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Thèse

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ℓ-independence for a system of Motivic Representations

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Soit X une variété algébrique propre et lisse sur un corps de nombres $F \subset \mathbb{C}$. On suppose que le motif de Hodge absolu $h^i(X)$ appartient à la catégorie Tannakienne engendrée par les motifs des variétés abélienne. Quitte à remplacer F par une extension finie, on peut supposer que, la représentation galoisienne ℓ-adique associée à M se factorise comme $\rho_{M,\ell} : \Gamma_F \to G_M(\mathbb{Q}_\ell)$, où G_M est le groupe de Mumford-Tate de M. Fixons une valuation v de F. La restriction $\rho_{M,\ell|\Gamma_{F_v}}$ définit une représentation $'W_v \to G_{M/\mathbb{Q}_\ell}$ du groupe de Weil-Deligne. J-P Serre et J-M Fontaine (indépendamment) ont fait des conjectures qui indiques que pour tout ℓ, la représentation $'W_v \to G_{M/\mathbb{Q}_\ell}$ est définie sur $\mathbb{Q}$ et pour ℓ variable elles forment un système compatible des représentations. Sous certaines hypothèses supplémentaire, nous montrons que ceci est vrai, si X a bonne réduction en v ou réduction semi-stable en v.

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ℓ-independence for a system of Motivic Representations

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Contents

Introduction

Fix a number field F with an embedding $\tau : F \hookrightarrow \mathbb{C}$. Let v be a non-archimedean valuation on F, $\bar{F}$ a fixed algebraic closure of F, $\bar{v}$ an extension of v to $\bar{F}$. We denote F_v the completion of F at v and $\bar{F}_v$ the localization of $\bar{F}$ at $\bar{v}$. The residue fields of F_v and $\bar{F}_v$ are denoted as k_v and $\bar{k}_v$, respectively. Let $p > 0$ be the characteristic of k_v. We denote $\Gamma_{F_v} := \mathrm{Gal}(\bar{F}_v/F_v) \subset \Gamma_F := \mathrm{Gal}(\bar{F}/F)$, $I_{F_v} \subset \Gamma_{F_v}$ is the inertia subgroup and $\phi_v \in \mathrm{Gal}(\bar{k}_v/k_v)$ the Frobenius automorphism. Fix an arithmetic Frobenius $\Phi_v \in \Gamma_{F_v}$, i.e. an element which induces ϕ_v.

Consider a proper and smooth algebraic variety X defined over F_v. The group Γ_{F_v} acts naturally on the etale cohomology groups $V_l^i := H^i_{\acute{e}t}(X_{/\bar{F}_v}, \mathbb{Q}_l)$, for each prime number l and all positive integers i. This action gives rise to the representations $\rho_l^i : \Gamma_{F_v} \to \mathrm{GL}(V_l^i)$. It is a major theme in arithmetic geometry to determine to what extent the properties of these representations are independent of l. In order to answer these l-independence questions, one often has to restrict the above representations to the Weil group W_{F_v} of F_v. This is the subgroup formed by those elements of Γ_{F_v} which induce an integral power of ϕ_v in $\mathrm{Gal}(\bar{k}_v/k_v)$. We endow W_{F_v} with the topology determined by the condition that $I_{F_v} \subset W_{F_v}$ is an open subgroup having the topology inherited from its topology as a Galois group.

In what follows we assume that $l \neq p$. Now let X have good reduction at v, i.e that X extends to a proper and smooth scheme over the ring of integers of F_v. This implies that the inertia subgroup I_{F_v} acts trivially on the etale cohomology groups V_l^i. It is well known from the works of P. Deligne on the Weil conjectures [Del80], that in this case the character of the representation of W_{F_v} on each V_l^i has values in $\mathbb{Q}$ and is independent of l. Since the action of inertia is trivial, this amounts to a statement on the action of the subgroup of Γ_{F_v}, generated by Φ_v. We will summarise this by saying that the ρ_l^i are *defined over* $\mathbb{Q}$ and form a *compatible system* of representations of W_{F_v}, for a fixed i and variable l.

If the algebraic variety X is defined over F, then the above situation has a natural generalization. By using the embedding $\tau : F \hookrightarrow \mathbb{C}$ we can consider the i-th singular cohomology $V^i = H^i_B(X(\mathbb{C}), \mathbb{Q})$ of $X(\mathbb{C})$. Then $V^i_l = V^i \otimes_{\mathbb{Q}} \mathbb{Q}_l$. Now let $H \subseteq \mathrm{GL}(V^i)$ be a linear algebraic group over $\mathbb{Q}$. Suppose that $\mathrm{Im}(\rho^i_l) \subseteq H(\mathbb{Q}_l)$. In most of the cases that we consider this H would be the Mumford-Tate group (see precise definition later) of the absolute Hodge motive $h^i(X)$. Then we can ask if the $H(\overline{\mathbb{Q}_l})$-conjugacy class of $\rho^i_l(\Phi_v)$ is defined over $\mathbb{Q}$ and if it is independent of l. Consider the case where $H = \mathrm{GL}(V^i)$. Then Deligne's theorem (cited above) becomes a special case of this problem.

We can ask similar questions in the case where the algebraic variety X does not have good reduction. To describe this situation we first need some general notions. Consider any arbitrary quasi-unipotent l-adic representation of the form

$$\xi_l : \Gamma_{F_v} \to H(\mathbb{Q}_l). \tag{1}$$

Grothendieck's l-adic monodromy theorem ([Del73, 8.2], [Ill94] or [ST68]) tells us that, for a sufficiently small open subgroup of I_{F_v}, this action can be described as exponential of a single endomorphism N'_l, the monodromy operator. The restriction of ξ_l to the Weil group W_{F_v} can then be encoded by giving N'_l together with a representation ξ'_l of W_v which is trivial on an open subgroup of the inertia group. We will refer to such a pair (N'_l, ξ'_l) as *a representation of the Weil-Deligne group* $'W_v$ of F_v. We often denote it simply as $'W_v \to H_{/\mathbb{Q}_l}$. In fact, $N'_l \in \mathrm{Lie}(H_{/\mathbb{Q}_l})$. To explain more about ξ'_l we need some preparation. Let $t_l : I_{F_v} \to \mathbb{Z}_l(1)$ be the surjection defined by $\sigma \mapsto \left(\frac{\sigma(\pi^{1/l^m})}{\pi^{1/l^m}}\right)_m$ for a prime element π of F_v. It is known that t_l is independent of the choice of π and its system of l^m-th roots π^{1/l^m}. Let $w \in W_{F_v}$ induce an integral power $\phi_v^{\alpha(w)}$ of ϕ_v in $\mathrm{Gal}(\bar{k}_v/k_v)$. Then, ξ'_l is given as :

$$\xi'_l(w) = \xi_l(w)\exp(-N'_l t_l(\Phi_v^{-\alpha(w)} w)). \tag{2}$$

For a fixed l, we say that the representation $'W_{F_v} \to H_{/\mathbb{Q}_l}$ is *defined over* $\mathbb{Q}$, if for every algebraically closed field $\Omega \supset \mathbb{Q}_l$, the base extension of (N'_l, ξ'_l) to Ω is conjugate under $H(\Omega)$ to all its images under $\mathrm{Aut}_{\mathbb{Q}}(\Omega)$. More precisely this means that for every $\sigma \in \mathrm{Aut}_{\mathbb{Q}}(\Omega)$, there exists a $g \in H(\Omega)$ such that

$$\sigma\xi'_{l/\Omega} = g \cdot \xi'_{l/\Omega} \cdot g^{-1} \quad \text{and} \quad \sigma(N'_l \otimes_{\mathbb{Q}_l} 1) = \mathrm{Ad}(g)(N'_l \otimes_{\mathbb{Q}_l} 1) \tag{3}$$

where $\xi'_{l/\Omega} : W_K \to H_{/\mathbb{Q}_l}(\Omega)$ is the extension of scalars and

$$N'_l \otimes_{\mathbb{Q}_l} 1 \in (\mathfrak{h} \otimes_{\mathbb{Q}} \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \Omega = \mathfrak{h} \otimes \Omega,$$

$\mathfrak{h}$ being the Lie-algebra of H.

Varying l over all primes different from p, we say that the representations (N'_l, ξ'_l) form a *compatible system of representations of* $'W_{F_v}$ if for every pair (l, l') of prime numbers and every algebraically closed field $\Omega \supset \mathbb{Q}_l, \mathbb{Q}_{l'}$, the base extension to Ω of the l-adic representation of $'W_{F_v}$ is $H(\Omega)$-conjugate to the base extension of the l'-adic representation. In terms of pairs (N'_l, ξ'_l) and $(N'_{l'}, \xi'_{l'})$, this means that there exists some $g \in H(\Omega)$ such that

$$\xi'_{l/\Omega} = g \cdot \xi'_{l'/\Omega} \cdot g^{-1} \quad \text{and} \quad N'_l \otimes_{\mathbb{Q}_l} 1 = \mathrm{Ad}(g)(N'_{l'} \otimes_{\mathbb{Q}_{l'}} 1) \in \mathfrak{h} \otimes \Omega. \tag{4}$$

These notions originated in the work of P. Deligne (see [Del73]). See also [Fon94].

Now return to the case of the algebraic variety X and the representations ρ^i_l. We want to use the above notions to study the case where X does not have good reduction at v. It is known that the representations ρ^i_l are quasi-unipotent ([GRO72]). Suppose again that X is defined over F and in (7) take H as $\mathrm{GL}(V^i)$ and ξ_l as ρ^i_l. Then it is conjectured by J-M Fontaine that for a fixed i and variable l, the (N'_l, ρ'^i_l) form a compatible system of representation of $'W_{F_v}$ defined over $\mathbb{Q}$, without any assumptions on the reduction of X. Here we refer to [Fon94, 2.4.3] conjecture C_{WD}, for a statement also covering p-adic representations.

The conjecture of Fontaine on the l independence of the representation of the Weil-Deligne group is actually hinged on the monodromy-weight conjecture, see [Ill94], [Ito04], [RZ82]. This subtle conjecture is somewhat more accessible under the hypothesis that X has a semi-stable reduction. If X is a scheme over $S := \mathrm{Spec}A$, where A is a discrete valuation ring, then we say that X has *semi-stable reduction* if etale locally on X and S, X is S-isomorphic to $\mathrm{Spec}A[T_1, ..., T_n]/(T_1, ...T_r - \pi)$ for some $r \geq 0$, and π being an uniformizing parameter. This condition is equivalent to the condition that X is regular, the generic fiber of X is smooth, and the closed fiber of X is a reduced divisor with normal crossings on X.

Now If X/F_v has semistable reduction over the ring of integers of F_v then it is known that ρ^i_l is unipotent on the inertia subgroup I_{F_v} ([Ill94, 3.3] or [RZ82]). In this situation the action of inertia on V^i_l is determined by the monodromy operator corresponding to ρ_l^i.

One may ask the above questions for motives instead of just algebraic varieties. At the core of the Grothendieck's theory of motives there is a conjectural universal cohomology $h(X)$ for algebraic varieties X/F, with values in a certain $\mathbb{Q}$-linear Tannakian category. The category of motives is intended to be the target of this functor h. Every (Weil) cohomolgy functor factors through h. The category of motives is generated by objects $h(X)$, for X running through proper and smooth algebraic varieties over F, and the Tate motive $\mathbb{Q}(1)$. There are various constructions of motives depending on how the morphisms are defined in this category. In Grothendieck's category of motives the morphisms are defined by algebraic cycles. However, many of the desired properties of this category depend on unknown properties (standard conjectures) of algebraic cycles. P. Deligne and J. Milne gave a construction of a category of motives, where morphisms are defined using absolute Hodge cycles. In this thesis we will deal with this unconditional theory of motives.

Let $\mathcal{M}_F$ denote the category of absolute Hodge motives as defined in [DMOS82]. It is a semisimple Tannakian category. The questions that we asked before are now stated in terms of a Mumford-Tate group of a motive in $\mathcal{M}_F$. A Mumford-Tate group G_M of any object M in $\mathcal{M}_F$ is defined as the automorphism group of a Betti realization functor restricted to the Tannakian category generated by M and the Tate motive (cf. definition 1.3.13 of this document). It is an linear algebraic group over $\mathbb{Q}$. Now for each prime number l, let $H_l(M)$ denote the l-adic realization of the motive M. Then the action of the Galois group Γ_F on $H_l(M)$ gives us a l-adic Galois representation

$$\rho_{M,l} : \Gamma_F \to \mathrm{GL}(H_l(M)).$$

It is known that $\rho_{M,l}$ factors through $G_M(\mathbb{Q}_l)$. This implies that the corresponding representations of the Weil-Deligne group of F_v factors through $G_{M/\mathbb{Q}_l}$. The above mentioned l-independence conjecture of Fontaine, for objects in the Tannakain category generated by M, is equivalent to the statement that the representations $'W_{F_v} \to G_{M/\mathbb{Q}_l}$ induced by $\rho_{M,l}$, form a compatible system (as defined before) *with values in* G_M. Here, v is fixed and l runs through set of primes different from p.

We say that a motive $M \in \mathrm{Ob}(\mathcal{M}_F)$ has good reduction at v if $\rho_{M,l}$ is trivial on the inertia subgroup for every $l \neq p$. For motives with good reduction J-P Serre [Ser94] has a more geometric formulation of the conjecture of Fontaine.

Conjecture. *If M is a motive with good reduction at v. Then there exists a class*

$$\mathrm{CL_M Fr_v} \in \mathrm{Conj}(G_M)(\mathbb{Q})$$

such that $\mathrm{Cl}(\rho_{M,l}(\Phi_v)) = \mathrm{CL_M Fr_v}$ *for every prime number* $l \neq p$.

Here $\mathrm{Conj}(G_M)$ is the universal categorical quotient of G_M for its action on itself by conjugation and $\mathrm{Cl} : G_M \to \mathrm{Conj}(G_M)$ is the corresponding quotient map. Serre has in fact conjectured this for any unconditional category of motives.

In this thesis, I study this conjecture and the motivic version of the conjecture of Fontaine, in the category of absolute Hodge motives $\mathcal{M}_F$. We will need to work in the Tannakian subcategory of motives $\mathcal{M}_F^{av}$ of $\mathcal{M}_F$ generated by the motives of abelian varieties. It is known that $\mathcal{M}_F^{av}$ contains the motives of $K3$-surfaces, unirational varieties of dimension ≤ 3, curves and Fermat hypersufaces (see [DMOS82, II.6.26]). Recall that we had an embedding $\tau : F \hookrightarrow \mathbb{C}$. A fundamental result of Deligne states that for varieties whose motives $M \in \mathcal{M}_F^{av}$, Hodge cycles relative to τ are absolutely Hodge ([DMOS82, II.6.27]). This implies in particular that for a motive $M \in \mathcal{M}_F^{av}$, the identity component G_M° of the Mumford-Tate group of the motive M coincides with the Mumford-Tate group of the Hodge structure $H_\tau(M)$. Here, H_τ is Betti realization functor and G_M is the automorphism group of H_τ restricted to the Tannakian subcategory generated by M.

Under some additional hypotheses described below, we prove special cases of the above mentioned conjectures. First we prove the conjecture of Serre for any algebraic variety whose motive belongs to $\mathcal{M}_F^{av}$. Second, we show the conjecture of Fontaine holds true for the representations $'W_v \to G_{M/\mathbb{Q}_l}$, if G_M is the Mumford-Tate group of a motive of an abelian variety, a K3-surface, a curve or a Fermat hypersuface, with semi-stable reduction at v.

First of all, in our main theorems (see Theorems A and D below) we need to assume that the base field is sufficiently large, to guarantee that the Mumford-Tate groups are connected and even that the Frobenius elements at the given place of F is *weakly neat*. The weakly neat condition means that 1 is the only root of unity that occurs as the quotient $\lambda\mu^{-1}$ of any eiegnvalues λ, μ of $\rho_{M,l}(\Phi_v)$ acting on $H_l(M)$. This is a variant of the concept of 'neat' elements that appears in classical literature on arithmetic groups, see [Bor01].

Secondly, in certain cases we prove the conjugacy only in a group which is larger than the Mumford-Tate group. Only certain factors of $G^{der}_{M/\bar{\mathbb{Q}}}$ of type D are affected by this modification. We denote this algebraic group by $G_M^\natural$. Since our questions deal with conjugations, it is sufficient to look at the adjoint groups. The adjoint group $G_M^{\natural ad}$ is a 'natural extension' of $G_{M/\bar{\mathbb{Q}}}^{ad}$ and $G_{M/\bar{\mathbb{Q}}}^{ad} \subset G_M^{\natural ad}$. There is an action of $G_M^{\natural ad}$ on $G_{M/\bar{\mathbb{Q}}}$, extending the adjoint action of $G_{M/\bar{\mathbb{Q}}}^{ad}$ on $G_{M/\bar{\mathbb{Q}}}^{der}$ and with $G_M^{\natural ad}$ acting trivially on the center of $G_{M/\bar{\mathbb{Q}}}$. The algebraic group $G_M^{\natural ad}$ also acts on the Lie algebra $\mathrm{Lie}(G_M) \otimes \bar{\mathbb{Q}}$, through the adjoint representation. An important fact to be noted is that *the adjoint of the Mumford-Tate group $G_{M/\bar{\mathbb{Q}}}^{ad}$ coincides with the identity component of $G_M^{\natural ad}$*. Enlarging the groups obviously weakens the notion of conjugacy. We will also need the universal categorical quotient for the adjoint action of $G_M^{\natural ad}$ on G_M, denoted here by $\mathrm{Conj}'(G_M)$.

The precise results of this thesis are the following :

Theorem A. *Let X be a smooth proper algebraic variety over F with good reduction at v and assume that the motive $M := h^i(X) \in \mathrm{Ob}\mathcal{M}_F^{av}$ for $i \in \mathbb{N}$. Suppose that G_M is connected and that there exists a prime number $l \neq p$ such that $\rho_{M,l}(\Phi_v)$ is weakly neat. Then there exists a conjugacy class*

$$\mathrm{Cl}_M\mathrm{Fr}_v \in \mathrm{Conj}'(G_M)(\mathbb{Q})$$

such that $\mathrm{Cl}(\rho_{M,l}(\Phi_v)) = \mathrm{Cl}_M\mathrm{Fr}_v$*, for every prime number $l \neq p$.*

In particular this theorem holds when X is an abelian variety, a $K3$-surface, an unirational variety of dimension ≤ 3, a curve, a Fermat hypersurface or any product of these algebraic varieties. This is theorem 4.2.6 of the thesis.

The Theorem A actually follows from the following theorem which establishes the result of the theorem over a finite extension of the base field, without the assumption of weak neatness or connectedness of the Mumford-Tate group of the motive M.

Theorem B. *Let X be as in theorem A. Then there exists a finite extension F' of F and a valuation v' on F' extending v and a conjugacy class*

$$\mathrm{Cl}_M\mathrm{Fr}_{v'} \in \mathrm{Conj}'(G_{M_{F'}})(\mathbb{Q})$$

such that $\mathrm{Cl}(\rho_{M,l}(\Phi_{v'})) = \mathrm{Cl}_M\mathrm{Fr}_{v'}$*, $\forall\, l$ with $v(l) = 0$.*

Note that here $\Phi_{v'} = \Phi_v^d$, where d is the residual degree of the extension $F'_{v'}/F_v$. We will first discuss how to prove the theorem B. It is theorem 4.2.1 of the main text.

Outline of the proof of Theorem B. The main steps of the proof are following

(a) As $M \in \mathrm{Ob}(\mathcal{M}_F^{av})$, by the existence of weak-Mumford Tate lifts (see [Noo06]) for abelian motives we have an abelian variety $B_{/\bar{F}}$ and an abelian variety $C_{/\bar{F}}$ with complex multiplication such that

$$M_{\bar{F}} \in \langle h^1(B_{/\bar{F}}), h^1(C_{/\bar{F}}), \mathbb{Q}(1)\rangle \tag{5}$$

This inclusion means that $M_{\bar{F}}$ belongs to the Tannakian subcategory of $\mathcal{M}_{\bar{F}}$, generated by $B_{/\bar{F}}$ and $C_{/\bar{F}}$ and the Tate motive $\mathbb{Q}(1)$.

Moreover we can assume that $B_{/\bar{F}}$ is a *tractable* abelian varietiey (see [Noo09], [Noo10] for the definition). Tractable abelian varities have also occured earlier in the works of Deligne (see [Del79, 2.3,2.4]).
Let $G_{B\times C}$ denote the Mumford-Tate group of the motive $h^1(B_{/\bar{F}} \times C_{/\bar{F}})$. Since $\mathcal{M}_F$ is a Tannakian category, (5) gives rise to a morphism

$$\theta : G_{B\times C} \to G_{M_{\bar{F}}}$$

(b) The Galois group $\mathrm{Gal}(\bar{F}/F)$ acts on the vector space of absolute Hodge cycles (on X, B and C) through a finite quotient (see [DMOS82, I.2.9]). This implies that we can find a a large enough finite extension F' of F, such that we have abelian varieties $B_{/F'}$ and $C_{/F'}$ with $B_{/F'} \otimes \bar{F} = B_{/\bar{F}}$ and $C_{/F'} \otimes \bar{F} = C_{/\bar{F}}$ satisfying the following:

$$M_{F'} \in \langle h^1(B_{/F'}), h^1(C_{/F'}), \mathbb{Q}(1)\rangle.$$

(c) Now denote by G_B and G_C the Mumford-Tate groups of the motives $h^1(B_{/F'})$ and $h^1(C_{/F'})$, respectively. By taking an appropriate finite extension of F we may assume that G_B and G_C are connected. After taking another finite base extension (denoted again by F') if necessary, we may suppose that $G_{M_{F'}}$ is connected and that the Mumford-Tate group of $h^1(B_{/F'} \times C_{/F'})$ is $G_{B\times C}$. Thus we have $G_{M_{F'}} = G_{M_{\bar{F}}}$. Then, for the representations of the Galois group $\Gamma_{F'} = \mathrm{Gal}(\bar{F}/F')$ it can be shown that the following diagram is commutative:

$$\begin{array}{ccc} & & (G_B \times G_C)(\mathbb{Q}_l) \\ & \nearrow^{(\rho_{B,l},\rho_{C,l})} & \uparrow i \\ \Gamma_{F'} & \xrightarrow{\rho_{B\times C,l}} & G_{B\times C}(\mathbb{Q}_l) \\ & \searrow_{\rho_{M,l}} & \downarrow \theta \\ & & G_{M_{F'}}(\mathbb{Q}_l) \end{array} \tag{6}$$

Here i denotes the map induced by the inclusion $G_{B\times C} \subseteq G_B \times G_C$

(d) Then we prove that the above inclusion $G_{B\times C} \subseteq G_B \times G_C$ induces a closed immersion of algebraic varieties

$$\mathrm{Conj}'(G_{B\times C}) \to \mathrm{Conj}'(G_B \times G_C).$$

(e) Next we use the fact that X have good reduction at v to show that the abelian variety $B_{/F'} \times C_{/F'}$ has potential good reduction at v. Here we need certain criteria (Néron-Ogg-Shafarevich) for potential good reduction from [ST68]. This implies that after making another base extension of F (which we again denote by F') if necessary we may suppose that $B_{/F'}$ and $C_{/F'}$ has good reduction at v'.

(f) Then for $* = B, C$, the results of ([Noo09, 2.2,2.4]) gives us conjugacy classes

$$\mathrm{Cl}_* \mathrm{Fr}_{v'} \in \mathrm{Conj}'(G_*)(\mathbb{Q})$$

such that $\mathrm{Cl}(\rho_{*,l}(\Phi_{v'})) = \mathrm{Cl}_* \mathrm{Fr}_{v'}$. By (4.4) and (d) it follows that $(\mathrm{Cl}_B \mathrm{Fr}_{v'}, \mathrm{Cl}_C \mathrm{Fr}_{v'})$ lies in $\mathrm{Conj}'(G_{B\times C})(\mathbb{Q})$. Now take $\mathrm{Cl}_M \mathrm{Fr}_{v'}$ to be the image of $(\mathrm{Cl}_B \mathrm{Fr}_{v'}, \mathrm{Cl}_C \mathrm{Fr}_{v'})$ under the map

$$\mathrm{Conj}'(G_{B\times C})(\mathbb{Q}) \to \mathrm{Conj}'(G_{M_{F'}})(\mathbb{Q})$$

induced by θ. Then the diagram (4.4) implies that $\mathrm{Cl}_M \mathrm{Fr}_{v'}$ is our required conjugacy class of the theorem.

□

Now we will show how to obtain Theorem A from Theorem B.

Outline of the proof of Theorem A. The following are the main steps of the proof

(a) Since G_M is assumed to be connected $G_M = G_{M_{F'}}$, where F' is a finite extension of F obtained in theorem B. Using some basic algebraic geometry and theorem B, we establish that for each $l \neq p$, the element $\mathrm{Cl}(\rho_{M,l}(\Phi_v)) \in \mathrm{Conj}'(G_M)(\bar{\mathbb{Q}})$.

(b) From (a) and the fact that $\rho_{M,l}(\Phi_v)$ is weakly neat and that the characteristic polynomial of $\rho_{M,l}(\Phi_v)$ has coefficients in $\mathbb{Q}$ and is independent of l (Deligne's theorem, cited before), we now deduce that $\mathrm{Cl}(\rho_{M,l}(\Phi_v))$ does not depend on l. Denote this common element by $\mathrm{Cl}_M \mathrm{Fr}_v$.

(c) Since $\mathrm{Cl}_M \mathrm{Fr}_v \in \mathrm{Conj}'(G_M)(\mathbb{Q}_l)$ for all $l \neq p$, we conclude that $\mathrm{Cl}_M \mathrm{Fr}_v \in \mathrm{Conj}'(G_M)(\mathbb{Q})$. This establishes the Theorem A.

□

We now describe the results obtained in the case where the algebraic variety X does not have good reduction at v.

Theorem C. *Let X be a smooth proper algebraic variety over F and assume that $M := h^i(X) \in \mathrm{Ob}\mathcal{M}_F^{av}$ for $i \in \mathbb{N}$. Suppose that G_M is connected. Then there exists a finite extension F' of F and a valuation v' extending v such that*

1. *for every $l \neq p$ the representation $(G_{M/\mathbb{Q}_l}, \rho'_{M,l}, N'_{M,l})$ of $'W_{v'}$ is defined over $\mathbb{Q}$ modulo the action $G_M^{\natural ad}$ and*
2. *for l running through primes different from p, these representations form a compatible system of representations of $'W_{v'}$ modulo the action of $G_M^{\natural ad}$.*

This is theorem 5.5.8 of the main text. Note that $'W_{v'}$ denotes the Weil-Deligne group of $F_{v'}$. As G_M is assumed to be connected, so $G_{M_{F'}} = G_M$. Here $N'_{M,l}$ denotes the monodromy operator corresponding to $\rho_{M,l}$. It is unchanged by finite base extensions. The morphism $\rho'_{M,l}$ is obtained from $\rho_{M,l}$ as in equation (8). Here of course it refers to the restriction to the Weil-group $W_{v'}$ of $F_{v'}$, which is a subgroup of the Weil-group W_v of F_v.

The statement of theorem C deserves a little explanation. To say that the representation $(G_{M/\mathbb{Q}_l}, \rho'_{M,l}, N'_{M,l})$ of $'W_{v'}$ is *defined over $\mathbb{Q}$ modulo the action $G_M^{\natural ad}$* means that base extension of this representation to any algebraically closed field $\Omega \supset \mathbb{Q}_l$ is conjugate under $G_M^{\natural ad}(\Omega)$ to all its images under $\mathrm{Aut}_{\mathbb{Q}}(\Omega)$. For varying l we say that these representations form a *compatible system of representations of $'W_{v'}$ modulo the action of $G_M^{\natural ad}$*, if for every pair of prime numbers (l, l') and every algebraically closed field $\Omega \supset \mathbb{Q}_l, \mathbb{Q}'_l$, the base extension to Ω of the l-adic representation of $'W_{v'}$ is $G_M^{\natural ad}(\Omega)$-conjugate to the base extension of the l'-adic representation. These conditions can be also formulated in the same ways as the equations (3) and (4). For details we refer the reader to §4 of chapter 4, of this document.

Outline of the Proof of Theorem C. The beginning of the proof is analogous to the proof of theorem B. So we will focus only on the new ideas involved.

(a) Let $B_{/F'}$ and $C_{/F'}$ be abelian varieties over a finite extension F' of F, as obtained in the proof of theorem B. Let v' be an extension of v to F'. The diagram (4.4) is commutative here, too. For any fixed l, using either Grothendieck's l-adic monodromy theorem or Semi-Stable reduction theorem [GRO72, IX,3.6] we may suppose that $\rho_{B,l}$ and $\rho_{B\times C,l}$ are unipotent on the inertia subgroup $I_{v'}$ of $\mathrm{Gal}(\bar{F}/F')$.

(b) As $C_{/F'}$ has potential good reduction, it follows that the monodromy operator corresponding to the galois representation $\rho_{C,l}$ is trivial. Thus using [Noo09, 2.2] we conclude that our theorem holds for the motive $h^1(C_{/F'})$. It is also known that the theorem holds for the motive $h^1(B_{/F'})$ [Noo10, 7.1]. From this we easily show that the theorem holds for the motive $h^1(B_{/F'} \times C_{/F'})$.

(c) From the fact that $B_{/F'}$ provides a weak-Mumford lift of M and that G_C is commutative, we obtain that $G^{\natural ad}_{B\times C} = G^{\natural ad}_{M}$. We denote this common algebraic group by $G^{\natural ad}$.

(d) Using again the l-adic monodromy theorem we may suppose that $\rho_{M,l}$ is unipotent on $I_{v'}$ and hence $\rho'_{M,l}$ is trivial on $I_{v'}$. Another consequence of the l-adic monodromy theorem is that, the map of Lie-algebras induced by θ, sends the monodromy operator $N'_{B\times C,l}$ (corresponding to $\rho_{B\times C,l}$) to the monodromy operator $N'_{M,l}$ (corresponding to $\rho_{M,l}$).

(e) Finally by studying the action of $G^{\natural ad}$, on $G_{B\times C}$, G_M and their Lie-algebras $\mathfrak{g}_{B\times C}$, $\mathfrak{g}_M$ respectively, we establish our theorem for the motive M.

□

Under certain additional hypothesis, the theorem C can be sharpened to give us the result over the base field F.

Theorem D. *Let X/F be an abelian variety, a K3-surface, a curve or a Fermat hypersuface, with semi-stable reduction at v. Let $M := h^i(X)$ for $i \in \mathbb{N}$, and assume G_M is connected. Suppose for some prime number l, the image $\rho'_{M,l}(\Phi_v)$ is weakly neat. Then*

1. *for every $l \neq p$ the representation $(G_{M/\mathbb{Q}_l}, \rho'_{M,l}, N'_{M,l})$ of $'W_v$ is defined over $\mathbb{Q}$ modulo the action $G^{\natural ad}_M$ and*
2. *for l running through primes different from p, these representations form a compatible system of representations of $'W_v$ modulo the action of $G^{\natural ad}_M$.*

This is theorem 5.5.10 of the main text.

Outline of the proof of Theorem D. The main steps in the proof are the following

(a) We begin by working over the finite extension F' of F and the valuation v' as obtained in theorem C. An useful fact here is that the monodromy operators are unchanged by finite base extensions. This is a consequence of the Grothendieck's l-adic monodromy theorem.

(b) As X has semi-stable reduction at v, thus $\rho_{M,l}$ is unipotent on the inertia subgroup I_v of $\mathrm{Gal}(\bar{F}/F)$ (see [RZ82] or [Ill94, 3.3]) for a fixed $l \neq p$. Thus we show that it suffices to study the behaviour of $\rho'_{M,l}$ at the arithmetic Frobenius Φ_v and the monodromy operator $N'_{M,l}$.

(c) Then we note that the characteristic polynomial of $\rho_{M,l}(\Phi_v)$ is in $\mathbb{Q}$ and is independent of l (see [Sai03] and [Och99]).

(d) By our hypothesis $\rho_{M,l}(\Phi_v)$ is weakly neat. Using some fundamental properties of weakly neat elements we now establish our theorem.

□

The Theorem A generalizes the main theorem of [Noo09] and Theorem D generalizes the main theorem of [Noo10]. We would also like to point out that the proofs of our results depends on the fact that the corresponding results are known for tractable abelian varieties, as shown by R.Noot in [Noo09], [Noo10].

Organisation of this document

The first chapter is devoted to study the properties of absolute Hodge cycles and the construction of the category of motives for absolute Hodge cycles, as defined by P.Deligne and J.Milne in [DMOS82]. Here we also discuss Tannakian categories, motivic galois groups and the Mumford-Tate group of a motive. At the end of this chapter we recall some facts about the Tannakian category of motives generated by abelian varieties(aka abelian motives). There are no new results here and so we state most of them without proofs.

In the second chapter we recall the definition of a tractable abelian variety as presented in [Noo09] and [Noo10]. Then we briefly discuss the notion of weak Mumford-Tate lifts of abelian motives as given in [Noo06]. Finally we recall the existence of Weak Mumford lifts that are also tractable abelian varieties.

The third chapter begins with some generalities on the action of algebraic groups on algebraic varieties and the existence of quotients for these actions. Then we construct the 'natural adjoint group' $G^{\natural ad}$ associated to any reductive algebraic group G defined over a field of characteristic 0 and describe its action. Finally we study weakly neat elements and prove some useful results about them.

In the fourth chapter we prove the first of our main results, theorem 4.2.1 and theorem 4.2.6. We begin this chapter by recalling some classical facts on good reduction of algebraic varieties (e.g Néron-Ogg-Shafarevich criterion). Then using the materials developed in the previous chapters we study the ℓ-independence problems for motives, in the good reduction case.

The final chapter treats the ℓ-independence problems for motives, in the semi-stable reduction case. We begin this chapter by recalling various facts about ℓ-adic monodromy. Then we describe the notions of compatible systems of representations of the Weil-Deligne groups and some related notions. The main results of this chapter are theorem 5.5.8 and theorem 5.5.10.

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Résumé

Fixons un corps de nombres F avec un plongement $\tau : F \hookrightarrow \mathbb{C}$. On choisit une clôture algébrique $\bar{F}$ de F. Soient v une valuation (discrète) de F et $\bar{v}$ une extension de v à $\bar{F}$. On note F_v la complétion de F en v, $\bar{F}_v$ la localisation de $\bar{F}$ en $\bar{v}$, k_v le corps résiduel de F en v et $\bar{k}_v$ le corps résiduel de $\bar{F}$ en $\bar{v}$. On suppose que la caractéristique de k_v est $p > 0$. On note $\Gamma_F = \mathrm{Gal}(\bar{F}/F)$ et $\Gamma_{F_v} := \mathrm{Gal}(\bar{F}_v/F_v)$. On a $\Gamma_{F_v} \subset \Gamma_F$. Soit $\phi_v \in \mathrm{Gal}(\bar{k}_v/k_v)$ l'automorphisme de Frobenius. On fixe un élément de Frobenius (arithmétique) $\Phi_v \in \Gamma_{F_v}$ qui induit ϕ_v sur $\bar{k}_v$.

Soit X une variété algébrique propre et lisse sur F_v. Le groupe Γ_{F_v} opère naturellement sur les groupes de cohomologie étale $V_l^i := H_t^i(X_{\bar{F}_v}, \mathbb{Q}_l)$, pour chaque nombre premier l et tous les entiers positifs i. Ainsi, nous avons des représentations $\rho_l^i : \Gamma_{F_v} \to \mathrm{GL}(V_l^i)$. Un thème majeur en géométrie arithmétique est de déterminer à quel point les propriétés de ces représentations, pour i fixé, sont indépendante de l. Pour ce faire, on restreint les représentations ci-dessus, au groupe de Weil W_{F_v} de F_v. C'est le sous-groupe formé par les éléments de Γ_{F_v} dont l'image dans $\mathrm{Gal}(\bar{k}_v/k_v)$ est une puissance entière de ϕ_v.

Dans ce qui suit, nous supposerons que $l \neq p$. Supposons d'abord que X a bonne réduction en v, c'est à-dire que X s'étend à un schéma propre et lisse sur l'anneau des entiers de F_v. Cela implique que le sous-groupe de l'inertie I_{F_v} de Γ_{F_v} opère trivialement sur les groupes V_l^i. Grâce aux travaux de P. Deligne sur les conjectures de Weil [Del80], on sait que dans ce cas le caractère de la représentation de W_{F_v} sur chaque V_l^i est à valeurs dans $\mathbb{Q}$ et est indépendant de l, quand i est fixé. Nous allons résumer cela en disant que, quand l varie et i est fixé, ρ_l^i sont *définis sur* $\mathbb{Q}$ et forment un *système compatible* de représentations de W_{F_v}. Puisque l'action de l'inertie est triviale, il suffit d'étudier l'action du sous-groupe de Γ_{F_v} engendré par un élément de Frobenius.

Si X est défini sur F, nous pouvons généraliser naturellement le problème

ci-dessus. Soit $\bar{\tau} : \bar{F} \hookrightarrow \mathbb{C}$, une extension du τ à $\bar{F}$. En utilisant le plongement τ on peut considérer

$$V^i := H^i_B(X(\mathbb{C}), \mathbb{Q}),$$

le i-ème groupe de cohomologie singulière de $X(\mathbb{C})$. On a un isomorphisme $V^i_l = V^i \otimes_{\mathbb{Q}} \mathbb{Q}_l$. Supposons $H \subseteq \mathrm{GL}(V^i)$ un groupe algébrique linéaire sur $\mathbb{Q}$, tel que $\mathrm{Im}(\rho^i_l) \subseteq H(\mathbb{Q}_l)$. Dans la plupart des cas que nous considérons, H sera le groupe de Mumford-Tate (voir la définition précise ci-dessous) du *motif de Hodge absolu* $h^i(X)$. Maintenant on peut se demander si la classe de conjugaison de $\rho^i_l(\Phi_v)$ dans $H(\mathbb{Q}_l)$ est définie sur $\mathbb{Q}$ et indépendante de l.

Considérons le cas où $H = \mathrm{GL}(V^i)$, alors le résultat de Deligne (cité ci-dessus) est un cas particulier de notre problème. Cela veut dire que la classe de conjugaison de l'image de Φ_v dans $\mathrm{GL}(V^i_l)$ est définie sur $\mathbb{Q}$ et indépendante de l.

Nous pouvons poser des questions similaires dans le cas où la variété algébrique X a mauvaise réduction en v. Pour traiter ces questions, nous rappelons quelques notions générales. Considérons une représentation l-adique quasi-unipotente

$$\xi_l : \Gamma_{F_v} \to H(\mathbb{Q}_l). \tag{7}$$

Le théorème de monodromie l-adique de A.Grothendieck (cf [Del73, 8.2]) affirme que, pour un sous-groupe ouvert suffisamment petit $J \subseteq I_{F_v}$, la restriction $\xi_{l|J}$ peut être décrite comme l'exponentiel d'un seul endomorphisme N'_l, *l'opérateur de monodromie*. Alors la restriction de ξ_l au groupe de Weil W_{F_v} est déterminée par N'_l et une représentation ξ'_l de W_{F_v} à valeurs dans H, qui est triviale sur J. Nous appelons un tel couple (N'_l, ξ'_l) une *représentation du groupe de Weil-Deligne* $'W_{F_v}$ de F_v. En fait, $N'_l \in \mathrm{Lie}(H_{/\mathbb{Q}_l})$.

Nous introduisons encore quelques notations. Soit $t_l : I_{F_v} \to \mathbb{Z}_l(1)$ la surjection définie par $\sigma \mapsto (\frac{\sigma(\pi^{1/l^m})}{\pi^{1/l^m}})_m$ pour une uniformisante π de F_v. On sait que t_l est indépendante du choix de π et du système de racines l^m-èmes π^{1/l^m}. Supposons que $w \in W_{F_v}$, alors w induit une puissance entière $\phi_v^{\alpha(w)}$ de ϕ_v. Avec ces notations ξ'_l est donné par

$$\xi'_l(w) = \xi_l(w)\exp(-N'_l t_l(\Phi_v^{-\alpha(w)} w)). \tag{8}$$

Pour l fixé, on dit que la représentation $'W_{F_v} \to H_{/\mathbb{Q}_l}$ est *définie sur* $\mathbb{Q}$, si pour tout corps algébriquement clos $\Omega \supset \mathbb{Q}_l$, l'extension de base de (N'_l, ξ'_l) à Ω est conjuguée sous $H(\Omega)$ à toutes ses images sous $\mathrm{Aut}_{\mathbb{Q}}(\Omega)$. Plus précisément, pour chaque $\sigma \in \mathrm{Aut}_{\mathbb{Q}}(\Omega)$, il existe un $g \in H(\Omega)$ tel que

$$\sigma\xi'_{l/\Omega} = g \cdot \xi'_{l/\Omega} \cdot g^{-1} \quad \text{et} \quad \sigma(N'_l \otimes_{\mathbb{Q}_l} 1) = \mathrm{Ad}(g)(N'_l \otimes_{\mathbb{Q}_l} 1)$$

où $\xi'_{l/\Omega} : W_{F_v} \to H_{/\mathbb{Q}_l}(\Omega)$ est déduit de ξ'_l par l'extension des scalaires et

$$N'_l \otimes_{\mathbb{Q}_l} 1 \in (\mathfrak{h} \otimes_{\mathbb{Q}} \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \Omega = \mathfrak{h} \otimes \Omega,$$

où $\mathfrak{h}$ est l'algèbre de Lie de H.

Pour l parcourant l'ensemble des nombres premiers différents de p, on dit que les couples (N'_l, ξ'_l) forment un *système compatible* de représentations de $'W_{F_v}$ si pour tout couple (l, l') de nombres premiers et tout corps algébriquement clos $\Omega \supset \mathbb{Q}_l, \mathbb{Q}_{l'}$, il existe $g \in H(\Omega)$ tel que

$$\xi'_{l/\Omega} = g \cdot \xi'_{l'/\Omega} \cdot g^{-1} \quad \text{et} \quad N'_l \otimes_{\mathbb{Q}_l} 1 = \mathrm{Ad}(g)(N'_l \otimes_{\mathbb{Q}_{l'}} 1) \in \mathfrak{h} \otimes \Omega$$

Ces notions trouvent leur origine dans les travaux de P. Deligne (voir [Del73], voir aussi [Fon94]).

Revenons maintenant à la variété algébrique X et les représentations ρ^i_l. Nous voulons utiliser les notions ci-dessus pour étudier le cas où X n'a pas bonne réduction en v. On sait d'après [GRO72] que les représentations ρ^i_l sont quasi-unipotentes. Supposons que X est définie sur F. Dans (7) prenons $H = \mathrm{GL}(V^i)$ et $\xi_l = \rho^i_l$. J-M Fontaine a conjecturé que pour i fixe et l variable, les (N'_l, ρ'^i_l) forment un système compatible défini sur $\mathbb{Q}$ des représentations de $'W_{F_v}$. Nous renvoyons à [Fon94, 2.4.3], la conjecture C_{WD}, qui couvre également les représentations p-adiques.

Cette conjecture de Fontaine est étroitement liée à la conjecture de monodromie-poids (voir [Ill94], [Ito04], [RZ82]). Cette conjecture subtile est un peu plus accessible sous l'hypothèse que X a réduction semi-stable. Soit A un anneau de valuation discrète et X un schéma défini sur $S := \mathrm{Spec}A$. On dit que X/S a *réduction semi-stable*, si localement pour la topologie étale sur X et sur S, X est S-isomorphe à $\mathrm{Spec}\frac{A[T_1,\cdots,T_R]}{(T_1,\cdots T_R - \pi)}$ pour certains $r \geq 0$ et π une uniformisante. Cette condition est équivalente à la condition suivante : X est régulier, la fibre générique de X est lisse, et la fibre spéciale de X est un diviseur réduit à croisements normaux dans X. La condition de réduction semi-stable implique en particulier que chaque ρ^i_l est unipotente sur tout le sous-groupe d'inertie I_{F_v}([Ill94, 3.3], [RZ82]). Dans ce cas $\rho'^i_{l\,|I_{F_v}}$ est trivial et l'action de l'inertie sur $H^i_{et}(X_{\bar{F}_v}, \mathbb{Q}_l)$ est déterminée par l'opérateur de monodromie N'_l.

On peut se poser les questions ci-dessus pour des motifs plutôt que pour des variétés algébriques. Au cœur de la théorie des motifs proposée par Grothendieck, il existe une cohomologie universelle (conjecturale) $h(X)$ pour les variétés algébriques X/F, à valeurs dans une catégorie Tannakienne $\mathbb{Q}$-linéaire. La cible de ce foncteur est la catégorie des *motifs*. Tout foncteur cohomologique (de Weil) se factorise à travers h. La catégorie des motifs est engendrée par des objets $h(X)$, pour X parcourant les variétés algébriques propres et lisse sur F, et le motif de Tate $\mathbb{Q}(1)$. Il existe des constructions différentes des motifs selon les définitions des morphismes. Dans la catégorie des motifs de Grothendieck, les morphismes sont définis par les cycles algébriques. Mais la plupart des propriétés désirées de cette catégorie dépende de certaines propriétés conjecturales des cycles algébriques, les conjectures standard. P. Deligne et J. Milne ont construit une variante de catégorie des motifs, où les morphismes sont définis en utilisant des *cycles de Hodge absolus*. Les analogues des conjectures standard pour les cycles absolu Hodge sont vrais (sur le corps F). Dans cette thèse, nous allons travailler avec la catégorie des motifs pour les cycles de Hodge absolus.

Notons $\mathcal{M}_F$ la catégorie des motifs pour les cycles de Hodge absolus, comme définie dans [DMOS82]. C'est une catégorie Tannakienne semi-simple. Maintenant nos questions peuvent être formulées en termes des groupes de Mumford-Tate des motifs dans $\mathcal{M}_F$. Le groupe de Mumford-Tate G_M d'un motif $M \in \mathrm{Ob}(\mathcal{M}_F)$ est défini comme le groupe des automorphismes du foncteur fibre défini par une réalisation de Betti restreint à la catégorie Tannakienne engendrée par M et le motif de Tate (cf. définition 1.3.13 de la thèse). C'est un groupe algébrique linéaire sur $\mathbb{Q}$. Pour chaque nombre premier l, notons $H_l(M)$ la réalisation l-adique d'un motif M. L'action de Γ_F sur $H_l(M)$ induit un morphisme

$$\rho_{M,l} : \Gamma_F \to \mathrm{GL}(H_l(M)).$$

On sait que $\rho_{M,l}$ se factorise par $G_M(\mathbb{Q}_l)$, ce qui implique que les représentations correspondantes du groupe de Weil-Deligne de F_v se factorisent également par $G_{M/\mathbb{Q}_l}$. Pour les objets dans la sous-catégorie Tannakienne engendrée par M, la conjecture de Fontaine citée ci-dessus sur l'indépendance de l est équivalente à ce que les représentations $'W_{F_v} \to G_{M/\mathbb{Q}_l}$ induites par $\rho_{M,l}$ forment un système compatible à *valeurs dans* G_M. Ici, v est fixé et l parcourt l'ensemble des nombres premiers différent que p.

On dit que un motif $M \in \mathrm{Ob}(\mathcal{M}_F)$ a bonne réduction en v si $\rho_{M,l}$ est trivial sur le groupe de l'inertie pour chaque nombre premier $l \neq p$. Pour un motif ayant bonne réduction J-P Serre a conjecturé dans [Ser94] que :

Conjecture. *Soit M un motif ayant bonne réduction en v. Alors il existe une classe* $\mathrm{CL_M Fr_v} \in \mathrm{Conj}(G_M)(\mathbb{Q})$ *telle que* $\mathrm{Cl}(\rho_{M,l}(\Phi_v)) = \mathrm{CL_M Fr_v}$ *pour tout nombre premier $l \neq p$.*

Nous avons noté $\mathrm{Conj}(G_M)$ le quotient catégorique universelle de G_M, pour son action sur lui-même par conjugaison et $\mathrm{Cl} : G_M \to \mathrm{Conj}(G_M)$ est l'application du quotient correspondant. En fait Serre a fait cette conjecture pour un motif quelconque.

Dans cette thèse, nous étudions cette conjecture et la version motivique de la conjecture de Fontaine, pour la catégorie $\mathcal{M}_F$. Nous considérons la sous-catégorie Tannakienne de motifs $\mathcal{M}_F^{av}$ de $\mathcal{M}_F$ engendrée par les motifs des variétés abéliennes et le motif de Tate. On sait que $\mathcal{M}_F^{av}$ contient les motifs des surfaces $K3$, des variétés uni-rationnelles de dimension ≤ 3, des courbes et des hypersurfaces de Fermat (voir [DMOS82, II.6.26]). Considérons le plongement $\tau : F \hookrightarrow \mathbb{C}$, un résultat fondamental de Deligne affirme que, pour les variétés dont les motifs $M \in \mathcal{M}_F^{av}$, les *cycles de Hodge relativement à* τ sont absolument Hodge ([DMOS82, II.6.27]). Cela implique que si $M \in \mathrm{Ob}\mathcal{M}_F^{av}$ la composante neutre G_M° du groupe de Mumford-Tate, coïncide avec le groupe de Mumford-Tate de la structure de Hodge $H_\tau(M)$. Pour les définitions précises nous renvoyons la lecture à [DMOS82] ou au chapitre 1 de cette thèse.

Sous certaines hypothèses supplémentaires, nous montrons des cas particuliers des conjectures indiquées ci-dessus. Il existe deux types d'hypothèses. Premièrement, dans les théorèmes principaux (théorèmes A et D, ci-dessous) nous devons supposer que le corps de base est suffisamment grand pour garantir que les groupes de Mumford-Tate des motifs soient connexes et que l'élément de Frobenius en v est *faiblement net*. La condition d'être faiblement net signifie que 1 est la seule racine de l'unité parmi les quotients $\lambda\mu^{-1}$ pour les valeurs propres λ, μ de $\rho_{M,l}(\Phi_v)$, agissant sur $H_l(M)$. C'est une variante de la notion d'élément 'net', qui apparaît dans la littérature classique sur les groupes arithmétiques (voir [Bor01]).

Deuxièmement, dans certains cas, nous montrons la conjugaison dans un groupe algébrique $G_M^\natural$ qui est un peu plus grand que le groupe de Mumford-Tate G_M. Seulement certains facteurs de type D de G_M^{der} sont affectés par cette modification. Comme nous traitons des questions sur les classes de conjugaison, il suffit de décrire le groupe adjoint. Le groupe adjoint $G_M^{\natural ad}$ est une extension "naturelle" de $G_{M/\mathbb{Q}}^{ad}$ et $G_{M/\mathbb{Q}}^{ad} \subset G_M^{\natural ad}$. Ce groupe $G_M^{\natural ad}$ opère sur $G_{M/\bar{\mathbb{Q}}}$. Cette opération étend l'action adjointe de $G_{M/\mathbb{Q}}^{ad}$ sur $G_{M/\bar{\mathbb{Q}}}^{der}$, avec $G_M^{\natural ad}$ agissant trivialement sur le centre de $G_{M/\bar{\mathbb{Q}}}$. Le groupe $G_M^{\natural ad}$ opère également sur $\mathrm{Lie}(G_M) \otimes \bar{\mathbb{Q}}$, par la représentation adjointe. *Le groupe* $G_{M/\bar{\mathbb{Q}}}^{ad}$ *coïncide avec la composante neutre de* $G_M^{\natural ad}$. Agrandir le groupe affaiblit évidemment la notion de conjugaison. Dans la suite, $\mathrm{Conj}'(G_M)$ désigne le quotient catégorique universel pour l'action adjointe de $G_M^{\natural ad}$ sur G_M.

Théorème A. *Soit X une variété algébrique propre et lisse sur F ayant bonne réduction en v et $M := h^i(X) \in \mathrm{Ob}(\mathcal{M}_F^{av})$. Supposons que le groupe de Mumford-Tate G_M est connexe et qu'il existe un nombre premier $l \neq p$, tel que $\rho_{M,l}(\Phi_v)$ est faiblement net. Il existe alors une classe*

$$\mathrm{Cl}_M\mathrm{Fr}_v \in \mathrm{Conj}'(G_M)(\mathbb{Q})$$

telle que $\mathrm{Cl}(\rho_{M,l}(\Phi_v)) = \mathrm{Cl}_M\mathrm{Fr}_v$ *pour tout $l \neq p$*

Ce théorème est vrai en particulier pour les motifs des variétés abéliennes, des surfaces $K3$, des variétés unirationnelles de dimension ≤ 3, des courbes et des hypersurfaces de Fermat. C'est le théorème 4.2.6 de la thèse.

Le théorème A est déduit du théorème suivant, qui établit le résultat sur une extension finie de F, sans l'hypothèse de faiblement net ou la connexité du groupe de Mumford-Tate G_M.

Théorème B. *Soient X et M comme dans le théorème A. Alors il existe une extension finie F' de F, une valuation v' de F' qui étend v et une classe*

$$\mathrm{Cl}_M\mathrm{Fr}_{v'} \in \mathrm{Conj}'(G_{M_{F'}})(\mathbb{Q})$$

telle que $\mathrm{Cl}(\rho_{M,l}(\Phi_{v'})) = \mathrm{Cl}_M\mathrm{Fr}_{v'}$, *pour tout $l \neq p$.*

Ici, $\Phi_{v'} \in \mathrm{Gal}(\bar{F}_v/F'_{v'})$ est un Frobenius (arithmétique) correspondant à $F'_{v'}$. On peut choisir $\Phi_{v'} = \Phi_v^d$, où d est le degré résiduelle de l'extension $F'_{v'}/F_v$. C'est le théorème 4.2.1 de la thèse.

Dans le cas où X a mauvaise réduction nous avons les résultats suivants.

Théorème C. *Soit X/F une variété algébrique propre et lisse avec $M := h^i(X) \in \mathrm{Ob}(\mathcal{M}_F^{av})$. Supposons G_M est connexe. Alors il existe une extension finie F' de F, une valuation v' de F' qui étend v, telles que*

- *(i) pour tout $l \neq p$ la représentation $(G_{M/\mathbb{Q}_l}, \rho'_{M,l}, N'_{M,l})$ de $'W_{v'}$ est définie sur $\mathbb{Q}$* modulo l'action de $G_M^{\natural ad}$ *et*
- *(ii) pour l parcourant l'ensemble des nombres premiers différents de p, ces représentations forment un système compatible de représentations de $'W_{v'}$ modulo l'action de $G_M^{\natural ad}$.*

C'est le théorème 5.5.8 de la thèse. Nous avons noté $'W_{v'}$ le groupe de Weil-Deligne de $F'_{v'}$.

Théorème D. *Soit X/F une variété abélienne, une surface K3, une courbe ou une hypersurface de Fermat avec réduction semi-stable en v. On note $M := h^i(X)$ et on suppose que G_M est connexe et qu'il existe un nombre premier $l \neq p$ tel que $\rho_{M,l}(\Phi_v)$ est faiblement net. Alors*

(i) *pour tout $l \neq p$ la représentation $(G_{M/\mathbb{Q}_l}, \rho'_{M,l}, N'_{M,l})$ de $'W_v$ est définie sur* $\mathbb{Q}$ modulo l'action de $G_M^{\natural ad}$ *et*

(ii) *pour l parcourant l'ensemble des nombres premiers différents de p, ces représentations forment un système compatible de représentations de $'W_v$ modulo l'action de $G_M^{\natural ad}$.*

On a noté $'W_v$ le groupe de Weil-Deligne de F_v. C'est le théorème 5.5.10 de la thèse.

Chapter 1

Motives for Absolute Hodge Cycles

In this chapter we present the theory of absolute Hodge cycles and the category of motives given by them. The main reference is [DMOS82].

1.1 Cohomology and absolute Hodge cycles

Throughout this thesis by an algebraic variety we mean a geometrically integral, separated scheme of finite type over a field, unless otherwise stated.

Let X be a smooth projective algebraic variety over a field F embeddable in $\mathbb{C}$. If $\tau : F \hookrightarrow \mathbb{C}$ is an embedding, then we get a complex analytic variety $\tau X = (X \times_{F,\tau} \mathbb{C})(\mathbb{C})$. Fix an algebraic closure $\bar{F}$ of F and let Γ_F denote the absolute Galois group $\mathrm{Gal}(\bar{F}/F)$. The following cohomology groups are of interest to us:

1. $H^r_\tau(X) = H^r_B(\tau X, \mathbb{Q})$, the singular cohomology (Betti) of τX corresponding to any embedding $\tau : F \hookrightarrow \mathbb{C}$. It is a finite dimensional $\mathbb{Q}$-vector space with a $\mathbb{Q}$-rational Hodge structure $H^r_\tau(X) \otimes \mathbb{C} = \bigoplus_{i+j=r} H^{i,j}_\tau$ (Hodge decomposition).
2. $H^r_l(X) = H^r_{et}(X_{\bar{F}}, \mathbb{Q}_l)$, the etale cohomology group for any prime number l. It is a finite dimensional $\mathbb{Q}_l$-vector space with a continuous action of Γ_F coming from the functoriality of etale cohomology.
3. $H^r_{dR}(X) = \mathbb{H}^r(X, \Omega^{\cdot}_X)$, the de Rham cohomology, which is by definition the hypercohomology of the complex $\Omega^{\cdot}_X$ of algebraic differential forms on X. This is a finite dimensional F-vector space with a decreasing F-linear filtration $F^i H_{dR}(X)$

There are the following comparison isomorphisms for these above mentioned cohomology groups, which is crucial for defining an absolute Hodge cycle.

$$I_{\infty,\tau} : H^r_\tau(X) \otimes \mathbb{C} \xrightarrow{\sim} H^r_{dR}(\tau X) \cong H^r_{dR}(X) \otimes_{F,\tau} \mathbb{C},$$

$$I_{l,\bar{\tau}} : H^r_\tau(X) \otimes \mathbb{Q}_l \xrightarrow{\sim} H^r_{et}(\bar{\tau}(X \otimes_F \bar{F}), \mathbb{Q}_l)$$

for every embedding $\tau : F \hookrightarrow \mathbb{C}$ and any extension $\bar{\tau} : \bar{F} \hookrightarrow \mathbb{C}$ of τ. Note that in the first comparison isomorphism $H^r_{dR}(\tau X)$ denotes the classical de Rham cohomology of the compex variety τX. $I_{\infty,\tau}$ relates the Hodge structure and the Hodge filtration in the following way

$$I_{\infty,\tau}(\bigoplus_{i' \geq i} H^{i',j}_\tau) = F^i H^r_{dR}(X) \otimes_{\tau,F} \mathbb{C}.$$

The $I_{l,\bar{\tau}}$ are related as follows: If $\tau' \in \Gamma_F$, then we have a commutative diagram

$$\begin{array}{ccc} & & H^r_l(X) \\ & \nearrow^{I_{l,\bar{\tau}}} & \uparrow \tau' \\ H^r_\tau(X) \otimes \mathbb{Q}_l & & \\ & \searrow_{I_{l,\bar{\tau}\tau'}} & \\ & & H^r_l(X) \end{array} \tag{1.1}$$

In each of these cohomolgy theories "Tate twists" can be defined, as below:

1. Let m be an integer, then define

$$H^r_\tau(X)(m) = (2\pi i)^m H^r_\tau(X) \subset H^r_\tau(X) \otimes \mathbb{C}.$$

   There is a shift $(i,j) \to (i-m, j-m)$ in the Hodge decomposition:

$$(H^r_\tau(X)(m))^{(i-m,j-m)} = H^{i-m,j-m}_\tau(X) \subset H^r_\tau(X) \otimes \mathbb{C}.$$

2.
$$H^r_l(X)(m) = H^r_l(X) \otimes \mathbb{Z}_l(1)^{\otimes m},$$

   where $\mathbb{Z}_l(1) = \varprojlim \mu_{l^n}$ as a Γ_F-module, μ_{l^n} being the group of roots of unity of degree l^n in $\bar{F}$.

3. $H^r_{dR}(X)_m = H^r_{dR}(X)$ as F-vector spaces, but the twist changes the index in the Hodge filtration as :

$$F^{i-m} H^r_{dR}(X)(m) = F^i H^r_{dR}(X).$$

The cohomlogy theories $H^r_*(X)$ for $* = \tau, l, dR$ as defined above satisfy the axioms of the Weil cohomology (with Tate twists) for finite dimensional cohomology groups.

1. If $CH^r(X)$ denotes the chow group of algebraic cycles of codimension r on X modulo linear equivalence, then we have natural homomorphisms (cycle maps)

$$cl^r_* : CH^r(X) \to H^{2r}_*(X)(r),$$

which are compatible with intersection product in the Chow ring and cup product in the cohomology rings. cl^r_* are also compatible with the comparison isomorphisms.

2. $H^r_*(X) = 0$ for $r \geq 2d$, $d = \dim X$, and there is a natural nondegenrate pairing

$$H^r_*(X) \times H^{2d-r}_*(X) \to H^{2d}_*(X) \xrightarrow{\mathrm{Tr}} \mathbb{Q}_*(-d)$$

for $r = 0, 1, ..., 2d$, where Tr is defined by $H^{2d}_*(X)(d) \to \mathbb{Q}_*$, $cl^d(pt) = 1$. This is the Poincaré duality.

3. If Y is another smooth projective variety then we have the Künneth isomorphism :

$$H^r_*(X \times Y) \xrightarrow{\sim} \sum_{r+s=n} H^r_*(X) \otimes H^s_*(Y)$$

For proofs and more details on these above facts see [DMOS82, I,§1]

Now set

$$H^r(X)(m) = H^r_{dR} \times \prod_l H^r_l(X)(m) \times \prod_\tau H^r_\tau(X)(m),$$

then we get a map

$$cl^r : CH^r(X) \otimes \mathbb{Q}_l \to H^{2r}(X)(r)$$

Remark 1.1.1. (a) It is known that the image of cl^r_τ consists of rational elements of type $(0,0)$ (*Hodge cycles*).

(b) The image of cl^r_l is a $\mathbb{Q}$-subspace of $H^r_l(X)^{\Gamma_F}$, i.e the $\mathbb{Q}_l$-subspace of elements fixed by Γ_F.

(c) The cycle maps are compatible with the comparison isomorphisms.

Definition 1.1.2. The original definition of absolute Hodge cycles as given in [DMOS82] is as follows:

(i) First assume that the base field F is algebraically closed embeddable in $\mathbb{C}$. Then an *absolute Hodge cycle* is an element $t \in H^{2r}_{dR}(X)(r) \times \prod_l H^{2r}_l(X)(r)$ satisfying the following conditions :

   (a) For each $\tau : F \hookrightarrow \mathbb{C}$ the element t lies in the rational subspace $H^{2r}_\tau(X)(r)$ given as the image of the comparison isomorphisms.

   (b) The first component t_{dR} of t lies in $F^0H^{2r}(X)(r) = F^rH^{2r}(X)$.

   If the above conditions are true for any fixed τ then t is said to be a *Hodge cycle relative to* τ.

(ii) Now suppose the base field F is not algebraically closed, an *absolute Hodge cycle* is then defined as an absolute Hodge cycle on $X \otimes_F \bar{F}$, that is fixed under the natural action of Γ_F.

Remark 1.1.3. We will often use the following alternative definition of an absolute Hodge cycle that appears in [Jan90, 2.10]. An *absolute Hodge cycle*(AHC) x in degree $2r$ is an element of the finite-dimensional $\mathbb{Q}$-vector space

$$C^r_{AH}(X) = \{(x_{dR}, x_l, x_\tau)_{l,\tau} \in H^{2r}(X)(r) | I_{\infty,\tau}(x_\tau) = x_{dR}, I_{l,\bar{\tau}}(x_\tau) = x_l$$

$$\text{for all } \tau : F \hookrightarrow \mathbb{C} \text{ and } \bar{\tau} : \bar{F} \hookrightarrow \mathbb{C} \text{ restricting to } \tau, x_{dR} \in F^0H^{2r}_{dR}(X)(r)\}$$

This definition is equivalent to the original definition as shown in [Pan94].

Example 1.1.4. (i) If $Z \in CH^r(X)$, then $cl^r(Z) = (cl^r_{dR}(Z), cl^r_l(Z), cl^r_\tau(Z)) \in CH^r(X)$ is an absolute Hodge cycle.

(ii) Let $d = \dim X$ and $\Delta \subset X \times X$ be the diagonal. Consider the Künneth decomposition

$$H^{2d}(X \times X)(d) = \bigoplus_{i=0}^{2d} H^{2d-i}(X) \otimes H^i(x)(d),$$

and the corresponding decomposition $cl^d(\Delta) = \sum_{i=0}^{2d} \pi^i$. The classes π^i are absolute Hodge cycles.

There is a natural action of the Galois group Γ_F on the $\mathbb{Q}$-vector space $C^r_{AH}(X_{/\bar{F}})$. The following result describes the nature of this action.

Proposition 1.1.5. *The action of the Galois group Γ_F on $C^r_{AH}(X_{/\bar{F}})$ is through a finite quotient.*

Proof. [DMOS82, I,2.9]. □

The following is an important result about absolute Hodge cycles on abelian varieties, proved by P.Deligne.

Theorem 1.1.6. *Let F be an algebraically closed field with an embedding $\tau : F \hookrightarrow \mathbb{C}$. Let X be an abelian variety over F. If t is a Hodge cycle relative to τ (1.1.2), then it is an absolute Hodge cycle.*

Proof. This is the main theorem [DMOS82, I,2.11]. □

Now, let X be of pure dimension d. We denote $H_*(X) = \bigoplus H_*^r(X)$, where $H_*^r(X)$ is any one of the cohomology groups defined in section 1. We write

$$\mathrm{Mor}_{AH}^m(X,Y) = C_{AH}^{d+m}(X \times Y)$$

then we have the following

$$\begin{aligned} \mathrm{Mor}_{AH}^m(X,Y) &\subset H^{2d+2m}(X \times Y)(m+d) \\ &= \bigoplus_{s+t=2d+2m} H^s(X) \otimes H^t(Y)(m+d) \\ &= \bigoplus_{t=s+2m} H^\vee(X) \otimes H^t(Y)(m) \\ &= \bigoplus_s \mathrm{Hom}(H^s(X), H^{s+2m}(Y)(m)) \end{aligned}$$

Using this description and the definition of absolute Hodge cycle, it can be shown that

Proposition 1.1.7. *An element $f \in \mathrm{Mor}_{AH}^m(X,Y)$ gives rise to*

(a) for each prime l, a homomorphism $f_l : H_l(X) \to H_l(Y)$ of graded vector spaces. This mean that f_l is a family of homomorphisms

$$f_l^r : H_l^r(X) \to H_l^r(Y)(m)$$

(b) a homomorphism $f_{dR} : H_{dR}(X) \to H_{dR}(Y)(m)$ of graded vector spaces;

(c) for each $\tau : F \hookrightarrow \mathbb{C}$, a homomorphism $f_\tau : H_\tau(X) \to H_\tau(Y)(m)$ of graded vector spaces.
These maps satisfy the following conditions

(d) for all $\gamma \in \Gamma_F$ and primes l, $\gamma f_l = f_l$;

(e) f_{dR} is compatible with the Hodge filtration on each homogeneous factor;

(f) for each $\tau : F \hookrightarrow \mathbb{C}$, the maps f_{dR}, f_l and f_τ are compatible with the comparison isomorphisms of section 1.

Conversely, when F is embeddable in $\mathbb{C}$ any family of maps f_l, f_{dR} as in (a) and (b) arises from an $f \in \mathrm{Mor}^m_{AH}(X \times Y)$ if

- *(f_l) and f_{dR} satisfy (d) and (e) respectively, and*
- *for every $\tau : F \hookrightarrow \mathbb{C}$, there exists an f_τ such that $(f_l), f_{dR}$ and f_τ satisfy the condition (f).*

Moreover f is unique.

Proof. [DMOS82, II,6.1]. □

1.2 Construction of the category of motives

Let $\mathcal{V}_F$ denote the category of smooth projective algebraic varieties over a field F of characteristic 0. Define $\mathcal{CV}_F$ to be the category whose objects are symbols $h(X)$, one for each $X \in \mathrm{Ob}(\mathcal{V}_F)$. If Y is another object of $\mathcal{V}_F$, then

$$\mathrm{Hom}(h(X), h(Y)) = \mathrm{Mor}^0_{AH}(X, Y)$$

Now, let Z be another object of $\mathcal{V}_F$, $f \in \mathrm{Hom}(h(X), h(Y))$ and $g \in \mathrm{Hom}(h(Y), h(Z))$. Then by proposition 1.1.7 we have

$$f = (f_{dR}, f_l, f_\tau)_{l,\tau} \in \mathrm{Mor}^0_{AH}(X, Y) \text{ and } g = (g_{dR}, g_l, g_\tau)_{l,\tau} \in \mathrm{Mor}^0_{AH}(Y, Z).$$

So now we define

$$g \circ f = (g_{dR} \circ f_{dR}, g_l \circ f_l, g_\tau \circ f_\tau)_{l,\tau}.$$

By the converse statement in proposition 1.1.7, $g \circ f$ is an element of

$$\mathrm{Hom}((h(X), h(Z)) = C^d_{AH}(X \times Z).$$

This defines the composition of morphisms in $\mathcal{CV}_F$.

We also have a graph map $\mathrm{Hom}(Y, X) \to \mathrm{Hom}(h(Y), h(X))$, which makes h a contravariant functor. There is a natural $\mathbb{Q}$-linear tensor law on $\mathcal{CV}_F$ for which $h(X) \otimes h(Y) = h(X \times Y)$, the commutativity and the associativity constraints are induced by the natural isomorphisms $X \times Y \cong Y \times X$, $(X \times Y) \times Z \cong X \times (Y \times Z)$, and $h(\mathrm{Spec}F)$ is the identity object.

We now present the construction of the category of motives for absolute Hodge cycles as defined in [DMOS82, II]. The construction follows in three steps

1. The category $\dot{\mathcal{M}_F}^+$ (the false effective motives) is the pseudo-abelian envelope of the $\mathbb{Q}$-linear, additive category $\mathcal{CV}_F$. Thus the objects of $\dot{\mathcal{M}_F}^+$ are pairs $(h(X), p)$, where $p \in \mathrm{End}(h(X))$ is a projector (i.e a morphism with $p^2 = p$) and

$$\mathrm{Hom}((h(X), p), (h(Y), q)) = \{f \in \mathrm{Hom}(h(X), h(Y)) | fp = qf\} / \sim \quad (1.2)$$

where $f \sim 0$ if $f \circ p = 0 = q \circ f$. The tensor law being given by

$$(h(X), p) \otimes (h(Y), p) = (h(X) \otimes h(Y), p \otimes q)$$

Since this category is pseudo-abelian, there exists a natural grading $h(X) = \oplus h^i(X)$, where for any object $X \in \mathrm{Ob}\mathcal{V}_F$ the objects $h^i(X)$ in $\dot{\mathcal{M}_F}^+$ are defined as pairs $(h(X), \pi^i)$, with π^i as in 1.1.4. This grading can be extended to give a grading on all the objects of $\dot{\mathcal{M}_F}^+$.

2. The category $\dot{\mathcal{M}_F}$ of false motives is obtained by adjoining to $\dot{\mathcal{M}_F}^+$ all inverse powers of the Lefschetz object $L = h^2(\mathbb{P}^1)$. Thus, objects of $\dot{\mathcal{M}_F}$ are pairs (M, m) , with $M \in \mathrm{Ob}\dot{\mathcal{M}_F}^+$ and $m \in \mathbb{Z}$. For false motives (M_1, m_1) and (M_2, m_2), morphisms are defined as

$$\mathrm{Hom}((M_1, m_1), (M_2, m_2)) = \mathrm{Hom}(M_1 \otimes L^{m-m_1}, M_2 \otimes L^{m-m_2}), m \geq m_1, m_2 \quad (1.3)$$

The definition is independent of m, since we have a canonical isomorphism

$$\mathrm{Hom}(M', N') \xrightarrow{\sim} \mathrm{Hom}(M' \otimes L, N' \otimes L)$$

The tensor law is

$$(M_1, m_1) \otimes (M_2, m_2) = (M_1 \otimes M_2, m_1 + m_2)$$

and there is a grading compatible with the tensor structure

$$(M, m)^r = M^{r-2m}$$

Here, M^{r-2m} is the $r - 2m$ th component of M in its grading in $\dot{\mathcal{M}_F}^+$. We shall identify $h(X)$ with its image under the embeddings

$$\mathcal{CV}_F \to \dot{\mathcal{M}_F}^+ \to \dot{\mathcal{M}_F},$$

$$h(X) \mapsto (h(X), id), \quad M \mapsto (M, 0)$$

The Tate motive T is $L^{-1} = (h(\text{Spec}F), 1)$. We abbreviate $M \otimes T^{\otimes m} = (M, m)$ by $M(m)$. The category $\dot{\mathcal{M}}_F$a rigid abelian tensor category, because we can define an internal Hom; for $d = \dim X$ define $\underline{\text{Hom}}(h(X), h(Y))$ as $h(X \times Y)(d)$, and this can be extended canonically to all of $\dot{\mathcal{M}}_F$. But it is not a Tannakian category, because the canonical rank function (defined for any rigid tensor category) has the value

$$\text{rk}\, h(X) = \Sigma \dim H^r_*(X)$$

This is the Euler-Poincaré characteristic of the variety and it need not be positive. But, in a Tannakian category it should coincide with the dimension of a vector space(the image of $h(X)$) under a fiber functor.

3. Let the commutativity constraint on $\dot{\mathcal{M}}_F$ be

$$\dot{\psi} : M \otimes N \xrightarrow{\approx} N \otimes M,\ \dot{\psi} = \oplus \dot{\psi}^{r,s},\ \dot{\psi}^{r,s} : M^r \otimes N^s \xrightarrow{\approx} N^s \otimes M^r$$

Here the morphisms $\dot{\psi}^{r,s}$ are those induced by $\dot{\psi}$ on the graded components of $M \otimes N$. As $\dot{\psi}$ is an isomorphism and the grading in $\dot{\mathcal{M}}_F$ is compatible with the tensor products, the $\dot{\psi}^{r,s}$ are also isomorphisms. Now, define a new commutativity constraint as :

$$\psi : M \otimes N \xrightarrow{\approx} N \otimes M,\ \psi = \oplus \psi^{r,s},\ \psi^{r,s} = (-1)^{r,s} \dot{\psi}^{r,s}$$

Then the category $\dot{\mathcal{M}}_F$ with the commutativity constraint $\dot{\psi}$ replaced by ψ, gives us the true category of motives $\mathcal{M}_F$

Here we list some fundamental properties of this category of motives.

Theorem 1.2.1. *(i) $\mathcal{M}_F$ is a neutral semisimple Tannakian category over $\mathbb{Q}$*

(ii) There exists a contravariant functor $h : \mathcal{V}_F \to \mathcal{M}_F$ such that $h(X \coprod Y) = h(X) \oplus h(Y)$, $h(X \times Y) = h(X) \otimes h(Y)$. Every effective motive is the image $(h(X), p)$ of an idempotent $p \in \text{End}(h(X))$ for some $X \in \text{Ob}(\mathcal{V}_F)$. Every motive in $\mathcal{M}_F$ is of the form $M(m)$ for some effective motive M and $m \in \mathbb{Z}$.

(iii) For all varieties X, Y, with X pure of dimension d,

$$C^{d+s-r}_{AH} = \text{Hom}(h(X)(r), h(Y)(s)).$$

Morphisms of motives can be expressed in terms of absolute Hodge cycles on varieties by means of (1.3) and (1.2) *(iv) For $* = \tau, \ell$, or dR, the functors H_* on $\mathcal{V}_F$ define $\mathbb{Q}_*$-linear fibre functors on $\mathcal{M}_F$, where $\mathbb{Q}_* = \mathbb{Q}, \mathbb{Q}_l$, or F.*

(v) for M, $N \in \mathrm{Ob}(\mathcal{V}_F)$, $\mathrm{Hom}(M, N)$ coincides with the $\mathbb{Q}$-vector space of families of maps $f_* : H_*(M) \to H_*(N)$, such that f_{dR} preserves the Hodge filtration, f_l is Γ_F-equivariant morphism, and such that the f_* are compatible under the comparison isomorphisms.

Proof. See [DMOS82, II,6.7]. See also [Jan92] for a correction in the proof of (i). □

Remark 1.2.2. The comparison isomorphisms of section 1 can be extended over all the motives in $\mathcal{M}_F$, see [Jan90]. For any $M \in \mathrm{Ob}\mathcal{M}_F$ we have

$$I_{\infty,\tau} : H_\tau(M) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H_{dR}(M) \otimes_{F,\tau} \mathbb{C}$$

and

$$I_{l,\tau} : H_\tau(M) \otimes_{\mathbb{Q}} \mathbb{Q}_l \xrightarrow{\sim} H_l(M)$$

for each extension $\bar{\tau} : \bar{F} \hookrightarrow \mathbb{C}$ of τ.

1.3 Motivic Galois groups

In this section first we recall a few important facts about rigid abelian tensor categories before moving on to neutral Tannakian category and finally to the motivic Galois group.

Let G be an affine group scheme over a field K and denote by $\mathrm{Rep}_K(G)$ the category of finite dimensional representations of G over K. Thus, an object of $\mathrm{Rep}_K(G)$ consists of a finite dimensional vector space V and a homomorphism $g \mapsto g_V : G \to \mathrm{GL}(V)$ of affine group schemes over K. Then $\mathrm{Rep}_K(G)$ is a rigid abelian tensor category and $\mathrm{End}(\mathbb{I}) = K$, where $\mathbb{I}$ denotes an identity object in $\mathrm{Rep}_K(G)$. This is an important example to keep in mind, as neutral Tannakian categories resemble this category.

Let ω (or ω^G) denote the forgetful functor $\mathrm{Rep}_K(G) \to \mathrm{Vec}_K$. The automorphism group of ω defines a functor from the category of K-algebras to sets : For any K-algebra R, $\mathrm{Aut}^{\otimes}(\omega)(R)$ are families (λ_V), $V \in \mathrm{Ob}(\mathrm{Rep}_K(G))$ and λ_V is an R-linear automorphism of $V \otimes R$ such that for V_1, V_2 and $W \in \mathrm{Ob}(\mathrm{Rep}_K(G))$, $\lambda_{V_1} \otimes \lambda_{V_2} = \lambda_{V_1 \otimes V_2}$, $\lambda_{\mathbb{I}}$ is the identity map on R and,

$$\lambda_W \circ (\alpha \otimes 1) = (\alpha \otimes 1) \circ \lambda_V : V \otimes R \to W \otimes R$$

for all G-equivariant morphisms $\alpha : V \to W$. Clearly, every $g \in G(R)$ defines an element of $\mathrm{Aut}^{\otimes}(\omega)(R)$, by its action on the objects of $\mathrm{Rep}_K(G)$

Proposition 1.3.1. *The natural morphism of functors $G \to \mathrm{Aut}^{\otimes}(\omega)$ is an isomorphism.*

Proof. [DMOS82, II,2.8]. □

A homomorphism $\phi : G \to G'$ of affine group schemes over K defines a tensor functor $\omega^{\phi} : \mathrm{Rep}_K(G') \to \mathrm{Rep}_K(G)$ such that $\omega^G \circ \omega^{\phi} = \omega^{G'}$, namely, $\omega^{\phi}(V, r_V) = (V, r_V \circ \phi)$, for (V, r_V) belonging to $\mathrm{Rep}_K(G')$.

Corollary 1.3.2. *Let G and G' be affine group schemes over K.Let Φ : $\mathrm{Rep}_K(G') \to \mathrm{Rep}_K(G)$ be a tensor functor such that $\omega^G \circ \Phi = \omega^{G'}$.Then there exists an unique homomorphism $\phi : G \to G'$ such that $\omega^{\phi} = \Phi$*

Proof. [DMOS82, II,2.9]. □

To see how the properties of G are reflected by $\mathrm{Rep}_K(G)$, we note the following two results.

Proposition 1.3.3. *Let G be an affine group scheme over K.*

(i) G is finite if and only if there exists an object X of $\mathrm{Rep}_K(G)$ such that every object of $\mathrm{Rep}_K(G)$ is a subquotient of an object X^n for some $n \geq 0$.

(ii) G is algebraic if and only if $\mathrm{Rep}_K(G)$ has a tensor generator X.

Proof. [DMOS82, II,2.20]. □

Remark 1.3.4. An object X of $\mathrm{Rep}_K(G)$ is a tensor generator if every object of $\mathrm{Rep}_K(G)$ is isomorphic to a subquotient of $P(X, X^{\vee})$ for some $P \in \mathbb{N}[t, s]$. Here $P(X, X^{\vee})$ is understood as a polynomial expression in X and $X^{\vee}$, with multiplication as tensor product in the category and sums as direct sums.

Proposition 1.3.5. *Let the notations be as above. Then*

(a) If $\mathrm{Rep}_K(G)$ is semisimple category, then ϕ is faithfully flat if and only if ω^{ϕ} is a fully faithful functor.

(b) ϕ is a closed immersion if and only if every object of $\mathrm{Rep}_K(G)$ is isomorphic to a subquotient of an object of the form $\omega^{\phi}(V')$, for $V' \in \mathrm{Ob}(\mathrm{Rep}_K(G))$.

Proof. [DMOS82, II,2.21 & 2.29]. □

Definition 1.3.6. A *Neutral Tannakian category* over K is a rigid abelian tensor category $(\mathcal{C}, \otimes)$, such that $K = \mathrm{End}(\mathbb{I})$ and there exists an exact faithful K-linear tensor functor $\omega : \mathbb{C} \to \mathrm{Vec}_K$. Any such functor is said to be a *fibre functor* on $\mathcal{C}$.

Now we present an important characterisation of rigid abelian $\otimes$-categories.

Theorem 1.3.7. *Let $(\mathcal{C}, \otimes)$ be a rigid abelian tensor category such that* $\mathrm{End}(\mathbb{I}) = K$*, and let $\omega : \mathcal{C} \to \mathrm{Vec}_K$ be any exact faithful K-linear tensor functor. Then*

(a) The functor $\mathrm{Aut}^{\otimes}(\omega)$ from the category of K-algebras to groups is representable by an affine group scheme G.

(b) The functor $\mathcal{C} \to \mathrm{Rep}_K(G)$ defined by ω is an equivalence of tensor categories.

Proof. [DMOS82, II,2.11].

□

Remark 1.3.8.

1. Here, $\mathrm{Aut}^{\otimes}(\omega)$ is the functor that associates to any K-algebra R, the automorphism group of $\omega \otimes R$, $\mathrm{Aut}^{\otimes}(\omega)(R) = \mathrm{Aut}(\omega \otimes R)$. The functor $\omega \otimes R$ is obtained from ω by composing it with the base change functor from K to R

$$\omega \otimes R : \mathcal{C} \to \mathrm{Vec}_K \to (R\text{-mod})$$

$$\omega \otimes R(T) = \omega(T) \otimes_K R$$

for $T \in \mathrm{Ob}(\mathcal{C})$

2. The equivalence in part (b) of the theorem is given by $T \to \omega(T)$.

Thus in view of theorem 1.3.7, every neutral Tannakian category is equivalent (in possibly many different ways) to the category of finite dimensional representations of an affine group scheme.

Example 1.3.9. Let $\mathcal{C}$ be the category $\mathrm{Hod}_{\mathbb{R}}$ of real Hodge structures. An object in $\mathcal{C}$ is a finite dimensional vector space over $\mathbb{R}$ together with a decomposition $V \otimes \mathbb{C} = \oplus_{p,q} V^{p,q}$ (Hodge decomposition) such that $V^{p,q}$ and $V^{q,p}$ are complex conjugate subspaces of $V \otimes \mathbb{C}$.

$$\omega : (V, (V^{p,q})) \mapsto V$$

is a fibre functor. The group scheme corresponding to $\mathrm{Hod}_{\mathbb{R}}$ and ω is the real algebraic group obtained from $\mathbb{G}_m$ by restriction of scalars from $\mathbb{C}$ to $\mathbb{R}$: $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$. The real Hodge structure $(V, V^{p,q}))$ corresponds to the representation of $\mathbb{S}$ on V such that an element $\lambda \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^{\times}$ acts on $V^{p,q}$ as multiplication $\lambda^{-p}\bar{\lambda}^{-q}$. We can write $V = \bigoplus V^n$, where $V^n \otimes \mathbb{C} = \oplus_{p,q=n} V^{p,q}$. The functor $(V, (V^{p,q})) \mapsto (V^n)$ from $\mathrm{Hod}_{\mathbb{R}}$ to the neutral Tannakian category of graded real vector spaces corresponds to a homomorphism $\mathbb{G}_m \to \mathbb{S}$ which on real points is $t \mapsto t^{-1} : \mathbb{R}^* \to \mathbb{C}^*$. The vector spaces V^n are Hodge structures of weight n.

Now, let $\mathbb{R}(1)$ denote the unique real Hodge structure with weight -2 and underlying vector space as $2\pi i\mathbb{R}$. For any integer m define $\mathbb{R}(m) = \mathbb{R}(1)^{\otimes m}$. A *polarization* on a real Hodge structure of weight n is a bilinear form

$$\phi : V \times V \to \mathbb{R}(-n)$$

such that the real valued form

$$(x, y) \mapsto (2\pi i)^n \phi(x, Cy),$$

where C denotes the element $i \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^*$, is positive definite and symmetric.

The *weight cocharacter* is the homomorphism of algebraic groups $w : \mathbb{G}_{m,\mathbb{R}} \to \mathbb{S}$, which is given on points by the natural inclusion

$$\mathbb{R}^* = \mathbb{G}_{m,\mathbb{R}}(\mathbb{R}) \hookrightarrow \mathbb{S}(\mathbb{R}) = \mathbb{C}^*$$

Definition 1.3.10. A $\mathbb{Q}$*-rational Hodge structure* is a finite dimensional vector space V over $\mathbb{Q}$, with a real Hodge structure on $V \otimes \mathbb{R}$, such that the weight decomposition is defined over $\mathbb{Q}$.

The *Tate structure* is defined to be the vector space $\mathbb{Q}(1) := 2\pi i \cdot \mathbb{Q} \subset \mathbb{C}$, with the Hodge structure $\mathbb{Q}(1) = \mathbb{Q}(1)^{-1,-1}$. It has weight -2.

For any integer n, a *polarization* on $\mathbb{Q}$-rational Hodge structure V is a bilinear pairing $\psi : V \to \mathbb{Q}(-n)$ such that $\psi \otimes \mathbb{R}$ is a polarization on the real Hodge structure $V \otimes \mathbb{R}$.

Definition 1.3.11. Let V be a $\mathbb{Q}$-rational Hodge structure and $h : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$ be the corresponding representation of $\mathbb{S}$. The Mumford-Tate group of V (notation $\mathrm{MT}(V)$) is the smallest algebraic subgroup $M \subseteq \mathrm{GL}(V)$, over $\mathbb{Q}$ such that h factors through $M_{\mathbb{R}}$.

In fact the category of all $\mathbb{Q}$-rational Hodge structures is Tannakian. The Mumford-Tate group of a $\mathbb{Q}$-rational Hodge structure V is the automorphism group $\mathrm{Aut}^{\otimes}(\omega)$, where $\omega : \langle V \rangle^{\otimes} \to \mathrm{Vec}_{\mathbb{Q}}$ is the forgetful functor from the Tannakian subcategory generated by the V to the category of vector spaces over $\mathbb{Q}$.

Proposition 1.3.12. *If V is a polarizable $\mathbb{Q}$-rational Hodge structure, then the Mumford-Tate group $\mathrm{MT}(V)$ is reductive.*

Proof. [DMOS82, I,3.6]. □

We shall apply the above theory to the category of absolute Hodge motives. First we fix a field F and an embedding $\tau : F \hookrightarrow \mathbb{C}$. Fix an algebraic closure $\bar{F}$ of F, and let $\bar{\tau} : \bar{F} \hookrightarrow \mathbb{C}$. Now consider the semisimple neutral Tannakian category $\mathcal{M}_F$ of absolute Hodge motives over F and the fibre functor $H_\tau : \mathcal{M}_F \to \mathrm{Vec}_\mathbb{Q}$. Using (1.3.7) we obtain an affine group scheme $G(F, \tau)$, such that H_τ defines an equivalence of categories $\mathcal{M}_F \simeq \mathrm{Rep}_\mathbb{Q}(G(F, \tau))$.

Definition 1.3.13. The affine group scheme $G(F, \tau)$ is called *motivic Galois group*. Let $M \in \mathrm{Ob}\mathcal{M}_F$ and denote by $\mathcal{M}_M$ the Tannakian subcategory of $\mathcal{M}_F$ generated by M and the Tate motive. Let G_M denote the corresponding quotient of $G(F, \tau)$ (i.e the group scheme representing $\mathrm{Aut}^\otimes H_\tau|_{\mathcal{M}_M}$). Then it follows from (1.3.3) that G_M is an algebraic group. We call G_M as the *Mumford-Tate group* of the motive M.

Theorem 1.3.14. *(a) $G(F, \tau)$ is pro-reductive affine group scheme over $\mathbb{Q}$. It is connected if F is algebraically closed and all Hodge cycles relative to τ are absolutely Hodge in $\mathcal{M}_F$.*

(b) If $\bar{\tau} : \bar{F} \hookrightarrow \mathbb{C}$ is an embedding restricting to τ, then there is an exact sequence of group schemes

$$1 \to G(\bar{F}, \bar{\tau}) \xrightarrow{i} G(F, \tau) \xrightarrow{\pi} \Gamma_F \to 1$$

where $G(\bar{F}, \bar{\tau})$ is the automorphism group of $H_{\bar{\tau}} : \mathcal{M}_{\bar{F}} \to Vec_\mathbb{Q}$, and Γ_F is regarded as a constant group scheme. If Hodge cycles relative to τ are absolute Hodge, then $G(\bar{F}, \bar{\tau}) = G^\circ(F, \tau)$, is the identity component of $G(F, \tau)$

Proof. [DMOS82, II,6.22 & 6.23]. □

Remark 1.3.15. (i) In fact $G(F, \tau) = \varprojlim G_M$, for $M \in \mathrm{Ob}(\mathcal{M}_F)$. As $\mathcal{M}_F$ is semisimple, G_M is a reductive algebraic group (see the proofs of [DMOS82, II,2.23 & 6.22]).

(ii) If all Hodge cycles relative to τ are absolute Hodge in $\mathcal{M}_M$, then $G^\circ_M = G_{M_{\bar{F}}}$ (see the proof of [DMOS82, II,6.23] and [Pan94]).

The Betti realization $H_\tau(M)$ carries a $\mathbb{Q}$-rational Hodge structure. Thus, by (1.3.9) we have a homomorphism of algebraic groups $h : \mathbb{S} \to G_{M/\mathbb{R}}$. The couple (G_M, h) is referred as the *Mumford-Tate datum* associated to the motive M.

The following proposition relates the Mumford-Tate group of the Hodge structure $H_\tau(M)$ and the Mumford-Tate group G_M of a motive M.

Proposition 1.3.16. *1. The Mumford-Tate group $MT(H_\tau(M))$ is contained in the identity component G_M° of G_M.*

2. Let G_l be the image of the homomorphism $\rho_l : \Gamma_F \to \mathrm{GL}(H_l(M))$ given by the action of Γ_F on $H_l(M)$. Then G_l is contained in $G_M(\mathbb{Q}_l)$.

3. If Hodge cycles are absolute Hodge in the category $\mathcal{M}_{\tau M}$, for a motive M and its base change τM to $\mathbb{C}$, then $G_M^\circ = MT(H_\tau(M))$ and $MT(H_\tau(M))(\mathbb{Q}_l)$ contains an open subgroup of G_l

Proof. [Pan94, 473-474] & cf lemma 4.2.3 of this document. □

We follow the notations as before.

Proposition 1.3.17. *Let $M \in \mathrm{Ob}\mathcal{M}_F$. If all Hodge cycles relative to the embedding τ are absolute Hodge cycles in the category $\mathcal{M}_M$. Then there exists a finite extension F' of F such that the Mumford-Tate group $G_{M_{F'}}$ is connected.*

Proof. Let $\bar{F}$ be an algebraic closure of F and $\bar{\tau} : \bar{F} \hookrightarrow \mathbb{C}$ be an extension of τ. Let $G_{M_{\bar{F}}}$ denote the Mumford-Tate group of $M_{\bar{F}}$. As $G_{M_{\bar{F}}}$ is reductive, it is the stabilizer of a line $L = \langle u \rangle$ in some tensor space $H_{\bar{\tau}}(M_{\bar{F}})^{\otimes n} \otimes H_{\bar{\tau}}(M_{\bar{F}})^{\vee\otimes m}$ ([DMOS82, I,3.1]). Since $G_{M_{\bar{F}}}$ is isomorphic to the Mumford-Tate group of the Hodge structure $H_\tau(M)$, thus it fixes all Hodge cycles relative to τ on all tensor products of the type $M_{\bar{F}}^{\otimes a} \otimes M_{\bar{F}}^{\vee\otimes b}$ and any element fixed by $G_{M_{\bar{F}}}$ in any such tensor spaces is a Hodge cycle relative to τ. Therefore u is a Hodge cycle on $N_{\bar{F}} := M_{\bar{F}}^{\otimes n} \otimes M_{\bar{F}}^{\vee\otimes m}$. By hypothesis Hodge cycles relative to τ are absolute Hodge cycle in $\mathcal{M}_M$, thus u is an absolute Hodge cycle on $N_{\bar{F}}$. Now the action of the Galois group $\mathrm{Gal}(\bar{F}/F)$ on the vector space of absolute Hodge cycles on the motive $N_{\bar{F}}$ is through a finite quotient ([Jan90, 2.19]). Thus let F' be a finite extension of F such that absolute Hodge cycles on $N_{\bar{F}}$ are invariant under the action of $\mathrm{Gal}(\bar{F}/F')$. This implies that u is an absolute Hodge cycle on $N_{F'} := M_{F'}^{\otimes n} \otimes M_{F'}^{\vee\otimes m}$, thus it is fixed by $G_{M_{F'}}$. This implies that $G_{M_{F'}} \subseteq G_{M_{\bar{F}}}$. By the remarks above we have $G_{M_{\bar{F}}} = G^\circ_{M_{F'}} \subseteq G_{M_{F'}}$. Thus $G_{M_{F'}} = G_{M_{\bar{F}}}$ is connected.

□

Remark 1.3.18. (a) An absolute Hodge cycle on a motive $M \in \mathrm{Ob}\mathcal{M}_F$ is an element of $H(M) = H_{dR}(M) \times \prod_l H_l(M) \times \prod_\tau H_\tau(M)$, satisfying similar conditions as in the definition1.1.3. See [Jan90, 2.10] for details.

(b) By theorem 1.2.1 every motive M is of the form $(h(X), p)(m)$ for $X \in \mathrm{Ob}\mathcal{V}_F$ and $p \in \mathrm{End}(h(X))$ and $m \in \mathbb{Z}$. Note that p is an absolute Hodge cycle in $C^d_{AH}(X \times X)$, where d is the dimension of X. If F' is any field extension of F then by $M_{F'}$ we mean the motive $(h(X_{F'}), p)(m) \in \mathrm{Ob}\mathcal{M}_{F'}$. This makes sense since because by definition of an absolute Hodge cycle, $p \in C^d_{AH}(X_{F'} \times X_{F'})$.

1.4 Motives of abelian varieties

In this section we discuss abelian motives. The Tannakian subcategory of $\mathcal{M}_F$, generated by the varieties of dimension zero, is called the category of *Artin motives*. Let $\mathcal{M}_F^{av}$ be the Tannakian category generated by the abelian varieties, Artin motives and the Tate motive. Let $\mathrm{Hod}_{\mathbb{Q}}$ denote the category of $\mathbb{Q}$-rational Hodge structures.

Theorem 1.4.1. *For any algebraically closed field F and any embedding $\tau : F \hookrightarrow \mathbb{C}$, the functor $H_\tau : \mathcal{M}_F^{av} \to \mathrm{Hod}_{\mathbb{Q}}$ is fully faithful.*

Proof. [DMOS82, II,6.25]. □

It is natural to ask, which varieties have their motives in $\mathcal{M}_F^{av}$. A partial answer is the following result of Deligne:

Proposition 1.4.2. *The motive $h(X) \in \mathrm{Ob}(\mathcal{M}_F^{av})$ if,*

1. *X is a curve .*
2. *X is an unirational variety of dimension ≥ 3.*
3. *X is a Fermat Hypersuface.*
4. *X is $K3$-surface.*

Proof. [DMOS82, II,6.26].

□

Corollary 1.4.3. *Every Hodge cycle on a variety that is a product of abelian varieties, zero dimensional varieties, and varities of type (1), (2), (3), (4), as above in the proposition, is absolutely Hodge.*

Proof. [DMOS82, II,6.27]. □

Remark 1.4.4. (a) If X is an algebraic variety of the type given in the corollary and $M := h^i(X)$, then from 1.3.16 it follows that $MT(H_\tau(M)) = G^0_M$. It also follows from (1.3.17) that after a finite base extension we can assume that G_M is connected.

(b) In case of an abelian variety A/F, the situation is quite elegant. Since, $h(A) = \bigoplus_i \bigwedge^i h^1(A)$, thus $G_{h^1(A)} = G_{h(A)}$. We will denote this reductive algebraic group by G_A

Chapter 2

Tractable abelian varieties

In this chapter we recall the notions of *Tractable* abelian varieties and *Weak Mumford-Tate lifts* as developed in [Noo06], [Noo09] and [Noo10].

2.1 Tractable abelain varieties

In [Noo10] the notion of admissible representation and tractable abelian varieties are defined, we recall them here.

Let K be a field of characteristic 0 and let $\overline{K}$ be an algebraic closure. Let G^s be a linear algebraic group over K such that $G^s_{\overline{K}}$ is almost simple of classical type, A, B, C or D. Let V^s be a faithful K-linear representation of G^s. We say that V^s is an *admissible* representation of G^s in the following cases.

1. $G^s_{\overline{K}}$ is of type A_n and $V^s \otimes_K \overline{K}$ is a multiple of the direct sum of the representations of highest weights ϖ_1 and ϖ_n.

2. $G^s_{\overline{K}}$ is of type B_n and $V^s \otimes_K \overline{K}$ is a multiple of the representation of highest weight ϖ_n .

3. $G^s_{\overline{K}}$ is of type C_n and $V^s \otimes_K \overline{K}$ is a multiple of the represntation of highest weight ϖ_1 .

4. $G^s_{\overline{K}}$ is of type D_n and $V^s \otimes_K \overline{K}$ is a multiple of the representation of highest weight ϖ_1 .

5. $G^s_{\overline{K}}$ is of type D_n and $V^s \otimes_K \overline{K}$ is a multiple of the direct sum of the representation of highest weights ϖ_{n-1} and ϖ_n .

In the first three cases we say that $(G^s,\ V^s)$ is of type A_n, B_n or C_n and in the last two cases we say that $(G^s,\ V^s)$ is of type $D_n^{\mathbf{H}}$ or of type $D_n^{\mathbf{R}}$ respectively.

Let A be an abelian variety over $\mathbb{C}$ and let G_A denote the Mumford-Tate group of the absolute Hodge motive $h^1(A)$. Denote $V := \mathrm{H}^1_B(A(\mathbb{C}), \mathbb{Q})$ and consider it as a representation of G_A. We say that A is *strictly tractable* if the following holds :

- there exists a totally real number field K and an almost simple linear algebraic group G^s over K such that $G_A^{der} = \mathrm{Res}_{K/\mathbb{Q}} G^s$,
- as a representation of G_A^{der}, the cohomolgy group V is the restriction of scalars of an admissible representation V^s of G^s,
- if $(G^s,\ V^s)$ is of type $D_n^{\mathbf{R}}$ then every character space in $V \otimes \overline{Q}$ for the action of the centre of $G_{A/\overline{\mathbb{Q}}}$ is an admissible representation of a factor of $G^{der}_{A/\overline{\mathbb{Q}}}$ and
- the above conditions do not hold for any proper abelian sub-variety of A.

We shall call A *tractable* if it is isogenous to a product $\prod_{i=1}^m A_i$ of strictly tractable abelian variety A_i and $G_A^{der} \cong \prod_{i=1}^m G_{A_i}^{der}$. If $K \subset \mathbb{C}$ is a sub-field, an abelian variety A/K is (strictly) tractable if $A_{\mathbb{C}}$ is so, and if G_A is connected.

2.2 Mumford-Tate lifts

In [Noo06] section 2, one comes across the notion of 'Mumford-Tate lifts' and 'Mumford-Tate decomposed abelian variety'. We recall these notions here.

Let $M \in \mathcal{M}_F^{av}$ be an abelian motive, with Mumford-Tate datum (G_M, h_M). This means that the Hodge structure on $V_M := H_\tau(M)$ is defined by the morphism of the real algebraic groups $h_M : \mathbb{S} \to G_{M/\mathbb{R}}$. We say that an abelian motive N with Mumford-Tate datum (G_N, h_N) provides a *Mumford-Tate lift* of M, if there exists a central isogeny $\pi : G_N \to G_M$, such that $\pi_{\mathbb{R}} \circ h_N = h_M$. We say that M is *Mumford-Tate liftable* if there exists an abelian motive $N/\mathbb{C}$ giving a Mumford-Tate lift of M and such that the morphism $\pi : G_N \to G_M$ is not an isomorphism. We say that M is *Mumford-Tate unliftable* if it is not

Mumford-Tate liftable.

If there exists a central isogeny $\pi^{der} : G_N^{der} \to G_M^{der}$ such that

$$p_N^{ad} \circ h_N = p_M^{ad} \circ h_M \tag{2.1}$$

where p_N^{ad} and p_M^{ad} denotes the projections $G_N \to G_N^{ad}$ and $G_M \to G_M^{ad}$, respectively, then N is said to be a *weak Mumford-Tate lift* of M. As, π^{der} is a central isogeny it induces an isomorphism $G_M^{ad} \to G_N^{ad}$, this is the reason that 2.1 is well defined. Finally, M is *essentially Mumford-Tate unliftable* if there does not exists any abelian motive $N/\mathbb{C}$ giving a weak Mumford-Tate lift of M for which π^{der} is not an isomorphism.

The following results give us the existence of the weak Mumford-Tate lifts for abelian varieties and motives and relates them with tractable abelian varieties.

Theorem 2.2.1. *For every abelian variety $A/\mathbb{C}$ there exists a weak Mumford-Tate lift $B/\mathbb{C}$ of A such that B is tractable*

Proof. See [Noo06, 2.12] and [Noo09, 1.8] . □

Corollary 2.2.2. *For $M \in \mathcal{M}_{\mathbb{C}}^{av}$, there exists a tractable abelian variety $B/\mathbb{C}$ which provides a weak Mumford-Tate lift for M.*

Proof. This is an immediate consequence of 2.2.1, cf. [Noo06, 2.15]. □

Now let us see some important properties of (weak) Mumford-Tate lifts. First a general result:

Proposition 2.2.3. *Suupose $A_{/\mathbb{C}}$ and $B_{/\mathbb{C}}$ are abelian varieties over $\mathbb{C}$. Let (G_A, h_A) and (G_B, h_B) be the associated Mumford-Tate data and assume that there exists an isomorphism $G_A^{ad} \cong G_B^{ad}$ such that $p_A^{ad} \circ h_A = p_B^{ad} \circ h_B$.*
Let $F \subset \mathbb{C}$ be an algebraically closed field. Then there exists an abelian variety A/F such that $A \otimes_F \mathbb{C} = A$ if and only if there exists an abelian variety B/F such that $B \otimes_F \mathbb{C}$.

Proof. [Noo06, 3.1].

□

Corollary 2.2.4. *The statement of the proposition holds if $B_{/\mathbb{C}}$ is a (weak) Mumford-Tate lift of $A_{/\mathbb{C}}$.*

Proof. This follows from the defintion of (weak) Mumford-Tate lifts and the proposition. □

Now let us turn to 'lifts' of motives. From now on for a motive $M \in \mathcal{M}_F$, $\langle M \rangle$ will denote the Tannakian sub category generated by M and $(\mathbf{CM})_F$ denotes the Tannakian sub category generated by motives of abelian varieties with complex multiplication.

Proposition 2.2.5. *Let A and B be abelian varieties over an algebraically closed field $F \subset \mathbb{C}$ such that $B_{/\mathbb{C}}$ provides a weak Mumford-Tate lift of $A_{/\mathbb{C}}$. Then $h^1(A) \in \langle h^1(B), (\mathbf{CM})_F \rangle$.*
Taking the Betti realization , this inclusion induces a a map between the corresponding Mumford-Tate groups. On the derived group, this map is the map $\pi^{der} : G_B^{der} \to G_A^{der}$ given by the structure of B as weak Mumford-Tate lift of A.

Proof. [Noo06, 3.4] □

Corollary 2.2.6. *If $M \in \mathcal{M}_F^{av}$. Then there exists a tractable abelian variety B over F such that*

$$M \in \langle h^1(B), \mathbf{CM}_F \rangle$$

Proof. [Noo06, 3.6]. □

Chapter 3

Quotients and Neatness

In this chapter, first we recall some generalities on categorical quotient. Then we define the 'natural adjoint' groups of reductive algebraic groups and a corresponding categorical quotient. Finally, we introduce *weakly neat elements* and establish some important facts about them.

3.1 Categorical quotient

Let G be a group scheme over a fixed base scheme S. Let $\mu : G \times_S G \to G$, $\beta : G \to G$ and $e : S \to G$, denote the S-morphisms defining multiplication, inverse and unit element.

Definition 3.1.1. A group scheme G/S as above, *acts* on a scheme X/S, if an S-morphism $\sigma : G \times_S X \to X$ is given, such that :

(a)

$$\begin{array}{ccc} G \times_S G \times_S X & \xrightarrow{1_G \times \sigma} & G \times_S X \\ \downarrow{\scriptstyle \mu \times 1_X} & & \downarrow{\scriptstyle \sigma} \\ G \times_S X & \xrightarrow{\sigma} & X \end{array}$$

(b) The composition

$$X \cong S \times_S X \xrightarrow{e \times 1_X} G \times_S X \xrightarrow{\sigma} X$$

equals 1_X

In other words for every S-scheme T, σ induces a left action of the group $G(T)$ on the set $X(T)$. We usually denote this action on points by

$$(g, x) \mapsto g \cdot x$$

Definition 3.1.2. For a scheme S-scheme T, let $f \in X(T)$. Then $\sigma \circ (1_G \times f)$ is a morphism from $G \times_S T$ to X. Define the morphism

$$\psi_f^G : G \times_S T \to X \times_S T$$

as $(\sigma \circ (1_G \times f), p_2)$. The image of ψ_f^G is called the *orbit* of f and is denoted by $O(f)$.

Now we can define the notion of a categorical quotient. Notations are as before.

Definition 3.1.3. Given an action σ of G/S on X/S, a pair (Y, π) consisting of a scheme Y/S and an S-morphism $\pi : X \to Y$ is called a *categorical quotient* (of X by G) if

(i) the diagram :

$$\begin{array}{ccc} G \times_S X & \xrightarrow{\sigma} & X \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\pi} & Y \end{array}$$

commutes,

(ii) given any pair (Z, ω) consisting of a scheme Z over S and a S-morphism $\omega : X \times Z$ such that $\omega \circ \sigma = \omega \circ p_2$, i.e (i) holds for Z and ω, then there is a unique morphism $\chi : Y \to Z$ such that $\omega = \chi \circ \pi$.

The pair (Y, π) will be called a *universal categorical quotient* if, for all morphisms $Y' \to Y$, we put $X' = X \times_Y Y'$ and let $\pi' : X' \to Y'$ denote p_2, then (Y', π') is a categorical quotient of X' by G.

The following is a well-known existence result on universal categorical quotient. We work over a field K over characteristic zero.

Theorem 3.1.4. *Let X be an affine scheme over K, let G be a reductive algebraic group , and let $\sigma : G \times X \to X$ be an action of G on X. Then an universal categorical quotient (Y, π) exists. If X is algebraic then Y is algebraic. Moreover X noetherian implies Y noetherian.*

Proof. [MFK94, 1.1]. □

Remark 3.1.5. Let $X = \operatorname{Spec} R$; then G acts dually on R. If $R_0 \subset R$ be the ring of invariants. Let $Y = \operatorname{Spec}(R_0)$ and let $\pi : X \to Y$ be induced by the inclusion of R_0 in R. Then (Y, π) is the universal categorical quotient for the action σ of G on X (cf [MFK94]).

3.2 The variety $\operatorname{Conj}' G$

Since we will work exclusively with smooth affine algebraic groups over a field of characteristic zero, thus henceforth we will suppress the terms 'smooth' and 'affine' and refer them as algebraic groups over the base field.

Let G be a reductive algebraic group over field K of characteristic zero, then the adjoint group G^{ad} is also reductive. The algebraic group G^{ad} acts on G by conjugation action. Thus by theorem 3.1.4 we have an universal categorical quotient $(\operatorname{Conj}(G), \operatorname{Cl})$. The pair $(\operatorname{Conj}(G), \operatorname{Cl})$ is also the universal categorical quotient $(\operatorname{Conj}(G), \operatorname{Cl})$ for the action of G on itself by conjugation.

Now assume further that G is connected. Let $\bar{K}$ be a separable algebraic closure of K. Then the derived algebraic group $G^{der}_{\bar{K}}$ over $\bar{K}$, is an almost direct product of almost simple subgroups G_i, for $i \in I$ a finite indexing set. Let $J \subset I$ such that for $i \in J$, $G_i \cong \mathrm{SO}(2k_i)_{\bar{K}}$ with $k_i \geq 4$. The Galois group $\Gamma_K := \operatorname{Gal}(\bar{K}/K)$ acts on $G^{der}_{\bar{K}}$. This action preserves the type of the almost simple factors of $G^{der}_{\bar{K}}$ and moreover it also preserves the factors of the form $\mathrm{SO}(2k_i)_{\bar{K}}$ (i.e it does not mix with factors of spin type). Thus J is stable under the action of Γ_K. Denote $G_i^\natural := \mathrm{O}(2k_i)_{\bar{K}}$, so that G_i is the identity component of $G_i^\natural$. Then for each $i \in J$, the algebraic group $G_i^\natural/G_i$ is isomorphic to the constant algebraic group $\{\pm 1\}$. Thus we get a constant algebraic group $\Lambda(G)_{\bar{K}} = \prod_{i\in J} G_i^\natural/G_i$ on which Γ_K acts by permutations of the factors. Now we know that that the category of finite groups endowed with a continuous action of Γ_K is equivalent to the category of etale algebraic groups over K. Thus we get an etale algebraic group $\Lambda(G)$ over K. Now define another algebraic group

$$G^{\natural ad} = \prod_{i\in J} G_i^{\natural ad} \times \prod_{i\in I\setminus J} G_i^{ad} \supset G^{ad}_{\bar{K}}$$

Denote by $G^\natural = \prod_{i\in J} G_i^\natural \times \prod_{i\notin J} G_i$. The reductive group $G^{\natural ad}$ has a natural action on $\mathcal{Z}(G_{\bar{K}}) \times G^\natural$. This gives an action of $G^{\natural ad}$ on $\mathcal{Z}(G_{\bar{K}}) \times \prod_{i\in I} G_i$. More explicitly on the $\bar{K}$-valued points this action can be described as follows:

$$(\bar{s}_i) \cdot (x, (g_i)) \mapsto \left(x, \left(s_i g_i s_i^{-1}\right)\right),$$

where $(\bar{s}_i) \in G^{\natural ad}(\bar{K})$ is a lifting of $(s_i) \in G^\natural(\bar{K})$, $x \in \mathcal{Z}(G_{\bar{K}})$ and $(g_i) \in \prod_{i\in I} G_i$. Since $G^{der}_{\bar{K}}$ is an almost direct product of G_i's, we get an action of

$G^{\natural ad}$ on $G_{\bar{K}} = \mathcal{Z}(G_{\bar{K}}) \times G^{der}_{\bar{K}}$ extending the action of $G^{ad}_{\bar{K}}$. Denote the categorical quotient of $G_{\bar{K}}$ under this action of $G^{\natural ad}$ as $\mathrm{Conj}'(G)_{\bar{K}}$. By the properties of categorical quotients ([Bor91, VI,6.10 & 6.16], this action induces an action of the constant algebraic group $\Lambda(G)_{\bar{K}} = G^{\natural ad}/G^{ad}_{\bar{K}}$ on $\mathrm{Conj}(G)_{\bar{K}}$, such that quotient for this action is isomorphic to $\mathrm{Conj}'(G)_{\bar{K}}$. This in turn gives us an action of $\Lambda(G)$ on $\mathrm{Conj}(G)$. The quotient of $\mathrm{Conj}(G)$ under this action, is denoted as $\mathrm{Conj}'(G)$ and the quotient map is again denoted as $\mathrm{Cl} : G \to \mathrm{Conj}'(G)$.

The variety $\mathrm{Conj}(G)_{\bar{K}}$ is the quotient of a maixmal torus of $G_{\bar{K}}$ for the action of the Weyl group W of G. The variety $\mathrm{Conj}'(G)_{\bar{K}}$ is also a quotient of a maximal torus for the action of a group $\tilde{W}$, which is an extension of $\Lambda(G)$ by W. See [Noo09] and also [Hum95] chapter 3 for more details on such 'quotients'.

Proposition 3.2.1. *Let M be an abelian motive and suppose Ω is an algebraic closed field containing $\mathbb{Q}_l$. Let $x, y \in G_M(\Omega)$ be conjugate under the action of $G^{\natural ad}_M$. Then x and y have the same characteristic polynomial acting on $H_\sigma(M)$*

Proof. Denote by V the vector space $H_\sigma(M)$ and consider the representation $\mathrm{GL}(V_{/\bar{\mathbb{Q}}})$, of the algebraic group $G_{M/\bar{\mathbb{Q}}}$. This representation gives us a representation of $G^{der}_{M/\bar{\mathbb{Q}}}$, which in turn gives a representation of $\prod G_i$, G_i's being the almost direct factors of $G^{der}_{M/\bar{\mathbb{Q}}}$. Now consider the decomposition $V_{/\bar{\mathbb{Q}}} = \oplus_k W_k$, where each W_k is an irreducible representation of $G^{der}_{M/\bar{\mathbb{Q}}}$. We can further decompose each $W_k = \otimes_i W_{ki}$, where W_{ki} is a representation of G_i (other G_j's acting trivially). Now, for the factors G_i of type $\mathrm{SO}(2m_i)_{/\bar{\mathbb{Q}}}$, the corresponding representation W_{ki} extends to a representation of $G^\natural_i := \mathrm{O}(2m_i)_{/\bar{\mathbb{Q}}}$. Repeating this procedure for all such G_i's, we see that W_k becomes a representation of $G^\natural_M = \prod_{i \in J} G^\natural_i \times \prod_{i \in I/J} G_i$ extending the representation of $\prod G_i$.

Now look at the Ω-valued points x and y of the algebraic group $G^\natural_M$. As G_M is reductive, we can write $x = z \cdot g$ and $y = z' \cdot g'$, with $z, z' \in \mathcal{Z}(G_M)(\Omega)$ and $g, g' \in G^{der}_M(\Omega)$. Since x and y are conjugate under the action of $G^{\natural ad}_M$, thus $z = z'$ and by the discussion in the preceding paragraph it is clear that g and g' have same characteristic polynomial. Since z is in the center it commutes with g, and therefore z and g can be simultaneously diagonalized. Thus, the eigenvalues of $z \cdot g$, are products of the eigenvalues of z and g. Similarly for z and g'. As $x = z \cdot g$ and $y = z \cdot g'$ have same eigenvalues they have the same characteristic polynomial. □

3.3 Neat and weakly neat elements

Let G be a reductive algebraic group over a field K of characteristic 0. Let Ω be an algebraically closed field containing K.

Definition 3.3.1. A semisimple element $g \in G(\Omega)$ is *neat* [Bor01] if the subgroup $\mathrm{Eig}(g)$ of Ω^* generated by the eigenvalues of g, in some faithful representation of G , does not contain any roots of unity other than 1.

Being neat is independent of the representation. Fix a faithful K-linear representation V of G.

Definition 3.3.2. A semisimple element $g \in G(\Omega)$ is said to be *weakly neat* ([Noo09], [Noo10]) if the the only root of unity amongst the quotients $\lambda\mu^{-1}$, with λ, μ being the eigenvalues of g, is 1.

Note that if g is neat then its weakly neat.

Let $\phi_n : G \to G$, denote the n th power map. Let Y/K denote any one of the algebraic varieties $\mathrm{Conj}(G)$ or $\mathrm{Conj}'(G)$. Clearly, ϕ_n is equivariant for the action of conjugation. Since Y is an universal categorical quotient, this implies that ϕ_n induces a map $\bar{\phi}_n : Y \to Y$ such that the following diagram is commutative :

$$\begin{array}{ccc} G & \xrightarrow{\phi_n} & G \\ {\scriptstyle \mathrm{Cl}}\downarrow & & \downarrow{\scriptstyle \mathrm{Cl}} \\ Y & \xrightarrow{\bar{\phi}_n} & Y \end{array} \tag{3.1}$$

The next two results are stated in [Noo09, 3.2] and [Noo10, 7.4]. For completeness we present their proofs here.

Proposition 3.3.3. *Let Ω be an algebraically closed field containing K. Let V be a faithful K-linear representation of G and $\alpha, \beta \in G(\Omega)$ be two weakly neat elements having same characteristic polynomial in the representation V. If $\bar{\phi}_n(\mathrm{Cl}(\alpha)) = \bar{\phi}_n(\mathrm{Cl}(\beta))$, then $\mathrm{Cl}(\alpha) = \mathrm{Cl}(\beta)$.*

Proof. It suffices to assume that $K = \Omega$. Let d be the dimension of the representation V. Firstly, consider an inclusion of tori $T \subseteq \mathbb{G}^d_{m/K}$. The symmetric group S_d acts on $\mathbb{G}^d_{m/K}$ by permuting the factors. Let T_{wn} be the set defined as:

$$T_{wn} = \{t \in T(\Omega) | t \text{ is weakly neat and } \mathcal{Z}_{S_d}(t) = \mathcal{Z}_{S_d}(T)\}$$

where $\mathcal{Z}_{S_d}(*)$ denotes centraliser of $*$. Note that $t \in T_{wn}$ implies $t^m \in T_{wn}$, for all positive integers $m \geq 1$. Now consider two elements $t_1, t_2 \in T_{wn}$ such that $\phi_n(t_1) = \phi_n(t_2)$ i.e ${t_1}^n = {t_2}^n$ and suppose that they have same characteristic polynomial. This implies that there exists $\tau \in S_d$ such that $\tau \cdot t_1 = t_2$. Thus,

$$\tau \cdot {t_1}^n = {t_2}^n = {t_1}^n$$

and from the definition of T_{wn} it follows that $\tau \in \mathcal{Z}_{S_d}(T)$. Hence, $\tau|_T = id$ and $t_1 = t_2$.

Now we are given that α is semisimple, thus it lies in a maximal torus $\mathbb{G}_m^d$ of the connected linear algebraic group $\mathrm{GL}(V)$. If $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{G}_m^d$, then we define a torus :

$$T^\alpha = \{(t_1, \cdots, t_d) \in \mathbb{G}_m^d | t_i = t_j \text{ if } \alpha_i = \alpha_j\}$$

We shall apply the preceding arguments to this torus. By construction it follows that $\alpha \in T^\alpha_{wn}$. According to the hypothesis $\bar{\phi}_n(\mathrm{Cl}(\alpha)) = \bar{\phi}_n(\mathrm{Cl}(\beta))$. From, the commutativity of diagram (3.1), it follows that $\mathrm{Cl}(\alpha^n) = \mathrm{Cl}(\beta^n)$. This mean that α^n and β^n are conjugate in $G(\Omega)$. Thus, there exists a weakly neat and semisimple element $\gamma \in G(\Omega)$, such that $\alpha^n = \gamma^n$, $\mathrm{Cl}(\beta) = \mathrm{Cl}(\gamma)$. Thus by the hypothesis characteristic polynomial of α is same as that of γ. As $\gamma^n = \alpha^n \in T^\alpha_{wn}$, it follows that $\gamma \in T^\alpha_{wn}$. Now from the discussion at the beginning it follows that $\alpha = \gamma$. Thus, $\mathrm{Cl}(\alpha) = \mathrm{Cl}(\beta)$.

□

In the course of the proof we have proved the following result for $G = \mathrm{GL}_d$

Proposition 3.3.4. *let Ω be an algebraically closed field, d a positive integer. Let $x, y \in \mathrm{GL}_d(\Omega)$ be two weakly neat elements such that $x^n = y^n$ for some positive integer n, and x and y have same characteristic polynomial. Then $x = y$*

Chapter 4

ℓ-independence for motives with good reduction

4.1 Models and criteria for good reduction

Let F be a field, v a discrete valuation of F and O_v the valuation ring of v; the residue field $O_v/\mathfrak{m}_v$ of v will be denoted by k_v. Let $\bar{F}$ be a separable algebraic closure of F and $\bar{v}$ an extension of v to $\bar{F}$. We denote the inertia group and decomposition group of $\bar{v}$ by $I_{\bar{v}}$ and $D_{\bar{v}}$, respectively. We have the inclusions $I_{\bar{v}} \subset D_{\bar{v}} \subset \mathrm{Gal}(\bar{F}/F)$. Let F_v be the completion of F with respect to the valuation v and $\bar{F}_{\bar{v}}$ is the localization of $\bar{F}$ at $\bar{v}$. Then $\mathrm{Gal}(\bar{F}_{\bar{v}}/F_v)$ can be identified with the decomposition group $D_{\bar{v}}$. We also have a canonical isomorphism

$$D_{\bar{v}}/I_{\bar{v}} \cong \mathrm{Gal}(\bar{k}_v/k_v)$$

where $\bar{k}_v$, the residue field of $\bar{v}$, is an algebraic closure of k_v. A *geometric Frobenius at* v is an element of $D_{\bar{v}}$ that induces on $\bar{k}_v$, the inverse of the Frobenius automorphism $x \mapsto x^{q_v}$, where q_v is the order of the field k_v.

Definition 4.1.1. Let F and v be as above. Let $S = \mathrm{Spec}(\mathcal{O}_v)$ and X be an algebraic variety over F. A *model* for X over S is a scheme $\mathcal{X} \xrightarrow{f} S$ whose generic fibre is isomorphic to X and f is surjective and flat.

Definition 4.1.2. Let F and v be as before. A smooth and proper algebraic variety X is said to have good reduction at v if there exists a proper model $\mathcal{X}$ over S such that the morphism $\mathcal{X} \to S$ is smooth.

There is the following classical result.

Proposition 4.1.3. *Let X be a smooth projective variety defined over a number field F. Then X has good reduction at all but finitely many valuations.*

Proof. [HS00, A.9.1.6]. □

In the case of an abelian variety good reduction can be expressed in terms of the *Néron model*. Let A be an abelian variety over F. The Néron minimal model $\mathcal{A}_v$ of relative to v is a smooth group scheme of finite type over $\mathcal{O}_v$, together with an isomorphism $\mathcal{A}_v \times_{\mathcal{O}_v} F \simeq A$, which represents the functor

$$Y \mapsto \mathrm{Hom}_F(Y \times_{\mathcal{O}_v} F, A)$$

on the category of schemes Y smooth over $\mathcal{O}_v$. The abelian variety A has good reduction at v if and only if $\mathcal{A}_v$ is proper over $\mathcal{O}_v$, i.e. is an abelian scheme over $\mathcal{O}_v$ (cf. [ST68]). In the case of an abelian variety we also have the criterion of *Néron-Ogg-Shafarevich*, for detecting good reduction at the given valuation. We shall describe it here. First we need to fix some more notations and defnitions.

A Galois extension E of F contained in $\bar{F}$ is unramified at v if and only if E is fixed by $I_{\bar{v}}$. More generally, if $\mathrm{Gal}(\bar{F}/F)$ acts on a set T, one says that T is *unramified* at v if $I_{\bar{v}}$ acts trivially on it; this does not depend on the choice of $\bar{v}$ because the inertia groups of two such choices are conjugate in $\mathrm{Gal}(\bar{F}/F)$. In other words, T is unramified at v if and only if the decomposition group $D_{\bar{v}}$ acts on T through its homomorphic image $\mathrm{Gal}(\bar{k}_v/k_v)$.

If $m \in \mathbb{Z}$ is prime to $\mathrm{char}(F)$, we put

$$A_m = \mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, A(\bar{F}))$$

Thus, A_m is the group of points of order dividing m in the group $A(\bar{F})$ of $\bar{F}$-points of A. It is known that A_m is a free $\mathbb{Z}/m\mathbb{Z}$-module of rank $2\dim(A)$ on which $\mathrm{Gal}(\bar{F}/F)$ acts continuously, i.e through a finite quotient by an open subgroup.

Now, for any prime number $l \neq \mathrm{char}(F)$, denote

$$T_l(A) = \lim_{\leftarrow} A_{l^n} = \mathrm{Hom}(\mathbb{Q}_l/\mathbb{Z}_l, A(\bar{F}))$$

This is a free module of rank $2\dim(A)$ over $\mathbb{Z}_l$; the group $\mathrm{Gal}(\bar{F}/F)$ acts continuously on $T_l(A)$.

Theorem 4.1.4. *Let A be an abelian variety over F. Let l be a prime number different from the residual characteristic* $\mathrm{char}(k_v)$*. The following properties are equivalent* *(a) A has good reduction at v.*

(b) A_m is unramified at v for all m prime to $\mathrm{char}(k_v)$.

(b') There exists infinitely many integers m, prime to $\mathrm{char}(k_v)$*, such that A_m is unramified at v.*

(c) $T_l(A)$ is unramified at v

Proof. [ST68, Theorem 1]. □

Let notations be as before. We say that the variety X/F has *potential good reduction* at v if there exists a finite extension F' of F and a prolongation v' of v to F' such that $X \times_F F'$ has good reduction at v'. If we consider abelian varieties, then using the Néron-Ogg-Shafarevich condition we can deduce some criteria for potential good reduction.

Let l be a prime number different from the residue characteristic, and let

$$\rho_l : \mathrm{Gal}(\bar{F}/F) \to \mathrm{Aut}(T_l) \tag{4.1}$$

denote the l-adic representation corresponding to the Galois module $T_l = T_l(A)$, for the abelian variety A/F.

Theorem 4.1.5. *(i) The abelian variety A has potential good reduction at v if and only if the image by ρ_l of the inertia group $I_{\bar{v}}$ is finite.*

(ii) When this is the case, the restriction of ρ_l to $I_{\bar{v}}$ is independent of l in the following sense : Its kernel is the same for all l, and its character has values in $\mathbb{Z}$ independent of l

Proof. [ST68, Theorem 2]. □

Corollary 4.1.6. *Suppose the residue field k is finite of characteristic p, and that, for some $l \neq p$, the image of* $\mathrm{Gal}(\bar{F}/F)$ *in* $\mathrm{Aut}(T_l)$ *is abelian. Then A has potential good reduction at v.*

Proof. [ST68] corollary 1 of theorem 2. □

To finish of this section, we discuss criterion for good reduction of a CM type abelian variety. We assume now that characteristic of F is 0. For an abelian variety A/F we denote by $\mathrm{End}(A)$ the F-endomorphisms of A and $\mathrm{End}^0(A) := \mathbb{Q} \otimes \mathrm{End}(A)$.

Definition 4.1.7. Let A be a simple abelian variety over a field F. We say that A is of *CM type* if $K := \mathrm{End}^0(A)$ is a CM-field of degree $2 \dim A$ over $\mathbb{Q}$. The abelian variety A is said to be with *complex multiplication by K* over F.

Now we have the general definition of complex multiplication.

Definition 4.1.8. An abelian variety A over a field F is said to have *complex multiplication* if it is isogeneous to a product of simple abelain varieties $\prod A_i^{e_i}$, where A_i 's are pairwise non-isogeneous simple abelian varieties with complex multiplication by K_i over the field F.

If we take the notations of the previous definition then it is known that there exists CM-fields $\tilde{K_i}$, such that $[\tilde{K_i} : \mathbb{Q}] = 2 \dim A^{e_i}$ and there are injections $\tilde{K_i} \hookrightarrow \mathrm{End}^0(A_i^{e_i})$. Now denote $K := \oplus \tilde{K_i}$ and let

$$R := K \cap \mathrm{End}(A).$$

Proposition 4.1.9. *Let A/F be an abelian variety with complex multiplication. The representation ρ_l attached to the Tate module T_l,* (4.1) *is a homomorphism of* $\mathrm{Gal}(\bar{F}/F)$ *into the group $R_l = \mathbb{Z}_l \otimes R$. In particular,* $\mathrm{Im}(\rho_l)$ *is a commutative group.*

Proof. [ST68, Thm.5, cor.2]. □

4.2 ℓ-independence in the good reduction case

In what follows, we will assume that the base field F is a number field, with a fixed embedding $\tau : F \hookrightarrow \mathbb{C}$. Fix an algebraic closure $\bar{F}$ of F. Let v be a given valuation on F. We also fix a geometric Frobenius element Fr_v in Γ_F. As described in section (1.3.16), the action of the absolute Galois group Γ_F of a field F on the l-adic realization of an absolute Hodge motive $M \in \mathrm{Ob}(\mathcal{M}_F)$ induces a map $\rho_{M,l} : \Gamma_F \to \mathrm{GL}(H_l(M))$. It is also known that that $Im(\rho_{M,l}) \subset G_M(\mathbb{Q}_l)$ (1.3.16), where G_M is the Mumford-Tate group of the motive M. We will give a proof of this last statement in one of the following lemmas (4.2.3). As G_M is reductive group, the construction (3.2) gives us an algebraic variety $\mathrm{Conj}'G_M$ over $\mathbb{Q}$. Recall that we also have the conjugacy class map $\mathrm{Cl} : G_M \to \mathrm{Conj}'G_M$. All the Mumford-Tate groups of motives considered hereby are assumed to be connected. This can be always achieved after a finite base extension (1.3.17).

Now, let X be a smooth, projective algebraic variety over F, such that the absolute Hodge motive $h(X) \in \mathrm{Ob}(\mathcal{M}_F^{av})$. Thus, for example X could be a curve, a Fermat hypersurface, a $K3$-surface, an abelian variety, or any product of these varieties (see 1.4.3).

Theorem 4.2.1. *Let $M := h^i(X)$ for some positive integer i. Assume that X has good reduction at v and that for some prime number l with $v(l) = 0$. Then there exists a finite extension F' of F and a valuation v' on F' extending v and a conjugacy class*

$$\mathrm{Cl}_M \mathrm{Fr}_{v'} \in \mathrm{Conj}'(G_M)(\mathbb{Q})$$

such that $\mathrm{Cl}(\rho_{M,l}(\mathrm{Fr}_{v'})) = \mathrm{Cl}_M \mathrm{Fr}_{v'}$, *for all l with $v(l) = 0$.*

The proof of this theorem will occupy most of this section. It follows from a series of lemmas that we state and prove in the course of the proof of the theorem. First, we state some important remarks.

Remark 4.2.2. (i) The theorem is known to hold over the base field F itself, in the case where X is a tractable abelian variety or an abelian variety with complex multiplication. We refer the reader to [Noo09, 2.2, 2.4].

(ii) We shall see in next theorem (4.2.6) that, under further assumptions the theorem can be sharpened to give us the result over the base field F itself. This generalizes the Theorem 1.8 of [Noo09], where the result is shown for X an abelian variety.

Proof of 4.2.1. Let $\bar{\tau} : \bar{F} \hookrightarrow \mathbb{C}$ be an extension of τ to an algebraic closure of F. By 2.2.2 and 2.2.6 we have a tractable abelian variety $B_{/\bar{F}}$ which provides a weak Mumford-Tate lift of M, such that the following holds

$$h^1(X_{/\bar{F}}) \in \langle h^1(B_{/\bar{F}}), (\mathbf{CM})_{\bar{F}} \rangle.$$

Here, $(\mathbf{CM})_{\bar{F}}$ denotes the motives of the abelian varieties over $\bar{F}$ having complex multiplication.

Thus, $h^i(X_{\bar{F}})$ is a subquotient of a polynomial expression (with coefficients in natural numbers and where $+$ should be interepreted as $\oplus$ and $\cdot$ as $\otimes$) in objects of the generating family. This implies that

$$h^i(X_{\bar{F}}) \in \langle h^1(B_{/\bar{F}}), h^1(C_1), \cdots, h^1(C_t), \mathbb{Q}(1) \rangle$$

for some finite number of abelian varities $C_1, \ldots, C_t$ having complex multiplication. Tannakian subactegories being closed under direct sums and subquotients, we have

$$h^i(X_{\bar{F}}) \in \langle h^1(B_{/\bar{F}}), h^1(C_{/\bar{F}}), \mathbb{Q}(1) \rangle \tag{4.2}$$

where $C_{/\bar{F}}$ is the fibred product over $\bar{F}$ of the C_i's , again an abelian variety with complex multiplication and $h^1(C_{/\bar{F}}) = h^1(C_1) \oplus \cdots \oplus h^1(C_t)$.

Let us denote by $G_{B_{\bar{F}}}$, $G_{C_{\bar{F}}}$ and $G_{(B\times C)_{\bar{F}}}$ the Mumford-Tate groups of the motives $h^1(B_{/\bar{F}})$, $h^1(C_{/\bar{F}})$ and $h^1(B_{/\bar{F}} \times C_{/\bar{F}})$ respectively. Then, from the properties of Tannakian categories (chapter 1, §1.4) , we get a map

$$\theta : G_{(B\times C)_{\bar{F}}} \longrightarrow G_M$$

Note that by our hypothesis $M \in \mathcal{M}_{\bar{F}}^{av}$ so Hodge cycles are absolute Hodge for M and its base change (see 1.4.2). Note that by our hypothesis G_M is connected, so that $G_M = G_{M_{\bar{F}}}$ (1.3.16) where $M_{\bar{F}}$ denotes $h^i(X_{/\bar{F}})$.
The relation 4.2 is actually true over a finite extension of F. To see this, note first that we can descend from $B_{/\bar{F}}$ and $C_{/\bar{F}}$ to abelian varieties $B, C/F'$ respectively, where F' is some finite extension of F. Now, $h^i(X_{/\bar{F}})$ is obtained from $h^1(B_{/\bar{F}})$, $h^1(C_{/\bar{F}})$ and Tate motive by taking direct sums, tensor products and sub-quotients. We are performing a finite number of operations on some idempotents in the endomorphism rings of the motives involved. These idempotents are by definition absolute Hodge cycles defined over $\bar{F}$. They are not (necessarily) invariant under the action of the Galois group $\mathrm{Gal}(\bar{F}/F')$. But (1.1.5), implies that the action of the Galois group $\mathrm{Gal}(\bar{F}/F)$ on the vector space of absolute Hodge cycles (on $B_{/\bar{F}} \times C_{/\bar{F}}$) is through a finite quotient. This means that by taking a big enough finite extension of F' (which we again denote by F'), we can make these absolute Hodge cycles invariant under the action of $\mathrm{Gal}(\bar{F}/F')$. So, from the definition of absolute Hodge cycles for non-algebraically closed fields (chapter 1, §1), we get that

$$h^i(X_{F'}) \in \langle h^1(B_{/F'}), h^1(C_{/F'}), \mathbb{Q}(1)\rangle \tag{4.3}$$

Let G_B, G_C and $G_{B\times C}$ denote the Mumford-Tate groups of the motives $h^1(B_{/F'}), h^1(C_{/F'})$ and $h^1(B_{/F'} \times C_{/F'})$ respectively. Taking F' to be sufficiently large we can assume that G_B, G_C and $G_{B\times C}$ are connected (1.3.17), and hence isomorphic to $G_{B_{\bar{F}}}$, $G_{C_{\bar{F}}}$ and $G_{(B\times C)_{\bar{F}}}$ respectively. Now, from (1.3.16) and the definition of Mumford-Tate group of Hodge structures (1.3.11) it follows easily that $G_{B\times C}$ is a closed subgroup of the algebraic group $G_B \times G_C$.

Lemma 4.2.3. *If M,B and C are as above then we get the following commutative diagram .*

$$\begin{array}{ccc} & & (G_B \times G_C)(\mathbb{Q}_l) \\ & \overset{(\rho_{B,l},\rho_{C,l})}{\nearrow} & \uparrow i \\ \Gamma_{F'} & \xrightarrow{\rho_{B\times C,l}} & G_{B\times C}(\mathbb{Q}_l) \\ & \underset{\rho_{M,l}}{\searrow} & \downarrow \theta \\ & & G_M(\mathbb{Q}_l) \end{array} \tag{4.4}$$

Proof. Note that here i denotes the map induced by the inclusion $G_{B\times C} \subseteq G_B \times G_C$. The first thing to observe is how the image of the Galois group is contained in the $\mathbb{Q}_l$-valued points of the Mumford-Tate groups. Then we will discuss the commutativity of the diagram.
Denote by $\mathcal{T}$ the Tannakian subcategory generated by M in $\mathcal{M}_F^{av}$ and $\mathcal{T}'$ the Tannakian category generated there by $N := h^1(B_{/F'} \times C_{/F'})$. Then by definition we have

$$G_M(\mathbb{Q}_l) = \mathrm{Aut}^{\otimes}(H_\tau \mid_{\mathcal{T}})(\mathbb{Q}_l)$$
$$= \{\phi \mid \phi : H_\tau \mid_{\mathcal{T}} \otimes \mathbb{Q}_l \longrightarrow H_\tau \mid_{\mathcal{T}} \otimes \mathbb{Q}_l \text{ is ismorphism of functors}\}$$
$$\text{and } G_{B\times C}(\mathbb{Q}_l) = \mathrm{Aut}^{\otimes}(H_\tau \mid_{\mathcal{T}'})(\mathbb{Q}_l)$$
$$= \{\phi \mid \phi : H_\tau \mid_{\mathcal{T}'} \otimes \mathbb{Q}_l \longrightarrow H_\tau \mid_{\mathcal{T}'} \otimes \mathbb{Q}_l \text{ is ismorphism of functors}\}$$

Let $\gamma \in \Gamma_{F'}$. Then we claim that γ induces an isomorphism of functors $\gamma_N^* : H_\tau|_{\mathcal{T}'} \otimes \mathbb{Q}_l \to H_\tau|_{\mathcal{T}'} \otimes \mathbb{Q}_l$ and similarly $\gamma_M^* : H_\tau|_{\mathcal{T}} \otimes \mathbb{Q}_l \to H_\tau|_{\mathcal{T}} \otimes \mathbb{Q}_l$. This is true since for any motive $N' \in \mathrm{Ob}(\mathcal{T}')$, the action of $\Gamma_{F'}$ on the l-adic realization $H_l(N')$ induces a map of vector spaces $\gamma_{N'}^* : H_l(N') \to H_l(N')$. Similarly for $M' \in \mathrm{Ob}(\mathcal{T})$, we get a map $\gamma_{M'}^* : H_l(M') \to H_l(M')$. Then, using the comparison isomorphism (1.2.2) we see that $H_l(N') = H_\tau(N') \otimes \mathbb{Q}_l$ and thus $\gamma_{N'}^* : H_\tau(N') \otimes \mathbb{Q}_l \to H_\tau(N') \otimes \mathbb{Q}_l$. If we have a morphism of motives $N_1 \xrightarrow{f} N_2$ for $N_1, N_2 \in \mathrm{Ob}(\mathcal{T}')$, then by 1.2.1(v), we know that the morphism f commutes with the action of the Galois group $\Gamma_{F'}$ on the l-adic realizations of motives. Thus we get a commutative diagram:

$$\begin{array}{ccc} H_\tau(N_1) \otimes \mathbb{Q}_l & \xrightarrow{\gamma_{N_1}^*} & H_\tau(N_1) \otimes \mathbb{Q}_l \\ \downarrow f_l & & \downarrow f_l \\ H_\tau(N_2) \otimes \mathbb{Q}_l & \xrightarrow{\gamma_{N_2}^*} & H_\tau(N_2) \otimes \mathbb{Q}_l \end{array} \qquad (4.5)$$

This shows that $\gamma_N^* \in G_{B\times C}(\mathbb{Q}_l)$. Similarly $\gamma_M^* \in G_M(\mathbb{Q}_l)$.

The map θ sends an isomorphism of functors $\phi \in G_{B\times C}(\mathbb{Q}_l)$ to the isomorphism of the same functors restricted to the full subcategory $\mathcal{T}$ of $\mathcal{T}'$. Thus, $\theta(\gamma_N^*) = \gamma_M^*$. By similar arguments it also follows that the upper triangle is commutative, since each of $h^1(B), h^1(C)$ generates a full subcategory in $\mathcal{T}'$, respectively. □

Now we return to the proof the theorem. Consider a valuation v', of F' extending v, where $B \times C/F'$ has good reduction. The idea of the proof is to find a conjugacy class $\mathrm{Cl}_{B\times C}\mathrm{Fr}_{v'} \in \mathrm{Conj}'(G_{B\times C})(\mathbb{Q})$, such that its image $\mathrm{Cl}_M\mathrm{Fr}_{v'} \in \mathrm{Conj}'(G_M)(\mathbb{Q})$ provides us the conjugacy class of the statement of the theorem. First, we need a few more lemmas.

Lemma 4.2.4. *The inclusion $G_{B\times C} \subseteq G_B \times G_C$ induces a closed immersion* $\mathrm{Conj}'(G_{B\times C}) \to \mathrm{Conj}'(G_B \times G_C)$.

Proof. Let $\mathrm{Conj}'(G_{B\times C}) \cong T_{B\times C}/\tilde{W}_1$ and $\mathrm{Conj}'(G_B \times G_C) \cong T_B \times T_C/\tilde{W}_2$ be the quotient of maximal tori $T_{B\times C}$ and $T_B \times T_C$ of $G_{B\times C}$ and $G_B \times G_C$ respectively, for the action of the finite groups $\tilde{W}_1$ and $\tilde{W}_2$ (chapter 3,§2). The algebraic groups W_1 and W_2 are extensions of the Weyl groups $W(G_{B\times C})$ and $W(G_B \times G_C)$ of $G_{B\times C}$ and $G_B \times G_C$ respectively, by the algebraic groups $\Lambda(G_{B\times C})$ and $\Lambda(G_B \times G_C)$ (chapter 3 §2). As G_C is commutative this yields that G_C^{der} and G_C^{ad} are trivial and

$$G_{B\times C}^{der} \cong (G_B \times G_C)^{der},\ G_{B\times C}^{ad} \cong (G_B \times G_C)^{ad},\ \text{and } G_{B\times C}^{\natural ad} \cong (G_B \times G_C)^{\natural ad}.$$

Since the Weyl group of any algebraic group is isomorphic to the Weyl group of its derived group, we have $W(G_{B\times C}) = W(G_B \times G_C)$. We also get that

$$\Lambda(G_{B\times C}) \cong G_{B\times C}^{\natural ad}/G_{B\times C}^{ad} \cong \Lambda(G_B \times G_C).$$

Therefore, $\tilde{W}_1 \cong \tilde{W}_2$ and we denote this common algebraic group by $\tilde{W}$. Since we have $G_{B\times C} \subseteq G_B \times G_C$, we can suppose that $T_{B\times C} \subseteq T_B \times T_C$. Denote the coordinate rings of $T_B \times T_C$ and $T_{B\times C}$ by R and S, respectively. Then we have a surjective morphism $R \to S$ of $\mathbb{Q}$-algebras, with $\tilde{W}$-action. Let $s \in S^{\tilde{W}}$, since we are working on $\mathbb{Q}$-algebras we can take the average of the elements of the $\tilde{W}$-orbit of any $r \in R$ mapping to s and this is also an element of $R^{\tilde{W}}$ mapping to s. Thus $R^{\tilde{W}} \to S^{\tilde{W}}$ is surjective, implying that $\mathrm{Conj}'(G_{B\times C}) \to \mathrm{Conj}'(G_B \times G_C)$ is a closed immersion.

□

Lemma 4.2.5. *Under the hypothesis of the theorem, the abelian variety $B\times C$ has potential good reduction at v.*

Proof. From the definition of a weak Mumford-Tate lift we know that the morphism $G_{B\times C} \to G_M$ induces isomorphism on the adjoint groups. Further, as X is smooth and has good reduction at v, the action of the inertia group on $H^i_{et}(X \times_F \bar{F}, \mathbb{Q}_l)$ is trivial. Hence, it follows that $\rho_{B\times C,l}(I_{\bar{v}})$ is contained in the centre of $G_{B\times C}(\mathbb{Q}_l)$. By proposition 1.3.12 we know that $G_{B\times C}$ is reductive. This means its center is a torus, a commutative group scheme. Then by corollary 4.1.6, B has potential good reduction at v. By combining proposition 4.1.9 and corollary 4.1.6, it follows that C has potential good reduction v. Thus, we can replace F by a finite extension F', extend v to v' and assume that $B \times C$ has good reduction at v'.

□

Thus taking F' to be a sufficiently large finite extension of F we can assume that B and C have good reduction at the valuation v' extending v to F'. Now by remarks 4.2.2 we have conjugacy classes $\mathrm{Cl}_B\mathrm{Fr}_{v'} \in \mathrm{Conj}'(G_B)(\mathbb{Q})$ and $\mathrm{Cl}_C\mathrm{Fr}_{v'} \in \mathrm{Conj}'(G_C)(\mathbb{Q})$ such that $\mathrm{Cl}(\rho_{B,l}(\mathrm{Fr}_{v'})) = \mathrm{Cl}_B\mathrm{Fr}_{v'}$ for l different from the residual characteristic of F' by v', and similarly for C. As $(\rho_{B,l}, \rho_{C,l})$ factors through $G_{B\times C}$, by diagram (4.4) and lemma 4.2.4 it follows that $(\mathrm{Cl}_B\mathrm{Fr}_{v'}, \mathrm{Cl}_C\mathrm{Fr}_{v'})$ lies in $\mathrm{Conj}'(G_{B\times C})(\mathbb{Q})$. Now take $\mathrm{Cl}_M\mathrm{Fr}_{v'}$ to be the image of $(\mathrm{Cl}_B\mathrm{Fr}_{v'}, \mathrm{Cl}_C\mathrm{Fr}_{v'})$, under the canonical map

$$\mathrm{Conj}'(G_{B\times C})(\mathbb{Q}) \to \mathrm{Conj}'(G_M)(\mathbb{Q}).$$

The the diagram (4.4) shows that $\mathrm{Cl}_M\mathrm{Fr}_{v'}$ is the required conjugacy class. This concludes the proof of the theorem.

□

Theorem 4.2.6. *Let the notations be as in the theorem 4.2.1. Assume further that $\rho_{M,l}(\mathrm{Fr}_v)$ is weakly neat for some $l \neq p$. Then there exists a conjugacy class*

$$\mathrm{Cl}_M\mathrm{Fr}_v \;\in\; \mathrm{Conj}'(G_M)(\mathbb{Q})$$

such that $\mathrm{Cl}(\rho_{M,l}(\mathrm{Fr}_v)) = \mathrm{Cl}_M\mathrm{Fr}_v$, for l different from the residual characteristic of F.

Proof. Let F'/F be a finite extension of the field F as given by the theorem 4.2.1. Let v' be an extension of the valuation v to F' and let

$$s := \mathrm{Cl}_M\mathrm{Fr}_{v'} \in \mathrm{Conj}'G_M(\mathbb{Q}) \subset \mathrm{Conj}'G_M(\bar{\mathbb{Q}}_l)$$

be the image of $\rho_{M,l}(\mathrm{Fr}_{v'})$ under the map Cl.

Now we follow the notations of the proposition 3.3.3. Thus $\bar{\phi}_n$ is the map induced by the nth power map $\phi_n : G_M \to G_M$, on $Y := \mathrm{Conj}'G_M$. Let $\tilde{s}$ be any pre-image of $\mathrm{Cl}_M\mathrm{Fr}_{v'}$ under the map of $\bar{\mathbb{Q}}_l$-valued points $Y(\bar{\mathbb{Q}}_l) \to Y(\bar{\mathbb{Q}}_l)$ induced by $\bar{\phi}_n$. First we claim that $\tilde{s}$ is $\bar{\mathbb{Q}}$-valued point of $Y(\bar{\mathbb{Q}}_l)$. To see this, let A be the co-ordinate ring of Y and let $\bar{\phi}_n^\sharp : A \to A$ denote the homomorphism induced on the rings by the morphism $\bar{\phi}_n$. As, $\bar{\phi}_n \circ \tilde{s} = s$, we get a commutative diagram

$$\begin{array}{ccc} \bar{\mathbb{Q}}_l & \xleftarrow{\tilde{s}} & A \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle \bar{\phi}_n^\sharp} \\ \mathbb{Q} & \xleftarrow[s]{} & A \end{array} \qquad (4.6)$$

where i is the inclusion map $\mathbb{Q} \to \bar{\mathbb{Q}}_l$. Since, $\bar{\phi}_n^\sharp$ is a finite homomorphism of $\mathbb{Q}$-algebras, $\tilde{s}(A)$ is a finite algebra over $s(A) = \mathbb{Q}$. Thus, $K := \mathrm{Frac}(\tilde{s}(A))$ is a finite field extension of $\mathbb{Q}$. Thus, the above diagram can be replaced by the following

$$\begin{array}{ccc} K & \xleftarrow{\tilde{s}} & A \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle \bar{\phi}_n^\sharp} \\ \mathbb{Q} & \xleftarrow[s]{} & A \end{array} \tag{4.7}$$

Therefore, $\tilde{s}$ is a K-valued point of Y. By embedding K into an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$, we see that $\tilde{s}$ is a $\bar{\mathbb{Q}}$-valued point of Y. This establishes our claim.

Now $\mathrm{Cl}(\rho_{M,l}(\mathrm{Fr}_v))$ is a $\mathbb{Q}_l$-valued point of Y mapped to s under the morphism $\bar{\phi}_n$. By, the preceding discussion $\mathrm{Cl}(\rho_{M,l}(\mathrm{Fr}_v)) \in Y(\bar{\mathbb{Q}})$. By the hypothesis $\rho_{M,l}(\mathrm{Fr}_v)$ is weakly neat. Now, as a consequence of the Weil Conjectures ([Del80, 3.3.9]) we know that the characteristic polynomial f of the linear map induced by $\rho_{M,l}(\mathrm{Fr}_v)$ on $H^i_{et}(X_{/\bar{F}}, \mathbb{Q}_l)$ is independent of l and is contained in $\mathbb{Q}[X]$. Thus if $l' \neq p$ is any other prime number, then we have another weakly neat element $\rho_{M,l'}(\mathrm{Fr}_v)$ with same characteristic polynomial as that of $\rho_{M,l}(\mathrm{Fr}_v)$. We also have an element $\mathrm{Cl}(\rho_{M,l'}(\mathrm{Fr}_v)) \in Y(\mathbb{Q}_l)$, such that

$$\bar{\phi}_n(\mathrm{Cl}(\rho_{M,l}(\mathrm{Fr}_v))) = \bar{\phi}_n(\mathrm{Cl}(\rho_{M,l'}(\mathrm{Fr}_v))).$$

By the above discussion $\mathrm{Cl}(\rho_{M,l'}(\mathrm{Fr}_v)) \in Y(\bar{\mathbb{Q}})$. Thus by proposition 3.3.3 we have $\mathrm{Cl}(\rho_{M,l}(\mathrm{Fr}_v)) = \mathrm{Cl}(\rho_{M,l'}(\mathrm{Fr}_v))$. By repeating the above procedure for all prime numbers different from p, we see that $\mathrm{Cl}(\rho_{M,l}(\mathrm{Fr}_v))$ is unique and we denote it by $\mathrm{Cl}_M\mathrm{Fr}_v$. Since we also have $\mathrm{Cl}_M\mathrm{Fr}_v \in Y(\mathbb{Q}_l)$ for all $l \neq p$. Thus, $\mathrm{Cl}_M\mathrm{Fr}_v$ is $\mathbb{Q}$-valued point of Y. In other words, $\mathrm{Cl}_M\mathrm{Fr}_v \in \mathrm{Conj}'G_M(\mathbb{Q})$.

□

Chapter 5

ℓ-independence for motives with semi-stable reduction

5.1 Monodromy

Let F be a complete discretely valued field, v denote the valuation of F and k_v the residue field of characteristic $p > 0$. For our purpose we will suppose that k_v is finite, unless otherwise stated. We fix an algebraic closure $\bar{F}$ of F and let $\bar{v}$ be the extension of v to $\bar{F}$. The residue field of $\bar{F}$ at $\bar{v}$ is denoted by $\bar{k}_v$ (which is also an algebraic closure of k_v).

The inertia group I_F of the extension $\bar{F}/F$ can be described as the subgroup of $\mathrm{Gal}(\bar{F}/F)$ defined by the following exact sequence:

$$1 \to I_F \to \mathrm{Gal}(\bar{F}/F) \to \mathrm{Gal}(\bar{k}_v/k_v) \to 1$$

Denote by μ_{l^n} the group of l^n-th roots of unity in $\bar{k}_v$. Let $\mathbb{Z}_l(1) = \varprojlim \mu_{l^n}$. Then the inertia group I_F fits into the following exact sequence ([Del73, §2])

$$1 \to P \to I_F \xrightarrow{t} \mathbb{Z}_{(p')}(1) \to 1$$

Where P is a pro-p-group and $\mathbb{Z}_{(p')}(1) = \prod_{l \neq p} \mathbb{Z}_l(1)$. Let $l \neq p$ be a prime number. We denote by $t_l : I_F \to \mathbb{Z}_l(1)$, the l-component of t.

Definition 5.1.1. Let $l \neq p$ be a prime number. An l-adic representation ξ_l of $\mathrm{Gal}(\bar{F}/F)$ is said to be *quasi-unipotent* if there exists an open subgroup J of the inertia group I_F such that the restriction of ξ_l to J is unipotent (i.e $\xi_l(\sigma)$ is a unipotent linear map for every $\sigma \in J$).

A general fact about quasi-unipotent representations is the following.

Theorem 5.1.2. *Let $\xi_l : \mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}(U)$ be any quasi-unipotent representation, for a finite dimensional $\mathbb{Q}_l$-vector space U. Then there exists a unique nilpotent morphism $N'_l : U(1) \to U$ such that if J is an open subgroup of I_F, where ξ_l is unipotent, then*

$$\xi_l(\sigma) = \exp(t_l(\sigma)N'_l) \text{ for all } \sigma \in J \tag{5.1}$$

Proof. This is [Del73, 8.2]. See also [Ill94] □

Remark 5.1.3. For any $\mathbb{Z}_l$-module U, $U(1)$ denotes $U \otimes_{\mathbb{Q}_l} \mathbb{Q}_l(1)$, where $\mathbb{Q}_l(1) = \mathbb{Q}_l \otimes_{\mathbb{Z}_l} \mathbb{Z}_l(1)$

Proposition 5.1.4. *Let the notations be as in the previous theorem. The monodromy operator N'_l is unchanged by any finite extension of the base field F.*

Proof. Let F' be any finite extension of F and $I_{F'}$ denote the corresponding inertia subgroup of $\Gamma_{F'} := \mathrm{Gal}(\bar{F}/F')$. Denote by $\xi_l|_{\Gamma_{F'}}$, the l-adic representation of $\Gamma_{F'}$, obtained by restricting ξ_l. By Galois theory we know that $\Gamma_{F'}$ is an open subgroup of $\mathrm{Gal}(\bar{F}/F)$. Thus the inertia group $I_{F'} = \Gamma_{F'} \cap I_F$ is an open subgroup of I_F. Let J be an open subgroup of I_F, where ξ_l is unipotent. Then, $J \cap I_{F'}$ is an open subgroup of both $I_{F'}$ and I_F. The representation $\xi_l|_{\Gamma_{F'}}$ is unipotent on $J \cap I_{F'}$. Thus, by the theorem there exists an unique nilpotent operator $N''_l : U(1) \to U$ such that (5.1) holds for every element $\sigma \in J \cap I_{F'}$, for $\xi_l|_{\Gamma_{F'}}$. But, ξ_l is also unipotent on $J \cap I_{F'}$. Thus, by the uniqueness of N'_l, we conclude that $N''_l = N'_l$. □

For any smooth proper variety X over F, we denote $V_l^i := H^i_{et}(X_{/\bar{F}}, \mathbb{Q}_l)$ the i-th l-adic cohomology of X.

Theorem 5.1.5. *The l-adic Galois representation $\rho_l^i : \mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}(V_l^i)$ is quasi-unipotent.*

Proof. [GRO72] □

Hence by theorem 5.1.2, the representation ρ_l^i satisfies (5.1).

5.2 Representations of Weil-Deligne group

Let the notations be as in the previous section. But, we now assume that F is a finite extension of $\mathbb{Q}_p$ and let $|k_v| = q_v$ be the order of k_v.

Definition 5.2.1. The *Weil group* W_F is the subgroup of $\mathrm{Gal}(\bar{F}/F)$, consisting of elements Ψ whose image in $\mathrm{Gal}(\bar{k}_v/k_v)$ is an integral power $\phi^{\alpha(\Psi)}$ of the Frobenius automorphism $\phi : a \mapsto a^{q_v}$, of $\bar{k}_v$.

Thus, we have defined a map $\alpha : W_F \to \mathbb{Z}$, which is a group homomorphism with kernel I_F. We now endow the Weil group W_F with the topology induced by the natural topology of I_F, for which I_F is open in W_F. The Weil group fits into the following diagram with exact rows :

$$\begin{array}{ccccccccc} 1 & \longrightarrow & I_F & \longrightarrow & W_F & \xrightarrow{\alpha} & \mathbb{Z} & \longrightarrow & 1 \\ & & \downarrow{=} & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & I_F & \longrightarrow & \mathrm{Gal}(\bar{F}/F) & \longrightarrow & \hat{\mathbb{Z}} & \longrightarrow & 1 \end{array} \tag{5.2}$$

The Galois group Γ_F is the completion of W_F for the topology of open subgroups of finite index. For a finite extension E of F in $\bar{F}$, the corresponding subgroup of W_F is W_E. We consider W_F as a constant group scheme over $\mathbb{Q}$.

Definition 5.2.2. The *Weil-Deligne group* $'W_F$ of F is the group scheme over $\mathbb{Q}$ defined as the semi-direct product of W_F with the additive group $\mathbb{G}_a$ over $\mathbb{Q}$, on which W_F acts as :

$$w \cdot x \cdot w^{-1} = q_v^{\alpha(w)} \cdot x$$

Definition 5.2.3. Let L be any field of characteristic 0. An *L-algebraic representation* of $'W_F$ is a triple (H, ξ', N), where H is a linear algebraic group over L, $\xi' : W_F \to H(L)$ a linear representation with an open subgroup of I_F in its kernel and $N \in \mathrm{Lie}(H)$ satisfying:

$$\mathrm{Ad}(\rho'(w))(N) = q_v^{\alpha(w)} \cdot N, \quad \text{for all } w \in W_F$$

We are interested here in the representations of the Weil-Deligne group with values in linear algebraic groups defined over $\mathbb{Q}_l$. Let H be a linear algebraic group over $\mathbb{Q}_l$ and ξ_l be a quasi unipotent l-adic repesentaton $\mathrm{Gal}(\bar{F}/F) \to H(\mathbb{Q}_l)$. Then we can construct a $\mathbb{Q}_l$-algebraic representation of $'W_F$ from ξ_l. To achieve this, restrict ξ_l to the Weil group W_F. Fix an arithmetic Frobenius $\Phi \in \mathrm{Gal}(\bar{F}/F)$, i.e a lift of the Frobenius automorphism ϕ. Then we define

$$\xi_l'(w) = \xi_l(w)\exp(-N_l' t_l(\Phi^{-\alpha(w)} w)). \tag{5.3}$$

Here N_l' is the monodromy operator as in (5.1). It is identified as an element of the $\mathrm{Lie}(H)$, by fixing an isomorphism $\mathbb{Q}_l \simeq \mathbb{Q}_l(1)$. Using the description of ξ_l as in (5.1), we see that ξ'_l is trivial on some open subgroup $J \subset I_F$. Thus, (H, ξ'_l, N'_l) indeed gives us a $\mathbb{Q}_l$-algebraic representation of $'W_F$. Moreover, according to [Del73, 8.11] the geometric isomorphism class of this representation is independent of the choice of Φ and the chosen isomorphism $\mathbb{Q}_l \simeq \mathbb{Q}_l(1)$.

5.3 Compatibility of representations of Weil-Deligne group

We keep the notations as in the previous section. Let (H, ξ', N) be a given L-algebraic representation of $'W_F$. Let K be a subfield of L. We say that the representation is *defined* over K if for every algebraically closed field Ω containing L the base extension of (H, ξ', N) to Ω is conjugate under $H(\Omega)$ to all its images under $\mathrm{Aut}_K(\Omega)$. More precisely this means that for every $\sigma \in \mathrm{Aut}_K(\Omega)$, there exists a $g \in H(\Omega)$ such that

$$\sigma\xi'_\Omega = g \cdot \xi'_\Omega \cdot g^{-1} \quad \text{and} \quad \sigma(N \otimes_L 1) = \mathrm{Ad}(g)(N \otimes_L 1)$$

where $\xi'_\Omega : W_F \to H(\Omega)$ is the extension of scalars and $N \otimes_L 1$ is the image of N in $\mathrm{Lie}(H) \otimes \Omega$.

Consider a a linear algebraic group H defined over K. Let K_1 and K_2 be two extensions of K and $(H_{/K_i}, \xi'_i, N_i)$ (for $i = 1, 2$) be K_i-algebraic representations of $'W_F$. We say $(H_{/K_1}, \xi'_1, N_1)$ and $(H_{/K_2}, \xi'_2, N_2)$ are *compatible* if they are defined over K and for every algebraically closed field $\Omega \supset K_1, K_2$, the base extensions of (H, ξ'_1, N_1) and (H, ξ'_2, N_2) to Ω are $H(\Omega)$-conjugate. More explicitly, there exists a $g \in H(\Omega)$ such that

$$\xi'_{1/\Omega} = g \cdot \xi'_{2/\Omega} \cdot g^{-1} \quad \text{and} \quad N_1 \otimes_{K_1} 1 = \mathrm{Ad}(g)(N_2 \otimes_{K_2} 1) \in \mathrm{Lie}(H) \otimes_K \Omega$$

5.4 Compatible systems modulo the action of $G^{\natural ad}$

Here we introduce a variant of the notions described in the previous section, for a system of $\mathbb{Q}_l$-algebraic representations of $'W_F$ (l different from the residual characteristic of F). The notations are as in the previous sections. For the Lie algebra of an algebraic group G, we use the Gothic letter $\mathfrak{g}$.

Fix an algebraic closure $\bar{\mathbb{Q}}$ of the rational numbers. Let G be a reductive algebraic group over $\mathbb{Q}$ and consider the algebraic group $G^{\natural ad}/\bar{\mathbb{Q}}$, as constructed in §3.2. It acts on $G_{/\bar{\mathbb{Q}}}$ by "conjugation" as defined there. For this action it is possible to define a map similar to the classical adjoint map in the theory of algebraic groups. We describe it here and we will still call it the adjoint map. First of all we need to make a base extension to an algebraically closed field $\Omega \supset \bar{\mathbb{Q}}$. Thus the action of $G^{\natural ad}$ on $G_{/\mathbb{Q}}$ induces an action

$$G^{\natural ad}_{/\Omega} \times G_{/\Omega} \to G_{/\Omega}.$$

If R is an Ω-algebra, $g \in G^{\natural ad}_{/\Omega}(R)$ and $x \in G_{/\Omega}(R)$, then we will denote the image of (g, x) under this action, by $g \cdot x \cdot g^{-1}$ to emphasis the fact that its an extension of the natural adjoint action of G^{ad} on G.

First we note that the Lie-algebra of an algebraic group H defined over a field K, is the tangent space of H at the identity element 1_H. Now let G and Ω be as above. Consider an element $g \in G^{\natural ad}(\Omega)$. Let R be an Ω-algebra. Denote the image of g under the map $G^{\natural ad}(\Omega) \to G^{\natural ad}(R)$ to be g_R. Then we get a morphism of groups $G_{/\Omega}(R) \to G_{/\Omega}(R)$, which is conjugation by g_R. This is functorial, and hence g induces a morphism of algebraic groups $G_{/\Omega} \to G_{/\Omega}$. This induces a morphism of Lie-algebras $\mathrm{Lie}(G_{/\Omega}) \to \mathrm{Lie}(G_{/\Omega})$. As $\mathrm{Lie}(G_{/\Omega}) = \mathfrak{g} \otimes \Omega$, so we get an element of $\mathrm{Aut}(\mathfrak{g} \otimes \Omega)$. Thus we have defined the adjoint map for this action

$$\mathrm{Ad} : G^{\natural ad}(\Omega) \to \mathrm{Aut}(\mathfrak{g} \otimes \Omega)$$

Now consider a $\mathbb{Q}_l$-algebraic representation $(G_{/\mathbb{Q}_l}, \xi', N)$ of $'W_F$, for some prime number l. For a fixed l we say that this representation is *defined over* $\mathbb{Q}$ *modulo the action of* $G^{\natural ad}$, if for every algebraically closed field $\Omega \supset \mathbb{Q}_l$, the base extension of $(G_{/\mathbb{Q}_l}, \xi', N)$ to Ω is conjugate under $G^{\natural ad}(\Omega)$ to all its images under $\mathrm{Aut}_{\mathbb{Q}}(\Omega)$. More precisely this means that for every $\sigma \in \mathrm{Aut}_{\mathbb{Q}}(\Omega)$, there exists a $g \in G^{\natural ad}(\Omega)$ such that

$$\sigma\xi'_\Omega = g \cdot \xi'_\Omega \cdot g^{-1} \quad \text{and} \quad \sigma(N \otimes_{\mathbb{Q}_l} 1) = \mathrm{Ad}(g)(N \otimes_{\mathbb{Q}_l} 1) \tag{5.4}$$

For any two prime numbers l_1 and l_2, we say that two representations $(G_{/\mathbb{Q}_{l_i}}, \xi'_i, N)$ (for $i = 1, 2$) are *compatible modulo the action of* $G^{\natural ad}$ if for every algebraically closed field $\Omega \supset \mathbb{Q}_{l_1}, \mathbb{Q}_{l_2}$, there exists some $g \in G^{\natural ad}(\Omega)$ such that

$$\xi'_{1/\Omega} = g \cdot \xi'_{2/\Omega} \cdot g^{-1} \quad \text{and} \quad N_1 \otimes_{\mathbb{Q}_{l_1}} 1 = \mathrm{Ad}(g)(N_2 \otimes_{\mathbb{Q}_{l_2}} 1) \in \mathfrak{g} \otimes \Omega. \tag{5.5}$$

If the above relations holds for all prime numbers l different from the residual characteristic of F, then we say that they form a *compatible system of representation modulo the action of* $G^{\natural ad}$.

5.5 ℓ-independence in the semi-stable reduction case

For an algebraic variety X defined over a field F, we will often denote $\bar{X}$ to be its base extension to a fixed algebraic closure of F. We begin this section by recalling some results on the traces of l-adic Galois representations.

Proposition 5.5.1. *Let F be a complete discretely valued field with a finite residue field of characteristic $p > 0$. Denote W_F the Weil group of F. Assume that either:*

1. *X is a proper smooth surface over F or*
2. *X is a smooth complete intersection variety over F*

Then for any positive integer i and any $w \in W_F$, $\operatorname{Tr}\left(w^*; H^i_{et}\left(\bar{X}, \mathbb{Q}_l\right)\right)$ is a rational integer which is independent of the choice of the prime number $l \neq p$.

Proof. [Och99, 2.4,2.5]. □

Corollary 5.5.2. *If w, F and X are as in the proposition then the characteristic polynomial* $\det\left(1 - w^*T; H^i_{et}\left(\bar{X}, \mathbb{Q}_l\right)\right)$ *has coefficients in $\mathbb{Q}$ and is independent of l.*

Proof. Let $\alpha_1, \cdots \alpha_m$ be the eigenvalues of $\rho(w)$. As w was chosen arbitrarily in the Weil group W_F in the proposition, $\beta_f = \operatorname{Tr}((w^f)*T; H^i_{et}(\bar{X}, \mathbb{Q}_l))$ is in $\mathbb{Q}$ and is independent of l, for every positive integer f. But $\beta_f = \sum_{s=1}^{m} \alpha_s^f$. Now let

$$p(T) := \det(1 - w^*T; H^i_{et}(\bar{X}, \mathbb{Q}_l)) = \sum_{r=0}^{m} (-1)^r a_r T^{m-r}$$

We know that the coefficients a_r are given by elementary symmetric polynomials in $\alpha_1, \cdots, \alpha_m$. Thus, by the Newton identities, a_r and the power sums β_f are related by the following recursion :

$$\beta_1 = a_1$$

$$\beta_f = \sum_{r=1}^{f} (-1)^{r-1} a_r \beta_{f-r} + (-1)^{f-1} f a_f$$

Thus it follows that the coefficients a_r in the characteristic polynomial $p(T)$ are in $\mathbb{Q}$ and are independent of l, since the same holds for β_f.

□

Remark 5.5.3. If X is an abelian variety over the field F, then we have a classical result that $\det\left(1 - w^*T; H^i_{et}\left(\bar{X}, \mathbb{Q}_l\right)\right)$ has coefficients in $\mathbb{Q}$ and is independent of l. See for example [Sai03, 3.5,0.4].

Let A be a henselian discrete valuation ring with field of fractions F. Denote $S := \operatorname{Spec} A$. Let s (resp. η) be the closed point (resp. generic point) of S.

Definition 5.5.4. We say a scheme X over S has *semi-stable reduction* if locally for etale topology on X and S, X is isomorphic to $\operatorname{Spec}\frac{A[T_1,\ldots,T_n]}{(T_1\ldots T_r-\pi)}$ for some $r \geq 0$, and π being an uniformizing parameter.

More precisely this means that if for every point of X there exists an etale neighbourhood U, such that U is isomorphic to $\operatorname{Spec}\frac{\tilde{A}[T_1,\cdots,T_n]}{T_1\cdots T_r-\pi}$, for some etale extension $\tilde{A}$ of A. This condition is equivalent to the condition that X is regular, the generic fiber of X is smooth, and the closed fiber of X is a reduced divisor with normal crossings on X (see [Ill94]).

There are some important remarks about semi-stable reduction, that we note here

Remark 5.5.5. (i) In the case of an abelian variety X/F, the usual definition for semi-stable reduction, found in literature is the following : Let G/S be the Néron minimum model of X and let G_s denote its closed fibre. Then the abelian variety X has semi-stable reduction if the identity component G_s° of G_s is a semi-abelian variety. Here semi-abelian variety means an extension of an abelian variety by a torus. In this report we will say that such an abelian variety has *semi-abelian reduction.*

(ii) If the abelian variety X has a semi-abelian reduction then by [Kün98, 4.6], it follows that after a finite extension of the base field, it also has semi-stable reduction in the sense of the definition 5.5.4.

(iii) The semi-stable reduction theorem of Grothendieck implies that any abelian variety X as above, has potential semi-abelian reduction. See [GRO72, IX,3.6] or [BLR90, §7.4,1].

(iv) If X is an abelian variety then the condition of semi-abelian reduction is equivalent to the fact that the action of the inertia subgroup of $\operatorname{Gal}(\bar{F}/F)$ on the Tate module $T_l(X)$ is unipotent. See [GRO72, IX,3.5,3.8].

Proposition 5.5.6. *Let A be a henselian discrete valuation ring with F as field of fractions. Let $\bar{F}$ be an algebraic closure F and $X \to \operatorname{Spec} A$ be* *a scheme having semi-stable reduction. Then for any prime number l, the action of the inertia group of* $\mathrm{Gal}(\bar{F}/F)$ *on the vector space* $H^i_{et}(X_{\bar{F}}, \mathbb{Q}_l)$ *is unipotent.*

Proof. See [Ill94, 3.3] and [RZ82] □

Remark 5.5.7. It follows from this proposition and the above remarks (in particular iv) that, if X is an abelian variety then semi-stable reduction implies semi-abelian reduction.

Now we look at the case of algebraic varieties defined over number fields. Let F be a number field embeddable in $\mathbb{C}$. We fix an embedding $\tau : F \hookrightarrow \mathbb{C}$. Let $\bar{F}$ be a fixed algebraic closure of F. Fix a valuation v of F and a valuation $\bar{v}$ of $\bar{F}$ extending v. Let F_v be the completion of F at v and $\bar{F}_{\bar{v}}$ be the completion of $\bar{F}$ at $\bar{v}$. Then $\bar{F}_{\bar{v}}$ is an algebraic closure of F_v. Let $\mathcal{O}_v$ be the ring of integers of F_v. To say that a scheme X/F has semi-stable reduction at v means that the scheme $X \times_F F_v$ has semi-stable reduction over $\mathcal{O}_v$. Fix an embedding $\bar{F} \hookrightarrow \bar{F}_{\bar{v}}$, then we have the following important isomorphism

$$H^i_{et}(X_{/\bar{F}}, \mathbb{Q}_l) \cong H^i_{et}(X_{/\bar{F}_{\bar{v}}}, \mathbb{Q}_l).$$

For the rest of this section we fix an *arithmetic* Frobeinus element Φ_v in $W_v \subset \mathrm{Gal}(\bar{F}_{\bar{v}}/F_v) \cong D_{\bar{v}} \subset \mathrm{Gal}(\bar{F}/F)$, where W_v is the Weil group of $\mathrm{Gal}(\bar{F}_{\bar{v}}/F_v)$. Denote $'W_v$ to be the Weil-Deligne group of F_v.

For any absolute Hodge motive $M \in \mathrm{Ob}\mathcal{M}_F$, we denote G_M its Mumford-Tate group w.r.t. the embedding τ (see chapter 1). This data defines, as in section 5.2, a $\mathbb{Q}_l$-algebraic representation $(G_{M/\mathbb{Q}_l}, \rho'_{M,l}, N'_{M,l})$ of $'W_v$ corresponding to the canonical l-adic Galois representation $\rho_{M,l} : W_v \to G_M(\mathbb{Q}_l)$ of M. The homomorphism $\rho'_{M,l}$ is given as

$$\rho'_{M,l}(w) = \rho_l(w)\exp(-N'_{M,l}t_l({\Phi_v}^{-\alpha(w)}w)) \tag{5.6}$$

In what follows we shall always assume that the Mumford-Tate group G_M is connected. Note that this can be always achieved for motives in $\mathcal{M}^{av}_F$ after a finite extension of the base field F (1.3.17).
All algebraic varieties X/F are assumed to be smooth and projective, and such that the absolute Hodge motive $h(X) \in \mathrm{Ob}(\mathcal{M}^{av}_F)$. Thus, as in the main theorem 4.2.6 of chapter 4, X could be a curve, an abelian variety, an unirational variety of dimension ≤ 3, a $K3$-surface, or any product of such varieties. Then we have the following result.

Theorem 5.5.8. *Let M be the absolute Hodge motive $h^i(X) \in \mathrm{Ob}\mathcal{M}^{av}_F$, then there exists a finite extension F' of F and a valuation v' extending v such that* *(i) for every $l \neq p$, the representation $(G_{M/\mathbb{Q}_l}, \rho'_{M,l}, N'_{M,l})$ of $'W_{v'}$ is defined over $\mathbb{Q}$ modulo the action $G_M^{\natural ad}$ and*

(ii) for l running through the primes different from p, these representations form a compatible system of representations of $'W_{v'}$ modulo the action of $G_M^{\natural ad}$.

Remark 5.5.9. 1. Since we assumed G_M to be connected, we have $G_M = G_{M_{F'}}$ for any finite extension F' of F. By abuse of notation, we denote the map $\rho'_{M,l}$ to be the restriction of $W_v \to G_M(\mathbb{Q}_l)$ to the subgroup $W_{v'}$. Strictly speaking we should denote it by $\rho'_{M_{F'},l}$. By proposition 5.1.4 monodromy opertor is unchanged by base extension.

2. The theorem is known for tractable abelian varieties, see [Noo10, 7.1].

Proof of theorem 5.5.8. Let B and C be as before in the proof of theorem 4.2.1, i.e B is a tractable abelian variety providing a weak Mumford-Tate lift of the abelian motive M, while C is a abelian variety of CM type, both defined over a finite extension of F. The relation (4.3) still holds in this situation after appropriate finite base extension. By preceding remarks we know that the theorem holds for B after some finite base extension. The theorem also holds for C (by theorem 4.2.6) as it has potential good reduction. Let F' be a finite extension of F such that both B and C verify the statement of the theorem over F'. Let v' be an extension of the valuation v to F'. By taking F' sufficiently large and using theorem 5.1.2 (or 5.5.5(iv)), we can also suppose that for any prime number $l \neq p$, the representations $\rho_{B,l}$ and $\rho_{B\times C,l}$ are unipotent on the inertia subgroup $I_{v'}$ of $\mathrm{Gal}(\bar{F}/F')$.

Denote the Weil-group of F' by $W_{v'}$. Then we have a map $\alpha' : W_{F'} \to \mathbb{Z}$ as in diagram (5.2). We also have a new arithmetic Frobenius for this extension $\Phi_{v'} = \Phi_v^d$, where d is the residual degree of the extension F'/F. By proposition 5.1.4 the monodromy operators are unchanged. We denote the restriction $\rho_{*,l}|_{W_{v'}}$ again by $\rho_{*,l}$ (for $* = B, C, B \times C$ or M). The equation (5.6) now reads as $\rho'_{*,l}(w) = \rho_{*,l}(w)\exp(N'_{*/l}t_l(\Phi_{v'}^{-\alpha'(w)}w))$ for any w in $W_{v'}$. Thus, $\rho'_{*,l}$ is same as the restriction $\rho'_{*,l}|_{W_{v'}}$

The lemma 4.2.3 holds here too. We follow the notations there. Let $\mathfrak{g}_B$, $\mathfrak{g}_C$, $\mathfrak{g}_{B\times C}$ and $\mathfrak{g}_M$ be the Lie-algebras of the Mumford-Tate groups G_B, G_C, $G_{B\times C}$ and G_M respectively. Let $N'_{B\times C,l}$, $N'_{B,l}$ be the monodromy operators corresponding to $B\times C$ and B respectively. Since C now has good reduction, the inertia subgroup acts trivially on the l-adic cohomology $H^1_{et}(C_{/\bar{F}}, \mathbb{Q}_l)$. Thus the monodromy operator corresponding to the motive $h^1(C)$ is trivial. Recall we have the inclusion of Mumford-Tate groups $G_{B\times C} \hookrightarrow G_B \times G_C$.

Since G_C is commutative this implies that $G^{\natural ad}_{B\times C} \cong G^{\natural ad}_{B}$.

Now fix a prime number $l \neq p$. Note that as $\rho_{B\times C,l}$ is unipotent on the inertia subgroup, thus from its definition, $\rho'_{B\times C,l}$ is trivial on the inertia subgroup $I_{v'}$. From (5.2) we have $W_{v'}/I_{v'} \simeq \mathbb{Z}$, and $\Phi_{v'}$ is a generator of $W_{v'}/I_{v'}$. This means that it suffices to verify the first equality of the equation (5.4) (as well as 5.5) on $\rho'_{B\times C,l/\Omega}$, at the arithmetic Frobenius $\Phi_{v'} \in W_{v'}$.

Since $\rho_{B\times C,l} = (\rho_{B,l}, \rho_{C,l})$, by theorem 5.1.2 we get that

$$\exp(N'_{B\times C,l}) = (\exp(N'_{B,l}), 1).$$

Let i be the inclusion of l-adic Lie groups $i : G_{B\times C}(\mathbb{Q}_l) \hookrightarrow (G_B \times G_C)(\mathbb{Q}_l)$ and let $\mathrm{Lie}(i) : \mathfrak{g}_{B\times C} \otimes \mathbb{Q}_l \hookrightarrow (\mathfrak{g}_B \oplus \mathfrak{g}_C) \otimes \mathbb{Q}_l$ be the corresponding map on Lie algebras. As $i \circ \exp = \exp \circ \mathrm{Lie}(i)$, we get that $N'_{B\times C,l} = (N'_B, 0) \in \mathfrak{g}_{B\times C} \otimes \mathbb{Q}_l$. It is now obvious from the fact that the theorem holds for B and C, that it is also true for $B \times C$.

Next we shall show that this implies that the theorem also holds for the abelian motive M after base extension to F'.

By theorem 5.1.2 we may suppose that $\rho_{M,l}$ is unipotent on the inertia subgroup. Thus, as before it suffices to verify the first equality of the equation (5.4) (as well as 5.5) on $\rho'_{M,l/\Omega}$, at the arithmetic Frobenius $\Phi_{v'} \in W_{v'}$.

Now let $\Omega \supset \mathbb{Q}_l$ be an algebraically closed field. We want to show that $(G_{M/\mathbb{Q}_l}, \rho'_{M,l}, N'_{M,l})$ is defined over $\mathbb{Q}$. Fix an automorphism $\sigma \in \mathrm{Aut}_{\mathbb{Q}}(\Omega)$ and let $g_\sigma \in G^{\natural ad}_{B\times C}(\Omega)$ be such that equation (5.4) holds true for $B \times C$. Thus,

$$\sigma\rho'_{B\times C,l/\Omega} = g_\sigma \cdot \rho'_{B\times C,l/\Omega} \cdot g_\sigma^{-1} \quad \text{and} \quad \sigma(N'_{B\times C,l} \otimes_{\mathbb{Q}_l} 1) = \mathrm{Ad}(g_\sigma)(N'_{B\times C,l} \otimes_{\mathbb{Q}_l} 1) \tag{5.7}$$

As $\theta \circ \rho_{B\times C/l} = \rho_{M,l}$, we get $\rho'_{M,l/\Omega}(\Phi_{v'}) = \theta(\rho'_{B\times C,l/\Omega}(\Phi_{v'}))$. This gives us

$$\sigma(\rho'_{M,l/\Omega}(\Phi_{v'})) = \theta(g_\sigma \cdot \rho'_{B\times C/l}(\Phi_{v'}) \cdot g_\sigma^{-1}), \tag{5.8}$$

where σ and g_σ are as in (5.7).

As B is a weak Mumford-Tate lift of M, we have $G^{ad}_{B\times C} \cong G^{ad}_B \cong G^{ad}_M$ (chapter 3,§2). This implies that

$$G^{\natural ad} := G^{\natural ad}_{B\times C} = G^{\natural ad}_B = G^{\natural ad}_M$$

and $\theta : G_{B\times C} \to G_M$ is equivariant for the conjugation action of $G^{\natural ad}$. In other words we have the following commutative diagram

$$\begin{array}{ccc} G^{\natural ad} \times G_{B\times C/\bar{\mathbb{Q}}} & \longrightarrow & G_{B\times C/\bar{\mathbb{Q}}} \\ \downarrow{\scriptstyle id\times\theta} & & \downarrow{\scriptstyle \theta} \\ G^{\natural ad} \times G_{M/\bar{\mathbb{Q}}} & \longrightarrow & G_{M/\bar{\mathbb{Q}}} \end{array} \tag{5.9}$$

Thus using (5.8) we get

$$\sigma(\rho'_{M,l}(\Phi_{v'})) = g_\sigma \cdot \theta(\rho'_{B\times C/l}(\Phi_{v'})) \cdot g_\sigma^{-1} = g_\sigma \cdot (\rho'_{M,l/\Omega}(\Phi_{v'})) \cdot g_\sigma^{-1}.$$

This establishes the first equality of equation 5.4 for the abelian motive M after base extension to F'.

Now using theorem 5.1.2 again, we get that

$$\theta(\exp(N'_{B\times C,l})) = \exp(N'_{M,l}).$$

Let $\mathrm{Lie}(\theta) : \mathfrak{g}_{B\times C} \otimes \mathbb{Q}_l \to \mathfrak{g}_M \otimes \mathbb{Q}_l$ be the map of Lie-algebras induced by θ. As $\theta \circ \exp = \exp \circ \mathrm{Lie}(\theta)$, we deduce that $\mathrm{Lie}(\theta)$ maps $N'_{B\times C,l}$ to $N'_{M,l}$. We also have the following isomorphisms of Lie algebras

$$\mathrm{Lie}\left(G^{ss}_{B\times C}\right) = \mathrm{Lie}\left(G^{ad}_{B\times C}\right) \cong \mathrm{Lie}\left(G^{ad}_{M}\right) \cong \mathrm{Lie}\left(G^{ss}_{M}\right).$$

The first equality follows from the fact that $G^{ss}_{B\times C} \to G^{ad}_{B\times C}$ is a finite map. Similarly the third equality. The second equality comes from the fact that $G^{ad}_{B\times C} \cong G^{ad}_M$. The action of $G^{\natural ad}$ on the Lie-algebras fits in the following commutative diagram

$$\begin{array}{ccc} G^{\natural ad}(\Omega) \times \mathfrak{g}^{ss}_{B\times C} \otimes \Omega & \longrightarrow & \mathfrak{g}^{ss}_{B\times C} \otimes \Omega \\ \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\ G^{\natural ad}(\Omega) \times \mathfrak{g}_M \otimes \Omega & \longrightarrow & \mathfrak{g}_M \otimes \Omega \end{array} \tag{5.10}$$

The isomorphism $\mathfrak{g}_{B\times C} \otimes \Omega \cong \mathfrak{g}_M \otimes \Omega$ maps $N'_{B\times C,l} \otimes_{\mathbb{Q}_l} 1$ to $N'_{M,l} \otimes_{\mathbb{Q}_l} 1$. Now let σ and g_σ be as before. From equation (5.7) and the commutativity of the preceding diagram, we get

$$\sigma(N'_{M,l} \otimes_{\mathbb{Q}_l} 1) = \mathrm{Ad}(g_\sigma)(N'_{M,l} \otimes_{\mathbb{Q}_l} 1)$$

Thus we have verified the equalities of (5.4) for M after base extension to F' for any $\sigma \in \mathrm{Aut}_{\mathbb{Q}}(\Omega)$. In other words, for a fixed l the representation $(G_{M/\mathbb{Q}_l}, \rho'_{M,l}, N'_{M,l})$ of $W_{v'}$ is defined over $\mathbb{Q}$.

By same arguments, using the diagrams above and the fact that equation (5.5) holds for $B \times C$ one shows that for varying l, the $(G_{M/\mathbb{Q}_l}, \rho'_{M,l}, N'_{M,l})$ forms a compatible system of representations of the Weil-Deligne group $'W_{v'}$ of $F'_{v'}$. □

Under some additional hypothesis the above theorem can be sharpened to give result over the base field F itself, for certain algebraic varieties.

Theorem 5.5.10. *Let X/F be either an abelian variety, a $K3$ surface, a Fermat hypersurface or a curve, with semi-stable reduction at v. Let M denote the absolute Hodge motive $h^i(X) \in \mathrm{Ob}\mathcal{M}_F^{av}$. Suppose that for some prime number l, the image $\rho'_{M,l}(\Phi_v)$ is weakly neat. Then*

(i) for every $l \neq p$, the representation $(G_{M/\mathbb{Q}_l}, \rho'_{M,l}, N'_{M,l})$ of $'W_v$ is defined over $\mathbb{Q}$ modulo the action of $G_M^{\natural ad}$ and

(ii) for l running through primes different from p, these representations form a compatible system of representations of $'W_v$ modulo the action of $G_M^{\natural ad}$.

Remark 5.5.11. 1. If X has good reduction at v then the monodromy $N'_{M,l}$ is trivial, as the inertia subgroup acts trivially on the l-adic cohomology groups $H^i_{et}(X_{/\bar{F}}, \mathbb{Q}_l)$. In this case the theorem reduces to the main result of chapter 4, theorem 4.2.6.

2. In the case where X is an abelian variety, semi-abelian reduction is sufficient to guarantee the theorem. See [Noo10, 3.5]

3. The condition that $\rho'_{M,l}(\Phi_v)$ is weakly neat is independent of the choice of l.

Proof of 5.5.10. First we note that since X has semistable reduction at v and as

$$H^i_{et}(X_{/\bar{F}}, \mathbb{Q}_l) \cong H^i_{et}(X_{/\bar{F}_{\bar{v}}}, \mathbb{Q}_l),$$

proposition 5.5.6 implies that $\rho'_{M,l}$ is trivial on the inertia subgroup. Thus as before, it suffices to verify the first equality of (5.4) (as well as of (5.5)) on $\rho'_{M,l/\Omega}(\Phi_v)$.

Let F' be a finite extension of F and v' be the extension of v to F' as obtained in theorem 5.5.8. By passing to F', we have $\Phi_{v'} = \Phi_v^d$, where d is the residual degree of the extension $F'_{v'}/F_v$. This implies that

$$\rho'_{M,l}(\Phi_{v'}) = \rho'_{M,l}(\Phi_v^d).$$

The monodromy operator $N'_{M,l}$ is unchanged by base extension.

Fix a prime number $l \neq p$. We want to show that the representation $(G_{M/\mathbb{Q}_l}, \rho'_{M,l}, N'_{M,l})$ of $'W_v$ is defined over $\mathbb{Q}$. Let $\Omega \supset \mathbb{Q}_l$ be an algebraically closed field and $\sigma \in \mathrm{Aut}_{\mathbb{Q}}(\Omega)$ be an automorphism. Then the theorem 5.5.8 gives us a $g_\sigma \in G_M^{\natural ad}(\Omega)$ such that equation (5.4) holds over F' i.e.

$$\sigma(\rho'_{M,l/\Omega}(\Phi_v^d)) = g_\sigma \cdot (\rho'_{M,l/\Omega}(\Phi_v^d)) \cdot g_\sigma^{-1} \text{ and}$$

$$\sigma(N'_{M,l} \otimes_{\mathbb{Q}_l} 1) = \mathrm{Ad}(g_\sigma)(N'_{M,l} \otimes_{\mathbb{Q}_l} 1).$$

So we have $(\sigma(\rho'_{M,l/\Omega}(\Phi_v)))^d = (g_\sigma \cdot (\rho'_{M,l/\Omega}(\Phi_v)) \cdot g_\sigma^{-1})^d$. By our hypothesis $\rho'_{M,l/\Omega}(\Phi_v)$ is weakly neat. So if we prove that $\sigma(\rho'_{M,l/\Omega}(\Phi_v))$ and $g_\sigma \cdot (\rho'_{M,l/\Omega}(\Phi_v)) \cdot g_\sigma^{-1}$ have same the characteristic polynomial, then we can use proposition 3.3.4 to conclude that $\sigma(\rho'_{M,l/\Omega}(\Phi_v)) = g_\sigma \cdot (\rho'_{M,l/\Omega}(\Phi_v)) \cdot g_\sigma^{-1}$. Since we already have $\sigma(N'_{M,l} \otimes_{\mathbb{Q}_l} 1) = \mathrm{Ad}(g_\sigma)(N'_{M,l} \otimes_{\mathbb{Q}_l} 1)$, it would follow that the representation $(G_{M/\mathbb{Q}_l}, \rho'_{M,l}, N'_{M,l})$ of $'W_v$ is defined over $\mathbb{Q}$.

We begin by noting that $g_\sigma \cdot \rho'_{M,l/\Omega}(\Phi_v) \cdot g_\sigma^{-1}$ and $\rho'_{M,l/\Omega}(\Phi_v)$ are conjugate under the action of $G^{\natural ad}$ and so by proposition 3.2.1 they have same characteristic polynomial. Let $\mathcal{B}$ be an ordered basis of the $\mathbb{Q}$-vector space $H_\tau(M)$ and let $\mathcal{B}_{\mathbb{Q}_l}$ denote the corresponding basis of

$$V_l := H_\tau(M) \otimes \mathbb{Q}_l.$$

If (a_{ij}) is the matrix of the transformation $\rho'_{M,l}(\Phi_v) : V_l \to V_l$ in the ordered basis $\mathcal{B}_{\mathbb{Q}_l}$, then

$$\sigma(\rho'_{M,l/\Omega}(\Phi_v)) : V_l \otimes_{\mathbb{Q}_l} \Omega \to V_l \otimes_{\mathbb{Q}_l} \Omega$$

is the linear transformation whose matrix in the basis $\mathcal{B}_\Omega$ is $(\sigma(a_{ij}))$. Using corollary 5.5.2 and remark 5.5.3, we conclude that $\sigma(\rho'_{M,l/\Omega}(\Phi_v))$ and $\rho'_{M,l/\Omega}(\Phi_v)$ have the same characteristic polynomial with coefficients in $\mathbb{Q}$. Thus $\sigma(\rho'_{M,l/\Omega}(\Phi_v))$ and $g_\sigma \cdot \rho'_{M,l/\Omega}(\Phi_v) \cdot g_\sigma^{-1}$ also have the same characteristic polynomial. This implies that

$$\sigma(\rho'_{M,l/\Omega}(\Phi_v)) = g_\sigma \cdot \rho'_{M,l/\Omega}(\Phi_v) \cdot g_\sigma^{-1}.$$

Thus, we have shown assertion (i) of the theorem.

Finally we want to show that for varying l the representations

$$(G_{M/\mathbb{Q}_l}, \rho'_{M,l}, N'_{M,l})$$

of W_v form a compatible system. For this, let l and l' be two different primes and Ω an algebraically closed field containing both $\mathbb{Q}_l$ and $\mathbb{Q}_{l'}$. Then by theorem 5.5.8 we know that there exists a $g \in G^{\natural ad}(\Omega)$, such that

$$\rho'_{M,l/\Omega}(\Phi_v^d) = g \cdot \rho'_{M,l'/\Omega}(\Phi_v^d) \cdot g^{-1} \text{ and} \tag{5.11}$$

$$N'_{M,l} \otimes_{\mathbb{Q}_l} 1 = \mathrm{Ad}(g)(N'_{M,l'} \otimes_{\mathbb{Q}_{l'}} 1).$$

As the mondoromy operators are unchanged by base extensions thus we just need to verify the first equality of equation (5.5). By using corollary 5.5.2 and remark 5.5.3 we know that $\rho'_{M,l/\Omega}(\Phi_v)$ and $\rho'_{M,l'/\Omega}(\Phi_v)$ have same characteristic polynomial with coefficients in $\mathbb{Q}$ and it is independent of the choice of l or l'. As $\rho'_{M,l/\Omega}(\Phi_v)$ and $\rho'_{M,l'/\Omega}(\Phi_v)$ are weakly neat, by proposition 3.3.4 and equation (5.11) we conclude that

$$\rho'_{M,l/\Omega}(\Phi_v) = g \cdot \rho'_{M,l'/\Omega}(\Phi_v) \cdot g^{-1}.$$

This establishes assertion (ii) of the theorem.

□

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