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# PROBABILITÉ DE SURVIE D'UN PROCESSUS DE BRANCHEMENT DANS UN ENVIRONNEMENT ALÉATOIRE MARKOVIENT

Yinna Ye

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*La pensée n'est qu'un éclair au milieu de la nuit. Mais c'est cet éclair qui est tout.*

— *Henri Poincaré*

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‘La pensée n’est qu’un éclair au milieu de la nuit. Mais c’est cet éclair qui est tout.’

Merci encore à tous ceux qui m’ont aidée, merci la France !

## REMERCIEMENTS

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# Résumé

L'objet de cette thèse est d'étudier la probabilité de survie d'un processus de branchement en environnement aléatoire markovien et d'étendre dans ce cadre les résultats connus en milieu aléatoire indépendant et identiquement distribué. Le coeur de l'étude repose sur l'utilisation des théorèmes limites locaux pour une marche aléatoire centrée  $(S_n)_{n \geq 0}$  sur  $\mathbb{R}$  à pas markoviens et pour  $(m_n)_{n \geq 0}$ , où  $m_n = \min(0, S_1, \dots, S_n)$ .

Pour traiter le cas d'un environnement aléatoire markovien, nous développons dans un premier temps une étude des théorèmes locaux pour une chaîne semi-markovienne à valeurs réelles en améliorant certains résultats déjà connus et développés initialement par E. L. Presman (voir [22] et [23]). Nous utilisons ensuite ces résultats pour l'étude du comportement asymptotique de la probabilité de survie d'un processus de branchement critique en environnement aléatoire markovien.

Les résultats principaux de cette thèse ont été annoncés dans les *Comptes Rendus de l'Académie des Sciences* ([21]). Un article plus détaillé est soumis pour publication dans la revue *Journal of Theoretical Probability*. Dans cette thèse, nous précisons les énoncés de ces théorèmes et détaillons leurs démonstrations.

**Mots clés :** Théorème limite local, chaîne de Markov, marche aléatoire à pas markoviens, processus de branchement, processus de branchement au milieu aléatoire



## RÉSUMÉ

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# Abstract

The purpose of this thesis is to study the survival probability of a branching process in markovian random environment and expand in this framework some known results which have been developed for a branching processus in i.i.d. random environment, the core of the study is based on the use of the local limit theorem for a centered random walk  $(S_n)_{n \geq 0}$  on  $\mathbb{R}$  with markovian increasements and for  $(m_n)_{n \geq 0}$ , where  $m_n = \min(0, S_1, \dots, S_n)$ .

In order to treat the case of a markovian random environment, we establish firstly a local limit theorem for a semi-markovian chain on  $\mathbb{R}$ , which improves certain results developed initially by E. L. Presman (see [22] and [23]). We then use these results to study the asymptotic behavior of a critical branching process in markovian environment.

The main results of this thesis are announced in *Comptes Rendus de l'Académie des Sciences* ([21]). A detailed paper is submitted for publication in the *Journal of Theoretical Probability*. In this thesis, we specify all the statements and the detailed proofs.

**Keywords :** Local limit theorem, markovian chain, random walk with markovian increasement, branching process, branching process in random environment

ABSTRACT

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# Notations

## Avertissement : constante variable

Dans les inégalités, les lettres  $C$  ou  $c$  désignent des constantes réelles positives dont la valeur est susceptible de changer d'une ligne à l'autre.

## Ensembles particuliers

- Si  $X$  est un espace topologique, la tribu borélienne  $\mathfrak{B}(X)$  est la plus petite tribu sur  $X$  qui contient tous les ensembles ouverts ; en particulier  $\mathfrak{B}(\mathbb{R})$  désigne la tribu borélienne de  $\mathbb{R}$ .
- $\mathcal{P}(E)$  est l'ensemble des parties de  $E$  et  $\mathcal{P}(\mathbb{R})$  désigne l'ensemble des probabilités sur  $\mathbb{R}$ .

## Abréviations

- *i.i.d* signifie *indépendantes et identiquement distribuées*
- *p.s.* signifie *presque-sûrement*
- *a.s* signifie *almost surely*

## Notation

- Pour toute matrice  $M = (m_{i,j})_{i,j}$  et tout entier  $n \geq 1$ , on note  $M^n$  la puissance  $n$ -ième de  $M$  ; de plus, l'élément de  $M^n$  situé à l'intersection de la ligne  $i$  et de la colonne  $j$  est noté  $m_{i,j}^{(n)}$ .
- Soit  $A$  un ensemble, alors  $A^c$  désigne le complémentaire de  $A$ .

## NOTATIONS

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# Introduction

L'objet de cette thèse est d'étudier la probabilité de survie d'un processus de branchement en environnement aléatoire markovien et d'étendre dans ce cadre les résultats connus en milieu aléatoire i.i.d.

Pour traiter le cas d'un environnement aléatoire markovien, nous développons dans un premier temps une étude des théorèmes locaux pour une marche aléatoire semi-markovienne sur  $\mathbb{R}$  en améliorant certains résultats déjà connus et développés initialement par E. L. Presman. Nous utilisons ensuite ces résultats pour l'étude du comportement asymptotique de la probabilité de survie d'un processus de branchement en milieu markovien.

## 0.1 Théorème limite local pour le minimum d'une marche aléatoire centrée sur $\mathbb{R}$

La question du théorème limite local pour le minimum d'une marche aléatoire sur  $\mathbb{R}$  consiste à déterminer un équivalent, lorsque  $n$  tend vers à l'infini, de la probabilité pour que le minimum d'une marche aléatoire jusqu'à l'instant  $n$  se trouve dans un intervalle borné de  $\mathbb{R}$ .

### 0.1.1 Cas d'une marche aléatoire à pas i.i.d.

Soit  $\{Y_i\}_{i \geq 1}$  une suite de v.a. i.i.d à valeurs dans  $\mathbb{R}$ , on pose

$$S_0 = 0, \quad S_n = Y_1 + \cdots + Y_n, \quad n \geq 1,$$

et

$$m_n = \min(S_0, S_1, \dots, S_n).$$

Notons que  $m_n$  est aussi l'opposé du maximum de la marche aléatoire  $(-S_n)_{n \geq 0}$ ; le maximum et le minimum d'une marche aléatoire centrée ont, au signe près, des comportements analogues, on trouve le plus souvent dans la littérature des résultats concernant la suite  $M_n := \max(0, -S_1, \dots, -S_n)$ .

Citons par exemple le résultat de M. V. Kozlov ([19]) :

**Théorème 0.1.1.** ([19]) Si  $\mathbb{E}Y_1 = 0$  et  $0 < \mathbb{E}Y_1^2 < +\infty$ , alors quand  $n \rightarrow +\infty$ ,

$$\mathbb{P}(M_n \leq x) \sim V(x)/\sqrt{n}, \quad x \geq 0,$$



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---

où  $V(x)$  est une fonction harmonique strictement positive sur  $\mathbb{R}^+$  pour la marche aléatoire centrée  $(S_n)_{n \geq 0}$  ( autrement dit,  $V(x)$  vérifie l'égalité :  $\forall x \in \mathbb{R}^+, V(x) = \mathbb{E}V(x + Y_1)$ ).

De plus, on a

$$V(x) \sim Cx, \quad x \rightarrow +\infty$$

et pour tout  $\lambda > 0$ ,

$$\mathbb{E}(e^{-\lambda M_n}) \sim \tilde{V}(\lambda)/\sqrt{n}, \quad n \rightarrow +\infty,$$

où  $\tilde{V}(\lambda) = \int_0^{+\infty} e^{-\lambda x} dV(x)$ .

La démonstration de ce théorème est basée sur la théorie de fluctuation des marches aléatoires.

### 0.1.2 Cas d'une marche aléatoire semi-markovienne

Dans le cas d'une marche aléatoire semi-markovienne, cette question a été abordée par E. L. Presman dans [22]. Cet auteur s'intéresse plus précisément au couple  $(M_n, S_n)$  et donne une version d'un théorème limite local pour ce couple (avec un comportement en  $\frac{1}{n^{3/2}}$ ) et en explicitant le comportement spatial à l'aide d'une gaussienne qui apparaît naturellement dans ce contexte. Il annonce seulement des résultats analogues pour le processus  $(M_n)_{n \geq 1}$  sans détailler la preuve. Nous utilisons l'approche qu'il a développée pour étendre dans un premier temps le Théorème 0.1.1 au cas d'une marche aléatoire semi-markovienne, puis montrer que la limite obtenue n'est pas dégénérée.

Soient  $(\Omega, \mathcal{F}, \mathbb{P})$  un espace probablisé,  $E = \{1, 2, \dots, N\}$  un ensemble fini et  $X = (X_n)_{n \geq 0}$  une chaîne de Markov irréductible apériodique sur  $E$ . On note  $P = (p_{i,j})_{i,j}$  la probabilité de transition de  $X$  et  $\nu$  l'unique mesure de probabilité invariante. Soit  $(F(i, j, dt))_{i,j \in E}$  une famille de mesures de probabilité sur  $\mathbb{R}$ . Considérons une suite de variables aléatoires  $(Y_n)_{n \geq 0}$  définie sur  $(\Omega, \mathcal{F}, \mathbb{P})$ , à valeurs dans  $\mathbb{R}$ , telle que  $(Y_n, X_n)_{n \geq 0}$  est une chaîne de Markov, dont le noyau de transition  $\tilde{P}$  est défini par :  $\forall (x, i) \in \mathbb{R} \times E$ ,  $\forall A \in \mathcal{B}(\mathbb{R})$  et  $\forall j \in E$ ,

$$\tilde{P}((x, i), A \times \{j\}) = \mathbb{P}(Y_{n+1} \in A, X_{n+1} = j / Y_n = x, X_n = i) = p_{i,j} F(i, j, A),$$

où l'on a posé  $S_0 = 0$  et  $S_n = Y_1 + \dots + Y_n$ , pour  $n \geq 1$  et  $m_n = \min(S_0, S_1, \dots, S_n)$ .

Le processus  $(S_n)_{n \geq 0}$  apparaît comme une généralisation, dans ce cadre markovien, des marches aléatoires à pas indépendants. Il est important de remarquer que ce processus n'est pas une chaîne de Markov ; pour pallier à cette difficulté, nous considérons plutôt le processus  $(S_n, X_n)_{n \geq 0}$  à valeurs dans  $\mathbb{R} \times E$  ; il n'est pas difficile de voir que cette suite est une chaîne de Markov, le fait de garder en mémoire la position  $X_n$  permettant de contrôler la loi de l'accroissement  $Y_n$ .

On introduit l'espace canonique

$$(\Omega = \{\mathbb{R} \times E\}^{\mathbb{N}}, (B(\mathbb{R}) \otimes \mathcal{P}(E))^{\otimes \mathbb{N}}, (S_n, X_n)_{n \geq 1}, (\mathbb{P}_{(x,i)})_{(x,i) \in \mathbb{R} \times E})$$

associé à la chaîne de Markov  $(S_n, X_n)_{n \geq 0}$ . Rappelons que pour tout  $(x, i) \in \mathbb{R} \times E$ , la mesure de probabilité  $\mathbb{P}_{(x,i)}$  est définie par : pour tout  $n \geq 1$  et toute fonction  $f$  :

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ALÉATOIRE CENTRÉE SUR  $\mathbb{R}$

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$(\mathbb{R} \times E)^{n+1} \rightarrow \mathbb{R}$  mesurable et bornée, on a

$$\begin{aligned} & \int_{\Omega} f((S_0(\omega), X_0(\omega)), (S_1(\omega), X_1(\omega)), \dots, (S_n(\omega), X_n(\omega))) \mathbb{P}_{(x,i)}(d\omega) \\ & := \sum_{(j_1, j_2, \dots, j_n) \in E^n} p_{i,j_1} \cdots p_{j_{n-1}, j_n} \int_{\mathbb{R}^n} f((x, i), (x + y_1, j_1), \dots, (x + y_1 + \dots + y_n, j_n)) \\ & \qquad \qquad \qquad F(i, j_1, dy_1) \cdots F(j_{n-1}, j_n, dy_n). \end{aligned}$$

L'espérance correspondant à la mesure de probabilité  $\mathbb{P}_{(x,i)}$  sera notée  $\mathbb{E}_{(x,i)}$ . Afin de ne pas alourdir le texte, la mesure  $\mathbb{P}_{(0,i)}$  sera notée  $\mathbb{P}_i$  et l'espérance correspondante  $\mathbb{E}_i$ .

Nous introduisons à présent les **hypothèses (H)** suivantes :

**H1** *Il existe  $\alpha > 0$ , tel que pour tout  $\lambda \in \mathbb{C}$  vérifiant  $|\Re \lambda| \leq \alpha$ , on ait,*

$$\max_{(i,j) \in E \times E} |\widehat{F}(i, j, \lambda)| < +\infty,$$

$$\text{où } \widehat{F}(i, j, \lambda) = \int_{\mathbb{R}} e^{\lambda t} F(i, j, dt).$$

**H2** *Il existe  $n_0 \geq 1$  et  $(i_0, j_0) \in E \times E$  tels que la mesure  $\mathbb{P}_{i_0}(X_{n_0} = j_0, S_{n_0} \in dx)$  possède une composante absolument continue par rapport à la mesure de Lebesgue sur  $\mathbb{R}$ .*

**H3**  $\sum_{i,j \in E \times E} \nu_i p_{i,j} \int_{\mathbb{R}} t F(i, j, dt) = 0$ , où  $\nu = (\nu_i)_{i \in E}$  est la mesure de probabilité  $P$ -invariante de  $(X_n)_{n \geq 0}$ .

L'hypothèse H1 précise simplement que les mesures  $(F(i, j, dt))_{i,j \in E}$  ont des moments exponentiels; elle est par exemple satisfaite lorsque ces mesures sont à support borné. L'hypothèse H2 est classique dans le cas i.i.d et conduit naturellement à la condition dite de "Cramer" portant sur le comportement à l'infini de la fonction caractéristique (plus précisément, on a alors  $\limsup_{t \rightarrow \infty} |\widehat{\mu}(t)| < 1$ , où  $\widehat{\mu}$  est la fonction caractéristique de la loi commune  $\mu$  dans le cas i.i.d). L'hypothèse H3 est une condition de centrage de la chaîne semi-markovienne. Le résultat de M. V. Kozlov se généralise alors comme suit :

**Théorème 0.1.2.** *Sous les hypothèses (H), pour tout  $(i, j) \in E \times E$  et tout  $x \geq 0$ , on a*

$$\lim_{n \rightarrow +\infty} \sqrt{n} \mathbb{P}_i(m_n \geq -x, X_n = j) = h_{i,j}(x) > 0,$$

où la fonction  $(x, i) \mapsto h_{i,j}(x)$  est une fonction harmonique <sup>(a)</sup> sur  $\mathbb{R}_+ \times E$  pour la chaîne semi-markovienne  $(S_n, X_n)_{n \geq 0}$ . De plus, il existe une constante  $\sigma^2 > 0$ , telle que

$$h_{i,j}(x) \sim \sqrt{\frac{2}{\sigma^2}} \nu_j x, \text{ lorsque } x \rightarrow +\infty.$$

La démonstration du Théorème 0.1.2 s'inspire largement de la factorization de E. L. Presman ([23]) et des méthodes d'analyse complexe; toutes ces notions seront développées dans le chapitre 1. Les éléments de démonstration et la comparaison avec le cas i.i.d seront discutés dans la section 0.3.

---

a. Plus précisément, pour tout  $j \in E$ , la fonction  $(x, i) \mapsto h_{i,j}(x)$  est  $\tilde{P}$ -harmonique et vérifie donc  $h_{i,j}(x) = \mathbb{E}_i[h_{X_1, j}(x + S_1)] = \mathbb{E}_i[h_{X_n, j}(x + S_n)]$ , pour tout  $n \geq 2$ .

## 0.2 Applications aux processus de branchement

Le théorème limite local pour le minimum d'une marche aléatoire s'applique à la résolution de problèmes concernant des processus de branchement en environnement aléatoire, en particulier dans l'étude du comportement asymptotique de la probabilité de survie correspondante.

Avant de décrire le processus de branchement en environnement aléatoire, nous rappelons quelques résultats classiques concernant le processus de Galton-Watson homogène. Nous présentons ensuite quelques modèles et résultats connus en environnement aléatoire stationnaire et ergodique (et non plus forcément i.i.d.). Dans la dernière partie de la section, on précise la notion de processus de branchement en environnement aléatoire markovien.

### 0.2.1 Processus de Galton-Watson homogène

L'étude du processus de Galton-Watson, introduit par Francis Galton en 1873 (voir aussi F. Galton [1889,1891]), est liée à de nombreuses applications dans les domaines de la physique et de la biologie. Le problème initial est celui de l'extinction des noms de familles nobles britanniques ; il a été étudié et développé par de nombreux auteurs depuis les années '60 (voir aussi T. E. Harris [15], D. Kendall [16], [17] et K. B. Athreya et P. E. Ney [3]). On veut modéliser à l'aide d'une suite de variables aléatoires  $\{Z_n\}_{n \geq 0}$  à valeurs entières l'extinction d'une famille donnée au fil des générations. Plus précisément, on pose  $Z_0 = 1$  et on note  $Z_n$  le nombre d'individus (mâles) de la génération  $n$  qui portent le nom de la famille considérée. Si  $(X_{n,i})_{n \geq 1}$  désigne le nombre de descendants masculins du  $i$ -ième individu mâle dans la génération  $n$ , on a donc, à la génération suivante

$$Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i}.$$

On suppose dans le cas classique abordé par F. Galton et H. Watson en 1875 que les variables aléatoires  $(X_{n,i})_{i \geq 1}$  sont mutuellement indépendantes, qu'elles sont de plus indépendantes de  $Z_n$  et qu'elles suivent la même loi de probabilité  $\mu := \sum_{k \geq 0} p_k \delta_k$  sur  $\mathbb{N}$ . Le

cas déterministe (i.e.  $p_k = 1$  pour un certain  $k \geq 0$ ) se traite de façon immédiate ; il en est de même lorsque  $p_0 + p_1 = 1$ . On suppose donc dans ce qui suit que  $p_0 + p_1 < 1$  et  $p_k \neq 1$ ,  $\forall k \geq 0$ .

La suite  $\{Z_n\}_{n \geq 0}$  est une chaîne de Markov sur  $\mathbb{N}$ , dont la probabilité de transition est donnée par

$$P(i, j) = \mathbb{P}\{Z_{n+1} = j / Z_n = i\} = \begin{cases} p_j^{*i}, & i \geq 1, j \geq 0; \\ \delta_{0,j}, & i = 0, j \geq 0. \end{cases}$$

où  $\delta_{i,j} = \begin{cases} 0, & \text{si } i \neq j, \\ 1, & \text{si } i = j. \end{cases}$  et  $\{p_k^{*i}\}_{k \geq 0}$  désigne la  $i$ -ième puissance de convolution de la probabilité  $\{p_k\}_{k \geq 0}$ . Cette chaîne de Markov est appelée *processus de Galton-Watson homogène*.

Soit  $g$  la fonction génératrice de la loi de probabilité  $\{p_k\}_{k \geq 0}$  définie par

$$\forall s \in [0, 1], \quad g(s) = \sum_{k=0}^{+\infty} s^k p_k.$$

La fonction génératrice de  $\{Z_n\}_{n \geq 0}$  est la  $n$ -ième composée itérée de  $g$  :

$$\forall s \in [0, 1], \quad g_n(s) = \mathbb{E}(s^{Z_n}) = \underbrace{g \circ \dots \circ g}_n(s),$$

Sous l'hypothèse  $\sum_{k \geq 0} k^2 p_k < +\infty$  et grâce au lemme d'Abel, on en déduit en particulier

$$m := \mathbb{E}(Z_1) = g'(1), \text{ et } \sigma^2 := \text{Var}(Z_1) = g''(1) + m - m^2.$$

Nous avons le

**Théorème 0.2.1** ([3]). *Si  $m \leq 1$ , alors  $\mathbb{P}(\lim_{n \rightarrow +\infty} Z_n = 0) = 1$ .  
Si  $m > 1$ , alors  $\mathbb{P}(\lim_{n \rightarrow +\infty} Z_n = 0) < 1$ .*

On dira que le processus de Galton-Watson est *sous-critique*, *critique*, ou *super-critique* selon que  $m < 1$ ,  $= 1$  ou  $> 1$ .

En 1938, A. Kolmogorov a précisé le comportement asymptotique de la probabilité de survie  $P(Z_n > 0)$  sous une hypothèse de moment d'ordre 2 sur  $Z_1$ . On a le

**Théorème 0.2.2** ([18]). *On suppose que la variable aléatoire  $Z_1$  est de carré intégrable et on pose  $\mathbb{E}(Z_1) = m$  et  $\text{Var}(Z_1) = \sigma^2$ .*

**i) Cas critique :** *si  $m = 1$ , alors*

$$\mathbb{P}(Z_n > 0) \sim \frac{2}{n\sigma^2}, \quad n \rightarrow +\infty.$$

**ii) Cas sous-critique :** *si  $m < 1$ , alors*

$$\mathbb{P}(Z_n > 0) \sim Cm^n, \quad n \rightarrow +\infty,$$

où  $C$  est une constante réelle strictement positive.

### 0.2.2 Quelques modèles classiques des processus de branchement en environnement aléatoire

Soit  $(E, \xi)$  un ensemble mesurable. On munit l'espace mesurable produit  $(\Omega, \mathcal{F}) = (E^{\mathbb{N}}, \xi^{\mathbb{N}})$  d'une probabilité  $\Pi$ . Tout élément  $\omega = (\omega_i)_{i \geq 0}$  de  $\Omega$  est une suite aléatoire à valeurs dans  $E$  et est appelé *processus d'environnement* de loi  $\Pi$ .

Supposons qu'à chaque point  $\theta$  de  $E$  soit associée une probabilité  $p_\theta$  sur  $\mathbb{N}$ , donnée par sa fonction génératrice

$$g_\theta(s) = \sum_{k=0}^{+\infty} p_\theta(k) s^k,$$

où  $(p_\theta(k))_{k \geq 0}$  est une suite de réels positifs tels que  $\sum_{k=0}^{+\infty} p_\theta(k) = 1$ .

On pose  $Z_0 = 1$  et on note  $(Z_n)_{n \geq 0}$ , le processus de Galton-Watson, non homogène dans le temps, dont la loi de reproduction des individus de la génération  $n \geq 1$  est

$$g_{\omega_n}(s) = \sum_{k=0}^{+\infty} p_{\omega_n}(k) s^k.$$

On note  $\mathbb{P}_\omega$  la probabilité sur  $\Lambda = \mathbb{N}^{\mathbb{N}}$  définissant la loi de  $(Z_n)_{n \geq 0}$  lorsque l'environnement  $\omega$  (c'est-à-dire toute la suite  $\omega = (\omega_i)_{i \geq 0}$ ) est fixé et  $\mathbb{P}$  la probabilité sur  $\Lambda \times \Omega$  définie par

$$\mathbb{P} = \int \mathbb{P}_\omega \otimes \delta_\omega d\Pi(\omega).$$

Pour toute variable aléatoire  $X : \Lambda \times \Omega \rightarrow \mathbb{R}^+$ , on a alors

$$\int X(x, \omega) d\mathbb{P}(x, \omega) = \int_\Omega \int_\Lambda X(x, \omega) \mathbb{P}_\omega(dx) d\Pi(\omega).$$

On note  $\mathbb{E}_\omega$  l'espérance sous la probabilité  $\mathbb{P}_\omega$  et  $\mathbb{E}$  celle sous la probabilité  $\mathbb{P}$ . Par définition, pour  $\omega \in \Omega$  fixé, nous avons

$$Z_0 = 1 \text{ et } Z_n = \sum_{i=1}^{Z_{n-1}} X_{n,i}, \quad n \geq 1,$$

où  $\{X_{n,i}\}$  sont des v.a. à valeurs dans  $\mathbb{N}$  mutuellement indépendantes et dont la loi correspond à la fonction génératrice  $g_{\omega_n}$ . Par conséquent,

$$\forall s \in [0, 1], \quad \mathbb{E}_\omega(s^{Z_n}) = g_{\omega_0} \circ \dots \circ g_{\omega_{n-1}}(s),$$

d'où

$$\mathbb{E}_\omega(Z_n) = g'_{\omega_0}(1) g'_{\omega_1}(1) \dots g'_{\omega_{n-1}}(1),$$

$$\mathbb{P}_\omega(Z_n = 0) = g_{\omega_0} \circ g_{\omega_1} \circ \dots \circ g_{\omega_{n-1}}(0)$$

et

$$\mathbb{P}_\omega(Z_n > 0) = 1 - g_{\omega_0} \circ g_{\omega_1} \circ \dots \circ g_{\omega_{n-1}}(0).$$

Nous allons considérer différents processus de branchement en environnement aléatoire selon la nature de la probabilité  $\Pi$  et donner un bref aperçu des résultats connus.

Lorsque  $\Pi$  est la probabilité produit :  $\Pi = \mathbb{Q}^{\otimes \mathbb{N}}$ , où  $\mathbb{Q}$  est une loi de probabilité sur  $E$ , on parle d'*environnement i.i.d.* Plus généralement, quand  $\Pi$  est ergodique et stationnaire pour le shift sur  $E^{\mathbb{N}}$ , on parle d'*environnement stationnaire et ergodique* ; la suite des fonctions génératrices  $\{g_n\}_{n \geq 0}$  est alors ergodique et stationnaire. Ce modèle a été étudié par K. B. Athreya et S. Karlin ([1], [2]). La dichotomie *critique ou sous-critique/super-critique* s'énonce dans ce cadre comme suit

**Théorème 0.2.3** ([1]). *Si  $\mathbb{E}[\ln g'_0(1)] \leq 0$ , alors*

$$\mathbb{P}_\omega(\lim_{n \rightarrow +\infty} Z_n = 0) = 1 \quad \text{\textit{\Pi - p.s.}}$$

*Si  $0 < \mathbb{E}[\ln g'_0(1)] < +\infty$  et  $\mathbb{E}[-\ln(1 - g_0(0))] < +\infty$ , alors*

$$\mathbb{P}_\omega(\lim_{n \rightarrow +\infty} Z_n = 0) < 1, \quad \text{\textit{\Pi - p.s.}}$$

On retrouve ainsi un résultat similaire à celui de Galton-Watson classique (cf. Théorème 0.2.1). On dit que le processus de branchement  $\{Z_n\}_{n \geq 0}$  est *sous-critique*, *critique* et *super-critique* selon que  $\mathbb{E}[\ln g'_0(1)] < 0$ ,  $= 0$  ou  $> 0$ .

Dans [11] et [12], J. Geiger et G. Kersting ont étudié dans les cas sous-critique et critique et en environnement i.i.d le comportement asymptotique de la probabilité de survie du processus  $\{Z_n\}_{n \geq 0}$ . Y. Guivarc'h, E. Le Page et Q. Liu ont étendu ces résultats sous des hypothèses significativement plus faibles et ont montré le

**Théorème 0.2.4** ([14]). *Supposons qu'il existe  $\varepsilon > 0$  tel que  $\mathbb{E}\left[\left(\frac{g''_0(1)}{g'_0(1)^2}\right)^\varepsilon\right] < +\infty$  avec de plus  $\mathbb{E}[\ln g'_0(1)] = 0$  et  $0 < \mathbb{E}[\ln g'_0(1)]^2 < +\infty$ . Il existe alors une constante  $C \in ]0, +\infty[$ , telle que*

$$\mathbb{P}(Z_n > 0) \sim \frac{C}{\sqrt{n}}, \quad \text{lorsque } n \rightarrow +\infty.$$

Soulignons que le fait que l'environnement soit aléatoire (i.e.  $\mathbb{P}(g_0 \neq g) > 0$ , pour tout  $g \in G$ ) modifie la décroissance à l'infinie de la probabilité de survie (cf. Théorème 0.2.2) et que l'on observe alors un ralentissement dans l'extinction du processus.

Pour tout nombre réel  $a > 0$ , notons  $\ln^+ a = \max(0, \ln a)$ .

**Théorème 0.2.5** ([14]). *Si  $\mathbb{E}[\ln g'_0(1)] < 0$ , il y a trois cas à considérer :*

**a) cas fortement sous-critique :** *si  $\mathbb{E}[g'_0(1) \ln g'_0(1)] < 0$  et  $\mathbb{E}[Z_1 \ln^+ Z_1] < +\infty$ , il existe une constante  $C \in ]0, +\infty[$ , telle que*

$$\mathbb{P}(Z_n > 0) \sim C(\mathbb{E}Z_1)^n, \quad \text{lorsque } n \rightarrow +\infty.$$

**b) cas intermédiairement sous-critique :** *si  $\mathbb{E}[g'_0(1) \ln g'_0(1)] = 0$  et  $\mathbb{E}[g''_0(1)] < +\infty$ , il existe une constante  $C \in ]0, +\infty[$ , telle que*

$$\mathbb{P}(Z_n > 0) \sim C(\mathbb{E}Z_1)^n n^{-1/2}, \quad \text{lorsque } n \rightarrow +\infty.$$

**c) cas faiblement sous-critique :** *si  $\mathbb{E}[g'_0(1) \ln g'_0(1)] > 0$  et  $\mathbb{E}g''_0(1) < +\infty$ , il existe une constante  $C \in ]0, +\infty[$  et  $\rho \in [0, 1]$ , telles que*

$$\mathbb{P}(Z_n > 0) \sim C\rho^n n^{-3/2}, \quad \text{lorsque } n \rightarrow +\infty.$$

Dans le cas où il n'y a pas d'environnement aléatoire (i.e.  $g_0 = g$   $\mathbb{P}$ -p.s.), seul le cas a) est à considérer et on retrouve l'énoncé du théorème 0.2.2 (cas sous-critique).

### 0.2.3 Processus de branchement en environnement aléatoire markovien

Soit  $X = (X_n)_{n \geq 0}$  une chaîne de Markov irréductible apériodique à valeurs dans un espace d'états fini  $E$ , de probabilité de transition  $P = (p_{i,j})_{i,j \in E}$  et de mesure invariante  $\nu$ . On note  $G$  l'ensemble des fonctions génératrices des probabilités sur  $\mathbb{N}$ , muni de la topologie de la convergence simple sur  $[0, 1]$ . Soient  $\Omega = \{G \times \mathbb{R} \times E\}^{\mathbb{N}}$ ,  $\mathcal{F} = (\mathcal{B}(G) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{P}(E))^{\mathbb{N}}$  et  $(F(i, j, dt))_{i,j \in E}$  une famille de mesures de probabilité sur  $G$ . Pour toute suite de fonctions génératrices  $(g_k)_{k \geq 0}$  telle que  $0 < g'_k(1) < +\infty$  pour tout  $k \geq 0$ , on pose

$$S_n = S_0 + \sum_{k=0}^{n-1} \ln g'_k(1), \quad n \geq 1, \quad \text{où } S_0 \text{ est une variable aléatoire fixée.}$$

Considérons la chaîne markovienne  $(\tilde{M}_n)_{n \geq 0} = (g_n, S_n, X_n)_{n \geq 0} \in \{G \times \mathbb{R} \times E\}^{\mathbb{N}}$ , dont le noyau de transition  $Q$  est donné par

$$Q\{(g_0, x, i), A \times \{j\}\} = p_{i,j} \int_G 1_A[(h, x + \ln g'_1(1))] F(i, j, dh),$$

pour tout  $(g_0, x, i) \in G \times \mathbb{R} \times E$  et tout  $A \in \mathcal{B}(G) \otimes \mathcal{B}(\mathbb{R})$ , tout  $j \in E$  (dans le chapitre 2, on considèrera plutôt le processus d'environnement  $M_n := (g_n, X_n)_{n \geq 0}$  qui contient en fait toute l'information nécessaire pour contrôler l'évolution du processus). La chaîne  $(\tilde{M}_n)_{n \geq 0}$  est appelée *processus d'environnement*.

Pour tout  $(g_0, x, i) \in G \times \mathbb{R} \times E$ , on note  $\mathbb{P}_{(g_0, x, i)}$  l'unique mesure de probabilité sur  $(\Omega, \mathcal{F})$  telle que pour tout  $n \geq 1$  et toute fonction  $f : (G \times \mathbb{R} \times E)^{n+1} \rightarrow \mathbb{R}$  mesurable et bornée, on ait

$$\begin{aligned} & \int_{\Omega} f(\tilde{M}_0(\omega), \tilde{M}_1(\omega), \dots, \tilde{M}_n(\omega)) \mathbb{P}_{(g_0, x, i)}(d\omega) \\ &= \sum_{(j_1, j_2, \dots, j_n) \in E^n} p_{i, j_1} \cdots p_{j_{n-1}, j_n} \int_{G^n} f((g_0, x, i), (g_0 \circ g_1, x + \ln g'_1(1), j_1), \dots, \\ & \quad (g_0 \circ g_1 \circ \cdots \circ g_n, x + \sum_{k=1}^n \ln g'_k(1), j_n)) F(i, j_1, dg_1) \cdots F(j_{n-1}, j_n, dg_n). \end{aligned}$$

Pour simplifier, nous noterons  $\mathbb{P}_i$  la mesure de probabilité  $\mathbb{P}_{(id, 0, i)}$  et  $\mathbb{E}_i$  l'espérance qui lui correspond.

Considérons un processus de branchement non homogène  $(Z_n)_{n \geq 0}$ , tel que  $Z_0 = 1$  dont la fonction génératrice de  $Z_n$  est donnée par

$$G_n(s) = g_0 \circ g_1 \circ \cdots \circ g_{n-1}(s), \quad 0 \leq s < 1.$$

Conditionnellement à l'environnement  $(\tilde{M}_n)_{n \geq 0}$ , la probabilité de survie à l'instant  $n$  de  $(Z_n)_{n \geq 0}$  est donc égale à

$$q_n = 1 - G_n(0).$$

On introduit les notations suivantes : pour tout  $n \geq 1$ ,  $0 \leq k \leq n-1$  et  $s \in [0, 1[$ , on pose

$$\begin{aligned} f_k(s) &:= \frac{1}{1 - g_k(s)} - \frac{1}{g'_k(1)(1-s)}, \\ g_{k,n} &:= g_k \circ g_{k+1} \circ \cdots \circ g_{n-1}, \quad g_{n,n} = id, \\ \eta_{k,n} &:= \begin{cases} f_k(g_{k+1,n}(0)), & 0 \leq k \leq n-1, \\ 1, & k = n, \\ 0, & k \geq n+1. \end{cases} \end{aligned}$$

En effet, si  $g''_k(1) < \infty$  pour tout  $k \geq 0$ , alors on peut prolonger le domaine de définition de  $f_k$  à  $[0, 1]$ , en posant que  $f_k(1) := \frac{g''_k(1)}{2(g'_k(1))^2}$  (voir aussi Remark 2.2.1).

Avec ces notations, on obtient la formule suivante :

$$q_n^{-1} = \sum_{k=0}^n \eta_{k,n} \exp(-S_k), \quad \mathbb{P}_i - \text{p.s.} \quad (1)$$

Supposons désormais que les hypothèses (H) sont satisfaites. Un processus de branchement vérifiant l'hypothèse H3 est appelé *critique*. D'après [1], l'hypothèse H3 entraîne que

$$\mathbb{P}_\nu(Z_n = 0) \rightarrow 1, \text{ lorsque } n \rightarrow +\infty.$$

Le support de  $\nu$  étant l'ensemble  $E$  tout entier, on en déduit aisément que

$$\forall i \in E, \quad \mathbb{P}_i(Z_n = 0) \rightarrow 1, \text{ lorsque } n \rightarrow +\infty.$$

De l'étude du minimum d'une marche aléatoire à pas markoviens, on déduit le résultat suivant qui précise la propriété d'extinction à l'infini des processus  $\{Z_n\}_{n \geq 0}$ .

**Théorème 0.2.6.** *Sous les hypothèses (H), pour tout  $(i, j) \in E \times E$ , il existe une constante  $\beta_{i,j} > 0$ , telle que*

$$\lim_{n \rightarrow +\infty} \sqrt{n} \mathbb{P}_i(Z_n > 0, X_n = j) = \beta_{i,j}. \quad (2)$$

## 0.3 Sur la factorisation de Wiener Hopf et son extension au cas markovien

### 0.3.1 Le cas où les variables $(Y_i)_{i \geq 1}$ sont i.i.d.

Considérons ici le cas où les variables aléatoires  $Y_1, Y_2, \dots$  sont indépendantes et identiquement distribuées et à valeurs réelles. On rappelle que  $S_0 = 0, S_n = S_0 + Y_1 + \dots + Y_n$  et on pose  $m_n = \min(S_0, S_1, \dots, S_n)$ . L'obtention d'un théorème limite local pour la suite  $(m_n)_{n \geq 1}$  nécessite dans un premier temps l'étude du comportement asymptotique de la queue de la distribution des temps de visite des demi-droites  $\mathbb{R}^+$  et  $\mathbb{R}^{*-}$ . Posons en effet  $T^+ := \inf\{n \geq 1 : S_n \geq 0\}$  et  $T^{*-} := \inf\{n \geq 1 : S_n < 0\}$ ; en précisant le premier instant



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$k \in \{0, \dots, n\}$  auquel la suite  $S_0, S_1, \dots, S_n$  atteint son minimum, on obtient l'expression suivante : pour tout réel  $\lambda > 0$

$$\mathbb{E}[e^{\lambda m_n}] = \sum_{k=0}^n \mathbb{E}\left[e^{\lambda S_k}; T^+ > k\right] \mathbb{P}[T^{*-} > n - k].^b \quad (3)$$

L'étude de la suite  $(\mathbb{E}[e^{\lambda m_n}])_{n \geq 0}$  nécessite donc au préalable celles du comportement asymptotique des suites  $(\mathbb{E}[e^{\lambda S_n}; T^+ > n])_{n \geq 0}$  et  $(\mathbb{P}[T^{*-} > n])_{n \geq 0}$ . Elles ont été menées par différents auteurs, on trouvera par exemple dans [20] les deux résultats suivants : il existe des constantes  $a > 0$  et  $b(\lambda) > 0$  ( dont on peut préciser la valeur à l'aide de quantités de nature probabiliste qui apparaissent dans ce contexte) telles que

$$\begin{aligned} \mathbb{P}[T^{*-} > n] &= \frac{a}{\sqrt{n}}(1 + \varepsilon(n)) \\ \text{et} \quad \mathbb{E}\left[e^{\lambda S_n}; T^+ > n\right] &= \frac{b(\lambda)}{n^{3/2}}(1 + \varepsilon(n)) \end{aligned}$$

avec  $\lim_{n \rightarrow +\infty} \varepsilon(n) = 0$ . On peut alors conclure qu'il existe une constante  $c(\lambda) > 0$  telle que

$$\mathbb{E}[e^{\lambda m_n}] \sim \frac{c(\lambda)}{\sqrt{n}} \quad \text{lorsque } n \rightarrow +\infty,$$

en utilisant le lemme élémentaire suivant dont on trouvera la référence dans [20] :

**Lemma 0.3.1.** *Soient  $(\alpha_n)_{n \geq 0}$  and  $(\beta_n)_{n \geq 0}$  deux suites de réels positifs telles que*

- $\lim_{n \rightarrow +\infty} \sqrt{n} \alpha_n = \alpha > 0$ ,
- $\sum_{n=0}^{+\infty} \beta_n = B < +\infty$ ,
- la suite  $(n\beta_n)_{n \geq 0}$  est bornée.

$$\text{Alors} \quad \lim_{n \rightarrow +\infty} \sqrt{n} \sum_{k=0}^n \alpha_k \beta_{n-k} = \alpha B.$$

On constatera en particulier qu'il n'est pas besoin pour appliquer ce lemme de préciser que  $\mathbb{E}\left[e^{\lambda S_n}; T^+ > n\right] \sim \frac{b(\lambda)}{n^{3/2}}$  mais seulement que la série correspondante est convergente et son terme général en  $O(\frac{1}{n})$ . Cependant, la question de la convergence de cette série n'est pas facile à trancher et dans le cas de la marche aléatoire à pas indépendants elle repose essentiellement sur l'utilisation de l'identité suivante, dite parfois de 'Spitzer'

$$\forall z \in \mathbb{C}, |z| < 1, \quad \sum_{n \geq 0} z^n \mathbb{E}[e^{\lambda S_n}; T^+ > n] = \exp\left(\sum_{n=1}^{+\infty} \frac{z^n}{n} \mathbb{P}[e^{\lambda S_n}; S_n < 0]\right). \quad (4)$$

En effet, une fois cette égalité acquise, on peut appliquer le théorème limite local classique pour une marche aléatoire centrée sur  $\mathbb{R}$  pour vérifier, avec un argument de monotonie sur  $[0, 1]$ , que

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b. nous devrions écrire en fait  $\mathbb{E}\left[e^{\lambda S_k} \times 1_{[T^+ > k]}\right]$  mais pour alléger les notations nous avons opté pour l'écriture  $\mathbb{E}\left[e^{\lambda S_k}; T^+ > k\right]$ . Nous adoptons ce principe dans la suite de cette thèse.

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- la série  $\sum_{n \geq 0} \mathbb{E}[e^{\lambda S_n}; T^+ > n] = \sum_{n=1}^{+\infty} \frac{1}{n} \mathbb{P}[e^{\lambda S_n}; S_n < 0]$  converge ;
- son terme général est en  $O(\frac{1}{n})$  ; ce fait s'obtient en utilisant le lemme élémentaire suivant

**Lemma 0.3.2.** *Soient  $(\alpha_n)_{n \geq 0}$  and  $(\beta_n)_{n \geq 0}$  deux suites de réels positifs telles que*

$$\sum_{n=1}^{+\infty} \alpha_n z^n = \exp\left(\sum_{n=1}^{+\infty} \beta_n z^n\right).$$

*Si la suite  $(n^{3/2} \beta_n)_{n \geq 1}$  est bornée, il en est de même pour la suite  $(n^{3/2} \alpha_n)_{n \geq 1}$ .*

L'égalité (4) repose essentiellement sur le fait que  $\mathbb{R}$  est un groupe abélien ; donnons à présent les grandes lignes formelles qui permettent de la démontrer.

La factorisation dite de Wiener-Hopf, et dont on trouvera une présentation dans les livres ([9] et [8]) est un outil essentiel pour l'étude des fluctuations d'une marche aléatoire sur  $\mathbb{R}$  de loi  $\mu$  ; de façon synthétique, elle peut s'écrire,

$$\delta_0 - \mu = (\delta_0 - \mu^+) * (\delta_0 - \mu^{*-})$$

où  $\mu^{*-}$  désigne la loi d'entrée dans  $\mathbb{R}^{*-}$  de la marche (autrement dit la loi de  $S_{T^{*-}}$ ),  $\mu^+$  désigne la loi d'entrée dans  $\mathbb{R}^+$  (autrement dit la loi de  $S_{T^+}$ ) et  $\delta_0$  la masse de Dirac en 0. Si l'on s'intéresse plus précisément à la loi des couples  $(T^{*-}, S_{T^{*-}})$  et  $(T^+, S_{T^+})$ , on utilise la version raffinée de type "espace-temps" de cette factorisation, à savoir

$$\delta_0 - z\mu = \left(\delta_0 - \mathbb{E}[z^{T^+} \delta_{S_{T^+}}]\right) * \left(\delta_0 - \mathbb{E}[z^{T^{*-}} \delta_{S_{T^{*-}}}\right]$$

valide pour tout  $z \in \mathbb{C}, |z| < 1$ . En appliquant cette formule aux caractères  $x \mapsto e^{\lambda x}$  de  $\mathbb{R}$ , (ce qui est possible dès que  $\lambda \in i\mathbb{R}$  et même pour  $|\operatorname{Re} \lambda|$  petit sous une hypothèse de moments exponentiels), cette identité s'écrit sous la forme

$$1 - z\varphi(\lambda) = \left(1 - \mathbb{E}[z^{T^+} e^{\lambda S_{T^+}}]\right) \left(1 - \mathbb{E}[z^{T^{*-}} e^{\lambda S_{T^{*-}}}\right] \quad (5)$$

où  $\varphi$  est la transformée de Laplace  $\lambda \mapsto \mathbb{E}[e^{\lambda Y_1}]$  de la loi  $\mu$ .

C'est alors que le caractère abélien de  $(\mathbb{R}, +)$  intervient de façon essentielle pour obtenir l'identité de Spitzer ; on peut en effet écrire formellement

$$\delta_0 - z\mu = \exp\left(-\sum_{n=1}^{+\infty} \frac{z^n}{n} \mu^{*n}\right), \quad (c)$$

et cette dernière expression se décompose en

$$\exp\left(-\sum_{n=1}^{+\infty} \frac{z^n}{n} \mu^{*n}\right) = \exp\left(-\sum_{n=1}^{+\infty} \frac{z^n}{n} \mu^{*n} 1_{\mathbb{R}^+}\right) * \exp\left(-\sum_{n=1}^{+\infty} \frac{z^n}{n} \mu^{*n} 1_{\mathbb{R}^{*-}}\right),$$

---

c. on utilise l'identité  $1 - z = \exp(\ln(1 - z)) = \exp\left(-\sum_{n \geq 1} \frac{z^n}{n}\right)$

### 0.3. SUR LA FACTORISATION DE WIENER HOPF ET SON EXTENSION AU CAS MARKOVIEN

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ce qui permet d'identifier les facteurs suivants

$$\begin{aligned}\delta_0 - \mathbb{E}[z^{T^{*-}} \delta_{S_{T^{*-}}} ] &= \exp\left(-\sum_{n=1}^{+\infty} \frac{z^n}{n} \mu^{*n} 1_{\mathbb{R}^{*-}}\right) \\ \text{et } \delta_0 - \mathbb{E}[z^{T^+} \delta_{S_{T^+}} ] &= \exp\left(-\sum_{n=1}^{+\infty} \frac{z^n}{n} \mu^{*n} 1_{\mathbb{R}^+}\right)\end{aligned}$$

(on utilise alors le fait que ces mesures sont portées respectivement par  $\mathbb{R}^{*-}$  et  $\mathbb{R}^+$ , ce qui permet de les identifier).

On introduit ensuite la suite  $(T_n^{*-})_{n \geq 0}$  des instants successifs de records strictement descendants de la marche  $(S_n)_{n \geq 0}$  définie par

$$T_0^{*-} = 0, \quad T_n^{*-} := \inf\{k > T_{n-1}^{*-} / S_k < S_{T_{n-1}^{*-}}\}, \quad \text{pour } n \geq 1 \text{ (d)};$$

à partir de l'identité précédente, on peut écrire, pour  $|z| < 1$ ,

$$\begin{aligned}\exp\left(\sum_{n=1}^{+\infty} \frac{z^n}{n} \mu^{*n} 1_{\mathbb{R}^{*-}}\right) &= \left(\delta_0 - \mathbb{E}[z^{T^{*-}} \delta_{S_{T^{*-}}} ]\right)^{-1} \\ &= \sum_{n \geq 0} \mathbb{E}[z^{T^{*-}} \delta_{S_{T^{*-}}} ]^{*n} \\ &= \sum_{n \geq 0} \mathbb{E}[z^{T_n^{*-}} \delta_{S_{T_n^{*-}}} ], \quad \text{les variables } (T_{k+1}^{*-} - T_k^{*-}, S_{T_{k+1}^{*-}} - S_{T_k^{*-}}) \\ &\quad \text{étant indépendantes et de même loi que } (T^{*-}, S_{T^{*-}}) \\ &= \sum_{k \geq 0} z^k \mathbb{E}[\exists n \geq 0 / k = T_n^{*-}; \delta_{S_k}] \\ &= \sum_{k \geq 0} z^k \mathbb{E}[S_k < S_{k-1} \cdots S_k < S_0; \delta_{S_k}] \\ &= \sum_{k \geq 0} z^k \mathbb{E}[Y_k < 0, Y_k + Y_{k-1} < 0, \dots, Y_k + \dots + Y_1 < 0; \delta_{S_k}] \\ &= \sum_{k \geq 0} z^k \mathbb{E}[T^+ > k; \delta_{S_k}].\end{aligned}$$

La dernière égalité provenant du fait que les  $k$ -uplets  $(Y_1, Y_2, \dots, Y_k)$  et  $(Y_k, Y_{k-1}, \dots, Y_1)$  ont la même loi. En évaluant ces mesures en la fonction  $x \mapsto e^{\lambda x}$  (avec  $\lambda \geq 0$ ), qui est un caractère de  $(\mathbb{R}, +)$  on obtient l'identité de Spitzer (4).

L'égalité  $\delta_0 - z\mu = \exp\left(-\sum_{n \geq 1} \frac{z^n}{n} \mu^{*n}\right)$  et le fait que  $(Y_1, \dots, Y_k)$  et  $(Y_k, \dots, Y_1)$  aient

la même loi jouent un rôle essentiel et disparaissent immédiatement quand les variables  $Y_i$  sont en dépendance markovienne. Nous devons donc utiliser un raisonnement alternatif qui évite cette identité, cette approche a été initiée par M.S. Bratiichuk de façon peu détaillée dans les années '90 ([5] et [6]), nous en expliquons les grandes lignes dans le paragraphe suivant.

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d. en particulier  $T_1^{*-} = T^{*-}$

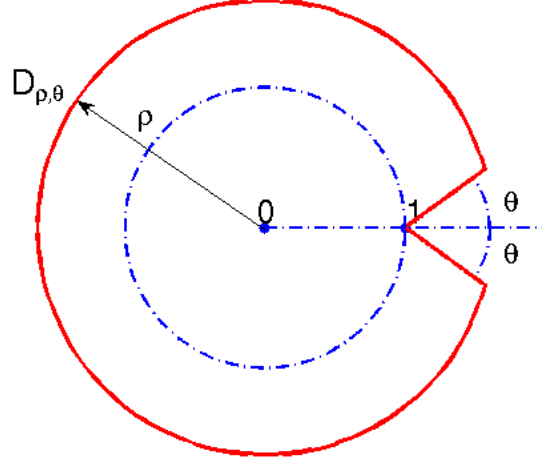


FIGURE 1 – Le domaine  $D_{\rho, \theta}$  dans le plan complexe

### 0.3.2 Stratégie de la démonstration dans le cadre markovien

Dans le chapitre 1 de cette thèse nous adaptions au cadre markovien l'étude décrite précédemment permettant d'évaluer le comportement asymptotique des suites  $\left(\mathbb{E}[e^{\lambda S_n}; T^+ > n]\right)_{n \geq 0}$  et  $(\mathbb{P}[T^{*-} > n])_{n \geq 0}$ ; l'étude du comportement de la suite  $(\mathbb{E}[e^{\lambda m_n}])_{n \geq 0}$  en découle via l'égalité (1.53) (voir lemme 1.3.1, chapitre 1) qui traduit en terme de fonction génératrice la formule de convolution de Spitzer (4).

Dans le cadre markovien, on suppose que les variables  $Y_i$  possèdent des moments exponentiels; leur transformée de Laplace n'est plus une fonction mais devient une matrice  $P(\lambda)$  admettant une valeur propre dominante simple  $k(\lambda)$ .

La fonction  $1 - z\hat{\mu}(\lambda)$  est modifiée en la matrice  $I - zP(\lambda)$  dont on cherche à contrôler le spectre en fonction de  $z$ . Vient ensuite l'identification des singularité de la résolvante  $(I - zP(\lambda))^{-1}$  de  $P(\lambda)$ . Cette étude est menée de façon détaillée dans la paragraphe 1.2 et repose sur un argument de factorisation (le théorème de préparation de Weierstrass) qui permet de ramener la singularité de la résolvante à celle d'un trinôme du second degré. On suit ici l'approche de ([5]) en détaillant l'analyse complexe qui y est menée.

Le paragraphe 1.3 est consacré à la factorisation de la matrice  $I - zP(\lambda)$ ; on commence par une version de type probabiliste pour  $|z| < 1$ ; on obtient la formule (1.54) suivante

$$I - zP(\lambda) = (I - \mathcal{P}B_z^*(\lambda))(I - \mathcal{N}^*C_z(\lambda))$$

(on renvoie le lecteur au paragraphe 1.3 pour la définition et la signification probabiliste de chacun des deux facteurs). On étend ensuite le domaine de validité de cette identité via des arguments généraux dus à E. L Presman, le but étant en particulier

- de démontrer l'analyticité de sa résolvante et de ses deux facteurs  $(I - \mathcal{P}B_z^*(\lambda))^{-1}$  et

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$(I - \mathcal{N}^*C_z(\lambda))^{-1}$  sur un domaine (voir le dessin 1) de type

$$D_{\rho,\theta} := \{z; z \neq 1, |\arg(z - 1)| > \theta > 0, |z| < \rho\}$$

avec  $\rho > 1$  et  $\theta \in ]0, \pi/2[$  ( $\arg$  désigne ici l'argument principal d'un nombre complexe);

- préciser le type de singularité de cette résolvante au voisinage de  $(z, \lambda) = (1, 0)$  et de voir comment cette singularité se propage à chacun des deux facteurs.

On montre en particulier que, pour  $\lambda > 0$  et  $z \in D_{\rho,\theta}$ , la fonction  $z \mapsto (I - \mathcal{P}B_z^*(\lambda))^{-1}$  admet une limite lorsque  $z \mapsto 1$  tandis que la fonction  $z \mapsto (I - \mathcal{P}B_z^*(0))^{-1}$  se comporte en  $\sqrt{1-z}$ . Ceci nous permet de conclure grâce au résultat suivant, de type théorème Tauberien, du à P. Flajolet et A. Odlyzko :

**Lemma 0.3.3** ([10]). *Soit  $z \mapsto G(z) = \sum_{n=0}^{+\infty} g_n z^n$  une fonction de la variable complexe satisfaisant les trois conditions suivantes*

- $G$  est analytique sur  $D_{\rho,\theta}$ ;
- $\lim_{\substack{z \in D_{\rho,\theta} \\ z \rightarrow 1}} \sqrt{1-z}G(z) = C > 0,$

alors

$$g_n \sim \frac{C}{\sqrt{\pi n}}, \quad n \rightarrow +\infty.$$

On trouvera dans [5] et [6] les l'éléments et étapes essentielles de l'analyse menée ici, mais de façon très peu détaillée; il était donc nécessaire de préciser un grand nombre d'arguments dans l'approche de M. S. Bratiichuk et en particulier ceux issus de la factorisation de E. L. Pressman. L'utilisation des estimations locales des potentiels mis en jeu dans cette étude dans le but d'obtenir des théorèmes de type "limite local" n'avait pas été amorcée par ces auteurs et les résultats obtenus dans ce cadre sont nouveaux. Dans le dernier chapitre nous appliquons ces techniques pour étudier la probabilité de survie pour des processus de branchement en environnement markovien dans le cas critique.

# Chapitre 1

## Théorème limite local pour le minimum d'une marche aléatoire centrée sur $\mathbb{R}$ et à pas markoviens

### 1.1 Introduction and main results

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and  $E$  be a finite set with  $N$  elements. Assume that  $X = (X_n)_{n \geq 0}$  is an irreducible and aperiodic Markov chain, defined on  $(\Omega, \mathcal{F}, \mathbb{P})$ , with values in  $E$  and with transition probability  $P = (p_{i,j})_{i,j \in E}$ . The chain  $X$  admits a unique invariant probability measure denoted by  $\nu$ . Let  $(F(i, j, dt))_{i,j \in E}$  be a family of probability measures on  $\mathbb{R}$ . Consider a sequence of  $\mathbb{R}$ -valued random variables  $(Y_n)_{n \geq 0}$  defined on  $(\Omega, \mathcal{F}, \mathbb{P})$ , such that  $(Y_n, X_n)_{n \geq 0}$  is a Markov chain on  $\mathbb{R} \times E$  with transition probability  $\tilde{P}$ , defined by:  
for any  $(u, i) \in \mathbb{R} \times E$ , any Borel set  $A \subset \mathbb{R}$  and  $j \in E$ ,

$$\tilde{P}\left((u, i), A \times \{j\}\right) = \mathbb{P}(Y_{n+1} \in A, X_{n+1} = j | Y_n = u, X_n = i) = p_{i,j} F(i, j, A).$$

Such a chain  $(Y_n, X_n)_{n \geq 0}$  is called a *semi-markovian chain*, since once the family  $(F(i, j, dt))_{i,j \in E}$  is fixed, the transitions of this chain is controlled by  $(X_n)_{n \geq 0}$ .

Set

$$S_0 = 0, S_n = S_0 + Y_1 + \cdots + Y_n \quad \text{and} \quad m_n = \min(S_0, S_1, \cdots, S_n).$$

We thus consider the canonical probability space  $\left((\mathbb{R} \times E)^{\mathbb{N}}, (\mathcal{B}(\mathbb{R}) \otimes \mathcal{P}(E))^{\otimes \mathbb{N}}, (\mathbb{P}_{(u,i)})_{(u,i) \in \mathbb{R} \times E}\right)$  associated with  $(Y_n, X_n)_{n \geq 0}$  and, for any  $(u, i) \in \mathbb{R} \times E$ , we denoted by  $\mathbb{E}_{(u,i)}$  the expectation with respect to  $\mathbb{P}_{(u,i)}$ . To simplify our notations, we will denote  $\mathbb{P}_{(0,i)}$  by  $\mathbb{P}_i$  and  $E_{(0,i)}$  by  $\mathbb{E}_i$ .

In the case when  $E$  reduces to one point, the random variable  $S_n$  is the sum of  $n$  independent and identically distributed random variables on  $\mathbb{R}$ . In this case, if  $(S_n)_{n \geq 0}$  is supposed to be centered, aperiodic with a finite variance, then for all continuous functions

$\varphi$  with compact support on  $\mathbb{R}_-$ , one gets

$$\lim_{n \rightarrow +\infty} \sqrt{n} \mathbb{E}(\varphi(m_n)) = C > 0,$$

with  $C$  a constant depending on  $\varphi$  (see [19] for instance).

The first goal of this paper is to extend the so-called local limit theorem for the process  $(m_n, X_n)_{n \geq 0}$  associated with the semi-markovian chain  $(S_n, X_n)_{n \geq 0}$  defined above. We assume once and for all the following **hypotheses (H)**:

**H1** *there exists  $\alpha > 0$ , such that for all  $\lambda \in \mathbb{C}$  with  $|\Re \lambda| \leq \alpha$ , we have*

$$\max_{(i,j) \in E \times E} |\widehat{F}(i,j,\lambda)| < +\infty, \quad \text{where } \widehat{F}(i,j,\lambda) = \int_{\mathbb{R}} e^{\lambda t} F(i,j,dt);$$

**H2** *there exist  $n_0 \geq 1$  and  $(i_0, j_0) \in E \times E$ , such that the measure  $\mathbb{P}_{i_0}(X_{n_0} = j_0, S_{n_0} \in dx)$  has an absolutely continuous component with respect to the Lebesgue measure  $dx$  on  $\mathbb{R}$ ;*

**H3**  $\mathbb{E}_\nu(S_1) = \sum_{(i,j) \in E \times E} \nu_i p_{i,j} \int_{\mathbb{R}} t F(i,j,dt) = 0$ , where  $\nu = (\nu_i)_{i \in E}$  is the stationary distribution of the Markov chain  $X = (X_n)_{n \geq 0}$ .

We have

**Theorem 1.1.1.** *Under the hypotheses (H), there exists a constant  $\sigma^2 > 0$ , such that for all  $(i,j) \in E \times E$  and  $\lambda > 0$ ,*

$$\sqrt{n} \mathbb{E}_i(e^{\lambda m_n}, X_n = j) \xrightarrow{n \rightarrow +\infty} \frac{H_{i,j}(\lambda)}{\sqrt{\pi}}, \quad (1.1)$$

where  $H_{i,j}(\lambda) > 0$  for all  $\lambda > 0$  and

$$\lim_{\lambda \rightarrow 0^+} \lambda H_{i,j}(\lambda) = \sqrt{\frac{2}{\pi \sigma^2}} \nu_j. \quad (1.2)$$

It will be also convenient to state this result under the following form:

**Theorem 1.1.2.** *For all  $(i,j) \in E \times E$ , one gets*

$$\lim_{n \rightarrow +\infty} \sqrt{n} \mathbb{P}_i(m_n \geq -x, X_n = j) = h_{i,j}(x), \quad (1.3)$$

where the functions  $(x,i) \mapsto h_{i,j}(x)$  are harmonic for  $(S_n, X_n)_{n \geq 0}$  and satisfy

- for any  $i, j \in E$ ,  $x \mapsto h_{i,j}(x)$  is increasing;
- $h_{i,j}(x) > 0$  for  $x \geq 0$ .

Moreover,

$$h_{i,j}(x) \sim x \sqrt{\frac{2}{\sigma^2}} \nu_j, \quad \text{as } x \rightarrow +\infty.$$

**Definition 1.1.1.** *The function  $(x,i) \mapsto h_{i,j}(x)$  are said to be harmonic (or  $\tilde{\mathbb{P}}$ -harmonic) for the process  $(S_n, X_n)_{n \geq 0}$ , if for a fixed  $j \in E$  we have for any  $(x,i) \in \mathbb{R} \times E$ ,  $h_{i,j}(x)$  satisfies*

$$h_{i,j}(x) = \mathbb{E}_i \left[ h_{X_1,j}(x + S_n) \right] = \mathbb{E}_i \left[ h_{X_n,j}(x + S_n) \right], \quad \forall n \geq 2.$$

As a corollary of the above theorem, we obtain the following recurrence property for the process  $(m_n)_{n \geq 0}$ :

$$\forall x > 0, \forall i \in E, \sum_{n \geq 0} \mathbb{P}_i(m_n \geq -x) = +\infty.$$

With similar arguments, we can also specify the asymptotic behavior, as  $n \rightarrow +\infty$ , of the sequence

$$\left( \mathbb{E}_i(e^{\lambda m_n - \mu S_n}, X_n = j) \right)_{n \geq 0}$$

for any  $\lambda > \mu > 0$ ; in the case when the  $(Y_n)$  are i.i.d (that is the case when  $E$  is reduced to one point), we know that  $\lim_{n \rightarrow +\infty} n^{3/2} \mathbb{E}(e^{\lambda m_n - \varepsilon S_n}, X_n = j)$  does exist and is  $> 0$ . In the markovian situation we study here, a similar result should hold with the same exponent  $3/2$  which appears after a derivation; unfortunately, as far as we understand, we are not able to decide whether or not this limit does not vanish. Nevertheless, the tools used to prove Theorem 1.1.1 and Theorem 1.1.2 allow us to state the following “transitional result”:

**Theorem 1.1.3.** *For  $0 < \varepsilon < \lambda$  small enough and for all  $(i, j) \in E \times E$ ,*

$$\sum_{n=0}^{+\infty} \mathbb{E}_i[e^{\lambda m_n - \varepsilon S_n}, X_n = j] < +\infty.$$

The local limit theorems 1.1.1, 1.1.2 and Theorem 1.1.3 have several simple consequences, which are of interest. These are natural generalizations of classical local limit theorems for  $(m_n)_{n \geq 0}$ , in the case when  $(S_n)_{n \geq 0}$  is a random walk on  $\mathbb{R}$  with i.i.d increments ([19],[20]). A typical such application is to study the asymptotic behavior of the survival probability of a critical branching process in an i.i.d random environment ([11], [14]). Analogous results, under appropriate conditions, hold therefore for a branching process in a markovian environment.

## 1.2 On the spectrum of the semi-markovian chain

Let  $M_N(\mathbb{C})$  denote the vector space containing all  $N \times N$  matrices with values in  $\mathbb{C}$ . Throughout the text, we consider that the norm on  $M_N(\mathbb{C})$  is sub-multiplicative; that is to say, for any  $A, B \in M_N(\mathbb{C})$ , we have

$$\|AB\| \leq \|A\| \times \|B\|.$$

For any  $\lambda \in \mathbb{C}$  such that  $|\Re \lambda| \leq \alpha$ , consider the  $\mathbb{C}$ -valued  $N \times N$  matrix  $P(\lambda)$  defined by

$$P(\lambda) = \left( P(\lambda)_{i,j} \right)_{i,j \in E}, \text{ with } P(\lambda)_{i,j} = p_{i,j} \widehat{F}(i, j, \lambda) = p_{i,j} \int_{\mathbb{R}} e^{\lambda t} F(i, j, dt).$$

Using Markov property, it is easy to verify that for any  $n \geq 1$ ,  $|\Re \lambda| < \alpha$ ,

$$P^n(\lambda) = \left( P^{(n)}(\lambda)_{i,j} \right)_{i,j} = \left( \mathbb{E}_i[e^{\lambda S_n}, X_n = j] \right)_{i,j}.$$



In particular,  $P(0)$  is equal to the transition matrix  $P$  of the Markov chain  $(X_n)_{n \geq 0}$  (and  $P^n(0) = P^n = (p_{i,j}^{(n)})_{i,j \in E}$ ). Its spectral radius is equal to 1 since  $P$  is stochastic; furthermore, since  $P(0)$  is aperiodic, the eigenvalue 1 is the unique eigenvalue with modulus 1 and it is simple with  $e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$  as a associated eigenvector. According to Perron-

Frobenius theorem, there thus exists a unique vector  $\nu = \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_N \end{pmatrix}$  with positive coefficients such that  $\sum_{i=1}^N \nu_i = 1$  and  ${}^t \nu P(0) = {}^t \nu$  (the vector  ${}^t \nu$  may be identified as a probability measure on  $E$ ). So we have

$$P = \Pi + R,$$

where

- $\Pi$  is a matrix of rank 1 given by

$$\Pi = (\Pi_{i,j})_{i,j \in E} = \begin{pmatrix} \nu_1 & \nu_2 & \cdots & \nu_N \\ \vdots & \vdots & & \vdots \\ \nu_1 & \nu_2 & \cdots & \nu_N \end{pmatrix},$$

- $R$  is a matrix with spectral radius  $< 1$ ,
- $\Pi$  and  $R$  satisfy the relation  $\Pi R = R \Pi = 0$ .

According to the analytical perturbation theory, for  $|\lambda|$  small enough,  $P(\lambda)$  has a unique eigenvalue  $k(\lambda)$  of modulus equal to the spectral radius of  $P(\lambda)$  and this eigenvalue is simple.

Therefore, there exists a unique vector  $\nu(\lambda) = \begin{pmatrix} \nu_1(\lambda) \\ \vdots \\ \nu_N(\lambda) \end{pmatrix}$  such that

$$\sum_{i=1}^N \nu_i(\lambda) = 1$$

and  ${}^t \nu(\lambda) P(\lambda) = k(\lambda) {}^t \nu(\lambda)$ ; we can thus also define a unique vector  $e(\lambda) = \begin{pmatrix} e_1(\lambda) \\ \vdots \\ e_N(\lambda) \end{pmatrix}$

such that  $P(\lambda)e(\lambda) = k(\lambda)e(\lambda)$  and  ${}^t \nu(\lambda)e(\lambda) = 1$ . More precisely, we have the following theorem:

**Theorem 1.2.1.** *Under hypotheses H1 and H2, there exist  $\gamma_0 < \frac{1}{3}$  and  $0 < \alpha_0 \leq \alpha$  such that*

1. *If  $\lambda \in \Delta_{\alpha_0} := \{\lambda \in \mathbb{C}; |\Re \lambda| \leq \alpha_0, |\Im \lambda| \leq \alpha_0\}$ , then*

$$P(\lambda) = k(\lambda)\Pi(\lambda) + R(\lambda), \tag{1.4}$$

where

- $k(\lambda) \in \mathbb{C}$  is the dominant eigenvalue of  $P(\lambda)$ , and satisfies

$$|1 - k(\lambda)| \leq \gamma_0;$$

- $\Pi(\lambda)$  is a rank 1 matrix, which corresponds to the spectral projector on the 1-dimensional eigenspace associated with  $k(\lambda)$  and is given by

$$\Pi(\lambda) = \left( e_i(\lambda) \nu_j(\lambda) \right)_{i,j \in E};$$

- $R(\lambda)$  is a matrix with spectral radius  $r(R(\lambda)) < 1 - 2\gamma_0$ .
- The matrices  $\Pi(\lambda)$  and  $R(\lambda)$  satisfy the following relation

$$\Pi(\lambda)R(\lambda) = R(\lambda)\Pi(\lambda) = 0. \quad (1.5)$$

Furthermore, the maps  $\lambda \mapsto k(\lambda)$ ,  $\lambda \mapsto \Pi(\lambda)$  and  $\lambda \mapsto R(\lambda)$  are analytic on the set  $\Delta_{\alpha_0}$ .

2. There exists  $\alpha'_0 \leq \alpha_0$  and  $\chi \in ]0, 1[$  such that if  $|\Re \lambda| \leq \alpha'_0$  and  $|\Im \lambda| \geq \alpha_0$ , the spectral radius of  $P(\lambda)$  satisfies the inequality

$$r(P(\lambda)) \leq \chi < 1. \quad (1.6)$$

The proof of this theorem will be stated in Appendix B.

It follows from this Theorem that, for any fixed  $\lambda \in \Delta_{\alpha_0}$ , the resolvent function  $z \mapsto (I - zP(\lambda))^{-1}$  is analytic on an open disc in  $\mathbb{C}$  with center 0 and radius  $1 + \varepsilon_0$  for some  $\varepsilon_0 > 0$ , excepted at the points  $z$  satisfying the equation  $zk(\lambda) = 1$ . In the following subsection, we will give an explicit expression of the solutions of this equation, in order to give some more information of the singular points of the holomorphic function  $z \mapsto (I - zP(\lambda))^{-1}$ .

The hypotheses (H) particularly allow us to control the local expansion at 0 of the eigenvalue  $k(\lambda)$ .

**Remark 1.2.1.** From now on and for all we will assume  $\alpha_0 = \alpha'_0$ ; by (1.4), for  $\lambda \in \mathbb{C}$  s.t.  $|\Re \lambda| \leq \alpha_0$ , one gets, for  $|z| \leq 1 + \varepsilon_0$  and  $zk(\lambda) \neq 0$

- if  $|\Im \lambda| \leq \alpha_0$  (i.e.  $\lambda \in \Delta_{\alpha_0}$ ) then

$$(I - zP(\lambda))^{-1} = \frac{zk(\lambda)}{1 - zk(\lambda)} \Pi(\lambda) + \sum_{n=0}^{+\infty} z^n R^n(\lambda). \quad (1.7)$$

- if  $|\Im \lambda| \geq \alpha_0$  then

$$r(P(\lambda)) \leq \chi \quad (1.8)$$

for some  $\chi \in ]0, 1[$ .

### 1.2.1 Local expansion of the spectral radius $k(\lambda)$ of $P(\lambda)$

In this section, for any function  $F$  from  $E \times E$  into the set  $\mathcal{P}(\mathbb{R})$  of probability measures on  $\mathbb{R}$  and any  $\lambda \in \mathbb{C}$ , we set

$$P(\lambda, F) := \left( P(\lambda, F)_{i,j} \right)_{i,j}, \text{ with } P(\lambda, F)_{i,j} := p_{i,j} \int_{\mathbb{R}} e^{\lambda t} F(i, j, dt),$$

where the matrix  $(p_{i,j})_{i,j \in E}$  is the transition probability of an irreducible and aperiodic Markov chain  $X = (X_n)_{n \geq 0}$  as defined at the beginning of Section 1.1.

When there is no risk of confusion about the function  $F$ , we can omit the sign  $F$  in this formula. We will assume that  $F$  satisfies H1, i.e. for some  $\alpha > 0$  and for all  $\lambda \in \mathbb{C}$  such that  $|\Re \lambda| \leq \alpha$ ,  $\sup_{(i,j) \in E \times E} |\widehat{F}(i, j, \lambda)| < +\infty$ , where  $\widehat{F}(i, j, \lambda) = \int_{\mathbb{R}} e^{\lambda t} F(i, j, dt)$ .

According to Rellich's analytic perturbation theory of linear operators (see N. Dunford and J. Schwartz 1958, VII.6, [7]), we have for  $\lambda \in \Delta_{\alpha_0}$ ,

$$P(\lambda, F) = k(\lambda, F)\Pi(\lambda, F) + R(\lambda, F),$$

where

- $k(\lambda, F) \in \mathbb{C}$  is the dominant eigenvalue of  $P(\lambda, F)$ , and satisfies  $|1 - k(\lambda, F)| \leq \gamma_0$  for  $0 < \gamma_0 < \frac{1}{3}$ ; in the particular case when  $\lambda = 0$ , we get  $k(0, F) = 1$ ;
- $\Pi(\lambda, F)$  is a projection ( i.e.  $\Pi^2(\lambda, F) = \Pi(\lambda, F)$  ) on the 1-dimensional eigenspace associated with  $k(\lambda, F)$ , and in the particular case when  $\lambda = 0$ ,

$$\Pi(0, F) = \left( \Pi_{i,j} \right)_{i,j \in E} = \begin{pmatrix} \nu_1 & \nu_2 & \cdots & \nu_N \\ \vdots & \vdots & & \vdots \\ \nu_1 & \nu_2 & \cdots & \nu_N \end{pmatrix},$$

with  $\sum_{i \in E} \nu_i = 1$  and  $\forall i \in E, \nu_i > 0$ .

- $R(\lambda, F)$  is a matrix with spectral radius  $< 1$  and satisfies the relation

$$\Pi(\lambda, F)R(\lambda, F) = R(\lambda, F)\Pi(\lambda, F) = 0.$$

In particular, the function  $\lambda \mapsto k(\lambda, F)$  is analytic on  $\Delta_{\alpha_0}$ ; we now compute the first term of its local expansion.

We introduce the *mean matrix*  $M(F)$  associated with  $F$  which is defined by

$$M(F) = \left( M(F)_{i,j} \right)_{i,j}, \text{ with } M(F)_{i,j} = p_{i,j} \int_{\mathbb{R}} t F(i, j, dt).$$

We have the

**Lemma 1.2.1.**  $k'(0, F) = {}^t \nu M(F) e = \sum_{i,j \in E} \nu_i p_{i,j} \int_{\mathbb{R}} t F(i, j, dt)$ .

In the sequel, we will denote

$$\gamma(F) := {}^t \nu M(F) e = \sum_{i,j \in E} \nu_i p_{i,j} \int_{\mathbb{R}} t F(i, j, dt).$$

*Proof.* Since  $P(\lambda, F) = k(\lambda, F)\Pi(\lambda, F) + R(\lambda, F)$ , with  $\Pi(\lambda, F)R(\lambda, F) = R(\lambda, F)\Pi(\lambda, F) = 0$  and  $\Pi(\lambda, F)^2 = \Pi(\lambda, F)$ , we have  $\Pi(\lambda, F)P(\lambda, F) = k(\lambda, F)\Pi(\lambda, F)$ . Using the fact that  $k(0, F) = 1$ , the derivation of the quantities in the two hand-sides of this equality at the point  $\lambda = 0$  leads to

$$\Pi'(0, F)P(0, F) + \Pi(0, F)P'(0, F) = k'(0, F)\Pi(0, F) + \Pi'(0, F).$$

Using thus the equality  $P(0, F)e = e$ , one gets

$$\begin{aligned} \Pi(0, F)P'(0, F)e &= k'(0, F)\Pi(0, F)e \\ &= k'(0, F)e. \end{aligned} \tag{1.9}$$

As  $P'(0, F)_{i,j} = p_{i,j} \int_{\mathbb{R}} tF(i, j, dt)$ , the equality (1.9) implies that

$$\sum_{i,j \in E} \nu_i p_{i,j} \int_{\mathbb{R}} tF(i, j, dt) = k'(0, F).$$

□

**Remark 1.2.2.** Note that we also have  $k'(0, F) = \lim_{n \rightarrow +\infty} \frac{\mathbb{E}_i[S_n]}{n}$  for any  $i \in E$ . It may be obtain by a classical procedure, getting the derivate at 0 of the two members of the equality

$$\mathbb{E}_i[S_n] = k(\lambda, F)^n (\Pi(\lambda, F)e)_i + (R(\lambda, F)^n e)_i.$$

Again, if one derivates twice these two members, letting  $\lambda = 0$ , one obtains

$$k''(0, F) = \lim_{n \rightarrow +\infty} \frac{\mathbb{E}_i[S_n^2]}{n}.$$

**Corollary 1.2.1.** Under the hypothesis H1 and H3, we have  $k'(0) = 0$ .

*Proof.* This is a direct consequence of Lemma 1.2.1, since we suppose here that

$${}^t \nu M(F)e = \sum_{i,j \in E} \nu_i p_{i,j} \int_{\mathbb{R}} tF(i, j, dt) = 0.$$

□

To compute  $k''(0, F)$ , we need first to “center” the function  $F$  in the following sense:

**Definition 1.2.1.** Suppose that  $F = (F(i, j, dt))_{i,j \in E}$  and  $F' = (F'(i, j, dt))_{i,j \in E}$  are two finite families of probability measures on  $\mathbb{R}$ . One says that  $F'$  is **a-equivalent** to  $F$ , if there exists a vector  $u = (u_i)_{i \in E}$ , such that for any  $i, j \in E$  satisfying  $p_{i,j} \neq 0$ , one has

$$F'(i, j, dt) = \delta_{u_j - u_i} * F(i, j, dt).$$

This notion of equivalence is relevant since we have the

**Property 1.2.1.** 1. If  $F$  and  $F'$  are a-equivalent and satisfy hypothesis H1, then  $k(\cdot, F) = k(\cdot, F')$  on  $\Delta_{\alpha_0}$ .

2. For any  $F : E \times E \rightarrow \mathcal{P}(\mathbb{R})$  satisfying H1, there exists a function  $\mathfrak{F} : E \times E \rightarrow \mathcal{P}(\mathbb{R})$  which is  $a$ -equivalent to  $F$  and such that  $M(\mathfrak{F})e = \gamma(F)e = \gamma(\mathfrak{F})e$ .

*Proof.* 1. By the equality  $F'(i, j, dt) = \delta_{u_j - u_i} * F(i, j, dt)$ , for any  $\lambda \in \Delta_{\alpha_0}$  and any  $i, j \in E$ , we have

$$P(\lambda, F')_{i,j} = e^{\lambda(u_j - u_i)} P(\lambda, F)_{i,j}.$$

Therefore,

$$\begin{aligned} P^{(n)}(\lambda, F')_{i,j} &= e^{\lambda(u_j - u_i)} P^{(n)}(\lambda, F)_{i,j} \\ &= e^{\lambda(u_j - u_i)} \left( k^n(\lambda, F) \Pi(\lambda, F)_{i,j} + R^{(n)}(\lambda, F)_{i,j} \right). \end{aligned} \quad (1.10)$$

Set  $\Pi(\lambda, F, u) := \left( \Pi(\lambda, F, u)_{i,j} \right)_{i,j}$  with  $\Pi(\lambda, F, u)_{i,j} := e^{\lambda(u_j - u_i)} \Pi(\lambda, F)_{i,j}$ .

According to (1.10), for any  $\lambda \in \Delta_{\alpha_0}$ ,

$$\frac{P^{(n)}(\lambda, F')}{k^n(\lambda, F)} \longrightarrow \Pi(\lambda, F, u) \neq 0, \text{ as } n \rightarrow +\infty.$$

So for any  $\lambda \in \Delta_{\alpha_0}$ ,  $|k(\lambda, F)|$  is equal to the spectral radius  $|k(\lambda, F')|$  of  $P(\lambda, F')$ ; there thus exists  $\theta = \theta(\lambda)$  in  $[0, 2\pi[$  such that

$$k(\lambda, F) = e^{i\theta} k(\lambda, F'). \quad (1.11)$$

Let  $e(\lambda, F')$  be a non-null eigenfunction of the matrix  $P(\lambda, F')$ , corresponding to the eigenvalue  $k(\lambda, F')$ :

$$P^{(n)}(\lambda, F') e(\lambda, F') = k^n(\lambda, F') e(\lambda, F'). \quad (1.12)$$

Using (1.10), (1.11) and (1.12), one gets for any  $i \in E$ ,

$$\begin{aligned} k^n(\lambda, F') e(\lambda, F')_i &= \\ &= e^{-\lambda u_i} \left[ k^n(\lambda, F') e^{in\theta} \sum_j e^{\lambda u_j} \Pi(\lambda, F)_{i,j} e(\lambda, F')_j + \sum_j e^{\lambda u_j} R^{(n)}(\lambda, F)_{i,j} e(\lambda, F')_j \right]. \end{aligned} \quad (1.13)$$

Let  $i_\lambda \in E$  such that  $e(\lambda, F')_{i_\lambda} \neq 0$ , then

$$0 \neq e(\lambda, F')_{i_\lambda} = e^{in\theta} a(\lambda)_{i_\lambda} + b(\lambda, n)_{i_\lambda},$$

where

- $a(\lambda) := \left( a(\lambda)_i \right)_i$  with  $a(\lambda)_i = e^{-\lambda u_i} \sum_j e^{\lambda u_j} \Pi(\lambda, F)_{i,j} e(\lambda, F')_j$ ;
- $b(\lambda, n) := \left( b(\lambda, n)_i \right)_i$  with  $b(\lambda, n)_i = e^{-\lambda u_i} k(\lambda, F')^{-n} \sum_j e^{\lambda u_j} R^{(n)}(\lambda, F)_{i,j} e(\lambda, F')_j$ .

Note that  $\forall i \in E, \lim_{n \rightarrow +\infty} b(\lambda, n)_i = 0$ , so that

$$\lim_{n \rightarrow +\infty} e^{in\theta} = \frac{e(\lambda, F')_{i_\lambda}}{a(\lambda)_{i_\lambda}} \neq 0.$$

We can thus conclude that  $\theta = 0$ , and so  $k(\lambda, F) = k(\lambda, F')$  for any  $\lambda \in \Delta_{\alpha_0}$ .

2. Set  $v(F) := M(F)e - \gamma(F)e = M(F)e - ({}^t\nu M(F)e)e$ . Since  ${}^t\nu v(F)$  is null, the vector  $\tilde{u} := \sum_{n=0}^{+\infty} P^n v(F)$  exists and satisfies

$$\tilde{u} - P\tilde{u} = v(F) = M(F)e - \gamma(F)e. \quad (1.14)$$

For any  $i, j \in E$ , let's define a function  $\mathfrak{F} : E \times E \rightarrow \mathcal{P}(\mathbb{R})$  by

$$\mathfrak{F}(i, j, dt) = \delta_{\tilde{u}_j - \tilde{u}_i} * F(i, j, dt).$$

Then one obtains

$$M(\mathfrak{F})e = M(F)e + P\tilde{u} - \tilde{u}. \quad (1.15)$$

Using (1.14) and (1.15), one has  $M(\mathfrak{F})e = \gamma(F)e$  and  $\gamma(\mathfrak{F}) = {}^t\nu M(\mathfrak{F})e = {}^t\nu M(F)e = \gamma(F)$ .  $\square$

**Remark 1.2.3.** *The proof proposed here holds when the state space is not finite, under some additive assumptions on the spectrum of the operator  $P$ . In the case when  $F$  is finite, the two matrices  $P(\lambda, F)$  and  $P(\lambda, F')$  have the same spectrum and Assertion 1 thus follows. Indeed the equality  $P(\lambda, F)V = \mu V$  with  $V = (V_i)_{i \in E}$  implies  $P(\lambda, F')W = \mu W$  with  $W = (W_i)_{i \in E} = (e^{-\lambda u_i} V_i)_{i \in E}$ .*

Thank to this property, we are now able to compute  $k''(0)$ . We first introduce the *inertial matrix*  $\Sigma(F)$  associated with  $F$ , defined by

$$\Sigma(F) := \left( \Sigma(F)_{i,j} \right)_{i,j}, \text{ with } \Sigma(F)_{i,j} := p_{i,j} \int_{\mathbb{R}} t^2 F(i, j, dt).$$

**Property 1.2.2.** *Let  $\mathfrak{F} : E \times E \rightarrow \mathcal{P}(\mathbb{R})$  be such that  $\mathfrak{F}$  is  $a$ -equivalent to  $F$  and*

$$M(\mathfrak{F})e = \gamma(F)e.$$

*Then*

$$k''(0, F) = k''(0, \mathfrak{F}) = {}^t\nu \Sigma(\mathfrak{F})e.$$

*Proof.* We have

$$\Pi(\lambda, \mathfrak{F})P(\lambda, \mathfrak{F}) = k(\lambda, \mathfrak{F})\Pi(\lambda, \mathfrak{F}), \quad (1.16)$$

where  $k(\lambda, \mathfrak{F})$  is the unique eigenvalue of  $P(\lambda, \mathfrak{F})$  of maximum modulus with

$$k(0, \mathfrak{F}) = 1$$

and  $\Pi(\lambda, \mathfrak{F})$  is the corresponding eigenvector.

Consider the following Taylor's formula:

$$\begin{aligned} k(\lambda, \mathfrak{F}) &= 1 + \lambda k'(0, \mathfrak{F}) + \frac{\lambda^2}{2} k''(0, \mathfrak{F}) + o(\lambda^2), \\ \Pi(\lambda, \mathfrak{F}) &= \Pi(0, \mathfrak{F}) + \lambda \Pi'(0, \mathfrak{F}) + \frac{\lambda^2}{2} \Pi''(0, \mathfrak{F}) + o(\lambda^2), \\ P(\lambda, \mathfrak{F}) &= P(0, \mathfrak{F}) + \lambda M(\mathfrak{F}) + \frac{\lambda^2}{2} \Sigma(\mathfrak{F}) + o(\lambda^2). \end{aligned}$$

By the identification of the coefficients of order  $\lambda^2$  (1.16), we get

$$\Pi(0, \mathfrak{F})\Sigma(\mathfrak{F}) + 2\Pi'(0, \mathfrak{F})M(\mathfrak{F}) + \Pi''(0, \mathfrak{F})P(0, \mathfrak{F}) = \Pi''(0, \mathfrak{F}) + 2k'(0, \mathfrak{F})\Pi'(0, \mathfrak{F}) + k''(0, \mathfrak{F})\Pi(0, \mathfrak{F}).$$

Multiplying the matrices in the two sides of this equation with  $e$  and using the facts  $P(0, \mathfrak{F})e = e$ ,  $M(\mathfrak{F})e = k'(0, \mathfrak{F})e$  and  $\Pi(0, \mathfrak{F})e = e$ , one gets

$$k''(0, \mathfrak{F}) = {}^t\nu\Sigma(\mathfrak{F})e.$$

And  $k''(0, F) = k''(0, \mathfrak{F})$  is a direct consequence of the fact that  $k'(\cdot, F) = k'(\cdot, \mathfrak{F})$  on  $\Delta_{\alpha_0}$ .  $\square$

**Corollary 1.2.2.** *For any  $F : E \times E \rightarrow \mathcal{P}(\mathbb{R})$  satisfying H1, we have  $k''(0, F) = 0$  if and only if  $F$  is  $a$ -equivalent to  $\delta_{\{0\}}$ .*

*Proof.* Suppose that  $F : E \times E \rightarrow \mathcal{P}(\mathbb{R})$  satisfies H1, from Property 1.2.2, there exists  $\mathfrak{F} : E \times E \rightarrow \mathcal{P}(\mathbb{R})$  such that

$$k''(0, F) = k''(0, \mathfrak{F}) = {}^t\nu\Sigma(\mathfrak{F})e = \sum_{i,j \in E} \nu_i p_{i,j} \int t^2 \mathfrak{F}(i, j, dt).$$

So that  $k''(0, F) = 0$  if and only if  $\forall (i, j) \in E^2, \mathfrak{F}(i, j, \cdot) = \delta_{\{0\}}$ .  $\square$

**Corollary 1.2.3.** *Under the hypotheses (H), we have*

$$\sigma^2 := k''(0) > 0.$$

*Proof.* Suppose that  $k''(0) = 0$ . By the definition of the semi-Markovian chain  $(S_n, X_n)_{n \geq 0}$ , we have for a fixed  $i_0 \in E$ , and any  $n \geq 1$ ,

$$\mathbb{P}_{i_0}(S_n \in dx) = \sum_{(i_1, \dots, i_n) \in E^n} \left[ \prod_{k=0}^{n-1} p(i_k, i_{k+1}) \right] F(i_0, i_1, dx) * F(i_1, i_2, dx) * \dots * F(i_{n-1}, i_n, dx). \quad (1.17)$$

According to Corollary 1.2.2 and the fact that the support of  $\nu$  is  $E$ , the measures  $F(i, j, dx)$  is a Dirac measure for any  $(i, j) \in E \times E$  such that  $p_{i,j} > 0$ . So by Formula (1.17), for every  $i_0 \in E$  and every  $n \geq 1$ , the law  $P_{i_0}(S_n \in dx)$  is discrete. However, the hypothesis H2 implies that  $\mathbb{P}_{i_0}(S_{n_0} \in dx)$  has an absolutely component with respect to the Lebesgue measure on  $\mathbb{R}$ . This leads to a contradiction. The proof is complete.  $\square$

### 1.2.2 The equation $zk(\lambda) = 1$ for $z \in \mathbb{C}$ and $|\Re \lambda| \leq \alpha_0$

We consider here the equation

$$zk(\lambda) = 1, \quad \text{for } z \in \mathbb{C} \text{ and } |\Re \lambda| \leq \alpha_0. \quad (1.18)$$

It is shown in the previous section that  $k''(0) > 0$  under our conditions (H). Since  $\lambda \mapsto k(\lambda)$  is analytic on the open set  $\Delta_{\alpha_0}$ , one may assume that  $k''(\lambda) > 0$  for any  $\lambda \in ]-\alpha_0, \alpha_0[$ . By the implicit function theorem, for  $z \in \mathbb{R}$ , the equation (1.18) has at most

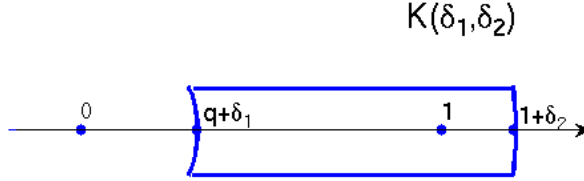


Figure 1.1: The region  $K(\delta_1, \delta_2)$  in the complex plane

two roots in a sub interval of  $[-\alpha_0, \alpha_0]$  ( still denoted by  $[-\alpha_0, \alpha_0]$  in order to simplify the notation). Set  $q = [\inf(k(-\alpha_0), k(\alpha_0))]^{-1}$ : one gets  $\min_{-\alpha_0 \leq \lambda \leq \alpha_0} k(\lambda) = k(0) = 1$ , since  $k'(0) = 0$ . The equation (1.18) with  $z \in [q, 1]$  has exactly one solution  $\lambda_-(z) \in [-\alpha_0, 0]$  and one solution  $\lambda_+(z) \in [0, \alpha_0]$ ; furthermore, these two solutions coincide if and only if  $z = 1$ , and  $\lambda_-(1) = \lambda_+(1) = 0$ .

For any  $\delta_1, \delta_2 > 0$  such that  $q + \delta_1 < 1$ , set (see the Figure 1.1)

$$K(\delta_1, \delta_2) := \{z : q + \delta_1 < |z| < 1 + \delta_2, \Re z > 0, |\Im z| < \delta_1\}.$$

We will describe in the following sections the local behavior of some functions of the complex variable  $z \in K(\delta_1, \delta_2)$  but with respect to the variable  $t := \sqrt{1-z}$ . In order to fix a principal determination of the function  $\sqrt{\cdot}$ , we introduce the subset  $K^*(\delta_1, \delta_2) \subset K(\delta_1, \delta_2)$  (see the Figure 1.2) defined by

$$K^*(\delta_1, \delta_2) := \{z, q + \delta_1 < |z| < 1 + \delta_2, \Re z > 0, |\Im z| < \delta_1, z \notin [1, 1 + \delta_2]\}.$$

Note that the map  $z \mapsto \sqrt{1-z}$  is well defined on  $K^*(\delta_1, \delta_2)$ .

By the analytic local inversion theorem, since  $k'(0) = 0$  and  $k''(0) > 0$ , one may choose  $\delta_1 \in ]0, 1-q[$  and  $\delta_2 > 0$  in such a way that the two functions  $z \mapsto \lambda_+(z)$  and  $z \mapsto \lambda_-(z)$ , defined a priori on  $]q + \delta_1, 1 + \delta_2[$ , admit an analytic expansion to the region  $K(\delta_1, \delta_2) \setminus \{1\}$  and these functions remain to be the solutions of (1.18) for  $z \in K(\delta_1, \delta_2) \setminus \{1\}$  and  $|\Re \lambda| \leq \alpha_0$ .

By the above, the functions  $z \mapsto \lambda_+(z)$  and  $z \mapsto \lambda_-(z)$  can be decomposed on  $K^*(\delta_1, \delta_2)$  as

$$\lambda_{\pm}(z) = \sum_{n=1}^{+\infty} (\pm 1)^n \alpha_n (1-z)^{n/2}, \quad (1.19)$$



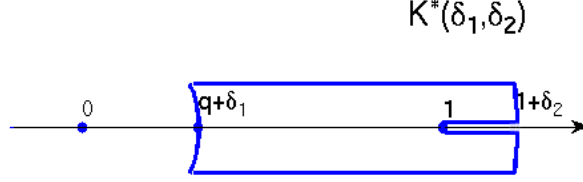


Figure 1.2: The region  $K^*(\delta_1, \delta_2)$  in the complex plane

where  $\alpha_n \in \mathbb{C}$  for any  $n \geq 1$ . On the other hand, for any  $\lambda$  in a neighborhood of 0, one has

$$k(\lambda) = 1 + \frac{k''(0)}{2!}\lambda^2 + \frac{k^{(3)}(0)}{3!}\lambda^3 + \dots \quad (1.20)$$

By the identification of the coefficients of the terms  $(1-z)$  and  $(1-z)^{3/2}$  in the two sides of the equality,

$$k(\lambda_+(z)) = \frac{1}{z} = \sum_{n=1}^{+\infty} (1-z)^n, \quad (1.21)$$

one obtains

$$\alpha_1 = \sqrt{\frac{2}{k''(0)}} \text{ and } \alpha_2 = -\frac{k^{(3)}(0)}{3(k''(0))^2}.$$

We can thus conclude that for any  $z \in \overline{K}(\delta_1, \delta_2)$ , the two solutions  $\lambda_-(z)$  and  $\lambda_+(z)$  of the equation (1.18) satisfy

$$\lambda_{\pm}(z) = \pm \sqrt{\frac{2}{k''(0)}}(1-z)^{1/2} - \frac{k^{(3)}(0)}{3(k''(0))^2}(1-z) + O((1-z)^{3/2}), \text{ as } z \rightarrow +\infty, z \notin [1, +\infty[. \quad (1.22)$$

### 1.2.3 On the spread-out property of the transition probability

Recall that  $N$  is the cardinal number of the finite set  $E = \{1, 2, \dots, N\}$ . We first introduce the

**Notations 1.2.1.** For any integer  $N \geq 1$ , let  $V_N$  denote the set of  $N \times N$  matrices whose coefficients are complex valued Radon measures on  $\mathbb{R}$ .

The set  $(V_N, +, \bullet)$  is an algebraic ring, when endowed with the sum  $+$  of Radon measures and the law  $\bullet$  defined by : for any  $B = (B_{i,j})_{i,j \in E}$  and  $C = (C_{i,j})_{i,j \in E}$  in  $V_N$

$$B \bullet C := \left( (B \bullet C)_{i,j} \right)_{i,j \in E},$$

with  $(B \bullet C)_{i,j}(dx) := \sum_{k \in E} B_{i,k} * C_{k,j}(dx)$ , where  $*$  denotes the convolution of measures.

For any  $n \geq 1$  we will set  $B^{\bullet n} = \underbrace{B \bullet \dots \bullet B}_{n \text{ times}} = (B_{i,j}^{\bullet n})_{i,j}$ .

For any  $[a, b] \subset \mathbb{R}$ , we denote by  $V_N[a, b]$  the subset of  $V_N$  of matrices whose coefficients  $\sigma$  are such that

$$\forall \lambda \in \mathbb{C} \quad \text{s.t.} \quad \Re \lambda \in [a, b] \quad \int_{\mathbb{R}} \exp(\lambda x) d|\sigma|(x) < +\infty.$$

The Laplace transform of such an element  $B$  of  $V_N[a, b]$  is defined by : for  $\lambda \in \mathbb{C}$  with  $\Re \lambda \in [a, b]$

$$\mathcal{L}(B)(\lambda)_{i,j} = \left( \mathcal{L}(B)(\lambda)_{i,j} \right)_{i,j} \quad \text{with} \quad \mathcal{L}(B)(\lambda)_{i,j} = \int_{\mathbb{R}} \exp(\lambda x) dB_{i,j}(x)$$

The classical properties of the Laplace transform lead to the following useful identity : for any  $B, C \in V_N[a, b]$  and  $\lambda \in \mathbb{C}$  with  $\Re \lambda \in [a, b]$ , one gets

$$\mathcal{L}(B \bullet C)(\lambda) = \mathcal{L}(B)(\lambda) \mathcal{L}(C)(\lambda) \tag{1.23}$$

In particular, in our context, since  $\mathcal{L}(M)(\lambda) = P(\lambda)$ , one may write for any  $n \geq 1$

$$\mathcal{L}(M^{\bullet n})(\lambda) = P(\lambda)^n.$$

Set  $M(dx) = \left( p_{i,j} F(i, j, dx) \right)_{i,j \in E}$ . Since the Markov chain  $X = (X_n)_{n \geq 0}$  is irreducible and  $(F(i, j, dt))_{i,j \in E}$  are probability measures on  $\mathbb{R}$ , one gets  $M_{i,j}^{\bullet k}(\mathbb{R}) > 0$  for any  $i, j \in E$  and  $k$  large enough. The hypothesis H2 implies that  $M_{i_0, j_0}^{\bullet n_0}(dx)$  has an absolutely continuous component. By Lemma A.0.5 of Appendix A, there exists  $k_1 \geq 1$  such that all the terms of  $M^{\bullet k_1}(dx)$  have absolutely continuous components. So one gets

$$\forall k \geq k_1, \quad M_{i,j}^{\bullet k}(dx) = \varphi_{k,i,j}(x) l(dx) + \theta_{k,i,j}(dx), \quad \text{(a)} \tag{1.24}$$

where for any  $(i, j) \in E \times E$ ,

- the function  $\varphi_{k,i,j}$  is nonnegative, belongs to  $\mathbb{L}^1(\mathbb{R}, dx)$  and satisfies

$$0 < \int \varphi_{k,i,j}(x) dx \leq 1;$$

---

a.  $l(dx)$  denotes here the Lebesgue measure on  $\mathbb{R}$ ; in order to simplify the notations we will omit the letter  $l$  and use the notation  $dx$ .

- $\theta_{k,i,j}(dx)$  is a singular measure with respect to the Lebesgue measure such that

$$0 \leq \theta_{k,i,j}(\mathbb{R}) < 1.$$

For  $|\Re \lambda| \leq \alpha_0$  and any  $k \geq 1$ , set

$$\begin{aligned} \Phi_k(dx) &= \left( \Phi_{k,i,j}(dx) \right)_{i,j} = \left( \varphi_{k,i,j}(x) dx \right)_{i,j}, & \Theta_k(dx) &= \left( \Theta_{k,i,j}(dx) \right)_{i,j} = \left( \theta_{k,i,j}(dx) \right)_{i,j}; \\ \mathfrak{L}(\Phi_k)(\lambda) &= \int_{\mathbb{R}} e^{\lambda u} \Phi_k(u) = \left( \widehat{\varphi}_{k,i,j}(\lambda) \right)_{i,j}, & \mathfrak{L}(\Theta_k)(\lambda) &= \int_{\mathbb{R}} e^{\lambda u} \Theta_k(du) = \left( \widehat{\theta}_{k,i,j}(\lambda) \right)_{i,j}. \end{aligned}$$

With these notations, we will write shortly in the sequel,

$$\forall k \geq 1, M^{\bullet k}(dx) = \Phi_k(dx) + \Theta_k(dx) \quad (1.25)$$

For every  $(i, j) \in E \times E$ , the measure  $\Phi_{k,i,j}(dx)$  is the absolutely continuous component of  $M_{i,j}^{\bullet k}(dx)$  and  $\Theta_{k,i,j}(dx)$  is its orthogonal component with respect to the Lebesgue measure; the functions  $\mathfrak{L}(\Phi_k)(\lambda)$  and  $\mathfrak{L}(\Theta_k)(\lambda)$  are their respective Laplace transforms (recall that the Laplace transform of  $M$  is  $\mathfrak{L}(M)(\lambda) = P(\lambda)$ ).

By (1.24) and the above notations, we have for any  $p \geq 1$  and  $k \geq k_1$ ,

$$M^{\bullet kp}(dx) = (\Phi_k(dx) + \Theta_k(dx))^{\bullet p} = \Phi_{kp}(dx) + \Theta_{kp}(dx). \quad (1.26)$$

Since the convolution of two measures s.t. one of them is absolutely continuous w.r. to the Lebesgue measure remains absolutely continuous, one may write

$$\Theta_{kp}(dx) \leq \Theta_k^{\bullet p}(dx). \quad (1.27)$$

We have the following lemma:

**Lemma 1.2.2.** *Let  $k_1 \geq 1$  such that (1.24) holds. There exists  $m_1 \geq 1$ , such that, for  $q \leq z \leq 1$ ,*

$$\|\mathfrak{L}(\Theta_{k_1}^{\bullet m_1})(\lambda_+(z))\| < z^{-k_1 m_1}. \quad (1.28)$$

*Proof.* Note first that, since  $z \in [q, 1]$ , one gets  $\lambda_+(z) \in \mathbb{R}^+$  so that the matrices  $\mathfrak{L}(\Theta_{k_1}^{\bullet n})(\lambda_+(z))$  are positive. Furthermore, for any  $n \geq 1$ , the inequality

$$\Phi_{k_1}^{\bullet n}(dx) + \Theta_{k_1}^{\bullet n}(dx) \leq M^{\bullet nk_1}(dx)$$

leads to the following one

$$\|\mathfrak{L}(\Theta_{k_1}^{\bullet n})(\lambda_+(z))\| \leq \left\| P^{nk_1}(\lambda_+(z)) \right\|,$$

which readily implies

$$\rho_{\Theta_{k_1}}(\lambda_+(z)) := \lim_{n \rightarrow +\infty} \left\| \mathfrak{L}(\Theta_{k_1}^{\bullet n})(\lambda_+(z)) \right\|^{1/n} \leq \lim_{n \rightarrow +\infty} \left\| P^{nk_1}(\lambda_+(z)) \right\|^{1/n} = k^{k_1}(\lambda_+(z)),$$

where  $\rho_{\Theta_{k_1}}(\lambda)$  denotes the spectral radius of  $\mathfrak{L}(\Theta_{k_1})(\lambda)$  for any  $\lambda \in \mathbb{C}$ . The equality  $zk(\lambda_+(z)) = 1$  thus leads to

$$\rho_{\Theta_{k_1}}(\lambda_+(z)) \leq z^{-k_1}. \quad (1.29)$$

Let us now prove that this inequality is strict. Otherwise, one should have

$$\rho_{\Theta_{k_1}}(\lambda_+(z)) = z^{-k_1} = k^{k_1}(\lambda_+(z)),$$

which should give  $1 = z^{k_1} k^{k_1}(\lambda_+(z)) = z^{k_1} \rho_{\Theta_{k_1}}(\lambda_+(z))$ . Since  $\rho_{\Theta_{k_1}}(\lambda_+(z))$  is an eigenvalue of  $\mathfrak{L}(\Theta_{k_1})(\lambda_+(z))$ , there would exist by the Perron-Frobenius theorem a non negative vector  $\alpha_+(z)$ , such that

$$\mathfrak{L}(\Theta_{k_1})(\lambda_+(z))\alpha_+(z) = \rho_{\Theta_{k_1}}(\lambda_+(z))\alpha_+(z) = z^{-k_1}\alpha_+(z).$$

By the definition of  $\mathfrak{L}(\Theta_{k_1})$ , one gets  $\mathfrak{L}(\Theta_{k_1})(\lambda_+(z)) = P^{k_1}(\lambda_+(z)) - \mathfrak{L}(\Phi_{k_1})(\lambda_+(z))$ , so we would get

$$\begin{aligned} 0 &= \Pi(\lambda_+(z)) \left( I - z^{k_1} \mathfrak{L}(\Theta_{k_1})(\lambda_+(z)) \right) \alpha_+(z) \\ &= \Pi(\lambda_+(z)) \left[ I - z^{k_1} P^{k_1}(\lambda_+(z)) + z^{k_1} \mathfrak{L}(\Phi_{k_1})(\lambda_+(z)) \right] \alpha_+(z). \end{aligned} \quad (1.30)$$

The equalities (1.4), (1.5) and the fact that  $zk(\lambda_+(z)) = 1$  give

$$\Pi(\lambda_+(z)) \left[ I - z^{k_1} P^{k_1}(\lambda_+(z)) \right] = [1 - z^{k_1} k^{k_1}(\lambda_+(z))] \Pi(\lambda_+(z)) = 0.$$

Consequently, (1.30) leads to the equality

$$0 = z^{k_1} \Pi(\lambda_+(z)) [\mathfrak{L}(\Phi_{k_1})(\lambda_+(z))] \alpha_+(z), \text{ for } q \leq z \leq 1. \quad (1.31)$$

However, since all the terms of matrix  $\mathfrak{L}(\Phi_{k_1})(\lambda_+(z))$  are strictly positive, the vector  $\mathfrak{L}(\Phi_{k_1})(\lambda_+(z)) \alpha_+(z)$  is strictly positive and the non-negative matrix  $\Pi(\lambda_+(z))$  has rank 1. We hence obtain

$$\Pi(\lambda_+(z)) [\mathfrak{L}(\Phi_{k_1})(\lambda_+(z))] \alpha_+(z) \neq 0.$$

This contradicts (1.31). So if we take  $m_1$  large enough, we can thus obtain (1.28).  $\square$

**From now on, we fix  $k_1, m_1 \geq 1$  such that (1.28) holds and we set  $n_1 := k_1 m_1$ .** We now fix  $\kappa > 0$  and denote  $\varphi_\kappa$  the density function of the  $\Gamma(2, \kappa)$ -distribution defined by  $\varphi_\kappa(x) = \kappa^2 x e^{-\kappa x} 1_{]0, +\infty[}$ ; for any  $s \in \mathbb{C}$  such that  $\Re s < \kappa$ , the Laplace transform  $\widehat{\varphi}_\kappa$  of  $\varphi_\kappa$  exists and one gets

$$\widehat{\varphi}_\kappa(s) = \frac{\kappa^2}{(s - \kappa)^2}. \quad (1.32)$$

Note that the family  $(\varphi_\kappa)_{\kappa > 0}$  is an approximation of the identity; this will be useful in the technical Lemma 1.2.3.

Throughout the whole texte, the convolution  $*$  of two  $\mathbb{R}$  to  $\mathbb{R}$  functions  $f, g$  is defined as follows: for any  $x \in \mathbb{R}$ ,

$$f * g(x) = \int_{\mathbb{R}} f(x + y)g(y)dy.$$

Consider the following matrix

$$\Phi_{n_1, \kappa}(dx) := \Phi_{n_1} * \varphi_\kappa(dx) = \left( \Phi_{n_1, \kappa, i, j}(dx) \right)_{i, j},$$

and  $\mathfrak{L}(\Phi_{n_1, \kappa})$  its Laplace transform defined for  $|\Re \lambda| \leq \alpha_0$ , where  $\Phi_{n_1, \kappa, i, j}(dx) = \varphi_{n_1, i, j} * \varphi_\kappa(x)dx$ , for any  $i, j \in E$ . One gets the

**Property 1.2.3.** *There exist  $\delta_1, \delta_2, \varepsilon_1 > 0$ ,  $0 < \gamma < 1$  and  $\kappa > 0$ , such that for all  $z \in \overline{K}(\delta_1, \delta_2)$ ,*

$$\|\Phi_{n_1}([- \infty, -x] \cup [x, +\infty])\| = O(e^{-\alpha_0 x}), \text{ for } x > 0; \quad (1.33)$$

$$|z|^{n_1} \|\mathfrak{L}(\Theta_{k_1}^{\bullet n_1})(s)\| \leq \gamma, \text{ for } -\varepsilon_1 \leq s \leq \varepsilon_1; \quad (1.34)$$

$$|z|^{n_1} \|\mathfrak{L}(\Phi_{n_1})(s) - \mathfrak{L}(\Phi_{n_1, \kappa})(s)\| \leq \frac{1 - \gamma}{2}, \text{ for } -\alpha_0 \leq s \leq \alpha_0. \quad (1.35)$$

*Proof.* 1) The first equality is derived from the fact that

$$\left\| \int_0^{+\infty} e^{\alpha_0 x} \Phi_{n_1}(dx) \right\| \leq \left\| \int_0^{+\infty} e^{\alpha_0 x} M^{\bullet n_1}(dx) \right\| < +\infty$$

$$\text{(resp. } \left\| \int_{-\infty}^0 e^{-\alpha_0 x} \Phi_{n_1}(dx) \right\| < +\infty).$$

Therefore, for  $x > 0$ ,

$$\|\Phi_{n_1}[x, +\infty[[ \leq \left\| e^{-\alpha_0 x} \int_x^{+\infty} e^{\alpha_0 t} \Phi_{n_1}(dt) \right\| \leq C e^{-\alpha_0 x}$$

(and  $\|\Phi_{n_1}[-\infty, x[[ \leq C e^{-\alpha_0 x}$ , for  $x < 0$ ).

2) The equality (1.26), the inequality (1.27) and Lemma 1.2.2 give, for  $q \leq z \leq 1$ ,

$$z^{n_1} \left\| \mathfrak{L}(\Theta_{n_1})(\lambda_+(z)) \right\| \leq z^{n_1} \left\| \mathfrak{L}(\Theta_{k_1}^{\bullet n_1})(\lambda_+(z)) \right\| < 1.$$

Recall that  $z \mapsto \lambda_+(z)$  is continuous on  $[q, 1]$  and  $s \mapsto \|P^{n_1}(s) - \mathfrak{L}(\Phi_{n_1})(s)\|$  is continuous on a neighborhood of 0, we can then choose some suitable  $\delta_1, \delta_2, \varepsilon_1 > 0$  and  $0 < \gamma < 1$ , such that (1.34) holds.

3) The inequality (1.35) is an immediate consequence of the following lemma, applied to the densities  $h = \varphi_{n_1, i, j}$  of  $M_{i, j}^{\bullet n_1}$  for any  $i, j \in E$ . □

**Lemma 1.2.3.** *Fix  $a < 0 < b$  and let  $h : \mathbb{R} \rightarrow \mathbb{R}$  be a Borel function, such that  $\forall s \in [a, b]$ ,*

$$\int_{\mathbb{R}} e^{sx} |h(x)| dx < +\infty.$$

Set  $h_\kappa = h * \varphi_\kappa$ , where  $h * \varphi_\kappa(x) = \int_0^{+\infty} h(x+y) \varphi_\kappa(y) dy$ . Then

$$\lim_{\kappa \rightarrow +\infty} \sup_{a \leq s \leq b} \int_{\mathbb{R}} e^{sx} |h(x) - h_\kappa(x)| dx = 0. \quad (1.36)$$

*Proof.* We first prove that

$$\lim_{y \rightarrow 0} \sup_{a \leq s \leq b} \int_{\mathbb{R}} e^{sx} |h(x+y) - h(x)| dx = 0. \quad (1.37)$$

Indeed, from the condition that  $h \in \mathbb{L}^1(\mathbb{R})$ , for a fixed  $\varepsilon > 0$ , one can choose a continuous function  $\psi_\varepsilon \in \mathbb{L}^1(\mathbb{R})$  with compact support  $[\alpha, \beta]$  such that

$$\int_{\mathbb{R}} (e^{at} + e^{bt}) |h(t) - \psi_\varepsilon(t)| dt < \varepsilon. \quad (1.38)$$

For  $a \leq s \leq b$  and  $|y| \leq 1$ , one thus gets

$$\int_{\mathbb{R}} e^{sx} |h(x+y) - \psi_\varepsilon(x+y)| dx \leq e^{-ys} \int_{\mathbb{R}} (e^{at} + e^{bt}) |h(t) - \psi_\varepsilon(t)| dt \leq e^{-ys} \varepsilon \leq (e^{-a} + e^b) \varepsilon.$$

Therefore,

$$\begin{aligned} \int_{\mathbb{R}} e^{sx} |h(x+y) - h(x)| dx &\leq \int_{\mathbb{R}} e^{sx} |h(x+y) - \psi_\varepsilon(x+y)| dx + \int_{\mathbb{R}} e^{sx} |\psi_\varepsilon(x+y) - \psi_\varepsilon(x)| dx \\ &\quad + \int_{\mathbb{R}} e^{sx} |\psi_\varepsilon(x) - h(x)| dx \\ &\leq 2(e^{-a} + e^b) \varepsilon + \int_{\alpha-1}^{\beta+1} (e^{ax} + e^{bx}) |\psi_\varepsilon(x+y) - \psi_\varepsilon(x)| dx. \end{aligned}$$

By the uniform continuity of  $\psi_\varepsilon$  on  $\mathbb{R}$ , one gets  $|\psi_\varepsilon(x+y) - \psi_\varepsilon(x)| \xrightarrow{y \rightarrow 0} 0$  uniformly on  $\mathbb{R}$  and by the dominated convergence theorem

$$\limsup_{y \rightarrow 0} \sup_{a \leq s \leq b} \int_{\mathbb{R}} e^{sx} |h(x+y) - h(x)| dx \leq 2(e^{-a} + e^b) \varepsilon.$$

One can conclude since  $\varepsilon$  is arbitrary.

We are now able to prove (1.36). Since  $\varphi_\kappa$  is a density, one gets

$$\int_{\mathbb{R}} e^{sx} |h(x) - h_\kappa(x)| dx \leq I_r(s, \kappa) + J_r(s, \kappa),$$

with

$$I_r(s, \kappa) := \int_0^r \varphi_\kappa(y) \left( \int_{\mathbb{R}} e^{sx} |h(x+y) - h(x)| dx \right) dy$$

and

$$J_r(s, \kappa) := \int_r^{+\infty} \varphi_\kappa(y) \left( \int_{\mathbb{R}} e^{sx} |h(x+y) - h(x)| dx \right) dy.$$

Fix  $\varepsilon > 0$ . By (1.37), one may choose  $r$  small enough in such a way that, for  $|y| \leq r$  and any  $s \in [a, b]$

$$\int_{\mathbb{R}} e^{sx} |h(x+y) - h(x)| dx \leq \varepsilon,$$

and since  $\varphi_\kappa$  is a density of probability, one gets  $\forall s \in [a, b], \forall \kappa > 0, \quad I_r(s, \kappa) \leq \varepsilon$ .

On the other hand,

$$\begin{aligned} J_r(s, \kappa) &\leq \int_r^{+\infty} e^{sy} \varphi_\kappa(y) \left( \int_{\mathbb{R}} e^{st} |h(t) - h(t-y)| dt \right) dy \\ &\leq \left[ \int_r^{+\infty} (1 + e^{|a|y}) \varphi_\kappa(y) dy \right] \times \sup_{a \leq s \leq b} \left( \int_{\mathbb{R}} e^{st} |h(t)| dt \right). \end{aligned}$$

Setting  $u = \kappa y$ , one obtains  $\int_r^{+\infty} e^{|a|y} \varphi_\kappa(y) dy = \int_{r\kappa}^{+\infty} u e^{u(\frac{|a|}{\kappa}-1)} du$ , and so, for  $\kappa > 2|a|$ ,

$$\int_r^{+\infty} e^{|a|y} \varphi_\kappa(y) dy \leq \int_{r\kappa}^{+\infty} u e^{-\frac{u}{2}} du;$$

then  $\limsup_{\kappa \rightarrow +\infty} \sup_{s \in [a,b]} J_r(s, \kappa) = 0$ . □

We now introduce the following matrices, if they exist

$$B(z, dx) := z^{n_1} \left( M^{\bullet n_1}(dx) - \Phi_{n_1, \kappa}(dx) \right),$$

$$\tilde{B}(z, dx) := \sum_{k=1}^{+\infty} B^{\bullet k}(z, dx)$$

and denote  $\mathfrak{L}(B)$  and  $\mathfrak{L}(\tilde{B})$  their Laplace transforms defined for  $|\Re \lambda| \leq \alpha_0$ .

**Lemma 1.2.4.** *There exist  $\delta_1, \delta_2$  and  $\varepsilon > 0$  such that*

1.  $\sup_{\substack{z \in \overline{K}(\delta_1, \delta_2) \\ |s| \leq \varepsilon}} \left\| \int_{\mathbb{R}} e^{su} \tilde{B}(z, du) \right\| < +\infty;$
2. *for  $z \in \overline{K}(\delta_1, \delta_2)$ ,  $|s| \leq \varepsilon$ ,  $\theta \in \mathbb{R}$ , the matrix  $I - \mathfrak{L}(B)(z, s + i\theta)$  is invertible and*

$$(I - \mathfrak{L}(B)(z, s + i\theta))^{-1} = I + \mathfrak{L}(\tilde{B})(z, s + i\theta).$$

*Proof.* **1)** For  $z \in \overline{K}(\delta_1, \delta_2)$  and  $|s| \leq \varepsilon$ , we have

$$\left\| \mathfrak{L}(B)(z, s) \right\| \leq |z|^{n_1} \left\| \mathfrak{L}(\Theta_{n_1})(s) \right\| + |z|^{n_1} \left\| \mathfrak{L}(\Phi_{n_1})(s) - \mathfrak{L}(\Phi_{n_1, \kappa})(s) \right\|.$$

From (1.27), (1.34) and (1.35), there exist  $\delta_1, \delta_2, \varepsilon > 0$  and  $0 < \gamma < 1$  such that

$$\left\| \int_{\mathbb{R}} e^{su} B(z, du) \right\| \leq \frac{1+\gamma}{2} < 1.$$

Therefore, for any  $z \in \overline{K}(\delta_1, \delta_2)$ ,  $|s| \leq \varepsilon$ ,

$$\left\| \mathfrak{L}(\tilde{B})(z, s) \right\| \leq \sum_{k \geq 0} \left\| \mathfrak{L}(B)(z, s) \right\|^k \leq \sum_{k \geq 0} \left( \frac{1+\gamma}{2} \right)^k < +\infty.$$

**2)** For any  $z \in \overline{K}(\delta_1, \delta_2)$  and  $\theta, s \in \mathbb{R}$ ,

$$\left\| \mathfrak{L}(B)(z, s + i\theta) \right\| \leq \left\| \mathfrak{L}(B)(z, s) \right\|.$$

By the first assertion, for any  $z \in \overline{K}(\delta_1, \delta_2)$ ,  $|s| \leq \varepsilon$  and  $\theta \in \mathbb{R}$ , the matrix

$$I - \mathfrak{L}(B)(z, s + i\theta)$$

is invertible, with inverse

$$(I - \mathfrak{L}(B)(z, s + i\theta))^{-1} = \sum_{k=0}^{+\infty} \mathfrak{L}(B^k)(z, s + i\theta) = I + \mathfrak{L}(\tilde{B})(z, s + i\theta).$$

□

### 1.2.4 The resolvent of $P(\lambda)$

We denote by  $V_N[-\alpha_0, \alpha_0]$  the algebra of  $N \times N$  matrices whose terms are Laplace transforms of Radon measures  $\sigma$  on  $\mathbb{R}$ , satisfying

$$\int_{\mathbb{R}} e^{\lambda x} d|\sigma|(x) < +\infty, \quad \text{for } |\Re \lambda| \leq \alpha_0.$$

**Theorem 1.2.2.** *There exist  $\delta_1, \delta_2$  and  $\varepsilon > 0$  such that*

1) *The function  $A(z, \lambda)$  defined by*

$$A(z, \lambda) := (I - zP(\lambda))^{-1} + \frac{\Pi_+(z)}{(\lambda - \lambda_+(z))\beta_+(z)} + \frac{\Pi_-(z)}{(\lambda - \lambda_-(z))\beta_-(z)} \quad (1.39)$$

*is analytic for  $(z, \lambda)$  in the open set*

$$E(\delta_1, \delta_2, \varepsilon) := \{(z, \lambda); z \in K(\delta_1, \delta_2), \lambda \in S_z(\varepsilon)\},$$

*with  $S_z(\varepsilon) := \{\lambda : \Re \lambda_-(z) - \varepsilon < \Re \lambda < \Re \lambda_+(z) + \varepsilon\}$ , where  $\beta_{\pm}(z) := zk'(\lambda_{\pm}(z))$  and  $\Pi_{\pm}(z) := \Pi(\lambda_{\pm}(z))$ .*

2) *For  $(z, \lambda) \in E(\delta_1, \delta_2, \varepsilon)$ , one gets*

$$(I - zP(\lambda))^{-1} = I - \frac{\Pi_+(z)}{(\lambda - \lambda_+(z))\beta_+(z)} - \frac{\Pi_-(z)}{(\lambda - \lambda_-(z))\beta_-(z)} + \int_{1]_{-\infty, 0]} e^{\lambda x} da_-(z, x) + \int_{0, +\infty[} e^{\lambda x} da_+(z, x), \quad (1.40)$$

*where  $da_+(z, \cdot)$  (resp.  $da_-(z, \cdot)$ ) is a matrix valued Radon measure on  $\mathbb{R}_+$  (resp.  $\mathbb{R}_-$ ) whose Laplace transform belongs to  $V_{N \times N}[-\alpha_0, \alpha_0]$ .*

*Furthermore, for  $x \geq 0$  (resp.  $x < 0$ ), the function  $z \mapsto a_+(z, x)$  (resp.  $z \mapsto a_-(z, x)$ ) is analytic on  $K(\delta_1, \delta_2)$ , and satisfy : for any  $z \in \overline{K}(\delta_1, \delta_2)$  :*

$$\|a_+(z, +\infty) - a_+(z, x)\| \leq C e^{-(\Re \lambda_+(z) + \varepsilon)x}, \quad x \geq 0, \quad (1.41)$$

$$\|a_-(z, -\infty) - a_-(z, x)\| \leq C e^{-(\Re \lambda_-(z) - \varepsilon)x}, \quad x < 0. \quad (1.42)$$

*Proof.* Throughout the present proof, the parameters  $\delta_1, \delta_2$  and  $\varepsilon$  will satisfy the conclusions of Lemma 1.2.4.

1) As we mentioned in Remark 1.2.1, for  $(z, \lambda)$  such that  $1 - zk(\lambda) \neq 0$ ,  $|\Re \lambda| \leq \alpha_0$  and  $|\Im \lambda| \leq \alpha_0$  (i.e;  $\lambda \in \Delta_{\alpha_0}$ ), the operator  $I - zP(\lambda)$  is invertible with inverse

$$(I - zP(\lambda))^{-1} = \frac{zk(\lambda)}{1 - zk(\lambda)} \Pi(\lambda) + \sum_{n=0}^{+\infty} z^n R^n(\lambda).$$

Recall that there exists real numbers  $\delta_1, \delta_2 > 0$  such that when  $z \in K(\delta_1, \delta_2)$ , the equation  $1 - zk(\lambda) = 0$  has two distinct roots  $\lambda_-(z)$  and  $\lambda_+(z)$ , given by

$$\lambda_{\pm}(z) = \pm \sqrt{\frac{2}{k''(0)} \sqrt{1 - z}} \pm \frac{k^{(3)}(0)}{3(k''(0))^2} (1 - z) + \sum_{k=3}^{+\infty} (\pm 1)^k \alpha_k (1 - z)^{k/2}. \quad (1.43)$$



So we can choose  $\delta_1, \delta_2$  and  $\varepsilon$  such that  $\Re \lambda_-(z) - \varepsilon < \Re \lambda_+(z) + \varepsilon$  for any  $z \in K(\delta_1, \delta_2)$ . The residue of the map  $\lambda \mapsto \frac{zk(\lambda)\Pi(\lambda)}{1-zk(\lambda)}$  at  $\lambda_+(z)$  (resp.  $\lambda_-(z)$ ) can be computed as

$$\text{Res} \left( \frac{zk(\lambda)\Pi(\lambda)}{1-zk(\lambda)}, \lambda_{\pm}(z) \right) = -\frac{\Pi_{\pm}(z)}{\beta_{\pm}(z)}.$$

Therefore, the function

$$(z, \lambda) \mapsto \frac{zk(\lambda)\Pi(\lambda)}{1-zk(\lambda)} + \frac{\Pi_+(z)}{\beta_+(z)(\lambda - \lambda_+(z))} + \frac{\Pi_-(z)}{\beta_-(z)(\lambda - \lambda_-(z))}$$

is analytic for  $(z, \lambda) \in E(\delta_1, \delta_2, \varepsilon)$ .

Moreover,  $\sup_{|\Re \lambda| \leq \alpha_0} r(R(\lambda)) < 1$ ; the function  $(z, \lambda) \mapsto \sum_{n=0}^{+\infty} z^n R^n(\lambda)$  is thus ana-

lytic on the domain  $E(\delta_1, \delta_2, \varepsilon)$  when  $\delta_1, \delta_2$  and  $\varepsilon$  are small enough.

At last, by Theorem 1.2.1 (2), one may choose  $\alpha_0$  small enough in such a way

$$\sup_{\substack{|\Re \lambda| \leq \alpha_0 \\ |\Im \lambda| \geq \alpha_0}} r(P(\lambda)) < 1$$

which leads to the analysis of the map  $(\lambda, z) \mapsto (I - zP(\lambda))^{-1}$  on the set  $\{(z, \lambda) \in E(\delta_1, \delta_2, \varepsilon) / |\Im \lambda| \geq \alpha_0\}$ ; the analysis of the maps  $(z, \lambda) \mapsto \frac{\Pi_+(z)}{\beta_+(z)(\lambda - \lambda_+(z))}$  and  $(z, \lambda) \mapsto \frac{\Pi_-(z)}{\beta_-(z)(\lambda - \lambda_-(z))}$  on this domain also hold and the proof of assertion 1) is achieved.

- 2) From the definitions of  $\lambda_{\pm}(z)$ , for  $q \leq z < 1$  and  $\Re \lambda_-(z) < \Re \lambda < \Re \lambda_+(z)$ , one gets  $zk(\Re \lambda) < 1$ ; since  $r(P(\lambda)) \leq r(P(\Re \lambda)) = k(\Re \lambda)$ , one thus obtains  $zr(P(\lambda)) < 1$  for such a  $z$  and so

$$(I - zP(\lambda))^{-1} = \sum_{n=0}^{+\infty} z^n P^n(\lambda) = \left( \sum_{n=0}^{+\infty} z^n \mathbb{E}_i(e^{\lambda S_n}, X_n = j) \right)_{i,j}. \quad (1.44)$$

For every  $(i, j) \in E \times E$ , we consider the following distribution functions:

$$\begin{aligned} \text{for } x \geq 0, \quad (a_+(z, x))_{i,j} &:= \sum_{n=1}^{+\infty} z^n \mathbb{P}_i(0 \leq S_n < x, X_n = j) - \frac{(\Pi_+(z))_{i,j}}{\lambda_+(z)\beta_+(z)}(1 - e^{-\lambda_+(z)x}); \\ \text{for } x < 0, \quad (a_-(z, x))_{i,j} &:= \sum_{n=1}^{+\infty} z^n \mathbb{P}_i(x \leq S_n < 0, X_n = j) + \frac{(\Pi_-(z))_{i,j}}{\lambda_-(z)\beta_-(z)}(1 - e^{-\lambda_-(z)x}). \end{aligned}$$

The measures  $a_+(z, x)$  and  $a_-(z, x)$  satisfy the following identities

$$\begin{aligned} \int 1_{[0, +\infty[}(x) e^{\lambda x} d(a_+(z, x))_{i,j} &= \sum_{n=1}^{+\infty} z^n \mathbb{E}_i(e^{\lambda S_n}, S_n \geq 0, X_n = j) + \frac{(\Pi_+(z))_{i,j}}{(\lambda - \lambda_+(z))\beta_+(z)}, \\ \int 1_{]-\infty, 0[}(x) e^{\lambda x} d(a_-(z, x))_{i,j} &= \sum_{n=1}^{+\infty} z^n \mathbb{E}_i(e^{\lambda S_n}, S_n < 0, X_n = j) + \frac{(\Pi_-(z))_{i,j}}{(\lambda - \lambda_-(z))\beta_-(z)}. \end{aligned}$$

Summing the two precedent equalities and using (1.44), we find the expected formula (1.40).

Now we prove the analysis of the functions  $z \mapsto a_+(z, \cdot)$  and  $z \mapsto a_-(z, \cdot)$ . By (1.39) and (1.40), we get

$$A(z, \lambda) = I + \int 1_{[0, +\infty[}(x) e^{\lambda x} da_+(z, x) + \int 1_{]-\infty, 0[}(x) e^{\lambda x} da_-(z, x).$$

Observe that the function  $x \mapsto a_+(z, x)$  is continuous and vanishes at  $x = 0$ ; applying the inversion formula for the Laplace integral transform ([24]), we obtain for  $x \geq 0$  and  $0 < \delta < \Re \lambda_+(z)$ ,

$$\begin{aligned} a_+(z, +\infty) - a_+(z, x) &= \sum_{n=1}^{+\infty} z^n \mathbb{P}_i(S_n \geq x, X_n = j) - \frac{\Pi(\lambda_+(z)) e^{-\lambda_+(z)x}}{\lambda_+(z) \beta_+(z)} \\ &= \frac{1}{2\pi i} \int_{\Re \lambda = \delta} e^{-\lambda x} \frac{A(z, \lambda)}{\lambda} d\lambda. \end{aligned} \quad (1.45)$$

On the other hand, the function  $(z, \lambda) \mapsto A(\lambda, z)$  is analytic on the set  $E(\delta_1, \delta_2, \varepsilon)$  and by Cauchy's theorem, one gets

$$\begin{aligned} a_+(z, +\infty) - a_+(z, x) &= \frac{1}{2\pi i} \int_{\Re \lambda = \Re \lambda_+(z) + \varepsilon} e^{-\lambda x} \frac{A(z, \lambda)}{\lambda} d\lambda \\ &= \frac{1}{2\pi} e^{-(\Re \lambda_+(z) + \varepsilon)x} \int_{\mathbb{R}} e^{-ix\theta} \frac{A(z, \Re \lambda_+(z) + \varepsilon + i\theta)}{\Re \lambda_+(z) + \varepsilon + i\theta} d\theta. \end{aligned}$$

To compute this last integral, we use the following lemma whose proof will be stated at the end of this subsection.

**Lemma 1.2.5.** *Let  $a \neq b$  two complex numbers such that  $\Re a > 0$  and  $\Re b > 0$ . For  $x \geq 0$ , one gets*

$$\int_{-\infty}^{+\infty} \frac{e^{ix\theta}}{(i\theta - a)(i\theta - b)} d\theta = 0.$$

By (1.39) and Lemma 1.2.5, one gets for  $x \geq 0$ ,

$$\begin{aligned} a_+(z, +\infty) - a_+(z, x) &= \frac{1}{2\pi} e^{-(\Re \lambda_+(z) + \varepsilon)x} \int_{\mathbb{R}} \frac{e^{-ix\theta} [I - zP(\Re \lambda_+(z) + \varepsilon + i\theta)]^{-1}}{\Re \lambda_+(z) + \varepsilon + i\theta} d\theta \\ &= \frac{1}{2\pi} e^{-(\Re \lambda_+(z) + \varepsilon)x} W_+(z, \varepsilon, x) \end{aligned}$$

with

$$W_+(z, \varepsilon, x) := \int_{\mathbb{R}} \frac{e^{-ix\theta} [I - zP(\Re \lambda_+(z) + \varepsilon + i\theta)]^{-1}}{\Re \lambda_+(z) + \varepsilon + i\theta} d\theta. \quad (1.46)$$

By a similar argument, one may write for  $x < 0$ ,

$$a_-(z, -\infty) - a_-(z, x) = \frac{1}{2\pi} e^{-(\Re \lambda_-(z) - \varepsilon)x} W_-(z, \varepsilon, x),$$

with

$$W_-(z, \varepsilon, x) := \int_{\mathbb{R}} \frac{e^{-ix\theta} [I - zP(\Re \lambda_-(z) - \varepsilon + i\theta)]^{-1}}{\Re \lambda_-(z) - \varepsilon + i\theta} d\theta. \quad (1.47)$$

Note that by definition of  $a_{\pm}$ , the functions  $x \mapsto W_{\pm}(z, \varepsilon, x)$  are left-continuous, for any  $z \in K(\delta_1, \delta_2)$ . One completes the proof by a simple application of the following:

**Property 1.2.4.** *We fix  $\varepsilon > 0$  and  $\delta_1, \delta_2 > 0$  small enough in such a way the conclusions of Lemma 1.2.4 hold for any  $z \in \overline{K}(\delta_1, \delta_2)$ . Set  $\lambda_{\pm}(z, \varepsilon) = \Re \lambda_{\pm}(z) \pm \varepsilon$ . Then, there exists a constant  $C = C(\varepsilon) > 0$  such that for  $x \geq 0$  (resp.  $x < 0$ ), one gets  $\forall z \in \overline{K}(\delta_1, \delta_2)$ ,*

$$\|W_{+}(z, x, \varepsilon)\| \leq C \quad (\text{resp. } \|W_{-}(z, x, \varepsilon)\| \leq C). \quad (1.48)$$

□

*Proof of Property 1.2.4.* Note first that by the choice of the constants  $\delta_1, \delta_2$  and  $\varepsilon$ , one gets  $|\lambda_{\pm}(z, \varepsilon)| \leq \varepsilon_1$  for any  $z \in \overline{K}(\delta_1, \delta_2)$ .

By Lemma 1.2.4 and the fact that  $\mathfrak{L}(M^{\bullet n_1})(\lambda) = P^{(n_1)}(\lambda)$ , we have for  $z \in \overline{K}(\delta_1, \delta_2)$  and  $|s| \leq \varepsilon$ , the matrices  $I - z^{n_1} P^{n_1}(s)$  and  $I - \mathfrak{L}(B)(z, s)$  are invertible; in addition, the identity

$$z^{n_1} P^{n_1}(s) = \mathfrak{L}(B)(z, s) + z^{n_1} \mathfrak{L}(\Phi_{n_1, \kappa})(s)$$

allows us to write

$$[I - z^{n_1} P^{n_1}(s)]^{-1} = [I - \mathfrak{L}(B)(z, s)]^{-1} + [I - z^{n_1} P^{n_1}(s)]^{-1} z^{n_1} \mathfrak{L}(\Phi_{n_1, \kappa})(s) [I - \mathfrak{L}(B)(z, s)]^{-1}. \quad (1.49)$$

Indeed, using the classical fact that for any  $N \times N$  matrices  $U$  and  $V$  such that  $I - U$  and  $I - V$  are invertible, setting  $W = U - V$ , one has

$$(I - U)^{-1} = (I - V)^{-1} + (I - U)^{-1} W (I - V)^{-1}.$$

Applying this identity to  $U = z^{n_1} P^{n_1}(s)$  and  $V = \mathfrak{L}(B)(z, s)$ , one obtains immediately the identity (1.49). Throughout this proof, in order to simplify the notations, we set  $\blacktriangle := \lambda_{+}(z, \varepsilon) + i\theta$  and  $U_{n_1}(z, \blacktriangle) := zP(\blacktriangle) + \dots + z^{n_1} P^{n_1}(\blacktriangle)$ , so that

$$\begin{aligned} [I - zP(\blacktriangle)]^{-1} &= I + U_{n_1}(z, \blacktriangle) [I - z^{n_1} P^{n_1}(\blacktriangle)]^{-1} \\ &= I + U_{n_1}(z, \blacktriangle) [I - \mathfrak{L}(B)(z, \blacktriangle)]^{-1} \\ &\quad + U_{n_1}(z, \blacktriangle) [I - z^{n_1} P^{n_1}(\blacktriangle)]^{-1} z^{n_1} \mathfrak{L}(\Phi_{n_1, \kappa})(\blacktriangle) [I - \mathfrak{L}(B)(z, \blacktriangle)]^{-1} \end{aligned}$$

and we may decompose  $W_{+}(z, \varepsilon, x)$  as  $W_{+}(z, \varepsilon, x) = W_{+1}(z, \varepsilon, x) + W_{+2}(z, \varepsilon, x) + W_{+3}(z, \varepsilon, x)$  with

$$\begin{aligned} W_{+1}(z, \varepsilon, x) &:= \int_{\mathbb{R}} \frac{e^{-i\theta x} I}{\blacktriangle} d\theta, \\ W_{+2}(z, \varepsilon, x) &:= \int_{\mathbb{R}} \frac{e^{-i\theta x} U_{n_1}(z, \blacktriangle) [I - \mathfrak{L}(B)(z, \blacktriangle)]^{-1}}{\blacktriangle} d\theta, \\ W_{+3}(z, \varepsilon, x) &:= \int_{\mathbb{R}} e^{-i\theta x} \frac{z^{n_1} U_{n_1}(z, \blacktriangle) [I - z^{n_1} P^{n_1}(\blacktriangle)]^{-1} \mathfrak{L}(\Phi_{n_1, \kappa})(\blacktriangle) [I - \mathfrak{L}(B)(z, \blacktriangle)]^{-1}}{\blacktriangle} d\theta. \end{aligned}$$

The fact that  $W_{+1}(z, \varepsilon, x)$  is bounded uniformly in  $z \in \overline{K}(\delta_1, \delta_2)$  and  $x \geq 0$  is a direct consequence of the following Lemma; indeed, one gets  $\int_{\mathbb{R}} \frac{e^{-i\theta x}}{\blacktriangle} d\theta = \pi(1 - \text{sgn}(x))e^{-\lambda_{+}(z, \varepsilon)x} = 0$ , since  $x \geq 0$ .

**Lemma 1.2.6.** *For any  $a > 0$  and any  $x \in \mathbb{R}$  one gets  $\int_{\mathbb{R}} \frac{e^{i\theta x}}{a + i\theta} d\theta = \pi e^{-ax} (1 + \operatorname{sgn}(x))$ .*

The proof of this lemma will be stated at the end of this subsection.

Now, we focus our attention on the term  $W_{+2}(z, \varepsilon)$ . By Lemma 1.2.4, the function  $z \mapsto [zP(\blacktriangle) + \dots + z^{n_1}P^{n_1}(\blacktriangle)][I - \mathfrak{L}(B)(z, \blacktriangle)]^{-1}$  is the Laplace transform at point  $\blacktriangle$  of the measure

$$\mu(z, dx) = [zM(dx) + \dots + z^{n_1}M^{n_1}(dx)] \bullet [\delta_{\{0\}}(dx)I + \tilde{B}(z, dx)].$$

By the definition of  $P$  and Lemma 1.2.4, for  $z \in [q + \delta_1, 1 + \delta_2]$ , the term  $\mu(z, \cdot)$  is a matrix of finite measures on  $\mathbb{R}$ , so we get

$$\sup_{z \in \overline{K}(\delta_1, \delta_2)} \|[zM(\mathbb{R}) + \dots + z^{n_1}M^{n_1}(\mathbb{R})][I + \tilde{B}(z, \mathbb{R})]\| < +\infty.$$

By the inversion formula for the Laplace integral transform, for any continuity point  $x \geq 0$  of the map  $t \mapsto \mu(z, [t, +\infty[)$ , one gets

$$e^{-\lambda_+(z, \varepsilon)x} W_{+2}(z, \varepsilon, x) = \mu(z, [x, +\infty]). \quad (1.50)$$

This equality holds in fact for any  $x \geq 0$  since the two members are left-continuous on  $\mathbb{R}$ . Therefore, for any  $x \geq 0$ , one gets

$$\|W_{+2}(z, \varepsilon, x)\| = \|e^{\lambda_+(z, \varepsilon)x} \mu(z, [x, +\infty])\| \leq \int_{-\infty}^{+\infty} e^{\Re \lambda_+(z, \varepsilon)t} \|zM(dt) + \dots + z^{n_1}M^{n_1}(dt)\| \bullet \tilde{B}(z, dt)\|.$$

Using Lemma 1.2.4 and the fact that  $\sup_{z \in \overline{K}(\delta_1, \delta_2)} \|P(\Re \lambda_+(z, \varepsilon))\| < +\infty$ , we obtain immediately

$$\sup_{\substack{z \in \overline{K}(\delta_1, \delta_2) \\ x \geq 0}} \|W_{+2}(z, \varepsilon, x)\| < +\infty.$$

We finally study the last term  $W_{+3}(z, x)$ . From the identity (1.32), one gets  $\|\mathfrak{L}(\Phi_{n_1, \kappa})(\blacktriangle)\| = \frac{\kappa^2}{|\blacktriangle - \kappa|^2} \|\mathfrak{L}(\Phi_{n_1})(\blacktriangle)\|$ , with

$$\sup_{z \in \overline{K}(\delta_1, \delta_2)} \|\mathfrak{L}(\Phi_{n_1})(\blacktriangle)\| \leq \|P(\lambda_+(z, \varepsilon))\|^{n_1} < +\infty.$$

On the other hand, by Lemma 1.2.4 one gets  $\sup_{z \in \overline{K}(\delta_1, \delta_2)} \|[I - \mathfrak{L}(B)(z, \blacktriangle)]^{-1}\| < \infty$ . Since

the matrices  $[I - z^{n_1}P^{n_1}(\blacktriangle)]^{-1}$  and  $zP(\blacktriangle) + \dots + z^{n_1}P^{n_1}(\blacktriangle)$  are clearly bounded in  $z \in \overline{K}(\delta_1, \delta_2)$ , there finally exists a constant  $C > 0$  such that

$$\forall z \in \overline{K}(\delta_1, \delta_2), \forall x \geq 0, \quad \|W_{+3}(z, \varepsilon, x)\| \leq C \sup_{z \in \overline{K}(\delta_1, \delta_2)} \int_{\mathbb{R}} \frac{1}{|\blacktriangle|} \times \frac{\kappa^2}{|\kappa - \blacktriangle|^2} d\theta < +\infty.$$

□

It remains to prove Lemmas 1.2.5 and 1.2.6.

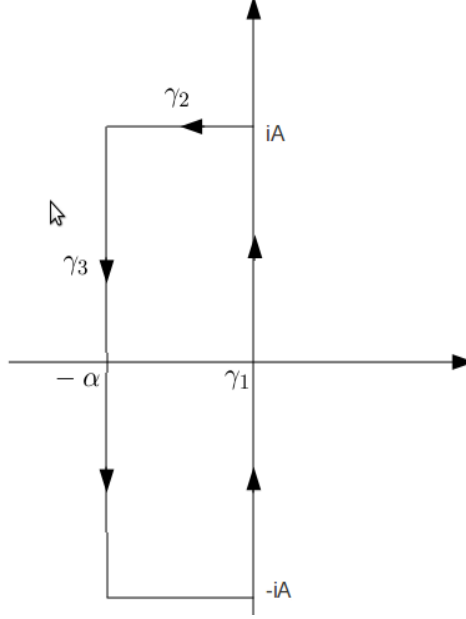


Figure 1.3: The closed path  $\gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$  of Lemma 1.2.5.

*Proof of Lemma 1.2.5.* For  $z \in \mathbb{C}$  and  $x \geq 0$ , set  $f(x, z) := \frac{e^{xz}}{(z-a)(z-b)}$ ; one gets

$$\int_{\gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4} f(x, z) dz = 0, \quad (1.51)$$

where  $\gamma_k$ ,  $1 \leq k \leq 4$ , are the paths defined as follows (see Figure 1.3) : for  $\alpha, A > 0$

$$\begin{aligned} \gamma_1 &= \{z = i\theta; -A \leq \theta \leq A\}, & \gamma_2 &= \{z = -t + iA; 0 \leq t \leq \alpha\}, \\ \gamma_3 &= \{z = -\alpha - i\theta, -A \leq \theta \leq A\}, & \gamma_4 &= \{z = t - iA; -\alpha \leq t \leq 0\}. \end{aligned}$$

In addition,

$$\begin{aligned} \left| \int_{\gamma_2} f(x, z) dz \right| &\leq \int_0^\alpha \left| \frac{e^{(-t+iA)x}}{(-t+iA-a)(-t+iA-b)} \right| dt \\ &= \int_0^\alpha \frac{e^{-tx} dt}{\sqrt{(t+\Re a)^2 + (A-\Im a)^2} \sqrt{(t+\Re b)^2 + (A-\Im b)^2}} \\ &\leq \frac{\alpha}{\sqrt{(\Re a)^2 + (A-\Im a)^2} \sqrt{(\Re b)^2 + (A-\Im b)^2}} \xrightarrow{A \rightarrow +\infty} 0. \end{aligned}$$

The same argument leads to

$$\left| \int_{\gamma_4} f(z) dz \right| = \left| \int_{-\alpha}^0 \frac{e^{(t-iA)x}}{(t-iA-a)(t-iA-b)} dt \right| \xrightarrow{A \rightarrow +\infty} 0.$$

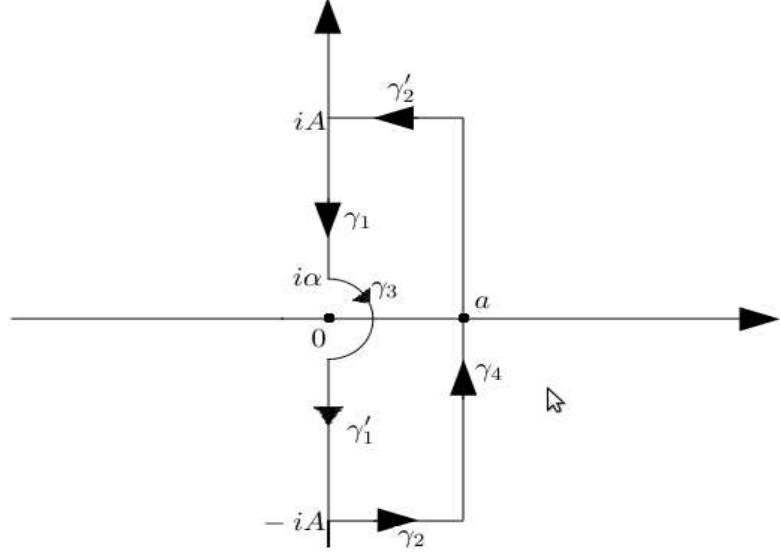


Figure 1.4: The closed path  $\gamma_1 \cup \gamma_1' \cup \gamma_2 \cup \gamma_2' \cup \gamma_3 \cup \gamma_4$  of Lemma 1.2.6.

On the other hand,

$$\begin{aligned} \left| \int_{\gamma_3} f(z) dz \right| &\leq e^{-\alpha x} \int_{-A}^A \frac{d\theta}{|\alpha + i\theta + a| |\alpha + i\theta + b|} \\ &\leq e^{-\alpha x} \int_{-\infty}^{+\infty} \frac{d\theta}{\sqrt{(\alpha + \Re a)^2 + (\theta - \Im a)^2} \sqrt{(\alpha - \Re b)^2 + (\theta - \Im b)^2}} \xrightarrow{\alpha \rightarrow +\infty} 0. \end{aligned}$$

$$\text{Then } \lim_{A \rightarrow +\infty} \int_{\gamma_1} f(x, z) dz = \int_{-\infty}^{+\infty} \frac{e^{i\theta x}}{(i\theta - a)(i\theta - b)} d\theta = 0. \quad \square$$

*Proof of Lemma 1.2.6.* For  $z \in \mathbb{C}$  and  $x \in \mathbb{R}$ , set  $g(x, z) := \frac{e^{xz}}{z}$ . For any fixed  $x > 0$ , one gets

$$\int_{\gamma_1 \cup \gamma_1' \cup \gamma_2 \cup \gamma_2' \cup \gamma_3 \cup \gamma_4} g(x, z) dz = 0, \quad (1.52)$$

where  $\gamma_k$ ,  $1 \leq k \leq 6$ , are the paths defined as follows (see Figure 1.4): for  $A > \alpha > 0$

- $\gamma_1$  is the oriented segment from  $iA$  to  $i\alpha$
- $\gamma_1'$  is the oriented segment from  $-i\alpha$  to  $-iA$
- $\gamma_2$  is the oriented segment from  $-iA$  to  $a - iA$

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- $\gamma'_2$  is the oriented segment from  $a + iA$  to  $iA$
- $\gamma_3$  is the clockwise oriented arc of circle from  $i\alpha$  to  $-i\alpha$
- $\gamma_4$  is the oriented segment from  $a - iA$  to  $a + iA$

One gets

$$\begin{aligned}
 1. \int_{\gamma_1 \cup \gamma'_1} g(x, z) dz &= -2i \int_{\alpha}^A \frac{\sin tx}{t} dt \xrightarrow{A \rightarrow +\infty} -i\pi \operatorname{sgn}(x), \\
 2. \left| \int_{\gamma_2 \cup \gamma'_2} g(x, z) dz \right| &\leq 2 \frac{e^{ax}}{A} \xrightarrow{A \rightarrow +\infty} 0, \\
 3. \int_{\gamma_3} g(x, z) dz &= -i \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{x\alpha e^{i\theta}} d\theta \xrightarrow{\alpha \rightarrow 0} -i\pi
 \end{aligned}$$

and equality (1.52) thus implies

$$\int_{\gamma_4} g(x, z) dz = ie^{ax} \int_{-A}^A \frac{e^{ix\theta}}{a + i\theta} d\theta \xrightarrow{A \rightarrow +\infty} i\pi(1 + \operatorname{sgn}(x))$$

and the Lemma follows.  $\square$

## 1.3 On the factorization of $I - zP(\lambda)$

### 1.3.1 Preliminaries and motivation

We first introduce the two following stopping times, which correspond to the first entrance time of the random walk  $(S_n)_{n \geq 1}$  inside one of the half-lines  $\mathbb{R}^+$ ,  $\mathbb{R}^{*+}$ ,  $\mathbb{R}^-$  and  $\mathbb{R}^{*-}$  :

$$\begin{aligned}
 T_+ &= \inf\{n \geq 1, S_n \geq 0\}; & T_+^* &= \inf\{n \geq 1, S_n > 0\}; \\
 T_- &= \inf\{n \geq 1, S_n \leq 0\}; & T_-^* &= \inf\{n \geq 1, S_n < 0\},
 \end{aligned}$$

with the convention  $\inf \emptyset = +\infty$ .

Recall that  $V_N[-\alpha_0, \alpha_0]$  is the algebra of  $N \times N$  matrices whose terms are Laplace transforms of Radon measures  $\sigma$  on  $\mathbb{R}$ , satisfying  $\int_{\mathbb{R}} e^{\lambda x} d|\sigma|(x) < +\infty$ , for  $|\Re \lambda| \leq \alpha_0$ . Let  $G \in V_N[-\alpha_0, \alpha_0]$ , defined by

$$G(\lambda) = \left( \int_{\mathbb{R}} e^{\lambda x} d\sigma_{i,j}(x) \right)_{1 \leq i, j \leq N}.$$

For  $|\Re \lambda| \leq \alpha_0$ , we set <sup>(b)</sup>

$$\begin{aligned}
 \mathcal{N}G(\lambda) &= \left( \int_{]-\infty, 0]} e^{\lambda x} d\sigma_{i,j}(x) \right)_{i,j}, & \mathcal{N}^*G(\lambda) &= \left( \int_{]-\infty, 0[} e^{\lambda x} d\sigma_{i,j}(x) \right)_{i,j}; \\
 \mathcal{P}G(\lambda) &= \left( \int_{[0, +\infty[} e^{\lambda x} d\sigma_{i,j}(x) \right)_{i,j}, & \mathcal{P}^*G(\lambda) &= \left( \int_{]0, +\infty[} e^{\lambda x} d\sigma_{i,j}(x) \right)_{i,j}.
 \end{aligned}$$

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b. the letter  $\mathcal{N}$  corresponds to the restriction of the Radon measure to the *negative* or *strictly negative* half line  $\mathbb{R}^-$  or  $\mathbb{R}^{*-}$  and the letter  $\mathcal{P}$  corresponds to the *positive* or *strictly positive* half line  $\mathbb{R}^+$  or  $\mathbb{R}^{*+}$

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For  $|z| < 1$ , we consider the following matrices of measures on  $\mathbb{R}$ :

$$B_z(dy) = \left( \sum_{n=1}^{+\infty} z^n \mathbb{P}_i \{S_1 \geq S_n, S_2 \geq S_n, \dots, S_{n-1} \geq S_n, S_n \in dy, X_n = j\} \right)_{i,j},$$

$$B_z^*(dy) = \left( \sum_{n=1}^{+\infty} z^n \mathbb{P}_i \{S_1 > S_n, S_2 > S_n, \dots, S_{n-1} > S_n, S_n \in dy, X_n = j\} \right)_{i,j},$$

$$C_z(dy) = \left( \sum_{n=1}^{+\infty} z^n \mathbb{P}_i \{S_1 \geq 0, S_2 \geq 0, \dots, S_{n-1} \geq 0, S_n \in dy, X_n = j\} \right)_{i,j},$$

$$C_z^*(dy) = \left( \sum_{n=1}^{+\infty} z^n \mathbb{P}_i \{S_1 > 0, S_2 > 0, \dots, S_{n-1} > 0, S_n \in dy, X_n = j\} \right)_{i,j}.$$

For  $\Re \lambda = 0$ , the related Laplace transforms of the above measures, denoted respectively by  $B_z(\lambda)$ ,  $B_z^*(\lambda)$ ,  $C_z(\lambda)$  and  $C_z^*(\lambda)$ , are defined as following:

$$\begin{aligned} B_z(\lambda) &= \int_{-\infty}^{+\infty} e^{\lambda y} B_z(dy), & B_z^*(\lambda) &= \int_{-\infty}^{+\infty} e^{\lambda y} B_z^*(dy); \\ C_z(\lambda) &= \int_{-\infty}^{+\infty} e^{\lambda y} C_z(dy), & C_z^*(\lambda) &= \int_{-\infty}^{+\infty} e^{\lambda y} C_z^*(dy). \end{aligned}$$

**Remark 1.3.1.** 1. Note that the series which appear in these formulas do converge for  $|z| < 1$  and that the matrices  $B_z(\lambda)$ ,  $B_z^*(\lambda)$ ,  $C_z(\lambda)$  and  $C_z^*(\lambda)$  belong to  $V_N[0, 0]$ .

2. From the definitions of  $\mathcal{N}$ ,  $\mathcal{N}^*$ ,  $\mathcal{P}$  and  $\mathcal{P}^*$ , these applications applying on the above matrices are equal to the Laplace transforms of the matrices of measures at the beginning of this page with adding respectively the conditions  $S_n \leq 0$  for  $\mathcal{N}$ ,  $S_n < 0$  for  $\mathcal{N}^*$ ,  $S_n \geq 0$  for  $\mathcal{P}$  and  $S_n > 0$  for  $\mathcal{P}^*$ .

Let us now explain briefly how we will use these waiting times to prove the local limit theorem for the process  $m_n := \min(0, S_1, \dots, S_n)$ . Indeed, the Laplace transform of  $m_n$  may be expressed in terms of the operators  $\mathcal{N}^*$  and  $\mathcal{P}$  and the matrices  $B_z^*$  and  $C_z$ ; we have the

**Lemma 1.3.1.** For  $\lambda > 0$  and  $|z| < 1$ ,

$$\sum_{n=0}^{+\infty} z^n \mathbb{E}_i(e^{\lambda m_n}; X_n = j) = \{[I + \mathcal{N}^* B_z^*(\lambda)][I + \mathcal{P} C_z(0)]\}_{i,j}. \quad (1.53)$$



*Proof.* Applying the Markov property to the process  $(S_n, X_n)$ , we get

$$\begin{aligned}
 & \sum_{n=0}^{+\infty} z^n \mathbb{E}_i(e^{\lambda m_n}; X_n = j) \\
 = & \sum_{n=0}^{+\infty} z^n \sum_{k=0}^n \mathbb{E}_i(e^{\lambda S_k}; S_0 > S_k, \dots, S_{k-1} > S_k, S_{k+1} \geq S_k, \dots, S_n \geq S_k, X_n = j) \\
 = & \sum_{n=0}^{+\infty} z^n \sum_{k=0}^n \sum_{l \in E} \mathbb{E}_i(e^{\lambda S_k}; S_1 > S_k, \dots, S_{k-1} > S_k, S_k < 0, X_k = l) \mathbb{E}_l(S_1 \geq 0, \dots, S_{n-k} \geq 0, X_{n-k} = j) \\
 = & \sum_{l \in E} \left[ \sum_{k=0}^{+\infty} z^k \mathbb{E}_i(e^{\lambda S_k}; S_1 > S_k, \dots, S_{k-1} > S_k, S_k < 0, X_k = l) \right] \left[ \sum_{p=0}^{+\infty} z^p \mathbb{E}_l(S_1 \geq 0, \dots, S_p \geq 0, X_p = j) \right] \\
 = & \left\{ [I + \mathcal{N}^* B_z^*(\lambda)] [I + \mathcal{P}C_z(0)] \right\}_{i,j}.
 \end{aligned}$$

□

We will have to study the regularity with respect to  $z$  and  $\lambda$  of each factor  $I + \mathcal{N}^* B_z^*(\lambda)$  and  $I + \mathcal{P}C_z(0)$ ; to do this, we will use a classical approach based on the so-called *Wiener-Hopf factorization*.

### 1.3.2 The initial probabilistic factorization

We have the

**Proposition 1.3.1.** *For  $\Re \lambda = 0$  and  $|z| < 1$ , one gets*

$$I - zP(\lambda) = (I - \mathcal{P}B_z^*(\lambda))(I - \mathcal{N}^*C_z(\lambda)), \quad (1.54)$$

$$(I - \mathcal{P}B_z^*(\lambda))^{-1} = I + \mathcal{P}C_z(\lambda), \quad (1.55)$$

$$(I - \mathcal{N}^*C_z(\lambda))^{-1} = I + \mathcal{N}^*B_z^*(\lambda). \quad (1.56)$$

*Proof.* We first check that

$$(I - \mathcal{N}^*C_z(\lambda))(I - zP(\lambda))^{-1} = I + \mathcal{P}C_z(\lambda), \quad (1.57)$$

and (1.54) will follow by (1.55). Note that, for  $\Re \lambda = 0$ ,  $r(P(\lambda)) \leq r(P(0)) = 1$ . So for  $|z| < 1$ ,  $(I - zP(\lambda))$  is invertible, with inverse

$$(I - zP(\lambda))^{-1} = I + \sum_{n=1}^{+\infty} z^n P^n(\lambda).$$

By the definition of  $P(\lambda)$ , we get

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$$\delta_{i,j} + \sum_{n=1}^{+\infty} z^n (P^n(\lambda))_{i,j} = \delta_{i,j} + \sum_{n=1}^{+\infty} z^n (P^n(\lambda))_{i,j} = \delta_{i,j} + E_1(z, \lambda)_{i,j} + E_2(z, \lambda)_{i,j},$$

where

$$E_1(z, \lambda) := \left( E_1(z, \lambda)_{i,j} \right)_{i,j} = \left( \mathbb{E}_i \left( \sum_{n=1}^{T_-^* - 1} z^n e^{\lambda S_n}; X_n = j \right) \right)_{i,j}$$

and

$$E_2(z, \lambda) := \left( E_2(z, \lambda)_{i,j} \right)_{i,j} = \left( \mathbb{E}_i \left( \sum_{n=T_-^*}^{+\infty} z^n e^{\lambda S_n}; X_n = j \right) \right)_{i,j}.$$

First, for any  $\Re \lambda = 0$ ,  $|z| < 1$  and  $i, j \in E$ , one gets

$$E_1(z, \lambda)_{i,j} = \mathbb{E}_i \left( \sum_{n=1}^{+\infty} z^n e^{\lambda S_n}; T_-^* \geq n + 1; X_n = j \right) = (\mathcal{P}C_z(\lambda))_{i,j}.$$

On the other hand, by the strong Markov property, one gets

$$\begin{aligned} & E_2(z, \lambda)_{i,j} \\ &= \mathbb{E}_i \left\{ z^{T_-^*} e^{\lambda S_{T_-^*}} \left[ \mathbb{E}_{X_{T_-^*}} \left( \sum_{n=0}^{+\infty} z^n e^{\lambda S_n}; X_n = j \right) \right] \right\} \\ &= \sum_{l \in E} \left\{ \left[ \mathbb{E}_i \left( \sum_{k=1}^{+\infty} z^k e^{\lambda S_k}; T_-^* = k; X_k = l \right) \right] \left[ \sum_{n=0}^{+\infty} \mathbb{E}_l \left( z^n e^{\lambda S_n}; X_n = j \right) \right] \right\} \\ &= \sum_{l \in E} \left\{ \left[ \sum_{k=1}^{+\infty} z^k \mathbb{E}_i(e^{\lambda S_k}; S_1 \geq 0, S_2 \geq 0, \dots, S_{k-1} \geq 0, S_k < 0; X_k = l) \right] \left[ \delta_{l,j} + \sum_{n=1}^{+\infty} \mathbb{E}_l(z^n e^{\lambda S_n}; X_n = j) \right] \right\} \\ &= \sum_{l \in E} (\mathcal{N}^* C_z(\lambda))_{i,l} ((I - zP(\lambda))^{-1})_{l,j} \\ &= (\mathcal{N}^* C_z(\lambda) (I - zP(\lambda))^{-1})_{i,j}. \end{aligned}$$

We hence obtain that  $(I - zP(\lambda))^{-1} = I + \mathcal{P}C_z(\lambda) + \mathcal{N}^* C_z(\lambda) (I - zP(\lambda))^{-1}$  which yields the equality (1.57).

We now prove (1.55) (and the proof of (1.54) will be complete, as we claimed above). Set  $F_z(\lambda) = (I - \mathcal{P}B_z^*(\lambda))(I + \mathcal{P}C_z(\lambda))$ ; we want to check that  $F_z(\lambda) = I$ . One gets

$$\left( F_z(\lambda) \right)_{i,j} = \delta_{i,j} + \left( \mathcal{P}C_z(\lambda) \right)_{i,j} - \left( \mathcal{P}B_z^*(\lambda) \right)_{i,j} - \left( \mathcal{P}B_z^*(\lambda) \mathcal{P}C_z(\lambda) \right)_{i,j}. \quad (1.58)$$

By the strong Markov property, we get

$$\begin{aligned}
 & \left( \mathcal{P}B_z^*(\lambda)\mathcal{P}C_z(\lambda) \right)_{i,j} \\
 &= \sum_{n=1}^{+\infty} z^n \mathbb{E}_i \left[ e^{\lambda S_n}; S_1 > S_n, S_2 > S_n, \dots, S_{n-1} > S_n \geq 0; \mathbb{E}_{X_n} \left( \sum_{k=1}^{+\infty} z^k e^{\lambda S_k}; S_1 \geq 0, \dots, S_k \geq 0, X_k = j \right) \right] \\
 &= \sum_{n \geq 1, k \geq 1} z^{n+k} \mathbb{E}_i [e^{\lambda S_{n+k}}; S_1 > S_n, \dots, S_{n-1} > S_n, S_{n+1} \geq S_n, \dots, S_{n+k} \geq S_n \geq 0, X_{n+k} = j] \\
 &= \sum_{m=2}^{+\infty} z^m \left[ \sum_{n=1}^{m-1} \mathbb{E}_i (e^{\lambda S_m}; S_1 > S_n, \dots, S_{n-1} > S_n, S_{n+1} \geq S_n, \dots, S_m \geq S_n \geq 0, X_m = j) \right].
 \end{aligned} \tag{1.59}$$

Therefore,

$$\begin{aligned}
 (F_z(\lambda))_{i,j} &= \delta_{i,j} + \sum_{m=1}^{+\infty} z^m \mathbb{E}_i (e^{\lambda S_m}; S_1 \geq 0, S_2 \geq 0, \dots, S_m \geq 0, X_m = j) \\
 &\quad - \sum_{m=1}^{+\infty} z^m \mathbb{E}_i (e^{\lambda S_m}; S_1 > S_m, S_2 > S_m, \dots, S_{m-1} > S_m \geq 0; X_m = j) \\
 &\quad - \sum_{m=2}^{+\infty} z^m \left[ \sum_{n=1}^{m-1} \mathbb{E}_i (e^{\lambda S_m}; S_1 > S_n, \dots, S_{n-1} > S_n, S_{n+1} \geq S_n, \dots, S_m \geq S_n \geq 0, X_m = j) \right] \\
 &= \delta_{i,j} + \sum_{m=1}^{+\infty} z^m \mathbb{E}_i (e^{\lambda S_m}; S_1 \geq 0, S_2 \geq 0, \dots, S_m \geq 0, X_m = j) - z \mathbb{E}_i (e^{\lambda S_1}; S_1 \geq 0, X_1 = j) \\
 &\quad - \sum_{m=2}^{+\infty} z^m \left[ \sum_{n=1}^m \mathbb{E}_i (e^{\lambda S_m}; S_1 > S_n, \dots, S_{n-1} > S_n, S_{n+1} \geq S_n, \dots, S_m \geq S_n \geq 0, X_m = j) \right] \\
 &= \delta_{i,j} + \sum_{m=1}^{+\infty} z^m \mathbb{E}_i (e^{\lambda S_m}; S_1 \geq 0, S_2 \geq 0, \dots, S_m \geq 0, X_m = j) \\
 &\quad - \sum_{m=1}^{+\infty} z^m \left[ \sum_{n=1}^m \mathbb{E}_i (e^{\lambda S_m}; S_1 > S_n, \dots, S_{n-1} > S_n, S_{n+1} \geq S_n, \dots, S_m \geq S_n \geq 0, X_m = j) \right]
 \end{aligned}$$

To prove  $F_z(\lambda) = I$ , we have to check that, for any  $m \geq 1$ ,

$$\begin{aligned}
 & \mathbb{E}_i (e^{\lambda S_m}; S_1 \geq 0, S_2 \geq 0, \dots, S_m \geq 0, X_m = j) \\
 &= \sum_{n=1}^m \mathbb{E}_i (e^{\lambda S_m}; S_1 > S_n, \dots, S_{n-1} > S_n, S_n \geq 0, S_{n+1} \geq S_n, \dots, S_m \geq S_n, X_m = j).
 \end{aligned}$$

Let us thus consider the random variables  $T_m, m \geq 1$ , defined by

$$T_m = \inf\{1 \leq n \leq m : S_n = \inf(S_1, \dots, S_m)\}.$$

We have the following equalities

$$\begin{aligned} & \mathbb{E}_i(e^{\lambda S_m}; S_1 \geq 0, S_2 \geq 0, \dots, S_m \geq 0, X_m = j) \\ &= \sum_{n=1}^m \mathbb{E}_i(e^{\lambda S_m}; S_1 \geq 0, S_2 \geq 0, \dots, S_m \geq 0, T_m = n, X_m = j) \\ &= \sum_{n=1}^m \mathbb{E}_i(e^{\lambda S_m}; S_1 > S_n, \dots, S_{n-1} > S_n, S_n \geq 0, S_{n+1} \geq S_n, \dots, S_m \geq S_n, X_m = j), \end{aligned}$$

which achieves the proof.

The proof of the equality (1.56) goes along the same lines.  $\square$

**Remark 1.3.2.** 1. When  $E$  reduces to one point, the sequence  $(S_n)_{n \geq 0}$  is a random walk on  $\mathbb{R}$  and Proposition 1.3.1 corresponds to the classical Wiener-Hopf factorization ([9]).

2. There is another way to express the matrices  $\mathcal{N}^*C_z(\lambda)$  and  $\mathcal{P}B_z^*(\lambda)$ ; for  $|z| < 1$  one gets

$$\mathcal{N}^*C_z(\lambda) = \left\{ \mathbb{E}_i \left( z^{T_-^*} e^{\lambda S_{T_-^*}}; X_{T_-^*} = j \right) \right\}_{i,j} \quad \text{when } \Re \lambda \geq 0$$

$$\mathcal{P}B_z^*(\lambda) = X^{-1} \left\{ \mathbb{E}_i \left( z^{\tilde{T}_+^*} e^{\lambda \tilde{S}_{\tilde{T}_+^*}}; X_{\tilde{T}_+^*} = j \right) \right\}_{i,j}^t X \quad \text{when } \Re \lambda \leq 0 \quad (c)$$

$$\text{where } X \text{ is the diagonal matrix } X := \begin{pmatrix} \nu_1 & & (0) \\ & \ddots & \\ (0) & & \nu_N \end{pmatrix}.$$

To explain briefly how to obtain for instance this “new” expression of  $\mathcal{N}^*C_z(\lambda)$ , we introduce the dual chain  $(\tilde{S}_n, \tilde{X}_n)$  of  $(S_n, X_n)$  whose transition probability is given by

$$\tilde{P}_{(i,x)}(\{j\} \times A) = \frac{\nu_j}{\nu_i} p_{j,i} F(A - x, j, i).$$

We also consider the  $N \times N$  matrix  $\tilde{C}_z^-$  defined by :

for  $|z| < 1$ ,  $|\Re \lambda| \leq \alpha_0$

$$\tilde{C}_z^- = \left( \sum_{n=1}^{+\infty} z^n \mathbb{E}_i(e^{\lambda \tilde{S}_n}, \tilde{S}_1 \leq 0, \tilde{S}_2 \leq 0, \dots, \tilde{S}_{n-1} \leq 0, \tilde{X}_n = j) \right)_{i,j}.$$

The remark (2) is a straightforward consequence of the

**Fact 1.3.1.** One gets  $\tilde{C}_z^- = X^{-1}(B_z^*)^t X$ .

*Proof.* We have the equality

$$\begin{aligned} & \mathbb{E}_i(e^{\lambda \tilde{S}_n}, \tilde{S}_1 \leq 0, \dots, \tilde{S}_{n-1} \leq 0, \tilde{X}_n = j) \\ &= \sum_{k_1, k_2, \dots, k_{n-1}} \int_{\mathbb{R}^n} \frac{\nu_{k_1}}{\nu_i} \frac{\nu_{k_2}}{\nu_{k_1}} \dots \frac{\nu_j}{\nu_{k_{n-1}}} e^{\lambda(\tilde{y}_1 + \dots + \tilde{y}_n)} \mathbf{1}_{[\tilde{y}_1 \leq 0]} \mathbf{1}_{[\tilde{y}_1 + \tilde{y}_2 \leq 0]} \dots \mathbf{1}_{[\tilde{y}_1 + \dots + \tilde{y}_n \leq 0]} \\ & \quad \times F(k_1, i, d\tilde{y}_1) P_{k_1, i} F(k_2, k_1, d\tilde{y}_2) P_{k_2, k_1} \dots F(j, k_{n-1}, d\tilde{y}_n) P_{j, k_{n-1}}. \end{aligned}$$

---

c. where, for any  $N \times N$  complex matrix  $A$ , we denote by  $A^t$  the transposed matrix of  $A$ .

Replacing in this equality  $\tilde{y}_k$  by  $y_{n+1-k}$  and  $\tilde{X}_k$  by  $X_{n-k}$  for all  $0 \leq k \leq n$ , we obtain

$$\begin{aligned} & \mathbb{E}_i(e^{\lambda \tilde{S}_n}, \tilde{S}_1 \leq 0, \dots, \tilde{S}_{n-1} \leq 0, \tilde{X}_n = j) \\ &= \frac{\nu_j}{\nu_i} \sum_{k_1, \dots, k_{n-1}} \mathbb{E}(e^{\lambda S_n}, S_n \leq S_{n-1}, \dots, S_n \leq S_1, X_0 = j, X_1 = k_{n-1}, \dots, X_{n-1} = k_1, X_n = i) \\ &= \frac{\nu_j}{\nu_i} \mathbb{E}_j(e^{\lambda S_n}, S_1 \geq S_n, S_2 \geq S_n, \dots, S_{n-1} \geq S_n, X_n = i). \end{aligned}$$

Therefore,  $\tilde{C}_z^-(\lambda)_{i,j} = \frac{\nu_j}{\nu_i} B_z^*(\lambda)_{j,i}$ . □

In the sequel, we will extend the factorization identity (1.54) to a larger set of parameters. We will first prove, by arguments of elementary type, that this identity is valid for  $|z| \leq 1$  and  $\Re \lambda \in [-\alpha_0, \alpha_0]$ . In a second step, we will extend this identity for  $\Re \lambda = 0$  and  $z$  in a neighbourhood of the unit disc, excepted the point 1 ; this is much more delicate and it relies on a general argument of algebraic type, due to Presman ([23]).

### 1.3.3 General factorization theory of Presman

Let  $\mathfrak{R}$  be an arbitrary algebraic ring with unit element  $e$  and  $\mathcal{I}$  be the identity operator in  $\mathfrak{R}$ . Let  $\mathfrak{N}$  be an additive operator defined on a two-side ideal  $\mathfrak{R}'$  of the ring  $\mathfrak{R}$ , such that  $\forall f, g \in \mathfrak{R}'$ ,

$$(\mathfrak{N}f)(\mathfrak{N}g) = \mathfrak{N}[(\mathfrak{N}f)g + f(\mathfrak{N}g) - fg]. \quad (1.60)$$

It is easy to check that the operator  $\mathfrak{P} = \mathcal{I} - \mathfrak{N}$  also satisfies the relation (1.60).

**Definition 1.3.1.** *We say that the element  $e - a$  of a ring  $\mathfrak{R}$  admits a **left canonical factorization** with respect to the operator  $\mathfrak{N}$  (l.c.f.  $\mathfrak{N}$ ) if  $a \in \mathfrak{R}'$  and if there exist  $b, c \in \mathfrak{R}'$  such that*

$$e - a = (e - \mathfrak{P}b)(e - \mathfrak{N}c) \quad (1.61)$$

$$(e - \mathfrak{P}b)^{-1} = e + \mathfrak{P}c \quad (1.62)$$

$$(e - \mathfrak{N}c)^{-1} = e + \mathfrak{N}b. \quad (1.63)$$

*In this case, we say that  $b$  and  $c$  provide a l.c.f.  $\mathfrak{N}$ . We call  $e - \mathfrak{P}b$  and  $e - \mathfrak{N}c$  respectively, **the positive and negative components of the l.c.f.  $\mathfrak{N}$** .*

The following lemma states the uniqueness of such a factorization once it exists.

**Lemma 1.3.2** ([23], lemma 1.1). *1. If  $b$  and  $c$  provide a l.c.f.  $\mathfrak{N}$  of the element  $e - a$  then*

(a) *the l.c.f.  $\mathfrak{N}$  is unique and is determined by any one of the elements  $\mathfrak{N}b$ ,  $\mathfrak{P}b$ ,  $\mathfrak{N}c$ ,  $\mathfrak{P}c$ ;*

(b) *for any  $d \in \mathfrak{R}$ , the equations*

$$x - \mathfrak{P}(xa) = d, \quad y - \mathfrak{N}(ay) = d \quad (1.64)$$

*have a unique solution, given by the formulas:*

$$x = d + \{\mathfrak{P}[da(e + \mathfrak{N}b)]\}(e + \mathfrak{P}c), \quad (1.65)$$

$$y = d + (e + \mathfrak{N}b)\mathfrak{N}[(e + \mathfrak{P}c)ad]; \quad (1.66)$$

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(c) for  $d = e$ , the elements  $x = e + \mathfrak{P}c$  and  $y = e + \mathfrak{N}b$  are solutions of equation (1.64);

(d)  $c_1 = c$  (resp.  $b_1 = b$ ) is the unique solution of the equation

$$(e + \mathfrak{P}C_1)(e - a) = e - \mathfrak{N}C_1 \quad (\text{resp. } (e - a)(e + \mathfrak{N}b_1) = e - \mathfrak{P}b).$$

2. Assume that  $e = d$ . If  $x'$  and  $y'$  are solutions of the equations (1.64), then  $x'(e-a)y' = e$ ; moreover, if any two of the three elements  $x'$ ,  $y'$ ,  $e - a$  are invertible, then  $b' = ay'$  and  $c' = x'a$  provide a l.c.f.  $\mathfrak{N}$  of the element  $e - a$ .

In the sequel, we shall assume that  $\mathfrak{R}$  is a Banach algebra and that the operator  $\mathfrak{P}$  is bounded. The function  $y(z)$  of the complex variable  $z$ , taking values in an arbitrary Banach algebra, will be called *analytic* in a neighborhood of the point  $z_0$  if, for sufficiently small  $|z - z_0|$ , it may be represented as a convergent series (in the sense of the algebra norm) of nonnegative powers of  $(z - z_0)$  with coefficient in the algebra.

Now, we assume that  $a$  depends analytically on the complex variable  $z$  in a neighbourhood of some  $z_0$  and describe the regularity of the two components of the l.c.f.  $\mathfrak{N}$ ; namely, we get the following

**Lemma 1.3.3** ([23], lemma 1.2). *Let  $a(z)$  be an analytic function in a neighborhood of the point  $z_0$ , taking values in an ideal  $\mathfrak{R}'$  of the Banach algebra  $\mathfrak{R}$  and suppose that  $b_0$  and  $c_0$  provide a l.c.f.  $\mathfrak{N}$  of the element  $e - a(z_0)$ . Then  $e - a(z)$  admits l.c.f.  $\mathfrak{N}$  in a neighborhood of the point  $z_0$ , where the elements  $b(z)$  and  $c(z)$  which provide the l.c.f.  $\mathfrak{N}$  of the element  $e - a(z)$  are analytic functions of  $z$  taking values in  $\mathfrak{R}'$ .*

We achieve this paragraph explaining how one will use this general result in our context.

Recall that the Banach  $\mathbb{C}$ -algebra  $V_N[-\alpha_0, \alpha_0]$  denotes the family of all the  $N \times N$  matrices whose terms are Laplace transforms of Radon measures  $\sigma$  on  $\mathbb{R}$ , with exponential moment of order  $\alpha_0$ . In particular,  $V_N[0, 0]$  denotes the matrices of Laplace transforms of Radon measures  $\sigma$  satisfying:  $\int_{\mathbb{R}} e^{\lambda x} d|\sigma|(x) < +\infty$ , for any  $\lambda \in \mathbb{C}$  such that  $\Re \lambda = 0$ . The operator  $\mathfrak{N}$  will be here the operator  $\mathcal{N}^*$  acting on  $V_N[-\alpha_0, \alpha_0]$  and  $\mathfrak{P}$  will be equal to  $\mathcal{P}$ , where  $\mathcal{N}^*$  and  $\mathcal{P}$  are the ones defined at the beginning of the subsection 1.3.1.

If  $\nu, \mu$  are two Radon measures on  $\mathbb{R}$ , we have the following identity :

$$\nu^{*-} * \mu^{*-} = (\nu^{*-} * \mu + \nu * \mu^{*-} - \nu * \mu)^{*-}. \quad (\text{d})$$

Taking into account this equality, we obtain that  $\mathcal{N}^*$  and  $\mathcal{P}$  both satisfy the identity (1.60) for any  $f, g \in V_N[a, b]$ .

For  $|z| < 1$  and  $|\Re \lambda| \leq \alpha_0$ , we will consider the following  $\mathbb{C}$ -valued  $N \times N$  matrices:

$$B_z^*(\lambda) := \left( \sum_{n=1}^{+\infty} z^n \int_{-\infty}^{+\infty} e^{\lambda y} d\mathbb{P}_i \{S_1 > S_n, S_2 > S_n, \dots, S_{n-1} > S_n, S_n \leq y, X_n = j\} \right)_{i,j},$$

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d. where, for any Radon measure  $\gamma$  on  $\mathbb{R}$ , we have denote by  $\gamma^{*-}$  its restriction to  $\mathbb{R}^{*-}$  defined by

$$\gamma^{*-}(dx) = 1_{] -\infty, 0[}(x) \gamma(dx).$$

$$C_z(\lambda) := \left( \sum_{n=1}^{+\infty} z^n \int_{-\infty}^{+\infty} e^{\lambda y} d\mathbb{P}_i \{S_1 \geq 0, S_2 \geq 0, \dots, S_{n-1} \geq 0, S_n \leq y, X_n = j\} \right)_{i,j}.$$

Recall now that  $P(\lambda)$  belongs to  $V_N[-\alpha_0, \alpha_0]$ , the matrices  $B_z^*$  and  $C_z$  are the ones defined in the subsection 1.3.1; furthermore, by Proposition 1.3.1, for any complex number  $z$  with modulus  $< 1$  and any  $\lambda \in \mathbb{C}$  such that  $\Re \lambda = 0$ , the operator  $I - zP(\lambda)$  admits a l.c.f  $\mathcal{N}^*$  on  $V_N[0, 0]$  provided with  $B_z^*$  and  $C_z$ .

The above general Presman's result are therefore applicable to  $z \mapsto A_z := zP(\lambda)$  with values in  $V_N[-\alpha_0, \alpha_0]$  for  $|z| < 1$  and analytic on the unit open disc of the complex plane.

In particular, the elements  $B_z^*$  and  $C_z$  belong to  $V_N[0, 0]$ . In fact, one may specify this last statement, with the following lemma due to Presman (lemma 1.3 in [23]) :

**Lemma 1.3.4.** *If  $I - A_z$  is an analytic function of  $z$  in a neighbourhood of the point  $z_0$ , taking values in the ring  $V_N[-\alpha_0, \alpha_0]$  and if in this neighbourhood  $I - A_z$ , as an element of  $V_N[0, 0]$ , admits a l.c.f. with respect to  $\mathcal{N}^*$  with corresponding elements  $B_z^*$  and  $C_z$ , then  $\mathcal{P}B_z^*$  (resp.  $\mathcal{N}^*C_z$ ) is analytic in  $z$  in this neighbourhood, with values in  $\mathcal{P}V_N[-\infty, \alpha_0]$  (resp.  $\mathcal{N}^*V_N[-\alpha_0, +\infty[)$ .*

In the sequel, we analyze the factorization of  $I - zP(\lambda)$  in a neighbourhood of the unit disc of the complex plane for some values of  $\lambda \in \mathbb{C}$  ; we thus introduce the

**Notation 1.3.1.** *We will denote by  $\mathbb{D}$  the closed unit ball in the complex number plane :*

$$\mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}.$$

*The open unit ball will be denoted  $\mathbb{D}^\circ$ .*

### 1.3.4 The factorization of $I - zP(\lambda)$ for $z \in \mathbb{D}^\circ$ and $\Re \lambda$ close to 0

We first state the the following

**Theorem 1.3.1.** *There exists  $\alpha_1 \in ]0, \alpha_0[$  such that for any  $z \in \mathbb{D}^\circ$ , one gets*

1. For  $-\alpha_1 \leq \Re \lambda \leq \alpha_1$

$$I - zP(\lambda) = (I - \mathcal{P}B_z^*(\lambda))(I - \mathcal{N}^*C_z(\lambda)), \quad (1.67)$$

2. For  $\Re \lambda \leq 0$

$$(I - \mathcal{P}B_z^*(\lambda))^{-1} = I + \mathcal{P}C_z(\lambda), \quad (1.68)$$

3. For  $\Re \lambda \geq 0$

$$(I - \mathcal{N}^*C_z(\lambda))^{-1} = I + \mathcal{N}^*B_z^*(\lambda). \quad (1.69)$$

*Furthermore, the maps  $z \mapsto \mathcal{P}B_z^*(\lambda)$  and  $z \mapsto \mathcal{N}^*C_z(\lambda)$  are analytic on  $\mathbb{D}^\circ$  with values  $\mathcal{P}V_N[-\infty, \alpha_1]$  (resp.  $\mathcal{N}^*V_N[-\alpha_1, +\infty[$ ).*

*Proof.* By the argument developped to establish Proposition 1.3.1, one checks easily that (1.55) (resp. (1.56)) is valid for  $|z| < 1$  and  $\Re \lambda \leq 0$  ( resp.  $\Re \lambda \geq 0$ ). So (1.68) and (1.69) are valid.

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The existence of the factorization (1.67) in  $V_N[0, 0]$  for any  $z \in \mathbb{D}^\circ$  is given by (1.54) of Proposition 1.3.1. The analysis of the different components  $B_z^*(\lambda)$  and  $C_z(\lambda)$  on  $\mathbb{D}^\circ$  for  $\Re \lambda = 0$  is a consequence of Lemma 1.3.3 ; we may also apply Lemma 1.3.4 and conclude that

$$\mathcal{P}B_z^* \in \mathcal{P}V_N[-\infty, \alpha_0] \quad \text{and} \quad \mathcal{N}^*C_z \in \mathcal{N}^*V_N[-\alpha_0, +\infty[.$$

From a classical property of Laplace transforms, for any  $z \in \mathbb{D}^\circ$ , the maps  $\lambda \mapsto I - zP(\lambda)$  and  $\lambda \mapsto (I - \mathcal{P}B_z^*(\lambda))(I - \mathcal{N}^*C_z(\lambda))$  are analytic on the strip  $\{|\Re \lambda| \leq \alpha_1\}$  for any  $\alpha_1 \in ]0, \alpha_0[$  and they coincide on the line  $\Re \lambda = 0$  ; they thus coincide as analytic functions on the strip  $\{|\Re \lambda| \leq \alpha_1\}$ . So (1.67) holds for  $-\alpha_1 \leq \Re \lambda \leq \alpha_1$  ; the analysis of the maps  $z \mapsto \mathcal{P}B_z^*(\lambda)$  and  $z \mapsto \mathcal{N}^*C_z(\lambda)$  on  $\mathbb{D}^\circ$  with values  $\mathcal{P}V_N[-\infty, \alpha_1]$  (resp.  $\mathcal{N}^*V_N[-\alpha_1, +\infty[$ ) is a direct consequence of Lemma 1.3.4.  $\square$

**Remark 1.3.3.** *Another way to prove the analysis of the maps  $z \mapsto \mathcal{P}B_z^*(\lambda)$  and  $z \mapsto \mathcal{N}^*C_z(\lambda)$  on  $\mathbb{D}^\circ$ , when  $\Re \lambda \in ]-\alpha_1, \alpha_1]$ , is to use the explicit form of the functions  $B_z^*$  and  $C_z$  and argue as follows :*

- for  $\Re \lambda = 0$ , it is a consequence of Lemma 1.3.3 as we said a few lines above ;
- when  $\Re \lambda > 0$ , it is a direct consequence of the identity

$$\mathcal{N}^*C_z(\lambda) = \left\{ \mathbb{E}_i \left( z^{T_-^*} e^{\lambda S_{T_-^*}} ; X_{T_-^*} = j \right) \right\}_{i,j} ;$$

- when  $\Re \lambda \in [-\alpha_1, 0[$ , we use (1.67) and (1.68) to write

$$\mathcal{N}^*C_z(\lambda) = I - (I + \mathcal{P}C_z(\lambda))(I - zP(\lambda))$$

with  $\mathcal{P}C_z(\lambda) = \left\{ \sum_{n \geq 0} z^n \mathbb{E}_i \left( e^{\lambda S_n} ; T_-^* > n, X_n = j \right) \right\}_{i,j}$ . The two factors on the right hand side of this last equality are clearly analytic in  $z \in \mathbb{D}^\circ$  and the result follows. The same argument holds for  $z \mapsto \mathcal{P}B_z^*(\lambda)$ .

#### 1.3.5 Expansion of the factorization outside the unit disc

Recall that  $\mathbb{D}$  the unit disc in the complex plane et by  $\partial\mathbb{D}$  its boundary.

We study here the extension of the preceding factorization when  $\Re \lambda = 0$  and  $z$  lives in a neighbourhood of  $\mathbb{D} \setminus \{1\}$ . We have the

**Theorem 1.3.2.** *There exists a neighbourhood  $U$  of  $\mathbb{D} \setminus \{1\}$  such that, for  $\Re \lambda = 0$ , the two maps  $z \mapsto B_z^*(\lambda)$  and  $z \mapsto C_z(\lambda)$  can be continuously expanded on  $U$  in such a way*

1. for any  $z \in U$ , the formulas (1.67), (1.68) and (1.69) hold.
2. the maps  $z \mapsto \mathcal{P}B_z^*$  and  $z \mapsto \mathcal{N}^*C_z$  are analytic on  $U$ , with values in  $\mathcal{P}V_N[-\infty, \alpha_0]$  and  $\mathcal{N}^*V_N[-\alpha_0, +\infty[$  respectively.

*Proof.* We fix  $\lambda$  s.t.  $\Re \lambda = 0$ ,  $z_0 \in \mathbb{C}$  with  $|z_0| = 1$ ,  $z_0 \neq 1$  and choose a sequence  $(z_n)_{n \geq 1}$  of complex numbers in  $\mathbb{D}^\circ$  which converges to  $z_0$ .



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By Remark 1.3.2, the two limits  $B_{z_0}^+(\lambda) := \lim_{n \rightarrow +\infty} \mathcal{P}B_{z_n}^*(\lambda)$  and  $C_{z_0}^{*-}(\lambda) := \lim_{n \rightarrow +\infty} \mathcal{N}^*C_{z_n}(\lambda)$  do exist ; furthermore, (1.67) holds at any point  $z_n$  and letting  $n \rightarrow +\infty$  one gets

$$I - z_0P(\lambda) = (I - B_{z_0}^+(\lambda))(I - C_{z_0}^{*-}(\lambda)).$$

Since  $z_0 \neq 1$ , the matrix  $I - z_0P(\lambda)$  is invertible, so is  $I - B_{z_0}^+(\lambda)$  ; by (1.68), the limit  $\lim_{n \rightarrow +\infty} \mathcal{P}C_{z_n}(\lambda)$  does also exists (and is equal to  $C_{z_0}^+(\lambda) := -I + (I - B_{z_0}^+(\lambda))^{-1}$ ).

Consequently,

$$C_{z_0}(\lambda) := \lim_{n \rightarrow +\infty} C_{z_n}(\lambda) = \lim_{n \rightarrow +\infty} \mathcal{N}^*C_{z_n}(\lambda) + \lim_{n \rightarrow +\infty} \mathcal{P}C_{z_n}(\lambda) = C_{z_0}^{*-}(\lambda) + C_{z_0}^+(\lambda)$$

does exist and one gets  $C_{z_0}^{*-} = \mathcal{N}^*C_{z_0}(\lambda)$  and  $C_{z_0}^+(\lambda) = \mathcal{P}C_{z_0}(\lambda)$ .

By the same argument, one shows that  $B_{z_0}^*(\lambda) := \lim_{n \rightarrow +\infty} B_{z_n}^*(\lambda)$  does exist and (1.69) holds at  $z_0$ .

Finally  $B_{z_0}^*(\lambda)$  and  $C_{z_0}(\lambda)$  provide a l.c.f  $\mathcal{N}^*$  of  $I - z_0P(\lambda)$  ; since  $z \mapsto I - z_0P(\lambda)$  is analytic in a neighbourhood of  $z_0$ , so are the maps  $z \mapsto \mathcal{P}B_z^*(\lambda)$  and  $z \mapsto \mathcal{N}^*C_z(\lambda)$  by Lemma 1.3.3, with values in  $\mathcal{P}V_N[-\infty, \alpha_0]$  and  $\mathcal{N}^*V_N[-\alpha_0, +\infty[$  respectively, by Lemma 1.3.4.  $\square$

In the sequel we will specify the neighbourhood  $U$  as follows: set

$$D_{\rho, \theta} := \{z; z \neq 1, |\arg(z - 1)| > \theta > 0, |z| < \rho\},$$

one may state the

**Corollary 1.3.1.** *There exist  $\rho > 1$  and  $\theta \in ]0, \pi/2[$  such that*

- *the formulas (1.67), (1.68) and (1.69) hold for  $\Re \lambda = 0$  and  $z \in D_{\rho, \theta} \cap \left(K(\delta_1, \delta_2)\right)^c$ ,*
- *for  $|\Re \lambda| \leq \alpha_0$ , the map  $z \mapsto \mathcal{P}B_z^*$  (resp.  $z \mapsto \mathcal{N}^*C_z$ ) is analytic on  $D_{\rho, \theta} \cap \left(K(\delta_1, \delta_2)\right)^c$ ; furthermore,  $I - \mathcal{P}B_z^*$  (resp.  $I - \mathcal{N}^*C_z$ ) is invertible (and their inverses are also analytic) on this domain.*

### 1.4 On the local behavior of the factors of the Laplace transform of the minimum

We know, by Lemma 1.3.1 that the Laplace transform of the minimum  $m_n$  may be decomposed as follows : for  $\lambda > 0$  and  $|z| < 1$ ,

$$\sum_{n=0}^{+\infty} z^n \mathbb{E}_i(e^{\lambda m_n}; X_n = j) = \{[I + \mathcal{N}^*B_z^*(\lambda)][I + \mathcal{P}C_z(0)]\}_{i,j}.$$

In this section, we will study the behavior of these two factors near  $z = 1$ . More precisely, we will first consider the case when  $|z| \leq 1$  and afterward investigate the case when  $|z| > 1$ .

## 1.4. ON THE LOCAL BEHAVIOR OF THE FACTORS OF THE LAPLACE TRANSFORM OF THE MINIMUM

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### 1.4.1 Preliminaries

As mentioned in the previous section, the matrices  $I + \mathcal{N}^* B_z^*(\lambda)$  and  $I + \mathcal{P}C_z(0)$  could be seen as the inverse of two factors for the matrix  $I - zP(\lambda)$ , we will first study the regularities of these quantities for  $z \in \overline{K}(\delta_1, \delta_2)$ . In the following, the constants  $\delta$  and  $\varepsilon$  are chosen small enough in such a way that, for  $z \in \overline{K}(\delta, 0)$ , one gets  $[\lambda_-(z) - \varepsilon, \lambda_+(z) + \varepsilon] \subset ]-\alpha_0, \alpha_0[$ . We have the

**Proposition 1.4.1.** *There exist  $\delta_1 > 0$  and  $\varepsilon > 0$  such that for  $z \in \overline{K}(\delta_1, 0)$ , one gets*

1. for  $\Re \lambda < \Re \lambda_+(z) + \varepsilon$  with  $\lambda \neq \lambda_+(z)$ ,

$$(I - \mathcal{P}B_z^*(\lambda))^{-1} = I + \mathcal{P}C_z(\lambda) = I - \frac{[I - \mathcal{N}^* C_z(\lambda_+(z))] \Pi_+(z)}{(\lambda_+(z) - \lambda) \beta_+(z)} + \int_0^{+\infty} e^{\lambda x} k_+(z, dx) \quad (1.70)$$

2. for  $\Re \lambda > \Re \lambda_-(z) - \varepsilon$  with  $\lambda \neq \lambda_-(z)$ ,

$$(I - \mathcal{N}^* C_z(\lambda))^{-1} = I + \mathcal{N}^* B_z^*(\lambda) = I - \frac{\Pi_-(z) [I - \mathcal{P}B_z^*(\lambda_-(z))]}{(\lambda_-(z) - \lambda) \beta_-(z)} + \int_{-\infty}^0 e^{\lambda x} k_-(z, dx) \quad (1.71)$$

where  $k_+(z, \cdot)$  (resp.  $k_-(z, \cdot)$ ) is a measure on  $[0, +\infty[$  (resp.  $] - \infty, 0]$ ) taking values in the vector space  $M_{N \times N}(\mathbb{C})$  of  $N \times N$  complex matrices, such that for  $z \in \overline{K}(\delta, 0)$ , one gets

$$\|k_+(z, x)\| \leq C e^{-(\lambda_+(z) + \varepsilon)x} \quad \text{for } x > 0, \quad (1.72)$$

$$\|k_-(z, x)\| \leq C e^{-(\lambda_-(z) - \varepsilon)x} \quad \text{for } x < 0, \quad (1.73)$$

where  $k_+(z, x) = k_+(z, ]x, +\infty[)$  for  $x > 0$  and  $k_-(z, x) = k_-(z, ]-\infty, x])$  for  $x < 0$ .

Furthermore, the following limits exist :

$$\lim_{|z| \uparrow 1} \frac{(I - \mathcal{N}^* C_z(\lambda_+(z))) \Pi_+(z)}{\beta_+(z)} = A_+ \quad \text{and} \quad \lim_{|z| \uparrow 1} \frac{\Pi_-(z) (I - \mathcal{P}B_z^*(\lambda_-(z)))}{\beta_-(z)} = A_-, \quad (1.74)$$

where  $A_+$  (resp.  $A_-$ ) is a  $N \times N$  matrix with non positive (resp. non negative) coefficients.

*Proof.* Since the probabilistic expression of  $\mathcal{N}^* C_z$  is quite simple, we first prove that (1.70) and (1.72) hold when  $z \in K(\delta, 0)$  for any  $0 < \delta < \alpha_0$ ; then, we will establish the existence of  $A_+$  in (1.74) when  $\delta$  is quite small (namely  $\delta \leq \delta_1$ ), which will allow us to prove that (1.70) and (1.72) holds in fact for  $z \in \overline{K}(\delta_1, 0)$  and  $\Re \lambda < \Re \lambda_+(z) + \varepsilon$ ,  $\lambda \neq \lambda_+(z)$ .

We first prove that equality (1.70) holds for  $z \in K(\delta, 0)$ ,  $0 < \delta < \alpha_0$ ; the same argument works to establish (1.71).

According to Theorem 1.3.1 and the definition of  $\mathcal{P}C_z(\lambda)$ , for  $q \leq |z| < 1$  and  $\Re \lambda \leq 0$ , one gets

$$\begin{aligned} (I - \mathcal{P}B_z^*(\lambda))^{-1} &= I + \mathcal{P}C_z(\lambda) = I + \sum_{n=1}^{+\infty} z^n \mathbb{E}_i(e^{\lambda S_n}, T_-^* > n, X_n = j) \\ &:= I + \int_0^{+\infty} e^{\lambda y} db_+(z, y). \end{aligned} \quad (1.75)$$

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By (1.67) and the Laplace inversion formula, for  $\lambda_-(z) < -\delta < 0$ , one may write for  $x > 0$ ,

$$b_+(z, x) - b_+(z, -\infty) = -\frac{1}{2\pi i} \int_{\Re \lambda = -\delta} e^{-\lambda x} \frac{(I - \mathcal{N}^* C_z(\lambda))(I - zP(\lambda))^{-1}}{\lambda} d\lambda.$$

Now we transfer the contour of integration to the straight line  $\Re \lambda = \lambda_+(z) + \varepsilon$ ; using Cauchy's formula on the convex open set  $\Omega = \{-\delta < \Re \lambda < \Re \lambda_+(z) + \varepsilon, |\Im \lambda| < \beta\}$  and the fact that  $\lambda \mapsto \frac{1}{\lambda}(I - \mathcal{N}^* C_z(\lambda))(I - zP(\lambda))^{-1}$  is analytic in  $\Omega \setminus \{0, \lambda_+(z)\}$ , we get for  $y > 0$ ,

$$\begin{aligned} b_+(z, y) - b_+(z, -\infty) &= -(I - \mathcal{N}^* C_z(0))(I - zP(0))^{-1} + \frac{e^{-\lambda_+(z)y} [I - \mathcal{N}^* C_z(\lambda_+(z))] \Pi_+(z)}{\beta_+(z) \lambda_+(z)} \\ &\quad - \frac{e^{-(\lambda_+(z)+\varepsilon)y}}{2\pi i} \int_{\Re \lambda = 0} e^{-\lambda y} \frac{[I - \mathcal{N}^* C_z(\lambda + \lambda_+(z) + \varepsilon)] [I - zP(\lambda + \lambda_+(z) + \varepsilon)]^{-1}}{\lambda + \lambda_+(z) + \varepsilon} d\lambda. \end{aligned} \quad (1.76)$$

As in the proof of Theorem 1.2.2, we set  $\lambda_+(z, \varepsilon) := \Re \lambda_+(z) + \varepsilon$  and, for  $x \geq 0$

$$k_+(z, x) := -\frac{e^{-\lambda_+(z, \varepsilon)x}}{2\pi} \int_{\mathbb{R}} e^{-i\theta x} \frac{[I - \mathcal{N}^* C_z(\lambda_+(z, \varepsilon) + i\theta)] [I - zP(\lambda_+(z, \varepsilon) + i\theta)]^{-1}}{\lambda_+(z, \varepsilon) + i\theta} d\theta. \quad (1.77)$$

Consequently, for  $z \in K(\delta, 0)$ ,  $\Re \lambda < \lambda_+(z, \varepsilon)$  and  $\lambda \neq \lambda_+(z)$ , one gets

$$(I - \mathcal{P}B_z^*(\lambda))^{-1} = I - \frac{[I - \mathcal{N}^* C_z(\lambda_+(z))] \Pi_+(z)}{(\lambda_+(z) - \lambda) \beta_+(z)} + \int_0^{+\infty} e^{\lambda x} k_+(z, dx). \quad (1.78)$$

Inequality(1.72) is thus a direct consequence of the following result, which is the analogous in the present context of Property 1.2.4

**Property 1.4.1.** *We fix  $\varepsilon > 0$  and  $\delta_1$  small enough in such a way that Theorem 1.2.2 is satisfied. We set*

- $\lambda_{\pm}(z, \varepsilon) = \Re \lambda_{\pm}(z) \pm \varepsilon$ ;
- $W'_+(z, \varepsilon, x) = \int_{\mathbb{R}} e^{-i\theta x} \frac{[I - \mathcal{N}^* C_z(\lambda_+(z, \varepsilon) + i\theta)] [I - zP(\lambda_+(z, \varepsilon) + i\theta)]^{-1}}{\lambda_+(z, \varepsilon) + i\theta} d\theta,$   
for  $x \geq 0$ ;
- $W'_-(z, \varepsilon, x) = \int_{\mathbb{R}} e^{-i\theta x} \frac{[I - zP(\lambda_-(z, \varepsilon) + i\theta)]^{-1} [I - \mathcal{P}B_z^*(\lambda_+(z, \varepsilon) + i\theta)]}{\lambda_-(z, \varepsilon) + i\theta} d\theta,$   
for  $x < 0$ ;

Then, there exists a constant  $C' = C'(\varepsilon) > 0$  such that for  $x \geq 0$  (resp.  $x < 0$ ), one gets

$$\forall z \in \overline{K}(\delta_1, 0), \quad \|W'_+(z, x, \varepsilon)\| \leq C' \quad (\text{resp. } \|W'_-(z, x, \varepsilon)\| \leq C'). \quad (1.79)$$

Let us now establish (1.74). Since for any  $|z| \leq 1$ ,

$$\mathcal{N}^* C_z(\lambda_+(z)) = \left\{ \mathbb{E}_i \left( z^{T_-^*} e^{\lambda_+(z) S_{T_-^*}}; X_{T_-^*} = j \right) \right\}_{i,j}$$

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and  $\lim_{z \rightarrow 1} \lambda_+(z) = 0$ , we obtain that for any  $z \in \overline{K}(0, \delta)$ ,

$$\|\mathcal{N}^* C_z(\lambda_+(z))\| \leq \left\| \left\{ \mathbb{E}_i \left( z^{T_-^*} e^{\Re \lambda_+(z) S_{T_-^*}}; X_{T_-^*} = j \right) \right\} \right\| < +\infty.$$

Moreover, by the second assertion of Theorem 1.2.2, we may choose  $\delta_1 \leq \delta$  and  $0 < \varepsilon_i < \alpha_0$ ,  $i = 1, 2$ , such that  $\|(I - zP(\lambda))^{-1}\| < +\infty$  for all  $z \in \overline{K}(\delta_1, 0)$  and  $\varepsilon_1 \leq \Re \lambda \leq \varepsilon_2$ .

Therefore, for any  $\varepsilon_1 \leq \Re \lambda \leq \varepsilon_2$  and  $|z| < 1$ , one gets

$$(I - \mathcal{P}B_z^*(\lambda))^{-1} = (I - \mathcal{N}^* C_z(\lambda))(I - zP(\lambda))^{-1}$$

and the limits as  $|z| \rightarrow 1$  of the two factors on the right hand side do exist ; this implies that  $(I - \mathcal{P}B_z^*(\lambda))^{-1}$  exists for  $z \in \overline{K}(\delta_1, 0)$  and  $\varepsilon_1 \leq \Re \lambda \leq \varepsilon_2$ , with the identity

$$(I - \mathcal{P}B_z^*(\lambda))^{-1} = (I - \mathcal{N}^* C_z(\lambda))(I - zP(\lambda))^{-1}. \quad (1.80)$$

In particular, letting  $|z| \rightarrow 1$  in (1.78), we obtain

$$\lim_{|z| \uparrow 1} \frac{[I - \mathcal{N}^* C_z(\lambda_+(z))]\Pi_+(z)}{\beta_+(z)} \text{ exists } (= A_+). \quad (1.81)$$

It remains to prove that (1.70) holds for  $|z| = 1$ ,  $\Re \lambda < \lambda_+(z, \varepsilon)$  and  $\lambda \neq \lambda_+(z)$ . Taking into account (1.80) and (1.81), we can confirm that for any  $\varepsilon_1 \leq \Re \lambda \leq \varepsilon_2$ , as  $|z| \rightarrow 1$ , the limits for the members in the equality (1.78) exist and (1.70) hold for  $|z| = 1$ . Since the different members in (1.70) exist as Laplace transforms (of certain measures) for  $\Re \lambda < \lambda_+(z, \varepsilon)$  and  $\lambda \neq \lambda_+(z)$  and any fixed  $z \in \overline{K}(0, \delta_1)$ , this equality (1.70) holds in fact for such values of  $z$  and  $\lambda$ .

The equalities (1.71), (1.73) and the existence of  $A_-$  may be proved with the same method.  $\square$

It remains to give the main lines of the proof of Property 1.4.1.

*Proof of Property 1.4.1.* We just give the main steps of the proof for  $W'_+(z, \varepsilon, x)$ , which is quite similar to the one of Property 1.2.4 ; we also set  $\blacktriangle := \lambda_+(z, \varepsilon) + i\theta$ , and decompose  $W'_+(z, \varepsilon, x)$  as  $W'_+(z, \varepsilon, x) = W'_{+1}(z, \varepsilon, x) + W'_{+2}(z, \varepsilon, x) + W'_{+3}(z, \varepsilon, x)$  with

$$W'_{+1}(z, \varepsilon, x) := \int_{\mathbb{R}} \frac{e^{-i\theta x}}{\blacktriangle} [I - \mathcal{N}^* C_z(\blacktriangle)] d\theta,$$

$$W'_{+2}(z, \varepsilon, x) := \int_{\mathbb{R}} \frac{e^{-i\theta x}}{\blacktriangle} [I - \mathcal{N}^* C_z(\blacktriangle)][zP(\blacktriangle) + \dots + z^{n_1} P^{n_1}(\blacktriangle)][I - \mathfrak{L}(B)(z, \blacktriangle)]^{-1} d\theta,$$

$$W'_{+3}(z, \varepsilon, x) := \int_{\mathbb{R}} \frac{e^{-i\theta x}}{\blacktriangle} [I - \mathcal{N}^* C_z(\blacktriangle)] z^{n_1} [zP(\blacktriangle) + \dots + z^{n_1} P^{n_1}(\blacktriangle)] \\ \times [I - z^{n_1} P^{n_1}(\blacktriangle)]^{-1} \mathfrak{L}(\Phi_{n_1, \kappa})(\blacktriangle) [I - \mathfrak{L}(B)(z, \blacktriangle)]^{-1} d\theta.$$

To check that  $W'_{+1}(z, \varepsilon, x)$  is bounded uniformly in  $z \in \overline{K}(\delta_1, \delta_2)$  and  $x \geq 0$ , one first uses Lemma 1.2.6 to get

$$W'_{+1}(z, \varepsilon, x) = \int_{\mathbb{R}} \frac{e^{-ix\theta} I}{\lambda_+(z, \varepsilon) + i\theta} d\theta - \left( \mathbb{E}_i \left[ z^{T_-^*} e^{\lambda_+(z, \varepsilon) S_{T_-^*}} \int_{\mathbb{R}} \frac{e^{i\theta(S_{T_-^*} - x)}}{\blacktriangle} d\theta; X_{T_-^*} = j \right] \right)_{i,j} = 0.$$

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To control  $W'_{+2}(z, \varepsilon, x)$ , one uses the fact that the function

$$z \mapsto [I - \mathcal{N}^* C_z(\blacktriangle)][zP(\blacktriangle) + \cdots + z^{n_1} P^{n_1}(\blacktriangle)][I - \mathfrak{L}(B)(z, \blacktriangle)]^{-1}$$

is the Laplace transform at point  $\blacktriangle$  of the measure

$$\mu'(z, dy) = N_z(dy) \bullet [zM(dy) + \cdots + z^{n_1} M^{n_1}(dy)] \bullet \tilde{B}(z, dy),$$

where  $N_z(dy) := \left\{ \delta_{i,j}(dy) - \mathbb{E}_i(z^{T^*}, S_{T^*} \in dy, X_{T^*} = j) \right\}_{i,j}$  and one may conclude as in the proof of Property 1.2.4.

The control of  $W'_{+3}(z, \varepsilon, x)$  is like the one of  $W_{+3}(z, \varepsilon, x)$  in Property 1.2.4. The proof for  $W'_-(z, \varepsilon, x)$  and  $x < 0$  goes along the same lines.  $\square$

In the following Proposition, we specify the type of regularity of  $(I - \mathcal{P}B_z^*(\lambda))^{-1}$  and  $(I - \mathcal{N}^*C_z(\lambda))^{-1}$  on the domain  $K^*(\delta_1, \delta_2)$  for small enough  $\delta_1, \delta_2 > 0$  (by Corollary 1.3.1, we already know that they are analytic on  $D_{\rho, \theta} \cap \left( K(\delta_1, \delta_2) \right)^c$  for some suitable  $\rho > 1$  and  $\theta > 0$ ).

We set

$$F_{\pm}(z, \lambda) = I + \frac{\lambda_{\pm}(z) - a_{\pm}}{\lambda - \lambda_{\pm}(z)} \Pi_{\pm}(z),$$

where  $a_+ = \alpha_0 + 1$  and  $a_- = -\alpha_0 - 1$ . Recall that for  $z \in K(\delta_1, \delta_2)$ , the matrices  $\Pi_{\pm}(z) := \Pi(\lambda_{\pm}(z))$  are rank 1 and given by

$$\Pi_{\pm}(\lambda) = \left( e_i(\lambda_{\pm}(z)) \nu_j(\lambda_{\pm}(z)) \right)_{i,j \in E},$$

with  $\nu(\lambda_{\pm}(z)) e(\lambda_{\pm}(z)) = 1$ .

Note that  $F_{\pm}(z, \lambda)$  are analytic with respect to  $z \in K^*(\delta_1, \delta_2)$ , excepted at  $\frac{1}{k(\lambda)}$  (so that  $\lambda \neq \lambda_{\pm}(z)$ ).

On the other hand, one gets

$$F_+^{-1}(z, \lambda) = I - \frac{\lambda_+(z) - a_+}{\lambda - a_+} \Pi_+(z)$$

(and similarly  $F_-^{-1}(z, \lambda) = I - \frac{\lambda_-(z) - a_-}{\lambda - a_-} \Pi_-(z)$ )<sup>(e)</sup>. Let us emphasize that  $F_{\pm}^{-1}(z, \lambda)$  are analytic on  $K^*(\delta_1, \delta_2)$  (even at point  $\frac{1}{k(\lambda)}$ !).

We now set  $B(z, \lambda) = F_+(z, \lambda)(I - zP(\lambda))F_-(z, \lambda)$ ; by the above, the matrix  $B(z, \lambda)$  is invertible, we denote by  $B^{-1}(z, \lambda)$  its inverse; we also set  $B_+(z, \lambda) = F_+(z, \lambda)(I - \mathcal{P}B_z^*(\lambda))$  and  $B_-(z, \lambda) = (I - \mathcal{N}^*C_z(\lambda))F_-(z, \lambda)$ .

---

e. Remark that for any column vector  $a$  and row vector  $b$  one gets, setting  $ba = \beta \in \mathbb{C}$

$$\det(I - ab) = 1 - \beta \text{ and } (I - ab)^{-1} = I + (1 - \beta)^{-1} ab.$$

One applies these formulae to  $a = -\frac{\lambda_+(z) - a_+}{\lambda - \lambda_+(z)} \begin{pmatrix} e_1(\lambda_+(z)) \\ \vdots \\ e_N(\lambda_+(z)) \end{pmatrix}$  and  $b = (\nu_1(\lambda_+(z)), \dots, \nu_N(\lambda_+(z)))$  to

obtain the announced expression of  $F_{\pm}^{-1}(z, \lambda)$

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For  $z \in \overline{K}(\delta_1, 0)$ , according to the relation (1.67), we have

$$B(z, \lambda) = B_+(z, \lambda)B_-(z, \lambda), \quad |\Re \lambda| \leq \alpha_0, \quad (1.82)$$

$$B^{-1}(z, \lambda) = B_-^{-1}(z, \lambda)B_+^{-1}(z, \lambda), \quad \lambda \in S_z(\varepsilon). \quad (1.83)$$

The regularity of  $B(z, \lambda)$ ,  $B_\pm(z, \lambda)$  and  $B^{-1}(z, \lambda)$ ,  $B_\pm^{-1}(z, \lambda)$  is described in the following

**Proposition 1.4.2.** *For  $\delta_1, \delta_2, \varepsilon > 0$  small enough,  $z \in K^*(\delta_1, \delta_2)$  and  $\lambda \in S_z(\varepsilon)$ , one gets*

$$B(z, \lambda) \in V[-\alpha_0, +\alpha_0] \quad \text{and} \quad B^{-1}(z, \lambda) = V[\lambda_-(z, \varepsilon), \lambda_+(z, \varepsilon)].$$

Furthermore, the maps

- $z \mapsto B(z, \lambda)$ ,  $z \mapsto B_-(z, \lambda)$ ,  $z \mapsto B_+(z, \lambda)$
- $z \mapsto B^{-1}(z, \lambda)$ ,  $z \mapsto B_-^{-1}(z, \lambda)$ ,  $z \mapsto B_+^{-1}(z, \lambda)$
- $z \mapsto \mathcal{P}B_z^*(\lambda)$ ,  $z \mapsto \mathcal{N}^*C_z(\lambda)$ ,

admit an analytic expansion on  $K^*(\delta_1, \delta_2)$  and with respect to the variable  $t = \sqrt{1-z}$  for  $z \in K^*(\delta_1, \delta_2)$ .

Furthermore, the maps  $z \mapsto (I - \mathcal{P}B_z^*(\lambda))^{-1}$  and  $z \mapsto (I - \mathcal{N}^*C_z(\lambda))^{-1}$  are analytic on  $K^*(\delta_1, \delta_2)$  excepted at point  $\frac{1}{k(\lambda)}$ ; in particular, they are analytic on  $D_{\rho, \theta}$ .

*Proof.* We first assume that  $\delta_1$  is chosen in such a way that the conclusions of Proposition 1.4.1 are valid. Since  $B_+(z, \lambda) \in V[-\alpha_0, \alpha_0]$ , by the formula (1.70) in Proposition 1.4.1, we find

$$B_+^{-1}(z, \lambda) = \left( I + \int_0^{+\infty} e^{\lambda x} k_+(z, dx) \right) F_+^{-1}(z, \lambda) + \frac{(I - \mathcal{N}^*C_z(\lambda_+(z)))\Pi_+(z)}{(\lambda - a_+)\beta_+(z)}.$$

The equality (1.74) thus implies that  $B_+^{-1}(z, \lambda)$  is bounded for  $z \in \overline{K}(\delta_1, 0)$  and  $\lambda \in S_z(\varepsilon)$ . The same holds for  $B_-^{-1}(z, \lambda)$ .

The relations (1.82) and (1.83) show that  $B^{\pm 1}(z, \lambda)$  admit a canonical factorization for all  $z$  on the unit circle such that  $|\Im z| < \delta_1$ . Since these functions are regular with respect to the variable  $t = \sqrt{1-z}$  for  $z \in K^*(\delta_1, \delta_2)$ , we may by Lemma 1.3.3 adapt the choice of  $\delta_1$  and  $\delta_2$  in such a way that the components of factorizations (1.82) and (1.83), regarded as functions of  $t$ , admit an analytic expansion with respect to the variable  $t$ . By the identity

$$\mathcal{P}B_z^*(\lambda) = I - B_+(z, \lambda)F_+^{-1}(z, \lambda), \quad (1.84)$$

one obtains the expected regularity of the functions  $z \mapsto \mathcal{P}B_z^*(\lambda)$ .

At last, for  $z \neq 1/k(\lambda)$ , one gets by the previous equality

$$(I - \mathcal{P}B_z^*(\lambda))^{-1} = B_+^{-1}(z, \lambda)F_+(z, \lambda), \quad (1.85)$$

with  $F_+(z, \lambda)$  well defined and analytic in  $z$  since  $\lambda \neq \lambda_\pm(z)$  and one concludes.

The same holds similarly for  $\lambda \mapsto \mathcal{N}^*C_z(\lambda)$  and  $\lambda \mapsto (I - \mathcal{N}^*C_z(\lambda))^{-1}$ .  $\square$

**1.4.2 On the regularity of the factors  $I + \mathcal{N}^* B_z^*(\lambda)$  and  $I + \mathcal{P}C_z(\lambda)$  on  $D_{\rho,\theta}$  for  $\lambda \in \mathbb{R}^*$**

In this section we fix  $\rho > 1$  and  $\theta \in ]0, \pi/2[$  such that the conclusions of Corollary 1.3.1 hold. Recall that the matrices  $A_-$  and  $A_+$  in the following texte are the ones defined in Proposition 1.4.1. We prove the

**Theorem 1.4.1.** 1. For  $\lambda > 0$  (resp.  $\lambda < 0$ ) close to 0, the function  $I + \mathcal{N}^* B_z^*(\lambda)$  (resp.  $I + \mathcal{P}C_z(\lambda)$ ) admits an analytic expansion on  $D_{\rho,\theta}$ .

2. We have

$$\lim_{\lambda \rightarrow 0^+} \lambda(I + \mathcal{N}^* B_1^*(\lambda)) = A_-, \quad (1.86)$$

$$\lim_{\lambda \rightarrow 0^-} \lambda(I + \mathcal{P}C_1(\lambda)) = A_+, \quad (1.87)$$

with

$$-\frac{k''(0)}{2} A_- A_+ = \Pi(0). \quad (1.88)$$

*Proof.* 1. First case : when  $z \in D_{\rho,\theta} \setminus K(\delta_1, \delta_2)$  and  $\lambda \in \mathbb{R}^*$ , this is a direct consequence of Corollary 1.3.1.

Second case : when  $z \in K(\delta_1, 0)$ , by the first assertion of Theorem 1.3.1, we have

$$(I - \mathcal{N}^* C_z(\lambda))^{-1} = I + \mathcal{N}^* B_z(\lambda), \quad \Re \lambda \geq 0, \quad (1.89)$$

$$(I - \mathcal{P}B_z^*(\lambda))^{-1} = I + \mathcal{P}C_z(\lambda), \quad \Re \lambda \leq 0. \quad (1.90)$$

Now, by Proposition 1.4.2, the quantities of left hand-side of the above formulae are proved to be analytic with respect to  $z \in K^*(\delta_1, \delta_2)$  for some  $\delta_2 > 0$  small enough and for  $z \neq \frac{1}{k(\lambda)}$ . Recall that  $z \neq \frac{1}{k(\lambda)} \Leftrightarrow \lambda \neq \lambda_{\pm}(z)$  with  $\lambda_{\pm}(z) \notin D_{\rho,\theta}$  when  $\lambda$  is close to 0. We hence obtain the expected result, using the fact that  $D_{\rho,\theta} \subset (D_{\rho,\theta} \setminus K(\delta_1, \delta_2)) \cup K^*(\delta_1, \delta_2)$ .

2. The equalities (1.86) and (1.87) are direct consequences of Proposition 1.4.1. Indeed, according to this Proposition, one gets

$$\lim_{z \rightarrow 1} \lambda_+(z)(I - \mathcal{P}B_z^*(0))^{-1} = -A_+, \quad (1.91)$$

$$\lim_{z \rightarrow 1} \lambda_-(z)(I - \mathcal{N}^* C_z(0))^{-1} = -A_-. \quad (1.92)$$

On the other hand, for  $q < z < 1$ , one gets

$$(1 - z)(I - zP(0))^{-1} = [\sqrt{1 - z}(I - \mathcal{N}^* C_z(0))^{-1}][\sqrt{1 - z}(I - \mathcal{P}B_z^*(0))^{-1}],$$

with  $(I - zP(0))^{-1} = \frac{z\Pi(0)}{1 - z} + \sum_{n=0}^{+\infty} z^n R^n(0)$ ; so

$$\lim_{z \rightarrow 1} [\sqrt{1 - z}(I - \mathcal{N}^* C_z(0))^{-1}][\sqrt{1 - z}(I - \mathcal{P}B_z^*(0))^{-1}] = \Pi(0).$$

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Since  $\lim_{z \rightarrow 1} \frac{\sqrt{1-z}}{\lambda_-(z)} = -\sqrt{\frac{k''(0)}{2}}$  and  $\lim_{z \rightarrow 1} \frac{\sqrt{1-z}}{\lambda_+(z)} = \sqrt{\frac{k''(0)}{2}}$  (see (1.22)), we hence obtain

$$-\frac{k''(0)}{2} A_- A_+ = \Pi(0) = \begin{pmatrix} \nu_1 & \nu_2 & \cdots & \nu_N \\ \nu_1 & \nu_2 & \cdots & \nu_N \\ & \cdots & & \\ \nu_1 & \nu_2 & \cdots & \nu_N \end{pmatrix},$$

which yields the result.  $\square$

### 1.4.3 On the regularity of the factors $I + \mathcal{N}^* B_z^*(0)$ and $I + \mathcal{P}C_z(0)$ on $D_{\rho,\theta}$

We prove here the

**Theorem 1.4.2.** *The functions  $\sqrt{1-z}(I + \mathcal{N}^* B_z^*(0))$  and  $\sqrt{1-z}(I + \mathcal{P}C_z(0))$  admit an analytic expansion on  $D_{\rho,\theta}$  and may be continuously extended on  $\overline{D_{\rho,\theta}}$ . Furthermore, one gets*

$$\lim_{z \rightarrow 1} \sqrt{1-z}(I + \mathcal{N}^* B_z^*(0)) = \sqrt{\frac{k''(0)}{2}} A_-, \quad (1.93)$$

$$\lim_{z \rightarrow 1} \sqrt{1-z}(I + \mathcal{P}C_z(0)) = -\sqrt{\frac{k''(0)}{2}} A_+. \quad (1.94)$$

*Proof.* First case : when  $z \in D_{\rho,\theta} \setminus K(\delta_1, \delta_2)$ , the analysis of  $z \mapsto \sqrt{1-z}(I + \mathcal{P}C_z(0))$  (resp.  $z \mapsto \sqrt{1-z}(I + \mathcal{N}^* B_z^*(0))$ ) is derived from Corollary 1.3.1 and the fact that  $z \mapsto (I - zP(0))^{-1}$  is analytic in  $D_{\rho,\theta} \setminus K(\delta_1, \delta_2)$ .

Second case : the map  $z \mapsto \sqrt{1-z}(I - \mathcal{P}B_z^*(0))^{-1}$  is the analytic expansion on  $K^*(\delta_1, \delta_2)$  of  $z \mapsto \sqrt{1-z}(I + \mathcal{P}C_z(0))$ , and, by (1.85), one gets

$$\sqrt{1-z}(I - \mathcal{P}B_z^*(0))^{-1} = \sqrt{1-z} B_+^{-1}(z, 0) F_+(z, 0).$$

By Proposition 1.4.2, the map  $z \mapsto B_+^{-1}(z, 0)$  is analytic on  $K^*(\delta_1, \delta_2)$  and one gets

$$\sqrt{1-z} F_+(z, 0) = \sqrt{1-z} \left( I - \frac{\lambda_+(z) - a_+}{\lambda_+(z)} \Pi_+(z) \right),$$

so that  $\lim_{z \rightarrow 1} \sqrt{1-z} F_+(z, 0)$  exists since  $\frac{\sqrt{1-z}}{\lambda_+(z)} \rightarrow \sqrt{\frac{k''(0)}{2}}$  as  $z \rightarrow 1$ . Hence,

$$z \mapsto \sqrt{1-z}(I + \mathcal{P}C_z(0))$$

is analytic on  $D_{\rho,\theta}$ .  $\square$

## 1.5 Proofs of the local limit theorems

This section is devoted to the proof of our local limit theorems 1.1.1, 1.1.2 and 1.1.3.



### 1.5.1 Preliminaries

In the previous section, we have described the local behavior near  $z = 1$  of a family of analytic functions, expressed in terms of Laplace transforms ; we thus need some argument which relies the type of singularity near  $z = 1$  of such a function to the asymptotics of its Taylor coefficients at the origin. The following lemma is a classical result in the theory of complex variables functions.

**Lemma 1.5.1** ([10]). *For  $\rho > 1$  and  $0 < \theta < \pi/2$ , set*

$$D_{\rho,\theta} = \{z; z \neq 1, |\arg(z - 1)| > \theta > 0, |z| < \rho\}.$$

*If a function  $z \mapsto G(z)$  satisfies simultaneously the following three conditions:*

- *$G$  is analytic on  $D_{\rho,\theta}$  and can be written as  $G(z) = \sum_{n=0}^{+\infty} g_n z^n$ ;*
- $\lim_{\substack{z \in D_{\rho,\theta} \\ z \rightarrow 1}} \sqrt{1 - z} G(z) = C > 0$ ,

*then*

$$g_n \sim \frac{C}{\sqrt{\pi n}}, \quad n \rightarrow +\infty.$$

*Proof.* For the sake of completeness, we detail here the proof. For every  $\varepsilon > 0$ ,  $r \in ]1, \rho[$  and  $\theta' > \theta$ , let's consider the arcs  $\gamma_0 = \gamma_0(\varepsilon, \theta')$ ,  $\gamma_1 = \gamma_1(\varepsilon, r')$ ,  $\gamma'_1 = \gamma'_1(\varepsilon, r')$  and  $\gamma_2 = \gamma_2(r)$  defined respectively by

$$\gamma_0 := \{z = 1 + \varepsilon e^{-it}; \theta' \leq t \leq 2\pi - \theta'\}; \quad (1.95)$$

$$\gamma_1 := \{z = 1 + t e^{i\theta'}; \varepsilon \leq t \leq r'\} \quad \text{and} \quad \gamma'_1 := \{z = 1 + (r' - t) e^{i(2\pi - \theta')}; 0 \leq t \leq r' - \varepsilon\}; \quad (1.96)$$

$$\gamma_2 := \{z = r e^{it}; \theta'' \leq t \leq 2\pi - \theta''\}, \quad (1.97)$$

where  $r'$  and  $\theta''$  verify the following system of equations:

$$\begin{cases} r \cos \theta'' = 1 + r' \cos \theta'; \\ r \sin \theta'' = r' \sin \theta'. \end{cases}$$

Define a closed path  $\gamma(\varepsilon, r)$ , composed by the curves  $\gamma_0, \gamma_1, \gamma_2$  and  $\gamma'_1$ , as showed in Figure 1.5. We now introduce the complex function  $F(z)$  defined by

$$F(z) = G(z) - \frac{C}{\sqrt{1 - z}} := \frac{\delta(z)}{\sqrt{1 - z}}.$$

Since  $z \mapsto G(z)$  is analytic on  $D_{\rho,\theta}^\circ$ , so is  $F$  on this set and one may write, for  $|z| < 1$

$$F(z) = \sum_{n=0}^{+\infty} f_n z^n$$

where  $f_n = \frac{1}{2\pi i} \int_{\gamma} \frac{F(z)}{z^{n+1}} dz$  doest not depend on  $\varepsilon, r$  and  $\theta$ . By hypothesis, one may adapt the choice of  $\rho$  and  $\theta$  in such a way that there exists some constant  $M > 0$  such that

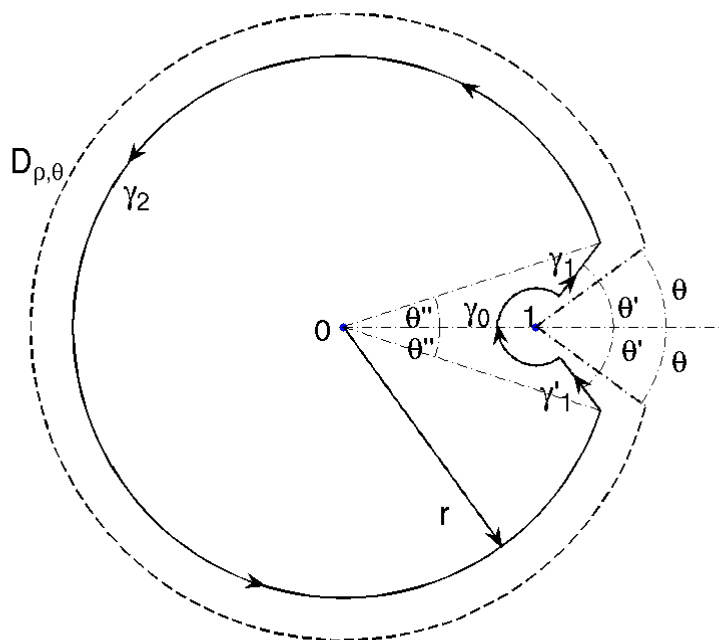


Figure 1.5: The closed path  $\gamma_0 \cup \gamma_1 \cup \gamma_2 \cup \gamma'_1$  of Lemma 1.5.1 and the open set  $D_{\rho, \theta}$ .

$|F(z)| \leq \frac{M}{|\sqrt{1-z}|}$  for  $z \in D_{\rho, \theta}^\circ$ ; one thus gets

$$\frac{1}{2\pi} \int_{\gamma_0} \frac{|F(z)|}{|z|^{n+1}} dz \leq \frac{M}{2\pi} \int_{\theta'}^{2\pi-\theta'} \frac{\sqrt{\varepsilon}}{(1+\varepsilon e^{it})^{n+1}} dt \leq \frac{M\sqrt{\varepsilon}}{|1-\varepsilon|^{n+1}}$$

and

$$\frac{1}{2\pi} \int_{\gamma_2} \frac{|F(z)|}{|z|^{n+1}} dz \leq \frac{M}{r^n \sqrt{r-1}}.$$

On the other hand,

$$\frac{1}{2\pi} \int_{\gamma_1 \cup \gamma_1'} \frac{|F(z)|}{|z|^{n+1}} dz \leq \frac{\delta(\varepsilon, r')}{\pi} \int_0^{r'} \frac{dt}{\sqrt{t}(1+t \cos \theta')^{n+1}},$$

with  $\delta(\varepsilon, r') := \sup_{z \in \gamma_1 \cup \gamma_1'} |\delta(z)|$ .

Set  $I_n(r') := \int_0^{r'} \frac{dt}{\sqrt{t}(1+t \cos \theta')^{n+1}}$ . Since  $\ln(1+u) \geq \frac{\ln r'}{r'} u$  as soon as  $0 \leq u \leq r'$ ,

for any  $t \in [0, r']$  one gets  $\ln(1+t \cos \theta') \geq \frac{\ln r'}{r'} t \cos \theta'$ , so that

$$I_n(r') = \int_0^{r'} \frac{e^{-(n+1)\ln(1+t \cos \theta')}}{\sqrt{t}} dt \leq \int_0^{r'} \frac{e^{-(n+1)\frac{\ln r'}{r'} t \cos \theta'}}{\sqrt{t}} dt.$$

Setting  $s = (n+1)t$ , one obtains  $I_n(r') \leq \frac{1}{\sqrt{n+1}} \int_0^{+\infty} \frac{e^{-\frac{\ln r'}{r'} s \cos \theta'}}{\sqrt{s}} ds$ , i.e.  $\sqrt{n} I_n(r') \leq M'$

for some constant  $M' \in ]0, +\infty[$ ; this readily implies  $\frac{1}{2\pi} \int_{\gamma_1 \cup \gamma_1'} \frac{|F(z)|}{|z|^{n+1}} dz \leq \delta(\varepsilon, r') M'$ . In summary one gets

$$\sqrt{n}|f_n| \leq \frac{M\sqrt{\varepsilon n}}{|1-\varepsilon|^{n+1}} + \frac{M\sqrt{n}}{r^n \sqrt{r-1}} + \delta(\varepsilon, r') M',$$

so that  $\sqrt{n}|f_n| \leq \frac{M\sqrt{n}}{r^n \sqrt{r-1}} + \delta(0, r') M'$  since  $\varepsilon$  may be chosen arbitrarily small. Letting now  $n \rightarrow +\infty$ , one gets, since  $r > 1$

$$\limsup_{n \rightarrow +\infty} \sqrt{n}|f_n| \leq \delta(0, r') M'$$

and one concludes that  $\sqrt{n}f_n \rightarrow 0$  as  $n \rightarrow +\infty$  noticing that  $\lim_{r' \rightarrow 0} \delta(0, r') = 0$ .

One achieves the proof writing  $g_n = f_n + Ca_n$  with  $a_n = \frac{2n!}{4^n (n!)^2} = \frac{1+o(n)}{\sqrt{\pi n}}$ , so that

$$g_n = f_n + Ca_n \sim \frac{C}{\sqrt{\pi n}} \quad \text{as } n \rightarrow +\infty.$$

□

### 1.5.2 Proof of Theorem 1.1.1

We fix  $\lambda > 0$  and set, for any  $i, j \in E$  and  $z \in \mathbb{D}^\circ$

$$G_{i,j}(z, \lambda) := \sum_{n=0}^{+\infty} z^n \mathbb{E}_i(e^{\lambda m_n}; X_n = j)$$

and

$$H_{i,j}(z, \lambda) := \sqrt{1-z} G_{i,j}(z, \lambda).$$

By lemma 1.3.1, we have

$$H_{i,j}(z, \lambda) = \{[I + \mathcal{N}^* B_z^*(\lambda)] \sqrt{1-z} [I + \mathcal{P}C_z(0)]\}_{i,j}.$$

By (1.94), we get

$$H_{i,j}(\lambda) := \lim_{z \rightarrow 1} H_{i,j}(z, \lambda) = -\sqrt{\frac{k''(0)}{2}} \{(I + \mathcal{N}^* B_1^*(\lambda)) A_+\}_{i,j}. \quad (1.98)$$

By (1.86) and (1.88), we obtain

$$\lim_{\lambda \rightarrow 0^+} \lambda H_{i,j}(\lambda) = -\sqrt{\frac{k''(0)}{2}} (A_- A_+)_{i,j} = \sqrt{\frac{2}{k''(0)}} (\Pi(0))_{i,j}. \quad (1.99)$$

On the other hand, since  $P_{i,j}^{(n)} \xrightarrow{n \rightarrow +\infty} \Pi(0)_{i,j} = \nu_j > 0$ , one gets

$$\lim_{\lambda \rightarrow 0^+} \lambda H_{i,j}(\lambda) = \sqrt{\frac{2}{k''(0)}} \nu_j > 0. \quad (1.100)$$

From (1.87), the coefficients of  $A_+$  are  $\leq 0$ , the function  $H_{i,j}$  is in fact the Laplace transform of a positive measure  $\mu_{i,j}$  on  $\mathbb{R}_-$  and this measure is  $\neq 0$  by (1.99); in particular, there exists an interval  $[a, b] \subset \mathbb{R}_-$  such that  $\mu_{i,j}([a, b]) > 0$ . Therefore, for all  $\lambda > 0$ , one gets

$$H_{i,j}(\lambda) = \int_{-\infty}^0 e^{\lambda x} d\mu_{i,j}(x) \geq \int_a^b e^{\lambda x} d\mu_{i,j}(x) \geq e^{\lambda a} \mu_{i,j}([a, b]) > 0. \quad (1.101)$$

Consequently, by the above, for any  $\lambda > 0$ , the function  $z \mapsto G_{i,j}(z, \lambda)$  is analytic on  $D_{\rho, \theta}$  and  $z \mapsto \sqrt{1-z} G_{i,j}(z, \lambda)$  is bounded on  $D_{\rho, \theta}$ . By Lemma 1.5.1, we obtain

$$\sqrt{n} \mathbb{E}_i(e^{\lambda m_n}, X_n = j) \xrightarrow{n \rightarrow +\infty} \frac{H_{i,j}(\lambda)}{\sqrt{\pi}}. \quad (1.102)$$

### 1.5.3 Proof of Theorem 1.1.2

In this paragraph, we specify the previous statement in terms of distribution function. The arguments that we use to obtain the limit of a sequence of the distribution functions are well developed in the i.i.d case (see for example [19]). So in the markovian case, it is natural to use a similar method. This is quite realizable, except that we are in a framework

of the matrices of Laplace transforms and those of the measures are somehow markovian dependant. Given the limit of the Laplace transforms, the existence of the limit distribution functions is stated in the extended continuity theorem (W. Feller [9]). Additionally, given the local expansion of the Laplace transform limit at 0, the Tauberian theorem (W. Feller [9]) allows us to obtain the asymptotic behavior of the limit distribution function at infinity. In the following text, we give a detailed explanation of this development.

We first introduce, for any any  $(i, j) \in E \times E$ , the distribution function  $h_{i,j} : \mathbb{R}_+ \rightarrow \mathbb{R}$  of the measure  $\mu_{i,j}$ , defined by

$$h_{i,j}(x) = \begin{cases} -\sqrt{\frac{k''(0)}{2\pi}} \{ [I + \mathcal{N}^* B_1^*(1_{[-x,0]})] A_+ \}_{i,j}, & x > 0; \\ -\sqrt{\frac{k''(0)}{2\pi}} (A_+)_{i,j}, & x = 0; \end{cases}$$

where  $\mathcal{N}^* B_1^*(1_{[-x,0]}) = \sum_{n=1}^{+\infty} z^n \mathbb{P}_i(S_1 > S_n, S_2 > S_n, \dots, S_{n-1} > S_n, -x \leq S_n \leq 0, X_n = j)$ , for  $x > 0$ . We will decompose the “ potential ”  $\mathcal{N}^* B_1^*(1_{[-x,0]})$  in terms of the ladder epochs  $\{\tau_j\}_{j \geq 0}$  of the random walk  $(S_n)_n$ , defined recursively by :

$$\tau_0 = 0 \quad \text{and} \quad \tau_j = \inf\{n; \text{ for all } n \geq \tau_{j-1}, S_n < S_{\tau_{j-1}}\}, \text{ for } j \geq 1.$$

For any  $x \in \mathbb{R}^{*+}$  and  $l \geq 0$ , we thus consider the matrix  $B_l^*(x)$  defined by

$$B_l(x) = \left( B_l(x)_{i,j} \right)_{i,j},$$

with  $B_l(x)_{i,j} = \sum_{k \in E} \mathbb{P}_i(S_{\tau_l} \geq -x, X_{\tau_l} = k)(A_+)_{k,j}$ .

One gets

$$h_{i,j}(x) = \sum_{l \geq 0} B_l^*(x)_{i,j} = \sum_{k \in E} \mathbb{E}_i \left[ \sum_{l \geq 0} 1_{[-x,0]}(S_{\tau_l}), X_{\tau_l} = k \right] (A_+)_{k,j}.$$

Notice that, for  $x$  large enough, one gets  $\mathbb{E}_i \left[ \sum_{l \geq 0} 1_{[-x,0]}(S_{\tau_l}), X_{\tau_l} = k \right] > 0$  for any  $i, k \in E$  since  $S_{\tau_l}$  is finite  $\mathbb{P}_i$ -a.s. ; so is  $h_{i,j}(x)$ , since by 1.88 at least one of the terms  $(A_+)_{k,j}$  is non negative. We will see that this property holds in fact for any  $x \geq 0$ .

First, one gets the

**Lemma 1.5.2.** *For any  $(i, j) \in E \times E$ , we have*

$$\sqrt{n} \mathbb{P}_i(m_n = 0, X_n = j) = \mathbb{P}_i(T_-^* > n, X_n = j) \longrightarrow \left( -\sqrt{\frac{k''(0)}{2\pi}} \right) (A_+)_{i,j}, \quad \text{as } n \rightarrow +\infty.$$

*Proof.* Indeed, (1.94) may be restated as follows

$$\sqrt{1-z} \left[ I + \sum_{n=1}^{+\infty} z^n \mathbb{P}_i(m_n = 0, X_n = j) \right] = \sqrt{1-z} [I + \mathcal{P}C_z(0)] \xrightarrow{z \rightarrow 1} -\sqrt{\frac{k''(0)}{2}} (A_+)_{i,j}$$

so that, by Lemma 1.5.1 (when  $-(A_+)_{i,j} > 0$ ),

$$\sqrt{n} \mathbb{P}_i(m_n = 0, X_n = j) \xrightarrow{n \rightarrow +\infty} -\sqrt{\frac{k''(0)}{2\pi}} (A_+)_{i,j}. \quad (1.103)$$

The same result holds when  $-(A_+)_{i,j} = 0$ , by Corollary 1 in [10].  $\square$

We will use the following

**Lemma 1.5.3.** *For any  $l \geq 1$ , any  $i, j \in E$  and  $x > 0$  such that  $h_{i,j}$  is discontinuous at  $x$ , we have*

$$\liminf_{n \rightarrow +\infty} \sqrt{n} \mathbb{P}_i(S_{\tau_l} \geq -x, \tau_l \leq n, \tau_{l+1} > n, X_n = j) \geq -\sqrt{\frac{k''(0)}{2\pi}} B_l(x)_{i,j}. \quad (1.104)$$

*Proof.* For any  $0 < \delta < 1$ , we have

$$\mathbb{P}_i(S_{\tau_l} \geq -x, \tau_l \leq n, \tau_{l+1} > n, X_n = j) \geq \mathbb{P}_i(S_{\tau_l} \geq -x, \tau_l \leq \delta n, \tau_{l+1} > n, X_n = j). \quad (1.105)$$

From Markov property, we have

$$\begin{aligned} \mathbb{P}_i(S_{\tau_l} \geq -x, \tau_l \leq \delta n, \tau_{l+1} > n, X_n = j) &= \sum_{k \in E} \mathbb{E}_i[(S_{\tau_l} \geq -x, \tau_l \leq \delta n, X_{\tau_l} = k) \mathbb{P}_k(\tau_{l+1} > n, X_{n-\tau_l} = j)] \\ &= \sum_{\substack{k \in E \\ 0 \leq p \leq \delta n}} \mathbb{E}_i[(S_p \geq -x, \tau_l = p, X_p = k) \mathbb{P}_k(\tau_{l+1} > n, X_{n-p} = j)]. \end{aligned}$$

In addition, one gets

$$\sqrt{n} \mathbb{P}_k(\tau_{l+1} > n, X_{n-p} = j) = \sqrt{n} \mathbb{P}_k(\tau_{l+1} > n-p, X_{n-p} = j) - \sqrt{n} \mathbb{P}_k(n-p < \tau_{l+1} \leq n, X_{n-p} = j).$$

Since  $0 \leq \mathbb{P}_k(n-p < \tau_{l+1} \leq n, X_{n-p} = j) \leq \mathbb{P}_k(n-p < \tau_{l+1} \leq n)$ , by Lemma 1.5.2, we hence obtain that

$$\sqrt{n} \mathbb{P}_k(n-p < \tau_{l+1} \leq n) = \sqrt{n} \mathbb{P}_k(\tau_{l+1} > n-p) - \sqrt{n} \mathbb{P}_k(\tau_{l+1} > n) \rightarrow 0, \quad \text{as } n \rightarrow +\infty.$$

So we have  $\lim_{n \rightarrow +\infty} \sqrt{n} \mathbb{P}_k(n-p < \tau_{l+1} \leq n, X_{n-p} = j) = 0$ . By lemma 1.5.2, we get

$$\lim_{n \rightarrow +\infty} \sqrt{n} \mathbb{P}_k(\tau_{l+1} > n, X_{n-p} = j) = -\sqrt{\frac{k''(0)}{2\pi}} (A_+)_{k,j}.$$

Using Fatou's lemma and the inequality (1.105), one concludes

$$\begin{aligned} \liminf_{n \rightarrow +\infty} \sqrt{n} \mathbb{P}_i(S_{\tau_l} \geq -x, \tau_l \leq n, \tau_{l+1} > n, X_n = j) &\geq \liminf_{n \rightarrow +\infty} \sqrt{n} \mathbb{P}_i(S_{\tau_l} \geq -x, \tau_l \leq \delta n, \tau_{l+1} > n, X_n = j) \\ &\geq \left( -\sqrt{\frac{k''(0)}{2\pi}} \right) \sum_{k \in E} \mathbb{P}_i(S_{\tau_l} \geq -x, X_{\tau_l} = k) (A_+)_{k,j} \\ &= \left( -\sqrt{\frac{k''(0)}{2\pi}} \right) B_l(x)_{i,j}. \end{aligned}$$

$\square$

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*Proof of Theorem 1.1.2.* From (1.102) and the extended continuity theorem (Thm 2a, XIII.1, W. Feller [9]), for any  $(i, j) \in E \times E$  and any  $x > 0$  such that  $h_{i,j}(\cdot)$  is continuous at  $x$ , one gets

$$\lim_{n \rightarrow +\infty} \sqrt{n} \mathbb{P}_i(m_n \geq -x, X_n = j) = h_{i,j}(x);$$

By Lemma 1.5.2, the same result holds for  $x = 0$ .

Now, fix  $x > 0$  such that  $h_{i,j}(\cdot)$  is discontinuous at  $x$ . The map  $x \mapsto h_{i,j}(x)$  being increasing and right-continuous on  $\mathbb{R}_+^*$ , the set of its points of discontinuity is countable and there thus exists a sequence  $(\varepsilon_k)_{k \geq 1}$  of non negative reals converging towards 0 and such that  $h_{i,j}$  is continuous at  $x + \varepsilon_k$  for any  $k \geq 1$ ; consequently, for any  $k \geq 1$  one gets

$$\sqrt{n} \mathbb{P}_i(m_n \geq -x, X_n = j) \leq \sqrt{n} \mathbb{P}_i(m_n \geq -x - \varepsilon_k, X_n = j)$$

and so

$$\limsup_{n \rightarrow +\infty} \sqrt{n} \mathbb{P}_i(m_n \geq -x, X_n = j) \leq h_{i,j}(x + \varepsilon_k).$$

The map  $h_{i,j}$  being right continuous, one gets

$$\limsup_{n \rightarrow +\infty} \sqrt{n} \mathbb{P}_i(m_n \geq -x, X_n = j) \leq h_{i,j}(x). \quad (1.106)$$

On the other hand, for any  $N \leq n$  and  $0 \leq l < N$ , one gets

$$\mathbb{P}_i(m_n \geq -x, X_n = j) \geq \sum_{l=0}^N \mathbb{P}_i(S_{\tau_l} \geq -x, \tau_l \leq n, \tau_{l+1} > n, X_n = j)$$

which readily implies, by Lemma 1.5.3

$$\begin{aligned} \liminf_{n \rightarrow +\infty} \sqrt{n} \mathbb{P}_i(m_n \geq -x, X_n = j) &\geq \sum_{l=0}^N \liminf_{n \rightarrow +\infty} \sqrt{n} \mathbb{P}_i(S_{\tau_l} \geq -x, \tau_l \leq n, \tau_{l+1} > n, X_n = j) \\ &\geq \left( -\sqrt{\frac{k''(0)}{2\pi}} \right) \sum_{l=0}^N B_l(x)_{i,j} \\ &= \left( -\sqrt{\frac{k''(0)}{2\pi}} \right) \sum_{l=0}^N \left( \sum_{k \in E} \mathbb{P}_i(S_{\tau_l} \geq -x, X_{\tau_l} = k) (A_+)_{k,j} \right) \\ &\xrightarrow{N \rightarrow +\infty} \left( -\sqrt{\frac{k''(0)}{2\pi}} \right) [(I + \mathcal{N}^* B_1^*(1_{[-x,0]})) A_+]_{i,j} = h_{i,j}(x). \end{aligned} \quad (1.107)$$

Combining (1.106) and (1.107), one gets the expected conclusion at  $x$ .

Now we are going to prove that for any  $j \in E$ , the function  $(x, i) \mapsto h_{i,j}(x)$  is harmonic with respect to  $(S_n, X_n)$  and positive on  $\mathbb{R} \times E$ . One gets

$$\sqrt{n+1} \mathbb{P}_i(m_{n+1} \geq -x, X_{n+1} = j) = \sqrt{\frac{n+1}{n}} \sum_{i_1 \in E} \int p_{i,i_1} \sqrt{n} \mathbb{P}_{i_1}(m_n \geq -x - y_1, X_{n+1} = j) F(i, i_1, dy_1)$$

with  $\mathbb{E}_i(x + |Y_1|) = x + \sum_{j \in E} p_{i,j} \int_{\mathbb{R}} |u| F(i, j, du) < +\infty$ . We now need the following lemma, whose proof will be stated later:

**Lemma 1.5.4.** *There exists a constant  $C > 0$  such that for all  $(i, j) \in E \times E$ ,*

$$\forall i, j \in E, \forall x \geq 0, \quad \sqrt{n} \mathbb{P}_i(m_n \geq -x, X_n = j) \leq C(x + 1). \quad (1.108)$$

By the dominated convergence theorem, one thus gets

$$\forall x \geq 0, \quad h_{i,j}(x) = \sum_{i_1 \in E} \int p_{i,i_1} h_{i_1,j}(y_1 + x) F(i, i_1, dy_1) = \mathbb{E}_i[h_{X_1,j}(x + Y_1)], \quad (1.109)$$

which means that  $(x, i) \mapsto h_{i,j}$  is harmonic for  $(S_n, X_n)$  on  $\mathbb{R}^+ \times E$ .

By equality (1.2) of Theorem 1.1.1, for  $\lambda > 0$ , one gets  $\lim_{\tau \rightarrow 0^+} \frac{H_{i,j}(\tau\lambda)}{H_{i,j}(\tau)} = \frac{1}{\lambda}$  and the classical Tauberian theorem (see for instance Thm 1, XIII.5, W. Feller [9]), we get

$$h_{i,j}(x) = \mu_{i,j}([-x, 0]) \sim \frac{H_{i,j}(1/x)}{\Gamma(2)} \sim \sqrt{\frac{2}{k''(0)}} \nu_j x \quad \text{as } x \rightarrow +\infty. \quad (1.110)$$

At last, assume that there exists  $(i_0, j_0) \in E \times E$  such that  $h_{i_0,j_0}(0) = 0$ . Iterating Formula (1.109), one gets for any  $n \geq 1$ ,

$$h_{i,j}(0) = \mathbb{E}_{i_0}[h_{X_n,j}(S_n)],$$

so that

$$h_{X_n,j_0}(S_n) = 0 \quad \mathbb{P}_{i_0} \text{ a.s.} \quad (1.111)$$

By (1.110), there exists  $M_{j_0} \geq 1$ , such that for  $x \geq M_{j_0}$ ,

$$\inf_{i \in E} h_{i,j_0}(x) \geq \frac{1}{2} \sqrt{\frac{2}{k''(0)}} \nu_{j_0} > 0 \quad (1.112)$$

and the central limit theorem for Markov chains ([13]) implies that for any  $i \in E$ ,

$$\mathbb{P}_i \left( \frac{S_n}{\sqrt{n}} \geq M_{j_0} \right) \xrightarrow{n \rightarrow +\infty} \frac{1}{\sqrt{\pi k''(0)}} \int_{M_{j_0}}^{+\infty} e^{-\frac{x^2}{2k''(0)}} dx := \alpha(M_{j_0}) > 0.$$

Setting  $B_n = \left\{ \omega : \frac{S_n(\omega)}{\sqrt{n}} \geq M_{j_0} \right\}$  and  $B = \limsup_{n \rightarrow +\infty} B_n$ , then for all  $i \in E$ , one thus may write

$$\mathbb{P}_i(B) = \lim_{m \rightarrow +\infty} \mathbb{P}_i \left( \bigcup_{n \geq m} B_n \right) \geq \lim_{m \rightarrow +\infty} \mathbb{P}_i(B_m) = \alpha(M_{j_0}) > 0.$$

For all  $\omega \in B$ , one gets  $\limsup_{x \rightarrow +\infty} [S_n(\omega)] = +\infty$  and so, by (1.112), one obtains

$$\limsup_{n \rightarrow +\infty} \inf_{i \in E} [h_{i,j_0}(x_0 + S_n) 1_B] \geq \frac{1}{2} \sqrt{\frac{2}{k''(0)}} \nu_{j_0} > 0 \quad \mathbb{P}_{i_0}\text{-a.s.}$$

This contradicts (1.111) since  $\mathbb{P}_i(B) > 0$ , for any  $i \in E$ . Then, for any  $i, j \in E$  and  $x \geq 0$  one gets  $h_{i,j}(x) \geq h_{i,j}(0) > 0$ .  $\square$



## 1.5. PROOFS OF THE LOCAL LIMIT THEOREMS

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It remains to prove Lemma 1.5.4; we will use the two following facts, whose proofs may be found in [19] :

**Fact 1.5.1** ([19]). *Let  $c, \nu \in \mathbb{R}_+^*$  and  $(a_n)_{n \geq 0}$  be a monotone sequence of non negative reals such that  $\sum_{n=0}^{+\infty} a_n s^n \leq c(1-s)^{-\nu}$  for any  $s \in [0, 1[$ . Then*

$$\forall n \geq 2, \quad a_n \leq ce(1-e^{-1})^{-\nu} 2^{1+\nu} n^{\nu-1}.$$

**Fact 1.5.2** ([19]). *Let  $H$  be a non-decreasing function on  $\mathbb{R}^+$  such that  $H(0) = 0$  and the integral  $\tilde{H}(\lambda) := \int_0^{+\infty} e^{-\lambda x} dH(x)$  does exist for any  $\lambda > 0$ . If there exist  $\delta, \gamma > 0$  such that*

$$\forall \lambda \in ]0, \delta], \quad \tilde{H}(\lambda) \leq c\lambda^{-\gamma},$$

then, for all  $x \geq \delta^{-1}$ , one gets  $H(x) \leq c e x^\gamma$ .

*Proof of Lemma 1.5.4.* Taking into account (1.99) and (1.88), we get for any  $i \in E$ ,

$$\lim_{\lambda \rightarrow 0} \lambda \sum_{j \in E} H_{i,j}(\lambda) = -\sqrt{\frac{k''(0)}{2}} \sum_{j \in E} (A_- A_+)_{i,j} = \sqrt{\frac{2}{k''(0)}} > 0,$$

which implies that there exist two constants  $\delta > 0$  and  $c > 0$  such that for any  $\lambda \in ]0, \delta]$  and  $s \in ]0, 1[$ ,

$$\sup_{i \in E} \sum_{n=0}^{+\infty} s^n \mathbb{E}_i(e^{\lambda m_n}) \leq c\lambda^{-1}(1-s)^{-1/2}.$$

For  $\lambda > 0$ , the sequence  $\left( \mathbb{E}(e^{\lambda m_n}) \right)_{n \geq 0}$  is decreasing with respect to  $n$  and the Fact 1.5.1 with  $\nu = 1/2$  leads to

$$\forall i \in E, \forall n \geq 2, \forall \lambda \in ]0, \delta], \quad \sqrt{n} \mathbb{E}_i(e^{\lambda m_n}) \leq ce(1-e^{-1})^{-1/2} 2^{3/2} \lambda^{-1}.$$

Applying now Fact 1.5.2 with  $\gamma = 1$ , we get, for all  $x \geq \delta^{-1} > 0$ ,  $n \geq 2$  and  $i, j \in E$ ,

$$\sqrt{n} \mathbb{P}_i(m_n \geq -x, X_n = j) \leq \sqrt{n} \mathbb{P}_i(m_n \geq -x) \leq c_1 x,$$

where  $c_1 = ce^2(1-e^{-1})^{-1/2} 2^{3/2}$ .

On the other hand, for  $0 \leq x < \delta^{-1}$ , one gets

$$\sqrt{n} \mathbb{P}_i(m_n \geq -x, X_n = j) \leq \sqrt{n} \mathbb{P}_i(m_n \geq -\delta^{-1}, X_n = j) \xrightarrow{n \rightarrow +\infty} h_{i,j}(\delta^{-1})$$

and one thus may write, , for any  $i, j \in E$  and  $x \geq 0$ ,

$$\sqrt{n} \mathbb{P}_i(m_n \geq -x, X_n = j) \leq c_1 x + c_2$$

where  $c_2 = \sup_{\substack{n \geq 1 \\ i, j \in E}} \mathbb{P}_i(m_n \geq -\delta^{-1}, X_n = j)$ . □

We end this section with the following elementary consequence of the above :

**Fact 1.5.3.** *There exists a constant  $c \geq 1$  such that, for any  $i, j \in E$  and  $x \geq 0$  one gets*

$$\frac{x+1}{c} \leq h_{i,j}(x) \leq c(x+1)$$

*Proof.* By (1.110), there exists  $c_1 > 0$  and  $x_1 \geq 0$  such that  $\frac{x+1}{c_1} \leq h_{i,j}(x) \leq c_1(x+1)$  for  $x \geq x_1$ . For  $0 \leq x \leq x_1$  one thus gets

$$\frac{h_{i,j}(0)}{c_1 h_{i,j}(x_1)}(1+x) \leq h_{i,j}(0) \leq h_{i,j}(x) \leq h_{i,j}(x_1) \leq c_1(x+1)(x_1+1).$$

and one set  $c := \max(c_1(x_1+1), c_1 \frac{h_{i,j}(x_1)}{h_{i,j}(0)})$ . □

#### 1.5.4 Proof of Theorem 1.1.3

*Proof.* By the Markov property and Fubini's theorem, we have, for  $0 < \varepsilon < \lambda$ ,

$$\begin{aligned} & \sum_{n=0}^{+\infty} z^n \mathbb{E}_i(e^{\lambda m_n - \varepsilon S_n}, X_n = j) \\ &= \sum_{n=0}^{+\infty} z^n [\delta_{i,j} + \sum_{k=1}^n \mathbb{E}_i(e^{\lambda S_k - \varepsilon S_n}, S_0 > S_k, \dots, S_{k-1} > S_k, S_{k+1} \geq S_k, \dots, S_n \geq S_k, X_n = j)] \\ &= \sum_{n=0}^{+\infty} z^n \left\{ \delta_{i,j} + \sum_{k=1}^n \sum_{l \in E} \mathbb{E}_i \left[ e^{(\lambda - \varepsilon) S_k}, S_0 > S_k, \dots, S_{k-1} > S_k, X_k = l \right] \times \right. \\ & \quad \left. \mathbb{E}_l [e^{-\varepsilon S_{n-k}}, S_1 \geq 0, \dots, S_{n-k} \geq 0, X_{n-k} = j] \right\} \\ &= \sum_{l \in E} \left[ \sum_{k=0}^{+\infty} z^k \mathbb{E}_i(e^{(\lambda - \varepsilon) S_k}; S_1 > S_k, \dots, S_{k-1} > S_k, S_k < 0, X_k = l) \right] \times \\ & \quad \left[ \sum_{p=0}^{+\infty} z^p \mathbb{E}_l(e^{-\varepsilon S_p}; S_1 \geq 0, \dots, S_p \geq 0, X_p = j) \right] \\ &= \left\{ (I + \mathcal{N}^* B_z^*(\lambda - \varepsilon))(I + \mathcal{P}C_z(-\varepsilon)) \right\}_{i,j}. \end{aligned}$$

So by the first assertion of Theorem 1.4.1, letting  $z \rightarrow 1$ , one obtains

$$\sum_{n=0}^{+\infty} \mathbb{E}_i(e^{\lambda m_n - \varepsilon S_n}, X_n = j) = \{(I + \mathcal{N}^* B_1^*(\lambda - \varepsilon))(I + \mathcal{P}C_1(-\varepsilon))\}_{i,j} < +\infty.$$

□



## Chapitre 2

# Applications à la probabilité de survie d'un processus de branchement en environnement aléatoire markovien

### 2.1 Model and result

Consider the following model:  $X = (X_n)_{n \geq 0}$  is an irreducible and aperiodic Markov chain on a finite space  $E$  with transition matrix  $P = (p_{i,j})_{i,j}$ . The chain  $X$  has a unique invariant probability  $\nu$ . We denote by  $G$  the set of generating functions of probability measures on  $\mathbb{N}$ , equipped with the topology of simple convergence on  $[0, 1]$  and by  $\mathcal{B}(G)$  its Borel  $\sigma$ -algebra. Let also  $\bar{F} := (\bar{F}(i, j, dt))_{i,j \in E}$  be a family of probability measures on  $G$ .

We consider now the Markov chain  $(M_n)_{n \geq 0} = (g_n, X_n)_{n \geq 0}$  with values in  $G \times E$  whose transition probability  $Q$  is given by : for any  $(g, i) \in G \times E$  and  $(A, j) \in \mathcal{B}(G) \times E$ ,

$$Q\{(g, i), (A \times \{j\})\} = p_{i,j} \bar{F}(i, j, A).$$

The Markov chain  $(M_n)_{n \geq 0}$  is called the *environment process*.

Let  $\Omega = (G \times E)^{\mathbb{N}}$  and  $\mathcal{F} = \bigotimes^{\mathbb{N}} (\mathcal{B}(G) \otimes \mathcal{P}(E))$ . For any  $(g_0, i) \in G \times E$ , we denote by  $\mathbb{P}_{(g_0, i)}$  the unique probability measure on  $(\Omega, \mathcal{F})$  defined by : for any  $n \geq 1$  and any bounded measurable function  $f : (G \times E)^{n+1} \rightarrow \mathbb{R}$ ,

$$\begin{aligned} & \int_{\Omega} f(M_0(\omega), M_1(\omega), \dots, M_n(\omega)) \mathbb{P}_{(g_0, i)}(d\omega) \\ = & \sum_{(j_1, j_2, \dots, j_n) \in E^n} p_{i, j_1} \cdots p_{j_{n-1}, j_n} \int_{G^n} f((g_0, i), (g_1, j_1), \dots, (g_n, j_n)) \bar{F}(i, j_1, dg_1) \cdots \bar{F}(j_{n-1}, j_n, dg_n). \end{aligned}$$

We denote by  $\mathbb{E}_{(g_0, i)}$  the corresponding expectation. To simplify the notations,  $\mathbb{P}_{(Id, i)}$  (resp.  $\mathbb{E}_{(Id, i)}$ ) will be denoted by  $\mathbb{P}_i$  (resp.  $\mathbb{E}_i$ ).

## 2.2. PRELIMINARIES AND NOTATIONS

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Given  $(M_n)_{n \geq 0}$ , we define now the branching process  $(Z_n)_{n \geq 0}$  such that  $Z_0 = 1$  and the generating function of  $Z_n$  is

$$g_0 \circ g_1 \circ \cdots \circ g_{n-1}(s) := G_n(s), \quad 0 \leq s < 1.$$

Therefore, given  $(M_n)_{n \geq 0}$ , the survival probability of the branching process  $(Z_n)_{n \geq 0}$  at time  $n$  is equal to

$$q_n := 1 - G_n(0).$$

Let's consider  $h : G \rightarrow \overline{\mathbb{R}}_+$ ,  $g \mapsto h(g) := \ln g'(1)$ . The image of the probability  $\overline{F}(i, j, dx)$  by the map  $h$  is denoted by  $F(i, j, dx)$ . We set  $F := (F(i, j, dt))_{i, j \in E}$  and one assumes that  $F$  satisfies the following hypotheses (H):

**H1** there exist  $\alpha > 0$ , such that for all  $\lambda \in \mathbb{C}$  satisfying  $|\Re \lambda| \leq \alpha$ , we have

$$\max_{(i, j) \in E \times E} |\widehat{F}(i, j, \lambda)| < +\infty, \quad \text{where } \widehat{F}(i, j, \lambda) = \int_{\mathbb{R}} e^{\lambda t} F(i, j, dt);$$

**H2** there exist  $n_0 \geq 1$  and  $(i_0, j_0) \in E \times E$ , such that the measure  $\mathbb{P}_{i_0}(X_{n_0} = j_0, S_{n_0} \in dx)$  has an absolutely continuous component with respect to the Lebesgue measure on  $\mathbb{R}$ ;

**H3**  $\sum_{(i, j) \in E \times E} \nu_i p_{i, j} \int_{\mathbb{R}} t F(i, j, dt) = 0$ , where  $\nu = (\nu_i)_{i \in E}$  is the stationary distribution of the Markov chain  $X = (X_n)_{n \geq 0}$ .

We prove here the following result:

**Theorem 2.1.1.** *Under hypotheses (H), for any  $(i, j) \in E \times E$ , there exists a constant  $\beta_{i, j} > 0$ , such that*

$$\lim_{n \rightarrow +\infty} \sqrt{n} \mathbb{P}_i(Z_n > 0, X_n = j) = \beta_{i, j}. \quad (2.1)$$

We will follow the argument developed by J. Geiger and G. Kersting in [11], splitting the quantity  $\mathbb{P}_i(Z_n > 0, X_n = j)$  into two parts

$$\mathbb{P}_i(Z_n > 0, X_n = j) = \mathbb{P}_i(Z_n > 0, m_n \geq -x, X_n = j) + \mathbb{P}_i(Z_n > 0, m_n < -x, X_n = j).$$

This allows us to link the asymptotic behavior of the survival probability  $\mathbb{P}_i(Z_n > 0, X_n = j)$  to the one of the variable  $m_n$ . In the first section, we introduce some useful notations; the proof of the Theorem is detailed in the second section, after some preliminary results concerning the behavior of the Markov walk  $(S_n)_{n \geq 1}$  conditioning to stay in a half line  $] -\infty, -x[$  or  $[-x, +\infty[$  for some  $x > 0$ .

## 2.2 Preliminaries and notations

Let us introduce some notations: set

$$\begin{aligned} f_k(s) &:= \frac{1}{1 - g_k(s)} - \frac{1}{g'_k(1)(1 - s)}, \quad 0 \leq s < 1, \\ g_{k, n} &:= g_k \circ g_{k+1} \circ \cdots \circ g_{n-1}, \quad 0 \leq k \leq n-1; \quad g_{n, n} = Id, \\ \eta_{k, n} &:= \begin{cases} f_k(g_{k+1, n}(0)), & 0 \leq k \leq n-1, \\ 1, & k = n, \\ 0, & k \geq n+1. \end{cases} \end{aligned}$$

## 2.2. PRELIMINARIES AND NOTATIONS

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And  $S_n := Y_1 + \dots + Y_n$ , for  $n \geq 1$ , with  $S_0 := 0$  and  $Y_n := \ln g'_{n-1}(1)$ , for  $n \geq 1$ . Then we can obtain the following formula ([11]):

$$q_n^{-1} = \exp(-S_n) + \sum_{k=0}^{n-1} \eta_{k,n} \exp(-S_k), \quad (2.2)$$

One first recall the following result, whose proof is detailed here :

**Lemma 2.2.1** ([11]). *Let  $g$  be a generating function of a probability law  $(p_k)_{k \geq 0}$  on  $\mathbb{N}$ , such that  $g''(1) < \infty$ . Define the function  $f : [0, 1[ \rightarrow \mathbb{R}$  by*

$$f(s) := \frac{1}{1-g(s)} - \frac{1}{g'(1)(1-s)}.$$

Then for  $0 \leq s < 1$ , one gets  $0 \leq f(s) \leq \frac{g''(1)}{g'^2(1)}$ .

*Proof.* By the convexity of the function  $g$  on  $[0, 1[$ , one gets  $1 - \frac{g'(s)(1-s)}{1-g(s)} \geq 0$ . Therefore,

$$g'(1)f(s) \leq g'(1)f(s) + \frac{s}{1-s} \left( 1 - \frac{g'(s)(1-s)}{1-g(s)} \right) = \frac{g'(1) - sg'(s)}{1-g(s)} - 1. \quad (2.3)$$

Set  $G(s) := \frac{g'(1) - sg'(s)}{1-g(s)}$  and let us decompose  $G(s)$  into  $G(s) = \sum_{k=1}^{+\infty} kr_k(s)$ , where  $r_k(s) := p_k(1-s^k)/(1-g(s))$  for  $k \geq 1$ . Note first that

$$\frac{r_{k+1}(s)}{r_k(s)} = \frac{p_{k+1}(1-s^{k+1})}{p_k(1-s^k)} = \frac{p_{k+1}}{p_k} \left( 1 + \frac{1}{\sum_{j=1}^k s^{-j}} \right),$$

and so the functions  $s \mapsto \frac{r_{k+1}(s)}{r_k(s)}$  are increasing on  $[0, 1[$ ; on the other hand, by definition of  $g$ , one gets  $\sum_{k \geq 1} r_k(s) = 1$  for any  $s \in [0, 1[$ .

It follows that  $r_1$  is decreasing on  $[0, 1[$  (otherwise, all the  $r_k$  would be increasing on  $[0, 1[$ , which contradicts the fact that their sum is always equal to 1).

We now prove that  $G$  is increasing on  $[0, 1[$ . Fix  $0 \leq s < t < 1$  and let  $N := \sup\{k \geq 1 : r_k(s) \geq r_k(t)\}$ . Noticing that  $r_k(s) < r_k(t)$  for some  $k \geq 1$  implies  $r_l(s) < r_l(t)$  for any  $l \geq k$ , there are two cases to consider :

1. If  $N = \infty$ , then  $r_k(s) \geq r_k(t)$  for any  $k \geq 1$ ; then  $r_k(s) = r_k(t)$  for any  $k \geq 1$  since  $\sum_{k \geq 1} r_k = 1$  on  $[0, 1[$ . So  $G(s) = G(t)$ .
2. If  $N < +\infty$ , setting  $R_k := \sum_{i \geq k+1} r_i$ , then  $r_k(s) < r_k(t)$  for any  $k > N$  and so  $R_k(s) < R_k(t)$  for any  $k \geq N$ . On the other hand, for  $k = 1, \dots, N$ , one gets  $r_k(s) \geq r_k(t)$  which implies  $R_k(s) = 1 - r_1(s) - \dots - r_k(s) \leq 1 - r_1(t) - \dots - r_k(t) = R_k(t)$ . It follows that  $G(s) \leq G(t)$  since  $G = 1 + \sum_{k \geq 1} R_k$ .

### 2.3. PROOF OF THEOREM 2.1.1

We hence obtain that for  $0 \leq s < 1$ ,

$$g'(1)f(s) \leq \lim_{u \uparrow 1} \frac{g'(1) - ug'(u)}{1 - g(u)} - 1 = \frac{g''(1)}{g'(1)}.$$

□

**Remark 2.2.1.** 1. If  $g''(1) < \infty$ , then the function  $f(s)$  admits an expansion by the continuity at  $s = 1$ . Indeed, one can prove easily that  $\lim_{s \uparrow 1} f(s) = \frac{g''(1)}{2(g'(1))^2}$ .

2. For any fixed  $k$ , the sequence  $(g_{k,n}(0))_{n \geq 0}$  is increasing on  $n$  and  $f_k$  is continuous on  $[0, 1[$ , which imply that the sequence  $(\eta_{k,n})_{n \geq 1}$  converges to a limit and we denote the limit by

$$\eta_{k,\infty} := \lim_{n \rightarrow +\infty} \eta_{k,n}.$$

### 2.3 Proof of Theorem 2.1.1

In this section, we prove Theorem 2.1.1, using the local limit theorem established in the previous chapter (cf. Theorem 1.1.2).

Recall that  $m_n = \min(0, S_1, \dots, S_n)$ . For any  $i \in E$  and  $x \geq 0$ , set  $h_i(x) = \sum_{j \in E} h_{i,j}(x)$ ,

where  $h_{i,j}$  is the same as the ones in Theorem 1.1.2. In the sequel, we will consider, for any  $i, j \in E$  and  $x \geq 0$ , the probabilities measures  $\widehat{\mathbb{P}}_{i,j,x}$  and  $\widehat{\mathbb{P}}_{i,x}$  (with corresponding expectations  $\widehat{\mathbb{E}}_{i,j,x}$  and  $\widehat{\mathbb{E}}_{i,x}$  respectively) defined on  $\sigma((g_0, X_0), (g_1, X_1), \dots)$  by :

1. the marginal distribution of  $\widehat{\mathbb{P}}_{i,j,x}$  and  $\widehat{\mathbb{P}}_{i,x}$  on  $\sigma(S_0, S_1, S_2, \dots)$  are given respectively by : for any  $n \geq 1$  and any Borel subset  $A \in \mathbb{R}^n$

$$\begin{aligned} \widehat{\mathbb{P}}_{i,j,x} [(S_0, \dots, S_n) \in A] &:= \frac{1}{h_{i,j}(x)} \mathbb{E}_i [1_A(S_0, \dots, S_n) h_{X_{n,j}}(x + S_n) 1_{[m_n \geq -x]}], \\ \widehat{\mathbb{P}}_{i,x} [(S_0, \dots, S_n) \in A] &:= \frac{1}{h_i(x)} \mathbb{E}_i [1_A(S_0, \dots, S_n) h_{X_n}(x + S_n) 1_{[m_n \geq -x]}]. \end{aligned}$$

2. the conditional distribution of the variables  $(g_0, X_0), (g_1, X_1), \dots$  under  $\widehat{\mathbb{P}}_{i,j,x}$  and  $\widehat{\mathbb{P}}_{i,x}$  given  $0 = S_0, S_1, S_2, \dots$  is given respectively by : for any  $n \geq 1$  and any Borel subsets  $A_0, A_1, \dots, A_n \in P(E) \times B(G)$  and  $s_1, s_2, \dots \in [-x, +\infty[$

$$\begin{aligned} \widehat{\mathbb{P}}_{i,j,x} [(g_k, X_k) \in A_k, 0 \leq k \leq n/S_l = s_l, l \geq 1] &:= \mathbb{P}_i [(g_k, X_k) \in A_k, 0 \leq k \leq n/S_l = s_l, l \geq 1], \\ \widehat{\mathbb{P}}_{i,x} [(g_k, X_k) \in A_k, 0 \leq k \leq n/S_l = s_l, l \geq 1] &:= \mathbb{P}_i [(g_k, X_k) \in A_k, 0 \leq k \leq n/S_l = s_l, l \geq 1]. \end{aligned}$$

One first proves the

**Proposition 2.3.1.** For any  $k \leq n$ , any  $x \geq 0$  and any bounded Borel function  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ , one gets

$$\lim_{n \rightarrow +\infty} \mathbb{E}_i [\varphi(S_0, \dots, S_k) | m_n \geq -x, X_n = j] = \widehat{\mathbb{E}}_{i,j,x} [\varphi(S_0, \dots, S_k)] \quad (2.4)$$

and

$$\lim_{n \rightarrow +\infty} \mathbb{E}_i [\varphi(S_0, \dots, S_k) | m_n \geq -x] = \widehat{\mathbb{E}}_{i,x} [\varphi(S_0, \dots, S_k)]. \quad (2.5)$$

### 2.3. PROOF OF THEOREM 2.1.1

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*Proof.* By the Markov property, we get

$$\begin{aligned}\mathbb{E}_i \left[ \varphi(S_0, \dots, S_k) / m_n \geq -x, X_n = j \right] &= \frac{\mathbb{E}_i [\varphi(S_0, \dots, S_k), m_n \geq -x, X_n = j]}{\mathbb{P}_i(m_n \geq -x, X_n = j)} \\ &= \frac{\mathbb{E}_i [\varphi(S_0, \dots, S_k), m_k \geq -x, \mathbb{P}_{(X_k, S_k)}(m_{n-k} \geq -x, X_{n-k} = j)]}{\mathbb{P}_i(m_n \geq -x, X_n = j)}\end{aligned}$$

From Theorem 1.1.2, we have for any  $x \geq 0$ ,

$$\lim_{n \rightarrow +\infty} \sqrt{n} \mathbb{P}_i(m_n \geq -x, X_n = j) = h_{i,j}(x)$$

and

$$\lim_{n \rightarrow +\infty} \sqrt{n} \mathbb{P}_{(X_k, S_k)}(m_{n-k} \geq -x, X_{n-k} = j) = h_{X_k, j}(x + S_k) \quad \mathbb{P}_i \text{-a.s.}$$

According to Lemma 1.5.4, there exists a constant  $C > 0$  such that, for any  $i \in E$  and  $k \leq n$ ,

$$\sqrt{n} \mathbb{P}_{(X_k, S_k)}(m_{n-k} \geq -x, X_{n-k} = j) 1_{[m_k \geq -x]} \leq C(x + S_k + 1) \quad \mathbb{P}_i \text{-a.s.}$$

with

$$\mathbb{E}_i |S_k| \leq \sum_{l=1}^k \mathbb{E}_i |Y_l| \leq k \sup_{l \in E} \mathbb{E}_l |Y_1| < +\infty.$$

by Hypothesis H1. According to the dominated convergence theorem and the definition of  $\widehat{\mathbb{E}}_{i,j,x}$ , we obtain for  $x \geq 0$ ,

$$\begin{aligned}\lim_{n \rightarrow +\infty} \mathbb{E}_i [\varphi(S_0, \dots, S_k) | m_n \geq -x, X_n = j] &= \frac{1}{h_{i,j}(x)} \mathbb{E}_i [\varphi(S_0, \dots, S_k) h_{X_k, j}(x + S_k); m_k \geq -x] \\ &= \widehat{\mathbb{E}}_{i,j,x} [\varphi(S_0, \dots, S_k)].\end{aligned}$$

The convergence (2.5) may be established with a similar argument.  $\square$

**Lemma 2.3.1.** *Assume that the conditions of Theorem 2.1.1 are satisfied, then for  $(i, j) \in E \times E$  and  $x \geq 0$ ,*

$$\lim_{n \rightarrow +\infty} \widehat{\mathbb{E}}_{i,j,x}(q_n) = \widehat{\mathbb{E}}_{i,j,x}(q_\infty)$$

$$\text{where } q_\infty := \left[ \sum_{k=0}^{+\infty} \eta_{k,\infty} \exp(-S_k) \right]^{-1}.$$

*Proof.* Since  $q_n$  is the survival probability of  $(Z_n)_{n \geq 0}$  at time  $n$ , so  $q_n \leq 1$ , which implies that  $q_\infty \leq 1$ . Therefore, for  $0 < \varepsilon < 1$ ,

$$\begin{aligned}|q_n - q_\infty| &= |q_n - q_\infty|^{1-\varepsilon} \times |q_n - q_\infty|^\varepsilon \leq 2^{1-\varepsilon} |q_n - q_\infty|^\varepsilon \\ &= 2^{1-\varepsilon} (q_n q_\infty)^\varepsilon \left| \frac{1}{q_n} - \frac{1}{q_\infty} \right|^\varepsilon \\ &\leq 2^{1-\varepsilon} |q_n^{-1} - q_\infty^{-1}|^\varepsilon.\end{aligned}$$



### 2.3. PROOF OF THEOREM 2.1.1

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It is thus sufficient to prove that for  $(i, j) \in E \times E$ ,  $x \geq 0$  and  $0 < \varepsilon < 1$ ,

$$\widehat{\mathbb{E}}_{i,j,x} |q_n^{-1} - q_\infty^{-1}|^\varepsilon \xrightarrow{n \rightarrow +\infty} 0. \quad (2.6)$$

Using (2.2), we get for  $0 \leq l \leq n-1$ ,

$$\begin{aligned} \widehat{\mathbb{E}}_{i,j,x} |q_n^{-1} - q_\infty^{-1}|^\varepsilon &\leq \widehat{\mathbb{E}}_{i,j,x} [\exp(-\varepsilon S_n)] + \sum_{k=0}^{l-1} \widehat{\mathbb{E}}_{i,j,x} \{ \exp(-\varepsilon S_k) |\eta_{k,n} - \eta_{k,\infty}|^\varepsilon \} \\ &\quad + 2 \sum_{k=l}^{+\infty} \widehat{\mathbb{E}}_{i,j,x} [\eta_{k,\infty}^\varepsilon \exp(-\varepsilon S_k)]. \end{aligned} \quad (2.7)$$

Let us first state the

**Fact 2.3.1.** *For any  $i \in E$ ,  $x \geq 0$  and  $\varepsilon > 0$ , we have  $\sum_{k=0}^{+\infty} \mathbb{E}_i(\exp(-\varepsilon S_k) 1_{[m_k \geq -x]}) < +\infty$ .*

Then  $\widehat{\mathbb{E}}_{i,j,x} [\exp(-\varepsilon S_n)] \xrightarrow{n \rightarrow +\infty} 0$  in (2.7). To control the two other terms on the right hand side of (2.7), it suffices to check that for any  $i, j \in E$  and  $x \geq 0$ ,

$$\sum_{k=0}^{+\infty} \widehat{\mathbb{E}}_{i,j,x} [\eta_{k,\infty}^\varepsilon \exp(-\varepsilon S_k)] < +\infty. \quad (2.8)$$

(Indeed, by (2.8), since  $\eta_{k,n} \xrightarrow{n \rightarrow +\infty} \eta_{k,\infty}$   $\mathbb{P}$ -a.s., one may thus apply the dominated convergence theorem and obtain  $\sum_{k=0}^{l-1} \widehat{\mathbb{E}}_{i,j,x} \{ \exp(-\varepsilon S_k) |\eta_{k,n} - \eta_{k,\infty}|^\varepsilon \} \xrightarrow{n \rightarrow +\infty} 0$  for any  $l \geq 1$ ; furthermore, the third term can be arbitrarily small by choosing  $l$  sufficiently large).

To verify (2.8), let us first notice that, according to Lemma 1.5.4, the fact that  $h_{i,j}$  is increasing and that  $\{m_{k+1} \geq -x\} \subseteq \{m_k \geq -x\}$ , we get

$$\begin{aligned} \widehat{\mathbb{E}}_{i,j,x} [\eta_{k,\infty}^\varepsilon \exp(-\varepsilon S_k)] &= \frac{1}{h_{i,j}(x)} \mathbb{E}_i \left[ \exp(-\varepsilon S_k) \eta_{k,\infty}^\varepsilon h_{X_{k+1},j}(x + S_{k+1}); m_{k+1} \geq -x \right] \\ &\leq \frac{C}{h_{i,j}(x)} \mathbb{E}_i [(1 + x + S_{k+1}^+) \eta_{k,\infty}^\varepsilon \exp(-\varepsilon S_k); m_{k+1} \geq -x] \\ &\leq \frac{C}{h_{i,j}(x)} \mathbb{E}_i [(1 + x + S_k^+ + |Y_k|) \eta_{k,\infty}^\varepsilon \exp(-\varepsilon S_k); m_{k+1} \geq -x] \\ &\leq \frac{C}{h_{i,j}(x)} \left( 1 + x + \frac{2}{\varepsilon} e^{\frac{\varepsilon}{2}(1+x)} \right) \mathbb{E}_i \left[ \eta_{k,\infty}^\varepsilon \left( \exp(-\varepsilon S_k) + \exp\left(-\frac{\varepsilon}{2} S_k\right) \right); m_k \geq -x \right] \\ &\quad + \frac{C}{h_{i,j}(x)} \mathbb{E}_i [|Y_k| \eta_{k,\infty}^\varepsilon \exp(-\varepsilon S_k); m_k \geq -x] \end{aligned}$$

where we have use the fact that for  $a, \alpha > 0$  one gets

$$(a + y^+) e^{-2\alpha y} \leq \left( a + \frac{1}{\alpha} e^{\alpha a} \right) (e^{-\alpha y} + e^{-2\alpha y}).$$

### 2.3. PROOF OF THEOREM 2.1.1

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Setting  $C_{i,j} = C_{i,j}(x, \varepsilon) := \frac{C}{h_{i,j}(x)} \left(1 + x + \frac{2}{\varepsilon} e^{\frac{\varepsilon}{2}(1+x)}\right) \times \sup_{l \in E} \mathbb{E}_l \left( (1 + |Y_0|) \eta_0^\varepsilon \right)$ , one thus gets, by the Markov property

$$\widehat{\mathbb{E}}_{i,j,x}[\eta_{k,\infty}^\varepsilon \exp(-\varepsilon S_k)] \leq C_{i,j} \left( \mathbb{E}_i \left[ \exp\left(-\frac{\varepsilon}{2} S_k\right); m_k \geq -x \right] + \mathbb{E}_i \left[ \exp(-\varepsilon S_k); m_k \geq -x \right] \right).$$

The proof of (2.8) is achieved using the Fact 2.3.1.  $\square$

*Proof of Fact 2.3.1.* For any  $\lambda > 0$ , one gets  $1_{[m_k \geq -x]} \leq \exp[\lambda(m_k + x)]$ . Therefore, for any  $i \in E$  and  $\varepsilon > 0$ , one may write

$$\mathbb{E}_i \left( \exp(-\varepsilon S_k) 1_{[m_k \geq -x]} \right) \leq e^{\lambda x} \mathbb{E}_i [\exp(-\varepsilon S_k + \lambda m_k)],$$

and, by Theorem 1.1.3, for  $\lambda > \varepsilon$ , one gets

$$\sum_{k=0}^{+\infty} \mathbb{E}_i (\exp(-\varepsilon S_k) 1_{[m_k \geq -x]}) \leq e^{\lambda x} \sum_{k=0}^{+\infty} \mathbb{E}_i [\exp(-\varepsilon S_k + \lambda m_k)] < +\infty.$$

$\square$

*Proof of Theorem 2.1.1.* As we claimed in the introduction, we decompose the quantity  $\sqrt{n} \mathbb{P}_i(Z_n > 0, X_n = j)$  as follows: we follow the argument developed by J. Geiger and G. Kersting in [11]. Note first that, for any  $x \geq 0$ , one gets

$$\begin{aligned} \sqrt{n} \mathbb{P}_i(Z_n > 0, X_n = j) &= \sqrt{n} \mathbb{P}_i(Z_n > 0, m_n \geq -x, X_n = j) \\ &\quad + \sqrt{n} \mathbb{P}_i(Z_n > 0, m_n < -x, X_n = j). \end{aligned} \tag{2.9}$$

We will study these two terms separately and obtain the two following lemmas :

**Lemma 2.3.2.** *Assume that the conditions of Theorem 2.1.1 are satisfied, then for any  $x \geq 0$  and  $(i, j) \in E \times E$ ,*

$$\lim_{n \rightarrow +\infty} \sqrt{n} \mathbb{P}_i(Z_n > 0, m_n \geq -x, X_n = j) = h_{i,j}(x) \widehat{E}_{i,j,x}(q_\infty).$$

**Lemma 2.3.3.** *Assume that the conditions of Theorem 2.1.1 are satisfied, then for any  $x \geq 0$  and  $(i, j) \in E \times E$ ,*

$$\lim_{x \rightarrow +\infty} \sup_{n \geq 1} \sqrt{n} \mathbb{P}_i(Z_n > 0, m_n < -x, X_n = j) = 0.$$

We now may achieve the Proof of Theorem 2.1.1. Applying these two lemmas, Theorem 2.1.1 can be proved easily:

Set

$$T_1(n, x) := \sqrt{n} \mathbb{P}_i(Z_n > 0, m_n \geq -x, X_n = j),$$

$$T_2(n, x) := \sqrt{n} \mathbb{P}_i(Z_n > 0, m_n < -x, X_n = j).$$

According to Lemma 2.3.3, for any  $\varepsilon > 0$ , there exists  $A \in \mathbb{R}^+$  such that  $\forall x \geq A$ ,

$$\sup_{n \geq 1} |T_2(n, x)| < \varepsilon/2. \tag{2.10}$$

### 2.3. PROOF OF THEOREM 2.1.1

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Lemma 2.3.2, (2.10) and (2.9), we have for  $x$  great enough,

$$\begin{aligned} h_{i,j}(x)\widehat{\mathbb{E}}_{i,j,x}(q_\infty) - \frac{\varepsilon}{2} &\leq \liminf_{n \rightarrow +\infty} \sqrt{n} \mathbb{P}_i(Z_n > 0, X_n = j) \leq \limsup_{n \rightarrow +\infty} \sqrt{n} \mathbb{P}_i(Z_n > 0, X_n = j) \\ &\leq h_{i,j}(x)\widehat{\mathbb{E}}_{i,j,x}(q_\infty) + \frac{\varepsilon}{2}. \end{aligned}$$

Letting  $x \rightarrow \infty$  in the above inequality, we get

$$\lim_{n \rightarrow +\infty} \sqrt{n} \mathbb{P}_i(Z_n > 0, X_n = j) = \lim_{x \rightarrow +\infty} h_{i,j}(x)\widehat{\mathbb{E}}_{i,j,x}(q_\infty),$$

which yields our result.  $\square$

It thus remains to prove Lemma 2.3.2 and Lemma 2.3.3. To prove Lemma 2.3.2, we will use the following relation: for  $m \leq n$  and  $\rho > 1$ ,

$$\begin{aligned} \sqrt{n} \mathbb{P}_i(Z_n > 0, m_n \geq -x, X_n = j) &= \sqrt{n} \mathbb{P}_i(Z_m > 0, m_n \geq -x, X_n = j) \\ &\quad + \Delta_{i,j}(n, \rho, x) + \Delta_{i,j}^{(1)}(m, n, \rho, x) + \Delta_{i,j}^{(2)}(m, n, \rho, x), \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} \Delta_{i,j}(n, \rho, x) &:= \sqrt{n} \mathbb{P}_i(Z_n > 0, m_n \geq -x, X_n = j) - \sqrt{n} \mathbb{P}_i(Z_n > 0, m_{\rho n} \geq -x, X_n = j), \\ \Delta_{i,j}^{(1)}(m, n, \rho, x) &:= \sqrt{n} \mathbb{P}_i(Z_n > 0, m_{\rho n} \geq -x, X_n = j) - \sqrt{n} \mathbb{P}_i(Z_m > 0, m_{\rho n} \geq -x, X_n = j), \\ \Delta_{i,j}^{(2)}(m, n, \rho, x) &:= \sqrt{n} \mathbb{P}_i(Z_m > 0, m_{\rho n} \geq -x, X_n = j) - \sqrt{n} \mathbb{P}_i(Z_m > 0, m_n \geq -x, X_n = j). \end{aligned}$$

We control all of these terms, using the following elementary facts :

**Fact 2.3.2.** *For every  $x \geq 0$ , there exists a constant  $C(x)$ , such that for  $n \geq 1$  and  $\rho > 1$ ,*

$$0 \leq \Delta_{i,j}(n, \rho, x) \leq C(x) \left(1 - \frac{1}{\sqrt{\rho}}\right) h_i(x).$$

*Proof.* One gets  $\{m_{\rho n} \geq -x\} \subset \{m_n \geq -x\}$  since  $\rho > 1$  and so

$$0 \leq \Delta_{i,j}(n, \rho, x) \leq \sqrt{n} [\mathbb{P}_i(m_n \geq -x) - \mathbb{P}_i(m_{\rho n} \geq -x)].$$

By Theorem 1.1.2, for any  $x \geq 0$ , there exists a constant  $C(x) > 0$  such that

$$\mathbb{P}_i(m_n \geq -x) - \mathbb{P}_i(m_{\rho n} \geq -x) \leq C(x) \left(1 - \frac{1}{\sqrt{\rho}}\right) h_i(x).$$

This achieves the proof.  $\square$

**Fact 2.3.3.** *For any  $x \geq 0$  and  $\rho > 1$ , one gets*

$$\limsup_{m \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \left| \Delta_{i,j}^{(1)}(m, n, \rho, x) \right| = 0.$$

### 2.3. PROOF OF THEOREM 2.1.1

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*Proof.* For  $m \leq n$ , one gets  $\{Z_n > 0\} \subseteq \{Z_m > 0\}$  and so

$$\begin{aligned} |\Delta_{i,j}^{(1)}(m, n, \rho, x)| &= \sqrt{n} \mathbb{P}_i(Z_m > 0, Z_n = 0, m_{\rho n} \geq -x, X_n = j) \\ &\leq \sqrt{n} \mathbb{P}_i(Z_m > 0, Z_n = 0, m_{\rho n} \geq -x) \\ &= \mathbb{P}_i(Z_m > 0, Z_n = 0/m_{\rho n} \geq -x) \sqrt{n} \mathbb{P}_i(m_{\rho n} \geq -x) \end{aligned} \quad (2.12)$$

Using Markov property

$$\mathbb{P}_i(Z_m > 0, Z_n = 0/m_{\rho n} \geq -x) = \frac{\mathbb{E}_i \left[ \int_{\mathbb{R}} (q_m - q_n), m_n \geq -x, S_n \in dy; \sqrt{n} \mathbb{P}_{X_n}(m_{\rho n-n} \geq -x - y) \right]}{\sqrt{n} \mathbb{P}_i(m_{\rho n} \geq -x)}.$$

According to Theorem 1.1.2 and Fact 1.5.3, for every  $x \geq 0$  there exists a constant  $C'(x)$  such that

$$\begin{aligned} \mathbb{E}_i(Z_m > 0, Z_n = 0/m_{\rho n} \geq -x) &\leq C'(x) \sqrt{\frac{\rho}{\rho-1}} \frac{1}{h_i(x)} \mathbb{E}_i[q_m - q_n, m_n \geq -x, h_{X_n}(x + S_n)] \\ &= C'(x) \sqrt{\frac{\rho}{\rho-1}} \widehat{\mathbb{E}}_{i,x}(q_m - q_n) \end{aligned}$$

and one concludes, letting first  $n \rightarrow +\infty$  and second  $m \rightarrow +\infty$  and using Lemma 2.3.1.  $\square$

**Fact 2.3.4.** For every  $x \geq 0$ , there exists a constant  $C_1(x)$  such that for any  $\rho > 1$ ,  $1 \leq m \leq n$  and  $(i, j) \in E \times E$ ,

$$|\Delta_{i,j}^{(2)}(m, n, \rho, x)| \leq C_1(x) \left(1 - \frac{1}{\sqrt{\rho}}\right) h_i(x).$$

The proof is similar to the one of Fact 2.3.2.

*Proof of Lemma 2.3.2.* By (2.11), for any  $n \geq 1$  and any  $m \leq n$ ,

$$\begin{aligned} &|\sqrt{n} \mathbb{P}_i(Z_n > 0, m_n \geq -x, X_n = j) - h_{i,j}(x) \widehat{\mathbb{E}}_{i,j,x}(q_\infty)| \\ &\leq |\sqrt{n} \mathbb{P}_i(Z_m > 0, m_n \geq -x, X_n = j) - h_{i,j}(x) \widehat{\mathbb{E}}_{i,j,x}(q_\infty)| \\ &\quad + |\Delta_{i,j}(n, \rho, x)| + |\Delta_{i,j}^{(1)}(m, n, \rho, x)| + |\Delta_{i,j}^{(2)}(m, n, \rho, x)|. \end{aligned}$$

Using Facts 2.3.2 and 2.3.4, there exists a constant  $C_2 = \sup_{x \geq 0} (C(x), C_1(x))$  such that

$$\begin{aligned} &|\sqrt{n} \mathbb{P}_i(Z_n > 0, m_n \geq -x, X_n = j) - h_{i,j}(x) \widehat{\mathbb{E}}_{i,j,x}(q_\infty)| \\ &\leq |\sqrt{n} \mathbb{P}_i(Z_m > 0, m_n \geq -x, X_n = j) - h_{i,j}(x) \widehat{\mathbb{E}}_{i,j,x}(q_\infty)| \\ &\quad + 2C_2 \left(1 - \frac{1}{\sqrt{\rho}}\right) h_i(x) + |\Delta_{i,j}^{(1)}(m, n, \rho, x)|. \end{aligned} \quad (2.13)$$

For any  $\varepsilon > 0$ , we can choose  $\rho_0 > 1$  close enough to 1 such that  $2C_2(1 - \frac{1}{\sqrt{\rho_0}})h_i(x) < \frac{\varepsilon}{3}$ . On the other hand, one gets

$$\begin{aligned} \sqrt{n} \mathbb{P}_i(Z_m > 0, m_n \geq -x, X_n = j) &= \mathbb{P}_i(Z_m > 0/m_n \geq -x, X_n = j) \sqrt{n} \mathbb{P}_i(m_n \geq -x, X_n = j) \\ &\xrightarrow{n \rightarrow +\infty} h_{i,j}(x) \widehat{\mathbb{E}}_{i,j,x}(q_m), \end{aligned}$$

### 2.3. PROOF OF THEOREM 2.1.1

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by Proposition 2.3.1 and Theorem 1.1.2.

So, letting  $n \rightarrow +\infty$  in the two sides of equality (2.13), we get

$$\begin{aligned} \limsup_{n \rightarrow +\infty} |\sqrt{n} \mathbb{P}_i(Z_n > 0, m_n \geq -x, X_n = j) - h_{i,j}(x) \widehat{\mathbb{E}}_{i,j,x}(q_\infty)| &\leq |h_{i,j}(x)| |\widehat{\mathbb{E}}_{i,j,x}(q_m) - \widehat{\mathbb{E}}_{i,j,x}(q_\infty)| \\ &+ \frac{\varepsilon}{3} + \limsup_{n \rightarrow +\infty} |\Delta_{i,j}^{(1)}(m, n, \rho, x)|. \end{aligned}$$

One concludes, letting  $m \rightarrow +\infty$  and using Facts 2.3.1 and 2.3.3.  $\square$

*Proof of Lemma 2.3.3.* For any  $k \geq 1$ ,  $i \in E$  and  $\mathbb{P}_i$ -a.s.

$$q_k = \mathbb{P}_i(Z_k > 0 / M_0, M_1 \cdots, M_{k-1}) \leq \mathbb{E}_i(Z_k / M_0, M_1 \cdots, M_{k-1}) = \exp(S_k),$$

where the last equality holds by the first moment estimation. Thus,

$$q_n = \min_{0 \leq k \leq n} q_k \leq \exp(m_n) \quad \mathbb{P}_i - a.s.$$

By Theorem 1.1.2, we obtain for any  $n \geq 1$  and  $x \geq 0$ ,

$$\begin{aligned} 0 \leq \sqrt{n} \mathbb{P}_i(Z_n > 0, m_n < -x, X_n = j) &\leq \sqrt{n} \mathbb{P}_i(Z_n > 0, m_n < -x) \\ &\leq \sqrt{n} \mathbb{E}_i(\exp(m_n); m_n < -x) \\ &\leq \sqrt{n} \sum_{k \geq [x]} \exp(-k) \mathbb{P}_i(-k - 1 \leq m_n < -k) \\ &\leq C_1 \sum_{k \geq [x]} h_i(k+1) \exp(-k) \\ &\leq C_1 \sum_{k \geq [x]} (k+2) \exp(-k) \xrightarrow{x \rightarrow +\infty} 0. \end{aligned}$$

$\square$

# Appendix



## Appendix A

# Absolutely continuous components for $k$ times convolution of a matrix of positive measures on $\mathbb{R}$

We use here the Notations 1.2.1 and we prove the

**Lemma A.0.4.** *Assume that  $M(dx) = (\mu_{i,j}(dx))_{1 \leq i,j \leq N}$  is a matrix of positive measures on  $\mathbb{R}$ . If the following two conditions hold simultaneously:*

1. *there exist  $(i_0, j_0) \in \{1, \dots, N\}^2$  and  $n_0 \geq 1$ , such that  $\mu_{i_0, j_0}^{(n_0)}(dx)$  has an absolutely continuous component;*
2. *there exists  $n_1 \geq 1$ , such that  $M^{\bullet n_1}(\mathbb{R}) > 0$ ,*

*then for any  $k \geq (n_0 + 1)n_1 n_0$ , one gets  $M^{\bullet k}(\mathbb{R}) > 0$  and there exists at least one absolutely continuous component term in  $M^{\bullet k}$ .*

*Proof.* There are two cases to consider.

**Case 1**  $i_0 = j_0$ . The matrix  $M^{\bullet n_0}$  has thus an absolutely continuous component term in its diagonal. Since  $M^{\bullet n_0 n_1} = (M^{\bullet n_0})^{n_1}$ , it is clear that there also exists an absolutely continuous component term on the diagonal of the matrix  $M^{\bullet n_0 n_1}$ . Moreover, one gets  $M^{\bullet n_1}(\mathbb{R}) > 0$ , so that  $M^{\bullet n_0 n_1}(\mathbb{R}) = (M^{\bullet n_1}(\mathbb{R}))^{n_0} > 0$ .

Consequently the matrix  $M^{\bullet n_0 n_1}$  has an absolutely continuous component term on its diagonal and  $M^{\bullet n_0 n_1}(\mathbb{R}) > 0$ . This implies that for any  $k \geq (n_1 + 1)n_0 n_1 > n_0 n_1$ , the matrix  $M^{\bullet k}$  has at least one absolutely continuous component term and  $M^{\bullet k}(\mathbb{R}) > 0$ .

**Case 2**  $i_0 \neq j_0$ . Set  $n'_1 = (n_1 + 1)n_0$ . The positivity of  $M^{\bullet n_0 n'_1}(\mathbb{R})$  can be obtained easily using the same argument as in Case 1. Remark that

$$\mu_{j_0, j_0}^{(n'_1)}(dx) = \sum_{l=1}^N \mu_{j_0, l}^{(n_0 n_1)}(dx) * \mu_{l, j_0}^{(n_0)}(dx).$$

Since  $M^{\bullet n_0 n_1}(\mathbb{R}) > 0$  and  $(M^{\bullet n_0}(dx))_{i_0, j_0}$  has an absolutely continuous component term, so has the measure  $\mu_{j_0, j_0}^{(n'_1)}(dx)$ . We are therefore in the first case and conclude easily.



□

In particular, we get the following lemma:

**Lemma A.0.5.** *If the hypotheses of lemma A.0.4 are valid, there exists  $k_1 \geq 1$  such that  $M^{\bullet k_1}(\mathbb{R}) > 0$  and all the terms of  $M^{\bullet k_1}$  have absolutely continuous components.*

*Proof.* Take  $k_1 = 4k_0$  with  $k_0 = (n_0 + 1)n_1n_0$ . The positivity of  $M^{\bullet k_1}(\mathbb{R})$  is an immediate consequence of lemma A.0.4. In addition,

$$M^{\bullet 2k_0} = M^{\bullet k_0} M^{\bullet k_0}.$$

By Lemma A.0.4, one has  $M^{\bullet k_0}(\mathbb{R}) > 0$  and  $M^{\bullet k_0}$  has an absolutely continuous component term  $\mu_{i'_0, j'_0}^{(k_0)}$ . So according to the above equality, we see that every term of  $M^{\bullet 2k_0}(e_{i'_0})$  and  $M^{\bullet 2k_0}(e_{j'_0})$  has an absolutely continuous component. It is thus clear that all the terms of the matrix  $M^{\bullet k_1} = M^{\bullet 4k_0}$  have an absolutely continuous component. □

## Appendix B

### Proof of Theorem 1.2.1

*Proof of Theorem 1.2.1.* 1. The first assertion is a direct consequence of the perturbation theorem (see Theorem 9 of Chapter 7 in [7] for instance).

2. To prove the second assertion, we will use the following lemmas:

**Lemma B.0.6.** *There exist  $\alpha_1 > 0$ ,  $\beta_1 > 0$  and  $\chi_1 \in ]0, 1[$  such that  $r(P(\lambda)) \leq \chi_1$  for any  $\lambda \in \mathbb{C}$  satisfying  $|\Re \lambda| \leq \alpha_1$  and  $|\Im \lambda| \geq \beta_1$ .*

**Lemma B.0.7.** *For any  $0 < a < b$ , there exist  $\alpha_{a,b} > 0$  and  $\chi_{a,b} \in ]0, 1[$  such that  $r(P(\lambda)) \leq \chi_{a,b}$  for any  $\lambda \in \mathbb{C}$  satisfying  $|\Re \lambda| \leq \alpha_{a,b}$  and  $a \leq |\Im \lambda| \leq b$ .*

Theorem 1.2.1 can thus be proved easily. Indeed, it is sufficient to fix  $a, b$  in Lemma B.0.7 in the following way :  $a = \alpha_0$ ,  $b = \beta_1$  given by Lemma B.0.6 and  $\alpha'_0 = \inf(\alpha_0, \alpha_1, \alpha_{a,b})$ ,  $\chi = \inf(\chi_1, \chi_{a,b})$ .

□

It remains to prove Lemma B.0.6 and Lemma B.0.7. We first need the following fact :

**Fact B.0.5.** *Fix  $\gamma > 0$  and let  $f : \mathbb{R} \mapsto \mathbb{R}$  be such that the function  $y \mapsto e^{\gamma|y|}f(y)$  belongs to  $L^1(\mathbb{R}, dx)$ . Then*

$$\lim_{\substack{|t| \rightarrow +\infty \\ t \in \mathbb{R}}} \sup_{|a| \leq \gamma} \left| \int_{\mathbb{R}} e^{(a+it)y} f(y) dy \right| = 0.$$

*Proof.* By a classical argument of density, for any  $\varepsilon > 0$ , one may choose a  $C^1$  function  $f_\varepsilon$  with compact support  $\subset [-M, M]$  such that

$$\int_{\mathbb{R}} e^{\gamma|y|} |f(y) - f_\varepsilon(y)| dy < \varepsilon. \quad (\text{B.1})$$

For any  $a \in [-\gamma, \gamma]$ , one has

$$\begin{aligned} \left| \int_{\mathbb{R}} e^{(a+it)y} f(y) dy \right| &\leq \left| \int_{\mathbb{R}} e^{(a+it)y} (f(y) - f_\varepsilon(y)) dy \right| + \left| \int_{\mathbb{R}} e^{(a+it)y} f_\varepsilon(y) dy \right| \\ &\leq \int_{\mathbb{R}} e^{\gamma|y|} |f(y) - f_\varepsilon(y)| dy + \frac{e^{\gamma M}}{|t|} \int_{\mathbb{R}} |f'_\varepsilon(y)| dy \end{aligned}$$

Using (B.1) and letting  $t \rightarrow +\infty$ , one can obtain the expected result.

□

*Proof of Lemma B.0.6.* Set  $M = \left( p_{i,j} F(i, j, dx) \right)_{i,j}$ . By Lemma A.0.5, there exists  $k_1 \geq 0$  such that all the terms of the matrix  $M^{\bullet k_1}$  have absolutely continuous components. Using the fact that

$$M_{i,j}^{\bullet k_1}(dx) = \varphi_{k_1,i,j}(x)dx + \theta_{k_1,i,j}(dx),$$

where for any  $(i, j) \in E \times E$ ,

- the function  $\varphi_{k,i,j}$  is strictly positive, belongs to  $\mathbb{L}^1(\mathbb{R}, dx)$  and satisfies

$$0 < \int \varphi_{k,i,j}(x)dx \leq 1;$$

- $\theta_{k,i,j}(dx)$  is a singular measure with respect to the Lebesgue measure such that

$$0 \leq \int \theta_{k,i,j}(dx) < 1.$$

Recall that the matrix containing the singular measures  $\theta_{k_1,i,j}$  is denoted by  $\Theta_{k_1}(dx)$  and its relative Laplace transform term by term is denoted by  $\mathfrak{L}(\Theta_k)(\lambda)$ , for  $|\Re \lambda| \leq \alpha_0$ .

By Lemma B.0.5, we have

$$\begin{aligned} \limsup_{\substack{|t| \rightarrow +\infty \\ t \in \mathbb{R}}} \sup_{|a| \leq \alpha_0} \|P^{k_1}(a + it)\| &\leq \lim_{\substack{|t| \rightarrow \infty \\ t \in \mathbb{R}}} \sup_{|a| \leq \alpha_0} \|\mathfrak{L}(H_{k_1})(a + it)\| + \limsup_{\substack{|t| \rightarrow +\infty \\ t \in \mathbb{R}}} \sup_{|a| \leq \alpha_0} \|\mathfrak{L}(\Theta_{k_1})(a + it)\| \\ &\leq \sup_{|a| \leq \alpha_0} \|\mathfrak{L}(\Theta_{k_1})(a)\|. \end{aligned}$$

Moreover,  $\|\mathfrak{L}(\Theta_{k_1})(0)\| = 1 - \delta$  with  $\delta \in ]0, 1[$ ; by continuity of the map  $x \in \mathbb{R} \mapsto \mathfrak{L}(\Theta_{k_1})(x)$ , there thus exists a real number  $\alpha_1$  such that

$$\sup_{|a| \leq \alpha_1} \|\mathfrak{L}(\Theta_{k_1})(a)\| \leq 1 - \delta/2 < 1.$$

Set  $\chi_1 = 1 - \delta/4$  and choose  $\beta_1 > 0$  such that for any  $\lambda \in \mathbb{C}$  satisfying  $|\Re \lambda| \leq \alpha_1$  and  $|\Im \lambda| \geq \beta_1$  one gets  $\|P^{k_1}(\lambda)\| \leq \chi_1$ , which implies  $r(P(\lambda)) \leq \chi_1^{1/k_1} < 1$ .  $\square$

*Proof of Lemma B.0.7.* Fix  $\lambda \in \mathbb{C}$  s.t.  $|\Re \lambda| \leq \alpha_1$  and  $|\Im \lambda| \in [a, b]$ . Since for any  $i, j \in E$  the measure  $P_{i,j}^{\bullet k_1}$  has an absolutely continuous component, one gets

$$|P_{i,j}^{k_1}(\lambda)| < P_{i,j}^{k_1}(\Re \lambda),$$

i.e.  $|P_{i,j}^{k_1}(\lambda)| \leq \rho_\lambda P_{i,j}^{k_1}(\Re \lambda)$  with  $0 < \rho_\lambda < 1$ ; by continuity of the map  $\lambda \mapsto |P_{i,j}^{k_1}(\lambda)|$ , one gets  $\rho_{a,b} := \sup_{\substack{|\Re \lambda| \leq \alpha_1 \\ |\Im \lambda| \in [a,b]}} \rho_\lambda \in ]0, 1[$ . There thus exists  $0 < \rho < 1$  such that

$$|P_{i,j}^{k_1}(\lambda)| \leq \rho P_{i,j}^{k_1}(\Re \lambda).$$

Therefore, for any  $\lambda$  such that  $|\Re \lambda| \leq \alpha_1$  and  $|\Im \lambda| \in [a, b]$ , one gets

$$r(P(\lambda)) \leq \rho_{a,b}^{1/k_1} k(\Re \lambda). \tag{B.2}$$

But, one gets

$$|k(\Re \lambda) - 1| \leq |\Re \lambda| \sup_{-\alpha_1 \leq u \leq \alpha_1} |k'(u)| \leq \alpha_1 M_{\alpha_1},$$

whith  $M_{\alpha_1} = \sup_{-\alpha_1 \leq u \leq \alpha_1} |k'(u)| < +\infty$ . Finally, for  $\alpha_1$  small enough, one gets

$$\chi_{a,b} := \sup_{\substack{|\Re \lambda| \leq \alpha_1 \\ |\Im \lambda| \in [a,b]}} r(P(\lambda)) \in ]0, 1[.$$

□



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