(ii) $\ker\partial$ *is factorially closed i.e.,* $ab \in \ker\partial \Rightarrow a, b \in \ker\partial$.
(iii) *If* $a \in A$ *is invertible, then* $a \in \ker\partial$.
(iv) *If* $\ker\partial = \ker\partial'$, *then there exist* $a, a' \in \ker\partial$ *such that* $a\partial = a'\partial'$.
(v) *If* $a \in \ker\partial$, *then* $a\partial$ *is again an LND.*
(vi) *If* $\partial(a) \in (a)$ *for some* $a \in A$, *then* $a \in \ker\partial$.
(vii) *The field extension* $\operatorname{Frac}(\ker\partial) \subseteq \operatorname{Frac} A$ *is purely transcendental of degree 1.*

The following definition is motivated by Lemma 2.1.4 (iv).

Definition 2.1.5. We say that two LNDs ∂ and ∂' on A are *equivalent* if $\ker\partial = \ker\partial'$. Geometrically this means that the general orbits of the associated $\mathbb{G}_a$-actions on $X = \operatorname{Spec} A$ coincide.

Let as before M and N be dual lattices. Let $\mathbf{k}(Y)$ be the field of rational functions of an algebraic variety Y. We consider a finitely generated effectively M-graded domain[1]

$$A = \bigoplus_{m\in\omega_M} A_m\chi^m, \qquad \text{where} \qquad A_m \subseteq \mathbf{k}(Y)\,. \tag{3}$$

A derivation ∂ on A is called *homogeneous* if it sends homogeneous elements into homogeneous elements. Hence ∂ sends homogeneous pieces of A into homogeneous pieces. A $\mathbb{G}_a$-action on an affine $\mathbb{T}$-variety is called *compatible* if the corresponding LND is homogeneous. In geometric terms, a $\mathbb{G}_a$-action is compatible if and only if it is normalized by the torus $\mathbb{T}$. Let

$$M_\partial = \{m \in \omega_M \mid \partial(A_m\chi^m) \neq 0\}\ .$$

The action of ∂ on homogeneous pieces of A defines a map $\partial_M : M_\partial \to \omega_M$ i.e., $\partial(A_m\chi^m) \subseteq A_{\partial_M(m)}\chi^m$. By Leibniz rule, for homogeneous elements $a \in A_m\chi^m \setminus \ker\partial$ and $a' \in A_{m'}\chi^{m'} \setminus \ker\partial$ we have

$$\partial(aa') = a\partial(a') + a'\partial(a) \in A_{\partial(m+m')}\,,$$

and so

$$\partial_M(m+m') = m + \partial_M(m') = m' + \partial_M(m)\,.$$

Thus $\partial_M(m) - m \in M$ is a constant function on M_∂. This leads to the following definition.

Definition 2.1.6. Let ∂ be a nonzero homogeneous derivation on A. The *degree* of ∂ is the lattice vector $\deg\partial$ defined by $\deg\partial = \deg\partial(f) - \deg(f)$ for any homogeneous element $f \notin \ker\partial$. With this notation the map $\partial_M : M_\partial \to \omega_M$ is just the translation by the vector $\deg\partial$.

We also say that an LND ∂ on A is *negative* if $\deg\partial \notin \omega_M$, *non-negative* if $\deg\partial \in \omega_M$, and *positive* if ∂ is non-negative and $\deg\partial \neq 0$.

The following well known fact shows that any LND on A decomposes into a sum of homogeneous derivations, some of which are locally nilpotent.

Lemma 2.1.7. *Let A be a finitely generated normal M-graded domain. For any derivation ∂ on A there is a decomposition $\partial = \sum_{e\in M}\partial_e$, where ∂_e is a homogeneous derivation of degree e. Moreover, let $\Delta(\partial)$ be the convex hull in $M_\mathbb{Q}$ of the set*

[1]Recall our convention regarding M-graded algebras in Remark 1.3.10 (i).

$\{e \in M : \partial_e \neq 0\}$. *Then* $\Delta(\partial)$ *is a bounded polyhedron and for every vertex* e *of* $\Delta(\partial)$, ∂_e *is locally nilpotent if* ∂ *is.*

PROOF. Letting $a_1, \cdots, a_r$ be a set of homogeneous generators of A we have $A \simeq \mathbf{k}^{[r]}/I$, where $\mathbf{k}^{[r]} = \mathbf{k}[x_1, \cdots, x_r]$ and I denotes the ideal of relations of $a_1, \cdots, a_r$. The M-grading and the derivation ∂ can be lifted to an M-grading and a derivation ∂' on $\mathbf{k}^{[r]}$, respectively.

The proof of Proposition 3.4 in [Fre06] can be applied to an M-grading, proving that $\partial' = \sum_{e\in M} \partial'_e$, where ∂'_e is a homogeneous derivation on $\mathbf{k}^{[r]}$. Furthermore, since $\partial'(I) \subseteq I$ and I is homogeneous, we have $\partial'_e(I) \subseteq I$. Hence ∂'_e induces a homogeneous derivation ∂_e on A of degree e, proving the first assertion.

The algebra A being finitely generated, the set $\{e \in M : \partial_e \neq 0\}$ is finite and so $\Delta(\partial)$ is a bounded polyhedron. Let e be a vertex of $\Delta(\partial)$ and $n \geq 1$. If $ne = \sum_{i=1}^{n} m_i$ with $m_i \in \Delta(\partial) \cap M$, then $m_i = e\ \forall i$. For $a \in A_m\chi^m$ this yields $\partial_e^n(a) = (\partial^n(a))_{m+ne}$, where $(\partial^n(a))_{m+ne}$ stands for the summand of degree $m+ne$ in the homogeneous decomposition of $\partial^n(a)$. Hence ∂_e is locally nilpotent if ∂ is so. □

In the following lemma we extend Lemma 1.8 in [FZ05a] to more general gradings. This lemma shows that any LND ∂ on a normal domain can be extended as an LND to a cyclic ring extension defined by an element of $\ker\partial$.

LEMMA 2.1.8. *Let* A *be a finitely generated normal domain and let* ∂ *be an LND on* A.

- (i) *Given a nonzero element* $v \in \ker\partial$ *and* $d > 0$, *we let* A' *denote the normalization of the cyclic ring extension* $A[u] \supseteq A$ *in its fraction field, where* $u^d = v$. *Then* ∂ *extends in a unique way to an LND* ∂' *on* A'.
- (ii) *Moreover, if* A *is* M*-graded and* ∂ *and* v *are homogeneous, with* $\deg v = dm$ *for some* $m \in M$, *then* A' *is* M*-graded as well, and* u *and* ∂' *are homogeneous with* $\deg u = m$ *and* $\deg\partial' = \deg\partial$.

PROOF. Actually (i) is contained in [FZ05a, Lemma 1.8] while the proof of (ii) is similar and so we omit it. □

Recall that $A = \bigoplus_{m\in\omega_M} A_m\chi^m$, where $A_m \subseteq \mathbf{k}(Y)$, $\mathbf{k}(Y)$ is a field of rational functions of an algebraic variety and $\mathrm{Frac}\,A = \mathbf{k}(Y)(M)$ [2]. The following lemma provides some useful extension of a homogeneous LND ∂ on A.

LEMMA 2.1.9. *For any homogeneous LND* ∂ *on* A, *the following hold:*

- (i) *The derivation* ∂ *extends in a unique way to a homogeneous* $\mathbf{k}$*-derivation on* $\mathbf{k}(Y)[M]$.
- (ii) *If* $\partial(\mathbf{k}(Y)) = 0$ *then the extension of* ∂ *as in* (i) *restricts to a homogeneous locally nilpotent* $\mathbf{k}(Y)$*-derivation on* $\mathbf{k}(Y)[\omega_M]$.

PROOF. Since $\mathrm{Frac}\,A = \mathbf{k}(Y)(M)$, any $f\chi^m$, $f \in \mathbf{k}(Y)$, can be written as $f_1\chi^{m_1}/f_2\chi^{m_2}$, where $f_1\chi^{m_1}, f_2\chi^{m_2} \in A$ homogeneous. Then ∂ extends to $\mathbf{k}(Y)[M]$ by the rule

$$\partial\left(\frac{f_1\chi^{m_1}}{f_2\chi^{m_2}}\right) = \frac{\partial(f_1\chi^{m_1})f_2\chi^{m_2} - f_1\chi^{m_1}\partial(f_2\chi^{m_2})}{f_2^2\chi^{2m_2}}\,.$$

[2]Recall that for a field $\mathbf{k}(Y)$ and a lattice M, $\mathbf{k}(Y)(M)$ denotes the function field of $\mathbf{k}(Y)[M]$.

To show (ii), suppose that $\partial(\mathbf{k}(Y))=0$. Assuming that $f\chi^m\in\mathbf{k}(Y)[\omega_M]$, we consider $r>0$ such that $A_{rm}\neq 0$. Letting $g\in A_{rm}$, we have $f^r\chi^{rm}=f'g\chi^{rm}$ for some $f'\in\mathbf{k}(Y)$. Thus $f^r\chi^{rm}$ is nilpotent an so is $f\chi^m$. □

In the setting as in the previous lemma, the extension of ∂ to $\mathbf{k}(Y)[M]$ will be still denoted by ∂. Following [FZ05a] we use the next definition.

Definition 2.1.10. With A as in (3), a homogeneous LND ∂ on A or, equivalently, a $\mathbb{G}_a$-action on $X=\operatorname{Spec} A$, is said to be of *fiber type* if $\partial(\mathbf{k}(Y))=0$ and of *horizontal type* otherwise.

Let A be a finitely generated domain and $X=\operatorname{Spec} A$. In this setting, ∂ is of fiber type if and only if the general orbits of the corresponding $\mathbb{G}_a$-action are contained in the closures of general orbits of the $\mathbb{T}$-action given by the M-grading. Otherwise, ∂ is of horizontal type.

2.2. Compatible $\mathbb{G}_a$-actions on toric varieties

In this section we consider more generally toric varieties defined over a field $\mathbf{k}$ of characteristic 0, not necessarily algebraically closed.

Let as before M and N be dual lattices of rank n. We also let $N_{\mathbb{Q}}=N\otimes\mathbb{Q}$, $M_{\mathbb{Q}}=M\otimes\mathbb{Q}$, and we consider the natural duality $M_{\mathbb{Q}}\times N_{\mathbb{Q}}\to\mathbb{Q}$, $(m,p)\mapsto\langle m,p\rangle$.

Notation 2.2.1. Let $\rho\in N$ and $e\in M$ be lattice vectors. We define $\partial_{\rho,e}$ as the homogeneous derivation of degree e on $\mathbf{k}[M]$ given by $\partial_{\rho,e}(\chi^m)=\langle m,\rho\rangle\cdot\chi^{m+e}$.

An easy computation shows that $\partial_{\rho,e}$ is indeed a derivation. Let H_ρ be the subspace of $M_{\mathbb{Q}}$ orthogonal to ρ, and H_ρ^+ be the halfspace of $M_{\mathbb{Q}}$ given by $\langle\cdot,\rho\rangle\geq 0$. The kernel $\ker\partial_{\rho,e}$ is spanned by all characters χ^m with $m\in M$ orthogonal to ρ, i.e., $\ker\partial_{\rho,e}=\mathbf{k}[H_\rho\cap M]$.

Let $\operatorname{Nil}(\partial_{\rho,e})$ be the subalgebra of $\mathbf{k}[M]$ where $\partial_{\rho,e}$ acts in a nilpotent way. Assume that $\langle e,\rho\rangle=-1$. For every $m\in H_\rho^+\cap M$, the character $\chi^m\in\operatorname{Nil}(\partial_{\rho,e})$ since $\partial_{\rho,e}^{\ell}(\chi^m)=0$, where $\ell=\langle m,\rho\rangle+1$. Thus, the derivation $\partial_{\rho,e}$ restricted to the subalgebra $\mathbf{k}[H_\rho^+\cap M]$ is a homogeneous LND. On the other hand, $\partial_{\rho,e}$ is not locally nilpotent in $\mathbf{k}[M]$, in fact for every $m\notin H_\rho^+\cap M$ the character $\chi^m\notin\operatorname{Nil}(\partial_{\rho,e})$.

Remark 2.2.2. If $\partial_{\rho,e}$ stabilizes a subalgebra $A\subseteq\mathbf{k}[H_\rho^+\cap M]$, then $\partial_{\rho,e}|_A$ is also a homogeneous LND on A of degree e and $\ker(\partial_{\rho,e}|_A)=A\cap\mathbf{k}[H_\rho\cap M]$.

For the rest of this section, we let σ be a pointed polyhedral cone in the vector space $N_{\mathbb{Q}}$ with dual cone $\omega\subseteq M_{\mathbb{Q}}$, and

$$A=\mathbf{k}[\omega_M]=\bigoplus_{m\in\omega_M}\mathbf{k}\chi^m$$

be the affine semigroup algebra of σ with the corresponding affine toric variety $X=\operatorname{Spec} A$. Since the cone σ is pointed, ω is of full dimension and the subalgebra $A\subseteq\mathbf{k}[M]$ is effectively graded by M.

To every ray $\rho\subseteq\sigma$ we can associate a facet $\tau\subseteq\omega$ given by $\tau=\omega\cap\rho^\perp$. As usual, we denote a ray and its primitive vector by the same letter ρ. Thus $\omega\subseteq H_\rho^+$ and $\tau\subseteq H_\rho$.

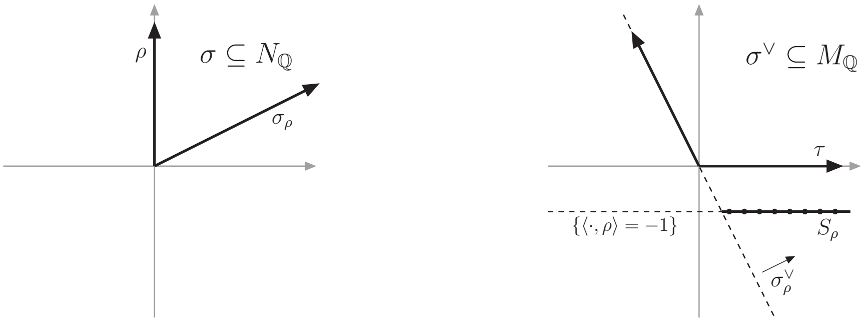


FIGURE 5. The set S_ρ and the cone σ_ρ

DEFINITION 2.2.3. Let σ_ρ be the cone spanned by all the rays of σ except ρ, so that $\omega = \sigma_\rho^\vee \cap H_\rho^+$. We also let

$$S_\rho = \sigma_\rho^\vee \cap \{e \in M \mid \langle e, \rho\rangle = -1\}\,.$$

This definition is illustrated in Figure 5, where $\rho \subseteq N_{\mathbb{Q}}$ is pointing upwards. Alternatively, we can define S_ρ as the set of lattice vectors $m \in M$ such that $\langle \rho, m\rangle = -1$ and $\langle \rho', m\rangle \geq 0$ for every other ray $\rho' \subseteq \sigma$.

LEMMA 2.2.4. *Let $e \in M$. Then $e \in S_\rho$ if and only if*

(i) $e \notin \omega_M$, *and*
(ii) $m + e \in \omega_M$, $\forall m \in \omega_M \setminus \tau_M$.

PROOF. Assume first that $e \in S_\rho$. Then (i) is evident. To show (ii), we let $m \in \omega_M \setminus \tau_M$. Then $m + e \in H_\rho^+$ because $\langle m + e, \rho\rangle = \langle m, \rho\rangle - 1$. Also $m \in \omega \subseteq \sigma_\rho^\vee$ yielding $m + e \in \sigma_\rho^\vee$. Thus $m + e \in \omega = \sigma_\rho^\vee \cap H_\rho^+$.

To show the converse, we let $e \in M$ be such that (i) and (ii) hold. Letting ρ_i, $i = 1, \cdots, \ell$ be all the rays of σ except ρ, for $m \in \omega_M \setminus \tau_M$ we have

$$\langle m + e, \rho_i\rangle = \langle m, \rho_i\rangle + \langle e, \rho_i\rangle \geq 0,\ \forall i \in \{1, \cdots, \ell\}\,.$$

If $m \in \rho_i^\perp \cap \omega_M$ then $\langle m, \rho_i\rangle = 0$ and so $\langle e, \rho_i\rangle \geq 0\ \forall i$. Thus $e \in \sigma_\rho^\vee$. Since $e \in \sigma_\rho^\vee \setminus \omega$, $\langle e, \rho\rangle$ is negative. We have $\langle e, \rho\rangle = -1$, otherwise $m + e \notin \omega$ for any $m \in \omega_M$ such that $\langle m, \rho\rangle = 1$. This yields $e \in S_\rho$. □

REMARK 2.2.5. Since $\rho \notin \sigma_\rho$ we have $S_\rho \neq \emptyset$. Furthermore, by the previous lemma, $e + m \in S_\rho$ whenever $e \in S_\rho$ and $m \in \tau_M$.

In the following lemma we provide a translation of Lemma 2.2.4 from the language of convex geometry to that of affine semigroup algebras.

LEMMA 2.2.6. *For every pair (ρ, e), where ρ is a ray of σ and e is a lattice vector in S_ρ, the homogeneous derivation $\partial_{\rho,e}$ restricts to a homogeneous LND on $A = \mathbf{k}[\omega_M]$ with kernel* $\ker \partial_{\rho,e} = \mathbf{k}[\tau_M]$ *and* $\deg \partial_{\rho,e} = e$.

PROOF. If $\sigma = \{0\}$, then σ has no, so the statement is trivial. We assume in the sequel that σ has at least one ray ρ. By Lemma 2.2.4 $\partial_{\rho,e}$ stabilizes A. Hence by Remark 2.2.2 (2), $\partial_{\rho,e}$ is a homogeneous LND on A with kernel $\mathbf{k}[\tau_M]$ and of degree e. □

The following theorem completes our classification, cf. [Dem70, Prop. 11] and [Oda88, Section 3.4].

THEOREM 2.2.7. *If $\partial \neq 0$ is a homogeneous LND on A, then $\partial = \lambda\partial_{\rho,e}$ for some ray ρ on σ, some lattice vector $e \in S_\rho$, and some $\lambda \in \mathbf{k}^*$.*

PROOF. The kernel $\ker\partial$ is a subsemigroup subalgebra of A of codimension 1. Since $\ker\partial$ is factorially closed (see Lemma 2.1.4), it follows that $\ker\partial = \mathbf{k}[\omega_M \cap H]$ for a certain codimension 1 subspace H of $M_\mathbb{Q}$.

If $\omega \cap H$ is not a facet of ω, then H divides the cone ω into two pieces. Since the action of ∂ on characters is a translation by a constant vector $\deg\partial$, only the characters from one of these pieces can reach H in a finite number of iterations of ∂, which contradicts the fact that ∂ is locally nilpotent.

In the case where $\omega \cap H = \tau$ is a facet of ω, we let ρ be the ray dual to τ. Since ∂ is an homogeneous LND, the translation by $e = \deg\partial$ maps $(\omega_M \setminus \tau_M)$ into ω_M. So by Lemma 2.2.4, $e \in S_\rho$ and $\partial = \lambda\partial_{\rho,e}$, as required. □

REMARK 2.2.8. In [Dem70] a similar result is proved for smooth, not necessarily affine, toric varieties. In *loc. cit.* the elements in the set

$$\mathcal{R} = \bigcup_{\rho\subseteq\sigma} -S_\rho$$

are called the roots of σ.

From our classification we obtain the following corollaries.

COROLLARY 2.2.9. *A homogeneous LND ∂ on a toric variety is uniquely determined, up to a constant factor, by its degree.*

PROOF. By Theorem 2.2.7 we have $\partial = \lambda\partial_{\rho,e}$ where $e = \deg\partial$. We claim that the ρ is uniquely determined by e. Indeed, the sets S_ρ and $S_{\rho'}$ are disjoint for $\rho \neq \rho'$. □

COROLLARY 2.2.10. *Every homogeneous LND ∂ on a toric variety X is of fiber type and negative.*

PROOF. The first claim is evident because $\mathbb{T}$ acts with an open orbit. By Theorem 2.2.7, any LND on a toric variety is of the form $\lambda\partial_{\rho,e}$. Its degree is $\deg\partial_{\rho,e} = e \in S_\rho$ and $S_\rho \cap \omega = \emptyset$, so ∂ is negative. □

COROLLARY 2.2.11. *Two homogeneous LNDs $\partial = \lambda\partial_{\rho,e}$ and $\partial' = \lambda'\partial_{\rho',e'}$ on A are equivalent if and only if $\rho = \rho'$. In particular, there is only a finite number of pairwise non-equivalent homogeneous LNDs on A.*

PROOF. The first assertion follows from the description of $\ker\partial_{\rho,e}$ in Lemma 2.2.6 and the second one from the fact that σ, being polyhedral, has only a finite number of rays. □

EXAMPLE 2.2.12. With $N = \mathbb{Z}^3$ we let σ be the cone in $N_\mathbb{Q}$ having rays $\rho_1 = (1,0,0)$, $\rho_2 = (0,1,0)$, $\rho_3 = (1,0,1)$, and $\rho_4 = (0,1,1)$. The dual cone $\omega \subseteq M_\mathbb{Q} = \mathbb{Q}^3$ is spanned by the lattice vectors $u_1 = (1,0,0)$, $u_2 = (0,1,0)$, $u_3 = (0,0,1)$, and $u_4 = (1,1,-1)$. Furthermore, these elements satisfy the relation $u_1 + u_2 = u_3 + u_4$ and the algebra $A = \mathbf{k}[\omega_M]$ is generated by $x_i = \chi^{u_i}$, $i = 1,\ldots,4$. Thus

$$A \simeq \mathbf{k}[x_1,x_2,x_3,x_4]/(x_1x_2 - x_3x_4)\,. \tag{4}$$

Corollary 2.2.11 shows that there are four non-equivalent homogeneous LNDs on A corresponding to the rays $\rho_i \subseteq \sigma$. By a routine calculation we obtain

$$S_{\rho_1} = \{(-1,b,c) \in M \mid b \geq 0, c \geq 1\}, \quad S_{\rho_2} = \{(a,-1,c) \in M \mid a \geq 0, c \geq 1\},$$

$$S_{\rho_3} = \{(a,b,c) \in M \mid a \geq 0, b+c \geq 0, a+c=-1\}, \text{ and}$$
$$S_{\rho_4} = \{(a,b,c) \in M \mid b \geq 0, a+c \geq 0, b+c=-1\}.$$

Letting $e_1 = (-1,0,1)$, $e_2 = (0,-1,1)$, $e_2 = (0,1,-1)$, $e_4 = (1,0,-1)$, $\partial_i = \partial_{\rho_i,e_i}$, and $m=(m_1,m_2,m_3)$, we have

$$\partial_1(\chi^m) = m_1 \cdot \chi^{m+e_1}, \quad \partial_2(\chi^m) = m_2 \cdot \chi^{m+e_2},$$
$$\partial_3(\chi^m) = (m_1+m_3)\cdot \chi^{m+e_3}, \quad \text{and} \quad \partial_4(\chi^m) = (m_2+m_3)\cdot\chi^{m+e_4},$$

Finally, under the isomorphism of (4) the four homogeneous LNDs on A are given by

$$\partial_1 = x_3\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_4}, \quad \partial_2 = x_3\frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_4},$$
$$\partial_3 = x_4\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_3}, \quad \text{and} \quad \partial_4 = x_4\frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3}.$$

2.3. Compatible $\mathbb{G}_a$-actions on $\mathbb{T}$-varieties of complexity 1

In this section we give a complete classification of homogeneous LNDs on $\mathbb{T}$-varieties of complexity 1 over an algebraically closed field $\mathbf{k}$ of characteristic 0. In the first part we treat the case of a homogeneous LNDs of fiber type, while in the second one we deal with the more delicate case of homogeneous LNDs of horizontal type.

We fix a lattice M of rank n, the torus $\mathbb{T}$, a smooth curve C and a proper σ-polyhedral divisor $\mathfrak{D}$ on C

$$\mathfrak{D} = \sum_{z\in C} \Delta_z \cdot z.$$

Letting $\mathbf{k}(C)$ be the function field of C, we consider the affine variety $X = \operatorname{Spec} A$, where

$$A = A[C,\mathfrak{D}] = \bigoplus_{m\in\omega_M} A_m\chi^m, \quad \text{with} \quad A_m = H^0(C, \mathcal{O}(\mathfrak{D}(m))) \subseteq \mathbf{k}(C).$$

We denote by $h_z = h_{\Delta_z}$ the support function of Δ_z so that

$$\mathfrak{D}(m) = \sum_{z\in C} h_z(m)\cdot z.$$

We also fix a homogeneous LND ∂ on A. In this context, we can interpret Definitions 2.1.6 and 2.1.10 as follows.

LEMMA 2.3.1. *With the notation as above, let ∂ be a homogeneous LND on A. Then the following hold.*

(*i*) *If ∂ is of fiber type, then ∂ is negative and* $\ker\partial = \bigoplus_{m\in\tau_M} A_m\chi^m$, *where τ is a facet of ω.*

(*ii*) *Assuming further that A is non-elliptic, ∂ is of fiber type if and only if ∂ is negative.*

PROOF. To prove (*i*) we let ∂ be a homogeneous LND of fiber type on A. By Lemma 2.1.9 we can extend ∂ to a homogeneous LND $\bar\partial$ on $\bar A = \mathbf{k}(C)[\omega_M]$ which is an affine semigroup algebra over $\mathbf{k}(C)$. Since $\partial(\mathbf{k}(C)) = 0$, $\bar\partial$ is a locally nilpotent $\mathbf{k}(C)$-derivation. It follows from Corollary 2.2.10 that $\deg\partial = \deg\bar\partial \notin \omega_M$, so ∂ is negative.

Furthermore, Lemma 2.2.6 and Theorem 2.2.7 show that $\ker \bar{\partial} = \mathbf{k}(C)[\tau_M]$, where τ is a facet of ω. Thus

$$\ker \partial = A \cap \ker \bar{\partial} = \bigoplus_{m\in\tau_M} (A_m \cap \mathbf{k}(C))\chi^m = \bigoplus_{m\in\tau_M} A_m\chi^m\,,$$

which proves (i).

To prove (ii) we assume further that A is non-elliptic. Let ∂ be a negative homogeneous LND on A. Let $\bar{\partial}$ be the extension of ∂ to $\mathbf{k}(C)[M]$ as in Lemma 2.1.9. Since ∂ is negative, $\partial(A_0) \subseteq A_{\deg \partial} = 0$. Since A is non-elliptic we have $\mathbf{k}(C) = \text{Frac}\ A_0$, so $\bar{\partial}(\mathbf{k}(C)) = 0$ and ∂ is of fiber type. □

REMARK 2.3.2. In the elliptic case, the second assertion in Lemma 2.3.1 does not hold, in general. Consider for instance the elliptic $\mathbf{k}$-domain $A = \mathbf{k}[x, y]$ graded via $\deg x = \deg y = 1$. Then the partial derivative ∂_x is a negative homogeneous LND of horizontal type on A.

2.3.1. Homogeneous LNDs of fiber type. In this subsection we consider a homogeneous LND ∂ on A of fiber type. Let as before $\bar{A} = \mathbf{k}(C)[\omega_M]$ be the affine semigroup $\mathbf{k}(C)$-algebra with cone $\sigma \in N_{\mathbb{Q}}$ over the field $\mathbf{k}(C)$. By Lemma 2.1.9, ∂ can be extended to a homogeneous locally nilpotent $\mathbf{k}(C)$-derivation on $\bar{A}$. To classify homogeneous LNDs of fiber type, we will rely on the classification of homogeneous LNDs on affine semigroup algebras from the previous section.

If σ has no ray then $\sigma = 0$ and $\omega = M_{\mathbb{Q}}$. By Lemma 2.3.1 in this case there are no homogeneous LND of fiber type. So we may assume in the sequel that σ has at least one ray, say ρ. Let τ be its dual facet, and let S_ρ be as defined in Lemma 2.2.4.

LEMMA 2.3.3. *For any $e \in S_\rho$,*

$$D_e := \sum_{z\in C} \max_{m\in\omega_M\setminus\tau_M} (h_z(m) - h_z(m + e)) \cdot z$$

is a well defined $\mathbb{Q}$-divisor on C.

PROOF. By Lemma 2.2.4, for all $m \in \omega_M \setminus \tau_M$, $m + e$ is contained in ω_M and thus $h_z(m)$ and $h_z(m + e)$ are well defined. Recall that for any $z \in C$, the function h_z is concave and piecewise linear on ω. Thus the above maximum is achieved by one of the linear pieces of h_z i.e., by one of the maximal cones in the normal quasifan $\Lambda(h_z)$ (see Definition 1.5.3).

For every $z \in C$, we let $\{\delta_{1,z}, \cdots, \delta_{\ell_z,z}\}$ be the set of all maximal cones in $\Lambda(h_z)$ and $g_{r,z},\ r \in \{1, \cdots, \ell_z\}$ be the linear extension of $h_z|_{\delta_{r,z}}$ to $M_{\mathbb{Q}}$. Since the maximum is achieved by one of the linear pieces we have

$$\max_{m\in\omega_M\setminus\tau_M} (h_z(m) - h_z(m + e)) = \max_{r\in\{1,\cdots,\ell_z\}} (-g_{r,z}(e))\,.$$

Since $g_{r,z}(e) \in \mathbb{Q}\ \forall(r, z)$, D_e is indeed a $\mathbb{Q}$-divisor. □

REMARK 2.3.4. An alternative description of D_e is as follows. Let the notation be as in the preceding proof. Since τ is a facet of ω, it is contained as a face in one and only one maximal cone $\delta_{r,z}$. We may assume that $\tau \subseteq \delta_{1,z}$. By the concavity of h_z we have $g_{1,z}(e) \le g_{r,z}(e)\ \forall r$ and so

$$D_e = -\sum_{z\in C} g_{1,z}(e) \cdot z\,.$$

NOTATION 2.3.5. We let

$$\Phi_e = H^0(C, \mathcal{O}_C(-D_e)), \quad \text{and} \quad \Phi_e^* = \Phi_e \setminus \{0\}\,.$$

We need the following lemma.

LEMMA 2.3.6. *Let $\rho \in \sigma$ be a ray, $\tau \in \omega$ be its dual facet and $e \in S_\rho$. If $\varphi \in \mathbf{k}(C)$, then $\varphi \in \Phi_e$ if and only if $\varphi A_m \subseteq A_{m+e}$ for any $m \in \omega_M \setminus \tau_M$.*

PROOF. If $\varphi \in \Phi_e$, then for every $m \in \omega_M \setminus \tau_M$,

$$\operatorname{div}(\varphi) \geq D_e \geq \sum_{z \in C} (h_z(m) - h_z(m+e)) \cdot z = \mathfrak{D}(m) - \mathfrak{D}(m+e)\,.$$

If $f \in \varphi A_m$ then $\operatorname{div}(f) + \mathfrak{D}(m) \geq \operatorname{div}(\varphi)$ and so $\operatorname{div}(f) + \mathfrak{D}(m+e) \geq 0$. Thus $\varphi A_m \subseteq A_{m+e}$.

To prove the converse, we let $\varphi \in \mathbf{k}(C)$ be such that $\varphi A_m \subseteq A_{m+e}$ for any $m \in \omega_M \setminus \tau_M$. With the notation of Remark 2.3.4, we let $m \in M$ be a lattice vector such that $\mathfrak{D}(m)$ is an integral divisor, and m and $m+e$ belong to $\operatorname{rel.int}(\delta_{1,z})$, for any $z \in C$.

For every $z \in \operatorname{Supp} \mathfrak{D}$, we let $f_z \in A_m$ be a rational function such that

$$\operatorname{ord}_z(f_z) = -h_z(m) = -g_{1,z}(m)\,.$$

By our assumption $\varphi f_z \in A_{m+e}$ and so

$$\operatorname{ord}_z(\varphi f_z) \geq -h_z(m+e) = -g_{1,z}(m+e)\,.$$

This yields $\operatorname{ord}_z(\varphi) \geq -g_{1,z}(m+e) + g_{1,z}(m) = -g_{1,z}(e)$ and so $\varphi \in \Phi_e$. This proves the lemma. □

There is a natural way to associate to a nonzero function $\varphi \in \Phi_e^*$ a homogeneous LND of fiber type on A. More precisely we have the following lemma.

LEMMA 2.3.7. *To any triple (ρ, e, φ), where ρ is a ray of σ, $e \in S_\rho$ is a lattice vector, and $\varphi \in \Phi_e^*$ is a nonzero function, we can associate a homogeneous LND $\partial_{\rho,e,\varphi}$ on $A = A[C, \mathfrak{D}]$ with kernel*

$$\ker \partial_{\rho,e,\varphi} = \bigoplus_{m \in \tau_M} A_m \chi^m, \qquad \textit{and} \qquad \deg \partial_{\rho,e,\varphi} = e\,.$$

PROOF. Letting $\bar{A} = \mathbf{k}(C)[\omega_M]$, we consider the $\mathbf{k}(C)$-LND $\partial_{\rho,e}$ on $\bar{A}$ as in Lemma 2.2.6. Since $\varphi \in \mathbf{k}(C)$, $\varphi\partial_{\rho,e}$ is again an $\mathbf{k}(C)$-LND on $\bar{A}$.

We claim that $\varphi\partial_{\rho,e}$ stabilizes $A \subseteq \bar{A}$. Indeed, let $f \in A_m \subseteq \mathbf{k}(C)$ be a homogeneous element so that $\operatorname{div} f + \mathfrak{D}(m) \geq 0$. If $m \in \tau_M$, then $\varphi\partial_{\rho,e}(f\chi^m) = 0$. If $m \in \omega_M \setminus \tau_M$, then

$$\varphi\partial_{\rho,e}(f\chi^m) = \varphi f \partial_{\rho,e}(\chi^m) = m_0 \varphi f \chi^{m+e}\,,$$

where $m_0 := \langle m, \rho \rangle \in \mathbb{Z}_{>0}$. By Lemma 2.3.6 $\varphi f \chi^{m+e} \in A$ and so does $m_0 \varphi f \chi^{m+e}$, proving the claim.

Finally $\partial_{\rho,e,\varphi} := \varphi\partial_{\rho,e}|_A$ is an homogeneous LND on A with kernel

$$\ker \partial_{\rho,e,\varphi} = \bigoplus_{m \in \tau_M} A_m \chi^m\,,$$

as desired. □

The following theorem gives the converse of Lemma 2.3.7 and so completes our classification of homogeneous LNDs of fiber type on $\mathbb{T}$-varieties.

THEOREM 2.3.8. *Every nonzero homogeneous LND ∂ of fiber type on $A = A[C,\mathfrak{D}]$ is of the form $\partial = \partial_{\rho,e,\varphi}$ for some ray $\rho \subseteq \sigma$, some lattice vector $e \in S_\rho$, and some function $\varphi \in \Phi_e$.*

PROOF. Since ∂ is of fiber type, $\partial(\mathbf{k}(C)) = 0$ and so ∂ can be extended to a $\mathbf{k}(C)$-LND $\bar{\partial}$ on the affine semigroup algebra $\bar{A} = \mathbf{k}(C)[\omega_M]$. By Theorem 2.2.7 we have $\bar{\partial} = \varphi\partial_{\rho,e}$ for some ray ρ of σ, some $e \in S_\rho$ and some $\varphi \in \mathbf{k}(C)$. Since A is stable under $\varphi\partial_{\rho,e}$, by Lemma 2.3.6, $\varphi \in \Phi_e$ and so $\partial = \varphi\partial_{\rho,e}|_A = \partial_{\rho,e,\varphi}$. □

COROLLARY 2.3.9. *Let as before $X = \operatorname{Spec} A$ be a $\mathbb{T}$-variety of complexity 1, ∂ be a homogeneous LND of fiber type on A, and let $f\chi^m \in A \setminus \ker\partial$ be a homogeneous element. Then ∂ is completely determined by the image $g\chi^{m+e} := \partial(f\chi^m) \in A_{m+e}\chi^{m+e}$.*

PROOF. By the previous theorem $\partial = \partial_{\rho,e,\varphi}$ for some ray ρ, some $e \in S_\rho$, and some $\varphi \in \Phi_e$, where $e = \deg\partial$ and ρ is uniquely determined by e, see Corollary 2.2.9.

In the proof of Lemma 2.3.7 it was shown that $\partial_{\rho,e,\varphi}(f\chi^m) = m_0\varphi f\chi^{m+e}$. Thus $\varphi = \frac{g}{m_0 f} \in \mathbf{k}(C)$ is also uniquely determined by our data. □

COROLLARY 2.3.10. *Two homogeneous LND $\partial = \partial_{\rho,e,\varphi}$ and $\partial' = \partial_{\rho',e',\varphi'}$ of fiber type on A are equivalent if and only if $\rho = \rho'$. In particular, there is a finite number of pairwise non-equivalent LNDs of fiber type on A.*

PROOF. The first assertion follows from the description of $\ker\partial_{\rho,e,\varphi}$ in Lemma 2.3.7. The second one follows from the fact that σ has a finite number of rays. □

Given a ray $\rho \subseteq \sigma$ and $e \in S_\rho$, it might happen that $\Phi_e^* = \emptyset$, so that there exist no homogeneous LND ∂ of fiber type on A with $\deg\partial = e$ and $\ker\partial = \bigoplus_{m\in\tau_M} A_m\chi^m$. In the following lemma we give a criterion for the existence of $e \in S_\rho$ such that $\dim\Phi_e$ is nonzero.

LEMMA 2.3.11. *Let $A = A[C,\mathfrak{D}]$, and let $\rho \subseteq \sigma$ be a ray dual to a facet $\tau \subseteq \omega$. There exists $e \in S_\rho$ such that $\dim\Phi_e$ is positive if and only if the curve C is affine or C is projective and $h_{\deg\mathfrak{D}}|_\tau \not\equiv 0$.*

PROOF. If C is affine, then for any $\mathbb{Z}$-divisor D the sheaf $\mathcal{O}_C(D)$ is generated by the global sections. It follows in this case that $\dim\Phi_e > 0$.

Let further C be a projective curve of genus g. If $\deg\lfloor -D_e\rfloor < 0$ then $\dim\Phi_e = 0$. On the other hand, by the Riemann-Roch theorem $\dim\Phi_e > 0$ if $\deg\lfloor -D_e\rfloor \geq g$ (see Lemma 1.2 in [Har77, Chapter IV]).

Letting $h = h_{\deg\mathfrak{D}} = \sum_{z\in C} h_z$, with the notation of Remark 2.3.4 we have $h|_\tau = \sum_{z\in C} g_{1,z}$ and $\deg(-D_e) = \sum_{z\in C} g_{1,z}(e)$. By the definition of proper σ-polyhedral divisor, $h(m) > 0$ for any m in the relative interior of ω.

If $h|_\tau \equiv 0$ then by the linearity of $g_{1,z}$ we obtain that $\deg(-D_e) < 0$, so $\deg\lfloor -D_e\rfloor < 0$ and $\dim\Phi_e = 0$.

If $h|_\tau \not\equiv 0$ then by the concavity of h, $h(m) > 0$ for all m in the relative interior of τ. By Remark 2.3.4, $\deg(-D_e)$ is linear on e and so, according to Remark 2.2.5, we can choose a suitable $e \in S_\rho$ so that $\deg\lfloor -D_e\rfloor \geq g$. Hence $\dim\Phi_e > 0$. □

We can now deduce the following corollary.

COROLLARY 2.3.12. *Let $A = A[C,\mathfrak{D}]$, and let $\rho \subseteq \sigma$ be a ray dual to a facet $\tau \subseteq \omega$. There exists a homogeneous LND of fiber type ∂ on A such that $\ker\partial = \bigoplus_{m\in\tau_M} A_m\chi^m$ if and only if C is affine or C is projective and $\rho \cap \deg\mathfrak{D} = \emptyset$.*

PROOF. Since $\rho \cap \deg \mathfrak{D} = \emptyset$ is equivalent to $h_{\deg \mathfrak{D}}|_\tau \not\equiv 0$, the corollary follows from Theorem 2.3.8 and Lemma 2.3.11. □

REMARK 2.3.13. By Corollaries 2.3.10 and 2.3.12, the equivalence classes of LNDs of fiber type on $A = A[C, \mathfrak{D}]$ are in one to one correspondence with the rays $\rho \subseteq \sigma$ if C is affine and with rays $\rho \subseteq \sigma$ such that $\rho \cap \deg \mathfrak{D} = \emptyset$ if C is projective.

2.3.2. Homogeneous LNDs of horizontal type. Let $A = A[C, \mathfrak{D}]$, where $\mathfrak{D}$ is a proper σ-polyhedral divisor on a smooth curve C. We consider a homogeneous LND ∂ of horizontal type on A. We also denote by ∂ its extension to a homogeneous $\mathbf{k}$-derivation on $\mathbf{k}(C)[M]$, where $\mathbf{k}(C)$ is the field of rational functions of C (see Lemma 2.1.9 (i)).

The existence of a homogeneous LND of horizontal type imposes strong restrictions on C, as we show in the next lemma.

LEMMA 2.3.14. *If there exists a homogeneous LND ∂ of horizontal type on $A = A[C, \mathfrak{D}]$, then $C \simeq \mathbb{P}^1$ in the case where A is elliptic and $C \simeq \mathbb{A}^1$ in the case where A is non-elliptic. In the latter case A_m is a free A_0-module of rank 1 for every $m \in \omega_M$ and so*

$$A_m = \varphi_m A_0 \quad \textit{for some} \quad \varphi_m \in A_m \quad \textit{such that} \quad \operatorname{div}(\varphi_m) + \lfloor \mathfrak{D}(m) \rfloor = 0\,.$$

PROOF. Let $\pi : X = \operatorname{Spec} A \dashrightarrow C$ be the rational quotient for the $\mathbb{T}$-action given by the inclusion $\pi^* : \mathbf{k}(C) \hookrightarrow K = \operatorname{Frac} A$. Since X is normal, the indeterminacy locus X_0 of π has codimension greater than 1, and so the general orbits of the $\mathbb{G}_a$-action corresponding to ∂ are contained in $X \setminus X_0$.

Since $\partial(\mathbf{k}(C)) \neq 0$, the general orbits of the $\mathbb{G}_a$-action on X are not contained in the fibers of π, so map dominantly onto C. Hence C being dominated by $\mathbb{A}^1$ we have $C \simeq \mathbb{P}^1$ in the elliptic case and $C \simeq \mathbb{A}^1$ in the non-elliptic case.

Thus, if C is affine then $A_0 = \mathbf{k}[t]$ and so A_m is a locally free A_0-module of rank 1 for any $m \in \omega_M$. By the primary decomposition, any locally free module over a principal ring is free and so $A_m \simeq A_0$ as a module (see also Ch. VII §4 Corollary 2 in [Bou70]). Now the last assertion easily follows. □

NOTATION 2.3.15. For the rest of this section we let $\mathbf{k}(C) = \mathbf{k}(t)$, so that $C = \mathbb{P}^1$ in the elliptic case, and $C = \mathbb{A}^1$ otherwise. We also let S_∂ be the set of all lattice vectors

$$S_\partial = \{m \in M \mid \ker \partial \cap A_m \chi^m \neq \{0\}\}\ ,$$

$L(\partial) \subseteq M$ be the sublattice spanned by S_∂, and $\eta(\partial)$ be the cone spanned by S_∂ in $M_\mathbb{Q}$. We write L and η instead of $L(\partial)$ and $\eta(\partial)$ whenever ∂ is clear from the context.

LEMMA 2.3.16. *Let $A = A[C, \mathfrak{D}]$, where $\mathfrak{D}$ is a proper σ-polyhedral divisor on C, and let ∂ be a homogeneous LND of horizontal type on A. With the notation as above, the following hold.*

(1) The kernel $\ker \partial$ *is a semigroup algebra given by* $\ker \partial = \bigoplus_{m \in \eta_L} \mathbf{k}\varphi_m \chi^m$*, where* $\varphi_m \in A_m$.

(2) In the non-elliptic case $\operatorname{div}(\varphi_m) + \mathfrak{D}(m) = 0$*, while in the elliptic one* $\operatorname{div}(\varphi_m) + \mathfrak{D}(m) = \lambda \cdot [z_\infty]$ *for some* $z_\infty \in \mathbb{P}^1$ *and some positive* $\lambda \in \mathbb{Q}$.

(3) The cone η is a maximal cone of the quasifan $\Lambda(\mathfrak{D})$ in the non-elliptic case, and of the quasifan $\Lambda(\mathfrak{D}|_{\mathbb{P}^1 \setminus \{z_\infty\}})$ in the elliptic one. In particular, $\operatorname{rank}(L) = n$.

(4) M is spanned by $\deg\partial$ and L. More precisely, any $m \in M$ can be uniquely written as $m = l + r\deg\partial$ for some $l \in L$ and some $r \in \mathbb{Z}$ with $0 \leq r < d$, where $d > 0$ is the smallest integer such that $d\deg\partial \in L$.

PROOF. Since $\mathbf{k} \subseteq \ker\partial$ we have $0 \in S_\partial$. If $m, m' \in S_\partial$ then $m + m' \in S_\partial$ and so S_∂ is a subsemigroup of ω_M.

For any $f \in \mathbf{k}(C) = \mathbf{k}(t)$ we have $\partial(f) = f'(t)\partial(t)$, where $\partial(t) \neq 0$ since ∂ is of horizontal type. Thus $\partial(f) = 0$ if and only if f is constant. Let us fix $m \in S_\partial$. If $\varphi_m, \varphi'_m \in \ker\partial \cap A_m\chi^m$ are nonzero, then $\varphi_m/\varphi'_m \in \ker\partial \cap \mathbf{k}(C) = \mathbf{k}$ and so $\varphi'_m = \lambda\varphi_m$ for some $\lambda \in \mathbf{k}^*$.

Hence $\ker\partial = \bigoplus_{m\in S_\partial} \mathbf{k}\varphi_m\chi^m$ and $\ker\partial$ is a semigroup algebra. Since $\ker\partial$ is normal, S_∂ is saturated, and so $S_\partial = \eta_L$, which proves (1).

To prove (2), we assume first that C is affine. Given $m \in \eta_L$, we let φ_m be as in Lemma 2.3.14. Since $\ker\partial$ is factorially closed, if $f\varphi_m\chi^m \in \ker\partial \cap A_m\chi^m$ for some $f \in A_0$, then $f \in \ker\partial \cap A_0 = \mathbf{k}$ and $\varphi_m\chi^m \in \ker\partial \cap A_m\chi^m$. The latter implies that $\varphi_m^r\chi^{rm} \in \ker\partial \cap A_{rm}\chi^{rm}$ $\forall r \geq 1$, and so

$$r\lfloor\mathfrak{D}(m)\rfloor = \lfloor r\mathfrak{D}(m)\rfloor, \quad \text{for all} \quad r \geq 1\,.$$

Hence $\mathfrak{D}(m)$ is an integral divisor, which yields (2) in the non-elliptic case.

In the case where $C = \mathbb{P}^1$, we may suppose that that $z_\infty = \infty$. Given $m \in \eta_L$, let us assume that

$$\mathrm{div}(\varphi_m) + \lfloor\mathfrak{D}(m)\rfloor \geq [0] + [\infty]\,,$$

so that $t\varphi_m \in A_m$ and $t^{-1}\varphi_m \in A_m$. We have

$$(t\varphi_m\chi^m)(t^{-1}\varphi_m\chi^m) = (\varphi_m\chi^m)^2 \in \ker\partial\,.$$

Thus $t\varphi_m\chi^m \in \ker\partial$, which contradicts (1). Henceforth

$$\mathrm{div}(\varphi_m) + \lfloor\mathfrak{D}(m)\rfloor = \lambda\cdot[z_\infty], \quad \text{for some} \quad \lambda \in \mathbb{Z}_{\geq 0}\,.$$

An argument similar to that employed in the non-elliptic case, yields

$$\mathrm{div}(\varphi_m) + \mathfrak{D}(m) = \lambda\cdot[z_\infty], \quad \text{for some} \quad \lambda \in \mathbb{Z}_{\geq 0}\,.$$

proving (2).

We have $\dim\ker\partial = \dim\eta$. Since ∂ is an LND, $\ker\partial$ has codimension 1 in A. Hence η is of full dimension in $M_\mathbb{Q}$. Furthermore, in the non-elliptic case (2) shows that $h_z|_\eta$ is linear $\forall z \in \mathbb{A}^1$, so that η is contained in a maximal cone δ in $\Lambda(\mathfrak{D})$.

Assume that $\eta \subsetneq \delta$. Let $m \in \delta \setminus \eta$ and $\varphi_m \in \mathbf{k}(t)$ be such that $\mathfrak{D}(m)$ is integral and $\mathrm{div}(\varphi_m) + \mathfrak{D}(m) = 0$. Letting $m' \in \eta_L$ be such that $m + m' \in \eta_L$, the linearity of $\mathfrak{D}$ implies

$$\varphi_m\chi^m\varphi_{m'}\chi^{m'} = \varphi_{m+m'}\chi^{m+m'} \in \ker\partial\,.$$

Hence $\varphi_m\chi^m \in \ker\partial$ which is a contradiction, proving (3) in the non-elliptic case. In the elliptic case a similar argument (with $z \in \mathbb{P}^1 \setminus \{z_\infty\}$) provides the result.

Finally, since ω_M spans M as a lattice and ∂ is a homogeneous LND, for any $m \in M$ we have $m + r\deg\partial \in L$ for some $r \in \mathbb{Z}$. Thus for $0 \geq r > -d$ the decomposition as in (4) is unique because of the minimality of d. $\square$

COROLLARY 2.3.17. *In the notation of Lemma 2.3.16, by (3) $\omega \subseteq N_\mathbb{Q}$ is a pointed polyhedral cone and by (1)*

$$\ker\partial = \bigoplus_{m\in\eta_L} \mathbf{k}\varphi_m\chi^m \simeq \mathbf{k}[\eta_L]$$

is an affine semigroup algebra. In particular $\ker\partial$ is finitely generated.

Let us consider two basic examples, one with a non-elliptic $\mathbb{T}$-action and the other one with an elliptic $\mathbb{T}$-action. They are universal in the sense of Lemma 2.3.21 below. We use both examples in our final classification, cf. Lemma 2.3.24 and Theorem 2.3.26.

Starting with an affine toric variety X and a homogeneous LND ∂ of fiber type (see Corollary 2.2.10), we can restrict the big torus action to an appropriate codimension 1 subtorus $\mathbb{T}$ so that ∂ becomes of horizontal type for the $\mathbb{T}$-action of complexity 1 on X. This is actually the case in our examples.

EXAMPLE 2.3.18. Letting $A = A[C, \mathfrak{D}]$, where $C = \mathbb{A}^1$, $p \in N_{\mathbb{Q}}$, and $\mathfrak{D} = (p + \sigma) \cdot [0]$ we have that $h_0 : \omega \to \mathbb{Q}$, $m \mapsto \langle m, p\rangle$ is linear and $h_z = 0\ \forall z \in \mathbf{k}^*$. Denoting by $h : M_{\mathbb{Q}} \to \mathbb{Q}$ the linear extension of h_0 to the whole $M_{\mathbb{Q}}$, for $m \in \omega_M$ we obtain

$$A_m = t^{-\lfloor h(m)\rfloor}\mathbf{k}[t] = \bigoplus_{r\geq -h(m)} \mathbf{k}t^r\,.$$

Letting $\widehat{N} = N \times \mathbb{Z}$, $\widehat{M} = M \times \mathbb{Z}$, and $\widehat{\sigma}$ be the cone in $\widehat{N}_{\mathbb{Q}}$ spanned by $(\sigma, 0)$ and $(p, 1)$, a vector $(m, r) \in \widehat{M}_{\mathbb{Q}}$ belongs to the dual cone $\widehat{\omega} := \widehat{\sigma}^\vee$ if and only if $m \in \omega$ and $r \geq -h(m)$. By identifying $\chi^{(0,1)}$ with t we obtain

$$A = \bigoplus_{(m,r)\in\widehat{\omega}_{\widehat{M}}} \mathbf{k}t^r\chi^m = \bigoplus_{(m,r)\in\widehat{\omega}_{\widehat{M}}} \mathbf{k}\chi^{(m,r)} = \mathbf{k}[\widehat{\omega}_{\widehat{M}}]\,.$$

Hence A is an affine semigroup algebra and so, we can apply the results of the previous section.

Since A_0 is spanned as affine semigroup algebra by the character $\chi^{(0,1)}$, the only facet of $\widehat{\omega}$ not containing the lattice vectors $(0, 1)$ is

$$\tau = \{(m, r) \in \widehat{M}_{\mathbb{Q}} \mid m \in \omega,\ r = -h(m)\}\,.$$

This is the face of $\widehat{\omega}$ dual to the ray ρ spanned by $(p, 1)$ in $\widehat{N}_{\mathbb{Q}}$.

In the notation of Lemma 2.2.4, picking $e' \in S_\rho$ and $\lambda \in \mathbf{k}^*$ we let $\partial = \lambda\partial_{\rho,e'}$ be the homogeneous LND with respect to the $\widehat{M}$-grading described in Lemma 2.2.6. Since $(0, 1) \notin \tau$, ∂ is of horizontal type with respect to the M-grading on A. Let $\deg_M$ stand for the corresponding degree function.

For any $e' = (e, s) \in M \times \mathbb{Z}$ we have $\deg_M \partial = e$ and $\ker \partial = \mathbf{k}[\tau_{\widehat{M}}]$. Therefore, in the notation of Lemma 2.3.16, $\eta = \omega$ and $L = \{m \in M : h(m) \in \mathbb{Z}\}$.

To be more concrete, we let $d > 0$ be the smallest integer such that $d \cdot p \in N$. Then $d \cdot h$ is an integer valued function on ω_M. Letting $m_1 \in M$ be a lattice vector such that $\{h(m_1)\} = \{\frac{1}{d}\}$, by a routine calculation we obtain

$$S_\rho = \left\{(e, s) \in \widehat{M} \mid e \in L - m_1,\ s = -h(e) - \tfrac{1}{d}\right\} \cap \sigma_\rho^\vee\,, \tag{5}$$

and

$$\partial(\chi^m \cdot t^r) = \lambda\,(r + h(m)) \cdot \chi^{m+e} \cdot t^{r-h(e)-1/d}, \quad \forall\ (m, r) \in \widehat{M} \tag{6}$$

where $\sigma_\rho \subseteq \widehat{N}_{\mathbb{Q}}$ is as defined in Lemma 2.2.4, $\lambda \in \mathbf{k}^*$, and ∂_t is the partial derivative with respect to t. Moreover, in this case $\sigma_\rho = \sigma \times \{0\}$ and so

$$S_\rho = \left\{(e, s) \in \widehat{M} \mid e \in \omega \cap (L - m_1),\ s = -h(e) - \tfrac{1}{d}\right\}\,.$$

EXAMPLE 2.3.19. Let $C=\mathbb{P}^1$, $p\in N_\mathbb{Q}$. Let Δ_∞ be a σ-tailed polyhedron, and let $\mathfrak{D}=(p+\sigma)\cdot[0]+\Delta_\infty\cdot[\infty]$. Under these assumptions $h_0:\omega\to\mathbb{Q}$, $m\mapsto\langle m,p\rangle$ is linear and $h_z=0$ $\forall z\in\mathbf{k}^*$. We let as before $h:M_\mathbb{Q}\to\mathbb{Q}$ denote the linear extension of h_0 to the whole $M_\mathbb{Q}$. We also suppose that $p+\Delta_\infty\subsetneq\sigma$ and so the sum $h_0+h_\infty\geq 0$ is not identically 0. Under these assumptions the σ-polyhedral divisor $\mathfrak{D}$ is proper. Letting $A=A[C,\mathfrak{D}]$, for any $m\in\omega_M$ we have

$$A_m=\bigoplus_{-h_0(m)\leq r\leq h_\infty(m)}\mathbf{k}t^r\,.$$

Let $\widehat{N}=N\times\mathbb{Z}$, $\widehat{M}=M\times\mathbb{Z}$, and let $\widehat{\sigma}$ be the cone in $\widehat{N}_\mathbb{Q}$ spanned by $(\sigma,0)$, $(p,1)$ and $(\Delta_\infty,-1)$. A vector $(m,r)\in\widehat{M}_\mathbb{Q}$ belongs to the dual cone $\widehat{\sigma}^\vee:=\widehat{\omega}$ if and only if $m\in\omega$, $r\geq -h_0(m)$ and $r\leq h_\infty(m)$. Thus by identifying $\chi^{(0,1)}$ with t we obtain:

$$A=\bigoplus_{(m,r)\in\widehat{\omega}_{\widehat{M}}}\mathbf{k}t^r\chi^m=\bigoplus_{(m,r)\in\widehat{\omega}_{\widehat{M}}}\mathbf{k}\chi^{(m,r)}=\mathbf{k}[\widehat{\omega}_{\widehat{M}}]\,.$$

Hence A is again an affine semigroup algebra, and so the results in the previous section can be applied.

We let as before $\rho\subseteq\widehat{\sigma}$ be the ray spanned by $(p,1)$. The facet dual to ρ is

$$\tau=\{(m,r)\in\widehat{M}_\mathbb{Q}\mid m\in\omega,\ r=-h(m)\}\,.$$

In the notation of Lemma 2.2.4, picking $e'\in S_\rho$ and $\lambda\in\mathbf{k}^*$ we let $\partial=\lambda\partial_{\rho,e'}$ be the homogeneous LND with respect to the $\widehat{M}$-grading described in Lemma 2.2.6. Again ∂ is of horizontal type with respect to the M-grading on A.

Furthermore, for any $e'=(e,r)\in M\times\mathbb{Z}$ we have $\deg_M\partial=e$ and $\ker\partial=\mathbf{k}[\tau_{\widehat{M}}]$. Therefore, in the notation of Lemma 2.3.16, $\eta=\omega$ and $L=\{m\in M: h(m)\in\mathbb{Z}\}$.

To be more concrete, we let d and m_1 be as in the previous example. By a routine calculation we obtain that S_ρ is as in (5) and ∂ is as in (6).

REMARK 2.3.20.

(1) In both examples, the homogeneous LND ∂ extends to a derivation on $\mathbf{k}(C)[M]$ given by (6).

(2) With the same formula (6), ∂ extends to a homogeneous LND on

$$A_M:=\bigoplus_{m\in M}t^{-\lfloor h(m)\rfloor}\mathbf{k}[t]\chi^m,\quad\text{where}\quad A\subseteq A_M\subseteq\mathbf{k}(C)[M]\,.$$

(3) In particular, if $p=0$, then ρ is the ray spanned by $(0,1)$, $d=1$, and $L=M$. Furthermore, we can choose $m_1=0$ so that $S_\rho=(M\times\{-1\})\cap\sigma_\rho^\vee$, and the homogeneous LND ∂ of horizontal type on A is given by $\partial=\lambda\chi^e\partial_t$, where $(e,-1)\in S_\rho$.

We return now to the general case. We recall that

$$A=A[C,\mathfrak{D}],\quad\text{where}\quad\mathfrak{D}=\sum_{z\in C}\Delta_z\cdot z$$

is a proper σ-polyhedral divisor on $C=\mathbb{A}^1$ or $C=\mathbb{P}^1$, h_z is the support function of Δ_z, and ∂ is a homogeneous LND of horizontal type on A.

In the next lemma we show that the subalgebra of A generated by the homogeneous elements whose degrees are contained in η, is as in the previous examples.

LEMMA 2.3.21. *With the notation of Lemma 2.3.16, we let* $A_\omega = \bigoplus_{m\in\eta_M} A_m\chi^m$. *Then* $A_\omega \simeq A[C,\mathfrak{D}_\omega]$ *as* M*-graded algebras, where*

(i) $\mathfrak{D}_\omega = (p+\omega)\cdot[0]$ *for some* $p \in N_\mathbb{Q}$*, in the case where* $C = \mathbb{A}^1$*, and*
(ii) $\mathfrak{D}_\omega = (p+\omega)\cdot[0] + \Delta_\infty\cdot[\infty]$ *for some* $p \in N_\mathbb{Q}$ *and some* $\Delta_\infty \in \mathrm{Pol}_\sigma(N_\mathbb{Q})$ *with* $p + \Delta_\infty \subsetneq \sigma$*, in the case where* $C = \mathbb{P}^1$.

PROOF. By Lemma 2.3.16 (3), the support functions h_z restricted to η are linear for all $z \in \mathbb{A}^1$ in the non-elliptic case and for all $z \in \mathbb{P}^1 \setminus \{z_\infty\}$ in the elliptic case. In the non-elliptic case this shows that $\mathfrak{D}_\omega = \sum_{z\in C}(p_z+\omega)\cdot z$, where $p_z \in N_\mathbb{Q}$. In the elliptic case, we may suppose that $z_\infty = \infty$ and so $\mathfrak{D}_\omega = \sum_{z\in\mathbb{A}^1}(p_z+\omega)\cdot z + \Delta_\infty\cdot[\infty]$, where $\Delta_\infty \in \mathrm{Pol}_\sigma(N_\mathbb{Q})$ and $p_z \in N_\mathbb{Q}$ $\forall z \in \mathbb{A}^1$.

By Lemma 2.1.4 (v), without loss of generality we may replace ∂ to assume that $\deg\partial \in \eta_M$. Letting $e = \deg\partial$ we consider the 2-dimensional finitely generated normal $\mathbb{Z}_{\geq 0}$-graded domain

$$B_e = \bigoplus_{r\in\mathbb{Z}_{\geq 0}} A_{re}\chi^{re}\,.$$

If C is affine then $(B_e, \partial|_{B_e})$ is a parabolic pair in the sense of Definition 3.1 in [FZ05a]. Now Corollary 3.19 in *loc. cit.* shows that, for any $r \in \mathbb{Z}_{\geq 0}$, the fractional part $\{\mathfrak{D}_\omega(re)\}$ is supported in at most one point[3]. While for C projective, $(B_e, \partial|_{B_e})$ is an elliptic pair in the sense of *loc. cit.* Then Theorem 3.3 in *loc. cit.* shows that B_e is an affine semigroup algebra. According to Example 5.1 in [Tim08], for any $r \in \mathbb{Z}_{\geq 0}$, the fractional part $\{\mathfrak{D}_\omega(re)\}$ is supported in at most two point.

Given $m \in L$, the derivation $\varphi_m\chi^m\partial$ on A with φ_m as in Lemma 2.3.16 (1) is again locally nilpotent. Applying the previous analysis to this LND shows that, for any $r \in \mathbb{Z}_{\geq 0}$, the fractional part $\{\mathfrak{D}_\omega(r\cdot(e+m))\}$ is supported in at most one point in the non-elliptic case and in at most two points in the elliptic case. By Lemma 2.3.16 (4) L and e span M. So the functions $h_z|_\eta$ are integral except for at most one value of z in the non-elliptic case and at most two values of z in the elliptic case. Furthermore, in the elliptic case one of the two values of $z \in \mathbb{P}^1$ such that h_z is not integral corresponds to $z = \infty$.

Without loss of generality, in both cases we may suppose that $z = 0$ is an exceptional value in $\mathbb{A}^1$, provided there is one. In particular $p_z \in N$ is a lattice vector for any $z \in \mathbf{k}^*$. Since any integral divisor on $\mathbb{A}^1$ and any integral divisor of degree 0 on $\mathbb{P}^1$ are principal, Theorem 1.5.15 shows that $\mathfrak{D}_\omega$ can always be chosen so that $p_z = 0$ $\forall z \in \mathbf{k}^*$. Now the result follows. □

REMARK 2.3.22.

(1) By Examples 2.3.18 and 2.3.19, the previous lemma shows that A_ω is an affine semigroup algebra, or equivalently, Spec A_ω is a toric variety.
(2) In the notation of Lemma 2.3.21, let $h(m) = \langle m, p\rangle$. By virtue of Lemma 2.3.16 (1) and (2), $L = \{m \in M : h(m) \in \mathbb{Z}\}$.

REMARK 2.3.23. Whatever is an isomorphism $A \simeq A[C,\mathfrak{D}]$, the proof of the previous lemma implies the following.

(1) If $C = \mathbb{A}^1$ then all $h_z|_\eta$ are linear and all but possibly one of them are integral.
(2) If $C = \mathbb{P}^1$ then all but possibly one of $h_z|_\eta$ are linear and all but possibly two of them are integral.

[3] The classification results in [FZ05a] are stated for surfaces over the field $\mathbb{C}$ but they are valid over any algebraically closed field of characteristic 0 with the same proofs.

(3) By virtue of Theorem 1.5.15, we may suppose, in both cases, that $h_z|_\eta = 0$ $\forall z \in \mathbf{k}^*$ and $h_0|_\eta$ is linear.

The following lemma provides the main ingredient in our classification of the homogeneous LNDs of horizontal type on $A = A[C, \mathfrak{D}]$.

LEMMA 2.3.24. *Let $\mathfrak{D}$ be a proper σ-polyhedral divisor on $C = \mathbb{A}^1$ or $C = \mathbb{P}^1$. Let η be a maximal cone in the quasifan $\Delta(\mathfrak{D})$ or $\Delta(\mathfrak{D}|_{\mathbb{A}^1})$, respectively, such that $h_z|_\eta = 0$ $\forall z \in \mathbf{k}^*$. Let ∂ be the derivation of degree e given by formula (6). Then ∂ extends to a homogeneous LND on $A = A[C, \mathfrak{D}]$ if and only if, for every $m \in \omega_M$ such that $m + e \in \omega_M$ the following hold.*

(i) If $h_z(m+e) \neq 0$, then $\lfloor h_z(m+e)\rfloor - \lfloor h_z(m)\rfloor \geq 1$ $\forall z \in \mathbf{k}^*$.
(ii) If $h_0(m+e) \neq h(m+e)$, then $\lfloor dh_0(m+e)\rfloor - \lfloor dh_0(m)\rfloor \geq 1 + dh(e)$.
(iii) If $C = \mathbb{P}^1$, then $\lfloor dh_\infty(m+e)\rfloor - \lfloor dh_\infty(m)\rfloor \geq -1 - dh(e)$.

Here h is the linear extension of $h_0|_\eta$ and $d > 0$ is the smallest integer such that dh is integral.

PROOF. Similarly as in Example 2.3.18, $h(m) = \langle m, p\rangle$ for some $p \in N_\mathbb{Q}$. Since each h_z is concave, $h_z(m) \leq 0$ for $z \in \mathbf{k}^*$ and $h_0(m) \leq h(m)$. Letting $A_M = \bigoplus_{m\in M} \varphi_m \mathbf{k}[t]\chi^m$, where $\varphi_m = t^{-\lfloor h(m)\rfloor}$ (see Remark 2.3.20) we have $A \subseteq A_M$. By virtue of this remark ∂ extends to a homogeneous LND on A_M. We still denote by ∂ this extension. Thus ∂ extends to a homogeneous LND on A if and only if ∂ stabilizes A.

To show that ∂ stabilizes A, let us start with the simplest case where $h = 0$.

Case $h = 0$. In this case, Remark 2.3.20 (3) shows that $L = M$, $d = 1$, and $r = -1$, and so $\partial = \lambda\chi^e\partial_t$. Furthermore, $h_z \leq 0$ $\forall z \in \mathbb{A}^1$ and in the elliptic case $h_\infty \geq 0$. For any $m \in \omega_M$ such that $m + e \in \omega_M$, the conditions in the lemma can be reduced to

(i') If $h_z(m+e) \neq 0$, then $\lfloor h_z(m+e)\rfloor - \lfloor h_z(m)\rfloor \geq 1$ $\forall z \in \mathbb{A}^1$.
(iii') If $C = \mathbb{P}^1$, then $\lfloor h_\infty(m+e)\rfloor - \lfloor h_\infty(m)\rfloor \geq -1$ $\forall m \in \omega_M$.

In this case $A_m = H^0(C, \mathcal{O}(\lfloor \mathfrak{D}(m)\rfloor)) \subseteq \mathbf{k}[t]$ and ∂ stabilizes A if and only if

$$f(t) \in A_m \Rightarrow f'(t) \in A_{m+e}\,, \forall m \in \omega_M\,,$$

or equivalently

$$\operatorname{div} f + \lfloor \mathfrak{D}(m)\rfloor \geq 0 \Rightarrow \operatorname{div} f' + \lfloor \mathfrak{D}(m+e)\rfloor \geq 0\,, \forall m \in \omega_M\,,$$

or else

$$\operatorname{ord}_z(f) + \lfloor h_z(m)\rfloor \geq 0 \Rightarrow \operatorname{ord}_z(f') + \lfloor h_z(m+e)\rfloor \geq 0\,, \forall m \in \omega_M \text{ and } \forall z \in C\,. \quad (7)$$

Next we show that (i') and (iii') hold if and only if (7) holds.

Let $z \in \mathbb{A}^1$ and let $m \in \omega_M$ such that $m+e \in \omega_M$. If $h_z(m+e) = 0$ the condition (7) holds since $f \in \mathbf{k}[t]$.

Assume $h_z(m+e) \neq 0$. Since $h_z \leq 0$ is concave, if $h_z(m) = 0$ then $h_z(m+re) \neq 0$ $\forall r > 1$ contradicting the fact that ∂ is an LND. Hence we may assume that $h_z(m) \neq 0$ so that $f \in (t-z)\mathbf{k}[t]$. In this setting $\operatorname{ord}_z(f') = \operatorname{ord}_z(f) - 1$ and so

$$\operatorname{ord}_z(f') + \lfloor h_z(m+e)\rfloor = \operatorname{ord}_z(f) + \lfloor h_z(m)\rfloor + (\lfloor h_z(m+e)\rfloor - \lfloor h_z(m)\rfloor - 1)\,. \quad (8)$$

Therefore (i') implies (7).

To show the converse, let us suppose that (7) holds. Assuming that C is affine, for every $m \in \omega_M$ we consider φ_m as in Lemma 2.3.16. Since by this lemma $\mathrm{ord}_z(\varphi_m) + \lfloor h_z(m) \rfloor = 0$, applying (7) and (8) to φ_m we obtain

$$\mathrm{ord}_z(\varphi_m) + \lfloor h_z(m) \rfloor + (\lfloor h_z(m+e) \rfloor - \lfloor h_z(m) \rfloor - 1) = \lfloor h_z(m+e) \rfloor - \lfloor h_z(m) \rfloor - 1 \geq 0\,,$$

proving (i') when C is affine. If C is projective, then for any $z \in \mathbb{A}^1$ and any $m \in \omega_M$ we can still find $\varphi_{m,z} \in A_m$ such that $\mathrm{ord}_z(\varphi_{m,z}) + \lfloor h_z(m) \rfloor = 0$. Thus again the previous argument applies.

In the elliptic case, we let $z = \infty$ and we fix $m \in \omega_M$. If f is constant, then (7) holds because $h_\infty(m) \geq 0$. Otherwise $\mathrm{ord}_\infty(f') = \mathrm{ord}_\infty(f) + 1$ and so

$$\mathrm{ord}_\infty(f') + \lfloor h_\infty(m+e) \rfloor = \mathrm{ord}_\infty(f) + \lfloor h_\infty(m) \rfloor + (\lfloor h_\infty(m+e) \rfloor - \lfloor h_\infty(m) \rfloor + 1)\,. \qquad (9)$$

Therefore (iii') implies (7).

To show the converse, we let as before $\varphi_{m,\infty} \in A_m$ be such that $\mathrm{ord}_\infty(\varphi_{m,\infty}) + \lfloor h_\infty(m) \rfloor = 0$. Applying (7) and (9) to $\varphi_{m,\infty}$ we obtain

$$\mathrm{ord}_\infty(\varphi_{m,\infty}) + \lfloor h_\infty(m) \rfloor + (\lfloor h_\infty(m+e) \rfloor - \lfloor h_\infty(m) \rfloor + 1) = \lfloor h_\infty(m+e) \rfloor - \lfloor h_\infty(m) \rfloor + 1 \geq 0\,,$$

proving (iii').

Next we assume that h is integral.

Case h integral. In this case we still have $d = 1$. We recall that $h(m) = \langle m, p \rangle$. Letting $\mathfrak{D}' = \mathfrak{D} - (p+\sigma) \cdot [0]$ if C is affine and $\mathfrak{D}' = \mathfrak{D} - (p+\sigma) \cdot [0] + (p+\sigma) \cdot [\infty]$ if C is projective, by Theorem 1.5.15 $A \simeq A[C, \mathfrak{D}']$. In this setting $A[C, \mathfrak{D}']$ is as in the previous case with $h'_0 = h_0 - h$, $h'_\infty = h_\infty + h$ and $h'_z = h_z$ $\forall z \in \mathbf{k}^*$.

This consideration shows that ∂ stabilizes A if and only if (i') and (iii') hold for $h'_z(m)$ $\forall z \in C$. For any $z \in \mathbf{k}^*$, (i') is equivalent to (i) in the lemma. Since

$$\lfloor h'_0(m+e) \rfloor - \lfloor h'_0(m) \rfloor - 1 = \lfloor h_0(m+e) \rfloor - \lfloor h_0(m) \rfloor - 1 - h(e)\,,$$

condition (i') for $z = 0$ is equivalent to (ii).

Similarly, if C is projective

$$\lfloor h'_\infty(m+e) \rfloor - \lfloor h'_\infty(m) \rfloor + 1 = \lfloor h_\infty(m+e) \rfloor - \lfloor h_\infty(m) \rfloor + 1 + h(e)\,,$$

and so (iii') is equivalent to (iii).

Now we turn to the general case.

General case. We may assume that h is not integral i.e., $d > 1$. We consider the normalization A' of $A[\sqrt[d]{\varphi_{de}}\chi^e]$, where $\varphi_{de} := t^{-h(de)}$ so that $A \subseteq A'$ is a cyclic extension. With the notation of Lemma 1.5.17 we have $A' = A[C', \mathfrak{D}']$ and $K'_0 = \mathbf{k}(C)[\sqrt[d]{\varphi_{de}}]$.

By the minimality of d we deduce that $\gcd(h(de), d) = 1$ and so $\sqrt[d]{\varphi_{de}} = t^{a+b/d}$, where $\gcd(b, d) = 1$. So $\mathbf{k}(C)' = \mathbf{k}(s)$, where $s^d = t$. Thus $C' \simeq \mathbb{A}^1$ if A is non-elliptic and $C' \simeq \mathbb{P}^1$ if A is elliptic. Let $p : C' \to C$, $z' \mapsto z'^d = z$ be the projection induced by the morphism $\mathbf{k}(C) \hookrightarrow K'_0$, $t \mapsto t = s^d$. By Lemma 1.5.17 we have

$$\mathfrak{D}' = d \cdot \Delta_0 \cdot [0] + \sum_{z' \in \mathbf{k}^*} \Delta_z \cdot z' \text{ if } C = \mathbb{A}^1\,,$$

and
$$\mathfrak{D}' = d\cdot\Delta_0\cdot[0] + d\cdot\Delta_\infty\cdot[\infty] + \sum_{z'\in\mathbf{k}^*}\Delta_z\cdot z' \text{ if } C=\mathbb{P}^1\,.$$
So $h'_0 = dh_0$, $h'_\infty = dh_\infty$ and $h'_{z'} = h_z$. Moreover $h'_0|_\eta$ is integral and A' is as in the previous case.

Recall that $A_M = \bigoplus_{m\in M}\varphi_m\mathbf{k}[t]\chi^m$, where $\varphi_m = t^{-\lfloor h(m)\rfloor}$. We define further
$$A'_M = \bigoplus_{m\in M}\varphi'_m\mathbf{k}[s]\chi^m, \qquad \text{where} \qquad \varphi'_m = -s^{dh(m)}\,.$$
Since $A_M\subseteq A'_M$ is a cyclic extension, by Lemma 2.1.8, $\partial: A_M\to A_M$ extends to a homogeneous LND $\partial' : A'_M\to A'_M$.

We claim that ∂ stabilizes A if and only if ∂' stabilizes A'. In fact the "only if" direction is a consequence of Lemma 2.1.8. If ∂' stabilizes A' then $\partial'(A)=\partial(A)\subseteq A_M\cap A' = A$, proving the claim.

We let h' be the linear extension of $h'_0|_\eta$. Clearly $h'=dh$. The previous case shows that ∂' stabilizes A' if and only if, for any $m\in\omega_M$ such that $m+e\in\omega_M$, the following conditions hold.

(i'') If $h'_{z'}(m+e)\neq 0$, then $\lfloor h'_{z'}(m+e)\rfloor - \lfloor h'_{z'}(m)\rfloor \geq 1$ $\forall z'\in\mathbf{k}^*$.
(ii'') If $h'_0(m+e)\neq h'(m+e)$, then $\lfloor h'_0(m+e)\rfloor - \lfloor h'_0(m)\rfloor \geq 1+h'(e)$.
(iii'') If $C=\mathbb{P}^1$, then $\lfloor h'_\infty(m+e)\rfloor - \lfloor h'_\infty(m)\rfloor \geq -1-h'(e)$.

Replacing in (i'')-(iii'') h' by dh, h'_0 by dh_0, h'_∞ by dh_∞, and $h'_{z'}$ by h_z for $z\in\mathbf{k}^*$, shows that ∂ stabilizes A if and only if (i)-(iii) of the lemma hold. Now the proof is completed. □

Remark 2.3.25. In the elliptic case, if $e\in\eta_M$, then (iii) in Lemma 2.3.24 holds. In fact
$$\begin{aligned}\lfloor dh_\infty(m+e)\rfloor - \lfloor dh_\infty(m)\rfloor &\geq dh_\infty(m+e) - 1 - dh_\infty(m)\\ &\geq dh_\infty(e) - 1 \geq -dh(e) - 1\,.\end{aligned}$$

In the following theorem we describe all the homogeneous LND of horizontal type on a $\mathbb{T}$-variety of complexity one. It is our main classification result which summarizes the previous ones.

Theorem 2.3.26. *Let $\mathfrak{D}$ be a proper σ-polyhedral divisor on $C=\mathbb{A}^1$ or $C=\mathbb{P}^1$, and let $A=A[C,\mathfrak{D}]$. Let $\eta\subseteq M_\mathbb{Q}$ be a polyhedral cone, and $e\in M$ be a lattice vector. Then there exists a homogeneous LND $\partial: A\to A$ of horizontal type with* $\deg\partial = e$ *and $\eta(\partial)=\eta$ if and only if the following conditions (i)-(v) hold.*

(i) *If $C=\mathbb{A}^1$, then η is a maximal cone in the quasifan $\Lambda(\mathfrak{D})$, and there exists $z_0\in C$ such that $h_z|_\eta$ is integral $\forall z\in C\setminus\{z_0\}$.*
(i') *If $C=\mathbb{P}^1$, then there exists $z_\infty\in\mathbb{P}^1$ such that (i) holds for $C_0:=\mathbb{P}^1\setminus\{z_\infty\}$.*

Without loss of generality, we may suppose that $z_0=0$, $z_\infty=\infty$ in the elliptic case, and $h_z(m)|_\eta = 0$ $\forall z\in\mathbf{k}^*$. Let h and d be as in Lemma 2.3.24, let m_1 be as in Example 2.3.18, and let L be as in Remark 2.3.22 (2).

(ii) *The lattice vector $(e, -\frac{1}{d}-h(e))$ belongs to S_ρ as defined in* (5).

For any $m\in\omega_M$ such that $m+e\in\omega_M$, the following hold.

(iii) *If $h_z(m+e)\neq 0$, then $\lfloor h_z(m+e)\rfloor - \lfloor h_z(m)\rfloor\geq 1$ $\forall z\in\mathbf{k}^*$.*
(iv) *If $h_0(m+e)\neq h(m+e)$, then $\lfloor dh_0(m+e)\rfloor - \lfloor dh_0(m)\rfloor \geq 1+dh(e)$.*
(v) *If $C=\mathbb{P}^1$, then $\lfloor dh_\infty(m+e)\rfloor - \lfloor dh_\infty(m)\rfloor\geq -1-dh(e)$.*

Moreover,

$$\ker \partial = \bigoplus_{m\in\eta_L} \mathbf{k}\varphi_m\chi^m \,,$$

where $\varphi_m \in A_m$ satisfy the relation

$$\operatorname{div}(\varphi_m) + \mathfrak{D}(m) = 0 \quad \text{if} \quad C = \mathbb{A}^1 \quad \text{or} \quad \operatorname{div}(\varphi_m)|_{C_0} + \mathfrak{D}(m)|_{C_0} = 0 \quad \text{if} \quad C = \mathbb{P}^1 \,.$$

PROOF. Let ∂ be a homogeneous LND of horizontal type on A with $\deg\partial = e$ and $\eta(\partial) = \eta$. Lemma 2.3.16 (3) and Remark 2.3.23 show that (i) and (i') hold. Lemma 2.3.21 and Examples 2.3.18 and 2.3.19 shows that (ii) holds. To conclude, Lemma 2.3.24 shows that (iii)-(v) hold.

To show the converse, assume that (i), (i') and (ii)-(v) are fulfilled. By Theorem 1.5.15, (i) and (i') imply that $A_\omega \simeq A[C, \mathfrak{D}_\omega]$ with $\mathfrak{D}_\omega$ as in Lemma 2.3.21. By Examples 2.3.18 and 2.3.19 and Remark 2.3.20 (2), (ii) shows that there exists a homogeneous LND $\partial : A_M \to A_M$ with $\deg\partial = e$. By Lemma 2.3.24 and its proof, (iii)-(v) imply that ∂ restricts to a homogeneous LND on A. Finally, by Lemma 2.3.16 (3), (i) and (i') imply that $\eta(\partial) = \eta$.

Moreover, Lemma 2.3.16 (1) and (2) give the desired description of $\ker\partial$. □

COROLLARY 2.3.27. *In the notation of Theorem 2.3.26, A admits a homogeneous LND ∂ of horizontal type such that $\eta(\partial) = \eta$ if and only if (i) and (i') in the theorem hold.*

PROOF. The *only if* part follows directly form Theorem 2.3.24.

Assume that (i) and (i') hold. By Theorem 2.3.24 and Examples 2.3.18 and 2.3.19, we only need to show that there exists $e \in M$ such that $\big(e, -\frac{1}{d} - h(e)\big) \in S_\rho$ and (iii)-(v) hold.

Let $(e', r') \in S_\rho$ (by Remark 2.2.5, this set is non-empty). By this remark $e = e' + m \ \forall m \in \eta_L$ is such that $(e, r' - h(m)) \in S_\rho$. In particular, we can assume that e belongs to the relative interior of η. In this setting, Remark 2.3.25 shows that (v) holds.

As in the proof of Lemma 2.3.3, for every $z \in \mathbb{A}^1$, we let $\{\delta_{0,z}, \cdots, \delta_{\ell_z,z}\}$ denote the set of all maximal cones in $\Lambda(h_z)$ and $g_{r,z}$, $r \in \{0, \cdots, \ell_z\}$ be the linear extension of $h_z|_{\delta_{r,z}}$ to $M_\mathbb{Q}$. We assume further that $\eta \subseteq \delta_{0,z} \, \forall z \in \mathbb{A}^1$.

Since the functions h_z are concave, the inequalities in (iii) and (iv) hold if they hold in every maximal cone on $\Lambda(h_z)$ except $\delta_{0,z}$ i.e.,

(iii') $\lfloor g_{r,z}(m+e)\rfloor - \lfloor g_z(m)\rfloor \geq 1 \ \forall z \in \mathbf{k}^*$, $\forall r \in \{1, \cdots, \ell_z\}$ and $\forall m \in \delta_{r,z} \cap M$.

(iv') $\lfloor dg_{r,0}(m+e)\rfloor - \lfloor dg_{r,0}(m)\rfloor \geq 1 + dh(e) \ \forall r \in \{1, \cdots, \ell_0\}$ and $\forall m \in \delta_{r,0} \cap M$.

These inequalities are fulfilled if the following hold

$$\begin{cases} g_{r,z}(e) \geq 1 \ \forall z \in \mathbf{k}^* \text{ and } \forall r \in \{1, \cdots, \ell_z\}, \quad \text{and} \\ g_{r,0}(e) \geq \frac{1}{d} + \lceil h(e) \rceil \ \forall r \in \{1, \cdots, \ell_0\} \,. \end{cases} \tag{10}$$

Since e belongs to the relative interior of η, we have $g_{r,z}(e) > g_{0,z}(e) \ \forall z \in \mathbb{A}^1$, $g_{0,0}(e) = h(e)$, and $g_{0,z} = 0 \ \forall z \in \mathbf{k}^*$. By the linearity of the functions $g_{r,z}$ we can choose e such that (10) holds, proving the corollary. □

COROLLARY 2.3.28. *In the notation on Theorem 2.3.26, two homogeneous LND ∂ and ∂' of horizontal type on A are equivalent if and only if $\eta(\partial) = \eta(\partial')$ and, in the elliptic case, $z_\infty(\partial) = z_\infty(\partial')$.*

PROOF. Indeed, the description of $\ker \partial$ given in Theorem 2.3.26 depends only on η in the non-elliptic case and on η and $z_\infty \in C$ in the elliptic one. □

COROLLARY 2.3.29. *The number of pairwise non-equivalent homogeneous LNDs of horizontal type on $A = A[C, \mathfrak{D}]$ is finite except in the case where A is elliptic and there exists a maximal cone η of $\Lambda(\mathfrak{D})$ such that all but possibly one $h_z|_\eta$ are integral.*

PROOF. Since $\Lambda(\mathfrak{D})$ has only a finite number of maximal cones, Corollary 2.3.28 gives the result in the case where A is non-elliptic. Furthermore, in the elliptic case by this corollary there is an infinite number of pairwise non-equivalent LNDs on A if and only if in Theorem 2.3.26 (i') we can choose $z_\infty \in \mathbb{P}^1$ arbitrarily. However the latter is indeed possible under the assumptions of the corollary. □

EXAMPLE 2.3.30. A combinatorial description of $\mathbf{k}^{[2]} = \mathbf{k}[x, y]$ with the grading induced by $\deg x = \deg y = 1$ is given by the proper σ-polyhedral divisor $\mathfrak{D} = (1 + \sigma) \cdot [0]$ on $\mathbb{P}^1$, where $\sigma = \mathbb{Q}_{\geq 0} \subseteq N_\mathbb{Q} \simeq \mathbb{Q}$. By Corollary 2.3.29 there exist an infinite number of pairwise non-equivalent LNDs on $\mathbf{k}^{[2]}$ homogeneous with respect to the given grading. Indeed, the derivations on the family

$$\partial_\lambda = \lambda \frac{\partial}{\partial x} + (1 - \lambda) \frac{\partial}{\partial y}$$

are homogeneous and pairwise non-equivalent for different values of λ.

In contrast, a combinatorial description of $\mathbf{k}^{[2]}$ with the grading induced by $\deg x = -\deg y = 1$ is given by the proper σ-polyhedral divisor $\mathfrak{D} = [0, 1] \cdot [0]$ on $\mathbb{A}^1$. By Corollary 2.3.29 there exist a finite number of pairwise non-equivalent LNDs homogeneous with respect to this grading. Indeed, by Corollary 2.3.27 the only such LNDs are the partial derivatives.

In the following example we study the existence of homogeneous LNDs on the M-graded algebra A of Example 1.5.11.

EXAMPLE 2.3.31. Let the notation be as in Example 1.5.11. Since $\sigma = \{0\}$, Lemma 2.3.1 shows that there is no homogeneous LND of fiber type on A. In contrast, let us show that there exist exactly 4 pairwise non-equivalent homogeneous LNDs on A.

Indeed, since h_0 is the only support function which is non-integral Corollaries 2.3.27 and 2.3.28 show that there are four non-equivalent homogeneous LNDs of horizontal type on A corresponding to the four maximal cones in $\Lambda(\mathfrak{D})$,

$$\delta_1 = \operatorname{cone}((1, 0), (-4, 1)), \quad \delta_2 = \operatorname{cone}((-4, 1), (-1, 0)),$$
$$\delta_3 = \operatorname{cone}((-1, 0), (8, -1)), \quad \delta_4 = \operatorname{cone}((8, -1), (1, 0)).$$

For the cones δ_1 and δ_2 the hypothesis of Lemma 2.3.24 are fulfilled i.e., $h_z|_{\delta_i} = 0$ $\forall z \in \mathbf{k}^*$ for $i = 1, 2$. Moreover, $e_1 = (-3, 1)$ and $e_2 = (-8, 1)$ satisfy conditions (i)-(iii) in this lemma for δ_1 and δ_2, respectively.

We let ∂_1 and ∂_2 be the respective LNDs defined in (6). Letting $m = (m_1, m_2) \in M$, by a routine calculation we obtain

$$\partial_1 \left(\chi^m t^r\right) = \left(r - \tfrac{1}{4} m_1 - m_2\right) \cdot \chi^{m+e_1} t^r, \qquad \text{and} \qquad \partial_2(\chi^m t^r) = r \cdot \chi^{m+e_2} t^r.$$

Furthermore, under the isomorphism (2) in Example 1.5.11, ∂_1 and ∂_2 can be extended to $\mathbf{k}^{[4]} = \mathbf{k}[x_1, x_2, x_3, x_4]$ as LNDs

$$\partial_1 = -\frac{1}{4} x_3 \frac{\partial}{\partial x_2} + x_1^2 x_2^3 \frac{\partial}{\partial x_4} \qquad \text{and} \qquad \partial_2 = x_3 \frac{\partial}{\partial x_1} - (2x_1 x_2^4 + 1) \frac{\partial}{\partial x_4}.$$

To obtain the derivations corresponding to δ_3 and δ_4 we let $C' = \operatorname{Spec} \mathbf{k}[s]$, $\Delta_1' = \{0\} \times [-1, 0]$, and $\mathfrak{D}' = \Delta_0 \cdot [0] + \Delta_1' \cdot [1]$. Theorem 1.5.15 shows that $A \simeq A[C', \mathfrak{D}']$. Under this new combinatorial description we have

$$u_1 = -s\chi^{(4,0)}, \quad u_2 = \chi^{(-1,0)}, \quad u_3 = (1-s)\chi^{(-4,1)}, \quad \text{and} \quad u_4 = s\chi^{(8,-1)} .$$

Now the assumptions of Lemma 2.3.24 are satisfied for δ_3 and δ_4. Moreover, $e_3 = (4, -1)$ and $e_4 = (9, -1)$ satisfy conditions (i)-(iii) in this lemma for δ_3 and δ_4, respectively.

We let ∂_3 and ∂_4 be the respective LNDs defined by (6). By a simple computation we obtain

$$\partial_3 \left(\chi^m s^r\right) = (r + m_2) \cdot \chi^{m+e_3} s^r, \quad \text{and} \quad \partial_4(\chi^m s^r) = \left(r - \tfrac{1}{4}m_1 - m_2\right) \cdot \chi^{m+e_4} s^{r+1} .$$

Furthermore, under the isomorphism (2) ∂_3 and ∂_4 are induced by the LNDs

$$\partial_3 = -x_4 \frac{\partial}{\partial x_1} + (2x_1x_2^4 + 1)\frac{\partial}{\partial x_3} \qquad \text{and} \qquad \partial_4 = \frac{1}{4}x_4\frac{\partial}{\partial x_2} - x_1^2x_2^3\frac{\partial}{\partial x_3}$$

on $\mathbf{k}^{[4]}$.

2.3.3. The surface case. A description of $\mathbb{C}^*$-surfaces was given in [FZ03] in terms of the DPD (Dolgachev-Pinkham-Demazure) presentation. In [FZ05a] this description was applied to classify the homogeneous LNDs on normal affine $\mathbb{C}^*$-surfaces (of both horizontal and fiber type). Here we relate both descriptions. Besides, we stress the difference that appears in higher dimensions.

In the case of dimension 2 the lattice N has rank 1, which makes things quite explicit (cf. e.g., [Süs08]).

We treat the elliptic case first. In this case σ is of full dimension, and so we can assume that $\sigma = \mathbb{Q}_{\geq 0} \subseteq N_{\mathbb{Q}} = \mathbb{Q}$. Let $A = A[C, \mathfrak{D}]$, where $\mathfrak{D}$ is a proper σ-polyhedral divisor on a smooth projective curve C. In this setting, $\mathfrak{D}$ is uniquely determined by the $\mathbb{Q}$-divisor $\mathfrak{D}(1)$ on C. Here $(C, \mathfrak{D}(1))$ coincides with the DPD presentation data. Since the only ray of σ is σ itself and $\deg \mathfrak{D}$ is σ-tailed, by Corollary 2.3.12 there is no homogeneous LND of fiber type on A.

Furthermore, if there is a homogeneous LND ∂ of horizontal type on A, then $\eta(\partial) = \omega$, and so by Remark 2.3.22 (1) $A = A_\omega$ is an affine semigroup algebra i.e., Spec A is an affine toric surface. This corresponds to Theorem 3.3 in *loc. cit.*

Next we consider a non-elliptic algebra A so that C is an affine curve. In *loc.cit.* this case is further divided into two subcases, the parabolic one which corresponds to $\sigma = \mathbb{Q}_{\geq 0}$, and the hyperbolic one which corresponds to $\sigma = \{0\}$.

In the parabolic case, the DPD presentation data is the same as in the elliptic one. In this case there is again just one ray $\rho = \sigma$ and $S_\rho = \{-1\}$. Moreover, since the support functions h_z are positively homogeneous on $\omega = \mathbb{Q}_{\geq 0}$, they are linear and so $D_{-1} = \mathfrak{D}(1)$ (see Lemma 2.3.3). By Theorem 2.3.8 the homogeneous LNDs of fiber type on A are in one to one correspondence with the rational functions

$$\varphi \in H^0(C, \mathcal{O}_C(\lfloor -\mathfrak{D}(1) \rfloor)) .$$

This corresponds to Theorem 3.12 in *loc. cit.*

If a graded parabolic 2-dimensional algebra A admits a homogeneous LND of horizontal type, then Spec A is a toric variety by the same argument as in the elliptic case. This yields Theorem 3.16 and Corollary 3.19 in *loc. cit.*

In the hyperbolic case the σ-polyhedral divisor $\mathfrak{D}$ is uniquely determined by the pair of $\mathbb{Q}$-divisors $(\mathfrak{D}(1), \mathfrak{D}(-1))$ which correspond to the pair (D_+, D_-) in the

DPD presentation data. Since $\mathfrak{D}$ is a proper polyhedral divisor, this pair satisfies $\mathfrak{D}(1)+\mathfrak{D}(-1)\leq 0$. In this case, by Lemma 2.3.1 there is no homogeneous LND of fiber type on A since $\sigma=\{0\}$. This corresponds to Lemma 3.20 in *loc. cit.*

The homogeneous LNDs of horizontal type are classified in Theorem 2.3.26 above. Specializing this classification to dimension 2 gives Theorem 3.22 in *loc. cit.* More precisely, conditions (*i*) and (*ii*) of 2.3.26 lead to (*i*) of Theorem 3.22 in *loc. cit.* while (*iii*) and (*iv*) in 2.3.26 lead to (*ii*) in Theorem 3.22 in *loc. cit.*

In contrast, in dimension 3 a new phenomena appear. For instance, there exist non-toric threefolds with an elliptic $\mathbb{T}$-action and a homogeneous LND of horizontal or fiber type, see Section 3.3.1 for an example of fiber type. With the notation as in Section 3.3.1, considering $C=\mathbb{P}^1$ and $\mathfrak{D}=\frac{1}{2}\Delta\cdot[0]+\frac{1}{2}\Delta\cdot[1]+\Delta'\cdot[\infty]$, where $\Delta'=\sigma\cap\{\langle(1,1),\cdot\rangle\geq 1\}\subseteq N_{\mathbb{Q}}$ gives a non-toric example with 2 equivalence classes of homogeneous LNDs of fiber type and 4 equivalence classes of homogeneous LNDs of horizontal type.

2.4. Compatible $\mathbb{G}_a$-actions of fiber type in arbitrary complexity

In this section we give a complete classification of compatible $\mathbb{G}_a$-actions on $\mathbb{T}$-varieties over an algebraically closed field $\mathbf{k}$ of characteristic 0.

One of the main results applied in the classification of $\mathbb{G}_a$-actions of fiber type in complexity one is 2.1.9 (*ii*) that allows to extend an LND of fiber type to a semigroup algebra over the field of $\mathbb{T}$-invariant rational functions. The same result will allows us to obtain a classification in the more general case of arbitrary complexity.

We fix a smooth semiprojective variety Y and a proper σ-polyhedral divisor

$$\mathfrak{D}=\sum_Z \Delta_Z\cdot Z \quad \text{on} \quad Y\,.$$

Letting $\mathbf{k}(Y)$ be the field of rational functions on Y and $\omega=\sigma^\vee$, we consider the affine variety $X=\operatorname{Spec} A$, where

$$A=A[Y,\mathfrak{D}]=\bigoplus_{m\in\omega_M} A_m\chi^m, \quad \text{with} \quad A_m=H^0\left(Y,\mathcal{O}(\mathfrak{D}(m))\right)\subseteq\mathbf{k}(Y)\,.$$

We denote by h_Z the support function of Δ_Z so that

$$\mathfrak{D}(m)=\sum_Z h_Z(m)\cdot Z, \quad \text{for all} \quad m\in\omega_M\,.$$

We also fix a homogeneous LND ∂ of fiber type on A, and we let $\bar{A}=\mathbf{k}(Y)[\omega_M]$ be the affine semigroup algebra of ω_M over $\mathbf{k}(Y)$. By Lemma 2.1.9 (*ii*) ∂ can be extended to a homogeneous locally nilpotent $\mathbf{k}(Y)$-derivation $\bar{\partial}$ on $\bar{A}$.

If σ has no ray i.e., $\sigma=\{0\}$, then $\bar{\partial}=0$ by Theorem 2.2 and so ∂ is trivial. In the sequel we assume that σ has at least one ray, say ρ. Let τ be its dual facet, and let S_ρ be as defined in Definition 2.2.3.

Definition 2.4.1. Similarly to Lemma 2.3.3, for any $e\in S_\rho$, we let D_e be the $\mathbb{Q}$-divisor on Y defined by

$$D_e:=\sum_Z \max_{m\in\omega_M\setminus\tau_M}\left(h_Z(m)-h_Z(m+e)\right)\cdot Z\,.$$

Remark 2.4.2. For every prime divisor Z on Y, we let $\{\delta_{1,Z},\cdots,\delta_{\ell_Z,Z}\}$ be the set of all maximal cones in $\Lambda(h_Z)$, where the facet τ is contained in $\delta_{1,Z}$. We also let

$g_{r,Z}$, $r \in \{1, \cdots, \ell_Z\}$ be the linear extension of $h_Z|_{\delta_{r,Z}}$ to $M_{\mathbb{Q}}$. The same argument as in Remark 2.3.4 shows that

$$D_e = -\sum_Z g_{1,Z}(e) \cdot Z\,.$$

The proofs of Lemma 2.4.3, Theorem 2.4.4, and Corollary 2.4.5 are analogous to the corresponding results in Section 2.3.1.

LEMMA 2.4.3. *For any $e \in S_\rho$ we define $\Phi_e = H^0(Y, \mathcal{O}_Y(-D_e))$. If $\varphi \in \mathbf{k}(Y)$ then $\varphi \in \Phi_e$ if and only if $\varphi A_m \subseteq A_{m+e}$ for any $m \in \omega_M \setminus \tau_M$.*

The following theorem gives a classification of LNDs of fiber type on normal affine $\mathbb{T}$-varieties analogous to Lemma 2.3.7 and Theorem 2.3.8. We let $\Phi_e^* = \Phi_e \setminus \{0\}$.

THEOREM 2.4.4. *To any triple (ρ, e, φ), where ρ is a ray of σ, $e \in S_\rho$, and $\varphi \in \Phi_e^*$, we can associate a homogeneous LND $\partial_{\rho,e,\varphi}$ of fiber type on $A = A[Y, \mathfrak{D}]$ of degree e with kernel*

$$\ker \partial_{\rho,e,\varphi} = \bigoplus_{m \in \tau_M} A_m \chi^m\,.$$

Conversely, every non-trivial homogeneous LND ∂ of fiber type on A is of the form $\partial = \partial_{\rho,e,\varphi}$ for some ray $\rho \subseteq \sigma$, some lattice vector $e \in S_\rho$, and some function $\varphi \in \Phi_e^*$.

COROLLARY 2.4.5. *Let as before ∂ be a homogeneous LND of fiber type on $A = A[Y, \mathfrak{D}]$, and let $f\chi^m \in A \setminus \ker\partial$ be a homogeneous element. Then ∂ is completely determined by the image $g\chi^{m+e} := \partial(f\chi^m) \in A_{m+e}\chi^{m+e}$.*

It might happen that Φ_e^* as above is empty. Given a ray $\rho \subseteq \sigma$, in the following theorem we give a criterion for the existence of $e \in S_\rho$ such that Φ_e^* is non-empty. The proof depends on the geometry of the variety Y and so it is not a direct generalization of the analogous result in Section 2.3.1.

THEOREM 2.4.6. *Let $A = A[Y, \mathfrak{D}]$, and let $\rho \subseteq \sigma$ be the ray dual to a facet $\tau \subseteq \omega$. Then there exists $e \in S_\rho$ such that $\dim \Phi_e$ is positive if and only if the divisor $\mathfrak{D}(m)$ is big for all lattice vectors $m \in \operatorname{rel.int}(\tau)$.*

PROOF. Let the notation be as in Remark 2.4.2. Assuming that $\mathfrak{D}(m)$ is big for all lattice vector $m \in \operatorname{rel.int}(\tau)$, we consider the linear map

$$G : M_{\mathbb{Q}} \to \operatorname{Div}_{\mathbb{Q}}(Y), \quad m \mapsto \sum_Z g_{1,Z}(m) \cdot Z\,,$$

so that $G(m) = \mathfrak{D}(m)$ for all $m \in \tau$ and $D_e = -G(e)$ for all $e \in S_\rho$. Choosing $m \in \operatorname{rel.int}(\tau) \cap (S_\rho + \mu)$ and $r \in \mathbb{Z}_{>0}$, we let $j = m - \frac{1}{r} \cdot \mu$. We consider the divisor

$$G(j) = G(m) - \tfrac{1}{r} \cdot G(\mu) = \mathfrak{D}(m) - \tfrac{1}{r} \cdot G(\mu)\,.$$

Since $\mathfrak{D}(m)$ is big and the big cone is open in $\operatorname{Div}_{\mathbb{R}}(Y)$ (see [Laz04, Def. 2.2.25]), by choosing r big enough, we may assume that $G(j)$ is big. Furthermore, possible increasing r, we may assume that $G(r \cdot j)$ has a section. Now,

$$r \cdot j = r \cdot m - \mu = (r-1) \cdot m + (m - \mu)\,.$$

Since $(r-1) \cdot m \in \tau_M$ and $m - \mu \in S_\rho$, we have $r \cdot j \in S_\rho$. Letting $e = r \cdot j \in S_\rho$ we obtain $D_e = -G(e)$ and so $\dim H^0(Y, O_Y(-D_e))$ is positive.

Assume now that there is $m \in \operatorname{rel.int}(\tau)$ such that $\mathfrak{D}(m)$ is not big. Since the set of big divisors is and open and convex subset in $\operatorname{Div}_{\mathbb{R}}(Y)$, the divisor $\mathfrak{D}(m)$ is not big whatever is $m \in \tau$. We let B be the algebra

$$B = \bigoplus_{m \in \tau_M} A_m \chi^m .$$

Under our assumption $\dim B < n + k - 1$. Since $\dim A = n + k$, by Lemma 2.1.4 (i) B cannot be the kernel of an LND on A. The latter implies, by Theorem 2.4.4 that there is no $e \in S_\rho$ such that $\dim \Phi_e$ is positive. □

Finally, we deduce the following corollary.

COROLLARY 2.4.7. *Two homogeneous LNDs of fiber type $\partial = \partial_{\rho,e,\varphi}$ and $\partial' = \partial_{\rho',e',\varphi'}$ on $A = A[Y, \mathfrak{D}]$ are equivalent if and only if $\rho = \rho'$. Furthermore, the equivalence classes of homogeneous LNDs of fiber type on A are in one to one correspondence with the rays $\rho \subseteq \sigma$ such that $\mathfrak{D}(m)$ is big $\forall m \in \operatorname{rel.int}(\tau)$, where τ is the facet dual to ρ.*

PROOF. The first assertion follows from the description of $\ker \partial_{\rho,e,\varphi}$ in Lemma 2.4.4. The second follows from the first one due to Theorem 2.4.6. □

2.5. Finitely generated rings of invariants

The generalized Hilbert's fourteenth problem can be formulated as follows.

Let $\mathbf{k} \subseteq L \subseteq K$ be field extensions, and let $A \subseteq K$ be a finitely generated $\mathbf{k}$-algebra. Is it true that the $\mathbf{k}$-algebra $A \cap L$ is also finitely generated?

In the case where $K = \operatorname{Frac} A$ and $\operatorname{Spec} A$ has a $\mathbb{G}_a$-action, we consider $L = K^{\mathbb{G}_a}$ so that $A \cap L$ is the subring of invariants of the $\mathbb{G}_a$-action. So $A \cap L = \ker \partial$, where ∂ is the associated LND on A. In this case the answer is known to be negative even for the polynomial rings in $n \geq 5$ variables.

Explicit counterexamples can be found in [Rob90], [Fre00] and [DF99] (see also [Fre06, Chapter 7]). For instance, Daigle and Freudenburg showed in [DF99] that $\ker \partial$ is not finitely generated for the LND

$$\partial = x_1^3 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_4} + x_1^2 \frac{\partial}{\partial x_5}$$

on $\mathbf{k}^{[5]} = \mathbf{k}[x_1, \ldots, x_5]$. Furthermore it is easy to see that ∂ is homogeneous of degree $(-1, 1)$ under the effective $\mathbb{Z}^2$-grading on $\mathbf{k}^{[5]}$ given by

$$\deg x_1 = (1,0), \quad \deg x_2 = (2,1), \quad \deg x_3 = (1,2),$$
$$\deg x_4 = (0,3), \quad \deg x_5 = (1,1) .$$

The corresponding $\mathbb{T}$-action on $\mathbb{A}^5$ is of complexity 3. In the following example, we describe the $\mathbb{T}$-variety $\mathbb{A}^5$ with the given action of $\mathbb{T}$ in terms of the Altmann-Hausen description. The combinatorial description $(Y, \mathfrak{D})$ below was obtained by a routine application of the method in [AH06, Section 11].

EXAMPLE 2.5.1. Let $\widetilde{N}$ be a lattice of rank 3 and fix an isomorphism $\widetilde{N} \simeq \mathbb{Z}^3$. We let $\Sigma \subseteq \widetilde{N}_{\mathbb{Q}}$ be the complete fan having the following six rays

$$\rho_1 = \operatorname{cone}(1,0,0), \quad \rho_2 = \operatorname{cone}(0,1,0), \quad \rho_3 = \operatorname{cone}(0,0,1),$$
$$\rho_4 = \operatorname{cone}(-1,-2,-3), \quad \rho_5 = \operatorname{cone}(-1,1,3), \quad \text{and} \quad \rho_6 = \operatorname{cone}(0,1,3) ,$$

and the following 8 maximal cones

$$\sigma_1 = \mathrm{cone}(\rho_1, \rho_3, \rho_4), \quad \sigma_2 = \mathrm{cone}(\rho_2, \rho_4, \rho_5), \quad \sigma_3 = \mathrm{cone}(\rho_3, \rho_5, \rho_6),$$

$$\sigma_4 = \mathrm{cone}(\rho_1, \rho_2, \rho_4), \quad \sigma_5 = \mathrm{cone}(\rho_1, \rho_2, \rho_6), \quad \sigma_6 = \mathrm{cone}(\rho_1, \rho_3, \rho_6),$$

$$\sigma_7 = \mathrm{cone}(\rho_2, \rho_5, \rho_6), \quad \text{and} \quad \sigma_8 = \mathrm{cone}(\rho_3, \rho_4, \rho_5)\,.$$

We consider now a lattice N of rank 2 and a fixed isomorphism $N \simeq \mathbb{Z}^2$. Letting $\sigma = \mathrm{cone}((1,0),(0,1))$ be the first quadrant of $N_\mathbb{Q}$ we define the σ-polyhedra

$$\Delta_3 = \mathrm{conv}\left((0,0), (2/3, -1/3)\right) + \sigma, \quad \Delta_4 = (0,1) + \sigma,$$

$$\Delta_5 = \mathrm{conv}((0,0),(1,-1)) + \sigma, \quad \text{and} \quad \Delta_6 = \mathrm{conv}((0,1),(1,-1)) + \sigma\,.$$

Let $\widetilde{\mathbb{T}}$ be the torus corresponding to the lattice $\widetilde{M} = \mathrm{Hom}(\widetilde{N}, \mathbb{Z})$, Y be the toric variety corresponding to the fan Σ with maximal torus $\widetilde{\mathbb{T}}$, and D_i be the $\widetilde{\mathbb{T}}$-invariant prime divisor on Y corresponding to the ray ρ_i, $i = 1, \cdots, 6$. We consider the σ-polyhedral divisor on Y

$$\mathfrak{D} = \Delta_4 \cdot D_4 + \Delta_5 \cdot D_5 + \Delta_6 \cdot D_6\,.$$

Letting $\mathbb{T}$ be the torus corresponding to the lattice $M = \mathrm{Hom}(N, \mathbb{Z})$, the $\mathbb{T}$-variety $X[Y, \mathfrak{D}]$ corresponds to $\mathbb{A}^5$ with the grading given in the counterexample due to Daigle and Freudenburg above.

On the other hand, for $\mathbb{T}$-actions of complexity 0, 1, or for LNDs of fiber type we have the following result.

THEOREM 2.5.2. *Let A be a normal finitely generated effectively M-graded algebra, where M is a lattice of finite rank, and let ∂ be a homogeneous LND on A. If the complexity of the corresponding $\mathbb{T}$-action on* Spec A *is 0 or 1, or the LND ∂ is of fiber type, then* $\ker\partial$ *is finitely generated.*

PROOF. If the complexity is 0, then by Lemma 2.2.6 and Theorem 2.2.7, $\ker\partial$ is an affine semigroup algebra, and so it is finitely generated.

If the complexity is 1 and ∂ is of horizontal type, then Corollary 2.3.17 shows again that $\ker\partial$ is an affine semigroup algebra.

In the case of arbitrary complexity and ∂ of fiber type, we let $A = A[C, \mathfrak{D}]$, where $\mathfrak{D}$ is a proper σ-polyhedral divisor on a semiprojective variety Y. In the notation of Theorem 2.4.4 we have $\partial = \partial_{\rho,e,\varphi}$, where $\rho \subseteq \sigma$ is a ray. Letting $\tau \subseteq \omega$ be the facet dual to ρ, Theorem 2.4.4 shows that $\ker\partial = \bigoplus_{m \in \tau_M} A_m \chi^m$.

Let $a_1, \ldots, a_r$ be a set of homogeneous generators of A. Without loss of generality, we assume further that $\deg a_i \in \tau_M$ if and only if $1 \le i \le s < r$. We claim that $a_1, \ldots, a_s$ generate $\ker\partial$. Indeed, let P be any polynomial such that $P(a_1, \ldots, a_r) \in \ker\partial$. Since $\tau \subseteq \omega$ is a face, $\sum m_i \in \tau_M$ for $m_i \in \omega_M$ implies that $m_i \in \tau\ \forall i$. Hence all the monomials composing $P(a_1, \ldots, a_r)$ are monomials in $a_1, \ldots, a_s$, proving the claim. □

REMARK 2.5.3. In the particular case where Spec A is rational, Theorem 2.5.2 is a consequence of the following theorem [Kur03].

THEOREM 2.5.4. *Let $X = \mathrm{Spec}\, A$ be a $\mathbb{T}$-variety, ∂ be a homogeneous LND on A, and ∂' be the extension of ∂ to a derivation on $K = \mathrm{Frac}\, A$. If $\ker\partial' \cap K^{\mathbb{T}}$ is a purely transcendental extension of $\mathbf{k}$ of degree at most 1, then $\ker\partial$ is finitely generated.*

In the following corollary we apply Theorem 2.5.4 to prove that the ring of invariants of any compatible $\mathbb{G}_a$-action on a rational $\mathbb{T}$-variety of complexity two is finitely generated.

COROLLARY 2.5.5. *Let* $X = \operatorname{Spec} A$ *be a normal rational affine* $\mathbb{T}$*-variety of complexity two. If* ∂ *is a homogeneous LND on* A*, then* $\ker \partial$ *is finitely generated.*

PROOF. If ∂ is of fiber type, then by Theorem 2.5.2 $\ker \partial$ is finitely generated. In the case where ∂ is of horizontal type, we denote by ∂' the extension of ∂ to the field of rational functions $K = \operatorname{Frac} A$. By Lüroth's Theorem $\ker \partial' \cap K^{\mathbb{T}}$ is a purely transcendental extension of $\mathbf{k}$ of degree 1. Hence $\ker \partial$ is finitely generated by Theorem 2.5.4. □

REMARK 2.5.6. To our best knowledge it is unknown whether Corollary 2.5.5 holds without the rationality hypothesis.

CHAPTER 3

The Makar-Limanov invariant

The Makar-Limanov invariant [KML97] (ML for short) is an important tool which allows, in particular, to distinguish certain varieties from the affine space. In this chapter, we consider a homogeneous version of the ML invariant.

For toric varieties and $\mathbb{T}$-varieties of complexity one we give an explicit expression of the latter invariant in terms of the classification developed in Chapter 2. The triviality of the homogeneous ML invariant implies that of the usual one. As an application we show a first example of a non-rational affine variety having a trivial ML invariant. This is a mayor shortcoming for the ML invariant.

Furthermore, we establish a birational characterization of affine varieties with trivial ML invariant and propose a field version of the ML invariant called the FML invariant. We conjecture that the triviality of the FML invariant implies rationality. We confirm this conjecture in dimension at most 3.

3.1. The homogeneous Makar-Limanov invariant

In this section we introduce the ML invariant and its homogeneous version, and show that there is a significant difference between these two invariants.

Definition 3.1.1. Let $X = \operatorname{Spec} A$ be a normal affine variety, and let $\mathrm{LND}(A)$ be the set of all LNDs on A. The *Makar-Limanov invariant* of A (or, equivalently, of X) is defined as

$$\mathrm{ML}(X) = \mathrm{ML}(A) = \bigcap_{\partial \in \mathrm{LND}(A)} \ker \partial\,.$$

Similarly, if A is effectively M-graded we let $\mathrm{LND}_{\mathrm{h}}(A)$ be the set of all homogeneous LNDs on A, $\mathrm{LND}_{\mathrm{fib}}(A)$ be the set of all homogeneous LNDs of fiber type on A, and $\mathrm{LND}_{\mathrm{hor}}(A)$ be the set of all homogeneous LNDs of horizontal type on A. We define

$$\mathrm{ML}_{\mathrm{h}}(X) = \mathrm{ML}_{\mathrm{h}}(A) = \bigcap_{\partial \in \mathrm{LND}_{\mathrm{h}}(A)} \ker \partial$$

the *homogeneous Makar-Limanov invariant* of A. We also let

$$\mathrm{ML}_{\mathrm{fib}}(A) = \bigcap_{\partial \in \mathrm{LND}_{\mathrm{fib}}(A)} \ker \partial, \quad \text{and} \quad \mathrm{ML}_{\mathrm{hor}}(A) = \bigcap_{\partial \in \mathrm{LND}_{\mathrm{hor}}(A)} \ker \partial\,.$$

Clearly,

$$\mathrm{ML}(A) \subseteq \mathrm{ML}_{\mathrm{h}}(A) \subseteq \mathrm{ML}_{\mathrm{fib}}(A), \quad \text{and} \quad \mathrm{ML}_{\mathrm{h}}(A) = \mathrm{ML}_{\mathrm{hor}}(A) \cap \mathrm{ML}_{\mathrm{fib}}(A)\,. \tag{11}$$

Remark 3.1.2.

(*i*) Let $X = \operatorname{Spec} A$ be an affine variety. Taking the kernel $\ker \partial$ on an LND ∂ on A is the same as taking the ring of invariants $H^0(X, \mathcal{O}_X)^{\mathbb{G}_\mathrm{a}}$ by the corresponding $\mathbb{G}_\mathrm{a}$-action, see Remark 2.1.3. Therefore, the above invariants can be expressed in terms of the $\mathbb{G}_\mathrm{a}$-actions on X.

(*ii*) Since two equivalent LNDs (see Definition 2.1.5) have the same kernel, to compute $\mathrm{ML}(A)$ or $\mathrm{ML_h}(A)$ it is sufficient to consider pairwise non-equivalent LNDs on A.

Now, we provide examples showing that, in general, the inclusions in (11) are strict and so, the homogeneous LNDs are not enough to compute the ML invariant.

EXAMPLE 3.1.3. Let $A = \mathbf{k}[x, y]$ with the grading given by $\deg x = 0$ and $\deg y = 1$. In this case, both partial derivatives $\partial_x = \partial/\partial x$ and $\partial_y = \partial/\partial y$ are homogeneous. Since $\ker \partial_x = \mathbf{k}[y]$ and $\ker \partial_y = \mathbf{k}[x]$ we have $\mathrm{ML_h} = \mathbf{k}$. Furthermore, it is easy to see that there is only one equivalence class of LNDs of fiber type. A representative of this class is ∂_y (see Corollary 2.3.10). This yields $\mathrm{ML_{fib}}(A) = \mathbf{k}[x]$. Thus $\mathrm{ML_h}(A) \subsetneq \mathrm{ML_{fib}}(A)$ in this case.

EXAMPLE 3.1.4. To provide an example where $\mathrm{ML}(A) \subsetneq \mathrm{ML_h}(A)$ we consider the Koras-Russell threefold $X = \operatorname{Spec} A$, where

$$A = \mathbf{k}[x, y, z, t]/(x + x^2y + z^2 + t^3)\,.$$

The ML invariant was first introduced in [KML97] to prove that $X \not\simeq \mathbb{A}^3$. In fact $\mathrm{ML}(A) = \mathbf{k}[x]$ while $\mathrm{ML}(\mathbb{A}^3) = \mathbf{k}$ [ML96]. In the recent paper [Dub09] Dubouloz shows that the cylinder over the Koras-Russell threefold has trivial ML invariant i.e., $\mathrm{ML}(A[w]) = \mathbf{k}$, where w is a new variable.

Let $A[w]$ be graded by $\deg A = 0$ and $\deg w = 1$, and let ∂ be a homogeneous LND on $A[w]$. If $e := \deg \partial \leq -1$ then $\partial(A) = 0$ and by Lemma 2.1.4 (*i*) we have that $\ker \partial = A$ and ∂ is equivalent to the partial derivative $\partial/\partial w$.

If $e \geq 0$ then $\partial(w) = aw^{e+1}$, where $a \in A$ and so, by Lemma 2.1.4 (*vi*) $w \in \ker \partial$. Furthermore, for any $a \in A$ we have $\partial(a) = bw^e$, for a unique $b \in A$. We define a derivation $\bar{\partial} : A \to A$ by $\bar{\partial}(a) = b$. Since $\partial^r(a) = \bar{\partial}^r(a)w^{re}$ the derivation $\bar{\partial}$ is LND. This yields $\mathrm{ML_h}(A[w]) = \mathrm{ML}(A) = \mathbf{k}[x]$ while $\mathrm{ML}(A[w]) = \mathbf{k}$.

REMARK 3.1.5. In Example 3.1.4, the $\mathbb{T}$-action on $X \times \mathbb{A}^1$ is of complexity three. On the contrary, in Section 3.2 we show that if X is a normal affine $\mathbb{T}$-variety of complexity zero i.e., a toric variety, then $\mathrm{ML}(X) = \mathrm{ML_h}(X)$.

To our best knowledge, it is unknown if the equality $\mathrm{ML}(X) = \mathrm{ML_h}(X)$ holds in complexity one or two. Nevertheless, Theorem 4.5 in [FZ05a] shows that it does hold for $\mathbf{k}^*$-surfaces.

In the following two sections we apply the results in Section 2.2 and 2.3 in order to compute $\mathrm{ML_h}(A)$ in the case where the complexity of the $\mathbb{T}$-action on Spec A is 0 or 1. We also give some partial results for the usual invariant $\mathrm{ML}(A)$ in this particular case.

3.2. ML-invariant of toric varieties

We treat now the case of affine toric varieties. Let $\sigma \subseteq N_{\mathbb{Q}}$ be a pointed polyhedral cone and $\omega \subseteq M_{\mathbb{Q}}$ be its dual cone.

PROPOSITION 3.2.1. *Let $A = \mathbf{k}[\omega_M]$ be an affine semigroup algebra so that $X =$ Spec A is a toric variety. Then*

$$\mathrm{ML}(A) = \mathrm{ML_h}(A) = \mathbf{k}[\theta_M]\,,$$

where $\theta \subseteq M_{\mathbb{Q}}$ is the maximal subspace contained in ω. In particular $\mathrm{ML}(A) = \mathbf{k}$ *if and only if σ is of complete dimension i.e., if and only if there is no torus factor in X.*

PROOF. By Corollary 2.2.11 and Theorem 2.2.7, the pairwise non-equivalent homogeneous LNDs on A are in one to one correspondence with the rays of σ. For any ray $\rho \subseteq \sigma$ and any $e \in S_\rho$ as in Lemma 2.2.4, the kernel of the corresponding homogeneous LND is $\ker \partial_{\rho,e} = \mathbf{k}[\tau_M]$, where $\tau \subseteq \omega$ is the facet dual to ρ.

Since $\theta \subseteq \omega$ is the intersection of all facets, we have $\mathrm{ML}_{\mathrm{h}}(A) = \mathbf{k}[\theta_M]$. Furthermore, the characters in $\mathbf{k}[\theta_M] \subseteq A$ are invertible functions on A and so, by Lemma 2.1.4 (iii), $\partial(\mathbf{k}[\theta_M]) = 0\ \forall \partial \in \mathrm{LND}(A)$. Hence $\mathbf{k}[\theta_M] \subseteq \mathrm{ML}(A)$, proving the lemma. □

3.3. ML-invariant of $\mathbb{T}$-varieties of complexity one

In this section we give a combinatorial description of the homogeneous ML invariant of $\mathbb{T}$-varieties of complexity one in terms of the Altmann-Hausen description. Let $A = A[C, \mathfrak{D}]$, where $\mathfrak{D}$ is a proper σ-polyhedral divisor on a smooth curve C.

We first compute $\mathrm{ML}_{\mathrm{fib}}(A)$. If A is non-elliptic (elliptic, respectively) we let $\{\rho_i\}$ be the set of all rays of ω (of all rays of ω such that $\rho \cap \deg \mathfrak{D} = \emptyset$, respectively). In both cases we let $\tau_i \subseteq M_{\mathbb{Q}}$ denote the facet dual to ρ_i and $\theta = \bigcap \tau_i$.

LEMMA 3.3.1. *With the notation as above,*

$$\mathrm{ML}_{\mathrm{fib}}(A) = \bigoplus_{m \in \theta_M} A_m \chi^m\,.$$

PROOF. By Corollary 2.3.12, for every ray ρ_i there is a homogeneous LND ∂_i of fiber type with kernel

$$\ker \partial_i = \bigoplus_{m \in \tau_i \cap M} A_m \chi^m\,.$$

By Corollary 2.3.10 any homogeneous LND of fiber type on A is equivalent to one of the ∂_i. Finally, taking the intersection $\bigcap_i \ker \partial_i$ gives the desired description of $\mathrm{ML}_{\mathrm{fib}}(A)$. □

REMARK 3.3.2. If A is non-elliptic, then $\theta \subseteq M_{\mathbb{Q}}$ is the maximal subspace contained in ω. In particular, if A is parabolic then $\theta = \{0\}$ and $\mathrm{ML}_{\mathrm{fib}}(A) = A_0$, and if A is hyperbolic then $\theta = M_{\mathbb{Q}}$ and $\mathrm{ML}_{\mathrm{fib}}(A) = A$.

If there is no LND of horizontal type on A, then $\mathrm{ML}_{\mathrm{hor}}(A) = A$ and $\mathrm{ML}_{\mathrm{h}}(A) = \mathrm{ML}_{\mathrm{fib}}(A)$. In the sequel we assume that A admits a homogeneous LND of horizontal type.

If A is non-elliptic, we let $\{\delta_i\}$ be the set of all cones in $M_{\mathbb{Q}}$ satisfying (i) in Theorem 2.3.26, and $\delta = \bigcap_i \delta_i$. If A is elliptic, we let $\{\delta_{i,z}\}$ be the set of all cones in $M_{\mathbb{Q}}$ satisfying (i') in Theorem 2.3.26 with $z_\infty = z$, $B = \{m \in \omega : h_{\deg \mathfrak{D}} = 0\}$, and $\delta = \bigcap_{i,z} \delta_{i,z} \cap B$.

LEMMA 3.3.3. *With the notation as before, if ∂ is a homogeneous LND on A of horizontal type, then*

$$\mathrm{ML}_{\mathrm{hor}}(A) = \bigoplus_{m \in \delta_L} \mathbf{k}\varphi_m \chi^m\,,$$

where $L = L(\partial)$ and $\varphi_m \in A_m$ satisfy the relation $\mathrm{div}(\varphi_m) + \mathfrak{D}(m) = 0$.

PROOF. We treat first the non-elliptic case. By Corollary 2.3.27 for every δ_i there is a homogeneous LND ∂_i of horizontal type with kernel

$$\ker \partial_i = \bigoplus_{m \in \delta_i \cap L_i} \mathbf{k}\varphi_m \chi^m\,,$$

where $L_i = L(\partial_i)$ and $\varphi_m \in A_m$ is such that $\operatorname{div}(\varphi_m) + \mathfrak{D}(m) = 0$. By Corollary 2.3.28, any homogeneous LND of horizontal type on A is equivalent to one of the ∂_i. Taking the intersection of all $\ker \partial_i$ gives the lemma in this case.

Let further A be elliptic, and let ∂ be a homogeneous LND of horizontal type on A. Let $z_0, z_\infty \in \mathbb{P}_1$, and η and L be as in Theorem 2.3.26 so that

$$\ker \partial = \bigoplus_{m \in \eta_L} \mathbf{k}\varphi_m \chi^m ,$$

where $\varphi_m \in A_m$ satisfies

$$\operatorname{div}(\varphi_m)|_{\mathbb{P}^1 \setminus \{z_\infty\}} + \mathfrak{D}(m)|_{\mathbb{P}^1 \setminus \{z_\infty\}} = 0 .$$

By permuting the roles of z_0 and z_∞ in Theorem 2.3.26 we obtain another LND ∂' on A. The description of $\ker \partial$ and $\ker \partial'$ shows that

$$\ker \partial \cap \ker \partial' = \bigoplus_{\eta_L \cap B} \mathbf{k}\varphi \chi^m ,$$

where $\varphi_m \in A_m$ is such that $\operatorname{div}(\varphi_m) + \mathfrak{D}(m) = 0$.

Now the lemma follows by an argument similar to that in the non-elliptic case. □

THEOREM 3.3.4. *In the notation of Lemmas 3.3.1 and 3.3.3, if there is no homogeneous LND of horizontal type on A, then*

$$\mathrm{ML}_{\mathrm{h}}(A) = \bigoplus_{m \in \theta_M} A_m \chi^m .$$

If ∂ is a homogeneous LND of horizontal type on A, then

$$\mathrm{ML}_{\mathrm{h}}(A) = \bigoplus_{m \in \theta \cap \delta_L} \mathbf{k}\varphi_m \chi^m ,$$

where $L = L(\partial)$ and $\varphi_m \in A_m$ is such that $\operatorname{div}(\varphi_m) + \mathfrak{D}(m) = 0$.

PROOF. The assertions follow immediately by virtue of (11) and Lemmas 3.3.1 and 3.3.3. □

In the following corollary we give a criterion of triviality of the homogeneous Makar-Limanov invariant $\mathrm{ML}_{\mathrm{h}}(A)$.

COROLLARY 3.3.5. *With the notation as above,* $\mathrm{ML}_{\mathrm{h}}(A) = \mathbf{k}$ *if and only if one of the following conditions hold.*

(*i*) *A is elliptic,* $\operatorname{rank}(M) \geq 2$, *and* $\deg \mathfrak{D}$ *does not intersect any of the rays of the cone ω.*

(*ii*) *A admits a homogeneous LND of horizontal type and $\theta \cap \delta = \{0\}$.*

In particular, in both cases $\mathrm{ML}(A) = \mathbf{k}$.

PROOF. By Lemma 3.3.1, (*i*) holds if and only if $\mathrm{ML}_{\mathrm{hor}}(A) = \mathbf{k}$. By Theorem 3.3.4, (*ii*) holds if and only if there is a homogeneous LND of horizontal type and $\mathrm{ML}_{\mathrm{h}}(A) = \mathbf{k}$. □

REMARK 3.3.6. It easily seen that $\mathrm{ML}_{\mathrm{h}}(A) = \mathbf{k}$ for A as in Example 2.3.31.

3.3.1. A non-rational threefold with trivial Makar-Limanov invariant. To exhibit such an example, we let σ be a pointed polyhedral cone in $M_{\mathbb{Q}}$, where $\operatorname{rank}(M) = n \geq 2$. We let as before $A = A[C, \mathfrak{D}]$, where $\mathfrak{D}$ is a proper σ-polyhedral divisor on a smooth curve C. By Remark 1.3.10 (*iii*), Frac $A = \mathbf{k}(C)(M)$ and so $X = \operatorname{Spec} A$ is birational to $C \times \mathbb{P}^n$.

By Corollary 3.3.5, if A is non-elliptic and $\mathrm{ML}(A) = \mathbf{k}$, then A admits a homogeneous LND of horizontal type. So $C \simeq \mathbb{A}^1$ and X is rational. On the other hand, if A is elliptic Corollary 3.3.5 (*i*) is independent of the curve C. So if (*i*) is fulfilled, then $\mathrm{ML}(A) = \mathbf{k}$ while X is birational to $C \times \mathbb{P}^n$. This leads to the following proposition.

Proposition 3.3.7. *Let $A = A[C, \mathfrak{D}]$, where $\mathfrak{D}$ is a proper σ-polyhedral divisor on a smooth projective curve C of positive genus. Suppose further that* $\deg \mathfrak{D}$ *is contained in the relative interior of σ. Then* $\mathrm{ML}(A) = \mathbf{k}$ *whereas* Spec A *is non-rational.*

Remark 3.3.8. It is evident that X in Proposition 3.3.7 is in fact stably non-rational i.e., $X \times \mathbb{P}^{\ell}$ is non-rational for all $\ell \geq 0$, cf. [Pop10, Example 1.22].

In the remaining of this section we give a concrete geometric example illustrating this proposition.

Example 3.3.9. Letting $N = \mathbb{Z}^2$ and $M = \mathbb{Z}^2$ with the canonical bases and duality, we let $\sigma \subseteq N_{\mathbb{Q}}$ be the first quadrant, $\Delta = (1,1) + \sigma$, and $h = h_{\Delta}$ so that $h(m_1, m_2) = m_1 + m_2$.

Furthermore, we let $A = A[C, \mathfrak{D}]$, where $C \subseteq \mathbb{P}^2$ is the elliptic curve with affine equation $s^2 - t^3 + t = 0$, and $\mathfrak{D} = \Delta \cdot P$ is the proper σ-polyhedral divisor on C with P being the point at infinity of C.

Since $C \not\simeq \mathbb{P}^1$ and $\deg \mathfrak{D} = \Delta$, A satisfies the assumptions of Corollary 3.3.7. Letting $\mathbf{k}(C)$ be the function field of C, by Theorem 1.5.5 we obtain

$$A_{(m_1,m_2)} = H^0(C, \mathcal{O}((m_1 + m_2)P)) \subseteq \mathbf{k}(C)\,.$$

The functions $t, s \in \mathbf{k}(C)$ are regular in the affine part of C, and have poles of order 2 and 3 on P, respectively. By the Riemann-Roch theorem $\dim H^0(C, \mathcal{O}(rP)) = r\ \forall r > 0$. Hence the functions $\{t^i, t^j s : 2i \leq r \text{ and } 2j + 3 \leq r\}$ form a basis of $H^0(C, \mathcal{O}(rP))$ (see [Har77] Chapter IV, Proposition 4.6).

In this setting the first gradded pieces are the $\mathbf{k}$-modules

$$\begin{gathered} A_{(0,0)} = A_{(1,0)} = A_{(0,1)} = \mathbf{k}\,, \\ A_{(2,0)} = A_{(1,1)} = A_{(0,2)} = \mathbf{k} + \mathbf{k}t\,, \\ A_{(3,0)} = A_{(2,1)} = A_{(1,2)} = A_{(0,3)} = \mathbf{k} + \mathbf{k}t + \mathbf{k}s\,, \\ A_{(4,0)} = A_{(3,1)} = A_{(2,2)} = A_{(1,3)} = A_{(0,4)} = \mathbf{k} + \mathbf{k}t + \mathbf{k}t^2 + \mathbf{k}s\,. \end{gathered}$$

It is easy to see that A admits the following set of generators.

$$u_1 = \chi^{(1,0)}, \qquad u_2 = \chi^{(0,1)}, \qquad u_3 = t\chi^{(2,0)}, \qquad u_4 = t\chi^{(1,1)}, \qquad u_5 = t\chi^{(0,2)},$$
$$u_6 = s\chi^{(3,0)}, \qquad u_7 = s\chi^{(2,1)}, \qquad u_8 = s\chi^{(1,2)}, \qquad u_9 = s\chi^{(0,3)}\,.$$

So $A \simeq \mathbf{k}^{[9]}/I$, where $\mathbf{k}^{[9]} = \mathbf{k}[x_1, \ldots, x_9]$, and I is the ideal of relations of u_i ($i = 1 \ldots 9$). Using a software for elimination theory it is possible to show that following list is a minimal set of generators of I.

$$
\begin{array}{cccc}
x_2x_4 - x_1x_5\,, & -x_4^2 + x_3x_5\,, & x_2x_3 - x_1x_4\,, & -x_5x_8 + x_4x_9\,, \\
-x_5x_7 + x_4x_8\,, & -x_5x_6 + x_4x_7\,, & -x_4x_8 + x_3x_9\,, & -x_4x_7 + x_3x_8\,, \\
-x_4x_6 + x_3x_7\,, & x_2x_8 - x_1x_9\,, & -x_8^2 + x_7x_9\,, & -x_7x_8 + x_6x_9\,, \\
x_2x_7 - x_1x_8\,, & -x_7^2 + x_6x_8\,, & x_2x_6 - x_1x_7\,, & -x_2^4x_5 - x_5^3 + x_9^2\,,
\end{array}
$$
$$
\begin{array}{cc}
-x_1x_2^3x_5 - x_4x_5^2 + x_8x_9\,, & -x_1^2x_2^2x_5 - x_3x_5^2 + x_7x_9\,, \\
-x_1^3x_2x_5 - x_3x_4x_5 + x_6x_9\,, & -x_1^4x_5 - x_3^2x_5 + x_6x_8\,, \\
-x_1^4x_4 - x_3^2x_4 + x_6x_7\,, & -x_1^4x_3 - x_3^3 + x_6^2\,.
\end{array}
$$

Furthermore, $A_m \subseteq \mathbf{k}[s,t]/(s^2 - t^3 + t)$ $\forall m \in \omega_M$ since $\mathfrak{D}$ is supported at the point at infinity P. The semigroup ω_M is spanned by $(1,0)$ and $(0,1)$, so letting $v = \chi^{(1,0)}$ and $w = \chi^{(0,1)}$ we obtain

$$A = \mathbf{k}[v, w, tv^2, tvw, tw^2, sv^3, sv^2w, svw^2, sw^3] \subseteq \mathbf{k}[s,t,v,w]/(s^2 - t^3 + t)\,.$$

Thus Spec A is birationally dominated by $C_0 \times \mathbb{A}^2$, where $C_0 = C \setminus \{P\}$.

Since $C \not\simeq \mathbb{P}^1$, by Lemma 2.3.14 there is no homogeneous LND of horizontal type on A. There are two rays $\rho_i \subseteq \sigma$ spanned by the vectors $(1,0)$ and $(0,1)$. Since $\deg \mathfrak{D} = \Delta$ is contained in the relative interior of σ, Corollaries 2.3.10 and 2.3.12 imply that there are exactly 2 pairwise non-equivalent homogeneous LNDs ∂_i of fiber type which correspond to the rays ρ_i, $i = 1, 2$, respectively.

The facet τ_1 dual to ρ_1 is spanned by $(0,1)$ and, in the notation of Lemma 2.3.7, $S_{\rho_1} = \{(-1, r) \mid r \geq 0\}$. Letting $e_1 = (-1,1)$ yields $D_{e_1} = 0$ and so $\Phi_{e_1} = \mathbf{k}$. We fix $\varphi_1 = 1 \in \Phi_{e_1}$. By the same lemma we can chose $\partial_1 = \partial_{\rho_1, e_1, \varphi_1}$ as

$$\partial_1\left(\chi^{(m_1,m_2)}\right) = m_1 \cdot \chi^{(m_1-1, m_2+1)}, \quad \text{for all} \quad (m_1, m_2) \in \sigma_M^\vee\,.$$

Likewise, the facet τ_2 dual to ρ_2 is spanned by $(1,0)$ and, in the notation of Lemma 2.3.7, $S_{\rho_2} = \{(r, -1) : r \geq 0\}$. Letting $e_2 = (1,-1)$ yields $D_{e_2} = 0$ and so $\Phi_{e_2} = \mathbf{k}$. We fix $\varphi_2 = 1 \in \Phi_{e_2}$. By Lemma 2.3.7 we can chose $\partial_2 = \partial_{\rho_2, e_2, \varphi_2}$ as

$$\partial_2\left(\chi^{(m_1,m_2)}\right) = m_2 \cdot \chi^{(m_1+1, m_2-1)}, \quad \text{for all} \quad (m_1, m_2) \in \sigma_M^\vee\,.$$

The kernels of ∂_1 and ∂ are given by

$$\ker \partial_1 = \bigoplus_{m \in \tau_1 \cap M} A_m \chi^m \quad \text{and} \quad \ker \partial_2 = \bigoplus_{m \in \tau_2 \cap M} A_m \chi^m\,.$$

Since $\tau_1 \cap \tau_2 = \{0\}$ we have

$$\mathrm{ML}(A) = \ker \partial_1 \cap \ker \partial_2 = A_{(0,0)} = \mathbf{k}\,.$$

This agrees with Corollary 3.3.7.

The LNDs ∂_i are induced, under the isomorphism $A \simeq \mathbf{k}^{[9]}/I$, by the following LNDs on $\mathbf{k}^{[9]}$:

$$\partial_1 = x_2\frac{\partial}{\partial x_1} + 2x_4\frac{\partial}{\partial x_3} + x_5\frac{\partial}{\partial x_4} + 3x_7\frac{\partial}{\partial x_6} + 2x_8\frac{\partial}{\partial x_7} + x_9\frac{\partial}{\partial x_8}\,,$$

and

$$\partial_2 = x_1\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_4} + 2x_4\frac{\partial}{\partial x_5} + x_6\frac{\partial}{\partial x_7} + 2x_7\frac{\partial}{\partial x_8} + 3x_8\frac{\partial}{\partial x_9}\,,$$

respectively.

We let below $X = $ Spec A, and we let $\pi : X \dashrightarrow C$ be the rational quotient for the $\mathbb{T}$-action on X. The comorphism of π is given by the inclusion $\pi^* : \mathbf{k}(C) \hookrightarrow$ Frac $A = \mathbf{k}(C)(u_1, u_2)$.

The orbit closure $\Theta=\overline{\pi^{-1}(0,0)}$ over $(0,0)\in C$ is general and it is isomorphic to $\mathbb{A}^2=\operatorname{Spec}\mathbf{k}[x_1,x_2]$. The restrictions to Θ of the $\mathbb{G}_a$-actions ϕ_i corresponding to ∂_i, $i=1,2$, respectively are given by

$$\phi_1|_\Theta:(t,(x_1,x_2))\mapsto(x_1+tx_2,x_2)\quad\text{and}\quad\phi_2|_\Theta:(t,(x_1,x_2))\mapsto(x_1,x_2+tx_1)\,.$$

Furthermore, there is a unique singular point $\bar{0}\in X$ corresponding to the fixed point of the $\mathbb{T}$-action on X. The point $\bar{0}$ is given by the augmentation ideal

$$A_+=\bigoplus_{\omega_M\backslash\{0\}}A_m\chi^m\,,$$

On the other hand, let $A=A[C,\mathfrak{D}]$, where $\mathfrak{D}$ is a proper σ-polyhedral divisor on a smooth projective curve C. By Theorem 2.5 in [KR82], if Spec A is smooth, then Spec $A\simeq\mathbb{A}^{n+1}$ (see also Proposition 3.1 in [Süs08]). In particular, Spec A is rational.

3.4. Birational geometry of varieties with trivial ML invariant

In this section we establish the following birational characterization of normal affine varieties with trivial ML invariant. Let $\mathbf{k}$ be an algebraically closed field of characteristic 0.

Theorem 3.4.1. *Let $X=\operatorname{Spec}A$ be an affine variety over $\mathbf{k}$. If $\mathrm{ML}(X)=\mathbf{k}$ then $X\simeq_{\mathrm{bir}}Y\times\mathbb{P}^2$ for some variety Y. Conversely, in any birational class $Y\times\mathbb{P}^2$ there is an affine variety X with $\mathrm{ML}(X)=\mathbf{k}$.*

Proof. As usual $\operatorname{tr.deg}_{\mathbf{k}}(K)$ denotes the transcendence degree of the field extension $\mathbf{k}\subseteq K$. Let $K=\operatorname{Frac}A$ be the field of rational functions on X so that $\operatorname{tr.deg}_{\mathbf{k}}(K)\geq 2$.

Since $\mathrm{ML}(X)=\mathbf{k}$, there exists at least 2 non-equivalent LNDs $\partial_1,\partial_2:A\to A$. We let $L_i=\operatorname{Frac}(\ker\partial_i)\subseteq K$, for $i=1,2$. By Lemma 2.1.4 (*vii*), $L_i\subseteq K$ is a purely transcendental extension of degree 1, for $i=1,2$.

We let $L=L_1\cap L_2$. By an inclusion-exclusion argument we have $\operatorname{tr.deg}_L(K)=2$. We let $\bar{A}$ be the 2-dimensional algebra over L

$$\bar{A}=A\otimes_{\mathbf{k}}L\,.$$

Since $\operatorname{Frac}\bar{A}=\operatorname{Frac}A=K$ and $L\subseteq\ker\partial_i$ for $i=1,2$, the LND ∂_i extends to a locally nilpotent L-derivation $\bar{\partial}_i$ by setting

$$\bar{\partial}_i(a\otimes l)=\partial_i(a)\otimes l,\quad\text{where}\quad a\in A,\ \text{and}\ l\in L\,.$$

Furthermore, $\ker\bar{\partial}_i=\bar{A}\cap L_i$, for $i=1,2$ and so

$$\ker\bar{\partial}_1\cap\ker\bar{\partial}_2=\bar{A}\cap L_1\cap L_2=L\,.$$

Thus the Makar-Limanov invariant of the 2-dimensional L-algebra $\bar{A}$ is trivial.

By the theorem in [ML, p. 41], $\bar{A}$ is isomorphic to an L-subalgebra of $L[x_1,x_2]$, where x_1,x_2 are new variables. Thus

$$K\simeq L(x_1,x_2),\quad\text{and so}\quad X\simeq_{\mathrm{bir}}Y\times\mathbb{P}^2\,,$$

where Y is any variety with L as the field of rational functions.

The second assertion follows from Lemma 3.4.2 bellow. This completes the proof. ☐

The following lemma provides examples of affine varieties with trivial ML invariant in any birational class $Y \times \mathbb{P}^n$, $n \geq 2$. It is a generalization of Section 3.3.1. Let us introduce some notation.

As before, we let N be a lattice of rank $n \geq 2$ and M be its dual lattice. We let $\sigma \subseteq N_{\mathbb{Q}}$ be a pointed polyhedral cone of full dimension. We fix $p \in \operatorname{rel.int}(\sigma) \cap M$. We let $\Delta = p + \sigma$ and $h = h_\Delta$ so that

$$h(m) = \langle p, m \rangle > 0, \quad \text{for all } m \in \omega \setminus \{0\}\,.$$

Furthermore, letting Y be a projective variety and H be a semiample and big Cartier $\mathbb{Z}$-divisor on Y, we let $A = A[Y, \mathfrak{D}]$, where $\mathfrak{D}$ is the proper σ-polyhedral divisor $\mathfrak{D} = \Delta \cdot H$, so that

$$\mathfrak{D}(m) = \langle p, m \rangle \cdot H, \quad \text{for all } m \in \omega\,.$$

Recall that $\operatorname{Frac} A = \mathbf{k}(Y)(M)$ so that $\operatorname{Spec} A \simeq_{\text{bir}} Y \times \mathbb{P}^n$.

Lemma 3.4.2. *With the above notation, the affine variety* $X = \operatorname{Spec} A[Y, \mathfrak{D}]$ *has trivial ML invariant.*

Proof. Let $\{\rho_i\}_i$ be the set of all rays of σ and $\{\tau_i\}_i$ the set of the corresponding dual facets of ω. Since rH is big for all $r > 0$, Theorem 2.4.6 shows that there exists $e_i \in S_{\rho_i}$ such that $\dim \Phi_{e_i}$ is positive, and so we can chose a non-zero $\varphi_i \in \Phi_{e_i}$. In this case, Theorem 2.4.4 shows that there exists a non-trivial locally nilpotent derivation $\partial_{\rho_i, e_i, \varphi_i}$, with

$$\ker \partial_{\rho_i, e_i, \varphi_i} = \bigoplus_{m \in \tau_i \cap M} A_m \chi^m\,.$$

Since the cone σ is pointed and has full dimension, the same holds for ω. Thus, the intersection of all facets reduces to one point $\bigcap_i \tau_i = \{0\}$ and so

$$\bigcap_i \ker \partial_{\rho_i, e_i, \varphi_i} \subseteq A_0 = H^0(Y, \mathcal{O}_Y) = \mathbf{k}\,.$$

This yields

$$\mathrm{ML}(A) = \mathrm{ML}_{\mathrm{h}}(A) = \mathrm{ML}_{\mathrm{fib}}(A) = \mathbf{k}\,.$$

□

Example 3.4.3. With the notation as in the proof of Lemma 3.4.2, we can provide yet another explicit construction. We fix isomorphisms $M \simeq \mathbb{Z}^n$ and $N \simeq \mathbb{Z}^n$ such that the standard bases $\{\mu_1, \cdots, \mu_n\}$ and $\{\nu_1, \cdots, \nu_n\}$ for $M_{\mathbb{Q}}$ and $N_{\mathbb{Q}}$, respectively, are mutually dual. We let σ be the first quadrant in $N_{\mathbb{Q}}$, and $p = \sum_i \nu_i$, so that

$$h(m) = \sum_i m_i, \text{ and } \mathfrak{D}(m) = \sum_i m_i \cdot H, \quad \text{where}$$
$$m = (m_1, \cdots, m_n), \text{ and } m_i \in \mathbb{Q}_{\geq 0}\,.$$

We let $\rho_i \subseteq \sigma$ be the ray spanned by the vector ν_i, and let τ_i be its dual facet. In this setting, $S_{\rho_i} = (\tau_i - \mu_i) \cap M$. Furthermore, letting $e_{i,j} = -\mu_i + \mu_j$ (where $j \neq i$) yields

$$h(m) = h(m + e_{i,j}), \quad \text{so that} \quad D_{e_{i,j}} = 0, \quad \text{and} \quad \Phi_{e_{i,j}} = H^0(Y, \mathcal{O}_Y) = \mathbf{k}\,.$$

Choosing $\varphi_{i,j} = 1 \in \Phi_{e_{i,j}}$ we obtain that $\partial_{i,j} := \partial_{\rho_i, e_{i,j}, \varphi_{i,j}}$ given by

$$\partial_{i,j}(f\chi^m) = \langle m, \nu_i \rangle \cdot f\chi^{m + e_{i,j}}, \quad \text{where} \quad i, j \in \{1, \cdots, n\},\ i \neq j$$

is a homogeneous LND on $A = A[Y, \mathfrak{D}]$ with degree $e_{i,j}$ and kernel

$$\ker \partial_{i,j} = \bigoplus_{\tau_i \cap M} A_m \chi^m .$$

As in the proof of Lemma 3.4.2 the intersection

$$\bigcap_{i,j} \ker \partial_{i,j} = \mathbf{k}, \quad \text{and so} \quad \mathrm{ML}(X) = \mathbf{k} .$$

We can give a geometrical description of X. Consider the $\mathcal{O}_Y$-algebra

$$\widetilde{A} = \bigoplus_{m \in \omega_M} \mathcal{O}_Y(\mathfrak{D}(m))\chi^m, \quad \text{so that} \quad A = H^0(Y, \widetilde{A}) .$$

In this case, we have

$$\widetilde{A} = \bigoplus_{r=0}^{\infty} \bigoplus_{\sum m_i = r,\ m_i \geq 0} \mathcal{O}_Y(rH)\chi^m \simeq \mathrm{Sym}\left(\bigoplus_{i=1}^{n} \mathcal{O}_Y(H)\right) .$$

And so $\widetilde{X} = \mathbf{Spec}_Y \widetilde{A}$ is the vector bundle associated to the locally free sheaf $\bigoplus_{i=1}^n \mathcal{O}_Y(H)$ (see Ch. II Ex. 5.18 in [Har77]). We let $\pi : \widetilde{X} \to Y$ be the corresponding affine morphism.

The morphism $\varphi : \widetilde{X} \to X$ induced by taking global sections corresponds to the contraction of the zero section to a point $\bar{0}$. We let $\theta := \pi \circ \varphi^{-1} : X \setminus \{\bar{0}\} \to Y$. The point $\bar{0}$ corresponds to the augmentation ideal $A \setminus \mathbf{k}$. It is the only attractive fixed point of the $\mathbb{T}$-action. The orbit closures of the $\mathbb{T}$-action on X are $\Theta_y := \overline{\theta^{-1}(y)} = \theta^{-1}(y) \cup \{0\}$, $\forall y \in Y$. Let $\chi^{\mu_i} = u_i$. Θ_y is equivariantly isomorphic to $\mathrm{Spec}\, \mathbf{k}[\omega_M] = \mathrm{Spec}\, \mathbf{k}[u_1, \cdots, u_n] \simeq \mathbb{A}^n$.

The $\mathbb{G}_a$-action $\phi_{i,j} : \mathbb{G}_a \times X \to X$ induced by the homogeneous LND $\partial_{i,j}$ restricts to a $\mathbb{G}_a$-action on Θ_y given by

$$\phi_{i,j}|_{\Theta_Y} : \mathbb{G}_a \times \mathbb{A}^n \to \mathbb{A}^n, \quad \text{where} \quad u_i \mapsto u_i + tu_j, \quad u_r \mapsto u_r,\ \forall r \neq i .$$

Moreover, the unique fixed point $\bar{0}$ is singular unless Y is a projective space and there is no other singular point. By Theorem 2.9 in [Lie09b] X has rational singularities if and only if $\mathcal{O}_Y$ and $\mathcal{O}_Y(H)$ are acyclic. The latter assumption can be fulfilled by taking, for instance, Y toric or Y a rational surface, and H a large enough multiple of an ample divisor.

3.5. A field version of the ML invariant

The main application of the ML invariant is to distinguish some varieties from the affine space. Nevertheless, this invariant is far from being optimal as we have seen in the previous section. Indeed, there is a large class of non-rational normal affine varieties with trivial ML invariant. To eliminate such a pathology, we propose below a generalization of the classical ML invariant.

Let A be a finitely generated normal domain. We define the FML invariant of A as the subfield of $K = \mathrm{Frac}\, A$ given by

$$\mathrm{FML}(A) = \bigcap_{\partial \in \mathrm{LND}(A)} \mathrm{Frac}(\ker \partial) .$$

In the case where A is M-graded we define $\mathrm{FML}_{\mathrm{h}}$ and $\mathrm{FML}_{\mathrm{fib}}$ in the analogous way.

Remark 3.5.1. Let $A = \mathbf{k}[x_1, \cdots, x_n]$ so that $K = \mathbf{k}(x_1, \cdots, x_n)$. For the partial derivative $\partial_i = \partial/\partial x_i$ we have $\operatorname{Frac}(\ker \partial_i) = \mathbf{k}(x_1, \cdots, \widehat{x_i}, \cdots, x_n)$, where $\widehat{x_i}$ means that x_i is omitted. This yields

$$\mathrm{FML}(A) \subseteq \bigcap_{i=1}^{n} \operatorname{Frac}(\ker \partial_i) = \mathbf{k}\,,$$

and so $\mathrm{FML}(A) = \mathbf{k}$. Thus, the FML invariant of the affine space is trivial.

For any finitely generated normal domain A there is an inclusion $\mathrm{ML}(A) \subseteq \mathrm{FML}(A)$. A priori, since $\mathrm{FML}(\mathbb{A}^n) = \mathbf{k}$ the FML invariant is stronger than the classical one in the sense that it can distinguish more varieties form the affine space that the classical one. In the next proposition we show that the classical ML invariant can be recovered from the FML invariant.

Proposition 3.5.2. *Let A be a finitely generated normal domain, then*

$$\mathrm{ML}(A) = \mathrm{FML}(A) \cap A\,.$$

Proof. We must show that for any LND ∂ on A,

$$\ker \partial = \operatorname{Frac}(\ker \partial) \cap A\,.$$

The inclusion "$\subseteq$" is trivial. To prove the converse inclusion, we fix an element $a \in \operatorname{Frac}(\ker \partial) \cap A$. Letting $b, c \in \ker \partial$ be such that $ac = b$, Lemma 2.1.4 (ii) shows that $a \in \ker \partial$. □

Let $A = A[Y, \mathfrak{D}]$ for some proper σ-polyhedral divisor $\mathfrak{D}$ on a normal semiprojective variety Y. In this case $K = \operatorname{Frac} A = \mathbf{k}(Y)(M)$, where $\mathbf{k}(Y)(M)$ corresponds to the field of fractions of the semigroup algebra $\mathbf{k}(Y)[M]$. It is a purely transcendental extension of $\mathbf{k}(Y)$ of degree $\operatorname{rank} M$.

Let ∂ be a homogeneous LND of fiber type on A. By definition, $\mathbf{k}(Y) \subseteq \operatorname{Frac}(\ker \partial)$ and so, $\mathbf{k}(Y) \subseteq \mathrm{FML}_{\mathrm{fib}}(A)$. This shows that the pathological examples as in Lemma 3.4.2 cannot occur. Let us formulate the following conjecture.

Conjecture 3.5.3. *Let X be an affine variety. If $\mathrm{FML}(X) = \mathbf{k}$ then X is rational.*

The following lemma proves Conjecture 3.5.3 in the particular case where $X \simeq_{\mathrm{bir}} C \times \mathbb{P}^n$, with C a curve.

Lemma 3.5.4. *Let $X = \operatorname{Spec} A$ be an affine variety such that $X \simeq_{\mathrm{bir}} C \times \mathbb{P}^n$, where C is a curve with field rational functions L. If C has positive genus then $\mathrm{FML}(X) \supseteq L$. In particular, if $\mathrm{FML}(X) = \mathbf{k}$ then C is rational.*

Proof. Assume that C has positive genus. We have $K = \operatorname{Frac} A = L(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ are new variables.

We claim that $L \subseteq \mathrm{FML}(A)$. Indeed, let ∂ be an LND on A and let $f, g \in L \setminus \mathbf{k}$. Since $\operatorname{tr.deg}_{\mathbf{k}}(L) = 1$, there exists a polynomial $P \in \mathbf{k}[x,y] \setminus \mathbf{k}$ such that $P(f,g) = 0$. Applying the derivation $\partial : K \to K$ to $P(f,g)$ we obtain

$$\frac{\partial P}{\partial x}(f,g) \cdot \partial(f) + \frac{\partial P}{\partial y}(f,g) \cdot \partial(g) = 0\,.$$

Since f and g are not constant we may suppose that $\frac{\partial P}{\partial x}(f,g) \neq 0$ and $\frac{\partial P}{\partial y}(f,g) \neq 0$. Hence $\partial(f) = 0$ if and only if $\partial(g) = 0$. This shows that one of the two following possibilities occurs:

$$L \subseteq \mathrm{Frac}(\ker \partial) \quad \text{or} \quad L \cap \mathrm{Frac}(\ker \partial) = \mathbf{k}\,.$$

Assume first that $L \cap \mathrm{Frac}(\ker \partial) = \mathbf{k}$. Then, by Lemma 2.1.4 (i) $\mathrm{Frac}(\ker \partial) = \mathbf{k}(x_1, \ldots, x_n)$ and so the field extension $\mathrm{Frac}(\ker \partial) \subseteq K$ is not purely transcendental. This contradits Lemma 2.1.4 (vii). Thus $L \subseteq \mathrm{Frac}(\ker \partial)$ proving the claim and the lemma. □

REMARK 3.5.5. We can apply Lemma 3.5.4 to show that the FML invariant carries more information than usual ML invariant. Indeed, let, in the notation of Lemma 3.4.2, Y be a smooth projective curve of positive genus. Lemma 3.4.2 shows that $\mathrm{ML}(A[Y, \mathfrak{D}]) = \mathbf{k}$. While by Lemma 3.5.4, $\mathrm{FML}(A[Y, \mathfrak{D}]) \supseteq \mathbf{k}(Y)$.

In the following theorem we prove Conjecture 3.5.3 in dimension at most 3.

THEOREM 3.5.6. *Let X be an affine variety of dimension* $\dim X \leq 3$. *If* $\mathrm{FML}(X) = \mathbf{k}$ *then X is rational.*

PROOF. Since $\mathrm{FML}(X)$ is trivial, the same holds for $\mathrm{ML}(X)$. If $\dim X \leq 2$ then $\mathrm{ML}(X) = \mathbf{k}$ implies X rational (see e.g., [ML, p. 41]). Assume that $\dim X = 3$. Lemma 3.4.1 implies that $X \simeq_{\mathrm{bir}} C \times \mathbb{P}^2$ for some curve C. While by Lemma 3.5.4, C is a rational curve. □

CHAPTER 4

Normal singularities with torus actions

In this chapter we give some classification results concerning the singularities of a normal $\mathbb{T}$-varieties in terms of the combinatorial description in Theorem 1.5.5 due to Altmann and Hausen. In particular, we give criteria for a $\mathbb{T}$-variety X to have rational, (minimal) elliptic, or Cohen-Macaulay singularities. This part of the thesis is taken from the preprint [Lie09b]. In a forthcoming joint work with H. Süß [LS10] we further generalize this results to give criteria for X to have $\mathbb{Q}$-Gorenstein, factorial or log-terminal singularities.

In all this chapter, we let as before, N be a lattice of rank n and $M=\operatorname{Hom}(N,\mathbb{Z})$ be its dual lattice, $N_\mathbb{Q}=N\otimes\mathbb{Q}$, $M_\mathbb{Q}=M\otimes\mathbb{Q}$, and we consider the natural duality $M_\mathbb{Q}\times N_\mathbb{Q}\to\mathbb{Q}$, $(m,p)\mapsto\langle m,p\rangle$.

We also let Y be a normal semiprojective variety, σ be a cone in $N_\mathbb{Q}$ with dual cone $\omega\in M_\mathbb{Q}$, and $\mathfrak{D}$ be a proper σ-polyhedral divisor on Y

$$\mathfrak{D}=\sum_Z \Delta_Z\cdot Z\,.$$

With these definitions we let $X=X[Y,\mathfrak{D}]$, $\widetilde{X}=\widetilde{X}[Y,\mathfrak{D}]$, $A=A[Y,\mathfrak{D}]$, $\widetilde{A}=\widetilde{A}[Y,\mathfrak{D}]$, and $\varphi:\widetilde{X}\to X$ be as in Theorems 1.5.5 and 1.5.7.

4.1. Divisors on $\mathbb{T}$-varieties

To formulate some of our classification results we need a combinatorial description of divisors on $\mathbb{T}$-varieties. In [FZ03] a characterization of $\mathbb{T}$-invariant divisors of an affine $\mathbf{k}^*$-surface is given, including formulas for the Canonical divisor, class group and Picard group. In [PS08] some of these results are generalized to the case of a $\mathbb{T}$-variety of arbitrary complexity. In this section we recall the needed results from [PS08] and add some minor generalizations.

Since the contraction morphism $\varphi:\widetilde{X}\to X$ in Theorem 1.5.7 is equivariant, the $\mathbb{T}$-invariant prime Weil divisors on X are in bijection with the $\mathbb{T}$-invariant prime Weil divisors on $\widetilde{X}$ not contracted by φ.

We first apply the orbit decomposition of the variety $\widetilde{X}$ in Proposition 7.10 and Corollary 7.11 of [AH06] to obtain a description of the $\mathbb{T}$-invariant prime Weil divisors in $\widetilde{X}$. There are 2 types of $\mathbb{T}$-invariant prime Weil divisors on $\widetilde{X}$:

(*i*) The horizontal type corresponding to families of $\mathbb{T}$-orbits closures of dimension $\operatorname{rank} M-1$ over Y; and

(*ii*) The vertical type corresponding to families of $\mathbb{T}$-orbits closures of dimension $\operatorname{rank} M$ over a prime divisor on Y.

Lemma 4.1.1. *Let $\mathfrak{D}=\sum_Z\Delta_Z\cdot Z$ be a proper σ-polyhedral divisor on a normal semiprojective variety Y. Letting $\widetilde{X}=\widetilde{X}[Y,\mathfrak{D}]$, the following hold.*

(i) The $\mathbb{T}$*-invariant prime Weil divisors on* $\widetilde{X}$ *of horizontal type are in bijection with the rays* $\rho \subseteq \sigma$*.*

(ii) The $\mathbb{T}$*-invariant prime Weil divisors on* $\widetilde{X}$ *of vertical type are in bijection with pairs* (Z, p) *where* Z *is a prime Weil divisor on* Y *and* p *is a vertex of* Δ_Z*.*

PROOF. The lemma follows from Proposition 7.10 and Corollary 7.11 of [AH06]. See also the proof of Proposition 3.13 in [PS08]. □

The following lemma is a reformulation of Proposition 3.13 in [PS08].

LEMMA 4.1.2. *Let* $\mathfrak{D} = \sum_Z \Delta_Z \cdot Z$ *be a proper* σ*-polyhedral divisor on a normal semiprojective variety* Y*. The following hold.*

(i) Let $\rho \subseteq \sigma$ *be an ray and let* $\tau \subseteq \omega$ *be its dual facet. The* $\mathbb{T}$*-invariant prime Weil divisors of horizontal type on* $\widetilde{X}$ *corresponding to* ρ *is not contracted by* φ *if and only if* $\mathfrak{D}(m)$ *is big for all* $m \in \operatorname{rel.int}(\tau)$*.*

(ii) Let Z *be a prime Weil divisor on* Y *and let* p *be a vertex of* Δ_Z*. The* $\mathbb{T}$*-invariant prime Weil divisors on* $\widetilde{X}$ *of vertical type corresponding to* (Z, p) *is not contracted by* φ *if and only if* $\mathfrak{D}(m)|_Z$ *is big for all* $m \in \operatorname{rel.int}(\operatorname{cone}(\Delta_Z - p)^\vee)$*.*

The following corollary gives a criterion as to when the morphism φ is an isomorphism in codimension one.

COROLLARY 4.1.3. *The morphism* $\varphi : \widetilde{X} \to X$ *is an isomorphism in codimension one if and only if the following conditions hold.*

(i) For every facet $\tau \subseteq \omega$*, the divisor* $\mathfrak{D}(m)$ *is big for all* $m \in \operatorname{rel.int}(\tau)$*.*

(ii) For every prime Weil divisor Z *on* Y *and every vertex* p *on* Δ_Z*, the divisor* $\mathfrak{D}(m)|_Z$ *is big for all* $m \in \operatorname{rel.int}(\operatorname{cone}((\Delta_Z - p)^\vee)$*.*

PROOF. We only need to prove that no $\mathbb{T}$-invariant Weil divisor is contracted by φ. The first condition ensures that no divisor of horizontal type is contracted and the second condition ensures that no divisor of vertical type is contracted. □

REMARK 4.1.4. In the case of a complexity one $\mathbb{T}$-action i.e., when Y is a smooth curve, the condition (ii) in Lemma 4.1.2 and Corollary 4.1.3 is trivially verified.

For one of our applications we need the following lemma concerning the Picard group of a $\mathbb{T}$-variety, see Proposition 3.1 in [PS08] for a particular case.

LEMMA 4.1.5. *Let* $X = \operatorname{Spec} X[Y, \mathfrak{D}]$*, where* $\mathfrak{D}$ *is a proper* σ*-polyhedral divisor on a normal semiprojective variety* Y*. If* Y *is projective then* $\operatorname{Pic}(X)$ *is trivial.*

PROOF. Let D be a Cartier divisor on X, and let f be a local equation of D in an open set $U \subseteq X$ containing $\bar{0}$. By [Bou65, §1, Exercise 16] we may assume that D and U are $\mathbb{T}$-invariant. Since $\bar{0}$ is an attractive fixed point, every $\mathbb{T}$-orbit closure contains $\bar{0}$ and so $U = X$, proving the lemma. □

4.2. Toroidal desingularization

In this section we elaborate a method to effectively compute an equivariant partial desingularization of an affine $\mathbb{T}$-variety in terms of the combinatorial data $(Y, \mathfrak{D})$. This partial desingularization has only toric singularities.

Let $\mathfrak{D}$ be a proper σ-polyhedral divisor on a normal semiprojective variety Y. For any projective morphism $\psi : \widetilde{Y} \to Y$ we can define the *pull back* of the σ-polyhedral divisor $\mathfrak{D}$ as

$$\psi^*\mathfrak{D} = \tfrac{1}{r}\psi^*(r \cdot \mathfrak{D})\,,$$

where r is a positive integer such that $r\mathfrak{D}(m)$ is an integral Cartier divisor for all $m \in \omega_M$.

The combinatorial description of $\mathbb{T}$-varieties in Theorem 1.5.5 is not unique. The following Lemma is a specialization of Corollary 8.12 in [AH06]. For the convenience of the reader, we provide a short argument.

Lemma 4.2.1. *Let $\mathfrak{D}$ be a proper σ-polyhedral divisor on a normal semiprojective variety Y. Then for any projective birational morphism $\psi : \widetilde{Y} \to Y$ the variety $X[Y,\mathfrak{D}]$ is equivariantly isomorphic to $X[\widetilde{Y},\psi^*\mathfrak{D}]$.*

Proof. We only need to show that

$$H^0(Y,\mathcal{O}_Y(\mathfrak{D}(m))) \simeq H^0(\widetilde{Y},\mathcal{O}_{\widetilde{Y}}(\psi^*\mathfrak{D}(m))),\ \text{for all } m \in \omega_M\,.$$

Letting $r \in \mathbb{Z}_{>0}$ be such that $r\mathfrak{D}(m)$ is an integral Cartier divisor $\forall m \in \omega_M$, we have

$$H^0(Y,\mathcal{O}_Y(\mathfrak{D}(m))) = \{f \in k(Y) \mid f^r \in H^0(Y, r\mathfrak{D}(m))\}, \quad \forall m \in \omega_M\,.$$

Since Y is normal and ψ is projective, by Zariski main theorem $\psi_*\mathcal{O}_{\widetilde{Y}} = \mathcal{O}_Y$ and by the projection formula, for all $m \in \omega_M$ we have

$$\begin{aligned} H^0(Y,\mathcal{O}_Y(\mathfrak{D}(m))) &\simeq \left\{f \in k(Z) \mid f^r \in H^0(\widetilde{Y},\mathcal{O}_{\widetilde{Y}}(\psi^* r\mathfrak{D}(m)))\right\} \\ &= H^0(\widetilde{Y},\mathcal{O}_{\widetilde{Y}}(\psi^*\mathfrak{D}(m)))\,. \end{aligned}$$

This completes the proof. □

Remark 4.2.2. In the previous Lemma, $\widetilde{X}[Y,\mathfrak{D}]$ is not equivariantly isomorphic to $\widetilde{X}[\widetilde{Y},\psi^*\mathfrak{D}]$, unless ψ is an isomorphism.

To restrict further the class of σ-polyhedral divisor we introduce the following notation.

Definition 4.2.3. We define the *support* of a σ-polyhedral divisor $\mathfrak{D}$ on a semiprojective variety Y as

$$\operatorname{Supp}\mathfrak{D} = \sum_{\Delta_Z \neq \sigma} Z\,.$$

We say that $\mathfrak{D}$ is an *SNC σ-polyhedral divisor* if Y is smooth, $\mathfrak{D}$ is proper, and $\operatorname{Supp}\mathfrak{D}$ is a simple normal crossing (SNC) divisor.

Remark 4.2.4. In the case of complexity one i.e., when Y is a curve, any proper σ-polyhedral divisor is SNC. Indeed, any normal curve is smooth and any divisor on a smooth curve is SNC.

Corollary 4.2.5. *For any $\mathbb{T}$-variety X there exists an SNC σ-polyhedral divisor on a smooth semiprojective variety Y such that $X = X[Y,\mathfrak{D}]$.*

Proof. By Theorem 1.5.5, there exists a proper σ-polyhedral divisor $\mathfrak{D}'$ on a normal semiprojective variety Y' such that $X = \operatorname{Spec} A[Y,\mathfrak{D}]$. Let $\psi : Y \to Y'$ be a resolution of singularities of Y such that $\operatorname{Supp}\mathfrak{D}'$ is SNC. By Chow Lemma we can assume that Y is semiprojective. By Lemma 4.2.1, $\mathfrak{D} = \psi^*\mathfrak{D}'$ is an SNC σ-polyhedral divisor such that $X = \operatorname{Spec} A[Y,\mathfrak{D}]$. □

Now we elaborate a method to effectively compute an equivariant partial desingularization of an affine $\mathbb{T}$-variety in terms of the combinatorial data $(Y, \mathfrak{D})$. A key ingredient for our results is the following example that is a generalization of Example 2.3.18.

EXAMPLE 4.2.6. Let H_i, $i \in \{1, \ldots, n\}$ be the coordinate hyperplanes in $Y = \mathbb{A}^n$, and let $\mathfrak{D}$ be the SNC σ-polyhedral divisor on Y given by

$$\mathfrak{D} = \sum_{i=1}^{n} \Delta_i \cdot H_i, \quad \text{where } \Delta_i \in \operatorname{Pol}_\sigma(N_\mathbb{Q})\,.$$

Letting $h_i = h_{\Delta_i}$ be the support function of Δ_i and $\mathbf{k}(Y) = \mathbf{k}(t_1, \ldots, t_n)$, for every $m \in \omega_M$ we have

$$\begin{aligned} H^0(Y, \mathcal{O}_Y(\mathfrak{D}(m))) &= \big\{f \in \mathbf{k}(Y) \mid \operatorname{div}(f) + \mathfrak{D}(m) \geq 0\big\} \\ &= \left\{f \in \mathbf{k}(Y) \mid \operatorname{div}(f) + \sum_{i=1}^{n} h_i(m) \cdot H_i \geq 0\right\} \\ &= \bigoplus_{r_i \geq -h_i(u)} \mathbf{k} \cdot t_1^{r_1} \cdots t_n^{r_n}\,. \end{aligned}$$

Let $N' = N \times \mathbb{Z}^n$, $M' = M \times \mathbb{Z}^n$ and σ' be the cone in $\widehat{N}_\mathbb{Q}$ spanned by $(\sigma, \overline{0})$ and (Δ_i, e_i), $\forall i \in \{1, \ldots, n\}$, where e_i is the i-th vector in the standard base of $\mathbb{Q}^n$. A vector $(m, r) \in M'$ belongs to the dual cone $\omega' := (\sigma')^\vee$ if and only if $m \in \omega$ and $r_i \geq -h_i(m)$.

With this definitions we have

$$A[Y, \mathfrak{D}] = \bigoplus_{m \in \omega_M} H^0(Y, \mathcal{O}_Y(\mathfrak{D}(m))) = \bigoplus_{(m,r) \in \omega' \cap M'} k \cdot t_1^{r_1} \cdots t_n^{r_n} \simeq k[\omega' \cap M']\,.$$

Hence $X[Y, \mathfrak{D}]$ is isomorphic as an abstract variety to the toric variety with cone $\sigma' \subseteq N'_\mathbb{Q}$. Since Y is affine $\widetilde{X} \simeq X$, and so $\widetilde{X}$ is also a toric variety.

DEFINITION 4.2.7. A normal variety X is called *toroidal* if for every $x \in X$ the formal neighborhood of x is isomorphic to the formal neighborhood of a point in a toric variety [KKMS73].

In the following proposition we show that $\widetilde{X}$ is a toroidal variety when $\mathfrak{D}$ is a SNC σ-polyhedral divisor.

PROPOSITION 4.2.8. *Let $\mathfrak{D} = \sum_Z \Delta_Z \cdot Z$ be a proper σ-polyhedral divisor on a normal semiprojective variety Y. If $\mathfrak{D}$ is SNC then $\widetilde{X} = \widetilde{X}[Y, \mathfrak{D}]$ is a toroidal variety.*

PROOF. For $y \in Y$ we consider the reduced fiber $\widetilde{X}_y$ over y for the morphism $\varphi : \widetilde{X} \to Y$. We let also $\mathfrak{U}_y$ be the formal neighborhood of $\widetilde{X}_y$.

We let $n = \dim Y$ and

$$S_y = \{Z \text{ prime divisor } \mid y \in Z \text{ and } \Delta_Z \neq \sigma\}\,.$$

Since $\operatorname{Supp} \mathfrak{D}$ is SNC, we have that $\operatorname{card}(S_y) \leq n$. Letting $j : S_y \to \{1, \ldots, n\}$ be any injective function, we consider the SNC σ-polyhedral divisor

$$\mathfrak{D}_y = \sum_{Z \in S_y} \Delta_Z \cdot H_{j(Z)}, \quad \text{on} \quad \mathbb{A}^n\,.$$

Since Y is smooth, $\mathfrak{U}_y$ is isomorphic to the formal neighborhood of the fiber over zero for the canonical morphism

$$\pi_y : \widetilde{X}[\mathbb{A}^n, \mathfrak{D}_y] = \mathbf{Spec}_{\mathbb{A}^n} \widetilde{A}[\mathbb{A}^n, \mathfrak{D}_y] \to \mathbb{A}^n .$$

Finally, Example 4.2.6 shows that $\widetilde{X}[\mathbb{A}^n, \mathfrak{D}_y]$ is toric for all y and so X is toroidal. This completes the proof. □

Remark 4.2.9. Since the contraction $\varphi : \widetilde{X}[Y, \mathfrak{D}] \to X[Y, \mathfrak{D}]$ in Theorem 1.5.7 is proper and birational, to obtain a full desingularization of X it is enough to have a desingularization of $\widetilde{X}$. If further $\mathfrak{D}$ is SNC, then $\widetilde{X}$ is toroidal and there exists a toric desingularization.

4.3. Higher direct images sheaves

Let $X = X[Y, \mathfrak{D}]$ be a $\mathbb{T}$-variety, with $\mathfrak{D}$ an SNC σ-polyhedral divisor on Y. In this section we apply the partial desingularization $\varphi : \widetilde{X}[Y, \mathfrak{D}] \to X[Y, \mathfrak{D}]$ to compute the higher direct images of the structure sheaf of any desingularization $\psi : W \to X$ in terms of the combinatorial data $(Y, \mathfrak{D})$. This allows us to provide information about the singularities of X.

Recall that the i-th direct image sheaf $R^i\psi_*\mathcal{O}_W$ is defined via

$$U \longrightarrow H^0(U, R^i\psi_*\mathcal{O}_W) := H^i\left(\psi^{-1}(U), \mathcal{O}_W|_{\psi^{-1}(U)}\right) .$$

The sheaves $R^i\psi_*\mathcal{O}_W$ are independent of the particular choice of a desingularization of X. Furthermore, X is normal if and only if $R^0\psi_*\mathcal{O}_W := \psi_*\mathcal{O}_W = \mathcal{O}_X$.

Definition 4.3.1. A variety X has *rational singularities* if there exists a desingularization $\psi : W \to X$, such that

$$\psi_*\mathcal{O}_W = \mathcal{O}_X, \quad \text{and} \quad R^i\psi_*\mathcal{O}_W = 0, \quad \forall i > 0 .$$

The following well known Lemma follows by applying the Leray spectral sequence. For the convenience of the reader we provide a short argument.

Lemma 4.3.2. *Let $\varphi : \widetilde{X} \to X$ be a proper surjective, birational morphism, and let $\psi : W \to X$ be a desingularization of X. If $\widetilde{X}$ has only rational singularities, then*

$$R^i\psi_*\mathcal{O}_W = R^i\varphi_*\mathcal{O}_{\widetilde{X}}, \quad \forall i \geq 0 .$$

Proof. We may assume that the desingularization ψ is such that $\psi = \varphi \circ \widetilde{\psi}$, where $\widetilde{\psi} : W \to \widetilde{X}$ is a desingularization of $\widetilde{X}$. The question is local on X, so we may assume that X is affine. Then, by [Har77, Ch. III, Prop. 8.5] we have[1]

$$R^i\psi_*\mathcal{O}_W = H^i(W, \mathcal{O}_W)^\sim \quad \text{and} \quad R^i\varphi_*\mathcal{O}_{\widetilde{X}} = H^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}})^\sim, \quad \forall i \geq 0 .$$

Since $\widetilde{X}$ has rational singularities

$$\widetilde{\psi}_*\mathcal{O}_W = \mathcal{O}_{\widetilde{X}}, \quad \text{and} \quad R^i\widetilde{\psi}_*\mathcal{O}_W = 0, \quad \forall i > 0 .$$

By Leray spectral sequence for $(p, q) = (i, 0)$ we have

$$H^i(W, \mathcal{O}_W) = H^i(\widetilde{X}, \widetilde{\psi}_*\mathcal{O}_W) = H^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}}), \quad \forall i \geq 0 ,$$

proving the Lemma. □

[1]As usual for a A-module M, $M^\sim$ denontes the associated sheaf on $X = \operatorname{Spec} A$.

Recall that $\omega \subseteq M_{\mathbb{Q}}$ is the cone dual to σ. In the following theorem for a $\mathbb{T}$-variety $X = X[Y, \mathfrak{D}]$ and a desingularization $\psi : W \to X$ of X we provide an expression for $R^i\psi_*\mathcal{O}_Z$ in terms of the combinatorial data $(Y, \mathfrak{D})$.

THEOREM 4.3.3. *Let $X = X[Y, \mathfrak{D}]$, where $\mathfrak{D}$ is an SNC σ-polyhedral divisor on Y. If $\psi : W \to X$ is a desingularization, then for every $i \geq 0$, the higher direct image $R^i\psi_*\mathcal{O}_W$ is the sheaf associated to*

$$\bigoplus_{u \in \omega_M} H^i(Y, \mathcal{O}(\mathfrak{D}(m)))$$

PROOF. Consider the proper birational morphism $\varphi : \widetilde{X} := \widetilde{X}[Y, \mathfrak{D}] \to X$. By Lemma 4.2.8 $\widetilde{X}$ is toroidal, thus it has only toric singularities wich are rational, see Theorem 1.6.6. By Lemma 4.3.2 we have

$$R^i\psi_*\mathcal{O}_W = R^i\varphi_*\mathcal{O}_{\widetilde{X}}, \quad \forall i \geq 0\,.$$

Since X is affine, we have

$$R^i\varphi_*\mathcal{O}_{\widetilde{X}} = H^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}})^{\sim}, \quad \forall i \geq 0\,,$$

see [Har77, Ch. III, Prop. 8.5]. Letting

$$\widetilde{A} = \widetilde{A}[Y, \mathfrak{D}] = \bigoplus_{m \in \omega_M} \mathcal{O}_Y(\mathfrak{D}(m))$$

we let π be the structure morphism $\pi : \widetilde{X} = \mathbf{Spec}_Y\, \widetilde{A} \to Y$. Since π is an affine morphism, we have

$$H^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = H^i(Y, \widetilde{A}) = \bigoplus_{m \in \omega_M} H^i(Y, \mathcal{O}_Y(\mathfrak{D}(m))), \quad \forall i \geq 0$$

by [Har77, Ch III, Ex. 4.1], proving the theorem. □

As an immediate consequence of Theorem 4.3.3, in the following theorem, we characterize $\mathbb{T}$-varieties having rational singularities.

THEOREM 4.3.4. *Let $X = X[Y, \mathfrak{D}]$, where $\mathfrak{D}$ is an SNC σ-polyhedral divisor on Y. Then X has rational singularities if and only if for every $m \in \omega_M$*

$$H^i(Y, \mathcal{O}_Y(\mathfrak{D}(m))) = 0, \quad \forall i \in \{1, \ldots, \dim Y\}\,.$$

PROOF. Since X is normal, by Theorem 4.3.3 we only have to prove that

$$\bigoplus_{m \in \omega_M} H^i(Y, \mathcal{O}_Y(\mathfrak{D}(m))) = 0, \quad \forall i > 0$$

This direct sum is trivial if and only if each summand is. Hence X has rational singularities if and only if $H^i(Y, \mathcal{O}_Y(\mathfrak{D}(m))) = 0$, for all $i > 0$ and all $m \in \omega_M$.

Finally, $H^i(Y, \mathscr{F}) = 0$, for all $i > \dim Y$ and for any coherent sheaf $\mathscr{F}$, see [Har77, Ch III, Th. 2.7]. Now the lemma follows. □

In particular, we have the following corollary.

COROLLARY 4.3.5. *Let $X = X[Y, \mathfrak{D}]$ for some SNC σ-polyhedral divisor $\mathfrak{D}$ on Y. If X has only rational singularities, then the structure sheaf $\mathcal{O}_Y$ is acyclic i.e., $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$.*

PROOF. This is the "only if" part of Theorem 4.3.4 for $m = 0$. □

Recall that a local ring is *Cohen-Macaulay* if its Krull dimension is equal to its depth. A variety is *Cohen-Macaulay* if all its local rings are, see Section 1.6. The following lemma is well known, see for instance [KKMS73, page 50].

LEMMA 4.3.6. *Let $\psi : W \to X$ be a desingularization of X. Then X has rational singularities if and only if X is Cohen-Macaulay and $\psi_*\omega_W \simeq \omega_X$* [2].

As in Lemma 4.3.2, applying the Leray spectral sequence the previous lemma is still valid if we allow W to have rational singularities. In the next proposition, we give a partial criterion as to when a normal $\mathbb{T}$-variety is Cohen-Macaulay.

PROPOSITION 4.3.7. *Let $X = X[Y, \mathfrak{D}]$, where $\mathfrak{D}$ is an SNC σ-polyhedral divisor on Y. Assume that following hold.*

(i) For every facet $\tau \subseteq \omega$, the divisor $\mathfrak{D}(m)$ is big for all $m \in \operatorname{rel.int}(\tau)$.
(ii) For every prime Weil divisor Z on Y and every vertex p on Δ_Z, the divisor $\mathfrak{D}(m)|_Z$ is big for all $m \in \operatorname{rel.int}(\operatorname{cone}((\Delta_Z - p)^\vee)$.

Then X is Cohen-Macaulay if and only if X has rational singularities.

PROOF. By Corollary 4.1.3, the contraction $\varphi : \widetilde{X} \to X$ is an isomorphism in codimension 1. Thus $\varphi_*\omega_{\widetilde{X}} \simeq \omega_X$. The result now follows from Lemma 4.3.6. □

For isolated singularities we can give a full classification whenever $\operatorname{rank} M \geq 2$.

COROLLARY 4.3.8. *Let $X = X[Y, \mathfrak{D}]$, where $\mathfrak{D}$ is an SNC σ-polyhedral divisor on Y. If $\operatorname{rank} M \geq 2$ and X has only isolated singularities, then X is Cohen-Macaulay if and only if X has rational singularities.*

PROOF. We only have to prove the "only if" part. Assume that X is Cohen-Macaulay and let $\psi : W \to X$ be a resolution of singularities. Since X has only isolated singularities we have that $R^i\psi_*O_W$ vanishes except possibly for $i = \dim X - 1$, see [Kov99, Lemma 3.3]. Now Theorem 4.3.3 shows that $R^i\psi_*O_W$ vanishes also for $i = \dim X - 1$ since $\dim Y = \dim X - \operatorname{rank} M$ and $\operatorname{rank} M \geq 2$. □

REMARK 4.3.9. In [Wat81] a criterion for X to be Cohen-Macaulay is given in the case of $\operatorname{rank} M = 1$. In this particular case, a condition for X to have rational singularities is given.

4.3.1. Complexity one. In this section we specialize Theorem 4.3.4 and Proposition 4.3.7 to the case of complexity one.

We let C be a smooth curve, and $\mathfrak{D}$ be the σ-polyhedral divisor on C

$$\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z\,.$$

The following proposition gives a simple characterization of T-varieties of complexity one having rational singularities.

PROPOSITION 4.3.10. *Let $X = X[C, \mathfrak{D}]$. Then X has rational singularities if and only if*

(i) C is affine, or
(ii) $C = \mathbb{P}^1$ and $\deg\lfloor\mathfrak{D}(m)\rfloor \geq -1$ for all $m \in \omega_M$.

[2] As usual ω_W and ω_X denote the canonical sheaf of W and X respectively.

Proof. If Y is affine, then the morphism $\varphi : \widetilde{X}[C,\mathfrak{D}] \to X$ is an isomorphism. By Lemma 4.2.8 X is toroidal and thus X has only toric singularities and toric singularities are rational.

If C is projective of genus g, we have $\dim H^1(C,\mathcal{O}_C) = g$. So by Corollary 4.3.5 if X has rational singularities then $C = \mathbb{P}^1$. Furthermore, for the projective line we have $H^1(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(D)) \neq 0$ if and only if $\deg D \leq -2$ [Har77, Ch. III, Th 5.1]. Now the corollary follows from Theorem 4.3.4. □

In the next proposition we provide a partial criterion for the Cohen-Macaulay property in the complexity one case. Recall that $\deg \mathfrak{D}$ is defined as the σ-polyhedron

$$\deg \mathfrak{D} = \sum_{z\in C} \Delta_z \,.$$

Proposition 4.3.11. *Let $X = X[C,\mathfrak{D}]$, where C is a smooth curve and $\mathfrak{D}$ is an proper σ-polyhedral divisor on C. If one of the following conditions hold,*

(*i*) *C is affine, or*
(*ii*) $\operatorname{rank} M = 1$

Then X is Cohen-Macaulay.

Moreover, if C is projective and $\deg \mathfrak{D}$ *does not intersect any of the rays of the cone $\omega = \sigma^\vee$, then X is Cohen-Macaulay if and only if X has rational singularities.*

Proof. If C is affine then $X = \widetilde{X}[C,\mathfrak{D}]$. Thus X has rational singularities and so X is Cohen-Macaulay. If $\operatorname{rank} M = 1$ then X is a normal surface. By Serre S_2 normality criterion any normal surface is Cohen-Macaulay, see Theorem 11.5 in [Eis95].

Since any proper σ-polyhedral divisor is SNC, the last assertion is the specialization of Proposition 4.3.7 to complexity one. Indeed, If C is projective, Proposition 4.3.7 (i) is equivalent to the condition $\deg \mathfrak{D}$ does not intersect any of the rays of the cone ω, while Proposition 4.3.7 (ii) is trivially satisfied in the case of complexity one. □

Remark 4.3.12. Corollary 4.3.8 and Proposition 4.3.11 give a full classification of isolated Cohen-Macaulay singularities on a $\mathbb{T}$-variety of complexity 1.

4.4. Quasihomogeneous surfaces singularities

In this section we study in more detail the particular case of a one dimensional torus action of complexity one i.e., the case of $\mathbf{k}^*$-surfaces. We characterize Gorenstein and elliptic singularities in terms of the combinatorial data as in Theorem 1.5.5.

Let $X = X[C,\mathfrak{D}]$ be a $\mathbf{k}^*$-surface, so that C is a smooth curve and $M \simeq \mathbb{Z}$. There are only two non-equivalent pointed polyhedral cones in $N_\mathbb{Q} \simeq \mathbb{Q}$ corresponding to $\sigma = \{0\}$ and $\sigma = \mathbb{Q}_{\geq 0}$, and any σ-polyhedral divisor $\mathfrak{D}$ on C is SNC.

With the notation of Theorem 1.5.7 suppose that C is affine. Then $X \simeq \widetilde{X}$ by Remark 1.5.8 and so X is toroidal by Lemma 4.2.8. In this case the singularities of X can be classified by toric methods. In particular they are all rational, see Section 1.6.

If C is projective, then $\sigma \neq \{0\}$ and so we can assume that $\sigma = \mathbb{Q}_{\geq 0}$. In this case $\mathfrak{D}(m) = m\mathfrak{D}(1)$. Hence $\mathfrak{D}$ is completely determined by $\mathfrak{D}_1 := \mathfrak{D}(1)$.

Furthermore,

$$A[C,\mathfrak{D}] = \bigoplus_{m\geq 0} A_m\chi^m, \quad \text{where} \quad A_m = H^0(C,\mathcal{O}_C(m\mathfrak{D}_1))\,.$$

and there is an unique atractive fixed point $\bar{0}$ corresponding to the augmentation ideal $\mathfrak{m}_0 = \bigoplus_{m>0} A_m\chi^m$.

This is exactly the setting studied in [FZ03], where all $\mathbf{k}^*$-surfaces are divided in three types: elliptic, parabolic and hyperbolic. In combinatorial language these correspond, respectively, to the cases where C is projective and $\sigma = \mathbb{Q}_{\geq 0}$, C is affine and $\sigma = \mathbb{Q}_{\geq 0}$, and finally C is affine and $\sigma = \{0\}$.

In particular, in [FZ03] invariant divisors on $\mathbf{k}^*$-surfaces are studied. The results in *loc.cit.* are stated only for the hyperbolic case. However, similar statements for the remaining cases can be obtained with essentially the same proofs. In the recent preprint [Süs08] some of the results in *loc.cit.* have been generalized to the case of $\operatorname{rank} M > 1$. Let us recall the necessary results from [FZ03, §4], see also [Süs08].

Let $X = X[C,\mathfrak{D}]$, where $\mathfrak{D}$ is a proper σ-polyhedral divisor on a projective smooth curve C, and let as before $\mathfrak{D}_1 = \mathfrak{D}(1)$. We can write

$$\mathfrak{D}_1 = \sum_{i=1}^{\ell} \frac{p_i}{q_i} z_i, \quad \text{where} \quad \gcd(p_i,q_i) = 1, \text{ and } q_i > 0\,.$$

In this case, with the notation of Theorem 1.5.7 the birational morphism

$$\theta := \pi\circ\varphi^{-1} : X \to C$$

is surjective and its indeterminacy locus consists of the unique fixed point corresponding to the augmentation ideal. The $\mathbf{k}^*$-invariant prime divisors are $D_z := \theta^{-1}(z)$, $\forall z \in C$. The total transforms are: $\theta^*(z) = D_z$ for all $z \notin \operatorname{Supp}\mathfrak{D}_1$, and $\theta^*(z_i) = q_i D_{z_i}$, for $i = 1,\ldots,\ell$. We let $D_i = D_{z_i}$ for $i = 1,\ldots,\ell$.

The canonical divisor of X is given by

$$K_X = \theta^*(K_C) + \sum_{i=1}^{\ell}(q_i - 1)D_i\,.$$

For a rational semi-invariant function $f\cdot\chi^m$, where $f \in K(C)$ and $m \in \mathbb{Z}$, we have

$$\operatorname{div}(f\cdot\chi^m) = \theta^*(\operatorname{div} f) + m\sum_{i=1}^{\ell} p_i D_i\,.$$

For our next result we need the following notation.

Notation 4.4.1. We let

$$m_G = \frac{1}{\deg\mathfrak{D}_1}\left(\deg K_C + \sum_{i=1}^{\ell}\frac{q_i-1}{q_i}\right), \tag{12}$$

and

$$D_G = \sum_{i=1}^{\ell} d_i z_i, \quad \text{where} \quad d_i = \frac{p_i m_G + 1}{q_i} - 1, \quad \forall i \in \{1,\ldots,\ell\}\,. \tag{13}$$

Recall that a normal variety X is *Gorenstein* if it is Cohen-Macaulay and the canonical divisor K_X is Cartier, see Section 1.6. By Serre S_2 normality criterion, all normal surface singularities are Cohen-Macaulay. In the following proposition we give a criterion for a $\mathbf{k}^*$-surface to have Gorenstein singularities.

PROPOSITION 4.4.2. *Let $X = X[C, \mathfrak{D}]$, where $\mathfrak{D}$ is a proper σ-polyhedral divisor on a smooth projective curve C. With the notation as in 4.4.1, the surface X has Gorenstein singularities if and only if m_G is integral and $D_G - K_C$ is a principal divisor on C.*

PROOF. By Lemma 4.1.5, X is Gorenstein if and only if K_X is a principal divisor i.e., there exist $m_G \in \mathbb{Z}$ and a principal divisor $D = \operatorname{div}(f)$ on C such that

$$K_X = \theta^*(K_C) + \sum_{i=1}^{\ell}(q_i - 1)D_i = \theta^* D + m_G \sum_{i=1}^{\ell} p_i D_i = \operatorname{div}(f \cdot \chi^{m_G}), .$$

Clearly $\operatorname{Supp}(K_C - D) \subseteq \{z_1 \ldots, z_\ell\}$. Letting

$$K_C - D = \sum_{i=1}^{\ell} d_i z_i$$

we obtain

$$\sum_{i=1}^{\ell} q_i d_i D_i = \sum_{i=1}^{\ell}(mp_i - q_i + 1)D_i \,.$$

Hence the d_i satisfy (12) in 4.4.1. Furthermore, since

$$\deg K_C = \deg(K_C - D) = \sum_{i=1}^{\ell} d_i \,,$$

m_G satisfies (13) in 4.4.1. So X is Gorenstein if and only if m_G is integral and $D = K_C - D_G$ is principal, proving the proposition. □

Let (X, x) be a normal surface singularity, and let $\psi : W \to X$ be a resolution of the singularity (X, x). One says that (X, x) is an *elliptic singularity*[3] if $R^1\psi_*\mathcal{O}_W \simeq \mathbf{k}$. An elliptic singularity is *minimal* if it is Gorenstein. e.g., [Lau77], [Wat80], and [Yau80].

In the following theorem we characterize quasihomogeneous (minimal) elliptic singularities of surfaces.

THEOREM 4.4.3. *Let $X = X[C, \mathfrak{D}]$ be a normal affine surface with an effective elliptic 1-torus action, and let $\bar{0} \in X$ be the unique fixed point. Then $(X, \bar{0})$ is an elliptic singularity if and only if one of the following two conditions holds:*

(i) $C = \mathbb{P}^1$, $\deg\lfloor m\mathfrak{D}_1 \rfloor \geq -2$ and $\deg\lfloor m\mathfrak{D}_1 \rfloor = -2$ for one and only one $m \in \mathbb{Z}_{>0}$.
(ii) C is an elliptic curve, and for every $m \in \mathbb{Z}_{>0}$, the divisor $\lfloor m\mathfrak{D}_1 \rfloor$ is not principal and $\deg\lfloor m\mathfrak{D}_1 \rfloor \geq 0$.

Moreover, $(X, \bar{0})$ is a minimal elliptic singularity if and only if (i) or (ii) holds, m_G is integral and $D_G - K_C$ is a principal divisor on C, where m_G and D_G are as in Notation 4.4.1.

[3]Some authors call such (X, x) a strongly elliptic singularity.

PROOF. Assume that C is a projective curve of genus g, and let $\psi : W \to X$ be a resolution of singularities. By Theorem 4.3.3,

$$R^1\psi_*\mathcal{O}_W = \bigoplus_{m\geq 0} H^1(C, \mathcal{O}_C(m\mathfrak{D}_1))\,.$$

Since $\dim R^1\psi_*\mathcal{O}_W \geq g = \dim H^1(C,\mathcal{O}_C)$, if X has an elliptic singularity then $g \in \{0,1\}$.

If $C = \mathbb{P}^1$ then $(X,\bar{0})$ is an elliptic singularity if and only if $H^1(C,\mathcal{O}_C(m\mathfrak{D}_1) = \mathbf{k}$ for one and only one value of m. This is the case if and only if (i) holds.

If C is an elliptic curve, then $H^1(C,\mathcal{O}_C) = \mathbf{k}$. So the singularity $(X,\bar{0})$ is elliptic if and only if $H^1(C, m\mathfrak{D}_1) = 0$ for all $m > 0$. This is the case if and only if (ii) holds.

Finally, the last assertion concerning maximal elliptic singularities follows immediately form Proposition 4.4.2. □

EXAMPLE 4.4.4. By applying the criterion of Theorem 4.4.3, the following combinatorial data gives rational $\mathbf{k}^*$-surfaces with an elliptic singularity at the only fixed point.

(i) $C = \mathbb{P}^1$ and $\mathfrak{D}_1 = -\frac{1}{4}[0] - \frac{1}{4}[1] + \frac{3}{4}[\infty]$. In this case $X = X[C,\mathfrak{D}]$ is isomorphic to the surface in $\mathbb{A}^3$ with equation

$$x_1^4x_3 + x_2^3 + x_3^2 = 0\,.$$

(ii) $C = \mathbb{P}^1$ and $\mathfrak{D}_1 = -\frac{1}{3}[0] - \frac{1}{3}[1] + \frac{3}{4}[\infty]$. In this case $X = X[C,\mathfrak{D}]$ is isomorphic to the surface in $\mathbb{A}^3$ with equation

$$x_1^4 + x_2^3 + x_3^3 = 0\,.$$

(iii) $C = \mathbb{P}^1$ and $\mathfrak{D}_1 = -\frac{2}{3}[0] - \frac{2}{3}[1] + \frac{17}{12}[\infty]$. In this case $X = X[C,\mathfrak{D}]$ is isomorphic to the surface

$$V(x_1^4x_2x_3 - x_2x_3^2 + x_4^2\ ;\ x_1^5x_3 - x_1x_3^2 + x_2x_4\ ;\ x_2^2 - x_1x_4) \subseteq \mathbb{A}^4\,.$$

This last example is not a complete intersection since otherwise $(X,\bar{0})$ would be Gorenstein i.e., minimal elliptic which is not the case by virtue of Theorem 4.4.3. In the first two examples the elliptic singlarities are minimal, since every normal hypersurface is Gorenstein.

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RÉSUMÉ

Une $\mathbb{T}$-variété est une variété algébrique munie d'une action effective d'un tore algébrique $\mathbb{T}$. Cette thèse est consacrée à l'étude de deux aspects des $\mathbb{T}$-variétés normales affines : les actions du groupe additif et la caractérisation des singularités.
Soit $X = \operatorname{Spec} A$ une $\mathbb{T}$-variété affine normale et soit ∂ une dérivation homogène localement nilpotente de l'algèbre affine intègre $\mathbb{Z}^n$-graduée A, alors ∂ engendre une action du groupe additif dans X. On donne une classification complète des couples (X, ∂) dans trois cas : pour les variétés toriques, dans le cas de complexité un, et dans le cas où ∂ est de type fibre. Comme application, on calcule l'invariant de Makar-Limanov (ML) homogène de ces variétés. On en déduit que toute variété d'invariant de ML trivial est birationnelle à $Y \times \mathbb{P}^2$, pour une certaine variété Y. Inversement, pour toute variété Y, il existe une $\mathbb{T}$-variété affine X d'invariant de ML trivial birationnelle à $Y \times \mathbb{P}^2$.
Dans la seconde partie concernant les singularités d'une $\mathbb{T}$-variété X, on calcule les images directes supérieures du faisceau structural d'une désingularisation de X. Comme conséquence, on donne un critère pour qu'une $\mathbb{T}$-variété ait des singularités rationnelles. On présente aussi une condition pour qu'une $\mathbb{T}$-variété soit de Cohen-Macaulay. Comme application, on caractérise les singularités elliptiques des surfaces quasi-homogènes.

ABSTRACT

A $\mathbb{T}$-variety is an algebraic variety endowed with an effective action of an algebraic torus $\mathbb{T}$. This thesis is devoted to the study of two aspects of normal affine $\mathbb{T}$-varieties: the additive group actions and the characterization of singularities.
Let $X = \operatorname{Spec} A$ be a normal affine $\mathbb{T}$-variety and let ∂ be a homogeneous locally nilpotent derivation on the normal affine $\mathbb{Z}^n$-graded domain A, so that ∂ generates an action of the additive group on X. We provide a complete classification of pairs (X, ∂) in three cases: for toric varieties, in the case where the complexity is one, and in the case where ∂ is of fiber type. As an application, we compute the homogeneous Makar-Limanov (ML) invariant of such varieties. We deduce that any variety with trivial ML-invariant is birationally decomposable as $Y \times \mathbb{P}^2$, for some variety Y. Conversely, given a variety Y, there exists an affine $\mathbb{T}$-variety X with trivial ML invariant birational to $Y \times \mathbb{P}^2$.
In the second part concerning singularities of a $\mathbb{T}$-variety X we compute the higher direct images of the structure sheaf of a desingularization of X. As a consequence, we give a criterion as to when a $\mathbb{T}$-variety has rational singularities. We also provide a condition for a $\mathbb{T}$-variety to be Cohen-Macaulay. As an application, we characterize quasihomogeneous elliptic singularities of surfaces.

MOTS-CLÉS

Actions du tore, actions du groupe additif, dérivations localement nilpotentes, variétés affines, invariant de Makar-Limanov, singularités rationnelles, singularités de Cohen-Macaulay, singularités elliptiques des surfaces.

CLASSIFICATION MATHÉMATIQUE

14R05, 14R20, 13N15, 14M25, 14J17, 14J50, 14E15.