



String Theory and Applications to Phenomenology and Cosmology

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LABORATOIRE DE PHYSIQUE THÉORIQUE ÉCOLE NORMALE SUPÉRIEURE

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Présentée par :

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et Cosmologiques**

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2. Which has become, by now, legendary to all those that have discussed physics with him.

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Abstract

This thesis treats applications of String Theory to problems of cosmology and high energy phenomenology. In particular, we investigate problems related to the description of the initial state of the universe, using the methods of perturbative String Theory. After a review of the string-theoretic tools that will be employed, we discuss a novel degeneracy symmetry between the bosonic and fermionic massive towers of states (MSDS symmetry), living at particular points of moduli space. We study the marginal deformations of MSDS vacua and exhibit their natural thermal interpretation, in connection with the resolution of the Hagedorn divergences of string thermodynamics. The cosmological evolution of a special, 2d thermal ‘Hybrid’ model is presented and the correct implementation of the full stringy degrees of freedom leads to the absence of gravitational singularities, within a fully perturbative treatment.

Keywords : String Theory, Conformal Field Theory, Orbifold Compactifications, String Thermodynamics, Hagedorn Phase Transition, Non-Singular String Cosmology.

Résumé

Cette thèse traite des applications de la Théorie des Cordes aux problèmes de la cosmologie et de la phénoménologie. En particulier, nous étudions des problèmes liés à la description de l’état initial de l’Univers, en utilisant les méthodes perturbatives de la Théorie des Cordes. Après une présentation des outils nécessaires, nous présentons une nouvelle symétrie de dégénérescence spectrale entre les états massifs bosoniques et fermioniques (appelée symétrie MSDS), se trouvant aux points particuliers de l’espace des modules. Nous étudions les déformations marginales des vides MSDS et mettons en évidence leur interprétation thermique, et leur lien avec la résolution des divergences de Hagedorn de la thermodynamique des cordes. L’évolution cosmologique d’un vide thermique bidimensionnel est présentée. On démontre que la prise en compte des tous les degrés de liberté au niveau des cordes mène à l’absence des singularités gravitationnelles, dans un traitement entièrement perturbatif.

Mots-clés : Théorie des Cordes, Théorie Conforme des Champs, Compactification sur des Orbifolds, Thermodynamique des Cordes, Transition de Phase de Hagedorn, Cosmologie des Cordes sans Singularité.

Résumé Détaillé

Cosmologie des Cordes et Sélection du Vide Initial

Certains des problèmes les plus difficiles ouverts de la cosmologie moderne sont directement liés à l'ère primordiale de notre Univers, où les notions de la théorie des champs ne sont plus valides, à cause des effets quantiques de la gravité, qui dominent la phase chaude et fortement courbée de l'Univers. Actuellement, la théorie des cordes ainsi que son extension non-perturbative, la théorie-M, sont les candidates les plus prometteuses pour formuler une théorie cohérente de la gravitation quantique. Par conséquent, une profonde compréhension de la physique à l'échelle de Planck nécessite un traitement purement cordique, ce qui peut aider à découvrir les implications de cette phase initiale pour la phénoménologie et la cosmologie aux basses énergies et pour les grands temps cosmologiques.

Des solutions cosmologiques peuvent être construites naturellement dans le cadre de la théorie des cordes perturbatives, comme des instabilités quantiques (ou thermiques) d'un fond qui est supposé initialement plat. En particulier, dans les vides de la théorie des cordes où la supersymétrie est spontanément brisée, un potentiel effectif de genre 1 non-nul, $V_{\text{eff}}(\mu_I) \neq 0$, induira une backréaction sur le fond initial.

Une correction du fond au niveau des arbres est alors nécessaire afin d'annuler le tadpole du dilaton et de restaurer l'invariance conforme au niveau d'une boucle (genre 1). Le même mécanisme peut aussi être appliqué dans le cas des vides thermiques, où la densité d'énergie libre finie, $\mathcal{F}(\beta)$ induit une backréaction similaire, avec des modules μ_I et, en particulier, la température $T = \beta^{-1}$, ayant acquis une dépendance temporelle.

Lorsque l'amplitude à une boucle (énergie libre) est finie, l'évolution induite peut être étudiée dans le cadre de la théorie des cordes perturbative et, ainsi, ce mécanisme est à la base de la cosmologie des cordes. Cependant, les tentatives naïves pour réaliser ce programme attrayant rencontrent typiquement deux obstacles majeurs :

- (i) les divergences du type Hagedorn, qui correspondent à une backréaction infinie et qui amènent la théorie vers un régime non-perturbatif
- (ii) Le problème de la singularité initiale, traditionnellement rencontré dans la cosmologie standard.

En outre, il existe des complications supplémentaires, associées au mécanisme de sélection du vide initial. Hors les exigences générales pour l'absence des divergences du type Hagedorn/tachyoniques et des singularités gravitationnelles, ainsi que la traçabilité perturbative de la théorie tout au long de l'évolution cosmologique, il y a d'autres questions à se poser. Ces questions concernent principalement l'arbitraire du mécanisme de brisure de la supersymétrie. On peut naturellement se demander s'il existe un principe fondamental pour briser

la supersymétrie. Idéalement, ce mécanisme serait dicté par des principes de symétrie.

Evidemment, des contraintes supplémentaires devront être imposées sur le vide initial, afin d’assurer la compatibilité avec les données d’observation aux grands temps cosmologiques. En particulier, le plus grand soin doit être apporté à la préparation de cet état initial, de sorte qu’il se decompactifie dynamiquement aux grands temps cosmologiques vers un espace-temps de Minkowski 3+1, avec brisure spontanée du spectre supersymétrique $\mathcal{N} = 1, 3$ générations de fermions chiraux et un groupe de jauge semi-réaliste GUT, tels que le $SO(10)$.

Une première étape cruciale dans la lutte contre ces problèmes a été la découverte d’une nouvelle symétrie de dégénérescence de Bose-Fermi dans tous les modes massifs de la théorie, présente aux points de symétrie étendue dans l’espace des modules de certaines compactifications spéciales à deux dimensions. Cette symétrie a été appelée “Symétrie Massive bose-fermi de dégénérescence spectrale” (MSDS) et peut être considérée comme une algèbre des courants élargie, différente de la supersymétrie ordinaire. Dans le cas le plus simple des modèles avec symétrie maximale, cette structure de dégénérescence se manifeste en terme des identités “généralisées” entre des fonctions- θ de Jacobi, ou en termes des identités entre des caractères de Kac-Moody $SO(24)$, par exemple :

$$V_{24} - S_{24} = 24 .$$

En particulier, la structure de dégénérescence du type ‘supersymétrique’ aux niveaux massifs garantit l’absence d’excitations tachyoniques aux points MSDS de l’espace des modules.

Compte tenu des remarques ci-dessus, il est très naturel de considérer la possibilité que l’univers se trouvait initialement dans un état chaud, comme un espace compact ($d \leq 2$) avec une courbure très proche de l’échelle des cordes. On pourrait envisager un scénario où la dynamique induit la décompactification de certaines des ces dimensions spatiales de sorte que, finalement, on obtienne un univers quadri-dimensionnel.

Il est clair que l’époque cosmologique initiale va être caractérisée par une structure non-géométrique de l’espace-temps, exigeant un traitement qui prend correctement en compte la totalité des degrés de liberté de la théorie des cordes. À cet égard, le haut degré de symétrie des vides MSDS nous invite à les considérer comme des candidats naturels pour décrire cette ère initiale.

Divergences du type Hagedorn

Afin de réaliser la connexion des vides MSDS définis aux basses dimensions ($d \leq 2$), avec les vides supersymétriques aux dimensions supérieures, il est important d’analyser leur

espace des modules et d'étudier leurs déformations marginales adiabatiques à l'échelle des arbres (avant de prendre en compte la back-réaction). Nous considérons ici des déformations du modèle- σ par des opérateurs du type courant-courant :

$$\int d^2z \lambda_{\mu\nu} J^\mu(z) J^\nu(\bar{z}) .$$

Une étude minutieuse des déformations :

$$\frac{SO(8,8)}{SO(8) \times SO(8)} ,$$

du réseau de compactification dans les modèles MSDS de symétrie maximale indique la présence de 2 modules indépendants qui contrôlent la brisure des supersymétries 'gauches' et 'droites'. Elles correspondent aux modules radiaux de Kähler associées à deux cycles toroïdaux spécifiques, X^0, X^1 .

Dans la limite de décompactification, le couplage aux charges de la R -symétrie est effectivement éliminé et on récupère un vide 4d, $\mathcal{N} = 8$ de type II et avec une description effective de supergravité jauge, la jauge étant induite par des flux géométriques bien-définis qui sont responsables pour la brisure de la supersymétrie.

On considère les deux autres grandes dimensions comme émergentes par ces déformations. Dans le cadre cosmologique, où les modules de déformation acquièrent une dépendance temporelle, il est naturel de considérer notre espace cosmologique (4d) comme étant créé dynamiquement d'un tel vide initial MSDS de deux dimensions. Bien sûr, une condition nécessaire pour cela est que le vide initial soit libre de divergences tachyoniques/Hagedorn, sous des déformations arbitraires des modules dynamiques.

Les tentatives visant à décrire la phase initiale de haute température (et/ou de haute courbure) de l'univers, $R \sim \ell_s$, butent généralement sur le problème des instabilités du type Hagedorn (/ tachyonique), ce qui empêche un traitement perturbatif de la back-réaction. Dans la description standard d'un système thermique en terme d'une trace thermique, ces divergences se produisent à cause de la croissance exponentielle en fonction des masses de la densité des états à une particule.

Dans l'image euclidienne, cependant, où le temps est compactifié sur un cycle (toroïdale) X^0 de rayon $R_0 = \beta/2\pi$, le même phénomène se manifeste de façon différente : certains états de corde, qui enroulent le cycle du temps euclidien, deviennent tachyoniques dès que le module thermique R_0 dépasse sa valeur critique (Hagedorn), R_H . De ce point de vue, les divergences Hagedorn ne doivent pas être interprétées comme des vraies pathologies de la théorie des cordes, mais, plutôt, comme des instabilités IR du fond euclidien et la transition de phase correspondante est entraînée par la condensation des ces tachyons.

La présence de ces condensats injecte de la charge de winding non-triviale $\langle \mathcal{O}_n \rangle \neq 0$ dans le vide, une propriété qui sera cruciale pour nos tentatives pour résoudre ces instabilités. En effet, le traitement dynamique correct de la transition de phase au niveau des cordes, en condensant le winding tachyon thermal reste un énorme problème ouvert. Cependant, une façon alternative de traiter le problème est de construire directement des vides thermiques stables (non-tachyoniques) avec des nombre de winding non-trivial, qui correspondent aux vides résultants qui décrivent la nouvelle phase après la transition.

En fait, les modèles thermiques MSDS correspondent aux températures supérieures à la valeur de Hagedorn sans produire de divergences. Cela est possible parce que le cycle du temps euclidien dans ces modèles est transpercé par des flux “gravito-magnétiques” non-triviaux, associés aux graviphotons et aux champs axio-vectoriels de jauge, $U(1)$. Leur présence affine l’ensemble thermique et rend l’énergie libre finie au point MSDS.

Pour illustrer ce point, on peut se dérouler le domaine fondamental, pour des valeurs suffisamment³ grandes de :

$$R_0^2 \equiv G_{00} - G_{0I} G^{IJ} G_{J0} ,$$

et de décomposer l’intégrale à ces orbites modulaires. La contribution de l’orbite $(0,0)$ dans l’espace des windings s’annule (à cause de la supersymétrie effective à $T = 0$) et on obtient l’énergie libre, donnée par l’intégrale suivante sur la bande :

$$\int_{||} \frac{d^2\tau}{\tau_2^2} (\dots) \frac{R_0}{\sqrt{\tau_2}} \sum_{\tilde{m}^0 \neq 0} e^{-\frac{\pi}{\tau_2} (R_0 \tilde{m}^0)^2} (-)^{(a+\bar{a})\tilde{m}^0} \sum_{m_I, n^I \in \mathbb{Z}} q^{\frac{1}{2}P_L^2} \bar{q}^{\frac{1}{2}P_R^2} (-)^{\bar{b}n^1} e^{2\pi i \tilde{m}^0 (G_{0I} Q_{(M)}^I - B_{0I} Q_{(N)}^I)} ,$$

où Q^I sont les 14 charges des $U(1)$ -transverses associées aux graviphotons et aux axio-vecteurs de jauge. Rappelons maintenant que la composante temporelle d’un potentiel de jauge du vide (constant) A_0 ne peut être éliminée par une transformation de jauge dans la présence de température non-nulle. Sa v.e.v. (valeur moyenne du vide) a une signification physique en tant que paramètre topologique du vide qui caractérise le système thermique.

Dans ce sens, l’amplitude thermique à une boucle (énergie libre) peut être réexprimé comme une trace thermique sur l’espace de Hilbert de la théorie initiale en 3 dimensions, avec supersymétrie $(4,0)$:

$$Z(\beta, \mu^I, \tilde{\mu}_I) = \text{Tr} \left[e^{-\beta H} e^{2\pi i (\hat{G}_0^I \hat{m}_I - \hat{B}_{0I} n^I)} \right] ,$$

avec $2\hat{m}_I, n^I$ les charges transversales entières. La trace thermique est ainsi déformée par la présence des flux thermiques associés aux graviphotons et aux axio-vecteurs. Les paramètres :

$$\hat{G}_0^I \equiv G_{0K} G^{KI} ,$$

3. Ceci est nécessaire afin d’assurer la convergence absolue.

$$\hat{B}_{0I} \equiv B_{0I} - \hat{G}_0^K B_{KI} ,$$

sont invariants d'échelle, non-fluctuants, et constituent des paramètres thermodynamiques du système thermique. Ces flux (globaux) peuvent être décrits en terme des condensats de champs de jauge, de 'field strength' non-nulle (localement), mais avec une valeur non-triviale de ligne de Wilson autour du cycle du temps euclidien.

Aux températures suffisamment basses, les états chargés sous ces champs $U(1)$ vont acquérir une masse et, effectivement, ils se découplent du système thermique, de sorte qu'on retrouve l'ensemble thermique canonique. Les tachyons 'potentiels' peuvent parfois être éliminés de cette même manière, étant eux-mêmes chargés sous ces champs $U(1)$.

Par une étude attentive du spectre thermique-BPS de basse énergie, il est possible d'obtenir des conditions qui garantissent l'absence de divergences Hagedorn pour toute déformation des modules transversaux (dynamiques). Si celles-ci sont satisfaites, une rotation discrète $O(8; \mathbb{Z}) \times O(8; \mathbb{Z})$ transforme les conditions pour les flux à la forme pratique suivante :

$$\hat{G}_0^k = \hat{B}_{0k} = 0 ,$$

$$\hat{G}_0^1 = 2\hat{B}_{01} = \pm 1 ,$$

où $k = 2, \dots, 7$ couvre les directions toroïdales, qui sont transversales au temps euclidien. En fait, ces conditions ont un sens géométrique assez simple : elles correspondent au cas où le cycle de la température couple de façon chirale aux charges 'gauches' de la R -symétrie et se factorise complètement du réseau toroïdal, qui reste couplé seulement au nombre fermionique 'droit', F_R . Les modèles sans tachyons thermiques, donc, correspondent à la factorisation suivante du réseau :

$$\Gamma_{(d,d)} \left[\begin{smallmatrix} a, \bar{a} \\ b, \bar{b} \end{smallmatrix} \right] = \Gamma_{(1,1)} \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] (R_0) \otimes \Gamma_{(d-1,d-1)} \left[\begin{smallmatrix} \bar{a} \\ \bar{b} \end{smallmatrix} \right] (G_{IJ}, B_{IJ}) .$$

En outre, dans la phase thermodynamique saturant les conditions ci-dessus, la trace thermique déformée se réduit tout simplement à l'indice de fermion droit :

$$Z = \ln \text{Tr} \left[e^{-\beta H} (-)^{F_R} \right] .$$

Modèles Hybrides et Cosmologie Non-Singulière

Il est facile de vérifier que les modèles de symétrie MSDS maximale ne satisfont pas aux conditions de stabilité présentées ci-dessus, même si des trajectoires sans tachyons⁴ peuvent

4. Des classes de modèles MSDS sans tachyons peuvent aussi être obtenu, en introduisant des orbifolds asymétriques de type \mathbb{Z}_2 . Dans ces derniers, les fluctuations des modules 'dangereux' sont éliminées par la structure particulière de l'orbifold.

être construites, les reliant aux vides de plus grande dimension, avec $\mathcal{N}_4 \leq 8$ supersymétrie. Cependant, une classe très intéressante de vides thermiques MSDS sans tachyons sont les modèles thermiques hybrides. Dans leur version froide, $T = 0$, ils sont définis comme des vides de supersymétrie $(4, 0)$, avec les supersymétries droites brisées spontanément à l'échelle des cordes et remplacées par la structure MSDS. Cette structure spéciale (anti-)chirale supporte un groupe de symétrie de jauge étendue, non-abélienne :

$$U(1)_L^8 \times SU(2)_{k=2,R}^8 .$$

La compactification d'une des directions longitudinales sur un cercle $S^1(R_0)$ et son identification avec la direction du temps euclidien nécessite un modding spécial par l'élément de Scherk-Schwarz $(-)^{F_L} \delta_0$, conformément à la connexion de spin-statistique. L'énergie libre de ces modèles thermiques s'écrit alors :

$$Z = \frac{V_1}{8\pi} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^{3/2}} \left[\frac{1}{2} \sum_{a,b} (-)^{a+b} \frac{\theta_b^{[a]4}}{\eta^4} \right] \Gamma_{E_8} (\bar{V}_{24} - \bar{S}_{24}) \Gamma_{(1,1)}^{[a]}(R_0) .$$

où :

$$\Gamma_{(1,1)}^{[a]}(R_0) = \frac{R_0}{\sqrt{\tau_2}} \sum_{\tilde{m}^0, n^0} e^{-\frac{\pi R_0^2}{\tau_2} |\tilde{m}^0 + \tau n^0|^2} (-)^{\tilde{m}^0 n^0 + a \tilde{m}^0 + b n^0} .$$

Les conditions d'absence de divergences tachyoniques sont saturées par la structure factorisée du réseau $\Gamma_{(1,1)}$ et l'énergie libre reste finie pour toute valeur des modules transversaux dynamiques.

En outre, la factorisation est préservée, puisque les modules de mixage (associés aux flux gravitomagnétiques) ne correspondent pas à des champs fluctuants. En outre, la présence de ces flux notamment restaure la T -dualité thermique⁵ et injecte un nombre de winding non-trivial dans le vide thermique.

Il est important de noter ici que l'intégrale des chemins euclidienne se réduit à l'expression de la trace thermique déformée :

$$\text{Tr} \left[e^{-\beta H} (-)^{F_R} \right] ,$$

seulement pour la région $R_0 > 1/\sqrt{2}$, où la convergence absolue est assurée. Afin d'obtenir l'expression analogue pour la région $R_0 < 1/\sqrt{2}$ on doit d'abord effectuer une double resommation de Poisson pour aller à la phase duale, puis déplier de nouveau le domaine fondamental.

Ainsi, la description en terme de la trace thermique est un exemple d'une expression de la théorie des champs qui échoue de prendre en compte la dualité thermique. Ceci permet de

5. La dualité correspond à $R_0 \rightarrow 1/(2R_0)$, avec un échange simultané de la chiralité des spineurs, $S_8 \leftrightarrow C_8$.

postuler que l'objet fondamental est l'intégrale de chemins euclidienne, qui est valable pour toutes les températures et qui contient les dualités de façon manifeste.

Un résultat important est que, une fois que la structure MSDS est préservée dans les modèles hybrides thermiques, la symétrie MSDS avec le 'level-matching' permettent le calcul explicite et exact de l'amplitude thermique à une boucle :

$$\frac{Z}{V_1} = 24 \times \left(R_0 + \frac{1}{2R_0} \right) - 24 \times \left| R_0 - \frac{1}{2R_0} \right| .$$

Ce résultat est exact au niveau de genre 1, sans aucune approximation en α' . Il est manifestement invariant sous la dualité thermique et la non-analyticité (structure conique) dans le second terme est induite par la présence des états de masse-nulle supplémentaires au point auto-dual fermionique $R = 1/\sqrt{2}$.

En outre, la préservation de la structure MSDS introduit de nombreuses annulations entre les deux tours des états massifs de la théorie à température nulle, de sorte que la trace thermique (déformée) puisse être réduite à un ensemble thermique canonique :

$$\text{Tr}|_{m^2=0} e^{-\beta H} ,$$

restreint à l'espace de Hilbert de masse nulle de la théorie froide $(4,0)$ initiale. Compte tenu de ce résultat, il n'est pas surprenant que l'équation d'état thermique définie par ces modèles :

$$\rho = P = 48\pi T^2 ,$$

est exactement celle du rayonnement (de masse-nulle) thermique en deux dimensions.

Cette observation implique la définition :

$$T = T_c e^{-|\sigma|} ,$$

de la température physique qui est invariante par la dualité en termes d'une variable thermique $\sigma \in (-\infty, +\infty)$, où maintenant :

$$T_c \equiv \frac{\sqrt{2}}{2\pi} ,$$

est la température critique (et maximale) de la théorie. On pourrait imaginer qu'avec l'augmentation du paramètre σ , le système se chauffe pour atteindre la température critique, où une transition de phase a lieu et, ultérieurement, le système refroidit dans la nouvelle phase.

La présence de cette transition de phase peut être motivée de la façon suivante. Considérons un état dynamique pur dans le secteur $S_8 \bar{V}_{24}$ et dans le point de symétrie étendue (critique), $\sigma = 0$. À ce stade, on observe la présence d'opérateurs localisés qui induisent des transitions

entre les états de moment purs et des états de winding purs. Par exemple, en prenant le mode zéro du courant :

$$J_-(z) = \psi^0 e^{-iX_L^0} ,$$

et en agissant sur l'opérateur vertex de l'état de moment pur :

$$J_-(z) e^{-\phi/2} S_{10,\alpha} e^{\frac{i}{2}X_L^0 + \frac{i}{2}X_R^0} \bar{V}_{24}(w) \sim \frac{1}{z-w} \gamma_{\alpha\dot{\beta}}^0 e^{-\phi/2} C_{10,\dot{\beta}} e^{-\frac{i}{2}X_L^0 + \frac{i}{2}X_R^0} \bar{V}_{24}(w) ,$$

on obtient l'état pur de winding de chiralité opposée dans le secteur $C_8 \bar{V}_{24}$. Par conséquent, le point de symétrie étendu $\sigma = 0$ est caractérisé par la présence des amplitudes non-triviales à trois points, qui conduisent des transitions entre les états de pur moment et les états de pur winding.

À ce point, il sera utile de souligner certaines hypothèses implicites dans cette description. Déjà, nous supposons une évolution adiabatique, avec des champs variants assez lentement et, en particulier, nous exigeons que le temps caractéristique de la transition est suffisamment court pour permettre une approximation du type 'delta' de Dirac, $\delta(\sigma) \sim \delta(x^0)$. Avec cette hypothèse, la transition de phase se déroule à un moment donné, $x^0 = 0$.

En outre, l'action effective totale devrait également contenir des contributions des autres modules de petite masse. En particulier, les 64 modules qui paramétrisent le coset des déformations, $SO(8,8)/SO(8) \times SO(8)$. Toutefois, dans toutes les phases du modèle hybride thermique, ces directions plates sont finalement levées par le potentiel effectif à 1-boucle et les modules ci-dessus se stabilisent au point de symétrie MSDS où ils sont entropiquement favorisés. Par conséquent, nous n'allons pas étudier leurs v.e.v. mais on va les considérer comme être effectivement gelées à leurs valeurs MSDS, en conservant la structure MSDS de la théorie.

Cette transition de phase a une description efficace en terme d'une brane de type espace, qui colle l'espace des moments avec l'espace dual des windings. L'action effective est :

$$S = \int d^2x e^{-2\phi} \sqrt{-g} \left(\frac{1}{2} R + 2(\nabla\phi)^2 \right) + \int d^2x \sqrt{-g} P \\ - \kappa \int dx^1 d\sigma e^{-2\phi} \sqrt{g_{11}} \delta(\sigma) ,$$

où, en plus du terme de gravité avec dilaton et du terme qui décrit le potentiel effectif thermique, l'action contient aussi une contribution effective d'une brane de type espace, localisée à la transition de phase à $\sigma = 0$. Ce terme représente de la pression négative localisée, sourcée par les 24 scalaires supplémentaires (complexes) de masse nulle, au point de la symétrie élargie.

Nous allons utiliser la paramétrization standard pour la métrique, pertinente pour deux dimensions :

$$ds^2 = -N^2(t) dt^2 + a^2(t) dx^2 , \tag{1}$$

où le facteur d'échelle sera paramétrisé comme $a = e^\lambda$. En supposant que la transition de phase est localisée dans le temps, à $t = 0$, on peut déduire des équations du mouvement du modèle σ , qui seront maintenant modifiées à cause de la présence de la brane :

$$\begin{aligned}\dot{\phi}^2 - \dot{\phi}\dot{\lambda} &= \frac{1}{2} N^2 e^{2\phi} \rho , \\ \ddot{\lambda} - \dot{\lambda} \left(2\dot{\phi} - \dot{\lambda} + \frac{\dot{N}}{N} \right) &= N^2 e^{2\phi} P , \\ \ddot{\phi} - 2\dot{\phi}^2 + \dot{\phi} \left(\dot{\lambda} - \frac{\dot{N}}{N} \right) &= \frac{1}{2} N^2 e^{2\phi} (P - \rho) - \frac{1}{2} N \kappa \delta(t) .\end{aligned}$$

On note la présence de la fonction δ de Dirac, qui apparaît seulement dans la troisième équation, ce qui implique une discontinuité (autour de la transition) seulement dans la première dérivée du dilaton, tandis que la fonction de laps, le facteur d'échelle, et leurs premières dérivées restent continues partout. Les équations ci-dessus peuvent être simplifiées si on se met dans la jauge conforme, $N = e^\lambda$ et en utilisant l'équation d'état, $\rho = P$.

Le couplage des cordes au point de la transition, $g_s(0)$, n'est pas un paramètre arbitraire, mais est lié à la tension de la brane, κ . En effet, les équations du mouvement au point de la transition donnent la relation suivante entre le couplage des cordes, le paramètre thermique $\Lambda \equiv 48\pi$ et la tension de la brane au point de la transition, $t = 0$:

$$g_s^2(0) \equiv e^{2\phi_0} = \frac{\beta_c^2 \kappa^2}{8\Lambda} .$$

Intuitivement, cela apparaît comme un équilibre entre les effets thermiques et la pression négative injectée au système par la brane, et qui est communiqué par le dilaton. Ainsi, la validité perturbative du modèle est assurée, à condition que le paramètre de la tension, κ , est suffisamment petit.

Si on impose la conservation de l'entropie thermique à travers la transition (c.à.d. l'absence de chaleur latente) on peut résoudre les équations cosmologiques dans la jauge conforme :

$$\begin{aligned}ds^2 &= \frac{4}{\kappa^2} \frac{e^{|\tau|}}{1 + |\tau|} \left(-d\tau^2 + dx^2 \right) , \\ g_s^2 &\equiv e^{2\phi(\tau)} = \frac{\pi \kappa^2}{192} \frac{1}{1 + |\tau|} .\end{aligned}$$

Ainsi, on peut voir que la présence de la transition de phase au point de la symétrie élargie provoque un rebond à la fois dans le facteur d'échelle et, en même temps, au dilaton, et l'évolution cosmologique contourne la singularité gravitationnelle, tout en restant dans le régime perturbatif, à condition que la tension est $\kappa^2 \ll 1$.

En effet, le scalaire de Riemann n'a pas de singularités nues :

$$R = \left(\frac{\kappa}{2}\right)^2 \frac{e^{-|\tau|}}{1+|\tau|} .$$

En outre, la singularité conique dans la dérivée du dilaton :

$$\ddot{\phi} = -\frac{\kappa}{2}\delta(t) ,$$

est résolue par la présence des états supplémentaires de masse nulle, localisés au point de la transition.

Comme mentionné ci-dessus, il est possible d'ajouter des dérivées de l'ordre supérieur, ainsi que des corrections de genre supérieur. Toutefois, celles-ci sont supprimées et sont, en effet, contrôllées par un développement perturbatif dans le paramètre de tension :

$$\kappa \sim g_{\text{str}} \ll 1 .$$

Ces corrections vont, éventuellement, étaler la brane dans le temps et lisser la transition.

On peut aussi présenter la solution dans le référentiel cosmologique :

$$ds^2 = -d\xi^2 + a^2(\xi)dy^2 ,$$

mais la solution ci-dessus ne peut être exprimée en terme des fonctions élémentaires. Elle implique le changement de variables ci-dessous :

$$\xi(\tau) = \frac{2|\tau|}{\kappa} \int_0^1 \frac{e^{|\tau|u/2}}{\sqrt{1+|\tau|}} du = \sqrt{\frac{8\pi}{e\kappa^2}} \left[\text{erfi} \left(\sqrt{\frac{1+|\tau|}{2}} \right) - \text{erfi} \left(\frac{1}{\sqrt{2}} \right) \right] ,$$

$$y = x/2 .$$

Le comportement asymptotique de la solution pour les petits temps cosmologiques, $|\kappa\xi| \ll 1$, a la forme suivante :

$$a(\xi) = \frac{4}{\kappa} \left[1 + \frac{1}{16}(\kappa\xi)^2 + \mathcal{O}(|\kappa\xi|^3) \right] ,$$

$$\frac{1}{g_{\text{str}}^2} = \frac{192}{\pi\kappa^2} \left[1 + \frac{|\kappa\xi|}{2} - \frac{1}{8}(\kappa\xi)^2 + \mathcal{O}(|\kappa\xi|^3) \right] .$$

Ce comportement illustre le rebond du facteur d'échelle, ainsi que la structure conique du dilaton.

De même, pour les grands temps cosmologiques, $|\kappa\xi| \gg 1$, on trouve :

$$a(\xi) = |\xi| \left[1 - \frac{1}{\ln(\kappa\xi)^2} + \dots \right] ,$$

$$\frac{1}{g_{\text{str}}^2} = \frac{192}{\pi\kappa^2} \left[\ln(\kappa\xi)^2 + \ln \ln(\kappa\xi)^2 + \text{const} + \dots \right] ,$$

et la cosmologie induite est celle d'un univers thermique du type Milne. La présence du 'dilaton courant' induit ainsi des corrections logarithmiques, comme affiché ci-dessus.

Le modèle hybride est le premier exemple dans la littérature où un traitement des degrés de liberté des cordes autour du point de symétrie élargie a amené à la découverte d'un mécanisme de cordes qui résout simultanément la singularité initiale ainsi que les divergences de Hagedorn.

On entend que les ingrédients de base de ce mécanisme qui protège l'évolution des singularités dans des vides plus généraux aux hautes dimensions sont déjà présents dans le modèle hybride. Évidemment, cette approche n'est que la première étape dans ce programme ambitieux de connecter l'ère initiale non-singulière de l'Univers avec la cosmologie standard aux grands temps cosmologiques.

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Chapitre 1

Introduction

One of the major breakthroughs in theoretical physics over the last century has been the development of the methods of quantum field theory. These efforts were eventually crowned with the enormous success of the Standard Model of high energy physics, describing most the interactions between elementary particles to an astonishing degree of accuracy. Despite its many successes, however, the Standard Model at the very best ignores a certain number of issues.

A first aesthetic problem is the inability to explain its 18 parameters, which are introduced into the model by hand. At the same time, the flavor sector is the least understood, with its quantum numbers being, again, assigned by hand and they do not seem to derive from some fundamental principle. One of the more serious, direct problems of the Standard Model is the absence of right-handed neutrinos. The difficulties arising in our attempts to describe Nature, however, extend much further beyond the Standard Model and its field-theoretic extensions. Perhaps the most serious deficit of these approaches is their inability to consistently incorporate gravity together with the other interactions, into a unified description.

A further problem, of a different nature, has to do with the fact that we essentially only have good control over a theory through a perturbative (asymptotic) expansion and only to the extent that we stay within its perturbative regime. Our understanding and treatment of non-perturbative phenomena is certainly far from complete and our insight is mostly based on those happy occasions when it is possible to describe a strongly coupled theory, in terms of a weakly-coupled dual theory. Hence, dualities are of fundamental importance to physics and are expected to play an important role in the structure of a fundamental theory that will presumably address the above problems.

A highly attractive candidate for such a fundamental theory (even if only in some limit) is String Theory. Though initially introduced for different reasons, in the context of dual models in the 70's, String Theory has since been the subject of extensive study, indicating far-reaching consequences for the nature of spacetime itself. The identification of a massless, spin-2 particle in its perturbative spectrum with the graviton has lead physicists to realize that String Theory is a theory of quantum gravity.

Even though there still exists no explicit proof for it being UV finite above the genus-2 level, there are strong arguments that it is indeed the case. The generalization from par-

ticles to strings spreads the interaction from spacetime points to world-surfaces, effectively smoothening the behaviour of amplitudes. String Theory has the extremely attractive property of coming only with one dimensionful parameter, the string length, ℓ_s . Provided that the geometric language of string perturbation theory is well-defined, one would expect the string length to provide a natural regulator and, hence, to render amplitudes finite in the ultraviolet limit.

The first attempts of quantizing relativistic strings yielded bosonic string theory which, however, did not include fermions in its spacetime spectrum. This marked the passage to superstrings, which naturally incorporated supersymmetry in their spacetime spectra. There exist five consistent, stable superstring theories in the critical, 10d case. These are the Type I, Type IIA and Type IIB theories of open and closed strings, as well as the $E_8 \times E_8$ and $SO(32)$ Heterotic (closed) string theories. Despite their apparent differences it was realized that they are all connected to each other in terms of dualities. This hinted to the fact that all of these superstring constructions are, in fact, different vacua of a single underlying theory, M-theory. Presently, String Theory and its non-perturbative extension, M-theory, are the most promising candidates for a consistent theory of quantum gravity.

To make contact with four-dimensions, one eventually must look for solutions of String Theory on spacetimes of the form $\mathbb{R}^{1,3} \times K_{(6)}$, where $K_{(6)}$ is a 6d compact space. For example, the geometrical data of the compact space enter the effective action as moduli fields and, hence, pose additional phenomenological problems unless they are stabilized, acquiring some non-vanishing mass. The process of compactification has a second caveat, namely, it gives rise to an enormous number of vacua without providing any dynamical mechanism or principle why any particular vacuum should be preferred to any other.

The Vacuum Selection problem is indeed a serious embarrassment for the (perturbative) theory and has lead, to some extent, to the disappointment of some of the leaders of the string community. A more optimistic approach is to study general properties of classes of string vacua and their implications for low-energy phenomenology. Starting from such properties of the observable sector, such as the presence of three net generations of chiral matter, one might try to investigate under what conditions they can be accommodated in string vacua. Furthermore, in order to gain more insight into the structure of phenomenologically attractive classes of vacua, it becomes necessary to study emergent structures and symmetries living at special points in the string moduli space.

At the same time, String Theory modifies the conventional field-theoretic notions about spacetime, geometry, dimensionality and even topology. These purely stringy phenomena typically arise as one tries to probe energies around the string scale. Around these high-curvature regions the conventional description of spacetime is expected to break down and a non-geometric picture arises, which may be studied using the underlying conformal field theory and string theory. One hopes that the application of String Theory in these regions where the effects of quantum gravity become dominant will be able to resolve some of the puzzles and problems posed by the phenomenological models of cosmology.

One of the most acute such problems concerns the initial phase of the universe and, in particular, the resolution of the initial ‘Big Bang’ singularity. The hope is that, by taking

correctly the effects of quantum gravity into account, the unphysical singularities will be resolved in the framework of the underlying theory. This necessitates the study of cosmological solutions arising at the perturbative string level as quantum (or thermal) instabilities from an initial string vacuum. This is intimately related to the problem of constructing exact string vacua with spontaneously broken supersymmetry, which can be used to describe this initial state of the universe.

However, the breaking of supersymmetry at the string level is usually accompanied by the presence of tachyonic modes, which pose additional problems to the perturbative incorporation of the backreaction to the effective potential. A related twin problem arises through the introduction of finite temperature effects in string theory. There, because of the exponentially growing density of states, thermodynamical quantities like the free energy diverge above a certain critical temperature, a problem known in the literature as the Hagedorn problem.

This thesis essentially discusses work published in references [46], [47], [48] and [91]. The main aspiration of this work has been to study the contribution of stringy effects arising in the early, high-temperature phase of the universe. The two obstacles that typically complicate matters, as mentioned above, are the Hagedorn divergences and the initial gravitation singularity. Attempts to systematically construct exact vacua with spontaneously broken supersymmetry, or which admit a natural thermal interpretation, while preserving the tachyon-free (stability) requirements consistent with a perturbative treatment, have lead to the discovery [45] of a novel Degeneracy Symmetry in the Spectra of Massive bosons and fermions ('MSDS').

The study and classification of such special vacua was performed in [46]. The problem of tachyon stability, the thermal interpretation of MSDS vacua, together with the general conditions leading to the resolution of the Hagedorn divergences, were discussed in [47]. In [48], the backreaction of a special 2d thermal 'Hybrid' vacuum on the initially flat background was exactly calculated. There, because of the remarkable cancellations introduced by the MSDS structure, the stringy contributions are under control and can be computed exactly around the extended symmetry point, associated to a thermal T -duality around the Euclidean time circle. The analysis hints at the presence of a phase transition, where states with non-trivial momenta are transformed into states with non-trivial windings. At the point of the transition, the temperature acquires its maximal (critical) value and the scale factor of the spacetime metric and the dilaton are found to bounce simultaneously and the cosmological evolution escapes the initial singularity, while remaining in the perturbative regime throughout its history.

The organization of this thesis is as follows. In Chapter 2 we briefly review some aspects of perturbative string theory. At the same time we introduce the conventions and notation to be used throughout the rest of the text. Chapter 3 discusses the process of compactification, introduces T -duality and briefly reviews exact vacuum constructions, such as compactifications on orbifolds, as well as fermionic and Gepner constructions. Subsequently, Chapter 4 discusses the beautiful relation between the presence of supersymmetry in spacetime and the enhancement of the local $N = 1$ superconformal algebra to a global $N = 2$ (or $N = 4$)

on the string worldsheet. In Chapter 5 we discuss the Scherk-Schwarz mechanism for breaking supersymmetry at the string level. In Chapter 6 we introduce MSDS vacua, we discuss their algebra and their classification. We then proceed in Chapter 7 to investigate their deformations and thermal interpretation in connection with the resolution of the Hagedorn divergences. In Chapter 8 we begin with short introduction to the classic problems of standard Big Bang cosmology and, subsequently, we introduce the Hybrid models and discuss their non-singular cosmology. Chapter 9 contains an independent aspect of the MSDS structure, as the spectral-flow algebra appearing in twisted sectors of orbifold vacua with a global $N = 4$ superconformal algebra realized internally in the bosonic side of the Heterotic string, hence, hinting to a possible deeper origin of these structures. Three appendices summarize useful definitions of modular functions, lattice identities, the equivalence between fermionic constructions and \mathbb{Z}_2 -type orbifolds and the technique of decomposition of modular invariant integrals into modular orbits, known as ‘unfolding’ of the fundamental domain.

Chapitre 2

Elements of String Theory

In this chapter we start with a brief review of string perturbation theory and quantization in 10 dimensions. The starting point is the quantization of the bosonic string and, subsequently, the requirement of having spacetime fermions in the string spectrum will necessitate the passage to the superstring formalism. As in the subsequent chapters, we will be interested in closed, oriented strings.

2.1 Brief Introduction to Bosonic String Theory

The Polyakov Action and its Symmetries

In this section we will give a lightning overview of the basic elements of bosonic string theory. We will adopt a rather direct line of approach and speedily introduce only the very essential elements, without making any attempt to give a complete picture in this rather vast subject. This section will partially serve to set the conventions to be used later on.

The starting point of string theory is to replace the quantum field-theoretic notion of point particles by extended 1-dimensional objects propagating in a D -dimensional spacetime \mathcal{M} . With the flow of proper time, these objects trace a 2d surface Σ , commonly referred to as the (string) worldsheet, which is embedded into the target spacetime.

One chooses local coordinates $\{\sigma_a\} = \{\sigma, t\}$ to parametrize points on the worldsheet, which are denoted as $X^\mu(\sigma, t)$. Physically, one may think of t as the proper time along the worldline of each point on the string and σ as the spatial parameter labeling each such point. Of course, the parametrization on the worldsheet is arbitrary and σ, t are not observables of the theory. This gives rise to the requirement that any physical string theory should be invariant under reparametrizations.

Consider the case of a closed string, where the two string ends are matched at the same point. As the ring-shaped string propagates in “time” t , it traces a 2d “tube-like” surface. This can be thought of as the string-analogue of a particle line in a Feynman diagram. Likewise, the matching of two incoming closed string tubes to create a single larger tube and its subsequent breaking to two outgoing string tubes is the analogue of a tree-level interaction diagram between 4 particles in quantum field theory. However, there are two

important differences from the field theoretical picture. The first is that in the string case there is no interaction vertex-point : the reason being that the string is an extended object and so, the interaction is expected to spread over the 2d surface. Thus, one might a priori expect some of the UV-divergences of field theory to be absent or at least softer in the string case. Secondly, in the string interaction no coupling constant is introduced ‘by hand’ as in the case of field theory but, rather, the string coupling itself is dynamical in the sense that it is identified with the vacuum expectation value (v.e.v.) of the dilaton. In this respect string theory dynamically determines the strength of its own interactions. Continuing along the same train of thought, one finds a similar analogy between the 2d surfaces with holes and loop diagrams in field theory.

In this formalism the spacetime coordinates $X^\mu(\sigma)$ and the induced metric $g_{ab}(\sigma)$ on the worldsheet become themselves dynamical fields, defining a map X from the 2d surfaces Σ parametrized by local coordinates σ_a , to the target spacetime :

$$X : \sigma_a \in \Sigma \longrightarrow X^\mu(\sigma) \in \mathcal{M}. \quad (2.1)$$

This map also defines the induced metric $g_{ab}(\sigma)$ on Σ as the pullback of the spacetime metric $G_{\mu\nu}(X(\sigma))$:

$$g_{ab}(\sigma) = G_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu. \quad (2.2)$$

One might then consider String theory essentially as the theory of consistent quantization of fields propagating on these 2d Riemann surfaces.

Let us motivate the structure of a general action for the fields X^μ, g_{ab} on a 2d surface, that is compatible with invariance under diffeomorphisms and which contains up to two derivatives. The positions $X^\mu(\sigma)$ on the string become scalar fields from the worldsheet point of view and we may begin with a general (Euclidean) action of the form :

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} \left(g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) + i E^{ab} \partial_a X^\mu \partial_b X^\nu B_{\mu\nu}(X) + \alpha' \phi(X) R^{(2)}(\sigma) + \mu \right). \quad (2.3)$$

One identifies the antisymmetric tensor field $B_{\mu\nu}$ with the Kalb-Ramond field and the scalar field ϕ with the dilaton, both of which arise naturally in the spectrum of massless string excitations. Here, we have also considered the possibility of adding a cosmological constant term μ . $R^{(2)}$ is the Ricci scalar curvature on the worldsheet and the imaginary i arises from the Euclidean rotation. Reparametrization invariance forces E^{ab} to be an antisymmetric tensor, proportional to the Levi-Civita antisymmetric tensor density¹.

The characteristic string scale will be defined in terms of the Regge slope parameter α' . The connection between the string tension T , the string length $(M_s)^{-1}$ and the α' is :

$$T = \frac{1}{4\pi\alpha'} \quad , \quad M_s = (\alpha')^{-1/2}. \quad (2.4)$$

1. In fact, the correct identification is to take $\sqrt{g}E^{ab} = \epsilon^{ab} = \pm 1$, depending on convention.

For general curved manifolds \mathcal{M} the above sigma model becomes non-linear and, hence, very difficult to quantize. Appart from very special cases, for example when \mathcal{M} has the structure of a group manifold², exact quantization in the interacting case is, in general, not possible and one resorts to a low curvature expansion in $\mathcal{O}(\sqrt{\alpha'}/R)$ around a flat background, with R being the characteristic curvature scale.

In this low-curvature and low-energy regime the purely stringy effects become effectively suppressed and one recovers the effective field theory description. This is a mixed blessing, however, in the sense that even though string theory does naturally reduce to the effective field theory description in this limit (as is desirable) but, at the same time, the novel inherently stringy phenomena arising from the extended nature of the string (in particular phenomena related to the winding around compact dimensions) are masked and can only be probed at scales of the order of the string length $\sqrt{\alpha'} \sim M_s^{-1}$. Consequently, attempts to study these interesting, inherently stringy phenomena are closely related to the problem of exact quantization in non-trivial backgrounds. This will be of major concern to us in later chapters.

The action (2.3) enjoys manifest diffeomorphism and general coordinate invariance. For purposes of exact quantization we will usually consider flat Minkowski backgrounds $G_{\mu\nu}(X) = \eta^{\mu\nu}$, in which case the relevant symmetry is Poincaré invariance. One may employ the reparametrization invariance in order to gauge-fix the metric to covariantly flat form :

$$\hat{g}_{ab}(\sigma) = e^{2\omega(\sigma)} \delta_{ab} . \quad (2.5)$$

The scale factor $\omega(\sigma)$ is the the only remaining degree of freedom of the 2d metric. In the limit $\mu = 0$ one recovers the (classical) symmetry under Weyl rescalings of the metric :

$$g_{ab}(\sigma) \rightarrow g'_{ab}(\sigma) = e^{2\Omega(\sigma)} g_{ab}(\sigma) . \quad (2.6)$$

Under this symmetry, the scale $\omega(\sigma)$ completely decouples (at least classically) from the action. In contrast to reparametrization invariance, the Weyl scaling of the metric is not a redundancy but a dynamical transformation of the metric g_{ab} and, thus, causes the distances between points to actually change. It is a highly non-trivial point about the dynamics of the theory to have an invariance also under Weyl scalings and it is a special property of 2 dimensions. The requirement that Weyl invariance is preserved at the quantum level, i.e. the requirement of cancellation of Weyl anomalies, gives rise to a set of consistency conditions on the central charge of the conformal field theory that remains as a residual symmetry, after gauge fixing. In this work we will take $\mu = 0$ and, hence, restrict ourselves to ‘critical’ string constructions³.

In general, the 2d Ricci term $\sqrt{g}R^{(2)}\phi$ breaks Weyl invariance because $\sqrt{g}R^{(2)}$ has an explicit dependence on the scale factor ω :

$$\sqrt{\hat{g}}\hat{R}^{(2)} = -2\Box\omega(\sigma). \quad (2.7)$$

2. These lead to Wess-Zumino-Witten models which can be solved exactly.

3. In fact, μ rather acts as a counterterm and is used to subtract the constant term in the anomaly of the trace $\langle T_a^a \rangle$ so that Weyl invariance is reinstated at the quantum level.

However, if the scalar $\phi(\sigma)$ is constant the $R^{(2)}$ -term becomes a total derivative and is, then, allowed to enter the action. Actually, for a 2-dimensional surface without boundary the integral is simply the topological Gauss-Bonnet term, proportional to the Euler characteristic $\chi = 2(1 - g_\Sigma)$, with g_Σ being the genus of the surface. By separating out the constant expectation value of the dilaton, $\phi(\sigma) = \langle \phi \rangle + \delta\phi(\sigma)$, we have :

$$\frac{\langle \phi \rangle}{4\pi} \int_{\Sigma} d^2\sigma \sqrt{g} R^{(2)} = \langle \phi \rangle \chi . \quad (2.8)$$

Gauge Fixing à la Faddeev-Popov

The path integral over the fields X^μ, g_{ab} naively diverges as a result of the local diffeomorphism and Weyl symmetry. One then fixes the gauge by using the gauge symmetry to take the metric to conformally flat form. The idea is to write the generic metric g_{ab} as the transformation of a representative \hat{g}_{ab} :

$$g_{ab} = T \cdot \hat{g}_{ab}, \quad (2.9)$$

where T denotes the gauge transformation. Following the Фаддеев-Попов (Faddeev-Popov or simply Φ - Π) procedure, we change integration variables from $\mathcal{D}g_{ab}$ into $\mathcal{D}T$, with T being (somewhat abstractly) the transformation parameter :

$$Z = \int \mathcal{D}X \mathcal{D}g_{ab} e^{-S[X,g]} = \int \mathcal{D}T \mathcal{D}X \cdot \Delta_{\Phi\Pi}[\hat{g}] e^{-S[X,\hat{g}]} . \quad (2.10)$$

The Jacobian of this change of integration variables is the Faddeev-Popov gauge invariant determinant :

$$\Delta_{\Phi\Pi}[\hat{g}] = \left| \frac{\partial \delta_T g_{ab}}{\partial \delta T} \right| = \det \begin{pmatrix} P & * \\ 0 & 1 \end{pmatrix} = \det P . \quad (2.11)$$

This is because the variation of the metric with respect to an infinitesimal diffeomorphism with parameter $\delta\sigma_a = \xi_a$ and a Weyl transformation with parameter $\delta\omega$ can be decomposed into two pieces :

$$\delta g_{ab} = \mathcal{L}_\xi g_{ab} + 2\delta\omega g_{ab} = 2\delta\tilde{\omega} g_{ab} - 2(P\xi)_{ab} , \quad (2.12)$$

where P is the operator :

$$(P\xi)_{ab} = \frac{1}{2} (\nabla_a \xi_b + \nabla_b \xi_a - g_{ab} \nabla_c \xi^c) , \quad (2.13)$$

mapping vectors into traceless symmetric rank 2 tensors. The latter is independent of the variation in the scale factor $\delta\omega$ and, hence, the Φ - Π determinant reduces simply to the determinant of P . The latter can be represented in exponential form by integration over

Grassmann ghost fields b_{ab}, c^a . Putting everything together and dropping the infinite volume factor arising from the integration over the gauge parameters we finally find :

$$Z = \int \mathcal{D}X \mathcal{D}b \mathcal{D}c \exp \left\{ -S[X, \hat{g}] - S_{\text{gh}}[b, c, \hat{g}] \right\} , \quad (2.14)$$

where now the ghost action is written :

$$S_{\text{gh}}[b, c, \hat{g}] = \frac{1}{2\pi} \int d^2\sigma \sqrt{\hat{g}} b_{ab} \nabla^a c^b . \quad (2.15)$$

The latter explicit form is obtained by using the symmetry properties of $(P \cdot c)^{ab}$ to force the b -ghost to be also symmetric. After dropping an irrelevant total derivative term, one arrives at the above ghost action.

The summation over all possible metrics must also take into account the various different worldsheet topologies. One is, then, lead to the Polyakov prescription for the string perturbative expansion, which takes the form :

$$\sum_{\text{topologies}} g_s^{-\chi} \int \mathcal{D}X \mathcal{D}b \mathcal{D}c e^{-S[X, \hat{g}] - S_{\text{gh}}[b, c, \hat{g}]} \prod_i \int d^2\sigma_i \sqrt{\hat{g}(\sigma_i)} \mathcal{V}_i(k_i, \sigma_i) , \quad (2.16)$$

where the string coupling g_s is naturally identified with the expectation value of the dilaton :

$$g_s \equiv e^{\langle \phi \rangle} . \quad (2.17)$$

Note that the location σ_i of the vertex operators $\mathcal{V}_i(k, \sigma)$, which describe the external on-shell states of the process, is integrated over the worldsheet in order to yield a diffeomorphism invariant amplitude.

The string loop expansion is then a topological expansion over Riemann surfaces Σ of distinct topology, characterized by the number of handles, boundaries and crosscaps, all of which contribute to the Euler characteristic χ . The dilaton is singled out in this way, being the field whose v.e.v. controls the strength of string interactions.

The Weyl Anomaly

Let us next return to the problem of examining the decoupling of the scale factor of the metric $\omega(\sigma)$ at the quantum level. Under a Weyl rescaling of the metric the path integral acquires a non-trivial Liouville factor :

$$Z[e^{2\Omega}g] = \exp \left\{ \frac{c}{24\pi} \int d^2\sigma \sqrt{g} (g^{ab} \partial_a \Omega \partial_b \Omega + R^{(2)} \Omega) \right\} Z[g] . \quad (2.18)$$

The anomaly coefficient c is identified with the central charge of the conformal field theory (CFT) on a flat worldsheet, which is the residual symmetry of the gauge-fixed action :

$$T(z)T(w) = \frac{c/4}{(z-w)^4} + \frac{2}{(z-w)^2} T(w) + \frac{1}{z-w} \partial T(w) + \dots , \quad (2.19)$$

where $T(z)$ is the holomorphic energy momentum tensor of the CFT in complex coordinates mapping the cylinder to the complex plane. The central charge is related to the conformal anomaly $\langle T^a_a \rangle = -\frac{c}{12}R^{(2)}$ arising from the coupling of the 2d CFT to a curved metric. In terms of the Laurent modes L_n of the Laurent expansion, $T(z) = \sum_{n \in \mathbb{Z}} \frac{L_n}{z^{n+2}}$, the above OPE takes the form of the (infinite-dimensional) Virasoro algebra :

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{n+m} . \quad (2.20)$$

Notice that $\{L_{-1}, L_0, L_{+1}\}$ close into an $SL(2, \mathbb{R})$ Lie subalgebra. They generate translations L_{-1} , dilatations (rescalings) L_0 and special conformal transformations L_1 . Together with the right-moving subalgebra they form $SL(2, \mathbb{C})$, which contains the globally defined generators on the Riemann sphere. They generate the Möbius (or rational) transformations :

$$z \rightarrow \frac{az + b}{cz + d} , \quad (2.21)$$

with $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$. Actually, this group is $PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/\mathbb{Z}_2$, because an overall change in sign leaves the transformation unaffected.

Furthermore, one may show that the absence of gravitational anomalies (anomalies under diffeomorphism invariance) requires the left-moving and right-moving central charges to match $c = \bar{c}$. The absence of conformal (Weyl) anomaly then requires the total central charge c receiving contribution from all degrees of freedom, including the ghosts, to cancel :

$$c_{\text{matter}} + c_{\text{ghost}} = 0 . \quad (2.22)$$

When this condition is applied to a CFT of D free bosons X^μ , it determines the (maximal) critical dimension, $D = 26$, of the bosonic string. The contribution to the central charge of a free boson is $c_X = +1$, whereas a bc -ghost system of conformal weight $(h, 1 - h)$ contributes $c_{bc} = -3(2h - 1)^2 + 1$. These may be easily derived by going to the flat metric in conformal coordinates :

$$z = e^{-i\sigma_1 + \sigma_2} , \quad (2.23)$$

that map the cylinder to the complex plane in the ‘radial frame’⁴. There, the operator product expansion (OPE) of the energy-momentum tensor with itself, $T(z)T(w)$, can be calculated in terms of free-field OPEs for the CFT of a free boson X and similarly for the bc -ghosts. From the ghost action (2.15) one reads the conformal weights $h_b = 2$ and $h_c = 1 - h_b = -1$. The anomaly cancellation then yields $c = D - 26 = 0$, from which the critical dimension of the bosonic string derives. Note that the maximal critical dimension of the bosonic string is fixed solely by the reparametrization properties of the worldsheet, which determine the kernel P and, hence, the conformal structure of the ghost action (2.15).

4. The time direction on the cylinder is identified with the modulus $|z|$ in the radial frame. Similarly, the spatial coordinate σ is given by the polar angle $\arg(z)$.

Mode Expansions

It will be useful to give the mode expansion of the free scalar field X . It is obtained by solving the Laplace equation subject to the relevant boundary conditions. The solution is decomposed into an holomorphic (left-moving) and an anti-holomorphic (right-moving) contribution, $X(z, \bar{z}) = X_L(z) + X_R(\bar{z})$. For the closed, oriented string, which is the case of interest in this manuscript, one imposes periodicity in σ . In fact, periodicity is imposed up to a possible winding term, when the scalar X is compact, i.e. taking values in S^1 :

$$X(\sigma + 2\pi, t) = X(\sigma, t) + 2\pi n R, \quad (2.24)$$

where $n \in \mathbb{Z}$ is the winding quantum number, counting the number of times the string encircles the compact dimension of radius R .

In the radial frame, the Fourier expansion turns into an expansion in Laurent modes, centered at $z = 0$. The (left-moving) free boson propagator is not a well-defined conformal object since its short distance behaviour is logarithmic, rather than a power-law singularity :

$$X_L(z)X_L(w) = -\frac{\alpha'}{2} \log(z - w) + \dots \quad (2.25)$$

The holomorphic current, however, $i\partial X(z)$ is a well-defined $(1, 0)$ conformal tensor and has a well-defined mode expansion. By integrating the latter, one obtains :

$$X_L(z) = \frac{x_0}{2} - i\frac{\alpha'}{2} P_L \log z + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \frac{\alpha_n}{z^n}, \quad (2.26)$$

$$X_R(\bar{z}) = \frac{x_0}{2} - i\frac{\alpha'}{2} P_R \log z + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \frac{\bar{\alpha}_n}{\bar{z}^n}. \quad (2.27)$$

The average position is x_0 and, in general, the left- and right- moving momenta $P_{L,R}$ need not be equal, unless the dimension parametrized by X is non-compact. The total momentum $P = \frac{1}{2}(P_L + P_R)$ satisfies the usual quantization :

$$\frac{1}{2}(P_L + P_R) = \frac{m}{R}, \quad (2.28)$$

where $m \in \mathbb{Z}$ is the momentum quantum number. The periodicity condition for $X = X_L + X_R$ gives :

$$X(e^{2\pi i} z, e^{-2\pi i} \bar{z}) = X(z, \bar{z}) + 2\pi \frac{\alpha'}{2} (P_L - P_R), \quad (2.29)$$

which, when compared with (2.24) yields :

$$\frac{1}{2}(P_L - P_R) = \frac{nR}{\alpha'}. \quad (2.30)$$

One then obtains the quantization of the momentum zero modes P_L, P_R in terms of the momentum and winding numbers :

$$P_{L,R} = \frac{m}{R} \pm \frac{nR}{\alpha'} . \quad (2.31)$$

In contrast to the point particle result, the string has tension $\sim (\alpha')^{-1}$ which tends to force the string to shrink in size (counterbalanced, in turn, by quantum fluctuations). The second term in (2.31) simply expresses the fact that it costs energy for the string to wrap around the compact dimension. In the limit of large radius $R/\sqrt{\alpha'} \gg 1$ the Kaluza-Klein modes $\sim 1/R$ are so densely packed that their mass spectrum becomes continuous, whereas the winding modes $\sim R$ become supermassive and essentially decouple, $n = 0$. One then recovers $P_L = P_R$, with the momentum now describing the centre-of-mass momentum of the string propagating in a non-compact dimension. Notice that the zero modes $P_{L,R}$ and, hence, the whole string spectrum is invariant under the inversion $R \rightarrow \alpha'/R$, together with the simultaneous exchange of momenta and windings $m \leftrightarrow n$. This is the first encounter with T -duality, the simplest perturbative duality which actually holds at all levels in perturbation theory.

In terms of the currents $J(z) = i\partial X(z)$, $\bar{J}(\bar{z}) = i\bar{\partial} X(\bar{z})$:

$$J(z) = \sum_{r \in \mathbb{Z}} \frac{J_r}{z^{r+1}} , \quad \bar{J}(\bar{z}) = \sum_{r \in \mathbb{Z}} \frac{\bar{J}_r}{\bar{z}^{r+1}} , \quad (2.32)$$

the zero modes are expressed as :

$$J_0 = \frac{\sqrt{\alpha'}}{\sqrt{2}} \left(\frac{m}{R} + \frac{nR}{\alpha'} \right) , \quad \bar{J}_0 = \frac{\sqrt{\alpha'}}{\sqrt{2}} \left(\frac{m}{R} - \frac{nR}{\alpha'} \right) . \quad (2.33)$$

This is the normalization that will appear later on in lattice sums, when we calculate the contribution to the torus partition function of compact scalars.

Equations of Motion for the Background Fields

The Weyl anomaly in T^a_a considered above was calculated under the implicit assumption that the 2d curved worldsheet metric is coupled to a free, exact CFT. Moreover, it was seen that the Gauss-Bonnet term in (2.3) containing the dilaton breaks Weyl invariance, even at the naive classical level, unless the dilaton field is constant. For more general backgrounds one has to explicitly calculate the anomaly by using weak field perturbation theory. The trace of the energy-momentum tensor for the sigma model (2.3) becomes renormalized⁵ as :

$$T^a_a = -\frac{1}{2}(\beta^G_{\mu\nu} g^{ab} + i\beta^B_{\mu\nu} E^{ab}) \partial_a X^\mu \partial_b X^\nu - \beta^\Phi R^{(2)} , \quad (2.34)$$

5. See, for example, [1] and references therein.

where β^G, β^B and β^Φ are the beta functions containing the information about the dependence of amplitudes on the scale factor. They are obtained as a perturbative expansion in α' :

$$\begin{aligned}\beta_{\mu\nu}^G &= R_{\mu\nu} - \frac{1}{4}H_{\mu\rho\sigma}H_{\nu}{}^{\rho\sigma} + 2\nabla_\mu\nabla_\nu\phi + \mathcal{O}(\alpha'), \\ \beta_{\mu\nu}^B &= -\frac{1}{2}\nabla^\rho(e^{-2\phi}H_{\mu\nu\rho}) + \mathcal{O}(\alpha'), \\ \beta^\Phi &= -\frac{\delta c}{12} + \frac{\alpha'}{8}\left(4(\nabla\phi)^2 - 4\nabla^2\phi - R^{(D)} + \frac{1}{12}H^2\right) + \mathcal{O}(\alpha')^2, \end{aligned} \quad (2.35)$$

where $H = dB$ is the 3-form field strength associated to the antisymmetric tensor :

More correctly, (2.35) should be considered a derivative expansion, since the dimensionless parameter controlling the perturbative expansion is $R/\sqrt{\alpha'}$, with R being the characteristic radius of curvature. The constant term in the beta function for the dilaton is given by the central charge deficit δc , which induces curvature in the form of a cosmological constant term.

Thus, the absence of Weyl anomalies for general $G_{\mu\nu}, B_{\mu\nu}, \phi$ backgrounds requires the vanishing of the sigma model beta functions $\beta_{\mu\nu}^G = \beta_{\mu\nu}^B = \beta^\Phi = 0$. In fact, it is straightforward to show that if $\beta_{\mu\nu}^G = \beta_{\mu\nu}^B = 0$, the sigma model describes a CFT with central charge $c = 12\beta^\Phi = \text{constant}$. This is consistent because the Bianchi identities give :

$$\begin{aligned}\partial_\nu\beta^\Phi &\sim \nabla_\nu\left(4(\nabla\phi)^2 - 4\nabla^2\phi - R + \frac{1}{12}H^2\right) \\ &= -2\nabla^\mu\left(R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\phi - \frac{1}{4}H_{\mu\rho\sigma}H_{\nu}{}^{\rho\sigma}\right) = 0, \end{aligned} \quad (2.36)$$

and, hence, β^Φ is indeed a c -number.

The above conditions for the absence of Weyl anomaly provide a set of equations of motion for the background fields. They can be reproduced from the variation of the string frame action :

$$S = \frac{1}{2\kappa^2} \int d^Dx \sqrt{-G} e^{-2\phi} \left(R + 4(\nabla\phi)^2 - \frac{1}{12}H^2 + \frac{2\delta c}{3}(\alpha')^{-1} \right) + \mathcal{O}(\alpha'), \quad (2.37)$$

where κ is the D -dimensional Newton's constant. The full action for the background fields receives α' -corrections from higher loops in the sigma model, as well as g_s -corrections from higher genera in which the dilaton appears as $\exp(\chi\phi)$.

Let us briefly motivate the presence of the central charge deficit term $\frac{2}{3}\delta c$ in the effective action (2.37). As mentioned above, β^Φ is a constant proportional to the total central charge of the CFT on the flat worldsheet and the dilaton equation of motion $\beta^\Phi = 0$ implies $c_{\text{matter}} = 26$. However, the constant term in β^Φ is only the flat contribution to the central charge, with the curved contribution arising from the operator valued $\mathcal{O}(\alpha')$ contribution. Let us be a bit more specific. Consider the target space to be the direct product $\mathcal{M}^D \times K$, where \mathcal{M}^D is a D -dimensional⁶ space of Lorentzian signature and K is an internal (compactified)

6. The concept of dimensionality of spacetime is only well-defined in the low curvature limit, as we will discuss in later chapters.

space of dimension N . The naive concept of dimension would imply $D + N = 26$ and this is indeed true for the contribution to the central charges of the spacetime and internal CFTs in the flat limit. We will take the spacetime and internal space to be parametrized by coordinates X^μ and X^I , respectively. Independently, the central charges of each of the two CFTs can be expressed as the anomalies of the two sigma models :

$$\begin{aligned} c_{\mathcal{M}} &= 12 \beta^\Phi|_{\mathcal{M}} = D + \frac{3\alpha'}{2} \left(4(\nabla\phi)^2 - 4\nabla^2\phi - R^{(D)} + \frac{1}{12}H^2 \right) (X^\mu) + \mathcal{O}(\alpha')^2, \\ c_K &= 12 \beta^\Phi|_K = N + \frac{3\alpha'}{2} \left(4(\nabla\phi)^2 - 4\nabla^2\phi - R^{(D)} + \frac{1}{12}H^2 \right) (X^I) + \mathcal{O}(\alpha')^2, \end{aligned} \quad (2.38)$$

where we explicitly include the dependence on the spacetime and internal coordinates X^μ , X^I , explicitly. Since c_K is the central charge of the internal CFT, we can write the curved contribution to the internal central charge as $\delta c_K = c_K - N$. The vanishing of the total central charge implies that the curved contributions cancel each other $\delta c_K = -\delta c_{\mathcal{M}}$. Then, the total beta function (including the ghost contribution) becomes :

$$\begin{aligned} \beta^\Phi &= \beta^\Phi|_{\mathcal{M}} + \beta^\Phi|_K - \frac{c_{\text{ghost}}}{12} \\ &= \frac{\delta c_K}{12} + \frac{\alpha'}{8} \left(4(\nabla\phi)^2 - 4\nabla^2\phi - R^{(D)} + \frac{1}{12}H^2 \right) (X^\mu) + \mathcal{O}(\alpha')^2. \end{aligned} \quad (2.39)$$

Thus, the central charge deficit arises by coupling the curved ‘spacetime’ (geometrical) sigma model to an internal, curved CFT. This mechanism can be used, for example, to generate tree-level cosmological solutions in string theory, with the role of the internal CFT being played by a gauged WZW model, as in [2].

BRST Quantization and the No-Ghost Theorem

Before ending this section, we will very briefly mention an important ingredient that plays a major role in the construction of consistent string amplitudes. This is the BRST quantization, which leads to a formulation of the No-Ghost theorem. A more general application of the Φ - Π procedure of gauge fixing would be to implement the gauge-fixing condition in terms of auxiliary fields (Lagrange multipliers) as follows. Imagine gauge transformations closing an algebra $[\delta_\alpha, \delta_\beta] = f_{\alpha\beta\gamma}\delta_\gamma$ and a set of gauge-fixing condition in the form $F^A(g) = 0$. The gauge fixing can be imposed through inserting a Dirac delta function $\delta(g - \hat{g})$, which has a simple exponential representation in terms of new dummy integration variables B_A , now acting as Lagrange multipliers :

$$\int \mathcal{D}X^\mu \mathcal{D}g_{ab} \mathcal{D}B_A \mathcal{D}b_A \mathcal{D}c^\alpha \exp(-S[X, g] - S_{\text{ghost}}[g, b, c] - S_{\text{gauge fix}}[g, B, F]). \quad (2.40)$$

The first two terms in the exponent are the gauge invariant sigma model action and the Φ - Π ghost action :

$$S_{\text{ghost}} = \int b_A (\delta_\alpha F^A(g)) c^\alpha. \quad (2.41)$$

The third term is the gauge-fixing action :

$$S_{\text{gauge fix}} = -i \int B_A F^A(g) . \quad (2.42)$$

The full action now enjoys a symmetry under the following BRST transformation [3], due to Becchi, Rouet and Stora :

$$\begin{aligned} \delta_{\text{BRST}}(g_{ab}) &= -i\epsilon c^\alpha \delta_\alpha g_{ab} , \\ \delta_{\text{BRST}}(B_A) &= 0 , \\ \delta_{\text{BRST}}(b_A) &= \epsilon B_A , \\ \delta_{\text{BRST}}(c^\alpha) &= \frac{1}{2} i\epsilon f_{\alpha\beta\gamma} c^\beta c^\gamma . \end{aligned} \quad (2.43)$$

BRST symmetry can be considered a remnant of the original symmetry that is preserved after gauge fixing. Its importance arises from the observation that physical states must be BRST invariant. This can be seen demanding that physical amplitudes are unaffected by a small variation in the gauge-fixing condition δF :

$$0 = \langle \Phi | \delta(S_{\text{ghost}} + S_{\text{gauge fix}}) | \Phi' \rangle = \langle \Phi | \int \{Q_B, b_A \delta F^A\} | \Phi' \rangle , \quad (2.44)$$

where the BRST charge Q_B is the generator of the BRST transformations. The requirement that this hold for arbitrary variation δF leads to the physical state condition :

$$Q_B |\Phi\rangle = 0 . \quad (2.45)$$

Next, the requirement that the BRST charge itself remains conserved even after the variation δF or, equivalently, the requirement that Q_B itself commutes with the variation of the Hamiltonian, leads to its nilpotency, $Q_B^2 = 0$. This has important consequences for the structure of physical states. Evidently, any state that is BRST-exact, $Q_B |\Phi\rangle$, is also automatically BRST-closed, i.e. annihilated by the BRST charge, and so it is physical. However, such states are null, in the sense that they are orthogonal to all other physical states (including themselves!) and, hence, decouple from physical amplitudes.

Since physical states differing by null states have equal amplitudes with any other physical state, one is lead to the notion of equivalence classes. The Hilbert space of physical states, therefore, will be defined as the cohomology of Q_B :

$$\mathcal{H}_{\text{physical}} = \frac{\mathcal{H}_{\text{closed}}}{\mathcal{H}_{\text{exact}}} . \quad (2.46)$$

The No-Ghost theorem (see, for example, [4]) is precisely the statement that the ‘transverse’ Hilbert space, which does not contain any longitudinal X^0, X^1, b, c excitations (and, hence, has a positive-definite inner product) is isomorphic to the BRST cohomology, $\mathcal{H}_{\text{physical}} = \mathcal{H}_{\text{BRST}}$. The BRST construction reflects another important fact. Namely, sufficient gauge symmetry must be present in the first place, if negative norm states are to decouple from

string amplitudes. This is also particularly clear in light-cone quantization, where the gauge symmetry (reparametrization invariance) is used to go to the lightcone gauge and, hence, fixing the longitudinal modes.

We conclude this brief introduction to bosonic string theory by giving the explicit form for the BRST current :

$$j_B(z) = cT_{\text{matter}}(z) + \frac{1}{2} : cT_{\text{ghost}} : (z) + \frac{3}{2} \partial^2 c(z). \quad (2.47)$$

The BRST charge Q is then the zero mode :

$$Q_B = \oint \frac{dz}{2\pi i} j_B(z) - \oint \frac{d\bar{z}}{2\pi i} \bar{j}_B(\bar{z}) , \quad (2.48)$$

with the anti-holomorphic BRST current being defined analogously. The conservation of the BRST charge is highly sensitive to anomalies in the gauge symmetries. As a result, a conformal anomaly in the theory will arise as a failure of the nilpotency of the BRST charge. In this case the $j_B j_B$ -OPE will contain a non vanishing simple-pole term :

$$j_B(z)j_B(w) = \dots - \frac{c_{\text{matter}} - 26}{12} \frac{1}{z-w} c\partial^3 c(w) + \dots , \quad (2.49)$$

which contributes a non-vanishing value to the anticommutator $\{Q_B, Q_B\}$. Similarly, the Tj_B OPE contains a fourth order pole and, thus, $j_B(z)$ is not a primary field unless $c_{\text{matter}} = 26$.

2.2 Bosonic String Spectrum

In section 2.1 we gave a speedy overview of some elements of bosonic string theory. Let us have a cursory look at its massless spectrum. The vertex operators of the closed bosonic string take the general form :

$$g_s \int dz d\bar{z} \mathcal{V}(z, \bar{z}) , \quad (2.50)$$

and can, subsequently, be used to perturb the sigma model. In this way, the various excitations of the string act as perturbations whose coherent contribution ‘builds up’ the background. In this sense, string theory is in principle expected to determine dynamically its own background, though in practice one is usually forced to treat small perturbations around some fixed setup. The unintegrated vertex operators $\mathcal{V}(z, \bar{z})$, are fields of definite conformal weight $(1, 1)$ so that the integrated insertions to the path integral are conformally invariant. Consider the theory in $D = 26$ flat dimensions. The lowest mass states are then :

$$: e^{ik \cdot X} : (0, 0) | 0 \rangle . \quad (2.51)$$

Their conformal weight is picked from the double pole in the $T(z)\mathcal{V}(w, \bar{w})$ OPE :

$$T(z)\mathcal{V}(w, \bar{w}) = \dots + \frac{h}{(z-w)^2} \partial \mathcal{V}(w, \bar{w}) + \frac{1}{z-w} \mathcal{V}(w, \bar{w}) + \dots , \quad (2.52)$$

where the free boson energy-momentum tensor is :

$$T(z) = -\frac{1}{\alpha'} : \partial X \cdot \partial X : (z) . \quad (2.53)$$

A straightforward calculation yields :

$$(h, \bar{h}) = \left(\frac{\alpha' k^2}{4}, \frac{\alpha' k^2}{4} \right) . \quad (2.54)$$

Throughout this manuscript we will adopt a standard ‘CFT convention’. Unless otherwise stated, in CFT calculations involving vertex operators on the sphere (genus 0), we will set $\alpha' = 2$ so that the chiral vertex operator $e^{iqX(z)}$, carrying a definite $U(1)$ charge q under $J(z) = i\partial X(z)$, will have conformal weight $(q^2/2, 0)$.

With this convention, the mass of $e^{ik \cdot X}$ is found to be :

$$m^2 \equiv m_L^2 + m_R^2 = -k^2 = -2 . \quad (2.55)$$

The separation into (equal) left- and right- moving masses $m_L^2 = m_R^2$ will be seen to arise naturally in the torus partition function. There, the value $m_L^2 = m_R^2 = -1$ will be precisely the mass level of the ground state, which is entirely determined by the (super-)reparametrization properties of the worldsheet theory, as they are encoded in the (super-)ghost structure.

The mass square in (2.55) is negative and the ground state of the bosonic string is, hence, tachyonic. This is a considerable embarrassment for the theory, because its presence signals an IR instability, with the tachyon rolling down its potential away from the unstable point. We will come back to this point later on, when we discuss the Hagedorn problem.

We next move to the first excited states, which can be constructed from linear combinations of the vertex operators :

$$\partial X^\mu \bar{\partial} X^\nu e^{ik \cdot X}(0, 0) |0\rangle . \quad (2.56)$$

In what follows we will suppress the normal ordering symbol. The conformal weight of these operators⁷ is $(1 + \frac{k^2}{2}, 1 + \frac{k^2}{2})$ and, hence, leads to massless states. Representation-wise, these states are tensor products of two vector representations of the $SO(1, D-1)$ Lorentz group. Decomposition into irreducible representations is straightforward and leads to a traceless, symmetric tensor of spin 2 (the graviton $G_{\mu\nu}$), an antisymmetric tensor (the Kalb-Ramond field $B_{\mu\nu}$), and the trace (the scalar dilaton Φ). This is expected from the beginning, since the massless excitations can be used as sources in the sigma model, with their fluctuations determining the background.

Higher mass levels can be constructed in a similar fashion.

7. There are certain ‘on-shell’ conditions that need to be imposed on the coefficients of the linear combination for such operators to define primary conformal fields. They are obtained by imposing the absence of higher order poles in the $T(z)\mathcal{V}(w, \bar{w})$ OPE.

2.3 Superstring Theory

Motivation and Worldsheet Fermions

Bosonic string theory suffers from several deficiencies, one of which is the presence of the tachyonic excitation. Within the framework of a string field theory, one should be able to trace the roll-down of the tachyon to a new stable vacuum, provided the latter exists. However, to date, it remains unclear whether bosonic string theory can exist as a theory in its own right. A far more serious problem is the total absence of spacetime fermions in the bosonic string spectrum. Clearly, states built out of oscillators transform under tensor (rather than spinor) representations of the Poincaré group. We present here the conventional approach to remedy this situation, in terms of the Ramond-Neveu-Schwarz (RNS) formulation of the superstring.

In the RNS formulation, spacetime fermions arise from a degenerate ground state, $|0\rangle_\alpha$, carrying a spinorial index of the spacetime little group. Assuming the presence of such an operator $\psi^\mu(z)$, its action on the vacuum state should produce the Dirac gamma matrix representation :

$$\psi^\mu(z)|0\rangle_\alpha = \frac{1}{\sqrt{2}} \gamma^\mu_{\alpha\beta} |0\rangle_\beta \quad , \quad z \rightarrow 0 \quad . \quad (2.57)$$

The degeneracy can only arise as long as the field $\psi^\mu(z)$ has zero modes. It is also necessary for $\psi^\mu(z)$ to carry a spacetime vector index, in order to reproduce the gamma matrix structure. Then, the antisymmetrized action with a second operator $\psi^\nu(z)$ must reproduce the Lorentz algebra :

$$\psi^{[\mu}\psi^{\nu]}(z)|0\rangle_\alpha = \frac{1}{4} [\gamma^\mu, \gamma^\nu]_{\alpha\beta} |0\rangle_\beta = \frac{1}{2} \sigma^{\mu\nu}_{\alpha\beta} |0\rangle_\beta \quad , \quad z \rightarrow 0 \quad . \quad (2.58)$$

We assume the singular part has been subtracted via some normal ordering scheme. The point now is that the antisymmetric combination of zero modes in the l.h.s. vanishes identically unless the fields $\psi^\mu(z)$ are fermionic and, hence, classically anticommuting. Therefore, one is lead naturally to the introduction of fermion fields $\psi^\mu(z)$ (necessarily carrying a Lorentz index) on the worldsheet. The idea is then that if worldsheet fermions can satisfy periodic boundary conditions, $\psi^\mu(\sigma + 2\pi, t) = +\psi^\mu(\sigma, t)$, their expansion will contain zero modes which can then give rise to the vacuum degeneracy structure required for spacetime fermions.

Therefore, the first ingredient we should include in a generalization of the bosonic sigma model action (2.3) is the addition of worldsheet fermions ψ^μ . One could also add ‘internal’ worldsheet fermions ψ^I , transforming according to some isometry of the internal space, but their zero-mode oscillators will give rise to spinors with respect to the internal group, rather than spacetime fermions. However, one immediately has to face an additional complication. The introduction of the new objects ψ^μ carrying a Lorentz index also leads to new states with negative norm. We, therefore, need to introduce a new gauge symmetry that will permit us to gauge away and decouple the new unphysical states, with a similar BRST cohomology construction as the one used for the bosonic string.

It turns out that the relevant new symmetry is super-reparametrization invariance, arising from the consistent Φ - Π gauge fixing of 2d worldsheet supergravity. The starting point is the $N = 1$ supergravity action in 2d, coupled to D superfields $\{X^\mu, \psi^\mu\}$:

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} \left[g^{ab} \partial_a X^\mu \partial_b X_\mu + \bar{\psi}_\mu \gamma^a \partial_a \psi^\mu - \bar{\chi}_a \gamma^b \gamma^a \psi^\mu \left(\partial_b X_\mu - \frac{1}{4} \bar{\chi}_b \psi_\mu \right) \right] . \quad (2.59)$$

For simplicity we assume a flat target space background. The auxiliary field of the off-shell scalar multiplet are eliminated by their equations of motion. The would-be kinetic term $\bar{\chi}_a \gamma^{abc} \nabla_b \chi_c$ of the worldsheet gravitino χ_a vanishes identically in two dimensions and only the ‘gamma-traceless’ (helicity $\pm \frac{3}{2}$) component of the gravitino, $\gamma^b \gamma^a \chi_b$, appears in the action. The spin connection does not enter explicitly in the derivative of the Majorana fermions ψ^μ , because of the Majorana spin-flip property.

This action is invariant under local supersymmetry, Weyl and super-Weyl transformations, 2d Lorentz transformations and reparametrizations. It is possible to use super-reparametrization and local Lorentz invariance to go to the superconformal gauge :

$$g_{ab} = e^{2\omega} \delta_{ab} \quad , \quad \chi_a = \gamma_a \zeta \quad , \quad (2.60)$$

where the conformal factor ω and the gamma-trace $\zeta \equiv \frac{1}{2} \gamma^a \chi_a$ (helicity $\pm \frac{1}{2}$) component of the gravitino completely decouple from the classical action. At the quantum level, they will give rise to a superconformal anomaly, similarly to the case of the bosonic string.

We will not present this process in detail but merely quote the most important results. After gauge fixing and some rescalings, the action of the free boson plus fermion takes the form :

$$S = \frac{1}{2\pi\alpha'} \int d^2z \partial X^\mu \bar{\partial} X_\mu + \frac{1}{4\pi} \int d^2z \left(\psi^\mu \bar{\partial} \psi_\mu + \bar{\psi}^\mu \partial \psi_\mu \right) . \quad (2.61)$$

It is easy to see that the residual symmetry is invariance under superconformal transformations (generated by T_F , as shown below). The two-point OPE of the worldsheet fermions is :

$$\psi^\mu(z) \psi^\nu(w) = \frac{\eta^{\mu\nu}}{z-w} \quad , \quad (2.62)$$

and the energy-momentum tensor is :

$$T(z) = -\frac{1}{\alpha'} \partial X^\mu \partial X_\mu - \frac{1}{2} \psi^\mu \partial \psi_\mu . \quad (2.63)$$

From these one can verify that the free fermion $\psi(z)$ has conformal weight $(\frac{1}{2}, 0)$, as expected from the structure of their kinetic term in the action. The superpartner of $T(z)$ is a tensor of weight $(\frac{3}{2}, 0)$ generating superconformal transformations. It is the worldsheet supercurrent arising from the Nöther procedure :

$$T_F(z) = i \sqrt{\frac{2}{\alpha'}} \psi^\mu \partial X_\mu(z) . \quad (2.64)$$

Taken together, the energy momentum tensor and the supercurrent form a 2d ‘gravity multiplet’ and their OPEs close into the $N = 1$ superconformal algebra :

$$\begin{aligned} T(z)T_F(w) &= \frac{3/2}{(z-w)^2} T_F(w) + \frac{1}{z-w} \partial T_F(w) , \\ T_F(z)T_F(w) &= \frac{\hat{c}}{(z-w)^3} + \frac{2}{z-w} T(w) , \end{aligned} \quad (2.65)$$

where $\hat{c} = \frac{2}{3}c$ is the CFT generalization of the naive number of free superfields in a superconformal field theory (SCFT).

The worldsheet fermions ψ^μ are 2d Weyl-Majorana spinors and can be taken to be real. This restricts their possible boundary conditions to a real phase :

$$\psi(\sigma + 2\pi, t) = e^{2\pi i\nu} \psi(\sigma, t) , \quad (2.66)$$

with $\nu = 0, \frac{1}{2}$. Grouping fermions together in pairs to make complex fermions would permit more general boundary conditions with $\nu \in [0, 1)$ being a real parameter, but this will be considered later. In fact, it is conventional to define :

$$a = 1 - 2\nu . \quad (2.67)$$

This will become associated to the helicity charge, or the R -symmetry charge of the gravitini. After a conformal transformation mapping the cylinder to the complex plane, the Fourier expansions subject to these conditions become the Laurent expansions :

$$\psi(z) = \sum_{r \in \mathbb{Z} + \nu} \frac{\psi_r}{z^{r+1/2}} . \quad (2.68)$$

The periodic boundary conditions $a = 1$ define the so-called Ramond (R) sector, whereas the anti-periodic boundary conditions $a = 0$ define the Neveu-Schwarz (NS) sector. The R-sector fermion operators ($\nu = 0$) have a branch cut, as they encircle the origin, $z \rightarrow e^{2\pi i} z$. On the other hand, the summation is offset by $\nu = \frac{1}{2}$ in the NS-sector such that the branch cut cancels. This has consequences for the assignment of boundary conditions in superstring constructions, since only theories with mutually local vertex operators lead to consistent amplitudes.

This implies, in particular, that the supercurrent has the same boundary conditions as the worldsheet fermions ψ^μ . In terms of Laurent modes, the $N = 1$ SCFT algebra (2.65) becomes :

$$\begin{aligned} \{G_r, G_s\} &= 2L_{r+s} + \frac{\hat{c}}{2} \left(r^2 - \frac{1}{4} \right) \delta_{r+s,0} , \\ [L_n, G_s] &= \left(\frac{n}{2} - s \right) G_{n+s} , \end{aligned} \quad (2.69)$$

where the mode expansion of the supercurrent is :

$$T_F(z) = \sum_{r \in \mathbb{Z} + \nu} \frac{G_r}{z^{r+3/2}} . \quad (2.70)$$

Notice that the $N = 1$ SCFT algebra has an (external) \mathbb{Z}_2 -automorphism under which the supercurrent transforms as $T_F \rightarrow -T_F$. This permits the general assignment of $\nu = 0, \frac{1}{2}$ boundary conditions, as in (2.70), without the need for the explicit realization in terms of the free worldsheet fermions.

The Ground States of NS and R Sectors

In the R-sector the supercurrent has zero-modes and the $N = 1$ algebra (2.69) implies :

$$\{G_0, G_0\} = 2L_0 - \frac{\hat{c}}{8} \geq 0 . \quad (2.71)$$

The last inequality follows from the requirement of having a unitary theory. This implies the following bound for the conformal weight of the states in the R-sector :

$$h_R \geq \frac{\hat{c}}{16} . \quad (2.72)$$

The R -ground state saturates this bound. In particular, a real (free) worldsheet fermion $\psi(z)$ has weight $(\frac{1}{16}, 0)$ in the R-sector, as one can verify from the zero-point energy. Let us repeat the argument here. The contribution to the vacuum takes the form :

$$E_R = \frac{1}{2} \sum_{n=0}^{\infty} \omega_n^B - \frac{1}{2} \sum_{n=0}^{\infty} \omega_n^F , \quad (2.73)$$

where $\omega_n^{B,F}$ are the frequencies of bosonic and fermionic oscillators, respectively. In the R-sector the bosonic and fermionic oscillators are integrally moded so that $E_R = 0$. However, there is a Casimir energy contribution arising from the conformal anomaly and the non-tensor properties of $T(z)$. This is proportional to the central charge of the system $E = L_0 - \frac{c}{24}$. Taking this offset into account we find the conformal weight of the R-ground state $h_R = \hat{c}/16$.

In the NS sector, the fermions are half-integrally moded and anti-periodic, and their contribution does not cancel against the bosonic. The Casimir energy in this case is :

$$E_{NS} = \frac{D}{2} \sum_{n=0}^{\infty} n - \frac{D}{2} \sum_{n=0}^{\infty} \left(n + \frac{1}{2} \right) = \frac{D}{2} \zeta(-1) - \left(-\frac{D}{4} \zeta(-1) \right) = -\frac{3D}{48} , \quad (2.74)$$

where we have used zeta-function regularization. This is justified because it manifestly preserves the gauge symmetries⁸ of the theory. The offset from the conformal anomaly, $+c/24 = 3D/48$, completely cancels this contribution so that the matter ground state in the NS-sector starts at weight $h_{NS} = 0$.

8. This is essentially equivalent to the statement that it is possible to choose a regulator which manifestly respects the gauge symmetry, in order to regularize the formally infinite expressions. Then, a suitable counterterm could be used to remove the divergent parts. The finite part cannot be similarly removed because the would-be counterterm would explicitly break the gauge symmetry.

Ghosts and Superghosts

The reparametrization ghosts b, c also have ‘superpartners’, which are the commuting super-ghosts β, γ . They contribute a similar term $\int \beta \bar{\partial} \gamma$ in the action. They arise from the Φ - Π procedure by gauge-fixing the super-reparametrization invariance. They contribute to the ghost energy momentum tensor and supercurrent as :

$$T_{\text{ghost}}(z) = -(\partial b)c - 2b\partial c - \frac{1}{2}(\partial\beta)\gamma - \frac{3}{2}\beta\partial\gamma , \quad (2.75)$$

$$T_{F, \text{ghost}}(z) = (\partial\beta)c + \frac{3}{2}\beta\partial c - 2b\gamma . \quad (2.76)$$

From the particular expression of the supercurrent (or, directly, by taking the OPE with $T(z)$) we instantly read the conformal dimensions of β, γ , which are $h - \frac{1}{2} = \frac{3}{2}$ and $1 - h + \frac{1}{2} = -\frac{1}{2}$, respectively. Hence, the commuting $\beta\gamma$ -system completes the anticommuting bc -system into a ghost supermultiplet.

The central charge of the super-ghost sector is $c_{\text{s-ghost}} = 3(2h - 2)^2 - 1$. For $h = 2$, we then find the relevant super-ghost contribution to be completely fixed by super-reparametrization invariance to the value $c_{\text{s-ghost}} = +11$. Combining this with the contribution of the bc -ghost sector, we find the critical matter central charge $c_{\text{matter}} = 15$ or $\hat{c}_{\text{matter}} = 10$. One then recovers the critical dimension, $D = 10$, of the superstring.

The ghost and super-ghost systems can be consistently bosonized. This implies that that the OPEs in the bc -theory can be reproduced by the introduction of vertex operators involving a scalar field σ with opposite-sign OPE $\sigma(z)\sigma(w) \sim \log(z - w)$. One may then express the b, c ghosts as :

$$b(z) = e^{-\sigma(z)} , \quad c(z) = e^{\sigma(z)} . \quad (2.77)$$

The OPE between $T_{bc}(z)$ and the ghost number current $j = bc$ has a cubic anomaly term $\sim \mathcal{O}(z^{-3})$, which has to be reproduced by the bosonization. This requires an appropriate shift of the σ -scalar energy momentum tensor, which amounts to a background charge :

$$T_{bc}(z) = \frac{1}{2}(\partial\sigma)^2 + \frac{3}{2}\partial^2\sigma . \quad (2.78)$$

The situation is more involved for the superghosts because their vacuum structure is more complicated. The reason is that arbitrary excitations are possible for the β, γ fields, because of their commuting nature. To this end, one introduces a scalar field $\phi(z)$, with the conventional sign of the $\phi(z)\phi(w) \sim -\log(z - w)$ OPE for scalars, and identifies it with the superghost number current $\partial\phi(z) = \beta\gamma(z)$. This implies that β, γ has charge -1 and $+1$, respectively, with respect to $\partial\phi$. This is, however, not enough to fully reproduce the $\beta\beta$, $\beta\gamma$ and $\gamma\gamma$ OPEs. Instead, it requires the introduction of an additional independent anticommuting η, ξ -system, identical to the bc -CFT, but with $h_\eta = 1$, $h_\xi = 0$. The correct bosonization of the superghosts is then :

$$\beta(z) = e^{-\phi} \partial\xi(z) , \quad \gamma(z) = e^{\phi} \eta(z) . \quad (2.79)$$

The reproduction of all OPEs again requires the modification of the energy-momentum tensor by the addition of a background charge. Furthermore, it is possible to bosonize the $\eta\xi$ -system as well, in terms of another scalar χ as $\eta = e^{-\chi}$, $\xi = e^{\chi}$. The fully bosonized energy-momentum tensor is then written as :

$$T_{\beta\gamma}(z) = -\frac{1}{2}(\partial\phi)^2 - \partial^2\phi + \frac{1}{2}(\partial\chi)^2 + \frac{1}{2}\partial^2\chi . \quad (2.80)$$

The effect of the background charge is similar to a linear dilaton CFT. In particular the vertex operator $e^{q\phi}$ has conformal weight :

$$e^{q\phi} \rightarrow h = -\frac{1}{2}q(q+2) . \quad (2.81)$$

Let us now perform the analogous calculation of zero-point energies for the ghost and super-ghost systems. The bc -ghosts are always integrally moded and are quantized with anticommutation relations. After taking into account the central charge offset, their contribution to the Casimir energy is :

$$h_{R,NS}^{bc} = -2 \times \frac{1}{2} \sum_{n=0}^{\infty} n + \frac{(-26)}{24} = -1 . \quad (2.82)$$

The super-ghosts, on the other hand are quantized with commutation relations, but the result will depend on the sector. In the R-sector, the modes are again integral and one finds :

$$h_R^{\beta\gamma} = +2 \times \frac{1}{2} \sum_{n=0}^{\infty} n + \frac{(+11)}{24} = +\frac{3}{8} . \quad (2.83)$$

In the NS-sector, the modes are half-integral and the analogous calculation yields :

$$h_{NS}^{\beta\gamma} = +2 \times \frac{1}{2} \sum_{n=0}^{\infty} \left(n + \frac{1}{2} \right) + \frac{(+11)}{24} = +\frac{1}{2} . \quad (2.84)$$

Of course, the calculation of divergent sums over zero-point energies is only mentioned here as a cross-check. The conformal approach provides the vacuum energies as a natural consequence.

Finally, let us comment on the structure of the ghost vacuum. The starting point in defining the vacuum is to notice the discrepancy between the $SL(2, \mathbb{C})$ -invariant⁹ vacuum $|0\rangle$, associated to the identity operator $1(z)$, and the true ground state. The latter is defined as the state annihilated by all lowering operators. Notice that $c(z)|0\rangle$ and $b(z)|0\rangle$ have to be regular as $z \rightarrow 0$, as a result of the regularity of the $c(z)1(0)$ and $b(z)1(0)$ OPEs. This implies :

$$\begin{aligned} c_n|0\rangle &= 0 \quad , \quad n \geq 2 \quad , \\ b_n|0\rangle &= 0 \quad , \quad n \geq -1 \quad . \end{aligned} \quad (2.85)$$

9. Indeed, the regularity of the $T(z)1(w)$ OPE indicates that $L_m|0\rangle = 0$ for $m \geq -1$. This, in particular, involves the $SL(2, \mathbb{R})$ subalgebra L_{-1}, L_0, L_1 .

Thus, the $SL(2, \mathbb{C})$ -invariant vacuum is not the true ground state of the bc -theory, since c_1 is a lowering operator which does not annihilate the state, $c_1|0\rangle \neq 0$. Therefore, at least for the bosonic theory, the ground state of the bc -system is identified with $c(0)|0\rangle$.

The situation is more complicated for the $\beta\gamma$ -system because its commuting character allows states to be built out of arbitrary β, γ -excitations. Repeating the previous argument for β, γ , we would again find :

$$\begin{aligned}\gamma_r|0\rangle &= 0 \quad , \quad r \geq \frac{3}{2} \quad , \\ \beta_r|0\rangle &= 0 \quad , \quad r \geq -\frac{1}{2} \quad .\end{aligned}\tag{2.86}$$

Now we encounter the complication. Evidently the $SL(2, \mathbb{C})$ -invariant vacuum $|0\rangle$ is not the ground state of the $\beta\gamma$ theory, since $\gamma_{1/2}$ lowers the weight. In contrast to the bc -ghost case, however, where the algebra was ‘fermionic’, the $\beta\gamma$ ground state is degenerate and cannot be simply obtained by $\gamma(z)|0\rangle$. This is because the spectrum is unbounded¹⁰ since, for example, further application of the lowering operator $\gamma_{1/2}$ does not terminate the spectrum, $\gamma_{1/2}(\gamma_{1/2}|0\rangle) \neq 0$. In order to define the NS-vacuum one needs the analogue of a delta function :

$$|0\rangle_{NS} \equiv \delta(\gamma(0))|0\rangle \quad ,\tag{2.87}$$

such that the application of $\gamma(z)$ and $\beta(z)$ produce a simple zero and a simple pole, respectively :

$$\begin{aligned}\gamma(z)\delta(\gamma(0)) &\sim z(\dots) \quad , \\ \beta(z)\delta(\gamma(0)) &\sim \frac{1}{z}(\dots) \quad ,\end{aligned}\tag{2.88}$$

hence, ensuring the vanishing of $\gamma_{1/2}|0\rangle_{NS}$ and terminating the representation. This is achieved by the coherent state $\delta(\gamma(z)) = e^{-\phi(z)}$, leading to a consistent definition of the NS-vacuum :

$$|0\rangle_{NS} \equiv e^{-\phi(0)}|0\rangle \quad ,\tag{2.89}$$

which satisfies all previous requirements.

In view of this construction, the definition of vertex operators is by no means unique. Rather, there are infinite inequivalent representations of the $\beta\gamma$ -algebra (called ‘pictures’), which are distinguished by their ϕ -ghost charge. In contrast to the mere shifting of the Fermi sea in the fermionic case, here a finite number of field operators cannot fill a state and the coherent state operators $e^{q\phi}$ interpolate between the various Bose sea levels.

10. This does not pose any problems in our case, because we are dealing with a free theory. In the absence of interactions, the transitions that would destabilize the vacuum are also absent.

Finally, the analogous argument can be carried out for the R-ground state, $|0\rangle_R = \Sigma(0)|0\rangle$. First notice that the R-vacuum is defined by :

$$\begin{aligned}\gamma_n|0\rangle_R &= 0 \quad , \quad n \geq 1 \quad , \\ \beta_n|0\rangle_R &= 0 \quad , \quad n \geq 0 \quad .\end{aligned}\tag{2.90}$$

We have included β_0 to the lowering operators and γ_0 to the raising operators, as is the case with the bosonic ghosts b_0, c_0 . This choice (Siegel gauge) is dictated by consistency of the string amplitudes. These conditions imply :

$$\begin{aligned}\gamma(z)\Sigma(0) &\sim z^{1/2}(\dots) \quad , \\ \beta(z)\Sigma(0) &\sim \frac{1}{z^{1/2}}(\dots) \quad .\end{aligned}\tag{2.91}$$

The R-ground state must then contain a factor with $-\frac{1}{2}$ superghost charge, $\Sigma(z) \sim e^{-\phi/2}$:

$$|0\rangle_R \equiv e^{-\phi/2}S(0)|0\rangle \quad ,\tag{2.92}$$

where $S(z)$ is the spin-field operator, introducing the required branch cut of the worldsheet fermions in the R-sector :

$$\psi^\mu(z)S_\alpha(w) = \frac{\gamma_{\alpha\beta}^\mu}{\sqrt{2}} \frac{S_\beta(w)}{(z-w)^{1/2}} + \dots \quad ,\tag{2.93}$$

where γ^μ are the conventionally normalized gamma matrices in D dimensions. As discussed in the beginning of the section in eq. (2.57), the R-vacuum is degenerate and transforms as a spacetime spinor. Hence, the associated spin-field will carry the relevant $SO(1, D-1)$ (Weyl) spinor index. We can obtain a simple free-field representation of the spin field by bosonizing the worldsheet fermions. First, one defines complex fermions as :

$$\begin{aligned}\Psi^{0,\pm}(z) &\equiv \frac{1}{\sqrt{2}} (\psi^0 \pm \psi^1) \quad , \\ \Psi^{a,\pm}(z) &\equiv \frac{1}{\sqrt{2}} (\psi^{2a} \pm i\psi^{2a+1}) \quad ,\end{aligned}\tag{2.94}$$

with $a = 1, \dots, 4$. These are bosonized in terms of free scalars $H^a(z)$:

$$\Psi^{a,\pm}(z) = e^{\pm iH^a(z)} \quad , \quad i\partial H^a(z) = \Psi^{a,+}\Psi^{a,-}(z) \quad .\tag{2.95}$$

The bosonization of the worldsheet fermions is similar to the bosonization of the bc -system considered above. However, unlike the ghosts, the conformal weight structure of the fermions is such that their bosonization does not introduce background charge. In order to reproduce the branch cut of (2.93), the spin field has to be taken to have $\lambda = \pm\frac{1}{2}$ helicity charges¹¹ :

$$S(z) = e^{i\lambda_a H^a(z)} \quad .\tag{2.96}$$

11. In fact, there are two spin-fields $S(z)$ and $C(z)$, corresponding to the two irreducible Weyl representations with opposite chiralities. They are constructed by keeping an even or odd number of minuses in the helicity charges, respectively.

This is the helicity basis of the spin-field, where the representation is encoded into the weight vector λ_a . It lacks in manifest covariance, but its main advantage is the explicit free-field representation which permits the straightforward calculation of OPEs. Its conformal weight is $(\frac{5}{2}, 0)$, in agreement with the calculation of the zero-point energy due to $D = 10$ worldsheet fermions, $\hat{c}/16 = 5/8$, in the R-sector.

Finally, by adding the bosonic ghost contribution $c(0)|0\rangle$ and using (2.81), it is easy to verify the conformal weights of the NS and R (ghost) ground states :

$$\begin{aligned} |0\rangle_{NS} &= c(0)e^{-\phi}(0)|0\rangle \quad , \quad h_{NS} = -\frac{1}{2} \quad , \\ |0\rangle_R &= c(0)e^{-\phi/2}S(0)|0\rangle \quad , \quad h_R = 0 \quad . \end{aligned} \quad (2.97)$$

Notice the negative $-\frac{1}{2}$ mass contribution in the NS sector. This is familiar tachyon of bosonic string theory. On the other hand, the R-sector starts already at the massless level.

BRST for the Superstring

Similar expressions hold for the right-movers. The bosonic no-ghost theorem generalizes with a similar construction to the superstring case and the modified BRST cohomology gives the physical positive definite spectrum. Picture changing and the consistent covariant quantization of the superstring is discussed in [20], [21]. The BRST charge Q_B is defined as the zero mode of the current :

$$j_B = cT_{\text{matter}} + \gamma T_{F,\text{matter}} + bc\partial c + \frac{3}{4}(\partial c)\beta\gamma + \frac{1}{4}c(\partial\beta)\gamma - \frac{3}{4}c\beta\partial\gamma - b\gamma^2 \quad . \quad (2.98)$$

The consistent definition of the BRST charge requires the BRST current to be integrally moded. This implies that ψ^μ, β and γ must have the same periodicity as the supercurrent T_F . The BRST charge is found to be nilpotent $Q_B^2 = 0$, as in the bosonic case, provided the superconformal anomalies cancel (total $c = 0$).

For a physical state $|\chi\rangle$, one imposes the additional (Siegel gauge) conditions $b_0|\chi\rangle = \beta_0|\chi\rangle = 0$. Their origin is kinematical and they consistently project out copies of the spectrum which would lead to delta functions in physical amplitudes. This leads to the additional conditions to be imposed on physical states :

$$\begin{aligned} L_0|\chi\rangle &= \{Q_B, b_0\}|\chi\rangle = 0 \quad , \\ G_0|\chi\rangle &= [Q_B, \beta_0]|\chi\rangle = 0 \quad . \end{aligned} \quad (2.99)$$

The first is the mass-shell condition, with L_0 being the zero mode of the total (matter plus ghost) energy-momentum tensor. These conditions reproduce the ones arising in covariant quantization, where one treats the super-Virasoro algebra (i.e. the residual symmetry) as a constraint algebra to be imposed on the physical spectrum.

2.4 Superstring Vertex Operators

Vertex Operators for Low-Lying States

Let us briefly describe here the construction of the simplest superstring vertex operators in the NS and R sectors. Depending on the choice of boundary conditions in the left- and right- movers, the theory contains four sectors : NS-NS, R-R, NS-R and R-NS. The first two give rise to spacetime bosons and the last two give rise to spacetime fermions, respectively.

We start with the NS-NS sector, where the (super-)ghost contribution lowers the vacuum energy by $-\frac{1}{2}$. The lowest state is the tachyon. Its mass is determined from the requirement (2.99) that $L_0 = 0$:

$$e^{ik \cdot X} |0\rangle_{NS} = e^{-\phi - \bar{\phi}} e^{ik \cdot X} c \bar{c} (0, 0) |0\rangle \longrightarrow \frac{k^2}{2} + h_{NS} = 0 . \quad (2.100)$$

Hence, $m^2 = -1$. Reinstating back the α' dependence, we have $m^2 = -2/\alpha'$.

In subsequent parts of the manuscript we will be somewhat cavalier about wording and will, hence, refer to the conformal weight of the vertex operator at zero momentum as the ‘mass level’ of the state. With this convention in mind, we will consider the tachyon ground state to be at $h = -\frac{1}{2}$ mass level. This definition of ‘mass level’ is consistent with the conventional one, since they coincide for the massless spectrum.

The next mass level in the NS-NS sector can be obtained by the action of fermion oscillators. For simplicity, we will only write the momentum independent part of the vertex operator.

$$e^{-\phi} \psi^\mu e^{-\bar{\phi}} \bar{\psi}^\nu c \bar{c} (0, 0) |0\rangle \quad , \quad h = 0 . \quad (2.101)$$

This is the massless level. As in the bosonic case, linear combinations of the above states have to be considered and conditions have to be imposed on the polarization vectors so that the vertex operators describe on-shell physical states. The tensor product decomposes again into massless irreducible representations of the spacetime little group. One obtains a symmetric, traceless tensor again corresponds to the massless spin-2 graviton $G_{\mu\nu}$, the antisymmetric tensor field $B_{\mu\nu}$ and the scalar dilaton ϕ . This is the gravitational multiplet.

We have seen that states in the R sector start at the massless level, at least (anti-)chirally. In the R-R sector, we have :

$$e^{-\phi/2} S^\pm e^{-\bar{\phi}/2} \bar{S}^\pm c \bar{c} (0, 0) |0\rangle \quad , \quad h = 0 . \quad (2.102)$$

The \pm in the spin-fields, S^\pm , indicates the choice of chirality. The relative chirality choice leads to distinct theories. Depending on the relative chirality the spinor product decomposes into the sum of forms of either odd or even rank. They will be discussed below, after the introduction of the GSO projection.

Finally consider the R-NS sector (NS-R is similarly treated). The conditions (2.99) imply that physical states have to be level matched, $L_0 = \bar{L}_0 = 0$:

$$e^{-\phi/2} S^\pm e^{-\bar{\phi}} \bar{\psi}^\mu c \bar{c} (0, 0) |0\rangle \quad , \quad h = 0 . \quad (2.103)$$

This decomposes into a helicity $\pm\frac{3}{2}$ state, the gravitino, and a spin $\pm\frac{1}{2}$ -fermion, the dilatino. The chirality of the gravitino is given by that of the spin-field $S^\pm(z)$.

The presence of spacetime supersymmetry is, in fact, the result of an enhancement of the local $N = 1$ worldsheet supersymmetry into a global $N = 2$ SCFT. Its spectral flow then guarantees the presence of spacetime supercharges, that map the bosonic matter representations into the fermionic ones and vice-versa. This will be discussed in Chapter 4.

The Gliozzi–Scherk–Olive Construction

We have seen that the fermions in the R sector are not single-valued because spin fields introduce branch cuts as worldsheet fermions move around them. The resulting correlation functions are rendered inconsistent, as the integration over the position arguments of various vertex operator insertions becomes ill-defined at the branch points. Locality of the operator products is, therefore, essential for a consistent string theory. The superstring solves this problem by introducing a consistent truncation onto states with even total (matter plus ghost) worldsheet fermion number F . We will motivate this here, although the GSO projection will be seen to arise from modular invariance constraints coming from higher genera.

The generic holomorphic part of vertex operators in the NS and R sectors is :

$$\text{NS} : P(\partial^n H, \partial^m X) e^{-\phi + i\alpha \cdot H} c(0)|0\rangle_L , \quad (2.104)$$

$$\text{R} : Q(\partial^n H, \partial^m X) e^{-\phi/2 + \frac{i}{2}\epsilon \cdot H} c(0)|0\rangle_L . \quad (2.105)$$

Here P, Q are polynomials built out of conformal currents $\partial H_a, \partial X_\mu$ and their derivatives. They encode all possible oscillator excitations. Since their operator products do not produce branch cuts, their presence is irrelevant to the present argument and will, hence, be ignored. Notice that we have chosen to factor out the $\frac{1}{2}$ out of the helicity charges so that the spin is parametrized by 5 signs, $\epsilon_a = \pm 1$. The charges in the NS sector, α_a , take values in $\alpha_a \in \{-1, 0, 1\}$, as is required by the bosonization of the worldsheet fermions. We have assumed a complexified basis for the fermions and the spinor representation index in the R sector is encoded, as before, in the choice of weight vectors $\lambda_a = \frac{1}{2}\epsilon_a$.

Now consider the OPE between two vertex operators, one being in the NS sector and the other in the R sector. The expansion will contain terms of the form :

$$\dots + \frac{1}{(z-w)^{1/2+\alpha \cdot \epsilon/2+p}} R(\partial^n H, \partial^m X) e^{-3\phi/2 + i(\alpha + \frac{1}{2}\epsilon) \cdot H} + \dots , \quad (2.106)$$

where $R(\partial^n H, \partial X)$ is again some polynomial of the currents and their derivatives. Also, $p \in \mathbb{N}$ is an integral contribution to the singularity arising from the oscillator contractions between P and Q . As z encircles w , there is a net ‘deficit’ phase :

$$\exp i\pi (1 + \alpha \cdot \epsilon) . \quad (2.107)$$

This must be equal to one for string amplitudes to be well defined. Because the signs ϵ_a are irrelevant modulo 2, the condition becomes :

$$\sum_a \alpha_a \in 2\mathbb{Z} + 1 . \quad (2.108)$$

This condition requires the sum of the fermionic charges, $\alpha_a = \oint \frac{dz}{2\pi i} i\partial H(z)$, to be odd. This can be reformulated in terms of the worldsheet fermion number as the zero mode of the ‘diagonal’ current :

$$\mathcal{F} \equiv \oint \frac{dz}{2\pi i} (-\partial\phi + i\partial H_0 + i\partial H_1 + i\partial H_2 + i\partial H_3 + i\partial H_4) . \quad (2.109)$$

Notice that we include also the super-ghost contribution, so that the NS vacuum (2.97) has negative worldsheet fermion number :

$$\begin{aligned} (-)^{\mathcal{F}} |0\rangle_{NS} &= -|0\rangle_{NS} , \\ (-)^{\mathcal{F}} |0\rangle_R &= \Gamma^D |0\rangle_R = \pm |0\rangle_R . \end{aligned} \quad (2.110)$$

More generally, the condition of mutual locality between the gravitino vertex operator and any other state with ghost charge q and helicity $\frac{1}{2}\alpha_a$ is :

$$-q + \sum_a \epsilon_a \alpha_a \in 2\mathbb{Z} . \quad (2.111)$$

This is the general form of the GSO condition. Essentially, it expresses the condition of preservation of the spacetime supersymmetry currents.

For the R vacuum, the worldsheet fermion number brings out the Γ^D -matrix¹² and gives the \pm -chirality of the R vacuum. One then imposes the consistent truncation of the theory down to states with even-even worldsheet fermion number :

$$(-)^{\mathcal{F}} |\chi\rangle = +|\chi\rangle , \quad (-)^{\bar{\mathcal{F}}} |\chi\rangle = +|\chi\rangle . \quad (2.112)$$

This is the Gliozzi-Scherk-Olive (GSO) projection [5] defining the supersymmetric (Type II) string. Indeed, imposing the projection independently on the left- and right- movers, as in (2.112), removes all states with even number of fermionic excitations from the NS sector. Acting on the R sector the GSO projection simply fixes the chirality of the spin-field. Modular invariance at one loop will require the GSO projection to act on the R sector as well and hence chirality will be fixed in the R sectors as well. The GSO-projection is precisely the condition that all vertex operator in the theory be local with respect to the vertex operator of the gravitino and, so, it is inherently related to the presence of spacetime supersymmetry.

The Type II Superstring

The construction we have described above is symmetric in the sense that it permits the combination of GSO-projected left- and right- moving NS and R sectors. This is the Type II construction. Let us briefly describe these four sectors, starting with NS-NS. Now the tachyon has been removed and the lowest mass states are the vectors $v \otimes \bar{v}$ constituting the massless gravity multiplet, (2.101). The overall chirality of the spinors is a matter of convention and

12. This is defined as the D -dimensional analogue of γ^5 in four dimensions.

only the relative chirality between the left- and right- moving R sector matters. Let us fix S to be the left-moving spinor representation (the opposite chirality will be denoted by C - for conjugate spinor). The R-NS sector will then contain tensor products of the type $S \otimes \bar{v}$, which are decomposed into the C -Majorana-Weyl (spin- $\frac{3}{2}$) gravitino and a (spin- $\frac{1}{2}$) C -fermion. For the NS-R sector there are two choices, depending on the right-moving chirality. Keeping the same chirality as with the left-movers, we obtain tensor products of the type $v \otimes \bar{S}$ and this leads to a chiral theory (Type IIB), with a C -gravitino and C -fermion. On the other hand, choosing the opposite chirality spinor $v \otimes \bar{C}$ yields a non-chiral theory (Type IIA), since now the gravitini appear with both chiralities. Finally, the massless R-R sector is $S \otimes \bar{S}$ in Type IIB theory, which decomposes into a scalar and forms of rank 2 and 4 (self-dual). In the Type IIA case, the massless states are generated by $S \otimes \bar{C}$ and decompose into a vectors and a rank 3 form.

Another important construction is that of the Heterotic string, which is a ‘hybrid’ marrying the structure of the superstring on the left-moving sector together with the structure of the bosonic string on the right-moving side. It will be easier to introduce them in the next section, from the point of view of their one-loop partition functions.

Finally, there are variants of the GSO-supersélection (2.112) that impose a correlated projection onto states that are even under the total (left- plus right- moving) worldsheet fermion number, $\mathcal{F} + \bar{\mathcal{F}}$. These constructions are called Type-0 strings but they are non-supersymmetric, their spectrum contains a tachyon and there are no spacetime fermions. We will not be consider them any further.

2.5 Vacuum Amplitude on the Torus

Moduli and Conformal Killing Vectors

In this section we introduce some of genus-one techniques that will be used throughout this manuscript. The calculation of the 1-loop vacuum amplitude (with no insertions) is useful because, apart from providing the vacuum energy in the case of setups with broken supersymmetry¹³, it gives rise to the (mass-generating) partition function encoding the information about the tree-level spectrum of the theory. This is seen most easily by cutting open the 1-loop amplitude. The information obtained is precisely about the tree-level states propagating around the loop. This, however, does not contain information about the interactions - one would need to cut open a two-loop double-donut to probe the interactions as well.

Let us return to our sketch of the topological expansion (2.16). The path integral prescription requires integration over all worldsheet metrics for a given topology. In the previous sections we described the process of gauge-fixing, which consistently sets the metric into conformally flat form. However, there is a catch. This logic implicitly assumes that all possible metrics can be connected to the conformally flat one through diffeomorphism trans-

13. In (Euclidean) thermal settings that we will discuss in subsequent chapters, the 1-loop amplitude expresses the free energy.

formations. This is not true in general, because topological obstructions may forbid some of the gauge transformations that would connect the arbitrary metric to the gauge-fixed one. In the case of the torus (genus 1, $\chi = 0$), for example, the obstructions take the form of periodicity conditions along the two non-contractible cycles, which are not in general respected by arbitrary diffeomorphism transformations.

Thus, the arbitrary metric can be gauge-transformed to conformally flat form only locally. The gauge-fixed metrics in different patches will generically differ and the gauge inequivalent metrics will be parametrized by quantities, μ_A , that we will refer to as moduli. They are precisely the remnants of the original general metric left unaffected by the diffeomorphisms and Weyl transformations of the gauge-fixing procedure and, thus, have to be integrated over in the path integral. This calls for a modification of the string S-matrix. For simplicity, we will only quote the result for the bosonic case. For the scattering of N states, with vertex operators $\mathcal{V}_i(k_i)$ the string S-matrix is :

$$\begin{aligned}
S(k_i) = & \sum_{\text{topologies}} g_s^{-\chi} \int_{\mathcal{F}} d^n \mu_A \int \mathcal{D}X \mathcal{D}b \mathcal{D}c e^{-S_{\text{matter}} - S_{\text{ghost}}} \\
& \times \left[\prod_{i-\text{unfixed}} \int d\sigma_i^a \right] \left[\prod_{A=1}^n \frac{1}{4\pi} \int d^2 \sigma_A \sqrt{\hat{g}} b^{ab} \partial_{\mu_A} \hat{g}_{ab} \right] \left[\prod_{i-\text{fixed}} c^a(\hat{\sigma}_i) \right] \prod_{i=1}^N \sqrt{\hat{g}(\sigma_i)} \mathcal{V}_i(k_i, \sigma_i) .
\end{aligned} \tag{2.113}$$

Let us comment on this structure. In addition to the gauge-inequivalent metrics which are parametrized by the moduli there exist also gauge transformations (Killing vectors) that leave the gauge-fixed metric form-invariant. Taken together, they build the conformal Killing group of the surface Σ_g . The Killing symmetry can then be used to gauge-fix the positions of some of the vertex operator insertions in the path integral.

The number of Killing vectors and moduli on a Riemann surface has topological origin and, hence, obeys topological constraints. In particular, the Riemann-Roch theorem relates the number of moduli, n , and the number of conformal Killing vectors, κ , to the Euler characteristic χ by :

$$\kappa - n = 3\chi . \tag{2.114}$$

One may understand the presence of the b and c insertions in the following way. Notice that to every unintegrated (fixed) vertex operator there corresponds a c -insertion, whereas every modulus results in a b -insertion. However, the b_{ab} and c^a ghosts in general have zero modes, which do not appear in the ghost action (2.15). Thus, unless a suitable number of ghost insertions is present, in order to soak up the zero modes, the path integral will vanish identically from the basic properties of Grassmann integration.

It is straightforward to see that the numbers of b and c insertions in (2.113) is precisely the one required to give a non-trivial result. The conformal Killing vectors are those gauge transformations leaving the metric invariant $\delta g_{ab} = 0$. From (2.12), this condition takes the form of the conformal Killing equation :

$$P\xi = 0 . \tag{2.115}$$

Hence, the conformal Killing vectors are precisely associated with the zero modes of the kernel, P , defined in (2.13).

The moduli, on the other hand, are precisely those variations $\delta^\perp g_{ab}$ of the metric that cannot be expressed as gauge (diffeomorphisms and Weyl) transformations. They are, therefore, orthogonal to the variation (2.12) with respect to the natural inner product :

$$\begin{aligned} 0 &= \int d^2\sigma \sqrt{g} \delta^\perp g_{ab} \delta g^{ab} \equiv \langle \delta^\perp g | \delta g \rangle = \langle \delta^\perp g | 2\delta\tilde{\omega} g - 2(P\xi) \rangle \\ &= 2\langle \delta^\perp g | \delta\tilde{\omega} g \rangle - 2\langle P^T \delta^\perp g | \xi \rangle . \end{aligned} \quad (2.116)$$

This has to vanish for arbitrary ξ , $\delta\tilde{\omega}$, therefore :

$$g^{ab} \delta^\perp g_{ab} = 0 \quad , \quad (P^T \delta^\perp g)_{ab} = 0 \quad , \quad (2.117)$$

where the transpose P^T is obtained via partial integration, $(P^T v)_a = -\nabla^b v_a$. The moduli are, therefore, associated to the traceless, symmetric zero-modes of P^T .

The above conditions for the conformal Killing vectors and the moduli as zero-modes of P and P^T , respectively, precisely coincide with the zero modes of b_{ab} , c^a , as seen from the ghost action :

$$S_{\text{ghost}} = \frac{1}{2\pi} \langle b | P c \rangle = \frac{1}{2\pi} \langle P^T b | c \rangle . \quad (2.118)$$

Thus, the number of b -zero modes is precisely equal to the number of moduli on the Riemann surface. Similarly, the number of c -zero modes equals the number of conformal Killing vectors. This is precisely the number of insertions in (2.113), as needed for the path integral to be non-trivial.

With every b -insertion, we obtain a derivative $\partial_{\mu_A} \hat{g}_{ab}$, because the total variation of the metric in the Φ -II procedure must also include the variation with respect to the moduli :

$$\delta g_{ab} = 2\delta\tilde{\omega} g_{ab} - 2(P\xi)_{ab} + \delta\mu^A \partial_{\mu_A} g_{ab} . \quad (2.119)$$

Putting everything together, the S-matrix structure (2.113) is in agreement with the considerations of BRST invariant vertex operators. Each fixed (unintegrated) vertex operator always comes multiplied by a $c\bar{c}$ -insertion, which renders it BRST invariant. The unfixed vertex operators have no such insertion and they are not BRST invariant. However, the action of Q_B on them produces a total derivative, which vanishes after integration. Also, notice that the action of Q_B on the b -insertion, $\langle b | \partial_{\mu_A} \hat{g} \rangle$, replaces b by the energy momentum tensor $\langle T | \partial_{\mu_A} \hat{g} \rangle$. However, this is precisely the total variation of the path integral with respect to the modulus μ_A :

$$\sum_{\text{topologies}} g_s^{-\chi} \int_{\mathcal{F}} d^m \mu \frac{\partial}{\partial \mu_A} \left[\int \mathcal{D}X \mathcal{D}b \mathcal{D}c e^{-S_{\text{matter}} - S_{\text{ghost}}} (\dots) \right] , \quad (2.120)$$

where the ellipsis denotes the remaining insertions with the particular insertion $\langle b | \partial_{\mu_A} \hat{g} \rangle$ removed. This becomes a surface integral living on the boundary of moduli space and, in most cases, vanishes.

There is another formulation, equivalent to (2.113), which is sometimes also useful. The idea is that we can absorb the coordinate dependence of the unfixed vertex operators into the transition functions between different patches and, hence, treat them effectively as moduli. The S-matrix then becomes :

$$S(k_i) = \sum_{\text{topologies}} g_s^{-\chi} \int_{\mathcal{F}} d^n \mu_A \left\langle \prod_{A=1}^n \mathcal{B}_A \prod_{i=1}^N c\bar{c} \mathcal{V}_i(k_i) \right\rangle , \quad (2.121)$$

where the \mathcal{B}_A -insertion is essentially the b -ghost insertion :

$$B_A = \frac{1}{4\pi} \int d^2 \sigma \sqrt{\hat{g}} b_{ab} \hat{g}^{ac} \hat{g}^{bd} \partial_{\mu_A} \hat{g}_{cd} , \quad (2.122)$$

where we implicitly assume that the moduli have been absorbed into the transition functions. The number of effective moduli in this framework is then $n_{\text{eff}} = n + 2N - \kappa = 2N - 3\chi$. In this formulation all vertex operator insertions are treated as fixed. This result has suitable generalization for the superstring.

The Metric on the Torus

Let us now return and apply the above to the 2-torus, T^2 . There are exactly two holomorphic differentials, corresponding to the two translations along the two non-contractible cycles and the conformal Killing group on the torus is, hence, $U(1) \times U(1)$. Because this volume is finite, we can choose not to fix the position of any vertex operator insertions and simply divide by the volume. This will permit one to calculate the 1-loop vacuum amplitude, that is, without any external-leg insertions. There are also 2 real moduli parameters τ_1, τ_2 , which we combine into a complex variable $\tau = \tau_1 + i\tau_2$.

We assume that $g_{ab}(\sigma)$ (and other fields) are doubly periodic in the coordinate region $\sigma^a \in [0, 1)$. Alternatively, one may extend this to the whole complex plane, $w = \sigma^1 + i\sigma^2$, with the double identification :

$$\begin{aligned} w &\sim w + 1 , \\ w &\sim w + i . \end{aligned} \quad (2.123)$$

We already mentioned that there are gauge inequivalent metrics, that is, metrics which cannot be connected via gauge (diffeomorphisms and Weyl) transformations and which are parametrized by the metric moduli. Nevertheless, one may perform a local Weyl rescaling in order to set the Ricci scalar to zero. In 2d this implies that the metric is Weyl-equivalent to the flat metric. By a coordinate transformation we would have been able to put the metric to the standard euclidean form $g_{ab} \propto \delta_{ab}$, were it not for the topological obstruction. In fact, one may use reparametrization invariance to set the metric to the standard form, $ds^2 \propto dw d\bar{w}$, however, this would spoil the original periodicity (2.123), so we would find instead :

$$\begin{aligned} w &\sim w + 1 , \\ w &\sim w + \tau , \end{aligned} \quad (2.124)$$

for some $\tau \in \mathbb{C}$. We have implicitly rescaled the coordinates and the metric (Weyl transformation) to preserve the first periodicity normalized to $w \sim w + 1$. Notice that in this approach the modulus τ has entered the periodicities, but not the metric. The converse is more convenient. Define new coordinates $\sigma^{a'}$:

$$w = \sigma^{1'} + \tau \sigma^{2'} , \quad (2.125)$$

such that the τ -dependence of the periodicity is absorbed. This reinstates the original double periodicity $\sigma^{a'} \sim \sigma^{a'} + 1$. However, the metric now explicitly contains the Teichmüller parameter τ :

$$ds^2 = \frac{1}{\tau_2} |d\sigma^{1'} + \tau d\sigma^{2'}|^2 , \quad g_{ab} = \frac{1}{\tau_2} \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix} . \quad (2.126)$$

Here we have also performed a Weyl rescaling in order to factor out the volume factor. Of course τ is restricted to takes values in the upper-half plane \mathbb{C}^+ in order for the metric to be positive definite.

The above argument that we can put the torus into flat form is a special case of a general uniformization theorem [6]. This states that every Riemann surface Σ_g is conformally equivalent to the quotient D/G , with $D = \mathbb{C} \cup \{\infty\}$, \mathbb{C} , or \mathbb{C}^+ and $G = \pi_1(\Sigma_g)$ being a freely acting discrete subgroup of the Möbius transformations $SL(2, \mathbb{R})$ preserving D . This implies that we can always conformally map Σ_g to a constant-curvature Riemann surface. In this way the sphere $\Sigma_0 = S^2$ becomes mapped to $\mathbb{C} \cup \{\infty\}$, with standard metric $ds^2 = dw d\bar{w} / (1 + w\bar{w})$ and positive constant curvature. The simple torus $\Sigma_1 = T^2$ is conformally mapped to \mathbb{C} / Λ_2 (with Λ_2 being a translation lattice), metric $ds^2 = dw d\bar{w}$ and vanishing curvature. For higher genera, $\Sigma_{g \geq 2}$ is mapped to \mathbb{C}^+ / G (G is called the Fuchsian subgroup) with metric $ds^2 = dw d\bar{w} / w^2$ and negative constant curvature.

Modular Transformations and the Fundamental Domain

In general, the Teichmüller space is defined as the quotient of the space of metrics with respect to the group of gauge transformations, continuously connected to the identity :

$$\text{Teichmüller} \equiv \frac{\text{metrics}}{\text{diffeomorphism} \times \text{Weyl}} . \quad (2.127)$$

For the sphere, there are no Teichmüller parameters, for the torus there are 2 (real), while for a Riemann surface of genus $g \geq 2$ there are $6(g - 1)$ Teichmüller parameters. However, there are in general additional gauge transformations that are not continuously connected to the identity (large coordinate transformations). They are typically generated by a discrete group of transformations Γ . These have to be factored out as well, so that the true moduli space \mathcal{M} is :

$$\mathcal{M} = \frac{\text{metrics}}{\text{diffeomorphism} \times \text{Weyl} \times \Gamma} . \quad (2.128)$$

Let us now return to the simple torus ($g = 1$) and comment on the physical meaning of τ_1 and τ_2 . The flat torus has been defined as the complex plane modded out by a 2d lattice \mathbb{C}/Λ_2 . This means that we can construct the torus from the parallelogram $(0, 0)$, $(0, 1)$, (τ_1, τ_2) , $(\tau_1 + 1, \tau_2)$, by wrapping and gluing together two opposite sides to obtain a cylinder. By joining together the two ends of the cylinder (two remaining sides) one obtains the torus. Consider first the case of the untwisted torus, $\tau_1 = 0$. The parallelogram is then orthogonal. By identifying the two vertical sides, we obtain a cylinder of perimeter equal to 1 and length equal to τ_2 . Subsequently, we glue together the two open ends of the cylinder and obtain the torus. Thus, the physical meaning of τ_2 is to fix the relative scale (ratio) between the ‘perimeters’ of the two non-contractible cycles of the torus. Now assume $\tau_1 \neq 0$ and notice that the points $(0, 0)$ and (τ_1, τ_2) are identified. This amounts to ‘twisting’ the cylinder, i.e. rotating the ends before gluing them, so that instead of gluing point $(\sigma, t) = (0, 0)$ with point $(0, \tau_2)$, we are gluing it (τ_1, τ_2) , instead.

As mentioned, in contrast to the transformations continuously connected to the identity, which cannot change the complex structure τ , there are global operations on the torus (large diffeomorphisms), called Dehn twists, which can change τ . For example, consider cutting the torus open on one of its cycles, rotating the ends by 2π and gluing them back together. This takes τ to $\tau + 1$. Points that were in the same neighborhood before the twist are still in the same neighborhood after the twist. However, this operation is clearly not continuously connected to the identity. Still, it corresponds to an equivalent torus (see the left diagram on Figure 2.1), because of the lattice periodicity (2.124).

A similar Dehn twist can be performed around the other non-trivial cycle of the torus. Cut open the second cycle, rotate its ends by 2π and glue them back together. To see what this translates into, one has to construct the ‘mirror’ image of the parallelogram. Simply take the original parallelogram, and rotate it counterclockwise so that its edge of length $|\tau|$ coincides with the vertical axis. This is similar to the original torus, seen through a mirror coordinate system $\tau_1 \leftrightarrow \tau_2$, with the edges of length $|1|$ and $|\tau|$ exchanged, which implies precisely the exchange of the two non-trivial cycles. We have to normalize correctly the side adjacent to the vertical axis. For this purpose, rescale the coordinates by $|\tau|$ so that the four vertices of the parallelogram are not at points $0, \frac{1}{\tau}, 1 + \frac{1}{\tau}$ and 1 . The new torus is now described by the effective Teichmüller parameter $\frac{1}{\tau}$. The Dehn twist is, again, a rotation by 2π around the ends of the cylinder, which means $\frac{1}{\tau} \rightarrow \frac{1}{\tau} + 1$. In terms of the ‘original’ τ parameter, this is precisely the transformation $\tau \rightarrow \frac{\tau}{\tau+1}$, corresponding to choosing the cell as in the right diagram on Figure 2.1.

These transformations leave the torus invariant and generate the modular group $\Gamma = PSL(2, \mathbb{Z})$, which is defined as the group of Möbius transformations with integer coefficients $a, b, c, d \in \mathbb{Z}$, modulo an overall sign :

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d} \quad , \quad ad - bc = 1 . \quad (2.129)$$



FIGURE 2.1 – The modular transformations $\tau \rightarrow \tau + 1$ (left) and $\tau \rightarrow \tau/(\tau + 1)$ (right).

It has two generators, which can be taken to be :

$$\begin{aligned} T &: \tau \rightarrow \tau + 1, \\ S &: \tau \rightarrow -\frac{1}{\tau}. \end{aligned} \tag{2.130}$$

The Teichmüller space for the torus is the upper half plane, \mathbb{C}^+ , and this has to be modded out by the modular group $PSL(2, \mathbb{Z})$. The resulting region, \mathcal{F} , is the fundamental domain, $\mathcal{F} = \{ \tau \in \mathbb{C}^+ : |\tau| \geq 1, |\tau_1| \leq \frac{1}{2} \}$. It is shown in Figure 2.2 below. The exclusion of the $\tau_2 \rightarrow 0$ region is important for the absence of UV divergences in string theory, because τ_2 will effectively become the analogue of the Schwinger parameter of field theory.

The Bosonic Torus Vacuum Amplitude

In order to construct the torus amplitude it will be useful to rewrite the expression (2.121) for the string S-matrix in a form valid even without insertions. Since there are 2 (real) conformal Killing vectors and 2 (real) moduli, we will always need a $\langle b\bar{b}c\bar{c} \rangle$ -ghost insertion in order to soak up the ghost zero-modes. We will consider all vertex operators to be integrated over and not will not use the Killing symmetry to fix any position.

The starting point is to calculate the B insertion. We will not give the details here but merely state that one varies the standard flat metric $ds^2 = dw d\bar{w}$ and from the periodicities one reads the induced variation of τ . This yields $\partial_\tau \hat{g}_{w\bar{w}} \sim i/\tau_2$. Then, one uses the fact that only the ghost zero modes $c_0 \bar{c}_0 b_0 \bar{b}_0$ contribute to the torus path integral and, hence, the B -insertion is independent of the position. The result is $B \sim ib_{w\bar{w}}(0)$. Finally, one divides by the (finite) volume of the conformal Killing group and integrates over all vertex operator insertions. The translation invariance of the vertex operators is, of course, necessary. The

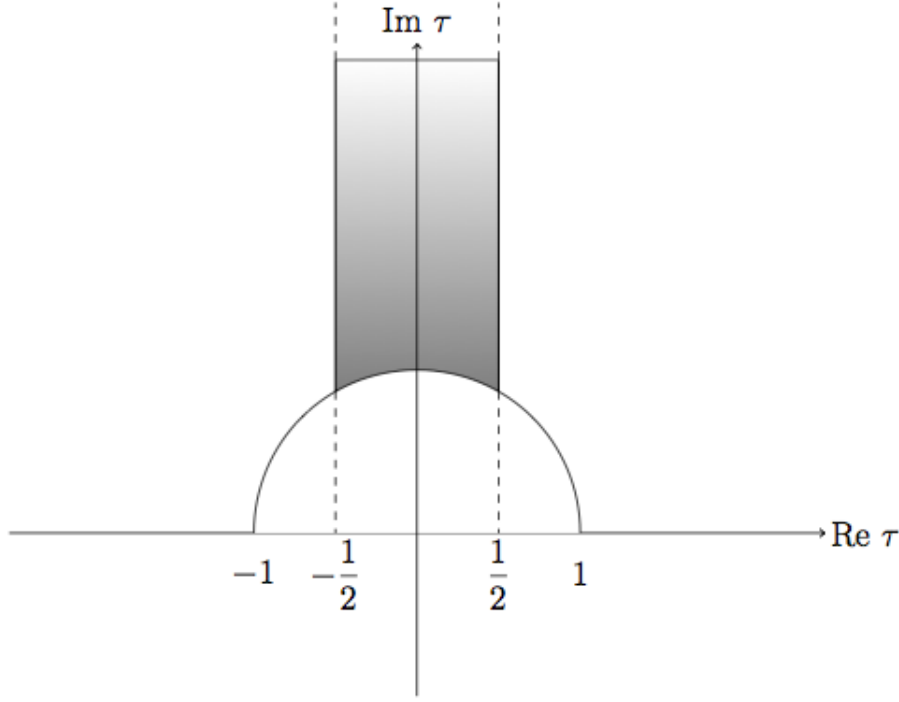


FIGURE 2.2 – The fundamental domain of the torus, $\mathcal{F} = \mathbb{C}^+ / PSL(2, \mathbb{Z})$.

final result is :

$$S_{g=1}(k_i) = \int_{\mathcal{F}} \frac{d^2\tau}{4\tau_2} \langle b(0)\bar{b}(0)c(0)\bar{c}(0) \rangle_{g=1} \left\langle \prod_{i=1}^N \int d^2w_i \mathcal{V}_i(k_i) \right\rangle_{g=1}. \quad (2.131)$$

This expression holds, in particular, for $N = 0$ vertex operator insertions, $\langle 1 \rangle_{g=1}$, which is precisely the vacuum amplitude that we want to calculate.

We first evaluate the path integral contribution of matter. There are two ways to perform the calculation. Let us take, for example, the simple case of a single free scalar X . The straightforward way is to carry out the gaussian path integration, which leads to the determinant of the Laplacian on the torus, subject to the periodicity conditions $\langle 1 \rangle_X \sim \det'^{-1/2} \square$. This is an infinite product of the non-vanishing eigenvalues, $\lambda_{mn} = \frac{(2\pi)^2}{\tau_2} |m - \tau n|^2$, which typically diverges and has to be regularized. The idea is then to express the determinant, $\det'^{-1/2} \square = \exp\left(\frac{1}{2}\zeta'_{T^2}(0)\right)$, in terms of the generalized zeta function of the torus, $\zeta_{T^2}(s)$, which encodes the eigenvalues of the Laplacian on the Riemannian manifold [7]. For the case of the torus, this is related to the Eisenstein series by :

$$\zeta_{T^2}(s) \equiv \sum_{(m,n) \neq (0,0)} \lambda_{m,n}^{-s} = (2\pi)^{-2s} E(\tau, s), \quad (2.132)$$

where the non-holomorphic Eisenstein series is defined for $\Re(s) > 1$ as :

$$E(\tau, s) \equiv \tau_2^s \sum_{(m,n) \in \mathbb{Z}/\{0\}} \frac{1}{|m + \tau n|^{2s}} . \quad (2.133)$$

By analytic continuation one may extend $E(\tau, s)$ to other values including, in particular, $s = 0$. Carefully separating out the zero modes and utilizing the periodicity in $y = n\tau_1$, one may obtain an integral representation by Fourier expansion. After some manipulations (see, for instance, [8]) one obtains the following expansion :

$$\begin{aligned} E(\tau, s) = & 2\tau_2^s \zeta(2s) + 2\sqrt{\pi} \tau_2^{1-s} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \zeta(2s - 1) \\ & + \frac{4\pi^s \sqrt{\tau_2}}{\Gamma(s)} \sum_{m,n=1}^{\infty} \left(\frac{m}{n}\right)^{s-1/2} (e^{2\pi i \tau_1 m n} + e^{-2\pi i \tau_1 m n}) K_{s-\frac{1}{2}}(2\pi \tau_2 m n) . \end{aligned} \quad (2.134)$$

Here, $K(x)$ is the modified Bessel function of the second kind. By expanding this around $s = 0$ we pick the coefficient of the linear term :

$$\zeta'_{T^2}(0) = \frac{\pi}{3} \tau_2 - \log(2\pi \tau_2) + 2 \sum_{m,n=1}^{\infty} \frac{1}{m} (q^{nm} + \bar{q}^{nm}) = -2 \log(\sqrt{\tau_2} \eta \bar{\eta}) , \quad (2.135)$$

where $q \equiv \exp(2\pi i \tau)$ and $\eta(\tau)$ is the Dedekind η -function (A.1). This leads to the result $\sqrt{\det' \square} = \sqrt{\tau_2} \eta(\tau) \bar{\eta}(\bar{\tau})$, so that

$$\langle 1 \rangle_X \sim \frac{1}{\sqrt{\tau_2} \eta \bar{\eta}} , \quad (2.136)$$

modulo a normalization factor that is proportional to the volume of the 1d ‘box’. This normalization can be fixed by comparison with the canonical result, by inserting a complete number of states and taking the limit $\tau_2 \rightarrow \infty$ so that only the ‘ground state’ contributes.

Instead, we will repeat the previous calculation in the canonical formalism. The torus can be seen as a cylinder whose endpoints are identified (after rotating them by τ_1). Take one of the two local coordinates, $\sigma^1 = t$, to be the analogue of ‘time’, assuming that it parametrizes the non-trivial cycle of length τ_2 . The cylinder with two points identified corresponds to a state propagating along the ‘time’ direction. The identification of the two ends implies that the initial and final states are the same and we are summing over the whole physical Hilbert space, which produces a trace. The generator of this translation by τ_2 is the Hamiltonian (transfer matrix) $H = 2\pi(L_0 + \bar{L}_0 - \frac{c+\bar{c}}{24})$. We also have to twist (rotate) the ends by τ_1 before gluing. This rotation is, in fact, a translation along the ‘spatial’ direction $\sigma^2 = \sigma$ corresponding to the other non-trivial cycle of the torus. The generator of translations along σ is the momentum operator $P = -2\pi i(L_0 - \bar{L}_0)$. Notice the non-conventional factors of 2π , which are there because we are using the non-conventional periodicity $\sigma^a \in [0, 1)$, instead

of the conventional $\sigma^a \in [0, 2\pi)$. Putting it all together, the torus path integral takes the general form :

$$\begin{aligned} Z(\tau, \bar{\tau}) &= \text{Tr} \left[e^{-2\pi\tau_2(L_0 + \bar{L}_0 - \frac{c+\bar{c}}{24})} e^{2\pi i\tau_1(L_0 - \bar{L}_0)} \right] \\ &= \text{Tr} \left[q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \right]. \end{aligned} \quad (2.137)$$

This is the spectrum-generating partition function. This ‘Hamiltonian’ representation is very useful because it allows one to formally consider the partition function as an expansion into Fourier modes $Z = \sum_{m,n} d_{mn} q^m \bar{q}^n$, where the exponent counts the mass level in CFT units and the coefficient d_{mn} of which term gives the degeneracy. Notice that the final result is periodic in τ_1 , as is expected of a rotation parameter.

The general state constructed out of the creation operators $\alpha_{n<0}$ of the scalar field X^μ (with d components) can be written as :

$$|\{N, \bar{N}\}, k\rangle = \prod_{\mu=1}^d \prod_{n=1}^{\infty} (\alpha_{-n}^\mu)^{N_n} (\bar{\alpha}_{-n}^\mu)^{\bar{N}_n} e^{ik \cdot X} |0\rangle, \quad (2.138)$$

where the occupation numbers $N_i \in \mathbb{N}$ completely specify the state. Its conformal weight is $h_{\{N\},l} = \frac{k^2}{2} + \sum_{n=1}^{\infty} n N_n$. We assume the states are properly normalized and compute the trace as follows :

$$Z_X(\tau, \bar{\tau}) = iV_{(d)} \int \frac{d^d k}{(2\pi)^d} e^{-2\pi\tau_2 k^2} q^{-c/24} \bar{q}^{-\bar{c}/24} \left[\prod_{n=1}^{\infty} \sum_{N=0}^{\infty} \sum_{\bar{N}=0}^{\infty} q^{nN} \bar{q}^{n\bar{N}} \right]^d \quad (2.139)$$

The integral is over the zero mode and $V_{(d)}$ is the volume of the d -dimensional ‘box’ we put the system into. The imaginary unit arises because the integral is written in Euclidean space, $k^0 \rightarrow ik^0$. Carrying out the integration and summation and using the definition (A.1) of the Dedekind eta function one finds :

$$Z_X(\tau, \bar{\tau}) = \frac{i(V_{(d)}/\alpha'^{d/2})}{(2\pi\sqrt{\tau_2} \eta\bar{\eta})^d}, \quad (2.140)$$

where we reinstate the α' -dependence. This coincides precisely with the result (2.136) derived from the path integral calculation, with the correct normalization. Notice that we used $c = d$ for this conformal system. The path integral (2.140) is invariant under the modular transformations (2.130), as required for the consistency of genus-1 amplitudes. Indeed, this is straightforward to see by using the modular transformation property (A.2) of the Dedekind function.

Incidentally, modular invariance can be used to yield the correct conformal anomaly. We could have derived the correct value $c = d$ of the central charge of d free scalars by demanding that the partition function be modular invariant. In fact, modular invariance

is very strongly related to unitarity and so reflects the general consistency of the theory. The presence of anomalies in the theory will generically manifest themselves as a breaking of modular invariance. In this sense, modular invariance at genus $g \geq 1$ provides the basic guiding principle in constructing consistent string vacua.

The contribution of the ghosts is straightforward to calculate in a similar way. Without insertions $\langle 1 \rangle \sim \det \nabla = 0$ for the ghost system because of the ghost zero modes. The ghost insertions (2.131) soak up precisely these zero modes and the finite result is :

$$\langle c_0 b_0 \bar{c}_0 \bar{b}_0 \rangle_{g=1} = (q\bar{q})^{26/24} \text{Tr} \left[(-)^{\mathcal{F}} c_0 b_0 \bar{c}_0 \bar{b}_0 q^{L_0} \bar{q}^{\bar{L}_0} \right] = \eta^2 \bar{\eta}^2 . \quad (2.141)$$

The $(-)^{\mathcal{F}}$ -operator is necessary in order to give the ghosts periodic (instead of anti-periodic) boundary conditions around the cylinder's (we cut the torus open) non-contractible cycle. This is required because of the anti-commuting statistics of the bc -ghosts.

However, in practice, it is easier to infer their contribution by invoking the No-Ghost theorem. Since the ghosts effectively cancel 2 longitudinal oscillators, their normalized contribution to the partition function is $\langle c_0 b_0 \bar{c}_0 \bar{b}_0 \rangle_{g=1} = \eta^2 \bar{\eta}^2$. The result for the bosonic string in $d = 26$ flat dimensions is then :

$$\mathcal{Z}_{1\text{-loop}} = i(\alpha')^{-13} V_{(26)} \int_{\mathcal{F}} \frac{d^2 \tau}{4\tau_2} \frac{1}{(2\pi\sqrt{\tau_2})^{26}} \frac{1}{\eta^{24} \bar{\eta}^{24}} . \quad (2.142)$$

Notice that we are using the complex normalization $d^2 \tau = 2d\tau_1 d\tau_2$. The vacuum amplitude is modular invariant, since the measure $d^2 \tau / \tau_2^2$ and the combination $\sqrt{\tau_2} \eta \bar{\eta}$ are separately modular invariant.

The integral over τ_1 essentially imposes level matching, which can easily be seen for $\tau_2 \gg 1$. The argument is more subtle¹⁴ at $\tau_2 \sim 1$, because of the circular shape of the lower boundary of \mathcal{F} . The integral (2.142) diverges exponentially as $\tau_2 \rightarrow \infty$, in the limit where the string becomes long, due to the presence of the bosonic string tachyon. This is an IR divergence signaling the instability of the bosonic string. The would-be UV divergence having already been taken care of by the theory, by restricting integration over $\mathcal{F} = \mathbb{C}^+ / PSL(2, \mathbb{Z})$ and, hence, excluding the integration in the dangerous UV-region $\tau_2 \rightarrow 0$. We will come back to the problem of tachyonic instabilities in later chapters.

The general form of the vacuum amplitude in the case of d -flat dimensions is :

$$\frac{iV(d)}{(2\pi\sqrt{\alpha'})^d} \int_{\mathcal{F}} \frac{d^2 \tau}{4\tau_2^2} \tau_2^{-d/2+1} Z_{\text{CFT}}(\tau, \bar{\tau}) , \quad (2.143)$$

where $Z_{\text{CFT}}(\tau, \bar{\tau})$ is the mass-generating¹⁵ partition function of the CFT in the lightcone.

14. This is crucial for the decoupling of unphysical states. The action of the BRST charges induces a total divergence and the corresponding surface terms cancel only because of the identification of the boundaries $|\tau| = 1$, $\tau_1 < 0$ and $\tau_1 > 0$. This would be impossible to achieve in field theory with the introduction of a naive cutoff.

15. This implies that it does not include the momentum contribution, which has already been integrated over.

The Partition Function For Fermions

We next move to the case of fermions. Take a complex fermion Ψ with arbitrary boundary conditions around the two non-contractible cycles of the torus :

$$\psi(\sigma^1 + 1, \sigma^2) = e^{i\pi(1-a)} \Psi(\sigma^1, \sigma^2) , \quad (2.144)$$

$$\psi(\sigma^1, \sigma^2 + 1) = e^{i\pi(1-b)} \Psi(\sigma^1, \sigma^2) . \quad (2.145)$$

Similar boundary conditions have to be specified for the for the right-movers, in terms of the parameters \bar{a}, \bar{b} . The first boundary condition is along the direction of propagation and, hence, affects the moding of the states and the zero-point constant in the Hamiltonian. The second boundary condition will be responsible for the insertion of the operator $\exp(i\pi b\mathcal{F})$ in the trace, \mathcal{F} being the worldsheet fermion number. The partition function of a single complex fermion can then be straightforwardly calculated :

$$Z_{\psi[b]}^a \equiv \text{Tr}_a \left[e^{i\pi b\mathcal{F}} q^{L_0[a]-1/24} \right] = \frac{\theta_{[b]}^a}{\eta} , \quad (2.146)$$

for any $a, b \in \mathbb{R}$. Here $\theta_{[b]}^a \equiv \theta_{[b]}^a(\tau, 0)$ are the Jacobi theta functions with characteristics (A.4). For the real fermions ψ^μ the only allowed boundary conditions are NS or R, corresponding to $a = 0$ or $a = 1$, respectively.

The consistent introduction of fermions involves the modification of the prescription for the S-matrix by requiring a further summation over all possible spin structures $[\frac{a}{b}]$, $[\frac{\bar{a}}{\bar{b}}]$ of the worldsheet fermions. It also requires the introduction of appropriate $\delta(\gamma)$, β -superghost insertions and integration over the now extended super-moduli space of the super-Riemann surface. We will not need to discuss this machinery here, but merely use light-cone arguments in order to write down the partition function for the superstring. The result is that the contribution of the β, γ -superghosts cancels the contribution of the longitudinal worldsheet fermions ψ^0, ψ^1 :

$$Z_{\beta, \gamma} = (-)^{b+\mu ab} \frac{\eta}{\theta_{[b]}^a} , \quad (2.147)$$

where $\mu = 0, 1$ specifies the chirality of the R vacuum. The spin-structure dependent cocycle $(-)^{b+\mu ab}$ arises from the trace over the superghost sector. This is because the NS and R ground states, $|0\rangle_{NS}$, $|0\rangle_R$, have non-trivial worldsheet fermion parity, according to (2.110). Thus, the $\exp(i\pi b\mathcal{F})$ operator insertion in the trace produces the cocycle factor :

$$\exp(i\pi b\mathcal{F})|0\rangle_a = (-)^{b+\mu ab}|0\rangle_a . \quad (2.148)$$

We now impose the GSO projection, (2.112), in a sector $a = 0, 1$ (NS or R). The relevant part is the one containing the 10 real fermions ψ^μ (2 longitudinal plus 8 transverse) and the superghosts β, γ :

$$\text{Tr}_a \left[\frac{1 + (-)^{\mathcal{F}}}{2} q^{L_0 - c/24} \right] = \frac{1}{2} \sum_{b=0,1} (-)^{b+\mu ab} \left(\frac{\eta}{\theta_{[b]}^a} \right) \frac{\theta_{[b]}^{5[a]}}{\eta^5} . \quad (2.149)$$

Together with the bosonic and b, c -ghost contributions, the full partition function becomes :

$$Z = \frac{1}{2} \sum_{a,b=0,1} \frac{1}{2} \sum_{\bar{a},\bar{b}=0,1} C_{a,\bar{a}} (-)^b \frac{\theta^4[a]}{\eta^{12}} (-)^{\bar{b}+\bar{\mu}\bar{a}\bar{b}} \frac{\bar{\theta}^4[\bar{a}]}{\bar{\eta}^{12}} . \quad (2.150)$$

We have only kept the relative spinor chirality, parametrized by $\bar{\mu} = 0, 1$ and we introduced a coefficient $C_{a,\bar{a}}$ depending on a, \bar{a} . By unitarity (implying a consistent particle interpretation), we see that this coefficient has to take values $C_{a,\bar{a}} = \pm 1$. The phase $C_{a,\bar{a}}$ can be further constrained by imposing modular invariance. Using the properties of theta functions under an S -modular transformation, (A.5), one can derive the necessary condition :

$$C_{a,\bar{a}} (-)^{b+\bar{b}} = C_{b,\bar{b}} (-)^{a+\bar{a}} , \quad (2.151)$$

which can be solved to give :

$$C_{a,b} = (-)^{a+\bar{a}} . \quad (2.152)$$

This is the correct spin-statistics cocycle which guarantees that bosons (NS-NS or R-R) have positive contributions to the 1-loop vacuum energy, whereas fermions (NS-R or R-NS) contribute with the opposite sign. The full modular invariant partition function of the 10d Type II superstring is then :

$$Z = \frac{1}{2} \sum_{a,b=0,1} (-)^{a+b} \frac{\theta^4[a]}{\eta^{12}} \frac{1}{2} \sum_{\bar{a},\bar{b}=0,1} (-)^{\bar{a}+\bar{b}+\bar{\mu}\bar{a}\bar{b}} \frac{\bar{\theta}^4[\bar{a}]}{\bar{\eta}^{12}} . \quad (2.153)$$

The choice of the chirality projection of the right-moving R-vacuum distinguishes between the Type IIA ($\bar{\mu} = 1$) and Type IIB theories ($\bar{\mu} = 0$). Let us mention that various conformal block contributions to the above partition function are not separately modular invariant (though they are modular covariant). This represents that the corresponding path integral is not well-defined under large coordinate transformations and the result is a global gravitational anomaly. Modular invariance of the combined blocks then implies the cancelation of the global anomalies.

By an explicit algebraic evaluation, the partition function (2.153) vanishes identically. This can be seen straightforwardly by employing the ‘abstrusa’ identity of Jacobi (A.8). This is a manifestation of the presence of spacetime supersymmetry in the superstring spectrum. There is a precise matching of bosonic to fermionic states at all mass levels and the contribution of bosons and fermions to the 1-loop vacuum amplitude cancel each other identically. The way spacetime supersymmetry arises from the worldsheet SCFT will be discussed in some detail in chapter 4. However, already at the level of superstring vertex operators one may immediately count $\mathcal{N}_4 = 2$ spacetime supersymmetries in 10d, each associated with one gravitino from each of the left- and right- moving sectors.

$\widehat{O}(N)$ Current Algebras

Let us mention here some elementary facts about (spin-1) current algebras. Consider the set of holomorphic $(1,0)$ -currents J^A present in a given CFT. Their OPE is constrained by holomorphicity and conformal invariance to take the form :

$$J^A(z)J^B(w) = \frac{\Delta^{AB}}{(z-w)^2} + if^{AB}_C \frac{J^C(w)}{z-w} + \dots , \quad (2.154)$$

with Δ^{AB} symmetric and f^{AB}_C antisymmetric in the first two indices. They have zero conformal weight, so Δ^{AB} and f^{AB}_C are constant c-numbers. The modes J_n^A close into an infinite-dimensional algebra :

$$[J_n^A, J_m^B] = if^{AB}_C J_{n+m}^C + n\Delta^{AB}\delta_{n+m,0} . \quad (2.155)$$

In addition, the zero modes J_0^A form a Lie algebra, \mathcal{G} , with structure constants f^{AB}_C :

$$[J_0^A, J_0^B] = if^{AB}_C J_0^C . \quad (2.156)$$

Furthermore, Δ^{AB} is identified with the natural Killing metric and, in the case of a simple algebra \mathcal{G} , one can choose a basis such that $\Delta^{AB} = k\delta^{AB}$, where δ^{AB} is the Kronecker delta. We have assumed that the long roots, α , are normalized to have $\alpha^2 = 2$. Such algebras arise naturally in non-linear Wess-Zumino-Witten (WZW) models. The quantization of the level, $k \in \mathbb{N}^+$, arises as a result of an underlying $SU(2)$ subalgebra with central extension $\sim k$. The level should be positive for a unitary theory.

There is a natural construction of a $(2,0)$ energy-momentum tensor entirely in terms of $(1,0)$ -current bilinears :

$$T_{\mathcal{G}}(z) = \frac{1}{2(k+\hat{h})} : J^a J^a : (z) , \quad (2.157)$$

where \hat{h} is the dual Coxeter number¹⁶ of the group. Thus, the affine (or Kac-Moody) current algebra $\hat{\mathcal{G}}$ also satisfies the Virasoro algebra with central extension :

$$c = \frac{k \dim \mathcal{G}}{k + \hat{h}} . \quad (2.158)$$

A very important case which arises frequently is $\widehat{SU}(2)_k$. For $\mathcal{G} = SU(2)$ one has $\hat{h} = 2$ and the central charge at level k becomes :

$$c_{SU(2)} = \frac{3k}{k+2} . \quad (2.159)$$

16. For $A-D-E$ algebras, which are simply-laced (i.e. all roots have the same length, whose square can then be normalized to 2), the dual Coxeter number is defined as $\hat{h} = -\frac{1}{2\dim \mathcal{G}} f^{AC}_D f^{BD}_C \delta_{AB}$.

Being larger than the Virasoro algebra, the affine algebra can be used to classify the spectrum in terms of its representations. The affine primary representations are defined as families of states created by acting on the invariant vacuum with those operators $\mathcal{O}_i(z)$ which transform tensorially :

$$J^A(z)\mathcal{O}_i(w) = \frac{(T^A)_{ij}}{z-w} \mathcal{O}_j(w) + \dots \quad (2.160)$$

The representation theory is analogous to the case of the Virasoro algebra. In particular, the operator $\mathcal{O}_i(z)$ transforms according to a representation of the zero-mode Lie subalgebra as :

$$J_0^A |\mathcal{O}\rangle_i = (T^A)_{ij} |\mathcal{O}\rangle_j , \quad (2.161)$$

where T^A are representation matrices of the Lie algebra \mathcal{G} . The absence of higher poles in the OPE (2.160) implies that the affine primary state is annihilated by all positive modes, $J_n^A |\mathcal{O}\rangle_i = 0$, ($n > 0$) and so the affine primaries are the ‘ground states’ of the representation. The descendant states are built from each affine primary by acting with the raising operators, J_n^A (with $n < 0$), thus, completing the representation. By a positivity argument one may show that only some of the representations of the Lie subalgebra can be unitary. For the case of $\widehat{SU}(2)_k$, these are the representations of spin $j \leq k/2$.

Now we focus on the CFT of N free fermions ψ^i . The global $O(N)$ symmetry gives rise to conserved holomorphic (and anti-holomorphic) currents :

$$J^{AB}(z) = i : \psi^A \psi^B : (z) . \quad (2.162)$$

Using the above realization in terms of free fermion bilinears, they can be seen to form an $\widehat{SO}(N)_{k=1}$ current algebra with central charge :

$$c = \frac{N(N-1)/2}{1+(N-2)} = \frac{N}{2} , \quad (2.163)$$

where we used the fact that $\hat{h}_{SO(N)} = N-2$. This is precisely the correct central charge for a system of N free fermions.

For $\widehat{SO}(N)$ at level $k=1$ with $N \in 2\mathbb{Z}$, there are four unitary (integrable) representations : the vacuum representation O_N , the vectorial V_N and the spinorials S_N and C_N of opposite chirality. The conformal weights of the affine primaries defining each representation are calculated by expressing L_0 in terms of $: J_0^{AB} J_0^{AB} :$ (which, upon action on the vacuum produces the normalized quadratic Casimir C_r of the representation r) using the Sugawara construction :

$$h_r = \frac{C_r}{k + \hat{h}} . \quad (2.164)$$

Thus, the four affine primaries of $\widehat{SO}(N)$ have conformal weights :

$$\begin{aligned} h_{O_N} &= 0 , \\ h_{V_N} &= \frac{(N-1)/2}{1+(N-2)} = \frac{1}{2} , \\ h_{S_N} &= h_{C_N} = \frac{N(N-1)/16}{1+(N-2)} = \frac{N}{16} . \end{aligned} \quad (2.165)$$

The vacuum representation O_N starts with the invariant vacuum $|0\rangle$. Its first excited state is obtained by the action of $J_{-1}^{AB}|0\rangle = i\psi_{-1/2}^A\psi_{-1/2}^B|0\rangle$. Since each raising operator $J_{n>0}^{AB}$ is built out of fermion bilinears, the entire vacuum representation is characterized by even $O(N)$ fermion parity.

The lowest state (primary) in the vectorial representation V_N starts with conformal weight $(\frac{1}{2}, 0)$ and carries an $O(N)$ vector index. This identifies it uniquely as the state $\psi_{-1/2}^A|0\rangle$ built out a NS-fermion acting on the vacuum. Since this affine primary is characterized by odd fermion parity, this property will be true for all excited states within this representation.

The (conjugate-) spinorial representations S_N, C_N start with conformal weight $(\frac{N}{16}, 0)$ and carry a spinorial $SO(N)$ index. The ‘ground states’ are identified with the R vacuum, or equivalently, with the spin fields $S(z)$ and $C(z)$, respectively, acting on the invariant vacuum $|0\rangle$. In bosonized form they are expressed as :

$$\begin{aligned} S(z) &= e^{\frac{i}{2}\epsilon \cdot H(z)} , \quad \epsilon_i = \pm 1 \quad \text{and} \quad \sum_{i=1}^{N/2} \epsilon_i \in 4\mathbb{Z} - \frac{N}{2} , \\ C(z) &= e^{\frac{i}{2}\epsilon \cdot H(z)} , \quad \epsilon_i = \pm 1 \quad \text{and} \quad \sum_{i=1}^{N/2} \epsilon_i \in 4\mathbb{Z} - \frac{N}{2} + 2 , \end{aligned} \quad (2.166)$$

where we are using a chirality convention according to which the S -spinors have even and the C -spinors carry odd fermion parity.

The Kac-Moody characters for the O_N, V_N, S_N and C_N representations are defined to be the traces :

$$\chi_r(v_i) \equiv \text{Tr}_r \left[q^{L_0 - c/24} e^{2\pi i v \cdot J_0} \right] , \quad r = O_N, V_N, S_N, C_N . \quad (2.167)$$

A straightforward calculation then yields :

$$\chi_{O_N} \equiv O_N = \frac{1}{2} \sum_{b=0,1} \prod_{i=1}^{N/2} \frac{\theta[\begin{smallmatrix} 0 \\ b \end{smallmatrix}](v_i)}{\eta} , \quad (2.168)$$

$$\chi_{V_N} \equiv V_N = \frac{1}{2} \sum_{b=0,1} \prod_{i=1}^{N/2} (-)^b \frac{\theta[\begin{smallmatrix} 0 \\ b \end{smallmatrix}](v_i)}{\eta} , \quad (2.169)$$

$$\chi_{S_N} \equiv S_N = \frac{1}{2} \sum_{b=0,1} \prod_{i=1}^{N/2} e^{-i\pi b N/4} \frac{\theta[\begin{smallmatrix} 1 \\ b \end{smallmatrix}](v_i)}{\eta} , \quad (2.170)$$

$$\chi_{C_N} \equiv C_N = \frac{1}{2} \sum_{b=0,1} \prod_{i=1}^{N/2} e^{-i\pi b(1+N/4)} \frac{\theta[\begin{smallmatrix} 1 \\ b \end{smallmatrix}](v_i)}{\eta} . \quad (2.171)$$

The above relations are useful because they permit one to explicitly decompose the spectrum in terms of the characters of the underlying current algebra and, hence, expose the representation content of the theory. We can decompose the 10d Type II partition function (2.153) into representation characters of the underlying $\widehat{SO}(8)_{k=1}$ current algebra.

$$\begin{aligned} Z_{IIA} &= \frac{1}{\eta^8 \bar{\eta}^8} (V_8 - C_8) (\bar{V}_8 - \bar{S}_8) , \\ Z_{IIB} &= \frac{1}{\eta^8 \bar{\eta}^8} (V_8 - C_8) (\bar{V}_8 - \bar{C}_8) . \end{aligned} \quad (2.172)$$

The $V_8 \bar{V}_8$ sector contains the massless gravitational multiplet. The two gravitini arise from $C_8 \bar{V}_8$, together with $V_8 \bar{C}_8$ in Type IIB (or $V_8 \bar{S}_8$ in Type IIA). Finally, we have the appearance of massless R-R fields. In Type IIB they arise from the product $C_8 \bar{C}_8$, which decomposes into $[0] + [2] + [4]$. Hence, Type IIB theory contains a zero form C , a 2-form $C_{\mu\nu}$ and a 4-form potential $C_{\mu\nu\rho\sigma}$ with self-dual field strength. In Type IIA the bispinor fields $C_8 \bar{S}_8$ decompose into $[1] + [3]$, which are a vector C_μ and a 3-form $C_{\mu\nu\rho}$.

The ‘Abstrusa’ identity of Jacobi, in terms of which spacetime supersymmetry manifests itself, is translated into the following identity between $\widehat{SO}(8)$ characters :

$$V_8 - S_8 = 0 . \quad (2.173)$$

This is a result of the triality property of $SO(8)$. The isomorphism between the vectorial and spinorial representations of $SO(8)$ is the result of the enhancement of the local worldsheet $N = 1$ supersymmetry to a global $N = 2$, whose spectral flow is responsible for the mapping between vectorial and spinorial representations.

2.6 The Heterotic Superstring

In this last section, we will briefly present the construction of the 10d Heterotic string. As its very name¹⁷ implies, it is a hybrid construction, mixing together the worldsheet structures of the Type II and the bosonic strings. The starting point is to take the string worldsheet to exhibit a local $N = 1$ SCFT in the left-moving sector as in Type II, while taking the right-moving sector to be as in the bosonic string.

The local worldsheet supersymmetry is then $N = (1, 0)$, which implies that worldsheet fermions with a spacetime vector index ψ^μ (and, hence, spacetime fermions as well) can only arise from the left-moving $N = 1$ sector. This would be inconsistent because the right-moving $N = 0$ sector does not contain enough local constraints to eliminate the extra negative norm states. As in the bosonic case, the right-moving sector in the Heterotic string does not contain superghosts and, hence, the vacuum structure there is as in the bosonic case, the ground state starting at the -1 mass level (in CFT units).

The left-moving worldsheet degrees of freedom (d.o.f.) $X^\mu, \psi^\mu, b, c, \beta, \gamma$ are identical to the Type II case and, moreover, the cancelation of the left-moving conformal anomaly yields again the critical dimension $d = 10$. The bosonic d.o.f., X^μ, b, c , give a contribution $\bar{c}_{X,b,c} = -16$ and so we are forced to introduce additional right-moving matter d.o.f. in order to cancel the right-moving conformal anomaly. There are two ways to achieve this. The first is the bosonic formulation, where one introduces 16 additional internal (right-moving) scalars $X^I(\bar{z})$. Modular invariance then forces these to be compactified on an anti-chiral (i.e. Euclidean) even, self-dual 16-dimensional lattice Γ_{16} . The lowest-weight primaries are the Cartan currents $\bar{J}^I(\bar{z}) = i\bar{\partial}X^I(\bar{z})$ and the vertex operators $\bar{J}_P(\bar{z}) = e^{iP_I X^I(\bar{z})}$, which have integer conformal weight because the P_I take values in Γ_{16} . Since \bar{J}_P are charged under the Cartan subalgebra, $[\bar{J}_0^I, \bar{J}_P] = P^I \bar{J}_P$, the compactified momenta P^I are identified with the root lattice of a Lie algebra.

It turns out that there are only two such 16d Euclidean root lattices, which are even and self-dual : they are the lattices associated to $Spin(32)/\mathbb{Z}_2$ and $E_8 \times E_8$. The roots of E_8 are simply the roots of $SO(16)$, $(\pm 1, \pm 1, 0, \dots, 0)$ and all permutations, plus the weights of the spinorial representation of $SO(16)$, namely $(\pm \frac{1}{2}, \dots, \pm \frac{1}{2})$ with an even number of $(-)$ -signs. The E_8 lattice is then

$$\Gamma_{E_8} \equiv \left\{ (n_1, \dots, n_8) / \text{all } n_i \in \mathbb{Z}, \text{ or all } n_i \in \mathbb{Z} + \frac{1}{2} \text{ and } \sum_i n_i \in 2\mathbb{Z} \right\}. \quad (2.174)$$

We will use the same symbol for the lattice sum, which is the associated contribution of the compact bosons in the partition function :

$$\Gamma_{E_8} = \sum_{P^I \in E_8} q^{\frac{1}{2}P^2} = \frac{1}{2} \sum_{\gamma, \delta=0,1} \theta^8[\gamma]_\delta. \quad (2.175)$$

17. Deriving from the greek word ‘ετέρωσις’, it is a term commonly used in genetics to describe a crossbred individual showing qualities superior to those of both parents.

The representation in terms of θ -functions to the 8th power shows immediately that the lattice sum is modular invariant, as expected for an even, self-dual lattice. Taking the square of this yields the $E_8 \times E_8$ lattice sum.

The $Spin(32)/\mathbb{Z}_2$ root lattice is constructed from the (O) and (S) conjugacy classes of $SO(32)$:

$$\Gamma_{Spin(32)/\mathbb{Z}_2} \equiv \left\{ (n_1, \dots, n_{16}) / \text{all } n_i \in \mathbb{Z}, \text{ or all } n_i \in \mathbb{Z} + \frac{1}{2} \text{ and } \sum_i n_i \in 2\mathbb{Z} \right\} . \quad (2.176)$$

The corresponding lattice sum also has a θ -function representation :

$$\Gamma_{Spin(32)/\mathbb{Z}_2} = \frac{1}{2} \sum_{\gamma, \delta=0,1} \theta^{16}[\gamma_\delta] . \quad (2.177)$$

Using the ‘Abstrusa’ identity (A.8) one can show that, algebraically, the two lattices $E_8 \times E_8$ and $Spin(32)/\mathbb{Z}_2$ have equal contribution to the partition function and, hence, the masses and state multiplicities for the two theories are identical. The two theories, however, are not identical. They differ in the quantum numbers and gauge groups, hence, they have different interactions. The gauge sector is, then, organized into representations of the corresponding level $k = 1$ Kac-Moody algebra.

The above representation of the lattice sums in terms of θ -functions hints at the possibility of a fermionic formulation of the Heterotic superstring. This will be equivalent to the bosonic formulation, through bosonization. The idea is to cancel the right-moving conformal anomaly by adding 32 real right-moving fermions, $\bar{\psi}^I$, transforming under the internal global $O(32)$ symmetry¹⁸.

Taking the fermions to be real, they will have either NS or R boundary conditions, similarly to the Type II case. Again, locality of operator products requires one to impose a generalized GSO-projection. For the internal fermions of the Heterotic, the contribution to the partition function is very different than the Type II case. Here, there is no SCFT present and no superghosts, which implies a very different vacuum structure (even under the GSO current). The full partition function takes the generic form :

$$Z = \frac{1}{2} \sum_{a,b=0,1} (-)^{a+b+\mu ab} \frac{\theta^4[\frac{a}{b}]}{\eta^{12}} \frac{1}{2^N \bar{\eta}^{24}} \sum_{h_i, g_i \in \frac{2n_i}{N_i}} \frac{1}{2} \sum_{\gamma, \delta=0,1} C[\gamma_\delta, h_i] \prod_{i=1}^{16} \bar{\theta}[\gamma_\delta + h_i] . \quad (2.178)$$

Here, we assume generalized \mathbb{Z}_N -boundary conditions for the 16 (complexified) $\bar{\Psi}^I$ fermions. We take these to be ‘twisted’ from the usual NS, R ($\gamma, \delta = 0, 1$) boundary conditions by h_i, g_i . In general, consider $(h_i, g_i) = \frac{2}{N_i}(n_i, m_i)$, with $N_i, n_i, m_i \in \mathbb{N}^+$ and $0 \leq n_i, m_i \leq N_i - 1$. The coefficients $C[\gamma_\delta, h_i]$ are phases necessary for modular invariance. In general, these are highly constrained by higher-genus modular invariance and cluster decomposition. We will assume

18. In generic compactifications, this can be broken down further to the product of orthogonal subgroups, by imposing different boundary conditions to some groups of fermions.

that the spin-structures $\begin{bmatrix} a \\ b \end{bmatrix}$ are completely uncorrelated to γ, δ, h_i, g_i in order to preserve the left-moving GSO projection.

Invariance under the S modular transformation essentially implies the appearance of γ, δ, h_i, g_i in $C[\begin{smallmatrix} \gamma, h_i \\ \delta, g_i \end{smallmatrix}]$ in combinations symmetric under $\gamma \leftrightarrow \delta$ and $h_i \leftrightarrow g_i$. The structure of the phase anomaly arising from the S -modular transformation requires :

$$\frac{1}{2} \sum_{i=1}^{16} (\gamma + h_i)(\delta + g_i) \in 2\mathbb{Z} . \quad (2.179)$$

In particular, this implies :

$$\begin{aligned} \sum_{i=1}^{16} h_i &= p_1 h_1 + p_2 h_2 + \dots \in 4\mathbb{Z} , \\ \sum_{i=1}^{16} g_i &= p_1 g_1 + p_2 g_2 + \dots \in 4\mathbb{Z} , \\ \sum_{i=1}^{16} h_i g_i &= p_1 h_1 g_1 + p_2 h_2 g_2 + \dots \in 4\mathbb{Z} , \end{aligned} \quad (2.180)$$

where $p_i = 0, 1, 2, \dots$ parametrize the twist-distribution by counting the number of complex fermions twisted by $\begin{bmatrix} h_i \\ g_i \end{bmatrix}$. These give, independently, for the multiplicities :

$$\begin{aligned} p_i &\in 0 \pmod{2N_i} , \\ p_i &\in 0 \pmod{N_i^2} . \end{aligned} \quad (2.181)$$

Now, combining the above conditions it is easy to show that if N_i is odd, then :

$$p_i = 0 \pmod{2N_i^2} \geq 18 , \quad (2.182)$$

and one would need to twist more complex fermions than there are available. For $N_i = 4$, the same conditions yield $p_i \in 16\mathbb{Z}$, which corresponds to the simultaneous twist of all 16 fermions by \mathbb{Z}_4 . Any other $N_i \geq 5$ will also not satisfy these conditions. The only possible cases of interest are, then, $N_i = 2$ and $N_i = 4$.

Let us now impose invariance under the T -modular transformation. In general, there is a subtlety having to do with the ‘+2 periodicity’ properties of θ -functions (A.10), which may give rise to additional phases upon modular transformations :

$$\begin{aligned} (\gamma, \delta) &\longrightarrow (\gamma, \gamma + \delta - 1) , \\ (h_i, g_i) &\longrightarrow (h_i, h_i + g_i) , \\ \bar{\theta}_{\begin{bmatrix} \gamma + h_i \\ \delta + \gamma - 1 + h_i + g_i \end{bmatrix}} &\longrightarrow \bar{\theta}_{\begin{bmatrix} \gamma + h_i \\ \delta + g_i \end{bmatrix}} \exp \left\{ -i\pi(\gamma + h_i) \left(\Delta_{\gamma + \delta, 0} + \epsilon_{h_i + g_i - \frac{2(N_i - 1)}{N_i}} \right) \right\} . \end{aligned} \quad (2.183)$$

Δ is the usual Kronecker function and ϵ is a discrete Heaviside function defined as :

$$\epsilon_k = \begin{cases} 1 & , \text{ for } k \geq 1 \\ 0 & , \text{ otherwise} \end{cases} . \quad (2.184)$$

Taking into account all the above conditions one may show that modular invariance implies :

$$C_{[\delta-\gamma+1, g_i-h_i]}^{\gamma, h_i} = C_{[\delta, g_i]}^{\gamma, h_i} \exp \left\{ -\frac{i\pi}{4} \sum_{i=1}^{16} h_i^2 \right\} , \quad (2.185)$$

but this can be solved by picking $C_{[\delta, g_i]}^{\gamma, h_i} \sim \exp \left(\frac{i\pi}{4} \sum_i h_i g_i \right)$. The fact that C must necessarily be a phase at most bilinear in the spin-structures follows from higher-loop modular invariance and factorization [10, 11], which implies :

$$C_{[\beta_i+\gamma_i]}^{\alpha_i} = C_{[\beta_i]}^{\alpha_i} C_{[\gamma_i]}^{\alpha_i} . \quad (2.186)$$

The $N_i = 4$ case implies twisting all 16 complex fermions by \mathbb{Z}_4 boundary conditions $h, g \in \{-\frac{1}{2}, 0, \frac{1}{2}, 1\}$. The relevant phase in this case is $C_{[\delta+g]}^{\gamma+h} = e^{4\pi i h g}$. Here, because $\mathbb{Z}_2 \subset \mathbb{Z}_4$, we can set $\alpha = \gamma + h$, $\beta = \delta + g$ and sum it directly¹⁹ over \mathbb{Z}_4 :

$$\Gamma_{\mathbb{Z}_4} = \frac{1}{4} \sum_{\alpha, \beta \in \{-\frac{1}{2}, 0, \frac{1}{2}, 1\}} e^{4\pi i \alpha \beta} \bar{\theta}^{16}[\alpha] . \quad (2.187)$$

The only other remaining case is the \mathbb{Z}_2 case, $N_i = 2$, which is uses entirely NS and R boundary conditions. One option is to give the same boundary conditions to all 16 complex fermions, in which case one obtains the $Spin(32)/\mathbb{Z}_2$ -lattice and the other option is to group the fermions in two groups of 8, with each group having independent boundary conditions, in which case one recovers the $E_8 \times E_8$ lattice sum. A final option is to attempt to break further each E_8 into two groups 4 fermions, in which case a balancing phase $C \sim (-)^{hg}$ is required for modular invariance :

$$\Gamma_{SO(8) \times SO(8)} = \frac{1}{2^2} \sum_{\gamma, \delta=0,1} \sum_{h, g=0,1} (-)^{hg} \bar{\theta}^4[\gamma] \bar{\theta}^4[\delta+g] = \Gamma_{E_8} . \quad (2.188)$$

However, it is a remarkable property of E_8 that, despite naively appearing broken down into $SO(8) \times SO(8)$, is in fact regenerated²⁰. Similarly, the \mathbb{Z}_4 -case is also reduced to the above ones, since :

$$\Gamma_{Spin(32)/\mathbb{Z}_2} = (\Gamma_{E_8})^2 = \Gamma_{\mathbb{Z}_4} = (\Gamma_{SO(8) \times SO(8)})^2 . \quad (2.189)$$

There are, hence, exactly two (supersymmetric) Heterotic theories in 10 dimensions, associated precisely to the $E_8 \times E_8$ and $Spin(32)/\mathbb{Z}_2$ lattice sums, which give rise to their gauge groups ($E_8 \times E_8$ and $SO(32)$, respectively) by means of the underlying current algebras.

19. Otherwise, we would have had to divide by the \mathbb{Z}_2 -group volume-factor.

20. This is a result of the triality property of $SO(8)$.

Appart from the usual graviton, antisymmetric tensor and dilaton, the NS sector also gives rise to vector bosons, $\psi^\mu(z)\bar{J}^A(\bar{z})$, with $\bar{J}^A(\bar{z})$ being a right-moving Kac-Moody $(0,1)$ -current. These are charged under the gauge group $E_8 \times E_8$ or $SO(32)$. The R sector contains the gravitino and spin- $\frac{1}{2}$ fermion, $S_8(z)\bar{\partial}X^\mu(\bar{z})$ (both Majorana-Weyl), and additional fermions in the adjoint of the gauge group $S_8(z)\bar{\psi}^A\bar{\psi}^B(\bar{z})$.

2.7 The Type I Superstring

In this section we will very briefly describe the Type I superstring. This is a theory of open plus closed unoriented strings. Consistency of the perturbation theory of open strings necessarily requires one to include the closed string sector as well. An intuitive way to see this is to consider open 1-loop amplitudes, where an open string splits into two open strings, which propagate and then recombine back into a single open string. The process can then be reinterpreted as a tree-level amplitude for the propagation of a closed string tube in the transverse channel.

The (oriented) open vacuum amplitude of Euler number $\chi = 0$ is the annulus A_2 . Considering also unoriented strings, which will become relevant here, one has to include contributions from the Klein bottle K_2 and the Möbius strip M_2 as well. The first modification is that, for open strings, one conventionally considers half the range of the spatial local coordinate $\sigma \in [0, \pi]$. All the above surfaces can be obtained from the upper half-plane \mathbb{C}^+ , parametrized by $w = t + i\sigma$, by appropriate identifications. The annulus A_2 is constructed as the region :

$$w \sim w + 2\pi it \quad , \quad \Re(w) \in [0, \pi] \quad . \quad (2.190)$$

The Teichmüller parameter here is real, $t \in [0, \infty)$. There is no modular group here, however, there exists one Killing vector associated to translations along the $\Im(w)$ -direction.

The Klein bottle K_2 is responsible for communicating the non-orientability in the closed sector. It can be obtained from the annulus by modding out with the freely-acting involution :

$$w \sim w + 2\pi \quad , \quad (2.191)$$

$$w \sim -\bar{w} + 2\pi it \quad . \quad (2.192)$$

Again there is no modular group, but there exists a Killing symmetry of translations in the $\Im(w)$ -direction.

Finally, the Möbius strip M_2 implements the orientation projection in the open sector. It is constructed from the strip $\Re(w) \in [0, \pi]$ by the parity twist Ω :

$$w \sim \pi - \bar{w} + 2\pi it \quad . \quad (2.193)$$

In order to introduce interactions in the open sector, one notices that the two ends of the open strings can carry quantum numbers, called Chan-Paton (CP) charges. The vacuum $|p; ij\rangle$ will then contain additional CP labels i, j , taking values from 1 to some number N .

They will be simply labels, without any influence on worldsheet dynamics. One typically uses the basis of N^2 Hermitian matrices λ^a (with $a = 1, \dots, N^2$) :

$$|p, a\rangle \equiv \sum_{i,j} |p; ij\rangle \lambda_{ij}^a . \quad (2.194)$$

Since the CP charges have trivial worldsheet dynamics, the only way open strings can interact with each other is if the associated CP charges at the joining endpoints are equal. This implies, in particular, that tree-level processes will come proportional to traces of CP matrices, $\text{Tr}[\lambda^a \lambda^b \dots]$, and the amplitudes will be invariant to global $U(N)$ transformations. However, it is a common phenomenon in string theory that continuous global symmetries on the worldsheet become local (gauge) symmetries in spacetime. Here, the presence of N^2 massless vector bosons in the open spectrum already implies this fact. This is straightforward to see by studying the structure of the interactions, but we will not discuss this here. Instead, we will quickly mention the minimal logical steps needed for the Type I vacuum amplitude.

The general structure of the annulus amplitude is :

$$Z_{A_2} = \int_0^\infty \frac{dt}{2t} \text{Tr} [e^{-2\pi t(L_0 - c/24)}] . \quad (2.195)$$

The trace is over the open string Hilbert space and includes in particular the zero modes, which give rise to the 10d volume factor. We are also implicitly assuming the (GSO-projected) sum over the spin-structures. In addition, the bosons are assumed to satisfy Neumann boundary conditions, $\partial_\sigma X = 0$, at both string endpoints (NN) :

$$\text{NN} : \quad \partial X - \bar{\partial} X = 0 \quad , \quad \text{at } \arg(z) = 0, \pi . \quad (2.196)$$

The bosonic contribution is calculated by noticing that :

$$L_0 = -\frac{c}{24} + \alpha' p^2 + N , \quad (2.197)$$

where the first term, $(-\frac{c}{24})$, is the anomaly contribution in the energy-momentum tensor, arising from the transformation mapping the disk to the cylinder. The next term contains the centre-of-mass momentum p^2 of the open string. Its normalization differs from the closed string by a factor of 4, exactly because we are considering the modified range $\sigma \in [0, \pi]$. Carrying out the momentum integration and performing the trace one obtains :

$$Z_{A_2} = \frac{iV_{10}N^2}{(2\pi\sqrt{\alpha'})^{10}} \int_0^\infty \frac{dt}{2t} \frac{1}{(2t)^5} \frac{1}{\eta^{12}(it)} \frac{1}{2} \sum_{a,b=0,1} (-)^{a+b} \theta^4 \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] (it) , \quad (2.198)$$

where the N^2 -factor is the CP multiplicity coming from the trace $\text{Tr}[1] = \sum_{i,j=1}^N \langle ij|ij\rangle = N^2$.

By expanding in q -modes it is easy to see that there are N^2 -massless vectors, which implies that the gauge group is $U(N)$, as long as we are considering the theory to be orientable.

The presence of supersymmetry guarantees the absence of tachyons and, hence, the absence of IR divergences as $\tau, t \rightarrow \infty$. This is the limit where the radius of the ‘cylinder’ becomes large (effectively giving a long strip) and the propagation is dominated by the lowest mass-level. In the UV limit, $\tau, t \rightarrow 0$, however, there is a fundamental difference between the torus and the annulus. In the case of the torus, the modular group restricts the integration region to the fundamental domain \mathcal{F} , which excludes the origin $\tau \sim 0$ and, hence, protects the theory against UV divergences. This is a very delicate procedure with no analogue in field theory. In the annulus, however, there is no modular group and the integration range includes the dangerous region $t \sim 0$, giving rise to apparent UV divergences.

The mechanism that string theory uses to deal with UV divergences is fundamentally different in the open sector. It turns out that the $t \sim 0$ divergence can be regarded as an IR divergence from the point of view of closed string theory and the latter are physical. As in field theory, infrared divergences typically imply that we are asking the wrong question or that we are expanding around the wrong vacuum. By going to the transverse channel, one can consider the limit $t \sim 0$ as the limit where the length of the cylinder grows like $1/t$. One can then look at the annulus amplitude as a closed string (tree-level) tube, propagating between two open-string boundaries (disks). These are tadpoles and generally indicate that we are expanding around a vacuum that does not solve the equations of motion. The divergence as $t \rightarrow 0$ is then, naturally, reinterpreted as an IR divergence because of the massless closed string states propagating between the two tadpoles. The tadpoles in the NS sector may be thought to arise because of the following term in the effective action, which generates 1-point functions for the metric and dilaton :

$$\sim \int d^{10}x e^{-\phi} \sqrt{G} . \quad (2.199)$$

This term gives a potential for the dilaton (after rescaling) and so flat space is no longer a solution to the equations of motion. Insisting on canceling the tadpoles within purely oriented open (plus closed) string theory would necessarily imply the breaking of some of the 10d Poincaré invariance.

In the R sector the situation is actually more serious. There, the tadpoles must be due to massless R-R states of the Type IIB theory, propagating between the two disk boundaries. Since Type IIB theory only has p -form potentials C_p of even degree, a constant tadpole would only arise from a 10-form, C_{10} . Its minimal coupling of the form :

$$\sim \mu_{10} \int C_{10} , \quad (2.200)$$

would then generate the required R-sector tadpole. However, C_{10} has no kinetic term, because the field strength vanishes identically in 10d and so its dynamics is trivial. The equation of motion would then require a vanishing C_{10} -charge for the vacuum, $\mu_{10} = 0$. This is precisely the requirement of tadpole cancellation that needs to be imposed.

It is remarkable that the inclusion of the non-oriented sector (K_2, M_2) gives a natural way to cancel these IR divergences. The full unoriented amplitude is obtained by inserting

the projection operator $\frac{1}{2}(1 + \Omega)$ into the trace. Evidently, this projection onto left-right symmetric states, $\Omega = 1$, can only take place in Type IIB theory. In the closed sector this sums the torus amplitude T_2 and the Klein bottle K_2 (with appropriate $\frac{1}{2}$ -factors because of the projection). In the open sector, one has to sum the annulus A_2 together with the Möbius strip M_2 .

Let us work out the action of the worldsheet parity Ω on the closed string oscillators. From $\Omega X(z, \bar{z})\Omega^{-1} = X(\bar{z}, z)$ we obtain :

$$\Omega \alpha_n \Omega^{-1} = \bar{\alpha}_n, \Omega \bar{\alpha}_n \Omega^{-1} = \alpha_n. \quad (2.201)$$

Similarly, for the fermions, the worldsheet parity acts as $\Omega \psi(w)\Omega^{-1} = \bar{\psi}(2\pi - \bar{w}) = e^{-2\pi i \nu} \bar{\psi}(\bar{w})$, where $\nu = 0, \frac{1}{2}$ corresponds to the R or NS boundary conditions, respectively. In terms of the modes we have :

$$\begin{aligned} \Omega \psi_r \Omega^{-1} &= e^{-2\pi i r} \bar{\psi}_r, \\ \Omega \bar{\psi}_r \Omega^{-1} &= e^{2\pi i r} \psi_r, \quad r \in \mathbb{Z} + \nu. \end{aligned} \quad (2.202)$$

Since only left-right symmetric states (NS-NS and R-R) contribute to the Klein bottle amplitude, the overall phase will not matter as it can be absorbed into the transformation of the vacuum and only the overall phase in the NS-NS and R-R sectors matters. The NS-NS phase is fixed to be positive by the graviton, while the R-R has to be negative because of the relative minus sign arising from exchanging the left and right R-sector (spinorial) ground states.

The Klein bottle amplitude is then :

$$\begin{aligned} Z_{K_2} &= \int_0^\infty \frac{dt}{2t} \text{Tr} \left[\frac{1}{2} \Omega e^{-2\pi t(L_0 + \bar{L}_0 - c/24 - \bar{c}/24)} \right] \\ &= \frac{iV_{10}}{(2\pi\sqrt{\alpha'})^{10}} \int_0^\infty \frac{dt}{4t} \frac{1}{t^5 \eta^{12}(2it)} \frac{1}{2} \sum_{a,b=0,1} (-)^{a+b} \theta^4[a]_b(2it). \end{aligned} \quad (2.203)$$

For the Möbius strip we need to work out the action of worldsheet parity Ω on the open string oscillators. To this end we impose NN boundary conditions on the bosonic mode expansion :

$$X(z, \bar{z}) = x - i\alpha' p \log |z|^2 + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\alpha_n}{n} \left(\frac{1}{z^n} + \frac{1}{\bar{z}^n} \right). \quad (2.204)$$

The worldsheet parity Ω that exchanges the string endpoints, $\sigma \rightarrow \pi - \sigma$, acts on the string coordinates as $\Omega X(z, \bar{z})\Omega^{-1} = X(\bar{z}, z)$. For the modes, this becomes :

$$\Omega \alpha_n \Omega^{-1} = (-)^n \alpha_n. \quad (2.205)$$

Similarly, for the fermions, the action is $\Omega \psi(w) \Omega^{-1} = \bar{\psi}(\pi - \bar{w})$. The open string worldsheet is defined with $w \in \mathbb{C}^+$ taking values in the upper half-plane. It is very convenient to use the doubling-trick to define the value of ψ on the lower-half plane as the value of $\bar{\psi}$ at its image in the upper half-plane, $\psi(\bar{w}) = \bar{\psi}(w)$, so that ψ is extended on the whole complex plane. Using this we can write the action of the worldsheet parity on the open string fermion oscillators as :

$$\Omega \psi_r \Omega^{-1} = e^{-i\pi r} \psi_r \quad , \quad r \in \mathbb{Z} + \nu \quad , \quad (2.206)$$

where $\nu = 0, \frac{1}{2}$ corresponds to the R or NS boundary conditions, respectively. The worldsheet parity Ω exchanges the CP charges and rotates the vacuum by a unitary matrix γ :

$$\Omega |ij\rangle = \epsilon_{NS} \gamma_{ii'} \gamma_{j'j} |j'i'\rangle \quad , \quad (2.207)$$

where ϵ_{NS} is an overall phase. The straightforward way to fix ϵ_{NS} is to consider the transformation of the gauge boson states :

$$\psi_{-1/2}^\mu |p; a\rangle_{NS} = \psi_{-1/2}^\mu \sum_{i,j} |p; ij\rangle \lambda_{ij}^a \quad . \quad (2.208)$$

Using (2.206) and imposing $\Omega^2 = 1$, it is straightforward to show that γ is constrained to be either symmetric or antisymmetric :

$$\gamma^T = \lambda \gamma \quad , \quad \lambda = \pm 1 \quad . \quad (2.209)$$

Moreover,

$$\epsilon_{NS}^2 = -1 \quad . \quad (2.210)$$

The symmetric case $\lambda = +1$ restricts the number of gauge bosons to $\frac{N(N-1)}{2}$ and corresponds to an orthogonal group $SO(N)$, whereas the antisymmetric case $\lambda = -1$ contains the $\frac{N(N+1)}{2}$ gauge bosons of the symplectic group $Sp(N)$. The requirement that these survive the orientifold projection, $\Omega = +1$, then gives :

$$\lambda^a = i \epsilon_{NS} \gamma (\lambda^a)^T \gamma^{-1} \quad . \quad (2.211)$$

The λ^a -matrices have to close into a Lie algebra so that the commutator should also satisfy the above condition :

$$[\lambda^a, \lambda^b] = \epsilon_{NS}^2 \gamma [\lambda^a, \lambda^b]^T \gamma^{-1} \quad , \quad (2.212)$$

so that consistency implies the value :

$$\epsilon_{NS} = +i \quad , \quad (2.213)$$

for the NS vacuum. The Möbius amplitude is then proportional to a CP trace :

$$\sum_{i,j} \langle ij | \Omega | ij \rangle = \epsilon_{NS} \text{Tr}[\gamma^{-1} \gamma^T] = i\lambda N . \quad (2.214)$$

Because of (2.206), each fermion oscillator contributes a factor $e^{-i\pi r}$, effectively replacing the Fourier modes q^r by $e^{-i\pi r} q^r$. It is then straightforward to calculate that :

$$\begin{aligned} Z_{M_2} &= \int_0^\infty \frac{dt}{2t} \text{Tr} \left[\frac{1}{2} \Omega e^{-2\pi t(L_0 - c/24)} \right] \\ &= \frac{iV_{10}(i\lambda N)}{(2\pi\sqrt{\alpha'})^{10}} \int_0^\infty \frac{dt}{4t} \frac{1}{(2t)^5} \frac{1}{\hat{\eta}^8} \frac{1}{2} \sum_{a,b=0,1} (-)^{a+b} \frac{\hat{\theta}_b^4[a]}{\hat{\eta}^4} , \end{aligned} \quad (2.215)$$

where the hatted characters are defined such that the oscillator contributions (but not the primaries!) effectively transform as $q \rightarrow (-q)$:

$$\begin{aligned} \hat{\eta} &\equiv q^{1/24} \prod_{n=1}^\infty (1 - (-q)^n) = e^{-i\pi/24} \eta(-q) , \\ \frac{\hat{\theta}_b^4[a]}{\hat{\eta}} &\equiv e^{i\pi \frac{ab}{2}} q^{\frac{a^2}{8} - \frac{1}{24}} \prod_n \left(1 + (-q)^{n + \frac{a-1}{2}} e^{i\pi b} \right) \left(1 + (-q)^{n - \frac{a+1}{2}} e^{-i\pi b} \right) = e^{-i\pi(\frac{a^2}{8} - \frac{1}{24})} \frac{\theta_b^4[a](-q)}{\eta(-q)} . \end{aligned} \quad (2.216)$$

Using the above relations, the Abstrusa identity (A.8), as well as the product representations (A.1), (A.7), one can calculate for the NS sector :

$$\sum_{b=0,1} (-)^b \frac{\hat{\theta}_b^4[0]}{\hat{\eta}^{12}} = e^{+i\pi \frac{4}{24} + i\pi \frac{8}{24}} \frac{\theta_2^4(-q)}{\eta^{12}(-q)} = i \frac{\theta_2^4(q^2) \theta_4^4(q^2)}{\theta_3^4(q^2) \eta^{12}(q^2)} , \quad (2.217)$$

with $q = e^{-2\pi t}$. Substituting this in (2.215), one finds :

$$Z_{M_2}^{NS} = - \frac{iV_{10}(\lambda N)}{(2\pi\sqrt{\alpha'})^{10}} \int_0^\infty \frac{dt}{8t} \frac{1}{(2t)^5} \frac{\theta_2^4(2it) \theta_4^4(2it)}{\theta_3^4(2it) \eta^{12}(2it)} , \quad (2.218)$$

Notice that, summing together the annulus and Möbius amplitude and keeping the massless contribution due to the gauge bosons, we recover the multiplicity $N(N - \lambda)/2$, which is precisely the dimension of $SO(N)$ for $\lambda = +1$ and $Sp(N)$ for $\lambda = -1$, as expected.

We next rewrite the Klein bottle, annulus and Möbius strip amplitudes in the transverse channel. For this purpose we need to invert and rescale the Teichmüller parameter t such that it describes the length of the propagating cylinder with or without crosscaps at the loop boundaries. There is a subtlety concerning this rescaling, because these manipulations involve divergent quantities and, in principle, one may naively give set them to any value (modulo

the infinite factor), by arbitrary rescaling. Instead, one has to respect the combinatorics and geometric structure of string perturbation theory, which dictates the correct prescription. We will sketch how this happens below.

For the annulus, the fundamental region is a rectangle with a horizontal edge of length π and vertical edge of length $2\pi t$, respecting $w \sim w + 2\pi it$. After identifying the two horizontal edges, we obtain a cylinder of length π and circular length $2\pi t$. We can set this in the transverse channel by a Weyl rescaling by a factor of $1/t$ so that the circular region has (normalized) length 2π . The length of the resulting cylinder is $s = \pi/t$.

Now consider the Klein bottle, whose fundamental region is the rectangle with horizontal edge 2π and vertical edge $2\pi t$. This respects the double periodicity, $w \sim w + 2\pi$ and $w \sim 2\pi - \bar{w} + 2\pi it$. In this choice for the fundamental region, the two vertical edges, $\sigma^1 = 0$ and $\sigma^1 = 2\pi$, are identified, forming a cylinder. The two horizontal edges, $\sigma^2 = 0$ and $\sigma^2 = 2\pi t$, are to be identified after reflection with respect to the middle vertical line $\sigma^1 = \pi$. This produces the usual representation of the Klein bottle as a twisted non-orientable 2d Riemann surface. This is the direct channel. There is, however, an alternative representation in terms of a cylinder with two crosscaps at the two ends. It is obtained by considering, instead, a rectangle with horizontal edge π and vertical edge $4\pi t$. This is, again, a fundamental region for the above identifications. Now, the two horizontal edges are again identified, giving rise to a cylinder of circular length $4\pi t$. Each of the two vertical edges is identified with itself after a (vertical) translation by half its length. This is precisely the definition of a crosscap. Finally, we rescale by $1/2t$, so that we obtain a cylinder of length $s = \pi/2t$ and circular region 2π .

Finally, we need to consider the Möbius strip. The situation is analogous to the Klein bottle. Take as fundamental region the rectangle with horizontal edge π and vertical edge $2\pi t$, which is compatible with the identification $w \sim \pi - \bar{w} + 2\pi it$. The two vertical edges are boundaries. The horizontal edges are then identified after a reflection with the respect to the middle vertical line $\sigma^1 = \pi/2$. This is precisely the usual representation of the Möbius strip as a strip of length π , with its ends twisted and glued. There is, again, an alternative representation, where one considers the rectangle of horizontal edge $\pi/2$ and vertical edge $4\pi t$. The two horizontal edges are identified and one obtains a cylinder of circular length $4\pi t$. One of the vertical edges, $\sigma^1 = 1$, is a boundary loop, whereas the second, $\sigma^1 = \pi/2$ is identified with itself after a (vertical) translation by half its length. Thus, the second vertical edge is a crosscap. Rescaling by $1/2t$ in order to set the circular length to its standard 2π value, we obtain a cylinder of length $s = \pi/4t$ with one open boundary at one end and one crosscap at the other.

By going to the transverse channel, where the length of the closed cylinder is parametrized by the s -variables defined above, and summing all the massless (divergent) tadpole contributions, one obtains in the NS sector :

$$\frac{iV_{10}}{2^{14}\pi^{11}(\alpha')^5} \left[N^2 + 2^{10} - 2^6 \lambda N \right] \int_0^\infty ds . \quad (2.219)$$

We then see that the tadpoles cancel²¹ only for $\lambda = 1$ and $N = 32$. This is the Type I superstring. The gauge group is determined to be $SO(32)$ by the requirement of tadpole cancellation, which is somewhat analogous to the modular invariance of closed strings. We will not be discussing open strings (nor their associated D-brane constructions) any further in this manuscript. An excellent review is [12], which also contains more specialized references.

21. An analogous result is obtained for the R-sector as well. Preservation of spacetime supersymmetry then fixes the phase in the Ω -transformation of the R-vacuum.

Chapitre 3

Compactification

In order to make contact with four-dimensional spacetime, it is important to have a consistent formulation of string theory in lower spacetime dimensions. The reduction of the 10d (or 26d) (super-)string theories to lower dimensions constitutes the problem of ‘compactification’. The name is somewhat misleading, since it suggests a geometric interpretation of the ‘internal’ space in the spirit of Kaluza-Klein reduction. Contrary to the field-theoretic intuition, one should rather not take this name seriously, since all that is actually required is a consistent, unitary CFT description for the ‘internal’ space with enough units of central charge to cancel the conformal anomaly.

It turns out that the geometric picture of field-theory is only recovered in the low curvature limit (in units of α'^{-1}). The fundamentally distinct ingredient in string theory, which has no field theory analogue is the notion of winding configurations, due to the extended nature of the string. This gives rise to points in moduli space where symmetry becomes enhanced, typically when both momentum and winding states have contributions of the same order of magnitude to the conformal weight of massless states.

In this chapter we start with the simplest compactifications admitting an exact CFT description. The toroidal case supports the maximal number of supersymmetries. We next consider compactifications on orbifolds. A convenient description that permits one to work with free fields is the fermionic or, in general, Gepner constructions.

3.1 Toroidal Compactification

The simplest compactification is the toroidal one, where we take the internal space to be a d -torus, T^d . The worldsheet dynamics are then especially simple because the relevant fields X^I are free. One then has to essentially extend the result obtained for the path integral of the free non-compact scalar, to the case of d free compact scalars, X^I . The periodicity in the two torus variables σ^a now only closes modulo windings around the compact dimensions :

$$\begin{aligned} X^I(\sigma^1 + 2\pi, \sigma^2) &= X^I(\sigma^1, \sigma^2) + 2\pi n^I , \\ X^I(\sigma^1, \sigma^2 + 2\pi) &= X^I(\sigma^1, \sigma^2) + 2\pi \tilde{m}^I , \end{aligned} \tag{3.1}$$

where $n^I, \tilde{m}^I \in \mathbb{Z}$ are winding numbers. The path integral breaks into the usual integral over the quantum fluctuations (2.136) and an instantonic sum over all topologically non-trivial configurations solving the classical equations of motion (and respecting the double periodicity) :

$$X_{\text{class}}^I(\sigma^1, \sigma^2) = n^I \sigma^1 + \tilde{m}^I \sigma^2 . \quad (3.2)$$

The instantonic piece will be proportional to the exponential of the classical action. Putting everything together we obtain the (modular invariant) contribution of d -compact scalars :

$$\Gamma_{(d,d)}(G, B) = \frac{\sqrt{\det G}}{(\sqrt{\alpha' \tau_2} \eta \bar{\eta})^d} \sum_{\tilde{m}^I, n^I \in \mathbb{Z}} e^{-\frac{\pi}{\alpha' \tau_2} (G_{IJ} + B_{IJ}) (\tilde{m}^I + \tau n^I) (\tilde{m}^J + \bar{\tau} n^J)} . \quad (3.3)$$

This is in the so-called ‘Lagrangian’ representation, as it comes directly from the path integral. The Hamiltonian representation is easily obtained by Poisson-resumming (A.13) the \tilde{m}^I -winding variables :

$$\Gamma_{(d,d)} = \frac{1}{(\eta \bar{\eta})^d} \sum_{m_I, n^I \in \mathbb{Z}} q^{\frac{1}{2} P_L^2} \bar{q}^{\frac{1}{2} P_R^2} , \quad (3.4)$$

where the left- and right-moving compact momenta, $P_{L,R}$, are given by :

$$(P_{L,R})_I = \frac{\sqrt{\alpha'}}{\sqrt{2}} \left(m_I + (\alpha')^{-1} (B_{IJ} \pm G_{IJ}) n^J \right) . \quad (3.5)$$

The square product $P_{L,R}^2$ is taken with respect to the metric :

$$P_{L,R}^2 = G^{IJ} (P_{L,R})_I (P_{L,R})_J . \quad (3.6)$$

These are precisely the $U(1)$ charges of the current algebra generated by $J^I(z) = i\sqrt{2} \partial X^I$, $\bar{J}^I(\bar{z}) = i\sqrt{2} \bar{\partial} X^I$. The states participating in the lattice sum (3.4) correspond to the affine primary vertex operators :

$$V_{P_L, P_R}(z, \bar{z}) = \exp \left(i \frac{1}{\sqrt{\alpha'}} P_{L,I} X_L^I(z) + i \frac{1}{\sqrt{\alpha'}} P_{R,I} X_R^I(\bar{z}) \right) . \quad (3.7)$$

The $U(1)$ charges with respect to J^I, \bar{J}^I are normalized as $\sqrt{\frac{\alpha'}{2}} (P_{L,R})_I$. Similarly, the Sugawara construction gives the energy-momentum tensor :

$$T(z) = \frac{G_{IJ}}{2\alpha'} : J^I J^J : (z) . \quad (3.8)$$

This precisely reproduces the weights appearing in the above lattice sum (3.4).

A very interesting phenomenon arises whenever $P_L^2 = 2$ (or $P_R^2 = 2$). In that case, the associated vertex operators become have conformal weight $(1, 0)$ or $(0, 1)$ and, hence,

correspond to (anti-)chiral currents. The abelian current algebra, $U_L(1)^d \times U_R(1)^d$, becomes enhanced with some of the $U(1)$ s combining to form non-abelian Lie algebras. This is precisely the case when a sub-lattice of the compactification lattice coincides with the root lattice of a simply-laced Lie algebra. This is a phenomenon particular to string theory, with no field-theoretic analogue. The simplest examples are the bosonic enhanced symmetry point, arising in $SU(2)_{k=1}$ theories at radius $R = \sqrt{\alpha'}$, and the fermionic enhanced symmetry point, arising in $SU(2)_{k=2}$ theories at the ‘fermionic’ radius $R = \sqrt{\alpha'/2}$. The latter symmetry is typically broken by boundary conditions, except for cases in which some of the supersymmetry is spontaneously broken as well.

The moduli space of toroidally compactified theories is explicitly known. One way to obtain it is to study the space of inequivalent theories parametrized by the background parameters G_{IJ}, B_{IJ} . Start by rewriting the Hamiltonian lattice representation (3.4) as :

$$q^{\frac{1}{2}P_L^2} \bar{q}^{\frac{1}{2}P_R^2} = e^{-\pi\tau_2(P_L^2+P_R^2)} e^{2\pi i\tau_1 m_I n^I} . \quad (3.9)$$

Notice that :

$$P_L^2 - P_R^2 = 2m_I n^I , \quad (3.10)$$

is independent of the moduli parameters and, in fact, is naturally preserved by $SO(d, d)$ rotations. However, as seen by (3.9), not all of the Lorentz rotations in $SO(d, d)$ are symmetries of the theory. Indeed, only the separate (chiral) rotations $SO(d)_L$ and $SO(d)_R$, preserving P_L^2 and P_R^2 , respectively, are true symmetries. Thus, the inequivalent moduli parametrize the coset space :

$$\frac{SO(d, d)}{SO(d) \times SO(d)} . \quad (3.11)$$

The dimension of this quotient is precisely d^2 , which coincides with the total number of G_{IJ} and B_{IJ} parameters. Another way to see this, is to note that we may start from some particular point in moduli space and then start to marginally deform the CFT by inserting the marginal $(1, 1)$ Cartan operators :

$$(G_{IJ} + B_{IJ}) \partial X^I(z) \bar{\partial} X^J(\bar{z}) \quad (3.12)$$

These marginal deformations of the current-current type induce a flow of the CFT along a line of physically inequivalent theories and the structure of this deformation is exactly parametrized by the coset (3.11). The conditions for a marginally operator to be exactly marginal¹ are given in [13].

1. By exactly marginal we imply that the deformation does not spoil the conformal weight $(1, 1)$ of the perturbing operator itself. This is typically the case if the marginal operator is uncharged, as in the current-current form described here.

3.2 T -Duality

The previous construction of the moduli space (3.11) is, in fact, incomplete. In the previous section we argued that, in order to obtain the space of inequivalent toroidal compactifications, we should divide by the independent (continuous) symmetries $SO(d)_L \times SO(d)_R$. There can exist, however, additional discrete symmetries corresponding to equivalent backgrounds and these have to be gauged as well. The full moduli space will then be obtained by modding out the coset (3.11) by a discrete group of symmetries, $\Gamma \subset SO(d, d; \mathbb{R})$.

The duality group for these toroidal (Narain) compactifications is the infinite discrete group $O(d, d; \mathbb{Z})$. This can be seen as follows. We rewrite $\frac{1}{2}P_{L,R}^2$ in matrix notation as :

$$\frac{1}{2}P_{L,R}^2 = \frac{1}{4} (m, n) (\mathcal{A} \pm \mathcal{J}) \begin{pmatrix} m \\ n \end{pmatrix}, \quad (3.13)$$

where m and n are d -dimensional column vectors of momenta and windings, respectively. The $2d$ -dimensional symmetric matrix \mathcal{A} carrying the moduli dependence is defined as follows :

$$\mathcal{A} = \begin{pmatrix} G^{-1} & G^{-1}B \\ -BG^{-1} & G - BG^{-1}B \end{pmatrix}. \quad (3.14)$$

The $O(d, d)$ -invariant metric \mathcal{J} is given by :

$$\mathcal{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (3.15)$$

In particular, the difference of squares of momenta is topological and is defined through \mathcal{J} in the matrix representation :

$$\frac{1}{2} (P_L^2 - P_R^2) = m^T n = (m, n) \mathcal{J} \begin{pmatrix} m \\ n \end{pmatrix}. \quad (3.16)$$

For a duality transformation to be a symmetry of the spectrum, (3.9), it must necessarily preserve the above expression. Hence, the (integer-valued) transformation matrix Ω acting on the momenta and windings,

$$\begin{pmatrix} m \\ n \end{pmatrix} \rightarrow \Omega \begin{pmatrix} m \\ n \end{pmatrix}, \quad (3.17)$$

must necessarily preserve the $O(d, d)$ -invariant metric \mathcal{J} :

$$\Omega^T \mathcal{J} \Omega = \mathcal{J}. \quad (3.18)$$

Thus, $\Omega \in O(d, d; \mathbb{Z})$. The background parameters then transform as :

$$\mathcal{A} \rightarrow \Omega \mathcal{A} \Omega^T, \quad (3.19)$$

in order to preserve the masses, $\frac{1}{2}(P_L^2 + P_R^2)$, invariant. Thus, the T -duality group of toroidal Type II compactifications is identified with $O(d, d; \mathbb{Z})$. The moduli space (disregarding the dilaton factor) associated to compactification on a T^d is then :

$$\frac{SO(d, d; \mathbb{R})}{SO(d; \mathbb{R}) \times SO(d; \mathbb{R}) \times O(d, d; \mathbb{Z})} . \quad (3.20)$$

This, in particular, includes rotations of the G, B moduli, integer shifts in B , as well as inversions generalizing the $R \rightarrow \alpha'/R$ duality of the simple S^1 compactification. The latter corresponds to the exchange of momenta and windings, $\Omega = \mathcal{J}$, together with the transformation $\mathcal{A} \rightarrow \mathcal{J}\mathcal{A}\mathcal{J} = \mathcal{A}^{-1}$. The latter becomes most apparent if one adopts a vielbein decomposition, $G_{IJ} = e_I^a e_J^a$, however we will not consider this here.

In Heterotic theories, the lattice becomes extended in order to include the gauge sector as well. The T -duality group is then $O(d, d + 16; \mathbb{Z})$, and the associated moduli space then becomes :

$$\frac{O(d, d + 16; \mathbb{R})}{O(d; \mathbb{R}) \times O(d + 16; \mathbb{R}) \times O(d, d + 16; \mathbb{Z})} . \quad (3.21)$$

As we mentioned before, points of enhanced symmetry correspond to points in moduli space where extra states become massless. This typically corresponds to an enhancement of the worldsheet current algebra, with the addition of extra currents, charged under the $U(1)$ s. There is a close connection between T -duality and enhanced symmetry. In fact, points of enhanced symmetry typically correspond to fixed points under the T -duality. As one deforms away from an enhanced symmetry point, some of the enhanced non-abelian symmetry gets broken down to $U(1)$ -factors and the Weyl group of the root lattice of the enhanced algebra becomes the T -duality group. The enhanced symmetry points are then associated to points that are T -self-dual. In this sense, T -duality is precisely the discrete remnant of enhanced gauge symmetry. This is an inherently stringy duality, valid for the interactions to all orders in perturbation theory, with no analogue in field theory (due to the absence of windings).

3.3 Orbifold Compactification

Toroidal compactifications support the maximal amount of unbroken supersymmetries. For example, in Heterotic toroidal compactifications in 4d, one has $\mathcal{N}_4 = 4$ supersymmetries, while in the Type II case, one obtains $\mathcal{N}_4 = 8$ supersymmetries (with four arising from the left-moving and another four from the right-moving side). For phenomenological purposes, one would like to reduce this number.

A particularly convenient way [14, 15] to achieve this are compactifications on singular surfaces, which are flat almost everywhere, except for localized curvature at their fixed points. These are obtained by taking the quotient \mathcal{M}/G of a manifold \mathcal{M} (e.g. a torus) by a discrete group G of isometries. If G is freely acting (no fixed points), then the quotient is still a smooth manifold, whereas if the orbifold action is non-freely-acting, there are conical singularities at the fixed points.

The simplest example encountered in string theory are toroidal orbifolds, where \mathcal{M} is taken to be a toroidal manifold. Orbifold compactifications have the appealing feature that their corresponding CFTs are exactly solvable, while at the same time, retaining some of the features of general compactifications on curved manifolds. Even though, generically, \mathcal{M}/G are singular spaces, their CFT description and the associated string theory is well-defined everywhere.

Here we will only write down the orbifold action for the bosons. The action on their fermionic superpartners is identical, as dictated by the preservation of the local $N = 1$ worldsheet SCFT, which is necessary for the elimination of ghost states and should remain unbroken at all times. The simplest example of orbifold action is a rotation and lattice translation :

$$X \sim \theta X + v , \quad (3.22)$$

where $v \in \Gamma$ belongs to the toroidal lattice Γ . For this to be well-defined on the torus, θ must be an automorphism of the lattice, $\theta v \in \Gamma$, for every $v \in \Gamma$. At the same time, the rotation matrix θ must also be an isometry (i.e. it must leave the metric invariant) :

$$(\theta e^a)_I (\theta e^a)_J = e_I^a e_J^a = G_{IJ} . \quad (3.23)$$

The group P of these isometries, $\theta \in P$, is called the point group of the orbifold. Gauging the discrete symmetry G of the orbifold group implies projecting down to the Hilbert space of states invariant under the orbifold action (3.22). However, the identification (3.22) means that there are additional ways in which the string boundary conditions may be satisfied. For example :

$$X(\sigma + 2\pi, t) = \theta X(\sigma, t) + v . \quad (3.24)$$

The case $\theta = \mathbf{1}$ (identity element in G) corresponds to the so-called ‘untwisted sector’, whereas the non-trivial elements $\theta \neq \mathbf{1}$ introduce new ‘twisted’ sectors, generating additional twisted states that were not present in the original toroidal theory.

One may see that the twisted sectors are, in fact, required for modular invariance. The partition function in the orbifold theory will be obtained by summing over all twisted sectors satisfying (3.24) with $\theta \equiv h \in G$. At the same time, we are to insert the projector :

$$\frac{1}{|G|} \sum_{g \in G} g , \quad (3.25)$$

eliminating non-invariant states under the G -action. Here, $|G|$ denotes the order of the orbifold group. One may then consider h as a twisting in the spatial variable and g as a twisting in the time variable of the torus. The partition function is then written as :

$$\frac{1}{|G|} \sum_{h, g \in G} Z_g^{[h]} = \frac{1}{|G|} \sum_{h, g \in G} \text{Tr}_{(h)} \left[g q^{L_0^{(h)} - c/24} \bar{q}^{\bar{L}_0^{(h)} - \bar{c}/24} \right] . \quad (3.26)$$

Performing T, S -modular transformations can be seen to take $Z[\frac{h}{g}]$ into $Z[\frac{h}{hg}]$ and $Z[\frac{g}{h-1}]$, respectively. Hence, the summation over all $h, g \in G$ is required in order to form a modular invariant partition function.

Consider now a compactification on T^6 , relevant for our four-dimensional spacetime. The $SO(8)$ -little group is then reduced into the product, $SO(2) \times SO(6)$, of the 4d helicity group, $SO(2)$, and the internal R -symmetry group of $\mathcal{N}_4 = 4$ supersymmetry, $R = SO(6)$. This can be seen immediately from the vertex operators of the gravitini, which transform under the spinorial representation, $\mathbf{4} \in SO(6)$. Since $\mathcal{N}_4 = 4$ multiplets can be classified into representations of the R -symmetry group, in order to break the $\mathcal{N}_4 = 4$ supersymmetry, one is lead to break the $SO(6)$ R -symmetry.

What discrete symmetries of the torus can be used to define an orbifold? There are translations, $SO(6)$ -rotations and gauge transformations in the Heterotic case. Out of these only the rotations act on the gravitini and, hence, on the $(0, 1)$ -currents generating the affine algebra of spacetime supersymmetry. In terms of the scalars $H^a(z)$ arising from the bosonization of ψ^I , (2.95), (2.96), the S_8 -spinor part is decomposed into the helicity and R -symmetry part as :

$$S_8(z) = \exp \left[+\frac{i}{2} H^1(z) \right] \exp \left[\pm \frac{i}{2} H^2(z) \pm \frac{i}{2} H^3(z) \pm \frac{i}{2} H^4(z) \right] , \quad (3.27)$$

where, conventionally, we assume a total even number of $(-)$ signs and choose to display only the positive helicity components. The $SO(6)$ -rotations in the complex basis of fermions (bosons) :

$$\Psi^A \rightarrow e^{2\pi i \phi^A} \Psi^A , \quad (3.28)$$

are represented in the bosonization basis as shifts :

$$H^A(z) \rightarrow H^A(z) + 2\pi \phi^A , \quad (3.29)$$

where $A = 2, 3, 4$. We do not allow $\phi^1 \neq 0$, because a non-trivial action on (X^μ, ψ^μ) would break the spacetime Lorentz symmetry. The condition that at least one supersymmetry current, say $\exp(\pm \frac{i}{2} \epsilon_a H^A(z))$ with $\epsilon_a = (+, +, +, +)$, survives the orbifold projection implies the condition :

$$\phi^2 + \phi^3 + \phi^4 = 0 \pmod{2} . \quad (3.30)$$

Under the G -action above, the spinorial $\mathbf{4}$ representation of $SO(6) \simeq SU(4)$ is decomposed into an invariant state $(+, +, +)$ plus the 3 non-invariant states $(+, -, -)$, $(-, +, -)$, $(-, -, +)$ which transform into each other forming a representation $\mathbf{3}$ of an $SU(3) \subset SU(4)$. Therefore, a necessary condition for the preservation of $\mathcal{N}_4 = 1$ supersymmetry is that the point group P is a subgroup of $SU(3)$ so that there exists a covariantly constant spinor on the orbifold.

This requirement, together with the requirement that P acts as an automorphism of the lattice is only possible [14, 15] for abelian orbifolds of the type \mathbb{Z}_N or $\mathbb{Z}_N \times \mathbb{Z}_M$. The cases that we will be considering are \mathbb{Z}_2 -type orbifolds.

Let us consider a \mathbb{Z}_2 -twist, $X \sim -X$, on a compact coordinate $X \in S^1$. In terms of the primary fields, $e^{iP_L X_L + iP_R X_R}$, the non-trivial orbifold element $g \in \mathbb{Z}_2$ reverses the sign of the momentum and winding quantum numbers, $P_{L,R} \rightarrow -P_{L,R}$, as well as provides a negative sign for every oscillator excitation :

$$g \cdot \mathcal{V}_{(m,n)}(z, \bar{z}) = (-)^{N+\bar{N}} \mathcal{V}_{(-m,-n)}(z, \bar{z}) , \quad (3.31)$$

with N, \bar{N} counting the left- and right-moving oscillators. The g -insertion in the trace receives contributions only from pure oscillator states (i.e. $m = n = 0$) and, in addition, oscillator contributions q^n effectively change sign.

In the twisted sector $h = 1$, one has to impose the twisted boundary condition $X(\sigma + 2\pi, t) = -X(\sigma, t)$. This leads to vanishing momentum zero modes $P_{L,R} = 0$ and half-integrally moded oscillators :

$$\partial X(z) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} \frac{\alpha_r}{z^{r+1}} . \quad (3.32)$$

The position zero mode x is now forced to have small fluctuations about the fixed points $x = 0, \pi R$. The ground states, $|\Sigma, 0\rangle$, $|\Sigma, \pi R\rangle$, are defined as the states annihilating all lowering operators :

$$\alpha_{n+1/2} |\Sigma\rangle = 0 \quad , \quad n \geq 0 . \quad (3.33)$$

Assume that $|\Sigma\rangle$ is generated from the $SL(2, \mathbb{C})$ -invariant vacuum, $|0\rangle$, by the action of a local operator $\Sigma(z)$ called a twist-field. Then, (3.33) implies the OPE :

$$\partial X(z) \Sigma(w, \bar{w}) = \frac{1}{(z-w)^{1/2}} \tau(w, \bar{w}) + \dots \quad (3.34)$$

Here, $\tau(w, \bar{w})$ is an excited twist-field. In general, [16], the twist-field operators Σ, τ have no simple free-field representation². However, for CFTs with holomorphic factorization, such as the free-fermionic theories or more general Gepner constructions, one may chirally represent the twist-fields in terms of fermion or parafermion blocks.

There are several ways to obtain the conformal weight, Δ , of the twist-field ground state $\Sigma(z)$. The orthodox way would be to use the mode expansion and oscillator algebra to calculate $\langle \Sigma | T(z) | \Sigma \rangle = \Delta / z^2$, similarly to a fermion in the R -sector. An alternative way would be to fermionize X and define the orbifold entirely in terms of a twist in the boundary conditions of one of the two associated fermions. Hence, one of the two fermions would have NS and the other R boundary conditions and, so, the ground state is necessarily $\Delta = 1/16$. This will be described more in the next section.

Finally, we describe here is a third way, which introduces a very interesting trick. As we have seen, the twisted sector does not depend on the radius R of the circle, since all fluctuations take place around the fixed points. This is a generic feature of non-freely-acting

2. Their correlation functions can still be derived by various conformal techniques. See, for example [19].

orbifolds. We can exploit this fact, in order to go to the ‘bosonic’ self-dual radius $R = \sqrt{2}$, where for convenience we take $\alpha' = 2$. This is the $SU(2)_{k=1}$ extended symmetry point mentioned in the previous section, characterized with the enhancement of the $U(1)$ current algebra of $J(z) = i\partial X(z)$:

$$J(z) = i\partial X(z) \quad , \quad J^\pm(z) = e^{\pm i\sqrt{2}X(z)} . \quad (3.35)$$

At the self-dual point, we are free to perform an $SU(2)$ rotation and choose instead :

$$J(z) = \sqrt{2} \sin(\sqrt{2}X(z)) . \quad (3.36)$$

This has the same transformation under the \mathbb{Z}_2 action as the current $i\partial X(z)$. It also has the same conformal weight and the correct normalization in order to properly reproduce the same JJ -OPE. It is now easy to see that :

$$\Sigma(z) = \cos\left(\frac{1}{2\sqrt{2}}X(z)\right) , \quad (3.37)$$

survives the orbifold projection and reproduces the desired OPE (3.34) with the current :

$$J(z)\Sigma(w) = \frac{1}{(z-w)^{1/2}} \tau(w) + \dots , \quad (3.38)$$

with :

$$\tau(z) = \frac{1}{\sqrt{2}} \sin\left(\frac{3}{2\sqrt{2}}X(z)\right) . \quad (3.39)$$

One may, then, easily read the conformal weight, $\Delta = 1/16$, of $\Sigma(z)$ by using the free-field representation (3.37). Of course, this trick relies on the fact that we can go to the special point $R = \sqrt{2}$, where the ‘basis rotation’ (3.36) can be performed and represent the twist-field operator there. We cannot, however, use the simple representation (3.37) in order to calculate correlation functions involving the twist-fields and untwisted vertex operators at different radii.

Putting everything together, we calculate the contribution to the partition function :

$$\begin{aligned} Z[g] &\equiv \text{Tr}_{(h)} \left[e^{i\pi g(N+\bar{N})} q^{L_0^{(h)}-1/24} \bar{q}^{\bar{L}_0^{(h)}-1/24} \right] = \frac{(q\bar{q})^{h^2/16-1/24}}{\prod_n (1 - q^{n-h/2} e^{i\pi g}) (1 - \bar{q}^{n-h/2} e^{-i\pi g})} \\ &= 2 \left(\frac{\eta \bar{\eta}}{\theta_{[1-h]}^{[1-h]} \bar{\theta}_{[1-g]}^{[1-h]}} \right)^{1/2} , \end{aligned} \quad (3.40)$$

with $(h, g) \neq (0, 0)$. The multiplicity 2 is precisely equal to the number of fixed points. The total partition function is obtained after summing over the twisted and untwisted contributions :

$$\frac{1}{2} \sum_{h,g=0,1} Z[g] = \frac{1}{2} \Gamma_{(1,1)}(R) + \sum_{(h,g) \neq (0,0)} \left| \frac{\eta}{\theta_{[1-h]}^{[1-h]}} \right| , \quad (3.41)$$

where $\Gamma_{(1,1)}(R)$ is the untwisted $(1,1)$ -lattice, dependent on the radius R of the circle.

In most subsequent discussions in this manuscript, we will use the word ‘twist’ to refer to non-freely-acting rotation orbifolds. Similarly, the word ‘shift’ will be employed to refer to freely-acting translations. This reflects the fact that freely-acting orbifolds do not modify the moduli space and, hence, they can be re-formulated as shifts in the background parameters.

For this simple orbifold twisting the coordinate on the S^1 , the resulting space is that of a line of length πR , with the fixed points $x = 0, \pi R$ having turned into boundaries. In higher-dimensional cases, there are additional constraints arising from the requirement that the orbifold action is an automorphism of the original lattice. For example, an orbifold can only twist one of the coordinates of a T^2 , if the torus is rectangular. This is reflected in the fact that the orbifold would then project out the off-diagonal moduli, G_{12}, B_{12} , which could otherwise have been used to ‘twist’³ the torus and violate the automorphism condition.

Another example is a \mathbb{Z}_2 orbifold exchanging the two coordinates, $X^1 \leftrightarrow X^2$, of a 2-torus. Going to ‘zweibein’ coordinates $X^a \equiv e_I^a X^I$:

$$\begin{aligned} X^{a=1} &= R_1 X^1 , \\ X^{a=2} &= R_2 (\hat{G} X^1 + X^2) , \end{aligned} \tag{3.42}$$

and demanding that this acts as an automorphism on the toroidal lattice leads to the condition $R_1^2 = R_2^2(1 - \hat{G}^2)$. The same condition may be obtained from the CFT point of view by considering the transformation properties of marginal operators, $G_{IJ} \partial X^I \bar{\partial} X^J$. Define the new coordinates, $X^\pm \equiv \frac{1}{\sqrt{2}}(X^1 \pm X^2)$, on which the \mathbb{Z}_2 acts diagonally :

$$X^\pm \rightarrow \pm X^\pm . \tag{3.43}$$

In the new basis the only invariant marginal operators are :

$$G_{++} \partial X^+ \bar{\partial} X^+ + G_{--} \partial X^- \bar{\partial} X^- , \tag{3.44}$$

and we can use these to arbitrarily deform the theory. The torus is then rectangular in the X^\pm coordinates. The condition $G_{+-} = G_{-+} = 0$ then exactly reproduces the above constraint between R_1, R_2 and \hat{G} .

This CFT approach of setting-up the orbifold at a particular point in moduli space and then using the remaining (unprojected) moduli to deform away and cover the truncated moduli space of the theory is exactly what will be used in the following section. There, orbifold theories can be constructed at the fermionic CFT points and can subsequently be deformed to yield the theory at the arbitrary point in the moduli space.

Let us next consider the generic \mathbb{Z}_2 orbifold action on the worldsheet fermions ψ^I . This is determined by the preservation of the local $N = 1$ SCFT algebra. The requirement that the worldsheet supercurrent $T_F(z)$ is globally well-defined⁴ dictates that the fermions must

3. We use the term ‘twist’ here not in the orbifold but in the sense of rotating the endpoints of a cylinder before gluing them together.

4. The contribution of $\psi^\mu \partial X_\mu$ to the worldsheet supercurrent, on which the orbifold acts trivially, necessitates this.

transform as :

$$\psi \rightarrow \theta^{-1,T} \psi . \quad (3.45)$$

We adopt the complex basis of (2.94) for the internal fermions Ψ^I , and we assume the internal bosons X^I are complexified in the same way. For the simple \mathbb{Z}_2 case, $\theta = -1$ and the boundary conditions of the complex fermions around the two non-trivial cycles of the torus are twisted as in (3.28) by $h, g \in \{0, 1\}$:

$$\Psi^A(\sigma^1 + 2\pi, \sigma^2) = e^{\pi i(1-a-h)} \Psi^A(\sigma^1, \sigma^2) , \quad (3.46)$$

$$\Psi^A(\sigma^1, \sigma^2 + 2\pi) = e^{\pi i(1-b-g)} \Psi^A(\sigma^1, \sigma^2) , \quad (3.47)$$

and the contribution of a complex worldsheet fermion Ψ^I with twisted boundary conditions $\begin{bmatrix} h \\ g \end{bmatrix}$ is :

$$\text{Tr}_{a,(h)} \left[e^{i\pi(b+g)\mathcal{F}} q^{L_0^{(h)}[a]-1/24} \right] = \frac{\theta \begin{bmatrix} a+h \\ b+g \end{bmatrix}}{\eta} . \quad (3.48)$$

Constraints from modular invariance severely restrict the choice of boundary conditions. We illustrate this in the case of a Heterotic \mathbb{Z}_2^N -orbifold where, for simplicity, we take the gauge group to be initially $SO(32)$. The (relevant part of the) partition function takes the generic form :

$$Z \sim \frac{1}{\eta^{12}\bar{\eta}^{24}} \frac{1}{2^N} \sum_{h_i, g_i} \frac{1}{2^2} \sum_{a, b, \gamma, \delta} (-)^{a+b} C \cdot \theta \begin{bmatrix} a \\ b \end{bmatrix} \prod_{i=1}^3 \theta \begin{bmatrix} a+h_i \\ b+g_i \end{bmatrix} \cdot \Gamma_{(6,6)} \begin{bmatrix} h_i \\ g_i \end{bmatrix} \cdot \prod_{I=1}^{32} \bar{\theta} \begin{bmatrix} \gamma+h_I \\ \delta+g_I \end{bmatrix} , \quad (3.49)$$

where, for simplicity, we group together with the same boundary conditions, $\begin{bmatrix} \gamma \\ \delta \end{bmatrix}$, all 16 complex gauge fermions. Other choices of boundary conditions, including the $E_8 \times E_8$ string can be effectively obtained by breaking the $O(32)$ worldsheet symmetry by appropriately twisting the boundary conditions of the gauge fermions. The phases C may contain (real) modular invariant phases in terms of the boundary conditions $(a, b, \gamma, \delta, h, g)$, as well as possible balancing phases needed for the restoration of modular invariance.

First perform an S -modular transformation, under which one picks up an additional phase :

$$\exp \left[\frac{i\pi}{2} \sum_i (a + h_i)(b + g_i) - \frac{i\pi}{2} \sum_I (\gamma + h_I)(\delta + g_I) \right] \quad (3.50)$$

which has to be absorbed into a transformation of C , if the partition function is to be modular invariant. However, due to the symmetric appearance of the boundary conditions indices in (3.50), the additional phase is itself modular invariant and, hence, cannot be completely absorbed in the transformation of C . For example, the best one can do is to try picking :

$$C \sim \exp \left[\frac{i\pi}{2} \sum_i a g_i - \frac{i\pi}{2} \sum_I \gamma g_I \right] . \quad (3.51)$$

Under the S -transform this produces the additional (balancing) phase :

$$\exp \left[-\frac{i\pi}{2} \sum_i (ag_i - bh_i) + \frac{i\pi}{2} \sum_I (\gamma g_I - \delta h_I) \right] . \quad (3.52)$$

Clearly, this is not sufficient to absorb the full phase (3.50). The parts of the phase containing hg -products cannot be absorbed into the transformation of C and, hence we must have :

$$\sum_i h_i g_i - \sum_I h_I g_I = 0 \pmod{4} . \quad (3.53)$$

When this requirement is satisfied, the C -transformation can absorb the remaining parts of the phase (3.50), provided that they are integer so that the sign difference in (3.52) does not matter :

$$\begin{aligned} \sum_i h_i &= 0 \pmod{2} , & \sum_i g_i &= 0 \pmod{2} , \\ \sum_I h_I &= 0 \pmod{2} , & \sum_I g_I &= 0 \pmod{2} . \end{aligned} \quad (3.54)$$

More generally, we could have considered a linear combination of twists h_A , for each complex fermion. The analogous expressions can be obtained simply by setting :

$$h_A \rightarrow N_A^\alpha h_\alpha , \quad N_A^\alpha \in \{0, \pm 1\} , \quad (3.55)$$

where A runs generically over all fermion slots $A \in \{i, I\}$, while α labels the generic twists in \mathbb{Z}_2^N . Similarly, from the T -modular transform one has :

$$\sum_i h_i^2 - \sum_I h_I^2 = 0 \pmod{4} , \quad (3.56)$$

in agreement with the above conditions. In all, we can summarize these as :

$$\begin{aligned} n_L(h_\alpha) - n_R(h_\alpha) &= 0 \pmod{4} , \\ n_L(h_\alpha; h_\beta) - n_R(h_\alpha; h_\beta) &= 0 \pmod{2} , \\ n_{L,R}(h_\alpha) &= 0 \pmod{2} . \end{aligned} \quad (3.57)$$

The first condition implies that the number of left-moving fermions twisted by h_α must differ from the analogous number of right-moving fermions twisted by h_α by a multiple of 4. The second condition expresses the fact that the number of left- minus the number of right-moving fermions twisted by both h_α and h_β is even. The third condition can be seen to derive from the second by replacing h_β with a or γ , respectively. For convenience, we consider only cases where all fermions can be complexified and we will not treat Ising-like situations. The same conditions will be encountered in the following section, arising through the framework of the fermionic construction.

Finally, we mention a few words about Heterotic orbifold models in four spacetime dimensions. The simplest compactification is on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$, with the standard action :

$$\begin{aligned}\theta_1 & : (\psi, X)^{4,5,8,9} \rightarrow -(\psi, X)^{4,5,8,9} , \\ \theta_2 & : (\psi, X)^{6,7,8,9} \rightarrow -(\psi, X)^{6,7,8,9} ,\end{aligned}\tag{3.58}$$

and the so-called ‘standard-embedding’ in the gauge sector, breaking⁵ $E_8 \rightarrow SO(10) \times U(1)^3$:

$$\begin{aligned}\theta_1 & : \bar{\psi}^{11,12,15,16} \rightarrow -\bar{\psi}^{11,12,15,16} , \\ \theta_2 & : \bar{\psi}^{13,14,15,16} \rightarrow -\bar{\psi}^{13,14,15,16} ,\end{aligned}\tag{3.59}$$

where we only write the non-trivial transformations on the coordinates. This orbifold preserves exactly one massless gravitino and so we have $\mathcal{N}_4 = 1$ unbroken supersymmetry. In a sense, the ‘choice’ $\mathbb{Z}_2 \times \mathbb{Z}_2$ is synonymous with having $SO(10)$ Grand Unification. The moduli space of current-current deformations is truncated down to :

$$\frac{SO(6, 6+16)}{SO(6) \times SO(6+16)} \longrightarrow \frac{SO(2, 2)}{SO(2) \times SO(2)} \times \frac{SO(2, 2)}{SO(2) \times SO(2)} \times \frac{SO(2, 2)}{SO(2) \times SO(2)} .\tag{3.60}$$

In particular, no continuous Wilson line moduli are left unprojected and the only possibility are the so-called discrete Wilson lines, which are discrete remnants of the Wilson line operators of the $\mathcal{N}_4 = 4$ level. The partition function (ommiting the trace over the zero-modes) is then :

$$\begin{aligned}Z &= \frac{1}{\eta^{12} \bar{\eta}^{24}} \frac{1}{2^5} \sum_{h_i, g_i} \sum_{a, b} \sum_{k, \ell} \sum_{\rho, \sigma} (-)^{a+b+\Phi} \theta_{[b]}^{[a]} \theta_{[b+g_1]}^{[a+h_1]} \theta_{[b+g_2]}^{[a+h_2]} \theta_{[b-g_1-g_2]}^{[a-h_1-h_2]} \\ &\times \Gamma_{(2,2)}^{[h_1]}[g_1] \Gamma_{(2,2)}^{[h_2]}[g_2] \Gamma_{(2,2)}^{[h_1+h_2]}[g_1+g_2] \cdot \bar{\theta}^{[k]}[\ell] \bar{\theta}^{[k+h_1]}[\ell+g_1] \bar{\theta}^{[k+h_2]}[\ell+g_2] \bar{\theta}^{[k-h_1-h_2]}[\ell-g_1-g_2] \cdot \bar{\theta}^8[\rho]_{[\sigma]} ,\end{aligned}\tag{3.61}$$

where Φ is a modular invariant phase in which one may include possible discrete torsions. The notion of discrete torsion reflects an arbitrariness in the relative phases (projections) of twisted sectors, which is undetermined by modular transformations. Indeed, consider the $\mathbb{Z}_2 \times \mathbb{Z}_2$ case. One then typically starts from the untwisted (unprojected) partition function, acts on it with the non-trivial elements of the full orbifold group and then completes the modular orbit by modular T, S -transformations. There is, however, a second (disconnected) orbit, which can be obtained from an element of the first by acting upon it with a non-trivial orbifold group element. The remaining elements in the second orbit are then obtained by the repeated action of the T, S -transformations. In this case the discrete torsion is a sign, $\epsilon = \pm 1$, which can be incorporated into the formalism by turning on or not the modular invariant cocycle $(-)^{h_1 g_2 - g_1 h_2}$:

$$Z = \frac{1}{2^2} \sum_{h_i, g_i} (-)^{\frac{1-\epsilon}{2} (h_1 g_2 - g_1 h_2)} Z_{[g_1, g_2]}^{[h_1, h_2]} .\tag{3.62}$$

5. Of course, enhancements may occur at special points in moduli space. For example, $SO(10)$ may be enhanced to E_6 .

Finally, let us briefly mention the possibility of asymmetric orbifolds [17], [18]. These are constructions where the orbifold acts differently on the left- and right-movers. They are only possible to construct at extended symmetry points, where the additional symmetries of the lattice permit one to independently rotate the left- from the right-movers. These are typically points where the CFT factorizes, as is the case in fermionic or, more generally, in Gepner models.

3.4 Tensor Products of Free CFTs : Fermionic Models

In this section we will briefly discuss a simple example of string constructions based on exactly solvable CFTs, in terms of free worldsheet fermions only. The key idea behind these ‘fermionic constructions’ is that all (internal) worldsheet degrees of freedom can be consistently fermionized at special points of moduli space. At these special points, the general constraints of ($g \geq 1$) modular invariance can be completely solved, [26]. This permits one to classify a very large number of theories which, in many cases, overlap with orbifold constructions. In fact, fermionic models are in some cases more general than (symmetric) orbifolds since they can naturally realize non-geometric constructions, such as the asymmetric orbifolds mentioned in the previous section.

The advantage of fermionic constructions lies in their uniform representation of worldsheet degrees of freedom in terms of free fermions which, in turn, permits easier algebraic control of their modular properties. Such models can be, subsequently, marginally deformed away from the fermionic points and cover the moduli space of the theory. In this sense, the fermionic models become useful starting points for the construction of phenomenological models and for the exploration of stringy effects.

As discussed in previous sections, consistent superstring constructions must realize a local $N = 1$ SCFT algebra on the string worldsheet, because this introduces precisely the constraints needed to eliminate unphysical states. This requirement, essentially, translates into the consistent definition of the worldsheet supercurrent $T_F(z)$.

Assuming that a complete fermionization is possible, start with a theory of N free-fermions enjoying an $O(N)$ -global (classification) symmetry on the worldsheet. Since we are fermionizing all worldsheet degrees of freedom, we must ensure that $T_F(z)$ can be realized in terms of free fermion fields or, equivalently, that local worldsheet supersymmetry can be realized (non-linearly) in terms of free fermions alone, first observed in [22] :

$$\delta\psi^A(z) = \epsilon\lambda f^{ABC} : \psi^B\psi^C : (z) . \quad (3.63)$$

The Nöther procedure then leads to the fermionized supercurrent :

$$T_F(z) = \frac{1}{3} i\lambda f^{ABC} : \psi^A\psi^B\psi^C : (z) , \quad (3.64)$$

which is a conformal field of weight $(\frac{3}{2}, 0)$. Here λ is a normalization parameter. Using the

explicit form (3.64), one finds :

$$\begin{aligned} T_F(z)T_F(w) = & \frac{2}{3} \frac{\lambda^2 f^{ABC} f^{ABC}}{(z-w)^3} + 2\lambda^2 f^{ACD} f^{BCD} \left(\frac{\psi^A \psi^B(w)}{(z-w)^2} - \frac{\psi^A \partial \psi^B(w)}{z-w} \right) \\ & + \lambda^2 f^{ABE} f^{CDE} \frac{\psi^A \psi^B \psi^C \psi^D}{z-w} + \dots \end{aligned} \quad (3.65)$$

Therefore, the definition (3.64) of $T_F(z)$ only closes into an $N = 1$ SCFT algebra :

$$T_F(z)T_F(w) = \frac{3c/2}{(z-w)^3} + \frac{2}{z-w} T(z) + \dots , \quad (3.66)$$

with $c = \frac{N}{2}$ if and only if :

$$\begin{aligned} f^{ABE} f^{CDE} + f^{ACE} f^{DBE} + f^{ADE} f^{BCE} &= 0 , \\ 2\lambda^2 f^{ACD} f^{BCD} &= \delta^{AB} . \end{aligned} \quad (3.67)$$

The first condition is a Jacobi identity, implying that f^{ABC} are the structure constants of a Lie algebra G , whereas the second imposes that G is semi-simple. The second condition also fixes the normalization :

$$\lambda = \frac{1}{2\sqrt{\hat{h}}}, \quad (3.68)$$

with \hat{h} being the dual Coxeter number of G .

We see that the worldsheet fermions ψ^A are forced to transform in the adjoint representation of some Lie group G . These conditions were derived in [23]. The requirement of local $N = 1$ SCFT actually forces the global $SO(N)$ symmetry to be gauged into a local symmetry G , such that the dimension of G equals the dimension of the fundamental representation of $SO(N)$:

$$\dim G = N . \quad (3.69)$$

The conformal anomaly of the (super-)ghost sector can be cancelled by fermionizing each internal coordinate. In $d = 4$ non-compact dimensions this implies $N = 18$ internal worldsheet fermions, ψ^A . Suppose, for the time, that we assign the same boundary conditions to all 18 (real) fermions so that the global $SO(18)$ stays unbroken. The possible choices for the fermionization group G are given by :

$$SU(2)^6 , \quad SU(3) \times SO(5) , \quad SU(4) \times SU(2) . \quad (3.70)$$

We will exclusively focus on the choice of maximal rank, $G = SU(2)^6$. In particular this gives the normalization $\lambda = 1/(2\sqrt{2})$ for each of the 6 $SU(2)$ s.

As mentioned above, the global $O(N)$ -current algebra must be gauged to a local super-Kac-Moody algebra \widehat{G} , which is the local gauge symmetry of the theory. This algebra is generated by the currents $J^A(z)$:

$$J^A(z)J^B(w) = \frac{k\delta^{AB}}{(z-w)^2} + if^{ABC} \frac{J^C(w)}{z-w} + \dots, \quad (3.71)$$

where the level turns out to be equal to the dual Coxeter number $k = \hat{h}$. For $SU(2)$ $\hat{h} = 2$ and, hence, the level is $k = 2$. Indeed, using (2.159), for an $SU(2)_{k=2}$ one obtains $c = 3/2$, which is precisely the central charge of a system of three free fermions.

This super-Kac-Moody algebra⁶ is realized by taking :

$$J^A(z) = \frac{1}{2}f^{ABC} : \psi^B \psi^C : (z). \quad (3.72)$$

Through the Sugawara construction (2.157), the local currents J^A together with $T(z)$ and $T_F(z)$ close into an $N = 1$ superconformal algebra.

The holomorphic contribution to the partition function then takes the form :

$$\frac{1}{2} \sum_{a,b} (-)^{a+b} \frac{\theta^{10} \begin{bmatrix} a \\ b \end{bmatrix}}{\eta^{12}} = \frac{1}{\eta^2} (V_{20} - S_{20}). \quad (3.73)$$

The theory is clearly non-supersymmetric from the left-moving side and, what is more, no massless fermions arise from the left-moving sector. This is a general result, [24], where it is shown that whenever the super-Kac-Moody algebra is non-abelian, its spectrum is strictly positive and, hence, massless fermions may only appear when the associated super-Kac-Moody algebra is abelian. By the way, this proves that a Heterotic theory with massless fermions charged under a non-abelian gauge group, can only have the non-abelian currents arising from the bosonic side (for discussions see [27]).

In order to construct theories with spacetime supersymmetry and massless fermions we need to break the local enhanced symmetry $SU(2)$ ⁶ down to its abelian factors $U(1)$ ⁶. Further gaugings G/H down to a subgroup $H \subset G$ are consistent, provided G/H is a symmetric space [23]. The way to do this is through the choice of boundary conditions. Separate each $SU(2)$ -triplet into its fermionic constituents $\{\psi^I, y^I, \omega^I\}$, with $I = 1, \dots, 6$. We assume that y^I, ω^I are the extra fermions arising from the fermionization of the bosonic coordinates $i\partial X^I = y^I \omega^I$. The conventional GSO projection then assigns to the internal worldsheet fermions ψ^I the same spin structures $[\frac{a}{b}]$ as the ψ^μ , whereas it assigns completely independent boundary conditions, say $[\frac{\gamma}{\delta}]$, to the auxiliary fermions y^I, ω^I . This is required in order for the full worldsheet supercurrent (internal plus spacetime part) to have a well-defined periodicity :

$$T_F(z) = i\psi^\mu \partial X^\mu(z) + i \sum_{I=1}^6 \psi^I y^I \omega^I(z). \quad (3.74)$$

6. The fermionic superpartners of the local currents $J^A(z)$ are the fermions ψ^A themselves.

The left-moving piece of the partition function is then :

$$\frac{1}{\eta^{12}} \frac{1}{2^2} \sum_{a,b=0,1} (-)^{a+b} \theta^4 \begin{bmatrix} a \\ b \end{bmatrix} \sum_{\gamma,\delta=0,1} \theta^6 \begin{bmatrix} \gamma \\ \delta \end{bmatrix} = \frac{1}{\eta^8} (V_8 - S_8) \frac{1}{2} \sum_{\gamma,\delta=0,1} \theta^6 \begin{bmatrix} \gamma \\ \delta \end{bmatrix} . \quad (3.75)$$

This is clearly supersymmetric, as can be verified from the factorization of the vanishing character difference $V_8 - S_8$.

The full modular invariance constraints for fermionic models with real boundary conditions (\mathbb{Z}_2^N) were solved in [26] by requiring modular invariance at genres one and two and the factorization of the double donut into a product of simple donuts (cluster decomposition). This then guarantees the modular invariance at higher genres. The set of conditions derived are a generalization of the conditions (3.57) for the \mathbb{Z}_2^N orbifold models.

We will briefly describe the solution. In the algebraic formalism of [26], a model is essentially determined by (i) a set of basis vectors $v_i \in \Xi$, generating a ‘parity’ group Ξ to be described below and (ii) by fixing the values of $\frac{N(N-1)}{2}$ independent generalized GSO-coefficients (GGSO), $C_{(\alpha|\beta)}$, which form an $N \times N$ phase matrix. N is the number of basis vectors of Ξ . The choice of basis fully determines all the possible sectors in the Hilbert space, while the phases $C_{(\alpha|\beta)}$ determine the GGSO projections on these states. Let us see how this happens.

The parity group Ξ is generated by a set of N basis vectors $\{b_i\}$, with $b_1 \equiv \mathbf{1}$ being conventionally associated to the vector with unit components, $\mathbf{1} = (1, 1, \dots | 1, \dots)$. There are 2^N independent boundary condition vectors $\xi \in \Xi$:

$$\xi = \sum_{i=1}^N n_i b_i \quad , \quad n_i \in \{0, 1\} . \quad (3.76)$$

The boundary condition vectors ξ encode the periodicities of each fermion as it is parallel transported around a non-contractible loop of the torus. The components are typically enclosed by curly brackets :

$$\xi = \{\alpha(\psi^\mu), \alpha(\psi^4), \dots, \alpha(\psi^9), \alpha(y^4), \alpha(\omega^4), \dots | \alpha(\bar{y}^4), \alpha(\bar{\omega}^4), \dots\} , \quad (3.77)$$

with $\alpha(\psi^i)$ denoting the periodicity of the fermion ψ^i :

$$\begin{aligned} \psi^i &\rightarrow -e^{-i\pi\alpha(\psi^i)} \psi^i \\ \bar{\psi}^i &\rightarrow -e^{+i\pi\alpha(\bar{\psi}^i)} \bar{\psi}^i . \end{aligned} \quad (3.78)$$

Since we are only treating real boundary conditions at this stage, $\alpha(\psi) \in \{0, 1\}$. More general boundary conditions will either twist the internal fermions ψ^I and will, hence, correspond to some \mathbb{Z}_N orbifold (other than \mathbb{Z}_2), or they will only affect the fermions y^I, ω^I arising from fermionization, in a way that leaves ψ^I and, thus, ∂X^I invariant. The latter correspond to translations of the coordinates X^I and, hence, to (freely-acting) translation orbifolds. These translation orbifolds can be incorporated much more effectively at the level of marginal deformations and so we will not be interested in them at this stage.

The choice of basis vectors b_i is restricted by the following modular invariance constraints :

$$\begin{aligned} b_i \cdot b_i &= 0 \pmod{8} , \\ b_i \cdot b_j &= 0 \pmod{4} , \\ \prod_f b_i(f) b_j(f) b_k(f) b_\ell(f) &= 0 \pmod{2} , \end{aligned} \quad (3.79)$$

and the additional constraints :

$$b_i(\psi^I) + b_i(y^I) + b_i(\omega^I) = b_i(\psi^\mu) \pmod{2} . \quad (3.80)$$

coming from the global definition of the worldsheet supercurrent. For the Type II string where there is a right-moving $N = 1$ SCFT, the analogous constraint have to be imposed on $\bar{\psi}^I, \bar{y}^I, \bar{\omega}^I, \bar{\psi}^\mu$.

One then specifies the weights of each spin structure in the partition function in terms of $\frac{N(N-1)}{2} + 1$ phases $C_{(1|1)} = \pm 1$ and $C_{(b_i|b_j)} = \pm 1$, for $i < j$. The partition function of the theory is then expressed entirely in terms of θ -functions :

$$Z = \frac{1}{2^N \eta^2 \bar{\eta}^2} \sum_{\alpha, \beta \in \Xi} C_{(\alpha|\beta)} \prod_{f_L} \left(\frac{\theta \left[\begin{smallmatrix} \alpha(f_L) \\ \beta(f_L) \end{smallmatrix} \right]}{\eta} \right)^{1/2} \prod_{f_R} \left(\frac{\bar{\theta} \left[\begin{smallmatrix} \alpha(f_R) \\ \beta(f_R) \end{smallmatrix} \right]}{\bar{\eta}} \right)^{1/2} . \quad (3.81)$$

The coefficients $C_{(\alpha|\beta)}$ for the arbitrary spin structures $(\alpha|\beta)$ can be computed in terms of the basis coefficients $C_{(b_i|b_j)}$ by using [26] :

$$\begin{aligned} C_{(0|\alpha)} &= (-)^{\alpha(\psi^\mu) + \alpha(\bar{\psi}^\mu)} , \\ C_{(\alpha|\beta+\gamma)} &= (-)^{\alpha(\psi^\mu) + \alpha(\bar{\psi}^\mu)} C_{(\alpha|\beta)} C_{(\alpha|\gamma)} , \\ C_{(\alpha|\beta)} &= e^{\frac{i\pi}{2} \alpha \cdot \beta} C_{(\beta|\alpha)} , \\ C_{(\alpha|\alpha)} &= e^{\frac{i\pi}{4} \alpha^2} C_{(\alpha|b_1)} . \end{aligned} \quad (3.82)$$

One may essentially obtain the same conditions by imposing only one-loop modular invariance, and a consistent particle interpretation (correct spin-statistics). This the starting point of [25]. However, it is much more satisfactory and, in fact, one of the beauties of string theory that its internal consistency (in the form of higher-loop modular invariance and unitarity) is enough to provide the correct spin-statistics connection (this was first argued in [10]). In this sense, the solution of the modular invariance constraints at higher-genuses [26] implies that spacetime bosons have positive contribution to the vacuum amplitude, whereas spacetime fermions contribute negatively and, hence, the correct spin-statistics connection arises automatically !

Equivalence to \mathbb{Z}_2^N -orbifolds

The fermionic construction in terms of real fermions, as presented above, can be shown to contain the \mathbb{Z}_2^N -orbifold constructions. Fermionic models are in fact more general, in the

sense that they are equivalent to asymmetric orbifolds. In Appendix B we demonstrate the equivalence in an explicit way, showing the exact mapping between a model defined in terms of the fermionic formulation (basis $\{b_i\}$ and GGSO coefficients $C_{(b_i|b_j)}$) and the equivalent orbifold model.

Before ending this section we will make a few clarifying remarks. Consider the triplet of internal fermions $\{\psi, y, \omega\}$ and assume that the orbifold action twists ψ as :

$$\psi \rightarrow -\psi . \quad (3.83)$$

Since the spacetime part $i\psi^\mu \partial X_\mu$ of the supercurrent does not transform, requiring that the twist preserves the $N = 1$ SCFT algebra forces one to consider an induced twist on one of the fermions y, ω :

$$\begin{aligned} y &\rightarrow -y , \\ \omega &\rightarrow +\omega . \end{aligned} \quad (3.84)$$

In terms of the coordinate $i\partial X(z) = iy\omega(z)$, this is the familiar \mathbb{Z}_2 -orbifold rotation :

$$X \rightarrow -X . \quad (3.85)$$

The fact that the twisted sectors of the orbifold (3.40) were expressed entirely in terms of Jacobi θ -functions already hints at the equivalence at the level of the partition function. In particular, consider the contribution of the (y, ω) pair to the partition function. We will show that it provides precisely the fermionic representation of the twisted $(1, 1)$ -lattice. Consider a symmetric action on the right-movers as well $\bar{\psi}, \bar{y}, \bar{\omega}$. The lattice contribution in the partition function then is due to $(y, \omega, \bar{y}, \bar{\omega})$:

$$\begin{aligned} \Gamma_{(1,1)}[g] &= \frac{1}{2} \sum_{\gamma, \delta} \theta_{(y)}^{1/2}[\gamma+h] \theta_{(\omega)}^{1/2}[\delta] \bar{\theta}_{(\bar{y})}^{1/2}[\gamma+h] \bar{\theta}_{(\bar{\omega})}^{1/2}[\delta] , \\ &= \frac{1}{2} \sum_{\gamma, \delta} \left| \theta_{[\delta+g]}^{[\gamma+h]} \theta_{[\delta]}^{[\gamma]} \right| . \end{aligned} \quad (3.86)$$

The subscript in each θ -function contribution labels the fermion it is associated with. Using the triple product identity (A.9), one may prove :

$$\frac{1}{2} \sum_{\gamma, \delta} \left| \frac{\theta_{[\delta+g]}^{[\gamma+h]} \theta_{[\delta]}^{[\gamma]}}{\eta^2} \right| = \left| \frac{2\eta}{\theta_{[1-g]}^{[1-h]}} \right| , \quad \text{for } (h, g) \neq (0, 0) . \quad (3.87)$$

The factor of 2 on the r.h.s. is easy to understand, because the sum over γ, δ traces 4 terms, but vanishes exactly twice. Once when $\gamma = \delta = 1$ and once when $\gamma + h = \delta + g = 1$. Let us ignore the ψ -fermion momentarily, in order to compare with the bosonic orbifold (3.41). Dividing by half because of the orbifold g -projection and summing over all twisted sectors h ,

we obtain recover precisely (3.41). Here, however, the untwisted lattice is expressed entirely in terms of the fermion characters :

$$\Gamma_{(1,1)}[0] = \frac{1}{2} \sum_{\gamma, \delta} \theta[\gamma] \bar{\theta}[\gamma] . \quad (3.88)$$

This is the precisely the fermionic representation of the untwisted lattice. To see this, notice that the bosonization at the 1-loop level implies the identity⁷ :

$$\theta[\gamma] \bar{\theta}[\gamma] = \frac{1}{\sqrt{2\tau_2}} \sum_{\tilde{m}, n \in \mathbb{Z}} e^{-\frac{\pi}{2\tau_2} |\tilde{m} + \tau n|^2 + i\pi(\tilde{m}n + \tilde{m}\gamma + n\delta)} . \quad (3.89)$$

This is already a $(1,1)$ -lattice at the fermionic radius $R = 1/\sqrt{2}$ (in units where $\alpha' = 1$), deformed by the Wilson line-like coupling to the charges γ, δ . Summing over γ, δ gives :

$$\Gamma_{(1,1)}[0] = \frac{1}{\sqrt{2\tau_2}} \sum_{\tilde{m}, n \in \mathbb{Z}} e^{-\frac{\pi}{2\tau_2} |\tilde{m} + \tau n|^2} , \quad (3.90)$$

which is precisely the Lagrangian representation of a $(1,1)$ -lattice at radius $R = 1/\sqrt{2}$. This is the fermionic point. Returning to the superstring case, the fermion ψ contributes a $\theta^{1/2} \begin{smallmatrix} a+h \\ b+g \end{smallmatrix}$ to the partition function. Working in this way, we can construct the partition function in the orbifold language at the fermionic point and then deform at will (here, by changing the radius). This gives a flavor of the equivalence between theories formulated at the fermionic point and orbifold constructions. For the exact mapping, one needs a way to translate the GGSO coefficients $C_{(\alpha|\beta)}$ to a specific choice of modular invariant phase Φ , which can be inserted in the partition function. This is done in Appendix B.

3.5 Gepner Models

Let us very briefly mention a class of generalized constructions, called Gepner models. Consider Heterotic string theory with $\mathcal{N}_4 = 1$ spacetime supersymmetry. As will be discussed in the next chapter, this implies an enhancement of the local worldsheet $N = 1$ SCFT to a global $N = 2$. One may even take a further step and extend the $N = 2$ SCFT to the right-moving bosonic side of the Heterotic string. This map [31], often referred to in the literature as the ‘Gepner map’, can most easily be obtained from a Type II theory with $N = (2, 2)$ SCFT, by embedding the Type II spin connection of the right-moving side into the gauge connection of the Heterotic. The left-moving extension of local $N = 1$ to $N = 2$ (and hence spacetime supersymmetry itself) is stable, because constraints coming from the preservation of the local $N = 1$ do not allow the breaking of $N = 2$ in the ‘bulk’ of moduli space. Whatever the breaking, a supersymmetric vacuum is not recovered unless one goes infinitely far, to the boundaries of moduli space. In the bosonic side of the Heterotic, however, there is no local

7. This is a special case of the more general identity (A.12).

SCFT algebra to respect. There are no pictures, no superghosts and no spacetime fermions arising from the right-moving side. The $N = (0, 2)$ enhancement is, therefore, unprotected and can be broken by deformations in the neighborhood of the point of enhancement.

At points where the right-moving $N = 2$ remains unbroken, one typically encounters enhancement of the gauge symmetry. One example of such enhancement is the $E_8 \times E_8$ Heterotic string itself. Under a single \mathbb{Z}_2 -orbifold the analogous enhancement would be $SO(12) \times SO(4) \rightarrow E_7$ (it can be performed continuously), whereas in the case of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds one may still have enhancement $SO(10) \times U(1) \rightarrow E_6$ for certain choice of discrete Wilson lines.

It is interesting to consider such $N = (2, 2)$ constructions. Solvable $N = (2, 2)$ CFTs with unitary representations can only exist for $c \geq 3$ or at discrete points, $c_i = 3k_i/(k_i + 2)$, defining the $N = 2$ minimal series. The total central charge, $\hat{c} = 6$, can be obtained by tensoring together the various $N = 2$ blocks :

$$\prod_i U(1) \otimes \left[\frac{SU(2)}{U(1)} \right]_{k_i} \quad (3.91)$$

The $U(1)$ factor corresponds to a free boson, while the $\left[\frac{SU(2)}{U(1)} \right]_k$ piece corresponds to a parafermion block. The total central charge then has to be saturated by the combined contribution of all $N = 2$ blocks :

$$\sum_i \frac{3k_i}{k_i + 2} = 9 . \quad (3.92)$$

We will not discuss the representation content any further. We only mention briefly that further conditions need to be imposed in order to guarantee modular invariance and spacetime supersymmetry. In particular, the GSO projection has to be defined as :

$$(-)^F \exp \left(2\pi i \sum_i Q_i \right) , \quad (3.93)$$

where F is the spacetime fermion number and Q_i is the associated $U(1)$ -charge of each $N = 2$ block. $N = (2, 2)$ Gepner constructions have the attractive property of lying in the same moduli space as geometric Calabi-Yau compactifications [32].

Chapitre 4

Spacetime vs Worldsheet Supersymmetry

In this chapter we review the conditions for spacetime supersymmetry. Part of the beauty and power of CFT techniques on the worldsheet lies in the fact that the presence of spacetime supersymmetry can be traced back to an extension of the local $N = 1$ superconformal algebra on the worldsheet. The result [29] is that $\mathcal{N}_4 = 1$ supersymmetry in spacetime arises from a global $N = 2$ SCFT on the worldsheet¹. Furthermore, an extended $\mathcal{N}_4 = 2$ spacetime supersymmetry is a consequence of a global worldsheet $N = 4$ SCFT. The case of $\mathcal{N}_4 = 4$ supersymmetry can only arise from a free (toroidal) theory.

4.1 Conditions for $\mathcal{N}_4 = 1$ Supersymmetry

We will start from the assumption of $\mathcal{N}_4 = 1$ spacetime supersymmetry and show that this implies the extension of the local $N = 1$ to a global, extended $N = 2$ SCFT on the worldsheet [27].

In the whole chapter the discussion and results refer to the supersymmetry arising chirally, from either the left- or right-moving sector of the string. Hence, we will only deal with the left-movers. The $\mathcal{N}_4 = 1$ supersymmetry algebra is realized by the zero modes $Q_\alpha, \bar{Q}_{\dot{\alpha}}$ of the supersymmetry currents :

$$\begin{aligned} J_\alpha(z) &= e^{-\phi/2} S_\alpha \Sigma(z) , \\ \bar{J}_{\dot{\alpha}}(z) &= e^{-\phi/2} C_{\dot{\alpha}} \Sigma^\dagger(z) . \end{aligned} \tag{4.1}$$

As introduced in the second chapter, $e^{-\phi/2}$ is the superghost dressing factor, so that the supersymmetry currents are both in the $(-\frac{1}{2})$ -ghost picture. Also Σ, Σ^\dagger are the weight- $(\frac{3}{8}, 0)$ ground states of the internal CFT in the R-sector. The currents must be preserved by GSO projection, so that Σ and Σ^\dagger must have opposite values of the worldsheet fermion parity.

1. The discussion in this chapter will be specialized to the Heterotic case, where the SCFT is only realized holomorphically. The results are easily extended to the Type II case.

The supersymmetry algebra arises from the current algebra OPE by taking the contour integral and picking the commutator of the supercharges :

$$J_\alpha(z)\bar{J}_{\dot{\beta}}(w) = \frac{1}{\sqrt{2}} \sigma_{\alpha\dot{\beta}}^\mu \frac{e^{-\phi}\psi_\mu(w)}{z-w} + \dots \quad (4.2)$$

The r.h.s. is simply the spacetime momentum operator in the (-1) -picture, as can be verified by acting upon it by the picture-changing operator, $e^\phi T_F(z)$, and picking the regular term. For the currents (4.1) to satisfy the SUSY algebra, the branch cuts from the superghost have to be cancelled by balancing branch cuts arising from the Σ, Σ^\dagger OPEs. This requires :

$$\begin{aligned} \Sigma(z)\Sigma(w) &= (z-w)^{3/4} (\dots) , \\ \Sigma(z)\Sigma^\dagger(w) &= \frac{1}{(z-w)^{3/4}} \left(1 + \frac{1}{2}(z-w)J(w) + \dots \right) . \end{aligned} \quad (4.3)$$

The ellipsis in the first equation denote single valued operators. The second term in the r.h.s. of the second equation, if non-vanishing (to be shown below), is a weight $(1,0)$ -object and, hence, a current. It will be shown to be precisely the $U(1)$ -current of an extended global $N=2$ SCFT. To this end, one calculates the four-point function :

$$G(z_1, z_2, z_3, z_4) = \langle \Sigma(z_1)\Sigma^\dagger(z_2)\Sigma(z_3)\Sigma^\dagger(z_4) \rangle = \left(\frac{z_{13}z_{24}}{z_{12}z_{14}z_{23}z_{34}} \right)^{3/4} f(z_i) , \quad (4.4)$$

where we use (4.3) to extract the behaviour of G at points where $z_{ij} \rightarrow 0$. Since the scale part $\left(\frac{z_{13}z_{24}}{z_{12}z_{14}z_{23}z_{34}} \right)^{3/4}$ fully captures the behaviour of G at the points where operators meet, as implied by the OPEs (4.3), $f(z_i)$ has to be a holomorphic function of its arguments.

In fact, one may do better. Under the globally defined² $SL(2, \mathbb{C})$ subgroup of conformal transformations :

$$z \rightarrow z' = \frac{az+b}{cz+d} , \quad ad-bc=1 , \quad (4.5)$$

the four-point amplitude transforms tensorially :

$$G(z') = \prod_i \left(\frac{\partial z'_i}{\partial z_i} \right)^{-\Delta_i} G(z) . \quad (4.6)$$

This can be used to fix $f(z_i)$ to be a function of the $SL(2, \mathbb{C})$ -invariant ratio x :

$$f(z_i) = f(x) , \quad x = \frac{z_{12}z_{34}}{z_{13}z_{24}} . \quad (4.7)$$

2. The $SL(2, \mathbb{C})$ is precisely that subgroup of the full 2d conformal group which is left unbroken by the conformal anomaly. It corresponds to the Virasoro generators $L_0, L_{\pm 1}$ for which the central extension of the Virasoro algebra vanishes.

This tends to a constant at the boundaries $x \rightarrow 0, 1, \infty$ and, thus, $f(x)$ is a constant everywhere. Using the OPEs (4.3) and the normalization $\langle \mathbf{1} \rangle = 1$ we see that $f = 1$. Now expand (4.4) in powers of z_{12} :

$$G(z_i) = \frac{1}{(z_{12}z_{34})^{3/4}} \left(1 + \frac{3}{4} \frac{z_{12}z_{34}}{z_{23}z_{24}} + \dots \right) . \quad (4.8)$$

Using the OPE (4.3) one can show that the non-vanishing term of order $z_{12}^{-1/4}$ indicates that the J -term in (4.3) does not vanish and, in fact :

$$\langle J(z_2) \Sigma(z_3) \Sigma^\dagger(z_4) \rangle = \frac{3}{2} \frac{z_{34}^{1/4}}{z_{23}z_{24}} . \quad (4.9)$$

Further taking the limits $z_{23} \rightarrow 0$ and $z_{24} \rightarrow 0$ one finds OPE's of the form :

$$\begin{aligned} J(z) \Sigma(w) &= \frac{3/2}{z-w} \Sigma(w) + \dots , \\ J(z) \Sigma^\dagger(w) &= \frac{-3/2}{z-w} \Sigma^\dagger(w) + \dots , \end{aligned} \quad (4.10)$$

indicating that the ground states Σ, Σ^\dagger carry $U(1)$ charge, with respect to $J(z)$. Finally, the limit $z_{34} \rightarrow 0$ fixes the normalization of the current :

$$J(z)J(w) = \frac{3}{(z-w)^2} . \quad (4.11)$$

The $U(1)$ current consistently factorizes and one may represent the current in terms of a free boson $H(z)$:

$$J(z) = i\sqrt{3} \partial H(z) . \quad (4.12)$$

The energy-momentum tensor then breaks into the energy momentum of the free boson, $-\frac{1}{2}(\partial H)^2$, and an H -independent piece. Since the ground state operators Σ, Σ^\dagger carry $U(1)$ charge $\pm\frac{3}{2}$, they will be proportional to :

$$\Sigma(z) = e^{i\frac{\sqrt{3}}{2}H(z)} , \quad \Sigma^\dagger(z) = e^{-i\frac{\sqrt{3}}{2}H(z)} . \quad (4.13)$$

The conformal weight of the $e^{\pm i\frac{\sqrt{3}}{2}H}$ -factors is precisely equal to $\frac{3}{8}$ and, hence, the ground states Σ, Σ^\dagger are completely fixed³ by their H -dependence.

Now notice that since the Σ, Σ^\dagger are ground states of the CFT in the R-sector, they must be annihilated by the action of the zero mode, G_0 , of the $N = 1$ worldsheet supercurrent

3. In general, an operator $\Phi(z, \bar{z})$ with $U(1)$ charge q can be written as $\Phi =: e^{i\frac{q}{\sqrt{3}}H} P(J) :$, where $P(J)$ is a polynomial in $J(z)$ and its derivatives. In the case of Σ, Σ^\dagger , the conformal weight of $P(J)$ turns out to be zero, so this has to be the identity operator.

$T_F(z)$. In terms of OPEs this is equivalent to :

$$\begin{aligned} T_F(z)\Sigma(w) &\sim \frac{1}{\sqrt{z-w}} (\dots) , \\ T_F(z)\Sigma^\dagger(w) &\sim \frac{1}{\sqrt{z-w}} (\dots) . \end{aligned} \quad (4.14)$$

The absence of more singular terms can also be seen as a consequence of BRST invariance of the gravitino vertex operator. In particular, $(e^\phi T_F)_{(z)} \cdot (e^{-\phi/2} S_\alpha \Sigma)_{(w)}$ must have no single-pole term. Since the $U(1)$ is decoupled, this implies that $T_F(z)$ has no definite charge but is, nevertheless, expanded into operators of definite charge :

$$T_F(z) = \sum_q e^{i\frac{q}{\sqrt{3}}H} \mathcal{T}_F^{(q)} . \quad (4.15)$$

By plugging in the explicit form (4.13), the branch cuts of (4.14) imply that only the charges $q = \pm 1$ contribute. This implies the decomposition $T_F(z) = T_F^+(z) + T_F^-(z)$, where :

$$T_F^\pm(z) \equiv e^{\pm i\frac{1}{\sqrt{3}}H(z)} \mathcal{T}_F^{(\pm 1)}(z) . \quad (4.16)$$

In terms of the $U(1)$ current, one finds :

$$J(z)T_F^\pm(w) = \pm \frac{1}{z-w} T_F^\pm(w) + \dots \quad (4.17)$$

This begins to look like the familiar $N = 2$ SCFT algebra. In order to complete the argument notice that there is an extra constraint. The full T_F still has to close into the local $N = 1$ SCFT :

$$T_F(z)T_F(w) = \frac{\hat{c}}{(z-w)^3} + \frac{1/3}{z-w} J^2(w) + \dots \quad (4.18)$$

Using (4.16), one calculates the same OPE and requires that $T_F T_F$ closes into the above $N = 1$ algebra :

$$\begin{aligned} T_F(z)T_F(0) &= z^{1/3} e^{i\frac{2}{\sqrt{3}}H(0)} \mathcal{T}_F^{(+1)}(z)\mathcal{T}_F^{(+1)}(0) + z^{1/3} e^{-i\frac{2}{\sqrt{3}}H(0)} \mathcal{T}_F^{(-1)}(z)\mathcal{T}_F^{(-1)}(0) \\ &+ \frac{1}{z^{1/3}} \left(1 + \frac{z^2}{18} J^2(0) \right) \left(\mathcal{T}_F^{(+1)}(z)\mathcal{T}_F^{(-1)}(0) + \mathcal{T}_F^{(-1)}(z)\mathcal{T}_F^{(+1)}(0) \right) . \end{aligned} \quad (4.19)$$

The above requirements fix :

$$\mathcal{T}_F^{(+1)}\mathcal{T}_F^{(-1)} \sim \mathcal{T}_F^{(-1)}\mathcal{T}_F^{(+1)} \sim \frac{\hat{c}/2}{z^{8/3}} + \dots \quad (4.20)$$

and the $N = 1$ algebra is satisfied with $\hat{c} = 6$, which is precisely the central charge of the internal six-dimensional CFT. It is now straightforward to see that $\{T(z), T_F^\pm(z)$ and $J(z)$

close into a global $N = 2$ SCFT algebra :

$$\begin{aligned}
T_F^+(z)T_F^-(w) &= \frac{\hat{c}}{(z-w)^3} + \frac{2}{(z-w)^2}J(w) + \frac{1}{z-w}(2T(w) + \partial J(w)) + \dots , \\
T_F^+(z)T_F^+(w) &\sim T_F^-(z)T_F^-(w) = \dots , \\
J(z)T_F^\pm(w) &= \pm \frac{1}{z-w}T_F^\pm(w) + \dots , \\
J(z)J(w) &= \frac{\hat{c}/2}{(z-w)^2} + \dots .
\end{aligned} \tag{4.21}$$

This is a remarkable result. The existence of $\mathcal{N}_4 = 1$ spacetime supersymmetry implies the enhancement of the local $N = 1$ SCFT to a global $N = 2$ SCFT on the worldsheet. The $N = 2$ SCFT algebra has a continuous $SO(2)$ (internal) automorphism : the rotation of the two supercharges. One may use this to impose arbitrary boundary conditions :

$$T_F^\pm(e^{2\pi i} z) = e^{\mp 2\pi i \alpha} T_F^\pm(z) . \tag{4.22}$$

This gives rise to isomorphic $N = 2$ algebras, parametrized by $\alpha \in \mathbb{R}$. They are connected by spectral-flow :

$$\begin{aligned}
J_n &\rightarrow J_n - 3\alpha\delta_{n,0} , \\
L_n &\rightarrow L_n - \alpha J_n + \frac{3}{2}\alpha^2\delta_{n,0} .
\end{aligned} \tag{4.23}$$

Looking at the moding of T_F , we see that $\alpha = 0$ corresponds to NS boundary conditions, whereas $\alpha = \pm \frac{1}{2}$ correspond to the R sector. Using the spectral flow, one may continuously interpolate between the two sectors. In particular, the spacetime supersymmetry currents (4.1) are precisely the spectral flow operators of the $N = 2$ algebra, interpolating between the NS and R sectors and, hence, connecting spacetime bosons with spacetime fermions

The converse is also true. Starting from an $N = 2$ SCFT, one may always construct the spectral flow operators (4.1) which generate spacetime supersymmetry.

4.2 $\mathcal{N}_4 = 2$ and $\mathcal{N}_4 = 4$ Spacetime Supersymmetry

The analysis for the case of $\mathcal{N}_4 = 2$ spacetime supersymmetry goes along the same lines. The result is that the internal CFT breaks into a $\hat{c} = 2$ SCFT made out of 2 free superfields, and an enhanced, global $N = 4$ SCFT for the remaining $\hat{c} = 4$ system. Finally, $\mathcal{N}_4 = 4$ spacetime supersymmetry can only arise if the internal CFT is made out of 6 free superfields and is, hence, necessarily toroidal. We will not discuss the extended $N = 2$ or $N = 4$ SCFT algebras or their representations here. A useful review is, for instance, [30].

Chapitre 5

Supersymmetry Breaking

In this chapter we describe some of the basic mechanisms for breaking supersymmetry in string theory, with special emphasis on the Scherk-Schwarz mechanism.

5.1 Explicit Breaking

We have already seen a mechanism that breaks supersymmetry explicitly by using non-freely-acting orbifolds. Let us take the simple example of a \mathbb{Z}_2 orbifold, rotating four internal super-coordinates. The twisting of the fermion boundary conditions, $\psi^I \rightarrow -\psi^I$, translates into the shift of the bosonized $H^A(z)$ -fields :

$$\begin{aligned} H^A(z) &\rightarrow H^A(z) & , \quad A = 1, 2 , \\ H^A(z) &\rightarrow H^A(z) + \pi & , \quad A = 3, 4 . \end{aligned} \tag{5.1}$$

The supersymmetry current in 4d, (3.27), has 4 ‘internal components’, $(+, +, +)$, $(+, -, -)$, $(-, +, -)$ and $(-, -, +)$, rotated into each other by the R -symmetry group. These are the four gravitini of an $\mathcal{N}_4 = 4$ supersymmetry. Under the \mathbb{Z}_2 -action described above, only $(-, +, -)$ and $(-, -, +)$ remain invariant and the resulting T^6/\mathbb{Z}_2 theory has reduced its supersymmetry by half, $\mathcal{N}_4 = 2$.

An additional \mathbb{Z}_2 twist :

$$\begin{aligned} H^A(z) &\rightarrow H^A(z) & , \quad A = 1, 3 , \\ H^A(z) &\rightarrow H^A(z) + \pi & , \quad A = 2, 4 , \end{aligned} \tag{5.2}$$

would project out $(-, +, -)$ and only leave out one gravitino invariant, $(-, -, +)$. Hence, the resulting compactification on a $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold has $\mathcal{N}_4 = 1$ spacetime supersymmetry.

Non-freely-acting orbifolds act on the worldsheet fermions and project out some of the currents generating spacetime supersymmetry (vertex operators of the gravitini). In this sense, the breaking is explicit, since states are projected out and cannot be recovered continuously in any limit. This method is, therefore, not very useful for breaking supersymmetry completely to $\mathcal{N}_4 = 0$, if only for phenomenological reasons.

5.2 Spontaneous Breaking

General Remarks

There are several ways to break supersymmetry spontaneously in string theory. Typically, in spontaneous breaking, Ward identities continue to exist in broken form and determine the short range properties of the theory. In a supergravity theory around a flat background, the breaking scale is essentially the mass of the gravitino, $m_{3/2}$, so that one effectively probes supersymmetric physics at energies much higher than this.

First of all, let us briefly mention a fundamental difference between the Heterotic and Type I theories. By comparing string scattering amplitudes with the associated supergravity results (or from anomaly considerations) one finds the relation between the gauge and gravitational couplings in, say, 4d :

$$\frac{1}{g^2} = \frac{k}{2} \left(\frac{M_P}{M_s} \right)^2, \quad (5.3)$$

where M_P is the Planck mass, $M_s = \sqrt{\alpha'}$ is the string scale and k is the associated level of the current algebra giving rise to the gauge interactions. In all attempts to obtain a realistic phenomenology, $k = 1$, otherwise one runs into exotic representations for massless chiral matter which are difficult to account for. Therefore, in the perturbative region, $g \lesssim 1$, and the string scale has to be of the order of the gravitational scale. Hence, it would be interesting from a phenomenological viewpoint (hierarchy problem) if supersymmetry could break spontaneously close to the TeV scale.

In Type I theories this is no longer the case, since gauge interactions in the open sector originate from the coupling of open strings to D-branes. The ratio between the volume of internal space wrapped by the D-branes and the total volume of the internal dimensions appears explicitly in the Yang-Mills coupling, making it possible to lower the string scale much below the Planck scale, while keeping the theory perturbative.

Supersymmetry Breaking with Fluxes

A very interesting type of compactifications is that in the presence of non-trivial fluxes for the p -form fields. These typically generate potentials that lift some of the flat directions of the moduli space. Such compactifications in the presence of non-trivial flux backgrounds typically break some or all of the supersymmetries. From the point of view of supergravity, they typically correspond to gauging some of the graviphotons to non-abelian structure (see [33] and references therein). In most cases, however, this has the problem of curving the background and necessitating a proper treatment of the backreaction and, furthermore, it typically leads to non-perturbative terms in the supergravity effective action (from the stringy point of view). The latter can presumably be treated by using various string dualities, but since our interest in this work is to have an exact perturbative description at the string level, we will disregard this possibility and focus primarily on special geometrical fluxes, which correspond to freely-acting orbifolds (e.g. stringy generalization of the Scherk-Schwarz

mechanism). More details on the correspondence between various gaugings of supergravity and compactification with non-trivial fluxes can be found in [33] (and references therein).

5.3 Scherk-Schwarz Breaking

In what follows we will describe a very particular ‘flat’ gauging, $f^{ABC} f^{ABD} = 0$, which has the attractive property that the potential it generates vanishes at its minimum and, hence, is compatible with a flat background. The starting point was the observation in field-theory by Scherk and Schwarz, [34], that it is possible to give masses to fields by coordinate-dependent compactifications.

Scherk-Schwarz mechanism in field theory

We will first start with the Scherk-Schwarz mechanism in field theory. Consider the theory of a scalar ϕ living on a $(d+1)$ -dimensional target space, $\mathcal{M}^{1,d-1} \times S^1$. Call x^μ the non-compact coordinates and y the compact coordinate on the circle of radius R . Because of the compact topology of the circle, one usually imposes periodic boundary conditions :

$$\phi(x^\mu, y + 2\pi R) = \phi(x^\mu, y) , \quad (5.4)$$

and then Fourier-expands in the compact coordinate :

$$\phi(x^\mu, y) = \sum_{m \in \mathbb{Z}} e^{im y/R} \phi_m(x^\mu) . \quad (5.5)$$

This is the usual procedure which gives rise to the Kaluza-Klein masses from the $(d+1)$ -dimensional kinetic term $\frac{1}{2} \partial_\mu \phi^T \partial^\mu \phi$.

Now assume, in addition, that the Lagrangian of the theory is invariant with respect to a (global) symmetry operator $O = \exp(iM\theta)$. Then, the periodicity boundary condition can close up to O -rotations :

$$\phi(x^\mu, y + 2\pi R) = e^{iM\theta} \phi(x^\mu, y) . \quad (5.6)$$

In order to Fourier-expand, let us make the field redefinition :

$$\phi(x^\mu, y) = e^{iM\theta y/2\pi R} \hat{\phi}(x^\mu, y) , \quad (5.7)$$

so that the new field, $\hat{\phi}(x^\mu, y)$ is now periodic in the compact direction and can be Fourier-expanded as in the periodic case above. The kinetic term of ϕ will then give rise to a mass term :

$$\int dy (\partial_\mu \phi^T \partial^\mu \phi + \partial_y \phi^T \partial_y \phi) = \sum_{m \in \mathbb{Z}} \left(\partial_\mu \hat{\phi}_m^T \partial^\mu \hat{\phi}_m - \hat{\phi}_m^T (P_m^T P_m) \hat{\phi}_m \right) , \quad (5.8)$$

where the mass matrix is given by :

$$P_m = \frac{m + q(\theta/2\pi)}{R} \quad , \quad m \in \mathbb{Z} . \quad (5.9)$$

We assumed, for simplicity, that the higher-dimensional symmetry generator, M , is diagonal and q is the charge of the state it acts upon. Thus, the Scherk-Schwarz mechanism deforms the Kaluza-Klein masses by an amount proportional to the charge of the state. Hence, this mechanism deforms also the zero modes.

Scherk-Schwarz mechanism in string theory

The Scherk-Schwarz mechanism can be naturally implemented at the string level, [35], [36], in order to generate masses for states charged under a symmetry Q . One may then use it, for example, to give mass to gauge bosons in Heterotic theories or even to gravitini (which are charged under the R -symmetry) and, hence, spontaneously break gauge symmetries or even supersymmetry, respectively. Since this mechanism will be central to the spontaneous breaking of supersymmetry in later chapters, we will develop it to some detail.

The starting point is to take a set of charged operators $O^a(z)$, closing into a global algebra :

$$O^a(z)O^b(w) = \dots + \frac{if^{abc}}{z-w} O^c(w) + \dots , \quad (5.10)$$

where the ellipsis denotes other (possibly singular) terms. Now we repeat the coordinate-dependent compactification presented above. The physical states are only periodic modulo a symmetry transformation. The vertex operators then become :

$$\mathcal{V}(z) = O^a(z) \exp \left(i \frac{\theta q}{2\pi R} Y(z) \right) . \quad (5.11)$$

The effect of this is to shift the momentum quantum number, m , in the Y -direction by $2\pi\theta q$:

$$\frac{m + q(\theta/2\pi)}{R} \pm \frac{nR}{\alpha'} . \quad (5.12)$$

Hence, the masses of charged states are shifted as well. This is the basic idea behind the stringy realization of the Scherk-Schwarz mechanism. Actually, this is not the full story. The worldsheet ‘backreacts’ to this θ -deformation and induces an additional momentum shift, proportional to the winding number, in order to preserve modular invariance. This will arise from the subsequent discussion.

We will now see the same mechanism arise in a different way. Namely, we will show that it is equivalent to turning on a non-trivial geometrical flux on the worldsheet. The starting point is to consider a perturbation of the sigma model by a marginal operator of the form :

$$-\frac{i}{2} G_{IJ} H_{KL}^J \left(\psi^I \psi^L - \frac{1}{2} X^I \partial X^L + \frac{1}{2} X^L \partial X^I \right) \bar{\partial} X^K . \quad (5.13)$$

This amounts to turning on an H -flux in Type II or an F -flux in Heterotic theories. This amounts to a rotation in the (I, J) -plane of the left-moving piece of the lattice and a translation (boost) in the right-moving sector. Due to the discreteness of the lattice, the rotation can only take place for discrete values of the H -flux, which correspond to angles of rotation that are automorphisms of the lattice. Although it looks like an interacting term in the sigma model, nevertheless, for constant value of the H -flux the model is exactly solvable. It can be seen, as in [37], [11], that the non-quadratic term can be absorbed into a field redefinition :

$$\begin{aligned}\psi^I &= (e^{iA})^I{}_J \hat{\psi}^J , \\ X^I &= (e^{iA})^I{}_J \hat{X}^J ,\end{aligned}\tag{5.14}$$

where :

$$A^I{}_J = \frac{1}{2} H^I{}_{JK} X^K .\tag{5.15}$$

In a more general setting, one can see the interacting term as a 2d gauge field $A_z dz + A_{\bar{z}} d\bar{z}$, minimally coupled to the worldsheet superfield, but without a kinetic term. This is always decomposable into a closed, co-closed part and a harmonic term. The first two can easily be absorbed by a gauge transformation. Absorbing the harmonic part, however, changes the boundary conditions of the fields :

$$\begin{aligned}\hat{X}(\sigma^1 + 2\pi, \sigma^2) &= e^{i\pi H^I{}_{JK} \tilde{m}^J} \hat{X}(\sigma^1, \sigma^2) , \\ \hat{X}(\sigma^1, \sigma^2 + 2\pi) &= e^{i\pi H^I{}_{JK} n^J} \hat{X}(\sigma^1, \sigma^2) , \\ \hat{\psi}(\sigma^1 + 2\pi, \sigma^2) &= e^{i\pi H^I{}_{JK} \tilde{m}^J} \hat{\psi}(\sigma^1, \sigma^2) , \\ \hat{\psi}(\sigma^1, \sigma^2 + 2\pi) &= e^{i\pi H^I{}_{JK} n^J} \hat{\psi}(\sigma^1, \sigma^2) ,\end{aligned}\tag{5.16}$$

where \tilde{m}, n are the winding numbers around the compact toroidal cycles. As mentioned above, not all rotations are automorphisms of the lattice. Equivalently, only those values of the flux for which the associated rotations which leave are discrete symmetries of the lattice and commute with the worldsheet supercurrent are allowed. This is, precisely, due to the fact that $(X^I \partial X^J - X^J \partial X^I)$ is not a well-defined conformal operator unless it corresponds to a discrete automorphism of the lattice. The quantization condition comes if one remembers that the (chiral) R -symmetry lattice is constructed out of real worldsheet fermions ψ at the fermionic point. The discrete deformation will, hence, correspond to a \mathbb{Z}_N -orbifold :

$$\frac{1}{4} NH \in \mathbb{Z} .\tag{5.17}$$

As an example, consider the case of \mathbb{Z}_2 -orbifold with non-trivial H_{12}^3 . The bosons are then periodic and their contribution to the partition function is unaffected. The spin structures of the fermions, on the other hand, are shifted by the winding numbers around the compact X^3 -direction :

$$\begin{aligned}a &\rightarrow a - 2n , \\ b &\rightarrow b - 2\tilde{m} .\end{aligned}\tag{5.18}$$

The cumulative result of the deformation is formally equivalent to turning on a very particular value of the Wilson line, around the compact X^3 -circle of radius R :

$$Z \sim \frac{R}{\sqrt{\tau_2} \eta^{12} \bar{\eta}^{12}} \sum_{\tilde{m}, n \in \mathbb{Z}} \left[\frac{1}{2} \sum_{a, b} (-)^{a+b} \theta^3 \begin{bmatrix} a \\ b \end{bmatrix} \theta \begin{bmatrix} a-2n \\ b-2\tilde{m} \end{bmatrix} \right] e^{-\frac{\pi R^2}{\tau_2} |\tilde{m} + \tau n|^2 + i\pi(\tilde{m}n - bn)} \Gamma_{(5,5)} \Gamma_{E_8 \times E_8} . \quad (5.19)$$

The Type II result is similar. Using the periodicity property of θ -functions, (A.10), one acquires the extra phase $(-)^{a\tilde{m}}$, which completes the modular invariant cocycle $(-)^{\tilde{m}n + a\tilde{m} + bn}$. The deformed partition function then becomes :

$$Z \sim \frac{1}{\eta^{12} \bar{\eta}^{12}} \frac{1}{2} \sum_{a, b} (-)^{a+b} \theta^4 \begin{bmatrix} a \\ b \end{bmatrix} \Gamma_{(1,1)} \begin{bmatrix} a \\ b \end{bmatrix} (R) \Gamma_{(5,5)} \Gamma_{E_8 \times E_8} , \quad (5.20)$$

where $\Gamma_{(1,1)} \begin{bmatrix} a \\ b \end{bmatrix}$ is the modular covariant $(1,1)$ -lattice, deformed by its coupling to the R -symmetry charges :

$$\Gamma_{(1,1)} \begin{bmatrix} a \\ b \end{bmatrix} (R) = \frac{R}{\sqrt{\tau_2}} \sum_{\tilde{m}, n \in \mathbb{Z}} e^{-\frac{\pi R^2}{\tau_2} |\tilde{m} + \tau n|^2} (-)^{\tilde{m}n + a\tilde{m} + bn} . \quad (5.21)$$

This coupling correlates the R -symmetry lattice with the compactification lattice. Formally, it looks like a Wilson line, shifting the masses of states charged under the R -symmetry. Poisson-resumming the $(1,1)$ -lattice to go to the Hamiltonian picture, the masses of the states are seen to become :

$$\frac{1}{2} P_{L,R}^2 = \frac{1}{4} \left(\frac{m - \frac{a+n}{2}}{R} \pm nR \right)^2 . \quad (5.22)$$

Therefore, states carrying non-trivial R -symmetry or winding charge have their masses shifted. This includes, in particular, the gravitini. Since these carry non-trivial R -symmetry charge, $a = 1$, and correspond to pure oscillator states, $m = n = 0$, they acquire masses proportional to $m_{3/2} \sim 1/R$. This is the Scherk-Schwarz mechanism adapted to the string level. It can be used to give masses to the gravitini or to gauge bosons and, hence, it is the natural mechanism for breaking spontaneously supersymmetry or even gauge symmetries.

The above discussion already suggests that the Scherk-Schwarz mechanism is equivalent to a freely-acting orbifold. For example, the \mathbb{Z}_2 -example encountered above is equivalent to moding out by $(-)^F \delta_3$, where F is the spacetime fermion number and δ_3 is an order 2 shift along the compact X^3 -cycle. Indeed, in the $[\frac{h}{g}]$ twisted sector, the boundary conditions of the compact scalar X^3 can also be satisfied as :

$$\begin{aligned} X^3(\sigma^1 + 2\pi, \sigma^2) &\sim X^3(\sigma^1, \sigma^2) + 2\pi nR + h\pi R , \\ X^3(\sigma^1, \sigma^2 + 2\pi) &\sim X^3(\sigma^1, \sigma^2) + 2\pi \tilde{m}R + g\pi R . \end{aligned} \quad (5.23)$$

In order to capture the alternating sign depending on the R -symmetry charge $(-)^F$ we also insert in the partition function the modular invariant cocycle :

$$(-)^{hb+ga+hg} . \quad (5.24)$$

All together this becomes :

$$\frac{1}{2} \sum_{h,g=0,1} \frac{R}{\sqrt{\tau_2}} \sum_{\tilde{m},n} e^{-\frac{\pi R^2}{\tau_2} |\tilde{m} + \frac{g}{2} + \tau(n + \frac{h}{2})|^2} (-)^{hb+ga+hg} . \quad (5.25)$$

Before we can perform the h, g -summation we first have to perform a double Poisson resummation (A.15) in order to take the lattice to its dual form :

$$\frac{1/R}{\sqrt{\tau_2}} \sum_{\tilde{m},n \in \mathbb{Z}} e^{-\frac{\pi}{\tau_2} (\frac{1}{R})^2 |\tilde{m} + \tau n|^2} \frac{1}{2} \sum_{h,g=0,1} (-)^{(\tilde{m}+a+h)(n+b+g) - (\tilde{m}+a)(n+b)} . \quad (5.26)$$

Perform first the summation over g . This imposes a constraint $(\tilde{m} + a + h) \in 2\mathbb{Z}$ in the \tilde{m} -summation. Because this is a projection, the $\frac{1}{2}$ -factor is also eliminated and one obtains :

$$\sum_{h=0,1} \frac{1/R}{\sqrt{\tau_2}} \sum_{\frac{1}{2}(\tilde{m}+a+h), n \in \mathbb{Z}} e^{-\frac{\pi}{\tau_2} (\frac{1}{R})^2 |\tilde{m} + \tau n|^2} (-)^{(\tilde{m}+a)(n+b)} . \quad (5.27)$$

By performing the h -summation, the summation over \tilde{m} is reset to be unconstrained :

$$\frac{1/R}{\sqrt{\tau_2}} \sum_{\tilde{m}, n \in \mathbb{Z}} e^{-\frac{\pi}{\tau_2} (\frac{1}{R})^2 |\tilde{m} + \tau n|^2} (-)^{(\tilde{m}+a)(n+b)} . \quad (5.28)$$

This is precisely of the form (5.21) with a redefinition of the radius, $R \rightarrow 1/R$, and a chirality change, $(-)^{ab}$. This demonstrates that the Scherk-Schwarz mechanism can be nicely formulated as a freely-acting orbifold.

Of course, the same mechanism can be applied to break gauge symmetries. The idea there is to correlate a charge of this symmetry with translations along a compact direction. The result is precisely the effect of a Wilson line which, depending on the presence or not of the associated marginal operator in the spectrum of the theory, can be continuous or discrete.

Chapitre 6

MSDS Constructions : Algebra & Vacuum Classification

In this chapter we discuss work published in [45] and [46]. After a brief motivational introduction, we discuss the properties of an exotic class of 2d string vacua exhibiting a novel degeneracy structure in their bosonic and fermionic spectra. This novel degeneracy structure, which was termed Massive Spectral bose-fermi Degeneracy Symmetry (MSDS), is reminiscent of the degeneracy structure of supersymmetric constructions in the massive sector. Their massless spectra, however, are non-supersymmetric and will be shown in Chapter 7 to correspond to a spontaneous breaking of supersymmetry, of the Scherk-Schwarz type. As we discuss in later chapters, the highly non-trivial cancellations between boson-fermion states taking place at high mass levels permit one to probe into the stringy behaviour around extended symmetry points and, in particular, to perform exact 1-loop vacuum calculations, which otherwise would have been extremely difficult.

6.1 High Curvature and High Temperature Phases of String Theory

In this section, we begin by briefly motivating the use of string theory in describing the very early state of the universe. As is well-known from classical cosmology, the naive extrapolation of the equations of motion of general relativity backwards in time, unavoidably encounters an unphysical, space-like singularity. On physical grounds, one would expect such gravitational singularities to arise as artifacts of our insisting on using classical physics in a region where quantum gravitational effects become dominant. It is then natural to assume that, once general relativity is replaced by some more fundamental theory of quantum gravity -such as string theory-, the above-mentioned pathologies will be eventually smeared out. We mention here the well-known example from ordinary quantum mechanics, where quantum fluctuations protect a particle from falling down the throat of a singular potential.

In field theory one typically works with a finite number of fields. From the string-theoretic perspective, an effective action is obtained after integrating out modes with masses of the

order of the string scale. However, in cosmological settings the moduli fields are expected to acquire time dependence and as one moves along the moduli space, one may encounter points where otherwise massive modes become massless. Our approximation then breaks down precisely around these points of extended symmetry, where the extra massless states appear, since one should certainly not expect the extra modes to be captured by the original effective field theory. In this sense, string theory connects distinct field theories in different regions of moduli space. One may then imagine that gravitational singularities are precisely signals of this break-down. From the point of view of the underlying string theory, where all modes are properly taken into account, it should be possible to connect the two distinct effective theories in a smooth way. For discussions, see [43], [48].

The study of string theory in the strong curvature and high temperature regime, relevant for the early stage of the universe, is expected to provide a fundamentally different picture of spacetime physics than the one implied by the field theory approximation. This is mainly due to the fact that classical notions, such as geometry, topology and even space-time dimensionality are only well-defined notions as effective low-energy approximation of string theory (see, for example, [38], [43] and references therein).

In the strong curvature and high temperature phases of string theory, where field-theoretic notions break down, new purely stringy phenomena are expected to take place. As discussed above and also in earlier chapters, the higher-derivative α' -expansion ceases to be trustworthy in such regions. The question then arises whether it is still possible to probe the physical properties of the theory by resumming the expansion, taking properly into account all relevant states. The answer is that, very often, in such regions of moduli space, there exist exact descriptions in terms of exactly solvable CFTs, which in many cases do not even require an underlying geometric interpretation. The new stringy phenomena one encounters are, essentially, due to the presence of extended symmetry points in the moduli space of string theory. A thorough understanding of string theory in these regimes may be the first step in dynamically connecting the early stringy phase of the universe with standard cosmological and phenomenological theories at late times.

Stringy Behaviour : an Example

Let us give a simple example of an inherently stringy behaviour where a geometrical interpretation is ambiguous. Consider a WZW model on the $SU(2)_k$ group manifold :

$$S_{WZW} = \frac{1}{4g^2} \int_{\Sigma} d^2\sigma \operatorname{Tr} (\partial_{\mu} g \partial^{\mu} g^{-1}) + \frac{ik}{8\pi} \int_{\partial\Sigma} d^3\sigma \operatorname{Tr} (U \wedge U \wedge U) , \quad (6.1)$$

where $g(X)$ is an element of the $SU(2)$ group and $U = g^{-1}dg$ is the Maurer-Cartan 1-form. The Wess-Zumino term is over any 3d boundary of the 2d manifold Σ defining the CFT. Since there can be many different boundaries for the same 2d manifold Σ , consistency requires that they should all yield the same path integral. This leads to a quantization condition ¹

1. This can be proven easily for the $SU(2)$ group and is then sufficient to extend the quantization to any $SU(2)$ subgroup of more general group manifolds.

on the coefficient k . Furthermore, for $g^2 = 4\pi/k$ the theory becomes conformally invariant at the quantum level, [44], and the Maurer-Cartan 1-forms $J = g^{-1}dg$, $\bar{J} = \bar{d}g g^{-1}$ become the chiral $(1,0)$ - and $(0,1)$ - conserved currents of an affine $SU(2)_k$ algebra, with k being identified as the level. We can parametrize the $SU(2)$ manifold by a round sphere S^3 with radius R and NS flux :

$$\begin{aligned} ds^2 &= R^2 (d\psi^2 + \sin^2 \psi d\Omega_2) , \\ B &= R^2 \left(\psi - \frac{1}{2} \sin 2\psi \right) \sin \theta d\theta \wedge d\phi . \end{aligned} \quad (6.2)$$

Here, $d\Omega_2 = d\theta^2 + \sin^2 \theta d\phi^2$ is the line element on S^2 . The beta functions for the metric and antisymmetric tensor vanish identically for this background, $\beta_G = \beta_B = 0$, whereas the beta function for the dilaton yields the (constant) central charge of the CFT, is $\beta_\Phi = c = 3k/(k+2)$, as is expected for a $SU(2)_k$ WZW model. Let us now consider the case $k = 1$. The sigma model description implies that the target space is a highly curved 3d sphere with one unit of H -flux :

$$\frac{1}{4\pi^2 \alpha'} \int_{S^3} H = \frac{R^2}{\alpha'} = 1 . \quad (6.3)$$

However, the central charge of the CFT is $c = 1$ and the target space can, hence, be equivalently parametrized as a compact scalar X taking values on a circle, S^1 , at the self-dual radius. This is an example of how geometry, topology and even dimensionality of the target space are not uniquely defined close to extended symmetry points. An unambiguous description is only recovered in the low curvature (large radius) limit, $k = R^2/\alpha' \rightarrow \infty$. There the central charge (effectively measuring the curvature, as discussed in Section 2.1) becomes effectively $c \simeq 3$ and the space is a 3-dimensional sphere with H -flux. Hence, we see that a geometrical description emerges in the large moduli limit. In the same way, we see that new non-compact dimensions do emerge in the decompactification limit, through large marginal deformations which may arise naturally in a cosmological setting.

Absence of Vacuum Selection Mechanism

Now let us return to the problem of choosing the initial string vacuum. String theory already is well-known to be plagued by a plethora of consistent vacua and no satisfactory selection mechanism has been proposed to pick the ‘physical’ vacuum. Determining the initial string vacuum in the context of string cosmology is a very ambitious program and will naturally be very sensitive to the details of each particular choice. Even so, let us try to list some of the generic properties which would be desirable for candidate vacua.

We would like the starting points to be exactly solvable CFTs, which should admit a (Euclidean) thermal interpretation so that we can use them for the description of thermal vacua. The properties of thermal vacua will be discussed in more detail in the next chapter. Such CFTs should ideally probe regions where the compactification radii are close to the

string scale and where the stringy phenomena appear. In thermal settings, we will be mostly interested in temperatures around the Hagedorn point, which is the temperature radius where the first (low-lying) string (winding) state becomes tachyonic in the Euclidean description of the theory. Ideally, good candidates would be free of tachyonic (or Hagedorn) divergencies so as to enable a perturbative treatment of the backreaction². Eventually, the initial vacuum should be continuously connected to 4d quasi-realistic vacua with $\mathcal{N}_4 = 1$ spontaneously broken supersymmetry at late cosmological times, containing 3 generations of chiral matter in representations of some grand unified (GUT) gauge group, such as the $SO(10)$. The latter requirement, however, will not be of major concern to us here, as we will be primarily interested in the more theoretical aspects of the initial vacuum construction. Namely, the two problems that we will be discussing are the Hagedorn instabilities of string theory at high temperature and the gravitational ‘Big Bang’ singularity, present in all field-theoretic treatments.

As these will be discussed thoroughly in the following chapters, here we will only briefly mention that the presence of tachyonic (or Hagedorn) instabilities, [40], signals a stringy phase transition into a new vacuum (thermodynamical phase), characterized by non-trivial winding number. Therefore, one could try to directly construct vacua with non-trivial winding number that are free of the above pathologies. Let us note that the absence of physical tachyons from the string spectrum is essentially equivalent [41], [42] (at least in 2d vacua) to an asymptotic degeneracy, $n_B - n_F \rightarrow 0$, in the difference between the numbers of bosonic and fermionic states at large mass levels. The desired initial vacua should then be such that supersymmetry is broken in the low-mass spectrum in such a way that at high mass levels the boson-fermion degeneracy is asymptotically restored. These considerations provided the motivation for the discovery and construction of vacua [45], [46], with a novel Massive Spectral boson-fermion Degeneracy Symmetry (MSDS). This family of string vacua satisfies several of the above requirements, and will be introduced below.

6.2 Maximally Symmetric MSDS Vacua

In the previous short section we essentially gave brief arguments motivating the construction of vacua with spontaneously broken supersymmetry, yet characterized by an asymptotic degeneracy structure :

$$n_B - n_F \rightarrow 0 \quad , \quad m^2 \rightarrow \infty \quad , \quad (6.4)$$

between the numbers n_B, n_F of bosonic and fermionic states, respectively, at high mass level m . Such constructions are typically tachyon-free, at least in the regions in moduli space where the above asymptotic degeneracy holds. The reason is that modular invariance, by means of an S -transformation, relates the degeneracy of the low-lying spectrum to the asymptotic

2. In thermal settings or settings where supersymmetry is spontaneously broken, the string vacuum is typically unstable and its backreaction, in the form of an effective potential coming from contributions of higher-genuses, may give rise to time-dependent, cosmological solutions.

degeneracy at high mass levels. Consider the generic form of the mass-generating partition function of the CFT :

$$Z_{CFT}(\tau, \bar{\tau}) = \sum_i (-)^{F_i} q^{\Delta_i - c/24} \bar{q}^{\bar{\Delta}_i - \bar{c}/24} . \quad (6.5)$$

Set $\tau = it$, and consider the limit $t \rightarrow 0$. Then, $q \rightarrow 1$ and the convergence is precisely dominated by the behaviour of degeneracies at high mass levels, $n_B - n_F$. The simplest way to make the sum converge is then to require asymptotic supersymmetry $n_B - n_F \rightarrow 0$. However, due to modular invariance, this is equal to the same expression with $t \rightarrow t' = 1/t$ and $q \rightarrow q' = \exp(-2\pi/t)$. Taking the limit $t \rightarrow 0$ now receives its dominant contribution from the lowest-mass states (IR) :

$$Z_{CFT} \sim \exp \left[-\frac{2\pi}{t} \min \left(\Delta + \bar{\Delta} - \frac{c + \bar{c}}{24} \right) \right] . \quad (6.6)$$

For this to converge, the low-lying spectrum has to start at least from the massless level. In supersymmetric theories this is automatically the case and is consistent, of course, with the exact degeneracy structure $n_B - n_F = 0$ at all mass levels. It is then clear that the presence of tachyons (states with negative mass square) will lead to IR divergences. Hence, asymptotic supersymmetry at high mass levels guarantees [41] the absence of tachyonic states in the low-lying spectrum³.

Our aim is then to construct such ‘minimal’ models with spontaneously broken supersymmetry, at specific points in moduli space admitting an exact CFT description, so that they can serve as laboratories where inherently stringy effects can be studied. Taken as starting points for more extensive studies (for example considering their marginal deformations, as in the next chapter) MSDS models were introduced as appealing candidates in an attempt to describe the initial non-geometrical era of the universe. Their construction and classification, which we will describe below, was given in [46] and opened the way for more elaborate constructions in terms of \mathbb{Z}_2^N -orbifolds which preserve the MSDS structure.

Introduction to MSDS Vacua

The MSDS vacua correspond to 2d compactifications on $\mathcal{M}^2 \times K$, where the compact space K is described by an exact $\hat{c} = 8$ CFT of free fermions. That is, we consider all 8 internal coordinates to be compactified at the ‘fermionic’ radius $R_i = \sqrt{\alpha'}/2$. This is precisely the point where the compact worldsheet bosons can be consistently fermionized.

As discussed in section 3.4, fermionization proceeds by realizing the abelian $U(1)^8$ current algebra of transverse bosons ∂X^I in terms of free worldsheet fermions which realize a global $G = \widehat{SO}(24)_{k=1}$ affine algebra. Then, preservation of the $N = 1$ worldsheet SCFT forces us

3. Asymptotic supersymmetry is a solution to the Kutasov-Seiberg theorem [41], however, it is not the only solution. Later work, [42], has shown that the absence of tachyonic modes actually implies an underlying ‘misaligned symmetry’ between the bosonic and fermionic degeneracy structure. Its subtle cancellations account for the convergence of the trace and also guarantee the absence of tachyons.

to gauge the original symmetry G down to a subgroup H , such that G/H is a symmetric space and with the fermions transforming in the adjoint representation of the semi-simple Lie algebra of H , of dimension $\dim H = 24$. The possible gaugings are $SU(2)^8$, $SU(5)$, $SO(7) \times SU(2)$, $G_2 \times Sp(4)$, $SU(4) \times SU(2)^3$, $SU(3)^3$. In these constructions we only use the gauging of maximal rank $H = SU(2)_{k=2}^8$, which will admits the ‘broader’ possible deformations.

The maximal MSDS vacua [45] are obtained by assigning the same spin-structures $[a]_b$ to all 24 (real) worldsheet fermions. The modular invariant partition functions for Type II theories is then :

$$\begin{aligned} Z_{\text{II}} &= \frac{1}{2^2} \sum_{a,b=0,1} (-)^{a+b} \frac{\theta[a]_b^{12}}{\eta^{12}} \sum_{\bar{a},\bar{b}=0,1} (-)^{\bar{a}+\bar{b}} \frac{\bar{\theta}[\bar{a}]_{\bar{b}}^{12}}{\bar{\eta}^{12}} \\ &= (V_{24} - C_{24}) (\bar{V}_{24} - \bar{C}_{24}) = 576 . \end{aligned} \quad (6.7)$$

Similarly, for Heterotic theories :

$$Z_{\text{het}} = \frac{1}{2} \sum_{a,b=0,1} (-)^{a+b} \frac{\theta[a]_b^{12}}{\eta^{12}} \Gamma_{H_R} = 24 \times (d_{H_R} + [\bar{j}(\bar{z}) - 744]) , \quad (6.8)$$

where Γ_{H_R} is an antichiral modular invariant lattice associated to the gauge group, H_R . The number of anti-chirally massless states, d_{H_R} , is equal to 1128 for $H_R = SO(48)$ and 744 for $H_R = SO(32) \times E_8$ or $H_R = E_8^3$. Notice the combination $(j(z) - 744)$ involving the Klein invariant function, (A.11), which is however eliminated⁴ by the τ_1 -integration (imposing level-matching).

By explicitly performing the expansion over Fourier modes in (6.7) and (6.8), we verify an astonishing property of MSDS vacua. Modulo the Klein j -invariant in the Heterotic case, the partition function is given by constant integer showing that there is an equal number of bosons and fermions $n_B = n_F$ at all massive levels, with the notable exception of the massless sector. That this observation carries on at all massive levels is the consequence of a remarkable identity between Jacobi θ -functions :

$$\frac{1}{2} \sum_{a,b} (-)^{a+b} \frac{\theta^{12}[a]_b}{\eta^{12}} = V_{24} - C_{24} = 24 , \quad (6.9)$$

which can be proven by using the ‘Abstrusa’ and ‘Triple Product’ identities, (A.8), (A.9). Thus, the MSDS structure arises from a generalized Jacobi identity $V_{24} - C_{24} = 24$, generalizing the ‘abstrusa’ identity $V_8 - C_8 = 0$ which expresses the triality of $SO(8)$ in the case of conventional supersymmetry.

Even though supersymmetry is broken in these constructions, they satisfy in an exact way the requirement of asymptotic supersymmetry at all massive levels and so, they are

4. This is essentially true modulo subtleties having to do with the integration near the boundaries of the fundamental domain, which we will not discuss here. For the purposes of this discussion let us note merely that these states are spurious and decouple from physical amplitudes.

free of tachyons⁵. In these maximal models where the $H_L \times H_R$ gauge symmetry remains unbroken, these are all bosonic. This is another example of the fact that the presence of extended gauge symmetry in Type II theories results in massive fermions arising from the same sector.

The generic structure of the partition function in generic MSDS models can be shown to take the form :

$$Z = m + n(\bar{j}(\bar{q}) - 744) , \quad (6.10)$$

where $m, n \in \mathbb{Z}$ and $m = n_B - n_F$ at the massless level. This is essentially the case because in models with vanishing physical state contribution, the physical states bequeath the same degeneracy structure to their unphysical descendants and the partition function can only be equal to a constant, modulo a rational function of the Klein modular invariant. The pole structure of Type II theories reduces the latter function to a constant, whereas in Heterotic theories this function can be at most linear.

We will argue below that MSDS can be regarded as a purely stringy ‘enhancement’ of the ordinary current algebra of supersymmetry. In the case of ordinary supersymmetry an enhanced worldsheet SCFT gave rise to a spectral flow, mapping bosonic states to fermionic and vice-versa. The spectral-flow operator was precisely identified with the currents generating spacetime supersymmetry. In the present case, the chiral nature of the $SO(24)$ character identity hints at the presence of a chiral $(1,0)$ -current, $j(z)$, responsible for the mapping of massive bosonic to massive fermionic representations, while leaving massless states invariant. Of course, as argued before, the presence of a non-abelian super Kac-Moody algebra implies that fermionic states are already massive and this is in accordance with the fact that the spinor representation, S_{24} , starts with conformal weight $(\frac{3}{2}, 0)$ and is, hence, massive.

Let us now construct the spectral-flow operator, $j(z)$, responsible for the mapping between the vectorial and spinorial representations of the $SO(24)$ current algebra. In ordinary supersymmetry, the relevant current algebra is $SO(8)$ and the vectorial representation, V_8 , gets mapped into the spinorial, S_8 . Let us see if something similar may arise more generally. The spectral-flow operators for conventional supersymmetry at the fermionic point can be expressed as a spin field⁶ :

$$j_{\text{SUSY}}(z) = e^{-\phi/2 - \frac{i}{2}H_0} C_8(z) \quad , \quad \tilde{j}_{\text{SUSY}}(z) = e^{-\phi/2 + \frac{i}{2}H_0} S_8(z) . \quad (6.11)$$

This is the case for $SO(8)$. Now let us see now if it is possible to do better for the generic $SO(N)$ representations. Disregarding the ghost and longitudinal dressing and using (A.17),

5. Note that this is only true as long as the MSDS structure is unbroken, hence, at a very particular point in moduli space. In the next chapter we study deformations away from this point and study the conditions for absence of tachyonic excitations.

6. Note that, according to our conventions, (2.166), the S -spinor of $SO(8)$ has an even number of minus signs, whereas the C -spinor is built out of an odd number of minus signs.

we calculate the action of the spectral-flow operator $S_N(z)$ on the spinorial state $C_N(z)$:

$$\begin{aligned}
S_N(z) C_N(w) &= \frac{1}{\sqrt{2}} \frac{\gamma^a \psi^a(w)}{(z-w)^{N/8-1/2}} \\
&+ \frac{1}{2\sqrt{2}} \frac{\gamma^{abc} \psi^a \psi^b \psi^c(w) + \gamma^a \partial \psi^a(w)}{(z-w)^{N/8-3/2}} \\
&+ \frac{1}{4\sqrt{2}} \frac{\gamma^{abcde} \psi^a \psi^b \psi^c \psi^d \psi^e + \gamma^{abc} \partial(\psi^a \psi^b \psi^c) + \gamma^a \partial^2 \psi^a}{(z-w)^{N/8-5/2}} + \dots, \quad (6.12)
\end{aligned}$$

with $a = 1, \dots, N$. Notice that the above OPE is correct for $N \in 4\mathbb{Z}$ only⁷. The first term in the OPE is, of course, the ground state of the vectorial representation, ψ^a of weight $(\frac{1}{2}, 0)$, giving rise to (holomorphically) massless states. The second term is the first excitation of the vectorial representation with weight $(\frac{3}{2}, 0)$, given by $\psi_{-3/2}^a \oplus J_{-1}^{ab} \psi_{-1/2}^c$, with $J^{ab} = i\psi^a \psi^b$ being the global $SO(N)$ currents of the Kac-Moody algebra. Similarly, the third term represents the second excited state, of weight $(\frac{5}{2}, 0)$, and so on. For $SO(8)$ the spinorial representation, C_8 , has $2^{8/2-1} = 8$ states and the spectral-flow operator S_8 transforms them all into the 8 states of the vectorial representation ψ^a .

Notice that the vectorial representation always carries weight $(\frac{1}{2}, 0)$, independently of the value of N . The same is not true for the spin-fields S_N, C_N whose conformal weight is $(\frac{N}{16}, 0)$ and they become massive for $N > 8$. Even so, the question that we would like to ask is whether it is possible for the $2^{N/2-1}$ states in the massive spinorial representation C_N to be mapped (isomorphically) into some massive descendant representation of the vectorial family.

Note first of all that the role of the superghost and longitudinal dressing factors is simply to correct the conformal weights of the vertex operator and remove branch cuts. Depending on the ghost picture, the total dressing can only provide corrections of the type $(z-w)^{k/2}$, with $k \in \mathbb{Z}$. This implies that the mutual locality constraint of operator products for the ‘internal’ part of the vertex operator translates to the condition $N \in 4\mathbb{Z}$. Since C_N has to be mapped to some massive descendant, the weight of the latter has to be equal to $\frac{N}{16}$. Use M to denote the M ’th descendant with $M = 0$ corresponding to the massless primary. Then,

$$\frac{N}{16} = \frac{2M+1}{2} + \frac{1}{2} \mathbb{Z}, \quad (6.13)$$

and we obtain the further condition $N \in 8\mathbb{Z}$. This already hints to the fact that there are $SO(8)$ subsectors whose spectral flow arises naturally because of the supersymmetric spectral-flow (triality).

Next we need to see if all states in C_N can be fit within a single vectorial descendant :

$$2^{N/2-1} = \sum_{p=0}^M \binom{N}{2p+1}. \quad (6.14)$$

7. For $N \in 4\mathbb{Z} + 2$, the charge structure would be such that the r.h.s. would start with the vacuum representation and, hence, the spinor would not be mapped to the vector representation.

It can be shown that this has no solution except for the pairs $(N, M) = (8, 0)$ and $(N, M) = (24, 1)$. The first is the case of supersymmetry, formally manifesting itself in terms of the triality of $SO(8)$. The second solution is the case of MSDS, associated to the $SO(24)$ affine algebra. The $2^{24/2-1}$ states in the massive spinorial C_{24} become mapped to the $24 + \binom{24}{3}$ states of the weight- $(\frac{3}{2}, 0)$ first vectorial descendant. At the level of characters, the 24 massless scalars in $V_8 \sim \psi^a$ remain unpaired, whereas all massive levels are bose-fermi degenerate and this is the origin of the simple relation $V_{24} - C_{24} = 24$.

Now let us try to write down the analogous superalgebra satisfied by the MSDS spectral-flow operators and compare with the case of ordinary supersymmetry. The two $(1, 0)$ -currents of opposite (longitudinal) helicity are :

$$j_\alpha(z) = e^{\phi/2 - \frac{i}{2}H_0} S_{24,\alpha}(z) \quad , \quad \tilde{j}_{\dot{\alpha}}(z) = e^{\phi/2 + \frac{i}{2}H_0} C_{24,\dot{\alpha}}(z) \quad . \quad (6.15)$$

The helicity charges are such that the currents respect the GSO-projection, i.e. they carry integral GSO charges. Directly evaluating their OPE one finds :

$$\begin{aligned} j_\alpha(z)j_\beta(0) &= \frac{e^\phi}{\sqrt{2} z^3} \Gamma_\mu \left[\Psi^\mu + \frac{z}{2} \partial \Psi^\mu + \frac{z^2}{4} \partial^2 \Psi^\mu + \dots \right] \\ &\times \left[c_{\alpha\beta} + \frac{z}{2} \gamma_{\alpha\beta}^{ab} \psi^a \psi^b + \frac{z^2}{4} \left(\gamma_{\alpha\beta}^{abcd} \psi^a \psi^b \psi^c \psi^d + \gamma_{\alpha\beta}^{ab} \partial(\psi^a \psi^b) + \frac{1}{2} c_{\alpha\beta} \partial \psi^a \psi^a \right) \right] + \dots \\ &= \dots + \frac{e^\phi}{\sqrt{2} z} \Gamma_\mu \left[\Psi^\mu \left(\gamma_{\alpha\beta}^{abcd} \psi^a \psi^b \psi^c \psi^d + \gamma_{\alpha\beta}^{ab} \partial(\psi^a \psi^b) + \frac{1}{2} c_{\alpha\beta} \partial \psi^a \psi^a \right) \right. \\ &\quad \left. + \frac{1}{4} \partial \Psi^\mu \gamma_{\alpha\beta}^{ab} \psi^a \psi^b + \frac{1}{4} \partial^2 \Psi^\mu c_{\alpha\beta} \right] + \dots \quad , \end{aligned} \quad (6.16)$$

where in the last two lines we display only the simple pole contribution, which is the one contributing to the $\{Q_\alpha, Q_\beta\}$ anticommutator of the charges. Note that α, β are regarded here as internal indices, playing the analogue of an $SO(24)$ R -symmetry that rotates the MSDS ‘supercharges’. Similarly, one finds :

$$\begin{aligned} j_\alpha(z)\tilde{j}_{\dot{\beta}}(0) &= \frac{e^\phi}{\sqrt{2} z^3} \left[1 + \frac{z}{2} \Gamma_{\mu\nu} \Psi^\mu \Psi^\nu + \frac{z^2}{4} \left(\Gamma_{\mu\nu} \partial(\Psi^\mu \Psi^\nu) + \frac{1}{2} \partial \Psi^\mu \Psi_\mu \right) + \dots \right] \\ &\times \left[\gamma_{\alpha\dot{\beta}}^a \psi^a + \frac{z}{2} \left(\gamma_{\alpha\dot{\beta}}^{abc} \psi^a \psi^b \psi^c + \gamma_{\alpha\dot{\beta}}^a \partial \psi^a \right) + \frac{z^2}{4} \left(\gamma_{\alpha\dot{\beta}}^{abcde} \psi^a \psi^b \psi^c \psi^d \psi^e + \gamma_{\alpha\dot{\beta}}^{abc} \partial(\psi^a \psi^b \psi^c) + \gamma_{\alpha\dot{\beta}}^a \partial^2 \psi^a \right) \right] + \dots \\ &= \dots + \frac{e^\phi}{4\sqrt{2} z} \left[\left(\Gamma_{\mu\nu} \partial(\Psi^\mu \Psi^\nu) + \frac{1}{2} \partial \Psi^\mu \Psi_\mu \right) \gamma_{\alpha\dot{\beta}}^a \psi^a + \Gamma_{\mu\nu} \Psi^\mu \Psi^\nu \left(\gamma_{\alpha\dot{\beta}}^{abc} \psi^a \psi^b \psi^c + \gamma_{\alpha\dot{\beta}}^a \partial \psi^a \right) \right. \\ &\quad \left. + \gamma_{\alpha\dot{\beta}}^{abcde} \psi^a \psi^b \psi^c \psi^d \psi^e + \gamma_{\alpha\dot{\beta}}^{abc} \partial(\psi^a \psi^b \psi^c) + \gamma_{\alpha\dot{\beta}}^a \partial^2 \psi^a \right] + \dots \quad , \end{aligned} \quad (6.17)$$

where, again, only the simple pole contribution is displayed in the last two lines. This represents the $\{Q_\alpha, \tilde{Q}_{\dot{\beta}}\}$ anticommutator.

Let us now compare the above with the case of ordinary supersymmetry in two dimensions. The first difficulty is that the r.h.s. appears in the non-canonical $(+1)$ -ghost picture.

Acting with the picture changing operator ($e^\phi T_F$) on the 0-picture translation operator ∂X^μ and keeping the zeroth-order term we find $e^\phi \Gamma_\mu \partial^2 \Psi^\mu$. This already appears in the $j_\alpha \tilde{j}_\beta$ -OPE, (6.16). This differs from the supersymmetric case in the presence of additional terms, built out of the global $SO(24)$ rotation currents $J^{ab}(z)$. Similarly, looking at the $j_\alpha \tilde{j}_\beta$ -OPE, (6.17), the r.h.s. contains (modulo other terms) a translation in the internal space and a 2d spacetime rotation. This spectral-flow algebra looks considerably different from the simple current algebra of supersymmetry, which is not very surprising.

Next we discuss the action of the MSDS operators on the spinorial representation C_{24} . The vertex operator of the massive, weight $(\frac{3}{2}, 0)$ spinorial representation C_{24} is :

$$e^{-3\phi/2 + \frac{i}{2}H_0} C_{24,\alpha}(z). \quad (6.18)$$

We act on this with the MSDS current and keep only the simple pole :

$$j_\alpha(z) e^{-3\phi/2 + \frac{i}{2}H_0} C_{24,\beta}(0) \rightarrow \dots + \frac{e^{-\phi}}{2\sqrt{2}z} \left(\gamma_{\alpha\beta}^{abc} \psi^a \psi^b \psi^c + \gamma_{\alpha\beta}^a \partial \psi^a \right) + \dots, \quad (6.19)$$

where we dropped an irrelevant term which depends on the longitudinal fermions, Ψ^μ , and hence decouples. Thus, the entire massive fermionic tower of states generated by the spinorial representation and its descendants is mapped into the bosonic tower of states generated by the first vectorial descendant. Vice-versa, by considering the action of the MSDS current on the massless vectorial, one obtains a regular OPE, implying that the 24 massless scalars remain unpaired. Of course, acting with the spectral-flow current j on the (massive) first descendant vertex operator in the vectorial representation :

$$V_{24,(1)}(z) = e^{-\phi} \left(\gamma_{\alpha\beta}^{abc} \psi^a \psi^b \psi^c + \gamma_{\alpha\beta}^a \partial \psi^a \right) \equiv e^{-\phi} \mathcal{D}\psi, \quad (6.20)$$

one immediately obtains a simple pole term with the (massive) spinorial representation, C_{24} , which completes the inverse argument.

The zero mode of $j_\alpha(z)$ defines a conserved MSDS charge, Q_{MSDS} , ensuring the mapping of massive states at the same mass level. What we have seen is that the SCFT algebra admits a spectral flow generated by the currents $j_\alpha(z)$. Comparing these with the currents generating spacetime supersymmetry in the $(+\frac{1}{2})$ -picture, one sees that the MSDS algebra can be regarded as an enhancement of the supersymmetry current algebra at the fermionic point of moduli space.

6.3 \mathbb{Z}_2^N -Orbifolds with MSDS Structure

Up to now the construction of vacua with MSDS structure has been motivated on the grounds of a ‘cosmological conjecture’, according to which, the universe in its very ‘early’ stage was two- (or even one-) dimensional, with at least 8 dimensions compactified at the string scale. Depending on the nature of the supersymmetry breaking mechanism, quantum and thermal corrections may generate a potential at the perturbative string level. Under

certain conditions, related to the specific details of the initial vacuum, this potential may drive some of the compact dimensions to decompactify so that one eventually obtains a higher-dimensional theory. If, in particular, some of the decompactifying dimensions are identified with the compact cycles controlling the supersymmetry breaking scale, as in the Scherk-Schwarz mechanism, a supersymmetric vacuum will be asymptotically recovered in the infinite radius limit.

The maximally symmetric MSDS vacua discussed in the previous section have supersymmetry broken spontaneously, as one may verify by noting that the spacetime gravitini are still present in the massive string spectrum. Indeed, the partition function (6.8) may be rewritten, e.g. for the $E_8 \times E_8$ as :

$$Z_{\text{het}} = \frac{1}{2} \sum_{a,b=0,1} (-)^{a+b} \frac{\theta_{[b]}^{[a]^4}}{\eta^{12}} \Gamma_{(8,8)[b]} (\Gamma_{E_8})^2 , \quad (6.21)$$

where we separate the internal (8,8)-lattice, coupled to the R -symmetry charges :

$$\Gamma_{(8,8)[b]} = \frac{1}{2} \sum_{\gamma,\delta} \theta^8[b] \bar{\theta}^8[\gamma] . \quad (6.22)$$

Decomposing this into $SO(8)$ -characters, we find the massive gravitino states :

$$\left(-C_8 \Gamma_{(8,8)[\binom{1}{+}] } - S_8 \Gamma_{(8,8)[\binom{1}{-}] } \right) \bar{O}_{16} \bar{O}_{16} , \quad (6.23)$$

where we use the linear combinations :

$$\Gamma_{(8,8)[\binom{a}{\epsilon}]} = \frac{1}{2} \sum_{b=0,1} (-)^{(\frac{1-\epsilon}{2})b} \Gamma_{(8,8)[b]} , \quad (6.24)$$

of the (8,8)-lattice with definite \mathbb{Z}_2 -GSO parity, $\epsilon = \pm 1$. Adopting a four-dimensional counting, we see that there are 4 gravitini of each chirality. In the limit of large deformations where supersymmetry is recovered in 4d, one expects to recover an $\mathcal{N}_4 = 4$ supersymmetric vacuum. Hence, anticipating the eventual decompactification to four dimensional supersymmetric vacua, we see that ‘maximal’ Heterotic MSDS vacua have spontaneously broken $\mathcal{N} = 4$ supersymmetry (in the Type II, $\mathcal{N} = 8$).

It would be phenomenologically desirable for the initial vacuum to dynamically decompactify and eventually flow to chiral 4d theories with spontaneously broken $\mathcal{N}_4 = 1$ supersymmetry. One way to reduce the number of eventually recovered supersymmetries is via non-freely-acting orbifolds which project out some of the gravitini. These orbifolds can be introduced in the initial MSDS era. However, their action will not generically preserve the MSDS structure. This opens the problem of constructing MSDS-preserving orbifolds of these prototype models. Similarly, it is also interesting and useful to consider the action of freely-acting orbifolds on the maximal MSDS models, since these correspond essentially to shifting the background parameters. For the case of \mathbb{Z}_2^N -orbifolds, all MSDS vacua were classified in [46].

We will not derive the classification here but only sketch the basic idea. In the untwisted sectors the orbifold simply introduces projections on a theory which originally enjoyed MSDS structure. Hence, in addition to the usual modular invariance constraints to be respected by the orbifold action, the only further requirement is the survival of at least one invariant spectral-flow operator, $j_\alpha(z)$. This is sufficient for the MSDS current to have a well-defined action on the spectrum. In the twisted sectors, the situation becomes more involved, giving rise to additional constraints. Essentially, the orbifold action has to be such that all chiral operator products involving j_α are local in the twisted sectors (no net branch cuts). Of course, since MSDS structure arises from a chiral algebra, any orbifold action correlating a chiral MSDS current with anti-chiral charges will immediately break⁸ MSDS.

The result is that whenever the boundary conditions and GGSO-projections respect the global, chiral definition of the MSDS spectral-flow operator and if, in addition to other consistency conditions, the boundary condition vectors b_a (defined via $\psi^a \rightarrow -e^{i\pi b_a} \psi^a$) satisfy the extra holomorphic constraints :

$$\begin{aligned} n_L(b) &= 0 \pmod{8} , \\ n_L(b_i \cap b_j) &= 0 \pmod{4} , \\ n_L(b_i \cap b_j \cap b_k) &= 0 \pmod{2} . \end{aligned} \tag{6.25}$$

the resulting string vacua preserve the MSDS structure. Essentially, this implies that real fermions have to be twisted in groups of 8. The derivation of these constraints as well as various examples of twist- and shift- orbifolds both in Heterotic and Type II theories can be found in [46]. Similar constructions are possible for more general \mathbb{Z}_N -orbifolds but we will not discuss them here.

Let us give a simple argument that permits one to derive the conditions (6.25). If we consider a general orbifold, the partition function will be decomposed into the sums of orbifold blocks :

$$Z = \frac{1}{|G|} \sum_{h_i, g_i} Z_{[g_i]}^{[h_i]} = \sum_{h_i} Z^{(h_i)} , \tag{6.26}$$

where $Z^{(h_i)}$ is the GSO- and orbifold-projected sector, twisted by $\{h_i\}$:

$$Z^{(h_i)} = \frac{1}{|G|} \sum_{g_i} Z_{[g_i]}^{[h_i]} . \tag{6.27}$$

Consider restricting these blocks to the holomorphic sector only. In general vacua, the $Z^{(h_i)}$ blocks are not modular invariant by themselves. The presence of MSDS structure, however, would imply that the $Z^{(h_i)}$ -blocks are constants, giving rise to chiral character identities. Hence, imposing the modular invariance directly on $Z^{(h_i)}$, before summing over the h_i -sectors, is a non-trivial condition which, can be seen to be equivalent to the condition

8. This is another way to see that MSDS structure (contrary to supersymmetry) breaks spontaneously after an infinitesimal deformation, because it can only be realized at the points of extended symmetry where the left- and right-moving CFTs factorize.

for the presence of MSDS structure. This is because every holomorphic modular invariant function can be expressed as a rational function of the Klein j -invariant, (A.11). However, the superconformal properties on the worldsheet⁹ determine this function to be a simple constant. Therefore, the conditions for MSDS structure are, essentially, obtained by the general conditions for modular invariance in models with complex fermions (general boundary conditions), by truncating these conditions to the holomorphic (or anti-holomorphic) sector.

Constructing orbifold MSDS models essentially involves the preservation of the spectral-flow in the twisted sectors. This is reflected by the presence of ‘twisted’ chiral MSDS identities, which are generalizations of the θ -function identity (6.9). Examples of such identities for a single \mathbb{Z}_2 are :

$$\begin{aligned} V_{16}O_8 - S_{16}C_8 &= 16 , \\ O_{16}V_8 - C_{16}S_8 &= 8 , \\ V_{16}C_8 - S_{16}O_8 &= 0 , \\ O_{16}S_8 - C_{16}V_8 &= 8 . \end{aligned} \tag{6.28}$$

In the case of $\mathbb{Z}_2 \times \mathbb{Z}_2$ -orbifolds the identities become even more involved (see [46]) :

$$\begin{aligned} V_8O_8O_8 - S_8C_8C_8 &= 8 , \\ O_8S_8O_8 - C_8V_8C_8 &= 8 , \\ V_{12}O_4O_4O_4 + O_{12}V_4V_4V_4 - S_{12}C_4C_4C_4 - C_{12}S_4S_4S_4 &= 12 , \\ V_{12}O_4V_4V_4 + O_{12}V_4O_4O_4 - S_{12}C_4S_4S_4 - C_{12}S_4C_4C_4 &= 4 , \\ V_{12}O_4S_4S_4 + O_{12}V_4C_4C_4 - S_{12}C_4V_4V_4 - C_{12}S_4O_4O_4 &= 0 , \\ V_{12}V_4S_4C_4 + O_{12}O_4C_4S_4 - S_{12}S_4V_4O_4 - C_{12}C_4O_4V_4 &= 4 . \end{aligned} \tag{6.29}$$

These identities can always be verified either by direct q -expansion¹⁰, or by using the Jacobi identities, (A.8), (A.9). For non- \mathbb{Z}_2 orbifolds θ -function identities relating different arguments need to be used as well (e.g. for \mathbb{Z}_4 -orbifolds one needs doubling formulas).

The above considerations permit the complete classification of Type II and Heterotic MSDS vacua. In particular, one may construct Heterotic MSDS models with an $SO(10)$ GUT gauge group and spontaneously broken $\mathcal{N} = 1$ supersymmetry in the presence of specific gravito/magnetic fluxes, [46]. The restrictions arising from imposing an unbroken MSDS structure are enormously constraining. Indeed, combined with the requirement of $SO(10)$ ‘visible’ gauge group, the MSDS constraints completely fix the holomorphic part of

9. This arises from the requirement that the pole structure in the q -expansion correctly reproduces the NS-vacuum.

10. Due to the modular properties of chiral characters, a function is uniquely determined by its first 2 terms in the q -expansion so it suffices to check only these.

the partition function. As an example, consider the following $T^8/\mathbb{Z}_2 \times \mathbb{Z}_2$ model :

$$Z = \frac{1}{\eta^{12} \bar{\eta}^{24}} \frac{1}{2^8} \sum_{\Gamma_\alpha, \Delta_\beta} \sum_{h_a, g_b} (-)^{a+b+HG+\Phi} \theta_{[b]}^a \theta_{[b+g_1]}^{a+h_1} \theta_{[b+g_2]}^{a+h_2} \theta_{[b-g_1-g_2]}^{a-h_1-h_2} \\ \times \Gamma_{(8,8)} \left[\begin{matrix} a, k \\ b, l \end{matrix} \middle| \begin{matrix} P, h_1, h_2, \psi \\ Q, g_1, g_2, \omega \end{matrix} \right] \underbrace{\bar{\theta}_{[l]}^5[k]}_{SO(10)} \underbrace{\bar{\theta}_{[l+g_1]}^{k+h_1} \bar{\theta}_{[l+g_2]}^{k+h_2} \bar{\theta}_{[l-g_1-g_2]}^{k-h_1-h_2}}_{U(1) \times U(1) \times U(1)} \times \underbrace{\bar{\theta}_{[\sigma]}^4[\rho] \bar{\theta}_{[\sigma+G]}^4[\rho+H]}_{SO(8) \times SO(8)}, \quad (6.30)$$

where $\Gamma_\alpha = (a, k, \rho)$, $\Delta_\beta = (b, l, \sigma)$, $h_a = (P, h_1, h_2, \psi, H)$, $g_b = (Q, g_1, g_2, \omega, G)$. The lattice can be expressed in terms of θ -functions :

$$\Gamma_{(8,8)} \equiv \theta_{[b+Q]}^{a+P} \theta_{[b+Q+g_1]}^{a+P+h_1} \theta_{[b+Q+g_2]}^{a+P+h_2} \theta_{[b+Q-g_1-g_2]}^{a+P-h_1-h_2} \times \bar{\theta}_{[l+\omega]}^5[k+\psi] \bar{\theta}_{[l+\omega+g_1]}^{k+\psi+h_1} \bar{\theta}_{[l+\omega+g_2]}^{k+\psi+h_2} \bar{\theta}_{[l+\omega-g_1-g_2]}^{k+\psi-h_1-h_2}. \quad (6.31)$$

The visible sector contains an $SO(10)$ gauge group¹¹, three $U(1)$'s and the hidden E_8 is decomposed into two $SO(8)$'s. This is a Heterotic theory and, so, MSDS conditions only fix the holomorphic part of the partition function.

There are 2^{29} choices for the modular invariant phase Φ , corresponding to possible GGSO projections. For certain choices of the projections the model is supersymmetric, whereas for others, supersymmetry can spontaneously break to an MSDS structure. A specific example is discussed further in Appendix B, where the translation between the fermionic and orbifold formalisms is explicitly performed. The particular model discussed there has the especially interesting property that it contains an abundance of massless fermionic states arising from the twisted sectors. The resulting partition function is negative and, hence, the generated genus-1 effective potential is positive at the MSDS point. This opens the possibility of constructing models in which the quantum instability naturally drives the system to decompactify some internal dimensions and, in the infinite limit, recover a supersymmetric vacuum.

The construction of MSDS models and the scan of their moduli space is only the first step in the ambitious project of connecting early string cosmology with late-time phenomenology. However, MSDS models are interesting for an additional (perhaps somewhat aesthetic) reason. They are the first examples of vacua where a purely stringy enhancement of the supersymmetry current algebra induces and completely determines the spontaneous supersymmetry breaking. Their most attractive feature is their continuous connection with $\mathcal{N} = 1$ superstring models in four dimensions [47]. We discuss their deformations in the next chapter.

11. The $SO(10)$ will presumably need to be broken at the string level at some later stage. We will not discuss this point here, but see [49] and references therein.

Chapitre 7

Deformations and Thermal Interpretation of MSDS Vacua

In the previous chapter we introduced MSDS constructions, as candidate vacua that may be used to describe the early non-geometrical era of the universe. The approach there was mostly algebraic, focusing on understanding the CFT nature of the MSDS spectral-flow and on obtaining a classification of its orbifolds. In this chapter, we proceed to examine the marginal deformations of these vacua and argue that they admit a natural thermal interpretation in the Euclidean. We then focus on the Hagedorn instabilities of high-temperature string theory and we discuss the conditions for a stable thermodynamical phase.

7.1 Finite Temperature

The usual prescription for studying finite temperature effects in field theory is to go to the Euclidean version of the theory, $t \rightarrow -i\tau$, where time is compactified on a circle of radius $R = 1/(2\pi T)$. This arises because the analytic continuation maps correlation functions in the quantum theory to thermal averages of the same theory at thermal equilibrium. The conventional thermal trace can be formulated, say in the case of a scalar, as a path integral over a periodic range $\tau \in [0, \beta]$:

$$Z(\beta) \equiv e^{-\beta F(\beta)} \equiv \text{Tr} [e^{-\beta H}] = \int_{\text{periodic}} \mathcal{D}\phi \exp \left(- \int_0^\beta d\tau \int d^{d-1}x \mathcal{L} \right), \quad (7.1)$$

where $F(\beta)$ is the Helmholtz free energy. Notice that the scalar $\phi(x)$ is forced to be periodic on the circle. For fermions, on the other hand, Grassman integration implies that the path integral has to be performed over anti-periodic configurations. The (anti-)periodicity of the thermal propagators :

$$G(\mathbf{x}, \tau) = \frac{1}{\beta} \sum_n \int \frac{d^{d-1}\mathbf{k}}{(2\pi)^{d-1}} e^{i\omega_n \tau + i\mathbf{k} \cdot \mathbf{x}} \tilde{G}(\mathbf{k}, \omega_n), \quad (7.2)$$

then implies that Matsubara frequencies are half-shifted for the fermions :

$$\omega_n = \frac{2\pi n}{\beta} \quad , \quad n \in \begin{cases} \mathbb{Z} & , \text{ bosons} \\ \mathbb{Z} + \frac{1}{2} & , \text{ fermions} \end{cases} . \quad (7.3)$$

Notice that this difference in the behaviour of bosons and fermions at finite temperature formally ‘breaks’ the supersymmetry of the Euclidean theory. Since this breaking is through boundary conditions, it formally corresponds to a spontaneous breaking of the Scherk-Schwarz type. This is exactly what we will find in the string case.

Furthermore, finite temperature only affects the IR properties of the theory, since at UV scales where $n/\beta \rightarrow 0$, the spectrum of thermal masses becomes asymptotically continuous and one recovers precisely the same behaviour as at zero-temperature.

We will in fact consider a slightly more general case. Consider a complex scalar or spinor coupled minimally to a constant $U(1)$ background¹ potential, A_μ . The path integral is still gaussian and can be performed explicitly (assuming we have put the system in a box) :

$$Z(\beta, gA_0) = \det^{\mp 1} (-\partial^2 + m^2) = N(\beta) \prod_{n, \mathbf{k}} \left[\left(\frac{2\pi n}{\beta} - gA_0 \right)^2 + \mathbf{k}^2 + m^2 \right]^{\mp 1} , \quad (7.4)$$

with the $(-)$ -sign for the scalar and $(+)$ -sign for the spinor. The expectation value of the gauge potential in the spatial directions, A_i , can always be gauged away. In the above expression it corresponds to a shift in the integration over the spatial momenta $k_i \rightarrow k_i + gA_i$. The same is not true, however, for the component A_0 of the gauge potential in the compact time direction. Indeed, there is a topological obstruction since a gauge transformation that would eliminate A_0 would effectively require changing the boundary conditions of the matter fields around the compact direction. Technically, this is reflected in the above expression by the fact that, contrary to the non-compact case, the summation over the discrete Matsubara modes cannot be shifted to absorb the A_0 -term.

Thus, A_0 is not a gauge artifact but has physical meaning as a vacuum parameter and should be regarded as such. Notice, furthermore, that the multiplicative factor has to be chosen as :

$$N(\beta, gA_0) = f(\beta gA_0) \prod_{n', \mathbf{k}} \beta^{\mp 2} \left(\frac{2\pi n'}{\beta} - gA_0 \right)^{\pm 2} , \quad (7.5)$$

in order to render the partition function dimensionless, with f being an arbitrary function. This yields for the bosonic case :

$$Z(\beta, gA_0) = f(\beta gA_0) \prod_{\mathbf{k}} \left(1 + \frac{\sinh^2 \frac{\beta \omega_{\mathbf{k}}}{2}}{\sin^2 \frac{\beta gA_0}{2}} \right)^2 , \quad (7.6)$$

1. At this stage we consider A_μ as a constant classical background, that is we do not give it a kinetic Yang-Mills term.

with the usual definition $\omega_{\mathbf{k}} \equiv \sqrt{\mathbf{k}^2 + m^2}$. The multiplicative function f is independent of the energy and is, hence, unphysical. It contributes a divergent, energy-independent term to the potential, which (in field theory) we normalize away². One finally finds :

$$\begin{aligned} Z(\beta, gA_0) &= \prod_{\mathbf{k}} \left| 2 \sinh \left(\frac{\beta(\omega_{\mathbf{k}} + igA_0)}{2} + i\pi \frac{1-\epsilon}{2} \right) \right|^{-2\epsilon}, \\ &= \prod_{\mathbf{k}} \frac{e^{-\epsilon\beta\omega_{\mathbf{k}}}}{|1 - \epsilon e^{-\beta(\omega_{\mathbf{k}} + igA_0)}|^{2\epsilon}} \equiv \text{Tr} \left[e^{-\beta H - 2\pi i \mu Q} \right], \end{aligned} \quad (7.7)$$

with $\epsilon = +1$ for bosons and $\epsilon = -1$ for fermions. We defined the dimensionless combination $\mu \equiv \beta g A_0 / 2\pi$ that couples to the $U(1)$ charge operator, Q .

From this one may easily calculate the free energy. Dropping the divergent (ground-state) term, we find :

$$\begin{aligned} F(\beta, \mu) &= \frac{2V\epsilon}{\beta} \int \frac{d^{d-1}\mathbf{k}}{(2\pi)^{d-1}} \log |1 - \epsilon e^{-\beta\omega_{\mathbf{k}} - 2\pi i \mu}| \\ &= \frac{V\epsilon}{\beta} \int \frac{d^{d-1}\mathbf{k}}{(2\pi)^{d-1}} \left[\log (1 - \epsilon e^{-\beta\omega_{\mathbf{k}} - 2\pi i \mu}) + (\mu \rightarrow -\mu) \right]. \end{aligned} \quad (7.8)$$

The free energy can be calculated by expanding the logarithm :

$$F(\beta, \mu) = -\frac{V\epsilon}{\beta} \sum_{n=1}^{\infty} \frac{\epsilon^n}{n} e^{-2\pi i \mu n} \int \frac{d^{d-1}\mathbf{k}}{(2\pi)^{d-1}} e^{-n\beta\omega_{\mathbf{k}}} + (\mu \rightarrow -\mu). \quad (7.9)$$

The integral on the r.h.s. can be evaluated straightforwardly by noticing its relation to a d -dimensional, Euclidean propagator. Consider the following phase space integral in d -dimensions with Euclidean signature :

$$I(b) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{ibk_0}}{k^2 + m^2} = \frac{1}{2} \int \frac{d^{d-1}\mathbf{k}}{(2\pi)^{d-1}} \frac{e^{-b\omega_{\mathbf{k}}}}{\omega_{\mathbf{k}}}, \quad (7.10)$$

where $k^2 = k_0^2 + \mathbf{k}^2$. The k_0 -integral has been evaluated by contour integration, encircling the pole at $k_0 = +i\omega_{\mathbf{k}}$. Now notice that the integral we need is simply $-2I'(n\beta)$. It is straightforward to calculate (7.10) by using the Schwinger trick :

$$I(b) = \frac{1}{2} \int_0^{\infty} d\alpha (2\pi\alpha)^{-d/2} \exp \left\{ -\frac{b^2}{2\alpha} - \frac{1}{2} m^2 \alpha \right\}. \quad (7.11)$$

Using this, we find the final result for the free energy :

$$F(\beta, \mu) = -V\epsilon \int_0^{\infty} \frac{d\alpha}{\alpha} (2\pi\alpha)^{-d/2} \sum_{n=1}^{\infty} \epsilon^n e^{-\frac{n^2\beta^2}{2\alpha} - \frac{m^2\alpha}{2} - 2\pi i \mu n} + (\mu \rightarrow -\mu). \quad (7.12)$$

2. In string theory this will be absent.

This expression can be also re-written as a sum over Bessel functions, but we will not discuss this here.

From (7.7), we notice that the free energy, as a function of μ has a symmetry, $F(\beta, \mu) = F(\beta, \frac{1}{2} - \mu)$. This implies that there is a finite range of gauge-inequivalent vacuum potentials, $0 \leq \mu \leq \frac{1}{2}$. One sees that in the Euclidean theory, A_0 becomes effectively transmuted from the zeroth component of a gauge potential to a scalar characterizing the thermal vacuum. For further discussion on these topological vacuum parameters, see [50], [51].

In particular, the thermal masses for the states charged under the $U(1)$ field are shifted. Notice from (7.12) that, at sufficiently low temperature, charged states become super-massive and, essentially decouple ($n = 0$). Hence, the modification from the canonical vacuum ($\mu = 0$) induced by non-trivial values for the vacuum potentials ($\mu \neq 0$) is only relevant for the high-temperature behaviour of the theory. We shall see in the next section, where the Hagedorn problem is discussed, that such non-trivial vacuum parameters refine the thermal ensemble and stabilize the would-be Hagedorn divergences.

Let us note, ‘en passant’, that one encounters similar expressions in open string theory, when one turns on a non-trivial, constant expectation value for a 2d gauge field, $\langle A_0 \rangle$, on the worldsheet. In that case, the partition function yields precisely the θ -functions with characteristics :

$$\text{Tr} \left[q^{L_0 - c/24} e^{2\pi i v Q} \right] = \frac{\theta_b^{[a]}(\tau, v)}{\eta(\tau)} . \quad (7.13)$$

7.2 Hagedorn Instability in String Theory

We now take the discussion of the previous section to the level of string theory. Here there is a fundamental difference from the field theory case, because string theory contains an infinite tower of states and there is no a priori reason why they should not thermalize. In the absence of a natural mechanism that regularizes the thermal trace, and hence suppresses the contribution of the infinite tower of states, one might attempt to use the ‘canonical’ definition for the thermal trace, $\mu = 0$. Summing over the densities of bosonic and fermionic states one finds :

$$F(\beta) = \frac{V}{\beta} \int \frac{d^{d-1} \mathbf{k}}{(2\pi)^{d-1}} \int_0^\infty dm \left[\rho_B(m) \log(1 - e^{-\beta \omega_{\mathbf{k}}}) - \rho_F(m) \log(1 + e^{-\beta \omega_{\mathbf{k}}}) \right] . \quad (7.14)$$

By using the behaviour of η and θ -functions under modular S -transforms, one may derive the dominant contribution to the asymptotic density of states in string theory, which is always exponential, [52], [53] :

$$\rho(m) \sim c m^{-d} e^{bm} . \quad (7.15)$$

The coefficient depends on the particular details of the model, d is the dimension of the space-time and b is a universal exponent that is uniquely determined by the super-reparametrization properties on the worldsheet. In particular, we have :

$$(2\pi)^{-1}b = \begin{cases} \sqrt{2\alpha'} & , \text{ Type II} \\ \left(1 + \frac{1}{\sqrt{2}}\right) \sqrt{\alpha'} & , \text{ Heterotic} \\ 2\sqrt{\alpha'} & , \text{ Bosonic} \end{cases} . \quad (7.16)$$

The exponentially growing density of states has serious consequences for the definition of the thermal trace since, above a particular temperature, $T_H = 1/b$, it will begin to dominate over exponential Boltzmann suppression factor leading to the divergence³ of the thermal trace and the free energy. This is called the Hagedorn divergence, [54].

Apart from the obvious embarrassment of the failure of the definition of the canonical ensemble, this also leads to a more practical problem from the point of view of string cosmology. Namely, the thermal analogue of the genus-1 effective potential in the case of spontaneously broken supersymmetry, is the free energy itself (i.e. the first-order thermal effective potential), acting as a source in the effective action because of the presence of thermal matter. Its divergence then, clearly, forbids a perturbative treatment of the backreaction and spoils any hope of calculating the induced cosmological behaviour.

The ‘conservative’ view to this problem is that the Hagedorn temperature is indeed a maximal temperature at which thermal equilibrium can exist for a system with an exponentially growing number of states. However, the Euclidean treatment actually suggests more. There the divergence arises because certain string states winding around the Euclidean time cycle become tachyonic once the thermal radius $R = 1/(2\pi T)$ exceeds its critical (Hagedorn) value⁴.

Indeed, consider the Type II theory compactified on a circle S^1 , associated with the Euclidean time direction. The correct incorporation of the spin-statistics connection then requires a very specific correlation between the spacetime fermion number and the winding number around the compact time cycle, [36]. The argument is essentially the requirement that spacetime fermions are anti-periodic around the Euclidean time cycle. This calls for a coordinate dependent compactification where the vertex operators have to be redefined, along the lines of (5.11). Let us see first what this would imply in field theory. Start by performing a field redefinition for the fermions :

$$\psi = \exp\left(i\pi F \frac{X^0}{2\pi R}\right) \hat{\psi} , \quad (7.17)$$

with F being the fermion number. From the integration measure we eventually we recover the spin-statistics cocycle $(-)^{\tilde{m}F}$:

$$\sum_{\tilde{m} \in \mathbb{Z}} \int \mathcal{D}\psi \dots e^{-S_{\tilde{m}} - S_{\text{quantum}}} = \sum_{\tilde{m} \in \mathbb{Z}} e^{i\pi F X_{\text{class}, \tilde{m}}/2\pi R} e^{-S_{\tilde{m}}} \int \mathcal{D}\hat{\psi} \dots e^{-S_{\text{quantum}}} , \quad (7.18)$$

3. In fact, this is as fast as the asymptotic density of states could grow, otherwise the thermal trace would never converge.

4. This can be seen essentially from (6.6).

where $X_{\text{class}, \tilde{m}} = 2\pi R \tilde{m}$, parametrizes all the topologically non-trivial configurations winding around the circle.

The analogous cocycle insertion at the string level should take into account both non-trivial torus cycles and, hence, by modular invariance :

$$\begin{cases} (-)^{(a+\bar{a})\tilde{m}+(b+\bar{b})n} & , \text{ Type II} \\ (-)^{\tilde{m}n+a\tilde{m}+bn} & , \text{ Heterotic} \end{cases} . \quad (7.19)$$

With this prescription, the genus-1 vacuum amplitude precisely produces the free energy :

$$F(\beta) = -\frac{V_{(9)}}{(2\pi\sqrt{\alpha'})^{10}} \int_{\mathcal{F}} \frac{d^2\tau}{4\tau_2^6} \left[\frac{1}{2} \sum_{a,b=0,1} (-)^{a+b} \frac{\theta^4[a]}{\eta^{12}} \frac{1}{2} \sum_{\bar{a},\bar{b}=0,1} (-)^{\bar{a}+\bar{b}} \frac{\bar{\theta}^4[\bar{a}]}{\bar{\eta}^{12}} \right. \\ \left. \sum_{\tilde{m},n \in \mathbb{Z}} e^{-\frac{\beta^2}{4\pi\alpha'\tau_2} |\tilde{m}+\tau n|^2} (-)^{(a+\bar{a})\tilde{m}+(b+\bar{b})n} \right] . \quad (7.20)$$

In order to see that this indeed reproduces the free energy of a gas of non-interacting strings, one may ‘unfold’ the fundamental domain, a technique that is presented in some detail in Appendix C. Essentially, the original sum is decomposed into modular orbits and the integral breaks into an integral of the $(\tilde{m}, n) = (0, 0)$ -orbit over \mathcal{F} and an integral of the zero winding sector, $(\tilde{m}, n) = (1, 0)$, over the strip.

The $(0, 0)$ -orbit has vanishing windings and, hence, the temperature lattice becomes effectively decoupled from the R -symmetry lattice of the fermions ψ^μ . This contribution then vanishes because of the underlying supersymmetric structure of the zero-temperature (undeformed) theory. The remaining term is the integral over the strip, in the zero winding sector and produces a result similar to field theory. In fact, by writing down this term one may easily recognize the Schwinger representation (7.12) of the vacuum amplitude (or free energy) :

$$F(\beta) = -\frac{1}{\beta} \log \text{Tr} [e^{-\beta H}] , \quad (7.21)$$

where the trace on the r.h.s. is over the Hilbert space of the 2nd quantized (supersymmetric) theory. The fact that one indeed reproduces the field-theoretic result for the free energy is not-trivial and, in fact, depends upon the fact that the unfolding procedure permits one to resum the Euclidean vacuum amplitude and express it as an integral over the strip in a sector with vanishing winding, as in field-theory. All the n -winding contributions has been effectively mapped into the high-frequency modes around the lower boundary of the strip.

Let us now return to the Hagedorn divergence. The Euclidean treatment suggests that this arises from the $\tau_2 \rightarrow \infty$ region and is, hence, an IR divergence. The convergence of the Euclidean path integral is governed by the low-lying states and one may easily trace the divergence down to the presence of a winding state ($n = 1$) that becomes tachyonic below some characteristic value of the thermal radius R_H . A few comments are in order here. First

of all, tachyons can only⁵ arise from the NS-NS sector, that is, the sector coupling to the $O_8\bar{O}_8$ fermion characters. The left- and right- GSO projection in this sector then forces the associated lattice contribution to have odd left- and odd right- GSO-parity :

$$O_8\bar{O}_8 \frac{1}{2^2} \sum_{b',\bar{b}'=0,1} (-)^{b'+\bar{b}'} \Gamma_{(1,1)}[{}^0_{b'+\bar{b}'}](R) , \quad (7.22)$$

where $R = \beta/2\pi$ and $\Gamma_{(1,1)}$ is defined as :

$$\Gamma_{(1,1)}[{}^{\alpha+\bar{\alpha}}_{\beta+\bar{\beta}}](R) = \frac{R}{\sqrt{\tau_2}} \sum_{\tilde{m},n \in \mathbb{Z}} e^{-\frac{\pi R^2}{\tau_2} |\tilde{m} + \tau n|^2} (-)^{(\alpha+\bar{\alpha})\tilde{m} + (\beta+\bar{\beta})n} . \quad (7.23)$$

The fermionic part indeed starts with weight $(-\frac{1}{2})$ in CFT units, as expected from the NS vacuum. The low-lying masses of the lattice part can be read from the Hamiltonian representation, after a Poisson-resummation :

$$\Gamma_{(1,1)}[{}^{\alpha+\bar{\alpha}}_{\beta+\bar{\beta}}](R) = \sum_{m,n \in \mathbb{Z}} q^{\frac{1}{2}P_L^2} \bar{q}^{\frac{1}{2}P_R^2} (-)^{(\beta+\bar{\beta})n} , \quad (7.24)$$

where :

$$\frac{1}{2}P_{L,R}^2 = \frac{1}{4} \left(\frac{m - \frac{\alpha+\bar{\alpha}}{2}}{R} \pm nR \right)^2 . \quad (7.25)$$

The NS-NS sector corresponds to $\alpha = \bar{\alpha} = 0$ and the GSO projection acting on the lattice picks precisely the odd winding sector, $n \in 2\mathbb{Z} + 1$. In addition, level-matching for the lowest state (no oscillators) imposes that $\frac{1}{2}(P_L^2 - P_R^2) = mn$ vanishes, hence, $m = 0$. The lowest conformal weight comes from the winding modes, $n = \pm 1$, and has to be compared to the $(-\frac{1}{2})$ weight of the NS-vacuum :

$$\frac{1}{2}P_{L,R}^2 - \frac{1}{2} = \frac{R^2}{4} - \frac{1}{2} . \quad (7.26)$$

This becomes negative for radii smaller than the critical Hagedorn radius $R_H = \sqrt{2}$, in accordance with (7.16). The fact that it is precisely a state of non-trivial winding that becomes tachyonic in the Euclidean description is of fundamental importance for any attempt to resolve the Hagedorn divergence, as we will argue below. For now, let us mention that it is no surprise that only those states that carry non-trivial winding charge around the Euclidean time direction become tachyonic. Winding modes are precisely those purely stringy states that distinguish this from any field-theoretic setup and it is, again, in string theory that one finds an exponentially growing density of states leading to the Hagedorn divergence.

5. As seen from the structure of the R-vacuum, fermions (or R-R states) are always at least massless, even at zero temperature.

Furthermore, notice that unphysical tachyons (those one would have normally projected out by ‘level-matching’) do not introduce divergences, as their contribution vanishes after the τ_1 -integration on the strip.

Returning to the Hagedorn divergence, the fact that it manifests itself as a tachyonic state in the Euclidean, does not imply necessarily a break-down of the thermodynamics or an internal pathology of the theory. Rather, the Hagedorn instability signals the onset of a phase transition at around the Hagedorn point to a new thermal vacuum, [55], [56]. From the worldsheet point of view, the transition can be mapped to a Kosterlitz-Thouless transition in the continuum limit of the XY model, [57], [58]. There, the condensation of worldsheet vortices (the $n = \pm 1$ winding modes) has to be taken into account and the new phase corresponds to a new (deformed) conformal field theory.

First of all, it can be argued [55] that the Hagedorn phase transition is a first order one. This is, essentially, because the effective potential for the winding modes, $n = \pm 1$, that become tachyonic, is trilinear. Let us see how this arises as a deformation of the CFT. Consider adding to the σ -model action the following perturbation, which corresponds to the vertex operators of the $n = \pm 1$ winding tachyons, ω, ω^* at the Hagedorn point $R = \sqrt{2}$:

$$\begin{aligned} S &= \frac{1}{2\pi} \int d^2z \left(\partial X^0 \bar{\partial} X^0 + \omega \psi^0 \bar{\psi}^0 e^{i(X_L - X_R)} + \omega^* \psi^0 \bar{\psi}^0 e^{-i(X_L - X_R)} \right), \\ &= -\frac{1}{2\pi} \int d^2z \left(\partial \tilde{X}^0 \bar{\partial} \tilde{X}^0 + 2|\omega| \psi^0 \bar{\psi}^0 \cos(\tilde{X}^0 + \theta) \right), \end{aligned} \quad (7.27)$$

where we taken constant v.e.v. for $\omega = |\omega|e^{i\theta}$ and use the dual coordinate :

$$\tilde{X}^0 \equiv X_L^0 - X_R^0. \quad (7.28)$$

Only the relevant terms are displayed in the σ -model action. In the second line of (7.27) we recover a sine-Gordon-like action, which precisely describes the dual of the continuous XY-model action after the condensation of vortices (for an interesting discussion, see [59]). This action can be derived more rigorously by starting from the well-known equivalence between the XY and sine-Gordon models and using renormalization group techniques to treat the long-range interaction between the vortices in the high-temperature regime above T_H . It is certainly not by accident that one could have derived the correct values for the Hagedorn temperature simply by studying when the free energy of vortice configurations gives rise to the Kosterlitz-Thouless transition.

The process of tachyon condensation typically involves large values for the condensates and the system is driven outside the perturbative domain. For further discussions see [60], where the properties of $\mathcal{N} = 4$ gauged supergravity are used to obtain the thermal effective potential and discuss high-temperature instabilities. Despite many efforts in the literature, however, a fully satisfactory dynamical description of this process is still lacking and constitutes a very important open problem.

An alternative way is to try to construct stable thermal vacua directly at the string perturbative level, characterized by non-trivial winding charge. This approach, [61], is based on deforming the background by other types of condensates in a way that is amenable

to perturbative treatment. The idea is to turn on particular (discrete) values of gravitomagnetic fluxes threading the Euclidean time and other spatial compact cycles. These fluxes will be related to condensates of graviphoton and axial vector gauge fields with vanishing field strength, but with non-trivial value of the Wilson line around the Euclidean time cycle.

This is formally identical to the Scherk-Schwarz mechanism described in section 5.3 and can be nicely implemented as a freely-acting orbifold. As is the case with Scherk-Schwarz deformations, the masses of charged states are shifted and, hence, this can be used to render the would-be winding tachyons massless, because the latter carry non-trivial winding charge. The MSDS vacua constructed in the previous chapter, including their thermal versions, fall precisely into these cases. Indeed, they are free of tachyons even though their thermal radius lies beyond the Hagedorn point. As we will see, their presence has the effect of refining the thermal ensemble and renders the free energy finite.

This opens the associated problem of studying the conditions of tachyon finiteness, that is, the conditions on the deformation parameters (gravitomagnetic fluxes) such that the resulting thermal vacuum will be free of Hagedorn instabilities and stable under fluctuations of the (dynamical) moduli.

7.3 Resolution of the Hagedorn Instability

The thermal interpretation of MSDS models and the more general problem of studying the conditions of absence of Hagedorn instability in terms of non-trivial condensates has been studied in considerable detail in [47]. Here we will briefly mention the key results. First of all, let us start from a general toroidal compactification on a (d, d) -lattice.

$$\Gamma_{(d,d)} \left[\begin{smallmatrix} a, \bar{a} \\ b, \bar{b} \end{smallmatrix} \right] = \frac{\sqrt{\det G_{\mu\nu}}}{(\sqrt{\tau_2})^d} \sum_{\tilde{m}^\mu, n^\nu \in \mathbb{Z}} e^{-\frac{\pi}{\tau_2} (G+B)_{\mu\nu} (\tilde{m} + \tau n)^\mu (\tilde{m} + \bar{\tau} n)^\nu + i\pi \mathcal{T}} . \quad (7.29)$$

We assume that the Euclidean time direction has been identified with the X^0 -compact cycle of the lattice. The lattice is assumed to have a coupling \mathcal{T} to the left- and right- moving R -symmetry charges, responsible for the spontaneous breaking of supersymmetry, à la Scherk-Schwarz. We assume there is an independent coupling of, say the X^1 -cycle, to the right-moving R -symmetry charges so that \mathcal{T} becomes :

$$\mathcal{T} = \left[(a + \bar{a})\tilde{m}^0 + (b + \bar{b})n^0 \right] + \left[\bar{a}\tilde{m}^1 + \bar{b}n^1 + \tilde{m}^1 n^1 \right] . \quad (7.30)$$

The full lattice can then be shown to break into two (coupled) sub-lattices :

$$\frac{R_0}{\sqrt{\tau_2}} \sum_{\tilde{m}^0, n^0 \in \mathbb{Z}} e^{-\frac{\pi R_0^2}{\tau_2} |\tilde{m}^0 + \tau n^0|^2} (-)^{(a+\bar{a})\tilde{m}^0 + (b+\bar{b})n^0} \sum_{m_I, n^I \in \mathbb{Z}} q^{\frac{1}{2}P_L^2} \bar{q}^{\frac{1}{2}P_R^2} (-)^{\bar{b}n^1} e^{2\pi i \tilde{m}^0 (G_{0I} Q_{(M)}^I - B_{0I} Q_{(N)}^I)} , \quad (7.31)$$

where we define the radius :

$$R_0^2 \equiv G_{00} - G_{0I} G^{IJ} G_{J0} . \quad (7.32)$$

It will be shown below to correspond precisely to the inverse temperature, $\beta = 2\pi R_0$. Here, $P_{L,R}^2 \equiv G_{IJ} P_{L,R}^I P_{L,R}^J$ are the canonical momenta of the transverse lattice :

$$P_{L,R}^I = \frac{1}{\sqrt{2}} G^{IJ} \left(m_J + \frac{\bar{a} + n^1}{2} \delta_{J1} + (B_{J\mu} \pm G_{J\mu}) n^\mu \right) , \quad (7.33)$$

with (+) and (−) corresponding to the left- and right- moving canonical momenta, respectively. We have also defined the Kac-Moody (non-integral) charges :

$$Q_{(M)}^I \equiv \frac{1}{\sqrt{2}} (P_L^I + P_R^I) \quad , \quad Q_{(N)}^I \equiv \frac{1}{\sqrt{2}} (P_L^I - P_R^I) . \quad (7.34)$$

In order to exhibit the thermal interpretation, we must decompose the lattice into modular orbits and unfold the fundamental domain as described in the previous section. The $(\tilde{m}^0, n^0) = (0, 0)$ -orbit term will yield a vanishing contribution and we are left with the $(\tilde{m}^0, 0) \neq (0, 0)$ orbit to be integrated over the strip. This will provide the familiar Schwinger representation of the free energy :

$$\frac{R_0}{\sqrt{\tau_2}} \sum_{\tilde{m}^0 \neq 0} e^{-\frac{\pi}{\tau_2} (R_0 \tilde{m}^0)^2} (-)^{(a+\bar{a})\tilde{m}^0} \times \sum_{m_I, n^I \in \mathbb{Z}} q^{\frac{1}{2} P_L^2} \bar{q}^{\frac{1}{2} P_R^2} (-)^{\bar{b}n^1} e^{2\pi i \tilde{m}^0 (G_{0I} Q_{(M)}^I - B_{0I} Q_{(N)}^I)} , \quad (7.35)$$

where the 14 transverse $U(1)$ charges $Q_{(M)}^I, Q_{(N)}^I$ (to be evaluated at $n^0 = 0$) are associated with the graviphoton, $G_{I\mu}$, and axial vector, $B_{I\mu}$, gauge fields. These condensates will precisely provide the deformation required to refine the thermal ensemble.

In their presence the thermal partition function becomes :

$$\mathcal{Z}(\beta, G_{0I}, B_{0I}) = \text{tr} \left[e^{-\beta H} e^{2\pi i (G_{0I} Q_{(M)}^I - B_{0I} Q_{(N)}^I)} \right] , \quad (7.36)$$

with the trace being performed over the Hilbert space of the zero-temperature, 3-dimensional $(4, 0)$ theory. This should be compared with the field theory result (7.7), calculated in the presence of a non-trivial v.e.v. for the $U(1)$ gauge potential.

The model describes a thermal ensemble at inverse temperature $\beta = 2\pi R_0$, which is further deformed by the non-trivial condensates of the vacuum gauge potentials associated to the the graviphoton and axial vector $U(1)$ -charges, $Q_{(M)}^I, Q_{(N)}^I$.

Notice that the argument of the phase in (7.36) is scale invariant. To make this invariance explicit and obtain full accordance with (7.7), we rewrite it in terms of the integral charges :

$$\hat{m}_I \equiv m_I + \frac{\bar{a} + n^1}{2} \delta_{1I} \quad , \quad n^I . \quad (7.37)$$

The resulting thermal trace is then :

$$\mathcal{Z}(\beta, \mu^I, \tilde{\mu}_I) = \text{tr} \left[e^{-\beta H} e^{2\pi i (\mu^I \hat{m}_I - \tilde{\mu}_I n^I)} \right] , \quad (7.38)$$

with the following definition for the vacuum expectation values :

$$\begin{aligned}\mu^I &\equiv \hat{G}_0^I \equiv G_{0K} G^{KI} , \\ \tilde{\mu}_I &\equiv \hat{B}_{0I} \equiv B_{0I} - \hat{G}_0^K B_{KI} .\end{aligned}\tag{7.39}$$

They are scale-invariant, non-fluctuating thermodynamical parameters of the statistical ensemble, associated to the Scherk-Schwarz ‘thermal’ fluxes. Let us note that the refinement of the thermal ensemble is only relevant at high temperatures. At temperatures sufficiently below the Hagedorn point, even with the other compact spatial cycles held fixed at the fermionic point, states charged under these $U(1)$ fields become massive and effectively decouple from the thermal system so that we recover the conventional thermal ensemble.

Next, we return to the problem of studying the conditions leading to stable thermal vacua, as discussed in [47]. As mentioned before, the only sector in danger of exciting tachyons is the sector coupling to the NS-NS vacuum, which is picked by imposing independent odd left- and right- moving GSO projections on the lattice :

$$O_8 \bar{O}_8 \frac{1}{2} \sum_{b', \bar{b}'=0,1} (-)^{b'+\bar{b}'} \Gamma_{(8,8)} \left[\begin{smallmatrix} 0,0 \\ b', \bar{b}' \end{smallmatrix} \right] (G_{IJ}, B_{IJ}) .\tag{7.40}$$

We are again looking for the lowest-lying states, which are in danger of becoming tachyonic. To this end, we analyze the contributions to the thermal masses. It is most convenient to work in the flat vielbein frame, $G_{\mu\nu} = e_\mu^a e_\nu^a$, in which the formula for the thermal masses (no oscillators) decomposes into the sum of d perfect squares :

$$\frac{1}{2} P_{L,R}^2 = \frac{1}{4} \sum_{a=0}^7 ((e^{*a})^\mu (\hat{m}_\mu + B_{\mu\nu} n^\nu) \pm e_\mu^a n^\mu)^2 ,\tag{7.41}$$

where $\hat{m}_\mu \equiv m_\mu + \frac{1}{2}(\bar{a} + n^1)\delta_{\mu,1}$. It can be proven in linear algebra⁶ that the decomposition of $G_{\mu\nu}$ into a lower triangular matrix e_μ^a is unique and, in fact, it is the physical basis in which the upper entry in the vielbein matrix contains the physical temperature radius R_0 . The first square in the decomposition (7.41), involving R_0 has the remarkable property of depending on the thermodynamical phase, $\{\beta, \mu^I, \tilde{\mu}_I\}$, alone :

$$\frac{1}{4} \left(\frac{m_0 - \mu^I \hat{m}^I + \tilde{\mu}_J n^J + \tilde{\mu}_I \mu^I n^0}{R_0} \pm R_0 n^0 \right)^2 .\tag{7.42}$$

Thus, this ‘pure’ thermal contribution to the masses is entirely unaffected by the fluctuation of the dynamical (transverse) moduli, which can (at most) contribute a non-negative contribution to the thermal masses. It is then clear that in order to construct tachyon-free thermal vacua for any deformation in the dynamical moduli, the low-lying thermal spectrum should be at least (anti-)chirally massless and protected precisely by this ‘frozen’ (non-dynamical) contribution.

6. This is known as the Choleksy theorem.

The conditions on the thermodynamical parameters $\mu, \tilde{\mu}$ saturating the condition for tachyon stability were derived in [47]. Here, we only give the result :

$$\left. \begin{array}{l} \hat{G}_0^k \in \mathbb{Z} \\ \hat{B}_{0k} \in \mathbb{Z} \end{array} \right\} , \quad \text{for } k = 2, \dots, d-1 ,$$

$$\hat{G}_0^1 \in 2\mathbb{Z} + 1 , \quad \hat{B}_{01} \in \mathbb{Z} + \frac{1}{2} . \quad (7.43)$$

Notice the requirement of the non-trivial value of the off-diagonal \hat{G}_{01} and \hat{B}_{01} . It can also shown (see the Appendix in [47]), that whenever the above conditions are met, one may employ a discrete $O(8, \mathbb{Z}) \times O(8, \mathbb{Z})$ transformation to rotate the fluxes to the form :

$$\hat{G}_0^k = \hat{B}_{0k} = 0 , \quad \hat{G}_0^1 = 2\hat{B}_{01} = \pm 1 . \quad (7.44)$$

In fact, in the latter form, the conditions for tachyon finiteness has a very simple geometric interpretation. They correspond precisely to the case when the temperature cycle completely factorizes from the remaining torus, while coupling chirally to F_L only :

$$\Gamma_{(d,d)}[{}^a_{b, \bar{b}}] \rightarrow \Gamma_{(1,1)}[{}^a_b](R_0) \cdot \Gamma_{(d-1, d-1)}[\bar{a}_{\bar{b}}](G_{IJ}, B_{IJ}) , \quad (7.45)$$

which is, indeed, equivalent to the complete factorization of the temperature cycle S^1 from the remaining compact manifold. The non-trivial coupling to F_L can be conveniently incorporated from the very beginning, in terms of a freely acting orbifold, where one mods out by $(-)^{F_L} \delta$, rather than the usual thermal deformation $(-)^{F_L + F_R} \delta$. Furthermore, whenever conditions (7.44) are met, the thermal trace simply reduces to the right-moving index :

$$\mathcal{Z} = \text{tr} \left[e^{-\beta H} (-1)^{F_R} \right] , \quad \text{with } \mu = 2\tilde{\mu} = 1 . \quad (7.46)$$

We will return to the conditions (7.44) and their effect arising as a deformation of the worldsheet CFT in Chapter 8 when we discuss the Hybrid construction.

7.4 Deformations and Stability

In [47], the structure of the moduli space of the ‘maximal’ MSDS models was analyzed. It is a highly non-trivial technical problem to start from a string vacuum at the special MSDS point and reinstate back the dependence of all G_{IJ}, B_{IJ} moduli. The complications arise precisely because at such multicritical extended symmetry points, the CFT holomorphically factorizes and the large number of dualities simultaneously do not allow an unambiguous identification of the moduli. The dualities are, in fact, even broken by the Scherk-Schwarz coupling of the full (8, 8)-lattice of MSDS models to the R -symmetry charges.

A priori, it is not at all trivial that MSDS models even have a thermal interpretation, in terms of the particular \mathbb{Z}_2 Scherk-Schwarz freely-acting orbifold introduced in the previous section. The calculation is rather technical and can be found in the appendix of ref. [47]. Here we will only sketch it. The idea is to start from the expression (6.22) of the $\Gamma_{(8,8)}$ -lattice

in terms of θ -functions at the MSDS point and then set it into lattice form by using standard bosonization techniques at genus-1, with the help of (A.12). The complication one encounters is then that the coupling to the R -symmetry charges appear explicitly as half-shifts in the windings and, in that representation, there is no simple way to isolate the moduli controlling the coupling to the R -symmetry charges.

However, by using the triality properties of each of the independent $SO(8)$'s embedded in the $\Gamma_{(8,8)}$ -lattice and the various dualities present at the MSDS point, we can eventually bring the $(8,8)$ -lattice to the standard Lagrangian form, arising from \mathbb{Z}_2 'thermal' Scherk-Schwarz breaking. The result is a deformed $E_8 \times E_8$ lattice of the form (7.29), with the deformation being due to the particular coupling to the R -symmetry charges, which break the full duality group down to 'transverse' subgroups. The values of the metric and anti-symmetric tensor parameters, G_{IJ}, B_{IJ} , are now completely fixed by the underlying MSDS algebra. The R -symmetry charges appear entirely in the phase and one recovers precisely the thermal coupling of the X^0 -cycle to the total fermion number, $F_L + F_R$. An independent coupling of the X^1 -cycle to F_R breaks the right supersymmetries down to $(4,0)$ at zero temperature. Thus, the MSDS models admit a natural thermal interpretation, simply by identifying Euclidean time with the direction X^0 coupling to the total fermion number, as required by spin-statistics.

Unfortunately, one may check that the maximally symmetric MSDS models discussed in the previous chapter have :

$$\begin{aligned}\hat{G}_0^I &= \left(\frac{3}{2}, \frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}, 0 \right) , \\ \hat{B}_{0I} &= \left(0, -\frac{3}{4}, -\frac{1}{4}, -\frac{3}{4}, 0, -\frac{1}{4}, 0 \right) .\end{aligned}\tag{7.47}$$

and these do not satisfy the conditions (7.44) that guarantee the absence of tachyonic modes. Hence, these models are not stable under arbitrary marginal deformations of the fluctuating, dynamical moduli. This is a result of the spontaneous breaking of the MSDS structure in the bulk, where an infinitesimal deformation typically destroys the asymptotic supersymmetric degeneracy of the spectrum at large mass levels and, hence, modular invariance no longer protects the low-lying spectrum against tachyonic excitations.

We will see below that the construction of non-tachyonic spectra with MSDS structure is possible, if one employs a combination of symmetric and asymmetric \mathbb{Z}_2 -type orbifolds to project out the dangerous moduli leading to the would-be tachyons. For now, however, let us mention that one may still identify continuous deformation directions in the maximal MSDS moduli space along which no tachyonic modes are ever encountered. For example, lowering the temperature by increasing the value of R_0 one eventually recovers, in the infinite limit, the $\mathcal{N}_L = 4$ left-moving supersymmetries in one higher dimension. Similarly, taking the limit $R_1 \rightarrow \infty$ which lowers the right-supersymmetry breaking scale, one eventually recovers the $\mathcal{N}_R = 4$ supersymmetries as well, in one higher dimension.

These marginal deformations, hence, provide the connection with 4d cold vacua, with spontaneously broken $\mathcal{N} \leq 8$ supersymmetry, induced by the well-defined geometrical fluxes

dictated by the initial (undeformed) MSDS algebra. The additional non-compact dimensions are seen to emerge from these large marginal deformations. In a cosmological setting, where the deformation moduli acquire time dependence, it becomes tempting to consider our 4d cosmological space as being dynamically generated from an initially non-singular, 2d MSDS vacuum. This provides the basis of our cosmological conjecture, as presented in [45–47]. Of course, before this ambitious programme can be carried out, one must first address the problem of identifying tachyon-free trajectories along which the evolution can be carried out with the methods of perturbative string theory and CFT.

Before concluding this section, let us also mention that the results of this section can essentially be carried out in the case of Heterotic MSDS vacua, defined on $T^8/\mathbb{Z}_2 \times \mathbb{Z}_2$ and with $SO(10)$ gauge interactions. Their marginal deformations connect them to $\mathcal{N} = 1$ vacua in four dimensions. However, the major complication, hindering our understanding of the region around the Hagedorn point, is that the previous conditions of absence of tachyons are only valid for Type II theories. There, precisely because the NS vacuum starts at conformal weight $-\frac{1}{2}$, the introduction of particular gravito-magnetic fluxes leading to the factorization of the thermal $(1, 1)$ -lattice (chirally coupled to F_L only) renders the would-be tachyon massless. This is because the chiral, odd GSO projection to an F_L -coupled $(1, 1)$ -lattice results in half-shifting both momenta and windings in the Hamiltonian representation. Alternatively, the absence of physical tachyons can be seen by the properties of the $\Gamma_{(1,1)}[a]_b$ -lattice with odd GSO b -projection. At the fermionic point, $R_0 = 1/\sqrt{2}$, its contributions can be decomposed in terms of $SO(2)$ characters :

$$\frac{1}{2} \sum_{b'} (-)^{b'} \frac{\theta[\frac{0}{b'}] \bar{\theta}[\frac{0}{b'}]}{\eta \bar{\eta}} = V_2 \bar{O}_2 + O_2 \bar{V}_2 . \quad (7.48)$$

Hence, the chiral, odd GSO-projection enforces a difference of $(\frac{1}{2})$ -units between the left- and right- moving conformal weights, thus, rendering the low-lying states in the NS-NS sector at least (anti-)chirally massless. Clearly, this eliminates physical tachyons only in the Type II case. In the Heterotic string, where the right-moving sector is bosonic, the vacuum starts at (-1) -units of conformal weight and, hence, the previous argument is no longer sufficient to resolve the tachyon instabilities.

Despite all that, there exist Heterotic models with MSDS structure which, apart from their interesting phenomenological structure, exhibit the attractive property of being stable under arbitrary $(8, 8)$ -deformations in the ‘geometrical’⁷ moduli, G_{IJ}, B_{IJ} . An example model is the Heterotic MSDS model (6.30), constructed in the previous chapter. We will not discuss their properties any further in this manuscript.

A very interesting Type II construction, utilizing MSDS structure and, at the same time, saturating the conditions (7.44) for the absence of thermal winding tachyons are the ‘Hybrid models’, also presented in [47]. However, we prefer to postpone their presentation

7. By abuse of language we will, at times, call backgrounds associated to the antisymmetric tensor field, B_{IJ} , as geometrical. This is not in contradiction to the stringy spirit, where T -duality can effectively ‘rotate’ metric backgrounds into antisymmetric tensor backgrounds.

until Chapter 8, where their cosmological evolution will be discussed in connection with the absence of gravitational ‘Big Bang’ singularities.

7.5 Tachyon-Free MSDS Orbifold Constructions

Before ending this chapter, we present an example of Type II, $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold constructions with MSDS structure, which remains free of tachyons for any deformation of the (untwisted) moduli. This construction was given in [47]. Of course, the classification of MSDS-preserving \mathbb{Z}_2^N -orbifolds given in [46] proves to be essential for the construction. For simplicity the vacua presented here can either be given a thermal or spacetime interpretation, depending on the choice of the Euclidean time direction.

The idea is to combine symmetric with asymmetric \mathbb{Z}_2 -orbifold shifts and twists in order to project out the moduli that could give rise to tachyonic modes. The resulting vacua will be stable (free of tachyons) under all possible deformations of the remaining (fluctuating) moduli. This is a highly non-trivial task, since the requirement of preservation of the MSDS structure severely restricts the possible orbifolds.

The starting point is to consider (a-)symmetric $\mathbb{Z}_2 \times \mathbb{Z}_2'$ twists of the maximally symmetric MSDS vacuum, acting on only four of the 8 internal coordinates :

$$\left. \begin{aligned} X_L^I(z) &\rightarrow (-)^g X_L^I(z) \\ X_R^I(\bar{z}) &\rightarrow (-)^{g+g'} X_R^I(\bar{z}) \end{aligned} \right\} , \quad \text{for } I = 5, 6, 7, 8 ,$$

with $g, g' \in \{0, 1\}$ denoting the elements of the two \mathbb{Z}_2 ’s, respectively. The asymmetric nature of the \mathbb{Z}_2' -orbifold has the effect of projecting out all moduli from this twisted $(4, 4)$ -lattice. The above orbifold action respects the conditions for MSDS structure, however, it still contains sectors with tachyonic modes, so our task is not yet complete. The additional necessary step is to introduce independent asymmetric $\mathbb{Z}_2^{(1)} \times \mathbb{Z}_2^{(2)}$ shifts acting on all 8 internal coordinates in the following ‘diagonal’ way :

$$\begin{aligned} X_L^I(z) &\rightarrow X_L^I(z) + \pi G_1 , \quad \text{for } I = 1, 2, 3, 4 , \\ X_R^I(\bar{z}) &\rightarrow X_R^I(\bar{z}) + \pi G_2 , \quad \text{for } I = 1, 2, 3, 4 , \\ X_L^J(z) &\rightarrow X_L^J(z) + \pi G_2 , \quad \text{for } I = 5, 6, 7, 8 , \\ X_R^J(\bar{z}) &\rightarrow X_R^J(\bar{z}) + \pi G_1 , \quad \text{for } I = 5, 6, 7, 8 . \end{aligned} \quad (7.49)$$

Notice that the $(4, 4)$ -lattice associated to the $X^{1,2,3,4}$ coordinates remains untwisted and, hence, contains fluctuating (and, thus, potentially dangerous) moduli. The partition function of the model is :

$$\begin{aligned} Z_{\text{Twisted}} &= \frac{1}{2^4 \eta^{12} \bar{\eta}^{12}} \sum_{h,g,h',g'} \sum_{a,b} (-)^{a+b} \theta_{[b+g]}^2 [a] \theta_{[b-g]}^{[a+h]} \theta_{[b-g]}^{[a-h]} \\ &\quad \times \sum_{\bar{a}, \bar{b}} (-)^{\bar{a}+\bar{b}} \bar{\theta}_{[\bar{b}]}^2 [\bar{a}] \bar{\theta}_{[\bar{b}+g+g']}^{[\bar{a}+h+h']} \bar{\theta}_{[\bar{b}-g-g']}^{[\bar{a}-h-h']} \Gamma_{(8,8)} [a, \bar{a}, h, h']_{b, \bar{b}, g, g'} . \end{aligned} \quad (7.50)$$

The full $\Gamma_{(8,8)}$ -lattice is then factorized into a shifted $\Gamma_{(4,4)}^{(1)}$ -lattice and a shifted/twisted $\Gamma_{(4,4)}^{(2)}$ -lattice :

$$\Gamma_{(8,8)}^{[a,\bar{a},h,h']_{b,\bar{b},g,g'}} = \frac{1}{2^2} \sum_{H_i, G_i} \Gamma_{(4,4)}^{(1)} \left[\begin{matrix} a,\bar{a} \\ b,\bar{b} \end{matrix} ; \begin{matrix} H_1, H_2 \\ G_1, G_2 \end{matrix} \right] \times \Gamma_{(4,4)}^{(2)} \left[\begin{matrix} a,\bar{a} \\ b,\bar{b} \end{matrix} ; \begin{matrix} H_1, H_2 \\ G_1, G_2 \end{matrix} \middle| \begin{matrix} h, h' \\ g, g' \end{matrix} \right] . \quad (7.51)$$

At the MSDS point, the shifted lattice can be expressed entirely in terms of free fermion characters :

$$\Gamma_{(4,4)}^{(1)} \left[\begin{matrix} a,\bar{a} \\ b,\bar{b} \end{matrix} ; \begin{matrix} H_1, H_2 \\ G_1, G_2 \end{matrix} \right] = \theta_{[b+G_1]}^{[a+H_1]} \theta_{[b-G_1]}^{[a-H_1]} \times \bar{\theta}_{[\bar{b}+G_2]}^{[\bar{a}+H_2]} \bar{\theta}_{[\bar{b}-G_2]}^{[\bar{a}-H_2]} . \quad (7.52)$$

Of course, its moduli can be reinstated following the procedure described in the appendix of [47]. Similarly, we can express the twisted $(4,4)$ -lattice in terms of free fermion characters :

$$\Gamma_{(4,4)}^{(2)} \left[\begin{matrix} a,\bar{a} \\ b,\bar{b} \end{matrix} ; \begin{matrix} H_1, H_2 \\ G_1, G_2 \end{matrix} \middle| \begin{matrix} h, h' \\ g, g' \end{matrix} \right] = \theta_{[b+G_2+g]}^{[a+H_2+h]} \theta_{[b-G_2]}^{[a-H_2]} \times \bar{\theta}_{[\bar{b}+G_1+g+g']}^{[\bar{a}+H_1+h+h']} \bar{\theta}_{[\bar{b}-G_1]}^{[\bar{a}-H_1]} (-)^{h'g'} . \quad (7.53)$$

Since the would-be (untwisted) moduli associated to this lattice are all projected out by the asymmetric action of \mathbb{Z}'_2 , the above expression remains ‘frozen’ at the fermionic point. Its non-vanishing components can be collected into a compact expression :

$$\Gamma_{(4,4)}^{(2)} \left[\begin{matrix} a,\bar{a} \\ b,\bar{b} \end{matrix} ; \begin{matrix} H_1, H_2 \\ G_1, G_2 \end{matrix} \middle| \begin{matrix} h, h' \\ g, g' \end{matrix} \right] = \left\{ \begin{array}{l} \theta_{[b+G_2]}^{[a+H_2]} \bar{\theta}_{[\bar{b}+G_1]}^{[\bar{a}+H_1]}{}^4, \text{ for } (h, g) = (h', g') = (0, 0) \\ \theta_{[b+G_2]}^{[a+H_2]}{}^4 \left(\frac{2\eta^3}{\bar{\theta}_{[1-h']}^2} \right)^2, \text{ for } \left\{ \begin{array}{l} (h, g) = (0, 0), (h', g') \neq (0, 0) \text{ and} \\ (H_1, G_1) = (\bar{a}, \bar{b}) \text{ or } (\bar{a} + h', \bar{b} + g') \end{array} \right. \\ \left| \frac{2\eta^3}{\bar{\theta}_{[1-g]}^2} \right|^4, \text{ for } \left\{ \begin{array}{l} (h, g) \neq (0, 0), (h', g') = (0, 0) \text{ and} \\ (H_1, G_1) = (\bar{a}, \bar{b}) \text{ or } (\bar{a} + h, \bar{b} + g) \text{ and} \\ (H_2, G_2) = (a, b) \text{ or } (a + h, b + g) \end{array} \right. \\ \left(\frac{2\eta^3}{\bar{\theta}_{[1-h]}^2} \right)^2 \bar{\theta}_{[\bar{b}+G_1]}^{[\bar{a}+H_1]}{}^4, \text{ for } \left\{ \begin{array}{l} (h, g) \neq (0, 0), (h', g') \neq (0, 0) \\ (h, g) = (h', g') \text{ and} \\ (H_2, G_2) = (a, b) \text{ or } (a + h, b + g) \end{array} \right. \\ \left(\frac{2\eta^3 \cdot 2\bar{\eta}^3}{\theta_{[1-g]}^2 \bar{\theta}_{[1-g-h']}^2} \right)^2, \text{ for } \left\{ \begin{array}{l} (h, g) \neq (0, 0), (h', g') \neq (0, 0) \\ (h, g) \neq (h', g') \text{ and} \\ (H_1, G_1) = (\bar{a}, \bar{b}) \text{ or } (\bar{a} + h + h', \bar{b} + g + g') \text{ and} \\ (H_2, G_2) = (a, b) \text{ or } (a + h, b + g) \end{array} \right. \end{array} \right. \quad (7.54)$$

It is easy to check that the model satisfies the MSDS conditions of [46]. Indeed, its partition function at the MSDS point is found to equal a constant integer, $Z_{\text{Twisted}} = 208$.

As mentioned already above, the asymmetric \mathbb{Z}'_2 action projects out the fluctuations of all moduli in the twisted $\Gamma_{(1,1)}^{(2)}$ -lattice and truncates the moduli space of the theory down

to :

$$\frac{SO(8, 8)}{SO(8) \times SO(8)} \xrightarrow{\mathbb{Z}_2 \times \mathbb{Z}'_2} \frac{SO(4, 4)}{SO(4) \times SO(4)} . \quad (7.55)$$

The only surviving marginal operators that can perturb the σ -model are, hence, those associated to the shifted $\Gamma_{(4,4)}^{(1)}$ -lattice.

In [47], it was proven that all sectors of the model remain at least (anti-)chirally massless for arbitrary deformations of the shifted $(4, 4)$ -lattice. In this sense, the vacuum is stable and no (physical) tachyonic states are produced, as long as only deformations with respect to the untwisted moduli are considered. The proof is straightforward but rather technical and will not be presented here. It suffices to mention that the action of the $\mathbb{Z}_2^{(1),(2)}$ shifts is such that each sector is protected against becoming tachyonic by the ‘frozen’ contribution of the twisted $\Gamma_{(4,4)}^{(2)}$ -lattice.

As mentioned above, the moduli dependence of the deformable, shifted $\Gamma_{(4,4)}^{(1)}$ -lattice can be reinstated. After a careful analysis of the moduli space one eventually realizes that the coupling to the R -symmetry charges is again controlled by the radial modulus associated to a compact cycle in the shifted T^4 . Decompactifying the latter, the coupling to the R -symmetry charges is effectively washed out and one recovers a 3d $\mathcal{N} = 2$ vacuum. As discussed in the previous sections, the new non-compact dimension may be interpreted as an emergent spatial dimension that can, in principle, be generated dynamically. Furthermore, with the introduction of additional orbifolds, and by decompactifying additional spatial dimensions, it is possible to reduce the supersymmetries down to $\mathcal{N} = 1$ in four dimensions. We end the discussion and the present chapter here by noting that the precise conditions leading to such dynamical, spontaneous decompactification can only be determined after studying the perturbative corrections to the effective potential, which constitutes in itself a highly non-trivial problem.

Chapitre 8

Non-Singular String Cosmology

In this chapter we take the discussion of Chapter 7 a step further and discuss the problem of constructing cosmological solutions within a perturbative string framework. As discussed in previous chapters, the basic idea is to start from an initial, unstable string vacuum and study how the induced thermal and quantum corrections to the effective potential drive the subsequent cosmological evolution. The chapter is organized as follows. In the first section we begin with a very brief introduction to the theoretical problems that standard cosmology does not address, with special attention to the initial singularity problem. This will provide the motivation for employing string theory to describe the early, high-temperature era of the universe. We will, then, describe a particular class of string toy-models, called the ‘Hybrid’ models, that were first introduced in ref. [47]. These constructions utilize the MSDS structure introduced in the previous chapters and their thermal versions are free of Hagedorn-type instabilities, by the mechanism discussed in Section 7.3. The last part of the chapter is dedicated to the Hybrid models and, in particular, to the discussion of a novel mechanism leading to the absence of initial gravitational singularities, proposed in ref. [48].

8.1 ‘Big Bang’ Cosmology

The basic hypothesis of cosmology, consistent with observation, is that the universe is, at least at large scales, spatially homogeneous and isotropic. One may show that there is a unique metric consistent with this cosmological principle, taken as kinematic input. This metric, which is independent of the underlying dynamics, is the called the Friedman-Lemaître-Robertson-Walker (FLRW) metric :

$$ds^2 = -N^2(t)dt^2 + a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega_{d-2}^2 \right), \quad (8.1)$$

where $N(t)$ is the lapse function, $a(t)$ is the scale factor of the universe, r is the radial comoving coordinate and we have included the line element, $d\Omega_{d-2}^2$, on the surface of the unit $(d-2)$ -sphere, S^{d-2} , for the remaining (hyper-spherical) comoving coordinates. The lapse function $N(t)$ is a gauge artifact, in the sense that it can always be removed by a time

reparametrization. The cosmological frame is defined by the choice $N = 1$. In addition, the parameter k takes discrete values $k = -1, 0, 1$ for a hyperbolic, flat and spherical space, respectively. It characterizes the spatial curvature of the universe, with $k = 0$ associated to a universe with vanishing spatial curvature (flat), $k = 1$ for a universe with positive spatial curvature ('closed') and $k = -1$ for a universe with negative curvature ('open'). We have explicitly kept the number, d , of (non-compact) dimensions arbitrary for future convenience.

Now imagine light being emitted 'radially' at some early time $t = 0$ from a point situated at comoving radial distance r from our observation point and eventually observed at the present cosmological time, t (for the purposes of this paragraph we set, $N = 1$). It is easy to relate the comoving distance r to the time elapsed by :

$$\int_0^t \frac{dt'}{a(t')} = \int_0^r \frac{dr'}{\sqrt{1 - kr'^2}} . \quad (8.2)$$

The proper distance that light has travelled at this time can then be related to the scale factor by :

$$d_H(t) = a(t) \int_0^r \frac{dr'}{\sqrt{1 - kr'^2}} = a(t) \int_0^t \frac{dt'}{a(t')} . \quad (8.3)$$

This defines the particle horizon, which is the surface of largest comoving r at $t = 0$, which is in causal connection with our present observation point at $r = 0$.

We will now setup the cosmological equations in the case of general relativity. For this one needs the Ricci tensor and scalar, parametrized in terms of N, a and their derivatives :

$$\begin{aligned} R^0_0 &= \frac{d-1}{N^2} \left[\frac{\ddot{a}}{a} - \frac{\dot{N}\dot{a}}{N a} \right] , \\ R^i_j &= \left\{ (d-2) \left[\frac{k}{a^2} + \frac{1}{N^2} \left(\frac{\dot{a}}{a} \right)^2 \right] + \frac{1}{N^2} \left[\frac{\ddot{a}}{a} - \frac{\dot{N}\dot{a}}{N a} \right] \right\} \delta^i_j , \\ R &= (d-1)(d-2) \frac{k}{a^2} + \frac{2(d-1)}{N^2} \left[\frac{d-2}{2} \left(\frac{\dot{a}}{a} \right)^2 - \frac{\dot{N}\dot{a}}{N a} + \frac{\ddot{a}}{a} \right] . \end{aligned} \quad (8.4)$$

Consider the standard Einstein-Hilbert action 'minimally' coupled to matter :

$$S = \frac{1}{\kappa_d^2} \int d^d x \sqrt{-g} \left(\frac{1}{2} R - 2\Lambda \right) + S_{\text{matter}} , \quad (8.5)$$

with

$$G_d \equiv \frac{\Gamma\left(\frac{d-1}{2}\right)}{4\pi^{(d-1)/2}} \kappa_d^2 , \quad (8.6)$$

being Newton's constant in d -dimensions. We have also included a cosmological constant Λ , although this will be later absorbed into the energy density of matter. Variation of the above action then yields the field equations for gravity coupled to the energy-momentum tensor of matter, T^{matter} :

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa_d^2 T_{\mu\nu}^{\text{matter}} - g_{\mu\nu} \Lambda . \quad (8.7)$$

Assuming that matter can be approximately described as a perfect fluid of energy density ρ and pressure P , the energy-momentum tensor simplifies to $T^0_0 = -\rho$, $T^i_j = P\delta^i_j$. The 00-component yields the so-called Friedmann-Lemaître (FL) equation :

$$\frac{1}{N^2} \left(\frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{2}{(d-1)(d-2)} (\kappa_d^2 \rho + \Lambda) . \quad (8.8)$$

The other diagonal components yield the acceleration equation (after some simplifications by the FL equation) :

$$\frac{1}{N^2} \left(\frac{\ddot{a}}{a} + \frac{\dot{N}}{N} \frac{\dot{a}}{a} \right) = - \frac{1}{(d-1)(d-2)} \left\{ \kappa_d^2 \left((d-3)\rho + (d-1)P \right) - 2\Lambda \right\} . \quad (8.9)$$

These simplify in the cosmological frame, $N = 1$. In what follows, we drop the cosmological constant term as this is absorbed into the definition of ρ . Using the above two equations, one may prove the continuity equation :

$$\dot{\rho} + (d-1) \left(\frac{\dot{a}}{a} \right) (\rho + P) = 0 . \quad (8.10)$$

It expresses the conservation of the thermal entropy per unit comoving volume. To see this, notice first that the thermodynamic identities following from $dF = -PdV - SdT$:

$$\begin{aligned} S &= \beta^2 \left(\frac{\partial F}{\partial \beta} \right)_V , \\ P &= - \left(\frac{\partial F}{\partial V} \right)_T = -\mathcal{F} , \\ \rho &= \mathcal{F} + \beta \frac{\partial \mathcal{F}}{\partial \beta} , \end{aligned} \quad (8.11)$$

where $\mathcal{F} \equiv F/V$ is the free energy per unit volume and we have used (7.12). Then, by factoring out the scale (volume) factor, a^{d-1} , from the definition of the free energy, $F = E - TS$, one obtains :

$$S = \beta a^{d-1} (\rho + P) . \quad (8.12)$$

Taking the derivative with respect to time and using :

$$\dot{P} = \dot{\beta} \frac{\partial P}{\partial \beta} = -\frac{\dot{\beta}}{\beta} (\rho + P) , \quad (8.13)$$

one readily finds :

$$\dot{S} = \beta a^{d-1} \left(\dot{\rho} + (d-1) \left(\frac{\dot{a}}{a} \right) (\rho + P) \right) = 0 , \quad (8.14)$$

which vanishes because of the equations of motion. Hence, the equations of motion imply the conservation of thermal entropy along the cosmological evolution.

We next assume an equation of state of the form $P = w\rho$, with w being approximately constant for each period of evolution. This permits the integration of (8.10) and yields :

$$\rho \sim a^{-(d-1)(1+w)} . \quad (8.15)$$

Already this implies (provided $w > -1$) that the energy density of the universe diverges as we approach the initial singularity, $a \rightarrow 0$. This is an unavoidable problem of our field-theoretic treatment and signals the breakdown of our approximation of neglecting the effects of quantum gravity. We will come back to this problem in subsequent sections.

Let us mention here some special cases of equation of state. A gas of ultra-relativistic (massless) particles has $P = \frac{1}{d-1}\rho$. This applies to the era of radiation domination. The state equation for ultra-relativistic matter arises by evaluating (7.12) in the massless limit (with $\mu = 0$). The result (dividing by 2 for real fields) is the d -dimensional generalization of the Stefan-Boltzmann law for N_B boson and N_F fermion fields :

$$F = -\frac{V}{\pi^{d/2}\beta^d} \Gamma(d/2) \zeta(d) \left[N_B + \left(1 - \frac{1}{2^{d-1}} \right) N_F \right] . \quad (8.16)$$

This precisely reproduces the state equation for radiation, with $w = 1/(d-1)$.

For matter¹ domination, $w = 0$, since massive particles exert negligible pressure, $P = 0$, as one may easily verify from (7.12).

The third case of interest is $w = -1$ and corresponds to constant energy density and pressure, $P = -p$, from (8.10). It corresponds to a cosmological constant.

Let us now set $N = 1$, for simplicity. Solving the FL equation with (8.15) and $k = 0$, yields :

$$a \sim \begin{cases} t^{\frac{2}{(d-1)(w+1)}} & w \neq -1 \\ e^{\sqrt{\frac{2\Lambda}{(d-1)(d-2)}} t} & w = -1 \end{cases} . \quad (8.17)$$

We can gather the behaviour of the energy density in each case in the following :

$$\begin{cases} w = 1/(d-1) & , & \rho \sim a^{-d} \\ w = 0 & , & \rho \sim a^{-(d-1)} \\ w = -1 & , & \rho = \text{constant} \end{cases} \quad (8.18)$$

The FL equation then, together with the above behaviour, implies that at early cosmological times radiation dominates over matter. Then, as the universe expands, there is an era where

1. By matter we essentially imply particles heavier than the electron.

matter begins to dominate, until eventually the universe reaches its later phase where the vacuum energy density (cosmological constant) will become more important. The reason why the energy density varies faster with the scale factor in the radiation era can be understood in terms of the gravitational redshift, which suppresses each massless particle by a factor $\sim 1/a$.

Using the standard definition, $H \equiv \dot{a}/a$, for the ‘Hubble parameter’ and defining the dimensionless quantity :

$$\Omega \equiv \frac{\rho}{\rho_{\text{crit}}} \equiv \frac{2}{(d-1)(d-2)} \frac{\kappa_d^2 \rho}{H^2} , \quad (8.19)$$

we can rewrite the FL equation as :

$$\Omega - 1 = \frac{k}{(aH)^2} . \quad (8.20)$$

Here, $\rho_{\text{crit}} \equiv (d-1)(d-2)H^2/(2\kappa_d^2)$ is the critical density for the universe to be spatially flat, $k = 0$. Cosmological observations indicate that, at present, the universe is dominated by the vacuum energy contribution to Ω . From :

$$(aH)^2 = \dot{a}^2 \sim a^{-(d-1)(w+1)+2} \sim \begin{cases} t^{-2+4/(d-1)(w+1)} & , \quad w \neq -1 \\ e^{2\sqrt{\frac{2\Lambda}{(d-1)(d-2)}} t} & , \quad w = -1 \end{cases} \quad (8.21)$$

it is easy to see that the universe is accelerating, $\ddot{a} > 0$. In the earlier era of radiation domination, however, the universe was decelerating, $\ddot{a} < 0$. This implies that, as we run backwards in time, the expansion rate of the universe was even larger and if extrapolated leads to a singularity, $a = 0$, the ‘Big Bang’. We will come back to this problem in the following sections.

Notice now that $k/(aH)^2$ grows in the radiation era, so that $\Omega - 1$ will rapidly grow with time unless it is exactly equal to 1. For radiation domination, entropy conservation gives $S = \beta a^{-w(d-1)} = \text{constant}$. Using this, the FL equation becomes :

$$\frac{k}{(a/\beta)^2} = \beta^2 H^2 (\Omega - 1) . \quad (8.22)$$

The l.h.s. is constant along the evolution and so must the r.h.s. With present observations one finds the bound :

$$\left| \frac{k}{(a/\beta)^2} \right| \lesssim 10^{-58} . \quad (8.23)$$

This requires an unnatural amount of fine-tuning in the value of Ω at early times :

$$|\Omega - 1| \lesssim 10^{-61} , \quad (8.24)$$

(at the Planck scale) in order to ensure that the currently measured value of Ω is already so close to unity (unless, of course, k is exactly zero). This is the ‘flatness’ problem of the Big Bang paradigm and is one of the unanswered questions of the model.

Another related problem of the Big Bang model has to do with the fact that the universe is homogeneous and isotropic to a high degree of accuracy. Since homogeneity and isotropy tend to become disrupted in the process of cosmological evolution as a result of gravitational attraction, even small inhomogeneities at early times would become enhanced and lead to macroscopically observable effects of anisotropy and inhomogeneity at present times. This implies that in its very early era, the universe was homogeneous and isotropic to an even greater extent and there is no mechanism to account for this fact within the Big Bang model.

The third major problem is the so-called ‘horizon’ problem. Around the time of ‘last scattering’², the universe was still so young ($\sim 10^5$ years old) that different positions on the surface of the particle horizon were not yet in causal connection at the time of emission. Yet, light observed in the spectrum of the Cosmic Microwave Background (CMB) exhibits a remarkable amount of homogeneity.

There exist also additional problems that standard Big Bang cosmology cannot address, such as the absence of topological defects (and other unwanted relics). Given the large energies involved in the early phases of the universe, one would have expected the creation of large numbers of topological defects, typically encountered in GUTs, such as monopoles, domain walls and cosmic strings. The fact that none of these have been observed (yet) poses a question as to the underlying mechanism that would have diluted their density to within experimental bounds.

Inflation is a mechanism that addresses at least some of the above problems. It was initially proposed as a potential solution to the horizon and flatness problem, by assuming that the radiation era was preceded by a period of accelerated expansion, where a increases faster than the radius of the horizon, H^{-1} . This implies that $(aH)^{-2}$ decreases rapidly with time, so that the flatness problem is resolved. At the same time, this provides an explanation for the horizon problem, since objects that were in causal contact before the period of inflation become rapidly separated (beyond the Hubble radius and, hence, their own horizons) and no longer appear to be in causal contact after inflation. We will not discuss inflation or its scenarios here but will directly proceed to discuss the construction of cosmological solutions at the perturbative string level.

8.2 String Cosmology

In the previous section we have briefly discussed some of the basic ideas and problems encountered in the Big Bang scenario. As we have already mentioned, the extrapolation of the equations of standard cosmology backwards in time inescapably encounters an initial singularity, where the energy density diverges, $\rho \rightarrow \infty$. Already this unphysical result is a

2. The time of last scattering is a time in cosmological evolution subsequent to the time of recombination (hydrogen formation) when the photons effectively decouple from the electron plasma and reach present observers undisturbed.

signal that our original approximation of ignoring effects of quantum gravity breaks down as we approach the Planck length. In these scales, the conventional description of spacetime itself is expected to break down and one needs to employ a theory of quantum gravity in order to keep track of the new phenomena, taking place in these regions. This provides the basic motivation and necessitates a framework to construct and study cosmological solutions at the perturbative string level.

Let us repeat here the basic idea behind such constructions. Vacua with spontaneously broken supersymmetry receive radiating corrections from higher genres, which appear as non-vanishing contributions to the effective potential of the theory. It is well-known that the 1-loop contribution to the effective potential is precisely equal to the vacuum amplitude, as a function of the various moduli fields. Consider, thus, such vacua where supersymmetry has been spontaneously broken and in a region of moduli space where the 1-loop effective potential is a finite function of the moduli, $V_{\text{eff.}}(\mu_I) \neq 0$. The same will apply to the case of thermal vacua with non-vanishing free energy, as a function of the thermodynamical parameters $\{\beta, \mu^I, \tilde{\mu}_I\}$ defining the phase, as in (7.38).

This non-vanishing 1-loop contribution triggers a backreaction on the underlying background. The genus-1 vacuum amplitude actually provides a tadpole for the dilaton, which needs to be cancelled by a correction of the tree-level background. This can be thought of as restoring conformal invariance at the 1-loop level, because the non-vanishing 1-loop amplitude can be interpreted as a conformal anomaly, in a way similar to how the conformal anomaly (and its subsequent cancellation) at tree level gives rise to the equations of motion for the background, (2.37) for the bosonic string case. It is precisely the same mechanism of minimization of β -functions as the one encountered at tree level, carried out at higher genres. Similar corrections to the background have to be taken into account at the level of higher genres. In all subsequent discussions, we will truncate these corrections at genus-1. For this to be consistent, we have to assume that we are in the perturbative regime where g_s is sufficiently small.

This instability of the background suffices to trigger a cosmological evolution, obtained via the above correction to the (flat) tree-level result, with the moduli (temperature, ...) acquiring time dependence. Hence, in principle, assuming various adiabatic conditions (slow variation of moduli with time), the finiteness of the 1-loop amplitude allows one to study the emergent cosmology within the framework of perturbative string theory.

This attractive programme lies at the core of (perturbative) string cosmology. However, before this can be realized one must first deal with two highly non-trivial obstacles. The first has already been encountered in the previous chapter and arises because of the presence of tachyonic states leading to the divergence of the genus-1 amplitude upon integration over the fundamental domain. As we argued there, this is formally equivalent to the presence of Hagedorn divergences and arise when winding states become tachyonic once certain radial moduli reach the string scale (Hagedorn temperature). The second problem is related to the initial singularity. This is the Big Bang curvature singularity, implied by the classical singularity theorems [62–64]. These are valid under the assumption that matter satisfies the

weak³ or null energy condition⁴. It is not likely that this problem can be resolved within a purely field-theoretical (or general relativistic) treatment. We will come back to this problem in the next section, where we introduce the Hybrid models.

We conclude this section by giving the cosmological equations of motion arising from string theory. The presence of the dilaton now modifies the form of the effective action. In terms of the lapse function and scale factor the gravity-dilaton action can be written (after integration by parts) as :

$$S = \frac{1}{\kappa_d^2} \int e^{-2\phi} N a^{d-1} \left[\frac{(d-1)(d-2)}{2} \frac{k}{a^2} + \frac{2(d-1)}{N^2} \left(\frac{\dot{a}}{a} \right)^2 \dot{\phi} - \frac{2}{N^2} \dot{\phi}^2 \right] + \int N a^{d-1} P . \quad (8.25)$$

Notice the absence of a dilaton dressing factor in the second integral (representing the matter contribution), hence, identifying it as a genus-1 contribution to the effective potential. For simplicity, we ignore the kinetic terms of other fields and consider them as fixed backgrounds.

Variation with respect to the lapse function, scale factor and dilaton yields, respectively :

$$\dot{\phi}^2 - (d-1) \left(\frac{\dot{a}}{a} \right) \dot{\phi} = \frac{1}{2} N^2 e^{2\phi} \rho , \quad (8.26)$$

$$\ddot{\phi} - \dot{\phi} \left(\dot{\phi} + \frac{\dot{N}}{N} \right) = \frac{(d-2)(d-3)}{4} \frac{k N^2}{a^2} + \frac{1}{2} N^2 e^{2\phi} P , \quad (8.27)$$

$$\frac{\ddot{a}}{a} + (d-2) \left(\frac{\dot{a}}{a} \right)^2 - \frac{\dot{a}}{a} \left(2\dot{\phi} + \frac{\dot{N}}{N} \right) = - \frac{(d-2)}{(d-1)} \frac{k N^2}{a^2} - \frac{1}{d-1} N^2 e^{2\phi} P . \quad (8.28)$$

From these one may derive the analogous conservation equation :

$$\dot{\rho} + (d-1) \left(\frac{\dot{a}}{a} \right) (\rho + P) = 2(d-1)(d-2) e^{-2\phi} \left[\left(\frac{k}{a^2} + \frac{1}{N^2} \left(\frac{\dot{a}}{a} \right)^2 \right) \dot{\phi} - \frac{d-3}{4} \frac{k}{a^2} \frac{\dot{a}}{a} \right] . \quad (8.29)$$

Notice that this does not have the simple form of (8.10), because the dilaton should also contribute to the (total) energy density and pressure. In 2d, however, which will be the case of interest for the Hybrid models of the next section, the r.h.s. of the above equation vanishes identically and we recover the same simple expression. For higher dimensions, one may perform a dilaton-dependent rescaling of the metric :

$$g_{\mu\nu} = \exp \left(\frac{4\phi}{d-2} \right) \tilde{g}_{\mu\nu} , \quad (8.30)$$

3. For an ideal fluid this implies that ρ and $\rho + P$ are non-negative.

4. For an ideal fluid this becomes $\rho + P \geq 0$.

to go to the ‘Einstein’ frame where the gravity-dilaton action takes the conventional form :

$$S_E = \frac{1}{\kappa_d^2} \int d^d x \sqrt{-\tilde{g}} \left(\frac{1}{2} \tilde{R} - \frac{2}{d-2} (\nabla\phi)^2 \right) + \dots , \quad (8.31)$$

but with the presence of a scalar dilaton with the opposite sign in the kinetic term.

In the stringy domain, new states appear together with various purely stringy, non-geometrical phenomena related to the presence of extended symmetries and dualities. This opens up new possibilities to address the initial singularity problem. For discussions, see [65–71] and also [2], [72] and references therein.

Departing from the effective field theory approximation and moving to the underlying string theory, we encounter an infinity of states, most of which carry masses of the order of the string scale. At sufficiently low energies, these are integrated out and one obtains an effective field theory description in terms of a finite number of fields. However, as we move along the moduli space of string theory, some of the masses are shifted and we may encounter points where new, purely stringy states (carrying non-trivial winding) become massless. These points are typically associated with enhanced symmetry, as we have already discussed in previous chapters.

Around these points, the original effective field theory approximation is no longer valid, precisely because the extra degrees of freedom can no longer be ignored. This signals a breakdown of the approximation and may lead to apparent singularities in field theory that can only be resolved within a fully stringy treatment, where all degrees of freedom are properly accounted for. In practice, incorporating the effect of these additional states in string theory typically requires a thorough understanding of their contribution to the genus-1 effective potential, around extended symmetry points. Already this is a hard problem because, as we will see below, lack of absolute convergence at the extended symmetry points poses problems in evaluating (even approximately) the relevant integral over the fundamental domain. This technical issue obscures even more attempts to properly treat these regions of moduli space, without introducing approximations and missing precisely the stringy effects. We will see that MSDS structure provides a way out of this problem (at least in some particular situations, such as that of the Hybrid model). The cancellations it introduces permit the exact calculation of the free energy as a function of temperature and make possible the study of the induced cosmology around the stringy region.

8.3 Hybrid Models in $2d$

Hybrid Model at $T = 0$

We will now discuss a particular class of tachyon-free models, with MSDS structure, first introduced in [47]. They take their name, ‘Hybrid models’, because they arise from mixing together supersymmetric and MSDS structures. At zero temperature, they are 2d Type II vacua, compactified on an eight-dimensional torus, T^8 , with special values for the radii, allowing the right-moving side to support MSDS structure. Hybrid models are $(4, 0)$ -supersymmetric constructions with 4 supersymmetries arising from the left-moving side,

while the right-supersymmetries are spontaneously broken at the string level and replaced by the MSDS structure. The genus-1 amplitude of the Hybrid model is then :

$$\begin{aligned}
Z &= \frac{V_2}{(2\pi)^2} \int_{\mathcal{F}} \frac{d^2\tau}{4\tau_2^2} \left[\frac{1}{2} \sum_{a,b} (-)^{a+b+ab} \frac{\theta^4[\frac{a}{b}]}{\eta^4} \right] \left[\frac{1}{2} \sum_{\gamma,\delta} \frac{\theta^8[\frac{\gamma}{\delta}]}{\eta^8} \right] \left[\frac{1}{2} \sum_{\bar{a},\bar{b}} (-)^{\bar{a}+\bar{b}+\bar{a}\bar{b}} \frac{\bar{\theta}^{12}[\frac{\bar{a}}{\bar{b}}]}{\bar{\eta}^{12}} \right] \\
&= \frac{V_2}{(2\pi)^2} \int_{\mathcal{F}} \frac{d^2\tau}{4\tau_2^2} \frac{1}{\eta^8} \Gamma_{E_8} (V_8 - S_8) (\bar{V}_{24} - \bar{S}_{24}). \tag{8.32}
\end{aligned}$$

The second line organizes the spectrum terms of $SO(2N)$ -characters in order to better display the origin of supersymmetric and MSDS structures. The spectrum contains 24×8 massless scalars from the $V_8 \bar{V}_{24}$ sector, with the same number of massless fermions arising from $S_8 \bar{V}_{24}$.

This model is easily obtained from the ‘maximal’ MSDS models, (6.7), by performing a \mathbb{Z}_2 -shift on all 8 internal left-moving coordinates, $X_L^I \rightarrow X_L^I + \pi$. In particular, this shift has the effect of breaking the extended left-moving gauge group $H_L = [SU(2)_L]_{k=2}^8$ down to the abelian subgroup, $U(1)_L^8$, while preserving the enhancement of the right-moving gauge group, $H_R = [SU(2)_R]_{k=2}^8$, which is tied together with the presence of MSDS structure.

Thermal Hybrid Vacuum

We now introduce thermal versions of the Hybrid model by the ‘thermal’ Scherk-Schwarz deformation introduced in the previous chapter. This will be done in a way so as to saturate the conditions (7.44) that will refine the thermal ensemble and guarantee the absence of Hagedorn-like singularities. To this end, we compactify one of the remaining longitudinal directions (associated with Euclidean time) to a circle of radius $R = \beta/2\pi$ and we mod out by the \mathbb{Z}_2 -orbifold, $(-)^{F_L} \delta$, where δ is the half-shift along the circle. This is relevant for a thermal interpretation, as discussed in the previous chapter.

This corresponds to the factorization of the $\Gamma_{(1,1)}[\frac{a}{b}]$ lattice of the Euclidean time circle, coupled only chirally to F_L . The genus-1 amplitude then expresses the free energy :

$$Z = \frac{V_1}{8\pi} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^{3/2}} \left[\frac{1}{2} \sum_{a,b} (-)^{a+b+ab} \frac{\theta^4[\frac{a}{b}]}{\eta^4} \right] \Gamma_{E_8} (\bar{V}_{24} - \bar{S}_{24}) \Gamma_{(1,1)}[\frac{a}{b}](R), \tag{8.33}$$

where

$$\Gamma_{(1,1)}[\frac{a}{b}](R) = \frac{R_0}{\sqrt{\tau_2}} \sum_{\tilde{m}^0, n^0} e^{-\frac{\pi R^2}{\tau_2} |\tilde{m}^0 + \tau n^0|^2} (-)^{\tilde{m}^0 n^0 + a \tilde{m}^0 + b n^0}, \tag{8.34}$$

and Γ_{E_8} is the chiral E_8 -lattice, which is modular invariant by itself. The spectrum becomes more transparent in the Hamiltonian representation, obtained by Poisson resumming \tilde{m} :

$$\frac{V_1}{8\pi} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^{3/2}} \Gamma_{E_8} (\bar{V}_{24} - \bar{S}_{24}) \sum_{m,n} \left(V_8 \Gamma_{m,2n} + O_8 \Gamma_{m+\frac{1}{2},2n+1} - S_8 \Gamma_{m+\frac{1}{2},2n} - C_8 \Gamma_{m,2n+1} \right). \tag{8.35}$$

In the above expression we defined the Hamiltonian lattice representation :

$$\Gamma_{m,n} \equiv q^{\frac{1}{2}P_L^2} \bar{q}^{\frac{1}{2}P_R^2} , \quad (8.36)$$

with the left- and right- moving momenta defined, as usual, by :

$$P_{L,R} \equiv \frac{1}{\sqrt{2}} \left(\frac{m}{R} \pm nR \right) . \quad (8.37)$$

This is a thermal vacuum and, hence, the 4 left-moving supersymmetries are spontaneously broken by the thermal deformation.

Let us now discuss the spectrum. Notice that the initially massless fermions, $S_8 \bar{V}_{24}$, now acquire masses and, hence, massless states may only arise from the scalar sector $V_8 \bar{V}_{24}$. Thermal winding tachyons can only arise from $O_8 \bar{O}_{24}$ -sector. However, states in the O_8 -sector now carry non-trivial winding and momentum quantum numbers around S^1 . As a result, the lowest mass states in the $O_8 \bar{V}_8$ -sector always have positive definite mass :

$$2m_{O\bar{V}}^2 = \left(\frac{1}{\sqrt{2}R} - \sqrt{2}R \right)^2 , \quad (8.38)$$

even after deformation of the dynamical moduli (transverse to the Euclidean time direction). One notices the appearance of extra massless states at the self-dual, ‘fermionic’ radius $R = 1/\sqrt{2}$. As expected, this is accompanied by enhancement of gauge symmetry. The ‘longitudinal’ $U(1)$ associated to the Euclidean time direction becomes enhanced as :

$$U(1)_L \times U(1)_R \longrightarrow [SU(2)_L]_{k=2} \times U(1)_R . \quad (8.39)$$

This enhancement of gauge symmetry takes place in the Euclidean description, where time is compactified. It arises because of the presence of string states, with no field-theoretic analogue, winding the Euclidean time direction, which become massless at the self-dual point.

It is now legitimate to pose a very interesting question. What is the analogous manifestation of this ‘enhanced symmetry’ in spacetime, where (real) time is non-compact and there are no states to wind around it? We will see below that the correct interpretation from the spacetime point of view is that of a phase transition.

Returning to the absence of tachyonic modes in this thermodynamical phase, notice that the conditions (7.44) for tachyon stability are only satisfied as long as the thermal $\Gamma_{(1,1)}[a_b]$ -lattice remains factorized from the remaining T^8 -torus. This is indeed the case, and the thermal lattice remains factorized for arbitrary deformations, since those moduli (associated to the gravito-magnetic fluxes) that would be responsible for the mixing do not correspond to fluctuating fields and, hence, the model is stable and free of Hagedorn instabilities.

In addition, these particular fluxes have the important effect of restoring the thermal T -duality :

$$R \rightarrow \frac{1}{2R} \quad \text{and} \quad S_8 \leftrightarrow C_8 , \quad (8.40)$$

which is typically present in Heterotic, but not in conventional Type II thermal vacua.

The second effect of these ‘gravito-magnetic’ fluxes is that they are responsible for injecting non-trivial winding charge into the thermal vacuum. The modified thermal deformation, $(-)^{F_L}\delta$, can then be shown to correspond precisely to the condensate of those operators characterized by non-trivial winding and, hence, correspond to the would-be winding tachyons. Notice, further, that the thermal duality is accompanied by the interchange of the chiralities of the spinors, so that the ‘Hybrid IIB’ and ‘Hybrid IIA’ phases are exchanged. As we discuss below, the interpretation of this duality map is that the thermal quanta of Hybrid IIA are mapped into vortices, carrying non-trivial winding numbers in the dual Hybrid IIB phase and vice-versa.

Free Energy in Hybrid Models

The free energy can be calculated by unfolding the fundamental domain for $R_0 > 1/\sqrt{2}$, along the lines described in Chapter 7. The result is an integral over the strip, [47], [48] :

$$\begin{aligned} \frac{Z_{1\text{-loop}}}{V_1} &= R \sum_{\tilde{m}^0 \neq 0} \int_{||} \frac{d^2\tau}{8\pi\tau_2^2} e^{-\frac{\pi(\tilde{m}^0 R)^2}{\tau_2}} \frac{1}{2\eta^{12}} \sum_{a,b} (-)^{a+b+ab} \theta^4_{[b]}[a] (-)^{\tilde{m}^0 a} \\ &\quad \times \frac{1}{2\bar{\eta}^{12}} \sum_{\bar{a},\bar{b}} (-)^{\bar{a}+\bar{b}+\bar{a}\bar{b}} \bar{\theta}^4_{[\bar{b}]}[\bar{a}] \Gamma_{(8,8)}[\bar{a}]_{[\bar{b}]} . \end{aligned} \quad (8.41)$$

This is identified with the Schwinger representation, (7.12), of the thermal partition function (thermal trace) of the form (7.46).

The presence of points with MSDS structure in the moduli space of Hybrid models has remarkable consequences, as it permits one to probe the contribution of the purely stringy (winding) states to the free energy. Indeed, it was shown in [47] that the strip integral for the free energy of the Hybrid model can be performed explicitly and without any (α' - or other) approximation, as long as the underlying right-moving MSDS structure is preserved. The result is that, because of MSDS structure, the deformed thermal trace (by the presence of the gravito-magnetic fluxes) over the Hilbert space of the initially (4,0)-supersymmetric string theory :

$$Z_{1\text{-loop}} = \log \text{Tr} \left[e^{-\beta H} (-)^{F_R} \right] \xrightarrow{\text{MSDS}} \log \text{Tr} \left[e^{-\beta H} \right] \Big|_{m^2=0} , \quad (8.42)$$

is reduced to the canonical (undressed) thermal trace, restricted over the massless Hilbert space of the (4,0)-theory. This miraculous fact provides an enormous simplification and is sufficient to give the behaviour of the winding states at the self-dual point. Let us see how this arises. The strip integral (8.41) becomes :

$$\frac{Z_{1\text{-loop}}}{V_1} = R \sum_{\tilde{m} \neq 0} \int_{||} \frac{d^2\tau}{8\pi(\text{Im}\tau)^2} e^{-\frac{\pi(\tilde{m}R)^2}{\tau_2}} \frac{\Gamma_{E_8}}{\eta^8} (V_8 - (-)^{\tilde{m}} S_8) (\bar{V}_{24} - \bar{S}_{24}) . \quad (8.43)$$

This expression implies that the initially massless bosons and fermions from sectors $V_8\bar{V}_{24}$ and $S_8\bar{V}_{24}$ are thermally excited, as in the ‘conventional’ thermal deformation. The difference lies in the sectors $V_8\bar{S}_{24}$ and $S_8\bar{S}_{24}$, which carry odd right-fermion number ($\bar{a} = 1$) and have string-scale masses, even at zero temperature. However, inspection of (8.43) shows that, due to the unbroken MSDS structure, $\bar{V}_{24} - \bar{S}_{24} = 24$ and level matching, the contribution of all massive boson and fermion string oscillators exactly cancel. This is an important observation, because it implies that only the thermally excited massless states give non-vanishing net contributions.

Notice that (8.41) is strictly valid only for radii larger than the fermionic value, $R > 1/\sqrt{2}$. The reason lies in the fact that the unfolding procedure, as well as the derived thermal interpretation, crucially require the absolute convergence of the integrand in order to interchange the orders of integration and summation. This requirement is, however, spoiled precisely at the extended symmetry point, $R = 1/\sqrt{2}$, where the presence of extra massless states produce terms that are no longer exponentially suppressed in the $\tau_2 \rightarrow \infty$ limit. Hence, the unfolding procedure and the thermal interpretation itself as a thermal trace (deformed or not) at temperature $\beta^{-1} = (2\pi R)^{-1}$ breaks down at the self-dual point and the above expressions are no longer valid for smaller radii.

Instead, in order to obtain the analogous expressions valid in the range $R < 1/\sqrt{2}$, one must first double-Poisson resum to the dual phase and then unfold, so as to ensure absolute convergence. However, in that case, the thermal interpretation changes and the expressions obtained now involve the ‘ T -dual’ temperature, $\tilde{\beta}^{-1} \equiv (2\pi\tilde{R})^{-1} = (\pi/R)^{-1}$. In other words, the field-theoretic expression (8.42) fails to capture the thermal duality, present in the full stringy description. Hence, it becomes clear that the fundamental object is not the (field-theoretic) thermal trace but, rather, the Euclidean stringy path integral, which is valid for all temperatures and which exhibits the dualities manifestly.

Putting everything together, the thermal 1-loop amplitude in the Hybrid models with unbroken MSDS structure is explicitly calculated as function of the thermal radius, [47] :

$$\frac{Z_{1\text{-loop}}}{V_1} = 24 \times \left(R + \frac{1}{2R} \right) - 24 \times \left| R - \frac{1}{2R} \right|. \quad (8.44)$$

It is important to stress that this is an exact result, without any α' - or large/small radius approximation. The above expression for the free energy is manifestly invariant under the thermal duality, $R \rightarrow 1/(2R)$. The second term involves the absolute value and is, hence, non-analytic. This conical structure can be traced back to the presence of the extra states becoming massless at the self-dual fermionic point, $R = 1/\sqrt{2}$. Since, as mentioned above, only massless states eventually contribute to the thermal trace, it comes as no surprise that this expression coincides with the free energy (in field theory) for a 2d gas of massless states. One may derive its equation of state :

$$\rho = P = 48\pi T^2, \quad (8.45)$$

which is precisely the equation of state, $w = 1$, for thermal massless radiation in 2d.

Physical, Duality-Invariant Temperature

The above discussion makes it possible to identify three distinct regions, depending on the value of the thermal radius. In the first region, $R > 1/\sqrt{2}$, and the free energy takes the usual form of 2d massless radiation :

$$Z_{1\text{-loop}} = \frac{24}{R} , \quad (8.46)$$

and the temperature is defined in the conventional way, $\beta = 2\pi R$. Up to now the situation is the one familiar from standard thermal treatments. As mentioned before, only the initially massless states in the $V_8\bar{V}_{24}$ and $S_8\bar{V}_{24}$ contribute to the thermal trace. They are the familiar states characterized by non-trivial thermal momenta, so this space will be referred to as the space of ‘pure momenta’.

The second region of interest contains the radii smaller than the fermionic value, $R < 1/\sqrt{2}$. There the free energy has the dual expression :

$$Z_{1\text{-loop}} = 24 \times (2R) , \quad (8.47)$$

and the inverse temperature now has to be defined by the T -dual expression, $\beta = 2\pi/(2R)$. The light states now occur in $V_8\bar{V}_{24}$ and $C_8\bar{V}_{24}$ sectors, which are characterized by non-trivial winding. Hence, we refer to this space as the ‘pure winding’ space. In this space, the light states are vortices, rather than the usual thermal quanta and if we insist on interpreting them as such, we are forced to pass to the dual description where these are mapped into thermal quanta, hence, the modification of the definition of temperature. The above expression is, then, in accordance with the physical thermodynamics requirement that the thermal partition function decreases with decreasing temperature.

Finally, there is a third region around the self-dual point, $R_0 \sim 1/\sqrt{2}$. This region is governed by the presence of the 24 (localized) additional massless states, arising from the $O_8\bar{V}_{24}$ -sector. Notice that are states carrying both non-trivial momentum and winding quantum numbers. We will see their significance below.

The above illustrates the necessity for three distinct, local effective field theories in order to describe the stringy thermal system. From the point of view of field theory, these regions are disconnected, even though they become connected in terms of the underlying string theory. However, to find the proper description that treats all these ‘disjoint’ spaces in a unified way and, hence, to be able to understand the physics of the ‘gluing’ in a way that permits one to study the cosmological evolution as the universe passes from one space to another is a highly non-trivial problem. A more detailed discussion can be found in [48]. In this sense, the Hybrid models can be seen as ‘prototype’ models where this connection can be carried out in a transparent way.

The first step in this direction is to introduce an ‘auxiliary’ thermal variable $\sigma \in (-\infty, +\infty)$:

$$R = \frac{1}{\sqrt{2}} e^\sigma , \quad (8.48)$$

in terms of which the physical temperature will be defined in a duality-invariant way, [48] :

$$T = T_c e^{-|\sigma|} \quad , \quad \text{where } T_c \equiv \frac{\sqrt{2}}{2\pi} . \quad (8.49)$$

In terms of this new parameter, the partition function of the theory takes the form :

$$Z_{1\text{-loop}} = (24\sqrt{2}) e^{-|\sigma|} = \Lambda T \quad , \quad \text{where } \Lambda = 48\pi . \quad (8.50)$$

As we will argue below, the physical picture as σ increases, is that the system starts to heats up until it reaches the critical temperature, undergoes a phase transition and then, subsequently, begins to cools down in the dual phase. The two dual phases can be distinguished at low temperature by looking at the chirality of the light thermally excited spinors, which come with opposite chiralities :

$$\begin{aligned} \text{Hybrid B phase : } \sigma > 0 &\longrightarrow S_8 , \\ \text{Hybrid A phase : } \sigma < 0 &\longrightarrow C_8 . \end{aligned} \quad (8.51)$$

In [48], it was proposed that, upon reaching the critical self-dual temperature, the system undergoes a phase transition from the space of light momenta to the space of windings (or vice-versa). At T_c new light modes appear and it is precisely these that are responsible for conducting the transition between the two dual phases.

We can motivate this picture in the following way. Consider a pure momentum state in the $S_8\bar{V}_{24}$ -sector, at the extended symmetry point $\sigma = 0$. One may see that, at this point, there exist localized (massless) operators inducing transitions between the spaces of pure momenta and pure windings. Indeed, taking, for example, the zero mode of the current $J_- = \psi^0 e^{-iX_L^0}$, which has the effect of lowering one unit of left-moving momentum, and acting with it on the pure momentum state we find :

$$J_-(z) e^{-\phi/2} S_{10,\alpha} e^{\frac{i}{2}X_L^0 + \frac{i}{2}X_R^0} \bar{V}_{24}(w) \sim \frac{1}{z-w} \gamma_{\alpha\dot{\beta}}^0 e^{-\phi/2} C_{10,\dot{\beta}} e^{-\frac{i}{2}X_L^0 + \frac{i}{2}X_R^0} \bar{V}_{24}(w) + \dots \quad (8.52)$$

Therefore, there exist non-trivial 3-point amplitudes inducing transitions between pure momentum and pure winding states of opposite chirality (in the $C_8\bar{V}_{24}$ -sector).

8.4 Non-Singular Hybrid Cosmology

There have been many works in the literature studying the induced cosmological evolution in the ‘intermediate era’, $\beta \gg \beta_H$. In particular, several of these constructions show the presence of attractor solutions to a radiation-like era, [73–80]. Other works include, e.g. [81–87]. There are several works in the literature that address the early hot phase of the universe and discuss the singularity problem from the point of view of string theory, [67–69]. For a comprehensive review, see [70] and references therein.

However, the usual approach to hot string gas cosmology typically analyzes the cosmology starting from the ‘intermediate’ era, where the temperature has already fallen sufficiently below the Hagedorn point, to allow for a perturbative treatment.

In [48], the attractive properties of MSDS models were employed in order to study the backreaction of the thermal Hybrid vacua on the initially flat metric and constant dilaton and, hence, derive the induced cosmology in the early, hot phase, around the Hagedorn point. The main goal of this work was to show that, by keeping properly track of the full stringy degrees of freedom, including the contributions of the extra massless states at the self-dual temperature, it is possible to obtain cosmological solutions that successfully avoid the initial gravitational singularity while, at the same time, remaining in the perturbative regime throughout the evolution.

In the previous section we have already explained the need for three distinct effective field theories in order to describe the full stringy thermal system. Returning to the Euclidean description, one notices that the region around the phase transition (self-dual) point at $\sigma = 0$ is described by a one-dimensional, extended, non-abelian gauge theory :

$$H_L \times H_R \equiv (SU(2) \times U(1)^8)_L \times (U(1) \times SU(2)^8)_R , \quad (8.53)$$

that is sourced by 24 localized complex massless scalars, χ_i . They give rise to a localized action :

$$S|_{\sigma=0} = \int dx^1 \sqrt{g_{11}} e^{-2\phi} \left(-g^{11} \frac{\partial \bar{\chi}_i}{\partial x^1} \frac{\partial \chi_i}{\partial x^1} \right) \Big|_{\text{at } \sigma(x^0)=0} , \quad (8.54)$$

that is restricted only to the region where $\sigma(x^0) = 0$. To study cosmological solutions we restrict ourselves to spatially homogeneous fields, g_{11} , ϕ (that is, independent of the x^1 -coordinate). The equation of motion of $\bar{\chi}_i$ then yields :

$$\chi_i = \alpha_i + \gamma_i \sqrt{g_{11}} x^1 . \quad (8.55)$$

Notice that masslessness implies the presence of the linear term, controlled by the γ_i -gradient parameters.

As proposed in [48], the phase transition admits an effective description in terms of an effective spacelike brane that consistently glues together the space of momenta with the dual space of windings. Adding the kinetic terms of the gravity-dilaton action, together with the thermal effective potential and the brane contribution gives an effective action of the form :

$$S = \int d^2x e^{-2\phi} \sqrt{-g} \left(\frac{1}{2} R + 2(\nabla\phi)^2 \right) + \int d^2x \sqrt{-g} P - \kappa \int dx^1 d\sigma e^{-2\phi} \sqrt{g_{11}} \delta(\sigma) . \quad (8.56)$$

The effect of the spacelike brane which lives only at the point of the phase transition, $\sigma = 0$, is to inject localized negative pressure to the system, which is precisely sourced by the 24 extra massless bosons at the extended symmetry point. The brane tension, κ is a positive

quantity, as can be seen by inspecting its microscopic origin, from (8.54). They are expressed in terms of the gradient coefficients as :

$$\kappa = \sum_i |\gamma_i|^2 > 0 . \quad (8.57)$$

At this point, it will be useful to mention some of the assumptions implicit in this description. As discussed also earlier, we are assuming an adiabatic evolution, with slowly varying fields and, in particular, we require that the characteristic time-scale of the transition is sufficiently short to allow an Dirac $\delta(\sigma) \sim \delta(x^0)$. With this assumption, the phase transition takes place at a given time, $x^0 = 0$.

At first order, we may neglect higher-derivative corrections which, in principle, have to be included to the action together with higher genus corrections to the thermal potential. We will see below that these are suppressed for most of the evolution and our approximation is self-consistent. These effects are controlled by the characteristic scale κ . However, even at the phase transition point, it is possible to have quantitative control due to the existence of an exact CFT description at the extended symmetry point.

Furthermore, the full effective action should also contain contributions of other light moduli. In particular, the 64 moduli parametrizing the deformation coset, $SO(8,8)/SO(8) \times SO(8)$. However, in all phases of the thermal Hybrid model, these flat directions are eventually lifted by the 1-loop effective potential and the above moduli become stabilized at the enhanced MSDS symmetry point, where they are ‘entropically’ favored. Hence, we will not study their background values but will consider them effectively frozen to their MSDS values, hence, preserving the MSDS structure of the theory.

We are now in the position to study the induced cosmological solution to the scale factor and dilaton. The energy density and pressure of the Hybrid phase are given by :

$$\rho = P = \frac{\Lambda}{\beta^2} , \quad (8.58)$$

with $\beta = \beta_c e^{|\sigma|}$, as discussed in the previous section. The critical value of the inverse proper temperature is $\beta_c = (2\pi)/\sqrt{2}$ and, for convenience, we defined the thermal parameter $\Lambda \equiv 48\pi$.

We will use the standard parametrization of the metric, relevant for 2d :

$$ds^2 = -N^2(t)dt^2 + a^2(t)dx^2 , \quad (8.59)$$

where the scale factor will be parametrized in terms of $a = e^\lambda$. Assuming the phase transition to be localized in time at $t = 0$ we may derive the ‘sigma model frame’ equations of motion,

which will now be modified because of the presence of the brane term :

$$\begin{aligned}
\dot{\phi}^2 - \dot{\phi}\dot{\lambda} &= \frac{1}{2} N^2 e^{2\phi} \rho , \\
\ddot{\lambda} - \dot{\lambda} \left(2\dot{\phi} - \dot{\lambda} + \frac{\dot{N}}{N} \right) &= N^2 e^{2\phi} P , \\
\ddot{\phi} - 2\dot{\phi}^2 + \dot{\phi} \left(\dot{\lambda} - \frac{\dot{N}}{N} \right) &= \frac{1}{2} N^2 e^{2\phi} (P - \rho) - \frac{1}{2} N \kappa \delta(t) .
\end{aligned} \tag{8.60}$$

Notice the presence of the Dirac δ -function appearing only in the third equation, implying a discontinuity (around the transition) in the first derivative of the dilaton only, while the lapse function, the scale factor and their first derivatives remain continuous everywhere. These can be simplified considerably by going to the conformal gauge $N = e^\lambda$ and by using the state equation, $\rho = P$.

At the point of the transition, the temperature has its maximal, critical value, β_c . The thermal entropy per unit comoving volume, $S = a\beta(\rho + P) = 2\Lambda(a/\beta)$, is then conserved across the brane, as a result of the continuity of the scale factor. Using the equations of motion, one arrives at the continuity equation :

$$\dot{\rho} + \dot{\lambda}(\rho + P) = \frac{2\kappa}{N} e^{-2\phi} \left(\dot{\lambda} - 2\dot{\phi} \right) \delta(t) . \tag{8.61}$$

Conservation of the thermal entropy across the transition then requires the r.h.s. to vanish at the critical point. Equivalently, the transition does not involve latent heat and is, in this respect, similar to a second order transition. Writing $\dot{\lambda}_\pm \equiv \dot{\lambda}(0\pm)$ and $\dot{\phi}_\pm \equiv \dot{\phi}(0\pm)$ for the derivatives of the scale factor and the dilaton, respectively, on the two sides of the brane, this condition becomes :

$$\dot{\lambda}_+ + \dot{\lambda}_- = 2 \left(\dot{\phi}_+ + \dot{\phi}_- \right) . \tag{8.62}$$

Using this with entropy conservation, one may already see the first hint that the gravitational singularity is absent :

$$\lambda = |\sigma| + \ln \frac{\beta_c S}{2\Lambda} \longrightarrow a \geq \beta_c \frac{S}{2\Lambda} . \tag{8.63}$$

As mentioned above, the equations of motion can be simplified in the conformal gauge, $N = a$:

$$\begin{aligned}
2\dot{\phi}^2 - 2\dot{\phi}\dot{\lambda} &= C e^{2\phi} , \\
\ddot{\lambda} - 2\dot{\phi}\dot{\lambda} &= C e^{2\phi} , \\
2\ddot{\phi} - 4\dot{\phi}^2 &= -\kappa e^{\lambda_0} \delta(t) .
\end{aligned} \tag{8.64}$$

where $e^{\lambda_0} \equiv \beta_c S / 2\Lambda$ is the value of the scale factor at the point of the transition and $C \equiv S^2 / 4\Lambda$ is a constant proportional to the number of light degrees of freedom. Integrating the third equation, one finds the discontinuity in the derivative of the dilaton :

$$\dot{\phi}_+ - \dot{\phi}_- = -\frac{1}{2}e^{\lambda_0}\kappa . \quad (8.65)$$

As expected, it is essentially controlled by the tension coefficient, κ . On the other hand, continuity of $\dot{\lambda}(0)$ and (8.62) imply :

$$\dot{\lambda}(0) = \dot{\phi}_+ + \dot{\phi}_- . \quad (8.66)$$

Furthermore, the conservation of the thermal entropy also implies :

$$\dot{\lambda} = \frac{\dot{\beta}}{\beta} , \quad (8.67)$$

and, hence, the derivative of the temperature, $\dot{\beta}(0)$, is also continuous. The fact that the system has its maximal temperature at the critical point then implies $\dot{\beta}(0) = \dot{\lambda}(0) = 0$. Since $\ddot{\lambda} = 2\dot{\phi}^2$, by the equations of motion, the scale factor bounces at the critical point and the initial Big Bang singularity is avoided.

Now notice that, in view of (8.66), this implies that the first derivative of the dilaton flips sign across the transition :

$$\dot{\phi}_- = -\dot{\phi}_+ = \frac{1}{4}e^{\lambda_0}\kappa > 0 . \quad (8.68)$$

The positivity of the tension parameter then guarantees that the dilaton bounces at the transition, after reaching its maximal value $g_s(0) = e^{\phi_0}$. Hence, $g_s(t) \leq g_s(0)$ and the system remains perturbative at all stages of the cosmological evolution, provided that $g_s(0)$ is sufficiently small.

The string coupling at the transition point, $g_s(0)$, is not an arbitrary parameter but is related to brane tension, κ . Indeed, the equations of motion at the transition point imply :

$$\dot{\phi}_-^2 = \frac{C}{2}e^{2\phi_0} , \quad (8.69)$$

which yields the relation between the string coupling, the thermal parameter Λ and the brane tension at the transition point :

$$g_s^2(0) \equiv e^{2\phi_0} = \frac{\beta_c^2 \kappa^2}{8\Lambda} . \quad (8.70)$$

Intuitively, this arises as a balance between the thermal effects and the negative pressure injected by the brane, as communicated by the dilaton. Thus, the perturbative validity of the model is ensured, provided that the tension parameter κ is sufficiently small, as discussed in the beginning of this section.

The solution of the equations of motion has a simple expression in the conformal frame. After some rescaling, it becomes [48] :

$$\begin{aligned} ds^2 &= \frac{4}{\kappa^2} \frac{e^{|\tau|}}{1+|\tau|} \left(-d\tau^2 + dx^2 \right), \\ g_s^2 &\equiv e^{2\phi(\tau)} = \frac{\pi\kappa^2}{192} \frac{1}{1+|\tau|}. \end{aligned} \quad (8.71)$$

This illustrates how the presence of the extended symmetry point (phase transition) induces a bounce, in the scale factor as well as in the dilaton, and the evolution evades the gravitational singularity, while remaining within the perturbative regime (provided, as mentioned above, that $\kappa^2 \ll 1$). Indeed, the metric has no naked singularities, as can be seen by inspection of the scalar curvature :

$$R = \left(\frac{\kappa}{2} \right)^2 \frac{e^{-|\tau|}}{1+|\tau|}. \quad (8.72)$$

Furthermore, the conical singularity in the dilaton derivative, $\ddot{\phi} = -\frac{\kappa}{2}\delta(t)$, is resolved by the presence of the extra massless states localized at the transition point.

As mentioned above, it is possible to add higher-derivative (and higher-genus) corrections. However, these are suppressed and are, in fact, controllable by a perturbative expansion in the tension parameter, $\kappa \sim g_{\text{str}} \ll 1$. These corrections will, essentially, ‘spread’ the brane in time and smoothen out the transition.

In the cosmological frame :

$$ds^2 = -d\xi^2 + a^2(\xi)dy^2, \quad (8.73)$$

the above solution cannot be expressed in terms of simple functions. It involves the following change of variables :

$$\begin{aligned} \xi(\tau) &= \frac{2|\tau|}{\kappa} \int_0^1 \frac{e^{|\tau|u/2}}{\sqrt{1+|\tau|}} du = \sqrt{\frac{8\pi}{e\kappa^2}} \left[\text{erfi} \left(\sqrt{\frac{1+|\tau|}{2}} \right) - \text{erfi} \left(\frac{1}{\sqrt{2}} \right) \right], \\ y &= x/2. \end{aligned} \quad (8.74)$$

The asymptotic behaviour of the above solution for early cosmological times, $|\kappa\xi| \ll 1$, is :

$$a(\xi) = \frac{4}{\kappa} \left[1 + \frac{1}{16}(\kappa\xi)^2 + \mathcal{O}(|\kappa\xi|^3) \right], \quad \frac{1}{g_{\text{str}}^2} = \frac{192}{\pi\kappa^2} \left[1 + \frac{|\kappa\xi|}{2} - \frac{1}{8}(\kappa\xi)^2 + \mathcal{O}(|\kappa\xi|^3) \right]. \quad (8.75)$$

This behaviour illustrates the bounce of the scale factor, as well as the conical structure of the dilaton. Similarly, for late cosmological times, $|\kappa\xi| \gg 1$, one finds :

$$a(\xi) = |\xi| \left[1 - \frac{1}{\ln(\kappa\xi)^2} + \dots \right], \quad \frac{1}{g_{\text{str}}^2} = \frac{192}{\pi\kappa^2} \left[\ln(\kappa\xi)^2 + \ln \ln(\kappa\xi)^2 + \text{const} + \dots \right], \quad (8.76)$$

and the resulting cosmology is that of a thermal Milne universe. The presence of the running dilaton induces logarithmic corrections as displayed above.

Of course, it remains an open and extremely interesting problem to understand further the dynamics of the proposed phase transitions. Before concluding this section, let us mention that the MSDS structure was crucial to probing the contribution of stringy states to the effective potential. Hybrid models are by no means to be regarded as phenomenological candidates by themselves, but as useful laboratories that permit one to study the necessary ingredients of the stringy mechanism that avoids the unphysical singularities. As discussed in [48], this is a necessary first step and will, hopefully, provide a guiding principle for future attempts to construct phenomenologically and cosmologically viable singularity-free models. In [48], we conjectured that the main ingredients of the stringy mechanism that avoids the gravitational singularities in more realistic models in higher dimensions are already present in the 2d Hybrid models. For very recent work in this direction, see [97].

Generalized solutions have also been discussed in [48], permitting the cosmological evolution to be triggered from an initially quasi-static thermal Rindler space, possessing geometrical entropy, which is turned into thermal after the transition. These models and their interesting implications will not be discussed here.

Chapitre 9

Emergent MSDS algebras & Spinor-Vector Duality

The MSDS structure was introduced in Chapter 6 as an enhanced algebra, arising at special points in the moduli space of 2d constructions, giving rise to an alternative spectral-flow, reminiscent (at massive levels) of that of supersymmetric theories. In this final chapter we discuss an entirely different aspect of MSDS algebras, namely, their emergence as internal spectral-flows in Gepner constructions. Embedding the spin connection of Type II theories into the gauge connection of $\mathcal{N}_4 = 2$ or $\mathcal{N}_4 = 1$ Heterotic vacua has the effect of realizing an internal, enhanced superconformal $N = 4$ or $N = 2$ algebra in the bosonic side, respectively. The presence of such unbroken SCFTs algebras leads to enhanced gauge symmetry, which then contains an ‘enhanced’ E_7 or E_6 factor, respectively.

The currents generating the spectral-flow associated with spacetime supersymmetry in Type II theories is then Gepner-mapped the spectral-flow generators of the internal extended SCFT. In [91], it was shown that the spectral-flow in the twisted sectors of Heterotic theories is none other than the MSDS spectral flow of Chapter 6 and is, in fact, responsible for the Spinor-Vector ‘duality’ map observed in [88].

9.1 Spinor-Vector Duality Map

Spinor-Vector duality was initially observed [88, 89] in the massless spectrum of $\mathcal{N}_4 = 1$ and $\mathcal{N}_4 = 2$ Heterotic $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ and T^4/\mathbb{Z}_2 models within the framework of fermionic constructions. Matter comes in twisted representations of the gauge group, which typically contains an $SO(10)$ or $SO(12)$ GUT factor, respectively. It was then noticed that for every model with twisted matter in a vectorial representation of these gauge groups, there existed always a ‘twin’ model with the same number of massless degrees of freedom, coming in a spinorial representation of the same gauge group. This exchange between vectorial and spinorial representations was later expanded in [90] and its true origin was explained, as arising precisely because of the enhancement of the GUT gauge groups to E_6 and E_7 factors, respectively.

In this section we will briefly display the duality map in a $\mathcal{N}_4 = 2$ compactification on $S^1 \times \tilde{S}^1 \times (T^4/\mathbb{Z}_2)$ of the $E_8 \times E_8$ Heterotic string. For this chapter only, where we will be exclusively dealing with Heterotic constructions, it will be convenient to slightly change our conventions and assume the bosonic side of the Heterotic string to be left-moving, whereas the right-movers will be associated with the local $N = 1$ worldsheet SCFT.

The \mathbb{Z}_2 -orbifold acts non-freely on the internal coordinates of T^4 and on their fermionic superpartners as :

$$g : \begin{cases} \bar{\psi}^I(\bar{z}) \rightarrow -\bar{\psi}^I(\bar{z}) \\ X^I(z, \bar{z}) \rightarrow -X^I(z, \bar{z}) \end{cases} \quad , \quad \text{for } I = 4, 5, 6, 7 \quad (9.1)$$

whereas the remaining two internal coordinates X^8, X^9 , parametrizing $S^1 \times \tilde{S}^1$, are invariant.

Similarly, we assume a standard embedding of the point group in the gauge sector, realized as a twist in the boundary conditions of two complex fermions associated to the gauge degrees of freedom¹ :

$$g : \Psi^A \rightarrow -\Psi^A \quad , \quad \text{for } A = 1, 2 \quad , \quad (9.2)$$

while the remaining 14 left-moving fermions remain untwisted. It is also convenient to define the following combinations of twisted characters :

$$\begin{aligned} Q_o &= \bar{O}_4 \bar{V}_4 - \bar{S}_4 \bar{S}_4 \quad , \quad Q_v = \bar{V}_4 \bar{O}_4 - \bar{C}_4 \bar{C}_4 \quad , \\ P_o &= \bar{O}_4 \bar{C}_4 - \bar{S}_4 \bar{O}_4 \quad , \quad P_v = \bar{V}_4 \bar{S}_4 - \bar{C}_4 \bar{V}_4 \quad . \end{aligned} \quad (9.3)$$

The partition function can be written in terms of \mathbb{Z}_2 -orbifold blocks, $Z_g^{[h]}$, as :

$$Z = \frac{1}{(\sqrt{\tau_2} \eta \bar{\eta})^2} \frac{1}{2} \sum_{h,g=0,1} Z_g^{[h]} \Gamma_{(1,1)}(\tilde{R}) \quad , \quad (9.4)$$

where $\Gamma_{(1,1)}(\tilde{R})$ is the $(1,1)$ -lattice associated to the spectator circle \tilde{S}^1 and which we will consider factorized from the other circle, S^1 . This assumption is only made in the interest of simplicity and the results of this section continue to hold even if the T^2 , which is invariant under the orbifold, is not factorized. The orbifold blocks have the form :

$$\begin{aligned} Z_g^{[h]} &= \left[\frac{1}{2} \sum_{\bar{a}, \bar{b}=0,1} (-)^{\bar{a}+\bar{b}+\bar{a}\bar{b}} C_1 \frac{\bar{\theta}^2[\frac{\bar{a}}{\bar{b}}] \bar{\theta}[\frac{\bar{a}+h}{\bar{b}+g}] \bar{\theta}[\frac{\bar{a}-h}{\bar{b}-g}]}{\bar{\eta}^4} \right] \Gamma_{(4,4)}^{[h]} \\ &\times \Gamma_{(1,1)}(R) \left[\frac{1}{2} \sum_{k,\ell=0,1} C_2 \frac{\theta^6[\frac{k}{\ell}] \theta[\frac{k+h}{\ell+g}] \theta[\frac{k-h}{\ell-g}]}{\eta^8} \right] \left[\frac{1}{2} \sum_{\rho,\sigma=0,1} \frac{\theta^8[\frac{\rho}{\sigma}]}{\eta^8} \right] \quad , \end{aligned} \quad (9.5)$$

where the $\Gamma_{(1,1)}(R)$ -lattice is, initially, a spectator :

$$\Gamma_{(1,1)}(R) = \frac{R}{\sqrt{2\tau_2}} \sum_{\tilde{m}, n \in \mathbb{Z}} e^{-\frac{\pi R^2}{2\tau_2} |\tilde{m} + \tau n|^2} \quad . \quad (9.6)$$

1. We use, for simplicity, the fermionic representation of the Heterotic string.

Here, C_1 and C_2 are modular invariant phases fixing the chiralities of the spinorial current algebra representations. In order to keep agreement with the conventions of [91], we will choose :

$$\begin{aligned} C_1 &= (-)^{\bar{a}\bar{b}}(-)^{(\bar{a}+h)(\bar{b}+g)} , \\ C_2 &= (-)^{k\ell}(-)^{(k+h)(\ell+g)} . \end{aligned} \quad (9.7)$$

At this point in moduli space, the model has enhanced $E_7 \times SU(2) \times E_8$ gauge symmetry. We will discuss later that this enhanced symmetry is, in fact, the result of the presence of an enhanced global $N_L = 4$ SCFT realized in the bosonic side of the Heterotic string. Decomposing the twisted spectrum in terms of $SO(12) \times SO(4) \times SO(16)$ characters, we notice that at this enhanced symmetry point, both spinorial and vectorial representations are present in the massless spectrum. This model is, hence, a Spinor-Vector ‘self-dual’ model.

The Spinor-Vector duality map can be seen to arise [90, 91] as the effect of discrete or continuous Wilson lines around a compact circle. This is precisely an application of the Scherk-Schwarz mechanism, giving mass to states carrying non-trivial gauge charges. Hence, for certain values of the Wilson line, the $SO(12)$ -spinorial representation will be massive, leaving a model with massless matter in a vectorial representation while, for others, the $SO(12)$ -vectorial representation will acquire mass instead, with the resulting model having matter in the spinorial representation. Of course, arbitrary values of the Wilson line (which at the $\mathcal{N}_4 = 2$ level are continuous) lead to the Coulomb branch where both the spinorial and vectorials can be thought of as massive.

To illustrate how this happens, consider turning on a general Wilson line along the X^8 compact direction, by perturbing the σ -model by the following marginal $(1, 1)$ -operator :

$$\delta S = \int d^2 z \, A_8^a \bar{\partial} X^8(\bar{z}) J^a(z) . \quad (9.8)$$

Here, $J^a(z)$ (with $a = 1, \dots, 16$) are the Cartan currents of $E_8 \times E_8$. For convenience, we factor out the volume dependence and define $y^a \equiv A_8^a/R$. The direct evaluation of the path integral yields the deformed partition function :

$$\begin{aligned} Z_g^{[h]} &= \left[\frac{1}{2} \sum_{\bar{a}, \bar{b}=0,1} (-)^{\bar{a}+\bar{b}+\bar{a}\bar{b}} C_1 \frac{\bar{\theta}^2[\bar{a}] \bar{\theta}[\bar{a}+h] \bar{\theta}[\bar{a}-h]}{\bar{\eta}^4} \right] \Gamma_{(4,4)}[h] \left[\sum_{\tilde{m}, n \in \mathbb{Z}} e^{-\frac{\pi R^2}{\tau_2} |\tilde{m} + \tau n|^2} \right. \\ &\quad \times \left. \frac{1}{2} \sum_{k, \ell=0,1} C_2 \left(\frac{\theta_{[\ell+g-2y^1 \tilde{m}]}^{[k+h-2y^1 n]} \theta_{[\ell-g-2y^2 \tilde{m}]}^{[k-h-2y^2 n]}}{\eta^2} \prod_{B=3}^8 \frac{\theta_{[\ell-2y^B \tilde{m}]}^{[k-2y^B n]}}{\eta} \right) \frac{1}{2} \sum_{\rho, \sigma=0,1} \prod_{C=9}^{16} \frac{\theta_{[\sigma-2y^C \tilde{m}]}^{[\rho-2y^C n]}}{\eta} e^{-i\pi \Xi_{\tilde{m}, n}(y)} \right] , \end{aligned} \quad (9.9)$$

where the phase :

$$\Xi_{\tilde{m}, n}(y) = \tilde{m} n \sum_{a=1}^{16} y^a y^a - n \left((\ell + g) y^1 + (\ell - g) y^2 + \ell \sum_{B=3}^8 y^B + \sigma \sum_{C=9}^{16} y^C \right) , \quad (9.10)$$

is necessary for modular invariance.

In [91], the general conditions for the choice of the Wilson line resulting in Spinor-Vector duality at the $\mathcal{N}_4 = 2$ level, were derived. The analogous conditions for $\mathcal{N}_4 = 1$ orbifold models were derived in [90], where the analogous Wilson lines are discrete, as the analogous would-be moduli are not invariant under the orbifold action.

As mentioned above, arbitrary values for the Wilson line break the enhancement down to its Cartan factors and only rational values of the Wilson line may preserve the enhancement. As an example, consider turning on a rational Wilson line, $y^a = p/q$, with $p < q$ being relatively prime. This can be shown to be equivalent to a freely-acting $\mathbb{Z}_{(1+p \bmod 2)q}$ orbifold. Here, we restrict our attention to the \mathbb{Z}_2 -case, namely, to specific discrete points $y^a \in \mathbb{Z}$ along the -otherwise continuous- Wilson line.

Using the periodicity properties (A.10), it is straightforward to show that the partition function reduces to :

$$\begin{aligned} Z_{[g]}^{[h]} &= \left[\frac{1}{2} \sum_{\bar{a}, \bar{b}=0,1} (-)^{\bar{a}+\bar{b}+\bar{a}\bar{b}} C_1 \frac{\bar{\theta}^2[\frac{\bar{a}}{\bar{b}}] \bar{\theta}[\frac{\bar{a}+h}{\bar{b}+g}] \bar{\theta}[\frac{\bar{a}-h}{\bar{b}-g}]}{\bar{\eta}^4} \right] \Gamma_{(4,4)}^{[h]} \\ &\times \tilde{\Gamma}_{(1,1)}^{[X]}(R) \left[\frac{1}{2} \sum_{k,\ell=0,1} C_2 \frac{\theta^6[\frac{k}{\ell}] \theta[\frac{k+h}{\ell+g}] \theta[\frac{k-h}{\ell-g}]}{\eta^8} \right] \left[\frac{1}{2} \sum_{\rho,\sigma=0,1} \frac{\theta^8[\frac{\rho}{\sigma}]}{\eta^8} \right], \end{aligned} \quad (9.11)$$

where $\tilde{\Gamma}_{(1,1)}^{[X]}(R)$ the shifted lattice in the Lagrangian representation :

$$\tilde{\Gamma}_{(1,1)}^{[X]}(R) = \frac{R}{\sqrt{2\tau_2}} \sum_{\tilde{m}, n \in \mathbb{Z}} e^{-\frac{\pi R^2}{2\tau_2} |\tilde{m} + \tau n|^2} (-)^{\tilde{m}X + nY}. \quad (9.12)$$

The shift is controlled by the X, Y -parameters, given by :

$$\begin{aligned} X &= (k+h)y^1 + (k-h)y^2 + k \sum_{B=3}^8 y^B + \rho \sum_{C=9}^{16} y^C + n \sum_{a=1}^{16} y^a y^a, \\ Y &= (\ell+g)y^1 + (\ell-g)y^2 + \ell \sum_{B=3}^8 y^B + \sigma \sum_{C=9}^{16} y^C. \end{aligned} \quad (9.13)$$

By Poisson resummation the shifted lattice can be written in the Hamiltonian representation :

$$\Gamma_{(1,1)}^{[X]}(R) = \sum_{m,n \in \mathbb{Z}} (-)^{nY} \Lambda_{2m+X, \frac{n}{2}}(2R), \quad (9.14)$$

where :

$$\Lambda_{M,N} \equiv \frac{q^{\frac{1}{2}P_L^2} \bar{q}^{\frac{1}{2}P_R^2}}{\eta \bar{\eta}} \quad , \quad P_{L,R} = \frac{M}{R} \pm NR. \quad (9.15)$$

To see how Spinor-Vector duality arises, focus only on the twisted sector $h = 1$ and notice that only states with $\rho = 0$, i.e. states which are “uncharged” under $SO(16)$, can contribute to the massless spectrum.

Let us pick directly the vectorial of $SO(12)$, by noticing that if it exists in the massless spectrum it must necessarily come from the sector :

$$P_o \Gamma_{(+)}^{h=1} \times \Lambda_{2m+X,n}(2R) \times V_{12} \times \{S_4 \oplus C_4\} \times \{O_{16} \oplus V_{16}\} , \quad (9.16)$$

where we have defined the linear combinations of the $(4, 4)$ -lattices with a definite \mathbb{Z}_2 -parity :

$$\Gamma_{(\pm)}^h \equiv \frac{1}{2} \left(\Gamma_{(4,4)[0]}^h \pm \Gamma_{(4,4)[1]}^h \right) , \quad (9.17)$$

where the symmetrically twisted lattice is :

$$\Gamma_{(4,4)[g]}^h = \begin{cases} \Gamma_{(4,4)}(G, B) & , \text{ for } (h, g) = (0, 0) \\ \left| \frac{2\eta}{\theta[1-g]} \right|^4 & , \text{ for } (h, g) \neq (0, 0) \end{cases} \quad (9.18)$$

Performing all projections, [91], one eventually finds that the surviving representation is :

$$P_o \Gamma_{(+)}^{h=1} \times \Lambda_{2m+X,n}(2R) \times V_{12} C_4 O_{16} . \quad (9.19)$$

Similarly, for the massless states in the vacuum representation of $SO(12)$:

$$P_o \Gamma_{(-)}^{h=1} \times \Lambda_{2m+X,n}(2R) \times O_{12} S_4 O_{16} . \quad (9.20)$$

This singlet representation of $SO(12)$ is always present in the massless spectrum whenever the vectorial one is. Finally, the spinorial of $SO(12)$ appears from the sector :

$$P_o \Gamma_{(+)}^{h=1} \times \Lambda_{2m+X,n}(2R) \times S_{12} O_4 O_{16} . \quad (9.21)$$

Notice, that because massless states only come from the even winding sector, the Y -shift introduces no modification to the projections.

The conditions for the low-lying states in these sectors to be massless can be found by imposing $X \in 2\mathbb{Z}$. Noting that, for these particular sectors, $h = 1$ and $\rho = 0$, one finds :

- If $\sum_{A=1}^2 y^A \in 2\mathbb{Z}$, then both $k = 0$ -sectors, V_{12} and O_{12} , are massless.
- If $\sum_{B=3}^8 y^B \in 2\mathbb{Z}$, then the spinorial sector, S_{12} , is massless.

Of course, there exist choices for the Wilson line where none, one or both conditions are simultaneously satisfied. If both are satisfied, one recovers the original Spinor-Vector self-dual models with enhanced E_7 -gauge symmetry. By violating both conditions, all charged hypermultiplets become massive. Finally, if only one of the two conditions is satisfied, the Wilson line higgses either the vectorial V_{12} (always followed by the vacuum O_{12}) representation, while keeping the spinorial S_{12} massless, or the opposite.

Let us mention that, in this class of models, where $\mathcal{N}_4 = 2$ supersymmetry is unbroken, the Wilson line deformation arises from perturbing the σ -model by a marginal operator in

the physical massless spectrum of the theory. Hence, it corresponds to a modulus within the $\frac{SO(16+2,2)}{SO(16+2) \times SO(2)}$ -moduli space and can, thus, take continuous values. This implies that models that are ‘twins’ under Spinor-Vector duality are continuously connected by these deformations.

As mentioned already, Spinor-Vector duality is directly related to the enhancement of gauge symmetry at the Spinor-Vector self-dual point. This can be seen by the structure of the massless representations and the above conditions, since at the self-dual points we have $X \in 2\mathbb{Z}$, independently of the values of k or h . This guarantees that the mapping

$$\left\{ \begin{array}{l} P_o \Gamma_{(-)}^{h=1} O_{12} S_4 O_{16} \\ P_o \Gamma_{(+)}^{h=1} V_{12} C_4 O_{16} \\ P_o \Gamma_{(+)}^{h=1} S_{12} O_4 O_{16} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} Q_v \Gamma_{(+)}^{h=0} O_{12} O_4 O_{16} \\ Q_v \Gamma_{(+)}^{h=0} O_{12} O_4 S_{16} \\ Q_v \Gamma_{(+)}^{h=0} C_{12} C_4 O_{16} \end{array} \right\}, \quad (9.22)$$

preserves conformal weights and, thus, the self-duality can only arise at enhanced symmetry points.

9.2 Spinor-Vector Duality from Twisted $N = 4$ SCFT Spectral-Flow

As discussed, Spinor-Vector duality arises starting from a Gepner point, where the internal bosonic CFT of the Heterotic string is enhanced to a global, extended SCFT algebra, and then turning on non-trivial Wilson lines to give masses to the vectorial or spinorial $SO(12)$ -representations in the twisted sectors. This is precisely analogous to the $N = 2$ spectral-flow of the SCFT, mapping the spacetime spinorial to spacetime vectorial representations of the $SO(1,3)$ little group.

In [91], Spinor-Vector duality was discussed again, from the CFT point of view. As already mentioned above, the duality map is a discrete remnant of the spectral flow of a spontaneously broken, left-moving, (global) extended SCFT algebra. The algebra responsible for this spectral-flow in the twisted sectors was identified, [91], with the MSDS algebra. In this section, we briefly review the relevant analysis.

All vacua containing massless vectorials of $SO(12)$ also contain massless hypermultiplets, which were singlets under the $SO(12)$ -factor. An interesting observation was that the total numbers of massless states in the twisted sector, vectorials V_{12} and singlets O_{12} on the one hand, and spinorials S_{12} on the other, were always found to be equal. This matching is not a numerical coincidence and actually hints at the presence of (at least) an unbroken, global $N = 4$ worldsheet (left-moving) superconformal symmetry, associated to the gauge symmetry enhancement in the self-dual case.

The presence of this unbroken $N = 4$ algebra can be seen as an embedding (Gepner map), [31], of the $N = 4$ worldsheet superconformal algebra of Type II theories into the bosonic (left-moving) sector of the Heterotic string. Its spectral flow transforms the spinorial representations of $SO(12)$ into the vectorials (always followed by the scalar) and vice-versa.

This can be seen by explicitly constructing the spectral-flow operators and evaluating its action on the vertex operators.

The discussion will use the example of the previous section. As discussed in Chapter 4, spacetime supersymmetry extends the local right-moving $N_R = 1$, $\hat{c} = 6$ superconformal system into a free $N_R = 2$, $\hat{c} = 2$ SCFT system and an $N_R = 4$ SCFT with $\hat{c} = 4$:

$$\{N = 1, \hat{c} = 6\} \longrightarrow \{N = 2, \hat{c} = 2\} \oplus \{N = 4, \hat{c} = 4\} . \quad (9.23)$$

In enhanced Gepner constructions, the left-moving, internal CFT also becomes enhanced to a direct sum of global $\{N_L = 4, \hat{c} = 4\} \oplus \{N_L = 2, \hat{c} = 2\}$ SCFTs. In particular, the free $\hat{c} = 2$ system will give rise to a compactification on T^2 . For more details on the $N = 4$ SCFTs, their spectral-flows and representation theory, we refer e.g. to [30].

To perform the Gepner map, one starts from a Type II theory with $N_4 = 2$ supersymmetries arising from the left-moving side. The generic vertex operator contains a factor :

$$e^{q\phi + is_0 H_0 + is_1 H_1 + irY + iQ\sqrt{2}H} , \quad (9.24)$$

where the spacetime part H_0, H_1 arises from the bosonization of the worldsheet fermions. Here, r is the $U(1)$ charge of the free $j(z) = i\partial Y(z)$ boson and Q is the ‘isospin’ charge with respect to the diagonal $SU(2)_{k=1}$ current $J^3(z) = \frac{i}{\sqrt{2}}\partial H(z)$ of the internal $N_L = 4$ SCFT. The currents generating spacetime supersymmetry are then of the form (4.1), with the difference that there are now two weight- $(\frac{3}{8}, 0)$ R-ground states $\Sigma^1(z), \Sigma^2(z)$. In terms of the $Y(z), H(z)$ -scalars, they are expressed as :

$$\begin{aligned} \Sigma^1(z) &= e^{\frac{i}{2}Y(z) + i\frac{1}{\sqrt{2}}H(z)} , \\ \Sigma^2(z) &= e^{\frac{i}{2}Y(z) - i\frac{1}{\sqrt{2}}H(z)} . \end{aligned} \quad (9.25)$$

Fermionizing $Y(z)$, we recover the 2 real fermions of the free, $\hat{c} = 2$ system. The GSO-projection is then suitably generalized, so as to ensure the well-defined (local OPEs) action of the supersymmetry currents on states (9.24) requires :

$$q + s_0 + s_1 + r + 2Q \in 2\mathbb{Z} \quad , \quad 2Q \in \mathbb{Z} , \quad (9.26)$$

and the $N_L = 4$ spectral-flow is, [30] :

$$\begin{aligned} J_n^3 &\rightarrow J_n^3 - \alpha\delta_{n,0} , \\ L_n &\rightarrow L_n - 2\alpha J_n^3 + \alpha^2\delta_{n,0} . \end{aligned} \quad (9.27)$$

Let us start from untwisted sector and consider the embedding of the left-moving spin connection of Type II into the ‘bosonic’ sector of the Heterotic string. The $SO(12)_{k=1}$ current algebra is realized by the free complex fermions Ψ^A (where $A = 1, \dots, 6$) and we can take one of these, $\Psi^6(z) \equiv e^{iY(z)}$, to be associated with the free boson $Y(z)$, of the $\hat{c} = 2$ system. Furthermore, the GSO-projection of Type II is carried through to the present case in a straightforward way. We list the invariant $(1, 0)$ -currents :

$$\Psi^A \Psi^B(z) \quad , \quad C_{12} e^{\pm i\frac{1}{\sqrt{2}}H}(z) \quad , \quad J^3(z) \quad , \quad J^\pm(z) , \quad (9.28)$$

where $J^\pm(z) = e^{\pm i\sqrt{2}H(z)}$ and $I = 1, \dots, 6$. The fermion bilinears transform in the adjoint **66** of $SO(12)$, whereas C_{12} is charged under the conjugate spinorial **$\overline{32}$** . Together with the three $SU(2)_{k=1}$ currents J^3, J^\pm (which are $SO(12)$ -singlets), these currents form the adjoint representation **133** of E_7 , which is the enhancement mentioned above.

For illustration purposes, we will henceforth be specific to the $\mathcal{N}_4 = 2$ orbifold model of the previous section with enhanced $E_7 \times SU(2) \times E_8$ gauge symmetry. This can be seen to arise through Gepner map from an initial $\mathcal{N}_4 = 4$ Type II compactification on $T^2 \times T^4/\mathbb{Z}_2$, in which 2 spacetime supersymmetries arise from each of the left- and right-moving sectors. It corresponds precisely to an $N = (4, 4) \oplus (2, 2)$ compactification. In terms of covariant conformal blocks, the Gepner map is realized by replacing :

$$\frac{1}{2} \sum_{a,b} (-)^{a+b+ab} \theta^2[a] \theta[b] \theta[a+h] \theta[b-g] \longrightarrow \frac{1}{2} \sum_{k,\ell} \theta^6[k] \theta[\ell] \theta[k+h] \theta[\ell-g] . \quad (9.29)$$

In the free-field description, the $\hat{c} = 4$ system realizing the $N_L = 4$ SCFT, is built out of 2 complex fermions $\Psi^{7,8}$ which are bosonized as $e^{\pm iH^j(z)}$, with $j = 7, 8$. In terms of free fields, the spectral-flow currents are constructed as :

$$C_{12}(z) e^{\pm \frac{i}{2}(H^7(z) - H^8(z))} \propto e^{\pm i \frac{1}{\sqrt{2}} H(z)} . \quad (9.30)$$

Notice that the spectral-flow current carries $\pm \frac{1}{2}$ units of J^3 -charge, as required by the $N_L = 4$ SCFT.

To make the connection with the $N_L = 4$ SCFT explicit, perform the linear field redefinition :

$$\begin{aligned} Y(z) &= H^6(z) , \\ X(z) &= (H^7(z) + H^8(z))/\sqrt{2} , \\ H(z) &= (H^7(z) - H^8(z))/\sqrt{2} , \end{aligned} \quad (9.31)$$

and notice that the scalars Y, X, H are still free. In this new basis, $i\partial Y(z)$ is identified with the $U(1)$ -charge of the $\hat{c} = 2$ system, while $J(z) = \frac{i}{\sqrt{2}} \partial H(z)$ is the Cartan charge of the $SU(2)_{k=1}$ current algebra of the $\hat{c} = 4$, $N_L = 4$ system. This completes the connection.

We next consider the action of the spectral-flow in the untwisted scalar spectrum. Start with the vectorial representation of $SO(12)$:

$$\Psi^A(z) e^{\pm iH^j(z)} e^{-\bar{\phi}(\bar{z}) \pm i\bar{H}_k(\bar{z})} \propto e^{\pm i\epsilon_j \frac{1}{\sqrt{2}} H(z)} , \quad (9.32)$$

where $j = 7, 8$ and $k = 3, 4$. The ϵ_j are defined as $\epsilon_7 = 1$ and $\epsilon_8 = -1$. Next, shift the $SU(2)_{k=1}$ charge Q by $\pm \frac{1}{2}$ units in order to make it vanish. For example, for $j = 7$:

$$e^{\pm iH^7(z)} \rightarrow e^{\pm \frac{i}{2}(H^7(z) + H^8(z))} . \quad (9.33)$$

As expected, the spectral-flow maps the $V_{12}V_4$ representation (9.32) into the $S_{12}S_4$ representation in the R-sector of the $N_L = 4$ SCFT :

$$S_{12}(z) e^{\pm \frac{i}{2}(H^7(z) + H^8(z))} e^{-\bar{\phi}(\bar{z}) \pm i\bar{H}_k(\bar{z})} . \quad (9.34)$$

The same transformation can be obtained by considering the action of the spectral-flow current (9.30) on the vertex operator (9.32).

We can now move to the twisted fermionic massless spectrum, which is relevant for the Spinor-Vector duality map. Following [91], we start from the $SO(12)$ vectorials :

$$\Psi^A e^{\pm \frac{i}{2}(H^7(z)-H^8(z))} \Gamma_{(+)}^1(z, \bar{z}) e^{-\frac{1}{2}\bar{\phi}(\bar{z}) + \frac{i}{2}\bar{H}_0(\bar{z}) \pm \frac{i}{2}(\bar{H}_1(\bar{z}) + \bar{H}_2(\bar{z}))} , \quad (9.35)$$

where $A = 1, \dots, 6$. The vertex operator now involves the contribution from the invariant twist-field, here denoted $\Gamma_{(+)}^{h=1}(z, \bar{z})$, and which starts with conformal weight $(\frac{1}{4}, \frac{1}{4})$. It is associated to the topological contribution of the $h = 1$ -twisted $\Gamma_{(4,4)}$ -lattice with definite (positive) \mathbb{Z}_2 -parity :

$$\Gamma_{(s)}^{h=1} = \frac{1}{2^2 \eta^4 \bar{\eta}^4} \sum_{g=0,1} \sum_{\gamma, \delta=0,1} (-)^{\left(\frac{1-s}{2}\right)g} \theta^2[\gamma]_{\delta} \theta^2[\gamma+1]_{\delta+g} \times \bar{\theta}^2[\gamma]_{\delta} \bar{\theta}^2[\gamma+1]_{\delta+g} . \quad (9.36)$$

The spectral-flow maps the $V_{12}C_4O_{16}$ representation, (9.35), into the spinorial, $S_{12}O_4O_{16}$. As before, this can be seen by shifting the $SU(2)_{k=1}$ charge by $\delta Q = \mp \frac{1}{4}$ units so that it vanishes :

$$e^{\pm \frac{i}{2}(H^7(z)-H^8(z))} \longrightarrow \mathbf{1}(z) , \quad (9.37)$$

where by the identity, $\mathbf{1}(z)$, we imply the vacuum representation and its higher excitations (with even 2d fermion parity). Taken together, they form the fermionic O_4 -representation and one recovers precisely the spinorial representation of $SO(12)$:

$$S_{12}(z) \mathbf{1}(z) \Gamma_{(+)}^1(z, \bar{z}) e^{-\frac{1}{2}\bar{\phi}(\bar{z}) + \frac{i}{2}\bar{H}_0(\bar{z}) \pm \frac{i}{2}(\bar{H}_1(\bar{z}) + \bar{H}_2(\bar{z}))} . \quad (9.38)$$

We will now explicitly construct the spectral-flow currents in the twisted sector and use them to derive the mapping. At this point it is important to note that, because we are dealing simply with a \mathbb{Z}_2 -twist, it is possible to represent the twist-field vertex operators, $\Gamma_{(\pm)}^{h_i}(z, \bar{z})$, associated to the twisted lattice in terms of level-one, free-fermions (organized in Kac-Moody characters). This has already been discussed in Section 3.3. Because the OPEs we will compute involve only the twisted sectors, the result will be valid at any point in the moduli space, and not restricted to the fermionic point, even though we are using the convenient representation in terms of free fields.

Performing the summation and projections in (9.36), we find the explicit form of this representation of the twisted vertex operators :

$$\begin{aligned} \Gamma_{(+)}^{h=1}(z, \bar{z}) = & \left\{ O_4 S_4 \bar{O}_4 \bar{S}_4 \oplus O_4 C_4 \bar{O}_4 \bar{C}_4 \oplus V_4 S_4 \bar{V}_4 \bar{S}_4 \oplus V_4 C_4 \bar{V}_4 \bar{C}_4 \right. \\ & \left. \oplus S_4 O_4 \bar{S}_4 \bar{O}_4 \oplus S_4 V_4 \bar{S}_4 \bar{V}_4 \oplus C_4 O_4 \bar{C}_4 \bar{O}_4 \oplus C_4 V_4 \bar{C}_4 \bar{V}_4 \right\} . \end{aligned} \quad (9.39)$$

$$\Gamma_{(-)}^{h=1}(z, \bar{z}) = \left\{ \begin{aligned} & O_4 S_4 \bar{V}_4 \bar{C}_4 \oplus O_4 C_4 \bar{V}_4 \bar{S}_4 \oplus V_4 S_4 \bar{O}_4 \bar{C}_4 \oplus V_4 C_4 \bar{O}_4 \bar{S}_4 \\ & \oplus S_4 O_4 \bar{C}_4 \bar{V}_4 \oplus S_4 V_4 \bar{C}_4 \bar{O}_4 \oplus C_4 O_4 \bar{S}_4 \bar{V}_4 \oplus C_4 V_4 \bar{S}_4 \bar{O}_4 \end{aligned} \right\}. \quad (9.40)$$

Out of these, only twisted ‘ground states’ with conformal weight $(*, \frac{1}{4})$ will be considered in the fusion rules, since massless states only arise from these.

We are now ready to construct the spectral-flow currents in the twisted sector. As argued [91], the untwisted spectral-flow operators (9.30) become extended (in the twisted sector) by a chiral dressing, $\Omega_{(\pm)}^I(z)$, with $I = 1, 2$, of conformal weight $\Delta_{1,(\pm)} = (\frac{1}{2}, 0)$, $\Delta_{2,(\pm)} = (1 \mp 1, 0)$, which acts on the twist-field contribution, $\Gamma_{(\pm)}^{h=1}(z, \bar{z})$. This dressing operator transforms as $\Omega_{(\pm)}^I \rightarrow \pm \Omega_{(\pm)}^I$ under the \mathbb{Z}_2 -orbifold. Its action on the twisted ground-state vertex operators relevant for the massless spectrum follows the fusion rules :

$$\begin{aligned} \Omega_{(\pm)}^1(z) \cdot \Gamma_{(r)}^1(w, \bar{w}) &= \frac{\Gamma_{(+)}^1(w, \bar{w})}{(z-w)^{\frac{1}{2} + \frac{1-r}{4}}} + \frac{\Gamma_{(-)}^1(w, \bar{w})}{(z-w)^{\frac{1-r}{4}}} + \dots \\ \Omega_{(+)}^2(z) \cdot \Gamma_{(r)}^1(w, \bar{w}) &= \Gamma_{(r)}^1(w, \bar{w}) + \dots \\ \Omega_{(-)}^2(z) \cdot \Gamma_{(r)}^1(w, \bar{w}) &= \frac{\Gamma_{(-r)}^1(w, \bar{w})}{(z-w)^{1-\frac{r}{2}}} + \dots \end{aligned} \quad (9.41)$$

Here, it is convenient to simplify our notation by suppressing the representation indices and the Dirac matrices of the transformation. In terms of free bosonic fields, $\Omega_{(\pm)}^I(z)$ can be represented as :

$$\begin{aligned} \Omega_{(\alpha)}^1(z) &= e^{\pm \frac{i}{2}(\Phi_1 - \alpha \Phi_2) \pm \frac{i}{2}(\Phi_3 - \alpha \Phi_4)}, \\ \Omega_{(+)}^2(z) &= \mathbf{1}(z), \\ \Omega_{(-)}^2(z) &= e^{\pm i(\frac{1+r}{2})\Phi_1 \pm i(\frac{1-r}{2})\Phi_2 \pm i(\frac{1+s}{2})\Phi_3 \pm i(\frac{1-s}{2})\Phi_4}, \end{aligned} \quad (9.42)$$

where the \pm -signs are arbitrary and independent, $\alpha = \pm 1$ is the \mathbb{Z}_2 -parity of the operator and $r, s = \pm 1$. The invariant spectral-flow operators responsible for the mapping between the various representations are given by the zero modes of the invariant current :

$$Q_{\text{s.f.}}^I = \oint \frac{dz}{2\pi i} \left[z^{1-I/2} (C_{12} C_4)(z) \Omega_{(+)}^I(z) + z^{I/2} (S_{12} S_4)(z) \Omega_{(-)}^I(z) \right], \quad (9.43)$$

where $I = 1, 2$. More explicitly we can write this as :

$$\begin{aligned} Q_{\text{s.f.}}^1 &= \oint \frac{dz}{2\pi i} z^{1/2} (C_{12} C_4 C_4 C_4 + S_{12} S_4 S_4 S_4)(z), \\ Q_{\text{s.f.}}^2 &= \oint \frac{dz}{2\pi i} (C_{12} C_4 \mathbf{1}_4 \mathbf{1}_4 + z S_{12} S_4 V_4 V_4)(z), \end{aligned} \quad (9.44)$$

where the explicit representation (9.42) was used.

Now consider the action of the spectral-flow charge on the massless $S_{12}O_4O_{16}$ -spinorial representation :

$$Q_{\text{s.f.}} \cdot \left[S_{12}O_4 \Gamma_{(+)}^1 e^{-\frac{1}{2}\bar{\phi} + \frac{i}{2}\bar{H}_0} \bar{S}_4 \bar{O}_4 \right] |0\rangle = \left[V_{12}C_4 \Gamma_{(+)}^1 + O_{12}S_4 \Gamma_{(-)}^1 \right] e^{-\frac{1}{2}\bar{\phi} + \frac{i}{2}\bar{H}_0} \bar{S}_4 \bar{O}_4 |0\rangle . \quad (9.45)$$

This illustrates that the spinorial representation of $SO(12)$ is precisely mapped into the vectorial, $V_{12}C_4O_{16}$, together with the accompanying singlet, $O_{12}S_4O_{16}$. This exact map between these representations is, hence, seen to arise from the spectral-flow of the twisted $N_L = 4$ SCFT. The matching of the numbers of massless d.o.f. can be summarized as :

| | | |
|-----------|---|--------|
| Spinorial | $S_{12}O_4 \rightarrow 2^{6-1} \times (4 \times 4)$ | (9.46) |
| Vectorial | $V_{12}C_4 \rightarrow 12 \times 2 \times (4 \times 4)$ | |
| Singlet | $O_{12}S_4 \rightarrow 1 \times 2 \times (8 \times 2 \times 4)$ | |

This ‘twisted’ spectral-flow leads to a number of ‘supersymmetric-like’ identities, realized internally in the left-moving sector. In the present case, however, the analogous identities are not those of ‘conventional’ supersymmetry but instead exhibit the degeneracy structure of MSDS constructions. The reason for this is that the relevant orbifold block² :

$$Z_g^{[1]} = \frac{1}{2^2 \eta^{12} \bar{\eta}^4} \left[\sum_{\ell=0,1} (-)^{\ell} \theta^6_{[\ell]} \theta^2_{[\ell+g]} \right] \left[\sum_{\gamma, \delta=0,1} \theta^2_{[\delta]} \theta^2_{[\delta+g]} \bar{\theta}^2_{[\delta]} \bar{\theta}^2_{[\delta+g]} \right] \quad (9.47)$$

assigns to the free fields precisely the boundary conditions needed to satisfy the conditions, (6.25), that guarantee that the MSDS spectral-flow charge has a well-defined action on the spectrum. This can be seen already from (9.44), which is precisely the invariant $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ -truncation of the maximal MSDS charge. The spectral-flow is, thus, identified with the MSDS spectral-flow of Chapter 6. Of course, since we are now in the bosonic side of the Heterotic string, giving rise to spacetime bosons only, the Kac-Moody characters of ‘vectors’ and ‘spinors’ are now summed, rather than subtracted.

The spectral-flow can be displayed in terms of character identities. For illustration, we focus on the contribution from the $Z_{(+)}^{h=1}$ -orbifold block with positive \mathbb{Z}_2 -parity, which couples to the P_0 twisted, right-moving fermion character (which is the only $h = 1$ sector giving rise to massless fermions) :

$$Z_{(+)}^{h=1}(\epsilon) = (O_{12}S_4 + \epsilon C_{12}V_4) \Gamma_{(-)}^{h=1} + (V_{12}C_4 + \epsilon S_{12}O_4) \Gamma_{(+)}^{h=1} . \quad (9.48)$$

We explicitly keep the dependence on $\epsilon = \pm 1$, which distinguishes the case of local worldsheet SCFT ($\epsilon = -1$), from the case when it is absent ($\epsilon = +1$) as in the case of the bosonic sector of the Heterotic string. As discussed above, these are distinguished by the fact that spacetime fermions only arise when local worldsheet supersymmetry is present.

2. The exact same block appears in the partition function of 2d MSDS orbifold models, constructed in [46].

In the case of Spinor-Vector duality, where $N_L = 4$ is embedded inside the ‘bosonic’ sector of the Heterotic string, there will be no cancellation between the vectorial and spinorial contributions to the partition function since both contribute with a positive sign. However, the spectral-flow can be displayed most explicitly by using the $\epsilon = -1$ identities. These illustrate the action of the spectral-flow on the representations in a particularly clear way and, in particular, reveal the MSDS structure.

Indeed, the presence of an underlying MSDS algebra can be seen by calculating the $\epsilon = -1$ contribution coupling to the various right-moving lattice pieces, $\bar{V}_4 \bar{C}_4, \bar{V}_4 \bar{S}_4, \dots \in \Gamma_{(+)}^{h=1}$:

$$\begin{aligned}
\bar{V}_4 \bar{C}_4 \cdot \left[(O_{12} S_4 - C_{12} V_4) O_4 S_4 + (V_{12} C_4 - S_{12} O_4) V_4 C_4 \right] &= 4 \bar{V}_4 \bar{C}_4 , \\
\bar{V}_4 \bar{S}_4 \cdot \left[(O_{12} S_4 - C_{12} V_4) O_4 C_4 + (V_{12} C_4 - S_{12} O_4) V_4 S_4 \right] &= 4 \bar{V}_4 \bar{S}_4 , \\
\bar{O}_4 \bar{C}_4 \cdot \left[(O_{12} S_4 - C_{12} V_4) V_4 S_4 + (V_{12} C_4 - S_{12} O_4) O_4 C_4 \right] &= 0 , \\
\bar{O}_4 \bar{S}_4 \cdot \left[(O_{12} S_4 - C_{12} V_4) V_4 C_4 + (V_{12} C_4 - S_{12} O_4) O_4 S_4 \right] &= 0 , \\
\bar{C}_4 \bar{V}_4 \cdot \left[(O_{12} S_4 - C_{12} V_4) S_4 O_4 + (V_{12} C_4 - S_{12} O_4) C_4 V_4 \right] &= 4 \bar{C}_4 \bar{V}_4 , \\
\bar{S}_4 \bar{V}_4 \cdot \left[(O_{12} S_4 - C_{12} V_4) C_4 O_4 + (V_{12} C_4 - S_{12} O_4) S_4 V_4 \right] &= 4 \bar{S}_4 \bar{V}_4 , \\
\bar{S}_4 \bar{O}_4 \cdot \left[(O_{12} S_4 - C_{12} V_4) C_4 O_4 + (V_{12} C_4 - S_{12} O_4) S_4 O_4 \right] &= 0 , \\
\bar{C}_4 \bar{O}_4 \cdot \left[(O_{12} S_4 - C_{12} V_4) S_4 V_4 + (V_{12} C_4 - S_{12} O_4) C_4 O_4 \right] &= 0 . \tag{9.49}
\end{aligned}$$

Of course, if we only restrict our attention to the massless spectrum, only the weight- $(*, \frac{1}{4})$ contributions in $Z_{(+)}^{h=1}$ become relevant and the relevant identities are ‘supersymmetric-like’, that is, come with a vanishing r.h.s.

It is very interesting that the MSDS spectral-flow appears in this, entirely unexpected, setting. The fact that MSDS spectral-flow is realized in twisted $N = 4$ SCFT sectors hints at a relation between these algebras.

Annexe A

Theta Functions and Lattice Identities

A.1 Theta Functions

In this Appendix we gather some useful definitions and identities involving modular forms. These are functions

The Dedekind η -function is defined as :

$$\eta(\tau) \equiv q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) . \quad (\text{A.1})$$

Its modular T and S -transformations are :

$$\eta(\tau + 1) = e^{\frac{i\pi}{12}} \eta(\tau) , \quad (\text{A.2})$$

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau} \eta(\tau) . \quad (\text{A.3})$$

Another very useful function encoding the the worldsheet fermion contribution to the torus amplitude are the Jacobi θ -functions :

$$\theta_{[b]}^a(\tau, v) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n - \frac{a}{2})^2} e^{2\pi i(v - \frac{b}{2})(n - \frac{a}{2})} . \quad (\text{A.4})$$

These are defined on the upper half-plane, $\tau \in \mathbb{C}^+$, for $a, b \in \mathbb{R}$. Under modular transformations they become :

$$\begin{aligned} \theta_{[b]}^a(\tau + 1, v) &= e^{-\frac{i\pi}{4}a(a-2)} \theta_{[a+b-1]}^a(\tau, v) , \\ \theta_{[b]}^a\left(-\frac{1}{\tau}, v\right) &= \sqrt{-i\tau} e^{\frac{i\pi}{2}ab + i\pi \frac{v^2}{\tau}} \theta_{[-a]}^b(\tau, v) . \end{aligned} \quad (\text{A.5})$$

The independent θ -functions for $a, b = 0, 1$ can also be written in the Jacobi notation :

$$\theta_{[0]}^0 \equiv \theta_3 \quad , \quad \theta_{[1]}^0 \equiv \theta_4 \quad , \quad \theta_{[0]}^1 \equiv \theta_2 \quad , \quad \theta_{[1]}^1 \equiv \theta_1 . \quad (\text{A.6})$$

A very useful product representation for the θ -functions is :

$$\frac{\theta_b^a(\tau, 0)}{\eta(\tau)} = e^{i\pi \frac{ab}{2}} q^{\frac{1}{8}a^2 - \frac{1}{24}} \prod_n \left(1 + q^{n + \frac{a-1}{2}} e^{i\pi b}\right) \left(1 + q^{n - \frac{a+1}{2}} e^{-i\pi b}\right) . \quad (\text{A.7})$$

There is a number of impressive identities relating different θ -functions. The most important ones are the ‘Aequatio Identica Satis Abstrusa’ :

$$\theta_3^4(\tau, v) - \theta_4^4(\tau, v) = \theta_2^4(\tau, v) - \theta_1^4(\tau, v) , \quad (\text{A.8})$$

and the ‘Triple Product’ identities :

$$\theta_2(\tau, 0)\theta_3(\tau, 0)\theta_4(\tau, 0) = 2\eta^3 . \quad (\text{A.9})$$

The θ -functions exhibit an additional periodicity in their characteristic arguments $[a]_b$, up to a possible phase :

$$\theta_b^{[a+2]}(\tau, v) = \theta_b^a(\tau, v) , \quad \theta_{[b+2]}^a(\tau, v) = e^{i\pi a} \theta_b^a(\tau, v) . \quad (\text{A.10})$$

Let us also introduce the Klein j -function, which is the unique holomorphic modular invariant function :

$$j(\tau) = \left(\frac{1}{2} \sum_{\gamma, \delta=0,1} \frac{\theta^8[\gamma]_\delta}{\eta^8} \right)^3 = \frac{1}{q} + 744 + 196884q + \dots \quad (\text{A.11})$$

The following identity [28] becomes useful in lattice bosonization at the fermionic point :

$$\theta[\gamma]_\delta \bar{\theta}[\gamma-h]_{\delta-g} = \frac{1}{\sqrt{2\tau_2}} \sum_{m,n \in \mathbb{Z}} e^{-\frac{\pi}{2\tau_2} |m + \frac{g}{2} + \tau(n + \frac{h}{2})|^2 + i\pi[mn - (\gamma - \frac{h}{2})m + (\delta - \frac{g}{2})n + \frac{h}{2}(\delta - \frac{g}{2})]} . \quad (\text{A.12})$$

Useful properties of modular series are sometimes obtained by Poisson resummation :

$$\sum_{\tilde{m}^i \in \mathbb{Z}} e^{-\pi A_{ij} \tilde{m}^i \tilde{m}^j + \pi B_i \tilde{m}^i} = \frac{1}{\sqrt{\det A}} \sum_{m_i \in \mathbb{Z}} e^{-\pi (m_i + \frac{i}{2} B_i) (A^{-1})^{ij} (m_j + \frac{i}{2} B_j)} . \quad (\text{A.13})$$

Finally, we include the definition of the non-holomorphic Eisenstein series :

$$E(\tau, s) \equiv \tau_2^s \sum_{(m,n) \in \mathbb{Z}/\{0\}} \frac{1}{|m + \tau n|^{2s}} . \quad (\text{A.14})$$

A.2 Double Poisson Resummation

Double Poisson resummation takes a lattice direction to its dual representation. It is obtained Poisson-resumming over both winding numbers \tilde{m}, n (defined in the Lagrangian representation) :

$$\frac{R}{\sqrt{\tau_2}} \sum_{\tilde{m}, n \in \mathbb{Z}} e^{-\frac{\pi R^2}{\tau_2} |\tilde{m} + \frac{g}{2} + \tau(n + \frac{h}{2})|^2 + i\pi(\tilde{m}A + nB)} = \frac{1/R}{\sqrt{\tau_2}} \sum_{\tilde{m}, n \in \mathbb{Z}} e^{-\frac{\pi(1/R)^2}{\tau_2} |n - \frac{B}{2} - \tau(\tilde{m} - \frac{A}{2})|^2 + i\pi[h(n - \frac{B}{2}) + g(\tilde{m} - \frac{A}{2})]} . \quad (\text{A.15})$$

It is a special case of the general $O(d, d; \mathbb{Z})$ duality transformation.

A.3 OPEs involving $SO(N)$ Spin-Fields

Most of the fusion rules in this manuscript can be verified by using the very useful OPEs involving the spin-fields of $SO(N)$:

$$\psi^a(z)S_\alpha(w) = \frac{\gamma_{\alpha\beta}^a}{\sqrt{2}} \frac{S_\beta(w)}{(z-w)^{1/2}} + \dots , \quad (\text{A.16})$$

and :

$$\begin{aligned} S_\alpha(z)S_\beta(w) = & \frac{c_{\alpha\beta}}{(z-w)^{N/8}} + \frac{\gamma_{\alpha\beta}^a}{\sqrt{2}} \frac{\psi^a(w)}{(z-w)^{N/8-1/2}} \\ & + \frac{\gamma_{\alpha\beta}^{ab}}{2} \frac{\psi^a\psi^b(w)}{(z-w)^{N/8-1}} + \frac{1}{2\sqrt{2}} \frac{\gamma_{\alpha\beta}^{abc} \psi^a\psi^b\psi^c(w) + \gamma_{\alpha\beta}^a \partial\psi^a(w)}{(z-w)^{N/8-3/2}} \\ & + \frac{1}{4} \frac{\gamma_{\alpha\beta}^{abcd} \psi^a\psi^b\psi^c\psi^d(w) + \gamma_{\alpha\beta}^{ab} \partial(\psi^a\psi^b)(w) + \frac{1}{2}c_{\alpha\beta}(\partial\psi^a)\psi^a(w)}{(z-w)^{N/8-2}} \\ & + \frac{1}{4\sqrt{2}} \frac{\gamma_{\alpha\beta}^{abcde} \psi^a\psi^b\psi^c\psi^d\psi^e + \gamma_{\alpha\beta}^{abc} \partial(\psi^a\psi^b\psi^c) + \gamma_{\alpha\beta}^a \partial^2\psi^a}{(z-w)^{N/8-5/2}} + \dots , \end{aligned} \quad (\text{A.17})$$

where $c_{\alpha\beta}$ is the charge conjugation matrix in the Dirac representation and the ellipses denote less singular terms.

Annexe B

Equivalence between Fermionic Constructions and \mathbb{Z}_2^N -type Orbifolds

In this Appendix we will demonstrate the equivalence of the fermionic and \mathbb{Z}_2^N -orbifold constructions. In the interest of clarity, we will perform the mapping explicitly within a particular model, even though the spirit of the equivalence generalizes straightforwardly to any fermionic model. The results of this Appendix provide an straightforward way to translate between the (somewhat counter-intuitive) language of the fermionic construction [26] and the more transparent, CFT language of orbifolds.

General setup

The equivalence will be illustrated in terms of a particular working model possessing either supersymmetric or MSDS structure and, hence, will be 2-dimensional. The reason for this choice is because it is, at the same time, instructive to see examples of models where different discrete torsions (or choices of background parameters) result in the spacetime supersymmetry being spontaneously broken down to MSDS. Since this is most relevant to phenomenology, the construction we present is a Heterotic one.

The procedure, of course, generalizes to any number of dimensions (and even to the Type II case) and is not in any way particular to the specific basis $\{b_i\}$ used here. The essential requirement is that only real boundary conditions are used, so that the associated orbifold is of the \mathbb{Z}_2^N -type, even though analogous dictionaries can be straightforwardly developed for more general boundary conditions as well. The latter case is however, more tedious and will be ignored here. The second requirement for the equivalence to hold is that the boundary condition assignments are such that all real fermions can be consistently complexified. This implies, essentially, the absence of ‘Ising’-type fermions.

The constraints of MSDS symmetry combined with the usual constraints of modular invariance and the global definition of the worldsheet supercurrent, are very restrictive on the possible boundary conditions that can be assigned to the worldsheet degrees of freedom. In fact, in the case of $\mathbb{Z}_2 \times \mathbb{Z}_2$ models, the holomorphic part of the most general basis of boundary conditions is uniquely fixed by the MSDS conditions [46]. The anti-holomorphic part can be

suitably chosen to take a standard form, useful for $SO(10)$ -GUT group phenomenology.

The basis vectors $\{b_i\}$ of the class of models we consider is the following¹ :

$$\begin{aligned}
v_1 \equiv \mathbf{1} &= \{\chi^{1,\dots,8}, y^{1,\dots,8}, \omega^{1,\dots,8} | \bar{y}^{1,\dots,8}, \bar{\omega}^{1,\dots,8}, \bar{\psi}^{1,\dots,5}, \bar{\eta}^{1,2,3}, \bar{\phi}^{1,\dots,8}\} \\
v_2 \equiv S &= \{\chi^{1,\dots,8}\} \\
v_3 \equiv H &= \{y^{1,\dots,8}, \omega^{1,\dots,8}\} \\
v_4 \equiv b_1 &= \{\chi^{3,4,5,6}, y^{3,4,5,6} | \bar{y}^{3,4,5,6}, \bar{\psi}^{1,\dots,5} \bar{\eta}^1\} \\
v_5 \equiv b_2 &= \{\chi^{1,2,5,6}, y^{1,2,5,6} | \bar{y}^{1,2,5,6}, \bar{\psi}^{1,\dots,5} \bar{\eta}^2\} \\
v_6 \equiv G &= \{\bar{y}^{1,\dots,8}, \bar{\omega}^{1,\dots,8}\} \\
v_7 \equiv z_1 &= \{\bar{\phi}^{1,2,3,4}\} \\
v_8 \equiv z_2 &= \{\bar{\phi}^{5,6,7,8}\}
\end{aligned} \tag{B.1}$$

We use here the symbols $\mathbf{1}, S, H, b_1, b_2, G, z_1, z_2$, since these have appeared in much of the free-fermion phenomenology literature and have become more or less conventional.

The general form for the partition function corresponding to this basis is :

$$\begin{aligned}
Z &= \frac{1}{\eta^{12} \bar{\eta}^{24}} \frac{1}{2^3} \sum_{\Gamma_{\alpha}, \Delta_{\beta}} \frac{1}{2^5} \sum_{h_a, g_b} (-)^{a+b+HG+\Phi} \theta_{[b]}^a \theta_{[b+g_1]}^{a+h_1} \theta_{[b+g_2]}^{a+h_2} \theta_{[b-g_1-g_2]}^{a-h_1-h_2} \times \Gamma_{(8,8)} \left[\begin{matrix} a, k \\ b, l \end{matrix} \middle| \begin{matrix} P, h_1, h_2, \psi \\ Q, g_1, g_2, \omega \end{matrix} \right] \\
&\quad \times \bar{\theta}_{[l]}^5 \bar{\theta}_{[l+g_1]}^{k+h_1} \bar{\theta}_{[l+g_2]}^{k+h_2} \bar{\theta}_{[l-g_1-g_2]}^{k-h_1-h_2} \times \bar{\theta}^4_{[\rho]} \bar{\theta}^4_{[\sigma+G]}^{\rho+H} ,
\end{aligned} \tag{B.2}$$

where the $\Gamma_{(8,8)}$ -lattice is :

$$\Gamma_{(8,8)} \equiv \theta_{[b+Q]}^{a+P} \theta_{[b+Q+g_1]}^{a+P+h_1} \theta_{[b+Q+g_2]}^{a+P+h_2} \theta_{[b+Q-g_1-g_2]}^{a+P-h_1-h_2} \times \bar{\theta}_{[l+\omega]}^5 \bar{\theta}_{[l+\omega+g_1]}^{k+\psi+h_1} \bar{\theta}_{[l+\omega+g_2]}^{k+\psi+h_2} \bar{\theta}_{[l+\omega-g_1-g_2]}^{k+\psi-h_1-h_2} . \tag{B.3}$$

For notational convenience, we utilize the following vector-like notation for the summation variables :

$$\Gamma_{\alpha} = (a, k, l) , \quad \Delta_{\alpha} = (b, l, \sigma) \tag{B.4}$$

$$h_a = (P, h_1, h_2, \psi, H) , \quad g_a = (Q, g_1, g_2, \omega, G) \tag{B.5}$$

with $\alpha = 1, 2, 3$ and $a = 1, \dots, 5$. The choice of GGSO coefficients corresponds to the various choices for the modular invariant phase Φ .

Relation of GGSO Coefficients to Modular Invariant Phases

The possible modular invariant phases that contribute to Φ can be classified into the following types :

$$A_{\alpha,a} (\Gamma_{\alpha} g_a + \Delta_{\alpha} h_a + h_a g_a) , \tag{B.6}$$

1. Here we adopt a notation in which only the periodic fermions in each basis vector, b_i , are denoted.

$$B_{a,b}(h_a g_b + h_b g_a) , \quad (\text{B.7})$$

$$C(\Gamma_1 \Delta_2 + \Gamma_2 \Delta_3 + \Gamma_3 \Delta_1 + \Delta_1 \Gamma_2 + \Delta_2 \Gamma_3 + \Delta_3 \Gamma_1) , \quad (\text{B.8})$$

and

$$D_{\alpha,\alpha} \Gamma_\alpha \Delta_\alpha . \quad (\text{B.9})$$

This gives a total of $3 \times 5 + \binom{5}{2} + 1 + 3 = 29$ phases which is exactly equal to the number of independent GSO coefficients $\frac{N(N-1)}{2} + 1$ for $N = 8$ basis elements. The coefficients A, B, C, D take binary values $\{0, 1\}$, depending on the absence $\{0\}$ or presence $\{1\}$ of the corresponding phase in the total modular invariant phase Φ of the model.

Using the basis :

$$\mathbf{X} = \begin{pmatrix} \Gamma_\alpha \\ h_a \end{pmatrix} , \quad \mathbf{Y} = \begin{pmatrix} \Delta_\alpha \\ g_a \end{pmatrix} , \quad (\text{B.10})$$

one may rewrite the modular invariant phase as a bilinear form $\Phi = \mathbf{X}^T \mathbf{M} \mathbf{Y}$, where the matrix \mathbf{M} is defined in terms of the phase coefficients :

$$\mathbf{M} = \left(\begin{array}{ccc|ccccc} D_1 & C & C & A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ * & D_2 & C & A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ * & * & D_3 & A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ \hline * & * & * & * & B_{12} & B_{13} & B_{14} & B_{15} \\ * & * & * & * & * & B_{23} & B_{24} & B_{25} \\ * & * & * & * & * & * & B_{34} & B_{35} \\ * & * & * & * & * & * & * & B_{45} \\ * & * & * & * & * & * & * & * \end{array} \right) , \quad (\text{B.11})$$

where the asterisks “*” denote dependent matrix elements that are irrelevant to our purpose. Suppose, furthermore, that we want to compare this phase matrix with the GGSO matrix $\mathbf{C}_{(\alpha|\beta)}$. In fact, it is more useful to consider the associated matrix $\mathbf{G}_{(\alpha|\beta)}$, which is defined as :

$$\mathbf{C}_{(\alpha|\beta)} \equiv (-)^{\mathbf{G}_{(\alpha|\beta)}} . \quad (\text{B.12})$$

Before attempting to relate \mathbf{C} with \mathbf{M} , however, one should notice that the latter is always symmetric, because of the $\Gamma_\alpha \leftrightarrow \Delta_\alpha$ and $h_a \leftrightarrow g_a$ exchange symmetry that results from invariance under the S -modular transformation. However, this is obviously not the case with \mathbf{C} . The discrepancy is due to the fact that \mathbf{C} captures the overall sign multiplying the four fundamental $\theta_{[0]}^0, \theta_{[1]}^0, \theta_{[0]}^1, \theta_{[1]}^1$ -functions, whereas the modular invariant phase $(-)^{\Phi}$ in expression (B.2) does not account for the extra phases that appear from the periodicity properties of the lower argument of the θ -functions :

$$\theta_{[y+2]}^x = e^{i\pi x} \theta_{[y]}^x , \quad \text{for } x, y \in \mathbb{R} . \quad (\text{B.13})$$

It is straightforward to see that these extra minus signs can arise from the extra phase :

$$\exp [i\pi(P + \psi)(g_1 + g_2 + g_1g_2) + i\pi(Q + \omega)(h_1g_2 + g_1h_2)] . \quad (\text{B.14})$$

Equivalently, we can express it as an “offset” matrix in the $\{v_i\}$ basis :

$$\mathbf{L}_{(\alpha|\beta)} = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} , \quad (\text{B.15})$$

where we now also included the contribution of the $(a + b + HG)$ in the phase. This is obviously not symmetric since the corresponding phase is not modular invariant. However, if added to $\mathbf{G}_{(\alpha|\beta)}$ the resulting matrix becomes symmetric, since the sum contains only the contribution from the (symmetric) modular invariant phase Φ .

In order to compare \mathbf{G} and the phase matrix \mathbf{M} , we finally need to express \mathbf{C} in the same \mathbf{X}, \mathbf{Y} basis as \mathbf{M} . For this purpose it will suffice to determine the matrix that transforms the $\{v_i\}$ basis into the \mathbf{X}, \mathbf{Y} one. It is straightforward to check that the transformation matrix is :

$$\mathbf{S} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} , \quad (\text{B.16})$$

and the two matrices are related by :

$$\mathbf{M} = \mathbf{S}^{-1}(\mathbf{G} + \mathbf{L})\mathbf{S}^{-1,T} . \quad (\text{B.17})$$

From this relation it is straightforward to compute the phase coefficients A, B, C, D in terms of the GSO coefficients and determine the modular invariant phase Φ that corresponds to a particular choice of $C_{(\alpha|\beta)}$.

A Particular Example

As an example, we study the model² corresponding to the following GGSO matrix :

$$G_{(\alpha|\beta)} = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}. \quad (\text{B.18})$$

From this, one may readily calculate the phase matrix \mathbf{M} :

$$\mathbf{M} = \left(\begin{array}{ccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array} \right), \quad (\text{B.19})$$

and determine the non-trivial phases that contribute to Φ :

$$D_1, A_{12}, A_{13}, A_{21}, A_{24}, A_{31}, A_{32}, A_{33}, A_{34}, B_{15}, B_{24}, B_{34}, B_{45}. \quad (\text{B.20})$$

This fixes the modular invariant phase describing this model to be :

$$\begin{aligned} \Phi &= a(g_1 + g_2) + b(h_1 + h_2) + k(Q + \omega) + l(P + \psi) + PG + QH \\ &\quad + \rho(Q + g_1 + g_2 + \omega) + \sigma(P + h_1 + h_2 + \psi) \\ &\quad + (h_1 + h_2)\omega + (g_1 + g_2)\psi + \psi G + \omega H. \end{aligned} \quad (\text{B.21})$$

Summing over the various sectors in the partition function (B.2) with the above modular invariant phase, we obtain :

$$Z = -240 + 2(\bar{j}(\bar{\tau}) - 744), \quad (\text{B.22})$$

which is indeed an MSDS model with extra massless fermions coming from the twisted sectors to give a negative contribution to the partition function at the MSDS point in moduli space. Having mapped the fermionic theory to its orbifold representation, one may easily consider deformations away from the fermionic point.

2. This particular model was first constructed by J. Rizos in the free-fermionic framework.

Finally, we will be interesting to separate the partition function into sectors :

$$Z[P, h_1, h_2, \psi, H] \equiv \frac{1}{\eta^{12} \bar{\eta}^{24}} \frac{1}{2^8} \sum_{\Delta_{\beta, g_b}} (-)^{a+b+HG+\Phi} \theta_{[b]}^a \theta_{[b+g_1]}^{a+h_1} \theta_{[b+g_2]}^{a+h_2} \theta_{[b-g_1-g_2]}^{a-h_1-h_2} \\ \times \Gamma_{(8,8)} \left[\begin{matrix} a, k \\ b, l \end{matrix} \middle| \begin{matrix} P, h_1, h_2, \psi \\ Q, g_1, g_2, \omega \end{matrix} \right] \bar{\theta}_{[l]}^5 \bar{\theta}_{[l+g_1]}^{k+h_1} \bar{\theta}_{[l+g_2]}^{k+h_2} \bar{\theta}_{[l-g_1-g_2]}^{k-h_1-h_2} \times \bar{\theta}_{[\sigma]}^4 \bar{\theta}_{[\sigma+G]}^4 . \quad (\text{B.23})$$

This facilitates the analysis of the various sectors, as shown below.

The Fully Untwisted sector $Z[0, 0, 0, 0, 0]$

The ‘fully’ untwisted sector of the model is $Z[0, 0, 0, 0, 0]$ for which we now proceed to determine its massless states. These only come from the $a = k = \rho = 0$ subsector and so the relevant phase vanishes $\Phi = 0$ for these states. Performing first the summation over $\{b, Q, \omega, \sigma \text{ and } G\}$ induces $\{\text{odd, even, even, even and even}\}$ projections, respectively. Furthermore, decomposition of the characters into invariant combinations under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold produces the following states :

$$\begin{aligned} & - O_2 V_2 O_2 O_2 \times O_{10} O_2 O_2 O_2 \times (\bar{V}_{10} \bar{V}_2 \bar{O}_2 \bar{O}_2 + \bar{O}_{10} \bar{O}_2 \bar{V}_2 \bar{V}_2) \times \bar{O}_{10} \bar{O}_2 \bar{O}_2 \bar{O}_2 \times \bar{O}_8 \bar{O}_8 : 48 \\ & - O_2 V_2 O_2 O_2 \times O_{10} O_2 O_2 O_2 \times \bar{O}_{10} \bar{O}_2 \bar{O}_2 \bar{O}_2 \times (\bar{V}_{10} \bar{V}_2 \bar{O}_2 \bar{O}_2 + \bar{O}_{10} \bar{O}_2 \bar{V}_2 \bar{V}_2) \times \bar{O}_8 \bar{O}_8 : 48 \\ & - O_2 O_2 V_2 O_2 \times O_{10} O_2 O_2 O_2 \times (\bar{V}_{10} \bar{O}_2 \bar{V}_2 \bar{O}_2 + \bar{O}_{10} \bar{V}_2 \bar{O}_2 \bar{V}_2) \times \bar{O}_{10} \bar{O}_2 \bar{O}_2 \bar{O}_2 \times \bar{O}_8 \bar{O}_8 : 48 \\ & - O_2 O_2 V_2 O_2 \times O_{10} O_2 O_2 O_2 \times \bar{O}_{10} \bar{O}_2 \bar{O}_2 \bar{O}_2 \times (\bar{V}_{10} \bar{O}_2 \bar{V}_2 \bar{O}_2 + \bar{O}_{10} \bar{V}_2 \bar{O}_2 \bar{V}_2) \times \bar{O}_8 \bar{O}_8 : 48 \\ & - O_2 O_2 O_2 V_2 \times O_{10} O_2 O_2 O_2 \times (\bar{V}_{10} \bar{O}_2 \bar{O}_2 \bar{V}_2 + \bar{O}_{10} \bar{V}_2 \bar{V}_2 \bar{O}_2) \times \bar{O}_{10} \bar{O}_2 \bar{O}_2 \bar{O}_2 \times \bar{O}_8 \bar{O}_8 : 48 \\ & - O_2 O_2 O_2 V_2 \times O_{10} O_2 O_2 O_2 \times \bar{O}_{10} \bar{O}_2 \bar{O}_2 \bar{O}_2 \times (\bar{V}_{10} \bar{O}_2 \bar{O}_2 \bar{V}_2 + \bar{O}_{10} \bar{V}_2 \bar{V}_2 \bar{O}_2) \times \bar{O}_8 \bar{O}_8 : 48 \\ & - V_2 O_2 O_2 O_2 \times O_{10} O_2 O_2 O_2 \times \bar{O}_{10} \bar{O}_2 \bar{O}_2 \bar{O}_2 \times \bar{O}_{10} \bar{O}_2 \bar{O}_2 \bar{O}_2 \times \bar{O}_8 \bar{O}_8 : 304 \end{aligned}$$

The counting of states goes as follows. For the first six lines, we have 2 holomorphic oscillator states from V_2 and $10 \times 2 + 2 \times 2$ from the antiholomorphic oscillators in $\bar{V}_{10} \bar{V}_2$ and $\bar{V}_2 \bar{V}_{10}$:

$$2 \times (10 \times 2 + 2 \times 2) = 48 . \quad (\text{B.24})$$

For the last line, the states arise from the 2 holomorphic oscillators in V_2 as well as the oscillators in the adjoint representation coming from the antiholomorphic \bar{O} -characters :

$$2 \times \left[\binom{10}{2} \times 2 + 1 \times 6 + \binom{8}{2} \times 2 \right] = 304 . \quad (\text{B.25})$$

It should be noted that the same last line also reproduces the correct multiplicity (two) for the antiholomorphic pole, $2/\bar{q}$. One verifies that the total number of massless states in this sector equals $6 \times 48 + 304 = 592$ and are all bosonic.

The Shifted Sector $Z[1, 0, 0, 0, 0]$

As an example of how massless fermions arise from the ‘twisted/shifted’ sectors, we present the characters corresponding to the massless states in the $Z[1, 0, 0, 0, 0]$ -sector. It is again straightforward to see that massless states come only from the $a = 1, k = 0$ subsector and are, therefore, fermions. The modular invariant phase relevant for these characters is :

$$\Phi \rightarrow G + g_1 + g_2 + l + \rho(\omega + g_1 + g_2 + Q) + \sigma . \quad (\text{B.26})$$

It is easy to see that for $\rho = 1$ the Q -projection renders the sector holomorphically massive. Taking $\rho = 0$, we notice that the l, σ, G, g_1 and g_2 -projections are now inversed because of the phase. This gives the following $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant characters :

$$\begin{aligned} & - (C_2 C_2 C_2 S_2 + S_2 S_2 S_2 C_2) \times O_{10} O_2 O_2 O_2 \times \bar{O}_{10} \bar{O}_2 \bar{O}_2 \bar{O}_2 \times \bar{V}_{10} \bar{O}_2 \bar{O}_2 \bar{O}_2 \times \bar{O}_8 \bar{V}_8 : 160 \\ & - (C_2 C_2 S_2 C_2 + S_2 S_2 C_2 S_2) \times O_{10} O_2 O_2 O_2 \times \bar{O}_{10} \bar{O}_2 \bar{O}_2 \bar{O}_2 \times \bar{O}_{10} \bar{V}_2 \bar{O}_2 \bar{O}_2 \times \bar{O}_8 \bar{V}_8 : 32 \\ & - (C_2 S_2 C_2 C_2 + S_2 C_2 S_2 S_2) \times O_{10} O_2 O_2 O_2 \times \bar{O}_{10} \bar{O}_2 \bar{O}_2 \bar{O}_2 \times \bar{O}_{10} \bar{O}_2 \bar{V}_2 \bar{O}_2 \times \bar{O}_8 \bar{V}_8 : 32 \\ & - (S_2 C_2 C_2 C_2 + C_2 S_2 S_2 S_2) \times O_{10} O_2 O_2 O_2 \times \bar{O}_{10} \bar{O}_2 \bar{O}_2 \bar{O}_2 \times \bar{O}_{10} \bar{O}_2 \bar{O}_2 \bar{V}_2 \times \bar{O}_8 \bar{V}_8 : 32 \end{aligned}$$

Similarly, the summing the number of states one obtains the contribution :

$$-2 \times 10 \times 8 - 2 \times (2 \times 8) \times 3 = -256 . \quad (\text{B.27})$$

The calculation of the spectrum of massless states in the remaining sectors proceeds in a completely analogous fashion.

Annexe C

Unfolding the Fundamental Domain

C.1 Decomposition into Modular Orbits

We are interested in calculating integrals of modular invariant quantities over the Teichmüller moduli space of the torus $\tau \in \mathbb{C}^+/\Gamma$. The symmetry group Γ is identified with the group of modular transformations $SL(2, \mathbb{Z})$. Its action is :

$$\tau \rightarrow \tau' = \frac{a\tau + b}{c\tau + d} \equiv \gamma \circ \tau \quad , \quad \text{where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \quad (\text{C.1})$$

It is generated by the translations T and inversions S :

$$T^m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \quad , \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (\text{C.2})$$

for any $m \in \mathbb{Z}$.

Consider the integral of a $(1, 1)$ -lattice over the fundamental domain \mathcal{F} :

$$I = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z(\tau, \bar{\tau}) \sum_{m, n \in \mathbb{Z}} e^{-\frac{\pi R^2}{\tau_2} |m + \tau n|^2}, \quad (\text{C.3})$$

where we have set $\alpha' = 1$ and $Z(\tau, \bar{\tau})$ is a modular invariant function. Let us separate out the $(0, 0)$ -orbit and use the absolute convergence in the remaining term in order to exchange integration with summation :

$$I = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z(\tau, \bar{\tau}) + \sum_{m, n \in \mathbb{Z}} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z(\tau, \bar{\tau}) e^{-\frac{\pi R^2}{\tau_2} |m + \tau n|^2} \quad (\text{C.4})$$

Consider performing a modular transformation (change of integration variable) $\tau = \gamma^{-1} \circ \tau'$:

$$I = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z(\tau, \bar{\tau}) + \sum_{m, n \in \mathbb{Z}} \int_{\gamma\mathcal{F}} \frac{d^2\tau'}{\tau_2'^2} Z(\tau', \bar{\tau}') e^{-\frac{\pi R^2}{\tau_2} |m + \tau n|^2}, \quad (\text{C.5})$$

where we used the fact that the measure $d^2\tau/\tau_2^2$ is invariant by itself and so is the function Z . Next, decompose (m, n) by factoring out the greatest common divisor $[m, n] = p$, so that $(m, n) = p(d, c)$, with $[c, d] = 1$. Thus, we have :

$$I = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z(\tau, \bar{\tau}) + \sum_{p=1}^{\infty} \sum_{[c,d]=1} \int_{\gamma\mathcal{F}} \frac{d^2\tau'}{\tau_2'^2} Z(\tau', \bar{\tau}') e^{-\frac{\pi R^2 p^2}{\tau_2} |d+c\tau|^2}. \quad (\text{C.6})$$

Let us now pick γ in the form :

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \quad (\text{C.7})$$

for some $a, b \in \mathbb{Z}$ relatively prime $[a, b] = 1$, so that $\det \gamma = 1$. However, under the above transformation, the imaginary part τ_2 transforms as :

$$\tau_2' = \frac{\tau_2}{|c\tau + d|^2}. \quad (\text{C.8})$$

Thus, we can write :

$$\begin{aligned} I &= \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z(\tau, \bar{\tau}) + \sum_{p=1}^{\infty} \sum_{[c,d]=1} \int_{\gamma\mathcal{F}} \frac{d^2\tau'}{\tau_2'^2} Z(\tau', \bar{\tau}') e^{-\frac{\pi R^2 p^2}{\tau_2}} \\ &= \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z(\tau, \bar{\tau}) + \sum_{p=1}^{\infty} \sum_{[c,d]=1} \int_{\gamma\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z(\tau, \bar{\tau}) e^{-\frac{\pi R^2 p^2}{\tau_2}}. \end{aligned} \quad (\text{C.9})$$

Note that the second line is simply obtained by a trivial renaming of the dummy (integration) variable.

It is easy to prove that the condition $\det M = 1$ for matrices $M \in SL(2, \mathbb{Z})$ implies that the elements of each row and column of M are relatively prime. Now, notice that modular transformation matrices with given lower row elements (c, d) define an equivalence class $\mathcal{C}_{c,d}$. In fact, all the matrices within the same equivalence class are related by translations. Indeed, take two arbitrary $SL(2, \mathbb{Z})$ matrices in the same equivalence class :

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \begin{pmatrix} a' & b' \\ c & d \end{pmatrix}. \quad (\text{C.10})$$

The condition of unit determinant :

$$da' - cb' = 1, \quad (\text{C.11})$$

is a diophantine equation for the pairs (a', b') , with the given coefficients c, d . It has infinite solutions, which can be constructed using the Euclidean division algorithm. In fact, for every solution (a, b) , it is easy to see that :

$$\begin{aligned} a' &= a + mc, \\ b' &= b + md, \end{aligned} \quad (\text{C.12})$$

with $m \in \mathbb{Z}$ is also a solution. In terms of matrices, this is written as :

$$\begin{pmatrix} a' & b' \\ c & d \end{pmatrix} = \begin{pmatrix} a + mc & b + md \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \quad (\text{C.13})$$

This implies that any two matrices $\gamma, \gamma' \in \mathcal{C}_{c,d}$ within the same equivalence class are related by translations $\gamma' = T^m \gamma$, for some $m \in \mathbb{Z}$.

Another important point is that the images $\gamma\mathcal{F}$ and $\gamma'\mathcal{F}$ of the fundamental domain \mathcal{F} under transformations $\gamma \in \mathcal{C}_{c,d}$ and $\gamma' \in \mathcal{C}_{c',d'}$ are disjoint $\gamma\mathcal{F} \cap \gamma'\mathcal{F} = \emptyset$ if the two equivalence classes are distinct $(c, d) \neq (c', d')$.

To see this, notice that for any $\tau \in \mathcal{F}$ the images $\hat{\gamma} \circ \tau$ are always distinct from $\hat{\gamma}' \circ \tau$, for any $\hat{\gamma}, \hat{\gamma}'$ within these distinct classes.

Since the various elements $\gamma \in \mathcal{C}_{c,d}$ within the same equivalence class are related under translations, one may always use this translation freedom so that the image $\gamma\mathcal{F}$ lies within the strip $-\frac{1}{2} \leq \tau_1 \leq \frac{1}{2}$. Thus, there is exactly one element $\hat{\gamma}$ in each class $\mathcal{C}_{c,d}$ that maps \mathcal{F} into the strip region¹.

First of all, notice that the only transformation that maps \mathcal{F} into itself is the identity. For example, consider the region $|\tau| = 1$ in \mathcal{F} . Under γ it transforms as :

$$\tau'_2 = \frac{\tau_2}{|c\tau + d|^2} \leq \frac{1}{(c-d)^2 + |cd|} \leq 1, \quad (\text{C.14})$$

where the last inequality follows trivially from the fact that the denominator is a positive integer. Thus, the region $|\tau| = 1$ necessarily falls within the complement $\mathcal{S} - \mathcal{F}$ of the fundamental domain (here \mathcal{S} is the strip $\{\tau \in \mathbb{C}^+ : |\tau_1| \leq \frac{1}{2}\}$), except for the transformations which saturate the inequality. These transformations belong to the equivalence classes $\mathcal{C}_{0,1}, \mathcal{C}_{1,0}, \mathcal{C}_{1,1}$. The first one $\mathcal{C}_{0,1}$ contains the identity, whereas the other two are eliminated by considering the mapping of points $\tau \in \mathcal{F}$ along the vertical boundary $\tau_1 = -\frac{1}{2}$. Thus, the only modular transformation mapping \mathcal{F} into itself is the identity $\in \mathcal{C}_{1,0}$.

Now, suppose we construct the image

$$\hat{\gamma}\mathcal{F} = \{(\hat{\gamma} \circ \tau) \in \mathbb{C}^+, \tau \in \mathcal{F}\} \quad (\text{C.15})$$

with respect to $\hat{\gamma} \in \mathcal{C}_{c,d}$. We will show that taking similar constructions of images with respect to distinct equivalence classes $\mathcal{C}_{c',d'}$ will create a partition of the strip. To see this, consider the image of \mathcal{F} with respect to some other equivalence class $\gamma \in \mathcal{C}_{c',d'}$:

$$\hat{\gamma}'\mathcal{F} = \{(\hat{\gamma}' \circ \tau) \in \mathbb{C}^+, \tau \in \mathcal{F}\} \quad (\text{C.16})$$

Assume now, that the images $\hat{\gamma}\mathcal{F}, \hat{\gamma}'\mathcal{F}$ of the two distinct equivalence classes have non-empty intersection. Consider one such element $w \in (\hat{\gamma}\mathcal{F} \cap \hat{\gamma}'\mathcal{F})$. Then, there will be points $\tau, \tau' \in \mathcal{F}$ such that :

$$w = \hat{\gamma} \circ \tau = \hat{\gamma}' \circ \tau'. \quad (\text{C.17})$$

1. Of course this map alone will not cover the whole strip.

However, because of the group composition law

$$\tau' = (\hat{\gamma}'^{-1} \cdot \hat{\gamma}) \circ \tau, \quad (\text{C.18})$$

the element $\hat{\gamma}'^{-1} \cdot \hat{\gamma} \in SL(2, \mathbb{Z})$ maps the fundamental domain into itself and, thus, has to be the identity matrix. Thus, $\hat{\gamma}' = \hat{\gamma}$ and we reach a contradiction. Therefore, transformation matrices belonging to different characteristic classes have disjoint images and provide a complete partition of the strip \mathcal{S} :

$$\mathcal{S} = \frac{\mathbb{C}^+}{\text{translations}} . \quad (\text{C.19})$$

In view of this result, (C.9) becomes :

$$I = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z(\tau, \bar{\tau}) + \int_{\mathcal{S}} \frac{d^2\tau}{\tau_2^2} Z(\tau, \bar{\tau}) e^{-\frac{\pi R^2 p^2}{\tau_2}} . \quad (\text{C.20})$$

Hence, the initial integral over the fundamental domain has been decomposed into the sum of two orbits. The $(0, 0)$ -orbit is an inherently stringy contribution over \mathcal{F} and the $(m, 0)$ -orbit ($m \neq 0$) is the ‘field-theoretic’ contribution integrated over the strip \mathcal{S} .

Of course, when Z is modular covariant (rather than invariant) the modular transformation will act on Z as well, but the generalization proceeds in a straightforward way. See, for example [92] for the bosonic case, and [94] for the superstring. For unfolding involving higher dimensional lattices, see the classic paper on thresholds, [93]. Another useful review is [95]. The evaluation of amplitudes can alternatively proceed without the unfolding procedure, by making use of a powerful theorem due to Rankin and Selberg (see, for example, [96] and references therein).

Annexe D

The ‘Abstrusa’ Identity of Jacobi

In this appendix we present an elementary proof of the ‘Abstrusa’ identity, (A.8), of Jacobi. In one of its generalized versions, the ‘Abstrusa’ identity, with shifts h_i, g_i and characteristics z_i takes the form :

$$\frac{1}{2} \sum_{a,b=0,1} (-)^{a+b} \prod_{i=1}^4 \theta_{[b+g_i]}^{[a+h_i]}(z_i) = e^{-i\pi h_1} \prod_{i=1}^4 \theta_{[1+g'_i]}^{[1+h'_i]}(z'_i) , \quad (\text{D.1})$$

provided $\sum_{i=1}^4 h_i = 0$. The transformed h'_i and z'_i are linear combinations of the h_i and z_i , respectively, defined in the next section. There exist similar identities for GSO projections of modified ‘chirality’.

D.1 A ‘direct’ proof

The starting point is the GSO-projected block (ignoring irrelevant Dedekind functions), with general shifts (h_i, g_i) :

$$I = \frac{1}{2} \sum_{a,b=0,1} (-)^{a+b} \prod_{i=1}^4 \theta_{[b+g_i]}^{[a+h_i]}(z_i) . \quad (\text{D.2})$$

We will take h_i, g_i to satisfy the constraint $\sum_i h_i = 0$, but otherwise arbitrary. Similarly, the characteristics z_i are taken to be arbitrary real numbers. The lower shifts, g_i can be always absorbed into the z_i by a shift $z_i \rightarrow z_i + \frac{1}{2}g_i$. Hence, we can effectively set g_i to zero and reinstate them at the end. Using the sum representation of θ -functions, the above quantity is written as :

$$I = \frac{1}{2} \sum_{a,b=0,1} (-)^{a+b} \sum_{n_i \in \mathbb{Z}} q^{\frac{1}{2} \sum_i (n_i - \frac{a}{2} - \frac{h_i}{2})^2} e^{2\pi i \sum_i (z_i - \frac{b}{2}) (n_i - \frac{a}{2} - \frac{h_i}{2})} \quad (\text{D.3})$$

Now, we perform the summation over $b = 0, 1$ (GSO-projection), which introduces the following constraint :

$$\sum_i \left(n_i - \frac{a + h_i}{2} \right) \in 2\mathbb{Z} , \quad (\text{D.4})$$

or, by using the initial constraint $\sum_i h_i = 0$, one finds :

$$\sum_i n_i \in 2\mathbb{Z} + 1 . \quad (\text{D.5})$$

This is a first constraint for the summation variables, $\{n_i\}$. We next define the auxiliary variables \tilde{n}_i as :

$$\tilde{n}_i \equiv n_i - \frac{a}{2} . \quad (\text{D.6})$$

These satisfy $\tilde{n}_i \in \mathbb{Z} + \frac{a}{2}$, and are, hence, either all integers or all half-integers, depending on the a -parity. The reason for this definition is that we want the a -summation to be absorbed into the definition of the \tilde{n}_i . Using the GSO-constraint derived above, we note that :

$$\frac{1}{2} \sum_i \tilde{n}_i \in \mathbb{Z} + \frac{1}{2} , \quad (\text{D.7})$$

and, hence, the \tilde{n}_i are not independent variables. Now we can rewrite I in terms of the \tilde{n}_i variables :

$$I = \sum_{a=0,1} (-)^a \sum_{n_i} q^{\frac{1}{2} \sum_i (\tilde{n}_i - \frac{h_i}{2})^2} e^{2\pi i \sum_i z_i (\tilde{n}_i - \frac{h_i}{2})} . \quad (\text{D.8})$$

The next step is to replace the explicit appearance of a in the phase, by \tilde{n} . Using (D.6), we note that :

$$(-)^a = e^{-i\pi(2\tilde{n}_1 - 2n_1)} = e^{-2\pi i \tilde{n}_1} , \quad (\text{D.9})$$

because $n_1 \in \mathbb{Z}$. Hence, we can express I as :

$$I = \sum_{a=0,1} \sum_{n_i} q^{\frac{1}{2} \sum_i (\tilde{n}_i - \frac{h_i}{2})^2} e^{2\pi i \sum_i z_i (\tilde{n}_i - \frac{h_i}{2}) - 2\pi i \tilde{n}_1} . \quad (\text{D.10})$$

Now we return to the constraints. We will perform a change of basis in the summation variables, such that the new variables will be independent. To this end, consider the symmetric matrix :

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} , \quad (\text{D.11})$$

which is chosen because of its property $A^2 = 4 \mathbf{1}_4$, with $\mathbf{1}_4$ being the 4-dimensional identity matrix. Consider now the transformation (written in matrix notation) :

$$\tilde{n} = \frac{1}{2} A \left(n' - \frac{1}{2} \mathbf{1} \right) , \quad (\text{D.12})$$

with $\mathbf{1}^T \equiv (1, 1, 1, 1)$. Written explicitly in terms of the $\{\tilde{n}_i\}$ -basis, the transformation is becomes :

$$\begin{aligned} n'_1 - \frac{1}{2} &= \frac{1}{2}(\tilde{n}_1 + \tilde{n}_2 + \tilde{n}_3 + \tilde{n}_4) , \\ n'_2 - \frac{1}{2} &= \frac{1}{2}(\tilde{n}_1 + \tilde{n}_2 - \tilde{n}_3 - \tilde{n}_4) , \\ n'_3 - \frac{1}{2} &= \frac{1}{2}(\tilde{n}_1 - \tilde{n}_2 + \tilde{n}_3 - \tilde{n}_4) , \\ n'_4 - \frac{1}{2} &= \frac{1}{2}(\tilde{n}_1 - \tilde{n}_2 - \tilde{n}_3 + \tilde{n}_4) . \end{aligned} \quad (\text{D.13})$$

Because of (D.7), the r.h.s. is a half-integer and, thus, the above transformation takes the \tilde{n} to the integral variables, $n'_i \in \mathbb{Z}$. The question now is whether the new variables, n'_i , are also independent. Taking their sum, one finds a constraint :

$$\sum_i n'_i = 2(\tilde{n}_1 + 1) = 2(n_1 + 1) - a \in 2\mathbb{Z} + a . \quad (\text{D.14})$$

However, upon summation over a , the sum $\sum_i n'_i \in \mathbb{Z}$, and the constraint is eliminated. For a proof of this see the complement at the end of this section.

The change of variables from \tilde{n} to n' in the sum is straightforward. Define the associated transformation (in matrix notation) of the shifts h_i and the characteristics z_i by :

$$\begin{aligned} h &= \frac{1}{2} A h' , \\ z &= \frac{1}{2} A z' . \end{aligned} \quad (\text{D.15})$$

Hence :

$$\left(\tilde{n} - \frac{h}{2} \right)^T \left(\tilde{n} - \frac{h}{2} \right) = \left(n' - \frac{\mathbf{1} + h'}{2} \right)^T \left(n' - \frac{\mathbf{1} + h'}{2} \right) , \quad (\text{D.16})$$

and :

$$z^T \left(\tilde{n} - \frac{h}{2} \right) = z'^T \left(n' - \frac{\mathbf{1} + h'}{2} \right) . \quad (\text{D.17})$$

Because of (D.14), we can also rewrite the pure phase as :

$$e^{-2\pi i \tilde{n}_1} = e^{-i\pi \sum_i n'_i} = e^{-i\pi \sum_i \left(n'_i - \frac{1+h'_i}{2} \right)} e^{-i\pi \frac{1}{2} \sum_i h'_i} = e^{-i\pi \sum_i \left(n'_i - \frac{1+h'_i}{2} \right)} e^{-i\pi h_1} . \quad (\text{D.18})$$

The first factor is precisely what is needed to contribute to the formation of a θ -function, while the second one is an overall phase.

Putting everything together, I becomes :

$$\begin{aligned} I &= e^{-i\pi h_1} \sum_{n'_i \in \mathbb{Z}} q^{\frac{1}{2} \sum_i \left(n'_i - \frac{1+h'_i}{2}\right)^2} e^{2\pi i \sum_i \left(z'_i - \frac{1}{2}\right) \left(n'_i - \frac{1+h'_i}{2}\right)} \\ &= e^{-i\pi h_1} \prod_i \theta_{[1+g'_i]}^{[1+h'_i]}(z'_i) . \end{aligned} \quad (\text{D.19})$$

This is the generalized ‘Abstrusa’ identity of Jacobi :

$$\frac{1}{2} \sum_{a,b=0,1} (-)^{a+b} \prod_{i=1}^4 \theta_{[b+g_i]}^{[a+h_i]}(z_i) = e^{-i\pi h_1} \prod_{i=1}^4 \theta_{[1+g'_i]}^{[1+h'_i]}(z'_i) . \quad (\text{D.20})$$

The original ‘abstrusa’ identity involves setting all shifts to zero, $h_i = g_i = 0$, and all characteristics equal to each other $z_i = z$. Then, the above identity takes the familiar form, (A.8) :

$$\theta_3^4(z) - \theta_4^4(z) - \theta_2^4(z) + \theta_1^4(z) = \theta_1(z)\theta_1^3(0) = 0 . \quad (\text{D.21})$$

We end this note on the proof of the Jacobi ‘abstrusa’ identity with a small comment on the fact that the summation in the n'_i basis is independent. There is a straightforward way to see that the new basis, $\{n'_i\}$, is unconstrained. Rewrite (D.13) in the original n -basis :

$$\begin{aligned} n'_1 - \frac{1}{2} &= \frac{1}{2}(n_1 + n_2 + n_3 + n_4) - a , \\ n'_2 - \frac{1}{2} &= \frac{1}{2}(n_1 + n_2 - n_3 - n_4) , \\ n'_3 - \frac{1}{2} &= \frac{1}{2}(n_1 - n_2 + n_3 - n_4) , \\ n'_4 - \frac{1}{2} &= \frac{1}{2}(n_1 - n_2 - n_3 + n_4) . \end{aligned} \quad (\text{D.22})$$

Now shift the original variables in the following order :

$$\begin{aligned} n_1 &\rightarrow n_1 - n_2 - n_3 - n_4 + 2a , \\ n_4 &\rightarrow n_4 - n_3 + a , \\ n_2 &\rightarrow n_2 + n_3 - n_4 . \end{aligned} \quad (\text{D.23})$$

The resulting basis has the form :

$$\begin{aligned} n'_1 - \frac{1}{2} &= \frac{1}{2} n_1 , \\ n'_2 - \frac{1}{2} &= \frac{1}{2} n_1 - n_4 , \\ n'_3 - \frac{1}{2} &= \frac{1}{2} n_1 - n_2 , \\ n'_4 - \frac{1}{2} &= \frac{1}{2} n_1 - n_2 - 2n_3 + n_4 + a . \end{aligned} \quad (\text{D.24})$$

This implies that n'_1 is arbitrary, because n_1 is an arbitrary odd number (because of the modified GSO constraint). Similarly, n'_2 and n'_3 are arbitrary because they are mapped isomorphically to n_4 and n_2 , respectively (which are arbitrary and independent in this basis). Finally, redefining $N_3 = 2n_3 - a$ and summing over n_3 and a sets N_3 to be an arbitrary integer so that n'_4 is also arbitrary and independent. This implies that the new variables n'_i are indeed arbitrary and independent.

Bibliographie

- [1] C. G. Callan, Jr., E. J. Martinec, M. J. Perry *et al.*, “Strings in Background Fields,” Nucl. Phys. **B262**, 593 (1985).
- [2] C. Kounnas, D. Lust, “Cosmological string backgrounds from gauged WZW models,” Phys. Lett. **B289**, 56-60 (1992). [hep-th/9205046].
- [3] C. Becchi, A. Rouet, R. Stora, “Renormalization of Gauge Theories,” Annals Phys. **98**, 287-321 (1976).
- [4] M. D. Freeman, D. I. Olive, “The Calculation Of Planar One Loop Diagrams In String Theory Using The BRS Formalism,” Phys. Lett. **B175**, 155 (1986).
- [5] F. Gliozzi, J. Scherk, D. I. Olive, “Supersymmetry, Supergravity Theories and the Dual Spinor Model,” Nucl. Phys. **B122**, 253-290 (1977).
- [6] H. M. Farkas, I. Kra, “Riemann Surfaces,” New York : Springer, 179-192 (1980).
- [7] S. Minakshisundaram, A. Pleijel, “Some properties of the eigenfunctions of the Laplace-operator on Riemannian manifolds,” Canadian Journal of Mathematics **1**, 242–256 (1949).
- [8] C.L. Siegel, “Analytische Zahlentheorie II,” Göttingen, (1963).
- [9] C.G.J. Jacobi, “Fundamenta Nova Theoriae Functionum Ellipticarum,” Königsberg, Germany : Regiomonti, Sumtibus fratrum Borntraeger, (1829). Reprinted in Gesammelte Werke, Band. 1. Providence, RI : Amer. Math. Soc., p. 147, (1969).
- [10] N. Seiberg, E. Witten, “Spin Structures in String Theory,” Nucl. Phys. **B276**, 272 (1986).
- [11] L. Alvarez-Gaume, G. W. Moore, C. Vafa, “Theta Functions, Modular Invariance and Strings,” Commun. Math. Phys. **106**, 1-40 (1986).
- [12] C. Angelantonj, A. Sagnotti, Phys. Rept. **371**, 1-150 (2002). [hep-th/0204089].
- [13] S. Chaudhuri, J. A. Schwartz, “A Criterion For Integrably Marginal Operators,” Phys. Lett. **B219**, 291 (1989).
- [14] L. J. Dixon, J. A. Harvey, C. Vafa, E. Witten, “Strings on Orbifolds,” Nucl. Phys. **B261**, 678-686 (1985).
- [15] L. J. Dixon, J. A. Harvey, C. Vafa, E. Witten, “Strings on Orbifolds. 2.,” Nucl. Phys. **B274**, 285-314 (1986).
- [16] S. Hamidi, C. Vafa, “Interactions on Orbifolds,” Nucl. Phys. **B279**, 465 (1987).

- [17] K. S. Narain, M. H. Sarmadi, C. Vafa, "Asymmetric Orbifolds," Nucl. Phys. **B288**, 551 (1987).
- [18] K. S. Narain, M. H. Sarmadi, C. Vafa, "Asymmetric orbifolds : Path integral and operator formulations," Nucl. Phys. **B356**, 163-207 (1991).
- [19] L. J. Dixon, D. Friedan, E. J. Martinec, S. H. Shenker, "The Conformal Field Theory of Orbifolds," Nucl. Phys. **B282**, 13-73 (1987).
- [20] D. Friedan, S. H. Shenker, E. J. Martinec, "Covariant Quantization of Superstrings," Phys. Lett. **B160**, 55 (1985).
- [21] D. Friedan, E. J. Martinec, S. H. Shenker, "Conformal Invariance, Supersymmetry and String Theory," Nucl. Phys. **B271**, 93 (1986).
- [22] P. Di Vecchia, V. G. Knizhnik, J. L. Petersen, P. Rossi, "A Supersymmetric Wess-Zumino Lagrangian in Two-Dimensions," Nucl. Phys. **B253**, 701 (1985).
- [23] I. Antoniadis, C. Bachas, C. Kounnas, P. Windey, "Supersymmetry Among Free Fermions and Superstrings," Phys. Lett. **B171**, 51 (1986).
- [24] D. Friedan, Z. -a. Qiu, S. H. Shenker, "Superconformal Invariance in Two-Dimensions and the Tricritical Ising Model," Phys. Lett. **B151**, 37-43 (1985).
- [25] H. Kawai, D. C. Lewellen, S. H. H. Tye, "Construction of Fermionic String Models in Four-Dimensions," Nucl. Phys. **B288**, 1 (1987).
- [26] I. Antoniadis, C. P. Bachas, C. Kounnas, "Four-Dimensional Superstrings," Nucl. Phys. **B289**, 87 (1987).
- [27] T. Banks, L. J. Dixon, D. Friedan, E. J. Martinec, "Phenomenology and Conformal Field Theory Or Can String Theory Predict the Weak Mixing Angle?," Nucl. Phys. **B299**, 613-626 (1988).
- [28] C. Kounnas, B. Rostand, "Deformations Of Superstring Solutions And Spontaneous Symmetry Breaking," Proceedings of 3rd Hellenic School on Elementary Particle Physics, Corfu, Greece, Sept. 13-23, (1989).
- [29] T. Banks, L. J. Dixon, "Constraints on String Vacua with Space-Time Supersymmetry," Nucl. Phys. **B307**, 93-108 (1988).
- [30] T. Eguchi, A. Taormina, "Extended Superconformal Algebras And String Compactifications," Lecture given at the Spring School in Superstrings, Trieste, Apr. 11-22 (1988).
- [31] D. Gepner, "Space-Time Supersymmetry in Compactified String Theory and Superconformal Models," Nucl. Phys. **B296**, 757 (1988).
- [32] E. Witten, "Phases of N=2 theories in two-dimensions," Nucl. Phys. **B403**, 159-222 (1993). [hep-th/9301042].
- [33] J. -P. Derendinger, C. Kounnas, P. M. Petropoulos, F. Zwirner, "Superpotentials in IIA compactifications with general fluxes," Nucl. Phys. **B715**, 211-233 (2005). [hep-th/0411276].
- [34] J. Scherk and J. H. Schwarz, "How to Get Masses from Extra Dimensions," Nucl. Phys. **B 153**, 61 (1979).

- [35] C. Kounnas, M. Porrati, “Spontaneous Supersymmetry Breaking in String Theory,” Nucl. Phys. **B310**, 355 (1988).
- [36] C. Kounnas, B. Rostand, “Coordinate Dependent Compactifications and Discrete Symmetries,” Nucl. Phys. **B341**, 641-665 (1990).
- [37] E. T. Tomboulis, “Gauge Fields On The World Sheet : Strings With Gauge And Supersymmetry Breaking,” Phys. Lett. **B198**, 165 (1987).
- [38] E. Kiritsis and C. Kounnas, “Dynamical topology change, compactification and waves in string cosmology,” Nucl. Phys. Proc. Suppl. **41** (1995) 311.
- [39] R. H. Brandenberger and C. Vafa, “Superstrings in the Early Universe,” Nucl. Phys. B **316**, 391 (1989).
- [40] R. Hagedorn, “Statistical thermodynamics of strong interactions at high-energies,” Nuovo Cim. Suppl. **3**, 147 (1965).
- [41] D. Kutasov and N. Seiberg, “ Number of degrees of freedom, density of states and tachyons in string theory and CFT,” Nucl. Phys. B **358**, 600 (1991).
- [42] K. Dienes, “Modular invariance, finiteness, and misaligned supersymmetry : New Constraints on the Numbers of Physical String States,” Nucl. Phys. B **429**, 533 (1994).
- [43] E. Kiritsis, C. Kounnas, “String gravity and cosmology : Some new ideas,” [gr-qc/9509017].
- [44] E. Witten, “Nonabelian Bosonization in Two-Dimensions,” Commun. Math. Phys. **92**, 455-472 (1984).
- [45] C. Kounnas, “Massive Boson-Fermion Degeneracy and the Early Structure of the Universe,” Fortsch. Phys. **56**, 1143-1156 (2008). [arXiv :0808.1340 [hep-th]].
- [46] I. Florakis, C. Kounnas, “Orbifold Symmetry Reductions of Massive Boson-Fermion Degeneracy,” Nucl. Phys. **B820**, 237-268 (2009). [arXiv :0901.3055 [hep-th]].
- [47] I. Florakis, C. Kounnas, N. Toumbas, “Marginal Deformations of Vacua with Massive boson-fermion Degeneracy Symmetry,” Nucl. Phys. **B834**, 273-315 (2010). [arXiv :1002.2427 [hep-th]].
- [48] I. Florakis, C. Kounnas, H. Partouche, N. Toumbas, “Non-singular string cosmology in a 2d Hybrid model,” Nucl. Phys. **B844**, 89-114 (2011). [arXiv :1008.5129 [hep-th]].
- [49] B. Assel, K. Christodoulides, A. E. Faraggi, C. Kounnas, J. Rizos, “Exophobic Quasi-Realistic Heterotic String Vacua,” Phys. Lett. **B683**, 306-313 (2010). [arXiv :0910.3697 [hep-th]].
- [50] A. Actor, “Chemical Potentials in Gauge Theories,” Phys. Lett. B **157** (1985) 53.
- [51] A. Actor, “Compactification at Finite Temperature in Noncompact Electrodynamics,” Annals Phys. **159** (1985) 445.
- [52] M. Axenides, S. D. Ellis, C. Kounnas, “Universal Behavior Of D-dimensional Superstring Models,” Phys. Rev. **D37**, 2964 (1988).
- [53] I. Antoniadis, J. R. Ellis, D. V. Nanopoulos, “Universality Of The Mass Spectrum In Closed String Models,” Phys. Lett. **B199**, 402 (1987).

- [54] R. Hagedorn, “Statistical thermodynamics of strong interactions at high-energies,” *Nuovo Cim. Suppl.* **3**, 147-186 (1965).
- [55] J. J. Atick, E. Witten, “The Hagedorn Transition and the Number of Degrees of Freedom of String Theory,” *Nucl. Phys.* **B310**, 291-334 (1988).
- [56] I. Antoniadis, C. Kounnas, “Superstring phase transition at high temperature,” *Phys. Lett.* **B261**, 369-378 (1991).
- [57] J. M. Kosterlitz, D. J. Thouless, “Ordering, metastability and phase transitions in two-dimensional systems,” *J. Phys. C* **C6**, 1181-1203 (1973).
- [58] J. Villain, “Theory of one- and two-dimensional magnets with an easy magnetization plane. II. The planar, classical, two-dimensional magnet,” *J. Phys. C* **C 36**, 581 (1975).
- [59] M. Maggiore, “The Atick-Witten free energy, closed tachyon condensation and deformed Poincare symmetry,” *Nucl. Phys.* **B647**, 69-100 (2002). [hep-th/0205014].
- [60] I. Antoniadis, J. P. Derendinger, C. Kounnas, “Nonperturbative temperature instabilities in $N=4$ strings,” *Nucl. Phys.* **B551**, 41-77 (1999). [hep-th/9902032].
- [61] C. Angelantonj, C. Kounnas, H. Partouche, N. Toumbas, “Resolution of Hagedorn singularity in superstrings with gravito-magnetic fluxes,” *Nucl. Phys.* **B809**, 291-307 (2009). [arXiv :0808.1357 [hep-th]].
- [62] R. Penrose, “Structure of space-time,” in *Battelle Rencontres, 1967 Lectures in Mathematics and Physics*, edited by C.M. DeWitt and J.A. Wheeler, pp. 121-235, Benjamin, New York, 1968.
- [63] S. W. Hawking and G. F. R. Ellis, “The large scale structure of space-time,” Cambridge University Press, Cambridge, 1973.
- [64] S. Hawking and G. F. R. Ellis, “Singularities in homogeneous world models,” *Phys. Lett.* **17**, 246 (1965).
- [65] R. H. Brandenberger and C. Vafa, “Superstrings in the early universe,” *Nucl. Phys. B* **316**, 391 (1989).
- [66] G. Veneziano, “Scale factor duality for classical and quantum strings,” *Phys. Lett. B* **265**, 287 (1991).
- [67] M. Gasperini, M. Giovannini, G. Veneziano, “Perturbations in a nonsingular bouncing universe,” *Phys. Lett.* **B569**, 113-122 (2003). [hep-th/0306113].
- [68] M. Gasperini, M. Maggiore, G. Veneziano, “Towards a nonsingular pre - big bang cosmology,” *Nucl. Phys.* **B494**, 315-330 (1997). [hep-th/9611039].
- [69] R. Brustein, M. Gasperini, G. Veneziano, “Duality in cosmological perturbation theory,” *Phys. Lett.* **B431**, 277-285 (1998). [hep-th/9803018].
- [70] M. Gasperini, G. Veneziano, “The Pre - big bang scenario in string cosmology,” *Phys. Rept.* **373**, 1-212 (2003). [hep-th/0207130].
- [71] A. Tseytlin and C. Vafa, “Elements of string cosmology,” *Nucl. Phys. B* **372** (1992) 443 [arXiv :hep-th/9109048].

- [72] C. Kounnas, N. Toumbas and J. Troost, “A wave-function for stringy universes,” JHEP **0708** (2007) 018 [arXiv :0704.1996 [hep-th]].
- [73] T. Catelin-Jullien, C. Kounnas, H. Partouche and N. Toumbas, “Thermal/quantum effects and induced superstring cosmologies,” Nucl. Phys. B **797** (2008) 137 [arXiv :0710.3895 [hep-th]].
- [74] T. Catelin-Jullien, C. Kounnas, H. Partouche and N. Toumbas, “Thermal and quantum superstring cosmologies,” Fortsch. Phys. **56** (2008) 792 [arXiv :0803.2674 [hep-th]].
- [75] T. Catelin-Jullien, C. Kounnas, H. Partouche and N. Toumbas, “Induced superstring cosmologies and moduli stabilization,” Nucl. Phys. B **820**, 290 (2009) [arXiv :0901.0259 [hep-th]].
- [76] I. Antoniadis and C. Kounnas, “The dilaton classical solution and the supersymmetry breaking evolution in an expanding universe,” Nucl. Phys. B **284**, 729 (1987).
- [77] F. Bourliot, C. Kounnas and H. Partouche, “Attraction to a radiation-like era in early superstring cosmologies,” Nucl. Phys. B **816** (2009) 227 [arXiv :0902.1892 [hep-th]].
- [78] F. Bourliot, J. Estes, C. Kounnas and H. Partouche, “Cosmological phases of the string thermal effective potential,” Nucl. Phys. B **830**, 330 (2010) [arXiv :0908.1881 [hep-th]].
- [79] J. Estes, C. Kounnas and H. Partouche, “Superstring cosmology for $N_4 = 1 \rightarrow 0$ superstring vacua,” arXiv :1003.0471 [hep-th].
- [80] N. Matsuo, “Superstring thermodynamics and its application to cosmology,” Z. Phys. C **36** (1987) 289.
- [81] J. Kripfganz and H. Perl, “Cosmological impact of winding strings,” Class. Quant. Grav. **5** (1988) 453.
- [82] M. J. Bowick and S. B. Giddings, “High temperature strings,” Nucl. Phys. B **325**, 631 (1989).
- [83] B. A. Bassett, M. Borunda, M. Serone and S. Tsujikawa, “Aspects of string-gas cosmology at finite temperature,” Phys. Rev. D **67** (2003) 123506 [arXiv :hep-th/0301180].
- [84] R. Easther, B. R. Greene, M. G. Jackson and D. N. Kabat, “String windings in the early universe,” JCAP **0502**, 009 (2005) [arXiv :hep-th/0409121].
- [85] T. Battefeld and S. Watson, “String gas cosmology,” Rev. Mod. Phys. **78**, 435 (2006) [arXiv :hep-th/0510022].
- [86] R. H. Brandenberger, “String gas cosmology,” arXiv :0808.0746 [hep-th].
- [87] B. Greene, D. Kabat and S. Marnerides, “Bouncing and cyclic string gas cosmologies,” arXiv :0809.1704 [hep-th].
- [88] A. E. Faraggi, C. Kounnas, J. Rizos, “Spinor-Vector Duality in fermionic $\mathbb{Z}_2 \times \mathbb{Z}_2$ heterotic orbifold models,” Nucl. Phys. B **774**, 208-231 (2007). [hep-th/0611251].
- [89] A. E. Faraggi, C. Kounnas, J. Rizos, “Spinor-vector duality in $N=2$ heterotic string vacua,” Nucl. Phys. B **799**, 19-33 (2008). [arXiv :0712.0747 [hep-th]].
- [90] T. Catelin-Jullien, A. E. Faraggi, C. Kounnas, J. Rizos, “Spinor-Vector Duality in Heterotic SUSY Vacua,” Nucl. Phys. B **812**, 103-127 (2009). [arXiv :0807.4084 [hep-th]].

- [91] A. E. Faraggi, I. Florakis, T. Mohaupt, M. Tsulaia, “Conformal Aspects of Spinor-Vector Duality,” [arXiv :1101.4194 [hep-th]].
- [92] B. McClain and B. D. B. Roth, “Modular Invariance For Interacting Bosonic Strings At Finite Temperature,” Commun. Math. Phys. **111**, 539 (1987).
- [93] L. J. Dixon, V. Kaplunovsky and J. Louis, “Moduli dependence of string loop corrections to gauge coupling constants,” Nucl. Phys. **B355**, 649-688 (1991).
- [94] K. H. O’Brien, C. I. Tan, “Modular Invariance of Thermopartition Function and Global Phase Structure of Heterotic String,” Phys. Rev. **D36**, 1184 (1987).
- [95] M. Trapletti, “On the unfolding of the fundamental region in integrals of modular invariant amplitudes,” JHEP **0302**, 012 (2003). [hep-th/0211281].
- [96] M. Cardella, “A Novel method for computing torus amplitudes for $Z(N)$ orbifolds without the unfolding technique,” JHEP **0905** (2009) 010. [arXiv :0812.1549 [hep-th]].
- [97] C. Kounnas, H. Partouche, N. Toubas, “Thermal duality and non-singular cosmology in d-dimensional superstrings,” [arXiv :1106.0946 [hep-th]].

Épilogue

“ Manche Männer bemühen sich lebenslang, das Wesen einer Frau zu verstehen. Andere befassen sich mit weniger schwierigen Dingen, z.B. der Relativitätstheorie... ”

Albert Einstein