UFR S.T.M.I.A.
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Roger NAKAD

Sous-variétés spéciales des variétés spinorielles complexes

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Rapporteur :	**Vestislav Apostolov**	Professeur, UQAM, Montréal
Membres du jury :		
Rapporteur :	**Sebastián Montiel**	Professeur, Grenade
Examinateurs :	**Jean Pierre Bourguignon**	Directeur de recherche (CNRS), IHÉS Professeur à l'École Polytechnique
	Oussama Hijazi	Directeur de Thèse, Professeur, Nancy I
	Emmanuel Humbert	Maître de conférences (HDR), Nancy I
	Andrei Moroianu	Chargé de recherche (CNRS), École Polytechnique, Palaiseau

Institut Élie Cartan Nancy, Laboratoire de Mathématiques, B.P. 239, 54506 Vandœuvre-lès-Nancy Cedex

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Introduction

Le sujet principal de cette thèse est d'exploiter les structures Spin^c dans le but d'étudier la géométrie de certaines sous-variétés. Nous commençons par établir des résultats de base pour l'opérateur de Dirac Spin^c. Ensuite, nous examinons les hypersurfaces des variétés Spin^c admettant des spineurs spéciaux.

La géométrie et la topologie d'une variété riemannienne compacte Spin sont fortement reliées aux propriétés du spectre d'un opérateur fondamental dit *l'opérateur de Dirac*. Cet opérateur différentiel linéaire d'ordre 1 agit sur les sections d'un fibré vectoriel appelé *le fibré des spineurs* qui est comparable à une "racine carrée" du fibré des formes différentielles mais il s'en distingue par sa dépendance par rapport à la métrique. C'est surtout à travers la formule de Schrödinger-Lichnerowicz [Lich63] que l'opérateur de Dirac se révèle porter des renseignements sur la géométrie et la topologie de la variété. Cette formule montre que la différence entre le carré de l'opérateur de Dirac et le laplacien spinoriel est proportionnelle à la courbure scalaire. Ainsi, avec la condition faible de positivité stricte de la courbure scalaire, A. Lichnerowicz [Lich63] en a déduit la trivialité du noyau de l'opérateur de Dirac. Ce fait, combiné avec le théorème de l'indice d'Atiyah-Singer [Atiy-Sing68], fournit une obstruction topologique à l'existence de métriques à courbure scalaire strictement positive.

Raffinant l'argument de A. Lichnerowicz, T. Friedrich [Fri80] a minoré le carré de toute valeur propre de l'opérateur de Dirac par un nombre proportionnel à l'infimum de la courbure scalaire. Le cas limite est caractérisé par l'existence d'un *spineur de Killing réel :* c'est une section du fibré des spineurs dont la dérivée covariante est proportionnelle à la multiplication de Clifford. L'existence d'un tel spineur entraine des restrictions sur la variété. Par exemple, celle-ci doit être d'Einstein, et en dimension 4 doit être à courbure sectionnelle constante. La caractérisation des variétés Spin admettant des spineurs de Killing réels [Bär93] fait apparaître en certaines dimensions d'autres exemples que la sphère. Ces exemples intéressent les physiciens en relativité générale où l'opérateur de Dirac joue un rôle important.

L'étude du spectre de l'opérateur de Dirac sur des sous-variétés des variétés riemanniennes Spin a été l'objet de plusieurs travaux. Même si la sous-variété est Spin, des difficultés apparaissent. Il est bien connu que la restriction du fibré des spineurs sur une sous-variété Spin est un fibré hermitien, donné par le produit tensoriel du fibré des spineurs extrinsèque de la sous-variété avec un certain fibré associé au fibré normal de l'immersion [Bär98, Gi-Mo00, HMZ01a, HMZ01b, Gin09]. En général, on n'a pas d'informations précises sur ce fibré hermitien à part dans le cas où le fibré normal est trivial, comme par exemple dans le cas des hypersurfaces. On sait dans ce cas, que la restriction du fibré des spineurs ambiant ou une partie du fibré des spineurs ambiant est le fibré des spineurs de l'hypersurface et l'opérateur de Dirac induit n'est autre que l'opérateur de Dirac de l'hypersurface.

T. Friedrich [Fri98] a caractérisé les surfaces simplement connexes immergées isométriquement dans $\mathbb{R}^3$ par l'existence d'un champ de spineur dit *spineur de Killing généralisé*. Ce spineur provient de la restriction à la surface d'un spineur parallèle de $\mathbb{R}^3$ et sa dérivée covariante est proportionnelle à la multiplication de Clifford d'un champ d'endomorphismes symétriques dit *le tenseur d'énergie-impulsion*. En 2000, O. Hijazi, S. Montiel et X. Zhang [HMZ01a, HMZ01b] ont minoré, à une constante près, la première valeur propre strictement positive de l'opérateur de Dirac défini sur le bord compact d'une variété Spin à courbure scalaire positive, par la courbure moyenne du bord supposée positive. Comme application du cas limite, ils donnent une preuve spinorielle élémentaire du théorème d'Alexandrov.

Actuellement, la géométrie spinorielle complexe Spin^c est devenue un domaine de recherche actif suite à la théorie de Seiberg-Witten [KM94, Wit94, Sei-Wit94, Fri00]. Cette théorie repose essentiellement sur le fait que toute variété riemannienne compacte orientée de dimension 4 possède une structure Spin^c. Les applications en topologie et en géométrie de dimension 4 de cette théorie sont célèbres : plusieurs théorèmes qui découlent des travaux de Donaldson retrouvent leurs preuves d'une manière assez élémentaire [Don96]. C. LeBrun [LeB95, LeB96] a obtenu des restrictions topologiques sur les variétés d'Einstein de dimension 4 et avec M.J. Gursky [GL98], ils ont calculé l'invariant de Yamabe pour certaines variétés de dimension 4 comme l'espace projectif complexe $\mathbb{C}P^2$.

D'un point de vue intrinsèque, les variétés Spin, presque complexes, complexes, kählériennes, de Sasaki et certaines variétés CR possèdent une structure Spin^c canonique. Par exemple, en utilisant les structures Spin^c, A. Moroianu [Moro99] a montré la conjecture de Lichnerowicz sur les variétés kählériennes Spin limites pour l'inégalité de Kirchberg en dimension complexe paire.

En 2006, O. Hijazi, S. Montiel et F. Urbano [HMU06] ont construit sur les variétés Kähler-Einstein à courbure scalaire positive, une structure Spin^c admettant des spineurs de Killing kählériens. La restriction de ces spineurs sur les sous-variétés lagrangiennes minimales fournit des informations topologiques et géométriques sur ces sous-variétés. Il semble que la restriction des spineurs Spin^c est un outil efficace et même incontournable pour étudier la géométrie et la topologie des sous-variétés et spécialement des hypersurfaces. C'est même un cadre plus naturel que la cadre Spin qui est devenu maintenant classique.

Dans un premier temps, on rappelle les notions de base de la géométrie spinorielle complexe en adoptant le point de vue de S. Montiel [Mon05], pour ensuite citer les résultats obtenus durant la thèse.

Structures Spin^c et survol sur les résultats obtenus

Sur une variété riemannienne compacte (M^n, g), de dimension n, on considère un fibré vectoriel complexe ΣM de rang l muni d'un produit scalaire hermitien $\langle .,. \rangle$ et d'une connexion ∇ compatible avec ce produit scalaire, i.e.

$$X(\langle \psi, \phi \rangle) = \langle \nabla_X \psi, \phi \rangle + \langle \psi, \nabla_X \phi \rangle \,,$$

pour tous $X \in \Gamma(TM)$ et $\psi, \phi \in \Gamma(\Sigma M)$. On suppose qu'il existe une application $C^\infty(M)$-linéaire

$$\gamma : TM \longrightarrow \mathrm{End}(\Sigma M),$$

qui à chaque vecteur X tangent à M, définit un endomorphisme $\gamma(X)$ de ΣM. On note $(\Sigma M, \langle .,. \rangle, \nabla, \gamma)$ tout fibré vectoriel complexe sur M vérifiant les conditions ci-dessus. Sous ces hypothèses générales, on définit un opérateur naturel elliptique d'ordre 1 par

$$\begin{array}{rcl} D : \Gamma(\Sigma M) & \longrightarrow & \Gamma(\Sigma M) \\ \psi & \longmapsto & D\psi := \displaystyle\sum_{j=1}^{n} \gamma(e_j) \nabla_{e_j} \psi, \end{array}$$

où $\{e_1, \cdots, e_n\}$ est un repère local g-orthonormé. Pour que l'opérateur D soit auto-adjoint par rapport au produit scalaire L^2 de $\Gamma(\Sigma M)$, il faut que l'application γ satisfasse les deux conditions suivantes :

$$\langle \gamma(X)\psi, \phi \rangle = - \langle \psi, \gamma(X)\phi \rangle \,, \tag{1}$$
$$\nabla_X(\gamma(Y)\psi) = \gamma(\nabla_X Y)\psi + \gamma(Y)\nabla_X \psi, \tag{2}$$

pour tous $X, Y \in \Gamma(TM)$ et $\psi, \phi \in \Gamma(\Sigma M)$. La seconde dérivée covariante ∇ est la dérivée covariante de Levi-Civita sur M. Si on demande que l'opérateur elliptique d'ordre deux D^2 ait un symbole principal égal à celui du laplacien $\triangle$ défini sur $\Gamma(\Sigma M)$, il faut imposer à γ la condition suivante :

$$\gamma(X)\gamma(Y) + \gamma(Y)\gamma(X) = -2g(X, Y), \tag{3}$$

pour tous $X, Y \in \Gamma(TM)$. Dans ce cas, on a

$$D^2 = \triangle + \frac{1}{2} \sum_{j,k=1}^{n} \gamma(e_j)\gamma(e_k)\mathcal{R}(e_j, e_k),$$

où $\mathcal{R}$ est le tenseur de courbure associé à la connexion ∇ définie sur ΣM. Un fibré de Dirac sur M est un fibré vectoriel complexe $(\Sigma M, \langle .,. \rangle, \nabla, \gamma)$ sur M vérifiant les conditions (1), (2) et (3). Dans ce cas, γ est appelée la multiplication de Clifford et D

l'opérateur de Dirac associé à ce fibré de Dirac.

L'exemple le plus simple d'un fibré de Dirac qu'on peut considérer sur une variété riemannienne (M^n, g) est le fibré $\Sigma M = \Lambda^* M \otimes_{\mathbb{R}} \mathbb{C}$ des formes complexes sur M. En effet, la métrique riemannienne de M induit un produit scalaire naturel sur les formes complexes de n'importe quel degré et la connexion de Levi-Civita s'étend de manière unique pour agir sur les formes. Cette connexion et ce produit scalaire sont compatibles. Pour tous $X \in \Gamma(TM)$ et $\omega \in \Gamma(\Lambda^* M \otimes_{\mathbb{R}} \mathbb{C})$, on définit une application γ par

$$\gamma(X)\omega = X^\flat \wedge \omega - X \lrcorner \omega,$$

où $\lrcorner$ est le produit intérieur et $\wedge$ le produit extérieur. Il est facile de démontrer que γ vérifie les équations (1), (2) et (3) et donc $\Lambda^* M \otimes_{\mathbb{R}} \mathbb{C}$ est un fibré de Dirac de rang 2^n. De plus, l'opérateur de Dirac associé à ce fibré n'est autre que l'opérateur d'Euler et son carré est le fameux opérateur dit le laplacien de Hodge, i.e.

$$D = d + \delta \quad \text{et} \qquad D^2 = \triangle^H,$$

où d est la différentielle extérieure, δ son adjoint par rapport au produit scalaire L^2 et $\triangle^H$ est le laplacien de Hodge agissant sur les formes différentielles.

On a appelé γ la multiplication de Clifford puisque la relation d'anticommutativité (3) qu'on a imposé est la même que celle qui définit l'algèbre de Clifford associée à un espace vectoriel muni d'un produit scalaire. Donc on peut étendre naturellement γ au fibré de Clifford $\mathbb{C}l(TM)$, i.e. au fibré vectoriel dont les fibres sont les algèbres de Clifford complexes construites sur l'espace tangent en tout point de la variété :

$$\gamma : \mathbb{C}l(TM) \longrightarrow \mathrm{End}(\Sigma M).$$

Par suite, en tout point $x \in M$, $(\gamma_x, \Sigma_x M)$ est une représentation complexe de l'algèbre de Clifford $\mathbb{C}l(T_x M) \simeq \mathbb{C}l_n$. En général, cette représentation n'est pas irréductible mais il est connu qu'une représentation complexe de $\mathbb{C}l_n$ est irréductible si et seulement si elle est de dimension complexe $2^{[\frac{n}{2}]}$ [LM89, Fri00, Hij01]. On en déduit que le rang l d'un fibré de Dirac est toujours supérieur ou égal à $2^{[\frac{n}{2}]}$.

La question est : sur une variété riemannienne (M^n, g) existe-t-il un fibré de Dirac ΣM de rang minimal, i.e. $l = 2^{[\frac{n}{2}]}$? En d'autre termes, existe t-il sur (M^n, g) un fibré de Dirac ΣM tel que, en tout point $x \in M$, $(\gamma_x, \Sigma_x M)$ est une représentation complexe irréductible de $\mathbb{C}l(T_x M)$? Si un tel fibré de Dirac existe, on l'appelle *un fibré des spineurs* et une section de ce fibré sera dite un champ de spineurs. Il faut noter que, si on a *un fibré des spineurs* $(\Sigma M, \langle ., . \rangle, \nabla, \gamma)$ sur M, on peut construire plusieurs autres en prenant le produit tensoriel de ΣM par un fibré complexe en droites $\mathcal{D}$ muni d'un produit scalaire et d'une connexion compatible avec ce produit scalaire. En effet, $\Sigma' M = \Sigma M \otimes \mathcal{D}$ est aussi *un fibré des spineurs.*

Une variété riemannienne orientée (M^n, g) possède une structure Spin^c si et seulement s'il existe sur M un fibré complexe en droites L tel que

$$[c_1(L)]_{mod\ 2} = \omega_2(M),$$

où $\omega_2(M)$ est la deuxième classe de Stiefel-Whitney de M et $c_1(L)$ est la première classe de Chern de L. Cela est équivalent à dire que le $\mathrm{SO}_n\times\mathbb{S}^1$-fibré principal $P_{\mathrm{SO}_n}M\times_M P_{\mathbb{S}^1}M$ admet un revêtement à deux feuillets non trivial. Ici, $P_{\mathrm{SO}_n}M$ désigne le SO_n-fibré des repères orthonormés de M et $P_{\mathbb{S}^1}M$ est le $\mathbb{S}^1$-fibré principal correspondant au fibré en droites L. Dans le cas particulier où le fibré en droites L a une racine carrée, i.e. $\omega_2(M)=0$, une structure Spin^c est dite une structure Spin.

Sur une variété riemannienne orientée (M^n,g), la donnée d'*un fibré des spineurs* permet de montrer que le fibré déterminant $\det\Sigma M$ admet une racine d'ordre $2^{[\frac{n}{2}]-1}$. On note L ce fibré complexe en droites, i.e.

$$L=(\det\Sigma M)^{2^{1-[\frac{n}{2}]}}.$$

Ce fibré en droites, appelé fibré auxiliaire, vérifie

$$[c_1(L)]_{mod\ 2}=\omega_2(M),$$

et donc la variété M admet une structure Spin^c. Inversement, si on fixe une structure Spin^c sur M, i.e. s'il existe sur M un fibré complexe en droites L tel que $[c_1(L)]_{mod\ 2}=\omega_2(M)$, on peut construire [Mon05], à un isomorphisme près, un unique *fibré des spineurs* ΣM. Par conséquent, *un fibré des spineurs* sera appelé le fibré des spineurs Spin^c associé à une structure Spin^c donnée. Quand la structure Spin^c est Spin, le fibré des spineurs Spin^c est dit le fibré des spineurs.

Lorsque M est une variété Spin, le fibré des spineurs peut être choisi de sorte que le fibré auxiliaire soit trivial. En effet, supposons qu'il existe un fibré complexe en droites E tel que $L=E^2$. Le fibré vectoriel $\Sigma^{'}M=\Sigma M\otimes E^{-1}$ est aussi *un fibré des spineurs* dont le fibré auxiliaire $L^{'}$ est relié à L par $L^{'}=L\otimes E^{-2}$, donc il est trivial puisque $L^{'}=L\otimes E^{-2}=L\otimes L^{-1}=1$.

Considérons une variété riemannienne (M^n,g) munie d'une structure Spin^c et ΣM son fibré des spineurs Spin^c. Localement, le fibré des spineurs existe toujours. On note $\Sigma^{'}M$ le fibré des spineurs dont le fibré auxiliaire $L^{'}$ est trivial et on rappelle qu'il existe un fibré complexe en droites $\mathcal{D}$ tel que $\Sigma^{'}M=\Sigma M\otimes\mathcal{D}$ et $L^{'}=L\otimes\mathcal{D}^2$. Localement et comme $L^{'}$ est trivial, on a $\mathcal{D}^2=L^{-1}$. Ainsi, $\Sigma M=\Sigma^{'}M\otimes L^{\frac{1}{2}}$. Cela signifie que même si le fibré des spineurs et $L^{\frac{1}{2}}$ peuvent ne pas exister globalement, leur produit tensoriel (le fibré des spineurs Spin^c) est défini globalement.

Supposons maintenant que M^n $(n=2m)$ admet une structure kählérienne. La complexification $TM\otimes_{\mathbb{R}}\mathbb{C}$ de l'espace tangent se décompose en une somme directe de deux sous-espaces propres associés aux valeurs propres $\pm i$ de l'extension de la structure complexe J à $TM\otimes_{\mathbb{R}}\mathbb{C}$. En effet,

$$TM\otimes_{\mathbb{R}}\mathbb{C}=T_{1,0}M\oplus T_{0,1}M,$$

où $T_{1,0}M=\overline{T_{0,1}M}=\{X-iJX,\ X\in\Gamma(TM)\}$. Cette décomposition induit une autre décomposition des p-formes complexes sur M $(p=0,...,m)$

$$\Lambda^pM\otimes_{\mathbb{R}}\mathbb{C}=\oplus_{r+s=p}\,\Lambda^{r,s}M,$$

où $\Lambda^{r,s}M = \Lambda^{r,0}M \otimes \Lambda^{0,s}M$ et $\Lambda^{r,0}M = \Lambda^r(T^*_{1,0}M)$. Le fibré complexe $\Lambda^{0,*}M = \oplus_{r=0}^m \Lambda^{0,r}M$, de rang 2^m, est muni d'un produit scalaire (l'extension de la métrique) et d'une connexion (l'extension de la connexion de Levi-Civita sur M) qui sont compatibles. De plus, l'application $\gamma : TM \longrightarrow \mathrm{End}(\Lambda^{0,*}M)$ définie par

$$\gamma(X)\psi = \frac{1}{\sqrt{2}}(X + iJX)^\flat \wedge \psi - \sqrt{2}X \lrcorner \psi,$$

satisfait les conditions (1), (2) et (3) pour tous $X \in \Gamma(TM)$, $\psi \in \Gamma(\Lambda^{0,r}M)$ et $r = 0, ..., m$. Par suite, toute variété kählérienne admet une structure Spin^c. Cette structure possède des spineurs parallèles (les fonctions complexes constantes).

Après ces préliminaires sur la géométrie spinorielle complexe, donnons une brève description des résulats obtenus.

Sur une variété Spin, le spectre de l'opérateur de Dirac a été largement étudié puisqu'il se révèle porter des renseignements subtils sur la géométrie et la topologie de la variété. Sur les variétés Spin^c, A. Moroianu and M. Herzlich ont montré une estimation de type Friedrich pour les valeurs propres de l'opérateur de Dirac [HM99] : sur une variété riemannienne (M^n, g) compacte et Spin^c , toute valeur propre λ de l'opérateur de Dirac satisfait

$$\lambda^2 \geqslant \lambda_1^2 := \frac{n}{4(n-1)} \inf_M (S - c_n|\Omega|), \tag{4}$$

où $c_n = 2[\frac{n}{2}]^{\frac{1}{2}}$, S est la courbure scalaire de (M^n, g) et $i\Omega$ est la 2-forme de courbure associée à une connexion fixée sur le fibré auxiliaire L de la structure Spin^c. Le cas d'égalité est caractérisé par l'existence d'un spineur de Killing Spin^c ψ vérifiant

$$\gamma(\Omega)\psi = i\frac{c_n}{2}|\Omega|\psi, \tag{5}$$

où $\gamma(\Omega)$ est l'extension de la multiplication de Clifford aux formes différentielles. Notons que l'inégalité (4) donne une information sur le spectre si $S - c_n|\Omega| > 0$. L'idée introduite par O. Hijazi [Hij95] est de modifier la connexion ∇ dans la direction d'un tenseur symétrique. Ainsi, on obtient une estimation optimale pour les valeurs propres de l'opérateur de Dirac en fonction d'un tenseur symétrique appelé le tenseur d'énergie-impulsion. Cette estimation implique l'inégalité (4) et a un intérêt même si $S - c_n|\Omega| \leqslant 0$. Le cas d'égalité dans cette nouvelle estimation est caractérisé par l'existence d'un *spineur de Killing* Spin^c *généralisé* vérifiant l'équation (5). Ce type de spineurs joue un rôle important dans l'étude extrinsèque des structures Spin^c.

Le tenseur d'énergie-impulsion apparaît dans différentes situations géométriques. En effet, sur les variétés Spin, J.P. Bourguignon et P. Gauduchon [BG92] ont montré que le tenseur d'énergie-impulsion intervient naturellement dans l'étude des variations du spectre de l'opérateur de Dirac. T. Friedrich et E.C. Kim [FK01] ont obtenu les équations de Dirac-Einstein comme les équations d'Euler-Lagrange d'une certaine fonctionnelle. Nous démontrons ces résultats sur les variétés Spin^c. Même si ce n'est pas un invariant géométrique, le tenseur d'énergie-impulsion est, à une constante près, la seconde forme fondamentale d'une immersion isométrique dans une variété Spin^c admettant un spineur parallèle. Enfin, sur les surfaces Spin^c, nous exprimons ce tenseur en termes d'invariants topologiques, comme le nombre d'Euler-Poincaré.

Les structures Spin^c sont des structures naturelles sur certaines classes de variétés, comme les variétés de dimension 2 ou 4, les variétés kählériennes, les variétés de Sasaki et certains types de variétés CR. Une hypersurface réelle d'une variété Spin^c a une structure Spin^c. Nous nous focalisons sur l'étude des structures Spin^c sur les hypersurfaces réelles de $\mathbb{C}P^2$ et $\mathbb{E}(\kappa,\tau)$. L'espace projectif complexe $\mathbb{C}P^2$, qui n'est pas une variété Spin, a une structure Spin^c naturelle, dite structure Spin^c canonique. Cette structure porte un spineur parallèle. *Lorsque nous restreignons ce spineur parallèle à une hypersurface M, nous obtenons un spineur de Killing* Spin^c *généralisé qui caractérisera l'immersion de M dans $\mathbb{C}P^2$.*

Les variétés $\mathbb{E}(\kappa,\tau)$ sont les variétés homogènes de dimension 3 dont le groupe d'isométries est de dimension 4 ($\mathbb{S}^2\times\mathbb{R}$, $\mathbb{H}^2\times\mathbb{R}$, Nil_3,...). Ces variétés sont Spin avec un champ de spineurs spécial ψ. J. Roth a prouvé que, sous certaines conditions supplémentaires, la restriction de ψ sur une surface, permet de caractériser l'immersion de la surface dans $\mathbb{E}(\kappa,\tau)$. Mais les variétés $\mathbb{E}(\kappa,\tau)$ ont également une structure Spin^c portant un spineur de Killing Spin^c, dont la restriction donne lieu à un champ de spineurs spécial ϕ, qui permet de caractériser l'immersion de M dans $\mathbb{E}(\kappa,\tau)$ *sans aucune hypothèse supplémentaire.* À partir de cette caractérisation nous obtenons *une preuve spinorielle de la correspondance généralisée de Lawson pour les surfaces à courbure moyenne constante dans* $\mathbb{E}(\kappa,\tau)$. Notons qu'il n'existe pas de preuve de ce résultat en utilisant les structures Spin sur $\mathbb{E}(\kappa,\tau)$.

Les spineurs sont devenus un outil efficace dans l'étude des hypersurfaces. En fait, O. Hijazi, S. Montiel et X. Zhang ont montré que si M^n est une hypersurface délimitant un domaine d'une variété Spin portant un spineur parallèle et si la courbure scalaire de la variété ambiante est positive et la courbure moyenne H positive, alors la première valeur propre strictement positive λ_1 de l'opérateur de Dirac extrinsèque satisfait

$$\lambda_1 \geqslant \frac{n}{2}\inf_M H.$$

Comme application, ils donnent une preuve spinorielle du théorème d'Alexandrov. Sur les variétés Spin^c, nous établissons une estimation similaire. Les hypersurfaces réelles compactes plongées dans $\mathbb{C}P^m$, à courbure moyenne constante et strictement positive, sont des exemples de variétés où l'égalité est atteinte.

Parmi toutes les variétés munies d'une structure Spin^c naturelle avec un champ de spineurs spécial, les variétés kählériennes, de Sasaki et, comme cela a été remarqué plus récemment, les variétés CR jouent un rôle central. Nous montrons que, sur une variété M, l'existence d'une structure Spin^c avec un champ de spineurs spécial, appelé *spineur pur* ou bien *spineur transversal*, est équivalente à l'existence d'une structure complexe ou bien d'une structure CR sur M. En outre, l'existence d'un champ de spineurs transversal sur une variété riemannienne Spin^c conditionne sa géométrie. Plus précisément, *l'existence d'un spineur parallèle ou de Killing transversal sur M implique que M est une variété feuilletée.*

Dans les sections suivantes, on détaille les résultats obtenus qui ont fait l'objet des chapitres 2, 3, 4, 5, 6 et 7.

Le spectre de l'opérateur de Dirac et le tenseur d'énergie-impulsion

L'étude du spectre de l'opérateur de Dirac, défini sur une variété Spin, a fait l'objet d'investigations intenses du fait qu'il contient des informations subtiles sur la topologie et la géométrie de la variété. La formule de Schrödinger-Lichnerowicz [Lich63] reliant le carré de l'opérateur de Dirac au laplacien spinoriel est donnée par :

$$D^2 = \triangle + \frac{1}{4}S.$$

Plusieurs conséquences découlent de cette formule. Tout d'abord, si la variété est à courbure scalaire strictement positive, le noyau de l'opérateur de Dirac est trivial. Dans ce cas, par le théorème d'Atiyah-Singer, les indices topologiques et analytiques de l'opérateur de Dirac sont nuls et toute valeur propre λ vérifie

$$\lambda^2 > \frac{1}{4}\inf_M S.$$

En 1980, T. Friedrich a minoré le carré de n'importe quelle valeur propre λ, à une constante près, par l'infimum de la courbure scalaire supposée positive. Plus précisément, toute valeur propre λ de l'opérateur de Dirac satisfait [Fri80]

$$\lambda^2 \geqslant \lambda_1^2 := \frac{n}{4(n-1)}\inf_M S. \tag{6}$$

Le cas d'égalité est caractérisé par l'existence d'un spineur de Killing. C'est une section ψ du fibré des spineurs qui vérifie, pour tout $X \in \Gamma(TM)$,

$$\nabla_X\psi = -\frac{\lambda_1}{n}X\cdot\psi,$$

où on a noté "$\cdot$" la multiplication de Clifford γ. Comme conséquence de l'existence des spineurs de Killing, la variété est d'Einstein [Fri80]. C. Bär [Bär93] a caractérisé toutes les variétés Spin simplement connexes admettant de tels spineurs.

L'inconvénient avec l'estimation de Friedrich est qu'elle n'apporte aucune information sur le spectre de l'opérateur de Dirac et sur la géométrie de la variété dans le cas où la courbure scalaire est négative ou nulle. D'où l'idée de O. Hijazi [Hij95] de modifier la connexion dans la direction d'un tenseur symétrique. Ainsi, sur le complémentaire des zéros d'un spineur ψ, O. Hijazi a défini un 2-tenseur symétrique ℓ^ψ, dit tenseur d'énergie-impulsion associé à ψ, par

$$\ell^\psi(X,Y) = g(\ell^\psi(X),Y) = \frac{1}{2}\mathrm{Re}\left\langle X\cdot\nabla_Y\psi + Y\cdot\nabla_X\psi, \frac{\psi}{|\psi|^2}\right\rangle,$$

pour tous $X, Y \in \Gamma(TM)$. Il montre que, pour tout spineur propre ψ associé à une valeur propre λ, on a [Hij95]

$$\lambda^2 \geqslant \lambda_1^2 := \inf_M(\frac{1}{4}S + |\ell^\psi|^2). \tag{7}$$

Il faut noter que C. Bär a montré que, comme le spineur ψ est un spineur propre, l'ensemble de ses zéros est de dimension de Hausdorff au plus égale à $n-2$ et donc de mesure nulle [Bär99]. La trace de ℓ^ψ étant égale à λ, l'inégalité (7) améliore celle de Friedrich puisque, par l'inégalité de Cauchy-Schwarz, $|\ell^\psi|^2 \geqslant \frac{1}{n}(\mathrm{tr}(\ell^\psi))^2 = \frac{1}{n}\lambda^2$, où $\mathrm{tr}(\ell^\psi)$ est la trace de ℓ^ψ. Le cas limite de (7) est caractérisé par l'existence d'un spineur de Killing généralisé, i.e. une section ψ du fibré des spineurs qui satisfait, pour tout $X \in \Gamma(TM)$, l'équation

$$\nabla_X\psi = -\ell^\psi(X)\cdot\psi.$$

N. Ginoux et G. Habib [GH09] ont montré que la variété d'Heisenberg est une variété compacte limite pour (7) mais que l'on n'a pas égalité dans (6). Inversement, si sur une variété Spin, on suppose qu'il existe un spineur ψ vérifiant

$$\nabla_X\psi = -E(X)\cdot\psi, \tag{8}$$

où E est un endomorphisme symétrique de TM, il est simple de voir par les propriétés de la multiplication de Clifford que $E = \ell^\psi$ et on est dans le cas limite de (7) [Mor02]. Si la dimension de M est égale à 2, T. Friedrich [Fri98] a prouvé que l'existence d'une paire (ψ, E) satisfaisant (8) est équivalente à l'existence d'une immersion locale de M dans l'espace euclidien $\mathbb{R}^3$ avec un tenseur de Weingarten égal à $2E$. Plus tard, G. Habib [Hab07] a étudié l'équation (8) pour un endomorphisme E qui n'est pas nécessairement symétrique. Il a montré que la partie symétrique de E est ℓ^ψ et la partie antisymétrique de E est q^ψ définie sur le complémentaire de l'ensemble des zéros de ψ par

$$q^\psi(X,Y) = \frac{1}{2}\mathrm{Re}\left\langle Y\cdot\nabla_X\psi - X\cdot\nabla_Y\psi, \frac{\psi}{|\psi|^2}\right\rangle,$$

pour tous $X, Y \in \Gamma(TM)$. Puis, il a établi l'estimation suivante

$$\lambda^2 \geqslant \inf_M(\frac{1}{4}S + |\ell^\psi|^2 + |q^\psi|^2), \tag{9}$$

pour toute valeur propre λ à laquelle est attachée un spineur propre ψ. Pour une meilleure compréhension du tenseur q^ψ, il a étudié les flots riemanniens et a montré que si le fibré normal porte un spineur parallèle, le tenseur q^ψ est le tenseur d'O'Neill du flot.

Dans le cadre des variétés Spin^c, la formule de Schrödinger-Lichnerowicz est donnée par [LM89]

$$D^2 = \triangle + \frac{1}{4}S + \frac{i}{2}\Omega\cdot.$$

Par l'inégalité de Cauchy-Schwarz, on a [HM99]

$$\langle i\Omega\cdot\psi, \psi\rangle \geqslant -\frac{c_n}{2}|\Omega||\psi|^2.$$

Cette inégalité combinée avec la formule de Schrödinger-Lichnerowicz permet de montrer l'inégalité (4) dite de type Friedrich Spin^c. Le cas d'égalité dans (4) est caractérisé par l'existence d'un spineur de Killing Spin^c vérifiant l'équation (5), i.e. une section $\psi \in \Gamma(\Sigma M)$ vérifiant pour tout $X \in \Gamma(TM)$,

$$\nabla_X\psi = -\frac{\lambda_1}{n}X\cdot\psi \quad \text{et} \quad \Omega\cdot\psi = i\frac{c_n}{2}|\Omega|\psi.$$

Dans ce cas aucune géométrie particulière n'est obtenue sur la variété Spin^c du fait que l'estimation dépend de la 2-forme de courbure associée à une connexion fixée sur le fibré auxiliaire.

On étend l'inégalité de Hijazi pour les valeurs propres de l'opérateur de Dirac sur une variété Spin^c.

Theorem 0.0.1. [*Nak10*] *Sur une variété riemannienne compacte* Spin^c *de dimension $n \geqslant 2$, toute valeur propre λ de l'opérateur de Dirac à laquelle est attachée un spineur propre ψ, satisfait*

$$\lambda^2 \geqslant \lambda_1^2 := \inf_M\left(\frac{1}{4}S - \frac{c_n}{4}|\Omega| + |\ell^\psi|^2\right). \tag{10}$$

Le cas d'égalité dans (10) est caractérisé par l'existence d'un spineur de Killing Spin^c généralisé vérifiant l'équation (5), i.e. une section $\psi \in \Gamma(\Sigma M)$ vérifiant pour tout $X \in \Gamma(TM)$,

$$\nabla_X\psi = -\ell^\psi(X)\cdot\psi \quad \text{et} \quad \Omega\cdot\psi = i\frac{c_n}{2}|\Omega|\psi.$$

La trace de ℓ^ψ étant égale à λ, l'inégalité (10) améliore l'inégalité (4). En guise d'exemple, la sphère $\mathbb{S}^3$, équipée d'une structure Spin^c spéciale, est une variété limite pour (10) mais on n'a pas l'égalité dans (4). Les spineurs de Killing Spin^c généralisés seront étudiés dans les sections suivantes. Sur les variétés Spin^c, l'inégalité (9) est donnée par

$$\lambda^2 \geqslant \inf_M \left(\frac{1}{4}S - \frac{c_n}{4}|\Omega| + |\ell^\psi|^2 + |q^\psi|^2\right). \tag{11}$$

Dans un premier temps, on ne considère la déformation de la connexion que dans la direction de l'endomorphisme symétrique ℓ^ψ.

Sur les variétés Spin, en utilisant la covariance conforme de l'opérateur de Dirac, O. Hijazi [Hij86] a montré que, en dimension $n \geqslant 3$, toute valeur propre de l'opérateur de Dirac vérifie

$$\lambda^2 \geqslant \lambda_1^2 := \frac{n}{4(n-1)}\mu_1, \tag{12}$$

où μ_1 est la première valeur propre de l'opérateur de Yamabe donné par

$$L := 4\frac{n-1}{n-2} \triangle + S,$$

où $\triangle$ est le laplacien agissant sur les fonctions. En dimension 2, C. Bär [Bär92] a montré que toute valeur propre de l'opérateur de Dirac Spin vérifie

$$\lambda^2 \geqslant \frac{2\pi\chi(M)}{\mathrm{Aire}(M,g)}, \tag{13}$$

où $\chi(M)$ est le nombre d'Euler-Poincaré de la surface M. Le cas limite de (12) et (13) est aussi caractérisé par l'existence d'un spineur de Killing. En terme de tenseur d'énergie-impulsion, O. Hijazi [Hij95] a montré que, sur de telles variétés, toute valeur propre de Dirac satisfait

$$\lambda^2 \geqslant \begin{cases} \frac{\pi\chi(M)}{\mathrm{Aire}(M,g)} + \inf\limits_M |\ell^\psi|^2 & \text{si} \quad n = 2, \\ \\ \frac{1}{4}\mu_1 + \inf\limits_M |\ell^\psi|^2 & \text{si} \quad n \geqslant 3. \end{cases} \tag{14}$$

Encore une fois, la trace de ℓ^ψ étant égale à λ, l'inégalité (14) implique les deux inégalités (12) et (13). Le cas limite de (14) est caractérisé par l'existence d'un champ de spineurs $\overline{\varphi}$ vérifiant, pour tout $X \in \Gamma(TM)$,

$$\overline{\nabla}_X \overline{\varphi} = -\ell^{\overline{\varphi}}(X) \,\bar{\cdot}\, \overline{\varphi}, \tag{15}$$

où $\overline{\varphi} = e^{-\frac{n-1}{2}u}\overline{\psi}$, le champ de spineurs ψ est un spineur propre associé à la première valeur propre de l'opérateur de Dirac et $\overline{\psi}$ est l'image de ψ par l'isométrie entre les fibrés des spineurs de (M^n, g) et $(M^n, \overline{g} = e^{2u}g)$.

En 1999, A. Moroianu et M. Herzlich ont montré que sur les variétés Spin^c de dimension $n \geqslant 3$, toute valeur propre de l'opérateur de Dirac satisfait [HM99]

$$\lambda^2 \geqslant \lambda_1^2 := \frac{n}{4(n-1)}\mu_1, \tag{16}$$

où μ_1 est la première valeur propre de l'opérateur de Yamabe tordu,

$$L^\Omega = L - c_n|\Omega|.$$

Le cas limite de (16) est caractérisé par l'existence d'un spineur de Killing Spin^c vérifiant l'équation (5). En fonction du tenseur d'énergie-impulsion, on montre :

Theorem 0.0.2. [*Nak10*] *Sur une variété riemannienne* Spin^c*, toute valeur propre* λ *de l'opérateur de Dirac à laquelle est attachée un spineur propre* ψ *satisfait*

$$\lambda^2 \geqslant \begin{cases} \frac{\pi\chi(M)}{\mathrm{Aire}(M,g)} - \frac{1}{2}\frac{\int_M |\Omega| v_g}{\mathrm{Aire}(M,g)} + \inf_M |\ell^\psi|^2 & si \quad n = 2, \\ \\ \frac{1}{4}\mu_1 + \inf_M |\ell^\psi|^2 & si \quad n \geqslant 3, \end{cases} \tag{17}$$

où μ_1 *est la première valeur propre de l'opérateur de Yamabe tordu.*

Comme corollaire du Théorème 0.0.2, on exprime, en dimension 4, l'estimation en fonction d'un invariant conforme (le nombre de Yamabe) et d'un invariant topologique. En effet, on a :

Corollary 0.0.1. *Sur une variété compacte* Spin^c *de dimension* 4 *dont la* 2*-forme de courbure* $i\Omega$ *est autoduale, toute valeur propre de l'opérateur de Dirac satisfait*

$$\lambda^2 \geqslant \frac{1}{4}\mathrm{vol}(M,g)^{-\frac{1}{2}}\Big(Y(M,[g]) - 4\pi\sqrt{2}\sqrt{c_1(L)^2}\Big) + \inf_M |\ell^\psi|^2,$$

où $c_1(L)$ *est le nombre de Chern du fibré auxiliaire* L *associé à la structure* Spin^c*.*

Le problème du tenseur d'énergie-impulsion est lié aux problèmes de variations du spectre de l'opérateur de Dirac. En effet :

Proposition 0.0.1. [*Nak11a*] *Soit* (M^n, g) *une variété riemannienne* Spin^c *et* $g_t = g + tk$ *une famille lisse de métriques. Pour tout champ de spineurs* $\psi \in \Gamma(\Sigma M)$*, on a*

$$\frac{d}{dt}\bigg|_{t=0} \int_M \mathrm{Re}\,\left\langle D^{M_t}\tau_0^t\psi, \tau_0^t\psi\right\rangle_{g_t} v_g = -\frac{1}{2}\int_M \langle k, \ell_\psi\rangle\, v_g,$$

où D^{M_t} *est l'opérateur de Dirac associé à la variété* $M_t = (M, g_t)$*,* $\ell_\psi(X) = |\psi|^2\,\ell^\psi(X) = \mathrm{Re}\,\langle X \cdot \nabla_X\psi, \psi\rangle$ *et* $\tau_0^t\psi$ *est l'image de* ψ *par l'isométrie* τ_0^t *entre les fibrés des spineurs* Spin^c *de* (M, g) *et de* (M, g_t)*.*

Cela a été prouvé par J.P. Bourguignon et P. Gauduchon [BG92, BGM05] pour les variétés Spin. En s'appuyant sur ce résultat, T. Friedrich et E.C. Kim [KF00] ont montré que, sur une variété Spin, les équations d'Einstein-Dirac sont les équations d'Euler-Lagrange d'une fonctionnelle. On étend ce résultat sur les variétés Spin^c :

Theorem 0.0.3. [*Nak11a*] *Soit* (M^n, g) *une variété riemannienne* Spin^c*. Une paire* (g_0, ψ_0) *est un point critique du lagrangien*

$$\mathcal{W}(g,\psi) = \int_U \Big(S_g + \epsilon\lambda|\psi|_g^2 - \epsilon\mathrm{Re}\,\langle D_g\psi, \psi\rangle\Big) v_g,$$

$(\lambda, \epsilon \in \mathbb{R})$ *pour tout ouvert* U *de* M *si et seulement si* (g_0, ψ_0) *est une solution du système suivant*

$$\begin{cases} D_g\psi = \lambda\psi, \\ \mathrm{ric}_g - \frac{1}{2}S_g\, g = \frac{\epsilon}{2}\ell_\psi, \end{cases}$$

où ric_g *est la courbure de Ricci de* M *considérée comme une forme bilinéaire symétrique.*

En général, nous ne pouvons pas calculer le tenseur d'énergie-impulsion puisqu'il dépend d'un spineur. Mais en petites dimensions et surtout en dimension 2, il peut être exprimé en termes de certains invariants topologiques :

Proposition 0.0.2. [*Ha-Na10*] *Sur une surface compacte munie d'une structure* Spin^c *quelconque, toute valeur propre de l'opérateur de Dirac à laquelle est attachée un spineur propre ψ satisfait*

$$\lambda^2 = \frac{S}{4} + |\ell^\psi|^2 + \Delta f + \left\langle \frac{i}{2}\Omega \cdot \psi, \frac{\psi}{|\psi|^2} \right\rangle,$$

où f est une fonction réelle définie par $f = \frac{1}{2}\ \ln|\psi|^2$.

Comme conséquence directe, nous obtenons [Ha-Na10]

$$\int_M \det(\ell^\psi) v_g \geqslant \frac{\pi\chi(M)}{2} - \frac{1}{4}\int_M |\Omega| v_g, \tag{18}$$

où $\chi(M)$ est le nombre d'Euler-Poincaré de la surface. L'égalité est atteinte si et seulement si Ω est nulle ou si la fonction $\Omega(e_1, e_2)$ a un signe constant pour tout repère g-orthonormé local $\{e_1, e_2\}$. On note que, dans le cas d'égalité, l'expression $\int_M |\Omega| v_g$ est un invariant topologique. Les surfaces de $\mathbb{S}^2 \times \mathbb{R}$ sont des exemples de surfaces où le cas d'égalité dans (18) est atteint. De plus, de la Proposition 0.0.2, on peut déduire facilement l'estimation en dimension 2 du Théorème 0.0.2.

En dimension 3, on montre :

Theorem 0.0.4. [*Ha-Na10*] *Soit (M^3, g) une variété riemannienne compacte munie d'une structure* Spin^c *quelconque. Toute valeur propre λ de l'opérateur de Dirac à laquelle est attachée un spineur propre ψ satisfait*

$$\lambda^2 \leqslant \frac{1}{\mathrm{vol}(M,g)} \int_M \Big(|\ell^\psi|^2 + \frac{S}{4} + \frac{|\Omega|}{2} \Big) v_g. \tag{19}$$

L'égalité est atteinte si et seulement si la norme de ψ est constante et $\Omega \cdot \psi = i|\Omega|\psi$.

La preuve de ce théorème repose essentiellement sur la formule de Schrödinger-Lichnerowicz et une écriture locale de la dérivée covariante de Levi-Civita spinorielle. Les hypersurfaces réelles compactes de $\mathbb{C}P^2$ sont des exemples de variétés de dimension 3 où l'égalité dans (19) est atteinte.

On se limite à ces résultats intrinsèques classiques et nous procédons à l'étude des structures Spin^c d'un point de vue extrinsèque.

Formule de Gauss Spin^c et interprétation géométrique du tenseur d'énergie-impulsion

Dans cette section, nous étudions les structures Spin^c sur les hypersurfaces et comme nous le verrons, ces structures fournissent un cadre naturel pour étudier certains problèmes extrinsèques riemanniens.

Soit $\mathcal{Z}$ une variété riemannienne Spin^c sans bord de dimension $n+1$. Soit $M \subset \mathcal{Z}$ une hypersurface orientée. Par restriction, l'hypersurface M est à son tour munie d'une structure Spin^c et le fibré des spineurs Spin^c sur M est donné par

$$\begin{cases} \Sigma M & \simeq & \Sigma\mathcal{Z}_{|_M} & \text{si } n \text{ est pair,} \\ \Sigma M & \simeq & \Sigma^+\mathcal{Z}_{|_M} & \text{si } n \text{ est impair.} \end{cases}$$

De plus, la multiplication de Clifford par un champ de vecteurs X, tangent à M, est donnée par

$$X \bullet \phi = (X \cdot \nu \cdot \psi)_{|_M}, \tag{20}$$

où $\psi \in \Gamma(\Sigma\mathcal{Z})$ (où $\psi \in \Gamma(\Sigma^+\mathcal{Z})$ si n est impair), ϕ est la restriction de ψ à M, "$\cdot$" est la multiplication de Clifford sur $\mathcal{Z}$, "$\bullet$" celle sur M et ν le vecteur normal unitaire entrant. La 1-forme de connexion définie sur le $\mathbb{S}^1$-fibré principal restreint $(P_{\mathbb{S}^1}M =: P_{\mathbb{S}^1}\mathcal{Z}_{|_M}, \pi, M)$ est donnée par

$$A = A^{\mathcal{Z}}{}_{|_M} : T(P_{\mathbb{S}^1}M) = T(P_{\mathbb{S}^1}\mathcal{Z})_{|_M} \longrightarrow i\mathbb{R}.$$

Ainsi la 2-forme de courbure $i\Omega$ sur le $\mathbb{S}^1$-fibré principal $P_{\mathbb{S}^1}M$ est donnée par $i\Omega = i\Omega^{\mathcal{Z}}{}_{|_M}$, qui n'est rien d'autre que la 2-forme de courbure du fibré en droites L, la restriction du fibré en droites $L^{\mathcal{Z}}$ sur M.

On note $\nabla^{\Sigma\mathcal{Z}}$ la dérivée covariante de Levi-Civita spinorielle sur $\Sigma\mathcal{Z}$ et par ∇ celle sur ΣM. Pour tout $X \in \Gamma(TM)$ et pour tout champ de spineurs $\psi \in \Gamma(\Sigma\mathcal{Z})$ on considère $\phi = \psi_{|_M}$. La formule de Gauss Spin^c est donnée par [Nak11a]

$$(\nabla_X^{\Sigma\mathcal{Z}}\psi)_{|_M} = \nabla_X\phi + \frac{1}{2}II(X) \bullet \phi, \tag{21}$$

où II est la seconde forme fondamentale de l'immersion. De plus, si $D^{\mathcal{Z}}$ et D sont les opérateurs de Dirac respectifs sur $\mathcal{Z}$ et M, en notant par le même symbole un spineur et sa restriction sur M, on a [Nak11a]

$$\widetilde{D}\phi = \frac{n}{2}H\phi - \nu \cdot D^{\mathcal{Z}}\phi - \nabla_\nu^{\Sigma\mathcal{Z}}\phi, \tag{22}$$

où $H = \frac{1}{n}\mathrm{tr}(II)$ est la courbure moyenne, $\widetilde{D} = D$ si n est pair et $\widetilde{D} = D \oplus (-D)$ si n est impair. Dans ce cas, les formes de courbure Ω et $\Omega^{\mathcal{Z}}$ sont reliées par

$$|\Omega^{\mathcal{Z}}|^2 = |\Omega|^2 + |\nu \lrcorner \Omega^{\mathcal{Z}}|^2, \tag{23}$$

$$(\Omega^{\mathcal{Z}} \cdot \psi)_{|_M} = \Omega \bullet \phi - (\nu \lrcorner \Omega^{\mathcal{Z}}) \bullet \phi. \tag{24}$$

Par la formule de Gauss Spin^c, le tenseur d'énergie-impulsion est, à une constante près, la seconde forme fondamentale de l'immersion. En effet :

Proposition 0.0.3. [*Nak11a*] *Soit $M^n \hookrightarrow (\mathcal{Z}, g)$ une hypersurface compacte orientée immergée isométriquement dans $(\mathcal{Z}, g)$, une variété riemannienne* $\mathrm{Spin^c}$ *admettant un spineur parallèle, de courbure moyenne H et de seconde forme fondamentale II. Le tenseur d'énergie-impulsion associé à $\phi =: \psi_{|_M}$ vérifie*

$$2\ell^\phi = II.$$

De plus, si la courbure moyenne H est constante, l'hypersurface M satisfait le cas d'égalité dans (10) si et seulement si

$$S^{\mathcal{Z}} - 2\ \mathrm{ric}^{\mathcal{Z}}(\nu,\nu) - c_n|\Omega| = 0. \tag{25}$$

Dans les conditions de la Proposition 0.0.3, l'hypersurface M admet un champ de spineurs ϕ vérifiant

$$\nabla_X \phi = -\frac{1}{2} II(X) \bullet \phi, \tag{26}$$

pour tout $X \in \Gamma(TM)$. Maintenant, soit (M^n, g) une variété riemannienne $\mathrm{Spin^c}$ admettant un spineur ϕ tel que, pour tout $X \in \Gamma(TM)$,

$$\nabla_X \phi = -\frac{1}{2} E(X) \bullet \phi, \tag{27}$$

où E est un champ d'endomorphismes symétriques. Il est naturel de se demander si le tenseur E peut être réalisé comme le tenseur de Weingarten d'une certaine immersion isométrique de M dans une variété $\mathrm{Spin^c}$ admettant un spineur parallèle. B. Morel [Mor02] a étudié cette question dans le cas des variétés Spin où le tenseur E est parallèle. C. Bär, P. Gauduchon et A. Moroianu [BGM05] l'ont étudié dans le cas où E est de Codazzi. Sur une variété $\mathrm{Spin^c}$, on a :

Theorem 0.0.5. [*Nak11a*] *Soit (M^n, g) une variété riemannienne* $\mathrm{Spin^c}$ *admettant un spineur ϕ satisfaisant (27) tel que E est de Codazzi. Alors, le cylindre généralisé $\mathcal{Z} := I \times M$ muni de la métrique $dt^2 + g_t$, où $g_t(X,Y) = g((\mathrm{Id} + tE)^2 X, Y)$, et de la structure* $\mathrm{Spin^c}$ *provenant de celle donnée sur M, possède un spineur parallèle dont la restriction sur M n'est autre que ϕ.*

Grâce à ce théorème, on classifie les variétés de dimension 3 qui satisfont le cas limite de (10). En effet :

Corollary 0.0.2. *Soit (M^3, g) une variété riemannienne compacte orientée et ϕ un spineur propre associé à la première valeur propre λ_1 de l'opérateur de Dirac tel que le tenseur d'énergie-impulsion associé à ce spineur est de Codazzi. La variété M est une variété limite pour (10) si et seulement si le cylindre généralisé $\mathcal{Z}^4$, muni de la structure* $\mathrm{Spin^c}$ *provenant de celle sur M, est une variété kählérienne de courbure scalaire positive et l'immersion de M dans $\mathcal{Z}$ est de courbure moyenne H constante.*

Cette question qui consiste à se demander si le tenseur E peut être réalisé comme le tenseur de Weingarten d'une certaine immersion isométrique de M dans une variété Spin^c, est aussi liée à un problème classique en géométrie riemannienne : quand une variété riemannienne peut-elle être isométriquement immergée dans une autre variété riemannienne fixée ? Par exemple, les équations de Gauss et les équations de Codazzi-Mainardi sont des conditions nécessaires et suffisantes pour immerger isométriquement une variété dans $\mathbb{R}^n$, $\mathbb{S}^n$ ou $\mathbb{H}^n$. Il s'avère que les structures Spin^c permettent l'étude des caractérisations des hypersurfaces des variétés kählériennes, de Sasaki et d'autres types de variétés. Dans la suite, nous présenterons quelques résultats dans cette direction.

Caractérisation Spin^c des surfaces de $\mathbb{E}(\kappa,\tau)$ et correspondance de Lawson généralisée

En dimension 3, la classification des variétés homogènes simplement connexes dont le groupe d'isométries est de dimension 4 est bien connue. Ces variétés, notées $\mathbb{E}(\kappa,\tau)$, ont la propriété qu'elles admettent une fibration riemannienne au dessus de la surface simplement connexe $\mathbb{M}^2(k)$ de courbure constante k et à courbure de fibration τ. Lorsque $\tau = 0$, la fibration est triviale et $\mathbb{E}(\kappa,\tau)$ n'est autre que l'espace produit $\mathbb{M}^2(k)\times\mathbb{R}$, soit $\mathbb{S}^2\times\mathbb{R}$ ou $\mathbb{H}^2\times\mathbb{R}$. Si $\tau \neq 0$, $\mathbb{E}(\kappa,\tau)$ sont les sphères de Berger, le groupe de Heisenberg Nil_3 ou le revêtement universel du groupe de Lie $\mathrm{PSL}_2(\mathbb{R})$. À l'exception des sphères de Berger et avec $\mathbb{R}^3$, $\mathbb{H}^3$, $\mathbb{S}^3$ et le groupe résoluble Sol_3, les variétés $\mathbb{E}(\kappa,\tau)$ définissent la géométrie de Thurston [Bon02, Mil76, Sco83].

Récemment, H. Rosenberg et U. Abresch ont donné un intérêt particulier aux surfaces de $\mathbb{E}(\kappa,\tau)$. La majorité de ces travaux concerne les surfaces minimales ou à courbure moyenne constante dans les deux espaces produits $\mathbb{S}^2\times\mathbb{R}$ et $\mathbb{H}^2\times\mathbb{R}$. Suite à ces travaux, de nouveaux exemples de surfaces minimales ont été obtenus comme les caténoïdes et les hélicoïdes. Au vu de ces résultats, il est naturel de penser que l'utilisation des spineurs pourrait permettre d'aborder des questions toujours ouvertes. En 2007, B. Daniel [Dan07] a donné une condition nécessaire et suffisante pour qu'une surface soit immergée isometriquement dans les $\mathbb{E}(\kappa,\tau)$. Cette condition est donnée par l'existence d'un triplet (E, T, f) composé d'un 2-tenseur symétrique E, d'un champ de vecteurs T et d'une fonction f définis sur M, et qui satisfont aux équations de Gauss et Codazzi correspondantes, ainsi qu'à deux relations supplémentaires.

La correspondance de Lawson est une correspondance naturelle entre surfaces à courbure moyenne constante dans $\mathbb{R}^3$, $\mathbb{S}^3$ et $\mathbb{H}^3$: *toute surface minimale simplement connexe de $\mathbb{S}^3$ est isométrique à une surface simplement connexe dans $\mathbb{R}^3$ à courbure moyenne égale à 1 et toute surface minimale simplement connexe dans $\mathbb{R}^3$ est isométrique à une surface simplement connexe à courbure moyenne constante 1 dans $\mathbb{H}^3$.*

Dans le cas des 3-espaces homogènes, B. Daniel [Dan07] a aussi montré une correspondance de Lawson. Par exemple, *toute surface minimale simplement connexe de*

Nil_3 *est isométrique à une surface simplement connexe à courbure moyenne constante* $\frac{1}{2}$ *dans* $\mathbb{H}^2\times\mathbb{R}$. J. Roth [Roth10] avait caractérisé les surfaces des 3-espaces homogènes via les spineurs. En effet, il utilise la structure Spin triviale existant sur ces variétés. Cette structure admet un spineur spécial provenant du spineur de Killing défini sur la fibration riemannienne. Notons qu'il n'existe pas de preuve spinorielle de la correspondance de Lawson démontrée par B. Daniel.

Les variétés homogènes de dimension 3 dont le groupe d'isométries est de dimension 4 possèdent une structure Spin^c naturelle. Cette structure est le relèvement de la structure Spin^c canonique sur $\mathbb{M}^2(\kappa)$ via la submersion $\mathbb{E}(\kappa,\tau)\longrightarrow\mathbb{M}^2(\kappa)$. La structure canonique sur $\mathbb{M}^2(k)$ admet un spineur parallèle et induit sur les $\mathbb{E}(\kappa,\tau)$ un spineur de Killing Spin^c de constante de Killing $\frac{\tau}{2}$, i.e. un champ de spineurs ψ satisfaisant

$$\nabla_X^{\mathbb{E}(\kappa,\tau)}\psi=\frac{\tau}{2}X\cdot\psi,$$

pour tout $X\in\Gamma(T(\mathbb{E}(\kappa,\tau)))$. Par la formule de Gauss Spin^c [Nak11a], la restriction de ψ à toute hypersurface M donne lieu à un champ de spineurs spécial ϕ. Plus précisément, le spineur ϕ satisfait l'équation

$$\nabla_X\phi=-\frac{1}{2}II(X)\bullet\phi+i\frac{\tau}{2}X\bullet\overline{\phi}, \tag{28}$$

pour tout $X\in\Gamma(TM)$. Ici, $\overline{\phi}=\phi_+-\phi_-$ désigne le conjugué de $\phi=\phi_++\phi_-$ par la décomposition du fibré des spineurs Spin^c en spineurs positifs et spineurs négatifs. Inversement, l'existence sur une surface d'une structure Spin^c admettant un champ de spineurs satisfaisant l'équation (28) permet d'immerger la surface dans $\mathbb{E}(\kappa,\tau)$. En effet :

Theorem 0.0.6. [*NR11*] *Soient* $\kappa,\tau\in\mathbb{R}$ *avec* $\kappa-4\tau^2\neq0$. *Considérons une surface riemannienne* (M^2,g) *simplement connexe et* E *un champ d'endomorphismes symétriques de* TM, *de trace égale à* $2H$. *Les assertions suivantes sont équivalentes :*

1. *Il existe une immersion isométrique* $F:(M^2,g)\longrightarrow\mathbb{E}(\kappa,\tau)$ *de seconde forme fondamentale* E, *de courbure moyenne* H *et telle que le vecteur vertical est donné par* $\xi=dF(T)+f\nu$, *où* ν *est le vecteur normal unitaire entrant,* f *est une fonction réelle sur* M *et* T *est la composante tangentielle de* ξ.

2. *La surface* M *est une variété* Spin^c *admettant un champ de spineurs non-trivial* ϕ *vérifiant, pour tout* $X\in\Gamma(TM)$,

$$\nabla_X\phi=-\frac{1}{2}E(X)\bullet\phi+i\frac{\tau}{2}X\bullet\overline{\phi}.$$

De plus, le fibré auxiliaire a une connexion de courbure donnée, dans un repère local orthonormé $\{e_1,e_2\}$, *par* $i\Omega(e_1,e_2)=-i(\kappa-4\tau^2)f=-i(\kappa-4\tau^2)<\phi,\frac{\overline{\phi}}{|\phi|^2}>$.

3. *La surface* M *est une variété* Spin^c *admettant un champ de spineurs non-trivial* ϕ *de norme constante et vérifiant*

$$D\phi=H\phi-i\tau\overline{\phi}.$$

De plus, le fibré auxiliaire a une connexion de courbure donnée, dans un repère local orthonormé $\{e_1, e_2\}$*, par* $i\Omega(e_1, e_2) = -i(\kappa - 4\tau^2)f = -i(\kappa - 4\tau^2) < \phi, \frac{\overline{\phi}}{|\phi|^2} >$.

Il faut noter que la caractérisation établie par J. Roth, en utilisant les structures Spin, impose des conditions supplémentaires.

Comme application, cette caractérisation Spin^c permet de redémontrer la correspondance de Lawson pour les surfaces de $\mathbb{E}(\kappa, \tau)$. Plus précisément, on a :

Theorem 0.0.7. [*NR11*] *Soient* $\mathbb{E}(\kappa_1, \tau_1)$ *et* $\mathbb{E}(\kappa_2, \tau_2)$ *deux variétés homogènes de dimension* 3 *dont le groupe d'isométries est de dimension* 4*. On suppose que* $\kappa_1 - 4\tau_1^2 = \kappa_2 - 4\tau_2^2$*. On note* ξ_1 *et* ξ_2 *les vecteurs verticaux de* $\mathbb{E}(\kappa_1, \tau_1)$ *et* $\mathbb{E}(\kappa_2, \tau_2)$ *respectivement. Considérons* (M^2, g) *une surface simplement connexe immergée isométriquement dans* $\mathbb{E}(\kappa_1, \tau_1)$ *de courbure moyenne constante* H_1 *de sorte que* $H_1^2 \geqslant \tau_2^2 - \tau_1^2$*. Soient* ν_1 *le vecteur normal unitaire entrant de l'immersion,* T_1 *la projection de* ξ_1 *sur* TM *et* $f = \langle \nu_1, \xi_1 \rangle$*. On choisit* $H_2 \in \mathbb{R}$ *et* $\theta \in \mathbb{R}$ *tels que*

$$\begin{aligned} H_2^2 + \tau_2^2 &= H_1^2 + \tau_1^2, \\ \tau_2 + iH_2 &= e^{i\theta}(\tau_1 + iH_1). \end{aligned}$$

Ainsi il existe une immersion isométrique F *de* (M^2, g) *dans* $\mathbb{E}(\kappa_2, \tau_2)$ *de courbure moyenne* H_2 *et telle que sur* M

$$\xi_2 = dF(T_2) + f\nu_2,$$

où ν_2 *est le vecteur normal unitaire entrant de l'immersion et* T_2 *est la composante tangentielle de* ξ_2*. De plus, les secondes formes fondamentales* II_1 *et* II_2 *sont reliées par*

$$II_2 - H_2\mathrm{Id} = e^{\theta J}(II_1 - H_1\mathrm{Id}).$$

Caractérisation Spin^c des hypersurfaces réelles de l'espace projectif complexe $\mathbb{C}P^2$

Une autre situation intéressante est lorsque la variété ambiante est kählérienne. Dans ce cas, nous considérons la structure Spin^c canonique admettant des spineurs parallèles. Par restriction, toute hypersurface M possède un spineur de Killing Spin^c généralisé. En petites dimensions et dans certains cas, l'existence d'un spineur de Killing Spin^c généralisé est une condition nécessaire et suffisante pour réaliser M comme une hypersurface de la variété kählérienne.

L'espace projectif complexe $\mathbb{C}P^2$ de dimension complexe 2 n'est pas une variété Spin mais il admet une structure Spin^c canonique provenant de la structure complexe. On va caractériser les hypersurfaces de $\mathbb{C}P^2$ par restriction des spineurs parallèles. En effet :

Theorem 0.0.8. [*NR11*] *Notons* (M^3, g) *une variété riemannienne simplement connexe admettant une structure de contact* $(\mathfrak{X}, \xi, \eta)$. *Soit* E *un champ de tenseurs symétriques de trace* $3H$. *On suppose que* E *satisfait une des équations de Codazzi correspondantes à* $\mathbb{C}P^2$. *Les assertions suivantes sont équivalentes :*

1. *Il existe une immersion isométrique de* (M^3, g) *dans* $\mathbb{C}P^2$ *de seconde forme fondamentale* E, *de courbure moyenne* H *telle que la restriction sur* M *de la structure complexe sur* $\mathbb{C}P^2$ *est donnée par* $J = \mathfrak{X}+\eta(\cdot)\nu$, *où* ν *est le vecteur normal unitaire entrant de l'immersion.*

2. *La variété* M *a une structure* Spin^c *admettant un spineur non-trivial* ϕ *vérifiant, pour tout* $X \in \Gamma(TM)$,
$$\nabla_X \phi = -\frac{1}{2}E(X) \bullet \phi \quad \text{et} \quad \xi \bullet \phi = -i\phi.$$
La 2-forme de courbure de la connexion sur le fibré auxiliaire est donnée par $i\Omega(e_1, e_2) = -6i$ *et* $i\Omega(e_i, e_j) = 0$ *sinon dans la base* $\{e_1, e_2 = \mathfrak{X}(e_1), e_3 = \xi\}$.

3. *La variété* M *a une structure* Spin^c *admettant un spineur non-trivial* ϕ *de norme constante vérifiant*
$$D\phi = \frac{3}{2}H\phi \quad \text{et} \quad \xi \bullet \phi = -i\phi.$$
La 2-forme de courbure de la connexion sur le fibré auxiliaire est donnée par $i\Omega(e_1, e_2) = -6i$ *et* $i\Omega(e_i, e_j) = 0$ *sinon dans la base* $\{e_1, e_2 = \mathfrak{X}(e_1), e_3 = \xi\}$.

Estimation de la première valeur propre de l'opérateur de Dirac sur les hypersurfaces et applications géométriques

Nous continuons d'explorer les structures Spin^c d'un point de vue extrinsèque. Cette fois-ci, on donne une estimation de la première valeur propre de l'opérateur de Dirac sur l'hypersurface et on compare cette estimation à celle de Friedrich (4). En utilisant l'inégalité de Reilly spinorielle, nous montrons :

Theorem 0.0.9. [*Nak3*] *Soient* $\mathcal{Z}^{n+1}$ *une variété riemannienne* Spin^c *vérifiant* $S^{\mathcal{Z}} \geqslant c_{n+1}|\Omega^{\mathcal{Z}}|$ *et* M^n *une hypersurface orientée compacte. On suppose que* M *a une courbure moyenne positive et que* M *est le bord d'un domaine* $\mathbb{D}$ *de* $\mathcal{Z}$. *La première valeur propre strictement positive* λ_1 *de* $\widetilde{D}$ *satisfait*
$$\lambda_1 \geqslant \frac{n}{2} \inf_M H. \tag{29}$$
On a égalité si et seulement si H *est constante et l'espace des spineurs propres associés à* λ_1 *est formé des restrictions des spineurs parallèles sur le domaine* $\mathbb{D}$.

Cela a été prouvé par O. Hijazi, S. Montiel et X. Zhang [HMZ01a, HMZ01b] sur les variétés Spin. Dans certains cas, cette inégalité démontre l'inégalité (4) de Friedrich Spin^c. En effet :

Proposition 0.0.4. *Soit (M^n, g) une variété compacte plongée dans une variété $\mathcal{Z}^{n+1}$ riemannienne* $\mathrm{Spin^c}$*. Si le tenseur d'Einstein* $\mathrm{ric}^{\mathcal{Z}} - \frac{S^{\mathcal{Z}}}{2} g_{\mathcal{Z}}$ *est semi-défini positif, l'inégalité (29) implique l'inégalité (4) de Friedrich* $\mathrm{Spin^c}$*.*

Caractérisation Spin^c des structures CR

Sur une variété différentiable M^n, une structure presque CR de dimension m et de codimension $k = n - 2m$ est la donnée d'un sous-fibré réel D de TM de rang $2m$ et d'un automorphisme J de D tel que $J^2 = -\mathrm{Id}$. Une structure presque CR est intégrable si et seulement si l'automorphisme J vérifie

$$[X, Y] - [JX, JY] \in \Gamma(\mathrm{D}) \quad \text{et} \quad J([X, Y] - [JX, JY]) = [JX, Y] + [X, JY],$$

pour tous $X, Y \in \Gamma(\mathrm{D})$. Dans ce cas, la structure presque CR est dite une structure CR de type (m, k).

Sur une variété différentiable M^n, on considère une structure CR de type hypersurface, i.e. une variété CR de type $(m, 1)$. Dans ce cas, la dimension de M est impaire et il existe une 1-forme globale θ, appelée une structure pseudohermitienne, vérifiant $\mathrm{D} = \ker\theta$. La forme de Levi est donnée par

$$G_\theta(X, Y) = d\theta(JX, Y) \quad \text{pour tous} \quad X, Y \in \Gamma(\mathrm{D}).$$

La structure CR est dite non dégénérée (resp. strictement pseudoconvexe) si la forme de Levi G_θ est non dégénérée (resp. définie positive). Si M est non dégénérée, on définit un champ de vecteurs T par

$$\theta(T) = 1 \quad \text{et} \quad T \lrcorner d\theta = 0.$$

Ce champ de vecteurs, appelé vecteur caractéristique de $d\theta$, est unique et non nul.

Comme pour les variétés kählériennes, toute variété strictement pseudoconvexe possède une structure $\mathrm{Spin^c}$ canonique. Il est donc naturel de se demander quand est-ce qu'une variété est kählérienne ou bien strictement pseudoconvexe.

Soit (M^n, g) une variété riemannienne $\mathrm{Spin^c}$ et ψ un champ de spineurs. En tout point $x \in M$, on définit le sous-espace du fibré tangent

$$\mathrm{D}_x = \{X \in T_xM \mid X \cdot \psi = iY \cdot \psi, \text{ pour un } Y \in T_xM \setminus \{0\}\}.$$

Un champ de spineurs non nul est appelé *un champ de spineurs pur* si et seulement si $\mathrm{D}_x = T_xM$, en tout point $x \in M$. Tout champ de spineurs pur définit une structure presque complexe J sur M et il est dit intégrable si et seulement si

$$\overline{Z} \cdot \nabla_{\overline{W}}\psi - \overline{W} \cdot \nabla_{\overline{Z}}\psi = 0,$$

pour tous $Z, W \in \Gamma(T_{1,0}M)$. En utilisant les spineurs purs, nous caractérisons les variétés kählériennes et hermitiennes.

Proposition 0.0.5. [*HN10*] *Soit M^n une variété différentiable. Il y a équivalence entre les 3 assertions suivantes :*

1. *M est une variété riemannienne* Spin^c *admettant un spineur pur intégrable ψ.*
2. *M est une variété riemannienne (M, g) admettant une structure complexe J telle que (M, J, g) est une variété hermitienne.*
3. *M est une variété riemannienne admettant une structure CR de type $(m, 0)$.*

Pour obtenir une structure kählérienne sur M, il faut supposer que le spineur pur est parallèle. En effet :

Theorem 0.0.10. [*HN10*] *Une variété riemannienne M admet une structure* Spin^c *avec un spineur pur parallèle si et seulement si elle est kählérienne.*

Soit (M^n, g) une variété riemannienne Spin^c. Un champ de spineurs ψ est dit *transversal* s'il définit une distribution D de rang constant tel qu'en tout point $x \in M$ la fibre est donnée par D_x. Un spineur transversal est appelé *m-transversal* si le rang de D est $2m$. De la définition d'un champ de spineurs m-transversal, si (M^n, g) est une variété riemannienne Spin^c portant un champ de spineurs m-transversal intégrable, alors elle admet une structure CR de type $(m, k = n - 2m)$ (voir [HN10]).

L'existence d'un champ de spineurs transversal sur une variété riemannienne Spin^c influence la géométrie et la topologie de la variété. En fait :

Theorem 0.0.11. [*HN10*] *Considérons (M^n, g) une variété riemannienne* Spin^c *admettant un champ de spineurs m-transversal ψ tel que ψ soit un champ de spineurs parallèle ou de Killing dans les directions orthogonales, i.e. il existe $\lambda \in \mathbb{R}$ tel que*

$$\nabla_Y \psi = \lambda Y \cdot \psi, \quad \textit{pour tout} \quad Y \in \Gamma(\mathrm{D}^\perp).$$

Alors la variété est feuilletée.

Introduction

In this thesis, we make use of Spin^c Geometry to study special submanifolds. We start by establishing basic results for the Spin^c Dirac operator and then move to examine hypersurfaces of Spin^c manifolds with special spinors.

The geometry and topology of a Riemannian compact Spin manifold are strongly related to the spectral properties of a fundamental operator called *the Dirac operator*. The Dirac operator is a first order differential operator acting on sections of a vector bundle called *the spinor bundle*. The spinor bundle is roughly speaking a "square root" of the bundle of differential forms but it differs by its dependence on the metric. An interesting tool when examining the Dirac operator is the Schrödinger-Lichnerowicz formula [Lich63]. This formula says that the difference between the square of the Dirac operator and the spinorial Laplacian is proportional to the scalar curvature. Thus, with the weak condition of the positivity of the scalar curvature, A. Lichnerowicz [Lich63] deduces that the kernel of the Dirac operator is trivial. This fact, combined with the Atiyah-Singer index theorem [Atiy-Sing68], provides a topological obstruction for the existence of positive scalar curvature metrics.

Refining the argument of A. Lichnerowicz, T. Friedrich [Fri80] proved a lower bound for the eigenvalues of the Dirac operator involving the infimum of the scalar curvature. The equality case is characterized by the existence of a *real Killing spinor:* it is a section of the spinor bundle whose covariant derivative is proportional to the Clifford multiplication. The existence of such spinors leads to restrictions on the manifold. For example, the manifold is Einstein and in dimension 4, it has constant sectional curvature. The classification of simply connected Riemannian Spin manifolds carrying real Killing spinors [Bär93] gives, in some dimensions, other examples than the sphere. These examples are relevant to physicists in general relativity where the Dirac operator plays a central role.

The study of the spectrum of the Dirac operator on submanifolds of Riemannian Spin manifolds has been extensively studied. Even if the submanifold is Spin, many problems appear. It is well known that the restriction of the spinor bundle to a Spin submanifold is a Hermitian fiber bundle given by the tensor product of the spinor bundle of the submanifold with a certain fiber bundle associated with the normal bundle of the immersion [Bär93, BHMM, BFGK, Gi-Mo00]. In general, it is not easy to have a control on such a Hermitian bundle except in the case where the normal bundle is trivial, for example in the case of hypersurfaces [Bär98, Gi-Mo00, HMZ01a, HMZ01b, Gin09]. We know that in this case, the restriction of the spinor bundle or a part of the spinor bundle of the ambient manifold is the spinor bundle of the hypersurface and the induced Dirac operator is the Dirac operator of the hypersurface.

T. Friedrich [Fri98] characterised simply connected surfaces isometrically immersed in $\mathbb{R}^3$ by the existence of a special spinor field called a *generalized Killing spinor field*. This spinor is the restriction to the surface of a parallel spinor on $\mathbb{R}^3$ and its covariant derivative is proportional to the Clifford multiplication of a field of symmetric endomorphism called *the energy-momentum tensor*. In 2000, O. Hijazi, S. Montiel and X. Zhang [HMZ01a, HMZ01b] gave a lower bound for the first positive eigenvalue of the Dirac operator defined on the compact boundary of a Riemannian Spin manifold of nonnegative scalar curvature. This lower bound involves the mean curvature of the boundary, assumed to be nonnegative. As an application of the limiting case, they give an elementary spinorial proof of the Alexandrov theorem.

Recently, complex Spin geometry became a field of active research with the advent of Seiberg-Witten theory [KM94, Wit94, Sei-Wit94, Fri00]. This theory is based on the fact that every oriented Riemannian compact 4-dimensional manifold has a Spin^c structure. Applications of the Seiberg-Witten theory to 4-dimensional geometry and topology are already notorious: several theorems arising from Donaldson theory found an elementary proof [Don96]. C. LeBrun [LeB95, LeB96] obtained topological restrictions on 4-dimensional Einstein manifolds and with M.J. Gursky [GL98], they calculated the Yamabe invariant for some 4-dimensional manifolds like the complex projective space. At the same time, the lift from classical Spin geometry to Spin^c geometry has led to many new questions and several results have now been proved.

From an intrinsic point of view, Spin, almost complex, complex, Kähler, Sasaki and some classes of CR manifolds have a canonical Spin^c structure. For example, using Spin^c structures, A. Moroianu [Moro99] proved the Lichnerowicz conjecture on Kähler Spin manifolds which are limiting manifolds for the Kirchberg inequality in even complex dimension [Kir86].

In 2006, O. Hijazi, S. Montiel and F. Urbano [HMU06] constructed on Kähler-Einstein manifolds with positive scalar curvature, a Spin^c structure carrying Kählerian Killing spinors. The restriction of these spinors to minimal Lagrangian submanifolds provides topological and geometric restrictions on these submanifolds. Hence, the restriction of Spin^c spinors is an effective tool to study the geometry and the topology of submanifolds. Moreover, from the extrinsic point of vue, it seems that it is more natural to work with Spin^c structures rather than Spin structures, which are by now very classic.

First, we introduce Spin^c structures on manifolds according to S. Montiel [Mon05]. Then, we quote the results obtained in my thesis.

Spin^c structures and results overview

On an oriented Riemannian manifold (M^n, g), we consider a complex vector bundle of rank l equipped with a Hermitian metric $\langle .,. \rangle$ and a connection ∇ which parallelizes the metric. We assume the existence of a $C^\infty(M)$-linear map,

$$\gamma : TM \longrightarrow \mathrm{End}(\Sigma M),$$

mapping vectors tangent to M onto endomorphisms of ΣM. We denote by $(\Sigma M, \langle .,. \rangle, \nabla, \gamma)$ any complex bundle over M endowed with the above data. Under these general hypotheses, we can define a natural first order elliptic operator $D : \Gamma(\Sigma M) \longrightarrow \Gamma(\Sigma M)$ by

$$D = \sum_{j=1}^{n} \gamma(e_j)\nabla_{e_j},$$

where $\{e_1, \cdots, e_n\}$ is any orthonormal local basis tangent to M. If we ask whether the operator D is a self-adjoint with respect to the L^2-scalar product of $\Gamma(\Sigma M)$, we must add the following two conditions on γ:

$$\langle \gamma(X)\psi, \phi \rangle = -\langle \psi, \gamma(X)\phi \rangle, \tag{30}$$
$$\nabla_X(\gamma(Y)\psi) = \gamma(\nabla_X Y)\psi + \gamma(Y)\nabla_X\psi, \tag{31}$$

where $X, Y \in \Gamma(TM)$, $\psi, \phi \in \Gamma(\Sigma M)$ and the second ∇ is the Levi-Civita connection on M. The operator D^2 is also an elliptic operator of second order and if we request that D^2 and the the Laplacian $\triangle$ defined on $\Gamma(\Sigma M)$ to have the same principal symbol, we should impose the following condition

$$\gamma(X)\gamma(Y) + \gamma(Y)\gamma(X) = -2g(X,Y), \tag{32}$$

for every $X, Y \in \Gamma(TM)$. A Dirac bundle is a complex fiber bundle $(\Sigma M, \langle .,. \rangle, \nabla, \gamma)$ such that the conditions (30), (31) and (32) are fulfilled. In this case, γ is called the Clifford multiplication and D the Dirac operator associated with this Dirac bundle.

We called γ the Clifford multiplication because the anticommutativity relation (32) that we imposed is the same as the one defining the Clifford algebra on a given metric vector space. Then, there exists a natural extension of γ to the bundle whose fibers are the Clifford algebras constructed on the tangent space at every point of the manifold:

$$\gamma : \mathbb{C}l(TM) \longrightarrow \mathrm{End}(\Sigma M).$$

Hence, at every point $x \in M$, Clifford multiplication provides a representation of the complex Clifford algebra $\mathbb{C}l(T_xM)$ on the vector space $\Sigma_x M$. This representation is not irreducible in general but it is known that it is irreducible if and only if it is of complex dimension $2^{[\frac{n}{2}]}$. We deduce that the rank l of a Dirac bundle is always greater than or equal to $2^{[\frac{n}{2}]}$, i.e. $l \geqslant 2^{[\frac{n}{2}]}$.

The question is: does there exist a Dirac bundle ΣM, such that the rank is minimal, i.e. $l = 2^{[\frac{n}{2}]}$? In other words, does there exist a Dirac bundle ΣM supplying an irreducible representation of the complex Clifford algebra at each point? If such a Dirac bundle exists, it is called *a spinor bundle*. We should point out that, if we have *a spinor bundle* $(\Sigma M, \langle .,. \rangle, \nabla, \gamma)$ over M, we can contruct many others by taking the tensor product of ΣM with a complex line bundle $\mathcal{D}$ endowed with a Hermitian metric and a metric connection, i.e. $\Sigma' M = \Sigma M \otimes \mathcal{D}$ is also *a spinor bundle*.

A Riemannian manifold has a Spin^c structure if and only if there exists a complex line bundle L on M such that

$$[c_1(L)]_{mod\ 2} = \omega_2(M),$$

where $\omega_2(M)$ is the second Stiefel-Whitney class of M and $c_1(L)$ is the first Chern class of L. In the particular case, when the line bundle has a square root, i.e. $\omega_2(M) = 0$, the manifold is called a Spin manifold.

Given *a spinor bundle* ΣM on a Riemannian manifold M, we can prove that the determinant line bundle $\det \Sigma M$, has a root of index $2^{[\frac{n}{2}]-1}$. We denote by L this root line bundle over M and we will call it the auxiliary line bundle. The auxiliary line bundle L satisfies $[c_1(L)]_{mod\ 2} = \omega_2(M)$ and so M has a Spin^c structure. Conversely, if we fix a Spin^c structure on M, i.e. if there exists a complex line bundle L on M such that $[c_1(L)]_{mod\ 2} = \omega_2(M)$, then we can construct [Mon05], up to isomorphism, a unique *spinor bundle* ΣM. Hence, a *spinor bundle* will be called the Spin^c bundle associated with a given Spin^c structure. In the particular case, when M is a Spin manifold, the Spin^c bundle is called the spinor bundle.

When M is a Spin manifold, the spinor bundle ΣM can be chosen such that the associated auxiliary line bundle L is trivial. Indeed, assume that there exists a line bundle E such that $L = E^2$. The fiber bundle $\Sigma' M = \Sigma M \otimes E^{-1}$ is also *a spinor bundle* whose auxiliary bundle L' is related to L by $L' = L \otimes E^{-2}$, hence it is trivial because $L' = L \otimes E^{-2} = L \otimes L^{-1} = 1$.

Consider a Riemannian Spin^c manifold (M^n, g) and ΣM its Spin^c bundle. Locally, the spinor bundle always exists. Hence, we denote by $\Sigma' M$ the possibly (globally) non-existent spinor bundle whose auxiliary line bundle L' is trivial and we recall that there exists a line bundle $\mathcal{D}$ such that $\Sigma' M = \Sigma M \otimes \mathcal{D}$ and $L' = L \otimes \mathcal{D}^2$. Locally, we have that $\mathcal{D}^2 = L^{-1}$ since L' is trivial. Thus $\Sigma M = \Sigma' M \otimes L^{\frac{1}{2}}$. This essentially means that, while the spinor bundle and $L^{\frac{1}{2}}$ may not exist globally, their tensor product (the Spin^c bundle) could be defined globally.

After these preliminaries about Spin^c geometry, I give a brief description of the results obtained during my thesis. These results will be developped in the next sections.

The study of the spectrum of the Dirac operator has been investigated on Spin manifolds since it contains subtle information on the geometry and topology of the manifold. On Spin^c manifolds, A. Moroianu and M. Herzlich proved a Friedrich type inequality [HM99]: on a compact Riemannian Spin^c manifold (M^n, g), any eigenvalue λ of the Spin^c Dirac operator satisfies

$$\lambda^2 \geqslant \lambda_1^2 := \frac{n}{4(n-1)} \inf_M (S - c_n|\Omega|), \tag{33}$$

where $c_n = 2[\frac{n}{2}]^{\frac{1}{2}}$, S is the scalar curvature of M and $i\Omega$ is the curvature form associated with a fixed connection on the auxiliary bundle. The equality case is characterized by the existence of a Spin^c Killing spinor ψ satisfying

$$\gamma(\Omega)\psi = i\frac{c_n}{2}|\Omega|\psi, \tag{34}$$

where $\gamma(\Omega)$ is the extension of the Clifford multiplication to differential forms. Note that Inequality (33) is of interest only in the case $S - c_n|\Omega| > 0$. The idea introduced by O. Hijazi [Hij95] is to modify the connection ∇ in the direction of a symmetric tensor field. Thus, we get an optimal lower bound for the eigenvalues of the Dirac operator involving a symmetric tensor field called the energy-momentum tensor. This lower bound improves the Friedrich lower bound (33) and is of interest although $S - c_n|\Omega|$ is negative or zero. The equality case of this new lower bound is characterized by the existence of a *generalized* Spin^c *Killing spinor* satisfying Equation (34). This type of spinors will play a key role in the study of extrinsic Spin^c structures.

Studying the energy-momentum tensor on a compact Riemannian Spin or Spin^c manifolds has been done by many authors, since it is related to several geometric situations. Indeed, on compact Spin manifolds, J.P. Bourguigon and P. Gauduchon [BG92] proved that the energy-momentum tensor appears naturally in the study of the variations of the spectrum of the Dirac operator. T. Friedrich and E.C. Kim [KF00] obtained the Einstein-Dirac equation as the Euler-Lagrange equation of a certain functional. We extend these last results to compact Spin^c manifolds [Nak11a]. Even if it is not a computable geometric invariant, the energy-momentum tensor is, up to a constant, the second fundamental form of an isometric immersion into a Spin^c manifold carrying a parallel spinor [Nak11a]. Finally, on Spin^c surfaces, we express this tensor in terms of topological invariants, such as the Euler-Poincaré number [Ha-Na10].

Spin^c structures are natural structures on some classes of manifolds, like manifolds of dimension 2 or 4, almost complex manifolds, Sasaki manifolds, some types of CR-manifolds. A real hypersurface of a Spin^c manifold can be endowed with a Spin^c structure. We investigate the study of Spin^c structures on real hypersurfaces of $\mathbb{C}P^2$ and $\mathbb{E}(\kappa, \tau)$. The complex projective space $\mathbb{C}P^2$, which is not a Spin manifold, has a natural Spin^c structure, called the canonical Spin^c structure, carrying a parallel spinor. *When we restrict the parallel spinor to a hypersurface M, we get a generalized* Spin^c *Killing spinor which will characterize the immersion of M into* $\mathbb{C}P^2$ [NR11].

The manifolds $\mathbb{E}(\kappa, \tau)$ are the homogeneous 3-manifolds with 4-dimensional isometry group ($\mathbb{S}^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}$, Nil$_3$,...). These manifolds are Spin, having a special spinor field ψ. J. Roth [Roth10] proved that, up to some additional assumptions, the restriction of ψ to a surface, allows to characterize the immersion of the surface into $\mathbb{E}(\kappa,\tau)$. But, the manifolds $\mathbb{E}(\kappa,\tau)$ have also a Spin^c structure carrying a Killing Spin^c spinor, whose restriction gives rise to a special spinor field, which allows the characterization of the immersion of M into $\mathbb{E}(\kappa,\tau)$ *without any additional assumption*. Moreover, from this characterization we get a *spinorial proof of the generalized Lawson correspondence for constant mean curvature surfaces in* $\mathbb{E}(\kappa,\tau)$ [NR11].

Spinors have now become an effective tool in the study of hypersurfaces. In fact, O. Hijazi, S. Montiel and X. Zhang [HMZ01a, HMZ01b] proved that, if M^n is a hypersurface bounding a domain of a Spin manifold carrying a parallel spinor and if the scalar curvature of the ambient manifold is nonnegative and the mean curvature H is nonnegative, then the first positive eigenvalue λ_1 of the extrinsic Dirac operator satisfies

$$\lambda_1 \geqslant \frac{n}{2}\inf_M H.$$

As an application, they give a spinorial proof of the Alexandrov theorem. On Spin^c manifolds, we establish a similar lower bound. Compact embedded hypersurfaces into the complex projective space $\mathbb{C}P^m$ with positive constant mean curvature are examples of manifolds satisfying the equality case.

Among all manifolds endowed with a natural Spin^c structure with a special spinor field, a central role is played by complex manifolds and recently by CR-manifolds. We prove that, on a manifold M, the existence of a Spin^c structure with a special spinor field, called *a pure spinor field* or *a transversal spinor field*, is equivalent to the existence of a complex structure or a CR-structure on M. Moreover, the existence of a transversal spinor field on a Riemannian Spin^c manifold gives constraints on its geometry. In fact, we find that *the existence of a parallel or a Killing transversal spinor field on M implies that M is a foliated manifold.*

In the following sections, we present, with some details, the results which are the subject of chapters 2, 3, 4, 5, 6 and 7.

Spectrum of the Spin^c Dirac Operator and the Energy-Momentum Tensor

Useful geometric informations have been obtained by estimating the first eigenvalue of the Dirac operator on a compact Riemannian Spin manifold (M^n, g). We will attempt to extend some lower bounds to the Spin^c Dirac operator. First and for simplicity, the Clifford multiplication γ will be denoted by "$\cdot$". On the complement set of zeroes of a spinor field ψ, O. Hijazi [Hij95] modified the spinorial Levi-Civita connection in the direction of the energy-momentum tensor, a symmetric 2-tensor ℓ^ψ defined by

$$\ell^\psi(X,Y) = g(\ell^\psi(X),Y) = \frac{1}{2}\mathrm{Re}\left\langle X\cdot\nabla_Y\psi + Y\cdot\nabla_X\psi, \frac{\psi}{|\psi|^2}\right\rangle,$$

for any $X, Y \in \Gamma(TM)$, to get a lower bound involving ℓ^ψ and the scalar curvature of the manifold. We extend the Hijazi inequality for the eigenvalues of the Spin^c Dirac operator:

Theorem 0.0.12. *[Nak10] On a compact Riemannian* Spin^c *manifold of dimension $n \geqslant 2$, any eigenvalue λ of the Dirac operator to which is attached an eigenspinor ψ satisfies*

$$\lambda^2 \geqslant \lambda_1^2 := \inf_M(\frac{1}{4}S - \frac{c_n}{4}|\Omega| + |\ell^\psi|^2). \tag{35}$$

The equality case in (35) is characterized by the existence of a *generalized* Spin^c *Killing spinor* satisfying Equation (34), i.e. a spinor field ψ satisfying for every $X \in \Gamma(TM)$,

$$\nabla_X\psi = -\ell^\psi(X)\cdot\psi \quad \text{and} \quad \Omega\cdot\psi = i\frac{c_n}{2}|\Omega|\psi.$$

Note that since the spinor field ψ is an eigenspinor, C. Bär showed that the zero set is contained in a countable union of $(n-2)$-dimensional submanifolds and has locally finite $(n-2)$-dimensional Hausdroff density [Bär99]. The trace of ℓ^ψ being equal to λ, Inequality (35) improves Inequality (33) since, by the Cauchy-Schwarz inequality, $|\ell^\psi|^2 \geqslant \frac{(tr(\ell^\psi))^2}{n}$, where $\mathrm{tr}(\ell^\psi)$ denotes the trace of ℓ^ψ. As an example, the sphere $\mathbb{S}^3$, equipped with a special Spin^c structure, is a limiting manifold for (35) but equality in (33) cannot occur. Generalized Killing spinors on Spin manifolds have been studied by many authors. In fact, assume that on a Spin manifold M, there exists a spinor field ψ such that for all $X \in \Gamma(TM)$,

$$\nabla_X\psi = -E(X)\cdot\psi, \tag{36}$$

where E is a symmetric 2-tensor defined on TM. It is easy to see that E must be equal to ℓ^ψ. If the dimension of M is equal to 2, T. Friedrich [Fri98] proved that the existence of a pair (ψ, E) satisfying (36) is equivalent to the existence of a local immersion of M into the Euclidean space $\mathbb{R}^3$ with Weingarten tensor equal to $2E$. Later, G. Habib [Hab07] studied Equation (36) for an endomorphism E not necessarily symmetric. He showed that the symmetric part of E is ℓ^ψ and the skew-symmetric part of E is q^ψ defined on the complement set of zeroes of ψ by

$$q^\psi(X,Y) = \frac{1}{2}\mathrm{Re}\left\langle Y\cdot\nabla_X\psi - X\cdot\nabla_Y\psi, \frac{\psi}{|\psi|^2}\right\rangle,$$

for all $X, Y \in \Gamma(TM)$. Then he established that, if ψ is an eigenspinor associated with an eigenvalue λ, one has

$$\lambda^2 \geqslant \inf_M(\frac{1}{4}S + |\ell^\psi|^2 + |q^\psi|^2). \tag{37}$$

For a better understanding of the tensor q^ψ, he studied Riemannian flows and proved that, if the normal bundle carries a parallel spinor, the tensor q^ψ is the O'Neill tensor of the flow. On Spin^c manifolds, the lower bound (37) is given by

$$\lambda^2 \geqslant \inf_M \Big(\frac{1}{4}S - \frac{c_n}{4}|\Omega| + |\ell^\psi|^2 + |q^\psi|^2\Big). \tag{38}$$

As a first step, I only consider the deformation of the connection in the direction of the symmetric endomorphism ℓ^ψ. In the next sections, we will study generalized Spin^c Killing spinors in more details.

We now study the variations of the spectrum of the Dirac operator:

Proposition 0.0.6. [*Nak11a*] *Let* (M^n, g) *be a Riemannian* Spin^c *manifold and* $g_t = g + tk$ *be a smooth 1-parameter family of metrics. For any spinor field* $\psi \in \Gamma(\Sigma M)$, *we have*

$$\frac{d}{dt}\Big|_{t=0} \int_M \mathrm{Re}\, \left\langle D^{M_t}\tau_0^t\psi, \tau_0^t\psi\right\rangle_{g_t} v_g = -\frac{1}{2}\int_M \langle k, \ell_\psi\rangle\, v_g, \tag{39}$$

where D^{M_t} *is the Dirac operator associated with* $M_t = (M, g_t)$, $\ell_\psi(X) = |\psi|^2\, \ell^\psi(X) = \mathrm{Re}\,\langle X\cdot\nabla_X\psi, \psi\rangle$ *and* $\tau_0^t\psi$ *is the image of* ψ *under the isometry* τ_0^t *between the* Spin^c *bundles of* (M, g) *and* (M, g_t).

This was proved by J.P. Bourguignon and P. Gauduchon for Spin manifolds [BG92]. Using this, we extend to Spin^c manifolds a result by T. Friedrich and E.C. Kim in [KF00] established for Spin manifolds:

Theorem 0.0.13. [*Nak11a*] *Let* M *be a Riemannian* Spin^c *manifold. A pair* (g_0, ψ_0) *is a critical point of the Lagrange functional*

$$\mathcal{W}(g,\psi) = \int_U \Big(S_g + \epsilon\lambda|\psi|_g^2 - \epsilon\mathrm{Re}\,\langle D_g\psi, \psi\rangle\Big) v_g,$$

$(\lambda, \epsilon \in \mathbb{R})$ *for all open subsets* U *of* M *if and only if* (g_0, ψ_0) *is a solution of the following system*

$$\begin{cases} D_g\psi = \lambda\psi, \\ \mathrm{ric}_g - \frac{S_g}{2}\, g = \frac{\epsilon}{2}\ell_\psi, \end{cases}$$

where ric_g *denotes the Ricci curvature of* M *considered as a symmetric bilinear form.*

In general, we cannot compute the energy-momentum tensor but in low dimensions and especially in dimension 2, it can be expressed in terms of some topological invariants:

Proposition 0.0.7. [*Ha-Na10*] *On a compact surface equipped with any* Spin^c *structure any eigenvalue* λ *of the Dirac operator to which is attached an eigenspinor* ψ *satisfies*

$$\lambda^2 = \frac{S}{4} + |\ell^\psi|^2 + \Delta f + \left\langle \frac{i}{2}\Omega\cdot\psi, \frac{\psi}{|\psi|^2}\right\rangle,$$

where f *is the real-valued function defined by* $f = \frac{1}{2}\,\ln|\psi|^2$.

As a direct consequence, we get [Ha-Na10]

$$\int_M \det(\ell^\psi) v_g \geqslant \frac{\pi\chi(M)}{2} - \frac{1}{4}\int_M |\Omega| v_g, \tag{40}$$

where $\chi(M)$ is the Euler-Poincaré number of the surface. Equality holds if and only if either Ω is zero or has constant sign. In the equality case, we get that $\int_M |\Omega| v_g$ is a topological invariant.

We limit ourselves to these intrinsic classical results and we proceed to study Spin^c structures from an extrinsic point of vue.

The Spin^c Gauss Formula and Geometric Interpretation of the Energy-Momentum Tensor

In this section, we study Spin^c structures on hypersurfaces and as we will see, these structures are natural framework to study some extrinsic Riemannian problems.

Let $\mathcal{Z}$ be an oriented $(n+1)$-dimensional Riemannian Spin^c manifold without boundary. Let $M \subset \mathcal{Z}$ be an oriented hypersurface. The hypersurface M inherits a Spin^c structure from that of $\mathcal{Z}$, and the Spin^c bundle of the hypersurface is given by [Nak11a]

$$\begin{cases} \Sigma M \simeq \Sigma\mathcal{Z}_{|_M} & \text{if } n \text{ is even,} \\ \Sigma M \simeq \Sigma^+\mathcal{Z}_{|_M} & \text{if } n \text{ is odd.} \end{cases}$$

Moreover Clifford multiplication by a vector field X, tangent to M, is given by

$$X \bullet \phi = (X \cdot \nu \cdot \psi)_{|_M}, \tag{41}$$

where $\psi \in \Gamma(\Sigma\mathcal{Z})$ (or $\psi \in \Gamma(\Sigma^+\mathcal{Z})$ if n is odd), ϕ is the restriction of ψ to M, "$\cdot$" is the Clifford multiplication on $\mathcal{Z}$, "$\bullet$" that on M and ν is the unit inner normal vector. The connection 1-form defined on the restricted $\mathbb{S}^1$-principal bundle $(P_{\mathbb{S}^1}M =: P_{\mathbb{S}^1}\mathcal{Z}_{|_M}, \pi, M)$, is given by

$$A = A^{\mathcal{Z}}{}_{|_M} : T(P_{\mathbb{S}^1}M) = T(P_{\mathbb{S}^1}\mathcal{Z})_{|_M} \longrightarrow i\mathbb{R}.$$

Then, the curvature 2-form $i\Omega$ on the $\mathbb{S}^1$-principal bundle $P_{\mathbb{S}^1}M$ is given by $i\Omega = i\Omega^{\mathcal{Z}}{}_{|_M}$, which is the curvature form of the auxiliary line bundle L, the restriction of the auxiliary line bundle $L^{\mathcal{Z}}$ to M. We denote by $\nabla^{\Sigma\mathcal{Z}}$ the spinorial Levi-Civita connection on $\Sigma\mathcal{Z}$ and by ∇ that on ΣM. For all $X \in \Gamma(TM)$ and for every spinor field $\psi \in \Gamma(\Sigma\mathcal{Z})$ we consider $\phi = \psi_{|_M}$ and we get the Spin^c Gauss formula [Nak11a]:

$$(\nabla^{\Sigma\mathcal{Z}}_X \psi)_{|_M} = \nabla_X \phi + \frac{1}{2} II(X) \bullet \phi, \tag{42}$$

where II denotes the Weingarten map. Moreover, let $D^{\mathcal{Z}}$ and D be the Dirac operators on $\mathcal{Z}$ and M, denoting by the same symbol any spinor and its restriction to M, we have [Nak11a]

$$\widetilde{D}\phi = \frac{n}{2} H\phi - \nu \cdot D^{\mathcal{Z}}\phi - \nabla^{\Sigma\mathcal{Z}}_{\nu}\phi, \tag{43}$$

where $H = \frac{1}{n}\mathrm{tr}(II)$ denotes the mean curvature and $\widetilde{D} = D$ if n is even and $\widetilde{D} = D \oplus (-D)$ if n is odd. Now, by the Spin^c Gauss formula, we interprete the energy-momentum tensor as the second fundamental form of the hypersurface M. Indeed,

Proposition 0.0.8. [*Nak11a*] *Let $M^n \hookrightarrow (\mathcal{Z}, g)$ be any compact oriented hypersurface isometrically immersed in an oriented Riemannian* Spin^c *manifold $(\mathcal{Z}, g)$ of Weingarten map II. Assume that $\mathcal{Z}$ admits a parallel spinor field ψ, i.e. $\nabla^{\Sigma\mathcal{Z}}\psi = 0$, then the energy-momentum tensor associated with $\phi =: \psi_{|_M}$ satisfies*

$$2\ell^{\phi} = II.$$

We point out here that under some additional assumptions, the hypersurface M is a limiting manifold for Inequality (35) (see [Nak11a]). Under the same conditions as Proposition 0.0.8, the hypersurface M carries a special spinor field ϕ satisfying the so-called generalized Spin^c Killing spinors equations, i.e.

$$\nabla_X \phi = -\frac{1}{2} II(X) \bullet \phi, \tag{44}$$

for all $X \in \Gamma(TM)$. Now, let M be a Riemannian Spin^c manifold carrying a generalized Spin^c Killing spinor ϕ, i.e. a spinor field satisfying, for all $X \in \Gamma(TM)$,

$$\nabla_X \phi = -\frac{1}{2} E(X) \bullet \phi,$$

where E is a field of symmetric endomorphisms. It is natural to ask whether the tensor E can be realized as the Weingarten tensor of some isometric immersion of M into a manifold $\mathcal{Z}$ carrying parallel spinors. This question is related to a classical problem in Riemannian geometry: when can a Riemannian manifold be isometrically immersed into a fixed Riemannian manifold. For example, the Gauss and the Codazzi-Mainardi equations are necessary and sufficient conditions for the existence of isometric immersions of manifolds into $\mathbb{R}^n$, $\mathbb{S}^n$ or $\mathbb{H}^n$. It turns out that Spin^c structures are natural structures to help characterize of hypersurfaces of Kähler, Sasaki and other manifolds. Next, we will present some results in this direction.

Spin^c Characterization of Surfaces into $\mathbb{E}(\kappa, \tau)$ and Generalized Lawson Correspondence

In dimension 3, the classification of simply connected homogeneous manifolds with 4-dimensional isometry groups is well known. These manifolds, denoted by $\mathbb{E}(\kappa, \tau)$, have the property that they admit a Riemannian fibration over the simply connected surface $\mathbb{M}^2(\kappa)$ of constant curvature k with bundle curvature τ. When $\tau = 0$, the fibration is trivial and $\mathbb{E}(\kappa, \tau)$ is nothing but the product space $\mathbb{M}^2(\kappa) \times \mathbb{R}$, namely $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$. If $\tau \neq 0$, $\mathbb{E}(\kappa, \tau)$ are the Berger spheres, the Heisenberg group Nil_3 or the universal cover of the Lie group $\mathrm{PSL}_2(\mathbb{R})$. Except the Berger spheres and with $\mathbb{R}^3$, $\mathbb{H}^3$, $\mathbb{S}^3$ and the solvable group Sol_3, the manifolds $\mathbb{E}(\kappa, \tau)$ define the geometry of Thurston [Bon02, Mil76, Sco83].

The manifold $\mathbb{E}(\kappa, \tau)$ admits a canonical Spin^c structure. This Spin^c structure is the lift of the canonical Spin^c structure on $\mathbb{M}^2(\kappa)$ via the submersion $\mathbb{E}(\kappa, \tau) \longrightarrow \mathbb{M}^2(\kappa)$.

We know that the canonical Spin^c structure on $\mathbb{M}^2(\kappa)$ carries a parallel spinor so the Spin^c structure on $\mathbb{E}(\kappa,\tau)$ admits a Killing spinor of Killing constant $\frac{\tau}{2}$, i.e. a spinor field ψ satisfying, for all $X \in \Gamma(T(\mathbb{E}(\kappa,\tau)))$,

$$\nabla_X^{\mathbb{E}(\kappa,\tau)}\psi = \frac{\tau}{2}X \cdot \psi.$$

With the help of the Spin^c Gauss formula (42), the restriction of ψ to any hypersurface M is a special spinor field ϕ. More precisely, the spinor field ϕ satisfies, for all $X \in \Gamma(TM)$,

$$\nabla_X\phi = -\frac{1}{2}II(X) \bullet \phi + i\frac{\tau}{2}X \bullet \overline{\phi}, \tag{45}$$

where $\overline{\phi} = \phi_+ - \phi_-$ is the conjugate of $\phi = \phi_+ + \phi_-$ given by the decomposition of the Spin^c bundle into positive and negative spinors. Conversely, the existence on a surface of a Spin^c structure carrying a spinor field satisfying Equation (45) allows to immerse this surface into $\mathbb{E}(\kappa,\tau)$. Indeed,

Theorem 0.0.14. [*NR11*] *Let $\kappa,\tau \in \mathbb{R}$ with $\kappa - 4\tau^2 \neq 0$. Consider a simply connected Riemannian surface (M^2, g) and a field of symmetric endomorphisms E of TM, with trace equal to $2H$. The following statements are equivalent:*

1. *There exists an isometric immersion $F : (M^2, g) \longrightarrow \mathbb{E}(\kappa,\tau)$ with Weingarten tensor E, mean curvature H and such that, over M, the vertical vector is given by $\xi = dF(T) + f\nu$, where ν is the unit inner normal vector to the surface, f is a real function on M and T the tangential part of ξ.*

2. *There exists a* Spin^c *structure on M carrying a non-trivial spinor field ϕ satisfying, for all $X \in \Gamma(TM)$,*

$$\nabla_X\phi = -\frac{1}{2}E(X) \bullet \phi + i\frac{\tau}{2}X \bullet \overline{\phi}.$$

   The auxiliary line bundle has a connection of curvature given in any orthonormal tangent frame $\{e_1, e_2\}$ by $i\Omega(e_1, e_2) = -i(\kappa - 4\tau^2)f = -i(\kappa - 4\tau^2) < \phi, \frac{\overline{\phi}}{|\phi|^2} >$.

3. *There exists a* Spin^c *structure on M carrying a non-trivial spinor field ϕ of constant norm and satisfying*

$$D\phi = H\phi - i\tau\overline{\phi}.$$

   The auxiliary line bundle has a connection of curvature given in any orthonormal tangent frame $\{e_1, e_2\}$ by $i\Omega(e_1, e_2) = -i(\kappa - 4\tau^2)f = -i(\kappa - 4\tau^2) < \phi, \frac{\overline{\phi}}{|\phi|^2} >$.

As an application, we get an elementary Spin^c proof of the generalized Lawson correspondence for constant mean curvature surfaces in $\mathbb{E}(\kappa,\tau)$, proved by B. Daniel [Dan07]. For example, via spinors, *we obtain the Lawson correspondence between minimal surfaces of* Nil_3 *and surfaces of* $\mathbb{H}^2\times\mathbb{R}$ *of constant mean curvature* $\frac{1}{2}$. More generally, we get:

Theorem 0.0.15. [*NR11*] *Let $\mathbb{E}(\kappa_1,\tau_1)$ and $\mathbb{E}(\kappa_2,\tau_2)$ be two 3-dimensional homogeneous manifolds with four dimensional isometry group and assume that $\kappa_1-4\tau_1^2=\kappa_2-4\tau_2^2$. We denote by ξ_1 and ξ_2 the vertical vectors of $\mathbb{E}(\kappa_1,\tau_1)$ and $\mathbb{E}(\kappa_2,\tau_2)$ respectively. Consider (M^2,g), a simply connected surface isometrically immersed into $\mathbb{E}(\kappa_1,\tau_1)$ with constant mean curvature H_1 so that $H_1^2\geqslant\tau_2^2-\tau_1^2$. Let ν_1 be the unit inner normal of the immersion, T_1 the tangential projection of ξ_1 and $f=\langle\nu_1,\xi_1\rangle$. We take $H_2\in\mathbb{R}$ and $\theta\in\mathbb{R}$ so that*

$$\begin{aligned} H_2^2+\tau_2^2 &= H_1^2+\tau_1^2,\\ \tau_2+iH_2 &= e^{i\theta}(\tau_1+iH_1). \end{aligned}$$

Then, there exists an isometric immersion F from (M^2,g) into $\mathbb{E}(\kappa_2,\tau_2)$ with mean curvature H_2 and so that over M,

$$\xi_2=dF(T_2)+f\nu_2,$$

where ν_2 is the unit inner normal vector of the immersion and T_2 is the tangential part of ξ_2. Moreover, the respective Weingarten tensors II_1 and II_2 are related by the following

$$II_2-H_2\mathrm{Id}=e^{\theta J}(II_1-H_1\mathrm{Id}).$$

Spin^c Characterization of Hypersurfaces into the Complex Projective Space $\mathbb{C}P^2$

Another interesting situation is when the ambient manifold is Kähler. In this case, we consider the canonical Spin^c structure which carries parallel spinor fields. By restriction to any hypersurface M, we get a generalized Spin^c Killing spinor ϕ. In low dimensions and in some cases, the existence of a generalized Spin^c spinor field ϕ on a 3-dimensional Spin^c manifold is a necessary and sufficient condition to realize M as a hypersurface of a Kähler manifold.

The complex projective space $\mathbb{C}P^2$ is not a Spin manifold. But, it carries always a natural Spin^c structure coming from the complex structure. This natural Spin^c structure admits a parallel spinor. We will attempt to characterize hypersurfaces of $\mathbb{C}P^2$ via restrictions of parallel spinor. We have:

Theorem 0.0.16. [*NR11*] *Denote by (M^3,g) a simply connected Riemannian manifold endowed with a contact metric structure $(\mathfrak{X},\xi,\eta)$. Let E be a field of symmetric endomorphisms on M with trace equal to $3H$. We assume that E satisfies one of the Codazzi equations corresponding to $\mathbb{C}P^2$. Then, the following statements are equivalent:*

1. *There exists an isometric immersion of (M^3,g) into $\mathbb{C}P^2$ with Weingarten tensor E, mean curvature H and so that, the restriction over M of the complex structure of $\mathbb{C}P^2$ is given by $J=\mathfrak{X}+\eta(\cdot)\nu$, where ν is the unit inner normal vector of the immersion.*

2. *The manifold M has a* $\mathrm{Spin^c}$ *structure carrying a non-trivial spinor ϕ satisfying, for all $X \in \Gamma(TM)$,*

$$\nabla_X \phi = -\frac{1}{2}E(X) \bullet \phi \quad \text{and} \quad \xi \bullet \phi = -i\phi.$$

The curvature 2-form of the connection on the auxiliary line bundle is given by $i\Omega(e_1, e_2) = -6i$ and $i\Omega(e_i, e_j) = 0$ elsewhere in the basis $\{e_1, e_2 = \mathfrak{X}(e_1), e_3 = \xi\}$.

3. *The manifold M has a* $\mathrm{Spin^c}$ *structure carrying a non-trivial spinor ϕ of constant norm and satisfying*

$$D\phi = \frac{3}{2}H\phi \quad \text{and} \quad \xi \bullet \phi = -i\phi.$$

The curvature 2-form of the connection on the auxiliary line bundle is given by $i\Omega(e_1, e_2) = -6i$ and $i\Omega(e_i, e_j) = 0$ elsewhere in the basis $\{e_1, e_2 = \mathfrak{X}(e_1), e_3 = \xi\}$.

An Estimate for the first Eigenvalue for the Hypersurface Dirac operator and a Geometric Application

We continue to explore $\mathrm{Spin^c}$ structures from an extrinsic point of vue. This time, we give a lower bound for the first eigenvalue of the hypersurface Dirac operator and we compare this lower bound to the Friedrich lower bound (33). Using the spinorial Reilly inequality, we prove:

Theorem 0.0.17. [*Nak3*] *Let $\mathcal{Z}^{n+1}$ be a Riemannian* $\mathrm{Spin^c}$ *manifold satisfying $S^{\mathcal{Z}} \geqslant c_{n+1}|\Omega^{\mathcal{Z}}|$ and M^n an oriented compact hypersurface. We assume that M has nonnegative mean curvature H and it bounds a compact domain $\mathbb{D}$ in $\mathcal{Z}$. Then, the first positive eigenvalue λ_1 of $\widetilde{D}$ satisfies*

$$\lambda_1 \geqslant \frac{n}{2} \inf_M H. \tag{46}$$

Equality holds if and only if H is constant and the eigenspace corresponding to λ_1 consists of restrictions to M of parallel spinors on the domain $\mathbb{D}$.

This was proved by O. Hijazi, S. Montiel and X. Zhang for Spin manifolds [HMZ01a, HMZ01b]. In some cases, this lower bound improves the well known Friedrich lower bound (33). In fact:

Proposition 0.0.9. *Let M be an embedded hypersurface on a Riemannian* $\mathrm{Spin^c}$ *manifold $\mathcal{Z}^{n+1}$. If the Einstein tensor $\mathrm{ric}^{\mathcal{Z}} - \frac{S^{\mathcal{Z}}}{2} g_{\mathcal{Z}}$ of $\mathcal{Z}$ is positive semidefinite, then the extrinsic lower bound (46) for the first eigenvalue of the Dirac operator $\widetilde{D}$ of M is sharper than the Friedrich inequality (33). The two lower bounds coincide if and only if the embedding is totally umbilical and the restricted* $\mathrm{Spin^c}$ *structure has a flat auxiliary line bundle.*

Spin^c Characterization of CR-structures

Strictly pseudoconvex nondegenerate CR-manifolds and Kähler manifolds have a canonical Spin^c structure. It is natural to ask when a Riemannian manifold is a strictly pseudoconvex CR-manifold or a Kähler manifold. Let (M^n, g) be a Riemannian Spin^c manifold and ψ a spinor field. Define at every $x \in M$ the following subspace of the tangent bundle

$$\mathrm{D}_x = \{X \in T_xM \mid X \cdot \psi = iY \cdot \psi, \text{ for some } \; Y \in T_xM \setminus \{0\}\}.$$

A nowhere zero spinor field ψ is called *a pure spinor field* if and only if $\mathrm{D}_x = T_xM$, for every $x \in M$. Every pure spinor field defines an almost complex structure J on M and a pure spinor field is called integrable if and only if

$$\overline{Z} \cdot \nabla_{\overline{W}}\psi - \overline{W} \cdot \nabla_{\overline{Z}}\psi = 0,$$

for every $Z, W \in \Gamma(T_{1,0}M)$. Using the notion of pure spinor fields, we characterize Kähler and Hermitian manifolds:

Proposition 0.0.10. [*HN10*] *Let M^n be a differentiable manifold. There is a correspondence between the following data:*

1. *M is a Riemannian* Spin^c *manifold carrying an integrable pure spinor field ψ.*
2. *M is a Riemannian manifold (M, g) having a complex structure J such that (M, J, g) is a Hermitian manifold.*
3. *M is a Riemannian manifold having a CR-structure of type $(m, 0)$.*

In order to get a Kähler structure on M, we should assume that the pure spinor ψ is parallel. Indeed,

Theorem 0.0.18. [*HN10*] *Let (M^n, g) be a Riemannian manifold. The manifold M has a* Spin^c *structure carrying a parallel pure spinor field if and only if M is Kähler.*

Let (M^n, g) be a Riemannian Spin^c manifold. A nowhere zero spinor field ψ is called *transversal spinor field* if it defines a distribution D of constant rank with fiber at every point x given by D_x. A transversal spinor is called *m-transversal* if the rank of D is $2m$. From the definition of m-transversal spinors, if (M^n, g) is a Riemannian Spin^c manifold carrying an m-transversal integrable spinor field ψ, then M admits a CR-structure of type $(m, k = n - 2m)$ (see [HN10]). The Heisenberg group H^{2m+1} of dimension $2m + 1$ is a strictly pseudoconvex nondegenerate CR-manifold of type $(m, 1)$ (see [DT]) and hence it carries an m-transversal integrable spinor field.

As for parallel and Killing spinors, the existence of a transversal spinor field on a Riemannian Spin^c manifold restricts the geometry and the topology of the manifold. In fact:

Theorem 0.0.19. [*HN10*] *Consider* (M^n, g) *a Riemannian* Spin^c *manifold carrying an* m*-transversal spinor field* ψ *such that* ψ *is a parallel or a Killing spinor field in the orthogonal directions, i.e. there exists* $\lambda \in \mathbb{R}$ *such that*

$$\nabla_Y \psi = \lambda Y \cdot \psi, \quad \textit{for every} \quad Y \in \Gamma(\mathrm{D}^\perp).$$

Then the manifold is foliated.

Chapter 1

Introduction to Complex Spin Geometry

1.1 The complex spin group and the spinor representation

The aim of this section is to introduce the complex Clifford algebra of the n-dimensional Euclidean space, in which the Spin^c group is constructed. Then, we will restrict the irreducible representations of the complex Clifford algebra to the complex spin group to get the complex spinor representation. For basic notions on Clifford algebras and Spin^c geometry, we refer to [LM89, Fri00, Hij01, Mon05, BHMM].

1.1.1 The complex Clifford algebra

We consider $\mathbb{R}^n$ (resp. $\mathbb{C}^n$) equipped with the canonical scalar product given by

$$\langle v, w\rangle := \sum_{j=1}^{n} v_j w_j,$$

for any v, $w \in \mathbb{R}^n$ (resp. $\mathbb{C}^n$).

Definition 1.1.1. *The real Clifford algebra* Cl_n *(resp. the complex Clifford algebra* $\mathbb{C}\mathrm{l}_n$*) is the unitary algebra generated by* $\mathbb{R}^n$ *(resp.* $\mathbb{C}^n$*) subject to the relations*

$$v \cdot w + w \cdot v = -2\langle v, w\rangle, \tag{1.1}$$

for any $v, w \in \mathbb{R}^n$ *(resp.* $\mathbb{C}^n$*).*

It is easy to check that if $(e_j)_{1\leqslant j\leqslant n}$ is an orthonormal basis of $\mathbb{R}^n$ (resp. $\mathbb{C}^n$), then

$$\{1, e_{i_1}\cdot \ldots \cdot e_{i_k},\ \ 1 \leqslant i_1 < \ldots < i_k \leqslant n,\ \ 0 \leqslant k \leqslant n\}$$

is a basis of Cl_n (resp. $\mathbb{C}\mathrm{l}_n$), thus $\dim_{\mathbb{R}} \mathrm{Cl}_n = 2^n$ (resp. $\dim_{\mathbb{C}} \mathbb{C}\mathrm{l}_n = 2^n$). The algebra Cl_n can be also viewed as the exterior algebra of $\mathbb{R}^n$ by the following isomorphism of vector spaces

$$\begin{array}{rcl} \Lambda^*\mathbb{R}^n & \longrightarrow & \mathrm{Cl}_n \\ e_{i_1} \wedge \ldots \wedge e_{i_k} & \longmapsto & e_{i_1} \cdot \ldots \cdot e_{i_k}. \end{array}$$

Similary $\Lambda^*\mathbb{C}^n = \mathbb{C}\mathrm{l}_n$. But, it is well known that $\Lambda^*\mathbb{C}^n \simeq \Lambda^*\mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{C}$, hence the complex Clifford algebra is isomorphic to the complexification of the real one, i.e. $\mathbb{C}\mathrm{l}_n \simeq \mathrm{Cl}_n \otimes_{\mathbb{R}} \mathbb{C}$.

The complex Clifford algebra is periodic with period 2 ($\mathbb{C}\mathrm{l}_{n+2} \simeq \mathbb{C}\mathrm{l}_n \otimes \mathbb{C}\mathrm{l}_2$) and the real Clifford algebra is also periodic but with period 8 ($\mathrm{Cl}_{n+8} \simeq \mathrm{Cl}_n \otimes \mathrm{Cl}_8$). For simplicity, we restrict ourselves to the study of the complex Clifford algebras. The algebra $\mathbb{C}\mathrm{l}_n$ decomposes into the direct sum of odd and even elements. In fact, the following map

$$\begin{array}{rcl} \beta : \mathbb{C}\mathrm{l}_n & \longrightarrow & \mathbb{C}\mathrm{l}_n \\ e_{i_1} \cdot \ldots \cdot e_{i_k} & \longmapsto & (-1)^k e_{i_1} \cdot \ldots \cdot e_{i_k} \end{array}$$

satisfies $\beta^2 = \mathrm{Id}$. Hence, it gives rise to the decomposition

$$\mathbb{C}\mathrm{l}_n = \mathbb{C}\mathrm{l}_n^0 \oplus \mathbb{C}\mathrm{l}_n^1,$$

where $\mathbb{C}\mathrm{l}_n^j = \{u \in \mathbb{C}\mathrm{l}_n, \;\; \beta(u) = (-1)^j u\}$, $j = 0$ or 1. The subspace $\mathbb{C}\mathrm{l}_n^0$ is called the even part and $\mathbb{C}\mathrm{l}_n^1$ is called the odd part of $\mathbb{C}\mathrm{l}_n$. It is easy to see that $\mathbb{C}\mathrm{l}_n^0$ is a subalgebra of $\mathbb{C}\mathrm{l}_n$, but not $\mathbb{C}\mathrm{l}_n^1$. The classification of complex Clifford algebras is given by:

Proposition 1.1.1. *The complex Clifford algebras are either isomorphic to $\mathbb{C}(2^m)$, or to $\mathbb{C}(2^m) \oplus \mathbb{C}(2^m)$. Indeed,*

$$\mathbb{C}\mathrm{l}_{2m} \simeq \mathbb{C}(2^m) \simeq \mathrm{End}(\Sigma_{2m}), \tag{1.2}$$

$$\mathbb{C}\mathrm{l}_{2m+1} \simeq \mathbb{C}(2^m) \oplus \mathbb{C}(2^m) \simeq \mathrm{End}(\Sigma_{2m}) \oplus \mathrm{End}(\Sigma_{2m}), \tag{1.3}$$

where $\mathbb{C}(2^m)$ denotes the ring of $2^m \times 2^m$ complex matrices and $\Sigma_{2m} \simeq \mathbb{C}^{2^m}$.

The above isomorphisms can be given explicitly in terms of the Pauli matrices defined by:

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

with the relations

$$g_1^2 = g_2^2 = 1 \quad \text{and} \quad g_1 g_2 + g_2 g_1 = 0.$$

The isomorphism (1.2) is then given by

$$e_j \longrightarrow iE \otimes \ldots \otimes E \otimes g_{\gamma(j)} \otimes \underbrace{T \otimes \ldots \otimes T}_{[\frac{j-1}{2}] \text{ times}},$$

where $\gamma(j) = 1$ if j is odd and 2 if j is even. For $n = 2m+1$, the isomorphism (1.3) is given by

$$e_j \longmapsto \begin{cases} (iE \otimes \ldots \otimes E \otimes g_{\gamma(j)} \otimes \underbrace{T \otimes \ldots \otimes T}_{[\frac{j-1}{2}] \text{ times}}, iE \otimes \ldots \otimes E \otimes g_{\gamma(j)} \otimes \underbrace{T \otimes \ldots \otimes T}_{[\frac{j-1}{2}] \text{ times}}), & \text{for } 1 \leqslant j \leqslant 2m, \\ (iT \otimes \ldots \otimes T, -iT \otimes \ldots \otimes T), & \text{for } j = 2m+1. \end{cases}$$

From Proposition 1.1.1, we deduce:

Theorem 1.1.1. *When $n = 2m$ is even, $\mathbb{C}\mathrm{l}_{2m}$ has a unique irreducible complex representation χ_{2m} of complex dimension 2^m,*

$$\chi_{2m} : \mathbb{C}\mathrm{l}_{2m} \longrightarrow \mathrm{End}(\Sigma_{2m}).$$

If $n = 2m + 1$ is odd, $\mathbb{C}\mathrm{l}_{2m+1}$ has two inequivalent irreducible representations both of complex dimension 2^m,

$$\chi^j_{2m+1} : \mathbb{C}\mathrm{l}_{2m+1} \longrightarrow \mathrm{End}(\Sigma^j_{2m}) \text{ for } j = 0 \text{ or } 1,$$

where $\Sigma^j_{2m} = \{\sigma \in \Sigma_{2m},\ \ \chi^j_{2m+1}(\omega_{\mathbb{C}})\sigma = (-1)^j\sigma\}$ and $\omega_{\mathbb{C}}$ is the complex volume element

$$\omega_{\mathbb{C}} = \begin{cases} i^m & e_1 \cdot \ldots \cdot e_n \qquad \text{if } n = 2m, \\ i^{m-1} & e_1 \cdot \ldots \cdot e_n \qquad \text{if } n = 2m+1. \end{cases}$$

Finally, we give the following isomorphism α, which is of particular importance to study the irreducibility of the spinor representation and for the identification of the Spin^c bundles in the context of immersions of hypersurfaces:

$$\begin{array}{rcl} \alpha : \mathbb{C}\mathrm{l}_n & \longrightarrow & \mathbb{C}\mathrm{l}^0_{n+1} \\ e_j & \longmapsto & e_j \cdot \nu, \end{array} \tag{1.4}$$

where $\mathbb{C}^n$ is embedded in $\mathbb{C}^{n+1}$ such that $(\mathbb{C}^n)^\perp$ is spanned by a unit inner vector ν.

1.1.2 The complex spin group and the spinor representation

The real spin group Spin_n is the multplicative subgroup of Cl^*_n generated by even products of vectors of length 1, i.e.

$$\mathrm{Spin}_n := \{v_1 \cdot \ldots \cdot v_{2k} \in \mathrm{Cl}_n \mid v_j \in \mathbb{R}^n \text{ such that } \langle v_j, v_j\rangle = 1\} \subset \mathrm{Cl}^0_n.$$

The spin group is a compact, connected, simply connected (for $n \geqslant 3$) Lie group of real dimension $\frac{n(n-1)}{2}$. The complex Clifford algebra contains the group Spin_n as well as the group $\mathbb{S}^1$ of all unit complex numbers. Together, they generate a group which we denote by Spin^c_n. Since $\mathbb{S}^1 \cap \mathrm{Spin}_n = \{\pm 1\}$, the complex spin group Spin^c_n is given by

$$\mathrm{Spin}^c_n = \mathrm{Spin}_n \times_{\mathbb{Z}_2} \mathbb{S}^1,$$

where

$$[a, z] = [a', z'] \Longleftrightarrow \begin{cases} (a, z) = (a', z') \\ \qquad \text{or} \\ (a, z) = (-a', -z') \end{cases}.$$

The complex spin group is not simply connected since $\pi_1(\mathrm{Spin}^c_n) = \mathbb{Z}$. Moreover, it can be identified with a subgroup of $\mathbb{C}\mathrm{l}^*_n$. In fact, we consider the map

$$\begin{array}{rcl} \jmath : \mathrm{Spin}_n \times \mathbb{S}^1 & \longrightarrow & \mathbb{C}\mathrm{l}^*_n \\ (a, z) & \longmapsto & \imath(a)z, \end{array}$$

where $\imath : \mathrm{Cl}_n \hookrightarrow \mathbb{C}\mathrm{l}_n$ is the canonical injection. We remark that $\ker \jmath = \mathbb{Z}_2$, hence $\mathrm{Spin}^c_n \simeq \mathrm{Im}\jmath$, and it is a subgroup of $\mathbb{C}\mathrm{l}^*_n$.

Example 1.1.1. The groups Spin_4^c and Spin_2^c. *We know that*

$$\mathrm{Spin}_2 \subset \mathrm{Cl}_2^0 \simeq \mathrm{Cl}_1 \simeq \mathbb{C}.$$

So $\mathrm{Spin}_2 \subset \mathbb{C}^*$. *But, it is a commutative connected Lie group of dimension* 1*. The only commutative connected Lie groups of dimension 1 are* $\mathbb{S}^1$ *or* $\mathbb{R}$*. Finally,* $\mathrm{Spin}_2 = \mathbb{S}^1$ *and* $\mathrm{Spin}_2^c \simeq \mathbb{S}^1 \times_{\mathbb{Z}_2} \mathbb{S}^1$*. Now, recall the following identifications:*

$$\begin{array}{ccc} \mathbb{R}^4 & \longrightarrow & \mathbb{C}^2 \longrightarrow \mathbb{H} \subset \mathbb{C}(2) \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} & \longmapsto & \begin{pmatrix} u = x_1 + ix_4 \\ v = x_2 + ix_3 \end{pmatrix} \longmapsto \mathbf{x} = x_1 E - ix_2 T + ix_3 g_2 + ix_4 g_1 = \begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix}. \end{array}$$

Note that $\det \mathbf{x} = u\bar{u} + v\bar{v} = (\mathbf{x}, \mathbf{x})$ *and* $\mathbf{x}^t \bar{\mathbf{x}} = (\mathbf{x}, \mathbf{x})\mathrm{Id}_2$*. We define the map*

$$\begin{array}{rcl} \xi : \mathrm{Spin}_4 \simeq SU_2 \times SU_2 & \longrightarrow & Gl(\mathbb{H}) \\ (a, b) & \longmapsto & \xi(a, b) : \mathbf{x} \longrightarrow b\mathbf{x}a^{-1} = \mathbf{X}. \end{array}$$

Since the Euclidean scalar product on $\mathbb{H} \simeq \mathbb{R}^4$ *is left invariant by* $\xi(a, b)$*, i.e.*

$$(\mathbf{X}, \mathbf{X}) = \det \mathbf{X} = \det \mathbf{x} = (\mathbf{x}, \mathbf{x}),$$

it follows that the surjective map $\xi : \mathrm{Spin}_4 \longrightarrow \mathrm{SO}_4$ *has kernel* $\mathbb{Z}_2$*. By the above identifications, the group* Spin_4 *can be realized as:*

$$\mathrm{Spin}_4 = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : \ a, b \in SU_2 \right\},$$

and the complex spin group as

$$\mathrm{Spin}_4^c = \left\{ \begin{pmatrix} za & 0 \\ 0 & zb \end{pmatrix} : \ a, b \in SU_2 \text{ and } z \in \mathbb{S}^1 \right\}.$$

Now, we define the adjoint map Ad by

$$\begin{array}{rcl} \mathrm{Ad} : \mathrm{Cl}_n^* & \longrightarrow & \mathrm{End}(\mathbb{R}^n) \\ w & \longmapsto & \mathrm{Ad}_w : v \longrightarrow \mathrm{Ad}_w(v) = w \cdot v \cdot w^{-1}. \end{array}$$

It is straighforward to check that, for all $x \in \mathbb{R}^n$ with $\|x\| = 1$, the map Ad_x is an endomorphism of $\mathbb{R}^n$ and $-\mathrm{Ad}_x$ is the reflection across $x^\perp$, hence an element of SO_n. By the Cartan-Dieudonné theorem, every element of SO_n is a product of an even number of reflections. We thus have the proposition:

Proposition 1.1.2. *The linear map* $\xi := \mathrm{Ad}|_{\mathrm{Spin}_n} : \mathrm{Spin}_n \longrightarrow \mathrm{SO}_n$ *is surjective and* Spin_n *is the double cover of* SO_n*, i.e. the following sequence is exact*

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}_n \xrightarrow{\xi} \mathrm{SO}_n \longrightarrow 1.$$

If $n \geqslant 3$, Spin_n is the universal cover of SO_n. Since $\mathbb{S}^1$ is the double cover of $\mathbb{S}^1$, the complex spin group is the double cover of $\mathrm{SO}_n \times \mathbb{S}^1$, this yields to the exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}_n^c \xrightarrow{\xi^c} \mathrm{SO}_n \times \mathbb{S}^1 \longrightarrow 1,$$

where $\xi^c = (\xi, \mathrm{Id}^2)$.

Example 1.1.2. *From Example 1.1.1, we have:*

$$\begin{array}{rcl} \xi^c : \mathrm{Spin}_4^c & \longrightarrow & \mathrm{SO}_4 \times \mathbb{S}^1 \subset GL(\mathbb{H}) \times \mathbb{S}^1 \\ \begin{pmatrix} za & 0 \\ 0 & zb \end{pmatrix} & \longmapsto & \big(\xi^c(za, zb) : \mathbf{x} \longrightarrow zb\mathbf{x}(za)^{-1} = b\mathbf{x}a^{-1}, z^2\big). \end{array}$$

It is well known that the Lie algebra $\mathfrak{spin}_n$ of Spin_n lies in $\mathbb{Cl}_n^*$ and is isomorphic to the Lie algebra $\mathfrak{so}_n$ of SO_n by the following isomorphism

$$\begin{array}{rcl} \xi_* : \mathfrak{spin}_n & \longrightarrow & \mathfrak{so}_n \simeq \Lambda^2\mathbb{R}^n \\ e_j \cdot e_k & \longmapsto & 2\, e_j \wedge e_k. \end{array}$$

The Lie algebra $\mathfrak{spin}_n^c = \mathfrak{spin}_n \oplus i\mathbb{R}$ of the group Spin_n^c is then isomorphic to $\mathfrak{so}_n \oplus i\mathbb{R}$.

$$\begin{array}{rcl} (\xi^c)_* : \mathfrak{spin}_n^c \simeq \mathfrak{spin}_n \oplus i\mathbb{R} & \longrightarrow & \mathfrak{so}_n \oplus i\mathbb{R} \\ e_j \cdot e_k + it & \longmapsto & 2\, e_j \wedge e_k + 2it. \end{array}$$

Using Theorem 1.1.1 and the isomorphism (1.4), we have:

Proposition 1.1.3. *For $n = 2m$, the restriction of χ_{2m} to $\mathbb{Cl}_n^0 \simeq \mathbb{Cl}_{n-1}$ splits into $\Sigma_{2m} = \Sigma_{2m}^+ \oplus \Sigma_{2m}^-$, where Σ_{2m}^+ and Σ_{2m}^- are two complex inequivalent irreducible representations of $\mathbb{Cl}_n^0$. For $n = 2m+1$, the restrictions of χ_{2m+1}^0 and χ_{2m+1}^1 to $\mathbb{Cl}_n^0$ are irreducible and equivalent.*

We define the complex spin representation ρ_n^c (resp. ρ_n) by the restriction of an irreducible representation of $\mathbb{Cl}_n$ to Spin_n^c (resp. Spin_n):

$$\rho_n^c := \begin{cases} \chi_{2m|_{\mathrm{Spin}_n^c}} & \text{if } n = 2m, \\ \chi^0_{2m+1|_{\mathrm{Spin}_n^c}} & \text{if } n = 2m+1, \end{cases}$$

$$\rho_n := \begin{cases} \chi_{2m|_{\mathrm{Spin}_n}} & \text{if } n = 2m, \\ \chi^0_{2m+1|_{\mathrm{Spin}_n}} & \text{if } n = 2m+1, \end{cases} .$$

The complex subalgebra generated by $\mathrm{Spin}_n^c \subset \mathbb{Cl}_n$ or by $\mathrm{Spin}_n \subset \mathbb{Cl}_n$ is the even part $\mathbb{Cl}_n^0$ of $\mathbb{Cl}_n$. Hence, two representations of $\mathbb{Cl}_n^0$ are irreducible or equivalent if and only if it is the case for their restrictions to Spin_n^c. By Proposition 1.1.3, we have:

Theorem 1.1.2. *When $n = 2m$, ρ_n^c decomposes into two inequivalent irreducible representations $(\rho_n^c)^+$ and $(\rho_n^c)^-$, i.e.*

$$\rho_n^c = (\rho_n^c)^+ + (\rho_n^c)^- : \mathrm{Spin}_n^c \to \mathrm{End}(\Sigma_{2m}).$$

The space Σ_{2m} decomposes into $\Sigma_{2m} = \Sigma_{2m}^+ \oplus \Sigma_{2m}^-$, where $\omega_{\mathbb{C}}$ acts on Σ_{2m}^+ as the identity and minus the identity on Σ_{2m}^-. If $n = 2m+1$, when restricted to Spin_n^c, the two representations $\chi^0_{2m+1|_{\mathrm{Spin}_n^c}}$ and $\chi^1_{2m+1|_{\mathrm{Spin}_n^c}}$ are equivalent and we simply choose $\Sigma_{2m} := \Sigma_{2m}^0$. The above facts are also true for ρ_n.

The natural inclusion $\mathfrak{i} : \mathrm{Spin}_n \hookrightarrow \mathrm{Spin}_n^c$, given by $\mathfrak{i}(a) = [a, 1]$ relates the spinor representations ρ_n and ρ_n^c. In fact, we have:

$$\rho_n^c([a, z]) = z\rho_n(a).$$

It is well known that $\det(\rho_n(a)) = 1$, thus $\det \rho_n^c([a, z]) = z^{2^{[\frac{n}{2}]}}$.

The complex vector space Σ_{2m} carries a natural Hermitian scalar product $\langle \cdot, \cdot \rangle$. This scalar product satisfies

$$\langle v \cdot \sigma_1, \sigma_2 \rangle = - \langle \sigma_1, v \cdot \sigma_2 \rangle \text{ for all } \sigma_1, \sigma_2 \in \Sigma_{2m} \text{ and } v \in \mathbb{R}^n.$$

Moreover, for this scalar product, the spinor representation ρ_n^c is unitary, i.e.

$$\langle [a, z] \cdot \sigma_1, [a, z] \cdot \sigma_2 \rangle = \langle \sigma_1, \sigma_2 \rangle \text{ for all } \sigma_1, \sigma_2 \in \Sigma_{2m} \quad \text{and} \quad [a, z] \in \mathrm{Spin}_n^c.$$

1.2 The Dirac operator on Riemannian Spin^c manifolds

We introduce Spin^c structures on a Riemannian manifold N which is needed to define globally a complex vector bundle ΣN (called the Spin^c bundle) such that at every point $x \in N$, the fiber is given by $\Sigma_x N = \Sigma_n$. On sections of the Spin^c bundle ΣN, we then define the Dirac operator and we give its basic properties.

1.2.1 Spin^c structures on manifolds

Let N^n be an oriented Riemannian manifold and $P_{\mathrm{SO}_n}N$ the SO_n-principal bundle of orthonormal tangent frames. A complex Spin^c structure on N is a Spin_n^c-principal bundle $P_{\mathrm{Spin}_n^c}N$ over N, an $\mathbb{S}^1$-principal bundle $P_{\mathbb{S}^1}N$ over N together with a twofold covering map $\Theta : P_{\mathrm{Spin}_n^c}N \longrightarrow P_{\mathrm{SO}_n}N \times_N P_{\mathbb{S}^1}N$ such that

$$\Theta(u[a, z]) = \Theta(u)\xi^c([a, z]),$$

for every $u \in P_{\mathrm{Spin}_n^c}N$ and $[a, z] \in \mathrm{Spin}_n^c$, i.e. the following diagram commutes

$$\begin{tikzcd}
P_{\mathrm{Spin}_n^c}N \times \mathrm{Spin}_n^c \arrow[r] \arrow[d, "\Theta\times\xi^c"] & P_{\mathrm{Spin}_n^c}N \arrow[r, "\pi"] \arrow[d, "\Theta"] & N \\
(P_{\mathrm{SO}_n}N \times_N P_{\mathbb{S}^1}N) \times (SO_n \times \mathbb{S}^1) \arrow[r] & P_{\mathrm{SO}_n}N \times_N P_{\mathbb{S}^1}N \arrow[ur, "\pi"'] &
\end{tikzcd}$$

The horizontal maps are respectively the action of Spin_n^c and $SO_n \times \mathbb{S}^1$ on the principal fiber bundles $P_{\mathrm{Spin}_n^c}N$ and $P_{\mathrm{SO}_n}N \times_N P_{\mathbb{S}^1}N$. We denote by $P_{\mathrm{SO}_n}N \times_N P_{\mathbb{S}^1}N$ the $SO_n \times \mathbb{S}^1$-principal fiber bundle obtained by the pull back of the $SO_n \times \mathbb{S}^1$-principal fiber bundle $(P_{\mathrm{SO}_n}N \times P_{\mathbb{S}^1}N, \pi, N \times N)$ via the diagonal map.

Equivalently, N has a Spin^c structure if and only if there exists an $\mathbb{S}^1$-principal bundle $P_{\mathbb{S}^1}N$ over N such that the transition functions $g_{\alpha\beta} \times l_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{SO}_n \times \mathbb{S}^1$ of the $\mathrm{SO}_n \times \mathbb{S}^1$-principal bundle $P_{\mathrm{SO}_n}N \times_N P_{\mathbb{S}^1}N$ admit a lift to Spin_n^c denoted by $\widetilde{g}_{\alpha\beta} \times \widetilde{l}_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{Spin}_n^c$, such that $\xi^c \circ (\widetilde{g}_{\alpha\beta} \times \widetilde{l}_{\alpha\beta}) = g_{\alpha\beta} \times l_{\alpha\beta}$, i.e. the following diagram commutes

$$\begin{tikzcd}
 & \mathrm{Spin}_n^c \arrow[d, "\xi^c"] \\
U_\alpha \cap U_\beta \subset N \arrow[ur, "\widetilde{g}_{\alpha\beta}\times\widetilde{l}_{\alpha\beta}"] \arrow[r, "g_{\alpha\beta}\times l_{\alpha\beta}"] & SO_n \times \mathbb{S}^1
\end{tikzcd}$$

This, anyhow, is equivalent to the second Stiefel-Whitney class $w_2(N)$ being equal, modulo 2, to the first Chern class $c_1(L)$ of the auxiliary line bundle L. It is the complex line bundle associated with the $\mathbb{S}^1$-principal fiber bundle via the standard representation of the unit circle, i.e. $L = P_{\mathbb{S}^1}N \times_\rho \mathbb{C}$, where ρ is the representation of $\mathbb{S}^1$ given by

$$\begin{aligned} \rho : \mathbb{S}^1 &\longrightarrow \mathrm{End}(\mathbb{C}) = \mathbb{C}^* \\ z &\longmapsto \rho(z) : w \to zw. \end{aligned}$$

Let $\Sigma N := P_{\mathrm{Spin}_n^c}N \times_{\rho_n^c} \Sigma_n$ be the vector bundle associated with the spinor representation ρ_n^c. It is called the Spin^c bundle and a section ϕ of ΣN will be called a spinor field. Any spinor field ϕ is locally given by

$$\phi|_U = [\widetilde{b \times s}, \sigma],$$

where $U \subset N$ is an open set of N, $\sigma \colon U \to \Sigma_n$ a function, $b = (e_1, \ldots, e_n)$ is a local orthonormal tangent frame, $s : U \longrightarrow P_{\mathbb{S}^1}N$ is a local section of $P_{\mathbb{S}^1}N$ and $\widetilde{b \times s}$ is the lift of the local section $b \times s : U \to P_{\mathrm{SO}_n}N \times_N P_{\mathbb{S}^1}N$ to $P_{\mathrm{Spin}_n^c}N$, i.e. the following diagram commutes

$$
\begin{tikzcd}
 & & P_{\mathrm{Spin}^c_n}N \arrow[d, "\Theta"] \\
U \subset N \arrow[urr, "\widetilde{b\times s}"] \arrow[rr, "b\times s"] & & P_{\mathrm{SO}_n}N \times_N P_{\mathbb{S}^1}N
\end{tikzcd}
$$

The transition functions of the Spin^c bundle ΣN are given by $\rho_n^c(\widetilde{g}_{\alpha\beta} \times \widetilde{l}_{\alpha\beta}) = \rho_n(\widetilde{g}_{\alpha\beta}) \times \widetilde{l}_{\alpha\beta}$.

Proposition 1.2.1. *The* Spin^c *bundle* ΣN *can be written as*

$$\Sigma N = \Sigma^{'} N \otimes L^{\frac{1}{2}},$$

where $\Sigma^{'} N$ *is the locally defined spinor bundle and* $L^{\frac{1}{2}}$ *is locally defined too but* ΣN *is globally defined. Moreover, the auxiliary line bundle* L *is a root of* $\det \Sigma M$ *of index* $2^{[\frac{n}{2}]-1}$*, i.e.*

$$L = (\det \Sigma N)^{2^{1-[\frac{n}{2}]}},$$

where $\det \Sigma N$ *is the complex line bundle whose transition functions are given by* $\det(\rho_n^c(\widetilde{g}_{\alpha\beta} \times \widetilde{l}_{\alpha\beta}))$.

Proof. It is clear that $\xi^c \circ (\widetilde{g}_{\alpha\beta} \times \widetilde{l}_{\alpha\beta}) = \xi(\widetilde{g}_{\alpha\beta}) \times (\widetilde{l}_{\alpha\beta})^2$ satisfies the cocycle condition but $\widetilde{l}_{\alpha\beta} = (l_{\alpha\beta})^{\frac{1}{2}}$ and $\xi(\widetilde{g}_{\alpha\beta})$ don't necessary satisfy the cocycle condition. Since $\xi(\widetilde{g}_{\alpha\beta}) = g_{\alpha\beta}$, the functions $\rho_n(\widetilde{g}_{\alpha\beta})$ present the locally defined spinor bundle, thus $\Sigma N = \Sigma^{'} N \otimes L^{\frac{1}{2}}$. Moreover, we have

$$\det(\rho_n^c(\widetilde{g}_{\alpha\beta} \times \widetilde{l}_{\alpha\beta})) = \det(\rho_n(\widetilde{g}_{\alpha\beta}) \times \rho(\widetilde{l}_{\alpha\beta})) = (\widetilde{l}_{\alpha\beta})^{2^{[\frac{n}{2}]}} = (l_{\alpha\beta})^{2^{[\frac{n}{2}]-1}},$$

which gives $L = (\det \Sigma N)^{2^{1-[\frac{n}{2}]}}$.

The tangent bundle $TN = P_{\mathrm{SO}_n}N \times_{\rho_0} \mathbb{R}^n$, where ρ_0 stands for the standard matrix representation of SO_n on $\mathbb{R}^n$, can be seen as the associated vector bundle

$$TN \simeq P_{\mathrm{Spin}^c_n}N \times_{pr_1 \circ \xi^c \circ \rho_0} \mathbb{R}^n,$$

where pr_1 is the first projection. One defines the Clifford multiplication "$\cdot$" at every point $x \in N$:

$$
\begin{array}{rcl}
T_xN \otimes \Sigma_x N & \longrightarrow & \Sigma_x N \\
[\widetilde{b \times s}, v] \otimes [\widetilde{b \times s}, \sigma] & \longrightarrow & [\widetilde{b \times s}, v] \cdot [\widetilde{b \times s}, \sigma] := [\widetilde{b \times s}, v \cdot \sigma = \chi_n(v)\sigma],
\end{array}
$$

where $\sigma \in \Sigma_n$ and $\chi_n = \chi_{2m}$ if n is even and $\chi_n = \chi^0_{2m+1}$ if n is odd. Clifford multiplication inherits the relations of the Clifford algebra, i.e. for $X, Y \in T_xN$ and $\phi \in \Sigma_x N$ we have

$$X \cdot Y \cdot \phi + Y \cdot X \cdot \phi = -2g(X, Y)\phi.$$

In even dimensions the Spin^c bundle splits into

$$\Sigma N = \Sigma^+ N \oplus \Sigma^- N,$$

where $\Sigma^\pm N = P_{\mathrm{Spin}^c_n} N \times_{(\rho^c_n)^\pm} \Sigma^\pm_n$. Clifford multiplication by a non-vanishing tangent vector interchanges $\Sigma^+ N$ and $\Sigma^- N$. The Spin^c_n-invariant scalar product on Σ_n and $\Sigma^\pm_n$ induces inner products on ΣN and $\Sigma^\pm N$ which we again denote by $\langle \cdot, \cdot \rangle$ and it satisfies

$$\langle X \cdot \psi, \phi \rangle = - \langle \psi, X \cdot \phi \rangle \,,$$

for every $X \in \Gamma(TN)$ and $\psi, \phi \in \Gamma(\Sigma N)$ or $\psi, \phi \in \Gamma(\Sigma^\pm N)$.

Example 1.2.1. *Every* Spin *manifold has a trivial* Spin^c *structure. In fact, denoting by $P_{\mathrm{Spin}_n} N$ the Spin_n-principal fiber bundle given by the* Spin *structure, we can enlarge the structure group via the inclusion $\mathfrak{i} : \mathrm{Spin}_n \hookrightarrow \mathrm{Spin}^c_n$, i.e. we define a Spin^c_n-principal bundle $P_{\mathrm{Spin}^c_n} N$ by $P_{\mathrm{Spin}^c_n} N := P_{\mathrm{Spin}_n} N \times_{\mathfrak{i}} \mathrm{Spin}^c_n$. This principal fiber bundle, with the trivial $\mathbb{S}^1$-principal fiber bundle, defines a* Spin^c *structure on N.*

1.2.2 The Levi-Civita connection on the Spin^c bundle

We fix a connection A on the $\mathbb{S}^1$-principal fiber bundle $P_{\mathbb{S}^1} N$,

$$A : T(P_{\mathbb{S}^1} N) \longrightarrow \mathrm{Lie}(\mathbb{S}^1) = i\mathbb{R}.$$

The connection A with the connection 1-form ω on $P_{\mathrm{SO}_n} N$ for the Levi-Civita connection ∇,

$$\omega : T(P_{\mathrm{SO}_n} N) \longrightarrow \mathrm{Lie}(\mathrm{SO}_n) = \mathfrak{so}_n,$$

defines a connection

$$\omega \times A : T(P_{\mathrm{SO}_n} N \times_N P_{\mathbb{S}^1} N) \longrightarrow \mathfrak{so}_n \oplus i\mathbb{R} = \mathfrak{spin}^c_n,$$

on the principal fiber bundle $P_{\mathrm{SO}_n} N \times_N P_{\mathbb{S}^1} N$. It is not difficult to see that this connection lifts to the 2-fold covering ξ^c as a connection $\widetilde{\omega}$ on the Spin^c_n-prinicipal bundle $P_{\mathrm{Spin}^c_n} N$, i.e.

$$\widetilde{\omega} = (\xi^c)^{-1}_* \circ (\omega \times A) \circ \Theta_*.$$

In fact, the following diagram commutes

$$\begin{tikzcd} T(P_{\mathrm{Spin}^c_n} N) \arrow[r, "\widetilde{\omega}"] \arrow[d, "\Theta_*"] & \mathfrak{spin}^c_n \arrow[d, "(\xi^c)_*"] \\ T(P_{\mathrm{SO}_n} N \times_N P_{\mathbb{S}^1} N) \arrow[r, "\omega \times A"] & \mathfrak{so}_n \oplus i\mathbb{R} \end{tikzcd}$$

Now, $\widetilde{\omega}$ induces a covariant derivative ∇ on ΣN and we would like to find its local form: let ϕ a spinor field given locally by $\phi|_U = [\widetilde{b \times s}, \sigma]$. The following diagram commutes

$$\begin{tikzcd} & T(P_{\mathrm{Spin}^c_n} N) \arrow[r, "\widetilde{\omega}"] \arrow[d, "\Theta_*"] & \mathfrak{spin}^c_n \arrow[d, "(\xi^c)_*"] \\ TU \subset TN \arrow[ur, "(\widetilde{b \times s})_*"] \arrow[r, "(b \times s)_*"] & T(P_{\mathrm{SO}_n} N \times_N P_{\mathbb{S}^1} N) \arrow[r, "\omega \times A"] & \mathfrak{so}_n \oplus i\mathbb{R} \end{tikzcd}$$

It is easy to show that, for every $X \in \Gamma(TU)$, we have

$$\omega(b_*(X)) = -\sum_{j<k} g(\nabla_X e_j, e_k)\ e_j \wedge e_k.$$

Hence we have

$$\begin{aligned}
\widetilde{\omega}((\widetilde{b \times s})_*(X)) &= (\xi^c)_*^{-1} \circ (\omega \times A) \circ (b \times s)_*(X) \\
&= (\xi^c)_*^{-1}(\omega(b_*(X)), A(s_*(X))) \\
&= -\sum_{j<k} g(\nabla_X e_j, e_k)(\xi^c)_*^{-1}(e_j \wedge e_k, A(s_*(X))) \\
&= -\frac{1}{2}\sum_{j<k} g(\nabla_X e_j, e_k) e_j \cdot e_k + \frac{1}{2}A(s_*(X)).
\end{aligned}$$

The covariant derivative ∇ on ΣN is given by

$$\nabla_X \phi = [\widetilde{b \times s}, X(\sigma) + (\rho_n^c)_*(\widetilde{\omega} \circ (\widetilde{b \times s})_*(X))\sigma],$$

where $X(\sigma)$ is the Lie derivative of σ in the direction of X. Since ρ_n^c is linear, i.e. $(\rho_n^c)_* = \rho_n^c$ we conclude

$$\begin{aligned}
\nabla_X \phi &= \Big[\widetilde{b \times s}, X(\sigma) - \frac{1}{2}\sum_{j<k} g(\nabla_X e_j, e_k) e_j \cdot e_k \cdot \sigma + \frac{1}{2}A(s_*(X))\sigma\Big] \\
&= X(\phi) + \frac{1}{4}\sum_{j=1}^{n} e_j \cdot \nabla_X e_j \cdot \phi + \frac{1}{2}A(s_*(X))\phi. \qquad (1.5)
\end{aligned}$$

The curvature of A is an imaginary valued 2-form denoted by $F_A = dA$, i.e. $F_A = i\Omega$, where Ω is a real valued 2-form on $P_{\mathbb{S}^1}N$. Using Equation (1.5) and the fact that the curvature $\mathcal{R}$ of ∇ is given by $\mathcal{R}(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, we get

$$\mathcal{R}(X, Y)\phi = \frac{1}{4}\sum_{j,k=1}^{n} \langle R(X, Y)e_j, e_k\rangle\ e_j \cdot e_k \cdot \phi + \frac{i}{2}\Omega(s_*(X), s_*(Y))\phi,$$

for any $X, Y \in \Gamma(TN)$. Here R is the curvature tensor of the Levi-Civita connection ∇. We know that Ω can be viewed as a real valued 2-form on N [Fri00, Ko-No96]. In this case, $i\Omega$ is the curvature form of the auxiliary line bundle L. Finally,

$$\mathcal{R}(X, Y)\phi = \frac{1}{4}\sum_{j,k=1}^{n} \langle R(X, Y)e_j, e_k\rangle\ e_j \cdot e_k \cdot \phi + \frac{i}{2}\Omega(X, Y)\phi. \qquad (1.6)$$

Using the Bianchi identity and Equation (1.6), we easily get the Ricci identity

$$\sum_{k=1}^{n} e_k \cdot \mathcal{R}(e_k, X)\phi = \frac{1}{2}\mathrm{Ric}(X) \cdot \phi - \frac{i}{2}(X \lrcorner \Omega) \cdot \phi, \qquad (1.7)$$

where Ric denotes the Ricci curvature considered as a field of endomorphism on TN.

Remark 1.2.1. *Consider a* Spin *manifold N with its trivial* Spin^c *structure (see Example 1.2.1). On the trivial $\mathbb{S}^1$-principal fiber bundle, we choose the trivial connection whose curvature form is zero, i.e. $i\Omega = 0$. Hence Formulas (1.5), (1.6) and (1.7) are well known on* Spin *manifolds.*

1.2.3 The Spin^c Dirac Operator

Definition 1.2.1. *The Dirac operator is the composition of the covariant derivative acting on sections of ΣN with the Clifford multiplication. Locally, it is given by*

$$\begin{aligned} D : \Gamma(\Sigma N) &\longrightarrow \Gamma(\Sigma N) \\ \phi &\longmapsto \sum_{j=1}^{n} e_j \cdot \nabla_{e_j}\phi. \end{aligned}$$

The Dirac operator is a first order partial differential operator, which is elliptic and formally self-adjoint with respect to $\int_N \langle ., . \rangle \, v_g$, if N is compact, where v_g denotes the volume element. It means that, for any $\phi, \psi \in \Gamma(\Sigma N)$, we have

$$\int_N \langle D\phi, \psi \rangle \, v_g = \int_N \langle \phi, D\psi \rangle \, v_g.$$

We should point out that, in even dimensions, the Dirac operator sends positive spinors to negative ones and conversely, i.e.

$$D : \Gamma(\Sigma^{\pm} N) \longrightarrow \Gamma(\Sigma^{\mp} N).$$

An important tool when examining the Dirac operator is the Schrödinger-Lichnerowicz formula. It relates the square of the Dirac operator to some geometric data, like the scalar curvature.

Theorem 1.2.1 (The Schrödinger-Lichnerowicz formula)**.** *Denoting by S the scalar curvature of a Riemannian* Spin^c *manifold N, we have*

$$D^2 = \nabla^*\nabla + \frac{1}{4} S \, \mathrm{Id}_{\Gamma(\Sigma N)} + \frac{i}{2}\Omega\cdot, \tag{1.8}$$

where $\Omega\cdot$ is the extension of the Clifford multiplication to differential forms and $\nabla^*\nabla$ is the Hodge-Laplacian defined on the Spin^c *bundle given in local normal coordinates by $\nabla^*\nabla = -\sum_{j=1}^{n} \nabla_{e_j}\nabla_{e_j}$.*

Proof. For normal coordinates at any point $x \in N$, we have

$$\begin{aligned} D^2 &= \Big(\sum_{j=1}^{n} e_j \cdot \nabla_{e_j}\Big)\Big(\sum_{k=1}^{n} e_k \cdot \nabla_{e_k}\Big) = \sum_{j,k=1}^{n} e_j \cdot e_k \cdot \nabla_{e_j}\nabla_{e_k} \\ &= -\sum_{j=1}^{n} \nabla_{e_j}\nabla_{e_j} + \sum_{j,k=1; j\neq k}^{n} e_j \cdot e_k \cdot \nabla_{e_j}\nabla_{e_k} \\ &= -\sum_{j=1}^{n} \nabla_{e_j}\nabla_{e_j} + \sum_{j<k}^{n} e_j \cdot e_k \cdot \big(\nabla_{e_j}\nabla_{e_k} - \nabla_{e_k}\nabla_{e_j}\big) \\ &= \nabla^*\nabla + \frac{1}{2}\sum_{j,k=1}^{n} e_j \cdot e_k \cdot \mathcal{R}_{e_j,e_k}. \end{aligned}$$

Using Equation (1.6), we get

$$\begin{aligned} D^2 = \nabla^*\nabla \;\; &+ \;\; \frac{1}{2}\sum_{j,k=1}^{n} e_j \cdot e_k \cdot \Big(\frac{1}{4}\sum_{l,m=1}^{n} \langle R(e_j,e_k)e_l, e_m\rangle\, e_l \cdot e_m \cdot \Big) \\ &+ \;\; \frac{i}{4}\sum_{j,k=1}^{n} \Omega(e_j,e_k)\; e_j \cdot e_k \cdot \end{aligned}$$

But an easy computation yields

$$\begin{aligned} \frac{1}{2}\sum_{j,k=1}^{n} e_j \cdot e_k \cdot \Big(\frac{1}{4}\sum_{l,m=1}^{n} \langle R(e_j,e_k)e_l, e_m\rangle\, e_l \cdot e_m \cdot \Big) \;\; &= \;\; \frac{1}{4}S, \\ \frac{i}{4}\sum_{j,k=1}^{n} \Omega(e_j,e_k) e_j \cdot e_k \cdot \;\; &= \;\; \frac{i}{2}\Omega\cdot, \end{aligned}$$

which finishes the proof.

1.3 Spin^c structures on complex manifolds

Now, we assume that N^n is a differentiable manifold carrying an almost complex structure. An almost complex structure on a differentiable manifold N^n is given by a $(1,1)$-tensor J satisfying $J^2 = -\mathrm{Id}_{TN}$. The pair (N, J) is then referred to as an almost complex manifold. An almost complex manifold should have an even real dimension, i.e. $n = 2m$. The integer m is called the complex dimension of the manifold N. The endomorphism J can be extended by $\mathbb{C}$-linearity to the complexified tangent bundle $T^{\mathbb{C}}N = TN \otimes_{\mathbb{R}} \mathbb{C}$, then

$$T^{\mathbb{C}}N = T_{1,0}N \oplus T_{0,1}N,$$

where $T_{1,0}N$ (resp. $T_{0,1}N$) denotes the eigenbundle of $T^{\mathbb{C}}N$ corresponding to the eigenvalue i (resp. $-i$) of J. The bundle $T_{1,0}N$ is given by

$$T_{1,0}N = \overline{T_{0,1}N} = \{X - iJX \;\; | X \in \Gamma(TN)\}.$$

Fix a Hermitian metric g compatible with the almost complex structure, i.e. a Riemannian metric g with the proprerty

$$g(JX, JY) = g(X, Y),$$

then $\ltimes(X, Y) = g(X, JY)$ is a real 2-form on N. We will call (N, J, g) a Hermitian manifold. On a Hermitian manifold (N, J, g), the almost complex structure is called a complex structure if and only if $T_{1,0}N$ is formally integrable, i.e. $[T_{1,0}N, T_{1,0}N] \subset T_{1,0}N$. This integrability condition is equivalent to say that the Nijenhuis tensor N^J vanishes. The Nijenhuis tensor N^J is the $(2,1)$-tensor defined by

$$N^J(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY],$$

for any $X, Y \in \Gamma(TN)$. A Kähler manifold is a Hermitian manifold (N, J, g) such that J is a complex structure and $\nabla J = 0$, where ∇ is the Levi-Civita connection on N.

Consider (N, J, g) a Hermitian manifold and $\{e_1, Je_1, \cdots, e_m, Je_m\}$ an orthonormal basis of TN. We point out that, as a complex vector space, TN is $\mathbb{C}$-isomorphic (resp. $\mathbb{C}$-anti-isomorphic) to $T_{1,0}N$ (resp. $T_{0,1}N$). We define a complex vector Z_j by

$$Z_j = \frac{1}{2}(e_j - iJe_j).$$

It is easy to check that $\{Z_1, \cdots, Z_m\}$ forms a basis of $T_{1,0}N$ and $\{\overline{Z_1}, \cdots, \overline{Z_m}\}$ forms a basis of $T_{0,1}N$. Moreover, we have for any $k, l = 1, \cdots, m$

$$\begin{aligned} g(Z_k, Z_l) &= g(\overline{Z_k}, \overline{Z_l}) = 0, \\ g(Z_k, \overline{Z_l}) &= g(\overline{Z_k}, Z_l) = \frac{1}{2}\delta_{kl}. \end{aligned}$$

Hence, the dual space $T^*_{1,0}N$ of $T_{1,0}N$ is $\mathbb{C}$-isomorphic to $T_{0,1}N$ by

$$\begin{aligned} T_{0,1}N &\longrightarrow T^*_{1,0}N \\ \overline{Z_j} &\longmapsto {Z_j}^* =: 2g(\overline{Z_j}, .) \end{aligned}$$

Similary, $T_{0,1}N$ is $\mathbb{C}$-isomorphic to $T^*_{0,1}N$. The set $\{{Z_1}^*, \cdots, {Z_m}^*\}$ forms a basis of $T^*_{1,0}N$ and $\{\overline{Z_1}^*, \cdots, \overline{Z_m}^*\}$ forms a basis of $T^*_{0,1}N$.

Proposition 1.3.1. *Every Hermitian manifold (N^n, J, g) has a canonical* $\mathrm{Spin^c}$ *structure whose* $\mathrm{Spin^c}$ *bundle is given by $\Sigma N = \Lambda^{0,*}N = \oplus_{r=0}^m \Lambda^{0,r}N$, where $\Lambda^{0,r}N = \Lambda^r(T^*_{0,1}N)$ is the bundle of complex r-forms of type $(0, r)$. The auxiliary line bundle of this canonical* $\mathrm{Spin^c}$ *structure is given by K_N^{-1}, where $K_N = \Lambda^m(T^*_{1,0}N)$ is the canonical bundle of N [Fri00].*

Proof. A Hermitian structure on N^{2m} produces a reduction of the SO_{2m} principal bundle $P_{\mathrm{SO}_{2m}}N$ to the subgroup U_m, where U_m is viewed canonically as a subgroup of SO_{2m}. We denote the U_m-reduction by $\mathrm{U}(N)$. Besides, there exists a homomorphism $F : \mathrm{U}_m \longrightarrow \mathrm{Spin}^c_{2m}$ such that the following diagram commutes

$$\begin{tikzcd} & \mathrm{Spin}^c_{2m} \arrow[d, "\xi^c"] \\ \mathrm{U}_m \arrow[ur, "F"] \arrow[r, "f"] & \mathrm{SO}_{2m} \times \mathbb{S}^1 \end{tikzcd}$$

Here, f is defined by $f(A) = (A, \det A)$, $A \in \mathrm{U}_m$ and the homomorphism F can be explicitly described. In fact,

$$F(A) = \Big[\prod_{j=1}^m \Big(\cos(\frac{\theta_j}{2}) + \sin(\frac{\theta_j}{2}) f_j J(f_j)\Big), e^{\frac{i}{2}\sum_{j=1}^m \theta_j}\Big],$$

where $\{f_1, \cdots, f_m\}$ is a unitary basis in $\mathbb{C}^m$ with respect to which A has diagonal form $A = (e^{i\theta_1}, \cdots, e^{i\theta_k})$ and J is the complex structure of $\mathbb{C}^m$. The lift F induces a Spin^c structure on N via the Spin^c_{2m}-principal fiber bundle

$$P_{\mathrm{Spin}^c_{2m}} N = \mathrm{U}(N) \times_{\mathrm{U}_m} \mathrm{Spin}^c_{2m}.$$

The corresponding $\mathbb{S}^1$-principal bundle is given by

$$P_{\mathbb{S}^1} N = \mathrm{U}(N) \times_{\det} \mathbb{S}^1.$$

By definition, the auxiliary line bundle of this canonical Spin^c structure is given by

$$L = P_{\mathbb{S}^1} N \times_\rho \mathbb{C} = \mathrm{U}(N) \times_{\rho\circ\det} \mathbb{C} = \Lambda^m(TN) = \Lambda^m(T^*_{0,1}N).$$

Now, using that $\mathbb{C}l_m \simeq \Sigma_{2m}$ we can easily check that $\rho^c_{2m} \circ F = \widetilde{\rho_m}$, where $\widetilde{\rho_m} : \mathrm{U}_m \longrightarrow \mathrm{End}(\mathbb{C}l_m)$ is the extension of the standard representation of U_m to $\mathbb{C}l_m$. Hence, the Spin^c bundle is given by

$$\Sigma N = \mathrm{U}(N) \times_{\rho^c_{2m}\circ F} \Sigma_{2m} \simeq \mathrm{U}(N) \times_{\widetilde{\rho}_m} \mathbb{C}l_m = \mathbb{C}l(TN) = \Lambda^*(TN) = \Lambda^*(T^*_{0,1}N) = \Lambda^{0,*}N.$$

For any other Spin^c structure the associated Spin^c bundle can be written as [Fri00]

$$\Sigma N = \Lambda^{0,*}N \otimes \mathcal{L},$$

where $\mathcal{L}^2 = K_N \otimes L$ and L is the auxiliary bundle associated with this Spin^c structure. In this case, the 2-form $\ltimes$ can be considered as an endomorphism of ΣN via Clifford multiplication and it acts on a spinor ψ locally by [Fri00]

$$\ltimes \cdot \psi = -\sum_{j=1}^m e_j \cdot Je_j \cdot \psi,$$

where $\{e_1, Je_1, \cdots, e_m, Je_m\}$ is any local frame of tangent vector fields. Moreover, we have the well-known orthogonal splitting

$$\Sigma N = \oplus_{l=0}^m \Sigma_l N,$$

where $\Sigma_l N$ denotes the eigensubbundle corresponding to the eigenvalue $i(m-2l)$ of $\ltimes$, with complex rank $\binom{m}{l}$. Moreover, for any $Z \in \Gamma(T_{1,0}N)$ and for any $\psi \in \Gamma(\Sigma_r N)$, we have $Z \cdot \psi \in \Gamma(\Sigma_{r+1}N)$ and $\overline{Z} \cdot \psi \in \Gamma(\Sigma_{r-1}N)$.

Remark 1.3.1. *If we choose the function f to be $f(A) = (A, \frac{1}{\det A})$, we get on N another* Spin^c *structure called the anti-canonical* Spin^c *structure for which $\Sigma N = \Lambda^{*,0}N$ and the auxiliary line bundle is given by K_N.*

Proposition 1.3.2. *Consider (N^{2m}, J, g) a Hermitian manifold. For any* Spin^c *structure, we have*

$$(\Sigma_0 N)^2 = K_N \otimes L,$$

where L is the auxiliary line bundle associated with the Spin^c *structure.*

Proof. First, we can prove that there is an isomorphism between $\Lambda^{0,r}N \otimes \Sigma_0 N$ and $\Sigma_r N$ for any $r = 0, \cdots, m$ [Fri00, Kir86]. Hence, $\Sigma N = \Lambda^{0,*}N \otimes \Sigma_0 N$. But, $\Sigma N = \Lambda^{0,*}N \otimes \mathcal{L}$, where $\mathcal{L}^2 = K_N \otimes L$. Finally, $(\Sigma_0 N)^2 = K_N \otimes L$.

For the canonical Spin^c structure and since $L = (K_N)^{-1}$, we deduce that $\Sigma_0 N$ is trivial. Hence, if M is Kähler, the canonical Spin^c structure carries parallel spinors lying in $\Sigma_0 N$ (the complex constant functions).

Remark 1.3.2. *We can view the canonical* Spin^c *structure on Kähler manifolds differently using the definition of* Spin^c *structure given in the Introduction. In fact, the complex fiber bundle* $\Lambda^{0,*}N = \oplus_{r=0}^m \Lambda^{0,r}N$*, of rank* 2^m*, has a scalar product (the extension of the metric) and a connection (the extension of the Levi-civita connection on* N*) which are compatible. Moreover, the map* $\gamma : TN \longrightarrow \mathrm{End}(\Lambda^{0,*}N)$ *defined by*

$$\gamma(X)\psi = \frac{1}{\sqrt{2}}(X + iJX)^\flat \wedge \psi - \sqrt{2}X \lrcorner \psi,$$

satisfies the conditions (30), (31) et (32) for every $X \in \Gamma(TN)$, $\psi \in \Gamma(\Lambda^{0,r}N)$ *and* $r = 0, \cdots, m$*. Hence, every Kähler manifold has a* Spin^c *structure.*

In 1997, A. Moroianu [Moro97] classified all simply connected Spin^c manifolds carrying parallel spinors:

Theorem 1.3.1. *A simply connected* Spin^c *manifold* N *carries a parallel spinor if and only if it is isometric to the Riemannian product* $N_1 \times N_2$ *of a simply connected Kähler manifold* N_1 *and a simply connected* Spin *manifold* N_2 *carrying a parallel spinor.*

In this case, the Spin^c structure of N is the product of the canonical Spin^c structure of N_1 and the Spin structure of N_2 and the parallel spinor is the product $\psi_1 \otimes \psi_2$ of a parallel spinor ψ_1 of the canonical Spin^c structure of N_1 (this spinor lies in $\Sigma_0 N_1$) and the parallel spinor of the Spin manifold N_2. Moreover, the connection on the auxiliary line bundle associated with this product Spin^c structure is the extension of the Levi-Civita connection to $(K_{N_1})^{-1}$. Hence, the curvature 2-form $i\Omega$ is given by $i\Omega = i\rho_{N_1}$, where ρ_{N_1} is the Ricci 2-form on N_1 defined by $\rho_{N_1}(X,Y) = \mathrm{ric}_{N_1}(X, JY)$ for every $X, Y \in \Gamma(TN_1)$. Here, ric_{N_1} is the Ricci curvature of N_1 considered as a bilinear symmetric form.

Chapter 2

Lower Bounds for the Eigenvalues of the Spin^c Dirac Operator[1]

2.1 Introduction

On a compact Riemannian Spin manifold (N^n, g) of dimension $n \geqslant 2$, T. Friedrich [Fri80] showed that any eigenvalue λ of the Dirac operator satisfies

$$\lambda^2 \geqslant \lambda_1^2 := \frac{n}{4(n-1)} \inf_N S, \tag{2.1}$$

The limiting case of (2.1) is characterized by the existence of a special spinor called a real Killing spinor. This is a section ψ of the spinor bundle satisfying, for every $X \in \Gamma(TN)$,

$$\nabla_X \psi = -\frac{\lambda_1}{n} X \cdot \psi.$$

In [Hij95], O. Hijazi defined, on the complement set of zeroes of any spinor field ϕ, a field of symmetric endomorphisms ℓ^ϕ associated with the field of quadratic forms, denoted also by ℓ^ϕ, called the energy-momentum tensor which is given, for any vector field X, by

$$\ell^\phi(X) = \mathrm{Re} \left\langle X \cdot \nabla_X \phi, \frac{\phi}{|\phi|^2} \right\rangle.$$

The associated symmetric bilinear form is then given for every $X, Y \in \Gamma(TN)$ by

$$\ell^\phi(X, Y) = \frac{1}{2} \mathrm{Re} \left\langle X \cdot \nabla_Y \phi + Y \cdot \nabla_X \phi, \frac{\phi}{|\phi|^2} \right\rangle.$$

Note that, if the spinor field ϕ is an eigenspinor, C. Bär showed that the zero set is contained in a countable union of $(n-2)$-dimensional submanifolds and has locally finite $(n-2)$-dimensional Hausdroff density [Bär99]. In 1995, O. Hijazi [Hij95] modified the connection ∇ in the direction of the endomorphism ℓ^ψ where ψ is an eigenspinor associated with an eigenvalue λ of the Dirac operator and established that

$$\lambda^2 \geqslant \inf_N \left(\frac{1}{4} S + |\ell^\psi|^2\right). \tag{2.2}$$

[1]This chapter is the subject of a published paper [Nak10]

The limiting case of (2.2) is characterized by the existence of a spinor field ψ satisfying for all $X \in \Gamma(TN)$,

$$\nabla_X \psi = -\ell^\psi(X) \cdot \psi. \tag{2.3}$$

The trace of ℓ^ψ being equal to λ, Inequality (2.2) improves Inequality (2.1) since, by the Cauchy-Schwarz inequality, $|\ell^\psi|^2 \geqslant \frac{(\mathrm{tr}(\ell^\psi))^2}{n}$, where $\mathrm{tr}(\ell^\psi)$ denotes the trace of ℓ^ψ. N. Ginoux and G. Habib showed in [GH09] that the Heisenberg manifold is a limiting manifold for (2.2) but equality in (2.1) cannot occur.

Using the conformal covariance of the Dirac operator, O. Hijazi [Hij86] showed that, on a compact Riemannian Spin manifold (N^n, g) of dimension $n \geqslant 3$, any eigenvalue of the Dirac operator satisfies

$$\lambda^2 \geqslant \lambda_1^2 := \frac{n}{4(n-1)}\mu_1, \tag{2.4}$$

where μ_1 is the first eigenvalue of the Yamabe operator given by

$$L := 4\frac{n-1}{n-2} \bigtriangleup + S,$$

where $\bigtriangleup$ is the Laplacian acting on functions. In dimension 2, C. Bär [Bär92] proved that any eigenvalue of the Dirac operator on N satisfies

$$\lambda^2 \geqslant \lambda_1^2 := \frac{2\pi\chi(N)}{Area(N,g)}, \tag{2.5}$$

where $\chi(N)$ is the Euler-Poincaré number of N. The limiting case of (2.4) and (2.5) is also characterized by the existence of a real Killing spinor. In terms of the energy-momentum tensor, O. Hijazi [Hij95] proved that, on such manifolds any eigenvalue of the Dirac operator satisfies the following

$$\lambda^2 \geqslant \lambda_1^2 := \begin{cases} \frac{\pi\chi(N)}{Area(N,g)} + \inf\limits_N |\ell^\psi|^2 & \text{if} \quad n = 2, \\ \\ \frac{1}{4}\mu_1 + \inf\limits_N |\ell^\psi|^2 & \text{if} \quad n \geqslant 3. \end{cases} \tag{2.6}$$

Again, the trace of ℓ^ψ being equal to λ, Inequality (2.6) improves Inequalities (2.4) and (2.5). The limiting case of (2.6) is characterized by the existence of a spinor field $\overline{\varphi}$ satisfying for all $X \in \Gamma(TN)$,

$$\overline{\nabla}_X \overline{\varphi} = -\ell^{\overline{\varphi}}(X) \mathbin{\bar{\cdot}} \overline{\varphi}, \tag{2.7}$$

where $\overline{\varphi} = e^{-\frac{n-1}{2}u}\overline{\psi}$, the spinor field ψ is an eigenspinor associated with the first eigenvalue of the Dirac operator and $\overline{\psi}$ is the image of ψ under the isometry between the spinor bundles of (N^n, g) and $(N^n, \overline{g} = e^{2u}g)$. Suppose that on a Spin manifold N, there exists a spinor field ϕ such that, for all $X \in \Gamma(TN)$,

$$\nabla_X \phi = -E(X) \cdot \phi, \tag{2.8}$$

where E is a symmetric 2-tensor defined on TN. It is easy to see that E must be equal to ℓ^ϕ. If the dimension of N is equal to 2, T. Friedrich [Fri98] proved that the existence of a pair (ϕ, E) satisfying (2.8) is equivalent to the existence of a local immersion of N into the Euclidean space $\mathbb{R}^3$ with Weingarten tensor equal to $2E$. In [Mor05], B. Morel showed that if N^n is a hypersurface of a manifold carrying a parallel spinor, then the energy-momentum tensor (associated with the restriction of the parallel spinor) is, up to a constant, the second fundamental form of the hypersurface. G. Habib [Hab07] studied Equation (2.8) for an endomorphism E, not necessarily symmetric. He showed that the symmetric part of E is ℓ^ϕ and the skew-symmetric part of E is q^ϕ defined on the complement set of zeroes of ϕ by

$$q^\phi(X,Y) = g(q^\phi(X),Y) = \frac{1}{2}\mathrm{Re}\left\langle Y\cdot\nabla_X\phi - X\cdot\nabla_Y\phi, \frac{\phi}{|\phi|^2}\right\rangle,$$

for all $X, Y \in \Gamma(TN)$. Then, he modifies the connection in the direction of $\ell^\psi + q^\psi$ where ψ is an eigenspinor associated with an eigenvalue λ and gets that

$$\lambda^2 \geqslant \inf_N(\frac{1}{4}S + |\ell^\psi|^2 + |q^\psi|^2). \tag{2.9}$$

The Heisenberg group and the solvable group are examples of limiting manifolds [Hab07]. For a better understanding of the tensor q^ϕ, he studied Riemannian flows and proved that, if the normal bundle carries a parallel spinor, the tensor q^ϕ plays the role of the O'Neill tensor of the flow. In this chapter, we prove the corresponding inequalities for $\mathrm{Spin^c}$ manifolds:

Theorem 2.1.1. *Let (N^n, g) be a compact Riemannian* $\mathrm{Spin^c}$ *manifold of dimension $n \geqslant 2$. Then, any eigenvalue of the Dirac operator to which is attached an eigenspinor ψ satisfies*

$$\lambda^2 \geqslant \inf_N \Big(\frac{1}{4}S - \frac{c_n}{4}|\Omega|_g + |\ell^\psi|^2 + |q^\psi|^2\Big), \tag{2.10}$$

where $c_n = 2[\frac{n}{2}]^{\frac{1}{2}}$ and $|\Omega|_g$ is the norm of Ω with respect to g.

We will only consider deformations of the connection in the direction of the symmetric endomorphism ℓ^ψ and hence under the same conditions as Theorem 2.1.1, one gets

$$\lambda^2 \geqslant \inf_N \Big(\frac{1}{4}S - \frac{c_n}{4}|\Omega|_g + |\ell^\psi|^2\Big). \tag{2.11}$$

In 1999, A. Moroianu and M. Herzlich [HM99] proved that on $\mathrm{Spin^c}$ manifolds of dimension $n \geqslant 3$, any eigenvalue of the Dirac operator satisfies

$$\lambda^2 \geqslant \lambda_1^2 := \frac{n}{4(n-1)}\mu_1, \tag{2.12}$$

where μ_1 is the first eigenvalue of the perturbed Yamabe operator defined by

$$L^\Omega = L - c_n|\Omega|_g.$$

The limiting case of (2.12) is characterized by the existence of a real Spin^c Killing spinor ψ satisfying $\Omega\cdot\psi = i\frac{c_n}{2}|\Omega|_g\psi$, i.e. a section ψ satisfying

$$\nabla_X\psi = -\frac{\lambda_1}{n}X\cdot\psi \quad\text{and}\quad \Omega\cdot\psi = i\frac{c_n}{2}|\Omega|_g\psi.$$

In terms of the energy-momentum tensor we prove:

Theorem 2.1.2. *Under the same conditions as Theorem 2.1.1, any eigenvalue λ of the Dirac operator to which is attached an eigenspinor ψ satisfies*

$$\lambda^2 \geqslant \begin{cases} \frac{1}{4}\mu_1 + \inf_N |\ell^\psi|^2 & \text{if} \quad n\geqslant 3,\\[2ex] \frac{\pi\chi(N)}{Area(N,g)} - \frac{1}{2}\frac{\int_N |\Omega|_g v_g}{Area(N,g)} + \inf_N |\ell^\psi|^2 & \text{if} \quad n=2,\end{cases} \tag{2.13}$$

where μ_1 is the first eigenvalue of the perturbed Yamabe operator.

Using the Cauchy-Schwarz inequality in dimension $n\geqslant 3$, we have that Inequality (2.13) implies Inequality (2.12). As a corollary of Theorem 2.1.2, we compare the lower bound to a conformal invariant (the Yamabe number) and to a topological invariant, in case of 4-dimensional manifolds whose auxiliary line bundle has self dual curvature (see Corollary 2.3.1 and Corollary 2.3.2). Finally, we study the limiting case of (2.11) and (2.13), and we give an example.

2.2 Eigenvalue estimates of the Spin^c Dirac operator

In this section, we prove the lower bound (2.10). This proof is based on the following Lemma given by A. Moroianu and M. Herzlich in [HM99]:

Lemma 2.2.1. *Let (N^n, g) be a Riemannian* Spin^c *manifold. For any spinor field $\psi\in\Gamma(\Sigma N)$ and a real 2-form Ω, we have*

$$\langle i\Omega\cdot\psi,\psi\rangle \ \geqslant -\frac{c_n}{2}|\Omega|_g|\psi|^2, \tag{2.14}$$

where $|\Omega|_g$ is the norm of Ω with respect to g, given by $|\Omega|_g^2 = \sum_{i<j}(\Omega_{ij})^2$, in any orthonormal local frame. Moreover, if equality holds in (2.14), then

$$\Omega\cdot\psi = i\frac{c_n}{2}|\Omega|_g\psi. \tag{2.15}$$

In this case, $\Omega = 0$ or Ω has maximal rank, i.e. $\Omega = 0$ or of rank n if n is even and $n-1$ if n is odd.

Proof. Consider Ω as a skew-symmetric Hermitian operator on $TN\otimes_{\mathbb{R}}\mathbb{C}$. Then $i\Omega$ is Hermitian, so that all its eigenvalues are real and $TN\otimes_{\mathbb{R}}\mathbb{C}$ decomposes as a direct sum of the corresponding eigenspaces. This easily shows that we may find an orthonormal basis $\{e_1, e_2,\cdots, e_n\}$ of TN such that

$$\Omega = \sum_{j=1}^{[\frac{n}{2}]} \alpha_j\ e_{2j-1}\wedge e_{2j}.$$

By the Cauchy-Schwarz inequality, we have

$$\langle ie_{2j-1}\cdot e_{2j}\cdot\psi,\psi\rangle^2 \leqslant |ie_{2j-1}\cdot e_{2j}\cdot\psi|^2|\psi|^2=|\psi|^4.$$

It follows that

$$\begin{aligned}\Big(\sum_{j=1}^{[\frac{n}{2}]}|\alpha_j|\Big)|\psi|^2 &= \sum_{j=1}^{[\frac{n}{2}]}|\alpha_j||\psi|^2\\ &\leqslant \Big(\sum_{j=1}^{[\frac{n}{2}]}|\alpha_j|^2\Big)^{\frac12}\Big(\sum_{j=1}^{[\frac{n}{2}]}|\psi|^4\Big)^{\frac12}\\ &= \Big[\frac{n}{2}\Big]^{\frac12}|\psi|^2\Big(\sum_{j=1}^{[\frac{n}{2}]}|\alpha_j|^2\Big)^{\frac12},\end{aligned}$$

and so

$$-\Big[\frac{n}{2}\Big]^{\frac12}\Big(\sum_{j=1}^{[\frac{n}{2}]}|\alpha_j|^2\Big)^{\frac12}|\psi|^2\leqslant-\sum_{j=1}^{[\frac{n}{2}]}|\alpha_j||\psi|^2\ \ \langle\ i\Omega\cdot\psi,\psi\ \rangle\,.$$

If equality holds in (2.14), then all above inequalities are actually equalities. Then, $\Omega\cdot\psi=i\frac{c_n}{2}|\Omega|_g\psi$. Moreover,

$$\sum_{j=1}^{[\frac{n}{2}]}|\alpha_j|=\Big[\frac{n}{2}\Big]^{\frac12}\Big(\sum_{j=1}^{[\frac{n}{2}]}|\alpha_j|^2\Big)^{\frac12},$$

and all the α_j's must have equal absolute values, thus showing that $\Omega=0$ or has maximal rank.

Proof of Theorem 2.1.1. Let E (resp. Q) be a symmetric (resp. skew-symmetric) 2-tensor defined on TN. For any spinor field ϕ, the modified connection

$$\widetilde{\nabla}_X\phi:=\nabla_X\phi+E(X)\cdot\phi+Q(X)\cdot\phi,$$

satisfies

$$|\widetilde{\nabla}\phi|^2=|\nabla\phi|^2+|E|^2|\phi|^2-2\sum_{i,j=1}^{n}E(e_i,e_j)\ell^\phi(e_i,e_j)|\phi|^2+|Q|^2|\phi|^2-2\sum_{i,j=1}^{n}Q(e_i,e_j)q^\phi(e_i,e_j)|\phi|^2,$$

where $\{e_1,...,e_n\}$ is any orthonormal local basis tangent to N. Let ψ be an eigenspinor corresponding to the eigenvalue λ of D. For $E=\ell^\psi$, $Q=q^\psi$, we get

$$|\widetilde{\nabla}\psi|^2=|\nabla\psi|^2-|\ell^\psi|^2|\psi|^2-|q^\psi|^2|\psi|^2.$$

By Lemma 2.2.1 and the Schrödinger-Lichnerowicz formula, it follows

$$\begin{aligned}\lambda^2\int_N|\psi|^2v_g &\geqslant \frac14\int_N S|\psi|^2v_g+\int_N(|\ell^\psi|^2+|q^\psi|^2)|\psi|^2v_g\\ &\quad+\int_N\Big\langle\frac{i}{2}\Omega\cdot\psi,\psi\Big\rangle v_g\\ &\geqslant \int_N\Big(\frac14 S-\frac{c_n}{4}|\Omega|_g+|\ell^\psi|^2+|q^\psi|^2\Big)|\psi|^2v_g.\end{aligned}$$

Finally,

$$\lambda^2 \geqslant \inf_N \Big(\frac{1}{4}S - \frac{c_n}{4}|\Omega|_g + |\ell^\psi|^2 + |q^\psi|^2\Big).$$

2.3 Conformal geometry and eigenvalue estimates

Before proving Theorem 2.1.2, we give some basic facts on conformal Spin^c geometry. The conformal class of g is the set of metrics $\overline{g} = e^{2u}g$, for a real function u on N. At a given point x of N, we consider a g-orthonormal basis $\{e_1, \ldots, e_n\}$ of T_xN. The corresponding $\overline{g}$-orthonormal basis is denoted by $\{\overline{e}_1 = e^{-u}e_1, \ldots, \overline{e}_n = e^{-u}e_n\}$. This correspondence extends to the Spin^c level to give an isometry between the corresponding Spin^c bundles. We put a " $-$ " above every object which is naturally associated with the metric $\overline{g}$, except for the scalar curvature where S (resp. S_u or S_h) denotes the scalar curvature associated with the metric g (resp. $\overline{g} = e^{2u}g = h^{\frac{4}{n-2}}g$). Then, for any spinor fields ψ and φ, one has

$$\langle\overline{\psi}, \overline{\varphi}\rangle = \langle\psi, \varphi\rangle \ ,$$

where $\langle .,.\rangle$ denotes the natural Hermitian scalar products on $\Gamma(\Sigma N)$ and on $\Gamma(\Sigma\overline{N})$. The corresponding Dirac operators satisfy

$$\overline{D}\,(\, e^{-\frac{(n-1)}{2}u}\ \overline{\psi}\,) = \ e^{-\frac{(n+1)}{2}u}\ \overline{D\psi}. \tag{2.16}$$

The norms of any real 2-form Ω with respect to g and $\overline{g}$ are related by

$$|\Omega|_{\overline{g}} \ = e^{-2u}|\Omega|_g. \tag{2.17}$$

O. Hijazi [Hij95] showed that on a Spin manifold the energy-momentum tensor verifies

$$|\ell^{\overline{\varphi}}|^2 = e^{-2u}\,|\ell^\varphi|^2 \ = e^{-2u}\,|\ell^\psi|^2, \tag{2.18}$$

where $\varphi = e^{-\frac{(n-1)}{2}u}\psi$. We extend the result to a Spin^c manifold and get the same relation.

Lemma 2.3.1. *Under the same conditions as Theorem 2.1.1, any eigenvalue λ of the Dirac operator to which is attached an eigenspinor ψ satisfies*

$$\lambda^2 \geqslant \frac{1}{4}\sup_u \inf_N (S_u e^{2u} - c_n|\Omega|_g) + \inf_N |\ell^\psi|^2.$$

Proof. For any spinor field ϕ and for any symmetric 2-tensor E defined on TN, the modified connection introduced in [Hij95]:

$$\nabla^E_X\phi = \nabla_X\phi + E(X)\cdot\phi,$$

verifies $|\nabla^E\phi|^2 = |\nabla\phi|^2 + |E|^2|\phi|^2 - 2\sum_{i,j=1}^n E(e_i,e_j)\ell^\phi(e_i,e_j)|\phi|^2$. Using the Schrödinger-Lichnerowicz formula on N, applied to the spinor field $\overline{\phi}$ with respect to the metric $\overline{g}$, yields

$$\begin{aligned}\int_N |\overline{\nabla}^E\overline{\phi}|^2 v_{\overline{g}} &= \int_N |\overline{D}\,\overline{\phi}|^2 v_{\overline{g}} - \int_N \frac{1}{4} S_u |\overline{\phi}|^2 v_{\overline{g}} + \int_N |E|^2 |\overline{\phi}|^2 v_{\overline{g}} \\ &\quad -2\int_N \sum_{i,j=1}^n E(\overline{e}_i,\overline{e}_j)\ell^{\overline{\phi}}(\overline{e}_i,\overline{e}_j)|\overline{\phi}|^2 v_{\overline{g}} - \int_N \left\langle \frac{i}{2}\Omega \cdot \overline{\phi}, \overline{\phi} \right\rangle v_{\overline{g}}. \quad (2.19)\end{aligned}$$

For the spinor $\varphi = e^{-\frac{(n-1)}{2}u}\,\psi$ with $D\psi = \lambda\psi$, one gets $\overline{D}\,\overline{\varphi} = \lambda e^{-u}\,\overline{\varphi}$, and hence by Lemma 2.2.1 and for $E = \ell^{\overline{\varphi}}$

$$\int_N \Big[\lambda^2 - (\frac{1}{4}S_u e^{2u} + |\ell^\psi|^2 - \frac{c_n}{4}|\Omega|_g)\Big] e^{-2u}|\overline{\varphi}|^2 v_{\overline{g}} \geqslant 0. \quad (2.20)$$

Lemma 2.3.2. *Let (N^n, g) be a compact Riemannian* Spin^c *manifold of dimension $n \geqslant 2$ and S (resp. S_u or S_h) the scalar curvature associated with the metric g (resp. $\overline{g} = e^{2u}g = h^{\frac{4}{n-2}}g$). We have*

$$\sup_u \inf_N (S_u e^{2u} - c_n|\Omega|_g) = \begin{cases} \frac{4\pi\chi(N) - 2\int_N |\Omega| v_g}{Area(N,g)} & \text{if} \quad n = 2, \\ \mu_1 & \text{if} \quad n \geqslant 3, \end{cases} \quad (2.21)$$

where μ_1 is the first eigenvalue of the perturbed Yamabe operator L^Ω.

Proof. For $n \geqslant 3$, let $\mathbf{h} > 0$ be an eigenfunction of L^Ω associated with the eigenvalue μ_1 such that $\int_N \mathbf{h}^2 v_g = 1$. For a conformal metric $\overline{g} = e^{2u}g = h^{\frac{4}{n-2}}g$, we have

$$S_h h^{\frac{4}{n-2}} - c_n|\Omega|_g = S_u e^{2u} - c_n|\Omega|_g = h^{-1}L^\Omega h.$$

So $\mu_1 = \mathbf{h}^{-1}L^\Omega\mathbf{h} = S_{\mathbf{h}}\mathbf{h}^{\frac{4}{n-2}} - c_n|\Omega|_g$. For any positive function H, we write $fH = \mathbf{h}$, where f is a positive function, and refering to [Hij91] we get

$$\mu_1 = \int_N (H^{-1}LH) f^2 H^2\ v_g - c_n\int_N |\Omega|_g f^2 H^2\ v_g + \int_N H^2|df|^2\ v_g.$$

Finally,

$$\mu_1 \geqslant \inf_N (H^{-1}L^\Omega H) = \inf_N (S_v e^{2v} - c_n|\Omega|_g),$$

where $e^{2v} = H^{\frac{4}{n-2}}$, then $\mu_1 = \sup_u \inf_N (S_u e^{2u} - c_n|\Omega|_g)$. For $n = 2$ and for every u we have $S_u e^{2u} = S + 2\triangle u$. The Stokes and Gau$\beta$-Bonnet theorems yield

$$\inf_N (S_u e^{2u} - 2|\Omega|_g) \leqslant \frac{\int_N \Big(S_u e^{2u} - 2|\Omega|_g\Big) v_g}{Area(N,g)} = \frac{4\pi\chi(N) - 2\int_N |\Omega|_g v_g}{Area(N,g)}.$$

Let u_0 be a solution of the following equation [Aub82]

$$2\triangle u = \frac{\int_N (S - 2|\Omega|_g) v_g}{Area(N,g)} - S + 2|\Omega|_g, \quad (2.22)$$

hence,

$$S_{u_0}e^{2u_0} - 2|\Omega|_g = 2\bigtriangleup u_0 + S - 2|\Omega|_g = \frac{4\pi\chi(N) - 2\int_N |\Omega|_g v_g}{Area(N,g)}.$$

Proof of Theorem 2.1.2. Combining Lemma 2.3.2 and Lemma 2.3.1, Theorem 2.1.2 follows.

Remark 2.3.1. *Inequality (2.11) improves Inequality (2.12), which itself implies the Friedrich* $\mathrm{Spin^c}$ *inequality (33). Equality holds in (33) if and only if equality holds in (2.12), i.e. if and only if the eigenspinor ψ associated with the first eigenvalue of D is a* $\mathrm{Spin^c}$ *Killing spinor satisfying $\Omega\cdot\psi = i\frac{c_n}{2}|\Omega|_g\psi$.*

Corollary 2.3.1. *Any eigenvalue of the Dirac operator on a compact Riemannian* $\mathrm{Spin^c}$ *manifold of dimension $n \geqslant 3$, satisfies*

$$\lambda^2 \geqslant \frac{1}{4}\mathrm{vol}(N,g)^{-\frac{2}{n}}\Big(Y(N,[g]) - c_n\|\Omega\|_{\frac{n}{2}}\Big) + \inf_N |\ell^\psi|^2,$$

where $Y(N,[g])$ is the Yamabe number given by

$$Y(N,[g]) = \inf_{\eta\neq 0}\frac{\int_N 4\frac{n-1}{n-2}|d\eta|^2 + S\eta^2}{\Big(\int_N |\eta|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}}.$$

Proof. Using the Hölder inequality, it follows

$$\mu_1 = \inf_{\eta\neq 0}\frac{\int_N 4\frac{n-1}{n-2}|d\eta|^2 + (S - c_n|\Omega|_g)\eta^2}{\int_N \eta^2} \geqslant \inf_{\eta\neq 0}\frac{\int_N 4\frac{n-1}{n-2}|d\eta|^2 + (S - c_n|\Omega|)\eta^2}{\Big(\int_N |\eta|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}\mathrm{vol}(N,g)^{\frac{2}{n}}}.$$

Using the Hölder inequality again, we deduce

$$\mu_1\ \mathrm{vol}(N,g)^{\frac{2}{n}} \geqslant \inf_{\eta\neq 0}\frac{\int_N 4\frac{n-1}{n-2}|d\eta|^2 + S\eta^2}{\Big(\int_N |\eta|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}} - c_n\Big(\int_N |\Omega|^{\frac{n}{2}}\Big)^{\frac{2}{n}} = Y(N,[g]) - c_n\|\Omega\|_{\frac{n}{2}}.$$

Finally, replacing in (2.13), we get the result.

Corollary 2.3.2. *On a compact 4-dimensional Riemannian* $\mathrm{Spin^c}$ *manifold with self-dual curvature form $i\Omega$, any eigenvalue of the Dirac operator satisfies*

$$\lambda^2 \geqslant \frac{1}{4}\mathrm{vol}(N,g)^{-\frac{1}{2}}\Big(Y(N,[g]) - 4\pi\sqrt{2}\sqrt{c_1(L)^2}\Big) + \inf_N |\ell^\psi|^2,$$

where $c_1(L)$ is the Chern number of the auxiliary line bundle L associated with the $\mathrm{Spin^c}$ *structure.*

Proof. It follows directly from Corollary 2.3.1 and the fact that if $n = 4$ and Ω self-dual, then $\int_N |\Omega|_g^2 v_g = 4\pi^2 c_1(L)^2$ (see [Fri00]).

2.4 Equality case

In this section, we study the limiting case of (2.11) and (2.13). An example is then given.

Proposition 2.4.1. *Under the same conditions as Theorem 2.1.1,*

$$\text{Equality in (2.11) holds} \iff \begin{cases} \nabla_X \psi = -\ell^\psi(X)\cdot\psi, \\ \Omega\cdot\psi = i\frac{c_n}{2}|\Omega|_g\psi, \end{cases} \tag{2.23}$$

for any $X \in \Gamma(TN)$ and where ψ is an eigenspinor associated with the first eigenvalue of the Dirac operator.

Proof. If equality in (2.11) is achieved, the two conditions follow directly. Now, suppose that $\nabla_X\psi = -\ell^\psi(X)\cdot\psi$ and $\Omega\cdot\psi = i\frac{c_n}{2}|\Omega|_g\psi$. The condition $\nabla_X\psi = -\ell^\psi(X)\cdot\psi$ implies that $|\psi|^2$ is constant. Denoting by $\mathcal{R}$ the curvature tensor on the Spin^c bundle associated with the connection ∇, one easily gets the following relation

$$\mathcal{R}_{X,Y}\,\psi + d\ell^\psi(X,Y)\cdot\psi + [\ell^\psi(X),\ell^\psi(Y)]\cdot\psi = 0,$$

where $d\ell^\psi$ is a 2-form with values in $\Gamma(TN)$ given by

$$d\ell^\psi(X,Y) = (\nabla_X\ell^\psi)Y - (\nabla_Y\ell^\psi)X.$$

Taking $Y = e_j$ and performing its Clifford multiplication by e_j yields by the Ricci identity (1.7) on a Spin^c manifold

$$\begin{aligned} -\frac{1}{2}Ric(X)\cdot\psi + \frac{i}{2}(X\lrcorner\Omega)\cdot\psi \ &+ \ \sum_j e_j\cdot d\ell^\psi(X,e_j)\cdot\psi \\ &+ \ \sum_j e_j\cdot[\ell^\psi(X),\ell^\psi(e_j)]\cdot\psi = 0. \end{aligned} \tag{2.24}$$

We then decompose the last two terms in (2.24) using that $X\cdot\alpha = X\wedge\alpha - X\lrcorner\alpha$ for any form α, it follows

$$\sum_j e_j\cdot d\ell^\psi(X,e_j)\cdot\psi = \sum_j [e_j\wedge d\ell^\psi(X,e_j)]\cdot\psi - [X(\mathrm{tr}\,\ell^\psi) + \mathrm{div}\,\ell^\psi(X)]\psi.$$

$$\sum_j e_j\cdot[\ell^\psi(X),\ell^\psi(e_j)]\cdot\psi = 2\,(\mathrm{tr}\,\ell^\psi)\,\ell^\psi(X)\cdot\psi - 2\sum_j g(X,\ell^\psi(e_j))\,\ell^\psi(e_j)\cdot\psi.$$

Taking the scalar product of (2.24) with ψ, and after seperating real and imaginary parts, yields for every vector field X the relation

$$\Big(X(\mathrm{tr}\,\ell^\psi) + \mathrm{div}\,\ell^\psi(X)\Big)|\psi|^2 = \frac{i}{2}\left\langle (X\lrcorner\Omega)\cdot\psi,\psi\right\rangle. \tag{2.25}$$

But, since Equality (2.15) holds, we compute

$$\begin{aligned} \langle (X\lrcorner\Omega)\cdot\psi,\psi\rangle &= \langle (X\wedge\Omega)\cdot\psi,\psi\rangle - \langle X\cdot\Omega\cdot\psi,\psi\rangle \\ &= \langle (X\wedge\Omega)\cdot\psi,\psi\rangle - i\left[\frac{n}{2}\right]^{\frac{1}{2}}|\Omega|_g\,\langle X\cdot\psi,\psi\rangle. \end{aligned}$$

After separating real and imaginary parts, $\langle (X\lrcorner\Omega)\cdot\psi,\psi\rangle$ must vanish. Using this and $\sum_{j=1}^{n} e_j\cdot(e_j\lrcorner\Omega)\cdot = 2\Omega\cdot$, Clifford multiplication of (2.24) with e_k, and for $X=e_k$, gives

$$-\frac{1}{2}S\psi - i\Omega\cdot\psi = \sum_{k,j} e_j\cdot(e_k\wedge d\ell^{\psi}(e_j,e_k))\cdot\psi - 2(\mathrm{tr}\ \ell^{\psi})^2\psi + 2|\ell^{\psi}|^2\psi.$$

An easy computation implies that $\sum\limits_{k,j} e_j\cdot(e_k\wedge d\ell^{\psi}(e_j,e_k))\cdot\psi = 0$, hence

$$-\frac{1}{2}S + \left[\frac{n}{2}\right]^{\frac{1}{2}}|\Omega|_g = -2(\mathrm{tr}\ \ell^{\psi})^2 + 2|\ell^{\psi}|^2, \tag{2.26}$$

which implies equality in (2.11).

Proposition 2.4.2. *On a compact Riemannian* Spin$^{\mathrm{c}}$ *manifold (N^n,g) of dimension $n\geqslant 3$, assume that the first eigenvalue λ_1 of the Dirac operator to which is attached an eigenspinor ψ satisfies the equality case in (2.13). Then, $|\ell^{\psi}|$ is constant and if $\mathbf{h}>0$ denotes an eigenfunction of the Yamabe operator corresponding to μ_1, then for any vector field X*

$$g(X,\ell^{\psi}(d\mathbf{h}) - \lambda_1 d\mathbf{h}) = g(\lambda_1 X - \ell^{\psi}(X), d\mathbf{h}) = 0. \tag{2.27}$$

Proof. If $n\geqslant 3$ and equality holds in (2.13), we consider the positive function $\mathbf{v}>0$ defined by $e^{2\mathbf{v}} = \mathbf{h}^{\frac{4}{n-2}}$ where $\mathbf{h}$ is an eigenfunction of the Yamabe operator corresponding to μ_1. Inequality (2.20) with $u=\mathbf{v}$ gives $|\ell^{\psi}|$ is constant, $\overline{\nabla}_X\overline{\varphi} = -\ell^{\overline{\varphi}}(X)\,\bar{\cdot}\,\overline{\varphi}$ and $\Omega\,\bar{\cdot}\,\overline{\varphi} = i\frac{c_n}{2}|\Omega|_{\overline{g}}\overline{\varphi}$. By Proposition 2.4.1, Equality (2.26) and (2.25) can be considered for the conformal metric $\overline{g} = e^{2\mathbf{v}}g = \mathbf{h}^{\frac{4}{n-2}}g$ to get

$$(\mathrm{tr}\ \ell^{\overline{\varphi}})^2 := f^2 = \frac{1}{4}S_{\mathbf{v}} - \frac{c_n}{4}|\Omega|_{\overline{g}} + |\ell^{\overline{\varphi}}|^2,$$

$$\mathrm{grad}\, f = -\mathrm{div}\,\ell^{\overline{\varphi}}.$$

It is straightforward to see that these two equalities give (2.27).

2.5 An example

If the lower bound (33) is achieved, automatically equality holds in (2.11). Here, we will give an example where equality holds in (2.11) but not in (33).

Let $(N^3,g)=(\mathbb{S}^3,can)$ be endowed with its unique Spin structure and consider a real Killing spinor ψ with Killing constant $\frac{1}{2}$. As the norm of ψ is constant, we may suppose that $|\psi|=1$. Let ξ be the Killing vector field on N defined by

$$ig(\xi,X) = \langle X\cdot\psi,\psi\rangle\,.$$

First, we have $id\xi(X,Y) = -\langle X \wedge Y \cdot \psi, \psi\rangle$ for any $X, Y \in \Gamma(TN)$. In fact,

$$\begin{aligned}
2id\xi(X,Y) &= X(\langle Y\cdot\psi,\psi\rangle) - Y(\langle X\cdot\psi,\psi\rangle) - \langle [X,Y]\cdot\psi,\psi\rangle \\
&= \langle Y\cdot\nabla_X\psi,\psi\rangle + \langle Y\cdot\psi,\nabla_X\psi\rangle - \langle X\cdot\nabla_Y\psi,\psi\rangle - \langle X\cdot\psi,\nabla_Y\psi\rangle \\
&= \frac{1}{2}\langle Y\cdot X\cdot\psi,\psi\rangle - \frac{1}{2}\langle X\cdot Y\cdot\psi,\psi\rangle - \frac{1}{2}\langle X\cdot Y\cdot\psi,\psi\rangle + \frac{1}{2}\langle Y\cdot X\cdot\psi,\psi\rangle \\
&= \langle Y\cdot X\cdot\psi,\psi\rangle - \langle X\cdot Y\cdot\psi,\psi\rangle \\
&= -2\langle X\wedge Y\cdot\psi,\psi\rangle .
\end{aligned}$$

Using the Hodge operator $*$ and the volume form which acts as the identity on the spinor bundle, we get

$$\begin{aligned}
id\xi(X,Y) &= ig(d\xi, X\wedge Y) \\
&= -\langle X\wedge Y\cdot\psi,\psi\rangle \\
&= \left\langle X\wedge Y\cdot\frac{*(X\wedge Y)\cdot *(X\wedge Y)}{\mid X\wedge Y\mid^2}\cdot\psi,\psi\right\rangle \\
&= \frac{1}{\mid X\wedge Y\mid^2}\left\langle \mid X\wedge Y\mid^2 v_g\cdot *(X\wedge Y)\cdot\psi,\psi\right\rangle \\
&= \langle *(X\wedge Y)\cdot\psi,\psi\rangle \\
&= ig\Big(\xi, *(X\wedge Y)\Big) \\
&= ig(*\xi, X\wedge Y),
\end{aligned}$$

hence $d\xi = *\xi$. We have

$$\begin{aligned}
2d\xi(X,Y) &= X(g(\xi,Y)) - Y(g(\xi,X)) - g(\xi,[X,Y]) \\
&= g(\nabla_X\xi,Y) + g(\xi,\nabla_X Y) - g(\nabla_Y\xi,X) - g(\xi,\nabla_Y X) \\
&\quad -g(\xi,\nabla_X Y) + g(\xi,\nabla_Y X) \\
&= g(\nabla_X\xi,Y) - g(\nabla_Y\xi,X) \\
&= 2g(\nabla_X\xi,Y).
\end{aligned}$$

Moreover, $d|\xi|^2 = -2d\xi(\xi,.) = -2\nabla_\xi\xi$. Indeed,

$$\begin{aligned}
d|\xi|^2(X) &= X(g(\xi,\xi)) \\
&= g(\nabla_X\xi,\xi) + g(\xi,\nabla_X\xi) \\
&= 2g(\nabla_X\xi,\xi) \\
&= -2g(\nabla_\xi\xi,X) \\
&= -2d\xi(\xi,X).
\end{aligned}$$

But $d\xi(\xi,.) = g(\nabla_\xi\xi,.) \simeq \nabla_\xi\xi$, so

$$d|\xi|^2 = -2d\xi(\xi,.) \simeq -2\nabla_\xi\xi.$$

Since $d\xi(\xi,.) = *\xi(\xi,.) = 0$, we conclude that $d|\xi|^2 = 0$ and then $|\xi| = \text{cte}$. Moreover,

$$d\xi\cdot\varphi = *\xi\cdot\varphi = -\xi\cdot\varphi \text{ for any spinor field } \varphi.$$

The Killing vector ξ is not zero because if $\xi = 0$ we get,

$$\langle \xi \cdot \psi, \psi \rangle = ig(\xi, \xi) = i|\xi|^2 = 0,$$

hence $\xi \cdot \psi \perp \psi$ and the vector space $T_xN \cdot \psi$ of dimension 3 is orthogonal to ψ. This is a contradiction because $\psi^\perp$ has complex dimension 1. Let$\{\ \xi/|\xi|, e_1, e_2\}$ be a local orthonormal frame of N and ϕ a spinor field generated by $\psi^\perp$. We have $\langle e_i \cdot \psi, \psi \rangle = ig(\xi, e_i) = 0$ hence $e_i \cdot \psi \perp \psi$, i.e, $e_i \cdot \psi = a_i \phi$ where a_1 and a_2 are no zero complex functions. Hence,

$$\xi \cdot \psi = -|\xi| e_1 \cdot e_2 \cdot \psi = -|\xi| e_1 \cdot (a_2 \phi) = |\xi| \frac{a_2}{a_1} \psi,$$

so there exists a complex function a such that $\xi \cdot \psi = a\psi$. By definition, we have $ig(\xi, \xi) = \langle \xi \cdot \psi, \psi \rangle = a$, hence $i|\xi|^2 = a$. On the one hand,

$$\xi \cdot \xi \cdot \xi \cdot \xi \cdot \psi = -|\xi|^2 (-|\xi|^2) \psi = |\xi|^4 \psi,$$

and on the other hand, we have

$$\begin{array}{rcl} \xi \cdot \xi \cdot \xi \cdot \xi \cdot \psi &=& |\xi|^2 |\xi|^2 \psi = |\xi|^2 (-ia) \psi \\ &=& (-ia)^2 \psi = -a^2 \psi \\ &=& -a(\xi \cdot \psi) = -\xi (a \cdot \psi) \\ &=& -\xi(\xi \cdot \psi) = |\xi|^2 \psi. \end{array}$$

Finally, $|\xi|^4 = |\xi|^2$ and since $|\xi|$ is constant, we get $|\xi| = 1$. As a conclusion,

$$a = i \quad \text{and} \quad \xi \cdot \psi = i\psi.$$

Let h be a real constant such that $h > 1$. We define the metric g^h on N, by:

$$\left\{ \begin{array}{lll} g^h(\xi, X) & = g(\xi, X) & \text{for any } X \in \Gamma(TN), \\ g^h(X, Y) & = h^{-2} g(X, Y) & \text{for } X, Y \perp \xi. \end{array} \right.$$

By the following isomorphism,

$$\begin{array}{rcl} (TN, g) & \longrightarrow & (TN, g^h) \\ Z & \longmapsto & Z^h = \left\{ \begin{array}{ll} Z & \text{if} \ \ Z = \xi, \\ hZ & \text{if} \ \ Z \perp \xi, \end{array} \right. \end{array}$$

if $\{\xi, e_1, e_2\}$ is a local orthonormal frame of (N, g), $\{\xi^h = \xi, e_1^h, e_2^h\}$ is a local g^h-orthonormal frame of (N, g^h).

Lemma 2.5.1. *The Levi-Civita connection ∇^h associated with the metric g^h is given by*

$$\begin{array}{rcl} \nabla^h_\xi \xi &=& \nabla_\xi \xi = 0, \\ g^h(\nabla^h_\xi e_i^h, e_j^h) &=& g(\nabla_\xi e_i, e_j) + (h^2 - 1) d\xi(e_i, e_j), \\ g^h(\nabla^h_{e_i^h} \xi, e_j^h) &=& d\xi(e_i^h, e_j^h) = h^2 g(\nabla_{e_i} \xi, e_j). \end{array}$$

Proof. By the Koszul formula, we get

$$\begin{aligned}
2g^h(\nabla^h_\xi \xi, e^h_i) &= \xi(g^h(\xi, e^h_i)) + \xi(g^h(\xi, e^h_i)) - e^h_i(g^h(\xi,\xi)) \\
&\quad + g^h(e^h_i, [\xi,\xi]) - g^h(\xi, [\xi, e^h_i]) - g^h(\xi, [\xi, e^h_i]) \\
&= \xi(hg(\xi, e_i)) + \xi(hg(\xi, e_i)) - he_i(g(\xi,\xi)) \\
&\quad + hg^h(e_i, 0) - 2hg^h(\xi, [\xi, e_i]) \\
&= -2hg^h(\xi, [\xi, e_i]).
\end{aligned}$$

Similary, $2g^h(\nabla_\xi \xi, e^h_i) = -2hg^h(\xi, [\xi, e_i])$. Again, by the Koszul formula, we have $g^h(\nabla_\xi\xi, \xi) = 0$ and $g^h(\nabla^h_\xi \xi, \xi) = 0$. Hence, $\nabla^h_\xi \xi = \nabla_\xi \xi = 0$ since $\nabla_\xi \xi = 0$. It follows

$$\begin{aligned}
2g^h(\nabla^h_{e^h_i} \xi, e^h_j) &= e^h_i(g^h(\xi, e^h_j)) + \xi(g^h(e^h_i, e^h_j)) - e^h_j(g^h(e^h_i, \xi)) \\
&\quad + g^h(e^h_j, [e^h_i, \xi]) - g^h(\xi, [e^h_i, e^h_j]) - g^h(e^h_i, [\xi, e^h_j]) \\
&= he_i(hg^h(\xi, e_j)) + \xi(h^2 g^h(e_i, e_j)) - he_j(hg^h(e_i, \xi)) \\
&\quad + h^2 g^h(e_j, [e_i, \xi]) - h^2 g^h(\xi, [e_i, e_j]) - h^2 g^h(e_i, [\xi, e_j]) \\
&= h^2 \Big[g^h(e_j, [e_i, \xi]) - g^h(\xi, [e_i, e_j]) - g^h(e_i, [\xi, e_j]) \Big] \\
&= h^2 \Big[h^{-2} g(e_j, [e_i, \xi]) + (1 - h^{-2}) g(\xi, e_j) g(\xi, [e_i, \xi]) \\
&\quad - h^{-2} g(\xi, [e_i, e_j]) - (1 - h^{-2}) g(\xi, \xi) g(\xi, [e_i, e_j]) \\
&\quad - h^{-2} g(e_i, [\xi, e_j]) - (1 - h^{-2}) g(\xi, e_i) g(\xi, [\xi, e_j]) \Big] \\
&= h^2 \Big[h^{-2} g(e_j, \nabla_{e_i} \xi) - h^{-2} g(e_j, \nabla_\xi e_i) - g(\xi, [e_i, e_j]) \\
&\quad - h^{-2} g(e_i, \nabla_\xi e_j) + h^{-2} g(e_i, \nabla_{e_j} \xi) \Big] \\
&= h^2 \Big[- g(\xi, [e_i, e_j]) - h^{-2} \xi(g(e_i, e_j)) \Big] \\
&= -h^2 g(\xi, [e_i, e_j]).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
2g(\nabla_{e_i} \xi, e_j) &= e_i(g(\xi, e_j)) + \xi(g(e_i, e_j)) - e_j(g(e_i, \xi)) \\
&\quad + g(e_j, [e_i, \xi]) - g(\xi, [e_i, e_j]) - g(e_i, [\xi, e_j]) \\
&= -g(\xi, [e_i, e_j]) + g(e_j, \nabla_{e_i} \xi) - g(e_j, \nabla_\xi e_i) \\
&\quad - g(e_i, \nabla_\xi e_j) + g(e_i, \nabla_{e_j} \xi) = -g(\xi, [e_i, e_j]).
\end{aligned}$$

Thus, $2g^h(\nabla^h_{e_i{}^h} \xi, e^h_j) = -h^2 g(\xi, [e_i, e_j]) = 2h^2 g(\nabla_{e_i} \xi, e_j)$ and finally

$$g^h(\nabla^h_{e_i{}^h} \xi, e^h_j) = h^2 g(\nabla_{e_i} \xi, e_j) = h^2 d\xi(e_i, e_j) = d\xi(e^h_i, e^h_j).$$

Again, by the Koszul formula,

$$\begin{aligned}
2g^h(\nabla^h_\xi e^h_i, e^h_j) &= \xi(g^h(e^h_i, e^h_j)) + e^h_i(g^h(\xi, e^h_j)) - e^h_j(g^h(\xi, e^h_i)) \\
&\quad + g^h(e^h_j, [\xi, e^h_i]) - g^h(e^h_i, [\xi, e^h_j]) - g^h(\xi, [e^h_i, e^h_j]) \\
&= h^2 g^h(e_j, [\xi, e_i]) - h^2 g^h(e_i, [\xi, e_j]) - h^2 g^h(\xi, [e_i, e_j]) \\
&= h^2\Big[h^{-2} g(e_j, [\xi, e_i]) + (1 - h^{-2}) g(\xi, e_j) g(\xi, [\xi, e_i])\Big] \\
&\quad - h^2\Big[h^{-2} g(e_i, [\xi, e_j]) + (1 - h^{-2}) g(\xi, e_i) g(\xi, [\xi, e_j])\Big] \\
&\quad - h^2 g(\xi, [e_i, e_j]) \\
&= g(e_j, [\xi, e_i]) - g(e_i, [\xi, e_j]) - h^2 g(\xi, [e_i, e_j]) \\
&= 2g(\nabla_\xi e_i, e_j) - 2g(\nabla_{e_i}\xi, e_j) - h^2 g(\xi, [e_i, e_j]) \\
&= 2g(\nabla_\xi e_i, e_j) + g(\xi, [e_i, e_j]) - h^2 g(\xi, [e_i, e_j]) \\
&= 2g(\nabla_\xi e_i, e_j) + (1 - h^2) g(\xi, [e_i, e_j]),
\end{aligned}$$

since we have

$$\begin{aligned}
2g(\nabla_{e_i}\xi, e_j) &= g(e_j, [e_i, \xi]) - g(\xi, [e_i, e_j]) - g(e_i, [\xi, e_j]) \\
&= g(e_j, \nabla_{e_i}\xi) - g(e_j, \nabla_\xi e_i) - g(\xi, [e_i, e_j]) \\
&\quad - g(e_i, \nabla_\xi e_i) + g(e_i, \nabla_{e_j}\xi) \\
&= -g(\xi, [e_i, e_j]).
\end{aligned}$$

Hence, $g^h(\nabla^h_\xi e^h_i, e^h_j) = g(\nabla_\xi e_i, e_j) + (h^2 - 1) d\xi(e_i, e_j)$. Similary, we get

$$g^h(\nabla^h_{e^h_i} {e_i}^h, {e_j}^h) = h g(\nabla_{e_i} e_i, e_j).$$

The following isomorphism

$$\begin{array}{rcl}
P_{\mathrm{SO}_3} N_g & \longrightarrow & P_{\mathrm{SO}_3} N_{g^h} \\
u = \{\xi, e_1, e_2\} & \longmapsto & u^h = \{\xi, e^h_1, e^h_2\},
\end{array}$$

can be extended in a natural way to

$$\begin{array}{rcl}
P_{\mathrm{Spin}_3} N_g & \longrightarrow & P_{\mathrm{Spin}_3} N_{g^h} \\
\tilde{u} & \longmapsto & \tilde{u}^h.
\end{array}$$

This isomorphism defines the following isomorphism of vector spaces:

$$\begin{array}{rcl}
\Sigma_g N & \longrightarrow & \Sigma_{g^h} N \\
\psi = [\tilde{u}, \sigma] & \longmapsto & \psi^h = [\tilde{u}^h, \sigma].
\end{array}$$

The Clifford multiplication with respect to g and g^h will be denoted by the same symbol. Hence, we have

$$\langle \psi_1, \psi_2 \rangle_{\Sigma_g N} = \left\langle \psi^h_1, \psi^h_2 \right\rangle_{\Sigma_{g^h} N},$$

$$(X \cdot \psi)^h = X^h \cdot \psi^h \text{ for every } X \in \Gamma(TN).$$

Lemma 2.5.2. *The spinorial Levi-Civita connection* ∇^h *of the* Spin *manifold* (N, g^h) *is given by*

$$\nabla^h_{X^h}\psi^h = \frac{h^2}{2}X^h \cdot \psi^h + i\Big((1-h^2)\xi\Big)(X^h)\psi^h,$$

for every $\psi^h \in \Gamma(\Sigma_{g^h}N)$.

Proof. For every $\psi^h \in \Gamma(\Sigma_{g^h}N)$, we compute

$$\begin{aligned}
\nabla^h_\xi\psi^h &= [\tilde{u}^h, \xi(\phi)] + \frac{1}{2}\sum_{i<j} g^h(\nabla^h_\xi e^h_i, e^h_j)e^h_i \cdot e^h_j \cdot \psi^h \\
&= [\tilde{u}^h, \xi(\phi)] + \frac{1}{2}g^h(\nabla^h_\xi e^h_1, e^h_2)e^h_1 \cdot e^h_2 \cdot \psi^h \\
&= [\tilde{u}^h, \xi(\phi)] + \frac{1}{2}\Big(g(\nabla_\xi e_1, e_2)e^h_1 \cdot e^h_2 \cdot \psi^h \\
&\quad +(h^2-1)d\xi(e_1, e_2)e^h_1 \cdot e^h_2 \cdot \psi^h\Big) \\
&= (\nabla_\xi\psi)^h + \frac{h^2-1}{2}\Big(d\xi(e_1, e_2)e_1 \cdot e_2 \cdot \psi\Big)^h \\
&= (\nabla_\xi\psi)^h + \frac{h^2-1}{2}(d\xi \cdot \psi)^h \\
&= (\nabla_\xi\psi)^h - \frac{h^2-1}{2}(\xi \cdot \psi)^h = \frac{1}{2}(\xi \cdot \psi)^h - \frac{h^2-1}{2}(\xi \cdot \psi)^h \\
&= (1 - \frac{h^2}{2})\xi \cdot \psi^h,
\end{aligned}$$

and

$$\begin{aligned}
\nabla^h_{e^h_1}\psi^h &= [\tilde{u}^h, e^h_1(\phi)] + \frac{1}{2}g^h(\nabla^h_{e^h_1}e^h_1, e^h_2)e^h_1 \cdot e^h_2 \cdot \psi^h \\
&\quad +\frac{1}{2}g^h(\nabla^h_{e^h_1}\xi, e^h_2)\xi \cdot e^h_2 \cdot \psi^h + \frac{1}{2}g^h(\nabla^h_{e^h_1}\xi, e^h_1)\xi \cdot e^h_1 \cdot \psi^h \\
&= h[\tilde{u}^h, e_1(\phi)] + \frac{1}{2}[hg(e_1, [e_1, e_2])e^h_1 \cdot e^h_2 \cdot \psi^h] + \frac{1}{2}[h^2 g(\nabla_{e_1}\xi, e_2)\xi \cdot e^h_2 \cdot \psi^h] \\
&= h(\nabla_{e_1}\psi)^h + \frac{h^2-h}{2}g(\nabla_{e_1}\xi, e_2)(\xi \cdot e_2 \cdot \psi)^h \\
&= h(\nabla_{e_1}\psi)^h + \frac{h^2-h}{2}d\xi(e_1, e_2)(\xi \cdot e_2 \cdot \psi)^h \\
&= h(\nabla_{e_1}\psi)^h + \frac{h^2-h}{2}(e_1 \cdot \psi)^h \\
&= \frac{h}{2}(e_1 \cdot \psi)^h + \frac{h^2-h}{2}(e_1 \cdot \psi)^h \\
&= \frac{h^2}{2}(e_1 \cdot \psi)^h.
\end{aligned}$$

Similary, we have $\nabla^h_{e^h_2}\psi^h = \frac{h^2}{2}(e_2 \cdot \psi)^h$.

We consider the trivial $\mathbb{S}^1$-principal bundle $(N \times \mathbb{S}^1, \pi, N)$ and let L be the trivial line bundle associated with this $\mathbb{S}^1$-principal fiber bundle via the standard representation. We know that if ∇^L denotes the covariant derivative on L, then there exists a real 1-form α and a global section l of L such that

$$\nabla^L_X l = i\alpha(X)l,$$

for every $X \in \Gamma(TM)$. We choose $\alpha = (1-h^2)\xi$ and we consider the connection $\nabla^{h,L} = \nabla^h \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^L$ on the bundle $\Sigma_{g^h} N \otimes L$. We have

$$\nabla^{h,L}_{X^h}(\psi^h \otimes l) = \frac{h^2}{2} X^h \cdot (\psi^h \otimes l) + 2i\Big((1-h^2)\xi\Big)(X^h)(\psi^h \otimes l).$$

Hence,

$$\nabla^{h,L}_{e_1^h}(\psi^h \otimes l) = \frac{h^2}{2} e_1^h \cdot (\psi^h \otimes l),$$
$$\nabla^{h,L}_{e_2^h}(\psi^h \otimes l) = \frac{h^2}{2} e_2^h \cdot (\psi^h \otimes l),$$
$$\nabla^{h,L}_{\xi}(\psi^h \otimes l) = (\frac{-3h^2}{2} + 2)\xi \cdot (\psi^h \otimes l).$$

The spinor field $\psi^h \otimes l$ is a section of $\Sigma_{g^h} N \otimes L$, which is the Spin^c bundle associated with the Spin^c structure whose auxiliary line bundle is given by L^2. This spinor field is an eigenspinor associated with the eigenvalue $\frac{h^2}{2} - 2$. In fact,

$$\begin{aligned}
D(\psi^h \otimes l) &= \xi \cdot \nabla^{h,L}_{\xi}(\psi^h \otimes l) + e_1^h \cdot \nabla^{h,L}_{e_1^h}(\psi^h \otimes l) + e_2^h \cdot \nabla^{h,L}_{e_2^h}(\psi^h \otimes l) \\
&= \Big(-\frac{h^2}{2} - \frac{h^2}{2} - (\frac{-3h^2}{2} + 2)\Big)(\psi^h \otimes l) \\
&= (\frac{h^2}{2} - 2)(\psi^h \otimes l).
\end{aligned}$$

It is clear that $\psi^h \otimes l$ is not a Spin^c Killing spinor field since $h > 1$, and hence (N, g^h) is not a limiting manifold for the Friedrich type Spin^c inequality. But, it is a limiting manifold for (2.11) since we will prove that

$$\begin{cases} \nabla^{h,L}_{X^h}(\psi^h \otimes l) = -\ell^{\psi^h \otimes l}(X^h) \cdot (\psi^h \otimes l), \\ d\alpha \cdot (\psi^h \otimes l) = i\frac{c_n}{2}|d\alpha|_{g^h}(\psi^h \otimes l), \end{cases}$$

where $id\alpha$ is the curvature form associated with the connection ∇^L of L. We have

$$id\xi(e_1^h, e_2^h) = ih^2 d\xi(e_1, e_2) = -h^2 \langle e_1 \cdot e_2 \cdot \psi, \psi\rangle = h^2 \langle \xi \cdot \psi, \psi\rangle = ih^2,$$

$$id\xi(\xi, e_1^h) = ihd\xi(\xi, e_1) = -ih \langle \xi \cdot e_1 \cdot \psi, \psi\rangle = -h \langle e_1 \cdot \psi, \psi\rangle = 0,$$
$$id\xi(\xi, e_2^h) = ihd\xi(\xi, e_2) = -ih \langle \xi \cdot e_2 \cdot \psi, \psi\rangle = -h \langle e_2 \cdot \psi, \psi\rangle = 0.$$

Hence,

$$d\xi = d\xi(\xi, e_1^h)\xi \wedge e_1^h + d\xi(\xi, e_2^h)\xi \wedge e_2^h + d\xi(e_1^h, e_2^h)e_1^h \wedge e_2^h = h^2\ e_1^h \wedge e_2^h.$$

The Clifford multiplication of $d\alpha$ by $\psi^h \otimes l$ is given by

$$\begin{array}{rcl} d\alpha\cdot(\psi^h\otimes l) &=& (1-h^2)d\xi\cdot(\psi^h\otimes l) \\ &=& (1-h^2)h^2 e_1^h\cdot e_2^h\cdot(\psi^h\otimes l) \\ &=& -i(1-h^2)h^2\ \psi^h\otimes l \\ &=& i(h^2-1)h^2\ \psi^h\otimes l. \end{array}$$

Hence, it follows that

$$|d\alpha|^2_{g^h} = (1-h^2)^2|d\xi|^2_{g^h} = (1-h^2)^2(d\xi(e_1^h,e_2^h))^2 = h^4(1-h^2)^2.$$

Since, $h>1$, $|d\alpha|_{g^h} = h^2(h^2-1)$. So, we have

$$d\alpha\cdot(\psi^h\otimes l) = ih^2(h^2-1)(\psi^h\otimes l) = i\frac{c_n}{2}|d\alpha|_{g^h}(\psi^h\otimes l).$$

Moreover, it is easy to check that $\ell^{\psi^h\otimes l}(e_1^h,e_1^h) = \ell^{\psi^h\otimes l}(e_2^h,e_2^h) = -\frac{h^2}{2}$. In fact,

$$\begin{array}{rcl} \ell^{\psi^h\otimes l}(e_1^h,e_1^h) &=& \frac{1}{2}\mathrm{Re}\left\langle \frac{h^2}{2}e_1^h\cdot e_1^h\cdot(\psi^h\otimes l)+\frac{h^2}{2}e_1^h\cdot e_1^h\cdot(\psi^h\otimes l), \frac{\psi^h\otimes l}{|\psi^h\otimes l|^2}\right\rangle \\ &=& -\frac{h^2}{2}. \end{array}$$

Similary, we have

$$\ell^{\psi^h\otimes l}(e_1^h,\xi) = \ell^{\psi^h\otimes l}(e_2^h,\xi) = \ell^{\psi^h\otimes l}(e_1^h,e_2^h) = 0,$$

$$\ell^{\psi^h\otimes l}(\xi,\xi) = \frac{3h^2}{2}-2.$$

Now, we compute

$$\begin{array}{rcl} -\ell^{\psi^h\otimes l}(e_1^h)\cdot(\psi^h\otimes l) &=& -\ell^{\psi^h\otimes l}(e_1^h,e_1^h)e_1^h\cdot(\psi^h\otimes l) \\ &=& \frac{h^2}{2}e_1^h\cdot(\psi^h\otimes l) \\ &=& \nabla^{h,L}_{e_1^h}(\psi^h\otimes l). \end{array}$$

Similary, we have

$$-\ell^{\psi^h\otimes l}(e_2^h)\cdot(\psi^h\otimes l) = \frac{h^2}{2}e_2^h\cdot(\psi^h\otimes l) = \nabla^{h,L}_{e_2^h}(\psi^h\otimes l),$$

$$-\ell^{\psi^h\otimes l}(\xi)\cdot(\psi^h\otimes l) = (\frac{-3h^2}{2}+2)\xi\cdot(\psi^h\otimes l) = \nabla^{h,L}_{\xi}(\psi\otimes l).$$

Finally, the manifold (N,g^h) is a limiting manifold for (2.11).

Chapter 3

The Hijazi Inequalities on Complete Riemannian Spin^c Manifolds[1]

3.1 Introduction

We recall that on a compact Riemannian Spin^c manifold (N^n, g) of dimension $n \geqslant 2$, any eigenvalue λ of the Dirac operator satisfies a Friedrich type inequality [HM99, Fri80]:

$$\lambda^2 \geqslant \frac{n}{4(n-1)} \inf_N (S - c_n |\Omega|), \tag{3.1}$$

Equality holds if and only if the eigenspinor ψ associated with the first eigenvalue λ_1 is a Spin^c Killing spinor satisfying $\Omega \cdot \psi = i\frac{c_n}{2}|\Omega|\psi$, i.e. for every $X \in \Gamma(TN)$ the eigenspinor ψ satisfies

$$\begin{cases} \nabla_X \psi = -\frac{\lambda_1}{n} X \cdot \psi, \\ \Omega \cdot \psi = i\frac{c_n}{2}|\Omega|\psi. \end{cases} \tag{3.2}$$

In Chapter 2, it is shown that on a compact Riemannian Spin^c manifold any eigenvalue λ of the Dirac operator to which is attached an eigenspinor ψ satisfies an Hijazi type inequality [Hij95] involving the energy-momentum tensor ℓ^ψ and the scalar curvature:

$$\lambda^2 \geqslant \inf_N (\frac{1}{4} S - \frac{c_n}{4}|\Omega| + |\ell^\psi|^2), \tag{3.3}$$

Equality holds in (3.3) if and only, for all $X \in \Gamma(TN)$, we have

$$\begin{cases} \nabla_X \psi = -\ell^\psi(X) \cdot \psi, \\ \Omega \cdot \psi = i\frac{c_n}{2}|\Omega|\psi, \end{cases} \tag{3.4}$$

where ψ is an eigenspinor associated with the first eigenvalue λ_1. By definition, the trace $tr(\ell^\psi)$ of ℓ^ψ, where ψ is an eigenspinor associated with an eigenvalue λ, is equal to λ. Hence, Inequality (3.3) improves Inequality (3.1) since, by the Cauchy-Schwarz inequality, $|\ell^\psi|^2 \geqslant \frac{1}{n}(tr(\ell^\psi))^2 = \frac{1}{n}\lambda^2$.

[1]This chapter is the subject of a published paper [Nak11b]

In the same spirit as in [Hij86], A. Moroianu and M. Herzlich (see [HM99]) generalized the Hijazi inequality [Hij86], involving the first eigenvalue of the Yamabe operator L, to the case of compact $\mathrm{Spin^c}$ manifolds of dimension $n \geqslant 3$: any eigenvalue λ of the Dirac operator satisfies

$$\lambda^2 \geqslant \frac{n}{4(n-1)}\mu_1, \tag{3.5}$$

where μ_1 is the first eigenvalue of the perturbed Yamabe operator defined by $L^\Omega = L - c_n|\Omega|_g = 4\frac{n-1}{n-2} \triangle + S - c_n|\Omega|_g$. The limiting case of (3.5) is equivalent to the limiting case in (3.1). The Hijazi inequality [Hij95], involving the energy-momentum tensor and the first eigenvalue of the Yamabe operator, is then proved in Chapter 2 for compact $\mathrm{Spin^c}$ manifolds. In fact, any eigenvalue of the Dirac operator to which is attached an eigenspinor ψ satisfies

$$\lambda^2 \geqslant \frac{1}{4}\mu_1 + \inf_N \ |\ell^\psi|^2. \tag{3.6}$$

Equality in (3.6) holds if and only, for all $X \in \Gamma(TN)$, we have

$$\begin{cases} \overline{\nabla}_X\overline{\varphi} = -\ell^{\overline{\varphi}}(X) \,\bar{\cdot}\, \overline{\varphi}, \\ \Omega \cdot \psi = i\frac{c_n}{2}|\Omega|_g\psi, \end{cases} \tag{3.7}$$

where $\overline{\varphi} = e^{-\frac{n-1}{2}u}\overline{\psi}$, the spinor field $\overline{\psi}$ is the image of ψ under the isometry between the spinor bundles of (N^n, g) and $(N^n, \overline{g} = e^{2u}g)$ and ψ is an eigenspinor associated with the first eigenvalue λ_1 of the Dirac operator. Again, Inequality (3.6) improves Inequality (3.5). In this chapter we examine these lower bounds on open manifolds, and especially on complete Riemannian $\mathrm{Spin^c}$ manifolds. We prove the following:

Theorem 3.1.1. *Let (N^n, g) be a complete Riemannian* $\mathrm{Spin^c}$ *manifold of finite volume. Then any eigenvalue λ of the Dirac operator to which is attached an eigenspinor ψ satisfies the Hijazi type inequality (3.3). Equality holds if and only if the eigenspinor associated with the first eigenvalue λ_1 satisfies (3.4).*

The Friedrich type inequality (3.1) is derived for complete Riemannian $\mathrm{Spin^c}$ manifolds of finite volume. This was proved by N. Grosse in [Nad08a] and [Nad08b] for complete Spin manifolds of finite volume. Using the conformal covariance of the Dirac operator we prove:

Theorem 3.1.2. *Let (N^n, g) be a complete Riemannian* $\mathrm{Spin^c}$ *manifold of finite volume and dimension $n > 2$. Any eigenvalue λ of the Dirac operator to which is attached an eigenspinor ψ satisfies the Hijazi type inequality (3.6). Equality holds if and only if Equation (3.7) holds.*

Now, the Hijazi type inequality (3.5) can be derived for complete Riemannian $\mathrm{Spin^c}$ manifolds of finite volume and dimension $n > 2$ and equality holds if and only if the eigenspinor associated with the first eigenvalue λ_1 satisfies (3.2). This was also proved by N. Grosse in [Nad08a] and [Nad08b] for complete Spin manifolds of finite volume and dimension $n > 2$. On complete manifolds, the Dirac operator is essentially self-adjoint and, in general, its spectrum consists of eigenvalues and the essential spectrum. For elements of the essential spectrum, we also extend to Spin^c manifolds the Hijazi type inequality (3.5) obtained by N. Grosse in [Nad08b] on Spin manifolds:

Theorem 3.1.3. *Let (N^n, g) be a complete Riemannian* Spin^c *manifold of dimension $n \geqslant 5$ with finite volume. Furthermore, assume that $S - c_n|\Omega|$ is bounded from below. If λ is in the essential spectrum of the Dirac operator $\sigma_{ess}(D)$, then λ satisfies the Hijazi type inequality (3.5).*

For the 2-dimensional case, N. Grosse proved in [Nad08a] that, for any Riemannian Spin surface of finite area, homeomorphic to $\mathbb{R}^2$ we have

$$\lambda^+ \geqslant \frac{4\pi}{\mathrm{Area}(M^2, g)}, \tag{3.8}$$

where $\lambda^+ = \inf_{\varphi \in C_c^\infty(N)} \frac{(D^2\varphi, \varphi)}{(\varphi,\varphi)}$ (in the compact case, λ^+ coincides with the first eigenvalue of the square of the Dirac operator). Recently, in [Bär09], C. Bär showed the same inequality for any connected 2-dimensional Riemannian manifold of genus 0, with finite area and equipped with a Spin structure which is *bounding at infinity.* A Spin structure on N is said to be *bounding at infinity* if N can be embedded into $\mathbb{S}^2$ in such a way that the Spin structure extends to the unique Spin structure of $\mathbb{S}^2$.

3.2 Preliminaries

In this section we briefly introduce basic notions concerning complete Riemannian Spin^c manifolds. Then we recall the refined Kato inequality which is crucial for the proof.

The Dirac operator on complete Riemannian Spin^c manifolds. Let (N^n, g) be a connected oriented Riemannian Spin^c manifold of dimension $n \geqslant 2$ and without boundary. We denote by $\Gamma(\Sigma N)$ the set of all spinors and those of compactly supported smooth spinors by $\Gamma_c(\Sigma N)$. Like in the compact case, the Spin^c bundle ΣN is equipped with a natural Hermitian scalar product, denoted by $\langle ., . \rangle$, satisfying

$$\langle X \cdot \psi, \varphi \rangle = - \langle \psi, X \cdot \varphi \rangle \quad \text{for every} \quad X \in \Gamma(TN) \quad \text{and} \quad \psi, \varphi \in \Gamma(\Sigma N),$$

where $X \cdot \psi$ denotes the Clifford multiplication of X and ψ. With this Hermitian scalar product we define an L^2-scalar product

$$(\psi, \varphi) = \int_N \langle \psi, \varphi \rangle \, v_g,$$

for any spinors ψ and φ in $\Gamma_c(\Sigma N)$. The Dirac operator is an elliptic and formally self-adjoint operator with respect to the L^2-scalar product, i.e. for all spinors ψ, φ at least one of which is compactly supported on N we have $(D\psi, \varphi) = (\psi, D\varphi)$.

For the Friedrich connection $\nabla^f_X\psi = \nabla_X\psi + \frac{f}{n}X\cdot\psi$ where f is real valued function one gets a Schrödinger-Lichnerowicz type formula:

$$(D-f)^2\psi = \triangle^f\psi + (\frac{S}{4} + \frac{n-1}{n}f^2)\psi + \frac{i}{2}\Omega\cdot\psi - \frac{n-1}{n}(2fD\psi + \nabla f\cdot\psi), \tag{3.9}$$

where $\triangle^f$ is the spinorial Laplacian associated with the connection ∇^f.

A complex number λ is an eigenvalue of D if there exists a nonzero eigenspinor $\psi\in\Gamma(\Sigma N)\cap L^2(\Sigma N)$ with $D\psi=\lambda\psi$. The set of all eigenvalues is denoted by $\sigma_p(D)$, the point spectrum. We know that, if N is closed, the Dirac operator has a pure point spectrum but on open manifolds, the spectrum might have a continuous part. In general the spectrum of the Dirac operator $\sigma(D)$ is composed of the point, the continuous and the residual spectrum. For complete manifolds, the residual spectrum is empty and $\sigma(D)\subset\mathbb{R}$. Thus, for complete manifolds, the spectrum can be divided into point and continuous spectrum. But often another decomposition of the spectrum is used: the one into discrete spectrum $\sigma_d(D)$ and essential spectrum $\sigma_{ess}(D)$.

A complex number λ lies in the essential spectrum of D if there exists a sequence of smooth compactly supported spinors ψ_i which are orthonormal with respect to the L^2-product and

$$\|(D-\lambda)\psi_i\|_{L^2}\longrightarrow 0.$$

The essential spectrum contains all eigenvalues of infinite multiplicity. In contrast, the discrete spectrum $\sigma_d(D) := \sigma_p(D)\diagdown\sigma_{ess}(D)$ consists of all eigenvalues of finite multiplicity. The proof of the next property can be found in [Nad08a]: on a Spin^c complete Riemannian manifold, 0 is in the essential spectrum of $D-\lambda$ if and only if 0 is in the essential spectrum of $(D-\lambda)^2$ and in this case, there is a normalized sequence $\psi_i\in\Gamma_c(\Sigma N)$ such that ψ_i converges L^2-weakly to 0 with $\|(D-\lambda)\psi_i\|_{L^2}\longrightarrow 0$ and $\|(D-\lambda)^2\psi_i\|_{L^2}\longrightarrow 0$.

Refined Kato inequalities. On a Riemannian manifold (N^n,g), the Kato inequality states that away from the zeroes of any section φ of a Riemannian or Hermitian vector bundle endowed with a metric connection ∇, we have:

$$|d(|\varphi|)|\leqslant|\nabla\varphi|. \tag{3.10}$$

This could be seen as follows $2|\varphi||d(|\varphi|)| = |d(|\varphi|)^2| = 2|\langle\nabla\varphi,\varphi\rangle|\leqslant 2|\varphi||\nabla\varphi|$. In [CGH00], refined Kato inequalities were obtained for sections in the kernel of first order elliptic differential operators P. They are of the form $|d(|\varphi|)|\leqslant k_P|\nabla\varphi|$, where k_P is a constant depending on the operator P and $0<k_P<1$. Without the assumption that $\varphi\in\ker P$, we get away from the zero set of φ

$$|d|\varphi||\leqslant|P\varphi| + k_P|\nabla\varphi|. \tag{3.11}$$

A proof of (3.11) can be found in [CGH00], [Nad08a], [Bran00] or [Nad08b]. In [CGH00] the constant k_P is determined in terms of the conformal weights of the differential operator P. For the Dirac operator D and for $D-\lambda$, where $\lambda\in\mathbb{R}$, we have $k_D = k_{D-\lambda} = \sqrt{\frac{n-1}{n}}$.

3.3 Proof of the Hijazi type inequalities

First, we follow the main idea of the proof of the original Hijazi inequality in the compact case ([Hij95], [Hij86]), and its proof in the Spin noncompact case obtained by N. Grosse [Nad08b]. We choose the conformal factor with the help of an eigenspinor and we use cut-off functions near its zero-set and near infinity to obtain compactly supported test functions.

Proof of Theorem 3.1.2. Let $\psi \in C^\infty(N,S) \cap L^2(N,S)$ be a normalized eigenspinor, i.e. $D\psi = \lambda\psi$ and $\|\psi\| = 1$. Its zero-set Υ is closed and lies in a closed countable union of smooth $(n-2)$-dimensional submanifolds which has locally finite $(n-2)$-dimensional Hausdorff measure [Bär99]. We can assume without loss of generality that Υ is itself a countable union of $(n-2)$-submanifolds described above. Fix a point $p \in N$. Since N is complete, there exists a cut-off function $\eta_i : N \to [0,1]$ which is zero on $N \setminus B_{2i}(p)$ and equal to 1 on $B_i(p)$, where $B_l(p)$ is the ball of center p and radius l. In between, the function is chosen such that $|\nabla\eta_i| \leqslant \frac{4}{i}$ and $\eta_i \in C_c^\infty(N)$. While η_i cuts off ψ at infinity, we define another cut-off near the zeros of ψ. Let $\rho_{a,\epsilon}$ be the function

$$\rho_{a,\epsilon}(x) = \begin{cases} 0 & \text{for } r < a\epsilon, \\ 1 - \delta \ln \frac{\epsilon}{r} & \text{for } a\epsilon \leqslant r \leqslant \epsilon, \\ 1 & \text{for } \epsilon < r, \end{cases}$$

where $r = d(x,\Upsilon)$ is the distance from x to Υ. The constant $0 < a < 1$ is chosen such that $\rho_{a,\epsilon}(a\epsilon) = 0$, i.e. $a = e^{-\frac{1}{\delta}}$. Then $\rho_{a,\epsilon}$ is continuous, constant outside a compact set and Lipschitz. Hence, for $\varphi \in \Gamma(\Sigma N)$ the spinor $\rho_{a,\epsilon}\varphi$ is an element in $H_1^r(\Sigma N)$ for all $1 \leqslant r \leqslant \infty$. Now, consider $\Psi := \eta_i\rho_{a,\epsilon}\psi \in H_1^r(\Sigma N)$. These spinors are compactly supported on $N \setminus \Upsilon$. Furthermore, $\overline{g} = e^{2u}g = h^{\frac{4}{n-2}}g$ with $h = |\psi|^{\frac{n-2}{n-1}}$ is a metric on $N \setminus \Upsilon$. Setting $\overline{\Phi} := e^{-\frac{n-1}{2}u}\overline{\Psi}$ ($\varphi = e^{-\frac{n-1}{2}u}\psi$), Equations (2.14), (2.17), (2.18) and the Schrödinger-Lichnerowicz formula imply

$$\begin{aligned}
\|\overline{\nabla}^{\ell^{\overline{\Phi}}}\overline{\Phi}\|^2_{\overline{g}} &= \|\overline{D}\ \overline{\Phi}\|^2_{\overline{g}} - \frac{1}{4}\int_{N-\Upsilon} \overline{S}|\overline{\Phi}|^2 v_{\overline{g}} - \int_{N-\Upsilon} |\ell^{\overline{\Phi}}|^2|\overline{\Phi}|^2 v_{\overline{g}} \\
&\quad - \int_{N-\Upsilon} \left\langle \frac{i}{2}\Omega \,\bar{\cdot}\, \overline{\Phi}, \overline{\Phi} \right\rangle v_{\overline{g}} \\
&\leqslant \|\overline{D}\ \overline{\Phi}\|^2_{\overline{g}} - \frac{1}{4}\int_N (\overline{S}e^{2u} - c_n|\Omega|_g)|\Psi|^2 e^{-u} v_g - \int_N |\ell^\Psi|^2|\Psi|^2 e^{-u} v_g \\
&= \|\overline{D}\ \overline{\Phi}\|^2_{\overline{g}} - \frac{1}{4}\int_N (h^{-1}L^\Omega h)|\Psi|^2 e^{-u} v_g - \int_N |\ell^\Psi|^2|\Psi|^2 e^{-u} v_g,
\end{aligned}$$

where $\nabla^{\ell^\varphi}_X\varphi$ is the spinor field defined in [Hij95] by $\nabla^{\ell^\varphi}_X\varphi := \nabla_X\varphi + \ell^\varphi(X)\cdot\varphi$ and where we used $|\overline{\Phi}|^2 v_{\overline{g}} = e^u|\Psi|^2 v_g$ and $\overline{S}e^{2u} - c_n|\Omega|_g = h^{-1}L^\Omega h$ (see [Nak10]). Using $\overline{D}\,\overline{\varphi} = \lambda e^{-u}\overline{\varphi}$ and $\langle\nabla(\eta_i\rho_{a,\epsilon})\,\bar{\cdot}\,\overline{\varphi}, \overline{\varphi}\rangle \in C^\infty(N, \imath\mathbb{R})$, we calculate

$$\|\overline{D}\ \overline{\Phi}\|^2_{\overline{g}} = \|\nabla(\eta_i\rho_{a,\epsilon})\,\bar{\cdot}\,\overline{\varphi}\|^2_{\overline{g}} + \lambda^2\int_N \eta_i^2\rho_{a,\epsilon}^2\, e^{-(n+2)u}|\varphi|^2 v_g. \tag{3.12}$$

Inserting (3.12) and $\|\overline{\nabla}^{\ell^{\overline{\Phi}}}\overline{\Phi}\|^2_{\overline{g}} \geqslant 0$ in the above inequality, we get

$$\|\nabla(\eta_i\rho_{a,\epsilon})\,\bar{\cdot}\,\overline{\varphi}\|^2_{\overline{g}} \geqslant \frac{1}{4}\int_N (h^{-1}L^\Omega h)|\Psi|^2 e^{-u}v_g \; + \; \int_N |\ell^\Psi|^2|\Psi|^2 e^{-u}v_g \\ - \; \lambda^2 \int_N \eta_i^2\rho_{a,\epsilon}^2|\varphi|^2 e^{-(n+2)u}v_g.$$

Moreover, we have $\|\nabla(\eta_i\rho_{a,\epsilon})\bar{\cdot}\,\overline{\varphi}\|^2_{\overline{g}} = \int\limits_M |\nabla(\eta_i\rho_{a,\epsilon})\cdot\psi|^2 e^{-u}v_g$. Thus, with $e^u = |\psi|^{\frac{2}{n-1}}$, the above inequality reads

$$\int\limits_N |\nabla(\eta_i\rho_{a,\epsilon})|^2|\psi|^{2\frac{n-2}{n-1}}v_g \;\; \geqslant \;\; \frac{1}{4}\int\limits_N \eta_i\rho_{a,\epsilon}|\psi|^{\frac{n-2}{n-1}}L^\Omega(\eta_i\rho_{a,\epsilon}|\psi|^{\frac{n-2}{n-1}})v_g - \lambda^2\int\limits_N \eta_i^2\rho_{a,\epsilon}^2|\psi|^{2\frac{n-2}{n-1}}v_g \\ -\frac{n-1}{n-2}\int\limits_N |\nabla(\eta_i\rho_{a,\epsilon})|^2|\psi|^{2\frac{n-2}{n-1}}v_g + \int_N |\ell^\psi|^2|\psi|^{2\frac{n-2}{n-1}}\eta_i^2\rho_{a,\epsilon}^2 v_g.$$

Hence, we obtain

$$\frac{2n-3}{n-2}\int\limits_N |\nabla(\eta_i\rho_{a,\epsilon})|^2|\psi|^{2\frac{n-2}{n-1}}v_g \geqslant \Big(\frac{\mu_1}{4} + \inf_N |\ell^\psi|^2 - \lambda^2\Big)\int\limits_N \eta_i^2\rho_{a,\epsilon}^2|\psi|^{2\frac{n-2}{n-1}}v_g,$$

where μ_1 is the infimum of the spectrum of the perturbed Yamabe operator. With $|\eta_i\nabla\rho_{a,\epsilon} + \rho_{a,\epsilon}\nabla\eta_i|^2 \leqslant 2\eta_i^2|\nabla\rho_{a,\epsilon}|^2 + 2\rho_{a,\epsilon}^2|\nabla\eta_i|^2$ we have

$$k\int\limits_N (\eta_i^2|\nabla\rho_{a,\epsilon}|^2 + \rho_{a,\epsilon}^2|\nabla\eta_i|^2)|\psi|^{2\frac{n-2}{n-1}}v_g \geqslant \Big(\frac{\mu_1}{4} + \inf_N |\ell^\psi|^2 - \lambda^2\Big)\|\eta_i\rho_{a,\epsilon}|\psi|^{\frac{n-2}{n-1}}\|^2,$$

where $k = 2\frac{2n-3}{n-2}$. Next, we examine the limits when a goes to zero. Recall that $\Upsilon \cap \overline{B_{2i}(p)}$ is bounded, closed and $(n-2)$-C^∞-rectifiable and has still locally finite $(n-2)$-dimensional Hausdorff measure. For fixed i we estimate

$$\int\limits_N |\nabla\rho_{a,\epsilon}|^2\eta_i^2|\psi|^{2\frac{n-2}{n-1}}v_g \leqslant \sup_{B_{2i}(p)} |\psi|^{2\frac{n-2}{n-1}} \int\limits_{B_{2i}(p)} |\nabla\rho_{a,\epsilon}|^2 v_g.$$

Furthermore, we set $\mathcal{B}_{\epsilon,p} := \{x \in B_\epsilon \mid d(x,p) = d(x,\Upsilon)\}$ with $B_\epsilon := \{x \in N \mid d(x,\Upsilon) \leqslant \epsilon\}$. For ϵ sufficiently small, each $\mathcal{B}_{\epsilon,p}$ is star-shaped. Moreover, there is an inclusion $\mathcal{B}_{\epsilon,p} \hookrightarrow B_\epsilon(0) \subset \mathbb{R}^2$ via the normal exponential map. Then, we can calculate

$$\int\limits_{B_\epsilon \cap B_{2i}(p)} |\nabla\rho_{a,\epsilon}|^2 v_g \leqslant \mathrm{vol}_{n-2}(\Upsilon \cap B_{2i}(p)) \sup_{x\in\Upsilon\cap B_{2i}(p)} \int\limits_{\mathcal{B}_{\epsilon,x}\setminus\mathcal{B}_{a\epsilon,x}} |\nabla\rho_{a,\epsilon}|^2 v_{g'} \\ \leqslant c\,\mathrm{vol}_{n-2}(\Upsilon \cap B_{2i}(p)) \int\limits_{B_\epsilon(0)\setminus B_{a\epsilon}(0)} |\nabla\rho_{a,\epsilon}|^2 v_{g_E} \\ \leqslant c'\int\limits_{a\epsilon}^{\epsilon} \frac{\delta^2}{r}dr = -c'\delta^2\ln a = c'\delta \to 0 \quad \text{for } a \to 0,$$

where vol_{n-2} denotes the $(n-2)$-dimensional volume and $g' = g_{|\mathcal{B}_{\epsilon,p}}$. The positive constants c and c' arise from $\mathrm{vol}_{n-2}(\Upsilon \cap B_{2i}(p))$ and the comparison of $v_{g'}$ with the volume element of the Euclidean metric. Furthermore, for any compact set $K \subset N$ and any positive function f, it holds $\rho^2_{a,\epsilon} f \nearrow f$ and thus by the monotone convergence theorem, we obtain, when $a \longrightarrow 0$,

$$\int_K \rho^2_{a,\epsilon} f v_g \longrightarrow \int_K f v_g.$$

When applied to the functions $\rho^2_{a,\epsilon}|\nabla\eta_i|^2|\psi|^{2\frac{n-2}{n-1}}$, with $K = B_{2i}(p)$ we get

$$\int\limits_{B_{2i}(p)} \rho^2_{a,\epsilon}|\nabla\eta_i|^2|\psi|^{2\frac{n-2}{n-1}} v_g \to \int\limits_{B_{2i}(p)} |\nabla\eta_i|^2|\psi|^{2\frac{n-2}{n-1}} v_g$$

as $a \to 0$ and thus,

$$k \int\limits_N |\nabla\eta_i|^2|\psi|^{2\frac{n-2}{n-1}} v_g \geqslant \Big(\frac{\mu_1}{4} + \inf_N |\ell^\psi|^2 - \lambda^2\Big) \int\limits_N \eta_i^2|\psi|^{2\frac{n-2}{n-1}} v_g.$$

Next, we have to study the limit when $i \to \infty$: since N has finite volume and $\|\psi\| = 1$, the Hölder inequality ensures that $\int\limits_N |\psi|^{2\frac{n-2}{n-1}} v_g$ is bounded. With $|\nabla\eta_i| \leqslant \frac{4}{i}$, we get the result. Equality is attained if and only if $\|\overline{\nabla}^{\ell^{\overline{\Phi}}}\overline{\Phi}\|^2_{\overline{g}} \longrightarrow 0$ for $i \to \infty$, $a \to 0$ and $\Omega\cdot\psi = i\frac{c_n}{2}|\Omega|_g\psi$. But we have

$$0 \leftarrow \|\overline{\nabla}^{\ell^{\overline{\Phi}}}\overline{\Phi}\|^2_{\overline{g}} = \|\eta_i\rho_{a,\epsilon}\overline{\nabla}^{\ell^{\overline{\Phi}}}\overline{\varphi} + \nabla(\eta_i\rho_{a,\epsilon})\,\overline{\cdot}\,\overline{\varphi}\|_{\overline{g}} \geqslant \|\eta_i\rho_{a,\epsilon}\overline{\nabla}^{\ell^{\overline{\varphi}}}\overline{\varphi}\|_{\overline{g}} - \|\nabla(\eta_i\rho_{a,\epsilon})\,\overline{\cdot}\,\overline{\varphi}\|_{\overline{g}}.$$

Since $\|\nabla(\eta_i\rho_{a,\epsilon})\overline{\cdot}\overline{\varphi}\|_{\overline{g}} \to 0$, we conclude that $\overline{\nabla}^{\ell^{\overline{\varphi}}}\overline{\varphi}$ has to vanish on $N \setminus \Upsilon$.

Remark 3.3.1. *By the Cauchy-Schwarz inequality, we have*

$$|\ell^\psi|^2 \geqslant \frac{1}{n}(\mathrm{tr}(\ell^\psi))^2 = \frac{1}{n}\lambda^2, \tag{3.13}$$

where $\mathrm{tr}(\ell^\psi)$ denotes the trace of ℓ^ψ. Hence the Hijazi type inequality (3.5) can be derived. Equality is achieved if and only if the eigenspinor ψ associated with the first eigenvalue λ_1 satisfies (3.2). In fact, if equality holds, then $\lambda^2 = \frac{n}{4(n-1)}\mu_1 = \frac{1}{4}\mu_1 + |\ell^\psi|^2$ and equality in (3.13) is satisfied. Hence, it is easy to check that

$$\ell^\psi(e_i, e_j) = 0 \textit{ for } i \neq j \textit{ and } \ell^\psi(e_i, e_i) = \pm\frac{\lambda}{n}.$$

Finally, $\ell^\psi(X) = \pm\frac{\lambda}{n}X$ and $\ell^{\overline{\varphi}}(X) = e^{-u}\ell^\psi(X) = \pm\frac{\lambda}{n}e^{-u}X$. By (3.7) we get that $\overline{\varphi}$ is a $\mathrm{Spin^c}$ *generalized Killing spinor and hence $\overline{\varphi}$ a* $\mathrm{Spin^c}$ *Killing spinor for $n \geqslant 4$ ([HM99, Theorem 1.1]). The function e^{-u} is then constant and ψ is a* $\mathrm{Spin^c}$ *Killing spinor. For*

$n=3$, we follow the same proof as in [HM99]. First, we suppose that $\lambda_1 \neq 0$, because if $\lambda_1 = 0$, the result is trivial. We consider the Killing vector $\overline{\xi}$ defined by

$$i\overline{g}(\overline{\xi}, X) = \langle X \bar{\cdot}\, \overline{\varphi}, \overline{\varphi}\rangle_{\overline{g}} \qquad \text{for every} \quad X \in \Gamma(TN).$$

In [HM99], it is shown that

$$\begin{array}{rcl} d\overline{\xi} &=& 2\lambda_1 e^{-u}(*\overline{\xi}), \\ \nabla|\overline{\xi}|^2 &=& 0, \\ \overline{\xi}\,\bar{\cdot}\,\overline{\varphi} &=& i|\overline{\xi}|^2\overline{\varphi}, \end{array}$$

where $*$ is the Hodge operator defined on differential forms. Since $*\overline{\xi}(\overline{\xi}, .) = 0$, the 2-form Ω can be written $\Omega = F\overline{\xi} + \overline{\xi}\wedge\alpha$, where α is a real 1-form and F a function. We have [HM99]

$$\begin{array}{rcl} \Omega(\overline{\xi}, .) &=& |\overline{\xi}|^2\alpha(.) = -4\lambda_1 d(e^{-u})(.), \\ \Omega\,\bar{\cdot}\,\overline{\varphi} &=& -iF\,\overline{\varphi} - i|\overline{\xi}|^2\alpha\,\bar{\cdot}\,\overline{\varphi}. \end{array} \tag{3.14}$$

But equality in (3.1) is achieved, so $\Omega\bar{\cdot}\,\overline{\varphi} = i\frac{c_n}{2}|\Omega|_{\overline{g}}\overline{\varphi}$, which implies that $\Omega\bar{\cdot}\,\overline{\varphi}$ is collinear to $\overline{\varphi}$ and hence $\alpha\bar{\cdot}\,\overline{\varphi}$ is collinear to $\overline{\varphi}$. Moreover, $d(e^{-u})(\overline{\xi}) = -\frac{1}{4\lambda_1}\Omega(\overline{\xi}, \overline{\xi}) = 0$, so $\alpha(\overline{\xi}) = 0$. It is easy to check that $\langle \alpha\bar{\cdot}\,\overline{\varphi}, \overline{\varphi}\rangle_{\overline{g}} = 0$ which gives $\alpha\bar{\cdot}\,\overline{\varphi} \perp \overline{\varphi}$. Because of $\alpha\bar{\cdot}\,\overline{\varphi} \perp \overline{\varphi}$ and $\alpha\bar{\cdot}\,\overline{\varphi}$ is collinear to $\overline{\varphi}$, we have $\alpha\bar{\cdot}\,\overline{\varphi} = 0$ and finally $\alpha = 0$. Using (3.14), we obtain $d(e^{-u}) = 0$, i.e. e^{-u} is constant, hence $\overline{\varphi}$ is a Killing Spin^c *spinor and finally ψ is also a* Spin^c *Killing spinor.*

Proof of Theorem 3.1.1. The proof of Theorem 3.1.1 is similar to Theorem 3.1.2. It suffices to take $\overline{g} = g$, i.e. $e^u = 1$. The Friedrich type inequality (3.1) is obtained from the Hijazi type inequality (3.5).

Next, we want to prove Theorem 3.1.3 using the refined Kato inequality.

Proof of Theorem 3.1.3. We may assume $\mathrm{vol}(N, g) = 1$. If λ is in the essential spectrum of D, then 0 is in the essential spectrum of $D-\lambda$ and of $(D-\lambda)^2$. Thus, there is a sequence $\psi_i \in \Gamma_c(\Sigma N)$ such that $\|(D-\lambda)^2\psi_i\| \to 0$ and $\|(D-\lambda)\psi_i\| \to 0$ while $\|\psi_i\| = 1$. We may assume that $|\psi_i| \in C_c^\infty(N)$. That can always be achieved by a small perturbation. Now let $\frac{1}{2} \leqslant \beta \leqslant 1$. Then $|\psi_i|^\beta \in H_1^2(N)$. First, we will show that the sequence $\|d(|\psi_i|^\beta)\|$ is bounded: by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \Big| \int_{|\psi_i|\neq 0} |\psi_i|^{2\beta-2}\left\langle (D-\lambda)^2\psi_i, \psi_i\right\rangle v_g \Big| &\leqslant \||\psi_i|^{2\beta-1}\|_{\{|\psi_i|\neq 0\}}\|(D-\lambda)^2\psi_i\| \\ &\leqslant \|\psi_i\|^{2\beta-1}\|(D-\lambda)^2\psi_i\| = \|(D-\lambda)^2\psi_i\|. \end{aligned}$$

Using (2.14) and the Schrödinger-Lichnerowicz type formula (3.9), we obtain

$$\begin{aligned}\|(D-\lambda)^2\psi_i\| \;\geqslant\; & \int\limits_{|\psi_i|\neq 0} |\psi_i|^{2\beta-2}|\nabla^\lambda\psi_i|^2 v_g + 2(\beta-1)\int\limits_{|\psi_i|\neq 0}|\psi_i|^{2\beta-3}\langle d|\psi_i|\cdot\psi_i, \nabla^\lambda\psi_i\rangle v_g \\ & + \int\left(\frac{S}{4}-\frac{c_n}{4}|\Omega| - \frac{n-1}{n}\lambda^2\right)|\psi_i|^{2\beta}v_g \\ & -2\frac{n-1}{n}\lambda \||\psi_i|^{2\beta-1}\|_{\{|\psi_i|\neq 0\}}\|(D-\lambda)\psi_i\|.\end{aligned}$$

The Cauchy-Schwarz inequality and the refined Kato inequality (3.10) for the connection ∇^λ imply

$$\begin{aligned}& \int\limits_{|\psi_i|\neq 0} |\psi_i|^{2\beta-2}|\nabla^\lambda\psi_i|^2 v_g + 2(\beta-1)\int\limits_{|\psi_i|\neq 0}|\psi_i|^{2\beta-3}\langle d|\psi_i|\cdot\psi_i, \nabla^\lambda\psi_i\rangle v_g \\ \geqslant\;& (2\beta-1)\int\limits_{|\psi_i|\neq 0}|\psi_i|^{2\beta-2}|d(|\psi_i|)|^2 v_g = (2\beta-1)\frac{1}{\beta^2}\int\limits_{|\psi_i|\neq 0}|d(|\psi_i|^\beta)|^2 v_g.\end{aligned}$$

Hence, we have

$$\begin{aligned}\|(D-\lambda)^2\psi_i\| \;\geqslant\; & (2\beta-1)\frac{1}{\beta^2}\int\limits_{|\psi_i|\neq 0}|d(|\psi_i|^\beta)|^2 v_g + \int\left(\frac{S}{4}-\frac{c_n}{4}|\Omega|-\frac{n-1}{n}\lambda^2\right)|\psi_i|^{2\beta}v_g \\ & -2\frac{n-1}{n}\lambda\|(D-\lambda)\psi_i\|.\end{aligned}$$

Since $S - c_n|\Omega|$ is bounded from below,

$$\int(S-c_n|\Omega|)|\psi_i|^{2\beta}v_g \geqslant \inf(S-c_n|\Omega|)\ \|\psi_i\|_{2\beta}^{2\beta} \geqslant \min\{\inf(S-c_n|\Omega|), 0\}$$

is also bounded. Thus, with $\|(D-\lambda)\psi_i\| \to 0$ we see that $\|d|\psi_i|^\beta\|$ is also bounded. Next we fix $\alpha = \frac{n-2}{n-1}$ and obtain

$$\begin{aligned}\frac{\mu_1}{4}-\frac{n-1}{n}\lambda^2 \;&\leqslant\; \left(\frac{\mu_1}{4}-\frac{n-1}{n}\lambda^2\right)\||\psi_i|^\alpha\|^2 \\ &\leqslant\; \frac{1}{4}\int |\psi_i|^\alpha L^\Omega|\psi_i|^\alpha v_g - \frac{n-1}{n}\lambda^2\||\psi_i|^\alpha\|^2 \\ &=\; \int|\psi_i|^{2\frac{n-2}{n-1}-2}\Big[\frac{n}{n-1}|d(|\psi_i|)|^2+\frac{1}{2}d^*d(|\psi_i|^2) \\ &\qquad + \left(\frac{S}{4}-\frac{c_n}{4}|\Omega|-\frac{n-1}{n}\lambda^2\right)|\psi_i|^2\Big]v_g,\end{aligned}$$

where we used the definition of μ_1 as the infimum of the spectrum of L^Ω and

$$|\psi_i|^\alpha d^*d(|\psi_i|^\alpha) = \frac{\alpha}{2}|\psi_i|^{2\alpha-2}d^*d(|\psi_i|^2) - \alpha(\alpha-2)|\psi_i|^{2\alpha-2}|d(|\psi_i|)|^2.$$

Next, using the following:

$$\frac{1}{2}d^*d\,\langle\psi_i,\psi_i\rangle \;\leqslant\; \left\langle D^2\psi_i,\psi_i\right\rangle - \frac{1}{4}(S-c_n|\Omega|)|\psi_i|^2 - |\nabla\psi_i|^2,$$

$$|\nabla^\lambda\psi_i|^2 = |\nabla\psi_i|^2 - 2\frac{\lambda}{n}\mathrm{Re}\,\langle(D-\lambda)\psi_i,\psi_i\rangle - \frac{\lambda^2}{n}|\psi_i|^2,$$

we have

$$\begin{aligned}\frac{\mu_1}{4} - \frac{n-1}{n}\lambda^2 \;&\leqslant\; \int |\psi_i|^{2\frac{n-2}{n-1}-2}\left(\frac{n}{n-1}|d(|\psi_i|)|^2 - |\nabla^\lambda\psi_i|^2\right)v_g\\ &\quad + \int |\psi_i|^{2\frac{n-2}{n-1}-2}\left\langle (D-\lambda)^2\psi_i,\psi_i\right\rangle v_g\\ &\quad + \int 2\left(1-\frac{1}{n}\right)\lambda|\psi_i|^{2\frac{n-2}{n-1}-2}\mathrm{Re}\,\langle(D-\lambda)\psi_i,\psi_i\rangle\, v_g.\end{aligned}$$

The limit of the last two summands vanish since

$$\left|\int |\psi_i|^{2\frac{n-2}{n-1}-2}\left\langle (D-\lambda)^2\psi_i,\psi_i\right\rangle v_g\right| \leqslant \|(D-\lambda)^2\psi_i\|\;\||\psi_i|^{\frac{n-3}{n-1}}\| \to 0,$$

$$\left|\int |\psi_i|^{2\frac{n-2}{n-1}-2}\mathrm{Re}\,\langle(D-\lambda)\psi_i,\psi_i\rangle\, v_g\right| \leqslant \|(D-\lambda)\psi_i\|\;\||\psi_i|^{\frac{n-3}{n-1}}\| \to 0.$$

For the other summand we use the Kato type inequality (3.11),

$$|d(|\psi|)| \leqslant |(D-\lambda)\psi| + k|\nabla^\lambda\psi|,$$

which holds outside the zero set of ψ and where $k=\sqrt{\frac{n-1}{n}}$. Thus, for $n\geqslant 5$, we can estimate

$$\begin{aligned}&\int |\psi_i|^{2\alpha-2}\left(\frac{n}{n-1}|d(|\psi_i|)|^2 - |\nabla^\lambda\psi_i|^2\right)v_g\\ &\quad= \int |\psi_i|^{2\alpha-2}\left(k^{-1}|d(|\psi_i|)| - |\nabla^\lambda\psi_i|\right)\left(k^{-1}|d(|\psi_i|)| + |\nabla^\lambda\psi_i|\right)v_g\\ &\quad\leqslant k^{-1}\int_{\{|d(|\psi_i|)|\geqslant k|\nabla^\lambda\psi_i|\}} |\psi_i|^{2\alpha-2}|(D-\lambda)\psi_i|\big(k^{-1}|d(|\psi_i|)| + |\nabla^\lambda\psi_i|\big)v_g\\ &\quad\leqslant 2k^{-2}\int_{\{|d(|\psi_i|)|\geqslant k|\nabla^\lambda\psi_i|\}} |\psi_i|^{2\alpha-2}|(D-\lambda)\psi_i||d(|\psi_i|)|v_g\\ &\quad\leqslant 2k^{-2}\frac{n-1}{n-3}\|(D-\lambda)\psi_i\|\;\|d(|\psi_i|^{\frac{n-3}{n-1}})\|.\end{aligned}$$

For $n\geqslant 5$, we have $1\geqslant \frac{n-3}{n-1}\geqslant\frac{1}{2}$ and, thus, $\|d|\psi_i|^{\frac{n-3}{n-1}}\|$ is bounded. Together with $\|(D-\lambda)\psi_i\|\to 0$ we obtain the following: for all $\epsilon>0$, there is an i_0 such that, for all $i\geqslant i_0$, we have

$$\int |\psi_i|^{2\frac{n-2}{n-1}-2}\left(\frac{n}{n-1}|d|\psi_i||^2 - |\nabla^\lambda\psi_i|^2\right)v_g \leqslant \epsilon.$$

Hence, we have $\frac{\mu_1}{4} \leqslant \frac{n-1}{n}\lambda^2$.

Chapter 4

The Energy-Momentum Tensor on Spin^c Manifolds[1]

4.1 Introduction

Studying the energy-momentum tensor on a Riemannian or semi-Riemannian Spin manifolds has been done by many authors, since it is related to several geometric constructions (see [Hab07], [BGM05], [Mor02] and [Fri98] for results in this topic). In this chapter we study the energy-momentum tensor on Riemannian and semi-Riemannian Spin^c manifolds. First, we prove that the energy-momentum tensor appears in the study of the variations of the spectrum of the Dirac operator:

Proposition 4.1.1. *Let (M^n, g) be a Riemannian* Spin^c *manifold and $g_t = g + tk$ a smooth 1-parameter family of metrics. For any spinor field $\psi \in \Gamma(\Sigma M)$, we have*

$$\frac{d}{dt}\Big|_{t=0} \int_M \mathrm{Re}\, \left\langle D^{M_t}\tau_0^t\psi, \tau_0^t\psi\right\rangle_{g_t} v_g = -\frac{1}{2}\int_M \langle k, \ell_\psi\rangle\, v_g, \tag{4.1}$$

where the Dirac operator D^{M_t} is the Dirac operator associated with $M_t = (M, g_t)$, $\ell_\psi(X) = |\psi|^2\ \ell^\psi(X) = \mathrm{Re}\,\langle X\cdot\nabla_X\psi, \psi\rangle$ and $\tau_0^t\psi$ is the image of ψ under the isometry τ_0^t between the Spin^c *bundles of (M, g) and (M, g_t).*

This was proven in [BG92] by J.P. Bourguignon and P. Gauduchon for Spin manifolds. Using this, we extend to Spin^c manifolds a result by T. Friedrich and E.C. Kim in [FK01] on Spin manifolds:

Theorem 4.1.1. *Let M be a Riemannian* Spin^c *manifold. A pair (g_0, ψ_0) is a critical point of the Lagrange functional*

$$\mathcal{W}(g,\psi) = \int_U \Big(S_g + \varepsilon\lambda|\psi|_g^2 - \varepsilon\mathrm{Re}\,\langle D_g\psi, \psi\rangle_g\Big) v_g,$$

[1]This chapter is the subject of two papers: one is published [Nak11a] and the other is submitted [Ha-Na10]

($\lambda, \varepsilon \in \mathbb{R}$) *for all open subsets* U *of* M *if and only if* (g_0, ψ_0) *is a solution of the following system*

$$\begin{cases} D_g\psi = \lambda\psi, \\ \mathrm{ric}_g - \frac{1}{2}S_g\, g = \frac{\varepsilon}{2}\ell_\psi, \end{cases}$$

where ric_g *denotes the Ricci curvature of* M *considered as a symmetric bilinear form.*

Now, we interprete the energy-momentum tensor as the second fundamental form of a hypersurface. In fact, we prove the following:

Proposition 4.1.2. *Let* $M^n \hookrightarrow (\mathcal{Z}, g)$ *be any compact oriented hypersurface isometrically immersed in an oriented Riemannian* Spin^c *manifold* $(\mathcal{Z}, g)$ *of mean curvature* H *and Weingarten map* II*. Assume that* $\mathcal{Z}$ *admits a parallel spinor field* ψ*, then the energy-momentum tensor associated with* $\phi =: \psi_{|_M}$ *satisfies*

$$2\ell^\phi = -II.$$

Moreover, if the mean curvature H *is constant, the hypersurface* M *satisfies the equality case in (2.11) if and only if*

$$S^{\mathcal{Z}} - 2\ \mathrm{ric}^{\mathcal{Z}}(\nu, \nu) - c_n|\Omega| = 0, \tag{4.2}$$

where $S^{\mathcal{Z}}$ *is the scalar curvature of* $\mathcal{Z}$ *and* $\mathrm{ric}^{\mathcal{Z}}$ *is the Ricci curvature of* $\mathcal{Z}$*.*

This was proven by B. Morel in [Mor02] for a compact oriented hypersurface of a Spin manifold carrying a non trivial parallel spinor but in this case the hypersurface M is directly a limiting manifold for (2.2) without the condition (4.2). Finally, we study generalized Killing spinors on Spin^c manifolds. They are characterized by the identity, for any tangent vector field X on M,

$$\nabla_X\psi = \frac{1}{2}E(X) \cdot \psi, \tag{4.3}$$

where E is a given symmetric endomorphism on the tangent bundle. It is straightforward to see that

$$2\ell^\psi(X, Y) = -\langle E(X), Y\rangle .$$

These spinors are closely related to the so-called T–Killing spinors studied by T. Friedrich and E.C. Kim in [FK01] on Spin manifolds. It is natural to ask whether the tensor E can be realized as the Weingarten tensor of some isometric embedding of M in a manifold $\mathcal{Z}^{n+1}$ carrying parallel spinors. B. Morel studied this problem in the case of Spin manifolds where the tensor E is parallel and in [BGM05], the authors studied the problem in the case of semi-Riemannian Spin manifolds where the tensor E is a Codazzi-Mainardi tensor. We establish the corresponding result for semi-Riemannian Spin^c manifolds:

Theorem 4.1.2. *Let* (M^n, g) *be a semi-Riemannian* Spin^c *manifold carrying a generalized* Spin^c *Killing spinor* ϕ *with a Codazzi-Mainardi tensor* E*. Then the generalized cylinder* $\mathcal{Z} := I \times M$ *with the metric* $dt^2 + g_t$*, where* $g_t(X, Y) = g((\mathrm{Id} - tE)^2X, Y)$*, equipped with the* Spin^c *structure arising from the given one on* M *has a parallel spinor whose restriction to* M *is just* ϕ*.*

A characterisation of limiting 3-dimensional manifolds for (2.11), having generalized Spin^c Killing spinors with Codazzi tensor is then given.

4.2 The Dirac operator on semi-Riemannian Spin^c manifolds

In this section, we collect some algebraic and geometric preliminaries concerning the Dirac operator on semi-Riemannian Spin^c manifolds. Details can be found in [Baum81] and [BGM05]. Let $r+s=n$ and consider on $\mathbb{R}^n$ the nondegenerate symmetric bilinear form of signature (r,s) given by

$$\langle v, w\rangle := \sum_{j=1}^{r} v_j w_j - \sum_{j=r+1}^{n} v_j w_j,$$

for any $v,\ w \in \mathbb{R}^n$. We denote by $\mathrm{Cl}_{r,s}$ the real Clifford algebra corresponding to $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$. This is the unitary algebra generated by $\mathbb{R}^n$ subject to the relations

$$e_j \cdot e_k + e_k \cdot e_j = \begin{cases} -2\delta_{jk} & \text{if} \quad j \leqslant r, \\ \ \ 2\delta_{jk} & \text{if} \quad j > r, \end{cases}$$

where $(e_j)_{1\leqslant j\leqslant n}$ is an orthonormal basis of $\mathbb{R}^n$ of signature (r,s), i.e. $\langle e_j, e_k\rangle = \varepsilon_j\delta_{jk}$ and $\varepsilon_j = \pm 1$. The complex Clifford algebra $\mathbb{C}\mathrm{l}_{r,s}$ is the complexification of $\mathrm{Cl}_{r,s}$ and it decomposes into even and odd elements $\mathbb{C}\mathrm{l}_{r,s} = \mathbb{C}\mathrm{l}^0_{r,s} \oplus \mathbb{C}\mathrm{l}^1_{r,s}$. The real spin group is defined by

$$\mathrm{Spin}(r,s) := \{v_1 \cdot ... \cdot v_{2k} \in \mathrm{Cl}_{r,s} \mid v_j \in \mathbb{R}^n \text{ such that } \langle v_j, v_j\rangle = \pm 1\}.$$

The spin group $\mathrm{Spin}(r,s)$ is the double cover of $\mathrm{SO}(r,s)$, in fact the following sequence is exact

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathrm{Spin}(r,s) \xrightarrow{\xi} \mathrm{SO}(r,s) \longrightarrow 1,$$

where $\xi = \mathrm{Ad}_{|_{\mathrm{Spin}(r,s)}}$ and Ad is defined by

$$\begin{array}{rcl} \mathrm{Ad} : \mathrm{Cl}^*_{r,s} & \longrightarrow & \mathrm{End}(\mathbb{R}^n) \\ w & \longmapsto & \mathrm{Ad}_w : v \longrightarrow \mathrm{Ad}_w(v) = w\cdot v\cdot w^{-1}. \end{array}$$

Here, $\mathrm{Cl}^*_{r,s}$ denotes the group of units of $\mathrm{Cl}_{r,s}$. Since $\mathbb{S}^1 \cap \mathrm{Spin}(r,s) = \{\pm 1\}$, we define the complex spin group by

$$\mathrm{Spin}^c(r,s) = \mathrm{Spin}(r,s) \times_{\mathbb{Z}_2} \mathbb{S}^1.$$

The complex spin group is the double cover of $\mathrm{SO}(r,s)\times\mathbb{S}^1$, this yields to the exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}^c(r,s) \xrightarrow{\xi^c} \mathrm{SO}(r,s)\times\mathbb{S}^1 \longrightarrow 1,$$

where $\xi^c = (\xi, \mathrm{Id}^2)$. When $n=2m$ is even, $\mathbb{C}\mathrm{l}_{r,s}$ has a unique irreducible complex representation χ_{2m} of complex dimension 2^m, $\chi_{2m} : \mathbb{C}\mathrm{l}_{r,s} \longrightarrow \mathrm{End}(\Sigma_{r,s})$. If $n=2m+1$ is odd, $\mathbb{C}\mathrm{l}_{r,s}$ has two inequivalent irreducible representations both of complex dimension 2^m, $\chi^j_{2m+1} : \mathbb{C}\mathrm{l}_{r,s} \longrightarrow \mathrm{End}(\Sigma^j_{r,s})$, for $j=0$ or 1, where $\Sigma^j_{r,s} = \{\sigma \in \Sigma_{r,s},\ \ \chi^j_{2m+1}(\omega_{r,s})\sigma = (-1)^j\sigma\}$ and $\omega_{r,s}$ is the complex volume element

$$\omega_{r,s} = \begin{cases} i^{m-s} & e_1\cdot ... \cdot e_n \qquad \text{if } n=2m, \\ i^{m-1+s} & e_1\cdot ... \cdot e_n \qquad \text{if } n=2m+1. \end{cases}$$

We define the complex spinorial representation ρ_n^c by the restriction of an irreducible representation of $\mathbb{C}\mathrm{l}_{r,s}$ to $\mathrm{Spin}^c(r,s)$:

$$\rho_n^c := \begin{cases} \chi_{2m}{}_{|_{\mathrm{Spin}^c(r,s)}} & \text{if } n = 2m \\ \chi^0_{2m+1}{}_{|_{\mathrm{Spin}^c(r,s)}} & \text{if } n = 2m+1 \end{cases}.$$

When $n = 2m$ is even, ρ_n^c decomposes into two inequivalent irreducible representations $(\rho_n^c)^+$ and $(\rho_n^c)^-$, i.e. $\rho_n^c = (\rho_n^c)^+ + (\rho_n^c)^- : \mathrm{Spin}^c(r,s) \to \mathrm{Aut}(\Sigma_{r,s})$. The space $\Sigma_{r,s}$ decomposes into $\Sigma_{r,s} = \Sigma^+_{r,s} \oplus \Sigma^-_{r,s}$, where $\omega_{r,s}$ acts on $\Sigma^+_{r,s}$ as the identity and minus the identity on $\Sigma^-_{r,s}$. If $n = r+s$ is odd, when restricted to $\mathrm{Spin}^c(r,s)$, the two representations $\chi^0_{2m+1}{}_{|_{\mathrm{Spin}^c(r,s)}}$ and $\chi^1_{2m+1}{}_{|_{\mathrm{Spin}^c(r,s)}}$ are equivalent and we simply choose $\Sigma_{r,s} := \Sigma^0_{r,s}$. The bundle $\Sigma_{r,s}$ carries a Hermitian symmetric bilinear $\mathrm{Spin}^c(r,s)$-invariant form $\langle \cdot, \cdot \rangle$, such that

$$\langle v \cdot \sigma_1, \sigma_2 \rangle = (-1)^{s+1} \langle \sigma_1, v \cdot \sigma_2 \rangle \text{ for all } \sigma_1, \sigma_2 \in \Sigma_{r,s} \text{ and } v \in \mathbb{R}^n.$$

Now, we give the following isomorphism α, which is of particular importance for the identification of the $\mathrm{Spin^c}$ bundles in the context of immersions of hypersurfaces:

$$\begin{aligned} \alpha : \mathbb{C}\mathrm{l}_{r,s} &\longrightarrow \mathbb{C}\mathrm{l}^0_{r+1,s} \\ e_j &\longmapsto \nu \cdot e_j, \end{aligned} \tag{4.4}$$

where we look at an embedding of $\mathbb{R}^n$ onto $\mathbb{R}^{n+1}$ such that $(\mathbb{R}^n)^\perp$ is spacelike and spanned by a spacelike unit vector ν.

Let N^n be an oriented semi-Riemannian manifold of signature (r,s) and let $P_{\mathrm{SO}(r,s)}N$ be the $\mathrm{SO}(r,s)$-principal bundle of positively space and time oriented orthonormal tangent frames. A complex $\mathrm{Spin^c}$ structure on N is a $\mathrm{Spin}^c(r,s)$-principal bundle $P_{\mathrm{Spin}^c(r,s)}N$ over N, an $\mathbb{S}^1$-principal bundle $P_{\mathbb{S}^1}N$ over N together with a twofold covering map $\Theta : P_{\mathrm{Spin}^c(r,s)}N \longrightarrow P_{\mathrm{SO}(r,s)}N \times_N P_{\mathbb{S}^1}N$ such that

$$\Theta(ua) = \Theta(u)\xi^c(a),$$

for every $u \in P_{\mathrm{Spin}^c(r,s)}N$ and $a \in \mathrm{Spin}^c(r,s)$, i.e. N has a $\mathrm{Spin^c}$ structure if and only if there exists an $\mathbb{S}^1$-principal bundle $P_{\mathbb{S}^1}N$ over N such that the transition functions $g_{\alpha\beta} \times l_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{SO}(r,s) \times \mathbb{S}^1$ of the $\mathrm{SO}(r,s) \times \mathbb{S}^1$-principal bundle $P_{\mathrm{SO}(r,s)}N \times_N P_{\mathbb{S}^1}N$ admit lifts to $\mathrm{Spin}^c(r,s)$ denoted by $\widetilde{g}_{\alpha\beta} \times \widetilde{l}_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{Spin}^c(r,s)$, such that $\xi^c \circ (\widetilde{g}_{\alpha\beta} \times \widetilde{l}_{\alpha\beta}) = g_{\alpha\beta} \times l_{\alpha\beta}$. This, anyhow, is equivalent to the second Stiefel-Whitney class $w_2(N)$ being equal, modulo 2, to the first chern class $c_1(L^N)$ of the auxiliary line bundle L^N. It is the complex line bundle associated with the $\mathbb{S}^1$-principal fiber bundle via the standard representation of the unit circle.

Let $\Sigma N := P_{\mathrm{Spin}^c(r,s)}N \times_{\rho_n^c} \Sigma_{r,s}$ be the $\mathrm{Spin^c}$ bundle associated with the spinor representation. A section of ΣN will be called a spinor field. Using the cocycle condition of the transition functions of the two principal fiber bundles $P_{\mathrm{Spin}^c(r,s)}N$ and $P_{\mathrm{SO}(r,s)}N \times_N P_{\mathbb{S}^1}N$, we can prove that

$$\Sigma N = \Sigma' N \otimes (L^N)^{\frac{1}{2}},$$

where $\Sigma' N$ is the locally defined spinorial bundle and $(L^N)^{\frac{1}{2}}$ is locally defined too but ΣN is globally defined. The tangent bundle $TN = P_{\mathrm{SO}(r,s)}N \times_{\rho_0} \mathbb{R}^n$, where ρ_0 stands for the standard matrix representation of $\mathrm{SO}(r,s)$ on $\mathbb{R}^n$, can be seen as the associated vector bundle $TN \simeq P_{\mathrm{Spin}^c(r,s)}N \times_{pr_1 \circ \xi^c \circ \rho_0} \mathbb{R}^n$ where pr_1 is the first projection. One defines the Clifford multiplication at every point $p \in N$:

$$\begin{array}{rcl} T_pN \otimes \Sigma_pN & \longrightarrow & \Sigma_pN \\ {[b,v]} \otimes [b,\sigma] & \longmapsto & [b,v]\cdot[b,\sigma] := [b, v\cdot\sigma = \chi_n(v)\sigma], \end{array}$$

where $b \in P_{\mathrm{Spin}^c(r,s)}N$, $v \in \mathbb{R}^n$, $\sigma \in \Sigma_{r,s}$ and $\chi_n = \chi_{2m}$ if n is even and $\chi_n = \chi^0_{2m+1}$ if n is odd. The Clifford multiplication can be extended to differential forms. Clifford multiplication inherits the relations of the Clifford algebra, i.e. for $X, Y \in T_pN$ and $\phi \in \Sigma_pN$ we have $X \cdot Y \cdot \phi + Y \cdot X \cdot \phi = -2\langle X, Y\rangle \phi$. In even dimensions the $\mathrm{Spin}^{\mathrm{c}}$ bundle splits into $\Sigma N = \Sigma^+N \oplus \Sigma^-N$, where $\Sigma^\pm N = P_{\mathrm{Spin}^c(r,s)}N \times_{(\rho^c_n)^\pm} \Sigma^\pm_{r,s}$. Clifford multiplication by a non-vanishing tangent vector interchanges Σ^+N and Σ^-N. The $\mathrm{Spin}^c(r,s)$-invariant nondegenerate symmetric sesquilinear form on $\Sigma_{r,s}$ and $\Sigma^\pm_{r,s}$ induces inner products on ΣN and $\Sigma^\pm N$ which we again denote by $\langle\cdot,\cdot\rangle$ and it satisfies

$$\langle X\cdot\psi, \phi\rangle = (-1)^{s+1}\langle \psi, X\cdot\phi\rangle,$$

for every $X \in \Gamma(TN)$ and $\psi, \phi \in \Gamma(\Sigma N)$. Additionally, given a connection 1-form A^N on $P_{\mathbb{S}^1}N$, $A^N : T(P_{\mathbb{S}^1}N) \longrightarrow i\mathbb{R}$ and the connection 1-form ω^N on $P_{\mathrm{SO}(r,s)}N$ for the Levi-Civita connection ∇^N, we can define the connection

$$\omega^N \times A^N : T(P_{\mathrm{SO}(r,s)}N \times_N P_{\mathbb{S}^1}N) \longrightarrow \mathfrak{so}_n \oplus i\mathbb{R} = \mathfrak{spin}^{\mathbb{C}}_n$$

on the principal fiber bundle $P_{\mathrm{SO}(r,s)}N \times_N P_{\mathbb{S}^1}N$ and hence a covariant derivative $\nabla^{\Sigma N}$ on ΣN [Fri00] given locally by

$$\begin{aligned} \nabla^{\Sigma N}_{e_k}\phi &= \left[\widetilde{b\times s}, e_k(\sigma) + \frac{1}{4}\sum_{j=1}^n \varepsilon_j\, e_j \cdot \nabla^N_{e_k}e_j \cdot \sigma + \frac{1}{2}A^N(s_*(e_k))\sigma\right] \\ &= e_k(\phi) + \frac{1}{4}\sum_{j=1}^n \varepsilon_j\, e_j \cdot \nabla^N_{e_k}e_j\cdot\phi + \frac{1}{2}A^N(s_*(e_k))\phi, \end{aligned} \tag{4.5}$$

where $\phi = [\widetilde{b\times s}, \sigma]$ is a locally defined spinor field, $b = (e_1, \ldots, e_n)$ is a local space and time oriented orthonormal tangent frame, $s : U \longrightarrow P_{\mathbb{S}^1}N$ is a local section of $P_{\mathbb{S}^1}N$ and $\widetilde{b\times s}$ is the lift of the local section $b \times s : U \to P_{\mathrm{SO}(r,s)}N \times_N P_{\mathbb{S}^1}N$ to the 2-fold covering $\Theta : P_{\mathrm{Spin}^c(r,s)}N \longrightarrow P_{\mathrm{SO}(r,s)}N \times_N P_{\mathbb{S}^1}N$. The curvature of A^N is an imaginary valued 2-form denoted by $F_{A^N} = dA^N$, i.e. $F_{A^N} = i\Omega^N$, where Ω^N is a real valued 2-form on $P_{\mathbb{S}^1}N$. We know that Ω^N can be viewed as a real valued 2-form on N [Fri00]. In this case $i\Omega^N$ is the curvature form of the auxiliary line bundle L^N. The curvature tensor $\mathcal{R}^{\Sigma N}$ of $\nabla^{\Sigma N}$ is given by

$$\mathcal{R}^{\Sigma N}(X,Y)\phi = \frac{1}{4}\sum_{j,k=1}^n \varepsilon_j\varepsilon_k \left\langle R^N(X,Y)e_j, e_k\right\rangle e_j\cdot e_k\cdot\phi + \frac{i}{2}\Omega^N(X,Y)\phi, \tag{4.6}$$

where R^N is the curvature tensor of the Levi-Civita connection ∇^N. In the Spin^c case, the Ricci identity translates, for every $X \in \Gamma(TN)$, to

$$\sum_{k=1}^{n} \varepsilon_k \, e_k \cdot \mathcal{R}^{\Sigma N}(e_k, X)\phi = \frac{1}{2}\mathrm{Ric}^N(X) \cdot \phi - \frac{i}{2}(X \lrcorner \Omega^N) \cdot \phi. \tag{4.7}$$

Here Ric^N denotes the Ricci curvature considered as a field of endomorphisms on TN. The Dirac operator maps spinor fields to spinor fields and is locally defined by

$$D^N \phi = i^s \sum_{j=1}^{n} \varepsilon_j e_j \cdot \nabla^{\Sigma N}_{e_j} \phi,$$

for every spinor field ϕ. The Dirac operator is an elliptic operator, formally selfadjoint, i.e. if ψ or ϕ has compact support, then $\int_N \left\langle D^N \phi, \psi \right\rangle v_g = \int_N \left\langle \phi, D^N \psi \right\rangle v_g$.

4.3 Semi-Riemannian Spin^c hypersurfaces and the Gauss formula

In this section, we study Spin^c structures of hypersurfaces, such as the restriction of a Spin^c bundle of an ambient semi-Riemannian manifold and the complex spinorial Gauss formula.

Let $\mathcal{Z}$ be an oriented $(n+1)$-dimensional semi-Riemannian Spin^c manifold and $M \subset \mathcal{Z}$ a semi-Riemannian hypersurface with trivial spacelike normal bundle. This means that there is a vector field ν on $\mathcal{Z}$ along M satisfying $\langle \nu, \nu \rangle = +1$ and $\langle \nu, TM \rangle = 0$. Hence, if the signature of M is (r, s), then the signature of $\mathcal{Z}$ is $(r+1, s)$.

Proposition 4.3.1. *The hypersurface M inherits a* Spin^c *structure from that on $\mathcal{Z}$, and we have*

$$\begin{cases} \Sigma M \simeq \Sigma \mathcal{Z}_{|M} & \text{if } n \text{ is even,} \\ \Sigma M \simeq \Sigma^+ \mathcal{Z}_{|M} & \text{if } n \text{ is odd.} \end{cases}$$

Moreover Clifford multiplication by a vector field X, tangent to M, is given by

$$X \bullet \phi = (\nu \cdot X \cdot \psi)_{|M}, \tag{4.8}$$

where $\psi \in \Gamma(\Sigma\mathcal{Z})$ (or $\psi \in \Gamma(\Sigma^+\mathcal{Z})$ if n is odd), ϕ is the restriction of ψ to M, "$\cdot$" is the Clifford multiplication on $\mathcal{Z}$, and "$\bullet$" that on M.

Proof. The bundle of space and time oriented orthonormal frames of M can be embedded into the bundle of space and time oriented orthonormal frames of $\mathcal{Z}$ restricted to M, by

$$\begin{array}{rrcl} \Phi : & P_{\mathrm{SO}(r,s)} M & \longrightarrow & P_{\mathrm{SO}(r+1,s)} \mathcal{Z}_{|M} \\ & (e_1, \cdots, e_n) & \longmapsto & (\nu, e_1, \cdots, e_n). \end{array} \tag{4.9}$$

The isomorphism α, defined in (4.4) yields the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Spin}^c(r,s) \arrow[r] \arrow[d, "\xi^c"] & \mathrm{Spin}^c(r+1,s) \arrow[d, "(\xi^c)_*"] \\
\mathrm{SO}(r,s)\times\mathbb{S}^1 \arrow[r] & \mathrm{SO}(r+1,s)\times\mathbb{S}^1
\end{tikzcd}$$

where the inclusion of $\mathrm{SO}(r,s)$ in $\mathrm{SO}(r+1,s)$ is that which fixes the first basis vector under the action of $\mathrm{SO}(r+1,s)$ on $\mathbb{R}^{n+1}$. This allows to pull back via Φ the principal bundle $P_{\mathrm{Spin}^c(r+1,s)}\mathcal{Z}_{|M}$ as a $\mathrm{Spin^c}$ structure for M, denoted by $P_{\mathrm{Spin}^c(r,s)}M$. Thus, we have the following commutative diagram:

$$\begin{tikzcd}
P_{\mathrm{Spin}^c(r,s)}M \arrow[r] \arrow[d, "\Theta"] & P_{\mathrm{Spin}^c(r+1,s)}\mathcal{Z}_{|M} \arrow[d, "\Theta"] \\
P_{\mathrm{SO}(r,s)}M\times_M P_{\mathbb{S}^1}\mathcal{Z}_{|M} \arrow[r] & P_{\mathrm{SO}(r+1,s)}\mathcal{Z}_{|M}\times_M P_{\mathbb{S}^1}\mathcal{Z}_{|M}
\end{tikzcd}$$

The $\mathrm{Spin^c}(r,s)$-principal fiber bundle $(P_{\mathrm{Spin}^c(r,s)}M,\pi,M)$ and the $\mathbb{S}^1$-principal fiber bundle $(P_{\mathbb{S}^1}M =: P_{\mathbb{S}^1}\mathcal{Z}|_M,\pi,M)$ define a $\mathrm{Spin^c}$ structure on M. Let $\Sigma\mathcal{Z}$ be the $\mathrm{Spin^c}$ bundle on $\mathcal{Z}$,

$$\Sigma\mathcal{Z} = P_{\mathrm{Spin}^c(r+1,s)}\mathcal{Z}\times_{\rho^c_{n+1}}\Sigma_{r+1,s},$$

where ρ^c_{n+1} stands for the spinorial representation of $\mathrm{Spin}^c(r+1,s)$. Moreover, for any spinor $\psi = [\widetilde{b\times s},\sigma]\in\Sigma\mathcal{Z}$ we can always assume that $pr_1\circ\Theta(\widetilde{b\times s}) = b$ is a local section of $P_{\mathrm{SO}(r+1,s)}\mathcal{Z}$ with ν for first basis vector where pr_1 is the projection into $P_{\mathrm{SO}(r+1,s)}\mathcal{Z}$. Then we have

$$\psi_{|M} = [\widetilde{b\times s}_{|U\cap M},\sigma_{|U\cap M}],$$

where the equivalence class is reduced to elements of $\mathrm{Spin}^c(r,s)$. It follows that one can realise the restriction to M of the $\mathrm{Spin^c}$ bundle $\Sigma\mathcal{Z}$ as

$$\Sigma\mathcal{Z}_{|M} = P_{\mathrm{Spin}^c(r,s)}M\times_{\rho^c_{n+1}\circ\alpha}\Sigma_{r+1,s}.$$

If $n=2m$ is even, it is easy to check that $\chi^0_{2m+1}\circ\alpha = \chi^0_{2m+1|_{\mathbb{C}\mathrm{l}^0_{r+1,s}}}$. Hence $\chi^0_{2m+1}\circ\alpha$ is an irreducible representation of $\mathbb{C}\mathrm{l}_{r,s}$ of dimension 2^m, as $\chi^0_{2m+1|_{\mathbb{C}\mathrm{l}^0_{r+1,s}}}$, and finally $\chi^0_{2m+1}\circ\alpha\cong\chi_{2m}$. We conclude that

$$\rho^c_{2m+1}\circ\alpha\cong\rho^c_{2m},\quad\text{and}\quad\Sigma\mathcal{Z}_{|M}\cong\Sigma M.$$

If $n=2m+1$ is odd, we know that χ^0_{2m+1} is the unique irreducible representation of $\mathbb{C}\mathrm{l}_{r,s}$ of dimension 2^m for which the action of the complex volume form is the identity. Since $n+1=2m+2$ is even, $\Sigma\mathcal{Z}$ decomposes into positive and negative parts,

$$\Sigma^{\pm}\mathcal{Z} = P_{\mathrm{Spin}^c(r+1,s)}\mathcal{Z}\times_{(\rho^c_{2m+1})^{\pm}}\Sigma^{\pm}_{r+1,s}.$$

It is easy to show that $\chi_{2m+2} \circ \alpha = \chi_{2m+2}|_{\mathbb{C}l^0_{r+1,s}}$, but $\chi_{2m+2} \circ \alpha$ can be written as the direct sum of two irreducible inequivalent representations, as $\chi_{2m+2}|_{\mathbb{C}l^0_{r+1,s}}$. Hence, we have

$$\chi_{2m+2} \circ \alpha = (\chi_{2m+2} \circ \alpha)^+ \oplus (\chi_{2m+1} \circ \alpha)^-,$$

where $(\chi_{2m+2} \circ \alpha)^{\pm}(\omega_{r,s}) = \pm \mathrm{Id}_{\Sigma_{r,s}}$. The representation χ^0_{2m+1} being the unique representation of $\mathbb{C}l_{r,s}$ of dimension 2^m for which the action of the volume form is the identity, we get $(\chi_{2m+2} \circ \alpha)^+ \cong \chi^0_{2m+1}$. Finally,

$$(\rho^c_{2m+2})^+ \circ \alpha \cong \rho^c_{2m+1} \quad \text{and} \quad \Sigma^+\mathcal{Z}_{|_M} \cong \Sigma M.$$

Now, Equation (4.8) follows directly from the above identification.

Remark 4.3.1. 1. *The algebraic remarks in the previous section show that, if n is odd, we can also get $\Sigma^-\mathcal{Z}_{|_M} \simeq \Sigma M$, where the Clifford multiplication by a vector field tangent to M is given by $X \bullet \phi = -(\nu \cdot X \cdot \psi)_{|_M}$.*
2. *The connection 1-form defined on the restricted $\mathbb{S}^1$-principal fiber bundle $(P_{\mathbb{S}^1}M =: P_{\mathbb{S}^1}\mathcal{Z}_{|_M}, \pi, M)$ is given by*

$$A = A^{\mathcal{Z}}{}_{|_M} : T(P_{\mathbb{S}^1}M) = T(P_{\mathbb{S}^1}\mathcal{Z})_{|_M} \longrightarrow i\mathbb{R}.$$

Then the curvature 2-form $i\Omega$ on the $\mathbb{S}^1$-principal bundle $P_{\mathbb{S}^1}M$ is given by $i\Omega = i\Omega^{\mathcal{Z}}{}_{|_M}$, which can be viewed as an imaginary 2-form on M and hence as the curvature form of the line bundle L, the restriction of the auxiliary line bundle $L^{\mathcal{Z}}$ to M.
3. *For every $\psi \in \Gamma(\Sigma\mathcal{Z})$ ($\psi \in \Gamma(\Sigma^+\mathcal{Z})$ if n is odd), the real 2-forms Ω and $\Omega^{\mathcal{Z}}$ are related by the following formulas:*

$$|\Omega^{\mathcal{Z}}|^2 = |\Omega|^2 + |\nu \lrcorner \Omega^{\mathcal{Z}}|^2, \tag{4.10}$$

$$(\Omega^{\mathcal{Z}} \cdot \psi)_{|_M} = \Omega \bullet \phi + (\nu \lrcorner \Omega^{\mathcal{Z}}) \bullet \phi. \tag{4.11}$$

In fact, we can write

$$\Omega^{\mathcal{Z}} = \sum_{i=1}^{n} \Omega^{\mathcal{Z}}(e_i, \nu)\ e_i \wedge \nu + \sum_{i<j}^{n} \Omega^{\mathcal{Z}}\ (e_i, e_j)\ e_i \wedge e_j = -(\nu \lrcorner \Omega^{\mathcal{Z}}) \wedge \nu + \Omega,$$

which is (4.10). When restricting the Clifford multiplication of $\Omega^{\mathcal{Z}}$ by ψ to the hypersurface M, we obtain

$$(\Omega^{\mathcal{Z}} \cdot \psi)_{|_M} = \big(\nu \cdot (\nu \lrcorner \Omega^{\mathcal{Z}}) \cdot \psi\big)_{|_M} + (\Omega \cdot \psi)_{|_M} = (\nu \lrcorner \Omega^{\mathcal{Z}}) \bullet \phi + \Omega \bullet \phi. \tag{4.12}$$

Proposition 4.3.2 (The Spin^c Gauss formula)**.** *We denote by $\nabla^{\Sigma\mathcal{Z}}$ the spinorial Levi-Civita connection on $\Sigma\mathcal{Z}$ and by ∇ that on ΣM. For all $X \in \Gamma(TM)$ and for every spinor field $\psi \in \Gamma(\Sigma\mathcal{Z})$, then*

$$(\nabla^{\Sigma\mathcal{Z}}_X \psi)_{|_M} = \nabla_X \phi - \frac{1}{2} II(X) \bullet \phi, \tag{4.13}$$

where II denotes the Weingarten map with respect to ν and $\phi = \psi_{|M}$. Moreover, let $D^{\mathcal{Z}}$ and D be the Dirac operators on $\mathcal{Z}$ and M. Denoting by the same symbol any spinor and its restriction to M, we have

$$\nu \cdot D^{\mathcal{Z}}\phi = \widetilde{D}\phi + \frac{i^s n}{2}H\phi - i^s\nabla^{\Sigma\mathcal{Z}}_{\nu}\phi, \tag{4.14}$$

where $H = \frac{1}{n}\mathrm{tr}(II)$ denotes the mean curvature and $\widetilde{D} = D$ if n is even and $\widetilde{D} = D \oplus (-D)$ if n is odd.

Proof. The Riemannian Gauss formula is given, for every vector fields X and Y on M, by

$$\nabla^{\mathcal{Z}}_X Y = \nabla_X Y + \langle II(X), Y\rangle\, \nu. \tag{4.15}$$

Let $\{e_1, e_2, ..., e_n\}$ a local space and time oriented orthonormal frame of M, such that $b = \{e_0 = \nu, e_1, e_2, ..., e_n\}$ is that of $\mathcal{Z}$. We consider ψ a local section of $\Sigma\mathcal{Z}$, $\psi = [\widetilde{b \times s}, \sigma]$ where s is a local section of $P_{\mathbb{S}^1}\mathcal{Z}$. Using (4.5), (4.15) and the fact that $X(\psi)|_M = X(\phi)$ for $X \in \Gamma(TM)$, we compute for $j = 1, ..., n$

$$\begin{aligned}
\left(\nabla^{\Sigma\mathcal{Z}}_{e_j}\psi\right)_{|M} &= e_j(\phi) + \frac{1}{4}\sum_{k=0}^{n}\varepsilon_k(e_k\cdot\nabla^{\mathcal{Z}}_{e_j}e_k\cdot\psi)_{|M} + \frac{1}{2}A^Z(s_*(e_j))\phi \\
&= e_j(\phi) + \frac{1}{4}\sum_{k=1}^{n}\varepsilon_k(e_k\cdot\nabla^{\mathcal{Z}}_{e_j}e_k\cdot\psi)_{|M} + \frac{1}{4}(\nu\cdot\nabla^{\mathcal{Z}}_{e_j}\nu\cdot\psi)_{|M} + \frac{1}{2}A(s_*(e_j))\phi \\
&= \nabla_{e_j}\phi + \frac{1}{4}\sum_{k=1}^{n}\varepsilon_k\,\langle II(e_j), e_k\rangle\,(e_k\cdot\nu\cdot\psi)_{|M} - \frac{1}{4}(\nu\cdot II(e_j)\cdot\psi)_{|M} \\
&= \nabla_{e_j}\phi - \frac{1}{2}(\nu\cdot II(e_j)\cdot\psi)_{|M} \\
&= \nabla_{e_j}\phi - \frac{1}{2}II(e_j)\bullet\phi.
\end{aligned}$$

Moreover $(D^{\mathcal{Z}}\psi)_{|M} = i^s\sum_{j=1}^n\varepsilon_j(e_j\cdot\nabla^{\Sigma\mathcal{Z}}_{e_j}\psi)_{|M} + i^s(\nu\cdot\nabla^{\Sigma\mathcal{Z}}_{\nu}\psi)_{|M}$, and by (4.13),

$$\begin{aligned}
i^s\sum_{j=1}^{n}\varepsilon_j(e_j\cdot\nabla^{\Sigma\mathcal{Z}}_{e_j}\psi)_{|M} &= i^s\sum_{j=1}^{n}\varepsilon_j\,(e_j\cdot\nabla_{e_j}\phi) - i^s\frac{1}{2}\sum_{j=1}^{n}\varepsilon_j\,(e_j\cdot\nu\cdot II(e_j)\cdot\psi)_{|M} \\
&= -i^s\nu\cdot\sum_{j=1}^{n}\varepsilon_j\,\nu\cdot e_j\cdot\nabla_{e_j}\phi + i^s\frac{1}{2}\sum_{j=1}^{n}\varepsilon_j\,(\nu\cdot e_j\cdot II(e_j)\cdot\psi)_{|M} \\
&= -\nu\cdot\widetilde{D}\phi - \frac{i^s}{2}\mathrm{tr}(II)(\nu\cdot\psi)_{|M}.
\end{aligned}$$

Proposition 4.3.3. *Let $\mathcal{Z}$ be an $(n+1)$-dimensional semi-Riemannian* Spin^c *manifold. Assume that $\mathcal{Z}$ carries a semi-Riemannian foliation by hypersurfaces with trivial spacelike normal bundle, i.e. the leaves M are semi-Riemannian hypersurfaces and there exists a vector field ν on $\mathcal{Z}$ perpendicular to the leaves such that $\langle\nu,\nu\rangle = 1$ and $\nabla^{\mathcal{Z}}_{\nu}\nu = 0$.*

Then the commutator of the leafwise Dirac operator and the normal derivative is given by

$$i^{-s}[\nabla^{\Sigma\mathcal{Z}}_\nu, \widetilde{D}]\,\phi = \mathfrak{D}^{II}\phi - \frac{n}{2}\,\nu\cdot\mathrm{grad}^M(H)\cdot\phi + \frac{1}{2}\,\nu\cdot\mathrm{div}^M(II)\cdot\phi + \frac{i}{2}\nu\cdot(\nu\lrcorner\Omega^{\mathcal{Z}})\cdot\phi.$$

Here grad^M *denotes the leafwise gradient,* $\mathrm{div}^M(II) = \sum_{i=1}^n \varepsilon_i\,(\nabla^M_{e_i}II)(e_i)$ *denotes the leafwise divergence of the endomorphism field* II *and* $\mathfrak{D}^{II}\phi = \sum_{i=1}^n \varepsilon_i\,\nu\cdot e_i\cdot\nabla_{II(e_i)}\phi$.

Proof. We choose a local oriented orthonormal tangent frame $\{e_1,\dots,e_n\}$ for the leaves and we may assume for simplicity that $\nabla^{\mathcal{Z}}_\nu e_j = 0$. Now, we compute

$$\begin{aligned}
i^{-s}[\nabla^{\Sigma\mathcal{Z}}_\nu, \widetilde{D}]\,\phi &= \sum \varepsilon_j\left(\nabla^{\Sigma\mathcal{Z}}_\nu(\nu\cdot e_j\cdot\nabla_{e_j}\phi) - \nu\cdot e_j\cdot\nabla_{e_j}\nabla^{\Sigma\mathcal{Z}}_\nu\phi\right)\\
&= \sum \varepsilon_j\,\nu\cdot e_j\cdot\left(\nabla^{\Sigma\mathcal{Z}}_\nu\nabla_{e_j}\phi - \nabla_{e_j}\nabla^{\Sigma\mathcal{Z}}_\nu\phi\right)\\
&\overset{(4.13)}{=} \sum \varepsilon_j\,\nu\cdot e_j\cdot\Big[\nabla^{\Sigma\mathcal{Z}}_\nu(\nabla^{\Sigma\mathcal{Z}}_{e_j} + \frac{1}{2}\nu\cdot II(e_j))\\
&\qquad -(\nabla^{\Sigma\mathcal{Z}}_{e_j} + \frac{1}{2}\nu\cdot II(e_j))\nabla^{\Sigma\mathcal{Z}}_\nu\Big]\phi\\
&= \sum \varepsilon_j\,\nu\cdot e_j\cdot\Big(\mathcal{R}^{\Sigma\mathcal{Z}}(\nu,e_j) + \nabla^{\Sigma\mathcal{Z}}_{[\nu,e_j]} + \frac{1}{2}\nu\cdot(\nabla^{\mathcal{Z}}_\nu II)(e_j)\Big)\phi\\
&\overset{(4.7)}{=} -\frac{1}{2}\nu\cdot\mathrm{Ric}^{\mathcal{Z}}(\nu)\cdot\phi + \frac{i}{2}\nu\cdot(\nu\lrcorner\Omega^{\mathcal{Z}})\cdot\phi\\
&\qquad +\sum \varepsilon_j\,\nu\cdot e_j\cdot\Big(\nabla^{\Sigma\mathcal{Z}}_{II(e_j)} + \frac{1}{2}\nu\cdot(\nabla^{\mathcal{Z}}_\nu II)(e_j)\Big)\phi\\
&\overset{(4.13)}{=} -\frac{1}{2}\nu\cdot\mathrm{Ric}^{\mathcal{Z}}(\nu)\cdot\phi + \frac{i}{2}\nu\cdot(\nu\lrcorner\Omega^{\mathcal{Z}})\cdot\phi\\
&\qquad +\sum \varepsilon_j\,\nu\cdot e_j\cdot\Big(\nabla_{II(e_j)} - \frac{1}{2}\nu\cdot II^2(e_j) + \frac{1}{2}\nu\cdot(\nabla^{\mathcal{Z}}_\nu II)(e_j)\Big)\phi\\
&= -\frac{1}{2}\nu\cdot\mathrm{Ric}^{\mathcal{Z}}(\nu)\cdot\phi + \frac{i}{2}\nu\cdot(\nu\lrcorner\Omega^{\mathcal{Z}})\cdot\phi + \mathfrak{D}^{II}\phi\\
&\qquad +\frac{1}{2}\sum \varepsilon_j\,e_j\cdot\Big(-II^2(e_j) + (\nabla^{\mathcal{Z}}_\nu II)(e_j)\Big)\phi.
\end{aligned}$$

The Riccati equation for the Weingarten map $(\nabla^{\mathcal{Z}}_\nu II)(X) = R^{\mathcal{Z}}(X,\nu)\nu + II^2(X)$ yields

$$\begin{aligned}
i^{-s}[\nabla^{\Sigma\mathcal{Z}}_\nu, \widetilde{D}]\,\phi &= -\frac{1}{2}\nu\cdot\mathrm{Ric}^{\mathcal{Z}}(\nu)\cdot\phi + \frac{i}{2}\nu\cdot(\nu\lrcorner\Omega^{\mathcal{Z}})\cdot\phi + \mathfrak{D}^{II}\phi\\
&\qquad +\frac{1}{2}\sum \varepsilon_j\,e_j\cdot(R^{\mathcal{Z}}(e_j,\nu)\nu)\cdot\phi\\
&= -\frac{1}{2}\nu\cdot\mathrm{Ric}^{\mathcal{Z}}(\nu)\cdot\phi + \frac{i}{2}\nu\cdot(\nu\lrcorner\Omega^{\mathcal{Z}})\cdot\phi + \mathfrak{D}^{II}\phi + \frac{1}{2}\mathrm{ric}^{\mathcal{Z}}(\nu,\nu)\phi\\
&= \mathfrak{D}^{II}\phi - \frac{1}{2}\sum \varepsilon_i\,\mathrm{ric}^{\mathcal{Z}}(\nu,e_j)\,\nu\cdot e_j\cdot\phi + \frac{i}{2}\nu\cdot(\nu\lrcorner\Omega^{\mathcal{Z}})\cdot\phi. \qquad (4.16)
\end{aligned}$$

The Codazzi-Mainardi equation for $X,Y,V \in TM$ is given by $\left\langle R^{\mathcal{Z}}(X,Y)V,\nu\right\rangle =$

$\left\langle (\nabla^M_X II)(Y), V \right\rangle - \left\langle (\nabla^M_Y II)(X), V \right\rangle$. Thus,

$$\begin{aligned}
\mathrm{ric}^{\mathcal{Z}}(\nu, X) &= \sum \varepsilon_j \left\langle R^{\mathcal{Z}}(X, e_j)e_j, \nu \right\rangle \\
&= \sum \varepsilon_j \left(\left\langle (\nabla^M_X II)(e_j), e_j \right\rangle - \left\langle (\nabla^M_{e_j} II)(X), e_j \right\rangle \right) \\
&= \mathrm{tr}(\nabla^M_X II) - \left\langle \mathrm{div}^M(II), X \right\rangle .
\end{aligned}$$

Plugging this into (4.16) we get

$$\begin{aligned}
i^{-s}[\nabla^{\Sigma\mathcal{Z}}_\nu, \widetilde{D}]\,\phi &= \mathfrak{D}^{II}\phi - \frac{1}{2}\sum \varepsilon_j \left(\mathrm{tr}(\nabla^M_{e_j} II) - \left\langle \mathrm{div}^M(II), e_j \right\rangle \right) \nu \cdot e_j \cdot \phi \\
&\quad + \frac{i}{2}\nu \cdot (\nu \lrcorner \Omega^{\mathcal{Z}}) \cdot \phi. \\
&= \mathfrak{D}^{II}\phi - \frac{1}{2}\sum \varepsilon_j\, e_j(\mathrm{tr}(II))\nu \cdot e_j \cdot \phi + \frac{1}{2}\nu \cdot \mathrm{div}^M(II) \cdot \phi \\
&\quad + \frac{i}{2}\nu \cdot (\nu \lrcorner \Omega^{\mathcal{Z}}) \cdot \phi. \\
&= \mathfrak{D}^{II}\phi - \frac{n}{2}\nu \cdot \mathrm{grad}^M(H) \cdot \phi + \frac{1}{2}\nu \cdot \mathrm{div}^M(II) \cdot \phi + \frac{i}{2}\nu \cdot (\nu \lrcorner \Omega^{\mathcal{Z}}) \cdot \phi.
\end{aligned}$$

4.4 The generalized cylinder on semi-Riemannian Spin^c manifolds

Let M be an n-dimensional smooth manifold and g_t a smooth 1-parameter family of semi-Riemannian metrics on M, $t \in I$ where $I \subset \mathbb{R}$ is an interval. We define the generalized cylinder by

$$\mathcal{Z} := I \times M,$$

with semi-Riemannian metric $g_{\mathcal{Z}} := \langle \cdot, \cdot \rangle = dt^2 + g_t$. The generalized cylinder is an $(n+1)$-dimensional semi-Riemannian manifold of signature $(r+1, s)$ if the signature of g_t is (r, s).

Proposition 4.4.1. *There is a 1-1-correspondence between the* Spin^c *structures on M and that on $\mathcal{Z}$.*

Proof. As explained in Section 4.3, Spin^c structures on $\mathcal{Z}$ can be restricted to Spin^c structures on M. Conversely, given a Spin^c structure on M, it can be pulled back to $I \times M$ via the projection $pr_2 : I \times M \longrightarrow M$ to give a Spin^c structure on $\mathcal{Z}$. In fact, the pull back of the Spin$^c(r,s)$-principal bundle $P_{\mathrm{Spin}^c(r,s)}M$ on M gives rise to a Spin$^c(r,s)$-principal bundle on $\mathcal{Z}$ denoted by $P_{\mathrm{Spin}^c(r,s)}\mathcal{Z}$

$$\begin{tikzcd}
P_{\mathrm{Spin}^c(r,s)}\mathcal{Z} \arrow[r] \arrow[d, "\pi"] & P_{\mathrm{Spin}^c(r,s)}M \arrow[d, "\pi"] \\
\mathcal{Z} = I \times M \arrow[r] & M
\end{tikzcd}$$

Enlarging the structure group via the embedding $\mathrm{Spin}^c(r,s) \hookrightarrow \mathrm{Spin}^c(r+1,s)$, which covers the standard embedding

$$\begin{array}{rcl} \mathrm{SO}(r,s)\times\mathbb{S}^1 & \hookrightarrow & \mathrm{SO}(r+1,s)\times\mathbb{S}^1 \\ (a,z) & \mapsto & \Big(\begin{pmatrix} 1 & 0 \\ 0 & \mathrm{a}\end{pmatrix}, z\Big), \end{array}$$

gives a $\mathrm{Spin}^c(r+1,s)$-principal fiber bundle on $\mathcal{Z}$, denoted also by $P_{\mathrm{Spin}^c(r+1,s)}\mathcal{Z}$. The pull back of the auxiliary line bundle L on M defining the $\mathrm{Spin}^{\mathrm{c}}$ structure on M, gives a line bundle $L^{\mathcal{Z}}$ on $\mathcal{Z}$ such that the following diagram commutes

$$\begin{tikzcd} L^{\mathcal{Z}} = pr_2^*(L) \arrow[r] \arrow[d, "\pi"] & L \arrow[d, "\pi"] \\ \mathcal{Z} = I\times M \arrow[r] & M \end{tikzcd}$$

The line bundle $L^{\mathcal{Z}}$ on $\mathcal{Z}$ and the $\mathrm{Spin}^c(r+1,s)$-principal fiber bundle $P_{\mathrm{Spin}^c(r+1,s)}\mathcal{Z}$ on $\mathcal{Z}$ yield the $\mathrm{Spin}^{\mathrm{c}}$ structure on $\mathcal{Z}$ which restricts to the given $\mathrm{Spin}^{\mathrm{c}}$ structure on M.

Remark 4.4.1. *If M is a Riemannian* $\mathrm{Spin}^{\mathrm{c}}$ *manifold and if we denote by $i\Omega$ the imaginairy valued curvature on the auxiliary line bundle L, we know that there exists a unique curvature 2-form, denoted by $i\Omega^{\mathcal{Z}}$, on the line bundle $L^{\mathcal{Z}} = pr_2^*(L)$ defined by $i\Omega^{\mathcal{Z}} = pr_2^*(i\Omega)$. Thus we have*

$$\Omega^{\mathcal{Z}}(X,Y) = \Omega(X,Y) \quad \textit{and} \quad \Omega^{\mathcal{Z}}(\nu,Y) = 0 \quad \textit{for any} \quad X,Y\in\Gamma(TM).$$

Proposition 4.4.2. [*BGM05*] *On a generalized cylinder $\mathcal{Z} = I\times M$ with semi-Riemannian metric $g^{\mathcal{Z}} = \langle\cdot,\cdot\rangle = dt^2 + g_t$ we define, at every $p\in M$ and $X,Y\in T_pM$, the first and second derivatives of g_t by*

$$\dot{g}_t(X,Y) := \frac{d}{dt}(g_t(X,Y)) \qquad \textit{and} \qquad \ddot{g}_t(X,Y) := \frac{d^2}{dt^2}(g_t(X,Y)).$$

Hence the following formulas hold:

$$\begin{aligned} \langle II(X),Y\rangle &= -\frac{1}{2}\dot{g}_t(X,Y), && (4.17)\\ \big\langle R^{\mathcal{Z}}(U,V)X,Y\big\rangle &= \big\langle R^{M_t}(U,V)X,Y\big\rangle && (4.18)\\ &\quad +\frac{1}{4}\Big(\dot{g}_t(U,X)\dot{g}_t(V,Y) - \dot{g}_t(U,Y)\dot{g}_t(V,X)\Big), \\ \big\langle R^{\mathcal{Z}}(X,Y)U,\nu\big\rangle &= \frac{1}{2}\Big((\nabla^{M_t}_Y\dot{g}_t)(X,U) - (\nabla^{M_t}_X\dot{g}_t)(Y,U)\Big), && (4.19)\\ \big\langle R^{\mathcal{Z}}(X,\nu)\nu,Y\big\rangle &= -\frac{1}{2}\Big(\ddot{g}_t(X,Y) + \dot{g}_t(II(X),Y)\Big), && (4.20) \end{aligned}$$

where $X,Y,U,V\in T_pM$, $p\in M$.

4.5 The variational formula for the Dirac operator on Spin^c manifolds

First we give some facts about parallel transport on Spin^c manifolds along a curve c. We consider a Riemannian Spin^c manifold N, we know that there exists a unique correspondence which, to a spinor field $\psi(t) = \psi(c(t))$ along a curve $c : I \longrightarrow N$ associates another spinor field $\frac{D}{dt}\psi$ along c, called the covariant derivative of ψ along c, such that

$$\frac{D}{dt}(\psi + \phi) = \frac{D}{dt}\psi + \frac{D}{dt}\phi, \quad \text{for any } \psi \text{ and } \ \phi \text{ along the curve } c,$$

$$\frac{D}{dt}(f\psi) = f\frac{D}{dt}\psi + (\frac{d}{dt}f)\ \psi, \text{ where } f \text{ is a differentiable function on } I,$$

$$\nabla^{\Sigma N}_{\dot c(t)}\psi = \frac{D}{dt}\phi, \text{ where } \phi(t) = \psi(c(t)).$$

A spinor field ψ along a curve c is called parallel when $\frac{D}{dt}\psi(t) = 0$ for all $t \in I$. Now, if ψ_0 is a spinor at the point $c(t_0)$, $t_0 \in I, (\psi_0 \in \Sigma_{c(t_0)}N)$ then there exists a unique parallel spinor ϕ along c such that $\psi_0 = \phi(t_0)$. The linear isometry $\tau_{t_0}^{t_1}$ defined by

$$\begin{array}{rcl} \tau_{t_0}^{t_1} : \Sigma_{c(t_0)}N & \longrightarrow & \Sigma_{c(t_1)}N \\ \psi_0 & \longmapsto & \phi(t_1), \end{array}$$

is called the parallel transport along the curve c from $c(t_0)$ to $c(t_1)$. The basic property of the parallel transport on a Spin^c manifold is the following: let ψ be a spinor field on a Riemannian Spin^c manifold N , $X \in \Gamma(TN)$, $p \in N$ and $c : I \longrightarrow N$ an integral curve through p, i.e. $c(t_0) = p$ and $\frac{d}{dt}c(t) = X(c(t))$, we have

$$(\nabla^{\Sigma N}_X \psi)_p = \frac{d}{dt}\Big(\tau_t^{t_0}(\psi(t))\Big)|_{t=t_0}. \tag{4.21}$$

Now, we consider g_t a smooth 1-parameter family of semi-Riemannian metrics on a Spin^c manifold M and the generalized cylinder $\mathcal{Z} = I \times M$ with semi-Riemannian metric $g^{\mathcal{Z}} = \langle \cdot, \cdot \rangle = dt^2 + g_t$. For $t \in I$ we denote by M_t the manifold (M, g_t). Let us write "$\cdot$" for the Clifford multiplication on $\mathcal{Z}$ and "$\bullet_t$" for that on M_t. Recall from Section 4.4 that Spin^c structures on M and $\mathcal{Z}$ are in 1-1-correspondence and $\Sigma\mathcal{Z}|_{M_t} = \Sigma M_t$ as hermitian vector bundles if $n = r+s$ is even and $\Sigma^+\mathcal{Z}|_{M_t} = \Sigma M_t$ if n is odd. For a given $x \in M$ and $t_0, t_1 \in I$, the parallel transport $\tau_{t_0}^{t_1}$ on the generalized cylinder $\mathcal{Z}$ along the curve $c : I \to I \times M, t \to (t, x)$ is given by

$$\tau_{t_0}^{t_1} : \Sigma_{c(t_0)}\mathcal{Z} \simeq \Sigma_x M_{t_0} \longrightarrow \Sigma_{c(t_1)}\mathcal{Z} \simeq \Sigma_x M_{t_1}.$$

This isomorphism satisfies

$$\tau_{t_0}^{t_1}(X \bullet_{t_0} \varphi) = (\zeta_{t_0}^{t_1} X) \bullet_{t_1} (\tau_{t_0}^{t_1}\varphi),$$

$$\left\langle \tau_{t_0}^{t_1}\psi, \tau_{t_0}^{t_1}\varphi \right\rangle = \langle \psi, \varphi \rangle ,$$

where $\zeta_{t_0}^{t_1} : T_{(x,t_0)}\mathcal{Z} \simeq T_x M_{t_0} \to T_{(x,t_1)}\mathcal{Z} \simeq T_x M_{t_1}$ is the parallel transport on $\mathcal{Z}$ along the same curve c, $X \in T_x M_{t_0}$ and $\psi, \varphi \in \Sigma_x M_{t_0}$.

Theorem 4.5.1. *On a Riemannian* Spin^c *manifold M, we consider g_t a smooth 1-parameter family of semi-Riemannian metrics. We denote by D^{M_t} the Dirac operator of M_t, and $\mathfrak{D}^{\dot g_t} = \sum_{i,j=1}^n \varepsilon_i\varepsilon_j \dot g_t(e_i,e_j)e_i \bullet_t \nabla^{\Sigma M_t}_{e_j}$. Then, for any smooth spinor field ψ on M_{t_0}, we have*

$$\frac{d}{dt}\Big|_{t=t_0} \tau_t^{t_0} D^{M_t}\tau_{t_0}^t\psi = -\frac{1}{2}\mathfrak{D}^{\dot g_{t_0}}\psi + \frac{1}{4}\,\mathrm{grad}^{M_{t_0}}(\mathrm{tr}_{g_{t_0}}(\dot g_{t_0}))\bullet_{t_0}\psi - \frac{1}{4}\,\mathrm{div}^{M_{t_0}}(\dot g_{t_0})\bullet_{t_0}\psi.$$

Proof. The vector field $\nu := \frac{\partial}{\partial t}$ is spacelike of unit length and orthogonal to the hypersurfaces $M_t := \{t\}\times M$. Denote by II_t the Weingarten map of M_t with respect to ν and by H_t the mean curvature. If X is a local coordinate field on M, then $\langle X,\nu\rangle = 0$ and $[X,\nu]=0$. Thus,

$$\begin{aligned} 0 &= d_\nu\langle X,\nu\rangle = \left\langle\nabla^{\mathcal{Z}}_\nu X,\nu\right\rangle + \left\langle X,\nabla^{\mathcal{Z}}_\nu\nu\right\rangle = \left\langle\nabla^{\mathcal{Z}}_X\nu,\nu\right\rangle + \left\langle X,\nabla^{\mathcal{Z}}_\nu\nu\right\rangle \\ &= -\langle II_t(X),\nu\rangle + \left\langle X,\nabla^{\mathcal{Z}}_\nu\nu\right\rangle = \left\langle X,\nabla^{\mathcal{Z}}_\nu\nu\right\rangle. \end{aligned}$$

Differentiating $\langle\nu,\nu\rangle = 1$ yields $\left\langle\nu,\nabla^{\mathcal{Z}}_\nu\nu\right\rangle = 0$. Hence $\nabla^{\mathcal{Z}}_\nu\nu = 0$, i.e. for $x\in M$ the curves $t\mapsto(t,x)$ are geodesics parametrized by arclength. So the assumptions of Proposition 4.3.3 are satisfied for the foliation $(M_t)_{t\in I}$. By Remark 4.4.1, the commutator formula of Proposition 4.3.3 gives for a section ϕ of ΣM_t, (or Σ^+M_t if n is odd)

$$i^{-s}[\nabla^{\Sigma\mathcal{Z}}_\nu, D^{M_t}]\,\phi = \mathfrak{D}^{II_t}\phi - \frac{n}{2}\,\mathrm{grad}^{M_t}(H_t)\bullet_t\phi \;+\; \frac{1}{2}\,\mathrm{div}^{M_t}(II_t)\bullet_t\phi. \tag{4.22}$$

From Proposition 4.4.2 we deduce

$$\mathrm{div}^{M_t}(II_t) = -\frac{1}{2}\mathrm{div}^{M_t}(\dot g_t), \qquad H_t = -\frac{1}{2n}\mathrm{tr}_{g_t}(\dot g_t) \qquad \text{and} \qquad \mathfrak{D}^{II_t} = -\frac{1}{2}\mathfrak{D}^{\dot g_t}.$$

Thus, Equation (4.22) can be rewritten as

$$i^{-s}[\nabla^{\Sigma\mathcal{Z}}_\nu, D^{M_t}]\,\phi = -\frac{1}{2}\mathfrak{D}^{\dot g_t}\phi + \frac{1}{4}\,\mathrm{grad}^{M_t}(\mathrm{tr}_{g_t}(\dot g_t))\bullet_t\phi - \frac{1}{4}\,\mathrm{div}^{M_t}(\dot g_t)\bullet_t\phi. \tag{4.23}$$

Now, if ϕ is parallel along the curves $t\mapsto(t,x)$, i.e. it is of the form $\phi(t,x) = \tau_{t_0}^t\psi(t_0,x)$ for some spinor field ψ on M_{t_0}, then using (4.21) at $t=t_0$, the left hand side of (4.23) could be written as

$$\begin{aligned} i^{-s}[\nabla^{\Sigma\mathcal{Z}}_\nu, D^{M_t}]\,\phi &= i^{-s}\nabla^{\Sigma\mathcal{Z}}_\nu D^{M_t}\,\phi = i^{-s}\,\frac{d}{dt}\Big|_{t=t_0}\tau_t^{t_0}D^{M_t}\,\phi \\ &= i^{-s}\,\frac{d}{dt}\Big|_{t=t_0}\tau_t^{t_0}D^{M_t}\tau_{t_0}^t\psi, \end{aligned} \tag{4.24}$$

which gives the variational formula for the Dirac operator.

Corollary 4.5.1. *Let (M^n,g) be a Riemannian* Spin^c *manifold, if we consider the family of metrics defined by $g_t = g+tk$, where k is a symmetric $(0,2)$-tensor, we have*

$$\frac{d}{dt}\Big|_{t=0}\tau_t^0 D^{M_t}\tau_0^t\psi = -\frac{1}{2}\mathfrak{D}^k\psi + \frac{1}{4}\,\mathrm{grad}^M(\mathrm{tr}_g(k))\cdot\psi - \frac{1}{4}\,\mathrm{div}^M(k)\cdot\psi, \tag{4.25}$$

where "$\cdot = \bullet_{t_0=0}$" is the Clifford multiplication on M.

This formula has been proved in [BG92] for Spin Riemannian manifolds and in [BGM05] for Spin semi-Riemannian manifolds.

4.6 The energy-momentum tensor on Spin^c manifolds

In this section, we study the energy-momentum tensor on Spin^c Riemannian manifolds from a geometric point of vue. We begin by giving the proofs of Proposition 4.1.1, Theorem 4.1.1 and Proposition 4.1.2.

Proof of Proposition 4.1.1. Using Equation (4.25) we calculate

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0}\int_M \mathrm{Re}\,\left\langle \tau_t^0 D^{M_t}\tau_0^t\psi,\psi\right\rangle_{g_t} v_g &= \frac{d}{dt}\Big|_{t=0}\int_M \mathrm{Re}\,\left\langle D^{M_t}\tau_0^t\psi,\tau_0^t\psi\right\rangle_{g_t} v_g \\
&= -\frac{1}{2}\int_M \mathrm{Re}\,\left\langle \mathfrak{D}^k\psi,\psi\right\rangle_g v_g \\
&= -\frac{1}{2}\sum_{i,j}\int_M k(e_i,e_j)\mathrm{Re}\,\left\langle e_i\cdot\nabla^{\Sigma M}_{e_j}\psi,\psi\right\rangle v_g \\
&= -\frac{1}{2}\int_M \langle k,\ell_\psi\rangle\, v_g.
\end{aligned}$$

Proof of Theorem 4.1.1. Let φ be a spinor field and k a symmetric (0,2)-tensor field on (M^n,g). At $t=0$, we compute

$$\begin{aligned}
\frac{d}{dt}\mathcal{W}(g+tk,\psi+t\varphi) &= \frac{d}{dt}\mathcal{W}(g+tk,\psi)+\frac{d}{dt}\mathcal{W}(g,\psi+t\varphi) \\
&= \frac{d}{dt}\int_U \left\{S_{g_t}+\varepsilon\lambda|\tau_0^t\psi|^2_{g_t}-\varepsilon\mathrm{Re}\,\left\langle D^{M_t}\tau_0^t\psi,\tau_0^t\psi\right\rangle_{g_t}\right\}v_{g_t} \\
&\quad+\frac{d}{dt}\int_U\{\varepsilon\lambda|\psi+t\varphi|^2_g-\varepsilon\mathrm{Re}\,\langle D_g(\psi+t\varphi),\psi+t\varphi\rangle_g\}v_g \\
&= \frac{d}{dt}\{\int_U S_{g_t}v_g+\int_U S_g v_{g_t}\}+\frac{d}{dt}\int_U\varepsilon\lambda|\psi|^2_g v_{g_t} \\
&\quad-\frac{d}{dt}\int_U\varepsilon\mathrm{Re}\,\left\langle D^{M_t}\tau_0^t\psi,\tau_0^t\psi\right\rangle_{g_t} v_g-\frac{d}{dt}\int_U\varepsilon\mathrm{Re}\,\langle D_g\psi,\psi\rangle_g v_{g_t} \\
&\quad+\frac{d}{dt}\int_U\varepsilon\lambda|\psi+t\varphi|^2_g v_g-\frac{d}{dt}\int_U\varepsilon\mathrm{Re}\,\langle D_g(\psi+t\varphi),\psi+t\varphi\rangle_g v_g
\end{aligned}$$

Moreover, the following holds

$$\frac{d}{dt}\Big|_{t=0} v_{g_t}=\frac{1}{2}\langle g,k\rangle\, v_g \quad\text{and}\quad \frac{d}{dt}\Big|_{t=0}\int_M S_{g_t}v_g=-\int_M\langle \mathrm{ric},k\rangle\, v_g.$$

Hence,

$$\begin{aligned}
&\frac{d}{dt}\mathcal{W}(g+tk,\psi+t\varphi) \\
&\quad= -\int_U\langle \mathrm{ric}_g,k\rangle_g v_g+\frac{1}{2}\int_U\langle S_g g,k\rangle_g v_g+\frac{1}{2}\int_U\left\langle\varepsilon\lambda|\psi|^2_g g,k\right\rangle_g v_g
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4}\int_U \langle \varepsilon\ell_\psi, k\rangle_g v_g - \frac{1}{2}\int_U \left\langle \varepsilon \mathrm{Re}\, \langle D_g\psi, \psi\rangle_g g, k\right\rangle_g v_g \\
&+2\int_U \mathrm{Re}\, \langle \varepsilon\lambda\psi, \varphi\rangle_g v_g - 2\int_U \mathrm{Re}\, \langle \varepsilon D_g\psi, \varphi\rangle_g v_g \\
= &\int_U \left\langle -\mathrm{ric}_g + \frac{1}{2}S_g g + \frac{\varepsilon\lambda}{2}|\psi|_g^2 g - \frac{\varepsilon}{2}\mathrm{Re}\, \langle D_g\psi, \psi\rangle_g g + \frac{\varepsilon}{2}\ell_\psi, k\right\rangle_g v_g \\
&+\int_U \mathrm{Re}\, \langle 2\varepsilon\lambda\psi - 2\varepsilon D_g\psi, \varphi\rangle_g v_g.
\end{aligned}
$$

Therefore, (g_0, ψ_0) is a critical point of the Lagrange functional $\mathcal{W}(g,\psi)$ for all open subsets U of M^n with compact closure if and only if it is a solution of the equations

$$
-\mathrm{ric}_g + \frac{S_g}{2}g + \frac{\varepsilon\lambda}{2}|\psi|_g^2 g - \frac{\varepsilon}{2}\mathrm{Re}\, \langle D_g\psi, \psi\rangle_g g + \frac{\varepsilon}{2}\ell_\psi = 0 \quad \text{and} \quad \lambda\psi = D_g\psi.
$$

Inserting the second equation into the first one yields $\mathrm{ric}_g - \frac{S_g}{2}g = \frac{\varepsilon}{2}\ell_\psi$.

Proof of Proposition 4.1.2. Let ψ be any parallel spinor field on $\mathcal{Z}$. Then Equation (4.13) yields

$$
\nabla_X\phi = \frac{1}{2}II(X)\bullet\phi. \tag{4.26}
$$

Let $\{e_1, ..., e_n\}$ be a positively oriented local orthonormal basis of TM. For $j = 1, ..., n$ we have

$$
\nabla_{e_j}\phi = \frac{1}{2}\sum_{k=1}^n II_{jk}\; e_k\bullet\phi.
$$

Taking Clifford multiplication by e_i and the scalar product with ϕ, we get

$$
\mathrm{Re}\, \left\langle e_i\bullet\nabla_{e_j}\phi, \phi\right\rangle = \frac{1}{2}\sum_{k=1}^n II_{jk}\mathrm{Re}\, \langle e_i\bullet e_k\bullet\phi, \phi\rangle\,.
$$

Since $\mathrm{Re}\, \langle e_i\bullet e_k\bullet\phi, \phi\rangle = -\delta_{ik}|\phi|^2$, it follows, by the symmetry of II

$$
\mathrm{Re}\, \left\langle e_i\bullet\nabla_{e_j}\phi + e_j\bullet\nabla_{e_i}\phi, \phi\right\rangle = -II_{ij}|\phi|^2.
$$

Therefore, $2\ell^\phi = -II$. Using Equation (4.14) it is easy to see that ϕ is an eigenspinor associated with the eigenvalue $-\frac{n}{2}H$ of $\widetilde{D}$. Since $S^{\mathcal{Z}} = S + 2\,\mathrm{ric}^{\mathcal{Z}}(\nu,\nu) - n^2H^2 + |II|^2$ we get

$$
\frac{1}{4}(S - c_n|\Omega|) + |\ell^\phi|^2 \;=\; \frac{1}{4}(S^{\mathcal{Z}} - 2\,\mathrm{ric}^{\mathcal{Z}}(\nu,\nu) - c_n|\Omega|) + n^2\frac{H^2}{4},
$$

hence M satisfies the equality case in (2.11) if and only if (4.2) holds.

Corollary 4.6.1. *Under the same conditions as Proposition 4.1.2, if* $n = 2$ *or* 3*, the hypersurface* M *satisfies the equality case in (2.11) if* $\mathrm{Ric}^{\mathcal{Z}}(\nu) = 0$ *and* $S^{\mathcal{Z}} \geqslant 0$.

Proof. Since $\mathcal{Z}$ has a parallel spinor, we have (see [Fri00])

$$|\mathrm{Ric}^{\mathcal{Z}}(\nu)| = |\nu \lrcorner \Omega^{\mathcal{Z}}|, \tag{4.27}$$

$$i(Y \lrcorner \Omega^{\mathcal{Z}}) \cdot \psi = \mathrm{Ric}^{\mathcal{Z}}(Y) \cdot \psi \quad \text{for every} \quad Y \in \Gamma(\Sigma\mathcal{Z}). \tag{4.28}$$

In Equation (4.28), replacing Y by e_j then taking Clifford multiplication by e_j and summing from $j = 1, ..., n+1$, we get

$$i\sum_{j=1}^{n+1} e_j \cdot (e_j \lrcorner \Omega^{\mathcal{Z}}) \cdot \psi = \sum_{j=1}^{n+1} e_j \cdot \mathrm{Ric}^{\mathcal{Z}}(e_j) \cdot \psi = -S^{\mathcal{Z}}\psi.$$

But $2\ \Omega^{\mathcal{Z}} \cdot \psi = \sum_{j=1}^{n+1} e_j \cdot (e_j \lrcorner \Omega^{\mathcal{Z}}) \cdot \psi$, hence we deduce that $\Omega^{\mathcal{Z}} \cdot \psi = i\frac{S^{\mathcal{Z}}}{2}\psi$. By (4.27) and (4.11) we obtain $\Omega \bullet \phi = i\frac{S^{\mathcal{Z}}}{2}\phi$. Since $n = 2$ or 3, we have $|\Omega| = \frac{S^{\mathcal{Z}}}{2}$ and Equation (4.2) is satisfied.

Corollary 4.6.2. *Under the same conditions as Proposition 4.1.2, if the restriction of the auxiliary line bundle $L^{\mathcal{Z}}$ is flat, i.e. L is a flat complex line bundle ($\Omega = 0$), the hypersurface M is a limiting manifold for (2.11).*

Proof. Since $\Omega = 0$, Equation (4.11) yields $i\frac{S^{\mathcal{Z}}}{2}\phi = \Omega^{\mathcal{Z}} \cdot \psi_{|_M} = (\nu \lrcorner \Omega^{\mathcal{Z}}) \bullet \phi$. But,

$$\begin{aligned} i(\nu \lrcorner \Omega^{\mathcal{Z}}) \bullet \phi &= i(\nu \cdot (\nu \lrcorner \Omega^{\mathcal{Z}}) \cdot \psi)_{|_M} = (\nu \cdot \mathrm{Ric}^{\mathcal{Z}}(\nu) \cdot \psi)_{|_M} \\ &= -\mathrm{ric}^{\mathcal{Z}}(\nu,\nu)\phi + \sum_{j=1}^{n} \mathrm{ric}^{\mathcal{Z}}(\nu, e_j)\ e_j \bullet \phi. \end{aligned} \tag{4.29}$$

Taking the real part of the scalar product of Equation (4.29) with ϕ yields $\frac{S^{\mathcal{Z}}}{2} = \mathrm{ric}^{\mathcal{Z}}(\nu,\nu)$, hence Equation (4.2) is satisfied.

Now, let M be a Riemannian Spin^c manifold having a generalized Killing spinor field ϕ with a symmetric endomorphism E on the tangent bundle TM. As mentioned in the introduction, it is straightforward to see that $2\ell^{\phi}(X,Y) = -\langle E(X), Y\rangle$. We will study these generalized Killing spinors when the tensor E is a Codazzi-Mainardi tensor, i.e. E satisfies

$$(\nabla^M_X E)(Y) = (\nabla^M_Y E)(X) \quad \text{for} \quad X, Y \in \Gamma(TM). \tag{4.30}$$

For this, we give the following lemma whose proof will be omitted since it is similar to Lemma 7.3 in [BGM05].

Lemma 4.6.1. [*BGM05*] *Let g_t be a smooth 1-parameter family of semi-Riemannian metrics on a* Spin^c *manifold $(M^n, g = g_0)$ and let E be a field of symmetric endomorphisms of TM. We consider the metric $g_{\mathcal{Z}} = \langle .,. \rangle = dt^2 + g_t$ on $\mathcal{Z}$ such that $g_t(X,Y) = g((\mathrm{Id} - tE)^2 X, Y)$ for all vector fields X, Y on M. We have $\langle R^{\mathcal{Z}}(U,\nu)\nu, V\rangle = 0$ for all vector fields U, V tangent to M and if E satisfies the Codazzi-Mainardi equation then $\langle R^{\mathcal{Z}}(U,V)W, \nu\rangle = 0$ for all U, V and W on $\mathcal{Z}$.*

Proof of Theorem 4.1.2. We define $\psi_{(0,x)} := \phi_x$ via the identification $\Sigma_x M \cong \Sigma_{(0,x)}\mathcal{Z}$ (resp. $\Sigma^+_{(0,x)}\mathcal{Z}$ for n odd) and $\psi_{(t,x)} = \tau_0^t \psi_{(0,x)}$. By Equation (4.17), the endomorphism E is the Weingarten tensor of the immersion of $\{0\} \times M$ in $\mathcal{Z}$ and hence by construction we have for all $X \in \Gamma(TM)$

$$\nabla^{\Sigma\mathcal{Z}}_X \psi|_{\{0\}\times M} = 0 \quad \text{and} \quad \nabla^{\Sigma\mathcal{Z}}_\nu \psi \equiv 0. \tag{4.31}$$

Since the tensor E satisfies the Codazzi-Mainardi equation, Lemma 4.6.1 yields

$$g_{\mathcal{Z}}(R^{\mathcal{Z}}(U,V)W,\nu) = 0,$$

for all U, V and $W \in \Gamma(\mathcal{Z})$ and $g_{\mathcal{Z}}(R^{\mathcal{Z}}(X,\nu)\nu, Y) = 0$ for all X and Y tangent to M. Hence $R^{\mathcal{Z}}(\nu, X) = 0$ for all $X \in \Gamma(TM)$. Let X be a fixed arbitrary tangent vector field on M. Using (4.6) and (4.31) we get

$$\nabla^{\Sigma\mathcal{Z}}_\nu \nabla^{\Sigma\mathcal{Z}}_X \psi = \mathcal{R}^{\Sigma\mathcal{Z}}(\nu, X)\psi = \frac{1}{2}R^{\mathcal{Z}}(X,\nu)\cdot\psi + \frac{i}{2}\Omega^{\mathcal{Z}}(X,\nu)\psi = 0.$$

Thus showing that the spinor field $\nabla^{\Sigma\mathcal{Z}}_X\psi$ is parallel along the geodesics $\mathbb{R}\times\{x\}$. Now, Equation (4.31) shows that this spinor vanishes for $t = 0$, hence it is zero everywhere on $\mathcal{Z}$. Since X is arbitrary, this shows that ψ is parallel on $\mathcal{Z}$.

Corollary 4.6.3. *Let (M^3, g) be a compact, oriented Riemannian manifold and ϕ an eigenspinor associated with the first eigenvalue λ_1 of the Dirac operator such that the energy-momentum tensor associated with ϕ is a Codazzi tensor. The manifold M is limiting for (2.11) if and only if the generalized cylinder $\mathcal{Z}^4$, equipped with the* Spin^c *structure arising from the given one on M, is Kähler of positive scalar curvature and the immersion of M in $\mathcal{Z}$ has constant mean curvature H.*

Proof. First, we should point out that every 3-dimensional compact, oriented, smooth manifold has a Spin^c structure. Now, if M^3 is a limiting manifold for (2.11), by Theorem 4.1.2, the generalized cylinder has a parallel spinor whose restriction to M is ϕ. Since $\mathcal{Z}$ is a 4-dimensional Spin^c manifold having parallel spinor, $\mathcal{Z}$ is Kähler [Atiy78]. Moreover, using (4.11), we have

$$\Omega \bullet \phi = i\frac{S^{\mathcal{Z}}}{2}\phi = i\frac{c_n}{2}|\Omega|\phi,$$

so $S^{\mathcal{Z}} \geqslant 0$. Finally $H = \frac{1}{n}\mathrm{tr}(II) = \frac{1}{n}\mathrm{tr}(-2\ell^\phi) = -\frac{2}{n}\lambda_1$, which is a constant. Now, if the generalized cylinder is Kähler and M is a compact hypersurface of constant mean curvature H, thus M is a compact hypersurface immersed in a Spin^c manifold having parallel spinor with constant mean curvature. Since $S^{\mathcal{Z}} \geqslant 0$ and $\nu\lrcorner\Omega^{\mathcal{Z}} = \mathrm{Ric}^{\mathcal{Z}}(\nu) = 0$, Corollary 4.6.1 gives the result.

4.7 The energy-momentum tensor in low dimensions Spin^c manifolds

On a compact Riemannian Spin surface, T. Friedrich and E.C. Kim proved that any eigenvalue λ of the Dirac operator satisfies the equality [FK01, Thm. 4.5]:

$$\lambda^2 = \frac{\pi\chi(M)}{Area(M)} + \frac{1}{Area(M)}\int_M |\ell^\psi|^2 v_g. \tag{4.32}$$

The proof of Equality (4.32) relies mainly on a local expression of the covariant derivative of ψ and the use of the Schrödinger-Lichnerowicz formula. This equality has many direct consequences. First, since the trace of ℓ^ψ is equal to λ, we have by the Cauchy-Schwarz inequality that $|\ell^\psi|^2 \geqslant \frac{1}{n}(\mathrm{tr}(\ell^\psi))^2 = \frac{1}{2}\lambda^2$, where $\mathrm{tr}(\ell^\psi)$ denotes the trace of ℓ^ψ. Hence, Equality (4.32) implies the Bär inequality [Bär92] given by

$$\lambda^2 \geqslant \lambda_1^2 := \frac{2\pi\chi(M)}{Area(M)}. \tag{4.33}$$

Another direct fact of Equality (4.32) is that $\int_M \det(\ell^\psi)v_g = \pi\chi(M)$, which gives an information on the energy-momentum tensor without knowing the eigenspinor nor the eigenvalue. Finally, for any closed surface M in $\mathbb{R}^3$ of constant mean curvature H, the restriction to M of a parallel spinor on $\mathbb{R}^3$ is a generalized Killing spinor with eigenvalue $-H$, with energy-momentum tensor equal to the Weingarten tensor II (up to the factor $-\frac{1}{2}$) [Mor02] and by (4.32), we have

$$H^2 = \frac{\pi\chi(M)}{Area(M)} + \frac{1}{4Area(M)}\int_M |II|^2 v_g.$$

Indeed, given any surface M carrying such a spinor field, T. Friedrich [Fri98] showed that the energy-momentum tensor associated with this spinor field satisfies the Gauss-Codazzi equations and hence M is locally immersed into $\mathbb{R}^3$.

Having a Spin^c structure on manifolds is a weaker condition than having a Spin structure because every Spin manifold has a trivial Spin^c structure. Additionally, any compact surface or any product of a compact surface with $\mathbb{R}$ has a Spin^c structure carrying special spinors. In the same spirit as in [Hij86], when using a suitable conformal change, we proved in Chapter 2 a Bär-type inequality for the eigenvalues of the Dirac operator on a compact surface endowed with any Spin^c structure. In fact, any eigenvalue λ of the Dirac operator satisfies

$$\lambda^2 \geqslant \lambda_1^2 := \frac{2\pi\chi(M)}{Area(M)} - \frac{1}{Area(M)}\int_M |\Omega| v_g, \tag{4.34}$$

Equality is achieved if and only if the eigenspinor ψ associated with the first eigenvalue λ_1 is a Spin^c Killing spinor satisfying $\Omega \cdot \psi = i|\Omega|\psi$. In this section, we give a formula corresponding to (4.32) for any eigenspinor ψ of the square of the Dirac operator on compact surfaces endowed with any Spin^c structure (see Theorem 4.7.1). It is motivated by the following two facts: first, when we consider eigenvalues of the square of the Dirac operator, another tensor field is of interest. It is the skew-symmetric tensor field q^ψ. This tensor was studied by G. Habib [Hab07] in the context of Riemannian flows. Second, we consider any compact surface M immersed in $\mathbb{S}^2 \times \mathbb{R}$ where $\mathbb{S}^2$ is the round sphere equipped with a metric of curvature one. The Spin^c structure on $\mathbb{S}^2 \times \mathbb{R}$, induced from the canonical one on $\mathbb{S}^2$ and the Spin structure on $\mathbb{R}$, admits a parallel spinor [Moro97]. The restriction to M of this Spin^c structure is still a Spin^c structure with a Spin^c generalized Killing spinor [Nak11a].

In Section 4.7.1 we deduce a formula for the integral of the determinant of $\ell^\psi + q^\psi$ and we establish a new proof of the Bär-type Inequality (4.34). In Section 4.7.2, we consider the 3-dimensional case and treat examples of hypersurfaces in $\mathbb{CP}^2$.

4.7.1 The 2-dimensional case

In this section, we consider compact surfaces endowed with any Spin^c structure. We have:

Theorem 4.7.1. *Let (M^2, g) be a Riemannian manifold and ψ an eigenspinor of the square of the Dirac operator D^2 with eigenvalue λ^2 associated with any* Spin^c *structure. Then, we have*

$$\lambda^2 = \frac{S}{4} + |\ell^\psi|^2 + |q^\psi|^2 + \Delta f + |Y|^2 - 2Y(f) + \left\langle \frac{i}{2}\Omega\cdot\psi, \frac{\psi}{|\psi|^2}\right\rangle,$$

where f is the real-valued function defined by $f = \frac{1}{2}\ln|\psi|^2$ and Y is a vector field on TM given by $g(Y,Z) = \frac{1}{|\psi|^2}\mathrm{Re}\,\langle D\psi, Z\cdot\psi\rangle$ for any $Z \in \Gamma(TM)$.

Proof. Let $\{e_1, e_2\}$ be an orthonormal frame of TM. Since the Spin^c bundle ΣM is of real dimension 4, the set $\{\frac{\psi}{|\psi|}, \frac{e_1\cdot\psi}{|\psi|}, \frac{e_2\cdot\psi}{|\psi|}, \frac{e_1\cdot e_2\cdot\psi}{|\psi|}\}$ is orthonormal with respect to the real product $\mathrm{Re}\,\langle\cdot,\cdot\rangle$. The covariant derivative of ψ can be expressed in this frame as

$$\nabla_X\psi = \delta(X)\psi + \alpha(X)\cdot\psi + \beta(X)e_1\cdot e_2\cdot\psi, \tag{4.35}$$

for all vector fields X, where δ and β are 1-forms and α is a $(1,1)$-tensor field. Moreover β, δ and α are uniquely determined by the spinor ψ. In fact, taking the scalar product of (4.35) respectively with $\psi, e_1\cdot\psi, e_2\cdot\psi, e_1\cdot e_2\cdot\psi$, we get $\delta = \frac{d(|\psi|^2)}{2|\psi|^2}$,

$$\alpha(X) = -\ell^\psi(X) - q^\psi(X) \quad \text{and} \quad \beta(X) = \frac{1}{|\psi|^2}\mathrm{Re}\,\langle\nabla_X\psi, e_1\cdot e_2\cdot\psi\rangle.$$

Using the Schrödinger-Lichnerowicz formula, it follows that

$$\lambda^2 = \frac{\Delta(|\psi|^2)}{2|\psi|^2} + |\alpha|^2 + |\beta|^2 + |\delta|^2 + \frac{1}{4}S + \left\langle \frac{i}{2}\Omega\cdot\psi, \frac{\psi}{|\psi|^2}\right\rangle.$$

Now it remains to compute the term $|\beta|^2$. We have

$$\begin{aligned}
|\beta|^2 &= \frac{1}{|\psi|^4}\mathrm{Re}\,\langle\nabla_{e_1}\psi, e_1\cdot e_2\cdot\psi\rangle^2 + \frac{1}{|\psi|^4}\mathrm{Re}\,\langle\nabla_{e_2}\psi, e_1\cdot e_2\cdot\psi\rangle^2 \\
&= \frac{1}{|\psi|^4}\mathrm{Re}\,\langle D\psi - e_2\cdot\nabla_{e_2}\psi, e_2\cdot\psi\rangle^2 + \frac{1}{|\psi|^4}\mathrm{Re}\,\langle D\psi - e_1\cdot\nabla_{e_1}\psi, e_1\cdot\psi\rangle^2 \\
&= g(Y,e_1)^2 + g(Y,e_2)^2 + \frac{|d(|\psi|^2)|^2}{4|\psi|^4} - g(Y, \frac{d(|\psi|^2)}{|\psi|^2}) \\
&= |Y|^2 - 2Y(f) + \frac{|d(|\psi|^2)|^2}{4|\psi|^4},
\end{aligned}$$

which gives the result by using the fact that $\Delta f = \frac{\Delta(|\psi|^2)}{2|\psi|^2} + \frac{|d(|\psi|^2)|^2}{2|\psi|^4}$. □

Remark 4.7.1. *Under the same conditions as in Theorem 4.7.1, if ψ is an eigenspinor of D with eigenvalue λ, we get*

$$\lambda^2 = \frac{S}{4} + |\ell^\psi|^2 + \Delta f + \left\langle \frac{i}{2}\Omega\cdot\psi, \frac{\psi}{|\psi|^2}\right\rangle.$$

In fact, in this case $Y = 0$ and

$$\begin{aligned} 0 &= \mathrm{Re}\,\langle D\psi, e_1\cdot e_2\cdot\psi\rangle = \mathrm{Re}\,\langle e_1\cdot\nabla_{e_1}\psi + e_2\cdot\nabla_{e_2}\psi, e_1\cdot e_2\cdot\psi\rangle \\ &= \mathrm{Re}\,\langle -e_2\cdot\nabla_{e_1}\psi + e_1\cdot\nabla_{e_2}\psi, \psi\rangle = 2q^\psi(e_2,e_1)|\psi|^2. \end{aligned} \tag{4.36}$$

This was proven by T. Friedrich and E.C. Kim in [FK01] for Spin *structures on M.*

In the following, we will give an estimate for the integral $\int_M \det(\ell^\psi+q^\psi)v_g$ in terms of geometric quantities, which has the advantage that it does not depend on the eigenvalue λ nor on the eigenspinor ψ. This is a generalization of the result of T. Friedrich and E.C. Kim in [FK01] for Spin structures.

Theorem 4.7.2. *Let M be a compact surface and ψ any eigenspinor of D^2 associated with eigenvalue λ^2. Then we have*

$$\int_M \det(\ell^\psi + q^\psi)v_g \geqslant \frac{\pi\chi(M)}{2} - \frac{1}{4}\int_M |\Omega|v_g. \tag{4.37}$$

Equality in (4.37) *holds if and only if either Ω is zero or has constant sign.*

Proof. As in the previous theorem, the spinor $D\psi$ can be expressed in the orthonormal frame of the Spin^c bundle. Thus the norm of $D\psi$ is equal to

$$\begin{aligned} |D\psi|^2 &= \frac{1}{|\psi|^2}\mathrm{Re}\,\langle D\psi,\psi\rangle^2 + \frac{1}{|\psi|^2}\sum_{i=1}^{2}\mathrm{Re}\,\langle D\psi, e_i\cdot\psi\rangle^2 + \frac{1}{|\psi|^2}\mathrm{Re}\,\langle D\psi, e_1\cdot e_2\cdot\psi\rangle^2 \\ &= (\mathrm{tr}\,\ell^\psi)^2|\psi|^2 + |Y|^2|\psi|^2 + \frac{1}{|\psi|^2}\mathrm{Re}\,\langle D\psi, e_1\cdot e_2\cdot\psi\rangle^2, \end{aligned} \tag{4.38}$$

where we recall that the trace of ℓ^ψ is equal to $\frac{1}{|\psi|^2}\mathrm{Re}\,\langle D\psi,\psi\rangle$. On the other hand, by (4.36) we have that $\frac{1}{|\psi|^2}\mathrm{Re}\,\langle D\psi, e_1\cdot e_2\cdot\psi\rangle^2 = 2|q^\psi|^2|\psi|^2$. Thus Equation (4.38) reduces to

$$\frac{|D\psi|^2}{|\psi|^2} = (\mathrm{tr}\,\ell^\psi)^2 + |Y|^2 + 2|q^\psi|^2.$$

Now, using the equality $\mathrm{Re}\,\langle D^2\psi,\psi\rangle = |D\psi|^2 - \mathrm{div}\xi$, where ξ is the vector field given by $\xi = |\psi|^2 Y$, we get

$$\lambda^2 + \frac{1}{|\psi|^2}\mathrm{div}\xi = (\mathrm{tr}\,\ell^\psi)^2 + |Y|^2 + 2|q^\psi|^2. \tag{4.39}$$

An easy computation leads to $\frac{1}{|\psi|^2}\mathrm{div}\xi = \mathrm{div}Y + 2Y(f)$ where we recall that $f = \frac{1}{2}\ln(|\psi|^2)$. Hence substituting this formula into (4.39) and using Theorem 4.7.1 yields

$$\frac{S}{4} + (\frac{i}{2}\Omega\cdot\psi, \frac{\psi}{|\psi|^2}) + \Delta f + \mathrm{div}Y = (\mathrm{tr}\ell^\psi)^2 + |q^\psi|^2 - |\ell^\psi|^2 = 2\det(\ell^\psi + q^\psi).$$

Finally, integrating over M and using the Gauss-Bonnet formula, we deduce the required result. Equality holds if and only if $\Omega\cdot\psi = i|\Omega|\psi$. In the orthonormal frame $\{e_1, e_2\}$, the 2-form Ω can be written as $\Omega = \Omega_{12}\ e_1\wedge e_2$, where Ω_{12} is a function defined on M. Using the decomposition of ψ into positive and negative spinors ψ^+ and ψ^-, we find that the equality is attained if and only if

$$\Omega_{12}\ e_1\cdot e_2\cdot\psi^+ + \Omega_{12}\ e_1\cdot e_2\cdot\psi^- = i|\Omega_{12}|\psi^+ + i|\Omega_{12}|\psi^-,$$

which is equivalent to saying that,

$$\Omega_{12}\psi^+ = -|\Omega_{12}|\psi^+ \quad \text{and} \quad \Omega_{12}\psi^- = |\Omega_{12}|\psi^-.$$

Now, if $\psi^+\neq 0$ and $\psi^-\neq 0$, we get $\Omega = 0$. Otherwise, it has constant sign. In the last case, we get that $\int_M |\Omega| v_g = 2\pi\chi(M)$, which means that the l.h.s. of this equality is a topological invariant. $\square$

Next, we will give another proof of the Bär-type Inequality (4.34) for the eigenvalues of any Spin^c Dirac operator. The following theorem was proved by the second author in [Nak10] using conformal deformation of the spinorial Levi-Civita connection.

Theorem 4.7.3. *Let M be a compact Riemannian surface. For any* Spin^c *structure on M, any eigenvalue λ of the Dirac operator D to which is attached an eigenspinor ψ satisfies*

$$\lambda^2 \geqslant \frac{2\pi\chi(M)}{Area(M)} - \frac{1}{Area(M)}\int_M |\Omega| v_g. \tag{4.40}$$

Equality holds if and only if the eigenspinor ψ is a Spin^c *Killing spinor satisfying $\Omega\cdot\psi = i|\Omega|\psi$.*

Proof. With the help of Remark (4.7.1), we have that

$$\lambda^2 = \frac{S}{4} + |\ell^\psi|^2 + \triangle f + \left\langle \frac{i}{2}\Omega\cdot\psi, \frac{\psi}{|\psi|^2}\right\rangle. \tag{4.41}$$

Using the Cauchy-Schwarz inequality, i.e. $|\ell^\psi|^2 \geqslant \frac{\lambda^2}{2}$ and $\langle i\Omega\cdot\psi,\psi\rangle \geqslant -\frac{c_n}{2}|\Omega||\psi|^2$, we easily deduce the result after integrating over M. Now the equality in (4.40) holds if and only if the eigenspinor ψ satisfies $\Omega\cdot\psi = i|\Omega|\psi$ and $|\ell^\psi|^2 = \frac{\lambda^2}{2}$. Thus, the second equality is equivalent to saying that $\ell^\psi(X) = \frac{\lambda}{2}X$ for all $X\in\Gamma(TM)$. Finally, a straightforward computation of the spinorial curvature of the spinor field ψ gives in a local frame $\{e_1, e_2\}$, after using the fact $\beta = -(*\delta)$, that

$$\begin{aligned}\frac{1}{2}R_{1212}\ e_1\cdot e_2\cdot\psi \ &= \ \Big(\frac{\lambda^2}{2} + e_1(\delta(e_1)) + e_2(\delta(e_2))\Big)e_2\cdot e_1\cdot\psi - \lambda\delta(e_2)e_1\cdot\psi \\ &\quad +\lambda\delta(e_1)e_2\cdot\psi + \Big(e_1(\delta(e_2)) - e_2(\delta(e_1))\Big)\psi.\end{aligned}$$

Thus the scalar product with $e_1\cdot\psi$ and $e_2\cdot\psi$ implies that $\delta = 0$. Finally, $\beta = 0$ and the eigenspinor ψ is a Spin^c Killing spinor satisfying $\Omega\cdot\psi = i|\Omega|\psi$. $\square$

Now, we will give some examples where equality holds in (4.40) or in (4.37). Some applications of Theorem 4.7.1 are also given.

Examples.

1. Let $\mathbb{S}^2$ be the round sphere equipped with the standard metric of curvature one. As a Kähler manifold, we endow the sphere with the canonical Spin^c structure of curvature form equal to $i\Omega = i\rho$, where ρ is the Ricci 2-form. Hence, we have $|\Omega| = |\rho| = 1$. Furthermore, we mentionned that for the canonical Spin^c structure, the sphere carries parallel spinors, i.e. an eigenspinor associated with the eigenvalue 0 of the Dirac operator D. Thus equality holds in (4.40). On the other hand, equality in (4.37) also holds, since the sign of the curvature form Ω is constant.

2. Let $f : M \to \mathbb{S}^3$ be an isometric immersion of a surface M^2 into the sphere equipped with its unique Spin structure and assume that the mean curvature H is constant. The restriction of a Killing spinor on $\mathbb{S}^3$ to the surface M defines a spinor field ϕ solution of the following equation [Hab-Roth10]
$$\nabla_X \phi = +\frac{1}{2} II(X) \bullet \phi - \frac{1}{2} J(X) \bullet \phi, \tag{4.42}$$
where II denotes the second fundamental form of the surface and J is the complex structure of M given by the rotation of angle $\frac{\pi}{2}$ on TM. It is easy to check that ϕ is an eigenspinor for D^2 associated with the eigenvalue $H^2 + 1$. Moreover $D\phi = -H\phi - e_1 \cdot e_2 \cdot \phi$, so that $Y = 0$. Moreover we have $\ell^\phi = -\frac{1}{2} II$ and $q^\phi = \frac{1}{2} J$. Hence, by Theorem 4.7.1, and since the norm of ϕ is constant, we obtain
$$H^2 + \frac{1}{2} = \frac{1}{4} S + \frac{1}{4} |II|^2.$$

3. On two-dimensional manifolds, we can define another Dirac operator associated with the complex structure J given by $\widetilde{D} = Je_1 \cdot \nabla_{e_1} + Je_2 \cdot \nabla_{e_2} = e_2 \cdot \nabla_{e_1} - e_1 \cdot \nabla_{e_2}$. Since $\widetilde{D}$ satisfies $D^2 = \widetilde{D}^2$, all the above results are also true for the eigenvalues of $\widetilde{D}$.

4. Let M^2 be a surface immersed in $\mathbb{S}^2 \times \mathbb{R}$. The product of the canonical Spin^c structure on $\mathbb{S}^2$ and the unique Spin structure on $\mathbb{R}$ define a Spin^c structure on $\mathbb{S}^2 \times \mathbb{R}$ carrying parallel spinors [Moro97]. Moreover, by the Schrödinger-Lichnerowicz formula, any parallel spinor ψ satisfies $\Omega^{\mathbb{S}^2\times\mathbb{R}} \cdot \psi = i\psi$, where $\Omega^{\mathbb{S}^2\times\mathbb{R}}$ is the curvature form of the auxiliary line bundle. Let ν be a unit normal vector field of the surface. We then write $\partial t = T + f\nu$, where T is a vector field on TM with $||T||^2 + f^2 = 1$. On the other hand, the vector field T splits into $T = \nu_1 + h\partial t$, where ν_1 is a vector field on the sphere. The scalar product of the first equation by T and the second one by ∂t gives $||T||^2 = h$ which means that $h = 1 - f^2$. Hence the normal vector field ν can be written as $\nu = f\partial t - \frac{1}{f}\nu_1$. As we mentionned before, the

$\mathrm{Spin^c}$ structure on $\mathbb{S}^2 \times \mathbb{R}$ induces a $\mathrm{Spin^c}$ structure on M with induced auxiliary line bundle. Next, we want to prove that the curvature form of the auxiliary line bundle of M is equal to $i\Omega(e_1, e_2) = -if$, where $\{e_1, e_2\}$ denotes a local orthonormal frame on TM. Since the spinor ψ is parallel, by Equation (1.7), we have that for all $X \in T(\mathbb{S}^2 \times \mathbb{R})$ the equality $\mathrm{Ric}^{\mathbb{S}^2\times\mathbb{R}} X \cdot \psi = i(X \lrcorner \Omega^{\mathbb{S}^2\times\mathbb{R}}) \cdot \psi$. Therefore, we compute

$$\begin{aligned}
(\nu \lrcorner \Omega^{\mathbb{S}^2\times\mathbb{R}}) \bullet \phi &= \nu \cdot (\nu \lrcorner \Omega^{\mathbb{S}^2\times\mathbb{R}}) \cdot \psi|_M = -i\nu \cdot \mathrm{Ric}^{\mathbb{S}^2\times\mathbb{R}}\ \nu \cdot \psi|_M \\
&= +\frac{1}{f} i\nu \cdot \nu_1 \cdot \psi|_M = -i\nu \cdot (\nu - f\partial t).\psi|_M \\
&= (i\psi + if\nu \cdot \partial t \cdot \psi)|_M.
\end{aligned}$$

Hence, by Equation (4.11), we get that $\Omega \bullet \phi = -i(f\nu \cdot \partial t \cdot \psi)|_M$. The scalar product of the last equality with $e_1 \cdot e_2 \cdot \psi$ gives

$$\Omega(e_1, e_2)|\phi|^2 = -f\mathrm{Re}\ \langle i\nu \cdot \partial t \cdot \psi, e_1 \cdot e_2 \cdot \psi\rangle\, |_M = -f\mathrm{Re}\ \langle i\partial t \cdot \psi, \psi\rangle\, |_M.$$

We now compute the term $i\partial t \cdot \psi$. For this, let $\{e_1', Je_1'\}$ be a local orthonormal frame of the sphere $\mathbb{S}^2$. The complex volume form acts as the identity on the $\mathrm{Spin^c}$ bundle of $\mathbb{S}^2 \times \mathbb{R}$, hence $\partial t \cdot \psi = e_1' \cdot Je_1' \cdot \psi$. But we have

$$\Omega^{\mathbb{S}^2\times\mathbb{R}} \cdot \psi = \rho \cdot \psi = \ltimes \cdot \psi = -e_1' \cdot Je_1' \cdot \psi.$$

Therefore, $i\partial t \cdot \psi = \psi$. Thus we get $\Omega(e_1, e_2) = -f$. Finally,

$$\langle i\Omega \bullet \phi, \phi\rangle = f\mathrm{Re}\ \langle \nu \cdot \partial t \cdot \psi, \psi\rangle\, |_M = -fg(\nu, \partial t)|\phi|^2 = -f^2|\phi|^2.$$

Hence, equality in Theorem 4.7.1 is just

$$H^2 = \frac{S}{4} + \frac{1}{4}|II|^2 - \frac{1}{2}f^2.$$

4.7.2 The 3-dimensional case

In this section, we will treat the 3-dimensional case.

Theorem 4.7.4. *Let (M^3, g) be a compact oriented Riemannian manifold. For any* $\mathrm{Spin^c}$ *structure on M, any eigenvalue λ of the Dirac operator to which is attached an eigenspinor ψ satisfies*

$$\lambda^2 \leqslant \frac{1}{\mathrm{vol}(M, g)} \int_M (|\ell^\psi|^2 + \frac{S}{4} + \frac{|\Omega|}{2}) v_g.$$

Equality holds if and only if the norm of ψ is constant and $\Omega \cdot \psi = i|\Omega|\psi$.

Proof. As in the proof of Theorem 4.7.1, the set $\{\frac{\psi}{|\psi|}, \frac{e_1\cdot\psi}{|\psi|}, \frac{e_2\cdot\psi}{|\psi|}, \frac{e_3\cdot\psi}{|\psi|}\}$ is orthonormal with respect to the real scalar product Re $\langle\cdot,\cdot\rangle$. The covariant derivative of ψ can be expressed in this frame as

$$\nabla_X \psi = \eta(X)\psi + \ell(X) \cdot \psi, \tag{4.43}$$

for all vector fields X, where η is a 1-form and ℓ is a $(1,1)$-tensor field. Moreover $\eta = \frac{d(|\psi|^2)}{2|\psi|^2}$ and $\ell(X) = -\ell^\psi(X)$. It follows that

$$\begin{aligned}\lambda^2 &= \frac{\Delta(|\psi|^2)}{2|\psi|^2} + |\ell^\psi|^2 + \frac{|d(|\psi|^2)|^2}{4|\psi|^4} + \frac{1}{4}S + (\frac{i}{2}\Omega\cdot\psi, \frac{\psi}{|\psi|^2}) \\ &= \Delta f - \frac{|d(|\psi|^2)|^2}{2|\psi|^4} + |\ell^\psi|^2 + \frac{1}{4}S + \left\langle \frac{i}{2}\Omega\cdot\psi, \frac{\psi}{|\psi|^2}\right\rangle.\end{aligned}$$

By the Cauchy-Schwarz inequality, we have $\frac{1}{2}\left\langle i\Omega\cdot\psi, \frac{\psi}{|\psi|^2}\right\rangle \leqslant \frac{1}{2}|\Omega|$. Integrating over M and using the fact that $|d(|\psi|^2)|^2 \geqslant 0$, we get the result. $\square$

Example 4.7.1. *Let M^3 be a compact 3-dimensional Riemannian manifold immersed in $\mathbb{CP}^2$ with constant mean curvature H. Since $\mathbb{CP}^2$ is a Kähler manifold, we endow it with the canonical* Spin^c *structure whose auxiliary line bundle has curvature equal to $6i\ltimes$. Moreover, by the Schrödinger-Lichnerowicz formula we have that any parallel spinor ψ satisfies $\Omega^{\mathbb{CP}^2}\cdot\psi = 12i\psi$. As in the previous example, we compute*

$$(\nu\lrcorner\Omega^{\mathbb{CP}^2})\bullet\phi = -i(\nu\cdot\mathrm{Ric}^{\mathbb{CP}^2}(\nu)\cdot\psi)_{|M} = 6i\phi.$$

Finally, $\Omega\bullet\phi = 6i\phi$. Using Equation (4.14), we have that $-\frac{3}{2}H$ is an eigenvalue of D. Since the norm of ϕ is constant, equality holds in Theorem 4.7.4 and hence

$$\frac{9}{4}H^2 + \frac{3}{2} = \frac{S}{4} + \frac{1}{4}|II|^2.$$

Chapter 5

Hypersurfaces of Spin^c Manifolds and Lawson Correspondence

5.1 Introduction

It is well-known that a conformal immersion of a surface in $\mathbb{R}^3$ can be characterized by a spinor field satisfying

$$D\phi = H\phi, \tag{5.1}$$

where D is the Dirac operator and H the mean curvature of the surface (see [Ku-Sc]). In [Fri98], T. Friedrich characterized surfaces in $\mathbb{R}^3$ in a geometrically invariant way. More precisely, consider an isometric immersion of a surface (M^2, g) into $\mathbb{R}^3$. The restriction to M of a parallel spinor of $\mathbb{R}^3$ satisfies, for all $X \in \Gamma(TM)$, the following relation

$$\nabla_X \phi = -\frac{1}{2} IIX \bullet \phi, \tag{5.2}$$

where ∇ is the spinorial connection of M, "$\bullet$" denotes the Clifford multiplication of M and II is the Weingarten map of the immersion. Hence, ϕ is a solution of the Dirac equation (5.1) with constant norm. Conversely, assume that a Riemannian surface (M^2, g) carries a spinor field ϕ, satisfying

$$\nabla_X \phi = -\frac{1}{2} EX \bullet \phi, \tag{5.3}$$

where E is a given symmetric endomorphism on the tangent bundle. It is straightforward to see that $E = 2\ell^\phi$. Then, the existence of a pair (ϕ, E) satisfying (5.3) implies that the tensor $E = 2\ell^\phi$ satisfies the Gauss and the Codazzi equations and by Bonnet's theorem, there exists a local isometric immersion of (M^2, g) into $\mathbb{R}^3$ with E as Weingarten map . T. Friedrich's result was extended by B. Morel [Mor05] for surfaces of the sphere $\mathbb{S}^3$ and the hyperbolic space $\mathbb{H}^3$.

Recently, J. Roth [Roth10] gave a spinorial characterization of surfaces isometrically immersed into 3-dimensional homogeneous manifolds with 4-dimensional isometry group. These manifolds, denoted by $\mathbb{E}(\kappa,\tau)$, are Riemannian fibrations over a simply connected 2-dimensional manifold $\mathbb{M}^2(\kappa)$ with constant curvature κ and bundle curvature τ. This fibration can be represented by a unit vector field ξ tangent to the fibers.

The manifolds $\mathbb{E}(\kappa,\tau)$ are Spin having a special spinor field ψ. This spinor is constructed using real or imaginary Killing spinors on $\mathbb{M}^2(\kappa)$. If $\tau \neq 0$, the restriction of ψ to a surface gives rise to a spinor field ϕ satisfying, for every vector field X tangent to M,

$$\nabla_X\phi = -\frac{1}{2}IIX\bullet\phi + i\frac{\tau}{2}X\bullet\overline{\phi} - i\frac{\alpha}{2}g(X,T)T\bullet\overline{\phi} + i\frac{\alpha}{2}fg(X,T)\overline{\phi}. \tag{5.4}$$

Here, $\alpha = 2\tau - \frac{\kappa}{2\tau}$, f is a real function and T is a vector field on M such that $\xi = T + f\nu$ is the decomposition of ξ into tangential and normal parts (ν is the normal vector field of the immersion). The spinor $\overline{\phi}$ is given by $\overline{\phi} := \phi^+ - \phi^-$, where $\phi = \phi^+ + \phi^-$ is the decomposition into positive and negative spinors. Up to some additional geometric assumptions on T and f, the spinor field ϕ allows to characterize the immersion of the surface into $\mathbb{E}(\kappa,\tau)$ [Roth10].

In this chapter, we consider Spin^c structures on $\mathbb{E}(\kappa,\tau)$ instead of Spin structures. As Sasakian manifolds, the manifolds $\mathbb{E}(\kappa,\tau)$ have a canonical Spin^c structure carrying a natural spinor field, namely, a real Killing spinor with Killing constant $\frac{\tau}{2}$. The restriction of this Killing spinor to M gives rise to a special spinor field satisfying

$$\nabla_X\phi = -\frac{1}{2}IIX\bullet\phi + i\frac{\tau}{2}X\bullet\overline{\phi}.$$

This spinor, with a curvature condition on the auxiliary line bundle, allows the characterization of the immersion of M into $\mathbb{E}(\kappa,\tau)$ without any additional geometric assumption on f or T (see Theorem 5.4.1). From this characterization, we get an elementary Spin^c proof of a generalized Lawson correspondence for constant mean curvature surfaces in $\mathbb{E}(\kappa,\tau)$ (see Theorem 5.5.1).

The second advantage of using Spin^c structures in this context is when we consider hypersurfaces of 4-dimensional manifolds. Indeed, any oriented 4-dimensional Kähler manifold has a canonical Spin^c structure with parallel spinors. In particular, the complex space forms $\mathbb{C}P^2$ and $\mathbb{C}H^2$. Then, using an analogue of Bonnet's Theorem for complex space forms, we prove a Spin^c characterization of hypersurfaces of the complex projective space $\mathbb{C}P^2$ and of the complex hyperbolic space $\mathbb{C}H^2$. This work generalizes to the complex case the results of [Mor05] and [La-Ro10].

5.2 Preliminaries

In this section, we give a short description of the complex space form $\mathbb{M}^2_{\mathbb{C}}(c)$ of complex dimension 2, the 3-dimensional homogeneous manifolds $\mathbb{E}(\kappa,\tau)$ with 4-dimensional isometry group and their hypersurfaces (see [Dan07, Sco83, Roth10]).

5.2.1 Basic facts about $\mathbb{E}(\kappa,\tau)$ and their hypersurfaces

We denote a 3-dimensional homogeneous manifold with 4-dimensional isometry group by $\mathbb{E}(\kappa,\tau)$. It is a Riemannian fibration over a simply connected 2-dimensional manifold $\mathbb{M}^2(\kappa)$ with constant curvature κ and such that the fibers are geodesic. We denote by τ the bundle curvature, which measures the defect of the fibration to be a Riemannian product. Precisely, we denote by ξ a unit vertical vector field, that is tangent to the fibers. The vector field ξ is a Killing vector field and satisfies for all vector field X,

$$\overline{\nabla}_X\xi = \tau X\wedge\xi, \tag{5.5}$$

where $\overline{\nabla}$ is the Levi-Civita connection on $\mathbb{E}(\kappa,\tau)$. When τ vanishes, we get a product manifold $\mathbb{M}^2(\kappa)\times\mathbb{R}$. If $\tau\neq 0$, these manifolds are of three types: they have the isometry group of the Berger spheres if $\kappa>0$, of the Heisenberg group Nil_3 if $\kappa=0$ or of $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$ if $\kappa<0$.

Note that if $\tau=0$, then $\xi=\frac{\partial}{\partial t}$ is the unit vector field giving the orientation of $\mathbb{R}$ in the product $\mathbb{M}^2(\kappa)\times\mathbb{R}$. The manifold $\mathbb{E}(\kappa,\tau)$, with $\tau\neq 0$, admits a local direct orthonormal frame $\{e_1,e_2,e_3\}$ with

$$e_3=\xi,$$

and such that the Christoffel symbols $\overline{\Gamma}^k_{ij}=\left\langle\overline{\nabla}_{e_i}e_j,e_k\right\rangle$ are given by

$$\begin{cases} \overline{\Gamma}^3_{12}=\overline{\Gamma}^1_{23}=-\overline{\Gamma}^3_{21}=-\overline{\Gamma}^2_{13}=\tau, \\ \overline{\Gamma}^1_{32}=-\overline{\Gamma}^2_{31}=\tau-\frac{\kappa}{2\tau}, \\ \overline{\Gamma}^i_{ii}=\overline{\Gamma}^i_{ij}=\overline{\Gamma}^i_{ji}=\overline{\Gamma}^j_{ii}=0, \quad \forall\, i,j\in\{1,2,3\}. \end{cases} \tag{5.6}$$

We call $\{e_1,e_2,e_3=\xi\}$ the canonical frame of $\mathbb{E}(\kappa,\tau)$.

Let M be an orientable surface of $\mathbb{E}(\kappa,\tau)$ with Weingarten tensor II associated with a unit normal inner vector ν. Moreover, we decompose ξ as $\xi=T+f\nu$, where the function f is the normal component of ξ and T is its tangential part. By a computation of the curvature tensor of $\mathbb{E}(\kappa,\tau)$, we deduce the Gauss and Codazzi equations:

$$K=\det(II)+\tau^2+(\kappa-4\tau^2)f^2, \tag{5.7}$$

$$\nabla_X IIY-\nabla_Y IIX-II[X,Y]=(\kappa-4\tau^2)f(g(Y,T)X-g(X,T)Y), \tag{5.8}$$

where K is the Gauss curvature of M. Moreover, from (5.5), we deduce

$$\nabla_X T=f(IIX-\tau JX) \quad\text{and}\quad df(X)=-g(IIX-\tau JX,T),$$

where J is the rotation of angle $\frac{\pi}{2}$ on TM and ∇ the Levi-Civita connection on (M^2,g). Now, we ask if the Gauss equation (5.7) and the Codazzi equation (5.8) are sufficient to get an isometric immersion of M into $\mathbb{E}(\kappa,\tau)$.

Definition 5.2.1 (Compatibility Equations). *Let E be a field of symmetric endomorphisms on a surface M, T a vector field on M and f a real-valued function on M so that $f^2 + \|T\|^2 = 1$. We say that (M, g, E, T, f) satisfies the compatibility equations for $\mathbb{E}(\kappa, \tau)$ if and only if for any $X, Y, Z \in \Gamma(TM)$,*

$$K = \det(E) + \tau^2 + (\kappa - 4\tau^2)f^2, \tag{5.9}$$

$$\nabla_X EY - \nabla_Y EX - E[X,Y] = (\kappa - 4\tau^2)f(g(Y,T)X - g(X,T)Y), \tag{5.10}$$

$$\nabla_X T = f(EX - \tau JX), \tag{5.11}$$

$$df(X) = -g(EX - \tau JX, T). \tag{5.12}$$

In [Dan09, Dan07], B. Daniel proved that these compatibility equations are necessary and sufficient for the existence of an isometric immersion F from M into $\mathbb{E}(\kappa, \tau)$ with Weingarten tensor $dF \circ E \circ dF^{-1}$ and so that $\xi = dF(T) + f\nu$.

5.2.2 Basic facts about $\mathbb{M}^2_{\mathbb{C}}(c)$ and their real hypersurfaces

Let $(\mathbb{M}^2_{\mathbb{C}}(c), J, \overline{g})$ be the complex space form of constant holomorphic sectional curvature $4c$ and complex dimension 2, that is for $c = 1$, $\mathbb{M}^2_{\mathbb{C}}(c)$ is the complex projective space $\mathbb{C}P^2$ and if $c = -1$, $\mathbb{M}^2_{\mathbb{C}}(c)$ is the complex hyperbolic space $\mathbb{C}H^2$. It is a well-known fact that the curvature tensor $\overline{R}$ of $\mathbb{M}^2_{\mathbb{C}}(c)$ is given by

$$\begin{aligned}\overline{g}\big(\overline{R}(X,Y)Z,W\big) &= c\Big\{\overline{g}(Y,Z)\overline{g}(X,W) - \overline{g}(X,Z)\overline{g}(Y,W) + \overline{g}(JY,Z)\overline{g}(JX,W) \\ &\quad -\overline{g}(JX,Z)\overline{g}(JY,W) - 2\overline{g}(JX,Y)\overline{g}(JZ,W)\Big\},\end{aligned} \tag{5.13}$$

for all X, Y, Z and W tangent vector fields to $\mathbb{M}^2_{\mathbb{C}}(c)$.

Let M^3 be an oriented real hypersurface of $\mathbb{M}^2_{\mathbb{C}}(c)$ endowed with the metric g induced by $\overline{g}$. We denote by ν the unit normal inner vector globally defined on M and by II the Weingarten tensor of this immersion. Moreover, the complex structure J induces on M an almost contact metric structure $(\mathfrak{X}, \xi, \eta)$, where $\mathfrak{X}$ is the $(1,1)$-tensor defined by $g(\mathfrak{X}X, Y) = \overline{g}(JX, Y)$ for all $X, Y \in \Gamma(TM)$, $\xi = -J\nu$ is a tangent vector field and η the 1-form associated with ξ, that is $\eta(X) = g(\xi, X)$ for all $X \in \Gamma(TM)$. Then, we see easily that, for every $X \in \Gamma(TM)$, the following holds:

$$\mathfrak{X}^2 X = -X + \eta(X)\xi, \quad g(\xi,\xi) = 1, \quad \text{and} \quad \mathfrak{X}\xi = 0. \tag{5.14}$$

Moreover, from the relation between the Riemannian connections $\overline{\nabla}$of $\mathbb{M}^2_{\mathbb{C}}(c)$ and ∇ of M, $\overline{\nabla}_X Y = \nabla_X Y + g(IIX, Y)\nu$, we deduce the two following identities:

$$(\nabla_X \mathfrak{X})Y = \eta(Y)IIX - g(IIX,Y)\xi \qquad \text{and} \qquad \nabla_X \xi = \mathfrak{X}IIX,$$

for every $X, Y \in \Gamma(TM)$. From the expression of the curvature of $\mathbb{M}^2_{\mathbb{C}}(c)$ given above, we deduce the Gauss and Codazzi equations. First, the Gauss equation says that for all $X, Y, Z, W \in \Gamma(TM)$,

$$\begin{aligned} g(R(X,Y)Z,W) \;=\;& c\Big\{g(Y,Z)\overline{g}(X,W) - g(X,Z)g(Y,W) + g(\mathfrak{X}Y,Z)g(\mathfrak{X}X,W) \\ &-g(\mathfrak{X}X,Z)g(\mathfrak{X}Y,W) - 2g(\mathfrak{X}X,Y)g(\mathfrak{X}Z,W)\Big\} \\ &+g(IIY,Z)g(IIX,W) - g(IIX,Z)g(IIY,W). \end{aligned} \tag{5.15}$$

The Codazzi equation is

$$d^{\nabla}II(X,Y) = c\big(\eta(X)\mathfrak{X}Y - \eta(Y)\mathfrak{X}X - 2g(\mathfrak{X}X,Y)\xi\big). \tag{5.16}$$

Now, we ask if the Gauss equation (5.15) and the Codazzi equation (5.16) are sufficient to get an isometric immersion of (M, g) into $\mathbb{M}^2_{\mathbb{C}}(c)$.

Definition 5.2.2 (Compatibility Equations). *Let (M^3, g) be a Riemannian manifold endowed with a contact metric structure $(\mathfrak{X}, \xi, \eta)$ and let E be a field of symmetric endomorphisms on M. We say that $(M, g, E, \mathfrak{X}, \xi, \eta)$ satisfies the compatibility equations for $\mathbb{M}^2_{\mathbb{C}}(c)$ if and only if, for any $X, Y, Z, W \in \Gamma(TM)$, we have*

$$\begin{aligned} g(R(X,Y)Z,W) \;=\;& c\Big\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W) + g(\mathfrak{X}Y,Z)g(\mathfrak{X}X,W) \\ &-g(\mathfrak{X}X,Z)g(\mathfrak{X}Y,W) - 2g(\mathfrak{X}X,Y)g(\mathfrak{X}Z,W)\Big\} \\ &+g(EY,Z)g(EX,W) - g(EX,Z)g(EY,W), \end{aligned} \tag{5.17}$$

$$d^{\nabla}E(X,Y) = c\big(\eta(X)\mathfrak{X}Y - \eta(Y)\mathfrak{X}X - 2g(\mathfrak{X}X,Y)\xi\big). \tag{5.18}$$

$$(\nabla_X\mathfrak{X})Y = \eta(Y)EX - g(EX,Y)\xi, \tag{5.19}$$

$$\nabla_X\xi = \mathfrak{X}EX. \tag{5.20}$$

In [PT08], P. Piccione and D.V. Tausk prove that the Gauss equation (5.17) and the Codazzi equation (5.18) together with (5.19) and (5.20) are necessary and sufficient for the existence of an isometric immersion from M into $\mathbb{M}^2_{\mathbb{C}}(c)$ such that the complex structure of $\mathbb{M}^2_{\mathbb{C}}(c)$ over M is given by $J = \mathfrak{X} + \eta(\cdot)\nu$.

5.2.3 Hypersurfaces and induced Spin^c structures

Here, we recall the relations between the extrinsic and the intrinsic Spin^c data. Let N be an oriented $(n+1)$-dimensional Riemannian Spin^c manifold and $M \subset N$ be an oriented hypersurface. The manifold M inherits a Spin^c structure induced from the one on N, and we have

$$\Sigma M \simeq \begin{cases} \Sigma N_{|_M} & \text{if } n \text{ is even,} \\ \\ \Sigma^+ N_{|_M} & \text{if } n \text{ is odd.} \end{cases}$$

Moreover, Clifford multiplication by a vector field X, tangent to M, is given by

$$X \bullet \phi = (X \cdot \nu \cdot \psi)_{|_M}, \tag{5.21}$$

where $\psi \in \Gamma(\Sigma N)$ (or $\psi \in \Gamma(\Sigma^+ N)$ if n is odd), ϕ is the restriction of ψ to M, "$\cdot$" is the Clifford multiplication on N, "$\bullet$" that on M and ν is the unit inner normal vector. The relation (5.21) differs from the relation (4.8) in Chapter 4, since we choose here the isomorphism (1.4) and not the isomorphism (4.4) to identify Clifford multiplications on N and M. In this case, for every $\psi \in \Gamma(\Sigma N)$ ($\psi \in \Gamma(\Sigma^+ N)$ if n is odd), the real 2-forms Ω and Ω^N are related by

$$(\Omega^N \cdot \psi)_{|_M} = \Omega \bullet \phi - (\nu \lrcorner \Omega^N) \bullet \phi. \tag{5.22}$$

We denote by $\nabla^{\Sigma N}$ the spinorial Levi-Civita connection on ΣN and by ∇ that on ΣM. For all $X \in \Gamma(TM)$, we have the Spin^c Gauss formula:

$$(\nabla^{\Sigma N}_X \psi)_{|_M} = \nabla_X \phi + \frac{1}{2} II(X) \bullet \phi, \tag{5.23}$$

where II denotes the Weingarten map of the hypersurface. Moreover, let D^N and D be the Dirac operators on N and M, after denoting by the same symbol any spinor and its restriction to M, we have

$$\widetilde{D}\phi = \frac{n}{2} H\phi - \nu \cdot D^N \phi - \nabla^{\Sigma N}_\nu \phi, \tag{5.24}$$

where $H = \frac{1}{n}\mathrm{tr}(II)$ denotes the mean curvature and $\widetilde{D} = D$ if n is even and $\widetilde{D} = D \oplus (-D)$ if n is odd.

5.3 Isometric immersions into $\mathbb{M}^2_{\mathbb{C}}(c)$ via spinors

In this section, we consider the canonical Spin^c structure on $\mathbb{M}^2_{\mathbb{C}}(c)$ carrying a parallel spinor field ψ. This spinor field lies in $\Sigma_0(\mathbb{M}^2_{\mathbb{C}}(c)) \subset \Sigma^+(\mathbb{M}^2_{\mathbb{C}}(c))$. The restriction of this Spin^c structure to any hypersurface M defines a Spin^c structure on M with a special spinor field. This spinor field characterizes the isometric immersion of M into $\mathbb{M}^2_{\mathbb{C}}(c)$.

5.3.1 Special spinor fields on $\mathbb{M}^2_{\mathbb{C}}(c)$ and their hypersurfaces

Assume that there exists an isometric immersion of (M^3, g) into $\mathbb{M}^2_{\mathbb{C}}(c)$ with Weingarten tensor II. We know that M has a contact metric structure $(\mathfrak{X}, \xi, \eta)$ such that $\mathfrak{X}X = JX - \eta(X)\nu$, for every $X \in \Gamma(TM)$.

Lemma 5.3.1. *The restriction ϕ of the parallel spinor ψ on $\mathbb{M}^2_{\mathbb{C}}(c)$ is a solution of the generalized Killing equation*

$$\nabla_X \phi + \frac{1}{2} II(X) \bullet \phi = 0. \tag{5.25}$$

Moreover, the spinor field ϕ satisfies $\xi \bullet \phi = -i\phi$. The curvature 2-form of the connection on the auxiliary line bundle associated with the induced Spin^c *structure is given by $i\Omega(X, Y) = 6ic \ltimes (X, Y)$ for every $X, Y \in \Gamma(TM)$, where $\ltimes$ is the Kähler form of $\mathbb{M}^2_{\mathbb{C}}(c)$.*

Proof. First, since ψ is parallel, we have $D^{\mathbb{M}^2_{\mathbb{C}}(c)}\psi = (\nabla^{\mathbb{M}^2_{\mathbb{C}}(c)})^*\nabla^{\mathbb{M}^2_{\mathbb{C}}(c)}\psi = 0$. Hence, by the Schrödinger-Lichnerowicz formula (1.8), we get

$$\Omega^{\mathbb{M}^2_{\mathbb{C}}(c)} \cdot \psi = 12ci\psi. \tag{5.26}$$

By the Spin^c Gauss formula (5.23), the restriction ϕ of the parallel spinor ψ on $\mathbb{M}^2_{\mathbb{C}}(c)$ satisfies

$$\nabla_X\phi = -\frac{1}{2}II(X) \bullet \phi.$$

Since the spinor ψ is parallel, by Equation (1.7), we have

$$\mathrm{Ric}^{\mathbb{M}^2_{\mathbb{C}}(c)}(X) \cdot \psi = i(X \lrcorner \Omega^{\mathbb{M}^2_{\mathbb{C}}(c)}) \cdot \psi,$$

for all $X \in T(\mathbb{M}^2_{\mathbb{C}}(c))$. Therefore, we compute,

$$\begin{aligned}
(\nu \lrcorner \Omega^{\mathbb{M}^2_{\mathbb{C}}(c)}) \bullet \phi &= (\nu \lrcorner \Omega^{\mathbb{M}^2_{\mathbb{C}}(c)}) \cdot \nu \cdot \psi|_M \\
&= -\nu \cdot (\nu \lrcorner \Omega^{\mathbb{M}^2_{\mathbb{C}}(c)}) \cdot \psi|_M \\
&= i\nu \cdot \mathrm{Ric}^{\mathbb{M}^2_{\mathbb{C}}(c)}\ \nu \cdot \psi_{|_M} \\
&= -6ci\phi.
\end{aligned}$$

By Equation (5.22), we get that

$$\Omega \bullet \phi = 6ci\phi. \tag{5.27}$$

Now, for any $X, Y \in \Gamma(TM)$, we have

$$\Omega(X,Y) = \Omega^{\mathbb{M}^2_{\mathbb{C}}(c)}(X,Y) = \rho(X,Y) = \mathrm{ric}^{\mathbb{M}^2_{\mathbb{C}}(c)}(X, JY) = 6c\overline{g}(X, JY) = 6c \ltimes (X,Y).$$

Let e_1 be a unit vector field tangent to M such that $\{e_1, e_2 = Je_1, \xi\}$ is an orthonormal basis of TM. In this basis, we have

$$\Omega \bullet \phi = \Omega(e_1,e_2)\ e_1 \bullet e_2 \bullet \phi + \Omega(e_1,\xi)\ e_1 \bullet \xi \bullet \phi + \Omega(e_2,\xi)\ e_2 \bullet \xi \bullet \phi.$$

But

$$\Omega(e_1, e_2) = -6c \qquad \text{and} \qquad \Omega(e_1,\xi) = \Omega(e_2,\xi) = 0.$$

Finally, $\Omega \bullet \phi = -6ce_1 \bullet e_2 \bullet \phi$. Using (5.27) and the fact that $e_1 \bullet e_2 \bullet \xi \bullet \phi = -\phi$, we conclude that $\xi \bullet \phi = -i\phi$.

Lemma 5.3.2. *Let E be a field of symmetric endomorphisms on a Riemannian* Spin^c *manifold M^3 of dimension 3, then*

$$\begin{aligned}
E(e_i) \bullet E(e_j) - E(e_j) \bullet E(e_i) &= 2(a_{j3}a_{i2} - a_{j2}a_{i3})e_1 \\
&\quad +2(a_{i3}a_{j1} - a_{i1}a_{j3})e_2 \\
&\quad +2(a_{i1}a_{j2} - a_{i2}a_{j1})e_3,
\end{aligned} \tag{5.28}$$

where $(a_{ij})_{i,j}$ is the matrix of E written in any local orthonormal frame $\{e_1, e_2, e_3\}$ of TM.

Remark 5.3.1. *Consider (M^3, g) a Riemannian* Spin^c *manifold endowed with a contact metric structure $(\mathfrak{X}, \xi, \eta)$. In the basis $\{e_1, \mathfrak{X}e_2, e_3 = \xi\}$, it is easy to check that the Gauss equation (5.17) and the Codazzi equation (5.16) are equivalent to*

$$\begin{cases} \mathrm{R}_{1332} = a_{12}a_{33} - a_{32}a_{13}, \\ \mathrm{R}_{1223} = a_{22}a_{13} - a_{32}a_{12}, \\ \mathrm{R}_{1221} = a_{22}a_{11} - a_{12}^2 + 4c, \\ \mathrm{R}_{1331} = a_{33}a_{11} - a_{13}^2 + c, \\ \mathrm{R}_{2113} = a_{23}a_{11} - a_{12}a_{13}, \\ \mathrm{R}_{2332} = a_{22}a_{33} - a_{23}^2 + c, \\ d^\nabla E(e_1, e_2) = -2ce_3, \\ d^\nabla E(e_1, e_3) = -ce_2, \\ d^\nabla E(e_1, e_3) = ce_1, \end{cases} \tag{5.29}$$

where R_{ijkl} denotes the curvature tensor of (M^3, g) and $i, j, k, l \in \{1, 2, 3\}$.

Proposition 5.3.1. *Let (M^3, g) be a Riemannian* Spin^c *manifold endowed with a contact metric structure $(\mathfrak{X}, \xi, \eta)$. Assume that there exists a non-trivial spinor ϕ satisfying*

$$\nabla_X \phi = -\frac{1}{2} EX \bullet \phi \quad \textit{and} \quad \xi \bullet \phi = -i\phi,$$

where E is a field of symmetric endomorphisms on M. We suppose that the curvature 2-form of the connection defined on the auxiliary line bundle associated with the Spin^c *structure is given by $i\Omega(e_1, e_2) = -6ic$ and $i\Omega(e_i, e_j) = 0$ elsewhere in the basis $\{e_1, e_2 = \mathfrak{X}e_1, e_3 = \xi\}$. Then, the Gauss and the Codazzi equations for $\mathbb{M}^2_{\mathbb{C}}(c)$ are satisfied (i.e. all the equations of System (5.29) are satisfied) if and only if one equation among System (5.29) is satisfied.*

Proof. We compute the spinorial curvature $\mathcal{R}$ on ϕ and we get

$$\mathcal{R}_{X,Y}\phi = -\frac{1}{2} d^\nabla E(X, Y) \bullet \phi + \frac{1}{4}(EY \bullet EX - EX \bullet EY) \bullet \phi.$$

In the basis $\{e_1, e_2 = \mathfrak{X}e_1, e_3 = \xi\}$, the Ricci identity (1.7) gives that

$$\begin{aligned} \frac{1}{2}\mathrm{Ric}(e_1) \bullet \phi &= e_2 \bullet \mathcal{R}(e_2, e_1)\phi + e_3 \bullet \mathcal{R}(e_3, e_1)\phi + \frac{i}{2}(e_1 \lrcorner \Omega) \bullet \phi \\ &= -\frac{1}{2} e_2 \bullet d^\nabla E(e_2, e_1) \bullet \phi + \frac{1}{4} e_2 \bullet (Ee_1 \bullet Ee_2 - Ee_2 \bullet Ee_1) \bullet \phi \\ &\quad -\frac{1}{2} e_3 \bullet d^\nabla E(e_3, e_1) \bullet \phi + \frac{1}{4} e_3 \bullet (Ee_1 \bullet Ee_3 - Ee_3 \bullet Ee_1) \bullet \phi \\ &\quad +\frac{i}{2}(e_1 \lrcorner \Omega) \bullet \phi. \end{aligned}$$

It is easy to check that $\mathrm{Ric}(e_1) = (\mathrm{R}_{1221} + \mathrm{R}_{1331})e_1 + \mathrm{R}_{1332}\ e_2 + \mathrm{R}_{1223}\ e_3$ and by Lemma 5.3.2 we have

$$\begin{aligned} & e_2 \bullet (Ee_1 \bullet Ee_2 - Ee_2 \bullet Ee_1) \bullet \phi + e_3 \bullet (Ee_1 \bullet Ee_3 - Ee_3 \bullet Ee_1) \bullet \phi \\ &\quad = 2(a_{11}a_{33} + a_{11}a_{12} - a_{13}^2 - a_{12}^2) e_1 \bullet \phi \\ &\qquad +2(a_{13}a_{33} - a_{32}a_{13}) e_2 \bullet \phi + 2(a_{22}a_{13} - a_{32}a_{12}) e_3 \bullet \phi. \end{aligned}$$

Finally, we get

$$\begin{array}{rl} & (\mathrm{R}_{1221}+\mathrm{R}_{1331})e_1\bullet\phi+\mathrm{R}_{1332}e_2\bullet\phi+\mathrm{R}_{1223}e_3\bullet\phi \\ = & -e_2\bullet d^{\nabla}E(e_2,e_1)\bullet\phi-e_3\bullet d^{\nabla}E(e_3,e_1)\bullet\phi \\ & +(a_{11}a_{33}+a_{11}a_{22}-a_{13}^2-a_{12}^2)e_1\bullet\phi \\ & +(a_{12}a_{33}-a_{32}a_{13})e_2\bullet\phi+(a_{22}a_{13}-a_{32}a_{12})e_3\bullet\phi \\ & -6ice_2\bullet\phi. \end{array} \tag{5.30}$$

Since $\xi\bullet\phi=-i\phi$, we have $-ie_2\bullet\phi=e_1\bullet\phi$ and Equation (5.30) becomes

$$\begin{array}{rl} & (\mathrm{R}_{1221}+\mathrm{R}_{1331}-a_{11}a_{33}-a_{11}a_{22}+a_{13}^2+a_{12}^2-5c)e_1\bullet\phi \\ & +(\mathrm{R}_{1332}-a_{12}a_{33}+a_{32}a_{13})e_2\bullet\phi \\ & +(\mathrm{R}_{1223}-a_{22}a_{13}+a_{32}a_{12})e_3\bullet\phi \\ = & -e_2\bullet d^{\nabla}E(e_2,e_1)\bullet\phi-e_3\bullet d^{\nabla}E(e_3,e_1)\bullet\phi \\ & +ce_1\bullet\phi. \end{array} \tag{5.31}$$

The same computation holds for the unit vector fields e_2 and e_3 and we get

$$\begin{array}{rl} & (\mathrm{R}_{2331}-a_{12}a_{33}+a_{13}a_{23})e_1\bullet\phi \\ & +(\mathrm{R}_{2332}+\mathrm{R}_{2112}-a_{22}a_{33}-a_{22}a_{11}+a_{13}^2+a_{12}^2-5c)e_2\bullet\phi \\ & +(\mathrm{R}_{2113}-a_{23}a_{11}+a_{12}a_{13}))e_3\bullet\phi \\ = & -e_1\bullet d^{\nabla}E(e_1,e_2)\bullet\phi-e_3\bullet d^{\nabla}E(e_3,e_2)\bullet\phi \\ & +ce_2\bullet\phi. \end{array} \tag{5.32}$$

Again, we have

$$\begin{array}{rl} & (\mathrm{R}_{3221}-a_{13}a_{22}+a_{23}a_{21})e_1\bullet\phi \\ & +(\mathrm{R}_{3112}-a_{32}a_{11}+a_{31}a_{12})e_2\bullet\phi \\ & +(\mathrm{R}_{3113}+\mathrm{R}_{3223}-a_{22}a_{33}-a_{11}a_{33}+a_{13}^2+a_{23}^2)e_3\bullet\phi \\ = & -e_1\bullet d^{\nabla}E(e_1,e_3)\bullet\phi-e_2\bullet d^{\nabla}E(e_2,e_3)\bullet\phi. \end{array} \tag{5.33}$$

Since $|\phi|$ is constant (say $|\phi|=1$), the set $\{\phi,e_1\bullet\phi,e_2\bullet\phi,e_3\bullet\phi\}$ is an orthonormal frame of ΣM with respect to the real scalar product Re $\langle .,.\rangle$. Hence, from Equation (5.31) we deduce

$$\begin{array}{rcl} \mathrm{R}_{1221}+\mathrm{R}_{1331}-(a_{11}a_{33}+a_{11}a_{22}-a_{13}^2-a_{12}^2+5c) & = & g(d^{\nabla}E(e_1,e_3),e_3) \\ & & -g(d^{\nabla}E(e_1,e_3),e_2)+c, \\ \mathrm{R}_{1332}-(a_{12}a_{33}-a_{32}a_{13}) & = & g(d^{\nabla}E(e_1,e_3),e_1), \\ \mathrm{R}_{1223}-(a_{22}a_{13}-a_{32}a_{12}) & = & g(d^{\nabla}E(e_1,e_2),e_1), \\ g(d^{\nabla}E(e_1,e_2),e_2) & = & -g(d^{\nabla}E(e_1,e_3),e_3). \end{array}$$

Similary, from Equations (5.32) and (5.33) we have

$$\begin{array}{rcl}
\mathrm{R}_{2331} - (a_{12}a_{33} - a_{13}a_{23}) & = & -g(d^\nabla E(e_2, e_3), e_2), \\
\mathrm{R}_{2332} + \mathrm{R}_{2112} - (a_{22}a_{33} + a_{22}a_{11} - a_{13}^2 - a_{12}^2 + 5c) & = & g(d^\nabla E(e_2, e_3), e_1) \\
& & +g(d^\nabla E(e_1, e_2), e_3) + c, \\
\mathrm{R}_{2113} - (a_{23}a_{11} - a_{12}a_{13}) & = & -g(d^\nabla E(e_1, e_2), e_2), \\
g(d^\nabla E(e_1, e_2), e_1) & = & g(d^\nabla E(e_2, e_3), e_3), \\
\mathrm{R}_{3221} - (a_{13}a_{22} - a_{23}a_{21}) & = & -g(d^\nabla E(e_2, e_3), e_3), \\
\mathrm{R}_{3112} - (a_{32}a_{11} - a_{31}a_{12}) & = & g(d^\nabla E(e_1, e_3), e_3), \\
\mathrm{R}_{3113} + \mathrm{R}_{3223} - (a_{22}a_{33} - a_{11}a_{33} + a_{13}^2 + a_{23}^2) & = & g(d^\nabla E(e_2, e_3), e_1) \\
& & -g(d^\nabla E(e_1, e_3), e_2), \\
g(d^\nabla E(e_2, e_3), e_2) & = & -g(d^\nabla E(e_1, e_3), e_1).
\end{array}$$

The last twelve equations imply that, if one of the equations in System (5.29) is satisfied, the Gauss and Codazzi equations for $\mathbb{M}_{\mathbb{C}}^2(c)$ are satisfied.

5.3.2 Spin^c characterization of Hypersurfaces of $\mathbb{M}_{\mathbb{C}}^2(c)$

Now, we give the main result of this section:

Theorem 5.3.1. *Consider (M^3, g) a Riemannian manifold endowed with a contact metric structure $(\mathfrak{X}, \xi, \eta)$, E a field of symmetric endomorphisms on M with trace equal to $3H$. Assume that E satisfies at least one equation in System (5.29). Then, the following statements are equivalent:*

1. *There exists an isometric immersion of (M^3, g) into $\mathbb{M}_{\mathbb{C}}^2(c)$ with Weingarten tensor E, mean curvature H and so that, over M, the complex structure of $\mathbb{M}_{\mathbb{C}}^2(c)$ is given by $J = \mathfrak{X} + \eta(\cdot)\nu$, where ν is the unit normal inner vector of the immersion.*

2. *There exists a* Spin^c *structure on M carrying a non-trivial spinor ϕ satisfying, for all $X \in \Gamma(TM)$,*

$$\nabla_X \phi = -\frac{1}{2} EX \bullet \phi \quad \textit{and} \quad \xi \bullet \phi = -i\phi.$$

   The curvature 2-form of the connection on the auxiliary line bundle is given by $i\Omega(e_1, e_2) = -6ic$ and $i\Omega(e_i, e_j) = 0$ elsewhere in the basis $\{e_1, e_2 = \mathfrak{X}e_1, e_3 = \xi\}$.

3. *There exists a* Spin^c *structure on M carrying a non-trivial spinor ϕ of constant norm and satisfying*

$$D\phi = \frac{3}{2} H\phi \quad \textit{and} \quad \xi \bullet \phi = -i\phi.$$

   The curvature 2-form of the connection on the auxiliary line bundle is given by $i\Omega(e_1, e_2) = -6c$ and $i\Omega(e_i, e_j) = 0$ elsewhere in the basis $\{e_1, e_2 = \mathfrak{X}e_1, e_3 = \xi\}$.

Proof. By Lemma 5.3.1, the first statement implies the second one. Using Proposition 5.3.1, to show that $2 \Longrightarrow 1$, it suffices to show that $\nabla_X\xi = \mathfrak{X}EX$ and $(\nabla_X\mathfrak{X})Y = \eta(Y)EX - g(EX,Y)\xi$ for every $X, Y \in \Gamma(TM)$. In fact, we simply compute the derivative of $\xi \bullet \phi = -i\phi$ in the direction of $X \in \Gamma(TM)$ to get

$$\nabla_X\xi \bullet \phi \;=\; \frac{i}{2}EX \bullet \phi + \frac{1}{2}\xi \bullet EX \bullet \phi.$$

Using that $-ie_2 \bullet \phi = e_1 \bullet \phi$, the last equation reduces to

$$\nabla_X\xi \bullet \phi - g(EX, e_1)e_2 \bullet \phi + g(EX, e_2)e_1 \bullet \phi = 0.$$

Finally $\nabla_X\xi = \mathfrak{X}EX$. Now, we compute the derivative of $-ie_2 \bullet \phi = e_1 \bullet \phi$ in the direction of e_1 to get

$$\nabla_{e_1}(\mathfrak{X}e_1) \bullet \phi - \frac{1}{2}e_2 \bullet Ee_1 \bullet \phi = i\nabla_{e_1}e_1 \bullet \phi - \frac{i}{2}e_1 \bullet Ee_1 \bullet \phi.$$

But, using that $\xi \bullet \phi = -i\phi$, we have

$$\frac{1}{2}e_2 \bullet Ee_1 \bullet \phi - \frac{i}{2}e_1 \bullet Ee_1 \bullet \phi = -a_{11}\xi \bullet \phi - a_{12}\phi.$$

Denoting by Γ_{ij}^k the Christoffel symbols of $\{e_1, \mathfrak{X}e_1, \xi\}$, we have $\nabla_{e_1}e_1 = \Gamma_{11}^1 e_1 + \Gamma_{11}^2 e_2 + \Gamma_{11}^3 e_3$. Moreover, using that $\nabla_{e_1}e_3 = \mathfrak{X}Ee_1$, we get

$$\Gamma_{11}^3 = g(\nabla_{e_1}e_1, e_3) = -g(e_1, \nabla_{e_1}e_3) = a_{12}.$$

Hence, $\nabla_{e_1}(\mathfrak{X}e_1) \bullet \phi = -a_{11}\xi \bullet \phi + \Gamma_{11}^1 e_2 \bullet \phi + \Gamma_{11}^2 e_2 \bullet \phi$. Finally

$$\nabla_{e_1}(\mathfrak{X}e_1) \bullet \phi - \mathfrak{X}(\nabla_{e_1}e_1) \bullet \phi = -a_{11}\xi \bullet \phi,$$

which is Equation (5.19) for $X = Y = e_1$. Similary, we compute the derivative of $-ie_2\bullet\phi = e_1\bullet\phi$ in the direction of e_2 and ξ to get Equation (5.19) for any $X, Y \in \Gamma(TM)$. It is easy to see that Assertion 2 implies Assertion 3. For $3 \Rightarrow 2$, since ϕ is of constant norm (say $|\phi| = 1$), the set $\{\phi, e_1 \bullet \phi, e_2 \bullet \phi, e_3 \bullet \phi\}$ is a local orthonormal frame of ΣM with respect to the real scalar product $\mathrm{Re}\,\langle .,.\rangle$. Hence, for every $X \in \Gamma(TM)$, we have

$$\nabla_X\phi = \eta(X)\phi + \ell(X) \bullet \phi, \tag{5.34}$$

where η is a 1-form and ℓ is a $(1,1)$-tensor field. Moreover, it is easy to check that $\eta = \frac{d(|\phi|^2)}{2|\phi|^2}$ and $\ell(X) = -\ell^\phi(X)$. Since ϕ is of constant norm we have $\eta = 0$. Moreover, $\ell(X) = -\ell^\phi(X)$ is symmetric and it suffices to consider $E = 2\ell^\phi$ to get the second assertion.

5.4 Isometric immersions into $\mathbb{E}(\kappa,\tau)$ via spinors

The manifold $\mathbb{E}(\kappa,\tau)$ has a Spin^c structure carrying a Killing spinor with Killing constant $\frac{\tau}{2}$. The restriction of this Spin^c structure to any surface M defines a Spin^c structure on M with a special spinor field. This spinor field characterizes the isometric immersion of M into $\mathbb{E}(\kappa,\tau)$.

5.4.1 Special spinor fields on $\mathbb{E}(\kappa,\tau)$ and their hypersurfaces

On Spin^c manifolds, A. Moroianu defined projectable spinors for arbitrary Riemannian submersions of Spin^c manifolds with 1-dimensional totally geodesic fibers [Moro96, Moro98]. These spinors will be used to get a Killing spinor on $\mathbb{E}(\kappa,\tau)$.

Proposition 5.4.1. *The canonical* Spin^c *structure on* $\mathbb{M}^2(\kappa)$ *induces a* Spin^c *structure on* $\mathbb{E}(\kappa,\tau)$ *carrying a Killing spinor with Killing constant* $\frac{\tau}{2}$.

Proof. By enlargement of the group structures, the two-fold covering $\Theta : P_{\mathrm{Spin}^c_2}\mathbb{M} \longrightarrow P_{\mathrm{SO}_2}\mathbb{M}\times_{\mathbb{M}} P_{\mathbb{S}^1}\mathbb{M}$, gives a two-fold covering

$$\Theta : P_{\mathrm{Spin}^c_3}\mathbb{M} \longrightarrow P_{\mathrm{SO}_3}\mathbb{M}\times_{\mathbb{M}} P_{\mathbb{S}^1}\mathbb{M},$$

which, by pull-back through $\pi : \overline{M} := \mathbb{E}(\kappa,\tau) \to \mathbb{M} := \mathbb{M}^2(\kappa)$, gives rise to a Spin^c structure on $\mathbb{E}(\kappa,\tau)$ [Moro98, Moro96] and the following diagram commutes

$$\begin{tikzcd} P_{\mathrm{Spin}^c_3}\overline{M} \arrow[r] \arrow[d, "\pi^*\Theta"] & P_{\mathrm{Spin}^c_3}\mathbb{M} \arrow[d, "\Theta"] \\ P_{\mathrm{SO}_3}\overline{M}\times_{\overline{M}} P_{\mathbb{S}^1}\overline{M} \arrow[r] & P_{\mathrm{SO}_3}\mathbb{M}\times_{\mathbb{M}} P_{\mathbb{S}^1}\mathbb{M} \end{tikzcd}$$

The next step is to relate the covariant derivatives of spinors on $\mathbb{M}$ and $\overline{M}$. We point out an important detail: since we are actually interested to get a Killing spinor on $\overline{M}$, the connection on $P_{\mathbb{S}^1}\overline{M}$ (which defines the covariant derivative of spinors on $\overline{M}$) that we will consider will be the pull-back connection if $\tau = 0$ and will not be the pull-back connection if $\tau \neq 0$. Hence, when $\tau = 0$, the connection A_0 on $P_{\mathbb{S}^1}\overline{M}$ is given by

$$A_0((\pi^*s)_*(X^*)) = A(s_*X) \qquad \text{and} \qquad A_0((\pi^*s)_*\xi) = 0. \tag{5.35}$$

Now, if $\tau \neq 0$, we consider a connection A_0 on $P_{\mathbb{S}^1}\overline{M}$ given by

$$A_0((\pi^*s)_*(X^*)) = A(s_*X) \qquad \text{and} \qquad A_0((\pi^*s)_*\xi) = i(2\tau - \frac{\kappa}{2\tau}), \tag{5.36}$$

where $e_3 = \xi$ is the vertical vector field on $\mathbb{E}(\kappa,\tau)$ if $\tau \neq 0$ or $e_3 = \frac{\partial}{\partial t}$ if $\tau = 0$, X^* is the horizontal left of a vector field X on $\mathbb{M}$, A is the connection defined on $P_{\mathbb{S}^1}\mathbb{M}$ and s a local section of $P_{\mathbb{S}^1}\mathbb{M}$. Recall that we have an identification of the pull back $\pi^*\Sigma\mathbb{M}$ with $\Sigma\overline{M}$ [Moro98, Moro96], and with respect to this identification, if X is a vector field and ψ a spinor field on $\mathbb{M}$, then

$$X^*\cdot\pi^*\psi = \pi^*(X\cdot\psi) \qquad \text{and} \qquad \xi\cdot\pi^*\psi = i\pi^*(\overline{\psi}). \tag{5.37}$$

The sections of $\Sigma\overline{M}$ which can be written as pull-back of sections of $\Sigma\mathbb{M}$ are called projectable spinors [Moro98, Moro96]. Now, we want to relate the covariant derivative $\nabla^{\mathbb{E}(\kappa,\tau)}$ of projectable spinors on $\mathbb{E}(\kappa,\tau)$ to the covariant derivative ∇ of spinors on $\mathbb{M}^2(\kappa)$. In fact, any spinor field ψ is locally written as $\psi = [\widetilde{b\times s},\sigma]$, where $b = (e_1, e_2)$

is a basis of $\mathbb{M}^2(\kappa)$, $s : U \longrightarrow P_{\mathbb{S}^1}\mathbb{M}$ is a local section of $P_{\mathbb{S}^1}\mathbb{M}$ and $\widetilde{b \times s}$ is the lift of the local section $b \times s : U \to P_{SO_2}\mathbb{M} \times_{\mathbb{M}} P_{\mathbb{S}^1}\mathbb{M}$ by the 2-fold covering. Then $\pi^*\psi$ can be expressed as $\pi^*\psi = [\pi^*(\widetilde{b \times s}), \pi^*\sigma]$. It is easy to see that the projection $\pi^*(\widetilde{b \times s})$ onto $P_{SO_3}\overline{M}$ is the canonical frame $(e_1^*, e_2^*, e_3 = \xi)$ and its projection onto $P_{\mathbb{S}^1}\overline{M}$ is just $\pi^*\sigma$. We have

$$\begin{aligned}
\nabla^{\mathbb{E}(\kappa,\tau)}_{e_1^*}\pi^*\psi &= [\pi^*(\widetilde{b \times s}), e_1^*(\pi^*\sigma)] + \frac{1}{2}\overline{g}(\overline{\nabla}_{e_1^*}e_1^*, e_2^*)e_1^* \cdot e_2^* \cdot \pi^*\psi \\
&\quad + \frac{1}{2}\sum_{j=1}^{2}\overline{g}(\overline{\nabla}_{e_1^*}e_j^*, e_3)e_j^* \cdot e_3 \cdot \pi^*\psi + \frac{1}{2}A_0((\pi^*s)_*e_1^*)\pi^*\psi \\
&\overset{(5.6)}{=} [\pi^*(\widetilde{b \times s}), \pi^*(e_1(\sigma))] + \frac{1}{2}g(\nabla_{e_1}e_1, e_2)\pi^*(e_1 \cdot e_2 \cdot \psi) \\
&\quad + \frac{\tau}{2}e_2^* \cdot e_3 \cdot \pi^*\psi + \frac{1}{2}A(s_*X)\pi^*\psi \\
&= \pi^*\Big([(\widetilde{b \times s}), (e_1(\sigma))] + \frac{1}{2}g(\nabla_{e_1}e_1, e_2)e_1 \cdot e_2 \cdot \psi \\
&\quad + \frac{1}{2}A(s_*X)\psi\Big) + \frac{\tau}{2}e_1^* \cdot \pi^*\psi \\
&= \pi^*(\nabla_{e_1}\psi) + \frac{\tau}{2}e_1 \cdot \pi^*\psi.
\end{aligned}$$

The same holds for e_2^*. Similary, if $\tau \neq 0$, we have

$$\begin{aligned}
\nabla^{\mathbb{E}(\kappa,\tau)}_{e_3}\pi^*\psi &= [\pi^*(\widetilde{b \times s}), e_3(\pi^*\sigma)] + \frac{1}{2}\overline{g}(\overline{\nabla}_{e_3}e_1^*, e_2^*)e_1^* \cdot e_2^* \cdot \pi^*\psi \\
&\quad + \frac{1}{2}\sum_{j=1}^{2}\overline{g}(\nabla_{e_3}e_j^*, e_3)e_j^* \cdot e_3 \cdot \pi^*\psi + \frac{1}{2}A_0((\pi^*s)_*e_3)\pi^*\psi \\
&\overset{(5.6)}{=} \frac{1}{2}\Big(\frac{\kappa}{2\tau} - \tau\Big)e_1^* \cdot e_2^* \cdot \pi^*\psi + \frac{i}{2}\Big(2\tau - \frac{\kappa}{2\tau}\Big)\pi^*\psi \\
&= \frac{1}{2}\Big(\frac{\kappa}{2\tau} - \tau\Big)e_3 \cdot \pi^*\psi + \frac{1}{2}\Big(2\tau - \frac{\kappa}{2\tau}\Big)e_3 \cdot \pi^*\overline{\psi}.
\end{aligned}$$

Now, the canonical $\mathrm{Spin^c}$ structure on $\mathbb{M}^2(\kappa)$ carries a parallel spinor $\psi \in \Gamma(\Sigma_0\mathbb{M}) \subset \Gamma(\Sigma^+\mathbb{M})$, so $\overline{\psi} = \psi$. Hence, the spinor $\pi^*\psi$ is a Killing spinor field on $\mathbb{E}(\kappa,\tau)$, since

$$\nabla^{\mathbb{E}(\kappa,\tau)}_{e_j^*}\pi^*\psi = \frac{\tau}{2}e_j^* \cdot \pi^*(\psi), \quad \text{for} \quad j = 1, 2 \;\; \text{and} \quad \nabla^{\mathbb{E}(\kappa,\tau)}_{\xi}\pi^*\psi = \frac{\tau}{2}\xi \cdot \pi^*\psi.$$

Now, if $\tau = 0$, a similar computation of $\nabla^{\mathbb{E}(\kappa,\tau)}_{e_3}\pi^*\psi$ gives that $\pi^*\psi$ is a parallel spinor field on $\mathbb{E}(\kappa,\tau)$.

Remark 5.4.1. *Every Sasakian manifold has a canonical* $\mathrm{Spin^c}$ *structure: in fact, giving a Sasakian structure on a manifold* (M^{2m+1}, g) *is equivalent to giving a Kähler structure on the cone over* M. *The cone over* M *is the manifold* $M \times_{r^2} \mathbb{R}^+$ *equipped with the metric* $r^2 g + dr^2$. *Moreover, there is a* 1-1*-correspondence between* $\mathrm{Spin^c}$ *structures on* M *and* *that on its cone [Moro97]. Hence, every Sasakian manifold has a canonical (resp. anti-canonical)* Spin^c *structure coming from the canonical one (resp. anti-canonical one) on its cone.*

In [Moro97], A. Moroianu classified all complete simply connected Spin^c *manifolds carrying real Killing spinors and he proved that the only complete simply connected* Spin^c *manifolds carrying real Killing spinors (other than the* Spin *manifolds) are the non-Einstein Sasakian manifolds endowed with their canonical (or anti-canonical)* Spin^c *structure.*

The manifold $\mathbb{E}(\kappa,\tau)$ is a complete simply connected non-Einstein manifold and hence the only Spin^c *structure on $\mathbb{E}(\kappa,\tau)$ carrying a Killing spinor is the canonical one (or the anti-canonical). Hence, the* Spin^c *structure on $\mathbb{E}(\kappa,\tau)$ described above, (i.e. the one coming from $\mathbb{M}^2(\kappa)$) is nothing but the canonical* Spin^c *structure on $\mathbb{E}(\kappa,\tau)$ coming from the Sasakian structure.*

From now on, we will denote the Killing spinor field $\pi^*\psi$ on $\mathbb{E}(\kappa,\tau)$ by ψ. Since, it is a Killing spinor, we have

$$(\nabla^{\mathbb{E}(\kappa,\tau)})^*\nabla^{\mathbb{E}(\kappa,\tau)}\psi = \frac{3\tau^2}{4}\psi \qquad \text{and} \qquad D^{\mathbb{E}(\kappa,\tau)}\psi = -\frac{3\tau}{2}\psi.$$

By the Schrödinger-Lichnerowicz formula, we get

$$\frac{i}{2}\Omega^{\mathbb{E}(\kappa,\tau)}\cdot\psi = \frac{3\tau^2}{2}\psi - \frac{(\kappa-\tau^2)}{2}\psi,$$

where $i\Omega^{\mathbb{E}(\kappa,\tau)}$ is the curvature 2-form of the auxiliary line bundle associated with the Spin^c structure. Finally,

$$\Omega^{\mathbb{E}(\kappa,\tau)}\cdot\psi = i(\kappa-4\tau^2)\psi. \tag{5.38}$$

5.4.2 Spin^c characterization of hypersurfaces of $\mathbb{E}(\kappa,\tau)$

Let $\kappa,\tau\in\mathbb{R}$ with $\kappa-4\tau^2\neq 0$ and M^2 a Riemannian surface immersed into $\mathbb{E}(\kappa,\tau)$. The vertical vector field ξ is written $\xi = T + f\nu$ where T be a vector field on M and f a real-valued function on M so that $f^2 + ||T||^2 = 1$. We endow $\mathbb{E}(\kappa,\tau)$ with the Spin^c structure described above, carrying a Killing spinor of Killing constant $\frac{\tau}{2}$.

Lemma 5.4.1. *The restriction ϕ of the Killing spinor ψ on $\mathbb{E}(\kappa,\tau)$ is a solution of the following equation*

$$\nabla_X\phi + \frac{1}{2}II(X)\bullet\phi - i\frac{\tau}{2}X\bullet\overline{\phi} = 0, \tag{5.39}$$

called the restricted Killing spinor equation. Moreover, $f = \frac{<\phi,\overline{\phi}>}{|\phi|^2}$ and the curvature 2-form of the connection on the auxiliary line bundle associated with the induced Spin^c *structure is given by $i\Omega(t_1,t_2) = -i(\kappa-4\tau^2)f$, in any local orthonormal frame $\{t_1,t_2\}$.*

Proof. We restrict the Spin^c structure on $\mathbb{E}(\kappa,\tau)$ to M. By the Spin^c Gauss formula (5.23), the restriction ϕ of the Killing spinor ψ on $\mathbb{E}(\kappa,\tau)$ satisfies

$$\nabla_X\phi + \frac{1}{2}II(X)\bullet\phi - \frac{\tau}{2}X\cdot\psi_{|_M} = 0.$$

Let $\{t_1, t_2, \nu\}$ be a local orthonormal frame of $\mathbb{E}(\kappa,\tau)$ such that $\{t_1, t_2\}$ is a local orthonormal frame of M and ν a unit normal inner vector field of the surface. The action of the volume forms on M and $\mathbb{E}(\kappa,\tau)$ gives

$$\begin{aligned} X\bullet\overline{\phi} &= i(X\bullet t_1\bullet t_2\bullet\phi)\\ &= i(X\cdot\nu\cdot t_1\cdot t_2\cdot\psi)_{|_M}\\ &= -i(X\cdot\psi)_{|_M}, \end{aligned}$$

which gives Equation (5.39). The vector field T splits into $T = \nu_1 + h\xi$ where ν_1 is a vector field generated by e_1 and e_2 and h a real function. The scalar product of T by $\xi = T + f\nu$ and the scalar product of $T = \nu_1 + h\xi$ by ξ gives $||T||^2 = h$ which means that $h = 1 - f^2$. Hence, the normal vector field ν can be written as $\nu = f\xi - \frac{1}{f}\nu_1$. As we mentionned before, the Spin^c structure on $\mathbb{E}(\kappa,\tau)$ induces a Spin^c structure on M with induced auxiliary line bundle. Next, we want to prove that the curvature 2-form of the connection on the induced auxiliary bundle is equal to $i\Omega(t_1,t_2) = -i(\kappa-4\tau^2)f$. Since ψ is a Killing spinor field, by the Ricci identity (1.7), we have

$$\mathrm{Ric}^{\mathbb{E}(\kappa,\tau)}(X)\cdot\psi - i(X\lrcorner\Omega^{\mathbb{E}(\kappa,\tau)})\cdot\psi = 2\tau^2 X\cdot\psi, \tag{5.40}$$

for all $X\in T(\mathbb{E}(\kappa,\tau))$. Therefore, we compute

$$\begin{aligned} (\nu\lrcorner\Omega^{\mathbb{E}(\kappa,\tau)})\bullet\phi &= (\nu\lrcorner\Omega^{\mathbb{E}(\kappa,\tau)})\cdot\nu\cdot\psi|_M\\ &= i(2\tau^2\psi + \nu\cdot\mathrm{Ric}^{\mathbb{E}(\kappa,\tau)}\ \nu\cdot\psi)_{|_M}. \end{aligned}$$

But, we have $\mathrm{Ric}^{\mathbb{E}(\kappa,\tau)}e_3 = 2\tau^2 e_3$, $\mathrm{Ric}^{\mathbb{E}(\kappa,\tau)}e_1 = (\kappa-2\tau^2)e_1$ and $\mathrm{Ric}^{\mathbb{E}(\kappa,\tau)}e_2 = (\kappa-2\tau^2)e_2$ [Dan07]. Hence,

$$\begin{aligned} \mathrm{Ric}^{\mathbb{E}(\kappa,\tau)}\nu &= f\mathrm{Ric}^{\mathbb{E}(\kappa,\tau)}e_3 - \frac{1}{f}\mathrm{Ric}^{\mathbb{E}(\kappa,\tau)}\nu_1 = 2\tau^2 fe_3 - \frac{1}{f}(\kappa-2\tau^2)\nu_1\\ &= 2\tau^2 fe_3 + (\kappa-2\tau^2)(\nu - fe_3)\\ &= -(\kappa-4\tau^2)fe_3 + (\kappa-2\tau^2)\nu. \end{aligned}$$

Using Equation (5.37), we conclude that

$$(\nu\lrcorner\Omega^{\mathbb{E}(\kappa,\tau)})\bullet\phi = -i(\kappa-4\tau^2)\phi - (\kappa-4\tau^2)f(\nu\cdot\psi)_{|_M}.$$

By Equation (5.22), we get $\Omega\bullet\phi = -(\kappa-4\tau^2)f(\nu\cdot\psi)_{|_M}$. The scalar product of the last equality with $t_1\bullet t_2\bullet\phi$ gives

$$\Omega(t_1,t_2)|\phi|^2 = f(\kappa-4\tau^2)(\psi, t_1\cdot t_2\cdot\nu\cdot\psi)|_M = -f(\kappa-4\tau^2)|\phi|^2.$$

We write in the frame $\{t_1, t_2, \nu\}$

$$\Omega^{\mathbb{E}(\kappa,\tau)}(t_1,t_2)t_1\cdot t_2\cdot\psi+\Omega^{\mathbb{E}(\kappa,\tau)}(t_1,\nu)t_1\cdot\nu\cdot\psi+\Omega^{\mathbb{E}(\kappa,\tau)}(t_2,\nu)t_2\cdot\nu\cdot\psi = i(\kappa-4\tau^2)\psi. \quad (5.41)$$

But we know that $\Omega^{\mathbb{E}(\kappa,\tau)}(t_1,t_2) = \Omega(t_1,t_2) = -(\kappa-4\tau^2)f$. For the other terms, we compute

$$\Omega^{\mathbb{E}(\kappa,\tau)}(t_1,\nu) = \Omega^{\mathbb{E}(\kappa,\tau)}(t_1,\frac{1}{f}e_3-\frac{1}{f}T) = -\frac{1}{f}g(T,t_2)\Omega^{\mathbb{E}(\kappa,\tau)}(t_1,t_2) = (\kappa-4\tau^2)g(T,t_2),$$

where the term $\Omega^{\mathbb{E}(\kappa,\tau)}(t_1,e_3)$ vanishes since, by Equation (5.40), we have $e_3\lrcorner\Omega^{\mathbb{E}(\kappa,\tau)} = 0$. Similarly, we find that $\Omega^{\mathbb{E}(\kappa,\tau)}(t_2,\nu) = -(\kappa-4\tau^2)g(T,t_1)$. By substituting these values into (5.41) and taking Clifford multiplication with $t_1\bullet t_2$, we get

$$T\bullet\phi = -f\phi+\overline{\phi}.$$

Finally, taking the real part of the scalar product of the last equation by ϕ, we get $f = \frac{<\phi,\overline{\phi}>}{|\phi|^2}$.

Remark 5.4.2. *Using also the equation $T\bullet\phi = -f\phi+\overline{\phi}$, we can deduce that*

$$g(T,t_1) = \mathrm{Re}\left\langle it_2\bullet\phi, \frac{\phi}{|\phi|^2}\right\rangle \quad \textit{and} \quad g(T,t_2) = -\mathrm{Re}\left\langle it_1\bullet\phi, \frac{\phi}{|\phi|^2}\right\rangle.$$

Proposition 5.4.2. *Let (M^2,g) be an oriented* Spin^c *surface carrying a non-trivial solution ϕ of the following equation*

$$\nabla_X\phi+\frac{1}{2}E(X)\bullet\phi-i\frac{\tau}{2}X\bullet\overline{\phi} = 0,$$

where E denotes a symmetric tensor field defined on M. Moreover, assume that the curvature 2-form of the asssociated auxiliary bundle satisfies $i\Omega(t_1,t_2) = -i(\kappa-4\tau^2)f = -i(\kappa-4\tau^2)\frac{<\phi,\overline{\phi}>}{|\phi|^2}$ in any local orthonormal frame $\{t_1,t_2\}$ of M. Then, there exists an isometric immersion of (M^2,g) into $\mathbb{E}(\kappa,\tau)$ with shape operator E, mean curvature H and such that, over M, the vertical vector is $\xi = dF(T)+f\nu$, where ν is the unit normal vector to the surface and T is the tangential part of ξ given by

$$g(T,t_1) = \mathrm{Re}\left\langle it_2\bullet\phi, \frac{\phi}{|\phi|^2}\right\rangle \quad \textit{and} \quad g(T,t_2) = -\mathrm{Re}\left\langle it_1\bullet\phi, \frac{\phi}{|\phi|^2}\right\rangle.$$

Proof. We compute the action of the spinorial curvature tensor $\mathcal{R}$ on ϕ. We have

$$\begin{aligned}\nabla_{t_1}\nabla_{t_2}\phi &= -\frac{1}{2}\nabla_{t_1}E(t_2)\bullet\phi+\frac{1}{4}E(t_2)\bullet E(t_1)\bullet\phi-\frac{\tau}{4}E(t_2)\bullet t_2\bullet\phi\\ &\quad -\frac{\tau}{2}\nabla_{t_1}t_1\bullet\phi+\frac{\tau}{4}t_1\bullet E(t_1)\bullet\phi-\frac{\tau^2}{4}t_1\bullet t_2\bullet\phi,\end{aligned}$$

as well as

$$\begin{aligned}\nabla_{t_2}\nabla_{t_1}\phi &= -\frac{1}{2}\nabla_{t_2}E(t_1)\bullet\phi+\frac{1}{4}E(t_1)\bullet E(t_2)\bullet\phi-\frac{\tau}{4}E(t_1)\bullet t_1\bullet\phi\\ &\quad -\frac{\tau}{2}\nabla_{t_2}t_2\bullet\phi+\frac{\tau}{4}t_2\bullet E(t_2)\bullet\phi-\frac{\tau^2}{4}t_2\bullet t_1\bullet\phi.\end{aligned}$$

So, taking into account that $[t_1,t_2]=\nabla_{t_1}t_2-\nabla_{t_2}t_1$, a straightforward computation gives

$$\mathcal{R}(t_1,t_2)\phi=-\frac{1}{2}(d^{\nabla}E)(t_1,t_2)\bullet\phi-\frac{1}{2}\det E\ t_1\bullet t_2\bullet\phi-\frac{\tau^2}{2}t_1\bullet t_2\bullet\phi.$$

On the other hand, it is well known that

$$\mathcal{R}(t_1,t_2)\phi=-\frac{1}{2}R_{1212}\ t_1\bullet t_2\bullet\phi+\frac{i}{2}\Omega(t_1,t_2)\phi.$$

Therefore, we have

$$(R_{1212}-\det E-\tau^2)t_1\bullet t_2\bullet\phi=(d^{\nabla}E(t_1,t_2)-if(\kappa-4\tau^2))\phi. \tag{5.42}$$

Now, let T a vector field of M given by

$$g(T,t_1)|\phi|^2=\mathrm{Re}\ \langle it_2\bullet\phi,\phi\rangle \quad\text{and}\quad g(T,t_2)|\phi|^2=-\mathrm{Re}\ \langle it_1\bullet\phi,\phi\rangle\,.$$

It is easy to check that $T\bullet\phi=-f\phi+\overline{\phi}$ and hence $f^2+\|T\|^2=1$. In the following, we will prove that the spinor field $\theta:=i\phi-if\overline{\phi}+JT\bullet\phi$ is zero. For this, it is sufficient to prove that its norm vanishes. Indeed, we compute

$$|\theta|^2=|\phi|^2+f^2|\phi|^2+||T||^2|\phi|^2-2\mathrm{Re}\ \left\langle i\phi,if\overline{\phi}\right\rangle+2\mathrm{Re}\ \langle i\phi,JT\bullet\phi\rangle\,. \tag{5.43}$$

Therefore, Equation (5.43) becomes

$$\begin{aligned}|\theta|^2 &= 2|\phi|^2-2f^2|\phi|^2+2\mathrm{Re}\ \langle i\phi,JT\bullet\phi\rangle\\ &= 2|\phi|^2-2f^2|\phi|^2+2g(JT,t_1)\mathrm{Re}\ \langle i\phi,t_1\bullet\phi\rangle+2g(JT,t_2)\mathrm{Re}\ \langle i\phi,t_2\bullet\phi\rangle\\ &= 2|\phi|^2-2f^2|\phi|^2+2g(JT,t_1)g(T,t_2)|\phi|^2-2g(JT,t_2)g(T,t_1)|\phi|^2\\ &= 2|\phi|^2-2f^2|\phi|^2-2||T||^2|\phi|^2=0.\end{aligned}$$

Thus, we deduce $if\phi=-f^2t_1\bullet t_2\bullet\phi-fJT\bullet\phi$, where we used the fact that $\overline{\phi}=it_1\bullet t_2\bullet\phi$. In this case, Equation (5.42) can be written as

$$(R_{1212}-\det E-\tau^2-(\kappa-4\tau^2)f^2)t_1\bullet t_2\bullet\phi=(d^{\nabla}E(t_1,t_2)+(\kappa-4\tau^2)JT)\bullet\phi.$$

This is equivalent to saying that both terms $R_{1212}-\det E-\tau^2-(\kappa-4\tau^2)f^2$ and $d^{\nabla}E(t_1,t_2)+(\kappa-4\tau^2)JT$ are equal to zero. In fact, these are the Gauss-Codazzi equations (5.9) and (5.10). In order to obtain the other two equations, we simply compute the derivative of $T\bullet\phi=-f\phi+\overline{\phi}$ in the direction of X in two ways. First, using that $iX\bullet\overline{\phi}=JX\bullet\phi$, we have

$$\begin{aligned}\nabla_XT\bullet\phi+T\bullet\nabla_X\phi &= \nabla_XT\bullet\phi-\frac{1}{2}T\bullet E(X)\bullet\phi+i\frac{\tau}{2}T\bullet X\bullet\overline{\phi}\\ &= \nabla_XT\bullet\phi-\frac{1}{2}T\bullet E(X)\bullet\phi+\frac{\tau}{2}T\bullet JX\bullet\phi.\end{aligned} \tag{5.44}$$

On the other hand, we have

$$\begin{aligned}
\nabla_X(T\bullet\phi) &= -X(f)\phi - f\nabla_X\phi + \nabla_X\overline{\phi}\\
&= -X(f)\phi + \frac{1}{2}fEX\bullet\phi + \frac{1}{2}EX\bullet\overline{\phi} - i\frac{\tau}{2}fX\bullet\overline{\phi} - \frac{i}{2}\tau X\bullet\phi\\
&= -X(f)\phi + \frac{1}{2}fEX\bullet\phi - \frac{1}{2}f\tau JX\bullet\phi\\
&\quad +\frac{1}{2}E(X)\bullet(T\bullet\phi + f\phi) - \frac{i}{2}\tau X\bullet\phi\\
&= -X(f)\phi + \frac{1}{2}fEX\bullet\phi + \frac{1}{2}EX\bullet(T\bullet\phi + f\phi)\\
&\quad -\frac{i}{2}\tau X\bullet\phi - \frac{1}{2}f\tau JX\bullet\phi. \qquad (5.45)
\end{aligned}$$

Take Equation (5.45) and substract (5.44) to get

$$-X(f)\phi + fE(X)\bullet\phi - g(T,E(X))\phi - \nabla_X T\bullet\phi - \frac{\tau}{2}T\bullet JX\bullet\phi = 0.$$

Taking the real part of the scalar product of the last equation with ϕ and using that $\langle iX\bullet\phi,\phi\rangle = -g(T,JX)|\phi|^2$, we get

$$X(f) = -g(T,E(X)) + \tau g(JX,T).$$

The imaginary part of the same scalar product gives $\nabla_X T = f(EX - \tau JX)$, which implies that there exists an immersion F from M into $\mathbb{E}(\kappa,\tau)$ with shape operator $dF\circ E\circ dF^{-1}$ and $\xi = dF(T) + f\nu$.

Now, we state the main result of this section, which characterizes any isometric immersion of a surface (M^2,g) into $\mathbb{E}(\kappa,\tau)$.

Theorem 5.4.1. *Let $\kappa,\tau\in\mathbb{R}$ with $\kappa-4\tau^2\neq 0$. Consider (M^2,g) a Riemannian surface. We denote by E a field of symmetric endomorphisms of TM, with trace equal to $2H$. The following statement are equivalent:*

1. *There exists an isometric immersion F of (M^2,g) into $\mathbb{E}(\kappa,\tau)$ with shape operator E, mean curvature H and such that, over M, the vertical vector is $\xi = dF(T)+f\nu$, where ν is the unit normal vector to the surface, f is a real function on M and T the tangential part of ξ.*

2. *There exists a* Spin^c *structure on M carrying a non-trivial spinor field ϕ satisfying, for all $X\in\Gamma(TM)$,*
$$\nabla_X\phi = -\frac{1}{2}EX\bullet\phi + i\frac{\tau}{2}X\bullet\overline{\phi}.$$
Moreover, the auxiliary bundle has a connection of curvature given, in any local orthonormal frame $\{t_1,t_2\}$, by $i\Omega(t_1,t_2) = -i(\kappa-4\tau^2)f = -i(\kappa-4\tau^2)\frac{<\phi,\overline{\phi}>}{|\phi|^2}$.

3. *There exists a* $\mathrm{Spin^c}$ *structure on* M *carrying a non-trivial spinor field* ϕ *of constant norm satisfying*
$$D\phi = H\phi - i\tau\overline{\phi}.$$
Moreover, the auxiliary bundle has a connection of curvature given, in any local orthonormal frame $\{t_1, t_2\}$*, by* $i\Omega(t_1,t_2) = -i(\kappa - 4\tau^2)f = -i(\kappa-4\tau^2)\frac{<\phi,\overline{\phi}>}{|\phi|^2}$.

Proof. Proposition 5.4.2 and Lemma 5.4.1 give the equivalence between the first two statements. If (2) holds, it is easy to check that in this case the Dirac operator acts on ϕ to give $D\phi = H\phi - i\tau\overline{\phi}$. Moreover, for any $X \in \Gamma(TM)$, we have

$$\begin{aligned} X(|\phi|^2) &= 2\mathrm{Re}\,\langle\nabla_X\phi,\phi\rangle \\ &= \mathrm{Re}\,\left\langle i\tau\ X\bullet\overline{\phi},\phi\right\rangle = \mathrm{Re}\,\langle JX\bullet\phi,\phi\rangle = 0. \end{aligned}$$

Hence, ϕ is of constant norm. Now, consider a non-trivial spinor field ϕ of constant length, which satisfies $D\phi = H\phi - i\tau\overline{\phi}$. Define the following 2-tensors on (M^2, g)

$$T^\phi_\pm(X,Y) = \mathrm{Re}\,\left\langle\nabla_X\phi^\pm, Y\bullet\phi^\mp\right\rangle\ .$$

First note that
$$\mathrm{tr}T^\phi_\pm = -\mathrm{Re}\,\left\langle D\phi^\pm,\phi^\mp\right\rangle = -H|\phi^\mp|^2\ . \tag{5.46}$$

Moreover, we have the following relations [Mor05]

$$T^\phi_\pm(t_1,t_2) = \tau|\phi^\mp|^2 + T^\phi_\pm(t_2,t_1), \tag{5.47}$$

$$\nabla_X\phi^+ = \frac{T^\phi_+(X)}{|\phi^-|^2}\bullet\phi^-, \tag{5.48}$$

$$\nabla_X\phi^- = \frac{T^\phi_-(X)}{|\phi^+|^2}\bullet\phi^+\ , \tag{5.49}$$

$$|\phi^+|^2T^\phi_+ = |\phi^-|^2T^\phi_-, \tag{5.50}$$

where the vector field $T^\phi_\pm(X)$ is defined by $g(T^\phi_\pm(X),Y) = T^\phi_\pm(X,Y)$ for any $Y\in\Gamma(TM)$. Now let $F^\phi := T^\phi_+ + T^\phi_-$. Thus, we have

$$\frac{F^\phi}{|\phi|^2} = \frac{T^\phi_+}{|\phi^-|^2} = \frac{T^\phi_-}{|\phi^+|^2}\ .$$

Hence $F^\phi/|\phi|^2$ is well defined on the whole surface M, and

$$\nabla_X\phi = \nabla_X\phi^+ + \nabla_X\phi^- = \frac{F^\phi(X)}{|\phi|^2}\bullet\phi, \tag{5.51}$$

where the vector field $F^\phi(X)$ is defined by $g(F^\phi(X),Y) = F^\phi(X,Y)$, for all $Y\in\Gamma(TM)$. Note that by Equation (5.47), the 2-tensor F^ϕ is not symmetric. It is easy to check that the energy-momentum tensor ℓ^ϕ associated with ϕ is given by

$$\ell^\phi(X,Y) = -\frac{1}{2|\phi|^2}\left(F^\phi(X,Y) + F^\phi(Y,X)\right)\ .$$

It is straigthforward to show that

$$\ell^\phi(t_1,t_1) = -F^\phi(t_1,t_1)/|\phi|^2 \quad , \quad \ell^\phi(t_2,t_2) = -F^\phi(t_2,t_2)/|\phi|^2 \ ,$$

$$\ell^\phi(t_1,t_2) = -F^\phi(t_1,t_2)/|\phi|^2 + \frac{\tau}{2} \quad \text{and} \quad \ell^\phi(t_2,t_1) = -F^\phi(t_2,t_1)/|\phi|^2 - \frac{\tau}{2}.$$

Taking into account these last relations in Equation (5.51), we conclude

$$\nabla_X\phi \ = \ -\ell^\phi(X)\bullet\phi + i\frac{\tau}{2}X\bullet\overline{\phi}.$$

5.5 Application: a spinorial proof of a Generalized Lawson Correspondence

In [Dan07], B. Daniel gave a generalized Lawson correspondence for constant mean curvature surfaces in $\mathbb{E}(\kappa,\tau)$. Namely, he proved the following:

Theorem 5.5.1. *Let $\mathbb{E}(\kappa_1,\tau_1)$ and $\mathbb{E}(\kappa_2,\tau_2)$ be two 3-dimensional homogeneous manifolds with four dimensional isometry group and assume that $\kappa_1 - 4\tau_1^2 = \kappa_2 - 4\tau_2^2$. We denote by ξ_1 and ξ_2 the vertical vectors of $\mathbb{E}(\kappa_1,\tau_1)$ and $\mathbb{E}(\kappa_2,\tau_2)$ respectively. We consider (M^2,g) a simply connected surface isometrically immersed into $\mathbb{E}(\kappa_1,\tau_1)$ with constant mean curvature H_1 so that $H_1^2 \geqslant \tau_2^2 - \tau_1^2$. Let ν_1 be the unit inner normal vector of the immersion, T_1 the tangential projection of ξ_1 and $f = \langle\nu_1,\xi_1\rangle$. We choose $H_2\in\mathbb{R}$ and $\theta\in\mathbb{R}$ so that*

$$H_2^2+\tau_2^2 = H_1^2+\tau_1^2, \qquad \textit{and} \qquad \tau_2+iH_2 = e^{i\theta}(\tau_1+iH_1).$$

Then, there exists an isometric immersion F from (M^2,g) into $\mathbb{E}(\kappa_2,\tau_2)$ with mean curvature H_2 and so that over M,

$$\xi_2 = dF(T_2) + f\nu_2,$$

where ν_2 is the unit normal vector of the immersion and T_2 the tangential part of ξ_2. Moreover, the respective Weingarten tensors II_1 and II_2 are related by the following

$$II_2 - H_2\mathrm{Id} = e^{\theta J}(II_1 - H_1\mathrm{Id}).$$

With the help of Theorem 5.4.1, we give an alternative proof of this results using spinors.

Proof of Theorem 5.5.1. Since M^2 is isometrically immersed into $\mathbb{E}(\kappa_1,\tau_1)$ there exists a spinor field ϕ_1 of constant norm satisfying

$$D\phi_1 = H_1\phi_1 - i\tau\overline{\phi_1},$$

associated with the Spin^c structure whose auxiliary line bundle has a connection of curvature given, in any local orthonormal frame $\{t_1,t_2\}$, by $i\Omega(t_1,t_2) = -i(\kappa-4\tau^2)f$, where $f = \frac{<\phi,\overline{\phi}>}{|\phi|^2}$. We deduce that

$$D\phi_1^+ = H_1\phi_1^- + i\tau_1\phi_1^-$$

$$D\phi_1^- = H_1\phi_1^+ - i\tau_1\phi_1^+.$$

We define $\phi_2 = \phi_1^+ + e^{i\theta}\phi_1^-$. First, we have

$$\begin{aligned} D\phi_2 &= D\phi_1^+ + e^{i\theta}D\phi_1^- \\ &= (H_1 + i\tau_1)\phi_1^- - ie^{i\theta}(\tau_1 + iH_1)\phi_1^+. \end{aligned}$$

Since $\tau_2 + iH_2 = e^{i\theta}(\tau_1 + iH_1)$, we deduce that $H_1 + i\tau_1 = e^{i\theta}(H_2 + i\tau_2)$ and so

$$D\phi_2 = H_2\phi_2 - i\tau_2\overline{\phi_2}.$$

Secondly,

$$\frac{< \phi_1, \overline{\phi}_1 >}{|\phi_1|^2} = \frac{< \phi_2, \overline{\phi}_2 >}{|\phi_2|^2}.$$

Now, since $\kappa_1 - 4\tau_1^2 = \kappa_2 - 4\tau_2^2$, the auxiliary line bundle has a connection of curvature given by $i\Omega(t_1, t_2) = -i(\kappa_2 - 4\tau_2^2)f$ and hence, by Theorem 5.4.1, there exists an isometric immersion F from (M^2, g) into $\mathbb{E}(\kappa_2, \tau_2)$ with mean curvature H_2 and so that the vertical vector field is given by $\xi_2 = dF(T_2) + f\nu_2$, where ν_2 is the unit normal inner vector of the surface and T_2 the tangential part of ξ_2.

Remark 5.5.1. *By the proof of Proposition 5.4.2, we have that*

$$g(T_2, t_1)|\phi_2|^2 = \mathrm{Re}\,\langle it_2 \bullet \phi_2, \phi_2\rangle \quad \textit{and} \quad g(T_2, t_2)|\phi_2|^2 = -\mathrm{Re}\,\langle it_1 \bullet \phi_2, \phi_2\rangle\,.$$

So, it is easy to see that $T_2 = e^{\theta J}(T_1)$.

Chapter 6

Eigenvalue Estimates for the Dirac Operator on Spin^C Hypersurfaces

6.1 Introduction

It is well known that the spectrum of the Dirac operator on a closed hypersurface of a Spin manifold detects informations on the geometry of such manifolds and their hypersurfaces ([HMZ01a, HMZ02, HMR02]). In this chapter, we will study the spectrum (lower and upper bounds) for the Dirac operator of a closed hypersurface in a Spin^c manifold.

In the first part of this chapter, we give an upper bound for the eigenvalues of the Dirac operator on a closed hypersurface in a Spin^c manifold carrying parallel or Killing spinors. We recall that a spinor field ψ on a Riemmanian Spin^c manifold $\mathcal{Z}$ is called a Killing spinor with Killing constant $\alpha \in \mathbb{C}$ if

$$\nabla_X^{\Sigma\mathcal{Z}}\psi = \alpha\ X \cdot \psi, \tag{6.1}$$

for all $X \in \Gamma(T\mathcal{Z})$. When $\alpha = 0$, the spinor ψ is called a parallel spinor. Now, we define

$$\mu = \mu(\mathcal{Z}, \alpha) := \dim_{\mathbb{C}}\{\psi, \psi \ \text{ is a Killing spinor on } \ \mathcal{Z} \ \text{ with Killing constant } \ \alpha\}.$$

We prove the following:

Theorem 6.1.1. *Let $\mathcal{Z}$ be a Riemannian* Spin^c *manifold. Let $\alpha \in \mathbb{R}$ and M be an n-dimensional closed Riemannian hypersurface. Then, there are at least $\mu(\mathcal{Z}, \alpha)$ eigenvalues $\lambda_1, \dots, \lambda_\mu$ of the Dirac operator $\widetilde{D}$ on M satisfying*

$$\lambda_j^2 \leq n^2\alpha^2 + \frac{n^2}{4\mathrm{vol}(M)}\int_M H^2 v_g, \tag{6.2}$$

where H denotes the mean curvature of M.

Kähler and Sasaki manifolds are examples of Spin^c manifolds carrying parallel or Killing spinors. We should point out that this theorem has been proved by C. Bär

[Bär98] for hypersurfaces of Spin manifolds.

In the second part of the chapter, under suitable boundary conditions and some curvature assumptions, a lower bound for the eigenvalues of the Dirac operator on the boundary is given. In fact, using the spinorial Reilly inequality, we prove:

Theorem 6.1.2. *Let $\mathcal{Z}^{n+1}$ be a Riemannian* Spin^c *manifold satisfying $S^{\mathcal{Z}} \geqslant c_{n+1}|\Omega^{\mathcal{Z}}|$ and M^n a compact hypersurface. We assume that M has nonnegative mean curvature H and it bounds a compact domain $\mathbb{D}$ in $\mathcal{Z}$. Then, the first positive eigenvalue λ_1 of $\widetilde{D}$ satisfies*

$$\lambda_1 \geqslant \frac{n}{2} \inf_M H. \tag{6.3}$$

Equality holds if and only if H is constant and the eigenspace corresponding to λ_1 consists of the restrictions to M of parallel spinors on the domain $\mathbb{D}$.

This was proved by O. Hijazi, S. Montiel and X. Zhang for Spin manifolds (see [HMZ01a], [HMZ02] and [HMR02]). Examples of the limiting case in (6.3) are then given where the limiting case of the upper bound holds too. In the last part, we compare the lower bound (6.3) to the well known Friedrich Spin^c lower bound [HM99, Fri80].

6.2 Preliminaries

In the case of a compact manifold without boundary, the Dirac operator is formally self-adjoint with respect to the L^2-product, so it has a real discrete spectrum. In the case of a manifold with boundary, a defect of symmetry appears, given by

$$\int_M \langle D\psi, \varphi\rangle \, v_g - \int_M \langle \psi, D\varphi\rangle \, v_g = -\int_{\partial M} \langle \nu\cdot\psi, \varphi\rangle \, s_g, \tag{6.4}$$

for $\psi, \varphi \in \Gamma(\Sigma M)$, where ν is the inner unit vector field along the boundary and v_g (resp. s_g) is the volume form of the manifold M (resp. the boundary ∂M).

Bounding Domains. Now we assume that the hypersurface M is the boundary of a compact domain $\mathbb{D}$ in the Spin^c manifold $\mathcal{Z}$ (the domain $\mathbb{D}$ could be the manifold $\mathcal{Z}$ itself). In this case, we will give an inequality called *spinorial Reilly inequality* relating the geometry of the domain $\mathbb{D}$ and that of its boundary $\partial\mathbb{D}$. For all spinor fields $\psi \in \Gamma(\Sigma\mathbb{D})$, we have

$$\begin{aligned}\int_{\partial\mathbb{D}} \Big(\Big\langle \widetilde{D}\varphi, \varphi \Big\rangle - \frac{n}{2} H|\varphi|^2 \Big) s_g \geqslant \int_{\mathbb{D}} \Big(\frac{1}{4}(S^{\mathcal{Z}} \quad &- \quad c_{n+1}|\Omega^{\mathcal{Z}}|)|\psi|^2 \\ &- \quad \frac{n}{n+1}|D^{\mathcal{Z}}\psi|^2 \Big) v_g,\end{aligned} \tag{6.5}$$

Moreover, equality occurs if and only if the spinor field ψ is a twistor-spinor and verifies

$$\Omega^{\mathcal{Z}} \cdot \psi = i\frac{c_{n+1}}{2}|\Omega^{\mathcal{Z}}|_g \psi, \tag{6.6}$$

i.e. if and only if it satisfies

$$\left\{ \begin{array}{c} \mathbb{P}\psi = 0, \\ \Omega^{\mathcal{Z}} \cdot \psi = i\frac{c_{n+1}}{2}|\Omega^{\mathcal{Z}}|_g\psi, \end{array} \right.$$

where $\mathbb{P}$ is the twistor operator acting on $\Sigma\mathcal{Z}$ locally given for all $X \in \Gamma(T\mathcal{Z})$ by $\mathbb{P}_X\psi = \nabla_X^{\Sigma\mathcal{Z}}\psi + \frac{1}{n+1}X \cdot D^{\mathcal{Z}}\psi$. The proof of this *spinorial Reilly inequality* on Spin^c manifolds having compact domains with boundary is similar to that on the spin case ([HMZ01a], [HMZ02], [HMR02]).

The Min-Max principle. We recall the Min-Max principle for the Dirac operator [Cha84] on a compact Riemannian Spin^c manifold M: let $(\lambda_k)_{k\geqslant 1}$ be the spectrum of the Dirac operator with $0 \leqslant |\lambda_1| \leqslant |\lambda_2| \leqslant ... \leqslant |\lambda_k| \leqslant ....$ For any natural integer $k \geqslant 1$, we have

$$\lambda_k^2 = \min_{E_k\subset\Gamma(\Sigma M)} \left\{ \max_{\psi\in E_k-\{0\}} \frac{\int_M \mathrm{Re}\ \langle D^2\psi, \psi\rangle\, v_g}{\int_M |\psi|^2 v_g} \right\},$$

where the minimum is taken on all k-dimensional vector subspaces E_k of $\Gamma(\Sigma M)$. Applying this theorem means choosing a subspace E_k of sections of ΣM called test-sections, on which the Rayleigh quotient $\frac{\int_M \mathrm{Re}\,\langle D^2\psi,\psi\rangle}{\int_M |\psi|^2 v_g}$ is evaluated.

6.3 Upper bounds for the Eigenvalues of the Dirac operator

In this section, we give an upper bound for the eigenvalues of the Dirac operator of a closed hypersurface in a Spin^c manifold carrying Killing spinors or parallel spinors. Examples are then given.

Proof of Theorem 6.1.1. First, note that the set of Killing spinors with Killing constant α is a vector space and since linearly independent Killing spinors are linearly independent at every point the space of restrictions of Killing spinors on $\mathcal{Z}$ to M, i.e.

$$\{\phi := \psi|_M,\ \psi \text{ is a spinor on } \mathcal{Z} \text{ satisfying } \nabla_X^{\Sigma\mathcal{Z}}\psi = \alpha\ X\cdot\psi \quad \forall X \in \Gamma(T\mathcal{Z})\}$$

is also μ-dimensional. Now, let ψ be a Killing spinor on $\mathcal{Z}$ with Killing constant $\alpha \in \mathbb{R}$. Such Killing spinors have constant length and we may assume that $|\psi| \equiv 1$. Using the Killing equation (6.1), we compute $D^{\mathcal{Z}}\psi = -(n+1)\alpha\ \psi$, and hence using (5.24) we get

$$\widetilde{D}\phi = n\alpha\ \nu\cdot\phi + \frac{n}{2}H\phi.$$

Now, we compute the Rayleigh quotient of $\widetilde{D}^2$

$$\begin{aligned}
\frac{\int_M \mathrm{Re}\left\langle \widetilde{D}^2\phi, \phi\right\rangle v_g}{\int_M |\phi| v_g} &= \frac{\int_M \mathrm{Re}\left\langle \widetilde{D}\phi, \widetilde{D}\phi\right\rangle v_g}{\mathrm{vol}(M)} \\
&= \frac{\int_M \mathrm{Re}\left\langle n\alpha\ \nu\cdot\phi + \frac{n}{2}H\phi, n\alpha\ \nu\cdot\phi + \frac{n}{2}H\phi\right\rangle v_g}{\mathrm{vol}(M)} \\
&= n^2\alpha^2 + \frac{n^2}{4}\frac{\int_M H^2 v_g}{\mathrm{vol}(M)}.
\end{aligned}$$

Since the Rayleigh quotient of $\widetilde{D}^2$ is bounded by $n^2\alpha^2 + \frac{n^2}{4}\frac{\int_M H^2}{\mathrm{vol}(M)}$ on a μ-dimensional space of spinors on M, the Min-Max principle implies the assertion.

Remarks and examples.
1. Since every Spin manifold has a trivial Spin^c structure, we obtain all estimates of C. Bär obtained in [Bär98] for hypersurfaces of Spin manifolds.
2. Simply connected Spin^c manifolds carrying parallel spinors are described in [Moro97]. A simply connected Riemannian Spin^c manifold M carrying a parallel spinor is isometric to the Riemannian product of a simply connected Kähler manifold M_1 with a simply connected Spin manifold M_2 carrying non trivial parallel spinors. Moreover, the Spin^c structure of M is the product of the canonical Spin^c structure of M_1 and the Spin structure of M_2 . We look at the most prominent examples:

• The only Spin^c structure on an irreducible non Ricci-flat Kähler manifold M which carries parallel spinors is the canonical one and in this case, $\mu(M, 0) = 1$ [Moro97]. Hence, Inequality (6.2) holds for the first eigenvalue of the Dirac operator $\widetilde{D}$ defined on any compact Riemannian hypersurface (the complex projective space $\mathbb{C}P^m$ with the Fubini-Study metric is an example of an irreducible Kähler not Ricci-flat manifold).

• The product of the canonical Spin^c structure on $\mathbb{S}^2$ (resp. $\mathbb{H}^2$) with the unique Spin structure on $\mathbb{R}$ gives a Spin^c structure on $\mathbb{S}^2 \times \mathbb{R}$ (resp. $\mathbb{H}^2 \times \mathbb{R}$). This Spin^c structure carries a parallel spinor coming from the tensor product of the parallel spinor on $\mathbb{S}^2$ with the parallel spinor on $\mathbb{R}$. Hence, the first eigenvalue λ_1 of the Dirac operator $\widetilde{D}$ of any compact Riemannian hypersurface of $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$ satisfies Inequality (6.2).

Now, we want to give some examples of Kähler Spin^c manifolds carrying parallel spinor and endowed with a Spin^c structure other than the canonical one. For this, we begin with the following lemma:

Lemma 6.3.1. *[HM99], [Atiy78] A 4-dimensional* Spin^c *manifold $\mathcal{Z}$ carries a parallel spinor if and only if it is Kähler.*

Proof. First, we recall some well-known facts on parallel spinors in dimension 4 [Atiy78]. By taking the projection of ψ onto $\Sigma^{\pm}\mathcal{Z}$ and changing the orientation of $\mathcal{Z}$ if necessary, we may assume that ψ is a section of $\Sigma^+\mathcal{Z}$. The equation

$$iX \cdot \psi = I(X) \cdot \psi,$$

defines a parallel, almost complex structure I on $\mathcal{Z}$, i.e. a Kähler structure. Conversely, every Kähler manifold admits at least one Spin^c structure carrying parallel spinors, namely the canonical Spin^c structure whose Spin^c bundle $\Lambda^{0,*}\mathcal{Z}$ obviously has constant functions as parallel spinors.

Corollary 6.3.1. *Let M be a 3-dimensional compact Riemannian manifold, isometrically immersed in a 4-dimensional Kähler manifold. Then the first eigenvalue of the Dirac operator $\widetilde{D}$ defined on M, endowed with a* Spin^c *structure coming from the restriction of any* Spin^c *structure on the Kähler manifold, satisfies*

$$\lambda_1^2 \leqslant \frac{9}{4}\frac{\int_M H^2 v_g}{\mathrm{vol}(M)}. \tag{6.7}$$

Proof. Since any 4-dimensional manifold has a Spin^c structure (hence many other Spin^c structures), we endow the Kähler manifold with one of these structures and we restrict it to M. By Theorem 6.1.1 and Lemma 6.3.1, we deduce the result. This is the case for the torus $\mathbb{T}^4$ (not simply connected Kähler manifold).

Corollary 6.3.2. *Let M be a compact Riemannian hypersurface isometrically immersed in a simply connected non-Einstein Sasakian manifold endowed with the canonical* Spin^c *structure. Then the first eigenvalue of the Dirac operator $\widetilde{D}$ satisfies*

$$\lambda_1^2 \leqslant \frac{n^2}{4} + \frac{n^2}{4}\frac{\int_M H^2 v_g}{\mathrm{vol}(M)}.$$

Proof. The result is trivial since simply connected non-Einstein Sasakian manifolds with the canonical Spin^c structure carry Killing spinors of normalized Killing constant $\frac{1}{2}$ and $\mu(\mathcal{Z}, \frac{1}{2}) = 1$ [Moro97].

Example:
3. Homogeneous 3-dimensional manifolds with a 4-dimensional isometry group, denoted by $E(\kappa, \tau \neq 0)$ are simply connected manifolds given by [DHM]:
- The Heisenberg group Nil_3 ($\kappa = 0$),
- The universal cover of the Lie group $PSL_2(\mathbb{R})$ ($\kappa < 0$),
- Berger spheres ($\kappa > 0$).

These manifolds are Sasaki non-Einstein [Boyer06] and hence, endowed with the canonical Spin^c structure, carry Spin^c Killing spinor fields of Killing constant $\frac{\tau}{2}$.

6.4 Lower bounds for the Eigenvalues of the hypersurface Dirac operator

In this section, we assume that the manifold $\mathcal{Z}$ has a compact domain $\mathbb{D}$ with boundary $M = \partial\mathbb{D}$ and we will use suitable boundary conditions for the Dirac operator $D^{\mathcal{Z}}$ to prove a lower bound for the eigenvalues of the extrinsic hypersurface Dirac operator $\widetilde{D}$. First, since the hypersurface M is compact, the Dirac operator $\widetilde{D}$ has a discrete spectrum. We denote by $\pi_+ : \Gamma(\Sigma M) \longrightarrow \Gamma(\Sigma M)$ the projection onto the subspace of $\Gamma(\Sigma M)$ spanned by the eigenspinors corresponding to the positive eigenvalues of $\widetilde{D}$. The projection π_+ provides an Atiyah-Patodi-Singer type boundary conditions for the Dirac operator $D^{\mathcal{Z}}$ of the domain $\mathbb{D}$.

Proof of Theorem 6.1.2. Since $S^{\mathcal{Z}} \geqslant c_{n+1}|\Omega^{\mathcal{Z}}|$ and the mean curvature H of the boundary is nonnegative, the following boundary problem

$$\begin{cases} D^{\mathcal{Z}}\psi = 0 & \text{on } \mathbb{D} \\ \pi_+\psi = \pi_+\varphi = \varphi & \text{on } M = \partial\mathbb{D}, \end{cases}$$

has a unique solution [HMZ01b], where φ is an eigenspinor on M corresponding to the first eigenvalue $\lambda_1 > 0$ of $\widetilde{D}$, i.e. $\widetilde{D}\varphi = \lambda_1\varphi$ and $\pi_+\varphi = \varphi$. From the Reilly inequality (6.5), we get

$$\int_M (\lambda_1 - \frac{n}{2}H)|\psi|^2 s_g \geqslant \frac{1}{4}\int_{\mathbb{D}} (S^{\mathcal{Z}} - c_{n+1}|\Omega^{\mathcal{Z}}|)|\psi|^2 v_g,$$

which implies (6.3). If the equality case holds in (6.3), then ψ is a harmonic spinor and a twistor spinor, hence parallel. Since $\pi_+\psi = \varphi$ along the boundary, ψ is a non-trivial parallel spinor and $\lambda_1 = \frac{n}{2}H > 0$. Futhermore, since ψ is parallel, we deduce by (5.24) that $\widetilde{D}\varphi = \frac{n}{2}H\varphi$. Hence we have $\varphi = \pi_+\psi = \psi$. Conversely, if H is a constant, the fact that the restriction to M of a parallel spinor on $\mathbb{D}$ is an eigenspinor with eigenvalue $\frac{n}{2}H$ is a direct consequence of (5.24).

Remark 6.4.1. *We should point out that, if equality holds in (6.3), we have $S^{\mathcal{Z}} = c_{n+1}|\Omega^{\mathcal{Z}}|$, i.e. the parallel spinor whose restriction to M is the eigenspinor associated with λ_1, is a parallel spinor satisfying (6.6).*

Examples:

4. We consider the complex projective space $\mathbb{C}P^m$ with the Einstein Fubini-Study metric, endowed with the canonical Spin^c structure whose auxiliary line bundle is $L = (K_{\mathbb{C}P^m})^{-1}$. It is a Spin^c structure carrying parallel spinors. Moreover, the curvature 2-form of the connection on L is the Ricci 2-form of $\mathbb{C}P^m$. Hence,

$$c_{2m}|\Omega^{\mathbb{C}P^m}| = c_{2m}\frac{S^{\mathbb{C}P^m}}{2m}| \ltimes^{\mathbb{C}P^m} | = S^{\mathbb{C}P^m}.$$

By Theorem (6.1.2), the first eigenvalue of the Dirac operator $\widetilde{D}$ of any compact hypersurface with positive constant mean curvature H, bounding a compact domain $\mathbb{D}$ in $\mathbb{C}P^m$, satisfies the equality case in (6.3), hence the equality case in (6.2). Compact embedded hypersurfaces are examples of manifolds viewed as a boundary of some enclosed domain in $\mathbb{C}P^m$.

5. The only embedded compact surface with constant mean curvature in $\mathbb{S}^2_+ \times \mathbb{R}$ is the standard rotational sphere [Ros02, DHM]. This is the Alexandrov theorem for

$\mathbb{S}^2_+ \times \mathbb{R}$. The vector field ∂t is a Killing vector field when we consider the product of the standard metric on $\mathbb{S}^2$ with the standard one on $\mathbb{R}$. The manifold $\mathbb{S}^2 \times \mathbb{R}$ admits rotations with respect to the vertical direction given by the canonical Killing vector field ∂t [DHM]. It is shown that there exists rotationally constant mean curvature spheres. For example, for $H > 0$, the rotational constant mean curvature H sphere in $\mathbb{S}^2 \times \mathbb{R}$ is given by

$$\mathrm{i}(u, v) = (-\cos\ k(u), \sin\ k(u) \cos\ v, \sin\ k(u) \sin\ v, h(u)),$$

where i is the embedding and

$$k(u) = 2\mathrm{Arctan}\ (\frac{2H}{\sqrt{1-u^2}}), \qquad h(u) = \frac{4H}{\sqrt{4H^2+1}}\mathrm{Arcsh}(\frac{u}{\sqrt{1-u^2+4H^2}}).$$

Now, the canonical Spin^c structure on $\mathbb{S}^2 \times \mathbb{R}$ carries a parallel spinor and we have $S^{\mathbb{S}^2\times\mathbb{R}} = c_3|\Omega^{\mathbb{S}^2\times\mathbb{R}}|$. Hence, the first eigenvalue of the Dirac operator $\widetilde{D}$ of the rotational sphere satisfies the equality case in (6.3) and hence the equality case in (6.2).

6.5 A geometric application

In this section, we prove that, under some additional assumptions, the extrinsic lower bound (6.3) is sharper than the corresponding intrinsic Friedrich lower bound.

Let M be an embedded hypersurface into a Riemannian Spin^c manifold $\mathcal{Z}$. If S denotes the scalar curvature of the induced metric on M, we have the Friedrich Spin^c inequality [HM99]

$$\lambda_1^2 \geqslant \frac{n}{4(n-1)} \inf_M (S - c_n|\Omega|). \tag{6.8}$$

A consequence from the Gauss formula for the embedding is that

$$S = S^{\mathcal{Z}} - 2\mathrm{ric}^{\mathcal{Z}}(\nu, \nu) + n^2H^2 - |II|^2,$$

and hence we get

$$S - c_n|\Omega| = S^{\mathcal{Z}} - 2\mathrm{ric}^{\mathcal{Z}}(\nu, \nu) + n^2H^2 - |II|^2 - c_n|\Omega|, \tag{6.9}$$

where $\mathrm{ric}^{\mathcal{Z}}$ is the Ricci tensor of $\mathcal{Z}$. From the last equation, it is clear that in general we cannot hope getting a relation between S and H allowing us to compare the Friedrich inequality (6.8) and Inequality (6.3).

Proposition 6.5.1. *Let M be an embedded hypersurface on a Riemannian* Spin^c *manifold $\mathcal{Z}^{n+1}$. If the Einstein tensor $\mathrm{ric}^{\mathcal{Z}} - \frac{S^{\mathcal{Z}}}{2} g_{\mathcal{Z}}$ of $\mathcal{Z}$ is positive semidefinite, then the extrinsic lower bound (6.3) for the first eigenvalue of the Dirac operator $\widetilde{D}$ of M is sharper than the Friedrich inequality (6.8). The two lower bounds coincide if and only if the embedding is totally umbilical and the restricted* Spin^c *structure has a flat auxiliary line bundle.*

Proof. Since the Einstein tensor $\mathrm{ric}^{\mathcal{Z}} - \frac{S^{\mathcal{Z}}}{2} g_{\mathcal{Z}}$ is positive semidefinite, we get by (6.9) that

$$S - c_n|\Omega| \leqslant n^2 H^2 - |II|^2 - c_n|\Omega|.$$

Using the Cauchy-Schwarz inequality and the fact that $|\Omega| \geqslant 0$, we get

$$S - c_n|\Omega| \leqslant n(n-1)H^2,$$

and the result follows. It is clear that the two lower bounds coincide if and only if $\Omega^M = 0$ and $II(X) = H\ X$ for $X \in \Gamma(TM)$.

Chapter 7

Spin^C Characterization of CR-structures

7.1 Introduction

It is well known that the existence of a special spinor on a Spin or Spin^c manifold imposes severe restrictions on the geometry and the topology of the manifold. For example, the existence of a Killing spinor (resp. parallel spinor) on a Spin manifold implies that the manifold is Einstein (resp. Ricci-flat) [Fri80, Hij01]. There is a notion of pure spinor, due to E. Cartan [Cart66], related to the existence of almost complex structures on Spin manifolds. There are also related to the notion of calibration introduced by Harvey-Lawson [Ha-La82, Dad83]. Moreover, distinguished differential forms are naturally associated to a given spinor field and, in particular, special Spin or Spin^c spinors on a given manifold give rise to special differential forms on any immersed hypersurface.

Pure spinors are also related to the Penrose formalism in General Relativity [Pen86a, Pen86b]. They are implicit in Penrose's notion of "flag planes", which correspond to the maximal isotropic spaces of a complex 4-dimensional orthogonal space. As the underlying real space has Lorentzian signature, pure spinors also exhibit a real structure: they determine a real null direction, or "flagpole", in Penrose's terminology [Pen75]. This correspondence means that pure spinors are useful for studying the properties of null congruences. It is well known that a vector field tangent to a congruence of null shear-free geodesics corresponds to a spinor field satisfying Sommers' equation [Somm76]. This equation is a generalisation of the twistor equation involving an additional vector field.

For instance, if (M^n, g) is a Riemannian Spin manifold, it admits a parallel pure spinor if and only if M is Kähler and Ricci-flat [LM89]. Note that, not all Kähler manifolds are Spin but they are always Spin^c. Hence, by considering Spin^c structures on manifolds, we characterize Kähler structures by the existence of a parallel pure Spin^c spinor (see Corollary 7.3.1). In this case, the manifold is not necessarily Ricci-flat.

CR-structures on manifolds attempt to intrinsically describe the property of being a hypersurface of a complex space form by studying the properties of holomorphic vector fields which are tangent to the hypersurface. Recently, it has been proved that every strictly pseudoconvex CR-manifold M^{2m+1} has a canonical Spin^c structure determined by the CR-structure [Pet05]. In this paper, we define an analogue of pure spinors on CR-manifolds, called transversal spinors. These spinors will characterize CR-structures on the manifold (see Corollary 7.3.1).

Analogous to the situation of parallel or Killing spinors, the existence of a transversal spinor on a Riemannian Spin^c manifold determines certain features of the geometry of the manifold. In fact, on a Spin^c manifold, we prove that the existence of a Killing or a parallel transversal spinor field in certain directions implies that the manifold is foliated (see Proposition 7.3.4).

7.2 CR-structures on manifolds

Let M^n be a differentiable manifold of real dimension n and $m \in \mathbb{N}$ an integer such that $1 \leqslant m \leqslant [\frac{n}{2}]$. An almost CR-structure on M^n of CR-dimension m and codimension $k = n-2m$ is a complex subbundle $T_{1,0}M$ of $T^{\mathbb{C}}M = TM \otimes_{\mathbb{R}} \mathbb{C}$ of complex rank m such that

$$T_{1,0}M \cap T_{0,1}M = \{0\},$$

where $T_{0,1}M = \overline{T_{1,0}M}$. In this case the almost CR-structure is called of type (m,k).

Definition 7.2.1. *On a differentiable manifold M^n, an almost CR-structure of type (m,k) is called a CR-structure of type (m,k) if $T_{1,0}M$ is formally integrable, i.e.*

$$[T_{1,0}M, T_{1,0}M] \subset T_{1,0}M.$$

It is easy to see that an almost CR-manifold (resp. a CR-manifold) M^n of type $(m,0)$ is an almost complex manifold (resp. a complex manifold) [Ko-No69]. CR-manifolds arise mainly as real hypersurfaces of complex manifolds. Indeed, let (N, J_N) be a complex manifold of complex dimension l and complex structure J_N. We consider a real hypersurface M of N, i.e. M is a hypersurface of N of real dimension $2l-1$. We set

$$T_{1,0}M := T_{1,0}N \cap (TM \otimes_{\mathbb{R}} \mathbb{C}),$$

where $T_{1,0}N$ is the holomorphic tangent bundle over N, i.e. $T_{1,0}N$ is the eigenbundle corresponding to the eigenvalue i of the extension of J_N to $T^{\mathbb{C}}M$. The bundle $T_{1,0}M$ defines a CR-structure on M of type $(2l-1, 1)$.

Proposition 7.2.1. *Having a CR-structure on M^n of type (m,k) is equivalent to having a real subbundle $H(M)$ of TM of real rank $2m$, a bundle automorphism J of $H(M)$ such that $J^2 = -\mathrm{Id}$ and, for every $X, Y \in \Gamma(H(M))$, we have*

$$[X,Y] - [JX,JY] \in \Gamma(H(M)) \quad \textit{and} \quad J([X,Y]-[JX,JY]) = [JX,Y]+[X,JY]. \tag{7.1}$$

In this case, $H(M)$ is called the maximal complex or the Levi distribution of the CR-structure.

Proof. Given $H(M)$ and J satisfying (7.1), we extend J by $\mathbb{C}$-linearity to $H(M)\otimes_{\mathbb{R}}\mathbb{C}$. The set $T_{1,0}M = \{Z \in H(M)\otimes_{\mathbb{R}}\mathbb{C}, JZ = iZ\}$ defines a CR-structure on M of type (m,k). Conversely, given a complex subbundle $T_{1,0}M$ of $TM\otimes_{\mathbb{R}}\mathbb{C}$ of rank m, we define the automorphism J_1 of $T_{1,0}M\oplus\overline{T_{1,0}M}$ which acts as multiplication by i (resp. $-i$) on $T_{1,0}M$ (resp. $\overline{T_{1,0}M}$), i.e.

$$\begin{aligned} J_1 : T_{1,0}M\oplus\overline{T_{1,0}M} &\longrightarrow T_{1,0}M\oplus\overline{T_{1,0}M}, \\ Z+\overline{W} &\longmapsto i(Z-\overline{W}). \end{aligned}$$

The real subbundle $H(M) := \mathrm{Re}\,(T_{1,0}M\oplus\overline{T_{1,0}M})$ of TM is of rank $2m$. The restriction J of J_1 to $H(M)$ satisfies (7.1).

Now, we will consider an oriented CR-manifold M^n of hypersurface type, i.e. an oriented CR-manifold M^n of type $(m,1)$. In this case, the dimension of M is odd, $n = 2m+1$ and there exists a global 1-form θ (not unique), called a pseudo-Hermitian structure on M such that $H(M) = \ker\theta$ (see[DT]). Given a pseudo-Hermitian structure θ on an oriented CR-manifold of type $(m,1)$, we define the Levi-form G_θ by

$$G_\theta(X,Y) = d\theta(JX,Y) \qquad \text{for every} \qquad X,Y\in\Gamma(H(M)).$$

We say that an oriented CR-manifold of type $(m,1)$ is nondegenerate if G_θ is nondegenerate for some choice of a pseudo-Hermitian structure θ on M. If G_θ is positive definite for some θ, the CR-manifold is said to be strictly pseudoconvex.

Remark 7.2.1. *Any two pseudo-Hermitian structures θ and $\hat\theta$ are related by*

$$\hat\theta = f\theta, \tag{7.2}$$

for some nowhere zero function $f : M \to \mathbb{R}$. Then, G_θ is nondegenerate if and only if $G_{\hat\theta}$ is nondegenerate. Hence nondegeneracy is a CR-invariant property, i.e. it is invariant under a transformation (7.2). But, strict pseudoconvexity is not a CR-invariant property. Indeed, if G_θ is positive definite, then $G_{-\theta}$ is negative definite.

Proposition 7.2.2. *Let M^n be an oriented nondegenerate CR-manifold of type $(m,1)$ and θ a pseudo-Hermitian structure on M. There is a unique globally defined nowhere zero tangent vector field T on M, such that*

$$\theta(T) = 1 \qquad \textit{and} \qquad T\lrcorner d\theta = 0.$$

The vector field T defines the characteristic direction of M and we have

$$TM = H(M)\oplus\mathbb{R}T.$$

Under the same conditions as in Proposition 7.2.2, we define a semi-Riemannian metric g_θ by

$$g_\theta(X,Y) = G_\theta(X,Y), \qquad g_\theta(X,T) = 0, \qquad g_\theta(T,T) = 1,$$

for any $X,Y\in\Gamma(H(M))$. This is called the Webster metric of the nondegenerate CR-manifold M. If θ is chosen in such a way that G_θ is positive definite, then g_θ is a Riemannian metric on M.

Example 7.2.1 (The Heisenberg group). *The set* $\mathbf{H_m} = \mathbb{C}^m \times \mathbb{R}$ *is a group with group law*

$$(z,t).(w,s) = (z+w, t+s+2\mathrm{Im}\,\langle z,w\rangle),$$

where $(z,t) = (z_1,\dots,z_m,t)$, $(w,s) = (w_1,\dots,w_m,s)$ *and* $\langle z,w\rangle = \sum_{j=1}^m z_j\overline{w_j}$. *We consider the complex vector fields on* $\mathbf{H_m}$,

$$T_j = \frac{\partial}{\partial z_j} + i\,\overline{z_j}\frac{\partial}{\partial t},$$

where $\frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j})$ *and* $z_j = x_j + iy_j$, $1 \leqslant j \leqslant m$. *We define* $T_{1,0}\mathbf{H_m}$ *as the space spanned by the* T_j*'s, i.e.*

$$T_{1,0}\mathbf{H_m} = \oplus_{j=1}^m \mathbb{C}T_j.$$

It is a CR-structure of type $(m,1)$. *Next, we consider the real* 1*-form* θ *on* $\mathbf{H_m}$ *defined by*

$$\theta = dt + i\sum_{j=1}^m \Big(z_j d\overline{z_j} - \overline{z_j}dz_j\Big).$$

Then, θ *is a pseudo-Hermitian structure on* $\mathbf{H_m}$ *and* G_θ *is nondegenerate and positive definite [DT]. Hence* $\mathbf{H_m}$ *is a strictly pseudoconvex CR-manifold of type* $(m,1)$. *We may easily check that* $T = \frac{\partial}{\partial t}$ *defines the characteristic direction.*

On an oriented strictly pseudoconvex CR-manifold M^{2m+1} of type $(m,1)$, the complexified tangent bundle $T^{\mathbb{C}}M$ can be written

$$T^{\mathbb{C}}M = T_{1,0}M \oplus T_{0,1}M \oplus \mathbb{C}T.$$

We set $\Lambda_H^{p,q}M = \Lambda^p(T^*_{1,0}M)\otimes\Lambda^q(T^*_{0,1}M)$. It is the complex bundle of forms of type (p,q). The bundle $K_M = \Lambda_H^{m,0}M$ is called the canonical bundle of M.

Proposition 7.2.3. [*Pet05*] *Every strictly pseudoconvex CR-manifold* M^{2m+1} *of hypersurface type has a canonical* Spin^c *structure whose* Spin^c *bundle can be identified with the bundle* $\Lambda_H^{0,*}M = \oplus_{r=0}^m \Lambda_H^{0,r}M$ *and whose auxiliary line bundle is given by* K_M^{-1}.

Obviously, Every strictly pseudoconvex CR-manifold M^{2m+1} of hypersurface type has also an anti-canonical Spin^c structure whose Spin^c bundle can be identified with the bundle $\Lambda_H^{*,0}M = \oplus_{r=0}^m \Lambda_H^{r,0}M$ and whose auxiliary line bundle is given by K_M.

For any other Spin^c structure, with auxiliary line bundle L, the Spin^c bundle can be written [Pet05]:

$$\Sigma M = \Lambda_H^{0,*}M \otimes \mathcal{L},$$

where $\mathcal{L}^2 = K_M \otimes L$. Moreover, the action of the 2-form $d\theta$ via Clifford multiplication gives the orthogonal splitting [Baum81]:

$$\Sigma M = \oplus_{r=0}^m \Sigma_r M,$$

where $\Sigma_r M$ is the eigenbundle corresponding to the eigenvalue $i(m-2r)$ of $d\theta$ [Baum81]. As in the complex case, we have $Z\cdot\psi \in \Gamma(\Sigma_{r+1}M)$ and $\overline{Z}\cdot\psi \in \Gamma(\Sigma_{r-1}M)$ for any $Z \in \Gamma(T_{1,0}M)$ and $\psi \in \Gamma(\Sigma_r M)$. We point out that for the canonical Spin^c structure, the subbundle $\Sigma_0 M$ trivial, i.e. $\Sigma_0 M = \Lambda_H^{0,0}M$.

7.3 CR-structures and complex structures via Spin^c spinors

Let (M^n, g) be a Riemannian Spin^c manifold. We call a nowhere zero spinor field ψ transversal if it defines a distribution D of constant rank with fiber at every point x given by D_x, where

$$\mathrm{D}_x = \{X \in T_xM \mid X \cdot \psi = iY \cdot \psi, \text{ for some } Y \in T_xM \setminus \{0\}\}.$$

Multiplying the defining equation $X \cdot \psi = iY \cdot \psi$ by i gives

$$Y \cdot \psi = -iX \cdot \psi.$$

Setting $JX = -Y$ leads to a well defined endomorphism J of D such that $J^2 = -\mathrm{Id}$. This means that the rank of D is even, say $2m$. Moreover, this almost structure is orthogonal. In fact, for every $X \in \Gamma(\mathrm{D})$, it follows

$$X \cdot JX \cdot \psi = -i|X|^2\psi \quad \text{and} \quad JX \cdot X \cdot \psi = i|JX|^2\psi.$$

Hence,

$$-2g(X, JX)\psi = X \cdot JX \cdot \psi + JX \cdot X \cdot \psi = i(|JX|^2 - |X|^2)\psi,$$

so that $g(X, JX) = 0$ and $|X| = |JX|$, i.e. J is orthogonal.

A transversal spinor is called m-transversal or of rank m if the rank of D is $2m$. When $\mathrm{D} = TM$, i.e. $n = 2m$, an $m = \frac{n}{2}$-transversal spinor is called a pure spinor. We should point out that our definitions of pure and transversal spinors will make no emphasis on isotropic subspaces (see [LM89]). From the definition of m-transversal spinors, we have:

Lemma 7.3.1. *Let (M^n, g) be a Riemannian* Spin^c *manifold carrying an m-transversal spinor field ψ, then M admits an almost CR-structure of type $(m, k = n - 2m)$ given by* (D, J).

Definition 7.3.1. *An m-transversal spinor field ψ is called integrable if and only if*

$$\overline{Z} \cdot \nabla_{\overline{W}}\psi - \overline{W} \cdot \nabla_{\overline{Z}}\psi = 0,$$

for every $Z, W \in \Gamma(T_{1,0}M)$, where $T_{1,0}M$ is the bundle of complex rank m defining the almost CR-structure (D, J) *of type $(m, l = n - 2m)$.*

Proposition 7.3.1. *Consider (M^n, g) a Riemannian* Spin^c *manifold admitting an m-transversal spinor field ψ. Then, ψ is integrable if and only if the almost CR-structure* (D, J) *is a CR-structure of type $(m, k = n - 2m)$.*

Proof. For any $X \in \Gamma(\mathrm{D})$, we have $(X + iJX) \cdot \psi = 0$. Differentiating this identity, we get

$$\nabla_Y X \cdot \psi + X \cdot \nabla_Y\psi + i\nabla_Y(JX) \cdot \psi + iJX \cdot \nabla_Y\psi = 0, \tag{7.3}$$

where $Y \in \Gamma(\mathrm{D})$. Substituting X with Y and Y with X gives

$$\nabla_X Y \cdot \psi + Y \cdot \nabla_X \psi + iJY \cdot \nabla_X \psi + i\nabla_X (JY) \cdot \psi = 0. \tag{7.4}$$

Take Equation (7.4) and substract (7.3) to get

$$\begin{aligned}[X,Y] \cdot \psi &= -i\nabla_X(JY) \cdot \psi + i\nabla_Y(JX) \cdot \psi - iJY \cdot \nabla_X \psi + iJX \cdot \nabla_Y \psi \\ &\quad + X \cdot \nabla_Y \psi - Y \cdot \nabla_X \psi. \end{aligned} \tag{7.5}$$

Substituting X with JX, and Y with JY gives

$$\begin{aligned}[JX,JY] \cdot \psi &= i\nabla_{JX} Y \cdot \psi - i\nabla_{JY}(X) \cdot \psi + iY \cdot \nabla_{JX} \psi - iX \cdot \nabla_{JY} \psi \\ &\quad + JX \cdot \nabla_{JY} \psi - JY \cdot \nabla_{JX} \psi. \end{aligned} \tag{7.6}$$

Finally, subtracting (7.6) from (7.5), we get

$$\begin{aligned}([X,Y] - [JX,JY]) \cdot \psi &= -i[X,JY]\psi - i[JX,Y] \cdot \psi \\ &\quad + (X + iJX) \cdot \nabla_{Y+iJY} \psi - (Y + iJY) \cdot \nabla_{X+iJX} \psi. \end{aligned}$$

If ψ is integrable, then

$$J([X,Y] - [JX,JY]) = [X,JY] + [JX,Y],$$

and, on the other hand, $[X,Y] - [JX,JY] \in \Gamma(\mathrm{D})$. Hence (D, J) is a CR-structure of type (m,k). Conversely, if the almost CR-structure (D, J) is a CR-structure, the spinor ψ is integrable.

Remark 7.3.1. *If $\mathrm{D} = TM$, i.e. if the spinor ψ is a pure spinor field, Proposition 7.3.1 means that the almost complex structure J is a complex structure on M if and only if the pure spinor field ψ is integrable.*

Next, we will consider separately m-transversal spinors for $n \neq 2m$ and for $n = 2m$ (pure spinors).

7.3.1 CR-structures via Spin^c structures

We will focus our attention on oriented Riemannian manifolds of dimension $n = 2m+1$ carrying an m-transversal integrable spinor ψ. In this case, $TM = \mathrm{D} \oplus \mathrm{D}^\perp$, where $\mathrm{D}^\perp$ is a trivial real line bundle over M and hence there exists a global 1-form θ, called a *transversal structure*, such that $\mathrm{D} = \ker \theta$. The transversal form is then given by $T_\theta(X,Y) = d\theta(JX,Y)$ for all $X, Y \in \Gamma(\mathrm{D})$.

Definition 7.3.2. *Let M be a Riemannian manifold of real dimension $2m+1$ carrying an m-transversal integrable spinor field ψ. The spinor ψ is called strictly pseudoconvex if T_θ is positive definite for some choice of θ.*

Proposition 7.3.2. *A differentiable manifold M^{2m+1} is a Riemannian* Spin^c *manifold carrying an m-transversal integrable strictly pseudoconvex spinor field if and only if M is a strictly pseudoconvex CR-manifold of type $(m,1)$.*

Proof. If M is a Riemannian Spin^c manifold carrying an m-transversal integrable spinor field ψ, then by Proposition 7.3.1, M has a CR-structure of type $(m,1)$. Since ψ is strictly pseudoconvex, T_θ is positive definite for some choice of θ. Hence G_θ is also positive definite. Now, if M is a strictly pseudoconvex CR-manifold of type $(m,1)$, the canonical Spin^c structure on (M, g_θ) carries an m-transversal integrable spinor field $\psi \in \Gamma(\Sigma_0 M = \Lambda^{0,0}_{\mathrm{H}} M)$. Moreover, ψ is strictly pseudoconvex since T_θ is positive definite.

Examples 7.3.1. *1. The Heisenberg group* $\mathbf{H_m}$ *of dimension* $2m+1$ *is a strictly pseudoconvex nondegenerate CR-manifold of type* $(m,1)$ *and hence it carries* m*-transversal integrable spinor fields.*
2. Every Sasakian manifold M^{2m+1} *is a strictly pseudoconvex CR-manifold and hence it carries* m*-transversal integrable spinor fields.*

Now we come to a point in which we can start adding familiar conditions on the spinors, such as being parallel in certain directions or being Killing in certain directions.

Parallelness conditions

We have three choices: ψ is parallel in all directions, ψ is parallel in horizontal directions D or ψ is parallel in the vertical direction $\mathrm{D}^\perp$. The first possibility would tell us that the manifold admits a parallel spinor, which would imply that locally M is a product of a Kähler manifold and a Spin manifold carrying parallel spinors [Moro97]. The second possibility gives the following:

Proposition 7.3.3. *Let* (M^n, g) *be a Riemannian* Spin^c *manifold admitting an* m*-transversal spinor* ψ *such that*

$$\nabla_Y \psi = 0, \quad \textit{for} \quad Y \in \Gamma(\mathrm{D}).$$

Then the orthogonal almost complex structure J *on* D *is parallel in the directions of* D*, i.e.*

$$\nabla J(X,Y) = 0, \quad \textit{for} \quad X, Y \in \Gamma(\mathrm{D}).$$

Proof. For every $X \in \Gamma(\mathrm{D})$, the covariant derivative of $X \cdot \psi = -iJX \cdot \psi$, gives $\nabla_Y(X \cdot \psi) = \nabla_Y X \cdot \psi = -i\nabla_Y(JX) \cdot \psi$. Hence,

$$\nabla_Y X \cdot \psi = -i\nabla_Y(JX) \cdot \psi. \tag{7.7}$$

Then $\nabla_Y X \in \Gamma(\mathrm{D})$ and $J(\nabla_Y X) = \nabla_Y(JX)$. This implies

$$\nabla J(X,Y) = \nabla_Y(JX) - J(\nabla_Y X) = 0.$$

The third possibility (ψ is parallel in the vertical direction $\mathrm{D}^\perp$) will be considered with the partial Killing condition.

Partial Killing condition

First, we need the following Lemma:

Lemma 7.3.2. *Let M^n be a Riemannian* $\mathrm{Spin^c}$ *manifold.*

1. *ψ is an m-transversal spinor if and only if for every $u \in \Gamma(\mathrm{D}^\perp)$, $u \cdot \psi$ is an m-transversal spinor.*

2. *Let $u \in \Gamma(TM)$ be such that for every $X \in \Gamma(\mathrm{D})$, we have*

$$X \cdot u \cdot \psi = -iJX \cdot u \cdot \psi.$$

Then $u \in \Gamma(\mathrm{D}^\perp)$.

Proof. 1. Using that $X \cdot\ u = -u \cdot X$ and $JX \cdot\ u = -u \cdot JX$ for every $X \in \Gamma(\mathrm{D})$, we get the result.

2. Assume that $X \cdot u \cdot \psi = -iJX \cdot u \cdot \psi$ for every $X \in \Gamma(\mathrm{D})$. Since $X \in \Gamma(\mathrm{D})$, we have $X \cdot \psi = -iJX \cdot \psi$. Thus,

$$-2g(u, X)\psi = 2ig(u, JX)\psi,$$

so that $g(u, X) = -ig(u, JX)$,i.e. $g(u, X) = g(u, JX) = 0$.

Proposition 7.3.4. *Let M be a Riemannian* $\mathrm{Spin^c}$ *manifold admitting an m-transversal spinor field ψ such that*

$$\nabla_u\psi = \lambda u \cdot \psi, \quad \textit{for all} \quad u \in \Gamma(\mathrm{D}^\perp) \quad \textit{and some} \quad \lambda \in \mathbb{R},$$

i.e. ψ is a Killing or a parallel spinor field in the directions orthogonal to the distribution D*. Then,*

1. *The orthogonal almost complex structure J on* D *is parallel in the directions orthogonal to* D*,*

$$\nabla J(X, u) = 0, \quad \textit{for} \quad X \in \Gamma(\mathrm{D}), u \in \Gamma(\mathrm{D}^\perp).$$

2. *The orthogonal distribution $\mathrm{D}^\perp$ is integrable:*

$$[\mathrm{D}^\perp, \mathrm{D}^\perp] \subset \mathrm{D}^\perp,$$

i.e. the manifold M is foliated.

Proof. 1. For every $X \in \Gamma(\mathrm{D})$, take the covariant derivative of $X \cdot \psi = -iJX \cdot \psi$, we get

$$\nabla_u X \cdot \psi + X \cdot \nabla_u\psi = -i\nabla_u(JX) \cdot \psi - iJX \cdot \nabla_u\psi,$$

for every $u \in \Gamma(\mathrm{D}^{\perp})$. If ψ is a parallel spinor field ($\lambda = 0$), the result holds. If ψ is a Killing spinor field ($\lambda \neq 0$), we have

$$\nabla_u X \cdot \psi + \lambda X \cdot u \cdot \psi = -i\nabla_u(JX) \cdot \psi - i\lambda JX \cdot u \cdot \psi.$$

By Lemma 7.3.2, we know that $X \cdot u \cdot \psi = -iJX \cdot u \cdot \psi$, so that

$$\nabla_u X \cdot \psi = -i\nabla_u(JX) \cdot \psi.$$

This means that $J(\nabla_u X) = \nabla_u(JX)$, i.e. $\nabla J(X, u) = 0$.

2. By Lemma 7.3.2 and since ψ is an m-transversal spinor field, $u \cdot \psi$ is also an m-transversal spinor field for every $u \in \Gamma(\mathrm{D}^{\perp})$. Hence,

$$X \cdot u \cdot \psi = -iJX \cdot u \cdot \psi.$$

Take the covariant derivative in the direction of $v \in \Gamma(\mathrm{D}^{\perp})$,

$$\begin{aligned} \nabla_v X \cdot u \cdot \psi + X \cdot \nabla_v u \cdot \psi + X \cdot u \cdot \nabla_v \psi &= -i\nabla_v(JX) \cdot u \cdot \psi - iJX \cdot \nabla_v u \cdot \psi \\ &\quad -iJX \cdot u \cdot \nabla_v \psi. \end{aligned} \tag{7.8}$$

The first assertion of this proposition implies that $\nabla_v X \in \Gamma(\mathrm{D})$, so $\nabla_v X \cdot u \cdot \psi = i\nabla_v(JX) \cdot u \cdot \psi$. Hence, if ψ is a parallel spinor field, Equation (7.8) gives

$$X \cdot \nabla_v u \cdot \psi = -iJX \cdot \nabla_v u \cdot \psi.$$

If ψ is a Killing spinor field, we have

$$\begin{aligned} X \cdot u \cdot \nabla_v \psi &= \lambda X \cdot u \cdot v \cdot \psi \\ &= -i\lambda JX \cdot u \cdot v \cdot \psi \\ &= -iJX \cdot u \cdot \nabla_v \psi, \end{aligned}$$

and hence Equation (7.8) yields to

$$X \cdot \nabla_v u \cdot \psi = -iJX \cdot \nabla_v u \cdot \psi. \tag{7.9}$$

If the role of u and v is exchanged, then

$$X \cdot \nabla_u v \cdot \psi = -iJX \cdot \nabla_u v \cdot \psi. \tag{7.10}$$

Subtracting (7.10) from (7.9), we conclude

$$X \cdot [u, v] \cdot \psi = -iJX \cdot [u, v] \cdot \psi,$$

which means that $[u, v]$ is orthogonal to every $X \in \Gamma(\mathrm{D})$.

7.3.2 Complex structures via Spin^c spinors

In this subsection, we study pure spinors on Riemannian Spin^c manifolds. We recall that in this case, $n = 2m$, i.e. $\mathrm{D} = TM$. Moreover, having a pure spinor on a Riemannian Spin^c manifold M^{2m} implies that M has an almost complex structure and, if the pure spinor is integrable, the almost complex structure is a complex structure.

Proposition 7.3.5. *Let M be a differentiable manifold of real dimension n. The following statements are equivalent:*

1. *M is a Riemannian* Spin^c *manifold carrying a pure spinor ψ.*
2. *M is a Riemannian manifold (M, g) with an orthogonal almost complex structure J such that (M, J, g) is a Hermitian manifold.*
3. *M is a Riemannian manifold having an almost CR-structure of type $(m, 0)$ for some $m \in \mathbb{N}$.*

Proof. Assume that (M^n, g) is a Riemannian Spin^c manifold carrying a pure spinor ψ. Hence, M has an orthogonal almost complex structure J. Moreover, for any $X_1, X_2 \in \Gamma(TM)$, we have

$$\begin{aligned} g(X_1 + X_2, X_1 + X_2) &= g(J(X_1 + X_2), J(X_1 + X_2)) \\ &= g(JX_1, JX_1) + g(JX_2, JX_2) + 2g(JX_1, JX_2) \\ &= g(X_1, X_1) + g(X_2, X_2) + 2g(JX_1, JX_2). \end{aligned}$$

But $g(X_1 + X_2, X_1 + X_2) = g(X_1, X_1) + g(X_2, X_2) + 2g(X_1, X_2)$. Hence, $g(X_1, X_2) = g(JX_1, JX_2)$ and the Riemannian metric g is compatible with the almost complex structure J. Conversely, if (M^n, J, g) is a Hermitian manifold ($n = 2m$), we consider the canonical Spin^c structure on M. The line subbundle $\Sigma_0 M$ of the complex spinorial bundle ΣM is trivial. Hence, it has a nowhere zero global spinor field ψ. Since $\psi \in \Gamma(\Sigma_0 M)$, we have $\overline{Z} \cdot \psi = 0$ for every $Z \in \Gamma(T_{1,0}M)$, which means that

$$(X + iJX) \cdot \psi = 0 \quad \text{for any} \quad X \in \Gamma(TM).$$

Then, the first two statements are equivalent. Now, the last two statements are equivalent because, if M^n is an almost complex manifold, we have that n is even ($n = 2m$). The eigenbundle $T_{1,0}M$ corresponding to the eigenvalue i of J gives the desired almost CR-structure of type $(\frac{n}{2} = m, 0)$. Conversely, it is easy to see that an almost CR-manifold of type $(m, 0)$ is an almost complex manifold ([Ko-No69], p.121). Denote by J the almost complex structure and define the metric h by $h(X, Y) = g(X, Y) + g(JX, JY)$, for any $X, Y \in \Gamma(TM)$. Then (M^{2m}, J, h) is a Hermitian manifold.

Proposition 7.3.6. *Let M^n be a differentiable manifold. The following statements are equivalent:*

1. *M is a Riemannian* Spin^c *manifold carrying an integrable pure spinor.*

2. *M is a Riemannian manifold (M, g) having a complex structure J such that (M, J, g) is a Hermitian manifold.*

3. *M is a Riemannian manifold having a CR-structure of type $(m, 0)$.*

Proof. Let (M^n, g) be a Spin^c manifold carrying an integrable pure spinor field ψ. By Proposition 7.3.5 and Proposition 7.3.1, M admits a complex structure J such that (M, J, g) is a Hermitian manifold. Now, if (M, J, g) is a Hermitian manifold, by Proposition 7.3.5, the canonical Spin^c structure on M admits a pure spinor field $\psi \in \Gamma(\Sigma_0 M)$. By Proposition 7.3.1 and since J is a complex structure, the pure spinor field is integrable. Hence, the first two statements are equivalent. The last two statements are also equivalent because if (M, J, g) is a Hermitian manifold with a complex structure J, the eigenbundle $T_{1,0}M$ of $TM \otimes \mathbb{C}$ corresponding to the eigenvalue i of J gives the desired almost CR-structure of type $(\frac{n}{2} = m, 0)$. Because $T_{1,0}M$ is integrable, the almost CR-structure is a CR-structure of type $(m, 0)$. Conversely, a Riemannian manifold having a CR-structure of type $(m, 0)$ is a complex manifold ([Ko-No69], p.121). Denoting by J the complex structure, we get again that the metric h, defined by by $h(X, Y) = g(X, Y) + g(JX, JY)$ for any $X, Y \in \Gamma(TM)$, is compatible with the complex structure.

Corollary 7.3.1. *Let (M^n, g) be a Riemannian manifold. The manifold M has a* Spin^c *structure carrying a parallel pure spinor field ψ if and only if M is Kähler.*

Proof. Assume that M is a Spin^c manifold carrying a parallel pure spinor. First of all, since $X|\psi|^2 =< \nabla_X\psi, \psi > + < \psi, \nabla_X\psi >= 0$, the norm of ψ is constant. Since ψ is parallel and pure, ψ is integrable. Hence, by Proposition 7.3.6, M is a complex manifold. It remains to show that $\nabla J = 0$. By taking the covariant derivative of $X \cdot \psi = -iJX \cdot \psi$, we get

$$\begin{aligned}
\nabla_Y(X \cdot \psi) &= \nabla_Y X \cdot \psi + X \cdot \nabla_Y \psi = \nabla_Y X \cdot \psi \\
&= -i\nabla_Y(JX) \cdot \psi - iJX \cdot \nabla_Y \psi \\
&= -i\nabla_Y(JX) \cdot \psi.
\end{aligned}$$

Hence, $\nabla_Y X \cdot \psi = -i\nabla_Y(JX) \cdot \psi$. But $\nabla_Y X \cdot \psi = -iJ(\nabla_Y X) \cdot \psi$. It gives $(\nabla_Y JX - J(\nabla_Y X)) \cdot \psi = 0$ so that $\nabla J = 0$ and M is Kähler. Now if M is a Kähler manifold, then the canonical Spin^c structure carries parallel spinors (constant functions) lying in the trivial subbundle $\Sigma_0 M = \Lambda^{0,0} M$, hence they are pure.

Corollary 7.3.1 was proved by A. Moroianu in [Moro97] to classify simply connected Spin^c manifolds carrying parallel spinors. Using the classification of Spin^c manifolds having parallel spinors [Moro97], we deduce:

Corollary 7.3.2. *Let M be a simply connected irreducible Kähler manifold. The only* Spin^c *structures on M carrying a parallel pure spinor are the canonical and the anti-canonical ones.*

Remark 7.3.2. *For transversal spinor fields, we should point out that if we replace* $-J$ *by* J*, we have to consider the anti-canonical* Spin^c *structure in all the proofs. In this case, the integrability condition of a transversal spinor* ψ *is given by*

$$Z \cdot \nabla_W \psi - W \cdot \nabla_Z \psi = 0,$$

for every $Z, W \in \Gamma(T_{1,0}M)$*, where* $T_{1,0}M$ *is the subbundle of* $T^{\mathbb{C}}M$ *given by the almost CR-structure* (D, J)*.*

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Le sujet principal de cette thèse est d'exploiter les structures Spin^c dans le but d'étudier la géométrie de certaines sous-variétés. Dans un premier temps, nous commençons par établir des résultats de base pour l'opérateur de Dirac Spin^c. On donne ainsi des inégalités de type Hijazi en terme du tenseur d'énergie-impulsion. Ce tenseur intervient dans l'étude des variations du spectre de l'opérateur de Dirac et dans les équations de Dirac-Einstein. L'étude des hypersurfaces des variétés Spin^c permet de mieux comprendre ce tenseur puisque ce dernier est le tenseur de Weingarten de l'immersion. Étant des structures naturelles sur les variétés homogènes $\mathbb{E}(\kappa, \tau)$ de dimension 3, les structures Spin^c permettent d'aborder des problèmes riemanniens sur les hypersurfaces de ces variétés. En effet, on donne une correspondance de Lawson pour les surfaces à courbure moyenne constante de $\mathbb{E}(\kappa, \tau)$. Finalement, on caractérise les structures complexes et CR sur une variété par les structures Spin^c admettant un champ de spineurs spécial appelé un spineur pur ou bien un spineur transversal.

Special submanifolds of Spin^c manifolds:

In this thesis, we aim to make use of Spin^c geometry to study special submanifolds. We start by establishing basic results for the Spin^c Dirac operator. We give then inequalities of Hijazi type involving the energy-momentum tensor. Studying the energy-momentum tensor on a Spin^c manifold is related to several geometric situations. Indeed, it appears in the study of the variations of the spectrum of the Dirac operator and in the Einstein-Dirac equation. The study of hypersurfaces of Spin^c manifolds allows us for a better understanding of this tensor since it is the second fundamental form of the immersion. Being natural structures on the 3-homogeneous manifolds $\mathbb{E}(\kappa, \tau)$, Spin^c structures will be investigated in the study of some Riemannian problems on hypersurfaces of these manifolds. In fact, we prove a Lawson correspondence for constant mean curvature surfaces in $\mathbb{E}(\kappa, \tau)$. Finally, we characterize complex structures and CR structures by Spin^c structures admitting a special spinor, called pure spinor or transversal spinor.

Discipline : Mathématiques
Mots clés : géométrie spinorielle complexe, tenseur d'énergie-impulsion, opérateur de Dirac, valeurs propres, hypersurfaces, géométrie extrinsèque, géométrie CR.

Institut Élie Cartan Nancy
Laboratoire de Mathématiques
B. P. 239 54506 Vandœuvre-lès-Nancy Cedex