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Sur les déformations isomonodromiques et la stabilité des équations différentielles

Bassem Ben Hamed

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Université de Sfax
Faculté des Sciences de Sfax

Université Paul Sabatier
Toulouse III

Thèse

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L'Université Paul Sabatier, Toulouse III

Et

L'Université de Sfax, Faculté des Sciences de Sfax

Intitulée

Sur les déformations isomonodromiques et la stabilité des équations différentielles

Par

Bassem Ben Hamed

Soutenue le 22 décembre 2006 devant le jury composé de

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Je dédie ce travail

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A tous mes amis, mes collègues et ceux qui m'aiment et qui croient en moi,

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Introduction

L'un des problèmes importants d'analyse depuis le $19^{\text{ème}}$ siècle était de chercher de "*bonnes fonctions transcendantes*" définies par des équations différentielles algébriques non-linéaires. Si l'équation est non-linéaire, on ne peut pas en général prévoir la position des singularités. Une équation différentielle algébrique non-linéaire est dite sans singularités mobiles, si la solution ne possède pas des points de ramifications (respectivement des singularités essentielles) qui changent de position lorsqu'on varie les conditions initiales. Ainsi, le problème est fomulé comme suit :

Trouver toutes les équations différentielles algébriques non-linéaires ne possèdant pas de singularités mobiles.

On dit dans ce cas qu'une telle équation satisfait la *propriété de Painlevé.*
Dans le cas où l'ordre de l'équation $n = 1$, le problème a été étudié et résolu par L. Fuchs et H. Poincaré. Toute équation satisfaisant la propriété de Painlevé pourra être transformée, par un changement de variables holomorphes en t et fractionnel linéaire en la variable de l'équation, en une équation de la fonction de Weierstrass $\wp$ ou une équation de Riccati.
Emile Picard a exprimé dans sa lettre envoyée à Mittag-Leffler en 1893, son opinion pessimiste de trouver des équations différentielles algébriques non-linéaires d'ordre $n \geq 2$ et satisfaisant la propriété de Painlevé. Ainsi, P. Painlevé a attaqué le problème avec des équations différentielles rationnelles et a démontré que de telles équations satisfaisant sa propriété se réduisent, par des transformations convenables à des équations s'intégrant par quadrature, à des équations linéaires, ou a la fameuse liste de ses six équations.

Considérons la sixième équation de Painlevé ($\mathbf{PVI}_\alpha$)

$$\begin{aligned} \frac{d^2\lambda}{dt^2} &= \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t}\right)\frac{d\lambda}{dt} \\ &\quad + \frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left[\alpha_0 - \alpha_1\frac{t}{\lambda^2} + \alpha_2\frac{t-1}{(\lambda-1)^2} + \left(\frac{1}{2} - \alpha_3\right)\frac{t(t-1)}{(\lambda-t)^2}\right]. \end{aligned} \tag{0.1}$$

paramétrisée par $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^4$. Bien que toute solution de $\mathbf{PVI}_\alpha$, pour des valeurs génériques α_i, soit transcendante (elle donne même une nouvelle fonction transcendante), il existe un très grand nombre de solutions qui sont algébriques en t. Leur classification, à l'exception du cas particulier

$$\alpha_0 = \frac{1}{2}(2\mu - 1)^2,\ \alpha_1 = \alpha_2 = \alpha_3 = 0,\ \mu \in \mathbb{R}$$

(voir (19, Dubrovin, Mazzocco) et (52, Mazzocco)), reste encore un problème ouvert (voir (50, Manin)).

Dans la première partie de cette thèse, nous allons présenter un cas particulier du problème de classification des solutions algébriques. Ce cas simple se produit quand une solution algébrique donnée satisfait chaque membre d'une famille non-triviale de $\mathbf{PVI}_\alpha$. Une telle famille non-triviale d'équations de $\mathbf{PVI}_\alpha$ contenant au

moins deux éléments distincts α' et α'', sera notée $\{\mathbf{PVI}_\alpha\}_\alpha$. Si une solution algébrique satisfait à la fois les équations $\mathbf{PVI}_{\alpha'}$ et $\mathbf{PVI}_{\alpha''}$ alors elle satisfait toute la famille d'équations $\{\mathbf{PVI}_\alpha\}_\alpha$ correspondante à la droite affine contenant α' et α''. Ainsi, toute famille non-triviale définie comme précédente, correspondant à un plan affine de l'espace des paramètres $\mathbb{C}^4\{\alpha\}$. Dans cette partie, nous donnons une classification de tous ces espaces affines avec leurs solutions algébriques associées. On pourra constater que les solutions coïncident avec les solutions obtenues récemment par Doran (voir (18)) qui a utilisé des déformations des surfaces elliptiques avec quatres fibres singulières et leurs équations de Picard-Fuchs associées. Par contre, la preuve de notre théorème n'utilise pas la notion d'équations de Picard-Fuchs.

Dans la suite, on va essayer de donner une explication partielle de cette coïncidence. Rappelons que chaque solution $(\lambda(t), \alpha)$ d'une équation $\mathbf{PVI}_\alpha$ donnée est gouvernée par une déformation isomonodromique d'un 2×2 système Fuchsian approprié possédant quatre points singuliers. Nous disons qu'une telle déformation est *géométrique* si le système fondamental de solutions est entièrement constituté d'intégrales Abéliennes, qui dépendent algébriquement du paramètre de déformation. Une déformation géométrique d'un système Fuchsien est isomonodormique et définit une solution algébrique $(\lambda(t), \alpha)$ d'une équation $\mathbf{PVI}_\alpha$ appropriée. Quand ceci est vrai, nous disons que la solution algébrique $(\lambda(t), \alpha)$ de l'équation $\mathbf{PVI}_\alpha$ est d'*origine géométrique*. Nous montrons que lorsque $\lambda(t)$ satisfait une famille d'équations $\mathbf{PVI}_\alpha$, alors ils existent α', α'' appartenant à la même famille, telles que $(\lambda(t), \alpha')$ et $(\lambda(t), \alpha'')$ sont d'origine géométrique.

La deuxième partie de cette thèse porte sur la stabilité et la stabilisation des systèmes dynamiques à retard. Ces systèmes dynamiques à retard constituent des modèles mathématiques basiques de phénomènes réels comme les réacteurs nucléaires, les systèmes d'ingénierie chimiques, les systèmes biologiques, les modèles dynamiques de population, et bien d'autres. Ils sont souvent une source d'instabilité et de dégradation de la performance dans beaucoup de problèmes de contrôle.

Pendant ces dernières décennies, le problème de stabilité des systèmes linéaires à retard a été soumis à plusieurs recherches considérables. Beaucoup de résultats signifiants ont été rapporté dans la littérature. Le lecteur pourra consulter la référence suivante : (21, Gu and al.).

Dans le troisième chapitre, on va présenter quelques définitions et notions de base sur les systèmes à retard. Le modèle choisi sera présenté, ainsi que l'existence et l'unicité des solutions pour les équations différentielles fonctionnelles (EDFR) associées. On introduit les méthodes des fonctionnelles de Lyapunov-Krasovskii et de fonction de Razumikhin, qui donnent des conditions suffisantes pour assurer la stabilité de ces systèmes à retard. Puis, on considère des classes de systèmes incertains à retard dans l'état et dans la commande. En utilisant des techniques de Lyapunov, on propose des classes de contrôleurs continus, qui assurent la stabilité globale uniforme exponentielle de ces systèmes en boucle fermée, en imposant quelques *conditions assorties* sur les incertitudes. La fonction de Laypunov quadratique du système nominal stable (c'est-à-dire, le système assosié en l'absence des incertitudes et du retard) est utilisé comme fonction de Lyapunov candidate du système global. Des exemples numériques seront donnés pour illustrer l'applicabilité des résultats obtenus.

Dans le quatrième chapitre, on va étudier la stabilité absolue d'une classe de systèmes à retard de type de Lurie. Cette classe est présentée comme une interconnexion du feed-back d'un système dynamique linéaire et d'une non-linéarité staisfaisant la condition du secteur. En utilisant quelques inégalités intégrales, on obtient une nouvelle condition suffisante de stabilité absolue présentée sous forme d'inégalités matricielles linéaires (LMI). Cette condition améliore celle donnée dans (26, Han). Par la suite, on utilisera cette nouvelle condition pour

construire un contrôleur basé sur un observateur dépendant du retard, tel que le système erreur soit présenté comme une interconnexion du feed-back d'un système linéaire et d'une non-linéarité multiple dépendante aussi du retard et satisfaisant la condition du secteur. Dans la conception de l'observateur, on va étendre les travaux de (2; 3, Arcak et Kokotovic) et (15, Fan et Arcak) dans le cas avec retard. Finalement, quelques exemples numériques seront donnés pour illustrer l'applicabilité des résultats obtenus.

Partie I

Déformations Isomonodormiques des Systèmes Fuchsiens et Solutions Algébriques des Equations de Painlevé

B. Ben Hamed, L. Gavrilov. On the algebraic solutions of the sixth Painlevé equation related to second order Picard-Fuchs equations, *Prépublication no 285 du Laboratoire de Mathématiques Emile Picard, arXiv :math.CA/0405308, 2004.*

B. Ben Hamed, L. Gavrilov. Families of Painlevé VI equations having a common solution, *Int. Math. Res. Not. 2005 No 60 pp. 3727-3752.*

Chapitre 1

Some basic facts on the Painlevé equations

The following six Painlevé equations :

$$
\begin{aligned}
\mathbf{PI} \ : \ \frac{d^2\lambda}{dt^2} &= 6\lambda^2 + t, \\
\mathbf{PII} \ : \ \frac{d^2\lambda}{dt^2} &= 2\lambda^3 + t\lambda + \alpha_0, \\
\mathbf{PIII} \ : \ \frac{d^2\lambda}{dt^2} &= \frac{1}{\lambda}(\frac{d\lambda}{dt})^2 - \frac{1}{t}\frac{d\lambda}{dt} + \frac{1}{t}(\alpha_0\lambda^2 + \alpha_1) + \alpha_2\lambda^3 + \frac{\alpha_3}{\lambda}, \\
\mathbf{PIV} \ : \ \frac{d^2\lambda}{dt^2} &= \frac{1}{2\lambda}(\frac{d\lambda}{dt})^2 + \frac{3}{2}\lambda^3 + 4t\lambda^2 + 2(t^2 - \alpha_0)\lambda + \frac{\alpha_1}{\lambda}, \\
\mathbf{PV} \ : \ \frac{d^2\lambda}{dt^2} &= (\frac{1}{2\lambda} + \frac{1}{\lambda - 1})(\frac{d\lambda}{dt})^2 - \frac{1}{t}\frac{d\lambda}{dt} + \frac{(\lambda-1)^2}{t}(\alpha_0\lambda + \frac{\alpha_1}{\lambda}) \\
&\quad + \alpha_2\frac{\lambda}{t} + \alpha_3\frac{\lambda(\lambda+1)}{\lambda-1}, \\
\mathbf{PVI} \ : \ \frac{d^2\lambda}{dt^2} &= \frac{1}{2}(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t})(\frac{d\lambda}{dt})^2 - (\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t})\frac{d\lambda}{dt} \\
&\quad + \frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}[\alpha_0 - \alpha_1\frac{t}{\lambda^2} + \alpha_2\frac{t-1}{(\lambda-1)^2} + (\frac{1}{2} - \alpha_3)\frac{t(t-1)}{(\lambda-t)^2}]
\end{aligned}
$$

where $\alpha_0,\ \alpha_1,\ \alpha_2,\ \alpha_3$ are complex constants, are known to appear in an universal way in many different branches of mathematics. We note that each equations **PJ** depends an 0, 1, 3, 2, 4, 4 parameters α_i respectively. In what follows, we are interested only in the sixth Painlevé equation because each **PJ**, **I,...,V** are obtained from **PVI** by the appropriate limit process, see (38, Prposotion 1.2.1, p.125).

The purpose of this chapter is to serve as an introduction to chapter 2. More details, together with proofs can be found in (5, Anosov and Bolibruch), (37, Ince), (38, Iwasaki and al.), (64, Sabbah). First, we describe the two fundamental methods which lead to the Painlevé equations. The first one is based on the so called Painlevé property (absence of movable critical points), see (60, Painlevé). The second one, due to R. Fuchs (21), is based on the notion of isomonodromic deformation. Next, we describe some algebraic solutions of the sixth Painlevé equation which are related to (isomonodromic) deformations of Picard-Fuchs equations of order two. Using the concept of monodromy preserving deformation, we derive the Garnier system written in the form of Hamiltonian system, which governs such deformation of second order Fuchsian equation with $n+3$ singularities. When $n = 1$, the Garnier system turns out to be equivalent to the sixth Painlevé equation. The

system thus obtained is free of movable branch points. Using the relation between the Garnier system and **PVI** equation, we obtain some algebraic solutions related to isomonodromic deformation of the singularity $\eta^2 + \zeta^4$ of type A_3, see (2, Arnold and al.).

1.1 Painlevé equations

One of the important problems of analysis in the 19th century was to find *good transcendental functions* defined by non-linear algebraic differential equations. A differential equation

$$F(t, y, \frac{dy}{dt}, ..., \frac{d^n y}{dt^n}) = 0 \tag{1.1}$$

defined in domain $D \subset \mathbb{C}$ is said to be algebraic if $F = F(t, y_0, y_1, ..., y_n)$ is polynomial in $\mathbf{y} = (y_0, y_1, ..., y_n)$ with coefficients meromorphic in $t \in D$, rational if it is algebraic and is of degree one with respect to y_n, and linear if it is algebraic and F is a linear $\mathbf{y}$.
Take $c := (c_0, ..., c_n) \in \mathbb{C}^{n+1}$ and $t_0 \in D$ so that $F(t_0, c_0, ..., c_n) = 0$, and denote by $\varphi(t) = \varphi(t; t_0, c)$ the holomorphic solution such that

$$\frac{d^i \varphi}{dt^i}(t_0) = c_i, \ i = 0, ..., n.$$

The function obtained by an analytic continuation $\varphi(t)$ is also dented by $\varphi(t)$.
If an equation is non-linear, we can in general predict neither where the singularities of solutions appear nor of what kind the singularities are. In such a case, we can hardly say that the function which is a solution of the equation is controlled by the differential equation. The following examples show that solutions may have branch points or essential singular points which change their position depending on integration constants.

Example 1.1.1

$$my'y^{m-1} = 1, \ m \in \mathbb{N}.$$

Solution : $y(t) = (t - c)^{1/m}$, $c \in \mathbb{C}$ *being an integration constant.* $y(t)$ *has an algebraic branch point over* $t = c$.

Example 1.1.2

$$y'' + (y')^2 = 0.$$

Solution : $y(t) = log(t - c_1) + c_2, \ c_1, c_2 \in \mathbb{C}$.

Example 1.1.3

$$yy'' + (y')^2(\frac{2y}{y'} - 1) = 0.$$

Solution : $y(t) = c_1 \ exp(-1/(t - c_2)), \ c_1, c_2 \in \mathbb{C}$.

Thus we face to seeking a non-linear differential equation such that the singularities (except poles) of the solutions are predictable.

An algebraic differential equation (1.1) is said to be *free of movable branch (resp. essential singular) points* if the solution $\varphi(t; t_0, c)$ has no branch (resp. essential singular point which changes its position when we vary (t_0, c) under the restriction $F(t_0, c) = 0$.

Problem 1.1.4 *Find all the algebraic differential equations free of movable branch points and movable essential singular points.*

We say that an algebraic differential equation enjoys the *Painlevé property* if (1.1) is free of movable branch points and movable essential singular points.

Equivalently, the Painlevé property can be defined as follows :

Definition 1.1.5 *We say that an algebraic differential equation (1.1) enjoys the Painlevé property if there is a finite set $\Delta \subset \mathbb{C}$ such that any germ of analytic solution in a neighborhood of any point $t_0 \in \mathbb{C}\backslash\Delta$ allows a meromorphic continuation along any path $\gamma \subset \mathbb{C}\backslash\Delta$ starting at t_0. (meromorphic continuation is defined by analog to analytic continuation)*

When $n = 1$, the problem was studied and solved by L. Fuchs and H. Poincaré; any equation of type (1.1) with the Painlevé property can be transformed, by a holomorphic change of the variable t and by a linear fractional change of the unknown with coefficients in $\mathcal{O}(D)$, into the equation of the Weierstrass $\wp$ function :

$$(\frac{dy}{dt})^2 = 4y^3 - g_2 y - g_3,\ g_2, g_3 \in \mathbb{C},$$

or into the Riccati equation :

$$\frac{dy}{dt} = a(t)y^2 + b(t)y + c(t),\ a(t), b(t), c(t) \in \mathcal{O}(D),$$

where $\mathcal{O}(D)$ stands for the ring of holomorphic function on D.

When the order n of (1.1) is one, only movable branch points appear, whereas when $n \geq 2$, movable essential singular points may appear. E. Picard pointed out this fact in his letter to Mittag-Leffler (1893), and expressed his pessimistic opinion that there might be very little hope of success to find non-linear differential equations with the Painlevé property in case $n \geq 2$. Despite of the negative prospect of E. Picard, P. Painlevé attacked the problem for rational differential equations of the form

$$\frac{d^2y}{dt^2} = R(t, y, \frac{dy}{dt}),$$

and showed by a huge amount of computation that any equation with the Painlevé property reduces, by an appropriate transformation of the variables, to an equation which can be integrated by quadrature, or to a linear equation, or to **PJ, J=I,...,VI**. Precisely speaking, P. Painlevé fond only **PI, PII, PIII** because of errors in his computations. His student, B.O. Gambier, added the equations **PIV, PV, PVI** to the list. See (22, Gambier) and (60, Painlevé).

Now, we present a second approach (due to R. Fuchs (21)) to the Painlevé equations based on isomonodromic deformations of Fuchsian equations (or systems).

Consider a linear ordinary differential equations

$$x'' + p_1(s)x' + p_2(s)x = 0, \quad ' = \frac{d}{ds} \tag{1.2}$$

where the p_j's are analytic in a domain $\mathbb{C}\backslash\Delta$.

Definition 1.1.6 (A characterization of Fuchsian equations) *The equation (1.2) is Fuchsian with regular singularities at $t_1, \ldots, t_m, t_{m+1} = \infty$ if and only if the coefficients p_j's have the following form :*

$$p_j(s) = \frac{a_j(s)}{\prod_{i=1}^{m}(s - t_i)^j},\ j = 1, 2,$$

where each $a_j(s)$ is a polynomial of degree at most $j(m-1)$.

To describe the multi-valuedness of solutions of (1.2), we can associate to (1.2) the conjugacy class of a subgroup of $GL(2,\mathbb{C})$, which will be called the monodromy of (1.2).
Recall that if $Y(x,t)$, where $t=(t_1,\dots,t_m)$, is a fundamental system of solutions of (1.2) in a neighborhood of $t_0\in\mathbb{C}\backslash\Delta$ and $\gamma\in\pi_1(\mathbb{C}\backslash\Delta,t_0)$ then the analytic continuation $\gamma_*Y(x,t)$ of $Y(x,t)$ along the path γ is also a fundamental system of solutions of (1.2), and we have $\gamma_*Y(x,t)=Y(x,t)\rho(t,\gamma)$, where

$$\begin{aligned}\rho(t,.) \quad : \quad & \pi_1(\mathbb{C}\backslash\Delta,t_0)\longrightarrow GL(2,\mathbb{C})\\ & \gamma\longmapsto\rho(t,\gamma)\end{aligned}$$

is a homomorphism of groups.

Definition 1.1.7 *The homomorphism $\rho(t,.)$ is called the monodromic representation of (1.2) and $Im(\pi_1(\mathbb{C}\backslash\Delta,t_0))\subset GL(2,\mathbb{C})$ is called the monodromy group of (1.2).*

Definition 1.1.8 *The deformation of the equation (1.2) is said to be isomonodromic if we can choose a fundamental system of solutions $Y(x,t)$ locally analytic in t and in x such that the monodromic representation $\rho(t,.)$ is independent of t.*

Consider a Fuchsian differential equation

$$x''+p_1(s)x'+p_2(s)x=0,\quad {}'=\frac{d}{ds} \tag{1.2}$$

with *five singular points, exactly one of which is apparent.* After a bi-rational change of the independent variable s and a linear change of the dependent variable x (involving s) we may suppose that the singular points are $0,1,t,\lambda,\infty$, where the singularity λ is apparent and the corresponding Riemann scheme is

$$\begin{pmatrix} 0 & 1 & t & \lambda & \infty\\ 0 & 0 & 0 & 0 & \alpha\\ \theta_0 & \theta_1 & \theta_t & n & \alpha+\theta_\infty\end{pmatrix}, n\in\mathbb{N}, 2\alpha+\sum_{i\in\{0,1,t,\infty\}}\theta_i+n=3$$

In what follows we shall always suppose that $n=2$ (which is satisfied generically)).
The coefficients p_1,p_2 are easily computed to be

$$p_1(s)=\frac{1-\theta_0}{s}+\frac{1-\theta_1}{s-1}+\frac{1-\theta_t}{s-t}-\frac{1}{t-\lambda}$$

$$p_2(s)=\frac{k}{s(s-1)}-\frac{t(t-1)K}{s(s-1)(s-t)}+\frac{\lambda(\lambda-1)\mu}{s(s-1)(s-\lambda)}$$

where μ is a constant and

$$k=\frac{1}{4}\{(\sum_{i\in\{0,1,t\}}\theta_i-1)^2-\theta_\infty^2\}$$

The compatibility condition for the singular point λ to be apparent reads

$$\begin{aligned}K=K(\lambda,\mu,t) \quad = \quad & \frac{1}{t(t-1)}[\lambda(\lambda-1)(\lambda-t)\mu^2-\{\theta_1(\lambda-1)(\lambda-t)\\ & +\theta_t\lambda(\lambda-t)+(\theta_0-1)\lambda(\lambda-1)\}\mu+k\lambda]\end{aligned}$$

From the discussion above it is seen that the Fuchs equation (1.2) depends on the parameters $\theta_0,\theta_1,\theta_t,\theta_\infty,\lambda,\mu,t$. Let us denote this equation by $E_\theta(\lambda,\mu,t)$.

Theorem 1.1.9 *(38, Theorem 4.1.2, p. 172) $(\lambda(t), \mu(t))$ is a solution of the Garnier system*

$$\begin{aligned} \frac{d\lambda}{dt} &= \frac{\partial K}{\partial \mu} \\ \frac{d\mu}{dt} &= -\frac{\partial K}{\partial \lambda}. \end{aligned} \tag{1.3}$$

if and only if the induced deformation of $E_\theta(\lambda, \mu, t)$ is isomonodromic.

It is straightforward to check that the sixth Painlevé system **PVI** with parameters

$$\alpha = (\frac{1}{2}\theta_\infty^2, \frac{1}{2}\theta_0^2, \frac{1}{2}\theta_1^2, \frac{1}{2}\theta_t^2) \tag{1.4}$$

is equivalent to the Garnier system. We get therefore the following.

Corollary 1.1.10 *If*

$$(t, \lambda, \mu) \to (t, \lambda(t), \mu(t))$$

is an isomonodromic deformation of $E_\theta(\lambda, \mu, t)$, then $\lambda(t)$ is a solution of **PVI** *equations with parameters given by (1.4).*

1.2 Algebraic solutions of PVI and Picard-Fuchs equations

It is known that for the generic values of the parameters α_i, $i = 0, \ldots, 3$, any solution of **PVI** is an "new transcendental function". It might be therefore surprising, that **PVI** still possesses a large amount of algebraic solutions. Their complete classification is still an open problem, see (50, Manin).

In this section we describe some algebraic solutions of Painlevé VI (**PVI**) equation related to deformations of Picard-Fuchs equations of special type. Recall that the **PVI** equation governs the isomonodromic deformations of the second order Fuchsian equations (1.2) with 5 singular points, one of which is apparent (38, Iwasaki and al.). Suppose that the solution of the Fuchsian equation (1.2) is given by an Abelian integral

$$x(s) = \int_{\gamma_s} \omega$$

where ω is a rational one-form on $\mathbb{C}^2$, $\Gamma_s \subset \mathbb{C}^2$ is a family of algebraic curves depending rationally on s, and $\gamma_s \subset \Gamma_s$ is a continuous family of closed loops. Then the equation (1.2) is said to be of Picard-Fuchs type and its monodromy group is conjugated to a subgroup of $\mathbf{Gl}_2(\overline{\mathbb{Q}})$ (generically $\mathbf{Gl}_2(\mathbb{Z})$). For this reason any continuous deformation

$$a \to \Gamma_{s,a}$$

of the family Γ_s induces an isomonodromic deformation of (1.2). If in addition $\Gamma_{s,a}$ depends algebraically in a, the coefficients of (1.2) are also algebraic functions in a, and hence they provide an algebraic solution of **PVI**.

More precisely, let $\lambda = \lambda(a)$, $t = t(a)$, be the apparent and non-apparent singularities of (1.2). This determine an algebraic function $t \to \lambda(t)$, which is a solution of **PVI**. It is simpler, however, to represent $\lambda = \lambda(t)$ in a parameterized form $\lambda = \lambda(a)$, $t = t(a)$.

Theorem 1.2.1 *The algebraic functions listed in Table 1.1 define an algebraic solutions of* **PVI**.

Sol'n	$t(a)$	$\lambda(a)$	**PVI**
1	$\frac{a^3(2-a)}{2a-1}$	$\frac{a^2(2-a)}{a^2-a+1}$	$(\frac{1}{8}, \frac{s}{8}, \frac{s}{8}, \frac{s}{8})$
2	$\frac{a^3(2-a)}{2a-1}$	$\frac{a(a-2)(2a^2+a+2)}{a^2-7a+1}$	$(\frac{1}{8}, \frac{1}{2}, 0, 0)$
3	$\frac{a^3(2-a)}{2a-1}$	$\frac{a^2(2-a)(a^2-7a+1)^2}{25a^6-75a^5+42a^4+41a^3+42a^2-75a+25}$	$(\frac{9}{8}, 0, 0, 0)$
4	$\frac{a^3(2-a)}{2a-1}$	$-\frac{a(a-2)^2}{5a^2-5a-1}$	$(\frac{1}{8}, 0, 0, \frac{1}{2})$
5	$\frac{a^3(2-a)}{2a-1}$	$\frac{a^2(2-a)(2a-1)}{a^2+5a-5}$	$(\frac{1}{8}, 0, \frac{1}{2}, 0)$

TABLE 1.1 – List of algebraic parameterized solutions $\lambda(t)$ of **PVI** equations

The meaning of these solutions is the following. Consider the family of elliptic curves

$$\Gamma_s = \{(\xi, \eta) \in \mathbb{C}^2 : f_a(\xi, \eta) = s\} \tag{1.5}$$

where

$$f_a(\xi, \eta) = \eta^2 + \frac{3}{2a-1}\xi^4 - \frac{4(a+1)}{2a-1}\xi^3 + \frac{6a}{2a-1}\xi^2$$

is a deformation of the singularity $\eta^2 + \xi^4$ of type A_3, see (2, Arnold and al.). The critical values of $f_a(\xi, \eta)$ are

$$0, 1, t = \frac{a^3(2-a)}{2a-1}.$$

Let $\gamma(s) \in H_1(\Gamma_s, \mathbb{Z})$ be a family of cycles depending continuously on $s \in \mathbb{C}$. The Abelian integral of first kind

$$\int_{\gamma(s)} \frac{d\xi}{\eta}$$

satisfies a Picard-Fuchs equation of second order depending on a parameter a, defining an isomonodromy deformation of the equation. This deformation corresponds then to an algebraic solution of **PVI** given by Sol'1. In a similar way, the Abelian integral of second kind

$$\int_{\gamma(s)} \frac{(3\xi^2 - 2(a+1)\xi)d\xi}{\eta}$$

satisfies a Picard-Fuchs equation of second order. This Abelian integral is of second kind in the sense that the differential form has no residues. In fact, if we put $\xi = \frac{1}{z}$ we obtain the following power serie :

$$\frac{d\xi}{\eta} = -\sqrt{-1}\sqrt{\frac{2a-1}{3}}[1 + \frac{2(a+1)}{3}z + \frac{2a^2+a+2}{3}z^2 + o(z^3)]$$

So,

$$\frac{(3\xi^2 - 2(a+1)\xi)d\xi}{\eta} = -\sqrt{-1}\sqrt{\frac{2a-1}{3}}[\frac{3}{z^2} + \frac{2a^2-5a-1}{3} + o(z)]$$

The isomonodromy deformation of this equation with respect to a is described by the Sol'2 of **PVI** equation.

A linear combinaison of the Abelian integrals of first and second kind

$$\int_{\gamma(s)} \frac{(54\zeta^2 - 36(a+1)\zeta - (a^2 - 7a + 1))d\zeta}{\eta}$$

$$\int_{\gamma(s)} \frac{(9\zeta^2 - 6(a+1)\zeta + a(2-a))d\zeta}{\eta}$$

$$\int_{\gamma(s)} \frac{(9\zeta^2 - 6(a+1)\zeta + (2a-1))d\zeta}{\eta}$$

satisfies also a Picard-Fuchs equations of second order. The isomonodromy deformations of these equations with respect to a is described by the Sol'3,4 and 5 of **PVI** equation.

It turns out that these solutions, up to symmetries, are in a one-to-one correspondence with the regular polyhedra in the three dimensional space. Our Sol'1 with

$$\alpha = (\frac{1}{8}, 0, 0, 0)$$

corresponds then to the tetrahedron solution of Dubrovin-Mazzocco ($\mu = +1/4$). It is identified to their solution (A_3) via the Okamoto type transformation (1.24),(1.25), see (19). It is remarkable that the same solution, but for

$$\alpha = (\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$$

was also found by Hitchin (31). This shows that Sol'1 is a common solution to the family of **PVI** equations with,

$$\alpha = (\frac{1}{8}, \frac{z}{8}, \frac{z}{8}, \frac{z}{8}) \ , \ z \in \mathbb{C} \tag{1.6}$$

It is clear that for transcendental values of z in Sol'1 the corresponding isomonodromic family of Fuchs equations (1.2) can not be of Picard-Fuchs type.

Next, we deduce the relevant Picard-Fuchs equation and establish the Theorem 1.2.1.

Consider the locally trivial smooth fibration

$$f^{-1}(\mathbb{C} \setminus \{0, 1, t\}) \to \mathbb{C} \setminus \{0, 1, t\}$$

whose fibers the affine curves Γ_s, (1.5), $s \in \mathbb{C} \setminus \{0, 1, t\}$. Each Γ_s is topologically a torus with two removed points. Hence $\dim H_1(\Gamma_s, \mathbb{Z}) = \dim H^1_{DR}(\Gamma_s, \mathbb{C}) = 3$. Therefore if $\gamma(s) \in H_1(\Gamma_s, \mathbb{Z})$ is a family of cycles depending continuously on s, then the Abelian integral

$$I(s) = \int_{\gamma(s)} \omega, \omega = P(\xi, \eta)d\xi + Q(\xi, \eta)d\eta, P, Q \in \mathbb{C}[\xi, \eta]$$

satisfies a Fuchsian differential equation of order three, whose coefficients are polynomials in s, a. In the case when the differential form ω has no residues, it satisfies a second order equation. Explicitly, if $\gamma_1(s)$, $\gamma_2(s)$, is a continuous family of cycles generating the homology group of the compactified elliptic curve Γ_s, then the equation reads

$$\det \begin{pmatrix} x & x' & x'' \\ \int_{\gamma_1(s)} \omega & (\int_{\gamma_1(s)} \omega)' & (\int_{\gamma_1(s)} \omega)'' \\ \int_{\gamma_2(s)} \omega & (\int_{\gamma_2(s)} \omega)' & (\int_{\gamma_2(s)} \omega)'' \end{pmatrix} = 0.$$

It follows from the Picard-Lefschetz formula and the moderate growth of the integrals, that the coefficients of the above differential equations are rational in s, a. A local analysis of the singularities shows for instance that

$$\det\begin{pmatrix} \int_{\gamma_1(s)}\omega & (\int_{\gamma_1(s)}\omega)' \\ \int_{\gamma_2(s)}\omega & (\int_{\gamma_2(s)}\omega)' \end{pmatrix} = \frac{p(s,a)}{s(s-1)(s-t)}$$

where $p(s,a)$ is a polynomial in s, a. If we put $\omega = \frac{dx}{y}$ then $\int_{\gamma_1(s)}\omega$ grows no faster than $s^{1/4-1/2}$ at ∞ (for a fixed a). Thus

$$\frac{p(s,a)}{s(s-1)(s-t)}$$

grows at infinity no faster than $s^{-1/2-1}$ and hence no faster than s^{-2}. It is expected therefore than $p(s,a)$ is of degree one in s and the corresponding root, which we denote by λ, is an apparent singularity for the Picard-Fuchs equation in consideration. We are therefore in a position to apply Theorem 1.1.9, provided that the deformation of the Fuchs equation with respect to the parameter a is isomonodromical. Indeed, the monodromy group of our equation is contained in $SL(2,\mathbb{Z})$ which shows that any deformation of this equation is isomonodromical. The Picard- Lefschetz formula shows that the monodromy group in question is generated by the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \tag{1.7}$$

To deduce explicit formula for the algebraic solution of **PVI** we need explicit formulae for the Picard-Fuchs equations.

Lemma 1.2.2 *Let $\gamma(s) \in H_1(\Gamma_s, \mathbb{Z})$ be a family of cycles depending continuously on s. The complete elliptic integrals of first and second kind*

$$x(s) = \int_{\gamma(s)} \frac{d\xi}{\eta}, \quad y(s) = \int_{\gamma(s)} \frac{(3\xi^2 - 2(a+1)\xi)d\xi}{\eta}$$

satisfy Picard-Fuchs equations of the form

$$a_0(s)x'' + a_1(s)x' + a_2(s)x = 0$$

$$b_0(s)y'' + b_1(s)y' + b_2(s)y = 0$$

where

$$\begin{aligned}
a_0(s) &= s(s-1)((2a-1)s + a^3(a-2))((a^2-a+1)s + a^2(a-2)) \\
a_1(s) &= 2(2a-1)(a^2-a+1)s^3 + (a^6 - 3a^5 + 9a^4 - 19a^3 + 9a^2 - 3a + 1)s^2 \\
&\quad +2a^2(a-2)(a^4 - 2a^3 - 2a + 1)s - a^5(a-2)^2 \\
a_2(s) &= (2a-1)[27(a^2-a+1)s^2 - (a-2)(2a^4 - a^3 - 60a^2 - a + 2)s \\
&\quad +a^2(a-2)^2(10a^2 + 11a + 10)]/144 \\
b_0(s) &= s(s-1)((2a-1)s + a^3(a-2))((a^2-7a+1)s - a(a-2)(2a^2+a+2)) \\
b_1(s) &= (2a-1)s[(a^2-7a+1)s^2 - 2a(a-2)(2a^2+a+2)s \\
&\quad -a(a-2)^2(a^4+a^3+a^2+a+1)] \\
b_2(s) &= -(2a-1)[9(a^2-7a+1)s^2 - (a-2)(10a^4 + 31a^3 - 12a^2 + 31a + 10)s \\
&\quad -a(a-2)^2(2a^2+a+2)^2]/144
\end{aligned}$$

The proof of the above Lemma is straightforward, see for instance (20, Dumortier and Li). It is seen that the roots of $a_0(s)$ are $0, 1$ and

$$\lambda = \frac{a^2(2-a)}{a^2-a+1}, t = \frac{a^3(2-a)}{2a-1}$$

which implies the algebraic Sol'1. In the same way the roots of $b_0(s)$ the Sol'2. The Riemann schemes of the Picard-Fuchs equations for $x(s)$, $y(s)$ are given by

$$\begin{pmatrix} 0 & 1 & \frac{a^3(2-a)}{2a-1} & \frac{a^2(2-a)}{a^2-a+1} & \infty \\ 0 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 2 & \frac{3}{4} \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 & \frac{a^3(2-a)}{2a-1} & \frac{a(a-2)(2a^2+a+2)}{a^2-7a+1} & \infty \\ 0 & 0 & 0 & 0 & \frac{1}{4} \\ 1 & 0 & 0 & 2 & -\frac{1}{4} \end{pmatrix}.$$

The Corollary after Theorem 1.1.9 implies that the curve Sol'1 is an integral curve of the **PVI** equation with parameters $\alpha = (\frac{1}{8}, 0, 0, 0)$, see (1.4). Similarly, the Fuchsian equation satisfied by the complete elliptic integral of second kind $y(s)$ provides the algebraic Sol'2 of **PVI** with $\alpha = (\frac{1}{8}, \frac{1}{2}, 0, 0)$.

It is remarkable that Sol'1 was found to be a solution of **PVI** by Hitchin (31)[p.177], but for $\alpha = (\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$. After taking the difference between these two equations we obtain the following affine equation of the integral curve

$$-\frac{t}{\lambda^2} + \frac{t-1}{(\lambda-1)^2} - \frac{t(t-1)}{(\lambda-t)^2} = 0$$

This also shows that Sol'1 is a common algebraic solution of **PVI** equations.

Lemma 1.2.3 *Let $\gamma(s) \in H_1(\Gamma_s, \mathbb{Z})$ be a family of cycles depending continuously on s. The Abelian integrals*

$$z(s) = \int_{\gamma(s)} \frac{(54\xi^2 - 36(a+1)\xi - (a^2-7a+1))d\xi}{\eta}$$

$$v(s) = \int_{\gamma(s)} \frac{(9\xi^2 - 6(a+1)\xi + a(2-a))d\xi}{\eta}$$

$$w(s) = \int_{\gamma(s)} \frac{(9\xi^2 - 6(a+1)\xi + (2a-1))d\xi}{\eta}$$

satisfy Picard-Fuchs equations of the form

$$c_0(s)z'' + c_1(s)z' + c_2(s)z = 0$$

$$d_0(s)v'' + d_1(s)v' + d_2(s)v = 0$$

$$e_0(s)w'' + e_1(s)w' + e_2(s)w = 0$$

where

$$
\begin{aligned}
c_0(s) &= s(s-1)((2a-1)s+a^3(a-2))((25a^6-75a^5+42a^4+41a^3+42a^2\\
&\quad -75a+25)s+a^2(a-2)(a^2-7a+1)^2)\\
c_1(s) &= 2(2a-1)(25a^6-75a^5+42a^4+41a^3+42a^2-75a+25)s^3+(25a^{10}-125a^9\\
&\quad +198a^8-192a^7+657a^6-1251a^5+657a^4-192a^3+198a^2-125a+25)s^2\\
&\quad +2a^2(a-2)(a^4-2a^3-2a+1)(a^2-7a+1)^2s-a^5(a-2)^2(a^2-7a+1)^2\\
c_2(s) &= -5[9(2a-1)(25a^6-75a^5+42a^4+41a^3+42a^2-75a+25)s^2\\
&\quad +(a-2)(2a-1)(10a^8-25a^7+28a^6-232a^5+868a^4-232a^3+28a^2\\
&\quad -25a+10)s-a^2(a-2)^2(2a-1)(a^2-7a+1)(2a^4+17a^3\\
&\quad +3a^2+17a+2)]/144
\end{aligned}
$$

$$
\begin{aligned}
d_0(s) &= s(s-1)((2a-1)s+a^3(a-2))((5a^2-5a-1)s+a(a-2)^2)\\
d_1(s) &= ((2a-1)s+a^3(a-2))((5a^2-5a-1)s^2+2a(a-2)^2s-a(a-2)^2)\\
d_2(s) &= -(2a-1)[(5a^2-5a-1)s^2+(a-2)^2(10a^3+15a^2+18a-5)s\\
&\quad -a(a-2)^2(10a^3+15a^2+4)]/144\\
e_0(s) &= s(s-1)((2a-1)s+a^3(a-2))((a^2+5a-5)s+a^2(a-2)(2a-1))\\
e_1(s) &= (2a-1)(s-1)((a^2+5a-5)s^2+2a^2(a-2)(2a-1)s+a^5(a-2)^2)\\
e_2(s) &= -(2a-1)[9(a^2+5a-5)s^2-(a-2)(2a-1)(5a^3-18a^2-15a-10)s\\
&\quad +a^2(a-2)^2(4a^3+15a+10)]/144
\end{aligned}
$$

It is seen that the roots of $c_0(s)$ are 0, 1 and

$$\lambda=\frac{a^2(2-a)(a^2-7a+1)^2}{25a^6-75a^5+42a^4+41a^3+42a^2-75a+25},\ t=\frac{a^3(2-a)}{2a-1}$$

which implies the algebraic Sol'3. In the same way the roots of $d_0(s)$ and $e_0(s)$ provide the Sol'4 and 5. The Riemann schemes of Picard-Fuchs equations for $z(s)$, $v(s)$ and $w(s)$ are given by

$$\begin{pmatrix} 0 & 1 & \frac{a^3(2-a)}{2a-1} & \frac{a^2(2-a)(a^2-7a+1)^2}{25a^6-75a^5+42a^4+41a^3+42a^2-75a+25} & \infty \\ 0 & 0 & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 2 & \frac{5}{4} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & \frac{a^3(2-a)}{2a-1} & -\frac{a(a-2)^2}{5a^2-5a-1} & \infty \\ 0 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & 2 & -\frac{1}{4} \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 & \frac{a^3(2-a)}{2a-1} & \frac{a^2(2-a)(2a-1)}{a^2+5a-5} & \infty \\ 0 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 1 & 0 & 2 & -\frac{1}{4} \end{pmatrix}$$

The Fuchsian equation satisfied by $z(s)$, $v(s)$ and $w(s)$ provides the algebraic Sol'3,4 and 5, of **PVI** with $\alpha=(\frac{9}{8},0,0,0)$, $(\frac{1}{8},0,0,\frac{1}{2})$ and $(\frac{1}{8},0,\frac{1}{2},0)$ respectively.

This completes the proof of Theorem 1.2.1.

Chapitre 2

Families of Painlevé VI equations having a common solution

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Families of Painlevé VI equations having a common solution [1]

Abstract : We classify all functions satisfying non-trivial families of Painlevé VI equations. Each family is parameterized by an affine space. This affine space, except for the so-called constant families, is generated by points of "geometric origin", associated either to deformations of elliptic surfaces with four singular fibers, or to deformations of three-sheeted covers of $\P^1$ with branching locus consisting of four points.

Key words : Isomonodromic deformations, Painlevé VI equation, Picard-Fuchs equations, elliptic surfaces.

Mathematical Subject Classification (2000) : 34Mxx.

2.1 Introduction

Consider the Painlevé VI ($\mathbf{PVI}_\alpha$) equation

$$\begin{aligned}\frac{d^2\lambda}{dt^2} &= \frac{1}{2}(\frac{1}{\lambda}+\frac{1}{\lambda-1}+\frac{1}{\lambda-t})(\frac{d\lambda}{dt})^2-(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{\lambda-t})\frac{d\lambda}{dt}\\ &+\frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}[\alpha_0-\alpha_1\frac{t}{\lambda^2}+\alpha_2\frac{t-1}{(\lambda-1)^2}+(\frac{1}{2}-\alpha_3)\frac{t(t-1)}{(\lambda-t)^2}].\end{aligned}$$

parameterized by $\alpha=(\alpha_0,\alpha_1,\alpha_2,\alpha_3)\in\mathbb{C}^4$. Although any solution of $\mathbf{PVI}_\alpha$, for generic α_i, is transcendental (and even a "new transcendental function"), there is a large amount of solutions which are algebraic in t. Their general classification is still an open problem (e.g. (50, Manin)), except in the particular case $\alpha_1=\alpha_2=\alpha_3=0$ (19, Dubrovin, Mazzocco), (52, Mazzocco). The present paper addresses the question of classifying *families* of algebraic solutions. The simplest case occurs when a given algebraic solution satisfies each member of a non-trivial family of $\mathbf{PVI}_\alpha$ equations. By a non-trivial family of $\mathbf{PVI}_\alpha$ equations we mean a set $\{\mathbf{PVI}_\alpha\}_\alpha$ containing at least two distinct elements corresponding to, say, α' and α''. Then this solution satisfies the $\mathbf{PVI}_\alpha$ equations corresponding to the affine line containing α' and α''. It follows that each non-trivial family as above corresponds to an affine plane in the parameter space $\mathbb{C}^4\{\alpha\}$. We classify all such affine spaces, together with their associated algebraic solutions (Theorem 2.2.1, Table 2.1). The proof of Theorem 2.2.1 does not use the notion of Picard-Fuchs equation. It turns out that the solutions $2A, 2B, \ldots, 5L$ on Table 2.1 coincide surprisingly with the solutions obtained earlier by Doran, who used deformations of elliptic surfaces with four singular fibers and the related Picard-Fuchs equations, see Theorem 2.3.1.

The second purpose of the paper is to give a partial explanation of the above coincidence. Recall that each solution $(\lambda(t),\alpha)$ of a given $\mathbf{PVI}_\alpha$ equation governs the isomonodromy deformation of an appropriate 2×2 Fuchsian system with four singular points. We say that such a deformation is *geometric*, if there is a fundamental matrix of solutions whose entries are Abelian integrals depending algebraically on the deformation parameter. A geometric deformation of a Fuchsian system is isomonodromic, and defines an algebraic solution $(\lambda(t),\alpha)$ of an appropriate $\mathbf{PVI}_\alpha$ equation. When this holds true, we say that the algebraic solution $(\lambda(t),\alpha)$ of $\mathbf{PVI}_\alpha$ is of *geometric origin*.

The solutions $(\lambda(t),\alpha')$ of geometric origin coming from deformations of elliptic surfaces with four singular fibers were computed by Doran (18) (this result is summarized in Theorem 2.3.1 and Table 2.4). We will prove

1. This work is done with Professor Lubomir Gavrilov.

that to each such $(\lambda(t), \alpha')$, we may associate a parameter $\alpha'' \neq \alpha'$, such that $(\lambda(t), \alpha'')$ is still of geometric origin, and governs the deformation of a ramified cover of $\P^1$ with four ramification points (Theorem 2.3.3 and Table 2.5). On its turn this already implies that $\lambda(t)$ is a common solution of the family $\{\mathbf{PVI}_\alpha\}_\alpha$, where α belongs to the affine line containing α' and α''. This explains why the solutions $\lambda(t)$ found by Doran reappeared in Theorem 2.2.1. Finally we note that the converse is also true (although there is no apparent reason for this). Namely, each affine space of $\mathbf{PVI}_\alpha$ equations on Table 2.1, except the families 0A,...,1F, is generated by points α' and α'' given in Table 2.4 and Table 2.5 respectively.

Acknowledgments. We thank the anonymous referees for the valuable recommendations. The solution 1A, as well its relation to the family 2A was pointed out to us by Philip Boalch. We acknowledge his interest and critical remarks.

2.2 Families of Painlevé VI equations having a common solution

Let $\lambda = \lambda(t)$ be a solution of the equations $\mathbf{PVI}_{\alpha'}$, $\mathbf{PVI}_{\alpha''}$, $\alpha' \neq \alpha''$. Then $\lambda(t)$ satisfies the implicit equation

$$\beta_0 - \beta_1 \frac{t}{\lambda^2} + \beta_2 \frac{t-1}{(\lambda-1)^2} - \beta_3 \frac{t(t-1)}{(\lambda-t)^2} = 0, \tag{2.1}$$

where $\beta = \alpha' - \alpha'' = (\beta_0, \beta_1, \beta_2, \beta_3)$, and hence it is an algebraic function. The function $\lambda(t)$ satisfies, moreover, the family $\{\mathbf{PVI}_\alpha\}_\alpha$, where α belongs to the affine line

$$\{\alpha' + s(\alpha' - \alpha'') : s \in \mathbb{C}\} \subset \mathbb{C}^4.$$

It is seen from this that the set of all α such that $\mathbf{PVI}_\alpha$ is satisfied by the function $\lambda(t)$ form an affine subspace of $\mathbb{C}^4$. We refer to the set of these $\mathbf{PVI}_\alpha$ equations as to a *family of Painlevé VI equations having a common solution.*

Theorem 2.2.1 *The list of all families of Painlevé VI equations having a common solution, together with the corresponding solution, is shown in Table 2.1, and Figures 2.1, 2.2, 2.3.*

Remark 2.2.2 *The meaning of the solutions 0A-0D is as follows. If we write down the* $\mathbf{PVI}_\alpha$ *equation as an equivalent hamiltonian non-autonomous system on* $\mathbb{C}^2$ *(e.g. (38, Theorem 1.5.2))) then* $\lambda = 0, 1, t$ *defines a solution of this system if and only if* $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 1/2$. *One may further complete canonically* $\mathbb{C}^2$ *to a surface* Σ *(the so called "space of initial conditions"), such that* $\mathbf{PVI}_\alpha$ *induces a foliation on* $\Sigma \times \{\mathbb{P}^1 \setminus \{0, 1, \infty\}\}$ *which is uniform with respect to the trivial fibration* $\Sigma \times \{\mathbb{P}^1 \setminus \{0, 1, \infty\}\} \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$ *(see (57, Okamoto) for details). Then* $\lambda = 0, 1, t, \infty$ *correspond to leaves of this foliation, for example, (57, p.45-47), if and only if* $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 1/2$, *or* $\alpha_4 = 0$, *respectively.*

Remark 2.2.3 *Each solution* $\lambda(t)$ *can be defined by a relation* $P(\lambda(t), t) \equiv 0$ *where* P *is an irreducible polynomial. This polynomial is given in Table 2.1. The solutions in each of the 6 series of families on Table 2.1 are equivalent up to a* S_4*-symmetry of Painlevé VI equation (see section 2.2.1).*

Remark 2.2.4 *The solutions* $1A, 1B, \ldots, 1F$ *depend on the parameter* $k \in \mathbb{C}$. *Therefore they are solutions in a different sense to all the others. However, for every* fixed *value of* k *they are common solutions of a family (affine line) of* $\mathbf{PVI}_\alpha$ *equations. It turns out that they are Okamoto equivalent to the solutions* $2A, 2B, 2C$ *which do not contain a parameter. More precisely, the solution* $1A$

$$\lambda^2 - kt = 0, \ \alpha = (a, k\,a, \frac{1}{8}, \frac{1}{8}),$$

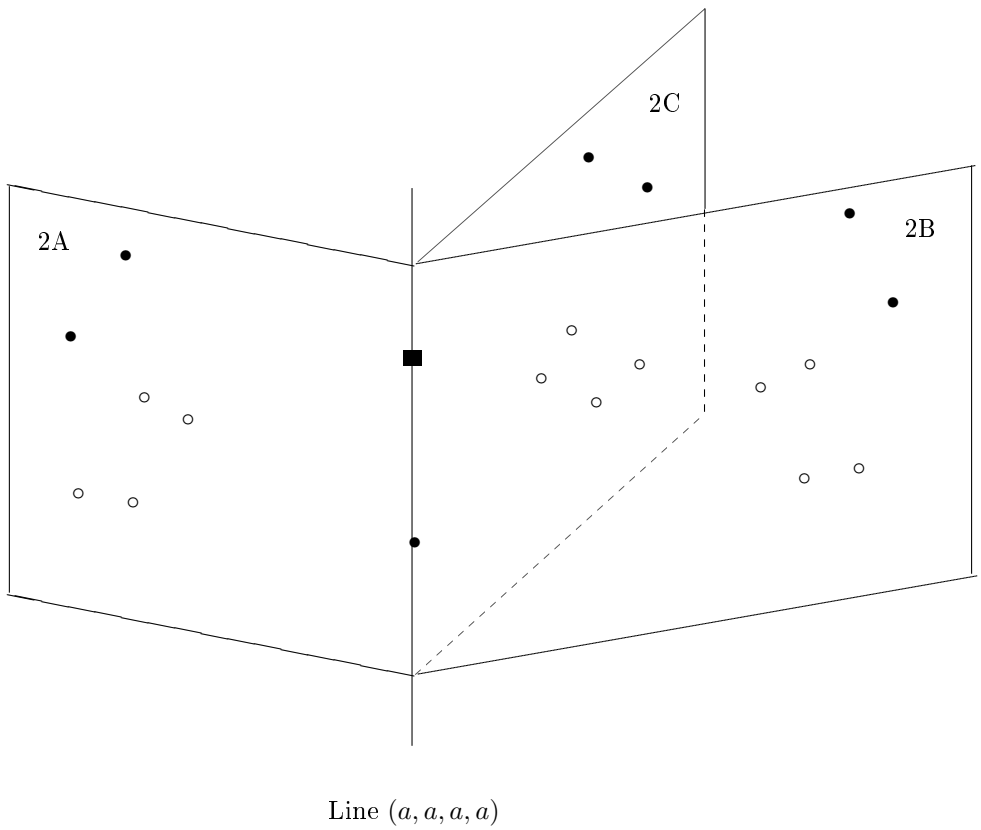

FIGURE 2.1 – The families of $\mathbf{PVI}_\alpha$ equations whose solution is $2A$, $2B$, or $2C$

is equivalent, after applying the transformation w_2 of Okamoto (58, p.363), to the solution $\lambda^2 - t = 0$ where the parameter α equals to

$$((\frac{\sqrt{2a}-\sqrt{2b}}{2\sqrt{2}})^2, (\frac{\sqrt{2a}-\sqrt{2b}}{2\sqrt{2}})^2, (1-\frac{\sqrt{2a}-\sqrt{2b}}{2\sqrt{2}})^2, (1-\frac{\sqrt{2a}-\sqrt{2b}}{2\sqrt{2}})^2),$$

and $b = k\,a$. Thus, up to Okamoto equivalence, the families of Painlevé VI equations having a common solution are represented (for instance) by the five families $0A, 2A, 3A, 4A, 5A$ on Table 2.1.

Outline of the Proof. *Denote by Γ_β the compactified and normalized algebraic curve defined by (2.1), with affine model*

$$\Gamma_\beta^{aff} = \{(\lambda, t) \in \mathbb{C}^2 : N(\lambda, t) = 0\}, \tag{2.2}$$

where

$$\begin{aligned} N(\lambda, t) &= \beta_0\lambda^2(\lambda-1)^2(\lambda-t)^2 - \beta_1 t(\lambda-1)^2(\lambda-t)^2 \\ &+ \beta_2(t-1)\lambda^2(\lambda-t)^2 - \beta_3 t(t-1)\lambda^2(\lambda-1)^2 = 0. \end{aligned}$$

Name	Solution of $\mathbf{PVI}_\alpha$ equation	Family of $\mathbf{PVI}_\alpha$ equations
0A	$\lambda = t$	$(a, b, c, \frac{1}{2})$
0B	$\lambda = 1$	$(a, b, 0, c)$
0C	$\lambda = 0$	$(a, 0, b, c)$
0D	$\lambda = \infty$	$(0, a, b, c)$
1A	$\lambda^2 - k\,t = 0$	$(a, k\,a, \frac{1}{8}, \frac{1}{8}), k = const. \neq 0$
1B	$(\lambda - 1)^2 + k\,(t-1) = 0$	$(a, \frac{1}{8}, k\,a, \frac{1}{8}), k = const. \neq 0$
1C	$(\lambda - t)^2 - k\,t(t-1) = 0$	$(a, \frac{1}{8}, \frac{1}{8}, k\,a), k = const. \neq 0$
1D	$-t(\lambda - 1)^2 + k\,(t-1)\lambda^2 = 0$	$(\frac{1}{8}, a, k\,a, \frac{1}{8}), k = const. \neq 0$
1E	$(\lambda - t)^2 + k\,(t-1)\lambda^2 = 0$	$(\frac{1}{8}, a, \frac{1}{8}, k\,a), k = const. \neq 0$
1F	$(\lambda - t)^2 - k\,t(\lambda - 1)^2 = 0$	$(\frac{1}{8}, \frac{1}{8}, a, k\,a), k = const. \neq 0$
2A	$\lambda^2 - t = 0$	(a, a, b, b)
2B	$\lambda^2 - 2\lambda + t = 0$	(a, b, a, b)
2C	$\lambda^2 - 2\lambda t + t = 0$	(b, a, a, b)
3A	$\lambda^4 - 6\lambda^2 t + 4\lambda t + 4\lambda t^2 - 3t^2 = 0$	$(a, 9a, a, a)$
3B	$3\lambda^4 - 4\lambda^3 - 4\lambda^3 t + 6\lambda^2 t - t^2 = 0$	$(9a, a, a, a)$
3C	$\lambda^4 - 4\lambda^3 + 6t\lambda^2 - 4t^2\lambda + t^2 = 0$	$(a, a, 9a, a)$
3D	$\lambda^4 - 4t\lambda^3 + 6t\lambda^2 - 4t\lambda + t^2 = 0$	$(a, a, a, 9a)$
4A	$\lambda^4 - 2t\lambda^3 - 2\lambda^3 + 6t\lambda^2$	$(a, \frac{1}{8}, a, a)$
	$-2t^2\lambda - 2t\lambda + t^3 - t^2 + t = 0$	
4B	$\lambda^4 - 2t\lambda^3 + 2t^2\lambda - t^3 = 0$	$(a, a, \frac{1}{8}, a)$
4C	$\lambda^4(t^2 - t + 1) - 2\lambda^3 t(t+1) + 6t^2\lambda^2$	$(\frac{1}{8}, a, a, a)$
	$-2\lambda t^2(t+1) + t^3 = 0$	
4D	$\lambda^4 - 2\lambda^3 + 2t\lambda - t = 0$	$(a, a, a, \frac{1}{18})$
5A	$-2\lambda^3 + 3t\lambda^2 + 3\lambda^2 - 6t\lambda + t^2 + t = 0$	$(4a, \frac{1}{18}, a, a)$
5B	$\lambda^3 - 3\lambda^2 + 3t\lambda - 2t^2 + t = 0$	$(a, \frac{1}{18}, 4a, a)$
5C	$\lambda^3 - 3t\lambda^2 + 3t\lambda + t^2 - 2t = 0$	$(a, \frac{1}{18}, a, 4a)$
5D	$2\lambda^3 - 3t\lambda^2 + t^2 = 0$	$(4a, a, \frac{1}{18}, a)$
5E	$\lambda^3 - 3t\lambda + 2t^2 = 0$	$(a, 4a, \frac{1}{18}, a)$
5F	$\lambda^3 - 3t\lambda^2 + 3t\lambda - t^2 = 0$	$(a, a, \frac{1}{18}, 4a)$
5G	$\lambda^3(2 - t) - 3t\lambda^2 + 3t^2\lambda - t^2 = 0$	$(\frac{1}{18}, a, 4a, a)$
5H	$\lambda^3(t+1) - 6t\lambda^2 + 3t(t+1)\lambda - 2t^2 = 0$	$(\frac{1}{18}, 4a, a, a)$
5I	$(1 - 2t)\lambda^3 + 3t\lambda^2 - 3t\lambda + t^2 = 0$	$(\frac{1}{18}, a, a, 4a)$
5J	$\lambda^3 - 3\lambda^2 + 3t\lambda - t = 0$	$(a, a, 4a, \frac{1}{18})$
5K	$\lambda^3 - 3t\lambda + 2t = 0$	$(a, 4a, a, \frac{1}{18})$
5L	$\lambda^3 - 3\lambda^2 + t = 0$	$(4a, a, a, \frac{1}{18})$

TABLE 2.1 – List of all algebraic solutions satisfying families of $\mathbf{PVI}_\alpha$ equations

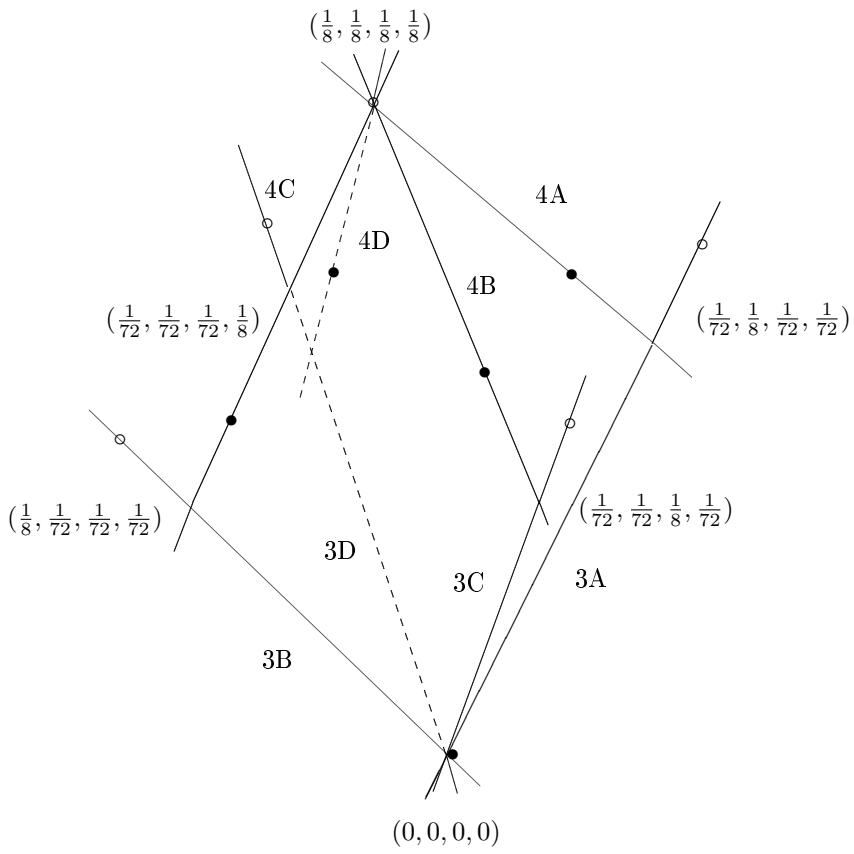

FIGURE 2.2 – The families of $\mathbf{PVI}_\alpha$ equations whose solution is $3A$, $3B$,..., $4C$, or $4D$

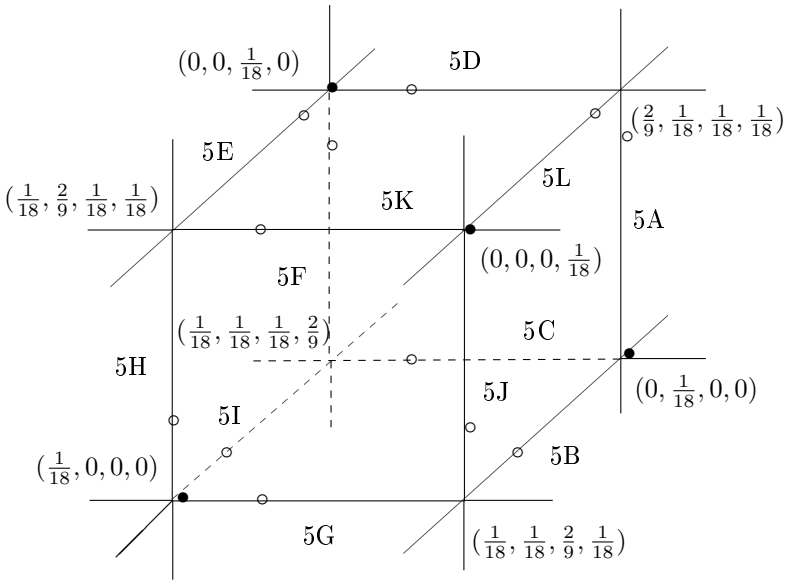

FIGURE 2.3 – The families of $\mathbf{PVI}_\alpha$ equations whose solution is $5A$, $5B$, ..., $5K$, or $5L$

In the case when Γ_β is irreducible, the relation $\{N(\lambda,t)=0\}$ defines an algebraic function $\lambda(t)$. If this function were a solution of some $\mathbf{PVI}_\alpha$ *equation, then the only ramification points of $\lambda(t)$ would be at $t=0,1,\infty$ (because* $\mathbf{PVI}_\alpha$ *satisfies the so called Painlevé property (38)). Equivalently, the pair (Γ_β,t) is a Belyi pair, which means that the only possible critical values of the map*

$$\pi:\Gamma_\beta\to\mathbb{C}\P^1:(\lambda,t)\to t \tag{2.3}$$

are 0, 1 or ∞. This means also that if $\Delta(t)$ is the discriminant of $N(\lambda,t)$ with respect to λ, then it is a polynomial whose only roots are at $t=0$ and $t=1$. A direct computation shows that this is impossible. The more difficult case is when $N(\lambda,t)$ is reducible over $\mathbb{C}$. Then Γ_β defines several algebraic functions and we have to apply the above to each of them. Finally we have to check whether the obtained function is actually a solution of some $\mathbf{PVI}_\alpha$ *equation. To check whether a given polynomial $N(\lambda,t)$ is reducible over $\mathbb{C}$ is a difficult task in general. We shall make use of the action of the symmetric group $\mathcal{S}_4$ (see section 2.2.1) on the set of curves Γ_β, parameterized by $\beta\in\mathbb{C}\P^3$.*

It turns out that, first, curves Γ_β with a trivial stabilizer under the action of $\mathcal{S}_4$ can not produce a solution of $\mathbf{PVI}_\alpha$*. The stabilizer of a curve acts on it as a group of automorphisms (symmetries) which imposes additional restrictions on β.*

The second ingredient of the proof is the study of the Puiseux expansion of $\lambda(t)$ in a neighborhood of $t=0,1,\infty$ (section 2.2.2). These expansions depend on the stabilizer of Γ_β only and imply the possible topological types of the solution $\lambda(t)$. Equivalently, to each solution $\lambda(t)$ we associate a Belyi pair and the Puiseux expansions determine their possible dessin d'enfant. The algebraic functions which we obtain in this way are, a posteriori, the solutions of the $\mathbf{PVI}_\alpha$ *presented in Table 2.1.*

Proof of Theorem 2.2.1.

2.2.1 The action of $\mathcal{S}_4$

The set of automorphisms of the projective line $\mathbb{C}\P^1$ which send four distinct points $(0,1,t,\infty)$ to the points $(0,1,\tilde{t},\infty)$ ($\tilde{t}=\tilde{t}(t)$ is uniquely defined) form a group isomorphic to $\mathcal{S}_4$ generated by the transpositions

$$x^1:s\mapsto 1-s, x^2:s\mapsto\frac{1}{s}, x^3:s\mapsto\frac{t-s}{t-1}. \tag{2.4}$$

Each x^i sends an isomonodromic family of Fuchsian systems with singular points at $0,1,t,\infty$ to an isomonodromic family of such systems with singular points at $0,1,\tilde{t},\infty$. Therefore x^i induce an action of $\mathcal{S}_4$ on the set of $\mathbf{PVI}_\alpha$ *equations, and hence on the set of curves Γ_β. Explicitly we have*

$$x^i:\Gamma_\beta\to\Gamma_{x^i_*(\beta)}:(\lambda,t)\to(x^i(\lambda),x^i(t)), i=1,2, \tag{2.5}$$

and

$$x^3:\Gamma_\beta\to\Gamma_{x^3_*(\beta)}:(\lambda,t)\to(x^3(\lambda),x^3(0))=(\frac{t-\lambda}{t-1},\frac{t}{t-1}), \tag{2.6}$$

where

$$\begin{cases} x^1_*:(\beta_0,\beta_1,\beta_2,\beta_3) & \to & (\beta_0,\beta_2,\beta_1,\beta_3),\\ x^2_*:(\beta_0,\beta_1,\beta_2,\beta_3) & \to & (\beta_1,\beta_0,\beta_2,\beta_3),\\ x^3_*:(\beta_0,\beta_1,\beta_2,\beta_3) & \to & (\beta_0,\beta_3,\beta_2,\beta_1),\end{cases} \tag{2.7}$$

which is the standard representation of $\mathcal{S}_4$ on $\mathbb{C}^4$ (upon identifying $\infty,0,1,t$ to $\beta_0,\beta_1,\beta_2,\beta_3$ respectively). The proof of the above facts is straightforward, see (38) for details.

2.2.2 The topological type of the projection $\Gamma_\beta \to \mathbb{C}\P^1$ in a neighborhood of the pre-image of $t = 0, 1, \infty$

Let Γ_β be the compactified and normalized curve defined by (2.2) (it is a disjoint union of Riemann surfaces). In this section we determine the topological type of the projection (2.3)

$$\Gamma_\beta \to \mathbb{C}\P^1$$

in a neighborhood of the pre-image of $t = 0, 1, \infty$ in Γ_β. In the projective space $\mathbb{C}\P^3$ with coordinates $[\beta_0 : \beta_1 : \beta_2 : \beta_3]$ consider the complex polyhedron W formed by the ten planes (2-faces)

$$W = \cup_{i \neq j}\{\beta_i = \beta_j\} \cup_k \{\beta_k = 0\}. \tag{2.8}$$

It has also 45 1-faces (projective lines) and 120 0-faces (points). We shall see in the process of the proof that the topological type of the projection in a neighborhood of the pre-image of $t = 0, 1, \infty$ is one and the same when β belongs to a given i-face, but does not belong to any other j-face with $j < i$. For this reason we shall use, until the end of this paper, the following convention. When we say that a point β belongs to a given face (satisfies some set of relations (2.8)), then this will mean that it does not belong to any other face of smaller dimension (does not satisfy any other relation from the list (2.8)). *The topological type in a neighborhood of the pre-image of any point is determined by a partition of the degree of the map which is 6. Thus a partition $(1+1+1+1+2)$ means that we have 5 pre-images, and that the multiplicity of π at each pre-image is $1, 1, 1, 1, 2$ respectively. Similarly, a partition $(1 + 1 + 2 + 2)$ means that have 4 pre-images with multiplicities $1, 1, 2, 2$ respectively etc. To formulate the result we note that the symmetric group $\mathcal{S}_4$ acts on the polyhedron W by its standard representation (2.7), as well on the set of curves Γ_β by (2.5). The subgroup $\mathcal{S}_3$ generated by x^1, x^2 permutes the ramification points $0, 1, \infty$ according to (2.4) without changing the topological type of the projection π over each of these points.*

Proposition 2.2.5 *The topological type of the projection (2.3) in a neighborhood of the pre-image of $t = 0, 1, \infty$ is one and the same when β belongs to a given face of the polyhedron W or it does not belong to W. This topological type is shown on Table 2.2 (one representative for each orbit of $\mathcal{S}_3 =< x^1, x^2 >$).*

Proof *The bi-rational transformations x^1, x^2 defined by (2.5) are compatible with the projection π and permute the points $t = 0, 1, \infty$. Therefore it suffices to consider the pre-image of 0. Let us consider in detail the "generic" case, when $\beta \notin W$. It follows from the Newton polygon of $N(\lambda, t)$, shown on fig. 2.4, that there are at least three Puiseux series in a neighborhood of $(0, 0)$ (for the terminology see for instance (45, Kirwan)). The first two correspond to the line segment $[(3, 0), (1, 2)]$ and have non-equivalent leading terms*

$$\lambda = c_1 t + \dots, \ \lambda = c_2 t + \dots,$$

where

$$(\beta_3 - \beta_1)c_{1,2}^2 + 2\beta_1 c_{1,2} - \beta_1 = 0,$$

provided that

$$\beta_1 \neq \beta_3, \ \beta_1^2 + \beta_1(\beta_3 - \beta_1) \neq 0.$$

The third one corresponds to the line segment $[(1, 2), (0, 4)]$ and has leading term

$$\lambda = c_3 t^{1/2} + \dots,$$

where

$$(\beta_0 - \beta_2)c_3^2 + \beta_3 - \beta_1 = 0,$$

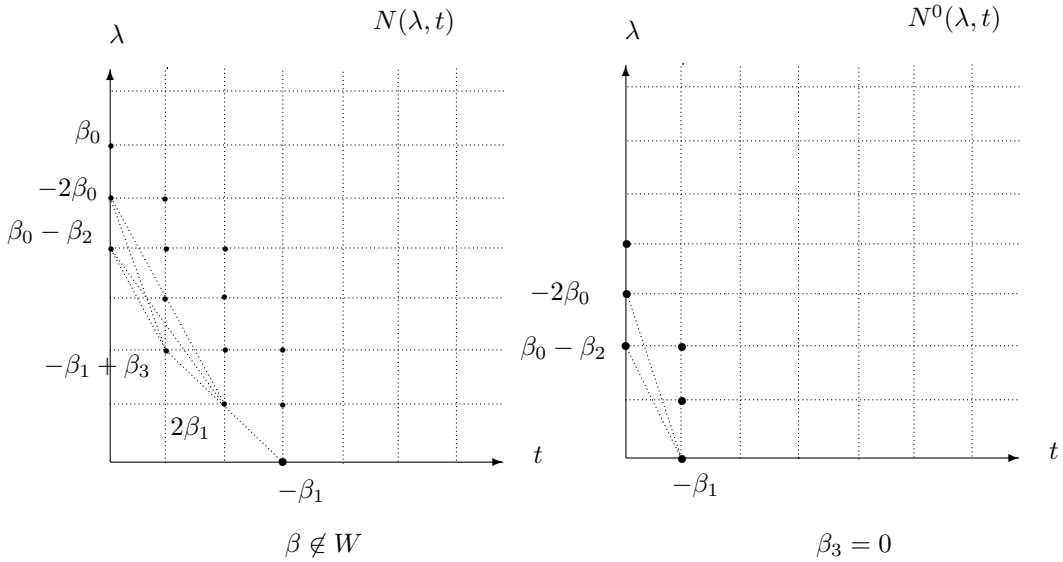

FIGURE 2.4 – The Newton polygon of $N(\lambda,t)$ and $N^0(\lambda,t)$

provided that

$$\beta_0\neq\beta_2,\ \beta_1\neq\beta_3.$$

Taking into consideration that

$$N(\lambda,0)=\lambda^4(\beta_0\lambda^2-2\beta_0\lambda+\beta_0-\beta_2),$$

we conclude that we have at least five pre-images of multiplicities at least $1,1,2,1,1$ *respectively. As the degree of the map* π *is six, then its topological type is exactly* $(1+1+1+1+2)$*. The topological type of the projection* π *over* 0 *and* 1 *is obtained by acting with the group* $\mathcal{S}_3$ *generated by* x^1, x^2.

In a similar way one verifies that when $\beta_0=\beta_2$*, or* $\beta_1=\beta_3$ *the multiplicities are* $(1+1+1+3)$*. If* $\beta_0=\beta_2$ *and* $\beta_1=\beta_3$ *the multiplicities are* $(1+1+2+2)$*. The case* $\beta_0=\beta_1=\beta_3$ *is the same as* $\beta_0=\beta_2$ *and the multiplicity is* $(1+1+1+3)$*. The case* $\beta_0=\beta_1=\beta_2=\beta_3$ *is of the same type as* $\beta_0=\beta_2$ *and* $\beta_1=\beta_3$*. The multiplicities of* π *over* 1 *and* ∞ *are obtained as before. This completes the study of faces of* W *for which* $\beta_i\neq 0$*. In the case* $\beta_3=0$ *we consider the curve*

$$\Gamma^0_\beta=\{(\lambda,t)\in\mathbb{C}^2:\beta_0-\beta_1\frac{t}{\lambda^2}+\beta_2\frac{t-1}{(\lambda-1)^2}=0,\lambda\neq 0,1\}.$$

The polynomial $N(\lambda,t)$ *is replaced by*

$$N^0(\lambda,t)=\beta_0\lambda^2(\lambda-1)^2-\beta_1t(\lambda-1)^2+\beta_2(t-1)\lambda^2$$

whose Newton polygon is shown on fig. 2.4. It follows that there is at least one Puiseux expansion with leading term

$$\lambda=ct^{1/2}+\dots,\ \beta_1+(\beta_2-\beta_0)c^2=0$$

provided that $\beta_1\neq 0$*,* $\beta_2\neq\beta_0$*. As*

$$N^0(\lambda,0)=\lambda^2(\beta_0(\lambda-1)^2-\beta_2),$$

then $t=0$ *has at least three pre-images, provided that* $\beta_0\beta_2\neq 0$*. We conclude that* $t=0$ *has exactly three pre-images with multiplicities* $2,1,1$ *respectively, provided that* β *belongs to the 2-face* $\beta_3=0$*. The remaining*

face of W	stabilizer	$t=0$	$t=1$	$t=\infty$
$\beta_i \neq \beta_j$	S_4	(1+1+1+1+2)	(1+1+1+1+2)	(1+1+1+1+2)
$\beta_0 = \beta_2$	$S_2 \times S_2$	(1+1+1+3)	(1+1+1+1+2)	(1+1+1+1+2)
$\beta_0 = \beta_2, \beta_1 = \beta_3$	D_4	(1+1+2+2)	(1+1+1+1+2)	(1+1+1+1+2)
$\beta_0 = \beta_1 = \beta_2$	S_3	(1+1+1+3)	(1+1+1+3)	(1+1+1+3)
$\beta_0 = \beta_1 = \beta_2 = \beta_3$	S_4	(1+1+2+2)	(1+1+2+2)	(1+1+2+2)
$\beta_3 = 0$	S_3	(1+1+2)	(1+1+2)	(1+1+2)
$\beta_0 = \beta_2, \beta_3 = 0$	S_2	(1+3)	(1+1+2)	(1+1+2)
$\beta_0 = \beta_1 = \beta_2, \beta_3 = 0$	S_3	(1+3)	(1+3)	(1+3)
$\beta_2 = \beta_3 = 0$	$S_2 \times S_2$	(2)	(1+1)	(2)
$\beta_2 = \beta_3 = 0, \beta_0 = \beta_1$	$S_2 \times S_2$	(2)	(1+1)	(2)

TABLE 2.2 – Multiplicity of π at the pre-images of $t = 0, 1, \infty$.

1-faces and 0-faces are studied in the same way. The result is summarized on Table 2.2. It worth noting that in all cases the computing of the leading term of the Puiseux expansion suffices to deduce the result.

We conclude this section by the following elementary claim which will be often useful in the computations.

Proposition 2.2.6 *Let $N_1(\lambda, t)$ be a polynomial of non-zero degree with respect to λ and of non-zero degree with respect to t, which divides $N(\lambda, t)$, and $\beta_1\beta_2\beta_3 \neq 0$. Then*

$$N_1(0,t) = c_0 t^{n_0}, c_0 \neq 0, 1 \leq n_0 \leq 3,$$

$$N_1(1,t) = c_1 (t-1)^{n_1}, c_1 \neq 0, 1 \leq n_1 \leq 3,$$

and

$$N_1(t,t) = c_2 t^{m_0}(t-1)^{m_1}, c_2 \neq 0, 1 \leq m_0, 1 \leq m_1, m_0 + m_1 \leq 3.$$

Proof *We have $N(0,t) = -\beta_1 t^3$. For a fixed $\lambda = c \sim 0$ the polynomial $N(c,t) \in \mathbb{C}[t]$ has exactly three roots which tend to zero when c tends to zero. Therefore the polynomial $N_1(c,t) \in \mathbb{C}[t]$ has at least one and at most three roots which tends to zero when c tends to zero, which proves the claim concerning $N_1(0,t)$. The claim concerning $N_1(1,t)$ is proved in the same way. As $N_1(0,0) = N_1(1,1) = 0$ then $N_1(t,t)$ is divided by $t(t-1)$ but also divides $N(t,t) = -\beta_3 t^3 (t-1)^3$.*

We are ready to compute the solutions of $\mathbf{PVI}_\alpha$ *corresponding to the faces of W. Let Γ be the Riemann surface of an irreducible component of Γ_β, which defines a solution of some* $\mathbf{PVI}_\alpha$ *equation. Then the only ramification points of the induced map*

$$\pi : \Gamma \to \mathbb{CP}^1 : (\lambda, t) \to t \tag{2.9}$$

are at $0, 1, \infty$, and Γ is connected. The pair (Γ, π) is called a Belyi pair, to which we associate a dessins d'enfant, *which is the graph obtained as a pre-image of the segment $[0,1]$ under the map π. The degree of the dessin is the degree of π (see (68)). The dessin d'enfant will be useful when describing the topological type of the projection π.*

2.2.3 The case $\beta \notin W$

We suppose that $\lambda(t)$ is an algebraic function, such that $N(\lambda(t), t) \equiv 0$, and consider the corresponding Belyi pair (Γ, π), (2.9). Let $\{\lambda_1, \ldots, \lambda_d\} = \pi^{-1}(t_0)$ where $t_0 \neq 0, 1, \infty$. The loops originating from t_0 and going clockwise once around $0, 1$ and ∞ induce permutations σ_0, σ_1 and σ_∞ of the points $\lambda_1, \ldots, \lambda_d$, such that $\sigma_0\sigma_1\sigma_\infty = 1$.

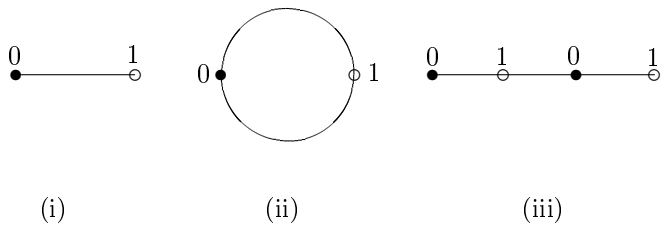

FIGURE 2.5 – Dessins of degree $1, 2$ and 3

According to Table 2.2 σ_0, σ_1 and σ_∞ are transpositions, unless one of them is the identity permutation. In the former case $(\sigma_0\sigma_1)^2 = 1$ and hence $\sigma_0\sigma_1 = \sigma_1\sigma_0$. Thus σ_∞ is a product of two disjoint transpositions, which contradicts to Table 2.2. On the other, if one of the permutations $\sigma_0, \sigma_1, \sigma_\infty$ is the identity, the group generated by them is either $\mathbf{Z}_2$ or is trivial. This shows that the degree of π is either two (because the covering (2.9) is connected), or one. The corresponding dessin d'enfants are shown on fig.2.5 (i) and (ii) (in particular $N(\lambda, t) \in \mathbb{C}[\lambda, t]$ is reducible). If the dessin is of degree one, then the solution is defined as $\lambda = P(t)$, where P is a polynomial (the coefficient of λ^6 in the polynomial $N(\lambda, t)$ is $\beta_0 \neq 0$). By Proposition 2.2.6 we conclude that either $\lambda = t$, or $\lambda = t^2$ or $\lambda = t^3$. But $\lambda - t$, $\lambda - t^2$, $\lambda - t^3$ can not divide $N(\lambda, t)$, provided that $\beta_i \neq 0$. If the dessin is of degree two, then $\lambda(t)$ is defined by $\lambda^2 + 2p(t)\lambda + q(t) = 0$. The functions p, q are polynomials in t, because the coefficient of λ^6 in the polynomial $N(\lambda, t)$ is $\beta_0 \neq 0$. Further, we may suppose (acting with an appropriate symmetry x^i on Γ, see section 2.2.1) that $\lambda(t)$ is ramified over 0 and ∞ only. By Proposition 2.2.6 $q(t)$ is a non-constant polynomial which divides t^3. As $p(t)^2 - q(t)$ is a non-constant monomial of odd degree, then $p(t) \equiv 0$. Proposition 2.2.6 implies that we have either $\lambda^2 = t$ or $\lambda^2 = t^3$. The polynomial $\lambda^2 - t^3$ can not divide $N(\lambda, t)$ while $\lambda^2 - t$ divides $N(\lambda, t)$ if and only if $\beta_0 = \beta_1$ and $\beta_2 = \beta_3$ (this case is excluded, as $\beta \notin W$). The curve Γ_β does not define a solution.

2.2.4 The face $\beta_1 = \beta_2$

The possible dessins d'enfant are determined as above. Namely, when one of the permutations $\sigma_0, \sigma_1, \sigma_\infty$ is identity, the dessin is of degree one or two. Up to a symmetry it is equivalent to the one shown on fig. 2.5 (i) or (ii). Reasoning as in the case $\beta \notin W$ we conclude that Γ_β does not define a solution.

If, on the other hand, σ_0, σ_1 are non-trivial transpositions we have one more case compared to section 2.2.3 : σ_3 is cyclic of order three, and σ_0, σ_1 are non-disjoined permutations. Taking into account that the covering (2.9) is connected, we conclude that the dessin is of degree three. Up to a symmetry, it is shown on 2.5 (iii) and $\lambda(t)$ satisfies $N_1(\lambda(t), t) \equiv 0$ where N_1 is an irreducible polynomial of degree three in λ dividing the polynomial $N(\lambda, t)$, defined after formula (2.2). We denote

$$N = N_1 N_2, \Gamma_1^{aff} = \{N_1(\lambda, t) = 0\},$$

$$\Gamma_2^{aff} = \{N_2(\lambda, t) = 0\}, \Gamma_\beta^{aff} = \Gamma_1^{aff} \cup \Gamma_2^{aff}.$$

As before, let $\Gamma_1, \Gamma_2, \Gamma_\beta$ be the corresponding compactified and normalized curves. The symmetry x^1 is an automorphism of Γ_β and hence it is either an automorphism of the curves Γ_1 and Γ_2, or it permutes these curves (we used that Γ_1 is irreducible). Suppose first that x^1 is an automorphism of Γ_1. Then the rational function

$$\frac{N_1(\lambda, t)}{\lambda(\lambda - 1)(\lambda - t)} = 1 + \frac{A_1}{\lambda} + \frac{B_1}{\lambda - 1} + \frac{C_1}{\lambda - t}$$

is invariant under the action of x^1 *too. Here* A_1, B_1, C_1 *are polynomials in* t *which divide* $t, t-1$ *and* $t(t-\lambda)$ *respectively (see (2.1)), and hence we have*

$$A_1(1-t) = -B_1(t),\ B_1(1-t) = -A_1(t),\ C_1(1-t) = -C_1(t).$$

Similarly, if

$$\frac{N_2(\lambda,t)}{\lambda(\lambda-1)(\lambda-t)} = 1 + \frac{A_2}{\lambda} + \frac{B_2}{\lambda-1} + \frac{C_2}{\lambda-t},$$

then

$$A_2(1-t) = -B_2(t),\ B_2(1-t) = -A_2(t),\ C_2(1-t) = -C_2(t).$$

We conclude that $C_1(t) = c_1 t(t-1)$, $C_2(t) = c_2 t(t-1)$, *which contradicts to* $C_1C_2 = -\beta_3 t(t-1)/\beta_1$, $\beta_1, \beta_3 \neq 0$. *Suppose now that the map* x^1 *exchanges the curves* Γ_1 *and* Γ_2. *Then we have*

$$A_1(1-t) = -B_2(t),\ B_1(1-t) = -B_2(t),\ C_1(1-t) = -C_2(t).$$

The polynomial $N(\lambda,t)$ *is of degree three with respect to* t *and*

$$A_1A_2 = -\frac{\beta_1}{\beta_0}t,\ B_1B_2 = -\frac{\beta_2}{\beta_0}(t-1),\ C_1C_2 = -\frac{\beta_3}{\beta_0}t(t-1).$$

Therefore without loss of generality we may suppose that $C_1(t) = c_1 t, C_2 = c_1(t-1)$ *and* $A_1(t) = a_1 t, B_2 = b_2(t-1)$ *or* $A_2(t) = a_2 t, B_1 = b_1(t-1)$, *where* $a_i, b_j \neq 0$. *In both cases the polynomials* $N_1(\lambda,t)$, $N_2(\lambda,t)$ *are of degree two in* t, *in contradiction to the fact that the degree of* $N(\lambda,t)$ *with respect to* t *is three. We conclude that the curve* Γ_β *does not define a solution.*

2.2.5 The face $\beta_0 = \beta_2$, $\beta_1 = \beta_3$

We have the identity

$$N(\lambda,t) = (\lambda^2 - 2\lambda + t)(\beta_0\lambda^2(\lambda-t)^2 - \beta_1 t^2(\lambda-1)^2).$$

Indeed,

$$\lambda^2 - 2\lambda + t = 0 \tag{2.10}$$

defines a solution of $\mathbf{PVI}_\alpha$, *e.g. (18), Table 2.1, solution* $2B$. *Its dessin is equivalent to the one on Figure 2.5 (ii). The function* $\lambda(t)$ *defined by*

$$\lambda(\lambda-t) - ct(\lambda-1) = 0,\ c = \pm\sqrt{\frac{\beta_1}{\beta_0}},$$

is ramified over $0, 1, \infty$ *only provided that* $c = 0, \pm 1$. *This is, however, impossible as* $\beta_0 \neq \beta_1, \beta_i \neq 0$. *The curve* Γ_β *defines the solution (2.10).*

2.2.6 The face $\beta_0 = \beta_1 = \beta_2$

According to Table 2.2 each of the permutations $\sigma_0, \sigma_1, \sigma_\infty$ *is either the identity, or is a cycle of length three.*

If one of the permutations $\sigma_0, \sigma_1, \sigma_\infty$ *is the identity then the group generated by* $\sigma_0, \sigma_1, \sigma_\infty$ *is either* $\mathbf{Z}_3$ *or the trivial one* $\{\mathbf{1}\}$, *and hence the degree of the corresponding dessin is one or three. The case of degree one does not lead to a solution (see section 2.2.3). The case of degree three is studied as in section 2.2.4 and does not lead to a solution too (provided that* $\beta_i \neq 0$*).*

If neither of the permutations $\sigma_0, \sigma_1, \sigma_\infty$ is the identity, then they are disjoint three-cycles. As the symmetric group $\mathcal{S}_3$ contains only two three-cycles we conclude that the degree of the projection π is at least four. Suppose that $\lambda(t)$ is defined by the polynomial N_1, $N_1(\lambda(t), t) \equiv 0$, where $N_1 \in \mathbb{C}[\lambda, t]$ is irreducible of degree four in λ. Then x^1, x^2 are automorphisms of

$$\Gamma_1 = \{N_1(\lambda, t) = 0\}.$$

It follows that the curve

$$\Gamma_2 = \{N_2(\lambda, t) = 0\}$$

defined by the polynomial $N_2 = N/N_1$ is also invariant. We have

$$\frac{N_2(\lambda, t)}{\lambda(\lambda - 1)} = 1 + \frac{A}{\lambda} + \frac{B}{\lambda - 1},$$

where A, B are polynomials in t of degree at most three. The $x^{1,2}$ invariance of the above expression implies

$$A(t)A(1/t) = 1,\ A(1/t)B(t) = -B(1/t),\ B(t) = -A(1-t),$$

with solutions

$$A(t) = t,\ B(t) = t - 1;$$

$$A(t) = t^3,\ B(t) = (t-1)^3;$$

$$A(t) = -t^2,\ B(t) = (t-1)^2.$$

The case $A(t) = t^3, B(t) = (t-1)^3$ does not lead to a solution as $N_1(\lambda, t)$ does depend on t. The case $A(t) = -t^2, B(t) = (t-1)^2$ implies $N_2(\lambda, t) = (\lambda - t)^2$, and hence $\beta_3 = 0$. Finally, in the case $A(t) = t, B(t) = t - 1$ we have $N_2(\lambda, t) = \lambda^2 - 2\lambda + 2\lambda t - t$ (which does not define a solution). The condition that $N_2(\lambda, t)$ divides $N(\lambda, t)$ leads to $\beta_3 = 9\beta_0$ and we get

$$N(\lambda, t) = \left(\lambda^2 - 2\,\lambda + 2\,\lambda\,t - t\right)\left(t^2 - 4\,\lambda^3 t + 6\,\lambda^2 t - 4\,\lambda\,t + \lambda^4\right).$$

The function $\lambda(t)$ defined by

$$t^2 - 4\,\lambda^3 t + 6\,\lambda^2 t - 4\,\lambda\,t + \lambda^4 = 0 \tag{2.11}$$

is indeed a solution of $\mathbf{PVI}_\alpha$, e.g. (18), Table 2.1, solution $3D$. In the case when the dessin corresponding to $\lambda(t)$ is of degree five we conclude that the polynomial $N_2(\lambda, t)$ is linear in λ. By Proposition 2.2.6 we get $N_2(\lambda, t) = \lambda - t^2$ which implies $\beta_1 = 0$. To resume, the curve Γ_β defines a solution, provided that $\beta_0 = \beta_1 = \beta_2 = \beta_3/9$.

2.2.7 The face $\beta_0 = \beta_1 = \beta_2 = \beta_3$

We have

$$N(\lambda, t) = \beta_0(\lambda^2 - 2\lambda + t)(\lambda^2 - 2\lambda t + t)(\lambda^2 - t)$$

and the three algebraic functions defined by $N(\lambda, t) = 0$ are solutions of suitable $\mathbf{PVI}_\alpha$ equations, e.g. (18), Table 2.1, solutions $2B, 2C, 2A$ respectively. The curve Γ_β defines three solutions.

2.2.8 The face $\beta_3 = 0$

Recall that in this case $N(\lambda, t) = (\lambda - t)^2 N^0(\lambda, t)$ *where*

$$N_0(\lambda, t) = \beta_0 \lambda^2 (\lambda - 1)^2 - \beta_1 t(\lambda - 1)^2 + \beta_2 (t - 1)\lambda^2.$$

Indeed, $\lambda = t$ *is the so called constant solution 0B (because, up to a symmetry, it coincides with* $\lambda = 0, 1, \infty$*). To the end of this section the polynomial* N *bill be replaced by* N^0 *and the curve* Γ_β *by* $\Gamma^0_\beta = \{N_0(\lambda, t) = 0\}$*. The same arguments as in section 2.2.3 show that the corresponding dessin d'enfant is of degree one or two.*

If the degree is one, then the solution is $\lambda = P(t)$ *for some non-constant polynomial* P*. Therefore* $N^0(P(t), t) \not\equiv 0$ *and* $P(t)$ *can not be a solution.*

If the degree is two, then $\lambda(t)$ *has exactly two ramification points. Without loss of generality we suppose that these points a* 0 *and* ∞*, and as in section 2.2.3 we conclude that* $\lambda(t)$ *is defined by* $\lambda^2 + 2p(t)\lambda + q(t) = 0$ *for some* $p, q \in \mathbb{C}[t]$*. The polynomial* $\lambda^2 + 2p(t)\lambda + q(t)$ *divides*

$$N_0(\lambda, t) = t(\beta_2 \lambda^2 - \beta_1 (\lambda - 1)^2) + \beta_0 \lambda^2 (\lambda - 1)^2 - \beta_2 \lambda^2$$

and hence $p(t) = c_1$ *and* $q(t) = c_2 t$ *for some constants* c_1, c_2*. Without loss of generality we suppose that the ramification points of* $\lambda(t)$ *are* 0 *and* ∞ *and hence the discriminant* $4(p^2 - q)$ *is a power of* t*. This implies that* $c_1 = 0$*. Finally, a direct computation shows that the identity* $N^0(\sqrt{-c_2 t}, t) \equiv 0$ *implies* $\beta_2 = 0$ *which is not true. The curve* Γ^0_β *does not define a solution.*

2.2.9 The face $\beta_3 = 0, \beta_0 = \beta_2$

It is easier to analyze the face $\beta_3 = 0, \beta_1 = \beta_2$*, which is equivalent to* $\beta_3 = 0, \beta_0 = \beta_2$ *after applying the transformation* x^2*. Suppose for a moment that* $\beta_3 = 0, \beta_1 = \beta_2$*. The dessin is of degree at most three, and hence* $N^0(\lambda, t)$ *is reducible. It follows that* $\lambda - c$ *divides* $N^0(\lambda, t)$ *for some constant* c*. As* $N^0(\lambda, t)$ *is linear in* t*, then* $\lambda - c$ *is deduced from the coefficient of* t *which equals to* $\beta_1(1 - 2\lambda)$*. Thus* $c = 1/2$ *and the condition that* $1 - 2\lambda$ *divides* $N^0(\lambda, t)$ *leads to* $\beta_0 = 4\beta_1 = 4\beta_2$*, in which case*

$$N^0(\lambda, t) = \beta_0 \left(2\,\lambda - 1\right) \left(2\,\lambda^3 - 3\,\lambda^2 + t\right).$$

The function $\lambda(t)$ *defined by* $\left(2\,\lambda^3 - 3\,\lambda^2 + t\right) = 0$ *is indeed a solution, see (18), Table 2.1, solutions 5L. Applying the transformation* $(x^2)^{-1} = x^2$ *of section 2.2.1 we get the solution (see (18), Table 2.1, solution 5K)*

$$\lambda^3 - 3\,\lambda t + 2\,t = 0$$

defined by Γ^0_β *with* $\beta_3 = 0, \beta_1 = 4\beta_0 = 4\beta_2$*. The curve* Γ^0_β *defines a solution provided that* $\beta_3 = 0, \beta_1 = 4\beta_0 = 4\beta_2$*.*

2.2.10 The face $\beta_3 = 0, \beta_0 = \beta_1 = \beta_2$

The polynomial $N^0(\lambda, t)$ *is irreducible and defines a solution, see (18), Table 2, solution 4D.*

2.2.11 The face $\beta_3 = 0, \beta_2 = 0$

The curve Γ^0_β defines the constant solution $\lambda = 1$, 0B, as well the solution 1A, $\beta_0 \lambda^2 - \beta_1 t = 0$.

2.2.12 The face $\beta_3 = 0, \beta_2 = 0, \beta_0 = \beta_1$

The curve Γ^0_β defines the constant solution $\lambda = 1$, 0B, and the solution $\lambda^2 = t$, 2A.

The results are summarized in Table 2.1. Theorem 2.2.1 is proved.

2.3 Algebraic solutions of $\mathbf{PVI}_\alpha$ and Picard-Fuchs equations

It was noted in the Remark after Theorem 2.2.1 that the families $1A, 1B, \ldots, 1F$ are Okamoto equivalent to the families $2A, 2B, 2C$. To this end we consider the remaining 23 families $2A - 5L$, see Table 2.1. To each of them corresponds an affine plane or line in the parameter space $\mathbb{C}^4\{\alpha\}$ which, as we shall prove bellow, is generated by special points α of geometric origin, see Tables 2.4 and 2.5. Indeed, observe that exactly the same 23 solutions $2A - 5L$ were already obtained by Doran, see Theorem 2.3.1 bellow, by making use of deformations of elliptic surfaces with four singular fibers. The corresponding special values of the parameter α are given in Table 2.4. The main result of this section is that exactly the same list of solutions can be obtained from deformations of ramified covers of $\P^1$ with four ramification points. The corresponding values of the parameters α are different and are shown on Table 2.5, see Theorem 2.3.3.

Recall that an elliptic surface is a complex compact analytic surface S with a projection $S \to \P^1$, such that the general fiber $f^{-1}(z) = \Gamma_z$ is an elliptic curve. Two elliptic surfaces are equivalent, if there is a bi-analytic map compatible with the projections, see (46, Kodaira).

We may suppose that the fiber Γ_z is written in the Weierstrass form

$$\Gamma_z = \{(x, y) \in \mathbb{C}^2 : y^2 = 4x^3 - g_2(z)x - g_3(z)\}$$

and consider the complete elliptic integrals of first and second kind :

$$\eta_1 = \int_{\gamma(z)} \frac{dx}{y}, \ \eta_2 = \int_{\gamma(z)} \frac{xdx}{y},$$

where $\gamma(z) \subset \Gamma_z$ is a continuous family of closed loops (representing a locally constant section $z \mapsto H_1(\Gamma_z, \mathbb{Z})$ of the associated homology bundle). Then η_1, η_2 satisfy the following Picard-Fuchs system (this goes back at least to (29, Griffiths), see (66, Sasai))

$$\triangle(z)\frac{d}{dz}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\frac{\triangle'_z}{12} & -\frac{3\delta}{2} \\ -\frac{g_2\delta}{8} & \frac{\triangle'_z}{12} \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \tag{2.12}$$

where

$$\triangle(g_2, g_3) = g_2^3 - 27g_3^2,$$

and

$$\delta(z) = 3g_3\frac{dg_2}{dz} - 2g_2\frac{dg_3}{dz}.$$

The singular points of the system correspond to singular fibers of the surface. The elliptic surfaces with four singular fibers were classified by (30, Herfurtner) who obtained 50 distinct case, but only 5 of them contain an additional parameter, see Table 2.3. They lead to non-trivial isomonodromic deformations of the above Picard-Fuchs system with four regular singular points. If we renormalize the singular points to be $0, 1, \infty, t$ then the zero λ of $\delta(z)$, considered as a function in t is a solution of an appropriate $\mathbf{PVI}_\alpha$ equation, see (59, Okamoto) for details. The result is summarized as follows.

Theorem 2.3.1 *(18, Theorem 3.13) All algebraic solutions $(\lambda(t), \alpha)$ of $\mathbf{PVI}_\alpha$ equation coming from moduli of elliptic surfaces with four singular fibers are shown on Table 2.1. The corresponding values of α together with the stabilizer of the solution and the $\mathbf{PVI}_\alpha$ equation under the action of the symmetric group S_4 are listed on Table 2.4.*

Remark 2.3.2 *The Picard-Fuchs system (2.12) has generically an infinite monodromy group. More precisely, let $z \mapsto (g_2(z), g_3(z))$ be a curve intersecting transversally the discriminant locus $\{g_2^3 - 27g_3^2 = 0\}$ of Γ_z at some smooth point $(g_2(z_0), g_3(z_0))$. The Picard-Lefschetz formula implies that the monodromy of the first homology group $H_1(\Gamma_z, \mathbb{Z})$ along a small loop which makes one turn about z_0 is represented by the unipotent matrix*

$$\begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}$$

and hence the monodromy group of (2.12) is infinite. When the curve

$$z \mapsto (g_2(z), g_3(z))$$

is chosen as on Table 2.3, the result also follows from (30, Table 1).

We will deduce a Picard-Fuchs system closely related to (2.12), but having a finite monodromy. Consider a ramified covering $\Gamma \to \P^1$ of degree three with branching locus consisting of four points, where Γ is a Riemann surface. We choose an affine model $\Gamma^{aff} = \{(x, z) \in \mathbb{C}^2 : f(x, z) = 0\}$ of Γ, where $f(x, z) = 4x^3 - g_2 x - g_3$, and $g_2 = g_2(z), g_3 = g_3(z)$ are suitable polynomials. Moreover, without loss of generality, we suppose that the covering $\Gamma \to \P^1$ is induced from the projection

$$\{(x, z) \in \mathbb{C}^2 : f(x, z) = 0\} \to \mathbb{C} : (x, z) \mapsto z. \tag{2.13}$$

Let $x_1(z), x_2(z)$ be two distinct roots of f. Then $\gamma(z) = x_1(z) - x_2(z)$ is a 0-cycle of the fiber $\{x : f(x, z) = 0\}$ and the Abelian integrals above are replaced by the algebraic functions

$$\eta_1(z) = \int_{\gamma(z)} x = x_1(z) - x_2(z), \eta_2(z) = \int_{\gamma(z)} x^2 = x_1^2(z) - x_2^2(z).$$

A straightforward computation shows that η_1, η_2 satisfy the following Picard-Fuchs system (see (28))

$$\triangle(z) \frac{d}{dz} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \frac{\triangle_z'}{6} & -3\delta \\ -\frac{g_2 \delta}{2} & \frac{\triangle_z'}{3} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}. \tag{2.14}$$

As before, if we renormalize the system (2.14) to have singular points at $0, 1, t, \infty$, then the root $\lambda(t)$ of $\delta(z)$ is a solution of a suitable $\mathbf{PVI}_\alpha$ equation, provided that the deformation is isomonodromic. The last property holds, strictly speaking, in the case when the system is non-resonant. In our case it holds too, because the deformation is isoprincipal in the sense of (42). This can be also checked by a direct computation. Thus, if we consider the fibration (2.13) and take for g_2, g_3 the expressions found by Herfutner, see Table 2.3, we get the 23 algebraic solutions 2A-5L shown on Table 2.1. The corresponding values for α are different, because the monodromy group of (2.14) is finite, see Remark 2.3.2. They are computed in Table 2.5. In this way we proved the following theorem.

Theorem 2.3.3 *The algebraic solutions $(\lambda(t), \alpha)$ of $\mathbf{PVI}_\alpha$ equation coming from deformations of the covering (2.13) with g_2, g_3 as on the Herfutner list, Table 2.3, are shown on Table 2.1. The corresponding values of α together with the stabilizer of the solution and the $\mathbf{PVI}_\alpha$ equation under the action of the symmetric group S_4 are listed on Table 2.5.*

Name	Deformation
1	$g_2(z,a) = 3(z-1)(z-a^2)^3$ $g_3(z,a) = (z-1)(z-a^2)^4(z+a)$
2	$g_2(z,a) = 12z^2(z^2+az+1)$ $g_3(z,a) = 4z^3(2z^3+3az^2+3az+2)$
3	$g_2(z,a) = 12z^2(z^2+2az+1)$ $g_3(z,a) = 4z^3(2z^3+3(a^2+1)z^2+6az+2)$
4	$g_2(z,a) = 3z^3(z+a)$ $g_3(z,a) = z^5(z+1)$
5	$g_2(z,a) = 3z^3(z+2a)$ $g_3(z,a) = z^4(z^2+3az+1)$

TABLE 2.3 – The Herfurtner list of "deformable" elliptic surfaces with four singular fibers

Remark 2.3.4 *Particular cases of the Picard-Fuchs system (2.14), in a more or less explicit way, were considered by many authors, e.g. (10, Boalch), (19, Dubrovin-Mazzocco), (31; 32, Hitchin), (44, Kitaev).*

To this end, for convenience of the reader, we explain how the solution 4A with $\alpha = (1/8, 1/8, 1/8, 1/8)$ *follows from (2.14) and the Table 2.3. The remaining solutions on Table 2.5 are computed in a similar way.*

The Picard-Fuchs system (2.14) *implies that the Abelian integral of first kind* η_1 *satisfies the following equation*

$$p_0(z,a)\eta_1'' + p_1(z,a)\eta_1' + p_2(z,a)\eta_1 = 0, \tag{2.15}$$

where

$$\begin{aligned} p_0(z,a) &= 144\delta\Delta^2, \\ p_1(z,a) &= 144\Delta(\delta\frac{d\Delta}{dz} - \Delta\frac{d\delta}{dz}), \\ p_2(z,a) &= 12\delta\frac{d^2\Delta}{dz^2} - 216\delta^3 g_2 - 12\Delta\frac{d\delta}{dz}\frac{d\Delta}{dz} - \delta(\frac{d\Delta}{dz})^2. \end{aligned}$$

Consider, for instance, the deformation 2 *from the Herfurtner list (Table 2.3)*

$$\begin{aligned} g_2 = g_2(z,a) &= 3z^3(z+a), \\ g_3 = g_3(z,a) &= z^5(z+1). \end{aligned}$$

We have

$$\begin{aligned} \Delta &= \Delta(g_2,g_3) = 27z^9((3a-2)z^2 + (3a^2-1)z + a^3), \\ \delta &= \delta(z,a) = -3z^7((3a-2)z+a). \end{aligned}$$

The Picard-Fuchs equation (2.15) takes the form

$$\begin{gathered} 144z^2((3a-2)z+a)((3a-2)z^2+(3a^2-1)z+a^3)^2\,\eta_1'' \\ +144z((3a-2)z^2+(3a^2-1)z+a^3)(3(3a-2)^2z^3 \\ +2(3a-2)(3a-1)(a+1)z^2+a(3a^3+7a^2-3)z+2a^4)\,\eta_1' \\ +[135(3a-2)^3z^5+(3a-2)^2(468a^2+267a-164)z^4 \end{gathered}$$

$$+2(3a-2)(189a^4+522a^3-48a^2-208a+10)z^3$$
$$-2a(270a^5-1269a^4+252a^3+460a^2-70)z^2$$
$$-a^4(243a^3-666a^2+176)z+27a^7]\,\eta_1=0$$

and has four regular singular points at ∞ *and the roots of* $(3a-2)z^2+(3a^2-1)z+a^3$ *(the roots of* Δ*), as well one apparent singularity at the root of* $(3a-2)z+a$ *(which is a root of* δ*). Re-normalizing the singular points to* $0,1,t,\infty$ *we get*

$$\lambda=\frac{a^2-a+1}{a^2(2-a)},\ t=\frac{2a-1}{a^3(2-a)},\ a\in\mathbb{C}. \tag{2.16}$$

The parameter a *defines an algebraic isomonodromic deformation of the Picard-Fuchs equation (2.15) with Riemann schema*

$$\begin{pmatrix} 0 & 1 & t & \lambda & \infty \\ -\frac{3}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{5}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 2 & \frac{3}{4} \end{pmatrix}.$$

Therefore the algebraic function $\lambda=\lambda(t)$ *determined implicitly by (2.16) is an algebraic solution of* $\mathbf{PVI}_\alpha$ *equation with*

$$\alpha_0=\frac{1}{8},\ \ \alpha_1=\frac{1}{8},\ \ \alpha_2=\frac{1}{8},\ \ \alpha_3=\frac{1}{8}.$$

(see (38) for details). Eliminating a *from (2.16) we get*

$$\lambda^4-2t\lambda^3-2\lambda^3+6t\lambda^2-2t^2\lambda-2t\lambda+t^3-t^2+t=0$$

which is an equation for the solution 4A with $\alpha=(1/8,1/8,1/8,1/8)$*. Together with the Doran's point* $\alpha=(0,1/8,0,0)$*, this implies that the solution* $\lambda(t)$ *satisfies also the implicit equation (2.1) :*

$$1+\frac{t-1}{(\lambda-1)^2}-\frac{t(t-1)}{(\lambda-t)^2}=0$$

corresponding to the affine line through $(1/8,1/8,1/8,1/8)$ *and* $(0,1/8,0,0)$ *described in Table 2.1, 4A. The solution (2.16) with* $\alpha=(1/8,1/8,1/8,1/8)$ *was found by Hitchin (31, section 6.1), (32, (34)).*

Remark 2.3.5 *If we repeat the same computation, but making use of the Picard-Fuchs system (2.12) then of course we obtain the same algebraic solution but with* $\alpha=(1/8,0,0,0)$*. This value has been erroneously computed by Doran (18) to be* $(1/18,0,0,0)$*. This led him to the wrong conclusion that the solution 4C is equivalent by an Okamoto transformation to the "cubic" solution* B_3 *of Dubrovin-Mazzocco (19, p.140) with* $\alpha=(25/18,0,0,0)$*, see (18, Remark 7). As the Okamoto transformations of* $\mathbf{PVI}_\alpha$ *act within the ring* $\mathbb{Z}[1/2,\sqrt{2\alpha_1},\sqrt{2\alpha_1},\sqrt{2\alpha_1},\sqrt{2\alpha_1}]$*, then no solution of* $\mathbf{PVI}_{(25/18,0,0,0)}$ *is equivalent to a solution of* $\mathbf{PVI}_{(1/8,0,0,0)}$*. M. Mazzocco kindly informed us for a missprint in the formula for the* B_3*-solution, (19, p.140). The corrected formula is reproduced in (18, formula (3.1)).*

Stabilizer of the solution	Name of the solution	$\mathbf{PVI}_\alpha$ equation	Stabilizer of $\mathbf{PVI}_\alpha$ equation
D_4	$2A$	$(0,0,\frac{1}{18},\frac{1}{18})$ $(\frac{1}{18},\frac{1}{18},0,0)$	$S_2 \times S_2$
	$2B$	$(\frac{1}{18},0,\frac{1}{18},0)$ $(0,\frac{1}{18},0,\frac{1}{18})$	
	$2C$	$(\frac{1}{18},0,0,\frac{1}{18})$ $(0,\frac{1}{18},\frac{1}{18},0)$	
D_4	$2A$ $2B$ $2C$	$(0,0,0,0)$	S_4
S_3	$3A$ $3B$ $3C$ $3D$	$(0,0,0,0)$	S_4
S_3	$4A$ $4B$ $4C$ $4D$	$(0,\frac{1}{8},0,0)$ $(0,0,\frac{1}{8},0)$ $(\frac{1}{8},0,0,0)$ $(0,0,0,\frac{1}{8})$	S_3
S_2	$5A$ $5B$ $5C$	$(0,\frac{1}{18},0,0)$	S_3
	$5D$ $5E$ $5F$	$(0,0,\frac{1}{18},0)$	
	$5G$ $5H$ $5I$	$(\frac{1}{18},0,0,0)$	
	$5J$ $5K$ $5L$	$(0,0,0,\frac{1}{18})$	

TABLE 2.4 – Solutions $(\lambda(t),\alpha)$ of $\mathbf{PVI}_\alpha$ equations related to the Picard-Fuchs system (2.12).

Stabilizer of the solution	Name of the solution	$\mathbf{PVI}_\alpha$ equation	Stabilizer of $\mathbf{PVI}_\alpha$ equation
D_4	$2A$	$(\frac{1}{8}, \frac{1}{8}, \frac{1}{18}, \frac{1}{18})$ $(\frac{1}{18}, \frac{1}{18}, \frac{1}{8}, \frac{1}{8})$	$S_2 \times S_2$
	$2B$	$(\frac{1}{18}, \frac{1}{8}, \frac{1}{18}, \frac{1}{8})$ $(\frac{1}{8}, \frac{1}{18}, \frac{1}{8}, \frac{1}{18})$	
	$2C$	$(\frac{1}{18}, \frac{1}{8}, \frac{1}{8}, \frac{1}{18})$ $(\frac{1}{8}, \frac{1}{18}, \frac{1}{18}, \frac{1}{8})$	
D_4	$2A$	$(\frac{1}{8}, \frac{1}{8}, \frac{1}{2}, \frac{1}{2})$ $(\frac{1}{2}, \frac{1}{2}, \frac{1}{8}, \frac{1}{8})$	$S_2 \times S_2$
	$2B$	$(\frac{1}{2}, \frac{1}{8}, \frac{1}{2}, \frac{1}{8})$ $(\frac{1}{8}, \frac{1}{2}, \frac{1}{8}, \frac{1}{2})$	
	$2C$	$(\frac{1}{2}, \frac{1}{8}, \frac{1}{8}, \frac{1}{2})$ $(\frac{1}{8}, \frac{1}{2}, \frac{1}{2}, \frac{1}{8})$	
S_3	$3A$	$(\frac{1}{8}, \frac{9}{8}, \frac{1}{8}, \frac{1}{8})$	S_3
	$3B$	$(\frac{9}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$	
	$3C$	$(\frac{1}{8}, \frac{1}{8}, \frac{9}{8}, \frac{1}{8})$	
	$3D$	$(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{9}{8})$	
S_3	$4A$	$(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$	S_4
	$4B$		
	$4C$		
	$4D$		
S_2	$5A$	$(\frac{1}{2}, \frac{1}{18}, \frac{1}{8}, \frac{1}{8})$	S_3
	$5B$	$(\frac{1}{8}, \frac{1}{18}, \frac{1}{2}, \frac{1}{8})$	
	$5C$	$(\frac{1}{8}, \frac{1}{18}, \frac{1}{8}, \frac{1}{2})$	
	$5D$	$(\frac{1}{2}, \frac{1}{8}, \frac{1}{18}, \frac{1}{8})$	
	$5E$	$(\frac{1}{8}, \frac{1}{2}, \frac{1}{18}, \frac{1}{8})$	
	$5F$	$(\frac{1}{8}, \frac{1}{8}, \frac{1}{18}, \frac{1}{2})$	
	$5G$	$(\frac{1}{18}, \frac{1}{8}, \frac{1}{2}, \frac{1}{8})$	
	$5H$	$(\frac{1}{18}, \frac{1}{2}, \frac{1}{8}, \frac{1}{8})$	
	$5I$	$(\frac{1}{18}, \frac{1}{8}, \frac{1}{8}, \frac{1}{2})$	
	$5J$	$(\frac{1}{8}, \frac{1}{8}, \frac{1}{2}, \frac{1}{18})$	
	$5K$	$(\frac{1}{8}, \frac{1}{2}, \frac{1}{8}, \frac{1}{18})$	
	$5L$	$(\frac{1}{2}, \frac{1}{8}, \frac{1}{8}, \frac{1}{18})$	

TABLE 2.5 – Solutions $(\lambda(t), \alpha)$ of $\mathbf{PVI}_\alpha$ equations related to the Picard-Fuchs system (2.14).

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Partie II

Stabilité et Stabilisation des Systèmes Dynamiques à Retard

B. Ben Hamed, M.A. Hammami. *Stablization of uncertain time-varying dynamical systems including control and state delay, Int. J. of Comp. and Num. Ana. and App. Vol. 5 No. 4 (2004) 301-315.*

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Chapitre 3

Stabilisation des systèmes dynamiques à retard

Pendant ces dernières décénnies, le problème de stabilisation des systèmes dynamiques à retard a été largement étudié, voir (21, Gu and al.), (19, Grecki and al.), (22, Hale), (23, Hale and Lunel), (31, Ivanescu), (38, Michiels), (43, Richard). Dans la plupart de ces travaux, l'étude des contrôleurs est basé sur la connaissance des liens d'incertitudes, des distribuances et des termes non linéaires qui dépendent de l'état et de l'éntrée du système. Cependant, si ces facteurs sont mal étudiés alors ils peuvent causer de sérieuses dégradation de la performance du système et même l'instabilité du système. Récemment, le meilleur rendement a été rapporté en utilsant des théorèmes de Lyapunov-Krasovskii (voir (11, Bliman), (4; 5, Ben Hamed and Hammami), (13, Cheres and al.), (37, Mao), (40, Oucheriah), (45, Shyu and Yan), (46, Sun)) et des techniques d'inégalités linéaires matricielles (LMI) (voir (9, Bliman), (12, Boyd and al.), (24, Han and Mehdi), (25, Han), (28, Hmamed), (34, Li and De Souza), (50, Wu and Mizukami)).

Dans ce chapitre, on présente quelques définitions et notions de base sur les systèmes à retard. le modèle choisi sera présenté, ainsi que l'existence et l'unicité des solutions pour les équations différentielles fonctionnelles (EDFR) associées. On introduit les méthodes des fonctionnelles de Lyapunov-Krasovskii et de fonction de Razumikhin qui donnent des conditions suffisantes pour assurer la stabilité de ces systèmes à retard.

Dans la dernière section de ce chapitre, on considère des classes de systèmes incertains à retard dans l'état et dans la commande. En utilisant des techniques de Lyapunov, on propose des classes de contrôleurs continus qui assurent la stabilité globale uniforme exponentielle et la stabilité globale uniforme pratique de ces systèmes en boucle fermés en imposant quelques conditions assorties sur les incertitudes. La fonction de Laypunov quadratique du système nominal stable (c'est à dire, le système assosié en l'absence des incertitudes et du retard) est utilisée comme fonction de Lyapunov candidate du système globale. Des exemples numériques sont donnés pour illustrer l'applicabilité des résultats obtenus.

3.1 Equations différentielles fonctionnelles

On note par $\mathcal{C} := \mathcal{C}([-r, 0], \mathbb{R}^n)$, $r \in \mathbb{R}_+^*$, *l'ensemble des fonctions continues de* $[-r, 0]$ *à valeurs dans* $\mathbb{R}^n$. *Pour tout* $A > 0$, *pour toute fonction continue* $\psi \in \mathcal{C}([t_0 - r, t_0 + A], \mathbb{R}^n)$ *et* $t_0 \leq t \leq t_0 + A$, *on définit* $\psi_t \in \mathcal{C}$ *par :* $\psi_t(\theta) = \psi(t + \theta)$, $-r \leq \theta \leq 0$.

La forme la plus générale d'une équation différentielle fonctionnelle à retard (EDFR) est

$$\dot{x}(t) = f(t, x_t), \tag{3.1}$$

où $x(t) \in \mathbb{R}^n$ *et* $f : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$. *Pour déterminer l'évolution de l'état* $x(t)$ *de l'équation (3.1), il est nécessaire de spécifier la condition initiale dans un intervalle de longueur* r

$$x_{t_0} = \phi, \tag{3.2}$$

où $\phi \in \mathcal{C}$ *est donnée. Autrement dit,* $x(t_0 + \theta) = \phi(\theta)$, $-r \leq \theta \leq 0$.
Pour tout $A > 0$, *une fonction* x *est dite solution de (3.1) sur l'intervalle* $[t_0 - r, t_0 + A)$ *si elle est continue sur cet intervalle et vérifie l'EDFR (3.1). Il est évident que si* x *est solution alors* (t, x_t) *appartient au domaine de définition de* f. *Si de plus, une solution vérifie la condition (3.2) alors on dit qu'elle passe par la condition initiale* (t_0, ϕ). *Dans ce cas, on la note* $x(t_0, \phi)(t)$.
Le théorème suivant assure l'existence et l'unicité des solutions des EDFR (3.1).

Théorème 3.1.1 *Soient* Ω *un ouvert de* $\mathbb{R} \times \mathcal{C}$, $f : \Omega \to \mathbb{R}^n$ *une fonction continue et* $f(t, \phi)$ *est Lipschitzienne par rapport à* ϕ *dans tout compact de* Ω. *Si* $(t_0, \phi) \in \Omega$ *alors il existe une unique solution de (3.1) passant par* (t_0, ϕ).

3.2 Stabilité des systèmes à retard

L'évolution des systèmes dépendent non-seulement de l'état actuel, mais aussi des états avec avance et des systèmes à états retardés. Comme les systèmes avec avance ne représentent pas des modèles réels, notre analyse de stabilité sera portée uniquement sur les systèmes à retard.

3.2.1 Concept de stabilité

Sans perte de généralités, on peut supposer dans tout le reste que $x = 0$ *est une solution triviale de l' EDFR (3.1), c'est à dire* $f(t, 0) = 0$ *pour tout* $t \in \mathbb{R}$. *En effet, on peut se ramener de l'étude de la stabilité d'une solution non triviale* $y(t)$, *par la transformation* $z(t) = x(t) - y(t)$, *à celle de la solution triviale. Le nouveau système*

$$\dot{z}(t) = f(t, z_t + y_t) - f(t, y_t),$$

possède la solution triviale $z = 0$.
Pour toute fonction $\phi \in \mathcal{C}$, *on définit la norme continue* $\|.\|_c$ *par*

$$\|\phi\|_c = \max_{-r \leq \theta \leq 0} \|\phi(\theta)\|,$$

où $\|.\|$ *représente la norme* $\|.\|_2$. *On définit ainsi la boule* $\mathcal{B}(0, a)$ *par :* $\mathcal{B}(0, a) = \{u \in \mathcal{C} \ : \ \|u\|_c < a\}$ *pour tout* $a > 0$.

Définition 3.2.1 *La solution* $x = 0$ *de l'équation (3.1) est dite stable si pour tous* $t_0 \in \mathbb{R}$, $\epsilon > 0$, *il existe* $\delta = \delta(t_0, \epsilon) > 0$ *tel que* $\phi \in \mathcal{B}(0, \delta)$ *implique* $x_t(t_0, \phi) \in \mathcal{B}(0, \epsilon)$ *pour* $t \geq t_0$.
Elle est dite asymptotiquement stable si elle est stable et pour tout $t_0 \in \mathbb{R}$, *il existe* $\delta_a = \delta_a(t_0) > 0$ *tel que* $\phi \in \mathcal{B}(0, \delta_a)$ *implique* $x_t(t_0, \phi) \to 0$ *quand* $t \to \infty$.
Elle est dite uniformément stable si elle est stable et $\delta(t_0, \epsilon)$ *est indépendante de* t_0.
Elle est dite uniformément asymptotiquement stable si elle est uniformément stable et il existe $\delta_a > 0$ *tel que*

pour tout $\eta > 0$, il existe $T = T(\eta)$, tel que $\phi \in \mathcal{B}(0, \delta_a)$ implique $x_t(t_0, \phi) \in \mathcal{B}(0, \eta)$ pour $t \geq t_0 + T$ et $t_0 \in \mathbb{R}$. Elle est dite globalement (uniformément) asymptotiquement stable si elle est (uniformément) asymptotiquement stable et δ_a peut être assez grand mais fini.

On remarque que la seule différence entre la stabilité des systèmes EDFR (3.1) et les systèmes sans retard est la forme de la condition initiale.
Pour quelques EDFR, les notions de stabilité et de stabilité uniforme sont les mêmes. En effet,

Lemme 3.2.2 *S'il existe $\omega > 0$ tel que $f(t + \omega, \phi) = f(t, \phi)$ pour tout $(t, \phi) \in \mathbb{R} \times \mathcal{C}$, alors la solution $x = 0$ est stable (resp. asymptotiquement stable) si et seulement si elle est uniformément stable (resp. uniformément asymptotiquement stable).*

3.2.2 Théorème de Lyapunov-Krasovskii

Dans cette partie, on donne des conditions suffisantes de stabilité de la solution triviale $x = 0$ de l'EDFR (3.1) qui généralise la méthode classique de Lyapunov pour les équations différentielles ordinaires.

Soient $V : \mathbb{R} \times \mathcal{C} \to \mathbb{R}$ une application continue et $x(t_0, \phi)$ la solution de l'équation (3.1) qui passe par (t_0, ϕ) alors on évalue la dérivée de $V(t, x_t)$ par rapport à t au point (t_0, ϕ) comme suit

$$\begin{aligned} \dot{V}(t_0, \phi) &= \frac{d}{dt} V(t, x_t)|_{\, t=t_0 \, , \, x_t=\phi} \\ &= \limsup_{h \to 0^+} \frac{1}{h}[V(t_0 + h, x_{t_0+h}(t_0, \phi)) - V(t_0, \phi)]. \end{aligned} \tag{3.3}$$

La fonctionnelle (3.3), $V(t, x_t)$, est appelée fonctionnelle de Lyapunov-Krasovskii.

Théorème 3.2.3 (21, Théorème de Lyapunov-Krasovskii)
Soit $f : \mathbb{R} \times \mathcal{C}_0 \to \mathcal{D}$ où $\mathcal{C}_0$ (resp. $\mathcal{D}$) est un ensemble borné de $\mathcal{C}$ (resp. $\mathbb{R}^n$). Supposons qu'il existe des fonctions $u, v, w : \mathbb{R}_+ \to \mathbb{R}_+$ continues, non-décroissantes et vérifiant : $u(s)$ et $v(s)$ strictement positives pour $s > 0$, et $u(0) = v(0) = 0$. S'il existe une fonction continue différentiable $V : \mathbb{R} \times \mathcal{C} \to \mathbb{R}$ telle que

$$u(\|\phi(0)\|) \leq V(t, \phi) \leq v(\|\phi\|_c),$$

et

$$\dot{V}(t, \phi) \leq -w(\|\phi(0)\|),$$

alors la solution triviale de (3.1) est uniformément stable. Si $w(s) > 0$ pour tout $s > 0$ alors cette solution est uniformément asymptotiquement stable. Si de plus, $\lim_{s \to \infty} u(s) = \infty$ alors elle est globalement uniformément asymptotiquement stable.

Le problème de stabilité uniforme ou uniforme asymptotique est donc réduit à trouver une fonctionnelle de Lyapunov-Krasovskii qui satisfait les conditions du Théorème 3.2.3 précédent.
Dans le cas où la fonctionnelle de Lyapunov-Krasovskii est quadratique et bornée, on obtient la forme réduite du Théorème de Lyapunov-Krasovskii suivante.

Proposition 3.2.4 *Le système (3.1) est asymptotiquement stable s'il existe une fonctionnelle de Lyapunov-Krasovskii quadratique bornée $V(\phi)$ telle que pour tout $\epsilon > 0$, on a*

$$V(\phi) \geq \epsilon \|\phi(0)\|^2,$$

et sa dérivée le long des trajectoires

$$\dot{V}(\phi) = \dot{V}(x_t)|_{\,x_t=\phi},$$

satisfait

$$\dot{V}(\phi) \leq -\epsilon\|\phi(0)\|^2.$$

3.2.3 Théorème de Razumikhin

La méthode des fonctionnelles de Lyapunov-Krasovskii exige que l'état du système $x(t)$ appartient à un intervalle de la forme $[t-r,t]$ qui rend l'application du théorème précédent plutôt difficile. On peut dépasser cette difficulté en utilisant le théorème suivant.

Théorème 3.2.5 (21, Théorème de Razumikhin)
Soit $f : \mathbb{R} \times \mathcal{C}_0 \to \mathcal{D}$ où $\mathcal{C}_0$ (resp. $\mathcal{D}$) est un ensemble borné de $\mathcal{C}$ (resp. de $\mathbb{R}^n$). Supposons qu'il existe des fonctions $u, v, w : \mathbb{R}_+ \to \mathbb{R}_+$ continues, non-décroissantes et vérifiant : $u(s)$ et $v(s)$ sont strictement positives pour tout $s > 0$, $u(0) = v(0) = 0$ et v est strictement croissante. S'il existe une fonction continue différentiable $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ telle que

$$u(\|x\|) \leq V(t,x) \leq v(\|x\|), \text{ pour tous } t \in \mathbb{R} \text{ et } x \in \mathbb{R}^n, \tag{3.4}$$

et la dérivée de V le long des trajectoires de (3.1) satisfait

$$\dot{V}(t,x) \leq -w(\|x(t)\|) \text{ si } V(t+\theta, x(t+\theta)) \leq V(t, x(t)), \tag{3.5}$$

pour $\theta \in [-r, 0]$, alors le système (3.1) est uniformément stable. Si $w(s) > 0$ pour tout $s > 0$ et il existe une fonction continue, non-décroissante et vérifiant : $p(s) > s$ pour tout $s > 0$ telle que la condition (3.5) devient

$$\dot{V}(t,x) \leq -w(\|x(t)\|) \text{ si } V(t+\theta, x(t+\theta)) \leq p(V(t, x(t))), \tag{3.6}$$

pour $\theta \in [-r, 0]$, alors le système (3.1) est uniformément asymptotiquement stable. Si de plus, $\lim_{s\to\infty} u(s) = \infty$ alors elle est globalement uniformément asymptotiquement stable.

La première condition du théorème est la condition classique de Lyapunov sur la fonction candidate et la deuxième condition est la condition de négativité de la dérivée de la fonction le long des trajectoires de l'EDFR (3.1).
Dans le cas où la fonction de Lyapunov est quadratique et bornée, on obtient la forme réduite du Théorème de Razumikhin suivante.

Proposition 3.2.6 *Le système (3.1) est asymptotiquement stable s'il existe une fonction de Lyapunov quadratique bornée V telle que pour tout $\epsilon > 0$, on a*

$$V(x) \geq \epsilon\|x\|^2, \tag{3.7}$$

et sa dérivée le long des trajectoires $\dot{V}(x(t))$ satisfait

$$\dot{V}(x(t)) \leq -\epsilon\|x(t)\|^2, \tag{3.8}$$

où

$$V(x(t+\zeta)) \leq p\, V(x(t)),\ -r \leq \zeta \leq 0, \tag{3.9}$$

pour toute constante $p > 1$.

Exemple 3.2.7

Considérons le système à retard suivant

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-r), \tag{3.10}$$

où A_0 et A_1 deux $n \times n$ matrices réelles données. La condition initiale du système (3.10) est de la forme

$$x_0 = \phi. \tag{3.11}$$

On considère la fonction de Lyapunov suivante

$$V(x) = x^T P x. \tag{3.12}$$

Ainsi, on obtient une simple condition de stabilité.

Proposition 3.2.8 *Le système (3.10) est asymptotiquement stable s'il existe un scalaire $\alpha > 0$ et une matrice réelle $P = P^T > 0$ telle que*

$$\begin{pmatrix} PA_0 + A_0^T + \alpha P & PA_1 \\ A_1^T P & -\alpha P \end{pmatrix} < 0. \tag{3.13}$$

Preuve *Considérons la dérivée de la fonction de Lyapunov (3.12) le long des trajectoires du système (3.10)*

$$\dot{V}(x(t)) = \frac{d}{dt} V(x(t)) = 2x^T(t) P [A_0 x(t) + A_1 x(t-r)].$$

Comme x_t satisfait la condition (3.9) alors on peut conclure que pour tout $\alpha > 0$, on a

$$\begin{aligned} \dot{V}(x(t)) &\leq 2x^T(t) P [A_0 x(t) + A_1 x(t-r)] \\ &\quad + \alpha [p\, x^T(t) P x(t) - x^T(t-r) P x(t-r)] \\ &= y_{0r}^T(t) \begin{pmatrix} PA_0 + A_0^T + \alpha p\, P & PA_1 \\ A_1^T P & -\alpha P \end{pmatrix} y_{0r}(t), \end{aligned} \tag{3.14}$$

où

$$y_{0r}(t) = (x^T(t) \;\; x^T(t-r))^T.$$

L'inégalité (3.13) implique que pour un réel suffisamment petit $\delta > 0$, on a

$$\begin{pmatrix} PA_0 + A_0^T + \alpha p\, P & PA_1 \\ A_1^T P & -\alpha P \end{pmatrix} < 0,$$

avec $p = 1 + \delta$. Ainsi, la condition (3.14) implique que la condition (3.8) de la Proposition 3.2.6 est satisfaite. D'où le système (3.10) est asymptotiquement stable. □

3.3 Stabilisation des systèmes à retard

Dans cette section, on propose des classes de contrôleurs continues qui assurent la stabilité globale uniforme exponentielle et la stabilité globale uniforme pratique de quelques systèmes incertains à retard en imposant quelques conditions assorties sur les incertitudes. La fonction de Laypunov quadratique du système nominal stable est utilisée comme fonction de Lyapunov candidate du système globale.

3.3.1 Classe de systèmes à retard dans la commande

On considère le système incertain à retard suivant

$$\begin{aligned}\dot{x}(t) &= [A+\triangle A(t)]x(t)+[B+\triangle B(t)]u(t)+[B_d+\triangle B_d(t)]u(t-h(t)) \\ &\quad +Bf(t,x(t),u(t-h(t))),\ t\geq t_0,\end{aligned} \tag{3.15}$$

où $x(t)\in\mathbb{R}^n$ et $u(t)\in\mathbb{R}^m$ représentent respectivement l'état et la commande du système, $h(t)$ est la fonction du retard bornée par $\bar{h}$ et $f: R\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^m$ est une application continue. Les matrices A, B et B_d sont des matrices constantes de dimensions convenables.
La condition initiale du système (3.15) est donnée par

$$x(t)=\phi(t),\ t\in[t_0-\bar{h},t_0],\ 0\leq h(t)\leq\bar{h}. \tag{3.16}$$

On suppose que la partie droite du système (3.15) est continue et satisfait les conditions qui assurent l'existence et l'unicité de la solution passant par une condition initiale fixée $\phi(t)$. On suppose aussi que la paire (A,B) est contrôllable.
D'abord, on suppose que les assertions suivantes sont vérifiées.

Propriètés 3.3.1 *Il existe des matrices continues $C(t)$, $D(t)$ et $E(t)$ de dimensions convenables telles que*

$$\triangle A(t)=BC(t),\ \triangle B(t)=BD(t),\ \triangle B_d(t)=BE(t). \tag{3.17}$$

Ces conditions (3.17) définissent ce qu'on appelle les conditions assorties des incertitudes et elles sont plutôt standard au problème de contrôle.
On suppose aussi que ces conditions assorties sont bornées

$$\|C(t)\|\leq\alpha_1,\ \|D(t)\|\leq\alpha_2,\ \|E(t)\|\leq\alpha_3, \tag{3.18}$$

$$\|f(t,x(t),u(t-h(t)))\|\leq\beta\|x(t)\|+\delta\|u(t-h(t))\|, \tag{3.19}$$

où $\alpha_1\geq 0$, $0<\alpha_2\leq\frac{1}{3}$, $\alpha_3\geq 0$, $\beta\geq 0$ et $\delta\geq 0$ sont des constantes positives.

Maintenant, on donne la définition de la stabilité globale uniforme exponentielle du système (3.15)-(3.16).

Définition 3.3.2 *Le système (3.15)-(3.16) est globalement uniformément exponentiellement stable avec taux $\gamma>0$ si pour tout $\varsigma>0$, il existe $k^*(\varsigma)>0$ tel que*

$$\|x(t)\|\leq k^*(\varsigma)e^{-\gamma(t-t_0)},\ \forall t\geq t_0, \tag{3.20}$$

quand $\sup\limits_{t_0-t\leq s\leq t_0}\|x(s)\|<\varsigma$.

Le développement du résultat principal de cette section exige le lemme suivant qui est une version modifié de (40, Lemma 1, p. 56), où l'idée de la preuve est assez semblable.

Lemme 3.3.3 *Soient g une fonction continue avec $g(t)\geq 0$ pour tout $t\geq t_0-\bar{h}$ et k^* un réel vérifiant*

$$k^*>\sup_{t_0-\bar{h}\leq s\leq t_0}g(s).$$

Supposons que

$$\dot{g}(t)\leq-\rho g(t)+\sigma\sup_{t-\bar{h}\leq s\leq t}g(s),\ \forall\, t\geq t_0, \tag{3.21}$$

où ρ et σ sont des constantes positives. Si $0 < \sigma < \rho$ alors

$$g(t) \leq k^* e^{-\gamma(t-t_0)}, \ \forall\, t \geq t_0,$$

où γ est l'unique solution de l'équation

$$-\gamma = -\rho + \sigma e^{\gamma \bar{h}}. \tag{3.22}$$

Preuve *Soit*

$$\dot{y}_n(t) = -\rho y_n(t) + \sigma \sup_{t-\bar{h} \leq s \leq t} y_n(s) + \varepsilon_n, \tag{3.23}$$

où (ε_n) est une suite de nombres positifs et décroissante vers zéro. Alors

$$y_n(t) = \frac{\varepsilon_n}{\rho - \sigma} + k^* e^{-\gamma(t-t_0)}, \ \forall\, t \geq t_0 - \bar{h}, \tag{3.24}$$

est une solution de (3.23) car γ satisfait (3.22).
Soit $\psi(\gamma) = \gamma - \rho + \sigma e^{\gamma \bar{h}}$. Si $0 < \sigma < \rho$ alors $\psi(0) < 0$ et $\psi(\rho) = \sigma e^{\rho \bar{h}} > 0$. Donc il existe un unique $\gamma > 0$ qui vérifie (3.22). D'après (3.24) et la définition de k^*, on a : $g(t) < y_n(t)$, $\forall\, t_0 - \bar{h} \leq t \leq t_0$.
Maintenant, supposons qu'il existe $t^* = \inf\{t : g(t) \geq y_n(t)\}$ alors

$$g(t^*) = y_n(t^*) \text{ et } \sup_{t^*-\bar{h} \leq s \leq t^*} g(s) \leq \sup_{t^*-\bar{h} \leq s \leq t^*} y_n(s).$$

En comparant (3.21) et (3.23), on a : $\dot{g}(t^*) < \dot{y}_n(t^*)$, or $g(t) < y_n(t)$, $\forall\, t < t^*$ et $g(t^*) < y_n(t^*)$ donc $\dot{g}(t^*) \geq \dot{y}_n(t^*)$, contradiction. Parsuite, $g(t) \leq y_n(t)$, $\forall\, t \geq t_0$ et comme $y_{n+1}(t) < y_n(t)$, $\forall\, t \geq t_0$ alors $g(t) \leq y(t)$ où $y(t) = \lim_{n \to +\infty} y_n(t)$. $\square$

Avant d'aller plus loin dans la conception du contrôleur, un résultat préliminaire est nécessaire. Comme la paire (A, B) est contrôllable alors la paire $(A + \mu I_n, B)$ l'est aussi pour toute constante $\mu \geq 0$, où I_n est la matrice identitée d'ordre n et l'équation matricielle de Riccati

$$[A + \mu I_n]^\top P + P[A + \mu I_n] - 2PBB^\top P = -Q, \tag{3.25}$$

possède une unique solution symétrique et définie positive $P = P^\top > 0$ pour toute matrice $Q = Q^\top > 0$.
On propose une classe de contrôleurs continues qui assure la stabilité globale uniforme exponentielle du système (3.15)-(3.16) en présence des incertitudes. Le contrôleur proposé est de la forme suivante

$$u(t) = -(1 + \omega) B^\top P x(t), \tag{3.26}$$

où ω est donnée par

$$\omega = \frac{1 - a - \sqrt{1 - 3a}}{a}, \tag{3.27}$$

avec $a = 3\alpha_2^2$.

Théorème 3.3.4 *Supposons que le système incertain (3.15)-(3.16) satisfait les Propriétés 3.3.1. Si*

$$Q > 3(\alpha_1 + \beta)^2 I_n + P B_d B_d^\top P, \tag{3.28}$$

et $\mu > 0$ est choisie telle que

$$\frac{9(1 + b)\|B^\top P\|^2}{(1 + \sqrt{1 - 3a})^2 \lambda_{\min}(P)} \leq \mu e^{-\mu \bar{h}}, \tag{3.29}$$

où b est donnée par $b = 3(\alpha_3 + \delta)^2$ et $\lambda_{\max}(P)$ (resp. $\lambda_{\min}(P)$) représente la valeur propre maximale (resp. minimale) de la matrice P.
Alors, le système en boucle fermée (3.15)-(3.16)-(3.26) est globalement uniformément exponentiellement stable avec taux $\gamma = \frac{\mu}{2} > 0$.

Preuve *Soit*

$$V(x(t)) = x^\top(t)Px(t), \tag{3.30}$$

la fonction de Lyapunov du système nominale, où P est la solution de l'équation matricielle de Ricatti (3.25) et $x(t)$ est la solution du système dynamique en boucle fermée (3.15)-(3.16)-(3.26) pour tout $t \geq t_0$. La fonction (3.30) est considérée comme la fonction de Lyapunov candidate pour le système en boucle fermée.
En utilisant (3.17) et (3.25), la dérivée de $V(x(t))$ le long des trajectoires du système (3.15)-(3.16) en boucle fermée avec (3.26) satisfait

$$\begin{aligned}
\dot{V}(x(t)) \quad = \quad & -x^\top(t)Qx(t) - 2\mu x^\top(t)Px(t) - [u(t-h(t)) - B_d^\top Px(t)]^\top \times \\
& [u(t-h(t)) - B_d^\top Px(t)] + u^\top(t-h(t))u(t-h(t)) \\
& + x^\top(t)PB_dB_d^\top Px(t) - 2\omega\|B^\top Px(t)\|^2 + 2x^\top(t)PB[C(t)x(t) \\
& + D(t)u(t) + E(t)u(t-h(t)) + f(t, x(t), u(t-h(t)))].
\end{aligned}$$

En utilisant (3.18), (3.19) et l'inégalitée suivante : $2tz \leq t^2 + z^2$ pour tout $(t, z) \in \mathbb{R}^2$, on obtient

$$\begin{aligned}
& 2x^\top(t)PB[C(t)x(t) + D(t)u(t) + E(t)u(t-h(t)) \\
& + f(t, x(t), u(t-h(t)))] \\
\leq \quad & 2\|B^\top Px(t)\|[(\alpha_1 + \beta)\|x(t)\| + \alpha_2\|u(t)\| \\
& + (\alpha_3 + \delta)\|u(t-h(t))\|] \\
\leq \quad & 3(\alpha_1 + \beta)^2\|x(t)\|^2 + (1 + a(1+\omega)^2)\|B^\top Px(t)\|^2 \\
& + b\|u(t-h(t))\|^2.
\end{aligned} \tag{3.31}$$

D'après (3.31) et en utilisant (3.27) et (3.28), on obtient l'inégalitée suivante

$$\dot{V}(x(t)) \leq -2\mu V(x(t)) + \frac{9(1+b)\|B^\top P\|^2}{(1+\sqrt{1-3a})^2}\|x(t-h(t))\|^2. \tag{3.32}$$

En regardant l'inégalitée

$$\|x(t-h(t))\|^2 \leq \sup_{t-\bar{h} \leq s \leq t} \|x(s)\|^2 \leq \frac{1}{\lambda_{\min}(P)} \sup_{t-\bar{h} \leq s \leq t} V(x(s)),$$

et en la substituant avec (3.29) à (3.32), on obtient

$$\dot{V}(x(t)) \leq -2\mu V(x(t)) + \mu e^{-\mu\bar{h}} \sup_{t-\bar{h} \leq s \leq t} V(x(s)).$$

On posant $\rho = 2\mu$, $\sigma = \mu e^{-\mu\bar{h}}$ et en appliquant le Lemme 3.3.3, on peut voir que $0 < \sigma < \rho$ et que $\gamma = \mu$ est l'unique solution de (3.22). Donc

$$V(x(t)) \leq k^* e^{-\mu(t-t_0)},$$

et

$$\|x(t)\| \leq \sqrt{\frac{k^*}{\lambda_{\min}(P)}} e^{-\frac{\mu}{2}(t-t_0)},$$

où $k^* > \sup_{t_0-\bar{h}\leq s\leq t_0} V(x(s))$.
D'où le système en boucle fermée (3.15)-(3.16)-(3.26) est globalement uniformément exponentiellement stable.
□

Exemple Numérique 3.3.5

Considérons le système incertain à retard (3.15) avec

$$A = \begin{pmatrix} -17.5 & 0 \\ 0.5 & -20 \end{pmatrix},\ \triangle A(t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0.9 \\ 0 \end{pmatrix},$$

$$\triangle B(t) = \begin{pmatrix} 0.09\sin(t) \\ 0 \end{pmatrix},\ B_d = \begin{pmatrix} 0.05 \\ 0 \end{pmatrix},\ \triangle B_d(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$f(t, x(t), u(t-h(t))) = 0.05\sin(u(t-h(t))),$$
$$h(t) = 0.02 + 0.025\sin(t).$$

D'après les Propriétés 3.3.1, on obtient

$$\alpha_1 = \alpha_3 = 0,\ \alpha_2 = 0.1,\ \beta = 0,\ \delta = 0.05,\ \bar{h} = 0.045.$$

On veut concevoir le contrôleur avec taux $\gamma = \frac{\mu}{2} = 10$ *et on posant* $Q = I_n$. *La solution de l'équation matricielle de Ricatti (3.25) est*

$$P = \begin{pmatrix} 3.4097 & 0.78567 \\ 0.78567 & 4.7513 \end{pmatrix},$$

avec $\lambda_{\min}(P) = 3.0474$ *et la condition (3.28) est satisfaite avec*

$$\begin{aligned} & Q - 3(\alpha_1+\beta)^2 I_n - PB_dB_d^\top P \\ = & \begin{pmatrix} 0.97093 & -6.6972\times 10^{-3} \\ -6.6972\times 10^{-3} & 0.99846 \end{pmatrix} > 0. \end{aligned}$$

La condition (3.29) est aussi satisfaite avec

$$\frac{9(1+b)\|B^\top P\|^2}{(1+\sqrt{1-3a})^2\lambda_{\min}(P)} = 7.7288,\ \mu e^{-\mu\bar{h}} = 8.1314.$$

D'après (3.27), le gain du contrôleur est : $\omega = 0.53536$.
Ainsi, le contrôleur (3.26) assure la stabilité globale uniforme exponentielle du système (3.15)-(3.16).

3.3.2 Classe de systèmes à retard dans l'état

Maintenant, on considère le système incertain à retard suivant

$$\begin{aligned} \dot{x}(t) &= [A+\triangle A(t)]x(t) + \triangle A_d(t)x(t-h(t)) + [B+\triangle B(t)]u(t) \\ &\quad + Bf_1(t,x(t)) + Bf_2(t,x(t-h(t))),\ t\geq t_0, \end{aligned} \tag{3.33}$$

où $x(t)\in\mathbb{R}^n$ *et* $u(t)\in\mathbb{R}^m$ *représentent respectivement l'état et la commande du système,* $h(t)$ *est la fonction du retard bornée par* $\bar{h}$ *et* $f_i : R\times\mathbb{R}^n\to\mathbb{R}^m$, $i=1,2$, *sont des applications continues. Les matrices* A *et* B

sont des matrices constantes de dimensions convenables et les matrices $\triangle A(t)$, $\triangle A_d(t)$, $\triangle B(t)$ représentent les incertitudes et sont bornées.
La condition initiale du système (3.33) est donnée par

$$x(t) = \phi(t),\ t \in [t_0 - \bar{h}, t_0]. \tag{3.34}$$

On suppose que la paire (A, B) est contrôllable.

Propriètés 3.3.6 *Il existe des matrices continues $C(t)$, $D(t)$ et $E(t)$ de dimensions convenables telles que*

$$\triangle A(t) = BC(t),\ \triangle A_d(t) = BD(t),\ \triangle B(t) = BE(t). \tag{3.35}$$

On suppose aussi la bornitude des conditions assorties précédentes

$$\begin{gathered} \|C(t)\| \leq \alpha_1,\ \|D(t)\| \leq \alpha_2, \\ \|f_1(t, x(t))\| \leq \beta_1 \|x(t)\|,\ \|f_2(t, x(t-h(t)))\| \leq \beta_2 \|x(t-h(t))\|, \end{gathered} \tag{3.36}$$

où $\alpha_i \geq 0,\ \beta_i \geq 0,\ i = 1, 2,$ sont des constantes positives.

Par commodité, on introduit les notations suivantes

$$\begin{aligned} \eta(t) &= \frac{1}{2}\lambda_{\min}(E(t) + E^\top(t)), \\ \sigma &= \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}, \end{aligned} \tag{3.37}$$

où P est la solution de l'équation matricielle de Ricatti (3.25) avec $\mu = 0$, c'est à dire

$$A^\top P + PA - 2PBB^\top P = -Q,$$

pour toute matrice $Q = Q^\top > 0$. On suppose aussi que

$$\eta(t) > -1,\ \forall\, t \geq t_0.$$

On propose une classe de contrôleurs continues qui assure la stabilité globale uniforme exponentielle du système (3.33)-(3.34) en présence des incertitudes. Le contrôleur proposé est de la forme suivante

$$u(t) = -\kappa(t) B^\top P x(t), \tag{3.38}$$

où la fonction $\kappa(t)$ est donnée par

$$\kappa(t) = \frac{2 + \delta_1^2(\alpha_1 + \beta_1)^2 + \delta_2^2 \sigma^2 (\alpha_2 + \beta_2)^2}{2(1 + \eta(t))}, \tag{3.39}$$

avec

$$\frac{1}{\delta_1^2} + \frac{q^2}{\delta_2^2} < \lambda_{\min}(Q), \tag{3.40}$$

où q est un nombre positif décrit dans la propiété suivante.

Propriètè 3.3.7 (24, Han, Mehdi), (40, Oucheriah) *Supposons qu'il existe un nombre positive $q > 1$, tel que l'inégalité suivante soit vérifiée*

$$V(x(t-h(t))) < q^2 V(x(t)),$$

où V est la fonction de Lyapunov donnée dans (3.30).

Comme

$$\begin{aligned}\|x(t-h(t))\|^2 &\leq \sup_{t-\bar{h}\leq s\leq t}\|x(s)\|^2 \leq \frac{1}{\lambda_{\min}(P)}\sup_{t-\bar{h}\leq s\leq t}V(x(s))\\ &\leq \frac{q^2}{\lambda_{\min}(P)}V(x(t)) \leq \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\,q^2\|x(t)\|^2.\end{aligned}$$

Ainsi, on obtient

$$\|x(t-h(t))\| \leq q\sigma\|x(t)\|. \tag{3.41}$$

Remarque 3.3.8 *Le contrôleur (3.38) est continu et linéaire par rapport à l'état. Par conséquent, l'existence de la solution du système (3.33)-(3.34) sous l'action du contrôleur (3.38) est assurée dans le sens usuel (voir (22, Hale)). Ce contrôleur linéaire peut être implémenté facilement dans plusieurs problèmes de contrôle. De plus, il assure la stabilité globale uniforme exponentielle du système (3.33)-(3.34), et ce fait est décrit dans le théorème suivant.*

Théorème 3.3.9 *Supposons que le système incertain (3.33)-(3.34) satisfait les Propriétés 3.3.6 et 3.3.7. Alors le système en boucle fermée (3.33)-(3.34)-(3.38) est globalement uniformément exponentiellement stable.*

Preuve *Soit $x(t)$ la solution du système dynamique en boucle fermée (3.33)-(3.34)-(3.38) pour tout $t \geq t_0$. Considérons la fonction $V(x(t))$ du système nominale, décrite dans (3.30), qui sera candidate de la fonction de Lyapunov du systèm globale. Alors pour tout $t \geq t_0$, on a*

$$\begin{aligned}\dot{V}(x(t)) &= -x^\top(t)Qx(t) + 2x^\top(t)PBB^\top Px(t) + 2x^\top(t)PBC(t)x(t)\\ &\quad +2x^\top(t)PBD(t)x(t-h(t)) - 2\kappa(t)x^\top(t)PB(I_m+E(t))B^\top Px(t)\\ &\quad +2x^\top(t)PBf_1(t,x(t)) + 2x^\top(t)PBf_2(t,x(t-h(t))).\end{aligned}$$

En regardant (3.37), (3.39) et en utilisant (3.40), on obtient

$$\begin{aligned}\dot{V}(x(t)) &\leq -x^\top(t)Qx(t) + [2-2\kappa(t)(1+\eta(t)]\|B^\top Px(t)\|^2 + 2x^\top(t)PB\times\\ &\quad [C(t)x(t)+D(t)x(t-h(t))+f_1(t,x(t))+f_2(t,x(t-h(t)))]\\ &\leq -x^\top(t)Qx(t) - [\delta_1^2(\alpha_1+\beta_1)^2+\delta_2^2\sigma^2(\alpha_2+\beta_2)^2]\|B^\top Px(t)\|^2\\ &\quad +2[(\alpha_1+\beta_1)+q\sigma(\alpha_2+\beta_2)]\|B^\top Px(t)\|\|x(t)\|\\ &= -x^\top(t)Qx(t) - [\delta_1(\alpha_1+\beta_1)\|B^\top Px(t)\| - \frac{1}{\delta_1}\|x(t)\|]^2\\ &\quad +\frac{1}{\delta_1^2}\|x(t)\|^2 - [\delta_2\sigma(\alpha_2+\beta_2)\|B^\top Px(t)\| - \frac{q}{\delta_2}\|x(t)\|]^2 + \frac{q^2}{\delta_2^2}\|x(t)\|^2\\ &\leq -x^\top(t)Qx(t) + [\frac{1}{\delta_1^2}+\frac{q^2}{\delta_2^2}]\|x(t)\|^2.\end{aligned} \tag{3.42}$$

En utilisant l'inégalitée (3.42) et on posant

$$\xi := \lambda_{\min}(Q) - [\frac{1}{\delta_1^2}+\frac{q^2}{\delta_2^2}], \tag{3.43}$$

on obtient

$$\dot{V}(x(t)) \leq -\xi\|x(t)\|^2.$$

Si les paramètres de gain δ_1 et δ_2 sont choisies telles que (3.40) est satisfaite, alors le réel $q > 1$ existe tel que $\xi > 0$.
Comme

$$\lambda_{\min}(P)\|x(t)\|^2 \leq V(x(t)) \leq \lambda_{\max}(P)\|x(t)\|^2,$$

où P est la solution de l'équation matricielle de Ricatti (3.25) avec $\mu = 0$. Alors, on obtient

$$\dot{V}(x(t)) \leq -\frac{\xi}{\lambda_{\max}(P)} V(x(t)).$$

Ainsi, on a

$$V(x(t)) \leq V(x(t_0)) \exp(-\frac{\xi(t-t_0)}{\lambda_{\max}(P)}).$$

Donc

$$\|x(t)\| \leq \sigma \|x(t_0)\| \exp(-\frac{\xi(t-t_0)}{2\lambda_{\max}(P)}).$$

D'où, le système en boucle fermée (3.33)-(3.34)-(3.38) est globalement uniformément exponentiellement stable avec taux $-\frac{\xi}{2\lambda_{\max}(P)}$ où ξ est donné par (3.43). $\square$

Dans la suite de cette sous-section, on propose une classe de contrôleurs continus composée d'une partie linéaire et d'une partie non-linéaire telle que la convergence pratique uniforme globale du système (3.33)-(3.34) soit assurée dans la présence des incertitudes. On commence par donné la définition suivante.

Définition 3.3.10 *Le système (3.33)-(3.34) est globalement uniformément convergent vers la boule (où globalement uniformément pratiquement convergent) de rayon $r > 0$ et avec taux $\gamma > 0$, s'il existe une constante positive $\delta > 0$ tel que l'inégualité suivante est vérifiée.*

$$\|x(t)\| \leq r + \delta \|x(t_0)\| e^{-\gamma(t-t_0)}, \ \forall\, t \geq t_0. \tag{3.44}$$

Propriètés 3.3.11 *On suppose que la condition (3.35) soit vérifiée. Dans la pratique, la deuxième ligne de la condition (3.36) prend la forme suivante*

$$\begin{aligned} \|f_1(t, x(t))\| &\leq \varrho_1 + \beta_1 \|x(t)\|, \\ \|f_2(t, x(t-h(t)))\| &\leq \varrho_2 + \beta_2 \|x(t-h(t))\|, \end{aligned} \tag{3.45}$$

où $\varrho_i > 0$, $\beta_i > 0$, $i = 1, 2$, sont des constantes positives.

Le contrôleur proposé est de la forme suivante

$$u(t, x(t)) = u_1(t, x(t)) + u_2(t, x(t)), \tag{3.46}$$

où

$$\begin{aligned} u_1(t, x(t)) &= -\kappa_1(t) B^\top P x(t), \\ u_2(t, x(t)) &= -\frac{\kappa_2(t) B^\top P x(t)}{\|B^\top P x(t)\| + \frac{\varepsilon}{2}}, \end{aligned} \tag{3.47}$$

pour un $\varepsilon > 0$ assez petit. Les fonctions de contrôle $\kappa_1(t)$ and $\kappa_2(t)$ sont données par

$$\begin{aligned} \kappa_1(t) &= \frac{2 + \delta_1^2(\alpha_1 + \beta_1)^2 + \delta_2^2 \sigma^2 (\alpha_2 + \beta_2)^2}{2(1 + \eta(t))}, \\ \kappa_2(t) &= \frac{\varrho_1 + \varrho_2}{1 + \eta(t)}, \end{aligned} \tag{3.48}$$

avec δ_i, $i = 1, 2$, satisfons la condition (3.40).

Remarque 3.3.12 *Le contrôleur (3.46) est constitué de deux parties. La première partie $u_1(t, x(t))$ est linéaire en l'état et la deuxième partie $u_2(t, x(t))$ est un contrôleur non-linéaire continu qui est utilisé pour compenser la partie non-linéaire et pour assurer la convergence globale uniforme pratique du système (3.33).*

On obtient le résultat suivant pour la conception de ce contrôleur continu.

Théorème 3.3.13 *Supposons que le système incertain à retard (3.33)-(3.34) satisfait les propriétés 3.3.7 et 3.3.11. Alors, le système en boucle fermé (3.33)-(3.34)-(3.46) est globalement uniformément pratiquement convergent avec rayon*

$$r(\varepsilon) := \sigma\sqrt{\frac{(\varrho_1 + \varrho_2)\varepsilon}{\xi}}, \tag{3.49}$$

où ξ est donné par (3.40). $\square$

Preuve *Pour tout $t \geq t_0$, on considère une solution $x(t)$ du système dynamique en boucle fermée (3.33)-(3.34)-(3.46). La dérivée de $V(x(t))$ décrite dans (3.30) le long des trajectoires du système en boucle fermée satisfait*

$$\begin{aligned}
\dot{V}(x(t)) &= -x^\top(t)Qx(t) + 2x^\top(t)PBB^\top Px(t) \\
&\quad +2x^\top(t)PBC(t)x(t) + 2x^\top(t)PBD(t)x(t-h(t)) \\
&\quad -2\kappa_1(t)x^\top(t)PB(I_m + E(t))B^\top Px(t) \\
&\quad -2\kappa_2(t)\frac{x^\top(t)PB(I_m + E(t))B^\top Px(t)}{\|B^\top Px(t)\| + \frac{\varepsilon}{2}} \\
&\quad +2x^\top(t)PBf_1(t, x(t)) + 2x^\top(t)PBf_2(t, x(t-h(t))),
\end{aligned}$$

En appliquant la preuve du Théorème 3.3.9 et en utilisant la Propriété 3.3.11, on obtient

$$\begin{aligned}
\dot{V}(x(t)) &\leq -x^\top(t)Qx(t) + [\frac{1}{\delta_1^2} + \frac{q^2}{\delta_2^2}]\|x(t)\|^2 \\
&\quad +2(\varrho_1 + \varrho_2)\|B^\top Px(t)\| \times [1 - \frac{\|B^\top Px(t)\|}{\|B^\top Px(t)\| + \frac{\varepsilon}{2}}] \\
&= -\xi\|x(t)\|^2 + \frac{(\varrho_1 + \varrho_2)\|B^\top Px(t)\|\varepsilon}{\|B^\top Px(t)\| + \frac{\varepsilon}{2}}.
\end{aligned} \tag{3.50}$$

Par conséquent, il suit d'après (3.50) et d'après l'inégalité suivante

$$0 \leq \frac{ab}{a+b} \leq a, \ \forall a, b > 0,$$

que

$$\dot{V}(x(t)) \leq -\xi\|x(t)\|^2 + (\varrho_1 + \varrho_2)\varepsilon. \tag{3.51}$$

En utilisant (3.51), on obtient

$$\dot{V}(x(t)) \leq -\frac{\xi}{\lambda_{\max}(P)}V(x(t)) + (\varrho_1 + \varrho_2)\varepsilon.$$

D'où

$$\begin{aligned}
\dot{V}(x(t)) &\leq \left[V(x(t_0)) - \frac{\varepsilon\lambda_{\max}(P)}{\xi}\right]e^{-\frac{\xi(t-t_0)}{\lambda_{\max}(P)}} \\
&\quad + \frac{\varepsilon\lambda_{\max}(P)}{\xi} \\
&\leq V(x(t_0))e^{-\frac{\xi(t-t_0)}{\lambda_{\max}(P)}} + \frac{\lambda_{\max}(P)(\varrho_1 + \varrho_2)\varepsilon}{\xi}.
\end{aligned}$$

Donc

$$\|x(t)\|^2 \leq \sigma^2 \|x(t_0)\|^2 e^{-\frac{\xi(t-t_0)}{\lambda_{\max}(P)}} + (r(\varepsilon))^2, \tag{3.52}$$

où $r(\varepsilon)$ est donné par (3.49).

Ainsi, d'après (3.52) et l'inégalité suivante

$$0 \leq \sqrt{a+b} \leq \sqrt{a} + \sqrt{b}, \ \forall a, b > 0,$$

on obtient

$$\|x(t)\| \leq \sigma \|x(t_0)\| e^{-\frac{\xi(t-t_0)}{2\lambda_{\max}(P)}} + r(\varepsilon).$$

Donc, le système en boucle fermée (3.33)-(3.34)-(3.46) est globalement uniformément pratiquement convergent avec rayon $r(\varepsilon)$ et taux $-\frac{\xi}{2\lambda_{\max}(P)}$ où ξ est donné par (3.40). □

Remarque 3.3.14 *Notons que si $\varrho_1 = \varrho_2 = 0$, alors on retrouve le résultat obtenu dans le Théorème 3.3.9.*

Aussi, si on suppose que $\varrho_1 = \varrho_1(t)$ et $\varrho_2 = \varrho_2(t)$ avec $\varrho_i(t)$, $i = 1, 2$, tend vers zéro quand t tend vers l'infini, alors toute les solutions convergent vers zéro.

Remarque 3.3.15 *La condition $\dot{h}(t) < 1$ est souvent imposée dans beaucoup de travaux pour assurer la stabilité de certains classes de systèmes dynamiques à retard. Dans ce travail, on suppose uniquement que la fonction du retard $h(t)$ est une fonction continue, positive et bornée. En outre, la classe de contrôleurs proposée est complétement indépendante du retard. Par conséquent, les résultats développés dans ce travail sont applicables à une classe de systèmes dynamiques avec quelques perturbations du retard.*

Exemple Numérique 3.3.16

Considérons le système incertain à retard (3.33) avec

$$\begin{gathered} A = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}, \ \triangle A(t) = \begin{pmatrix} \cos(t) & \sin(t) \\ 0 & 0 \end{pmatrix}, \\ \triangle A_d(t) = \begin{pmatrix} \cos(t) & \frac{1}{2}\sin(2t) \\ 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \triangle B(t) = \begin{pmatrix} t \\ 0 \end{pmatrix}, \\ f_1(t, x(t)) = \tfrac{1}{1+t}(1 + x_1(t) + x_2(t)), \\ f_2(t, x(t-h(t))) = \tfrac{1}{2+t}(2 + x_2(t-h(t))), \end{gathered} \tag{3.53}$$

où $x(t) = (x_1(t), x_2(t)) \in \mathbb{R} \times \mathbb{R}$ *et* $t \geq 0$.

D'après les Propiétés 3.3.6, 3.3.7 et 3.3.11, on obtient

$$\begin{gathered} \alpha_1 = 1, \ \alpha_2 = 1.414, \\ \varrho_1 = \varrho_2 = 1, \ \beta_1 = 2, \ \beta_2 = 0.5, \\ 1 + \eta(t) = 1 + t. \end{gathered}$$

Pour une matrice définie positive symétrique donnée

$$Q = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix},$$

avec $\lambda_{\min}(Q) = 2 > 0$, *la solution de l'équation matricielle de Riccati (3.25) est*

$$P = \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix},$$

avec $\lambda_{\min}(P) = 2.382 > 0$ *et* $\sigma = 1.392$.

Si les paramètres de gain sont choisis comme suit

$$\delta_1 = 1.2, \ \delta_2 = 1,$$

alors un nombre suffisament petit $q = 1.1 > 1$ *existe tel que*

$$\xi = 9.55 \times 10^{-2} > 0,$$

satisfait la condition (3.40).

D'après (3.47) et (3.48), le contrôleur proposé du système incertain à retard (3.53) est donné par

$$\begin{aligned} u(t, x(t)) \quad = \quad & -11.03(3x_1(t) + x_2(t)) \\ & -\frac{2(3x_1(t) + x_2(t))}{(1+t)[|3x_1(t) + x_2(t)| + 5 \times 10^{-3}]} \end{aligned} \tag{3.54}$$

Ce contrôleur assure la globale uniforme pratique convergence avec rayon $r = 0.636 > 0$ *et taux* $\gamma = 1.03 \times 10^{-2} > 0$.

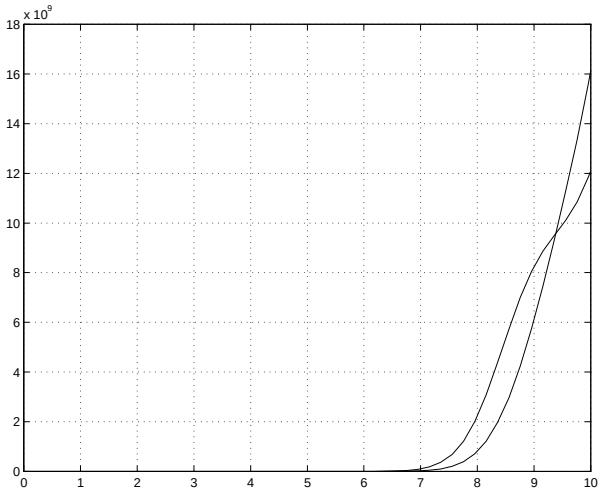

FIGURE 3.1 – Evolution des états x_1 et x_2 du système en boucle ouverte (3.53)

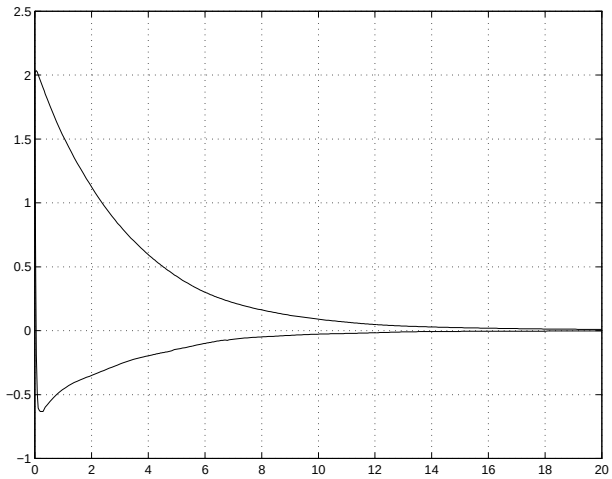

FIGURE 3.2 – Evolution des états x_1 et x_2 du système en boucle fermée (3.53)-(3.54)

Les résultats des simulations de cet exemple sont affichés dans les Figures 3.1 and 3.2. Dans la Figure 3.1, l'évolution des états x_1 et x_2 sans contrôle, est donnée et dans la Figure 3.2 l'action du contrôle est présenté. On constate, d'après la Figure 3.2, que le système en boucle fermée est en effet globalement uniformément pratiquement convergent.

3.3.3 Classe de systèmes à retard dans l'état et la commande

Enfin, on va appliquer les résultats obtenus pour les deux classes de système (3.15) et (3.33) pour concevoir un contrôleur du système incertain à retard suivant, où le retard apparaît à la fois dans l'état et dans la commande. On considère le système à retard suivant

$$\dot{x}(t) \quad = \quad Ax(t) + Bu(t) + Bf(t, x(t-h(t)), u(t-h(t))),\ t \geq t_0, \tag{3.55}$$

où $x(t) \in \mathbb{R}^n$ et $u(t) \in \mathbb{R}^m$ représentent respectivement l'état et la commande du système, $h(t)$ est la fonction du retard bornée par $\bar{h}$ et $f : R \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ est une application continue. Les matrices A et B sont des matrices constantes de dimensions convenables. On suppose que la paire (A, B) est contrôllable.
La condition initiale du système (3.55) est donnée par

$$x(t) = \phi(t),\ t \in [t_0 - \bar{h}, t_0]. \tag{3.56}$$

On suppose la condition de bornitude suivante

$$\|f(t, x(t-h(t)), u(t-h(t)))\| \leq \alpha\|x(t-h(t))\| + \beta\|u(t-h(t))\|, \tag{3.57}$$

où $\alpha > 0$ et $\beta > 0$ sont des constantes.
On pose

$$a = \|B^\top P\|,\ \sigma = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}},$$

où P est la solution de l'équation matricielle de Ricatti (3.25) avec $\mu = 0$.
On propose une classe de contrôleurs continues de la forme

$$u(t) = -\omega B^\top P x(t), \tag{3.58}$$

où ω est donnée par

$$\omega = \frac{1 - a\sigma^2\delta^2\alpha\beta - \sqrt{1 - 2a\sigma^2\delta^2(\alpha + a\beta)\beta}}{a^2\sigma^2\delta^2\beta^2}, \tag{3.59}$$

avec

$$\frac{q^2}{\delta^2} < \lambda_{\min}(Q), \tag{3.60}$$

où q est un nombre positif décrit dans la Propriété 3.3.7.

Proposition 3.3.17 *On suppose que le système (3.55)-(3.56) satisfait la condition (3.57). Si*

$$0 < a\delta^2(\alpha + a\beta)\beta \leq \frac{1}{2}\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}, \tag{3.61}$$

alors le système en bucle fermée (3.55)-(3.56)-(3.58) est globalement uniformément exponentiellement stable.

Preuve *La dérivée de la fonction de Lyapunov candidate $V(x(t))$ donnée dans (3.30), le long des trajectoires du système (3.55)-(3.56) satisfait*

$$\begin{aligned}\dot{V}(x(t)) &= 2x^\top(t)PAx(t) + 2x^\top(t)PBu(t) \\ &\quad +2x^\top(t)PBf(t, x(t-h(t)), u(t-h(t))) \\ &\leq -x^\top(t)Qx(t) + (2-2\omega)\|B^\top Px(t)\|^2 \\ &\quad +2q\sigma(\alpha + a\omega\beta)\|B^\top Px(t)\|\|x(t)\|. \end{aligned} \tag{3.62}$$

En utilisant (3.59) et (3.61), on obtient $\omega > 1$ et

$$2 - 2\omega = -\delta^2\sigma^2(\alpha + a\omega\beta)^2.$$

et en la substituant à (3.62), on obtient

$$\begin{aligned}\dot{V}(x(t)) &\leq -x^\top(t)Qx(t) + \frac{q^2}{\delta^2}\|x(t)\|^2 \\ &\quad -[\delta\sigma(\alpha + a\omega\beta)\|B^\top Px(t)\| - \frac{q}{\delta}\|x(t)\|]^2 \\ &\leq -x^\top(t)Qx(t) + \frac{q^2}{\delta^2}\|x(t)\|^2. \end{aligned} \tag{3.63}$$

En regardant l'inégalitée (3.63) et on posant

$$\xi := \lambda_{\min}(Q) - \frac{q^2}{\delta^2},$$

on obtient

$$\dot{V}(x(t)) \leq -\xi\|x(t)\|^2.$$

Si le paramètre δ est choisi tel que la condition (3.60) est satisfaite alors il existe un réel $q > 1$ tel que $\xi > 0$. En utilisant la preuve du Théorème 3.3.9, on peut conclure que le système en boucle fermée (3.55)-(3.56)-(3.58) est globalement uniformément exponentiellement stable. □

Chapitre 4

Absolute Stability and Application to Design of Observer-Based Controller for Nonlinear Time-Delay Systems

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Absolute Stability and Application to Design of Observer-Based Controller for Nonlinear Time-Delay Systems[1]

Abstract : *In this paper we presents a new sufficient condition for absolute stability of delay Lure system. This condition improve the one given in (26, Han). We use this new criterion to construct an observer-based control for a class of nonlinear time-delay systems. Some examples are given to illustrate the results of this paper.*

Key words : *Delay system, Absolute stability, Lure system, LMI, $\mathcal{S}$-procedure, Lyapunov-Krasovskii functional.*

Mathematics Subject Classification (2000) : *93Bxx, 93Dxx, 39Bxx.*

4.1 Introduction

Time-delay systems constitute basic mathematical models of real phenomena such as nuclear reactors, chemical engineering systems, biological systems, and population dynamics models. They are often a source of instability and degradation in control performance in many control problems, see (23, Hale and Lunel).

During the last two decades, the problem of stability of linear time-delay systems has been subject of considerable research efforts. Many significant results have been reported in the literature. For the recent progress, the reader is referred to (21, Gu and al.) and the references therein.

Since, the introduction of absolute stability by Lur'e (1957), the problem of absolute stability of a class of nonlinear control systems with a fixed matrix in the linear part of the system and one or multiple uncertain nonlinearities satisfying the sector constraints has been studied in (1, Aizerman and Gantmacher), (32, Khalil), (35, Liao), (36, Lur'e), (41, Popov), (51, Yakubovich and al.).

Due to time-delay occurring in practical systems, the problem of absolute stability for systems with delay has been also studied in (10; 11, Bliman), (17, Gan and Ge), (18, Gan and Han), (27, He and al.), (42, Popov and Halanay), (44, Somolines). However, the results mentioned above are delay-independent. When the time-delay is small, these results are often overly conservative; especially, they are not applicable to closed-loop systems which are open-loop unstable and are stabilized using delayed inputs, due to either delayed measurements or delayed actuator action in the input channels. Recently, in (26, Han), (52, Yu and al.), (53, Zhang and al.), the authors has studied the delay-dependent absolute stability for uncertain time-delay systems, which motivates the present work.

In this paper, we will deal with the problem of absolute stability for a class of time-delay systems which can be represented as a feedback connection of a linear dynamical system and a nonlinearity satisfying a sector constraint. Delay-dependent absolute stability criteria will be derived with employing some integral inequalities, and a new criteria are obtained and formulated in the from of linear matrix inequalities (LMI) which improve the one given in (24, Han). We use this new criterion to construct a delay-dependent dynamic observer-based output feedback control such that the observer error system is presented as the feedback interconnection of a lin-

1. This work is done with Abdallah Ben Abdallah and Mohamed Chaabane.

ear system and a state-dependent multivariable sector-bounded nonlinearity. In the observer design, we extend the works of (2; 3, Arcak and Kokotovic) and (15, Fan and Arcak) in the case without delay. Finally, some numerical examples are given to illustrate the applicability of the results.

Notation : *Throughout this paper, $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^n$ denotes the n dimensional Euclidean space, and $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices. I is the identity matrix matrix. The set $\mathcal{C}_{n,h} := \mathcal{C}([-h, 0], \mathbb{R}^n)$ is the space of continuous functions mapping the interval $[-h, 0]$ to $\mathbb{R}^n$, where h is a positive constant. The notation $A > 0$ denote that the matrix A is positive definite.*

4.2 Absolute stability

We consider the following delay-system

$$\begin{aligned} \dot{x}(t) &= A_0x(t) + A_1x(t-h) + B\omega(t), \\ y(t) &= C_0x(t) + C_1x(t-h), \\ \omega(t) &= -\varphi(t, y(t)), \end{aligned} \tag{4.1}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $y(t) \in \mathbb{R}^p$ the measured output, and the nonlinear function $\varphi(.,.) : \mathbb{R}_+ \times \mathbb{R}^p \to \mathbb{R}^p$ is assumed to be continuous and belongs to sector $[0, K]$ i.e. $\varphi(.,.)$ satisfies

$$\varphi^\top(t, y)\,[\varphi(t, y) - Ky] \leq 0,\ \forall (t, y) \in \mathbb{R}_+ \times \mathbb{R}^p, \tag{4.2}$$

where K is a positive definite matrix. The matrices A_0, A_1, B, C_0, and C_1 are real matrices with appropriate dimensions.
The initial condition of (4.1) is given by

$$x(t) = \phi(t),\ t \in [-h, 0],\ \phi \in \mathcal{C}_{n,h}.$$

It is assumed that the right-hand side of (4.1) is continuous and satisfies enough smoothness conditions to ensure the existence and uniqueness of the solution through every initial condition ϕ.

We first introduce the following definition.

Definition 4.2.1 *The system (4.1) is said to be absolutely stable in the sector $[0, K]$ if the system is globally uniformly asymptotically stable for any nonlinear function $\varphi(t, y(t))$ satisfying (4.2).*

The development of the work in this paper requires the following lemma which is a Proposition 3 in Reference (53, Zhang and al.).

Proposition 4.2.2 *Let $x(t) \in \mathbb{R}^n$ be a vector-valued function with first-order continuous-derivative entries. Then, the following integral inequality holds for any matrices $M_1, M_2 \in \mathbb{R}^{n\times n}$ and $X = X^\top > 0$, and a scalar $h > 0$:*

$$-\int_{t-h}^{t} \dot{x}^\top(s)X\dot{x}(s)ds \quad \leq \quad \xi^\top(t)\Upsilon\xi(t) + h\xi^\top(t)\Gamma^\top X^{-1}\Gamma\xi(t), \tag{4.3}$$

where

$$\Upsilon := \begin{bmatrix} M_1^\top + M_1 & -M_1^\top + M_2 \\ * & -M_2^\top - M_2 \end{bmatrix},\ \Gamma^\top := \begin{bmatrix} M_1^\top \\ M_2^\top \end{bmatrix},\ \xi(t) := \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}.$$

Proof *See (53, Zhang and al.).*

Remark 4.2.3 *The inequality (4.3) is called an integral inequality. It plays a key role in the derivation of a criterion for absolute stability in this paper. Note that the free matrices M_1 and M_2 make the LMI (4.4) proposed in Theorem 4.2.4 more feasible. Using the inequality (4.3), we will generalize the absolute stability criterion given in (26, Han).*

Under the sector condition (4.2), we will give a sufficient condition for absolute stability of system (4.1). We have the following theorem.

Theorem 4.2.4 *For given scalar $h > 0$, the system (4.1) with nonlinear function satisfying (4.2) is absolutely stable if there exist a scalar $\varepsilon > 0$, a positive definite matrices $P > 0,\ Q > 0,\ R > 0$, and real matrices $M_1,\ M_2 \in \mathbb{R}^{n\times n}$ such that the LMI*

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & hA_0^\top R & hM_1^\top \\ * & \Xi_{22} & \Xi_{23} & hA_1^\top R & hM_2^\top \\ * & * & -2\epsilon\, I & hB^\top R & 0 \\ * & * & * & -hR & 0 \\ * & * & * & * & -hR \end{bmatrix} < 0, \tag{4.4}$$

where
$\Xi_{11} = A_0^\top P + PA_0 + Q + M_1^\top + M_1,$
$\Xi_{12} = PA_1 - M_1^\top + M_2,$
$\Xi_{13} = PB - \epsilon\, C_0^\top K,$
$\Xi_{22} = -Q - M_2^\top - M_2,$
$\Xi_{23} = -\epsilon\, C_1^\top K,$
holds.

Proof *We consider the Lyapunov-Krasovskii functional condidate*

$$\begin{aligned} V(t,x_t) &= x^\top(t)Px(t) + \int_{t-h}^{t} x^\top(s)Qx(s)ds \\ &\quad + \int_{-h}^{0}\int_{t+\theta}^{t} \dot{x}^\top(s)R\dot{x}(s)dsd\theta, \end{aligned} \tag{4.5}$$

where the matrices P, Q, and R are positive definite. The derivative of V along the trajectories of system (4.1) is given by

$$\begin{aligned} \dot{V}(t,x_t) &= 2\dot{x}^\top(t)Px(t) + x^\top(t)Qx(t) - x^\top(t-h)Qx(t-h) \\ &\quad + h\dot{x}^\top(t)R\dot{x}(t) - \int_{t-h}^{t} \dot{x}^\top(s)R\dot{x}(s)ds. \end{aligned} \tag{4.6}$$

Applying the integral inequality (4.3) to the term of the right-hand side of (4.6) for any matrices $M_1,\ M_2 \in \mathbb{R}^{n\times n}$ yields the following inequality

$$\dot{V}(t) \le \eta^\top(t)\ \Pi\ \eta(t), \tag{4.7}$$

where

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} \\ * & \Pi_{22} & \Pi_{23} \\ * & * & \Pi_{33} \end{bmatrix},\ \eta(t) := \begin{bmatrix} x(t) \\ x(t-h) \\ \omega(t) \end{bmatrix},$$

with
$\Pi_{11} = A_0^\top P + PA_0 + Q + M_1^\top + M_1 + hA_0^\top RA_0 + hM_1^\top R^{-1}M_1,$
$\Pi_{12} = PA_1 - M_1^\top + M_2 + hA_0^\top RA_1 + hM_1^\top R^{-1}M_2,$
$\Pi_{13} = PB + hA_0^\top RB,$
$\Pi_{22} = -Q - M_2^\top - M_2 + hA_1^\top RA_1 + hM_2^\top R^{-1}M_2,$
$\Pi_{23} = hA_1^\top RB$
$\Pi_{33} = hB^\top RB.$

A sufficient condition for absolute stability of the system (4.1) is that there exist real matrices $P > 0$, $Q > 0$, and $R > 0$ such that

$$\dot{V}(t) \leq \eta^\top(t)\Pi\eta(t) < 0, \tag{4.8}$$

for all $\eta(t) \neq 0$. The sector condition (4.2) implies

$$u^\top(t)u(t) + u^\top(t)\left[KC_0x(t) + KC_1x(t-h)\right] \leq 0. \tag{4.9}$$

Using the $\mathcal{S}$-procedure, we can find $\epsilon > 0$ such that

$$\eta^\top(t)\,\Pi\,\eta(t) - 2\epsilon\,\omega^\top(t)\omega(t) - 2\epsilon\,\omega^\top(t)\left[KC_0x(t) + KC_1x(t-h)\right] < 0, \tag{4.10}$$

for all $\eta(t) \neq 0$. Rewrite (4.10) as

$$\eta^\top(t)\,\Sigma\,\eta(t) < 0, \tag{4.11}$$

where

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} \\ * & \Sigma_{22} & \Sigma_{23} \\ * & * & \Sigma_{33} \end{bmatrix},$$

with
$\Sigma_{ij} = \Pi_{ij},\ (i,j = 1,2),$
$\Sigma_{13} = \Pi_{13} - \epsilon\, C_0^\top K,$
$\Sigma_{23} = \Pi_{23} - \epsilon\, C_1^\top K,$
$\Sigma_{33} = \Pi_{33} - 2\epsilon\, I.$

Using Shur complement, the LMI ($\Sigma < 0$) is equivalent to the LMI (4.4). This completes the proof.

Remark 4.2.5 *In (52, Yu and al.), the authors considered the absolute stability of system (4.1) for the case of $C_1 = 0$. The idea in (52, Yu and al.) was to first transform the system to a system with a distributed delay, then to apply an inequality to some cross terms. However, from the proof process of Theorem 4.2.4, one can clearly see that neither model transformation nor bounding technique for cross terms is involved.*

4.3 Stabilization of nonlinear delay system

This section present the delay-dependent stabilization condition obtained by using the absolute stability proposed in section 4.2.

Consider the following nonlinear control delay system

$$\begin{aligned} \dot{x}(t) &= A_0x(t) + A_1x(t-h) + B\omega(t) + Gu(t), \\ y(t) &= C_0x(t) + C_1x(t-h), \\ \omega(t) &= -\psi(t, y(t)), \end{aligned} \tag{4.12}$$

where $A_0,\ A_1,\ B,\ G,\ C_0,\ C_1$ are real matrices with appropriate dimensions, and the nonlinearity $\psi(t,y)$ belong in the sector $[0,K]$, $K>0$.
The initial condition of (4.12) is given by

$$x(t) = \phi(t),\ t \in [-h, 0],\ \phi \in \mathcal{C}_{n,h}.$$

It is assumed that the right-hand side of (4.12) is continuous and satisfies enough smoothness conditions to ensure the existence and uniqueness of the solution through every initial condition ϕ.
The closed-loop system with the state control feedback

$$u(t) = Nx(t), \tag{4.13}$$

is given by

$$\dot{x}(t) = (A_0 + GN)\,x(t) + A_1x(t-h) + B\omega(t). \tag{4.14}$$

The following theorem gives a sufficient condition for stabilization of the system by means a state feedback when the nonlinearity $\psi(t,y)$ belong in the sector $[0,K]$.

Theorem 4.3.1 *For given numbers $h>0,\ \lambda_i \in \mathbb{R},\ i=1,2$, if there exist a scalar $\epsilon>0$, a positive definite matrices $\overline{P}>0,\ \overline{Q}>0,\ \overline{R}>0$, and a matrix $Y \in \mathbb{R}^{n\times r}$ such that the LMI*

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & h\lambda_1\overline{R} \\ * & \Xi_{22} & \Xi_{23} & \Xi_{24} & h\lambda_2\overline{R} \\ * & * & -2\epsilon\, I & \Xi_{34} & 0 \\ * & * & * & -h\overline{R} & 0 \\ * & * & * & * & -h\overline{R} \end{bmatrix} < 0, \tag{4.15}$$

where
$\Xi_{11} = \overline{P}(A_0+\lambda_1 I)^\top + (A_0+\lambda_1 I)\overline{P} + GY + Y^\top G^\top + \overline{Q}$,
$\Xi_{12} = A_1\overline{P} + (\lambda_2-\lambda_1)\overline{P}$,
$\Xi_{13} = B - \epsilon\overline{P}C_0^\top K$,
$\Xi_{14} = h\overline{P}A_0^\top + hY^\top G^\top$,
$\Xi_{22} = -\overline{Q} - 2\lambda_2\overline{P}$,
$\Xi_{23} = -\epsilon\overline{P}C_1^\top K$,
$\Xi_{24} = h\overline{P}A_1^\top$,
$\Xi_{34} = hB^\top\overline{R}$,
holds. Then the origin of the controlled system (4.12) is stabilizable by the linear state feedback (4.13) where

$$N = Y\overline{P}^{-1}.$$

Proof *Let $h>0$, and λ_1,λ_2 are fixed reals. Using Theorem 1, the closed-loop system is stable if there exist a positive definite matrices $P>0,\ Q>0,\ R>0$, and $M_1,\ M_2 \in \mathbb{R}^{n\times n}$ such that the LMI (4.4) with replacing*

A_0 by $A_0 + GN$ holds, then the origin of system (4.14) is globally uniformly asymptotically stable. This is equivalent to the feasibility of the following LMI

$$T^\top \, \Xi \, T = \Xi_T < 0,$$

where $T = diag\{P^{-1}, P^{-1}, I, R^{-1}, R^{-1}\}$. Denoting $\overline{P} = P^{-1}$, $\overline{Q} = P^{-1}QP^{-1}$, $\overline{R} = R^{-1}$, $NP^{-1} = Y$, and picking $M_i = \lambda_i P$, $i = 1, 2$, we obtain the desired LMI (4.15).

Remark 4.3.2 *The criterion in Theorem 4.3.1 does not require any assumptions about the systems matrices, e.g., the pairs (A_0, G) and $(A_0 + A_1, G)$ need not stabilizable. So, systems designed based on this criterion that have both state and input delay can be stabilized, even when (A_0, G) nor $(A_0 + A_1, G)$ is stabilizable.*

4.4 Observer-based control for a class of nonlinear delay system

In this section, a learning control approch, combined with state estimation, is developed to perform the stabilization of the system proposed in section 4.3 where the state information is not available.

4.4.1 System description

Now, we consider the time delay system of the form

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + A_1 x(t-h) + B\omega(t), \\ y(t) &= C_0 x(t) + C_1 x(t-h), \\ \omega(t) &= -\gamma(H_0 x(t) + H_1 x(t-h)), \end{aligned} \tag{4.16}$$

where $x(t) \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}^r$ is the measured output, $h > 0$ is the constant delay, and the multivariable nonlinearity $\gamma(.) : \mathbb{R}^p \to \mathbb{R}^p$ is decentralized in the sense that $\gamma_i(v)$, $v \in \mathbb{R}^p$, $1 \le i \le p$, depends only on v_i ; that is

$$\gamma(v) = \begin{bmatrix} \gamma_1(v_1) \\ \vdots \\ \gamma_p(v_p) \end{bmatrix}. \tag{4.17}$$

We suppose that the nonlinearities $\gamma(.)$ satisfy

$$a_i \le \frac{\gamma_i(t) - \gamma_i(t')}{t - t'} \le b_i, \ \forall \, t, t' \in \mathbb{R}, \ t \ne t', \ \forall \, 1 \le i \le p. \tag{4.18}$$

If $\gamma_i(.)$, $1 \le i \le p$, is continuous differentiable, then its slope is restricted by

$$\frac{d\gamma_i}{dv_i}(v_i) \in [a_i, b_i] \text{ for all } 1 \le i \le p, \ v_i \in \mathbb{R}.$$

The scalar function γ_i staisfies (4.18) with $a_i = 0$, $1 \le i \le p$. (If $a_i \ne 0$, we can define a new functions $\widetilde{\gamma}_i(t) := \gamma_i(t) - a_i t$ which satisfies (4.18) with $\widetilde{a}_i = 0$, $\widetilde{b}_i = b_i - a_i$, $1 \le i \le p$, and we rewrite (4.16) as

$$\begin{aligned} \dot{x}(t) &= \widetilde{A}_0 x(t) + \widetilde{A}_1 x(t-h) + B\widetilde{\omega}(t), \\ y(t) &= C_0 x(t) + C_1 x(t-h), \\ \widetilde{\omega}(t) &= -\widetilde{\gamma}(H_0 x(t) + H_1 x(t-h)), \end{aligned}$$

with

$$\begin{aligned}\widetilde{A}_0 &= A_0 - B\ diag\{a_1,\dots,a_p\}H_0,\\ \widetilde{A}_1 &= A_1 - B\ diag\{a_1,\dots,a_p\}H_1.)\end{aligned}$$

The matrices $A_i \in \mathbb{R}^{n\times n}$, $C_i \in \mathbb{R}^{r\times n}$, $B \in \mathbb{R}^{n\times p}$ *and* $H_i \in \mathbb{R}^{p\times n}$, $0 \le i \le 2$, *are constant matrices. The initial condition of (4.16) is given by*

$$x(t) = \phi(t),\ t \in [-h, 0],\ \phi \in \mathcal{C}_{n,h}.$$

It is assumed that the right-hand side of (4.16) is continuous and satisfies enough smoothness conditions to ensure the existence and uniqueness of the solution through every initial condition ϕ.

4.4.2 Observer design

In this subsection, observer is designed for a class of delay-dependent nonlinear systems. The approach is to represent the observer error system as the feedback interconnection of a linear system and a state-dependent multivariable sector-bounded nonlinearity. We extend a design proposed in (2; 3, Arcak and Kokotovic) and (15, Fan and Arcak) in the case without delay.

With the assumption (4.18), our observer has the following form

$$\begin{aligned}\dot{\widehat{x}}(t) &= A_0\widehat{x}(t) + A_1\widehat{x}(t-h) + L(\widehat{y}(t) - y(t)) + B\widehat{\omega}(t),\\ \widehat{y}(t) &= C_0\widehat{x}(t) + C_1\widehat{x}(t-h),\\ \widehat{\omega}(t) &= -\gamma(S(\widehat{y}(t) - y(t)) + H_0\widehat{x}(t) + H_1\widehat{x}(t-h)),\end{aligned} \tag{4.19}$$

Our task is to design the matrices $S \in \mathbb{R}^{p\times r}$ *and* $L \in \mathbb{R}^{n\times r}$ *to make the observer error* $e(t) = x(t) - \widehat{x}(t)$ *approach zero. At this point we assume that the solution of (4.16) does not escape to infinity in finite time. From (4.16) and (4.18), the dynamics of the observer error* $e(t)$ *is governed by*

$$\dot{e}(t) = (A_0 + LC_0)e(t) + (A_1 + LC_1)e(t-h) - B[\gamma(v(t)) - \gamma(w(t))], \tag{4.20}$$

where

$$\begin{aligned}v(t) &= H_0x(t) + H_1x(t-h),\\ w(t) &= S(\widehat{y}(t) - y(t)) + H_0\widehat{x}(t) + H_1\widehat{x}(t-h).\end{aligned}$$

To represent the observer error system (4.20) as the feedback interconnection of a linear system and multivariable sector nonlinearity, we view $\gamma(v(t)) - \gamma(w(t))$ *as a function of* t *and*

$$z(t) := v(t) - w(t) = (H_0 + SC_0)e(t) + (H_1 + SC_1)e(t-h).$$

That is, a state-dependent multivariable nonlinearity in t *and* $z(t)$ *:*

$$\psi(t, z(t)) := \gamma(v(t)) - \gamma(w(t)). \tag{4.21}$$

We rewrite the observer error (4.20) as

$$\begin{aligned}\dot{e}(t) &= (A_0 + LC_0)e(t) + (A_1 + LC_1)e(t-h) + B\Omega(t),\\ z(t) &= (H_0 + SC_0)e(t) + (H_1 + SC_1)e(t-h),\\ \Omega(t) &= -\psi(t, z(t)).\end{aligned} \tag{4.22}$$

First, we start showing under the condition given in (4.18) that $\psi(t, z(t))$ satisfies a multivariable sector property. Indeed, since

$$\begin{aligned}\psi_i(t, z(t)) &= \gamma_i(v(t)) - \gamma_i(w(t)) = \gamma_i(v_i(t)) - \gamma_i(w_i(t)) = \psi_i(t, z_i(t)) \\ &= \int_0^1 [\frac{\partial \gamma}{\partial s}]_{s=v_i+\lambda(w_i(t)-v_i(t))}(v_i(t) - w_i(t))d\lambda, \ \forall\, 1 \leq i \leq p.\end{aligned}$$

Thus, from (4.18), each component $\psi_i(.,.)$ satisfies the following sector condition

$$\psi_i(t, z_i(t))[\psi_i(t, z_i(t)) - b_i z_i(t)] \leq 0, \ \forall\, 1 \leq i \leq p. \tag{4.23}$$

Taking $K = diag\{b_1, \ldots, b_p\} > 0$, it can be seen that (see (32, Khalil))

$$\psi^\top(t, z(t))[\psi(t, z(t)) - Kz(t)] \leq 0. \tag{4.24}$$

The convergence of observer is a consequence of Theorem 4.2.4. We have the following result.

Theorem 4.4.1 *For given numbers λ_i, $i = 1, 2$, if there exist a positive scalar $\epsilon > 0$, a positive definite matrices $P > 0$, $Q > 0$, $\overline{R} > 0$, $Y \in \mathbb{R}^{n\times r}$, and $S \in \mathbb{R}^{p\times r}$ such that the LMI*

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & h\lambda_1\overline{R} \\ * & \Xi_{22} & \Xi_{23} & \Xi_{24} & h\lambda_2\overline{R} \\ * & * & -2\epsilon\, I & \Xi_{34} & 0 \\ * & * & * & -h\overline{R} & 0 \\ * & * & * & * & -h\overline{R} \end{bmatrix} < 0, \tag{4.25}$$

with
$\Xi_{11} = (A_0 + \lambda_1 I)^\top P + P(A_0 + \lambda_1 I) + YC_0 + C_0^\top Y + Q,$
$\Xi_{12} = P(A_1 + (\lambda_2 - \lambda_1)I) + YC_1,$
$\Xi_{13} = PB - \epsilon\,(H_0 + SC_0)^\top K,$
$\Xi_{14} = hA_0^\top P + hC_0^\top Y^\top,$
$\Xi_{22} = -Q - 2\lambda_2 P,$
$\Xi_{23} = -\epsilon\,(H_1 + SC_1)^\top K,$
$\Xi_{24} = hA_1^\top P + hC_1^\top Y^\top,$
$\Xi_{34} = hB^\top P,$
holds. Then the system (4.19) is an observer of system (4.16). The gain matrix S is given by the resolution of the previous LMI and

$$L = P^{-1}Y.$$

Proof *A sufficient condition for the convergence of the observer is given by Theorem 4.2.4 with replacing A_i by $A_i + LC_i$, and H_i by $H_i + SC_i$, $i = 1, 2$. That is the existence of $\varepsilon > 0$, a positive definite matrices $P > 0$, $Q > 0$, $R > 0$, and real matrices M_1, $M_2 \in \mathbb{R}^{n\times n}$ such that*

$$\Delta := \begin{bmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} & \Delta_{14} & hM_1^\top \\ * & \Delta_{22} & \Delta_{23} & \Delta_{24} & hM_2^\top \\ * & * & -2\epsilon\, I & \Delta_{34} & 0 \\ * & * & * & -hR & 0 \\ * & * & * & * & -hR \end{bmatrix} < 0$$

where
$\Delta_{11} = (A_0 + LC_0)^\top P + P(A_0 + LC_0) + Q + M_1^\top + M_1,$

$\Delta_{12} = P(A_1 + LC_1) - M_1^\top + M_2,$
$\Delta_{13} = PB - \epsilon\,(H_0 + SC_0)^\top K,$
$\Delta_{14} = h(A_0 + LC_0)^\top R,$
$\Delta_{22} = -Q - M_2^\top - M_2,$
$\Delta_{23} = -\epsilon\,(H_1 + SC_1)^\top K,$
$\Delta_{24} = h(A_1 + LC_1)^\top R,$
$\Delta_{34} = hB^\top R.$

This LMI is equivalent to

$$Z^T \Delta Z = \Delta_Z < 0,$$

where $Z = diag(I, I, I, R^{-1}P, R^{-1}P)$. *If we choose* $M_i = \lambda_i P, i = 1, 2$, $Y = PL$, *and* $\overline{R} = PR^{-1}P$, *we obtain the desired LMI (4.25).*

4.4.3 Observer-based control feedback

Our goal is to find a sufficient condition for to the existence of an observer-based control feedback witch robust stabilize the following system

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + A_1 x(t-h) + B\omega(t) + Gu(t), \\ y(t) &= C_0 x(t) + C_1 x(t-h), \\ \omega(t) &= -\gamma(H_0 x(t) + H_1 x(t-h)), \end{aligned} \tag{4.26}$$

where $u(t) \in \mathbb{R}^m$ *is the control input, and the nonlinearity* $\gamma(.)$ *satisfies the condition (4.17)-(4.18) with* $a_i = 0$. *Consider the closed-loop system (4.26) with the control feedback*

$$u(t) = N\hat{x}(t), \tag{4.27}$$

and the dynamic of observer is

$$\begin{aligned} \dot{\hat{x}}(t) &= (A_0 + GN)\hat{x}(t) + A_1\hat{x}(t-h) + L(\hat{y}(t) - y(t)) + B\hat{\omega}(t), \\ \hat{y}(t) &= C_0\hat{x}(t) + C_1\hat{x}(t-h), \\ \hat{\omega}(t) &= -\gamma(H_0\hat{x}(t) + H_1\hat{x}(t-h) + S(\hat{y}(t) - y(t))). \end{aligned} \tag{4.28}$$

It follows that

$$\begin{aligned} \dot{e}(t) &= (A_0 + LC_0)e(t) + (A_1 + LC_1)e(t-h) + B\Omega(t), \\ \Omega(t) &= -\psi(t, z(t)), \end{aligned} \tag{4.29}$$

with

$$\begin{gathered} \psi(t, z(t)) = \gamma(v(t)) - \gamma(w(t)), \\ v(t) = H_0 x(t) + H_1 x(t-h), \\ w(t) = S(\widehat{y}(t) - y(t)) + H_0\widehat{x}(t) + H_1\widehat{x}(t-h), \\ z(t) = v(t) - w(t) = (H_0 + SC_0)e(t) + (H_1 + SC_1)e(t-h). \end{gathered}$$

Here, the objective of this subsection is to develop a procedure to design an observer-based controller for the time-delay system (4.26), such that the resulting closed subsystem given by

$$\dot{x}(t) = (A_0 + GN)x(t) + A_1 x(t-h) - B\gamma(v(t)) - GNe(t), \tag{4.30}$$

is globally asymptotically stable.

Theorem 4.4.2 *For a given numbers $\lambda_1, \lambda_2 \in \mathbb{R}$, if there exist a positive scalars $\epsilon_1 > 0, \epsilon_2 > 0$, a positive definite matrices $\overline{P}_x, \overline{Q}_x, \overline{R}_x, P_e, Q_e, R_e$ and two matrices $N_1, N_2 \in \mathbb{R}^{n\times n}$ such that*

$$\Delta_T := T^\top \Delta T = \begin{bmatrix} \Delta_{11T} & \Delta_{12T} & \Delta_{13T} & h\overline{P}_x A_0^\top + hY^\top G^\top & h\lambda_1 \overline{R}_x \\ * & \Delta_{22T} & \Delta_{23T} & h\overline{P}_x A_1^\top & h\lambda_2 \overline{R}_x \\ * & * & -2\epsilon_2 I & hB^\top & 0 \\ * & * & * & -h\overline{R}_x & 0 \\ * & * & * & * & -h\overline{R}_x \end{bmatrix} < 0, \tag{4.31}$$

and

$$\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & P_e B - \epsilon_1 (H_0 + SC_0)^\top K & hA_{N0}^\top R_e & hN_1^\top \\ * & \Lambda_{22} & -\epsilon_1 (H_1 + SC_1)^\top K & hA_1^\top R_e & hN_2^\top \\ * & * & -2\epsilon_1 I & hB^\top R_e & 0 \\ * & * & * & -hR_e & 0 \\ * & * & * & * & -hR_e \end{bmatrix} < 0, \tag{4.32}$$

where

$$\begin{aligned}
\Lambda_{11} &= P_e A_0 + A_0^\top P_e + Q_e + \overline{Y} C_0 + C_0^\top \overline{Y}^\top + N_1^\top + N_1 \\
\Lambda_{12} &= P_e A_1 + \overline{Y} C_1 - N_1^\top + N_2, \\
\Lambda_{22} &= -Q_e - N_2^\top - N_2, \\
\Delta_{11} &= (A_0 + \lambda_1 I)\,\overline{P}_x + \overline{P}_x\,(A_0 + \lambda_1 I)^\top + \overline{Q}_x + GY + Y^\top G^\top \\
\Delta_{12T} &= A_1 \overline{P}_x + (\lambda_2 - \lambda_1)\overline{P}_x, \\
\Delta_{13T} &= B - \epsilon_2 \overline{P}_x H_0^\top K, \\
\Delta_{22T} &= -\overline{Q}_x - 2\lambda_2 \overline{P}_x, \\
\Delta_{23T} &= -\epsilon_2 \overline{P}_x H_1^\top K.
\end{aligned}$$

Then the system (4.26) is stabilizable by the dynamic feedback (4.27)-(4.28), where the matrices gain is given by

$$L = P_e^{-1}\overline{Y},\ N = \overline{P}_x^{-1} Y.$$

Proof *Define the Lyapunov-Krasovskii functional V as follows*

$$V(t) := V_x(t, x_t) + a\, V_e(t, e_t), \tag{4.33}$$

where

$$\begin{aligned}
V_x(t, x_t) &= x^\top(t) P_x x(t) + \int_{t-h}^{t} x^\top(s) Q_x x(s) ds \\
&\quad + \int_{-h}^{0} \int_{t+\theta}^{t} \dot{x}^\top(s) R_x \dot{x}(s) ds d\theta,
\end{aligned} \tag{4.34}$$

and

$$\begin{aligned} V_e(t,e_t) &= e^\top(t)P_e e(t) + \int_{t-h}^{t} e^\top(s)Q_e e(s)ds \\ &\quad + \int_{-h}^{0}\int_{t+\theta}^{t} \dot{e}^\top(s)R_e\dot{e}(s)dsd\theta, \end{aligned} \tag{4.35}$$

with $x_t,\ e_t \in \mathcal{C}_{n,h}$, *and* a *is a postive real will be chosen.*

The Lyapunov-Krasovskii functional (4.33) is legitimate functional, see (23, Hale and Lunel) and (43, Richard).

The corresponding derivate of the functional (4.34) is given by

$$\begin{aligned} \dot{V}_x(t,x_t) &= 2\dot{x}^\top(t)P_x x(t) + x^\top(t)Q_x x(t) - x^\top(t-h)Q_x x(t-h) \\ &\quad + h\dot{x}^\top(t)R_x\dot{x}(t) - \int_{t-h}^{t} \dot{x}^\top(s)R_x\dot{x}(s)ds. \end{aligned} \tag{4.36}$$

Applying the integral inequality (4.3) to the term of the right-hand side of (4.36), the inequalities

$$-2e^\top(t)(GN)^\top P_x x(t) \le \frac{1}{b}x^\top(t)P_x x(t) + be^\top(t)(GN)^\top P_x GNe(t),$$

and

$$-2he^\top(t)(GN)^\top R_x\Gamma\eta(t) \le hbe^\top(t)(GN)^\top R_x GNe(t) + \frac{h}{b}\eta^\top(t)\Gamma^\top R_x\Gamma\eta(t),$$

we obtain, for any matrices $M_1,\ M_2 \in \mathbb{R}^{n\times n}$ *and any* $b > 0$, *the following inequality*

$$\begin{aligned} \dot{V}_x(t,x_t) &\le \eta^\top(t)\left(\Pi + \frac{h}{b}\Gamma^\top R_x\Gamma\right)\eta(t) + \frac{1}{b}x^\top(t)P_x x(t) \\ &+ e^\top(t)(GN)^\top\left(bP_x + h(1+b)R_x\right)GNe(t), \end{aligned} \tag{4.37}$$

where

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} \\ * & \Pi_{22} & \Pi_{23} \\ * & * & \Pi_{33} \end{bmatrix},\ \eta(t) := \begin{bmatrix} x(t) \\ x(t-h) \\ -\gamma(v(t)) \end{bmatrix},\ \Gamma^\top := \begin{bmatrix} A_{N_0}^\top \\ A_1^\top \\ B^\top \end{bmatrix},$$

with $A_{N_0} := A_0 + GN$, *and*

$\Pi_{11} = A_{N_0}^\top P_x + P_x A_{N_0} + Q_x + M_1^\top + M_1 + hA_{N_0}^\top R_x A_{N_0} + hM_1^\top R_x^{-1}M_1,$
$\Pi_{12} = P_x A_1 - M_1^\top + M_2 + hA_{N_0}^\top R_x A_1 + hM_1^\top R_x^{-1}M_2,$
$\Pi_{13} = P_x B + hA_{N_0}^\top R_x B,$
$\Pi_{22} = -Q_x - M_2^\top - M_2 + hA_1^\top R_x A_1 + hM_2^\top R_x^{-1}M_2,$
$\Pi_{23} = hA_1^\top R_x B,$
$\Pi_{33} = hB^\top R_x B.$

Now, the corresponding derivate of functional (4.35) is given by

$$\begin{aligned} \dot{V}_e(t,e_t) &= 2\dot{e}^\top(t)P_e e(t) + e^\top(t)Q_e e(t) - e^\top(t-h)Q_e e(t-h) \\ &\quad + h\dot{e}^\top(t)R_e\dot{e}(t) - \int_{t-h}^{t} \dot{e}^\top(s)R_e\dot{e}(s)ds. \end{aligned} \tag{4.38}$$

Applying the integral inequality (4.3) to the term of the right-hand side of (4.38) for any $N_1, N_2 \in \mathbb{R}^{n\times n}$ yields the following inequality

$$\dot{V}_e(t, e_t) \leq \zeta^\top(t)\Theta\zeta(t), \tag{4.39}$$

where

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} \\ * & \Theta_{22} & \Theta_{23} \\ * & * & \Theta_{33} \end{bmatrix}, \ \zeta(t) = \begin{bmatrix} e(t) \\ e(t-h) \\ -\psi(t,z) \end{bmatrix},$$

with

$$\begin{aligned}
\Theta_{11} &= P_e(A_0 + LC_0) + (A_0 + LC_0)^\top P_e + Q_e + N_1^\top + N_1 + hN_1^\top R_e^{-1} N_1 \\
&\quad + h(A_0 + LC_0)^\top R_e (A_0 + LC_0), \\
\Theta_{12} &= P_e(A_1 + LC_1) - N_1^\top + N_2 + h(A_0 + LC_0)^\top R_e (A_1 + LC_1) \\
&\quad + hN_1^\top R_e^{-1} N_2, \\
\Theta_{13} &= P_e B + h(A_0 + LC_0)^\top R_e B, \\
\Theta_{22} &= -Q_e - N_2^\top - N_2 + hN_2^\top R_e^{-1} N_2 + h(A_1 + LC_1)^\top R_e (A_1 + LC_1), \\
\Theta_{23} &= h(A_1 + LC_1)^\top R_e B, \\
\Theta_{33} &= hB^\top R_e B.
\end{aligned}$$

Using (4.37) and (4.39), we obtain

$$\begin{aligned}
\dot{V}(t) &\leq \zeta^\top(t)\Theta\zeta(t) + e^\top(t)(GN)^\top \left(abP_x + a(b+1)hR_x\right) GNe(t) \\
&\quad + a\eta^\top(t)\left(\Pi + \frac{h}{b}\Gamma^\top R_x \Gamma\right)\eta(t) + \frac{a}{b}x^\top(t)P_x x(t).
\end{aligned}$$

A sufficient condition for stability is

$$V_e(t) \leq \zeta^\top(t)\Theta\zeta(t) < 0, \ \textit{and} \ \eta^\top(t)\Pi\eta(t) < 0,$$

Indeed, in this case by applying the $\mathcal{S}$-procedure we can choose $b > 0$ large enough such that

$$\eta^\top(t)\left(\Pi + \frac{h}{b}\Gamma^\top R_x \Gamma\right)\eta(t) + \frac{1}{b}x^\top(t)P_x x(t) < 0.$$

Then we chose $a > 0$ small such that

$$\zeta^\top(t)\Theta\zeta(t) + e^\top(t)(GN)^\top \left(abP_x + a(b+1)hR_x\right) GNe(t) < 0.$$

This implies the negativity of $\dot{V}(t)$, and hence the stability.

Using Theorem 4.4.1, $V_e(t, e_t) \leq \zeta^\top(t)\Theta\zeta(t) < 0$ is equivalent to the following LMI

$$\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} & hA_0^\top R_e & hN_1^\top \\ * & \Lambda_{22} & \Lambda_{23} & hA_1^\top R_e & hN_2^\top \\ * & * & \Lambda_{33} & hB^\top R_e & 0 \\ * & * & * & -hR_e & 0 \\ * & * & * & * & -hR_e \end{bmatrix} < 0,$$

where

$$\begin{aligned}
\Lambda_{11} &= P_e(A_0+LC_0)+(A_0+LC_0)^\top P_e+Q_e+N_1^\top+N_1\\
\Lambda_{12} &= P_e(A_1+LC_1)-N_1^\top+N_2,\\
\Lambda_{13} &= P_eB-\epsilon_1(H_0+SC_0)^\top K,\\
\Lambda_{22} &= -Q_e-N_2^\top-N_2,\\
\Lambda_{23} &= -\epsilon_1(H_1+SC_1)^\top K,\\
\Lambda_{33} &= -2\epsilon_1 I.
\end{aligned}$$

Suppose that we have determine the matrix N. In this case and for a given ϵ_1, we can transform as in proof of Theorem 4.4.1, $\Lambda<0$ into LMI in $P_e>0, Q_e>0, \overline{R}_e>0, \overline{Y}=P_eL, S, N_1, N_2$.

To establish a sufficient condition in term of LMI, we use the sector condition

$$\gamma(v(t))^\top[\gamma(v(t))-Kv(t)]\leq 0, \tag{4.40}$$

and the $\mathcal{S}$-procedure implies

$$\begin{aligned}
&\eta^\top(t)\Pi\eta(t)-2\epsilon\gamma(v(t))^\top\gamma(v(t))\\
&-2\epsilon\gamma(v(t))^\top K[H_0x(t)+H_1x(t-h)]<0,
\end{aligned} \tag{4.41}$$

for all $\eta(t)\neq 0$. This give the following LMI

$$\Delta=\begin{bmatrix}
\Delta_{11} & \Delta_{12} & \Delta_{13} & hA_{N0}^\top R_x & hM_1^\top\\
* & \Delta_{22} & \Delta_{23} & hA_1^\top R_x & hM_2^\top\\
* & * & \Delta_{33} & hB^\top R_x & 0\\
* & * & * & -hR_x & 0\\
* & * & * & * & -hR_x
\end{bmatrix}<0,$$

where

$$\begin{aligned}
\Delta_{11} &= P_xA_{N0}+A_{N0}^\top P_x+Q_x+M_1^\top+M_1\\
\Delta_{12} &= P_xA_1-M_1^\top+M_2,\\
\Delta_{13} &= P_xB-\epsilon H_0^\top K,\\
\Delta_{22} &= -Q_x-M_2^\top-M_2,\\
\Delta_{23} &= -\epsilon H_1^\top K,\\
\Delta_{33} &= -2\epsilon I.
\end{aligned}$$

As in the proof of Theorem 4.3.1, for a given $\lambda_1,\lambda_2\in\mathbb{R}$, this LMI is equivalent to the existence of $\epsilon_2>0$ such that the following LMI in $\overline{P}_x>0, \overline{Q}_x>0, \overline{R}_x>0, Y\in\mathbb{R}^{n\times p}$:

$$\Delta_T:=T^\top\Delta T=\begin{bmatrix}
\Delta_{11T} & \Delta_{12T} & \Delta_{13T} & h\overline{P}_xA_0^\top+hY^\top G^\top & h\lambda_1\overline{R}_x\\
* & \Delta_{22T} & \Delta_{23T} & h\overline{P}_xA_1^\top & h\lambda_2\overline{R}_x\\
* & * & \Delta_{33T} & hB^\top & 0\\
* & * & * & -h\overline{R}_x & 0\\
* & * & * & * & -h\overline{R}_x
\end{bmatrix}<0,$$

where

$$\begin{aligned}
\Delta_{11} &= (A_0+\lambda_1 I)\,\overline{P}_x + \overline{P}_x\,(A_0+\lambda_1 I)^\top + \overline{Q}_x + GY + Y^\top G^\top \\
\Delta_{12} &= A_1\overline{P}_x + (\lambda_2-\lambda_1)\overline{P}_x, \\
\Delta_{13} &= B - \epsilon_2\overline{P}_x H_0^\top K, \\
\Delta_{22} &= -\overline{Q}_x - 2\lambda_2\overline{P}_x, \\
\Delta_{23} &= -\epsilon_2\overline{P}_x H_1^\top K, \\
\Delta_{33} &= -2\epsilon_2 I.
\end{aligned}$$

holds. The gain matrix N is given by

$$N = \overline{P}_x^{-1} Y. \tag{4.42}$$

This completes the proof.

Remark 4.4.3 *Two steps to solve the LMI's in Theorem 4.4.2. The first step is the resolution of LMI $\Delta_T < 0$ and we determine the gain matrix N. The second step is the resolution of the LMI $\Lambda < 0$ and determine the gain matrices S and L.*

4.5 Numerical examples

Example 4.5.1

We consider the time-delay system (4.1) where the nonlinearity satisfy condition (4.2) with

$$A_0 = \begin{bmatrix} -2.02 & -1 \\ 0.5 & 0.22 \end{bmatrix}, A_1 = \begin{bmatrix} 0.49 & 1 \\ -0.1 & -0.68 \end{bmatrix}, B = \begin{bmatrix} -0.5 & 0 \\ 0 & 0.2 \end{bmatrix},$$

$$C_0 = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.5 \end{bmatrix}, C_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}, K = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}. \tag{4.43}$$

This example has been studied in (26, Han) where the nonlinearity belong the sector $[K_1, K_2]$. However, one can easily formulate a criterion along the idea in (26, Han) in the sector $[0, K]$. By new criterion, the numerical example result is $h_{\max} = 2.023$.
Applying Theorem 4.2.4, the maximum allowed time-delay is computed as $h_{\max} = 2.615$. The solutions of LMI (4.4) are

$$P = \begin{bmatrix} 114.126 & 89.035 \\ 89.035 & 277.363 \end{bmatrix}, \; Q = \begin{bmatrix} 112.482 & 30.266 \\ 30.266 & 114.814 \end{bmatrix},$$

$$R = \begin{bmatrix} 41.825 & 100.196 \\ 100.196 & 368.121 \end{bmatrix},$$

$$M_1 = \begin{bmatrix} -8.139 & -33.797 \\ -14.423 & -127.778 \end{bmatrix}, \; M_2 = \begin{bmatrix} 13.901 & 35.819 \\ 37.239 & 136.017 \end{bmatrix},$$

and $\epsilon = 545.12 > 0$.
It is a gain to show that the criteria developed in Theorem 4.2.4 is less conservative than the corresponding criteria in (26, Han).

Example 4.5.2

We consider the time-delay system (4.12) where the nonlinearity $\psi(.,.)$ satisfy condition (4.2), the matrices A_0, A_1, B, C_0, C_1 are given by (4.43), and the matrix G is given by

$$G = \begin{bmatrix} 0.8 \\ -1.2 \end{bmatrix}.$$

If we consider the Example 4.5.1 with $h = 3.3 > h_{\max} = 2.615$, then the open-loop system is unstable.
Using Theorem 4.3.1, we find a control state feedback such that the closed-loop system is stable. For $\lambda_1 = -0.07$ and $\lambda_2 = 0.09$, the solutions of LMI (4.15) are

$$\overline{P} = \begin{bmatrix} 0.178 & 0.01 \\ 0.01 & 0.084 \end{bmatrix}, \overline{Q} = \begin{bmatrix} 0.194 & 0.018 \\ 0.018 & 0.237 \end{bmatrix},$$

$$\overline{R} = \begin{bmatrix} 1.58 & -0.451 \\ -0.451 & 1.976 \end{bmatrix}, Y = \begin{bmatrix} -0.021 & 0.278 \end{bmatrix},$$

and $\epsilon = 1.2$. So, the origin of controlled system (4.12) is stabilizable by the linear state feedback (4.13) where

$$N = \begin{bmatrix} -0.322 & 3.348 \end{bmatrix}.$$

4.6 Conclusion

The problem of absolute stability of a class of time-delay systems with sector-bounded nonlinearity have been considered. New delay-dependent stability criteria with the sector condition have been proposed. This criteria have been formulated in the form of linear matrix inequalities (LMI), and derived with employing some integral inequalities. We are used this new criterion to construct a delay-dependent dynamic observer-based output feedback control such that the observer error system have been presented as the feedback interconnection of a linear system and a state-dependent multivariable sector-bounded nonlinearity. Finally, some numerical examples are given to illustrate the applicability of the results.

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