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Arbres, Processus de branchement non markoviens et Processus de Lévy

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École Doctorale Paris Centre

THÈSE DE DOCTORAT

Discipline : Mathématiques

présentée par

Mathieu RICHARD

**Arbres, Processus de branchement non
markoviens et Processus de Lévy**

dirigée par Amaury LAMBERT

Soutenue le 5 décembre 2011 devant le jury composé de :

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*Cette thèse est dédiée à tous ceux qui m'ont
soutenu lors de ces trois dernières années,
en particulier Élodie.*

*"En essayant continuellement, on finit par
réussir. Donc plus ça rate, plus on a de
chances que ça marche."
Les Shadocks*

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Résumé

Résumé

Dans cette thèse, nous nous intéressons à trois développements des arbres de ramification ("splitting trees") introduits par Geiger & Kersting (1997), et aux processus de branchement de Crump-Mode-Jagers (CMJ) qui y sont associés. Ces arbres aléatoires modélisent une population où tous les individus ont des durées de vie indépendantes et identiquement distribuées et qui donnent naissance à taux constant b durant leurs vies à des copies d'eux-mêmes. Le processus comptant le nombre d'individus vivants au cours du temps est un processus CMJ binaire et homogène qui peut être vu comme une généralisation du processus de vie et de mort markovien dans lequel les durées de vie sont exponentielles.

Dans un premier chapitre, nous considérons un modèle île-continent, généralisant celui de Karlin et McGregor, et dans lequel des individus portant des types immigrent à taux θ vers une île et y fondent des familles qui évoluent indépendamment et suivant le mécanisme décrit précédemment. Différentes hypothèses sont faites sur la façon dont les types sont choisis (soit chaque nouvel immigrant est d'un type différent des précédents, soit il est de type i avec une proba p_i , etc.) et nous déterminons les proportions asymptotiques de chacun des types dans la population totale. Dans le cas "nouvel immigrant=nouveau type", la limite suit une distribution *GEM* de paramètre θ/b et nous remarquons qu'elle ne dépend que de ce rapport et pas de loi de la durée de vie des individus.

Dans un second temps, nous étudions un autre modèle de population dans des mutations pouvant se produire à la naissance des individus avec une certaine probabilité. Nous considérons un modèle dit à une infinité d'allèles, c'est-à-dire que chaque mutant est d'un type (ou allèle) jamais rencontré auparavant, et neutre car quels que soient leurs types, les individus évoluent tous de la même manière. Nous étudions la partition allélique de la population en considérant son spectre de fréquence qui décrit le nombre de types d'âge donné et portés par un nombre donné d'individus. Nous obtenons des résultats concernant son comportement asymptotique en utilisant les caractéristiques aléatoires de Jagers & Nerman. Nous donnons également la convergence en loi des abondances des plus grandes familles et des âges des plus vieilles familles.

Dans le dernier chapitre, nous nous intéressons à des processus de Lévy spectralement positifs (ou sans sauts négatifs), ne dérivant pas vers $+\infty$ et que l'on conditionne à rester positifs en un nouveau sens. Pour cela, un processus X partant de $x > 0$ est conditionné à atteindre des hauteurs arbitrairement grandes avant de toucher 0 où le terme hauteur est à comprendre au sens du processus des hauteurs de Duquesne & Le Gall (2002). La loi du processus conditionné est définie à l'aide d'une h -transformée via une martingale. Lorsque X est à variation finie, l'argument principal est que X peut être vu comme le processus de contour d'un arbre de ramification et ainsi conditionner le processus de Lévy revient à conditionner l'arbre à atteindre des générations arbitrairement grandes. Lorsque

X est à variation infinie, le processus des hauteurs est défini à l'aide de temps locaux et la martingale est construite à partir du processus d'exploration de Duquesne et Le Gall, qui est un processus de Markov à valeurs mesures.

Mots-clefs

Processus de branchement, Processus de Lévy, Modèles de population, Immigration, Mutation, Conditionnement.

Trees, Non Markov branching processes and Lévy processes

Abstract

In this dissertation, we focus on three developments of splitting trees introduced by Geiger & Kersting (1997), and on Crump-Mode-Jagers (CMJ) branching processes associated with them. These random trees model a population where all individuals have i.i.d. lifelengths and during their lives, they give birth at constant rate b to copies of themselves. The process counting the number of extant individuals through time is a binary and homogeneous CMJ process, that can be seen as a generalization of the Markovian birth and death process in which lifetimes are exponentially distributed.

First, we consider a mainland-island model, which generalizes that of Karlin and McGregor and in which individuals carrying types immigrate at rate θ to an island and start families that evolve independently and according to the previously described mechanism. Various assumptions are made about how types are chosen (either each new immigrant is of a different type, or it is of type i with probability p_i , etc.) and we determine the asymptotic relative abundances of each type in the total population. In the "new immigrant = new type" case, the limiting distribution follows a *GEM* distribution with parameter θ/b and we note that it only depends on this ratio and not on the lifelength distribution.

Second, we study another population model where mutations can occur at birth of individuals with a certain probability. We consider a model with infinitely many alleles, that is, each new mutant is of a type (or allele) never encountered before, and neutral because individuals all behave in the same way regardless of their types. We study the allelic partition of the population by looking at its frequency spectrum which describes the number of types of a given age and carried by a given number of individuals. Using random characteristics techniques, we obtain results about its asymptotic behavior. We also prove the distribution convergence of sizes of largest families and of ages of the oldest ones.

In the last chapter, we focus on spectrally positive Lévy processes that do not drift to $+\infty$ and that we condition to stay positive in a new way. A process X starting from $x > 0$ is conditioned to reach arbitrarily large heights before hitting 0 where the term height has to be understood in the meaning of the height process of Duquesne & Le Gall (2002). The law of the conditioned process is defined with an h -transform via a martingale. When X

has finite variation, the main argument is that X can be seen as the contour process of a splitting tree and thus to condition the Lévy process is equivalent to condition the tree to reach arbitrarily large generations. When X has infinite variation, the height process is defined via local times and the martingale is constructed from the exploration process defined by Duquesne and Le Gall.

Keywords

Branching processes, Lévy processes, Population dynamics, Immigration, Mutation, Conditioning.

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Chapitre 1

Introduction

Le présent document regroupe les travaux réalisés durant trois années de thèse sous la direction d'Amaury Lambert au Laboratoire de Probabilités et Modèles Aléatoires (PMA) de l'Université Pierre et Marie Curie (Paris 6).

Il est décomposé en trois chapitres formés chacun d'un article (en anglais) publié, en révision ou en voie de soumission. Deux sujets principaux sont abordés. Les deux premiers chapitres portent sur deux modèles de population neutres, liés à certains processus de branchement non markoviens et aux arbres de ramification. Dans le troisième chapitre, on s'intéresse à un nouveau conditionnement de processus de Lévy sans sauts négatifs à rester positifs.

L'introduction est organisée comme suit. Dans la section 1.1, nous définissons brièvement les notions importantes utilisées dans ce manuscrit : les processus de Lévy, les processus de Crump-Mode-Jagers et les arbres de ramification. Dans la section 1.2, nous énonçons les résultats principaux obtenus dans les différents chapitres.

1.1 Définitions des notions principales

1.1.1 Processus de Lévy

Nous commençons par quelques définitions et propriétés sur les processus de Lévy. Pour le lecteur intéressé par plus de détails, nous conseillons la lecture de [5] et de [44].

Un *processus de Lévy* $X = (X_t, t \geq 0)$ est un processus à accroissements indépendants et stationnaires et dont les trajectoires sont continues à droite et admettent des limites à gauche (càdlàg). Il est dit *spectralement positif* (ou sans sauts négatifs) lorsqu'il est réel et qu'il n'effectue que des sauts positifs. Il est alors caractérisé par sa transformée de Laplace

$$\mathbb{E} \left[e^{-\lambda X_t} \right] = e^{t\psi(\lambda)}, \quad t \geq 0, \lambda > 0$$

où l'exposant de Laplace ψ vérifie l'équation de Lévy-Khintchine

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0, \infty)} \left(e^{-\lambda r} - 1 + \lambda r \mathbf{1}_{\{r < 1\}} \right) \Lambda(dr)$$

avec $\alpha \in \mathbb{R}$, $\beta \geq 0$ et Λ une mesure sur $(0, \infty)$ telle que $\int_{(0, \infty)} (1 \wedge x^2) \Lambda(dx) < \infty$. C'est la *mesure de Lévy* associée au processus X .

Les processus de Lévy spectralement positifs (et leurs équivalents spectralement négatifs) ont de nombreuses applications : ils peuvent modéliser le capital d'une agence d'assurance, le niveau d'un barrage ou être utilisés en théorie des files d'attente (voir

[59, 63, 44]). Ces processus vérifient de nombreuses propriétés [5, ch. VII]. Le problème de double sortie a notamment une solution s'écrivant simplement lorsque les sauts ne sont que positifs en utilisant la *fonction d'échelle* associée à X , qui est l'unique fonction continue et croissante $W : [0, \infty] \rightarrow [0, \infty]$, solution de

$$\int_0^\infty e^{-\lambda x} W(x) dx = \frac{1}{\psi(\lambda)}, \quad \lambda > \eta \quad (1.1)$$

où η est la plus grande racine de ψ . On a alors

$$\mathbb{P}\left(T_{(-\infty, 0)} < T_{(a, +\infty)} | X_0 = x\right) = \frac{W(a-x)}{W(a)}, \quad 0 < x < a$$

où pour B un borélien de \mathbb{R} , $T_B = \inf\{t \geq 0, X_t \in B\}$.

1.1.2 Processus de Crump-Mode-Jagers

Définis indépendamment par K. Crump et C. Mode [16] et par P. Jagers [34], les processus de branchement généraux, ou processus de Crump-Mode-Jagers (CMJ), sont les processus de branchement les plus généraux possibles. Ce sont des processus de branchement en temps continu et à espace d'états discret qui comptent le nombre d'individus vivants au cours du temps. À chaque individu x dans la population sont associés une variable aléatoire λ_x et un processus ponctuel ξ_x sur $[0, +\infty)$ qui ne sont pas forcément indépendants mais tels que les couples $(\lambda_x, \xi_x)_x$ sont indépendants et identiquement distribués (i.i.d.). En fait, λ_x est la durée de vie de x et ξ_x son processus de reproduction, c'est-à-dire qu'il décrit les dates de naissance de ses enfants qui vont évoluer indépendamment et suivant la même dynamique. Le processus de Crump-Mode-Jagers est alors le processus en temps continu et à espace d'états discret qui compte le nombre d'individus vivants au cours du temps.

Ces processus sont très généraux car les temps de vie ne sont pas forcément exponentiels comme dans les processus de branchement markoviens [3, ch. III], les naissances n'ont pas nécessairement lieu à la mort des individus comme dans les processus de Bellman-Harris ou de Sevastyanov ([3, ch. IV] ou [35, p.124]) et le nombre d'enfants d'un individu peut même dépendre de son âge puisque les processus de reproduction et les durées de vie ne sont pas supposés indépendants. Toutes ces raisons font que les processus CMJ peuvent être plus intéressants et pertinents en vue d'applications biologiques. À noter que même les processus de Bienaymé-Galton-Watson (BGW) peuvent être considérés comme des processus CMJ en choisissant des durées de vie constantes égales à 1 et des processus de reproduction ayant leurs atomes en 1.

De nombreux problèmes, concernant notamment les probabilités d'extinction, les taux de croissance des populations et d'autres théorèmes limites, ont été étudiés pour les processus CMJ par des techniques utilisant en particulier la théorie du renouvellement et les caractéristiques aléatoires (voir par exemple [35, 55, 36, 37, 56]). Cependant, même si on arrive à prouver l'existence de limites (en plusieurs sens possibles), celles-ci ne sont en général pas calculables explicitement.

Récemment, A. Lambert [48] a obtenu de nouvelles propriétés dans le cas particulier des processus CMJ *binaires* et *homogènes* : les naissances n'arrivent qu'une par une et à taux constant $b \in (0, \infty)$ durant la vie des individus. La distribution commune des durées de vie est notée $\Lambda(\cdot)/b$ où Λ est une mesure positive sur $(0, \infty]$ de masse b . Elle est appelée *mesure de longévité*.

Si $\Xi(t)$ désigne le nombre d'individus vivants au temps t pour ce cas particulier de processus CMJ, le processus Ξ n'est en général pas markovien sauf si la mesure de longévité est bde^{-dr} pour $d > 0$ ou $b\delta_\infty$. Le processus Ξ est alors respectivement un processus de naissance et de mort avec taux de naissance b et taux de mort d , ou un processus markovien de naissance pure à taux b , également appelé processus de Yule. On considère donc une généralisation du processus de naissance et de mort markovien au sens où les naissances sont poissonniennes et les durées de vie sont i.i.d. mais pas nécessairement exponentielles.

On peut associer naturellement au processus Ξ un arbre généalogique appelé *arbre de ramification*, traduction française de "splitting tree" défini par J. Geiger et G. Kersting dans [26] et [28] (voir aussi [48]). Il est à noter qu'il existe une définition plus générale d'un arbre de ramification (voir [48]) qui inclut le cas où le taux de naissance b est infini. Nous ne l'utiliserons que très peu et le préciserons quand ce sera le cas.

L'objet essentiel à l'étude d'un arbre de ramification est le processus de contour qui lui est associé. Lorsque l'arbre est fini, ce processus de contour défini par A. Lambert dans [48] parcourt à vitesse -1 tous les individus dans l'arbre une et une seule fois. Il démarre à la date de mort de l'ancêtre puis le "visite" en revenant dans le temps jusqu'à arriver à la date de naissance de sa plus jeune fille; le processus fait alors un saut d'amplitude égale à la durée de vie de celle-ci et parcourt alors sa vie de la même manière. Quand le processus atteint la date de naissance d'un individu, la visite de celui-ci est terminée et la visite de sa mère reprend au niveau où elle s'était arrêtée précédemment. Le parcours de l'arbre s'achève à la fin de la visite de l'ancêtre, le processus de contour prend alors la valeur 0.

Lorsque l'arbre est infini, on peut définir de la même manière le contour de l'arbre tronqué à un niveau chronologique t . Un exemple de processus de contour est donné sur la figure 1.1.

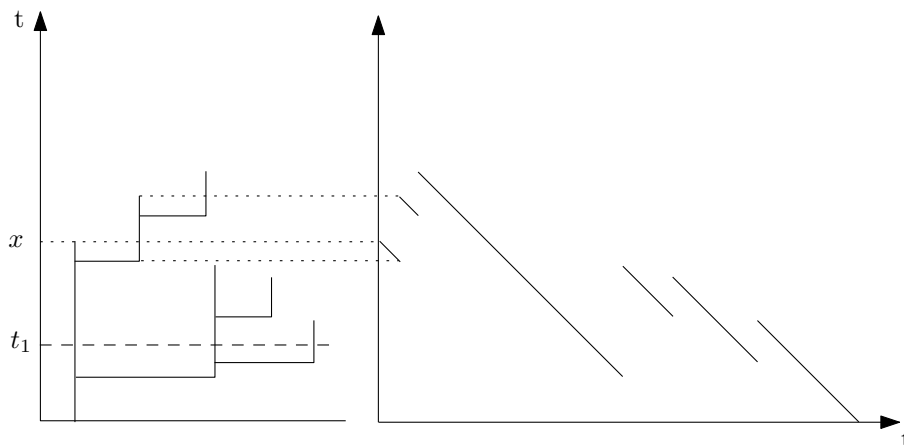


FIGURE 1.1 – À gauche, un arbre de ramification dont l'ancêtre a une durée de vie x . L'axe vertical représente le temps et l'axe horizontal indique les relations de filiation entre individus. Par exemple, au temps t_1 , 3 individus sont vivants. À droite, le processus de contour associé.

A. Lambert a montré que le processus de contour d'un arbre de ramification tronqué au niveau t est un processus de Lévy spectralement positif, réfléchi sous t , tué lorsqu'il

atteint 0 et d'exposant de Laplace

$$\psi(\lambda) := \lambda - \int_{(0, \infty]} (1 - e^{-\lambda r}) \Lambda(dr).$$

Les processus de Lévy spectralement positifs ayant de nombreuses propriétés connues, des propriétés sur les arbres de ramification et sur le processus Ξ en ont été déduites. En particulier, on peut expliciter les marginales unidimensionnelles de Ξ à partir de la fonction d'échelle W du processus de contour définie par (1.1) et montrer les propriétés suivantes lorsque l'arbre est surcritique, c'est-à-dire lorsque $m := \int_{(0, \infty]} r \Lambda(dr) > 1$.

Proposition (Lambert [48]). *Soit η la plus grande racine de ψ et*

$$\text{Ext} := \left\{ \Xi(t) \xrightarrow[t \rightarrow \infty]{} 0 \right\}$$

l'événement d'extinction de l'arbre. On a alors $\mathbb{P}(\text{Ext}) = 1 - \eta/b$ et sur Ext^c ,

$$e^{-\eta t} \Xi(t) \xrightarrow[t \rightarrow \infty]{(d)} \mathcal{E}$$

où (d) dénote la convergence en distribution et où \mathcal{E} est une variable aléatoire (v.a.) exponentielle de paramètre $\psi'(\eta)$.

La racine η est appelée *paramètre malthusien* associé à Ξ car $\Xi(t)$ croît comme $e^{\eta t}$ lorsque $t \rightarrow \infty$.

Dans les chapitres 2 et 3, nous étudions deux développements de ces arbres de ramification pour des modèles de dynamique des populations : un modèle où des individus immigreront à taux poissonnien et un modèle où des mutations neutres peuvent se produire à la naissance. La plupart des résultats obtenus généralisent des propriétés déjà connues si les durées de vie sont exponentielles. Cependant, comme les processus considérés ne sont pas markoviens, des propriétés habituellement utilisées comme les équations de Kolmogorov ne peuvent plus l'être et certains théorèmes limites ne peuvent plus être obtenus à l'aide des martingales habituelles, ce qui nous oblige à faire appel à des techniques plus avancées. En écologie, les modèles considérés se rapprochent du cadre de la théorie neutre de la biodiversité [32].

1.2 Principaux résultats obtenus

1.2.1 Principaux résultats du chapitre 2

Dans le chapitre 2, qui a fait l'objet de la publication [62], on étudie un modèle d'immigration du type "île-continent" qui généralise celui considéré par S. Karlin et J. McGregor [41] à des temps de vie qui ne sont pas forcément exponentiels. Ce modèle décrit une méta-communauté (ou continent) qui délivre des immigrants à une communauté locale comme dans la théorie de la biologie insulaire [54]. Considérons une île initialement déserte. À chaque temps d'arrivée d'un processus de Poisson de paramètre θ , un individu émigre du continent vers cette île et fonde une famille qui évolue comme $(\Xi(t), t \geq 0)$ un processus CMJ binaire, homogène (*i.e.* avec taux de naissance b et durées de vie distribuées selon $\Lambda(\cdot)/b$) et surcritique, cela indépendamment des individus déjà présents sur l'île. Voir la figure 1.2.

Le processus Ξ est supposé surcritique, soit donc $\eta > 0$ le paramètre malthusien qui lui est associé. On note $I(t)$ le nombre total d'individus présents sur l'île au temps t . On

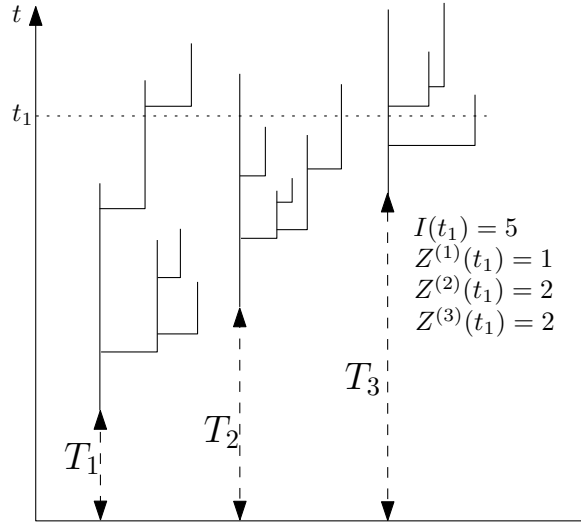


FIGURE 1.2 – Un arbre de ramification avec immigration. Sur cet exemple, au temps t_1 , trois populations, fondées aux temps T_1, T_2 et T_3 , sont présentes.

obtient alors des propriétés pour le processus $(I(t), t \geq 0)$ similaires à celles énoncées dans [48] pour le processus sans immigration. Soit W la fonction d'échelle associée au processus de contour et définie par l'équation (1.1).

Théorème. *Si $t \geq 0$, $I(t)$ est distribuée selon une loi binomiale négative de paramètres $1 - W(t)^{-1}$ et θ/b . i.e. si $s \in [0, 1]$, sa série génératrice est*

$$G_t(s) := \mathbb{E} \left[s^{I(t)} \right] = \left(\frac{W(t)^{-1}}{1 - s(1 - W(t)^{-1})} \right)^{\theta/b}.$$

De plus,

$$e^{-\eta t} I(t) \xrightarrow[t \rightarrow \infty]{} I \quad p.s.$$

où I suit une distribution Gamma de paramètres θ/b et $\psi'(\eta)$, dont la densité par rapport à la mesure de Lebesgue est

$$g(x) = \frac{\psi'(\eta)^{\theta/b} x^{\theta/b-1} e^{-\psi'(\eta)x}}{\Gamma(\theta/b)}, \quad x > 0$$

où Γ désigne la fonction Gamma d'Euler.

La population totale $I(t)$ est donc aussi d'ordre $e^{\eta t}$ lorsque $t \rightarrow \infty$. Dans [42], D.G. Kendall avait prouvé la première assertion de ce théorème dans le cas d'un processus markovien de naissance et mort avec immigration. Pour prouver la convergence presque sûre dans la deuxième assertion, on montre la propriété de log-intégrabilité suivante pour le processus sans immigration conditionné à ne pas s'éteindre.

Proposition. *Si on note $\log^+ x := \log x \vee 0$ pour $x > 0$,*

$$\mathbb{E} \left[\left(\log^+ \sup_{t \geq 0} (e^{-\eta t} \Xi(t)) \right)^2 \middle| \text{Ext}^c \right] < \infty.$$

On suppose ensuite que les individus portent des types (ou appartiennent à des espèces) qui sont transmis à leurs descendants et que notre modèle est neutre : tous les individus sont échangeables puisque aucun type n'a d'avantage reproductif sur les autres (ils ont tous le même taux de naissance et la même distribution de durée de vie). Différentes hypothèses décrites ci-après sont faites sur les types portés par les immigrants.

Par plus vieille famille, on entendra la famille fondée par le premier migrant et l'abondance relative sera la taille d'une famille renormalisée par la taille de la population totale. On s'intéresse alors aux questions suivantes. Quelle est l'abondance relative au temps t de la plus vieille famille, seconde plus vieille famille, ... quand $t \rightarrow \infty$? Quelle est l'abondance relative asymptotique d'un type donné?

Modèle I : Dans ce modèle, on suppose qu'à chaque fois qu'un migrant arrive sur l'île, il est d'un type différent de ceux déjà présents.

Pour $i \geq 1$, soit $(Z^{(i)}(t), t \geq 0)$ l'effectif de la i -ème plus vieille famille parmi les familles survivantes (rappelons que nous avons supposé que le processus Ξ est surcritique et donc la probabilité de survie d'une famille est η/b). On obtient alors le théorème suivant qui généralise un résultat de S. Tavaré [66] qui considérait le cas d'un processus de naissance pure pour lequel la mesure de longévité est $\Lambda := b\delta_\infty$ et auquel il ajoutait une immigration à taux poissonnien.

Théorème. *On a*

$$\lim_{t \rightarrow \infty} \left(\frac{Z^{(1)}(t)}{I(t)}, \frac{Z^{(2)}(t)}{I(t)}, \dots \right) = (P_1, P_2, \dots) \quad p.s.$$

où (P_1, P_2, \dots) suit une distribution GEM de paramètre θ/b , c'est-à-dire pour $i \geq 1$,

$$P_i \stackrel{(d)}{=} B_i \prod_{j=1}^{i-1} (1 - B_j)$$

où $(B_i)_{i \geq 1}$ est une suite de v.a. i.i.d. de loi Beta(1, θ/b).

Il est important de noter que dans ce théorème, la limite ne dépend que du ratio immigration-naissance θ/b et pas de la loi des durées de vie. La variable GEM (pour Griffiths, Engen et McCloskey) qui apparaît à la limite est une v.a. à valeurs dans les partitions de masse de $[0, 1]$: moralement, on tire une loi Beta sur $[0, 1]$ et ce qui se trouve à sa gauche correspond à la proportion P_1 . On tire ensuite une autre loi Beta indépendante sur la partie restante ; P_2 correspond alors à la partie gauche et ainsi de suite. Une v.a. GEM de paramètre $\alpha > 0$ est aussi une permutation *biaisée par la taille* d'une v.a. de Poisson-Dirichlet de paramètre α ($PD(\alpha)$) qui est à valeurs dans les partitions de masse de $[0, 1]$ dont les blocs sont rangés par ordre décroissant. Plus précisément, une variable $PD(\alpha)$ est distribuée comme $(Y_i/\Sigma, i \geq 1)$ où $Y_1 > Y_2 > \dots$ sont les points d'un processus ponctuel de Poisson sur $(0, \infty)$ d'intensité $\alpha y^{-1} e^{-y} dy$ et $\Sigma := \sum_{i \geq 1} Y_i$. Pour plus de détails, voir [6] ou

[58]. Le fait de réorganiser les blocs de manière "biaisée par la taille" se comprend sur notre modèle en remarquant que la plus vieille famille ayant été fondée en premier, elle a plus de chances d'occuper une proportion asymptotique plus importante que les autres familles.

Modèle II : Soit $(p_i, i \geq 1)$ une mesure de probabilité sur \mathbb{N}^* . Dans ce modèle, un migrant est de type $i \geq 1$ avec probabilité p_i . On s'intéresse alors aux abondances relatives de chaque type dans la population totale. Pour $i \geq 1$, soit $I_i(t)$ le nombre d'individus de type i vivants au temps t .

Théorème. Pour $i \geq 1$, posons $\theta_i := \frac{\theta p_i}{b}$. On a alors

$$\lim_{t \rightarrow \infty} \left(\frac{I_1(t)}{I(t)}, \frac{I_2(t)}{I(t)}, \dots \right) = (P'_1, P'_2, \dots) \quad p.s.$$

où pour $i \geq 1$,

$$P'_i \stackrel{(d)}{=} B'_i \prod_{j=1}^{i-1} (1 - B'_j)$$

et $(B'_i)_{i \geq 1}$ est une suite de v.a. indépendantes telles que B'_i est de loi Beta $\left(\theta_i, \frac{\theta}{b} \sum_{j \geq i+1} p_j \right)$.

En particulier, P'_i suit une distribution Beta $(\theta_i, \theta/b - \theta_i)$.

À nouveau, la limite ne dépend pas de la loi des durées de vie mais seulement des taux θ et b et des probabilités p_i . On peut également remarquer que si on ne considère qu'un nombre fini n de types possibles, on a

$$\sum_{i=1}^n P'_i = \sum_{i=1}^n \frac{I_i}{I} = 1, \quad \sum_{i=1}^n \theta_i = \frac{\theta}{b}$$

et la densité jointe du n -uplet (P'_1, \dots, P'_n) est

$$\frac{\Gamma(\theta/b)}{\prod_{i=1}^n \Gamma(\theta_i)} \left(\prod_{i=1}^{n-1} x_i^{\theta_i-1} \mathbf{1}_{\{x_i > 0\}} \right) (1 - x_1 - \dots - x_{n-1})^{\theta_{n-1}} \mathbf{1}_{\{x_1 + \dots + x_{n-1} < 1\}}$$

qui est la densité jointe d'une distribution de Dirichlet $\text{Dir}(\theta_1, \dots, \theta_n)$.

Modèle III : Dans ce troisième et dernier modèle, on considère un spectre continu des types possibles et l'on modifie quelque peu la dynamique d'immigration. Soit f une fonction positive représentant la densité des abondances des types dans la métacommunauté, c'est-à-dire que la probabilité d'y tirer un individu dont l'abondance appartient à $[x, x+dx]$ est $\frac{xf(x)}{\theta} dx$ où $\theta := \int_0^\infty xf(x) dx$ est supposé fini. Dans ce modèle, quand un individu migre vers l'île, il débute une nouvelle population qui évolue comme un processus de branchement avec immigration dont le taux d'immigration est proportionnel à l'abondance de son espèce dans la métacommunauté.

Plus précisément, soit Π un processus ponctuel de Poisson sur $\mathbb{R}_+ \times \mathbb{R}_+$ d'intensité $dt \otimes xf(x) dx$. On a alors $\Pi = ((T_i, \Delta_i), i \geq 1)$ où la suite croissante $T_1 < T_2 < \dots$ représente les temps d'arrivées d'un processus de Poisson d'intensité θ et $(\Delta_i, i \geq 1)$ est une suite de v.a. i.i.d. de densités $\theta^{-1}xf(x)$ et indépendante de $(T_i, i \geq 1)$. Au temps T_i , une nouvelle famille débute et évolue suivant un processus de branchement avec immigration défini précédemment, mais avec un taux d'immigration aléatoire Δ_i .

Un cas particulier de densité d'abondance f est utilisé dans plusieurs articles de biologie. En effet, I. Volkov et al. [68] et G. Watterson [69] considèrent $f(x) = \frac{e^{-ax}}{x}$ comme un équivalent continu de la série logarithmique des distributions proposée par R. Fisher et al. [25]. Avec ce cas particulier de fonction f , les espèces avec de faibles abondances apparaissent souvent mais ont un faible taux d'immigration.

Soit F la fonction définie par

$$F(v) := \int_0^\infty dx \frac{xf(x)}{\theta} \int_v^\infty \frac{e^{-\psi'(\eta)t} t^{x/b-1} \psi'(\eta)^{x/b}}{\Gamma(x/b)} dt.$$

En fait, F est la queue de distribution d'un mélange d'une loi Gamma et d'une v.a. distribuée selon la densité $\theta^{-1}xf(x)dx$. Soit $(Z^i(t), t \geq 0)$ la i -ème plus vieille famille et on appelle encore $I(t)$ la taille de la population totale au temps t . Dans ce cas, on obtient également les marginales unidimensionnelles de I :

$$\mathbb{E} \left[s^{I(t)} \right] = \exp \left(- \int_0^t du \left(\theta - \int_0^\infty \frac{xf(x)}{(W(u)(1-s) + s)^{x/b}} dx \right) \right), \quad s \in [0, 1]$$

et un résultat sur les abondances relatives des différents types.

Théorème. Si $\int_0^\infty x^2 f(x) dx < \infty$, on a

$$\left(\frac{Z^1(t)}{I(t)}, \frac{Z^2(t)}{I(t)}, \dots \right) \xrightarrow{t \rightarrow \infty} \left(\frac{\sigma_1}{\sigma}, \frac{\sigma_2}{\sigma}, \dots \right) \text{ p.s.}$$

où pour $i \geq 1$, $\sigma_i := e^{-\eta T_i} G_i$ avec $(G_i)_{i \geq 1}$ une suite de v.a. i.i.d. de queue de distribution F et indépendante du processus de Poisson $(T_i)_{i \geq 1}$ et où $\sigma = \sum_{i \geq 1} \sigma_i$ est fini p.s.

1.2.2 Principaux résultats du chapitre 3

Dans ce chapitre, on s'intéresse à un autre modèle de population où l'immigration est remplacée par des mutations pouvant avoir lieu à la naissance des individus. Plus précisément, la dynamique des naissances et des morts reste la même que dans le chapitre 2 : chaque individu vit un temps distribué selon $\Lambda(\cdot)/b$ et donne naissance à taux b au cours de sa vie. La différence est que, indépendamment de tous les individus, chaque enfant peut être un clone de sa mère (*i.e.* porte le même type ou allèle) avec probabilité $1 - p \in (0, 1)$, ou un enfant mutant avec probabilité p . De plus, quand un enfant mutant apparaît, il est d'un type jamais porté auparavant. On considère donc un modèle dit à *infinité d'allèles*. Il est également neutre car comme précédemment, tous les individus, quel que soit leur type, évoluent de la même façon. Voir la figure 1.3.

On suppose que $m := \int_{(0, \infty]} r \Lambda(dr) > 1$; le processus $(\Xi(t), t \geq 0)$ est donc surcritique. Comme on s'intéresse à des propriétés asymptotiques de ce modèle, dans la plupart des résultats qui suivent, on se placera sous la probabilité conditionnelle

$$\mathbb{P}^* := \mathbb{P}(\cdot | \text{Ext}^c).$$

Dans la suite, on appellera *famille* tous les individus partageant un allèle commun et le *processus clonal* $(\Xi_c(t), t \geq 0)$ sera le processus décrivant la famille des individus de même type que l'ancêtre. On voit que ce processus Ξ_c est un processus CMJ binaire homogène dont la mesure de longévité est $\Lambda_c := (1 - p)\Lambda$. On peut donc associer à ce processus un exposant de Laplace

$$\psi_c(\lambda) := p\lambda + (1 - p)\psi(\lambda)$$

de plus grande racine η_c et de fonction d'échelle W_c . Ce processus est sous-critique, critique ou surcritique si $(1 - p)m$ est respectivement strictement inférieur, égal ou strictement supérieur à 1.

Spectre des fréquences

Dans un premier temps, on étudie la partition allélique de la population au cours du temps. Pour cela, on s'intéresse au *spectre des fréquences* $(M_t^i, i \geq 1)$ où pour $i \geq 1$ et $t \geq 0$, M_t^i est le nombre de familles ayant i représentants vivants au temps t (voir la figure

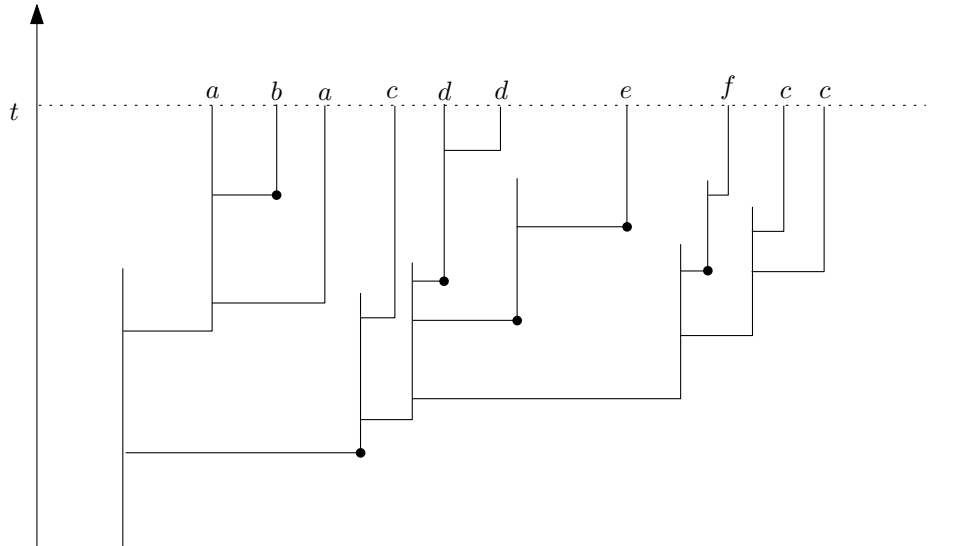


FIGURE 1.3 – Un arbre de ramification avec mutations à la naissance. Les points correspondent aux mutations arrivant à la naissance et les différentes lettres représentent les allèles des individus vivants au temps t . Le spectre des fréquences $(M_t^i, i \geq 1)$ vaut $(3, 2, 1, 0, \dots)$ car trois allèles (b, e, f) sont portés par seulement un individu, a et d sont portés par deux individus et c est le seul porté par trois.

1.3). L'avantage d'étudier la partition allélique via ce spectre des fréquences est que l'on n'a pas besoin de donner un nom aux différents types. Le plus célèbre exemple de spectre des fréquences est la *formule d'échantillonnage d'Ewens* [24] qui décrit la distribution de ce spectre dans le cadre d'un coalescent de Kingman avec mutations poissonniennes. Cela peut également être vu comme le cas limite quand $N \rightarrow \infty$ d'un modèle de Wright-Fisher à N individus et avec des mutations à la naissance se produisant avec une probabilité de l'ordre de $1/N$.

Les deux propriétés suivantes donnent un résultat plus précis sur la partition allélique car on tient également compte des "âges" des types, c'est-à-dire depuis combien de temps ils sont apparus dans la population. Soit $M_t := \sum_{i \geq 1} M_t^i$ le nombre d'allèles présents au temps t et pour $a > 0$, soit $M_t^{i,a}$ le nombre de familles ayant i représentants vivants au temps t et apparues après le temps $t - a$ c'est-à-dire d'âge inférieur à a au temps t .

On rappelle que W_c est la fonction d'échelle associée au processus clonal et on note

$$J := \eta \int_0^\infty e^{-\eta u} \ln(W_c(u)) du$$

et

$$J^{i,a} := \int_0^a e^{-\eta u} \left(1 - \frac{1}{W_c(u)}\right)^{i-1} \frac{W_c'(u)}{W_c^2(u)} du.$$

On peut tout d'abord calculer explicitement la valeur moyenne du spectre allélique pour des types dont les âges sont compris entre $a - da$ et a .

Proposition. *Pour $0 < a < t$ et $i \geq 1$, si $M_t^{i,da}$ désigne le nombre de types portés par i individus au temps t et d'âges dans $[a - da, a]$, on a*

$$\mathbb{E}[M_t^{i,da}] = \frac{p}{b(1-p)} W'(t-a) \left(1 - \frac{1}{W_c(a)}\right)^{i-1} \frac{W_c'(a)}{W_c^2(a)} da.$$

On obtient également des résultats de convergence pour le nombre de types présents au temps t (i), pour le spectre des fréquences (ii) et pour les abondances relatives des types dans la population totale (iii).

Proposition. Sous \mathbb{P}^* ,

(i)

$$e^{-\eta t} M_t \xrightarrow[t \rightarrow \infty]{} \frac{p}{1-p} J \mathcal{E} \quad \text{p.s.}$$

(ii)

$$e^{-\eta t} M_t^{i,a} \xrightarrow[t \rightarrow \infty]{} \frac{p}{1-p} J^{i,a} \mathcal{E} \quad \text{p.s.}$$

(iii)

$$\frac{M_t^{i,a}}{M_t} \xrightarrow[t \rightarrow \infty]{} \frac{J^{i,a}}{J} \quad \text{p.s.}$$

où \mathcal{E} est une v.a. exponentielle de paramètre $\psi'(\eta)$.

Âges des familles pour un processus clonal (sous)critique

On s'intéresse aux âges des différentes familles présentes au temps t quand $t \rightarrow \infty$. Tout d'abord, lorsque le processus clonal est critique, c'est-à-dire quand $(1-p)m = 1$, on obtient que sous \mathbb{P}^* , l'âge d'un allèle au temps t est de l'ordre de $c_t := t - \ln t/\eta$ lorsque $t \rightarrow \infty$. Plus précisément, on a

Proposition. On suppose que $(1-p)m = 1$ et que $\psi_c''(0) = (1-p) \int_{(0,\infty]} r^2 \Lambda(dr) < \infty$. Pour $-\infty < \underline{a} < \bar{a} \leq \infty$, soit $\nu_t(\underline{a}, \bar{a})$ le nombre de types présents au temps t et d'âges compris entre $\underline{a} + c_t$ et $\bar{a} + c_t$.

Alors, sous \mathbb{P}^* , $\nu_t(\underline{a}, \bar{a})$ converge en loi vers une v.a. de Poisson mélangée de paramètre

$$\frac{m-1}{\psi'(\eta)\eta} \left(e^{-\eta \underline{a}} - e^{-\eta \bar{a}} \right) E$$

avec E une v.a. exponentielle de moyenne 1.

On obtient ensuite la convergence du processus ponctuel des âges des familles au temps t . Soit $\mathcal{M}(\mathbb{R})$ l'ensemble des mesures σ -finies sur \mathbb{R} et finies sur \mathbb{R}^+ que l'on munit de la topologie semi-vague induite par les applications $\nu \mapsto \int_{\mathbb{R}} f(x) \nu(dx)$ pour toute fonction f continue bornée et telle que $f \mathbf{1}_{\mathbb{R}^+}$ ait un support borné.

Théorème. Sous les mêmes hypothèses que dans la proposition précédente, soit ν_t le processus ponctuel

$$\nu_t(dx) := \sum_{k \geq 1} \delta_{A_t^k - c_t}(dx)$$

où $A_t^1 \geq A_t^2 \geq \dots$ est la suite décroissante des âges des familles vivantes au temps t . Sous \mathbb{P}^* , quand $t \rightarrow \infty$, ν_t converge en loi dans $\mathcal{M}(\mathbb{R})$ muni de la topologie semi-vague vers un processus ponctuel de Poisson mélangé d'intensité

$$\frac{m-1}{\psi'(\eta)} E e^{-\eta x} dx$$

où E est de loi exponentielle de paramètre 1.

Pour un processus clonal sous-critique (*i.e.* $(1-p)m < 1$), il n'y a pas de propriétés équivalentes dans le cas général. Cependant, en supposant l'existence de moments exponentiels de Λ , on peut montrer qu'il existe $\alpha_c < 0$ tel que $\psi_c(\alpha_c) = 0$, et alors les âges des plus vieilles familles sont d'ordre $c_t := \frac{\eta}{\eta - \alpha_c} t$. Plus précisément, avec des notations identiques à celles du cas critique, on obtient les résultats suivants.

Proposition. *On suppose que $(1-p)m < 1$ et $-\infty < \underline{a} < \bar{a} \leq \infty$. Alors, sous \mathbb{P}^* , $\nu_t(\underline{a}, \bar{a})$ converge en loi vers une v.a. de Poisson mélangé de paramètre*

$$\frac{K}{\eta - \alpha_c} \left(e^{-(\eta - \alpha_c)\underline{a}} - e^{-(\eta - \alpha_c)\bar{a}} \right) E$$

avec E une v.a. exponentielle de moyenne 1 et où K est une constante explicite ne dépendant que de p , de ψ et de ψ_c .

Sous les mêmes hypothèses, on obtient

Théorème. *Sous \mathbb{P}^* , quand $t \rightarrow \infty$, ν_t converge en loi dans $\mathcal{M}(\mathbb{R})$ muni de la topologie semi-vague vers un processus ponctuel de Poisson mélangé d'intensité*

$$KE e^{-(\eta - \alpha_c)x} dx.$$

Tailles des grandes familles pour un processus clonal surcritique

Dans une dernière partie, on s'intéresse à des propriétés similaires sur les tailles asymptotiques des différentes familles lorsque le processus clonal est surcritique : $(1-p)m > 1$. On sait alors que conditionnellement à la survie de Ξ_c , $e^{-\eta c t} \Xi_c(t)$ converge p.s. vers une v.a. exponentielle quand $t \rightarrow \infty$. C'est pourquoi on s'intéresse à \mathcal{N}_t le nombre de types portés par plus de $Ae^{\eta c t}$ au temps t pour $A > 0$. La proposition suivante confirme le fait que les grandes familles ont bien des tailles d'ordre $e^{\eta c t}$ lorsque $t \rightarrow \infty$.

Proposition. *Si $(1-p)m > 1$,*

$$\mathbb{E}[\mathcal{N}_t] \xrightarrow{t \rightarrow \infty} \frac{\eta c p}{b(1-p)} \int_0^\infty W'(x) \exp(-A\psi'_c(\eta_c)e^{\eta c x}) dx + \frac{\eta c}{b(1-p)} e^{-A\psi'_c(\eta_c)}.$$

On aimerait obtenir des résultats plus précis (comme la convergence en distribution de \mathcal{N}_t), mais les techniques utilisées dans le paragraphe précédent (comme les processus CMJ comptés avec des caractéristiques dépendant du temps) ne semblent pas s'appliquer dans ce cadre.

1.2.3 Principaux résultats du chapitre 4

Dans le chapitre 4, dont les résultats proviennent de l'article en révision [61], on s'intéresse à une certaine classe de processus de Lévy conditionnés à rester positifs à l'aide de leurs processus des hauteurs.

Soit $X = (X(t), t \geq 0)$ un processus de Lévy spectralement positif ne dérivant pas vers $+\infty$. On note \mathbb{P}_x la loi du processus partant de x . Sa transformée de Laplace est donnée par

$$\mathbb{E}_0 \left[e^{-\lambda X_t} \right] = e^{t\psi(\lambda)}$$

avec

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0, \infty)} \Lambda(dr)(e^{-\lambda r} - 1 + \lambda r)$$

où $\alpha \geq 0, \beta \geq 0$ et $\int_{(0,\infty)} \Lambda(dr)(r \wedge r^2) < \infty$. Le processus X dérive vers $-\infty$ (resp. est récurrent) ssi $\alpha > 0$ (resp. $\alpha = 0$). On notera $(\mathcal{F}_t, t \geq 0)$ sa filtration naturelle associée, $(I_t, t \geq 0)$ son processus infimum passé et T_0 son temps d'atteinte de $(-\infty, 0]$ qui est égal au temps d'atteinte de 0 dans ce cas puisque X ne fait aucun saut négatif.

D'autres conditionnements à rester positifs que celui proposé dans cette thèse ont déjà été étudiés pour des processus de Lévy. La première façon intuitive de conditionner un processus de Lévy général (et donc pas nécessairement spectralement positif) partant de x a été donnée par L. Chaumont [13, 14] puis par L. Chaumont et R. Doney [15] : le processus X est conditionné à ne pas atteindre $(-\infty, 0]$ jusqu'au temps \mathbf{e}/ε où \mathbf{e} est une v.a. exponentielle de paramètre 1, indépendante de X et pour $\varepsilon > 0$ arbitrairement petit. Plus précisément, les auteurs définissent une famille de mesures $\mathbb{P}_x^\uparrow, x > 0$ via une transformée harmonique de Doob (ou h -transformée) limite de

$$\mathbb{P}_x^\uparrow(\Theta, t < \zeta) = \lim_{\varepsilon \rightarrow 0} \mathbb{P}_x(\Theta, t < \mathbf{e}/\varepsilon | X_s > 0, 0 \leq s \leq \mathbf{e}/\varepsilon)$$

pour $t \geq 0, \Theta \in \mathcal{F}_t$ et où ζ est le temps de mort de X c'est-à-dire le temps d'atteinte par X d'un certain point cimetièrre.

Il est à noter que dans le cas particulier où X est spectralement positif et dérive vers $-\infty$ (i.e. lorsque $\mathbb{E}[X_1] < 0$), \mathbb{P}_x^\uparrow est une sous-probabilité i.e. $\mathbb{P}_x^\uparrow(\zeta < \infty) = 1$ alors que lorsqu'il est récurrent ($\mathbb{E}[X_1] = 0$), c'est une probabilité.

Dans [31], K. Hirano a considéré deux façons différentes de conditionner un processus de Lévy général à rester positif. La première est proche de celle de L. Chaumont puisque le processus est conditionné à ne pas atteindre $(-\infty, 0]$ avant le temps s pour s arbitrairement grand. Quant à la deuxième méthode, X est conditionné à atteindre des niveaux arbitrairement grands avant de devenir négatif. Plus précisément, sous l'existence de moments exponentiels de X et en rappelant que $T_{(s,+\infty)}$ est son temps d'atteinte de $(s, +\infty)$, K. Hirano montre que les deux limites

$$\lim_{s \rightarrow \infty} \mathbb{P}_x(\Theta | T_0 > s) \quad \text{et} \quad \lim_{s \rightarrow \infty} \mathbb{P}_x(\Theta | T_0 > T_{(s,+\infty)}), \quad \Theta \in \mathcal{F}_t$$

existent pour tout $t \geq 0$ et en déduit deux conditionnements différents à rester positif via des h -transformées.

Dans ce travail, on cherche à conditionner des processus spectralement positifs à atteindre des hauteurs arbitrairement grandes avant le temps T_0 . Le mot hauteur ne doit pas être confondu avec le terme "niveau" utilisé dans le deuxième conditionnement de Hirano : il fait en fait référence au processus des hauteurs $(H_t, t \geq 0)$ défini par J.F. Le Gall et Y. Le Jan [50] puis par T. Duquesne et J.F. Le Gall [19]. On distingue deux cas : X est à trajectoires soit à variation finie soit à variation infinie. Dans le premier cas, H_t est le "nombre de records" au temps t du processus X retourné en t . Dans le deuxième cas, la définition équivalente de H_t que l'on rappelle ci-après nécessite l'utilisation de temps locaux.

Dans les deux cas, pour $t \geq 0$ et $\Theta \in \mathcal{F}_t$, on montre que la limite

$$\lim_{a \rightarrow \infty} \mathbb{P}_x \left(\Theta, t < T_0 \mid \sup_{s \leq T_0} H_s \geq a \right)$$

existe. Comme on suppose que le processus X est récurrent ou dérive vers $-\infty$, l'événement par lequel on conditionne est évanescant, c'est-à-dire qu'il est de probabilité nulle lorsque $a \rightarrow \infty$.

Cas de la variation finie

On s'intéresse tout d'abord au cas où X a des trajectoires à variation finie. Son exposant de Laplace s'écrit alors sous la forme

$$\psi(\lambda) = \lambda - \int_{(0,\infty)} (1 - e^{-\lambda r}) \Lambda(dr)$$

et

$$m := \int_{(0,\infty)} r \Lambda(dr) \leq 1$$

puisqu'on considère uniquement des processus de Lévy récurrents ($m = 1$) ou dérivant vers $-\infty$ ($m < 1$). Sans perte de généralité, on s'est restreint à une dérive égale à 1 pour pouvoir utiliser le fait que X est le processus de contour d'un arbre de ramification défini précédemment. Lorsque la dérive d est différente de 1, on peut toujours le voir comme un processus de contour, le contour s'effectuant non plus à vitesse -1 mais à vitesse $-d$.

Définition. La hauteur H_t du processus X au temps t est le nombre (fini p.s.) de sauts de son infimum futur

$$H_t := \#\{0 \leq s \leq t; X_{s-} < \inf_{s \leq r \leq t} X_r\}.$$

On pose

$$\{s_t^1 < \dots < s_t^{H_t}\} := \{0 \leq s \leq t < T_0; X_{s-} < \inf_{s \leq r \leq t} X_r\},$$

et les records de l'infimum futur sont $\rho_i^t := \inf_{s_t^i \leq r \leq t} X_r - X_{s_t^i-}$ pour $1 \leq i \leq H_t$ et $\rho_0^t = \inf_{0 \leq r \leq t} X_r$.

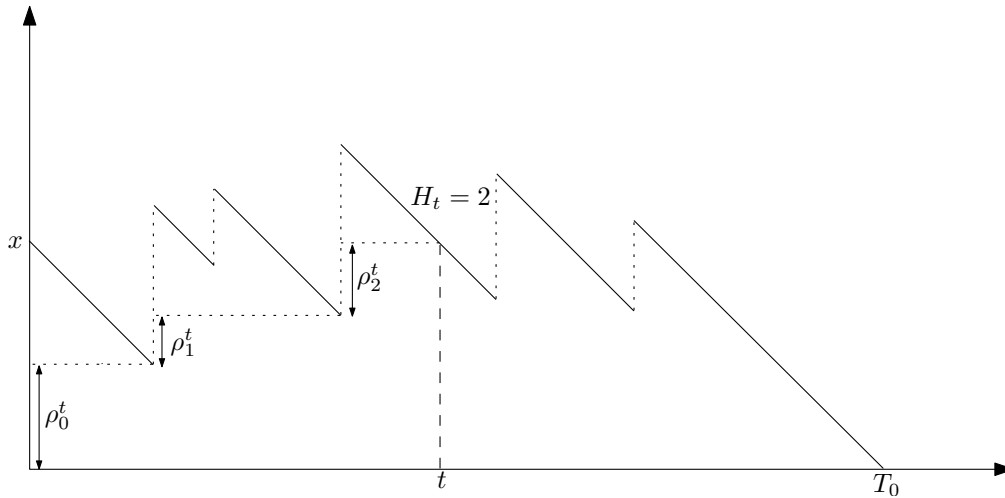


FIGURE 1.4 – Une trajectoire de X partant de x et tué quand il touche 0 sur laquelle sont représentés les sauts de l'infimum futur au temps t : ρ_0^t, ρ_1^t et ρ_2^t .

Au contraire des deux premiers chapitres où on utilise des propriétés du contour pour en déduire des résultats sur l'arbre de ramification et sur le processus Ξ , on se sert ici de propriétés connues des arbres aléatoires pour prouver des résultats sur les processus de Lévy. Plus précisément, si on voit X comme le processus de contour d'un arbre de ramification, H_t correspond alors à la hauteur généalogique dans l'arbre de l'individu visité au temps t par le processus de contour, et l'on montre que conditionner X à atteindre la

hauteur a avant T_0 revient à conditionner l'arbre (critique ou sous-critique) à atteindre la génération a . On en déduit

Théorème. (i) On suppose que $b := \Lambda(\mathbb{R}^+)$ est fini et que

$$\left(m = 1 \text{ et } \int_{(0,\infty)} z^2 \Lambda(dz) < \infty \right) \text{ ou } \left(m < 1 \text{ et } \int_{[1,\infty)} z \log(z) \Lambda(dz) < \infty \right).$$

Alors, pour $t \geq 0$ et $\Theta \in \mathcal{F}_t$,

$$\lim_{a \rightarrow \infty} \mathbb{P}_x \left(\Theta, t < T_0 \mid \sup_{s \leq T_0} H_s \geq a \right) = \frac{1}{x} \mathbb{E}_x [M_{t \wedge T_0} \mathbf{1}_\Theta]$$

où

$$M_t = \sum_{i=0}^{H_t} \rho_i^t m^{-i}.$$

En particulier, si $m = 1$, $M_t = X_t$. De plus, le processus $(M_{t \wedge T_0}, t \geq 0)$ est une (\mathcal{F}_t) -martingale.

(ii) Si $b = \infty$, le conditionnement par rapport à l'événement $\{\sup_{s \in (0, T_0)} H_s \geq a\}$ est trivial car pour $a \geq 0$, $\mathbb{P}_x(\sup_{s \in (0, T_0)} H_s \geq a) = 1$.

On obtient également un résultat sur un arbre de ramification conditionné à atteindre des générations arbitrairement grandes. Soit \mathbb{T} un arbre de ramification de mesure de longévité Λ finie et soit \mathbf{P} sa loi. Les arbres considérés ici sont sous-critiques ou critiques ce qui est le cas si m est respectivement strictement inférieur ou égal à 1. Chaque individu de l'arbre est étiqueté par une suite finie d'entiers : la racine de l'arbre, c'est-à-dire l'unique ancêtre, a l'étiquette \emptyset et si un individu de l'arbre de la génération i est étiqueté par la suite d'entiers $u_1 \dots u_i$, on notera successivement par $u_1 \dots u_i 1$ sa plus jeune fille, par $u_1 \dots u_i 2$ sa seconde plus jeune fille et ainsi de suite pour toutes ses filles rangées dans l'ordre inverse de leurs dates de naissance.

Pour $n \geq 0$, soit \mathcal{Z}_n le nombre d'individus vivants à la génération n dans l'arbre de ramification \mathbb{T} et soit

$$\mathbf{P}^n := \mathbf{P}(\cdot \mid \mathcal{Z}_n > 0)$$

la loi de l'arbre conditionné à survivre au moins n générations. Sur l'événement $\{\mathcal{Z}_n > 0\}$, on distingue une lignée pour l'arbre tronqué ne prenant en compte que les individus des n premières générations. Cette lignée particulière, notée \mathbb{B}^n , est la première lignée à atteindre la génération n pour l'ordre lexicographique lié à l'étiquetage défini précédemment (voir la figure 1.5). On peut montrer que cette lignée distinguée est caractérisée par la suite $(A_k^n, R_k^n)_{0 \leq k \leq n-1}$ et par A_n^n , où A_k^n (resp. R_k^n) est l'âge (resp. le temps qu'il lui reste à vivre) de l'individu de la génération k dans \mathbb{B}^n quand il donne naissance à sa fille appartenant à \mathbb{B}^n , et où A_n^n est la durée de vie de l'individu de la génération n dans \mathbb{B}^n . Un exemple de telle suite est représenté sur la figure 1.5.

Dans le théorème suivant, on montre que quand $n \rightarrow \infty$, l'arbre sous la loi \mathbf{P}^n converge vers un arbre possédant une unique épine dorsale d'individus ayant des durées de vie biaisées par la taille et sur laquelle sont greffés des arbres "normaux" (sous-critiques ou critiques).

Théorème. On suppose que la mesure Λ vérifie les mêmes hypothèses que dans le théorème précédent. Alors, quand $n \rightarrow \infty$, la loi de \mathbb{B}^n sous \mathbf{P}^n converge au sens des troncatures

Cas de la variation infinie

On obtient ensuite un résultat similaire si X est à variation infinie et ne dérive pas vers $+\infty$. Son exposant de Laplace

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,\infty)} \Lambda(dr)(e^{-\lambda r} - 1 + \lambda r)$$

doit vérifier

$$\beta > 0 \text{ ou } \int_{(0,1)} r\Lambda(dr) = \infty,$$

ce qui est vrai sous l'hypothèse

$$\int_{[1,+\infty)} \frac{d\lambda}{\psi(\lambda)} < \infty. \quad (1.2)$$

On définit à présent le processus des hauteurs H , équivalent du processus de la partie précédente quand X est à variation infinie. Il a été introduit par J.F. Le Gall et Y. Le Jan [50] pour le cas $\beta = 0$ et dans le cas général par T. Duquesne et J.F. Le Gall [19]. Soit $X^{(t)}$ le processus retourné au temps t associé à X

$$X_s^{(t)} := X_{t-s} - X_{(t-s)-} \quad s \in [0, t],$$

et soit $S^{(t)}$ le processus supremum de $X^{(t)}$.

Définition. *Le processus des hauteurs $H = (H_t, t \geq 0)$ associé à X est tel que $H_0 := 0$ et pour $t > 0$, H_t est le temps local au niveau 0 et au temps t du processus $S^{(t)} - X^{(t)}$.*

Ce processus est en un certain sens l'équivalent continu du processus des hauteurs $(H_n, n \geq 0)$ qui code un arbre de Galton-Watson. Pour $n \geq 0$, H_n désigne la hauteur généalogique (ou génération) du n -ième sommet de l'arbre rencontré lors du parcours en profondeur de ce dernier. Dans [19], les auteurs montrent que ces deux processus sont reliés via des théorèmes limites et que le processus $(H_t, t \geq 0)$ code la généalogie d'un processus de branchement à espace d'états continu dont le mécanisme de branchement est ψ .

L'équivalent de la suite $(\rho_i^t, i \leq H_t)$, définie dans le cas à variation finie, est la mesure aléatoire ρ_t qui est une légère modification du processus d'exploration défini dans [19] et auquel on ajoute un poids I_t en 0.

Définition. *Pour $t \geq 0$, soit ρ_t la mesure positive aléatoire vérifiant que pour toute fonction f mesurable et positive,*

$$\langle \rho_t, f \rangle = \int_{[0,t]} d_s I_s^t f(H_s) + I_t f(0)$$

où $d_s I_s^t$ est la mesure de Stieltjes associée à la fonction croissante $s \mapsto I_s^t := \inf_{[s,t]} X$.

Théorème. *On suppose que la condition (1.2) est vérifiée. Alors, pour $t \geq 0$ et $\Theta \in \mathcal{F}_t$,*

$$\lim_{a \rightarrow \infty} \mathbb{P}_x \left(\Theta, t < T_0 \mid \sup_{s \leq T_0} H_s \geq a \right) = \frac{1}{x} \mathbb{E}_x \left[\int_0^{H_t} \rho_t(dz) e^{\alpha z}; t \leq T_0, \Theta \right]$$

où on rappelle que $\alpha = \psi'(0)$ est l'opposé de la dérivée de X . De plus, si

$$M_t := \int_0^{H_t} \rho_t(dz) e^{\alpha z},$$

$(M_{t \wedge T_0}, t \geq 0)$ est une martingale. En particulier, comme dans le cas de la variation finie, si $\alpha = 0$ (cas récurrent), alors $M_t = X_t$.

Nouvelle probabilité \mathbb{P}_x^\uparrow

Que X soit à variation finie ou infinie, on peut définir une nouvelle mesure de probabilité \mathbb{P}_x^\uparrow via les martingales $M_{\cdot \wedge T_0}$.

Définition. Pour $x > 0$, \mathbb{P}_x^\uparrow est la mesure de probabilité sur (Ω, \mathcal{F}_t) définie par

$$\mathbb{P}_x^\uparrow(\Theta) = \frac{1}{x} \mathbb{E}_x [M_{t \wedge T_0} \mathbf{1}_\Theta], \quad \Theta \in \mathcal{F}_t.$$

Cette définition est cohérente puisque $M_{\cdot \wedge T_0}$ est une martingale. De plus, il est à noter que $(X_t, t \geq 0)$ n'est plus markovien sous cette nouvelle probabilité. On peut voir cette mesure comme la loi du processus X partant de x et conditionné à rester positif. Contrairement aux conditionnements à rester positif déjà mentionnés, \mathbb{P}_x^\uparrow est une probabilité (et non une sous-probabilité) et sa définition ne requiert pas l'existence de moments exponentiels de X . Dans les deux cas étudiés, on montre la propriété suivante.

Proposition. Sous \mathbb{P}_x^\uparrow , en probabilité,

$$X_t \xrightarrow[t \rightarrow \infty]{} +\infty.$$

Chapitre 2

Théorèmes limites pour des arbres de ramification avec immigration

Ce chapitre est basé en grande partie sur l'article "*Limit theorems for supercritical age-dependent branching processes with neutral immigration*" paru dans le volume **43-1** de *Advances in Applied Probability* en 2011 [62].

Abstract: We consider a branching process with Poissonian immigration where individuals have inheritable types. At rate θ , new individuals singly enter the total population and start a new population which evolves like a supercritical, homogeneous, binary Crump-Mode-Jagers process: individuals have i.i.d. lifetimes durations (non necessarily exponential) during which they give birth independently at constant rate b . First, using spine decomposition, we relax previously known assumptions required for a.s. convergence of total population size.

Then, we consider three models of structured populations: either all immigrants have a different type, or types are drawn in a discrete spectrum or in a continuous spectrum. In each model, the vector (P_1, P_2, \dots) of relative abundances of surviving families converges a.s. In the first model, the limit is the GEM distribution with parameter θ/b .

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2.1 Introduction

We want to study and give some properties about several birth-death and immigration models where immigration is structured. All individuals behave independently of the others, their lifetimes are i.i.d. but non-necessarily exponential and each individual gives birth at constant rate b during her life. We will consider the supercritical case i.e., the mean number of children of an individual is greater than 1. In the absence of immigration, if $\Xi(t)$ denotes the number of extant individuals at time t , the process $(\Xi(t), t \geq 0)$ is a particular case of Crump-Mode-Jagers (or CMJ) processes [35, Chapter 6], also called general branching processes. Here, Ξ is a binary (births arrive singly) and homogeneous (constant birth rate) CMJ process. Now, we assume that at each arrival time of a Poisson process with rate θ , a new individual enters the population and starts a new population independently of the previously arrived ones. This immigration model extends to general lifetimes the mainland-island model of S. Karlin and J. McGregor [41]. In that case, the total population process Ξ is a linear birth-and-death process with immigration. For more properties about this process, see [67] or [70] and references therein. In the context of ecology [10], this model can be used as a null model of species diversity, in the framework of the neutral theory of biodiversity [32].

We first give the asymptotic behavior of the process $(I(t), t \geq 0)$ representing the total number of extant individuals on the island at time t . Specifically, there exists $\eta > 0$ (the Malthusian parameter associated with the branching process Ξ) such that $e^{-\eta t} I(t)$ converges almost surely. S. Tavaré [66] proved this result in the case of a linear birth process with immigration. The case of general CMJ processes was treated by P. Jagers [35] under the hypothesis that the variance of the number of children per individual is finite. We manage to relax this assumption in the case of homogeneous (binary) CMJ-processes thanks to spine decomposition of splitting trees [28, 47, 48] which are the genealogical trees generated by those branching processes. In passing, we obtain technical results on the log-integrability of $\sup_t e^{-\eta t} \Xi(t)$.

Then, we consider models where individuals bear clonally inherited types. These models intend to model a metacommunity (or mainland) which delivers immigrants to the island as in the theory of island biogeography [54]. However, we made specific assumptions about the spectrum of abundances in the metacommunity. In Model I, there is a discrete spectrum with zero macroscopic relative abundances: when an immigrant enters the population, it is each time of a new type. In Model II, we consider a discrete spectrum with nonzero macroscopic relative abundances: the type of each new immigrant is chosen according to some probability $(p_i, i \geq 1)$. In Model III, we consider a continuous spectrum of possible types but to enable a type to be chosen several times from the metacommunity, we change the immigration model: at each immigration time, an individual belonging to a species with abundance in $(x, x + dx)$ is chosen with probability $\frac{x f(x)}{\theta} dx$ (where f is a positive function representing the abundance density and such that $\theta := \int_0^\infty x f(x) dx < \infty$) and it starts an immigration process with immigration rate x . The particular case of abundance density $f(x) = \frac{e^{-ax}}{x}$ appears in many papers. I. Volkov et al. [68] and G. Watterson [69] considered $f(x)$ as a continuous equivalent of the logarithmic series distribution proposed by R. Fisher et al. [25] as a species abundance distribution. In this particular case, species with small abundances are often drawn but they will have a small immigration rate.

In the three models, we get results for the abundances P_1, P_2, \dots of different types as time t goes to infinity: the vector (P_1, P_2, \dots) rescaled by the total population size converges almost surely. More precisely, in Model I which is an extension of S. Tavaré's

result [66] to general lifetimes, we consider the abundances of the surviving families ranked by decreasing ages and the limit follows a *GEM distribution* with parameter θ/b . This distribution appears in other contexts: P. Donnelly and S. Tavaré [17] proved that for a sample of size n whose genealogy is described by a Kingman coalescent with mutation rate θ , the frequencies of the oldest, second oldest, etc. alleles converge in distribution as $n \rightarrow \infty$ to the GEM distribution with parameter θ ; S. Ethier [22] showed that it is also the distribution of the frequencies of the alleles ranked by decreasing ages in the stationary infinitely-many-neutral-alleles diffusion model.

In a sense, the surviving families that we consider in our immigration model are "large" families because their abundances are of the same order as the population size. A. Lambert [49] considered "small" families: he gave the joint law of the number of species containing k individuals, $k = 1, 2, \dots$

In Section 2.2, we describe the models we consider and state results we prove in other sections. Section 2.3 is devoted to proving a result about the process Ξ , Section 2.4 to proving a property of the immigration process $(I(t), t \geq 0)$ while in Section 2.5, we prove theorems concerning the relative abundances of types in the three models.

2.2 Preliminaries and statement of results

We first define splitting trees which are random trees satisfying:

- individuals behave independently from each other and have i.i.d. lifetime durations (possibly infinite),
- conditional on her birthdate α and her lifespan ζ , each individual reproduces according to a Poisson point process on $(\alpha, \alpha + \zeta)$ with intensity b ,
- births arrive singly.

We denote the common distribution of lifespan ζ by $\Lambda(\cdot)/b$ where Λ is a positive measure on $(0, \infty]$ with mass b called the *lifespan measure* [48].

The total population process $(\Xi(t), t \geq 0)$ belongs to a large class of processes called Crump-Mode-Jagers or CMJ processes. In these processes, also called general branching processes [35, ch.6], a typical individual reproduces at ages according to a random point process ξ on $[0, \infty)$ (denote by $\mu := \mathbb{E}[\xi]$ its intensity measure) and it is alive during a random time ζ . Then, the CMJ-process is defined as

$$\Xi(t) = \sum_x \mathbf{1}_{\{\alpha_x \leq t < \alpha_x + \zeta_x\}}, \quad t \geq 0$$

where for any individual x , α_x is her birth time and ζ_x is her lifespan. In this work, the process Ξ is a homogeneous (constant birth rate) and binary CMJ-process and we get

$$\mu(dx) = dx \int_{[x, \infty]} \Lambda(dr).$$

We assume that the mean number of children per individual $m := \int_{(0, \infty]} r \Lambda(dr)$ is greater than 1 (supercritical case).

For $\lambda \geq 0$, define $\psi(\lambda) := \lambda - \int_{(0, \infty]} (1 - e^{-\lambda r}) \Lambda(dr)$. The function ψ is convex, differentiable on $(0, \infty)$, $\psi(0^+) = 0$ and $\psi'(0^+) = 1 - \int_0^\infty r \Lambda(dr) < 0$. Then there exists a unique positive real number η such that $\psi(\eta) = 0$. It is seen by direct computation that this real number is a *Malthusian parameter* [35, p.10], i.e. it is the finite positive solution of $\int_0^\infty e^{-\eta r} \mu(dr) = 1$ and is such that $\Xi(t)$ grows like $e^{\eta t}$ on the survival event (see forthcoming Proposition 2.1). From now on, we define

$$c := \psi'(\eta)$$

which is positive because ψ is convex.

Another branching process appears in splitting trees: if we denote by \mathcal{Z}_n the number of individuals belonging to generation n of the tree, then $(\mathcal{Z}_n, n \geq 0)$ is a Bienaymé-Galton-Watson process started at 1 with offspring generating function

$$f(s) := \int_{(0, \infty]} b^{-1} \Lambda(dr) e^{-br(1-s)} \quad 0 \leq s \leq 1.$$

To get results about splitting trees and CMJ-processes, A. Lambert [47, 48] used tree contour techniques. He proved that the contour process Y of a splitting tree was a *spectrally positive* (i.e. with no negative jumps) Lévy process whose Laplace exponent is ψ . Lambert obtained result about the law of the population in a splitting tree alive at time t . More precisely, for $x > 0$, let $\tilde{\mathbb{P}}_x$ be the law of the process $(\Xi(t), t \geq 0)$ conditioned to start with a single ancestor living x units of time. Then,

$$\tilde{\mathbb{P}}_x(\Xi(t) = 0) = W(t-x)/W(t) \quad (2.1)$$

and conditional on being nonzero, under $\tilde{\mathbb{P}}_x$, $\Xi(t)$ has a geometric distribution with success probability $1/W(t)$ i.e. for $n \in \mathbb{N}^*$,

$$\tilde{\mathbb{P}}_x(\Xi(t) = n) = \left(1 - \frac{W(t-x)}{W(t)}\right) \left(1 - \frac{1}{W(t)}\right)^{n-1} \frac{1}{W(t)} \quad (2.2)$$

where W is the *scale function* [5, ch.VII] associated with Y : this is the unique absolutely continuous increasing function $W : [0, \infty] \rightarrow [0, \infty]$ satisfying

$$\int_0^\infty e^{-\lambda x} W(x) dx = \frac{1}{\psi(\lambda)} \quad \lambda > \eta. \quad (2.3)$$

The two-sided exit problem can be solved thanks to this scale function:

$$\mathbb{P}\left(T_0 < T_{(a, +\infty)} | Y_0 = x\right) = \frac{W(a-x)}{W(a)}, \quad 0 < x < a \quad (2.4)$$

where for a Borel set B of \mathbb{R} , $T_B = \inf\{t \geq 0, Y_t \in B\}$.

We now give some properties, including asymptotic behavior, about the CMJ-process Ξ .

Proposition 2.1. *We denote by Ext the event $\left\{\lim_{t \rightarrow \infty} \Xi(t) = 0\right\}$.*

(i) *We have*

$$\mathbb{P}(\text{Ext}) = 1 - \eta/b \quad (2.5)$$

and conditional on Ext^c,

$$e^{-\eta t} \Xi(t) \xrightarrow[t \rightarrow \infty]{a.s.} E \quad (2.6)$$

where E is an exponential random variable with parameter c.

(ii) *If, for $x > 0$, $\log^+ x := \log x \vee 0$,*

$$\mathbb{E} \left[\left(\log^+ \sup_{t \geq 0} \left(e^{-\eta t} \Xi(t) \right) \right)^2 \middle| \text{Ext}^c \right] < \infty \quad (2.7)$$

We remind that $c = \psi'(\eta)$ and then depends on the measure $\Lambda(\cdot)$. The proof of the last assertion requires involved arguments using spine decomposition of splitting trees. Proposition 2.1, which will be proved in Section 2.3, is known in a particular case: if the lifetime $\Lambda(\cdot)/b$ has an exponential density with parameter d , $(\Xi(t), t \geq 0)$ is a Markovian birth and death process with birth rate b and death rate $d < b$. In that case, $\eta = b - d$, $c = 1 - d/b = \mathbb{P}(\text{Ext}^c)$ and integrability of $\sup_{t \geq 0} (e^{-\eta t} \Xi(t))$ stems from Doob's maximal inequality.

We now define the immigration model: let θ be a positive number and $0 = T_0 < T_1 < T_2 < \dots$ be the points of a Poisson process of rate θ . At each time T_i , we assume that a new individual immigrates and starts a new population whose size evolves like Ξ , independently of the other populations. That is, if for $i \geq 1$ we call $(Z^i(t), t \geq 0)$ the size of the i th oldest family (the family which was started at T_i) then $Z^i(t) = \Xi_i(t - T_i) \mathbf{1}_{\{t \geq T_i\}}$ where Ξ_1, Ξ_2, \dots are copies of Ξ and $(\Xi_i, i \geq 1)$ and $(T_i, i \geq 1)$ are independent. See Figure 2.1 for a graphical representation of a splitting tree with immigration. This immigration model is a generalization of S. Karlin and J. McGregor's model [41] in the case of general lifetimes.

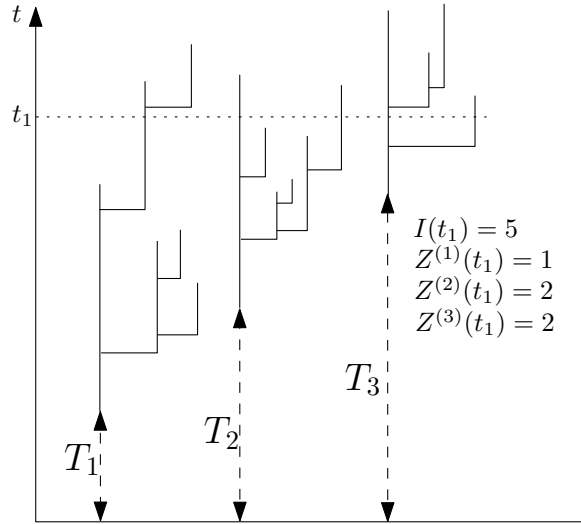


Figure 2.1: splitting trees with immigration. The vertical axis is time, horizontal axis shows filiation. At time t_1 , three populations are extant.

For $i \geq 1$, define $(Z^{(i)}(t), t \geq 0)$ as the i th oldest family among the surviving populations and $T^{(i)}$ its birthdate. In particular, by thinning of Poisson point process, $(T^{(i)}, i \geq 1)$ is a Poisson point process with parameter $\theta\eta/b$ thanks to (2.5).

We are now interested in the joint behavior of the surviving families $(Z^{(i)}(t), t \geq 0)$ for $i \geq 1$:

$$\begin{aligned} e^{-\eta t} Z^{(i)}(t) &= e^{-\eta T^{(i)}} e^{-\eta(t-T^{(i)})} Z^{(i)}(t) \\ &\stackrel{(d)}{=} e^{-\eta T^{(i)}} e^{-\eta(t-T^{(i)})} \Xi_{(i)}(t - T^{(i)}) \mathbf{1}_{\{T^{(i)} \leq t\}} \end{aligned}$$

where the notation $\Xi_{(i)}$ is obvious (see just above). As in (2.6), denote by $E_i := \lim_{t \rightarrow \infty} e^{-\eta t} \Xi_{(i)}(t)$ for $i \geq 1$. Thus E_1, E_2, \dots are i.i.d. exponential r.v. with parameter c . Moreover, the sequences $(E_i, i \geq 1)$ and $(T^{(i)}, i \geq 1)$ are independent. It follows that $e^{-\eta t} Z^{(i)}(t) \rightarrow e^{-\eta T^{(i)}} E_i$ a.s. as $t \rightarrow \infty$. We record this in the following

Proposition 2.2.

$$e^{-\eta t}(Z^{(1)}(t), Z^{(2)}(t), \dots) \xrightarrow[t \rightarrow \infty]{} (e^{-\eta T^{(1)}} E_1, e^{-\eta T^{(2)}} E_2, \dots) \quad a.s.$$

where the E_i 's are independent copies of E and independent of the $T^{(i)}$'s.

For $t \geq 0$, let $I(t)$ be the size of the total population at time t

$$I(t) = \sum_{i \geq 1} Z^i(t).$$

The process $(I(t), t \geq 0)$ is a non-Markovian continuous-time branching process with immigration.

Theorem 2.3. (i) For t positive, $I(t)$ has a negative binomial distribution with parameters $1 - W(t)^{-1}$ and θ/b . i.e. for $s \in [0, 1]$, its generating function is

$$G_t(s) := \mathbb{E} [s^{I(t)}] = \left(\frac{W(t)^{-1}}{1 - s(1 - W(t)^{-1})} \right)^{\theta/b}.$$

(ii) We have

$$I := \lim_{t \rightarrow \infty} e^{-\eta t} I(t) = \sum_{i \geq 1} e^{-\eta T^{(i)}} E_i \quad a.s.$$

and I has a Gamma distribution $\Gamma(\theta/b, c)$ i.e the density of I with respect to Lebesgue measure is

$$g(x) = \frac{c^{\theta/b} x^{\theta/b-1} e^{-cx}}{\Gamma(\theta/b)}, \quad x > 0.$$

Notice that point (i) still holds when Ξ is subcritical or critical, that is, if m is respectively less than or equal to 1. It is a generalization of a result by D.G. Kendall [42] which was the particular Markovian case of a birth, death and immigration process. The proof we give in Section 2.4 uses equations (2.1) and (2.2) about the law of $\Xi(t)$.

There exist other proofs of the almost sure convergence in (ii), but they require stronger assumptions. For example, P. Jagers [35, p.198] gave a proof for the convergence of general branching processes with immigration under the hypothesis that the variance of the number of children per individual $\xi(\infty)$ is finite. In our case, this is only true if $\int_{(0, \infty]} r^2 \Lambda(dr) < \infty$. In the particular Markovian case described previously, the proof is also easier since $(e^{-\eta t} \Xi(t), t \geq 0)$ is a non-negative martingale [3, p.111], $(e^{-\eta t} I(t), t \geq 0)$ is a non-negative submartingale and both converge a.s. In the proof we give in Section 2.4, the only assumption we use about the measure Λ is that its mass is finite. The proof is based on Proposition 2.1(ii).

In the following, we will consider different kinds of metacommunity where immigrants are chosen and will give results about abundances of surviving populations. In Model I, there is a discrete spectrum with zero macroscopic relative abundances: when a new family is initiated, it is of a type different from those of previous families. The following theorem yields the asymptotic behaviors of the fractions of the surviving subpopulations ranked by decreasing ages in the total population:

Theorem 2.4 (Model I).

$$\lim_{t \rightarrow \infty} I(t)^{-1}(Z^{(1)}(t), Z^{(2)}(t), \dots) = (P_1, P_2, \dots) \quad a.s.$$

where the law of (P_1, P_2, \dots) is a GEM distribution with parameter θ/b . In other words, for $i \geq 1$

$$P_i \stackrel{(d)}{=} B_i \prod_{j=1}^{i-1} (1 - B_j)$$

and $(B_i)_{i \geq 1}$ is a sequence of i.i.d. random variables with law $\text{Beta}(1, \theta/b)$ whose density with respect to Lebesgue measure is

$$\frac{\theta}{b} (1-x)^{\theta/b-1} \mathbf{1}_{[0,1]}(x).$$

This result was proved by S. Tavaré [66] in the case where $\Lambda(dr) = \delta_\infty(dr)$ (pure birth process); it is the exponential case defined previously with $b = 1$ and $d = 0$. His result is robust because we see that in our more general case, the limit distribution does not depend on the lifespan distribution but only on the immigration-to-birth ratio θ/b . In biogeography, a typical question is to recover data about population dynamics (immigration times, law of lifespan duration) from the observed diversity patterns. In this model, we see that there is a loss of information about the lifespan duration. However, the ratio θ/b can be estimated thanks to the species abundance distribution. We will prove Theorem 2.4 in Subsection 2.5.1.

In Model II, we consider a discrete spectrum with nonzero macroscopic relative abundances. Contrary to Model I where types were always new, they are now given a priori and types of immigrants are independently drawn according to some probability $p = (p_i, i \geq 1)$. When a population is initiated (i.e. at each time of the θ -Poisson point process), it is of type i with probability $p_i > 0$.

Theorem 2.5 (Model II). *For $i \geq 1$, denote by $I_i(t)$ the number of individuals of type i at time t and set $\theta_i := \frac{\theta p_i}{b}$. Then*

$$\lim_{t \rightarrow \infty} I(t)^{-1} (I_1(t), I_2(t), \dots) = (P'_1, P'_2, \dots) \quad a.s.$$

where for $i \geq 1$

$$P'_i \stackrel{(d)}{=} B'_i \prod_{j=1}^{i-1} (1 - B'_j)$$

and $(B'_i)_{i \geq 1}$ is a sequence of independent random variables such that

$$B'_i \sim \text{Beta} \left(\theta_i, \frac{\theta}{b} \sum_{j \geq i+1} p_j \right).$$

In particular, for $i \geq 1$, P'_i has a Beta distribution $\text{Beta}(\theta_i, \theta/b - \theta_i)$.

The proof of this theorem will be done in Subsection 2.5.2. In this model, the limit only depend on θ/b and the metacommunity spectrum $(p_i, i \geq 1)$.

Remark 2.6. If the number of possible types n is finite, then

$$\sum_{i=1}^n P'_i = \sum_{i=1}^n \frac{I_i}{I} = 1, \quad \sum_{i=1}^n \theta_i = \frac{\theta}{b}$$

and the joint density of (P'_1, \dots, P'_n) is

$$\frac{\Gamma(\theta/b)}{\prod_{i=1}^n \Gamma(\theta_i)} \left(\prod_{i=1}^{n-1} x_i^{\theta_i-1} \mathbf{1}_{\{x_i > 0\}} \right) (1 - x_1 - \dots - x_{n-1})^{\theta/b-1} \mathbf{1}_{\{x_1 + \dots + x_{n-1} < 1\}}.$$

This is the joint density of a Dirichlet distribution $\text{Dir}(\theta_1, \dots, \theta_n)$.

In Model III, we consider a continuous spectrum of possible types and we slightly modify the immigration process: when an individual arrives on the island, it starts a new population with an immigration rate proportional to its abundance on the metacommunity. More precisely, let Π be a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity $dt \otimes xf(x)dx$ where f is a nonnegative function such that $\theta := \int_0^\infty xf(x)dx$ is finite. Then, write $\Pi := ((T_i, \Delta_i), i \geq 1)$ where $T_1 < T_2 < \dots$ are the times of a θ -linear Poisson point process and $(\Delta_i, i \geq 1)$ is a sequence of i.i.d random variables whose density is $\theta^{-1}xf(x)dx$ which is independent of $(T_i, i \geq 1)$. At time T_i , a new population starts out and it evolves like the continuous-time branching process with immigration defined for the two previous models with an immigration rate Δ_i . The interpretation of this model is as follows: for $x > 0$, $f(x)$ represents the density of species with abundance x in the metacommunity and at each immigration time, an individual of a species with abundance in $(x, x + dx)$ is chosen with probability $\frac{xf(x)}{\theta}dx$ proportional to its abundance in the metacommunity.

If $(Z^i(t), t \geq 0)$ is the i -th oldest family,

$$Z^i(t) = I_{\Delta_i}^i(t - T_i)\mathbf{1}_{\{t \geq T_i\}}, \quad t \geq 0$$

where the $I_{\Delta_i}^i$'s are independent copies of I_Δ , which, conditional on Δ , evolves like the immigration process of the first two models with an immigration rate Δ . According to Theorem 2.3(ii), we know that

$$e^{-\eta t} I_\Delta(t) \xrightarrow[t \rightarrow \infty]{} G \text{ a.s.}$$

where conditional on Δ , $G \sim \text{Gamma}(\Delta/b, c)$. We denote by F its tail distribution

$$F(v) := \mathbb{P}(G \geq v) = \int_0^\infty dx \frac{xf(x)}{\theta} \int_v^\infty \frac{e^{-ct}t^{x/b-1}c^{x/b}}{\Gamma(x/b)} dt.$$

Hence, we also have

Proposition 2.7. *For $i \geq 1$,*

$$e^{-\eta t} Z^i(t) \xrightarrow[t \rightarrow \infty]{} \sigma_i := e^{-\eta T_i} G_i \text{ a.s.}$$

where $(G_i, i \geq 1)$ is a sequence of i.i.d. r.v. with the same distribution as G and independent of $(T_i, i \geq 1)$.

We again denote by $I(t)$ the total population at time t : $I(t) := \sum_{i \geq 1} Z^i(t)$ and we obtain a result similar to Theorem 2.3 concerning the asymptotic behavior of $I(t)$.

Proposition 2.8. *If $\int_0^\infty x^2 f(x)dx < \infty$ we have*

$$e^{-\eta t} I(t) \xrightarrow[t \rightarrow \infty]{a.s.} \sigma := \sum_{i \geq 1} \sigma_i$$

and the Laplace transform of σ is

$$\mathbb{E}[e^{-s\sigma}] = \exp\left(-\frac{\theta}{\eta} \int_0^\infty \frac{F(v)}{v} (1 - e^{-sv}) dv\right).$$

Moreover,

$$\mathbb{E}[\sigma] = \frac{1}{\eta bc} \int_0^\infty x^2 f(x)dx < \infty.$$

The proof of this result will be done in Subsection 2.5.3. An immediate consequence of Propositions 2.7 and 2.8 deals with the asymptotic abundances of the different types.

Proposition 2.9 (Model III). *If $\int_0^\infty x^2 f(x) dx < \infty$, we have*

$$\left(\frac{Z^1(t)}{I(t)}, \frac{Z^2(t)}{I(t)}, \dots \right) \xrightarrow{t \rightarrow \infty} \left(\frac{\sigma_1}{\sigma}, \frac{\sigma_2}{\sigma}, \dots \right) \text{ a.s.}$$

Notice that in this model, the limit only depends on the lifespan measure via the Malthusian parameter η and the constant c .

2.3 Proof of the properties of the process Ξ

2.3.1 Some useful, technical lemmas

Thereafter, we state some lemmas that will be useful in the subsequent proof of Proposition 2.1.

Lemma 2.10. *Let Y_1, Y_2, \dots be a sequence of i.i.d positive random variables with finite expectation. Then, if $S := \sup_{n \geq 1} \left(\frac{1}{n} \sum_{i=1}^n Y_i \right)$,*

$$\mathbb{E} \left[(\log^+ S)^k \right] < \infty, \quad k \geq 1.$$

Proof. According to Kallenberg [40, p.184], for $r > 0$,

$$r \mathbb{P}(S \geq 2r) \leq \mathbb{E}[Y_1; Y_1 \geq r].$$

Hence, choosing $r = e^s/2$, we have for $s \geq 0$,

$$\mathbb{P}(\log^+ S \geq s) \leq 2e^{-s} \mathbb{E}[Y_1]$$

and

$$\mathbb{E}[(\log^+ S)^k] = \int_0^\infty k s^{k-1} \mathbb{P}(\log^+ S \geq s) ds \leq 2\mathbb{E}[Y_1] k \int_0^\infty s^{k-1} e^{-s} ds < \infty.$$

This completes the proof. \square

Lemma 2.11. *Let A be a homogeneous Poisson process with parameter ρ . If we denote $S := \sup_{t > 0} (A_t/t)$, then for $a > 0$,*

$$\mathbb{P}(S > a) = \frac{\rho}{a} \vee 1.$$

In particular,

$$\forall k \geq 1, \quad \mathbb{E} \left[(\log^+ S)^k \right] < \infty.$$

Proof. If $a < \rho$, since $\lim_{t \rightarrow \infty} A_t/t = \rho$, $\mathbb{P}(S > a) = 1$. Let now a be a real number greater than ρ . Then

$$\mathbb{P}(S \leq a) = \mathbb{P}(\forall t \geq 0, at - A_t \geq 0).$$

According to Bertoin [5, chap.VII], since $(at - A_t, t > 0)$ is a Lévy process with no positive jumps and with Laplace exponent $\phi(\lambda) = \lambda a - \rho(1 - e^{-\lambda})$, we have as in (2.4)

$$\mathbb{P}(S \leq a) = \frac{H(0)}{H(\infty)}$$

where H is the scale function associated with $(at - A_t, t > 0)$. We compute $H(0)$ and $H(\infty)$ using Tauberian theorems [5, p.10]:

- since $\phi(\lambda) \sim \lambda(a - \rho)$ as $\lambda \rightarrow 0$, then $H(\infty) = (a - \rho)^{-1}$,
- since $\phi(\lambda) \sim \lambda a$ as $\lambda \rightarrow \infty$, then $H(0) = a^{-1}$.

Hence,

$$\mathbb{P}(S \leq a) = \frac{a - \rho}{a} = 1 - \frac{\rho}{a}, \quad a > \rho.$$

Then,

$$\begin{aligned} \mathbb{E}[(\log^+ S)^k] &= \int_0^\infty kr^{k-1} \mathbb{P}(\log^+ S \geq r) dr \\ &\leq \int_0^{\log^+ \rho} kr^{k-1} dr + \int_{\log^+ \rho}^\infty kr^{k-1} \frac{\rho}{e^r} dr < \infty \end{aligned}$$

and the proof is completed. \square

Notice that Lemma 2.11 is also a direct consequence of Theorem 2.1 in [39].

2.3.2 Proof of Proposition 2.1(i)

A. Lambert proved in [47] that $\mathbb{P}(\text{Ext}) = 1 - \eta/b$ and in [48] that, conditional on Ext^c ,

$$e^{-\eta t} \Xi(t) \xrightarrow[t \rightarrow \infty]{(d)} E$$

where E is an exponential random variable with parameter c . To obtain a.s. convergence, we use [55, Thm 5.4] where O. Nerman gives sufficient conditions for convergence of CMJ processes to hold almost surely. Here, the two conditions of his theorem are satisfied. Indeed, the second one holds if there exists on $[0, \infty)$ an integrable, bounded, non-increasing positive function h such that

$$\mathbb{E} \left[\sup_{t \geq 0} \left(\frac{e^{-\eta t} \mathbf{1}_{\{t < \zeta\}}}{h(t)} \right) \right] < \infty$$

where we recall that ζ is the lifespan duration of a typical individual in the CMJ-process Ξ . Then, choosing $h(t) = e^{-\eta t}$, this condition is trivially satisfied.

The first one holds if there exists a non-increasing Lebesgue integrable positive function g such that

$$\int_0^\infty \frac{1}{g(t)} e^{-\eta t} \mu(dt) < \infty. \quad (2.8)$$

Taking $g(t) = e^{-\beta t}$ with $\eta > \beta > 0$ and recalling that $\mu(dt) = \int_{(t, \infty)} \Lambda(dr) dt$, we have

$$\begin{aligned} \int_0^\infty \frac{1}{g(t)} e^{-\eta t} \mu(dt) &= \int_0^\infty e^{(\beta - \eta)t} \int_t^\infty \Lambda(dr) dt = \int_{(0, \infty]} \Lambda(dr) \int_0^r e^{(\beta - \eta)t} dt \\ &\leq C \int_{(0, \infty]} \Lambda(dr) = Cb < \infty \end{aligned}$$

and condition (2.8) is fulfilled.

2.3.3 Proof of Proposition 2.1(ii)

We will say that a process G satisfies condition (C) if

$$\mathbb{E} \left[\left(\log^+ \sup_{t \geq 0} e^{-\eta t} G_t \right)^2 \right] < \infty$$

and our aim is to prove that, conditional on non-extinction, the homogeneous CMJ-process $(\Xi(t), t \geq 0)$ satisfies it.

According to Theorem 4.4.1.1 in [47], conditional on non-extinction of $(\Xi(t), t \geq 0)$,

$$\Xi(t) = \Xi_t^\infty + \Xi_t^d + \Xi_t^g$$

where

- Ξ_t^∞ is the number of individuals alive at time t and whose descendance is infinite. In particular, $(\Xi_t^\infty, t \geq 0)$ is a Yule process with rate η .
- Ξ_t^d is the number of individuals alive at time t descending from trees grafted on the right hand side of the Yule tree (right refers to the order of the contour of the planar splitting tree)

$$\Xi_t^d := \sum_{i=1}^{\tilde{N}_t} \tilde{\Xi}_i(t - \tilde{T}_i)$$

where

- $(\tilde{\Xi}_i, i \geq 1)$ is a sequence of i.i.d. splitting trees conditional on extinction and independent of Ξ^∞ . We know that such trees have the same distribution as subcritical splitting trees with lifespan measure $\tilde{\Lambda}(dr) = e^{-\eta r} \Lambda(dr)$ (cf. [47]).
- Conditionally on $(\Xi_t^\infty, t \geq 0)$, $(\tilde{N}_t, t \geq 0)$ is a non homogeneous Poisson process with mean measure $(b - \eta)\Xi_t^\infty dt$ and independent of $(\tilde{\Xi}_i)_i$. We denote its arrival times by $\tilde{T}_1, \tilde{T}_2, \dots$
- Ξ_t^g is the number of individuals alive at time t descending from trees grafted on the left hand side of the Yule tree (left also refers to the contour order).

More specifically, let (A, R) be a couple of random variables with joint law given by

$$\mathbb{P}(A + R \in dz, R \in dr) = e^{-\eta r} dr \Lambda(dz), \quad 0 < r < z \quad (2.9)$$

and let $((A_{i,j}, R_{i,j}), i \geq 0, j \geq 1)$ be i.i.d random variables distributed as (A, R) . We consider the arrival times

$$T_{i,j} = \tau_i + A_{i,1} + A_{i,2} + \dots + A_{i,j}, \quad i \geq 0, j \geq 1$$

where $0 = \tau_0 < \tau_1 < \tau_2 < \dots$ are the splitting times of the Yule tree, that is, on each new infinite branch, we start a new A -renewal process independent of the others. We define for $t \geq 0$,

$$\Xi_t^g := \sum_{i,j} \hat{\Xi}_{i,j}(t - T_{i,j}) \mathbf{1}_{\{T_{i,j} \leq t\}}$$

where $(\hat{\Xi}_{i,j}, i \geq 0, j \geq 1)$ is a sequence of i.i.d. splitting trees independent of Ξ^∞ , conditioned on extinction and such that the unique ancestor of $\hat{\Xi}_{i,j}$ has lifetime $R_{i,j}$.

We denote by $\hat{N}_t := \#\{(i,j), T_{i,j} \leq t\}$ the number of graft times before t .

This spine decomposition is represented on Figure 2.2.

In the following, we will prove that $(\Xi_t, t \geq 0)$ satisfies condition (C). To do this, using Minkowski inequality and the inequality

$$\forall x, y \geq 0, \log^+(x + y) \leq \log^+ x + \log^+ y + \log 2,$$

we only have to check that the three processes Ξ^∞, Ξ^g and Ξ^d satisfy (C).

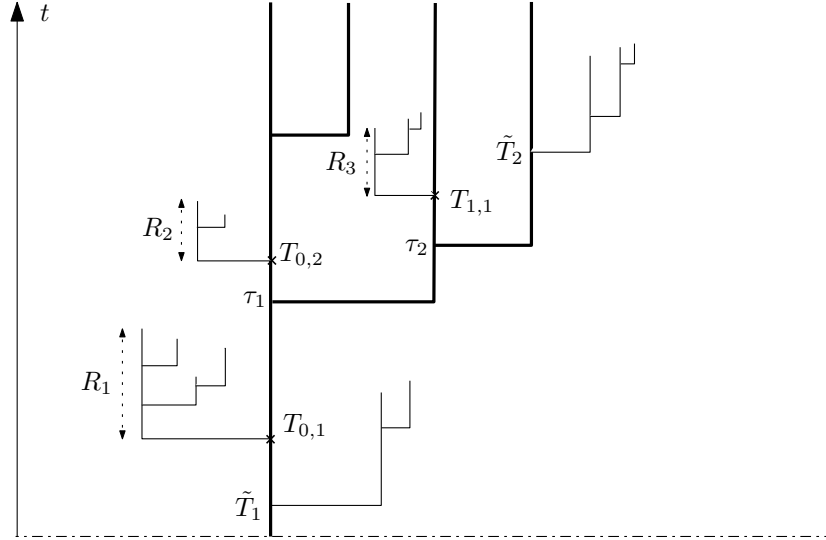


Figure 2.2: Spine decomposition of a splitting tree. The thick lines represent the Yule tree Ξ^∞ on which we graft on the left (at times $T_{0,1}, \dots$) the trees conditioned on extinction whose ancestors have lifetime durations distributed as R and on the right (at time \tilde{T}_1, \dots) the trees conditioned on extinction.

Proof of condition (C) for Ξ^∞

Since $(\Xi_t^\infty, t \geq 0)$ is a η -Yule process, $(e^{-\eta t} \Xi_t^\infty, t \geq 0)$ is a non-negative martingale [3, p.111] and so by Doob's inequality [60, p.54],

$$\mathbb{E} \left[\sup_{t \geq 0} (e^{-\eta t} \Xi_t^\infty)^2 \right] \leq 4 \sup_{t \geq 0} \mathbb{E}[(e^{-\eta t} \Xi_t^\infty)^2]. \quad (2.10)$$

Moreover, $\mathbb{E}[(\Xi_t^\infty)^2] = 2(e^{2\eta t} - e^{\eta t})$ (again in [3]) and

$$\mathbb{E}[(e^{-\eta t} \Xi_t^\infty)^2] = 2e^{-2\eta t} (e^{2\eta t} - e^{\eta t}) \xrightarrow{t \rightarrow \infty} 2.$$

Hence, the supremum in the right hand side of (2.10) is finite and

$$\mathbb{E} \left[\left(\sup_{t \geq 0} (e^{-\eta t} \Xi_t^\infty) \right)^2 \right] < \infty.$$

From now on, we will set $M := \sup_{t \geq 0} e^{-\eta t} \Xi_t^\infty$. Since $\mathbb{E}[M^2] < \infty$, (C) is trivially satisfied by Ξ^∞ .

Proof of condition (C) for Ξ^d

We recall that

$$\Xi_t^d = \sum_{i=1}^{\tilde{N}_t} \tilde{\Xi}_i(t - \tilde{T}_i).$$

Denote by Y_i the total progeny of the conditioned splitting tree $\tilde{\Xi}_i$, that is, the total number of descendants of the ancestor plus one. Then, a.s for all $t \geq 0$ and $i \geq 1$, we have $\tilde{\Xi}_i(t - \tilde{T}_i) \leq Y_i$ and

$$\Xi_t^d \leq \sum_{i=1}^{\tilde{N}_t} Y_i \text{ a.s. } t \geq 0.$$

Hence, almost surely for all t ,

$$e^{-\eta t} \Xi_t^d \leq e^{-\eta t} \sum_{i=1}^{\tilde{N}_t} Y_i = \left(e^{-\eta t} \tilde{N}_t \right) \left(\frac{1}{\tilde{N}_t} \sum_{i=1}^{\tilde{N}_t} Y_i \right)$$

and, thanks to Minkowski's inequality,

$$\begin{aligned} \mathbb{E} \left[\left(\log^+ \sup_{t \geq 0} \left(e^{-\eta t} \Xi_t^d \right) \right)^2 \right]^{1/2} &\leq \mathbb{E} \left[\left(\log^+ \sup_{t \geq 0} \left(e^{-\eta t} \tilde{N}_t \right) \right)^2 \right]^{1/2} \\ &\quad + \mathbb{E} \left[\left(\log^+ \sup_{t > 0} \left(\frac{1}{\tilde{N}_t} \sum_{i=1}^{\tilde{N}_t} Y_i \right) \right)^2 \right]^{1/2}. \end{aligned} \quad (2.11)$$

We first consider the second term in the right hand side of (2.11):

$$\mathbb{E} \left[\left(\log^+ \sup_{t > 0} \left(\frac{1}{\tilde{N}_t} \sum_{i=1}^{\tilde{N}_t} Y_i \right) \right)^2 \right] \leq \mathbb{E} \left[\left(\log^+ \sup_{n \geq 1} \left(\frac{1}{n} \sum_{i=1}^n Y_i \right) \right)^2 \right]$$

since $(\tilde{N}_t, t \geq 0)$ is integer-valued. Thanks to Lemma 2.10, this term is finite because $\mathbb{E}[Y_1]$ is finite. Indeed, Y_1 is the total progeny of a subcritical branching process and it is known [30, p.32] that its mean is finite.

We are now interested in the first term in the r.h.s. of (2.11). We work conditionally on $\Xi^\infty =: (f(t), t \geq 0)$. Since we have $e^{-\eta t} \int_0^t f(s) ds \leq M$ using the supremum M of $(e^{-\eta t} \Xi_t^\infty, t \geq 0)$,

$$e^{-\eta t} \tilde{N}_t = e^{-\eta t} \int_0^t f(s) ds \frac{\tilde{N}_t}{\int_0^t f(s) ds} \leq M \frac{\tilde{N}_t}{\int_0^t f(s) ds}.$$

Moreover,

$$(\tilde{N}_t, t \geq 0) \stackrel{(d)}{=} \left(N'_{\int_0^t f(s) ds}, t \geq 0 \right)$$

where N' is a homogeneous Poisson process with parameter $b - \eta$.

Hence, using Minkowski's inequality,

$$\begin{aligned} \mathbb{E} \left[\left(\log^+ \sup_{t \geq 0} \left(e^{-\eta t} \tilde{N}_t \right) \right)^2 \right]^{1/2} &\leq \log^+ M + \mathbb{E} \left[\left(\log^+ \sup_{t > 0} \left(\frac{N'_{\int_0^t f(s) ds}}{\int_0^t f(s) ds} \right) \right)^2 \right]^{1/2} \\ &= \log^+ M + \mathbb{E} \left[\left(\log^+ \sup_{t > 0} \left(\frac{N'_t}{t} \right) \right)^2 \right]^{1/2} \end{aligned}$$

and the second term of the r.h.s. is finite using Lemma 2.11.

Hence, $(\tilde{N}_t, t \geq 0)$ satisfies (C) since $\mathbb{E}[M^2] < \infty$ and Ξ^d as well, which ends this paragraph.

Proof of condition (C) for Ξ^g

We have

$$\Xi_t^g = \sum_{i=1}^{\hat{N}_t} \hat{\Xi}_i(t - \hat{T}_i)$$

As in the previous section,

$$e^{-\eta t} \Xi_t^g \leq \left(e^{-\eta t} \hat{N}_t \right) \left(\frac{1}{\hat{N}_t} \sum_{i=1}^{\hat{N}_t} \hat{Y}_i \right) \text{ a.s.}$$

where \hat{Y}_i is the total progeny of the conditioned CMJ-process $(\hat{\Xi}_i(t), t \geq 0)$.

Hence,

$$\begin{aligned} \mathbb{E} \left[\left(\log^+ \sup_{t \geq 0} \left(e^{-\eta t} \Xi_t^g \right) \right)^2 \right]^{1/2} &\leq \mathbb{E} \left[\left(\log^+ \sup_{t \geq 0} \left(e^{-\eta t} \hat{N}_t \right) \right)^2 \right]^{1/2} \\ &\quad + \mathbb{E} \left[\left(\log^+ \sup_{n > 0} \left(\frac{1}{n} \sum_{i=1}^n \hat{Y}_i \right) \right)^2 \right]^{1/2} \end{aligned} \quad (2.12)$$

We first prove that the second term in the r.h.s. is finite using Lemma 2.10. We only have to check that $\mathbb{E}[\hat{Y}_1]$ is finite. We recall that \hat{Y}_1 is the total progeny of a splitting tree whose ancestor has random lifespan R_1 and conditioned on extinction. Conditioning on R_1 , it is also the total progeny of a subcritical Bienaymé-Galton-Watson process starting from a Poisson random variable with mean R_1 . Hence, $\mathbb{E}[\hat{Y}_1]$ is finite if $\mathbb{E}[R_1]$ is finite. As a consequence of (2.9), we have that $\mathbb{P}(R_1 \in dr) = e^{-\eta r} \int_r^\infty \Lambda(dz) dr$ and

$$\begin{aligned} \mathbb{E}[R_1] &= \int_{(0, \infty)} \Lambda(dz) \int_0^z r e^{-\eta r} dr \\ &= - \int_{(0, \infty)} \Lambda(dz) \frac{z e^{-\eta z}}{\eta} + \int_{(0, \infty)} \Lambda(dz) \frac{1 - e^{-\eta z}}{\eta^2} \\ &= \frac{\psi'(\eta) - 1}{\eta} + \frac{1}{\eta} = \frac{\psi'(\eta)}{\eta} < \infty. \end{aligned}$$

We are now interested in the first term of the r.h.s. of (2.12). We need to make calculations on \hat{N}_t which is the total number of times of graftings $T_{i,j}$ less than or equal to t . Recall that for $i \geq 0, j \geq 1$, $T_{i,j} = \tau_i + \bar{A}_{i,j}$ where $\bar{A}_{i,j} := A_{i,1} + \dots + A_{i,j}$ and that τ_i is the birth time of individual i and that \hat{N}_t is the sum of the numbers of graftings before t on each of the Ξ_t^∞ branches. For $i \geq 0$, denote by $\alpha_1^i, \alpha_2^i, \dots$, the birth times of the daughters of individual i and $\alpha_0^i = \tau_i$. For $k \geq 1$, denote by $\tilde{\tau}_k^i := \alpha_k^i - \alpha_{k-1}^i$ the interbirth times. In particular, $(\tilde{\tau}_k^i, i \geq 0, k \geq 1)$ are i.i.d. exponential r.v. with parameter η since we consider a η -Yule tree.

We enlarge the probability space by redefining the renewal processes $(\bar{A}_{i,j}, j \geq 1)$ in terms of a doubly indexed sequence of i.i.d. A -renewal processes $(\mathcal{A}_{i,k}, i \geq 0, k \geq 1)$. We define the process $(\bar{A}_{i,j}, j \geq 1)$ recursively by concatenation of the $\mathcal{A}_{i,k}$'s as in Figure 2.3. To simplify notation, we only define $(\bar{A}_{0,j}, j \geq 1)$ which will be denoted by $(\bar{A}_j, j \geq 1)$. Then,

$$\bar{A}'_1 := \inf\{t > \alpha_1 | \mathcal{A}_1 \cap [t, +\infty) \neq \emptyset\}, \quad C_1 := \#\mathcal{A}_1 \cap [0, \alpha_1] + 1,$$

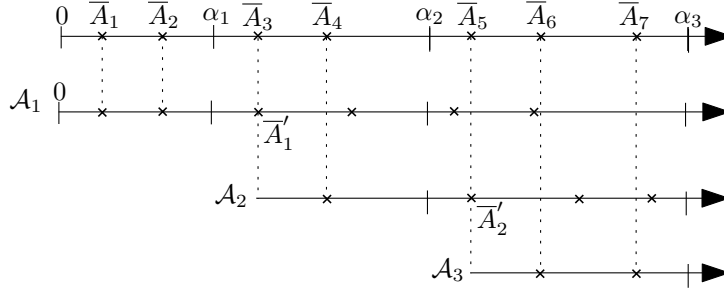


Figure 2.3: Construction of the renewal process $(\bar{A}_j, j \geq 1)$ by concatenation of the renewal processes \mathcal{A}_k .

and

$$\bar{A}_j := \inf\{t \geq 0 \mid \#\mathcal{A}_1 \cap [0, t] = j\}, \quad j = 1, \dots, C_1 + 1.$$

Moreover, for $l \geq 1$, if one knows C_l and \bar{A}'_l , let r_l be the unique integer such that \bar{A}'_l belongs to $]\alpha_{r_l}, \alpha_{r_l+1}]$ and define

$$\bar{A}'_{l+1} := \bar{A}'_l + \inf\{t > \alpha_{r_l+1} - \bar{A}'_l \mid \mathcal{A}_{r_l+1} \cap [t, +\infty) \neq \emptyset\},$$

$$C_{l+1} := \#\mathcal{A}_{r_l+1} \cap [0, \alpha_{r_l+1} - \bar{A}'_l] + 1,$$

and

$$\bar{A}_{C_1+\dots+C_{l+1}j} := \bar{A}'_l + \inf\{t \geq 0 \mid \#\mathcal{A}_{r_l+1} \cap [0, t] = j\}, \quad j = 1, \dots, C_{l+1}.$$

Then, $(\bar{A}_j, j \geq 1)$ is a A -renewal process because we concatenated the independent renewal processes (\mathcal{A}_k) stopped at the first renewal time after a given time. Indeed, one can see a renewal process as the range of a compound Poisson process whose jumps are distributed as A_1 . The renewal process stopped at the first point after t is then the range of a compound Poisson process stopped at the first hitting time T of $[t, \infty)$, which is a stopping time. Then $(\bar{A}_j, j \geq 1)$, which is the range of the concatenation of independent compound Poisson processes killed at stopping times, is a compound Poisson process, by the strong Markov property. In conclusion, $(\bar{A}_j, j \geq 1)$ is a A -renewal process.

According to previous computations, we have

$$\hat{N}_t \leq \sum_{i \geq 0, l \geq 1} C_l^i \mathbf{1}_{\{\alpha_{r_l, i} \leq t\}}$$

where C_l^i and $r_{l, i}^i$ are the analogous notations as C_l and r_l for individual i . Then if

$$D_k^i := \#\mathcal{A}_{i, k} \cap [0, \tilde{\tau}_k^i] + 1 \quad k \geq 1, i \geq 0,$$

since $\alpha_{r_{l, i+1}^i} - \bar{A}'_{l, i} \leq \alpha_{r_{l, i+1}^i} - \alpha_{r_{l, i}^i} = \tilde{\tau}_{r_{l, i}^i}^i$, then $C_l^i \leq D_{r_{l, i}^i}^i$ and

$$\hat{N}_t \leq \sum_{i \geq 0, l \geq 1} D_{r_{l, i}^i}^i \mathbf{1}_{\{\alpha_{r_{l, i}^i}^i \leq t\}} \leq \sum_{i \geq 0, k \geq 1} D_k^i \mathbf{1}_{\{\alpha_k^i \leq t\}} \text{ a.s.} \quad (2.13)$$

Moreover, the random variables $(D_k^i, i \geq 0, k \geq 1)$ are independent and identically distributed as 1 plus the value of a A -renewal process at an independent exponential time E with parameter η

$$D := \sup\{j \geq 1, A_1 + \dots + A_j \leq E\} + 1.$$

The sum of the r.h.s. of (2.13) has $2\Xi_t^\infty - 1$ terms. Indeed, each individual of the Yule tree contributes to $1 + n_i$ terms in the sum where n_i is the number of daughters of individual i born before t . Then, there is $\Xi_t^\infty + \sum_{i=0}^{\Xi_t^\infty - 1} n_i$ terms in the sum and $\sum_i n_i$ is the number of descendants of the individual 0 born before t which equals $\Xi_t^\infty - 1$.

Hence, using Minkowski's inequality,

$$\begin{aligned} \mathbb{E} \left[\left(\log^+ \sup_{t \geq 0} \left(e^{-\eta t} \hat{N}_t \right) \right)^2 \right]^{1/2} &\leq \mathbb{E} \left[\left(\log^+ \sup_{t \geq 0} \left(e^{-\eta t} (2\Xi_t^\infty - 1) \right) \right)^2 \right]^{1/2} \\ &+ \mathbb{E} \left[\left(\log^+ \sup_{n > 0} \left(\frac{1}{n} \sum_{i=1}^n D_i \right) \right)^2 \right]^{1/2} \end{aligned} \quad (2.14)$$

where $(D_i, i \geq 1)$ are i.i.d. r.v. distributed as D . The first term of (2.14) is smaller than $\mathbb{E}[(\log^+(2M))^2]^{1/2}$ and the second term is finite by another use of Lemma 2.10 if $\mathbb{E}[D_1] < \infty$. Denote by $(C_t, t \geq 0)$ the renewal process whose arrival times are distributed as A . A little calculation from (2.9) gives us that $\mathbb{E}[A] = \frac{m-1}{\eta} > 0$. Thus, according to Theorem 2.3 of Chapter 5 in [20], we have

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E}[C_t]}{t} = \frac{\eta}{m-1}$$

and so there exists $\kappa > 0$ such that $\mathbb{E}[C_t] \leq \kappa t$ for $t \geq 0$. Then,

$$\mathbb{E}[D_1] = 1 + \mathbb{E}[C_E] = 1 + \int_0^\infty \eta e^{-\eta t} dt \mathbb{E}[C_t] \leq 1 + \kappa \int_0^\infty \eta e^{-\eta t} dt = 1 + \frac{\kappa}{\eta} < \infty.$$

Finally, the r.h.s. of (2.14) and (2.12) are finite and Ξ^g satisfies condition (C).

2.4 Proof of the almost sure convergence of the total population

2.4.1 Some preliminary lemmas

Before proving Theorem 2.3, we state some results that we will need in the following. We start with some properties about W the scale function associated with ψ and defined by (2.3).

Lemma 2.12. (i) $W(0) = 1$,

(ii) $e^{-\eta t} W(t) \rightarrow c^{-1}$ as $t \rightarrow \infty$,

(iii) W is differentiable and $W \star \Lambda = bW - W'$ where \star is convolution product.

Proof. (i) We have

$$\int_0^\infty e^{-\lambda t} W(t) dt = \frac{1}{\psi(\lambda)} \underset{\lambda \rightarrow \infty}{\sim} \frac{1}{\lambda}$$

because $\psi(\lambda) = \lambda - b + \int_{(0, \infty]} e^{-\lambda r} \Lambda(dr)$. Then, by a Tauberian theorem [5, p10], $\lim_{t \rightarrow 0} W(t) = 1$.

(ii) (From [48]) For $\lambda > 0$, using a Taylor expansion and $\psi(\eta) = 0$,

$$\psi(\lambda + \eta) \underset{\lambda \rightarrow 0}{\sim} \lambda\psi'(\eta) = \lambda c.$$

Then

$$\int_0^\infty W(t)e^{-\eta t}e^{-\lambda t} dt \underset{\lambda \rightarrow 0}{\sim} \frac{1}{\lambda c}$$

and another Tauberian theorem entails that $W(t)e^{-\eta t}$ converges to $1/c$ as $t \rightarrow \infty$.

(iii) We first compute the Laplace transform of $W \star \Lambda$. Let $\lambda > \eta$

$$\begin{aligned} \int_0^\infty e^{-\lambda t} W \star \Lambda(t) dt &= \int_0^\infty e^{-\lambda t} W(t) dt \int_{(0, \infty]} e^{-\lambda r} \Lambda(dr) \\ &= \frac{1}{\psi(\lambda)} (\psi(\lambda) - \lambda + b) \end{aligned}$$

Integrating by parts and using (i) and (ii),

$$\int_0^\infty e^{-\lambda t} W'(t) dt = \left[e^{-\lambda t} W(t) \right]_0^\infty + \lambda \int_0^\infty e^{-\lambda t} W(t) dt = -1 + \frac{\lambda}{\psi(\lambda)}$$

and so the Laplace transform of $bW - W'$ is $b/\psi(\lambda) + 1 - \lambda/\psi(\lambda)$ which equals that of $W \star \Lambda$. This completes the proof. □

The following lemma deals with the convergence of random series:

Lemma 2.13. *Let $(\zeta_i, i \geq 1)$ be a sequence of i.i.d. positive random variables such that $\mathbb{E}[\log^+ \zeta_1]$ is finite and let $(\tau_i, i \geq 1)$ be the arrival times of a Poisson point process with parameter ρ independent of $(\zeta_i, i \geq 1)$. Then for any $r > 0$, the series $\sum_{i \geq 1} e^{-r\tau_i} \zeta_i$ converges a.s.*

Proof. We have

$$\sum_{i \geq 1} e^{-r\tau_i} \zeta_i \leq \sum_{i \geq 1} \exp \left(-i \left[r \frac{\tau_i}{i} - \frac{\log^+ \zeta_i}{i} \right] \right) \tag{2.15}$$

We use the following consequence of Borel-Cantelli's lemma: if ξ_1, ξ_2, \dots are i.i.d. non-negative random variables,

$$\limsup_{n \rightarrow \infty} \frac{\xi_n}{n} = 0 \text{ or } \infty \text{ a.s.}$$

according to whether $\mathbb{E}[\xi_1]$ is finite or not. We use it with $\xi_i = \log^+ \zeta_i$. Hence, since $\mathbb{E}[\log^+ \zeta_1]$ is finite, $\lim_{i \rightarrow \infty} \log^+ \zeta_i / i = 0$ a.s. Moreover, by the strong law of large numbers, τ_i / i converges almost surely to $1/\rho$ as i goes to infinity. Then,

$$r \frac{\tau_i}{i} - \frac{\log^+ \zeta_i}{i} \xrightarrow{i \rightarrow \infty} \frac{r}{\rho} > 0 \text{ a.s.}$$

So the series in (2.15) converges a.s. □

2.4.2 Proof of Theorem 2.3(i)

In order to find the law of $I(t)$ the total population at time t , we use the fact that it is the sum of a Poissonian number of population sizes. More specifically, if we denote by N_t the number of populations at time t , $(N_t, t \geq 0)$ is a Poisson process with parameter θ and conditionally on $\{N_t = k\}$, the k -tuple (T_1, \dots, T_k) has the same distribution as $(U_{(1)}, \dots, U_{(k)})$ which is the reordered k -tuple of k independent uniform random variables on $[0, t]$. Hence, conditionally on $\{N_t = k\}$,

$$I(t) \stackrel{(d)}{=} \sum_{i=1}^k \Xi_i(t - U_{(i)}) \stackrel{(d)}{=} \sum_{i=1}^k \Xi_i(t - U_i) \stackrel{(d)}{=} \sum_{i=1}^k \Xi_i(U_i)$$

since all $U_{(i)}$'s appear in the sum and the U_i 's are independent from the Ξ_i 's. Hence, conditionally on $\{N_t = k\}$, $I(t)$ has the same distribution as a sum of k i.i.d. r.v. with law $\Xi(U)$. Then,

$$\begin{aligned} G_t(s) &= \sum_{k \geq 0} \mathbb{E} \left[s^{I(t)} \mid N_t = k \right] \mathbb{P}(N_t = k) \\ &= \sum_{k \geq 0} \mathbb{E} \left[s^{\Xi_1(U_1)} \right]^k \frac{(t\theta)^k}{k!} e^{-\theta t} \\ &= e^{-\theta t} \exp \left(t\theta \mathbb{E} \left[s^{\Xi_1(U_1)} \right] \right) \end{aligned} \quad (2.16)$$

We now compute the law of $\Xi(t)$ for $t > 0$ and then we will compute the law of $\Xi_1(U_1)$. Using (2.1), (2.2) and Lemma 2.12(iii), we have

$$\begin{aligned} \mathbb{P}(\Xi(t) = 0) &= \int_{(0, \infty]} \tilde{\mathbb{P}}_r(\Xi(t) = 0) \frac{\Lambda(dr)}{b} = \int_{(0, \infty]} \frac{W(t-x)}{W(t)} \frac{\Lambda(dr)}{b} \\ &= \frac{1}{bW(t)} W \star \Lambda(t) = 1 - \frac{W'(t)}{bW(t)} \end{aligned}$$

and for $n \in \mathbb{N}^*$,

$$\begin{aligned} \mathbb{P}(\Xi(t) = n) &= \int_{(0, \infty]} \tilde{\mathbb{P}}_r(\Xi(t) = n) \frac{\Lambda(dr)}{b} \\ &= \frac{1}{bW(t)} \left(1 - \frac{1}{W(t)} \right)^{n-1} \left(b - W(t)^{-1} W \star \Lambda(t) \right) \\ &= \left(1 - \frac{1}{W(t)} \right)^{n-1} \frac{W'(t)}{bW(t)^2}. \end{aligned}$$

We now compute $\mathbb{P}(\Xi_1(U_1) = n)$

$$\begin{aligned} \mathbb{P}(\Xi_1(U_1) = 0) &= \frac{1}{t} \int_0^t \mathbb{P}(\Xi_1(u) = 0) du = 1 - \frac{1}{tb} \int_0^t \frac{W'(u)}{W(u)} du \\ &= 1 - \frac{\log W(t)}{tb} \end{aligned}$$

because $W(0) = 1$. For $n > 0$,

$$\begin{aligned} \mathbb{P}(\Xi_1(U_1) = n) &= \frac{1}{t} \int_0^t \mathbb{P}(\Xi_1(u) = n) du = \frac{1}{t} \int_0^t \left(1 - \frac{1}{W(u)} \right)^{n-1} \frac{W'(u)}{bW(u)^2} du \\ &= \frac{1}{bt} \int_{W(t)^{-1}}^1 \frac{(1-u)^{n-1}}{n} du = \frac{(1 - 1/W(t))^n}{bnt}. \end{aligned}$$

We are now able to compute the generating function of $\Xi_1(U_1)$. For $s, t > 0$,

$$\begin{aligned} \mathbb{E} \left[s^{\Xi_1(U_1)} \right] &= \frac{1}{bt} \sum_{n \geq 1} \frac{s^n}{n} \left(1 - \frac{1}{W(t)} \right)^n + 1 - \frac{\log W(t)}{tb} \\ &= 1 - \frac{1}{bt} \left[\log \left(1 - s(1 - 1/W(t)) \right) + \log W(t) \right] \\ &= 1 - \frac{1}{bt} \log \left(W(t) + s(1 - W(t)) \right). \end{aligned}$$

Finally for $t, s > 0$, according to (2.16),

$$G_t(s) = e^{-\theta t} \exp \left(t\theta \mathbb{E} \left[s^{\Xi_1(U_1)} \right] \right) = (W(t) + s(1 - W(t)))^{-\theta/b}.$$

which is the p.g.f. of a negative binomial distribution with parameters $1 - W(t)^{-1}$ and θ/b .

Remark 2.14. We denote by M_t the number of alive families at time t and for $i \geq 1$, M_t^i is the number of families of size i at time t . Then, $(M_t^i, i \geq 1)$ is the frequency spectrum of the total population at time t (see Paragraph 3.3). Using similar computations to those of the previous proof, we can prove that M_t and M_t^i are Poisson r.v. Indeed, for $s \in [0, 1]$,

$$\mathbb{E} \left[s^{M_t} \right] = e^{-\theta t} \exp \left(t\theta \mathbb{E} \left[s^{\mathbf{1}_{\{\Xi_1(U_1) > 0\}}} \right] \right) = \exp \left(t\theta(s - 1) \mathbb{P}(X_1(U_1) > 0) \right) = W(t)^{(s-1)\theta/b}$$

and

$$\mathbb{E} \left[s^{M_t^i} \right] = e^{-\theta t} \exp \left(t\theta \mathbb{E} \left[s^{\mathbf{1}_{\{\Xi_1(U_1) = i\}}} \right] \right) = \exp \left(\frac{\theta}{b} \frac{1}{i} \left(1 - \frac{1}{W(t)} \right)^i (s - 1) \right)$$

2.4.3 Proof of Theorem 2.3(ii)

We first prove the almost sure convergence. Splitting $I(t)$ between the surviving and the non-surviving populations, we have

$$e^{-\eta t} I(t) = \sum_{i \geq 1} e^{-\eta t} Z^{(i)}(t) + \sum_{i \geq 1} e^{-\eta t} \Xi_i(t - T_i) \mathbf{1}_{\{t \geq T_i\} \cap \text{Ext}_i} \quad (2.17)$$

where for $i \geq 1$, Ext_i denotes the extinction of the process Ξ_i . We will show that for each of these two terms, we can exchange summation and limit, so that in particular, the second term vanishes as $t \rightarrow \infty$.

We first treat the second term of the r.h.s. of (2.17). We have

$$C_t := \sum_{i \geq 1} \Xi_i(t - T_i) \mathbf{1}_{\{t \geq T_i\} \cap \text{Ext}_i} \leq \sum_{i \geq 1} \mathbf{1}_{\{t \geq T_i\}} Y_i \mathbf{1}_{\text{Ext}_i} \text{ a.s.}$$

where Y_i is the total progeny of the i -th population which does not survive. Moreover, $\mathbb{E}[Y_1 \mathbf{1}_{\text{Ext}_1}] \leq \mathbb{E}[Y_1]$, Y_1 is the total progeny of a subcritical Bienaymé-Galton-Watson process so its mean is finite. Hence, since the r.h.s. in the previous equation is a compound Poisson process with finite mean, it grows linearly and $e^{-\eta t} C_t$ vanishes as $t \rightarrow \infty$.

To exchange summation and limit in the first term of the r.h.s. of (2.17), we will use the dominated convergence theorem. By Proposition 2.2, we already know that $e^{-\eta t} Z^{(i)}(t)$ a.s. converges as t goes to infinity to $e^{-\eta T^{(i)}} E_i$. Hence, it is sufficient to prove that

$$\sum_{i \geq 1} \sup_{t \geq 0} \left(e^{-\eta t} Z^{(i)}(t) \right) < \infty \text{ a.s.} \quad (2.18)$$

Since

$$\sup_{t \geq 0} \left(e^{-\eta t} Z^{(i)}(t) \right) = e^{-\eta T^{(i)}} \sup_{t \geq 0} \left(e^{-\eta t} \Xi_{(i)}(t) \right),$$

we have

$$\sum_{i \geq 1} \sup_{t \geq 0} \left(e^{-\eta t} Z^{(i)}(t) \right) = \sum_{i \geq 1} e^{-\eta T^{(i)}} J_i$$

where $J_i := \sup_{t \geq 0} \left(e^{-\eta t} \Xi_{(i)}(t) \right)$ for $i \geq 1$ and J_1, J_2, \dots are i.i.d. Thus, using Lemmas 2.13, this series a.s. converges if $\mathbb{E}[\log^+ J_1]$ is finite, which is checked thanks to Proposition 2.1(ii). Then we get (2.18) and using the dominated convergence theorem,

$$I := \lim_{t \rightarrow \infty} e^{-\eta t} I(t) = \sum_{i \geq 1} e^{-\eta T^{(i)}} E_i \text{ a.s.}$$

In order to find the law of I , we compute its Laplace transform. For $a > 0$, using part(i) of this theorem,

$$\mathbb{E} \left[e^{-ae^{-\eta t} I(t)} \right] = G_t(e^{-ae^{-\eta t}}) = \left(e^{-ae^{-\eta t}} + (1 - e^{-ae^{-\eta t}}) W(t) \right)^{-\theta/b}$$

and

$$e^{-ae^{-\eta t}} + (1 - e^{-ae^{-\eta t}}) W(t) \underset{t \rightarrow \infty}{\sim} 1 + ae^{-\eta t} W(t) \xrightarrow{t \rightarrow \infty} 1 + \frac{a}{c}$$

using Lemma 2.12 (ii). Then,

$$\mathbb{E} \left[e^{-aI} \right] = \left(\frac{c}{a+c} \right)^{\theta/b}$$

which is the Laplace transform of a Gamma($\theta/b, c$) random variable.

2.5 Other proofs

2.5.1 Proof for Model I

To prove Theorem 2.4, we follow Tavaré's proof [66]. For $r \geq 1$, we first compute the joint distribution of

$$\left(\sigma_1, \dots, \sigma_r, \sum_{j>r} \sigma_j \right) = \left(e^{-\eta T^{(1)}} E_1, \dots, e^{-\eta T^{(r)}} E_r, e^{-\eta T^{(r)}} I' \right)$$

where

$$I' := \sum_{j>r} e^{-\eta(T^{(j)} - T^{(r)})} E_j$$

is independent of $E_1, \dots, E_r, T^{(1)}, \dots, T^{(r)}$ and has the same distribution as I by the strong Markov property and because $(E_j, j \geq 1)$ are i.i.d. r.v. and independent of $(T^{(j)}, j \geq 1)$. Then, since we know that $(T^{(i)})_{i \geq 1}$ is a Poisson process of rate $\theta\eta/b$, that $(E_i)_{i \geq 1}$ is an independent sequence of i.i.d. exponential variables with parameter c and that I' is a Gamma r.v. with parameters θ/b and c (Theorem 2.3(ii)), one can compute the joint density of this r -tuple as follows

$$f(y_1, \dots, y_{r+1}) = \frac{(\theta/b)^r c^{\theta/b} (y_{r+1})^{\theta/b-1} e^{-c(y_1 + \dots + y_{r+1})}}{\Gamma(\theta/b) (y_{r+1} + y_r) \cdots (y_{r+1} + \dots + y_1)}$$

for $y_1, \dots, y_{r+1} > 0$. Then, since $I = \sigma_1 + \dots + \sigma_r + \sum_{j>r} \sigma_j$ we can compute the joint density of $(\sigma_1/I, \dots, \sigma_r/I, I)$

$$g(z_1, \dots, z_{r+1}) = \frac{(\theta/b)^r (1 - z_1 - \dots - z_r)^{\theta/b}}{(1 - z_1) \dots (1 - z_1 - \dots - z_r)} \cdot \frac{c^{\theta/b} (z_{r+1})^{\theta/b-1} e^{-cz_{r+1}}}{\Gamma(\theta/b)}$$

for $z_1, \dots, z_{r+1} > 0$ such that $z_1 + \dots + z_r < 1$.

Then, I is independent of (P_1, \dots, P_r) and the left-hand term in the last equation is the joint density of this r -tuple. This joint density is exactly that of the vector defined in the statement of Theorem 2.4.

2.5.2 Proof for Model II

In this section, we prove Theorem 2.5. We recall that in Model II, immigrants are of type i with probability p_i . Denote by $N^i(t)$ the number of immigrants of type i which arrived before time t . Then, $(N^i(t), t \geq 0)$ is a Poisson process with parameter θp_i and the processes $(N^i, i \geq 1)$ are independent. Hence, $I_1(t), I_2(t), \dots$ are independent and their asymptotic behaviors are the same as $I(t)$ in Theorem 2.3, replacing θ with θp_i . Then we have

$$e^{-\eta t} I_i(t) \xrightarrow[t \rightarrow \infty]{} I_i \quad \text{a.s.} \quad i \geq 1$$

where the I_i 's are independent and I_i has a Gamma distribution $\Gamma(\theta_i, c)$ (recall that $\theta_i = \theta p_i/b$).

Moreover,

$$e^{-\eta t} I(t) \xrightarrow[t \rightarrow \infty]{} I \quad \text{a.s.}$$

where $I \sim \Gamma(\theta/b, c)$. Therefore, for $r \geq 1$,

$$\lim_{t \rightarrow \infty} I(t)^{-1} (I_1(t), \dots, I_r(t)) = \left(\frac{I_1}{I}, \dots, \frac{I_r}{I} \right) \quad \text{a.s.}$$

In order to investigate the law of this r -tuple, we prove that

$$I = \sum_{i \geq 1} I_i \quad \text{a.s.} \quad (2.19)$$

First, by Fatou's lemma,

$$\liminf_{t \rightarrow \infty} e^{-\eta t} \sum_{i \geq 1} I_i(t) \geq \sum_{i \geq 1} \liminf_{t \rightarrow \infty} e^{-\eta t} I_i(t) \quad \text{a.s.}$$

and so

$$I \geq \sum_{i \geq 1} I_i \quad \text{a.s.}$$

Second,

$$\mathbb{E} \left[\sum_{i \geq 1} I_i \right] = \sum_{i \geq 1} \frac{\theta_i}{c} = \frac{\theta}{bc} = \mathbb{E}[I].$$

The last two equations yield (2.19). For $1 \leq i \leq r$, we can write

$$\frac{I_i}{I} = \frac{I_i}{I_1 + \dots + I_r + I^*}$$

where I^* is independent of $(I_i, 1 \leq i \leq r)$ and has a Gamma distribution $\Gamma(\theta/b - \bar{\theta}_r, c)$ with $\bar{\theta}_r := \sum_{i=1}^r \theta_i$. Hence, one can compute the joint density of the r -tuple $(I_1/I, \dots, I_r/I)$ as follows

$$f(x_1, \dots, x_r) = \frac{\Gamma(\theta/b)}{\Gamma(\theta/b - \bar{\theta}_r) \prod_{i=1}^r \Gamma(\theta_i)} x_1^{\theta_1-1} \dots x_r^{\theta_r-1} (1 - x_1 - \dots - x_r)^{\theta/b - \bar{\theta}_r - 1}$$

for $x_1, \dots, x_r > 0$ satisfying $x_1 + \dots + x_r < 1$. This joint density is exactly that of (P'_1, \dots, P'_r) defined in the statement of the theorem.

2.5.3 Proof for Model III

We begin with a technical lemma which will be useful in this paragraph.

Lemma 2.15. *Let $(T_i, i \geq 1)$ be the arrival times of a Poisson process with parameter ρ and ζ_1, ζ_2, \dots be i.i.d. r.v. independent of the T_i 's. Denote by g the density of ζ_1 with respect to Lebesgue measure and by $F(v) := \mathbb{P}(\zeta_1 \geq v)$ its tail distribution. Then for $r > 0$, $(e^{-rT_i} \zeta_i, i \geq 1)$ are the points (in some order) of a non homogeneous Poisson point process on $(0, \infty)$ with intensity measure $\frac{\rho}{r} \frac{F(v)}{v} dv$.*

Proof. We first study the collection $\Pi = \{(T_i, \zeta_i), i \geq 1\}$. This is a Poisson point process on $(0, \infty) \times (0, \infty)$ with intensity measure $\rho g(y) dt dy$. Then, $(e^{-rT_i} \zeta_i, i \geq 1)$ is a Poisson point process whose intensity measure is the image of $\rho g(y) dt dy$ by $(t, y) \mapsto e^{-rt} y$. We now compute it. Let h be a non-negative mapping. Changing variables, we get

$$\begin{aligned} \int_0^\infty \int_0^\infty h(e^{-rt} y) \rho g(y) dt dy &= \frac{\rho}{r} \int_0^\infty h(v) dv \int_0^1 g\left(\frac{v}{u}\right) \frac{du}{u^2} \\ &= \frac{\rho}{r} \int_0^\infty h(v) \frac{F(v)}{v} dv \end{aligned}$$

and the proof is completed. \square

We are now able to prove Proposition 2.8. We begin its proof by showing the almost sure convergence. In order to do that, as in the proof of Theorem 2.3(ii), we will use the dominated convergence theorem for the sum

$$e^{-\eta t} I(t) = \sum_{i \geq 1} e^{-\eta t} I_{\Delta_i}^i(t - T_i) \mathbf{1}_{\{t \geq T_i\}}.$$

As in the proof of Theorem 2.3(ii), this sum is bounded by $\sum_{i \geq 1} e^{-\eta T_i} \sup_{t \geq 0} (e^{-\eta t} I_{\Delta_i}^i(t))$ which, according to Lemma 2.13, is a.s. finite if

$$\mathbb{E} \left[\log^+ \sup_{t \geq 0} (e^{-\eta t} I_{\Delta}^1(t)) \right] < \infty.$$

However, $I_{\Delta}^1(t) = \sum_{i \geq 1} \Xi^i(t - \tilde{T}_i) \mathbf{1}_{\{t \geq \tilde{T}_i\}}$ where conditionally on Δ , $(\tilde{T}_i, i \geq 1)$ is a Poisson process with parameter Δ . Hence,

$$\sup_{t \geq 0} (e^{-\eta t} I_{\Delta}(t)) \leq \sum_{i \geq 1} e^{-\eta \tilde{T}_i} \sup_{t \geq 0} (e^{-\eta t} \Xi^i(t)) = \sum_{i \geq 1} e^{-\eta \tilde{T}_i} J_i$$

where J_1, J_2, \dots is an i.i.d. sequence of random variables independent from $\tilde{T}_1, \tilde{T}_2, \dots$ distributed as $\sup_{t \geq 0} (e^{-\eta t} \Xi(t))$ where $(\Xi(t), t \geq 0)$ is a homogeneous CMJ-process. According to Proposition 2.1(ii), we know that $\mathbb{E}[(\log^+ J_1)^2] < \infty$.

We define for $i \geq 1$, $\varsigma_i := e^{-\eta \tilde{T}_i} J_i$ and $\varsigma := \sum_{i \geq 1} \varsigma_i$ and we have to prove that $\mathbb{E}[\log^+ \varsigma]$ is finite. To do that, we first work conditionally on Δ . According to Lemma 2.15, we know that $(\varsigma_i, i \geq 1)$ are the points of a non homogeneous Poisson process on $(0, \infty)$ with intensity measure $\frac{\Delta}{\eta} \frac{L(v)}{v} dv$ where $L(v) := \mathbb{P}(J \geq v)$. Then, using the inequality

$$\log^+(x + y) \leq \log^+ x + \log^+ y + \log 2, \quad x, y \geq 0,$$

we have

$$\mathbb{E}[\log^+ \varsigma] \leq \log 2 + \mathbb{E} \left[\log^+ \sum_{i \geq 1} \varsigma_i \mathbf{1}_{\{\varsigma_i \leq 1\}} \right] + \mathbb{E} \left[\log^+ \sum_{i \geq 1} \varsigma_i \mathbf{1}_{\{\varsigma_i > 1\}} \right] \quad (2.20)$$

We first consider the second term of the r.h.s.

$$\mathbb{E} \left[\log^+ \sum_{i \geq 1} \varsigma_i \mathbf{1}_{\{\varsigma_i \leq 1\}} \right] \leq \mathbb{E} \left[\sum_{i \geq 1} \varsigma_i \mathbf{1}_{\{\varsigma_i \leq 1\}} \right] = \int_0^1 v \frac{\Delta}{\eta} \frac{L(v)}{v} dv \leq \frac{\Delta}{\eta}. \quad (2.21)$$

Then, we compute the third term of the r.h.s. of (2.20): if $A := \sup_i \varsigma_i$,

$$\begin{aligned} \mathbb{E} \left[\log^+ \sum_{i \geq 1} \varsigma_i \mathbf{1}_{\{\varsigma_i > 1\}} \right] &\leq \mathbb{E} \left[\log^+ (A \cdot \#\{i \geq 1 | \varsigma_i > 1\}) \right] \\ &\leq \mathbb{E} [\log^+ A] + \mathbb{E} [\log^+ \#\{i \geq 1 | \varsigma_i > 1\}] \end{aligned}$$

Furthermore, the number of ς_i greater than 1 has a Poisson distribution with parameter $\int_1^\infty \frac{\Delta}{\eta} \frac{L(v)}{v} dv$. Since $\mathbb{E}[\log^+ J] < \infty$,

$$\int_0^\infty \mathbb{P}(\log^+ J \geq s) ds = \int_0^\infty \mathbb{P}(J \geq e^s) ds = \int_1^\infty \frac{L(v)}{v} dv < \infty.$$

Then,

$$\mathbb{E} [\log^+ \#\{i \geq 1 | \varsigma_i > 1\}] \leq \mathbb{E} [\#\{i \geq 1 | \varsigma_i > 1\}] = \frac{\Delta}{\eta} \int_1^\infty \frac{L(v)}{v} dv \leq C \Delta \quad (2.22)$$

where C is a finite constant which does not depend on Δ .

We now want to study A :

$$\mathbb{P}(A \leq x) = \mathbb{P}(\#\{i \geq 1 | \varsigma_i > x\} = 0) = \exp \left(- \int_x^\infty \frac{\Delta}{\eta} \frac{L(v)}{v} dv \right), \quad x > 0.$$

So that

$$\mathbb{P}(A \in dx) = \frac{\Delta}{\eta} \frac{L(x)}{x} \exp \left(- \int_x^\infty \frac{\Delta}{\eta} \frac{L(v)}{v} dv \right) dx.$$

Then,

$$\begin{aligned} \mathbb{E}[\log^+ A] &= \int_1^\infty \log x \frac{\Delta}{\eta} \frac{L(x)}{x} \exp \left(- \int_x^\infty \frac{\Delta}{\eta} \frac{L(v)}{v} dv \right) dx \\ &\leq \frac{\Delta}{\eta} \int_1^\infty \log x \frac{L(x)}{x} dx = \frac{\Delta}{\eta} \int_0^\infty u L(e^u) du \\ &\leq \frac{\Delta}{\eta} \int_0^\infty u \mathbb{P}(\log^+ J \geq u) du \leq C' \Delta \end{aligned} \quad (2.23)$$

where C' is a finite constant since $\mathbb{E}[(\log^+ J)^2] < \infty$ according to Proposition 2.1(ii). Hence, with (2.21), (2.22) and (2.23) we have

$$\mathbb{E}[\log^+ \varsigma | \Delta] \leq \log 2 + C'' \Delta.$$

Then, $\mathbb{E}[\log^+ \varsigma]$ is finite because $\mathbb{E}[\Delta] = \theta^{-1} \int_0^\infty x^2 f(x) dx < \infty$. and, using the dominated convergence theorem, $e^{-\eta t} I(t)$ a.s. converges toward $\sigma = \sum_{i \geq 1} e^{-\eta T_i} G_i$ as $t \rightarrow \infty$.

We now compute the law of the limit σ . Recall that $\sigma_i := \exp(-\eta T_i) G_i$ for $i \geq 1$. Then, using Lemma 2.15, $(\sigma_i)_{i \geq 1}$ are the points of a non homogeneous Poisson point process on $(0, \infty)$ with intensity measure $\frac{\theta}{\eta} \frac{F(y)}{y} dy$ where $F(y) = \mathbb{P}(G \geq y)$. To compute the Laplace transform of σ , we use the exponential formula for Poisson point processes

$$\mathbb{E}[e^{-s\sigma}] = \exp\left(-\frac{\theta}{\eta} \int_0^\infty \frac{F(v)}{v} (1 - e^{-sv}) dv\right), \quad s > 0,$$

and to get the expectation of σ , we differentiate the last displayed equation at 0:

$$\mathbb{E}[\sigma] = \frac{\theta}{\eta} \int_0^\infty F(v) dv = \frac{\theta}{\eta} \mathbb{E}[G]$$

and

$$\mathbb{E}[G] = \theta^{-1} \int_0^\infty x f(x) \frac{x}{b} \frac{1}{c} dx < \infty.$$

Hence,

$$\mathbb{E}[\sigma] = \frac{1}{\eta bc} \int_0^\infty x^2 f(x) dx < \infty$$

which ends the proof of Proposition 2.8.

Remark 2.16. Thanks to similar calculations as in Theorem 2.3(i), we can compute the generating function of $I(t)$

$$\mathbb{E}[s^{I(t)}] = \exp\left(-\int_0^t du \left(\theta - \int_0^\infty \frac{x f(x)}{(W(u)(1-s) + s)^{x/b}} dx\right)\right), \quad s \in [0, 1].$$

and we can deduce the law of σ in another way.

Chapitre 3

Arbres de ramification avec mutations neutres à la naissance

Abstract: *We consider a supercritical, binary and homogeneous Crump-Mode-Jagers process where individuals can give birth to clonal (resp. mutant) children with probability $1 - p$ (resp. p). It is an infinitely many alleles model with neutral mutations.*

We are first interested in the frequency spectrum of the total population that describes the number of types with a given size and a given age through time. We compute the expected value of this allelic partition and obtain some convergence results by using classical properties of CMJ processes counted by random characteristics.

We then get asymptotic results, as $t \rightarrow \infty$, about the sizes of large families when the clonal process (that counts the number of individuals of the ancestral type) is supercritical and about the ages of old families when it is subcritical or critical.

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3.1 Introduction

We consider a general branching model with neutral mutations occurring at birth. We suppose that individuals carry types (or alleles), have i.i.d. (and not necessarily exponential) life lengths and give birth at constant rate b during their lives to children who can be mutants with probability p or clones of their parents with probability $1 - p$. We are working with an infinite alleles model, that is, when a mutation occurs, the type of the mutant child was never encountered before. Moreover, mutations are neutral because

they do not imply advantages or disadvantages (all individuals have identically distributed dynamics). We call a *family* the set of all individuals carrying a same type.

Without mutation, the model is linked to a genealogical tree called *splitting tree* [26, 28, 48]. Moreover, if $\Xi(t)$ denotes the number of alive individuals at time t , the process $\Xi := (\Xi(t), t \geq 0)$ is a *Crump-Mode-Jagers process* (or general branching process) [35] which is binary (births occur singly) and homogeneous (constant birth rate). We are interested in asymptotic results as $t \rightarrow \infty$, so we suppose that Ξ is supercritical and we work conditionally on its survival event.

We are first interested in the allelic partition of the population and more precisely in properties of the *frequency spectrum* $(M_t^{i,a}, i \geq 1)$ where $M_t^{i,a}$ is the number of distinct types carried by exactly i individuals at time t and that appeared after time $t - a$ (or equivalently the types whose ages are less than a at time t).

This kind of question was first studied by Ewens [23] who discovered the well known ‘sampling formula’ named after him and which describes the law of the allelic partition for a Wright-Fisher model with neutral mutations. In [29], Griffiths and Pakes are interested in the case of a Bienaymé-Galton-Watson (BGW) process where children can be mutants with a probability u : the authors obtained asymptotic results about the number of alleles and the frequency spectrum at generation n as $n \rightarrow \infty$. See also Pakes [57] for similar properties concerning continuous-time Markov branching processes.

In the two papers [7, 8], Bertoin considers an infinite alleles model with neutral mutations in a subcritical or critical BGW-process where individuals independently give birth to a random number of clonal and mutant children according to the same joint distribution. In [7], he defines a tree of alleles where all individuals of a common type are gathered in clusters and specifies the law of the allelic partition of the total population by describing the joint law of the sizes of the clusters and the numbers of their mutant children. A tree of alleles is also built by Taïb in [65] for CMJ-processes; we use this construction in Section 3.4. In [8], Bertoin obtains the convergence of the sizes of allelic families for a large initial population and a small mutation rate.

Here, we obtain two kinds of results concerning the frequency spectrum $(M_t^{i,a}, i \geq 1)$. We first compute the expected value of the number of types carried by i individuals at time t and with age in $[a - da, a]$. As $t \rightarrow \infty$ and on the survival event of Ξ , we also obtain the asymptotic behaviors of the frequency spectrum and of the relative abundances of types.

We also treat another kind of question: if the process counting individuals of ancestral types (that we call *clonal process*) is subcritical or critical, what are the asymptotic behaviors of the ages of oldest types at time t as $t \rightarrow \infty$? If the clonal process is supercritical, what are those of the sizes of largest families as $t \rightarrow \infty$?

In two related papers [11, 12], Champagnat and Lambert consider a model really close to ours: the authors also work with a supercritical Crump-Mode-Jagers process but individuals independently experience mutations at Poisson times during their lives. In [11], explicit formulas about the expected frequency spectrum at time t are stated. In [12], the authors are interested in large and old families; they obtain asymptotic results about the sizes of the largest families and about the ages of the oldest ones as $t \rightarrow \infty$. Their results are analogous to ours, stated in Sections 3.5 and 3.6.

Branching processes with mutation have also been used in carcinogenesis (see [33, 64, 21]). Indeed, they can model the evolution of cancerous cells undergoing a medical treatment and becoming resistant after having experienced a certain number of mutations. It

is of interest to obtain information about the arrival time of the first resistant cell or about the number of resistant cells.

The main technique we use in the sequel is CMJ processes counted by random characteristics and time-dependent random characteristics developed by Jagers and Nerman (see [35, 36, 37]). Most of our results follow ideas of Taïb [65] who considered CMJ-processes with more general kind of mutations at birth (for example, they can depend on the age of the mother) but in our particular case, some of the limits he obtained can be explicitly computed thanks to a recent paper of Lambert [48] giving for example the one-dimensional marginals of Ξ and its asymptotic behavior.

The paper is organized as follows: in Section 3.2, we expose the model that we consider and give some of its properties that will be useful to get our main results. Section 3.3 is devoted to the study of the frequency spectrum. In Section 3.4, we define the allele tree we use in Sections 3.5 and 3.6 to respectively obtain results about ages and sizes of families as $t \rightarrow \infty$. In Section 3.7, we conclude by giving some results linked to Yule's problem where mutants come at a rate proportional to the number of alive families.

3.2 Preliminaries

As a population model, we consider *splitting trees* [26, 28, 48] with neutral mutations happening at birth, that is,

- At time $t = 0$, there is only one progenitor.
- All individuals have i.i.d. reproduction behaviors.
- Conditional on her birthdate α and her lifespan ζ , each individual gives birth to children at constant rate $b \in (0, \infty)$ during $(\alpha, \alpha + \zeta]$.
- Births occur singly.
- When a birth event occurs, independently of other individuals, a child is a mutant with probability p and a clone of her mother with probability $1 - p$ (to avoid trivialities, we suppose that $p \in (0, 1)$).

The model that we consider has *neutral mutations* because all individuals behave in the same way regardless of type, and is an *infinitely many alleles* model, that is, the type of a new mutant was never encountered before (see Figure 3.1).

We denote the lifespan distribution by $\Lambda(\cdot)/b$ where Λ is a finite positive measure on $(0, +\infty]$ with total mass b and called *lifespan measure* [48].

Without mutation, if $\Xi(t)$ is the number of extant individuals at time t , then the process $(\Xi(t), t \geq 0)$ is a Crump-Mode-Jagers (CMJ) process or general branching process (see [35, 36] and references therein). Here, due to aforementioned assumptions, we consider the special case of a binary and homogeneous CMJ-process. We use the formalism of the previous citations. As explained in the introductory chapter, in CMJ-processes, one associates with each individual x a non-negative r.v. λ_x (its life length), and a point process ξ_x called birth point process. One assumes that the sequence $(\lambda_x, \xi_x)_x$ is i.i.d. but λ_x and ξ_x are not necessarily independent. In our particular case, the common distribution of life spans is $\Lambda(\cdot)/b$ and conditional on her lifespan, the birth point process of an individual is distributed as a Poisson point process during its life. If we denote by ξ the birth point process of the ancestor, then its characteristic measure is

$$\mu(dt) := \mathbb{E}[\xi(dt)] = dt\Lambda((t, +\infty]). \quad (3.1)$$

In the same way, if ξ_m (resp ξ_c) is the point process of the mutant (resp. clonal) children of the ancestor, then by thinning of Poisson point processes, ξ_m and ξ_c are independent, $\xi = \xi_m + \xi_c$ and with obvious notation, $\mu_m(dt) = p\mu(dt)$ and $\mu_c(dt) = (1-p)\mu(dt)$.

Thereafter, we use some results of Z. Taïb [65] who considers a similar model with more general birth point processes and mutations. However, most of his convergence theorems gives non explicit limits. In our particular case, some limits are computable because of the knowledge of the one-dimensional marginals and the asymptotic behavior of Ξ (see below formulas (3.5), (3.7) and (3.8)).

We first check that the general assumptions (C.1-4) of the appendix in [65] hold in our case: we must have

- (i) $\mu(\mathbb{R}^+) > 1$ (supercritical case).
- (ii) There is $\eta > 0$ such that $\hat{\mu}(\eta) := \int_0^\infty e^{-\eta u} \mu(du) = 1$.
- (iii) $\beta := \int_0^\infty u e^{-\eta u} \mu(du) < \infty$.
- (iv) μ is not supported by any lattice $k\mathbb{Z}$ for $k > 0$.

First, point (iv) is straightforward with (3.1). Then, we set

$$\psi(\lambda) := \lambda - \int_{(0,\infty]} (1 - e^{-\lambda r}) \Lambda(dr) \quad (3.2)$$

and we suppose that

$$m := \int_{(0,\infty]} r \Lambda(dr) > 1$$

which implies (i). Moreover, ψ is a convex function satisfying $\psi(0^+) = 0$, $\psi'(0^+) = 1 - m < 0$. Hence, there exists a unique $\eta > 0$ such that $\psi(\eta) = 0$. Using (3.1),

$$\int_0^\infty e^{-\eta u} \mu(du) = \int_0^\infty e^{-\eta u} du \int_{(u,\infty)} \Lambda(dr) = \int_{(0,\infty]} \Lambda(dr) \frac{1 - e^{-\eta r}}{\eta} = \frac{\eta - \psi(\eta)}{\eta} = 1. \quad (3.3)$$

Then, η satisfies (ii) and is the *Malthusian parameter* of the population process because $\Xi(t)$ grows like $e^{\eta t}$ on the non-extinction event. More precisely, according to Proposition 2.1, we have

$$\mathbb{P}(\text{Ext}) = 1 - \eta/b \quad (3.4)$$

and on Ext^c ,

$$e^{-\eta t} \Xi(t) \xrightarrow[t \rightarrow \infty]{} \mathcal{E} \quad \text{a.s.} \quad (3.5)$$

where \mathcal{E} follows an exponential distribution with parameter $\psi'(\eta)$.

Finally, let us check that (iii) holds

$$\begin{aligned} \beta &= \int_0^\infty u e^{-\eta u} du \int_{(u,\infty)} \Lambda(dr) = \int_{(0,\infty]} \Lambda(dr) \int_0^r u e^{-\eta u} du \\ &= \int_{(0,\infty]} \Lambda(dr) \left(\frac{-r e^{-\eta r}}{\eta} + \frac{1 - e^{-\eta r}}{\eta^2} \right) = \frac{\psi'(\eta) - 1}{\eta} + \frac{\eta - \psi(\eta)}{\eta^2} \\ \beta &= \frac{\psi'(\eta)}{\eta} < \infty. \end{aligned}$$

Let W be the unique increasing continuous function $(0, \infty) \rightarrow (0, \infty)$ satisfying

$$\int_0^\infty W(x) e^{-\lambda x} dx = \frac{1}{\psi(\lambda)}, \quad \lambda > \eta. \quad (3.6)$$

In fact, W is the *scale function* (see [5]) of the contour of a splitting tree defined in [48] and which is a spectrally positive Lévy process. Thanks to this function, we deduce the one-dimensional marginals of Ξ (see Section 2.4.2)

$$\mathbb{P}(\Xi(t) = 0) = 1 - \frac{W'(t)}{bW(t)} \quad (3.7)$$

and for $n \in \mathbb{N}^*$,

$$\mathbb{P}(\Xi(t) = n) = \left(1 - \frac{1}{W(t)}\right)^{n-1} \frac{W'(t)}{bW(t)^2}. \quad (3.8)$$

In particular, if T denotes the time of extinction of the tree

$$T := \inf\{t \geq 0; \Xi(t) = 0\},$$

we have for $t \geq 0$

$$\mathbb{P}(T > t) = \mathbb{P}(\Xi(t) \neq 0) = \frac{W'(t)}{bW(t)}. \quad (3.9)$$

In the sequel, an important role will be played by the clonal process $(\Xi_c(t), t \geq 0)$ (c for clonal) where $\Xi_c(t)$ is the number of extant individual at time t carrying the same type as the ancestor. Since mutations occur independently of $(\Xi(t), t \geq 0)$, by thinning of Poisson process, the process Ξ_c defines a splitting tree with lifespan measure $\Lambda_c = (1-p)\Lambda$. Moreover, conditional on its arrival time, each type evolves like Ξ_c and independently of other types (this property is the basis of the tree of alleles that we construct in Section 3.4). Then, the mutations have an effect on the births by decreasing the birth rate while in [11, 12], since they occur at a Poissonian rate on the trees, the mutations can be seen as an additional death rate for the clonal process.

Hence, only replacing Λ by Λ_c in (3.2), we can define

$$\psi_c(\lambda) := \lambda - (1-p) \int_{(0,\infty)} (1 - e^{-\lambda r}) \Lambda(dr) = p\lambda + (1-p)\psi(\lambda) \quad (3.10)$$

and the clonal process is subcritical, critical or supercritical if $(1-p)m$ is less than, equal to, or greater than 1 according to [48]. Let η_c be the largest root of ψ_c . When the clonal process is subcritical or critical, $\eta_c = 0$ while in the supercritical case, $\eta_c > 0$. Moreover, by definition of η_c , we have $p\eta_c + (1-p)\psi(\eta_c) = 0$ using (3.10). Then

$$\psi(\eta_c) = \frac{p\eta_c}{p-1} \leq 0$$

and $\eta_c < \eta$.

In all the previous cases, if Ext_c denotes the extinction event of Ξ_c ,

$$\mathbb{P}(\text{Ext}_c) := \mathbb{P}\left(\Xi_c(t) \xrightarrow[t \rightarrow \infty]{} 0\right) = 1 - \frac{\eta_c}{b(1-p)}, \quad (3.11)$$

since $b(1-p)$ is the birth rate of clonal children. Moreover, in the supercritical case, we have an equivalent result of (3.5).

Like previous formulas, we can also define the scale function W_c solution to

$$\int_0^\infty W_c(x) e^{-\lambda x} dx = \frac{1}{\psi_c(\lambda)}, \quad \lambda > \eta_c, \quad (3.12)$$

and which gives the one-dimensional marginals of Ξ_c and the law of T_c defined as the survival time of the clonal process.

3.3 Frequency spectrum

We denote by M_t the number of types carried by at least one alive individual at time t . For $i \in \mathbb{N}^*$ and $a > 0$, let $M_t^{i,a}$ be the number of types whose ages are less than a and carried by i individuals at t . An example is represented on Figure 3.1.

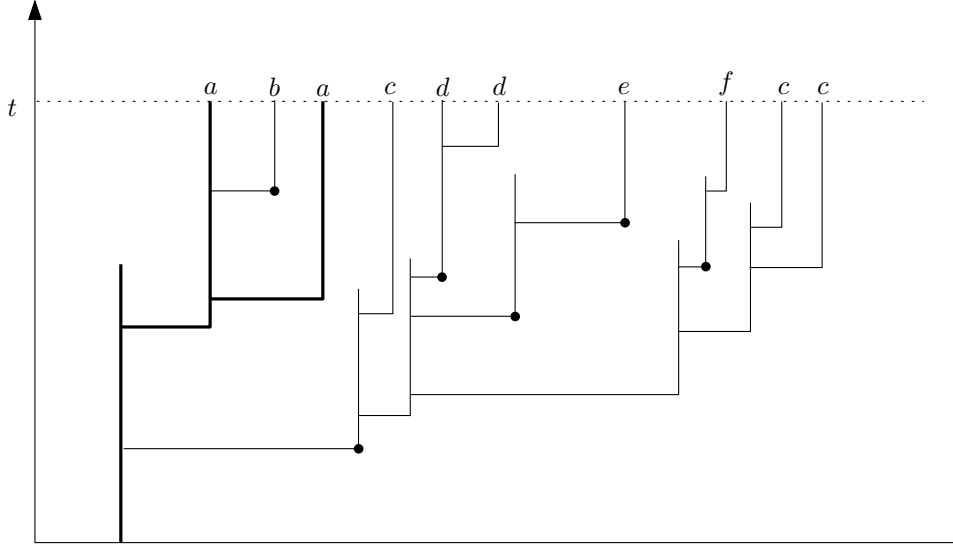


Figure 3.1: An example of a splitting tree with mutations and of the allelic spectrum at time t . Vertical axis is time and horizontal axis shows filiation. Full circles represent mutations occurring at birth and thick lines, the clonal splitting tree of the ancestor up to time t . The different letters are the alleles of alive individuals at time t . In this example, the frequency spectrum $(M_t^i, i \geq 1)$ is $(3, 2, 1, 0, \dots)$ because three alleles (b, e, f) are carried by one individual, a and d are carried by two individuals and c is the only allele carried by three individuals.

3.3.1 Expected frequency spectrum

We first give an exact expression of the expected frequency spectrum at any time t . For $0 < a < t$ and $i \geq 1$, we denote by $M_t^{i,da}$ the number of types carried by i individuals at time t and with ages in $[a - da, a]$. The following proposition yields its expected value.

Proposition 3.1. *For $0 < a < t$ and $i \geq 1$, we have*

$$\mathbb{E}[M_t^{i,da}] = \frac{p}{b(1-p)} W'(t-a) \left(1 - \frac{1}{W_c(a)}\right)^{i-1} \frac{W'_c(a)}{W_c^2(a)} da. \quad (3.13)$$

Proof. Conditional on $\Xi(t-a)$, thanks to the branching property and classical properties of Poisson point processes, $M_t^{i,da}$ is the sum of $\Xi(t-a)$ i.i.d. r.v. distributed as the number of atoms in the interval $[t-a, t-a+da]$ of a Poisson point process with parameter $bp\mathbb{P}(\Xi_c(a) = i)$. Hence,

$$\mathbb{E}[M_t^{i,da}] = \mathbb{E}[\Xi(t-a)] bp \mathbb{P}(\Xi_c(a) = i) da.$$

According to (3.8), $\mathbb{E}[\Xi(t-a)] = W'(t-a)/b$ and

$$\mathbb{P}(\Xi_c(a) = i) = \left(1 - \frac{1}{W_c(a)}\right)^{i-1} \frac{W'_c(a)}{b(1-p)W_c(a)^2}$$

and we obtain the desired result. \square

Then, we deduce the expected value of the frequency spectrum.

Corollary 3.2. *For $a \leq t$ and $i \geq 1$,*

$$\begin{aligned} \mathbb{E}[M_t^{i,a}] &= \frac{p}{b(1-p)} \int_0^a W'(t-x) \left(1 - \frac{1}{W_c(x)}\right)^{i-1} \frac{W'_c(x)}{W_c^2(x)} dx \\ &\quad + \frac{1}{b(1-p)} \left(1 - \frac{1}{W_c(t)}\right)^{i-1} \frac{W'_c(t)}{W_c^2(t)} \mathbf{1}_{\{t=a\}}. \end{aligned} \quad (3.14)$$

Moreover, for $a < t$, as $t \rightarrow \infty$,

$$e^{-\eta t} \mathbb{E}[M_t^{i,a}] \longrightarrow \frac{\eta}{b} \frac{pJ^{i,a}}{(1-p)\psi'(\eta)} \quad (3.15)$$

where

$$J^{i,a} := \int_0^a e^{-\eta u} \left(1 - \frac{1}{W_c(u)}\right)^{i-1} \frac{W'_c(u)}{W_c^2(u)} du.$$

The first assertion of this corollary still holds when $m \leq 1$, that is, when Ξ is sub-critical or critical. In [57], Pakes obtained a similar result: in Lemma 3.1.2, he computed the expected frequency spectrum for a Markov branching process where children can be mutants of their mothers with probability u .

Proof. We obtain (3.14) by integrating (3.13) on $(0, a)$. The second term of the r.h.s. corresponds to $\mathbb{P}(\Xi_c(t) = i)$ the probability that the progenitor has i alive clonal descendants at time t .

We get the convergence result (3.15) by using the dominated convergence theorem. We first need the asymptotic behavior of W' . Recall from Lemma 2.12 that as $t \rightarrow \infty$, $W(t)$ is equivalent to $e^{\eta t}/\psi'(\eta)$. Moreover, according to (3.4) and (3.9),

$$\frac{W'(t)}{bW(t)} = \mathbb{P}(T > t) \xrightarrow[t \rightarrow \infty]{} \mathbb{P}(\text{Ext}^c) = \frac{\eta}{b}$$

and it implies that as $t \rightarrow \infty$, $W'(t-x)$ is equivalent to

$$\frac{\eta}{\psi'(\eta)} e^{\eta(t-x)}.$$

Then, we obtain the convergence of the integral because $\left(1 - \frac{1}{W_c(x)}\right)^{i-1} \frac{W'_c(x)}{W_c^2(x)} \leq b(1-p)$ and there exists $C > 0$ such that $e^{-\eta t} W'(t-x) \leq C$ for t large enough. \square

3.3.2 Convergence results

We are now interested in improvements of the convergence result (3.15). The following proposition yields the asymptotic behavior as $t \rightarrow \infty$ of the frequency spectrum ($M_t^{i,a}$, $i \geq 1$). We also obtain the convergence of the relative abundances $M_t^{i,a}/M_t$ of families of sizes i and ages less than a .

Proposition 3.3. *Under $\mathbb{P}^* := \mathbb{P}(\cdot | \text{Ext}^c)$,*

(i)

$$e^{-\eta t} M_t \xrightarrow[t \rightarrow \infty]{} \frac{p}{1-p} J \mathcal{E} \quad \text{a.s.}$$

(ii)

$$e^{-\eta t} M_t^{i,a} \xrightarrow[t \rightarrow \infty]{} \frac{p}{1-p} J^{i,a} \mathcal{E} \quad \text{a.s.}$$

(iii)

$$\frac{M_t^{i,a}}{M_t} \xrightarrow[t \rightarrow \infty]{} \frac{J^{i,a}}{J} \quad \text{a.s.}$$

where

$$J := \eta \int_0^\infty e^{-\eta u} \ln(W_c(u)) du$$

and where \mathcal{E} is the exponential r.v. with parameter $\psi'(\eta)$ defined by (3.5).Notice that point (ii) is consistent with (3.15) since $\mathbb{P}(\text{Ext}^c) = \eta/b$ and $\mathbb{E}[\mathcal{E}] = 1/\psi'(\eta)$.

Remark 3.4. Unfortunately, we can not explicitly compute the integrals J and $J^{i,a}$ appearing in the three limits in Proposition 3.3 because in general the scale function W_c is unknown. However, even if W_c is computable, it is not easier: when the life lengths are exponentially distributed with parameter d (in that case Ξ is simply a Markovian birth and death process), we have from [48, p393]

$$W_c(x) = \begin{cases} \frac{d-(1-p)be^{((1-p)b-d)x}}{d-(1-p)b} & \text{if } d - (1-p)b \neq 0 \\ 1 + dx & \text{if } d = (1-p)b \end{cases} \quad (3.16)$$

and integrals are still uncomputable. The example of a pure-birth process ($d = 0$) is treated in Subsection 3.3.3.

Proof of Proposition 3.3. We first prove point (ii). We follow the proof of Theorem 3.3 in [65] using a random characteristic. We denote by $\tau_1 < \tau_2 < \dots$ the successive birth times of the children of the ancestor, by ρ_1, ρ_2, \dots the independent sequence of i.i.d. Bernoulli r.v. with parameter $1-p$ saying if a child is a clone or a mutant and by Ξ_c^1, Ξ_c^2, \dots the i.i.d. clonal population processes of the mutant children. Then, for $u \geq a$, we set the characteristic

$$\chi^{(i)}(u) := \sum_{j \geq 1} (1 - \rho_j) \mathbf{1}_{\{u-a \leq \tau_j < u\}} \mathbf{1}_{\{\Xi_c^j(u-\tau_j)=i\}},$$

that is, $\chi^{(i)}(u)$ is the number of mutant children of the ancestor born between $u-a$ and u and whose types are carried by i individuals at u . For $u \leq a$, we set $\chi^{(i)}(u) = 0$.

In the same way, for any individual k and $u > a$, denote by $\chi_k^{(i)}(u)$ the number of mutant children of k born between $u-a$ and u units of time after the birth of k and whose types are carried by i individuals u units of time after the birth of k . Hence, for $t \geq 0$,

$$M_t^{i,a} = Z_t^{\chi^{(i)}} := \sum_{k=1}^{y_t} \chi_k^{(i)}(t - \alpha_k) + \mathbf{1}_{\{t \leq a\}} \mathbf{1}_{\{\Xi_c(t)=i\}} \quad (3.17)$$

where α_k denotes the birth time of k and

$$y_t := \#\{\text{individuals born before } t\}.$$

The second term in the r.h.s. of (3.17) pertains to the ancestor which is not counted by the characteristic. However, this term vanishes until $t > a$ and so disappears at the limit $t \rightarrow \infty$.

According to Theorem 5 in the appendix of [65], if there exists $\eta' < \eta$ such that

$$\int_0^\infty e^{-\eta't} \mu(dt) < \infty \quad (3.18)$$

and such that

$$\mathbb{E} \left[\sup_{u \geq 0} e^{-\eta'u} \chi^{(i)}(u) \right] < \infty, \quad (3.19)$$

then, under Ext^c ,

$$\lim_{t \rightarrow \infty} \frac{M_t^{i,a}}{y_t} = \mathbb{E}[\widehat{\chi}^{(i)}(\eta)] \quad \text{a.s.}$$

where $\widehat{\chi}^{(i)}(\eta) := \int_0^\infty \eta e^{-\eta u} \chi^{(i)}(u) du$ is the Laplace transform of the function $\chi^{(i)}$ at point η . Moreover, conditioning by the point process ξ_m which is independent from the Ξ_c^j 's, we have

$$\begin{aligned} \mathbb{E}[\widehat{\chi}^{(i)}(\eta)] &= \int_0^\infty \eta e^{-\eta u} du \mathbb{E} \left[\sum_{j \geq 1} (1 - \rho_j) \mathbf{1}_{\{u-a \leq \tau_j < u\}} \mathbf{1}_{\{\Xi_c^j(u-\tau_j)=i\}} \right] \\ &= \int_0^\infty \eta e^{-\eta u} du \mathbb{E} \left[\int_{u-a}^u \mathbb{P}(\Xi_c(u-t) = i) \xi_m(dt) \right] \\ &= \int_0^\infty \eta e^{-\eta u} du \int_{u-a}^u \mathbb{P}(\Xi_c(u-t) = i) \mu_m(dt) \\ &= \int_0^\infty \mu_m(dt) e^{-\eta t} \int_0^a \eta e^{-\eta u} \mathbb{P}(\Xi_c(u) = i) du \end{aligned}$$

using Fubini-Tonelli's theorem and a change of variables. Since $\mu_m = p\mu$ and η satisfies (3.3), $\int_0^\infty e^{-\eta t} \mu_m(dt) = p$. Then, thanks to (3.8),

$$\lim_{t \rightarrow \infty} \frac{M_t^{i,a}}{y_t} = \frac{p\eta}{b(1-p)} \int_0^a e^{-\eta u} \left(1 - \frac{1}{W_c(u)}\right)^{i-1} \frac{W_c'(u)}{W_c^2(u)} du \quad \text{a.s.} \quad (3.20)$$

We still have to find $\eta' < \eta$ such that conditions (3.18) and (3.19) are satisfied. First, as in (3.3), for $\eta' \in (0, \eta)$, $\int_0^\infty e^{-\eta u} \mu(du) = 1 - \psi(\eta')/\eta' < \infty$. Second, the characteristic $\chi^{(i)}$ is stochastically dominated by a Poisson process with parameter b , say $(N_t, t \geq 0)$. Then, to prove (3.19), it is sufficient to show that $\mathbb{E} \left[\sup_{t \geq 0} e^{-\eta't} N_t \right]$ is finite. At the end of the proof of Proposition 5.1 in [11], the authors prove that for κ large enough, the process $A := \left((e^{-\eta't} N_t + \kappa)^2, t \geq 0 \right)$ is a supermartingale, which implies that for $y \geq 0$,

$$\mathbb{P} \left(\sup_{t \geq 0} e^{-\eta't} N_t \geq y \right) = \mathbb{P} \left(\sup_{t \geq 0} A_t \geq (y + \kappa)^2 \right) \leq \frac{\mathbb{E}[A_0]}{(y + \kappa)^2} = \frac{\kappa}{(y + \kappa)^2},$$

where the last inequality holds thanks to Exercise 1.15 p58 in [60]. Then,

$$\mathbb{E} \left[\sup_{t \geq 0} e^{-\eta't} N_t \right] = \int_0^\infty dy \mathbb{P} \left(\sup_{t \geq 0} e^{-\eta't} N_t \geq y \right) < \infty.$$

Furthermore, using again [65, Thm 5] with the characteristic $\chi'(u) = \mathbf{1}_{\{0 \leq u \leq \lambda\}}$, on Ext^c

$$\frac{\Xi_t}{y_t} = \frac{Z_t^{\chi'}}{y_t} \xrightarrow{t \rightarrow \infty} \mathbb{E}[\widehat{\chi}'(\eta)] = \int_0^\infty \eta e^{-\eta x} dx \int_{(x, \infty)} \frac{\Lambda(dr)}{b} = \frac{\eta}{b} \quad (3.21)$$

because $\psi(\eta) = 0$. Notice that in that case, condition (3.19) is easily satisfied because $\chi'(u) \leq 1$ for $u \geq 0$. Finally, using together (3.5), (3.20) and (3.21) we get point (ii).

According to [65, Thm 3.3], on Ext^c ,

$$\frac{M_t}{y_t} \xrightarrow{t \rightarrow \infty} \hat{\mu}_m(\eta) \left(1 - \eta \int_0^\infty e^{-\eta u} \mathbb{P}(\Xi_c(u) = 0) \right) du = p\eta \int_0^\infty e^{-\eta u} \frac{W_c'(u)}{b(1-p)W_c(u)} du.$$

We then get (i) by using again (3.5) and (3.21) and by an integration by parts (remind that $W_c(0) = 1$ according to Lemma 2.12). Point (iii) is straightforward from (i) and (ii). \square

We denote by M_t^i the number of families of size i at time t , that is, $M_t^i = \lim_{a \rightarrow \infty} M_t^{i,a}$. Then, M_t^i can be counted by the random characteristic

$$\bar{\chi}(u) := \sum_{j \geq 1} (1 - \rho_j) \mathbf{1}_{\{\tau_j < u\}} \mathbf{1}_{\{\Xi_c^j(u - \tau_j) = i\}}$$

and similar computations as in the proof of Proposition 3.3 lead to the following result.

Proposition 3.5. *Under \mathbb{P}^* , with probability 1 as $t \rightarrow \infty$,*

$$\frac{M_t^i}{M_t} \longrightarrow \frac{1}{i} \frac{\int_0^\infty e^{-\eta u} \left(1 - \frac{1}{W_c(u)} \right)^i du}{\int_0^\infty e^{-\eta u} \ln(W_c(u)) du}$$

3.3.3 A computable example: the pure birth process

Suppose that the life lengths are a.s. infinite, *i.e.* distributed as $\Lambda(\cdot)/b = \delta_\infty$. Hence, each individual lives an infinite time and gives birth at rate b : Ξ is then a Markovian pure birth process and direct computations give $\psi(\lambda) = (\lambda - b) \mathbf{1}_{\{\lambda > 0\}}$, $\eta = b$ and from (3.16), $W_c(x) = e^{(1-p)bx}$. Obviously, $\mathbb{P}(\text{Ext}) = 0$ and $\mathbb{P}^* = \mathbb{P}$. Then, by a change of variables and integrating by parts

$$\begin{aligned} J^{i,a} &= (1-p)b \int_0^a e^{-bu} \left(1 - e^{-b(1-p)u} \right)^{i-1} e^{-b(1-p)u} du \\ &= (1-p) \int_{e^{-ba}}^1 \left(1 - y^{1-p} \right)^{i-1} y^{1-p} dy \\ &= \frac{e^{-ba} \left(1 - e^{-ba(1-p)} \right)}{i} + \int_{e^{-ba}}^1 \frac{\left(1 - y^{1-p} \right)^i}{i} dy. \end{aligned}$$

Moreover,

$$J = \eta \int_0^\infty e^{-\eta u} \ln(W_c(u)) du = b^2(1-p) \int_0^\infty e^{-bu} u du = 1-p.$$

This can be recorded in the following proposition.

Proposition 3.6. *As $t \rightarrow \infty$,*

(i)

$$\frac{M_t^{i,a}}{M_t} \longrightarrow \frac{1}{1-p} \left(\frac{e^{-ba} \left(1 - e^{-ba(1-p)} \right)}{i} + \int_{e^{-ba}}^1 \frac{\left(1 - y^{1-p} \right)^i}{i} dy \right) \quad \text{a.s.}$$

(ii)

$$\frac{M_t^i}{M_t} \longrightarrow \frac{1}{i(1-p)} \int_0^1 \left(1 - y^{1-p} \right)^i dy = \frac{1}{i(1-p)} \sum_{k=0}^i (-1)^k \frac{1}{1 + (1-p)k} \quad \text{a.s.}$$

3.4 Tree of alleles

As done by Taïb in [65, Ch.4], we can define a tree of alleles by aggregating all individuals of the same type into clusters. In [7], Bertoin does an equivalent construction for Bienaymé-Galton-Watson process with neutral mutations. Then, a type can be considered as an individual whose life lengths is its time of extinction and which gives birth at the times of births of mutant children of any individual of this type. More precisely, recalling that M_t is the number of extant types at time t , we have

Proposition 3.7. *The process $(M_t, t \geq 0)$ is a Crump-Mode-Jagers process where the pairs of life spans and birth point processes are distributed as*

$$\left(T_c, \left(\tilde{N}_{\int_0^t \Xi_c(s) ds}, t \geq 0 \right) \right)$$

where \tilde{N} is a Poisson process with parameter bp , independent from Ξ_c and where $T_c := \inf\{t \geq 0, \Xi_c(t) = 0\}$ is the survival time of the ancestral type.

Proof. The result stems from the branching property of Ξ and the thinning of Poisson processes already used in Section 3.2: conditional on the ancestral process Ξ_c , mutant trees are grafted at rate $bp\Xi_c$ on it. \square

Remark 3.8. According to (3.11), notice that $\mathbb{P}(T_c = \infty) = \eta_c/(b(1-p))$ and then a type may have an infinite number of "children" if $\eta_c > 0$, that is, if Ξ_c is supercritical.

The fact that this process counting the number of types over time is a CMJ-process is very useful because it enables us to use results about CMJ-processes counted with random characteristics (as in Section 3.3) or with time-dependent random characteristics (see Sections 3.5 and 3.6).

We denote by y'_t the number of different types arrived before time t . Notice that it is different from M_t that counts the types that are *alive* at time t . The process y' is the equivalent, in the allele process, of the process y defined in Section 3.3 and that counts the individuals born up to time t .

Proposition 3.9. *Under \mathbb{P}^* ,*

$$e^{-\eta t} y'_t \xrightarrow[t \rightarrow \infty]{} y'_\infty := \frac{bp}{\eta} \mathcal{E} \quad \text{a.s.}$$

Proof. According to [65, Thm 3.2], on Ext^c ,

$$\frac{y'_t}{y_t} \xrightarrow[t \rightarrow \infty]{} \int_0^\infty e^{-\eta u} \mu_m(du) = p \quad \text{a.s..}$$

Then, using (3.5) and (3.21), we have the result. \square

Let us say that the type of the progenitor is of *kind 0*. Then, recursively, for $k \geq 1$, we call a type of *the k-th kind* if it is carried by a mutant child of an individual carrying an allele of kind $k - 1$. Equivalently, they are types that have been affected by k more mutations than the ancestral type.

For $n \geq 0$, we denote by Z_n the number of types of the n -th kind.

Proposition 3.10. *The process $(Z_n, n \geq 0)$ is a supercritical Bienaymé-Galton-Watson process with values in $\mathbb{N} \cup \{+\infty\}$, starting at 1 and with offspring r.v. K satisfying,*

$$G_K(s) := \mathbb{E} [s^K] = \frac{1 - ps}{1 - p} - \frac{\phi_c(pb(1 - s))}{b(1 - p)}, \quad s \in [0, 1] \quad (3.22)$$

where $\phi_c : (0, \infty) \rightarrow (\eta_c, \infty)$ is the bijective inverse of ψ_c

$$\psi_c(\phi_c(\lambda)) = \lambda, \quad \lambda > 0.$$

In particular,

$$G_K(1^-) = \mathbb{P}(K < \infty) = 1 - \frac{\eta_c}{b(1 - p)}.$$

In order to prove this result, we need the following lemma.

Lemma 3.11. *We denote by $T_{0,c} := \int_0^{T_c} \Xi_c(s) ds$ the total length of the ancestral tree. Then, for $\lambda > 0$,*

$$\mathbb{E} [e^{-\lambda T_{0,c}}] = 1 - \frac{\phi_c(\lambda) - \lambda}{b(1 - p)}.$$

Proof. According to [48], $T_{0,c}$ is distributed as the hitting time of 0 of a spectrally positive Lévy process $(Y_t, t \geq 0)$ with Laplace exponent ψ_c and such that Y_0 is distributed as $\Lambda_c(\cdot)/(b(1 - p)) = \Lambda(\cdot)/b$. Thanks to [5, Thm VII.1], conditional on $Y_0 = y$, the hitting time T_0 of 0 by Y satisfies

$$\mathbb{E} [e^{-\lambda T_0} | Y_0 = y] = e^{-y\phi_c(\lambda)}.$$

Then, integrating the latter against $\frac{\Lambda_c(\cdot)}{b(1 - p)}$, we have

$$\begin{aligned} \mathbb{E} [e^{-\lambda T_{0,c}}] &= (b(1 - p))^{-1} \int_{(0, \infty]} \Lambda_c(dy) e^{-y\phi_c(\lambda)} = \frac{\psi_c(\phi_c(\lambda)) - \phi_c(\lambda) + b(1 - p)}{b(1 - p)} \\ &= 1 + \frac{\lambda - \phi_c(\lambda)}{b(1 - p)} \end{aligned}$$

which completes the proof. \square

Proof of Proposition 3.10. By Proposition 3.7, since $(M_t, t \geq 0)$ is a general branching process, $(Z_n, n \geq 0)$ is a BGW-process. We now only have to find the law of the offspring distribution K . In fact, conditional on $T_{0,c}$ (the total length of the ancestral tree), the number of children, in the tree of alleles, of the ancestral type follows a Poisson distribution with parameter $pbT_{0,c}$. Then, for $s \in [0, 1]$, $G_K(s) = \mathbb{E} [e^{-pb(1-s)T_{0,c}}]$ and using Lemma 3.11 with $\lambda = pb(1 - s)$, for $s < 1$,

$$G_K(s) = 1 + \frac{pb(1 - s) - \phi_c(pb(1 - s))}{b(1 - p)} = \frac{1 - ps}{1 - p} - \frac{\phi_c(pb(1 - s))}{b(1 - p)}$$

Moreover, using (3.11), $\mathbb{P}(K < \infty) = \mathbb{P}(T_{0,c} < \infty) = \mathbb{P}(\text{Ext}_c) = 1 - \eta_c/((1 - p)b)$.

It remains to prove that the BGW process is supercritical. Differentiating (3.22) at $s = 1$, we have

$$\mathbb{E}[K; K < \infty] = G'_K(1^-) = \frac{p}{1 - p}(\phi'_c(0) - 1) = \frac{p}{1 - p} \left(\frac{1}{\psi'_c(0)} - 1 \right)$$

Then, if $\psi'_c(0) = 1 - (1 - p)m = 0$, *i.e.* if Ξ_c is critical, $\mathbb{E}[K; K < \infty] = \mathbb{E}[K] = \infty$. In other cases, we have

$$\mathbb{E}[K; K < \infty] = \frac{p - p(1 - (1 - p)m)}{(1 - p)(1 - (1 - p)m)} = 1 + \frac{m - 1}{1 - (1 - p)m}.$$

Then, since $m > 1$, if Ξ_c is subcritical, then $(1 - p)m < 1$ and $\mathbb{E}[K] > 1$ while if Ξ_c is supercritical, $\mathbb{E}[K; K < \infty] < 1$ but $\mathbb{E}[K] = \infty$ because $\mathbb{P}(K = \infty) > 0$. \square

Remark 3.12. Notice that since p_∞ , the surviving probability of Z , is the unique fixed point less than 1 of G_K (see [3, p7]), we have by (3.22)

$$b(1 - pp_\infty) - \phi_c(pb(1 - p_\infty)) = b(1 - p)p_\infty$$

which implies that

$$\phi_c(pb(1 - p_\infty)) = b(1 - p_\infty).$$

Then, recalling that ϕ_c is the inverse mapping of ψ_c and using (3.10), we have $\psi(b(1 - p_\infty)) = 0$ and $p_\infty = 1 - \eta/b$.

Remark 3.13. In [7], Bertoin considers a subcritical or critical BGW process where each individual gives birth to a random number of clonal and mutant children distributed as the pair $(\xi^{(c)}, \xi^{(m)})$. In his Theorem 1, he computes the joint law of the total progeny of the BGW tree and of the total number of different alleles. He also gives the joint distribution of the total number of individuals carrying the ancestral type and of the number of types of the first kind. These results can be applied in our case with $(\xi^{(c)}, \xi^{(m)})$ such that

$$\mathbb{E} \left[s_1^{\xi^{(c)}} s_2^{\xi^{(m)}} \right] = \int_{(0, \infty)} \frac{\Lambda(dz)}{b} e^{(s_1 - 1)b(1 - p)z} e^{(s_2 - 1)bpz}.$$

For $n \geq 0$, let \mathcal{A}_n be the set of alleles of n -th kind. It is also the set of the individuals belonging to generation n in the tree of alleles defined above. For any type x , we denote by $\zeta'(x)$ the total length of the subtree describing individuals with type x , that is, the sum of the life lengths of all individuals carrying allele x .

Proposition 3.14. *If we denote by*

$$S_n := \sum_{x \in \mathcal{A}_n} \zeta'(x)$$

the total length of the subtrees of n -th kind, the process $(S_n, n \geq 0)$ is a Jirina process with branching mechanism

$$F(\lambda) := \frac{p}{1 - p} (\phi_c(\lambda) - \lambda).$$

Proof. A Jirina process [38] X is a time-homogeneous Markov chain with values in \mathbb{R}^+ , satisfying the branching property with respect to the initial condition, (*i.e.* if X^a and X^b are two independent copies of X respectively starting from a and b , then $X^a + X^b$ has the same distribution as X starting from $a + b$). Then, one can prove that there exists a function F such that

$$\mathbb{E} \left[e^{-\lambda X_n} | X_0 = z \right] = e^{-zF_n(\lambda)}, \quad n \geq 0$$

where $F_n := F \circ \dots \circ F$ is the n -th iterate of F . This function F is the *branching mechanism* of the Jirina process X .

First, $S_0 = T_{0,c}$ and using Proposition 3.7, conditional on $S_0 = z$,

$$S_1 \stackrel{(d)}{=} \sum_{s \leq z} \Delta_s$$

where $(\Delta_s, s \geq 0)$ is a Poisson point process with intensity $\text{Leb} \otimes \Theta$ and where Θ is a positive measure on $(0, \infty)$ with total mass bp and such that $\frac{\Theta(\cdot)}{bp}$ is the distribution of $T_{0,c}$. Hence, by the exponential formula for Poisson point processes, for $\lambda > 0$,

$$\begin{aligned} \mathbb{E} \left[e^{-\lambda S_1} | S_0 = x \right] &= \exp \left(-x \int_0^\infty (1 - e^{-\lambda y}) \Theta(dy) \right) = \exp \left(-x(bp - bp\mathbb{E}[e^{-\lambda T_{0,c}}]) \right) \\ &= \exp \left(-x \left(\frac{p}{1-p} (\phi_c(\lambda) - \lambda) \right) \right) \end{aligned}$$

using Lemma 3.11. Then, if we denote by $S_1(x)$ the value of S_1 conditional on $S_0 = x$, the process $(S_1(x), x \geq 0)$ is a subordinator (*i.e.* a Lévy process with non-decreasing paths) with Laplace exponent F .

Moreover, conditional on all total lengths of subtrees describing types in \mathcal{A}_n ,

$$S_{n+1} = \sum_{x \in \mathcal{A}_n} \sum_{s \leq \zeta'(x)} \Delta_s^x$$

where $(\Delta_s^x, s \geq 0)$ are i.i.d. Poisson point processes distributed as $(\Delta_s, s \geq 0)$. Then, we have

$$S_{n+1} = \sum_{s \leq S_n} \Delta_s^n$$

with $(\Delta_s^n, s \geq 0)$ a Poisson point processes distributed as $(\Delta_s, s \geq 0)$ and independent from S_n .

Hence, by induction on n , S_n is the value at S_0 of the composition (in Bochner's sense) of n i.i.d. subordinators with Laplace exponent F . Then, S is a Jirina process with branching mechanism F (for more precision, see Section 2.4 in [48]). \square

3.5 Ages of old families in (sub)critical case

In this paragraph, we are interested in ages of types when $t \rightarrow \infty$.

3.5.1 Critical case

In this section, we prove that the ages of old families are of order $t - \ln t / \eta$ as $t \rightarrow \infty$ when the clonal process is critical, that is, when $(1-p)m = 1$. We also assume that

$$\psi_c''(0) = (1-p) \int_{(0,\infty]} r^2 \Lambda(dr) < \infty,$$

that

$$\lim_{x \rightarrow \infty} x^2 \int_{[x,\infty)} \Lambda(dr) = 0 \tag{3.23}$$

and that

$$\lim_{x \rightarrow \infty} x^2 \int_{[x,\infty)} r \Lambda(dr) = 0. \tag{3.24}$$

Proposition 3.15. *Let $c_t := t - \ln t/\eta$ and $-\infty < \underline{a} < \bar{a} \leq \infty$. If $\nu_t(\underline{a}, \bar{a})$ is the number of types alive at time t and with ages between $\underline{a} + c_t$ and $\bar{a} + c_t$, then*

$$\mathbb{E}[\nu_t(\underline{a}, \bar{a})] \xrightarrow{t \rightarrow \infty} \frac{\eta}{b} \frac{m-1}{\psi'(\eta)\eta} \left(e^{-\eta \underline{a}} - e^{-\eta \bar{a}} \right)$$

Proof. By summing (3.13) over $i \geq 1$ and by integrating it on $(\underline{a} + c_t, \bar{a} + c_t)$, we have

$$\mathbb{E}[\nu_t(\underline{a}, \bar{a})] = \frac{p}{b(1-p)} \int_{\underline{a}+c_t}^{\bar{a}+c_t} W'(t-x) \frac{W'_c(x)}{W_c(x)} dx.$$

for t large enough. Indeed, by the definition of c_t , there exists t_0 such that $\underline{a} + c_t \geq 0$ and $\bar{a} + c_t \leq t$ for $t \geq t_0$ and the expected number of types of age t does not appear in the last display.

Notice that since $(1-p)m = 1$, $p/(1-p) = m-1$. Then, by a change of variables,

$$\mathbb{E}[\nu_t(\underline{a}, \bar{a})] = \frac{m-1}{b} \int_{\underline{a}}^{\bar{a}} W'(t-c_t-x) \frac{W'_c(x+c_t)}{W_c(x+c_t)} dx.$$

As stated above, $W'(t)$ is equivalent to $\frac{\eta}{\psi'(\eta)} e^{\eta t}$ as $t \rightarrow \infty$. We also need the asymptotic behavior of the tail distribution of T_c since $\mathbb{P}(T_c > t) = \frac{W'_c(t)}{b(1-p)W_c(t)}$. Notice that since we have for $\lambda > 0$

$$\int_0^\infty W_c(x) e^{-\lambda x} dx = \frac{1}{\psi_c(\lambda)} \underset{0}{\sim} \frac{2}{\psi_c''(0)\lambda^2}$$

because $\psi_c(0) = \psi_c'(0) = 0$ (remind that the clonal process is critical), by a Tauberian theorem [5, p.10], we have

$$W_c(u) \underset{\infty}{\sim} \frac{2}{\psi_c''(0)} u.$$

However, it is not possible to obtain an asymptotic behavior of W'_c with the help of a Tauberian theorem because W'_c is not necessarily monotone.

The two hypotheses (3.23) and (3.24) imply that those of [35, Thm. 6.6.11] hold and we have

$$\lim_{t \rightarrow \infty} t \mathbb{P}(T_c > t) = \frac{2 \int_0^\infty r \mu_c(dr)}{V}$$

where $V := \text{Var} [\xi_c((0, \infty))]$ is the variance of the number of clonal children of the ancestor. Then, using that, conditional on the lifespan of the ancestor is z , $\xi_c((0, \infty))$ is a Poisson r.v. with parameter $b(1-p)z$, we have $V = b(1-p)^2 \int_{(0, \infty)} r^2 \Lambda(dr)$. Moreover,

$$\int_0^\infty r \mu_c(dr) = (1-p) \int_{(0, \infty)} \frac{r^2}{2} \Lambda(dr)$$

using that $\mu_c = (1-p)\mu$ and (3.1) the definition of μ . Thus,

$$\mathbb{P}(T_c > t) \underset{t \rightarrow \infty}{\sim} \frac{1}{tb(1-p)} \tag{3.25}$$

and $W'_c(t)/W_c(t)$ is equivalent to $1/t$ as $t \rightarrow \infty$.

Hence, recalling that $c_t = t - \ln t/\eta$, as $t \rightarrow \infty$,

$$W'(t-c_t-x) \frac{W'_c(x+c_t)}{W_c(x+c_t)} \rightarrow \frac{\eta}{\psi'(\eta)} e^{-\eta x}$$

To complete the proof, we use the dominated convergence theorem. Using the previously mentioned convergence results, there exist $C_1, C_2 > 0$ such that

$$W'(x) \leq C_1 e^{\eta x} \quad \text{and} \quad \frac{W'_c(x)}{W(x)} \leq \frac{C_2}{x}, \quad x > 0.$$

Then, for all $t \geq t_0$ and $x \geq \underline{a}$, we have

$$W'(t - c_t - x) \frac{W'_c(x + c_t)}{W_c(x + c_t)} \leq C_1 e^{-\eta x} t \frac{C_2}{x + c_t} \leq C_1 C_2 e^{-\eta x} \frac{t}{\underline{a} + c_t} \leq C_3 e^{-\eta x}$$

because by assumption $x + c_t \geq \underline{a} + c_t > 0$ for all $t \geq t_0$ and $x \geq \underline{a}$.

Since the r.h.s. of the last displayed equation is integrable on $(\underline{a}, +\infty)$, the proof is complete. \square

We now obtain a more precise result about the behavior of $\nu_t(\underline{a}, \bar{a})$ using a time-dependent random characteristic as done by Taïb.

Proposition 3.16. *With the same assumptions as in Proposition 3.15, for $-\infty < \underline{a} < \bar{a} \leq \infty$, under \mathbb{P}^* , $\nu_t(\underline{a}, \bar{a})$ converges in distribution to a mixed Poisson r.v. with parameter*

$$\frac{m-1}{\psi'(\eta)\eta} \left(e^{-\eta \underline{a}} - e^{-\eta \bar{a}} \right) E$$

where E is an exponential r.v. with mean 1.

Proof. We follow the proof of Proposition 4.2 in [65]. For $-\infty < \underline{a} < \bar{a} \leq \infty$, let χ_t be the time-dependent characteristic defined by

$$\chi_t(u) = \mathbf{1}_{\{0 \leq u < T_c\}} \mathbf{1}_{[\underline{a}+c_t, \bar{a}+c_t]}(u), \quad u \in \mathbb{R}.$$

Then if $\sigma_1, \sigma_2, \dots$ are the birth times of types,

$$\nu_t(\underline{a}, \bar{a}) = \sum_{k=1}^{y'_t} \chi_{t,k}(t - \sigma_k)$$

where $\chi_{t,k}$ is the equivalent of χ_t for type k (see (3.17)). According to Theorem 7 in Appendix of [65], if (d) stands for the convergence in distribution,

$$\nu_t(\underline{a}, \bar{a}) \xrightarrow[t \rightarrow \infty]{(d)} Z$$

where Z is characterized by

$$\phi_Z(\theta) := \mathbb{E} \left[e^{i\theta Z} \right] = \mathbb{E} \left[e^{y'_\infty f_\eta(\theta)} \right], \quad \theta \in \mathbb{R}$$

if

$$f_\eta(\theta) := \lim_{t \rightarrow \infty} \eta e^{\eta t} \int_0^\infty \left(\mathbb{E} \left[e^{i\theta \chi_t(u)} \right] - 1 \right) e^{-\eta u} du, \quad \theta \in \mathbb{R}. \quad (3.26)$$

The hypotheses of Theorem 7 hold in our case thanks to application C in [36, p.63]. Moreover, for $u \geq 0$, we have

$$\mathbb{E} \left[e^{i\theta \chi_t(u)} \right] - 1 = \left(e^{i\theta} - 1 \right) \mathbb{P}(T_c > u) \mathbf{1}_{[\underline{a}+c_t, \bar{a}+c_t]}(u).$$

Then,

$$\begin{aligned} f_\eta(\theta) &= (e^{i\theta} - 1) \lim_{t \rightarrow \infty} \eta e^{\eta t} \int_{\underline{a}+c_t}^{\bar{a}+c_t} e^{-\eta u} \mathbb{P}(T_c > u) du \\ &= (e^{i\theta} - 1) \lim_{t \rightarrow \infty} \eta e^{\eta(t-c_t)} \left[\int_{\underline{a}}^{\infty} e^{-\eta u} \mathbb{P}(T_c > u + c_t) du \right. \\ &\quad \left. - \int_{\bar{a}}^{\infty} e^{-\eta u} \mathbb{P}(T_c > u + c_t) du \right] \quad (3.27) \end{aligned}$$

by a change of variables. We need the asymptotic behavior of $\mathbb{P}(T_c > u)$ as $u \rightarrow \infty$. According to (3.9), we have

$$\mathbb{P}(T_c > u) = \frac{W'_c(u)}{b(1-p)W_c(u)}.$$

Hence, using (3.25), $\mathbb{P}(T_c > u + c_t)$ is equivalent to $(b(1-p)c_t)^{-1}$ as $t \rightarrow \infty$ and recalling that $c_t = t - \ln t/\eta$ and using the dominated convergence theorem,

$$\begin{aligned} \lim_{t \rightarrow \infty} e^{\eta(t-c_t)} \int_{\underline{a}}^{\infty} e^{-\eta u} \mathbb{P}(T_c > u + c_t) du &= \lim_{t \rightarrow \infty} t \int_{\underline{a}}^{\infty} e^{-\eta u} \mathbb{P}(T_c > u + c_t) du \\ &= \frac{e^{-\eta \underline{a}}}{\eta b(1-p)}. \end{aligned}$$

Coming back to (3.27), we have

$$f_\eta(\theta) = \frac{(e^{i\theta} - 1)(e^{-\eta \underline{a}} - e^{-\eta \bar{a}})}{b(1-p)}$$

and thanks to Proposition 3.9,

$$\phi_Z(\theta) = \mathbb{E} \left[e^{y'_\infty f_\eta(\theta)} \right] = \mathbb{E} \left[\exp \left(\frac{p}{(1-p)\psi'(\eta)\eta} (e^{i\theta} - 1) (e^{-\eta \underline{a}} - e^{-\eta \bar{a}}) E \right) \right]$$

where E is an exponential r.v. with mean 1. The proof is complete since $p/(1-p) = m-1$ \square

Remark 3.17. Our definition of $f_\eta(\theta)$ in (3.26) is different from the definition given in the appendix of [65]. In fact, there is a missing α in the definition of $f_\alpha(\theta)$ in the statement of Theorem 7 and in the paper [37] where it is proved. The first display at page 55 in [37] contradicts Lemma 1 in the same article. Moreover, choosing $\chi_t(u) = e^{-\eta u} \chi(u)$ as a time-dependent characteristic (where χ is a "typical" random characteristic) in Theorem 7 leads to an inconsistency with convergence results about CMJ-processes counted with non time-dependent characteristics.

Let $\mathcal{M}(\mathbb{R})$ be the set of non-negative σ -finite measures on \mathbb{R} and finite on \mathbb{R}^+ , equipped with the *left-vague topology* induced by the maps $\nu \mapsto \int_{\mathbb{R}} f(x) \nu(dx)$ for all bounded continuous functions f such that there exists $x_0 \in \mathbb{R}$ satisfying $\forall x \leq x_0, f(x) = 0$.

Theorem 3.18. *We suppose that $(1-p)m = 1$ and that $\psi''_c(0)$ is finite. Let ν_t be the point process defined by*

$$\nu_t(dx) := \sum_{k \geq 1} \delta_{A_t^k - c_t}(dx)$$

where $A_t^1 \geq A_t^2 \geq \dots$ is the decreasing sequence of ages of alive families at t . Then, as $t \rightarrow \infty$, ν_t converges in \mathbb{P}^* -distribution in $\mathcal{M}(\mathbb{R})$ equipped with the left-vague topology to a mixed Poisson point process with intensity measure

$$\frac{m-1}{\psi'(\eta)} E e^{-\eta x} dx \quad (3.28)$$

where E has an exponential distribution with parameter 1.

Proof. Let f be a continuous bounded function such that there exists x_0 such that $f = 0$ on $(-\infty, x_0]$ and let \mathcal{P} be a mixed Poisson point measure with intensity measure defined by (3.28). We have to show that

$$\int_{\mathbb{R}} f(x) \nu_t(dx) \xrightarrow[t \rightarrow \infty]{(d)} \int_{\mathbb{R}} f(x) \mathcal{P}(dx)$$

or for all continuous and bounded functions F on \mathbb{R}

$$\mathbb{E}^* \left[F \left(\int_{\mathbb{R}} f(x) \nu_t(dx) \right) \right] \xrightarrow[t \rightarrow \infty]{} \mathbb{E} \left[F \left(\int_{\mathbb{R}} f(x) \mathcal{P}(dx) \right) \right] \quad (3.29)$$

First, as explained in [37] or [65], the proof of Proposition 3.16 can be modified by changing the characteristic χ_t to obtain that $\nu_t \left(\bigcup_{j=1}^J (g_j, d_j] \right)$ converges in \mathbb{P}^* -distribution to a mixed Poisson r.v. with parameter

$$\frac{m-1}{\psi'(\eta)\eta} \sum_{j=1}^J (e^{-\eta g_j} - e^{-\eta d_j}) E$$

where $(g_1, d_1], \dots, (g_J, d_J]$ are disjoint intervals of $(-\infty, +\infty]$. That implies that (3.29) holds for all piecewise constant functions \tilde{f} with compact support.

We now use arguments of proof of [12, Cor 4.4]. We fix $\varepsilon > 0$. According to the previous proposition,

$$\mathbb{P}^*(\nu_t([x_0, +\infty)) \geq N_0) \xrightarrow[t \rightarrow \infty]{} \mathbb{P}(G_0 \geq N_0) \leq \frac{\mathbb{E}[G_0]}{N_0}.$$

where G_0 is a mixed Poisson r.v. with parameter $(m-1)e^{-\eta x_0}/(\psi'(\eta)\eta)E$. Then, it is a geometric r.v. with success probability

$$\frac{\eta\psi'(\eta)}{\eta\psi'(\eta) + (m-1)e^{-\eta x_0}}$$

and in particular, $\mathbb{E}[G_0] < \infty$.

Then, there exist τ and N_0 large enough such that

$$\sup_{t \geq \tau} \mathbb{P}^*(\nu_t([x_0, +\infty)) \geq N_0) \leq \varepsilon \quad (3.30)$$

and

$$\mathbb{P}(\mathcal{P}([x_0, +\infty)) \geq N_0) \leq \varepsilon.$$

Moreover, since $\nu_t([M, +\infty))$ converges in distribution to a geometric r.v. G_1 such that

$$\mathbb{P}(G_1 = 0) = \frac{\eta\psi'(\eta)}{(m-1)e^{-\eta M} + \eta\psi'(\eta)} \xrightarrow[M \rightarrow \infty]{} 1$$

enlarging τ if necessary, and for M large enough, we have

$$\sup_{t \geq \tau} \mathbb{P}^*(\nu_t([M, +\infty)) \geq 1) \leq \varepsilon \quad (3.31)$$

and

$$\mathbb{P}(\mathcal{P}([M, +\infty)) \geq 1) \leq \varepsilon.$$

Inequalities (3.30) and (3.31) imply that for $t \geq \tau$,

$$\left| \mathbb{E}^* \left[F \left(\int_{\mathbb{R}} f d\nu_t \right) \right] - \mathbb{E}^* \left[F \left(\int_{[x_0, M]} f d\nu_t \right); \nu_t([x_0, +\infty)) \leq N_0 \right] \right| \leq 2\|F\|_{\infty} \varepsilon.$$

The same inequality holds replacing \mathbb{P}^* by \mathbb{P} and $d\nu_t$ by $d\mathcal{P}$.

Since F is continuous, there is $\delta > 0$ such that

$$\sup\{|F(x) - F(y)|; |x| \leq \|f\|_{\infty} N_0, |y| \leq \|f\|_{\infty} N_0, |x - y| \leq \delta\} \leq \varepsilon$$

By density, let \tilde{f} be a piecewise constant function on $[x_0, M]$ such that $\|f - \tilde{f}\|_{L^{\infty}(x_0, M)} \leq \delta/N_0$. Hence, if $t \geq \tau$,

$$\left| \mathbb{E}^* \left[F \left(\int_{[x_0, M]} f d\nu_t \right) - F \left(\int_{[x_0, M]} \tilde{f} d\nu_t \right); \nu_t([x_0, +\infty)) \leq N_0 \right] \right| \leq \varepsilon$$

and again the same inequality holds replacing \mathbb{P}^* by \mathbb{P} and $d\nu_t$ by $d\mathcal{P}$.

Finally, recalling that (3.29) holds for \tilde{f} and putting together the previous inequalities, we obtain that for t large enough

$$\left| \mathbb{E}^* \left[F \left(\int_{\mathbb{R}} f(x) \nu_t(dx) \right) \right] - \mathbb{E} \left[F \left(\int_{\mathbb{R}} f(x) \mathcal{P}(dx) \right) \right] \right| \leq (6\|F\|_{\infty} + 3)\varepsilon.$$

This ends the proof. \square

3.5.2 Subcritical case

In this paragraph, we consider the case where the clonal process is subcritical, that is, $m(1-p) < 1$. All the proofs of the previous part can be done in this case. The only change is that we consider another correction c'_t because the asymptotic behavior of $\mathbb{P}(T_c > u)$ as $u \rightarrow \infty$ is not the same. More precisely, as done by Z. Taïb in [65], we have to assume that ψ_c (or ψ) is defined on an interval $(\lambda_0, +\infty)$ with $\lambda_0 < 0$ large enough to have $\alpha_c \in (\lambda_0, 0)$ such that $\psi_c(\alpha_c) = 0$ and such that

$$\psi'_c(\alpha_c) = 1 - \int_{(0, \infty]} r e^{-\alpha_c r} \Lambda_c(dr) > -\infty. \quad (3.32)$$

Notice that it is possible to find such a λ_0 because the function ψ_c is convex, $\psi_c(0) = 0$ and $\psi'_c(0) = 1 - (1-p)m > 0$ since the clonal process is subcritical. These conditions hold in the case of a Markovian birth and death process with birth rate b and death rate d .

We use [35, Thm 6.7.10], whose conditions hold thanks to (3.32). Then, $e^{-\alpha_c t} \mathbb{P}(T_c > t)$ converges as $t \rightarrow \infty$ to a limit which is positive if

$$\mathbb{E} \left[\int_0^{\infty} e^{-\alpha_c t} \xi_c(dt) \log\{\xi_c((0, \infty))\} \right] < \infty.$$

Like a previous proof in the critical case in Subsection 3.5.1, by conditioning by the lifespan of the ancestor and by using classical properties about Poisson point processes, we have

$$\mathbb{E} \left[\int_0^\infty e^{-\alpha_c t} \xi_c(dt) \log \{ \xi_c((0, \infty)) \} \right] \leq \int_{(0, \infty)} \Lambda(dz) \frac{1 - e^{-\alpha_c z}}{\alpha_c} \left((1-p) + b(1-p)^2 z \right)$$

which is finite thanks to (3.32). Finally, there exists $K_c > 0$ such that

$$\mathbb{P}(T_c > t) \underset{t \rightarrow \infty}{\sim} K_c e^{\alpha_c t}. \quad (3.33)$$

We now compute K_c . First, for $\lambda > 0$,

$$\int_0^\infty W_c(u) e^{-\lambda u} du = \frac{1}{\psi_c(\lambda)} \underset{0}{\sim} \frac{1}{\psi'_c(0)\lambda}.$$

Then, using a Tauberian theorem, we deduce that

$$W_c(u) \underset{\infty}{\sim} \frac{1}{\psi'_c(0)} = \frac{1}{1 - (1-p)m}$$

and using (3.33),

$$W'_c(u) \underset{u \rightarrow \infty}{\sim} \frac{b(1-p)}{1 - (1-p)m} K_c e^{\alpha_c u}. \quad (3.34)$$

Moreover, for $\lambda > \alpha_c$, we have

$$\int_0^\infty W'_c(u) e^{-\lambda u} du = \frac{\lambda}{\psi_c(\lambda)}$$

with the convention $0/\psi_c(0) = 1/\psi'_c(0) < \infty$. Then,

$$\int_0^\infty (W'_c(u) e^{-\alpha_c u}) e^{-\lambda u} du = \frac{\lambda + \alpha_c}{\psi_c(\lambda + \alpha_c)} \underset{\lambda \rightarrow 0}{\sim} \frac{\alpha_c}{\lambda \psi'_c(\alpha_c)}.$$

But, using [9, Thm. 1.7.6] with (3.34), we get

$$\int_0^\infty (W'_c(u) e^{-\alpha_c u}) e^{-\lambda u} du \underset{\lambda \rightarrow 0}{\sim} \frac{b(1-p)K_c}{1 - (1-p)m} \frac{1}{\lambda}.$$

Hence, putting all pieces together,

$$\mathbb{P}(T_c > u) \underset{\infty}{\sim} K_c e^{\alpha_c u}$$

with

$$K_c = \frac{\alpha_c(1 - (1-p)m)}{\psi'_c(\alpha_c)b(1-p)}.$$

Then for $\underline{a} > -\infty$, following the proof of Proposition 3.16, we have to find c'_t such that

$$\lim_{t \rightarrow \infty} e^{\eta(t-c'_t)} \int_{\underline{a}}^\infty e^{-\eta u} \mathbb{P}(T_c > u + c'_t) du$$

exists in $(0, +\infty)$. Hence, as $t \rightarrow \infty$, c'_t has to satisfy $e^{\eta(t-c'_t)} e^{\alpha_c c'_t} \rightarrow 1$. Choosing

$$c'_t := \frac{\eta}{\eta - \alpha_c} t,$$

we have

$$\lim_{t \rightarrow \infty} e^{\eta(t-c'_t)} \int_{\underline{a}}^\infty e^{-\eta u} \mathbb{P}(T_c > u + c'_t) du = K_c \frac{e^{-(\eta - \alpha_c)\underline{a}}}{\eta - \alpha_c}.$$

We are now able to state the counterparts of Proposition 3.15, Proposition 3.16 and Theorem 3.18 in this case and whose proofs are the same.

Proposition 3.19. *We suppose that $(1-p)m < 1$. Let $c'_t := \frac{\eta}{\eta - \alpha_c}t$ and $-\infty < \underline{a} < \bar{a} \leq \infty$. If $\nu'_t(\underline{a}, \bar{a})$ is the number of types alive at time t and with ages between $\underline{a} + c'_t$ and $\bar{a} + c'_t$, then*

$$\mathbb{E}[\nu'_t(\underline{a}, \bar{a})] \xrightarrow{t \rightarrow \infty} \frac{\eta}{b} \frac{p\alpha_c(1 - (1-p)m)}{(1-p)\psi'(\eta)\psi'_c(\alpha_c)(\eta - \alpha_c)} \left(e^{-\eta\underline{a}} - e^{-\eta\bar{a}} \right).$$

Proposition 3.20. *If $(1-p)m < 1$ and for $-\infty < \underline{a} < \bar{a} \leq \infty$, under \mathbb{P}^* , $\nu'_t(\underline{a}, \bar{a})$ converges in distribution to a mixed Poisson r.v. with parameter*

$$\frac{p\alpha_c(1 - (1-p)m)}{(1-p)\psi'(\eta)\psi'_c(\alpha_c)(\eta - \alpha_c)} \left(e^{-(\eta - \alpha_c)\underline{a}} - e^{-(\eta - \alpha_c)\bar{a}} \right) E$$

where E is an exponential r.v. with parameter 1.

Theorem 3.21. *We keep the assumptions of Proposition 3.20. Let ν'_t be the point process defined by*

$$\nu'_t(dx) := \sum_{k \geq 1} \delta_{A_t^k - c'_t}(dx)$$

where $A_t^1 \geq A_t^2 \geq \dots$ is the decreasing sequence of ages of alive families at t . Then, as $t \rightarrow \infty$, ν'_t converges in \mathbb{P}^* -distribution in $\mathcal{M}(\mathbb{R})$ equipped with the left-vague topology to a mixed Poisson point process with intensity measure

$$\frac{p\alpha_c(1 - (1-p)m)}{(1-p)\psi'(\eta)\psi'_c(\alpha_c)} E e^{-(\eta - \alpha_c)x} dx.$$

Remark 3.22. In [12], the authors consider mutations that occur at a Poissonian rate θ on a supercritical splitting tree. Their Propositions 3.4 and 3.6 are consistent with the ones stated in this section. Indeed, if their clonal process is subcritical ($\eta < \theta$), under \mathbb{P}^* , as $t \rightarrow \infty$, the ages of the old families are of order $\frac{\eta}{\theta}t$ while in the critical case ($\eta = \theta$), they are of order $t - \frac{\log t}{\eta}$ like in our critical case.

3.6 Sizes of large families in the supercritical case

In this section, we suppose that the clonal process is supercritical that is, $(1-p)m > 1$. Then, as in (3.4),

$$\mathbb{P}(\Xi_c(t) \rightarrow 0) = 1 - \frac{\eta_c}{(1-p)b}$$

and on $\{\Xi_c(t) \rightarrow 0\}^c$, $e^{-\eta_c t} \Xi_c(t)$ a.s. converges as $t \rightarrow \infty$ to an exponential r.v. with parameter $\psi'_c(\eta_c)$. Therefore, for $A > 0$, we are interested in the behavior of \mathcal{N}_t the number of types carried by more than $Ae^{\eta_c t}$ extant individuals at time t .

Proposition 3.23. *If $(1-p)m > 1$,*

$$\mathbb{E}[\mathcal{N}_t] \xrightarrow{t \rightarrow \infty} \frac{\eta_c p}{b(1-p)} \int_0^\infty W'(x) \exp(-A\psi'_c(\eta_c)e^{\eta_c x}) dx + \frac{\eta_c}{b(1-p)} e^{-A\psi'_c(\eta_c)}.$$

Proof. According to (3.14), if we set $n_t := Ae^{\eta_c t}$

$$\begin{aligned} \mathbb{E}[\mathcal{N}_t] &= \frac{1}{b(1-p)} \sum_{i \geq n_t} \left\{ p \int_0^t W'(t-x) \left(1 - \frac{1}{W_c(x)}\right)^{i-1} \frac{W'_c(x)}{W_c^2(x)} dx \right. \\ &\quad \left. + \left(1 - \frac{1}{W_c(t)}\right)^{i-1} \frac{W'_c(t)}{W_c^2(t)} \right\} \\ &= \frac{1}{b(1-p)} \left\{ p \int_0^t W'(t-x) \left(1 - \frac{1}{W_c(x)}\right)^{\lfloor n_t \rfloor} \frac{W'_c(x)}{W_c(x)} dx \right. \\ &\quad \left. + \left(1 - \frac{1}{W_c(t)}\right)^{\lfloor n_t \rfloor} \frac{W'_c(t)}{W_c(t)} \right\} \end{aligned} \quad (3.35)$$

where $\lfloor \cdot \rfloor$ is the floor function. We first investigate the asymptotic behavior of the second term of the r.h.s. As previous computations, we have that

$$W_c(u) \approx \frac{e^{\eta_c u}}{\psi'_c(\eta_c)} \quad \text{and} \quad W'_c(u) \approx \frac{\eta_c e^{\eta_c u}}{\psi'_c(\eta_c)}.$$

Then, $W'_c(t)/W_c(t) \rightarrow \eta_c$ and

$$\left(1 - \frac{1}{W_c(t)}\right)^{\lfloor n_t \rfloor} = \exp \left\{ \lfloor Ae^{\eta_c t} \rfloor \ln \left(1 - W_c^{-1}(t)\right) \right\} \xrightarrow{t \rightarrow \infty} e^{-A\psi'_c(\eta_c)}.$$

We now compute the limit of the first term in the r.h.s of (3.35). By a change of variables, we have

$$\int_0^t W'(t-x) \left(1 - \frac{1}{W_c(x)}\right)^{\lfloor n_t \rfloor} \frac{W'_c(x)}{W_c(x)} dx = \int_0^t W'(x) \left(1 - \frac{1}{W_c(t-x)}\right)^{\lfloor n_t \rfloor} \frac{W'_c(t-x)}{W_c(t-x)} dx \quad (3.36)$$

and by similar computations as above,

$$\left(1 - \frac{1}{W_c(t-x)}\right)^{\lfloor n_t \rfloor} \frac{W'_c(t-x)}{W_c(t-x)} \mathbf{1}_{\{x < t\}} \xrightarrow{t \rightarrow \infty} \eta_c \exp \left\{ -A\psi'_c(\eta_c) e^{\eta_c x} \right\}$$

According to (3.9), for all $0 \leq x \leq t$, $\frac{W'_c(t-x)}{W_c(t-x)} \leq b(1-p)$. Moreover, since $W_c(u)e^{-\eta_c u}$ converges as $u \rightarrow \infty$ and since $\ln(1-z) \leq -z$ for $z \in [0, 1)$, there exists $A_1 > 0$ such that for all $t \geq 0$ and $x < t$,

$$\left(1 - \frac{1}{W_c(t-x)}\right)^{\lfloor n_t \rfloor} \leq \exp(-A_1 e^{\eta_c x})$$

Then, there exist A_1 and A_2 such that

$$W'(x) \frac{W'_c(t-x)}{W_c(t-x)} \left(1 - \frac{1}{W_c(t-x)}\right)^{\lfloor n_t \rfloor} \mathbf{1}_{\{x < t\}} \leq A_2 W'(x) \exp \left\{ -A_1 e^{\eta_c x} \right\}.$$

Since the r.h.s. on the last display is integrable on $(0, +\infty)$ (remind that $W'(x)$ behaves like $e^{\eta_c x}$), we have the result by using the dominated convergence theorem. \square

Remark 3.24. We would like to state similar results to those of Section 3.5 concerning the convergence in distribution of \mathcal{N}_t as $t \rightarrow \infty$ and the convergence of an associated point measure. As in [65, Thm 4.6], \mathcal{N}_t can be counted by the time-dependent characteristic

$$\chi'_t(u) := \mathbf{1}_{\{\Xi_c(u) > Ae^{\eta t}\}}$$

and it is possible to prove the convergence of

$$\lim_{t \rightarrow \infty} e^{\eta t} \int_0^\infty \left(\mathbb{E} \left[e^{i\theta \chi'_t(u)} \right] - 1 \right) e^{-\eta u} du, \quad \theta \in \mathbb{R}$$

but we have doubts about the validity of the technical assumptions of the previously cited convergence theorem and which are verified neither in [65] nor in [37].

Remark 3.25. Proposition 3.23 is consistent with [12, Prop 3.3] where Champagnat and Lambert consider Poissonian mutations occurring during life.

They also obtain results about the sizes of the largest families when the clonal process is subcritical or critical. In our case, it seems possible to get similar results because we also know the asymptotic behaviors of W_c and W'_c . However, we cannot directly use dominated convergence theorem as in (3.36) because the limit is not integrable on $(0, +\infty)$. With more accurate estimates, it would seem that the sizes of oldest families are of order $(\eta t^2 - t \log t)/(\psi''(0)/2)$ in the critical case and of order $(-\eta/\log((1-p)m))t$ in the subcritical case if we suppose that the hypothesis about the existence of α_c , defined in Section 3.5.2, holds.

3.7 Yule's problem

We conclude this chapter by making some remarks about Yule's problem [72], mentioned in [30, p. 105]. In taxonomy, *i.e.* in biological classification of living and fossil organisms, the lowest level in the hierarchy is the species and the level immediately superior is the genus.

In the original problem of Yule, the number of species in a genus behaves as a Yule process (or pure birth Markovian process, see Section 3.3.3) and new genera appear at a rate proportional to the number of present genera, that is, if M_t denotes this number at time t , $(M_t, t \geq 0)$ is another Yule process.

To be consistent with the previous models we have considered, genera will be named by families and species will be named individuals. We have already treated two cases where families behave like splitting trees. In Chapter 2, new families came at a Poissonian rate θ and in the beginning of this chapter, mutant families appear with a rate proportional to the number of extant individuals. In the first case, according to Remark 2.14, the number of families M_t at time t follows a Poisson r.v. and grows linearly as $t \rightarrow \infty$ while in the second case, according to Propositions 3.3 and 3.5, M_t and M_t^i behave like $e^{\eta t}$ as $t \rightarrow \infty$.

Then, one can wonder what are their rates of growth in the intermediate case, that is, when new families come with a rate proportional to the number of extant families as in Yule's problem.

More precisely, we suppose that all families behave in an i.i.d. manner as homogeneous, binary and supercritical CMJ-process with lifespan measure Λ . Moreover, during its presence, each family gives birth to new families at rate \tilde{b} . Then, if we again denote by M_t the number of extant families at time t , the process $M := (M_t, t \geq 0)$ is a splitting tree with lifespan measure $\tilde{\Lambda}$ such that $\frac{\tilde{\Lambda}(\cdot)}{\tilde{b}}$ is the probability law of T the time of extinction of a family. We first investigate some of the characteristics of this process and then we

will state asymptotic results about M_t and M_t^i the number of families represented by i individuals at time t .

Proposition 3.26. *For $\lambda > 0$,*

$$\tilde{\psi}(\lambda) := \lambda - \int_{(0, \infty]} (1 - e^{-\lambda r}) \tilde{\Lambda}(dr) = \lambda \left(1 - \frac{\tilde{b}}{b} \int_0^\infty e^{-\lambda x} \frac{W'(x)}{W(x)} dx \right).$$

Moreover, M is supercritical and its associated Malthusian parameter $\tilde{\eta}$ is greater than η iff $J > b/\tilde{b}$ where J is the integral defined in Proposition 3.3.

Proof. For $\lambda > 0$, thanks to Fubini's theorem, we have

$$\tilde{\psi}(\lambda) = \lambda - \lambda \tilde{b} \mathbb{E} \left[\frac{1 - e^{-\lambda T}}{\lambda} \right] = \lambda \left(1 - \tilde{b} \int_0^\infty e^{-\lambda x} \mathbb{P}(T \geq x) dx \right)$$

and we obtain the desired result by using (3.9).

The fact that M is supercritical is obvious because T is infinite with a positive probability and then with a positive probability, a family gives birth to an infinite number of daughter families. The relation between $\tilde{\eta}$ and η is clear since $\tilde{\eta}$ is the greatest root of $\tilde{\psi}$ and since $\tilde{\psi}(\eta)$ is negative iff $\left(1 - \frac{\tilde{b}}{b} \int_0^\infty e^{-\eta x} \frac{W'(x)}{W(x)} dx \right)$ is negative. \square

Notice that the last condition in the last proposition holds if $\tilde{b} > b$. Indeed, since T stochastically dominates ζ the life span of the ancestral individual, we have

$$J = \int_0^\infty e^{-\eta x} \frac{W'(x)}{W(x)} dx = \int_0^\infty e^{-\eta x} \mathbb{P}(T \geq x) b dx \geq \int_0^\infty e^{-\eta x} \mathbb{P}(\zeta \geq x) b dx = 1$$

by the definition of η . In fact, if M has a longer lifespan distribution and a greater rate of birth than Ξ , it seems evident that it grows faster.

An immediate consequence of the last proposition yields the asymptotic behavior of M as in (3.5).

Corollary 3.27. *Under the survival event of M ,*

$$e^{-\tilde{\eta}t} M_t \xrightarrow[t \rightarrow \infty]{} \tilde{\mathcal{E}} \quad \text{a.s.}$$

where $\tilde{\mathcal{E}}$ follows an exponential distribution with parameter

$$\tilde{\psi}(\tilde{\eta}) = \tilde{\eta} \left(1 - \frac{\tilde{b}}{b} \int_0^\infty x e^{-\tilde{\eta}x} \frac{W'(x)}{W(x)} dx \right).$$

In the following proposition, we give the asymptotic behavior of the frequency spectrum $(M_t^i, i \geq 1)$ as $t \rightarrow \infty$.

Proposition 3.28. *Under the survival event of M , for $i \geq 1$,*

$$e^{-\tilde{\eta}t} M_t^i \xrightarrow[t \rightarrow \infty]{} \tilde{\mathcal{E}} \cdot \frac{\tilde{b}}{b} \int_0^\infty e^{-\tilde{\eta}x} \left(1 - \frac{1}{W(x)} \right)^{i-1} \frac{W'(x)}{W(x)} dx \quad \text{a.s.}$$

Proof. First, for $t \geq 0$ and $i \geq 1$, we have

$$M_t^i = \sum_{j=1}^{\tilde{y}_t} \mathbf{1}_{\{\Xi_j(t-\tilde{\sigma}_j)=i\}}$$

where \tilde{y}_t is the number of families arrived before t , where $\tilde{\sigma}_j$ is the date of creation of the family Ξ_j . Hence, M_t^i can be viewed as M_t counted via the random characteristic

$$\tilde{\chi}(u) = \mathbf{1}_{\{\Xi(u)=i\}}$$

and according to the previously used [65, Thm 5],

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{M_t^i}{\tilde{y}_t} &= \lim_{t \rightarrow \infty} \frac{\sum_j \tilde{\chi}_j(t - \tilde{\sigma}_j)}{\tilde{y}_t} = \tilde{\eta} \int_0^\infty e^{-\tilde{\eta}u} \mathbb{P}(\Xi(u) = i) du \\ &= \tilde{\eta} \int_0^\infty e^{-\tilde{\eta}u} \left(1 - \frac{1}{W(u)}\right)^{i-1} \frac{W'(u)}{bW(u)} du \end{aligned} \quad (3.37)$$

using (3.8). The two conditions (3.18) and (3.19) are trivially satisfied because for $u \geq 0$, $\tilde{\chi}(u) \leq 1$. Moreover, as in (3.21),

$$\frac{M_t}{\tilde{y}_t} \xrightarrow[t \rightarrow \infty]{} \frac{\tilde{\eta}}{\tilde{b}} \quad \text{a.s.}$$

(we retrieve this result by summing (3.37) on $i \geq 1$ and by using the definition of $\tilde{\eta}$). The proof is complete by putting together the last displayed equation, equation (3.37) and Corollary 3.27. \square

Chapitre 4

Processus de Lévy conditionnés à rester positifs via leurs processus des hauteurs

Ce chapitre est l'objet de l'article *Lévy process conditioned by its height process* [61] en cours de révision pour les *Annales de l'Institut Henri Poincaré*.

Abstract: *In the present work, we consider spectrally positive Lévy processes $(X_t, t \geq 0)$ not drifting to $+\infty$ and we are interested in conditioning these processes to reach arbitrarily large heights (in the sense of the height process associated with X) before hitting 0.*

This way we obtain a new conditioning of Lévy processes to stay positive. The (honest) law of this conditioned process is defined as a Doob h -transform via a martingale. For Lévy processes with infinite variation paths, this martingale is $(\int \tilde{\rho}_t(dz)e^{\alpha z} + I_t) \mathbf{1}_{\{t \leq T_0\}}$ for some α and where $(I_t, t \geq 0)$ is the past infimum process of X , where $(\tilde{\rho}_t, t \geq 0)$ is the so-called exploration process defined in [19] and where T_0 is the hitting time of 0 for X .

When the process X is a compensated compound Poisson process, the previous martingale is defined through the jumps of the future infimum process of X . The computations are easier in this case because X can be viewed as the contour process of a (sub)critical splitting tree. We can also give an alternative characterization of our conditioned process in the vein of spine decompositions.

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4.1 Introduction

In this paper, we consider Lévy processes $(X_t, t \geq 0)$ with no negative jumps (or spectrally positive), not drifting to $+\infty$ and conditioned to reach arbitrarily large heights (in the sense of the height process H defined in [19]) before hitting 0. Let \mathbb{P}_x be the law of X conditional on $X_0 = x$ and $(\mathcal{F}_t, t \geq 0)$ be its natural filtration.

Many papers deal with conditioning Lévy processes in the literature. In seminal works by L. Chaumont [13, 14] and then in [15], for general Lévy processes, L. Chaumont and R. Doney construct a family of measures $\mathbb{P}_x^\uparrow, x > 0$ of Lévy processes starting from x and conditioned to stay positive defined via a h -transform and it can be obtained as the limit

$$\mathbb{P}_x^\uparrow(\Theta, t < \zeta) = \lim_{\varepsilon \rightarrow 0} \mathbb{P}_x(\Theta, t < \mathbf{e}/\varepsilon | X_s > 0, 0 \leq s \leq \mathbf{e}/\varepsilon)$$

for $t \geq 0$, $\Theta \in \mathcal{F}_t$ and for \mathbf{e} an exponential r.v. with parameter 1 independent from the process X and where ζ is the killing time of X . In the spectrally positive case, when $\mathbb{E}[X_1] < 0$, \mathbb{P}_x^\uparrow is a sub-probability while, if $\mathbb{E}[X_1] = 0$, it is a probability. In [31], K. Hirano considers Lévy processes drifting to $-\infty$ conditioned to stay positive. More precisely, under exponential moment assumption, he is interested in two types of conditioning events: either the process X is conditioned to reach $(-\infty, 0]$ after time s or to reach level $s > 0$ before $(-\infty, 0]$. Then, at the limit $s \rightarrow \infty$, in both cases, he defines two different conditioned Lévy processes which can be described via h -transforms. In [5, ch. VII], J. Bertoin considers spectrally negative Lévy processes, *i.e.* with no positive jumps, and also constructs a family of conditioned processes to stay positive via the scale function associated with X .

Here, we restrict ourselves to study spectrally positive Lévy processes and consider a new way to obtain a Lévy process conditioned to stay positive without additional assumptions and contrary to [15], the law of the conditioned process is honest. The process X is conditioned to reach arbitrarily large heights before $T_0 := \inf\{t \geq 0; X_t = 0\}$. The term *height* should not be confused with the *level* used in the previously mentioned conditioning of Hirano. It has to be understood in the sense of the height process H associated with X and defined below. More precisely, for $t \geq 0, \Theta \in \mathcal{F}_t$, we are interested in the limit

$$\lim_{a \rightarrow \infty} \mathbb{P}_x \left(\Theta, t < T_0 \mid \sup_{0 \leq s \leq T_0} H_s \geq a \right). \quad (4.1)$$

In the following, we will consider three different cases for the Lévy process X : a Lévy process with finite variation and infinite Lévy measure, a Lévy process with finite variation and finite Lévy measure and finally a Lévy process with infinite variation.

In the first case, as it is stated in Theorem 4.3, the conditioning in (4.1) is trivial because $\mathbb{P}_x(\sup_{0 \leq t \leq T_0} H_t \geq a) = 1$ for all positive a .

In the second case, X is simply a compensated compound Poisson process whose Laplace exponent can be written as

$$\psi(\lambda) = \lambda - \int_{(0, \infty)} (1 - e^{-\lambda r}) \Lambda(dr)$$

where Λ is a finite measure on $(0, \infty)$ such that $m := \int_{(0, \infty)} r \Lambda(dr) \leq 1$ (without loss of generality, we suppose that the drift is -1). Thus, X is either recurrent or drifts to $-\infty$ and its hitting time T_0 of 0 is finite a.s. In this finite variation case, the height H_t at time t is the (finite) number of records of the future infimum, that is, the number of times s such that

$$X_{s-} < \inf_{[s, t]} X.$$

The process X is then conditioned to reach height a before T_0 . In the limit $a \rightarrow \infty$ in (4.1), one obtains a h -transform defined via a martingale which depends on the jumps of the future infimum. In the particular case $m = 1$, this martingale is $X_{\cdot \wedge T_0}$ and we recover the same h -transform as the one obtained in [15]. The key result used in our proof is due to A. Lambert [48]. Indeed, the process X can be seen as a contour process of a splitting tree [28]. These random trees are genealogical trees where each individual lives independently of other individuals, gives birth at rate $\Lambda(\mathbb{R}^+)$ to individuals whose lifetimes are distributed as $\Lambda(\cdot)/\Lambda(\mathbb{R}^+)$. Then, to consider X conditioned to reach height n before T_0 is equivalent to look at a splitting tree conditional on having alive descendance at generation n .

Notice that we only consider the case when the drift α of X equals 1. However, the case $\alpha \neq 1$ can be treated in the same way because X is still the contour process of a splitting tree but visited at speed α .

We also obtain a more precise result about conditional subcritical and critical splitting trees. For $n \in \mathbb{N}$, set \mathbb{P}^n the law of a splitting tree conditional on $\{\mathcal{Z}_n \neq 0\}$ where \mathcal{Z}_n denotes the number of extant individuals in the splitting tree belonging to generation n . In fact, $(\mathcal{Z}_n, n \geq 0)$ is a Galton-Watson process. We are interested in the law of the tree under \mathbb{P}^n as $n \rightarrow \infty$. We obtain that under a $x \log x$ -condition on the measure Λ , the limiting tree has a unique infinite spine where individuals have the size-biased lifelength distribution $m^{-1}z\Lambda(dz)$ and typical finite subtrees are grafted on this spine.

The spine decomposition with a size-biased spine that we obtain is similar to the construction of size-biased splitting trees marked with a uniformly chosen point in [26] where all individuals on the line of descent between root and this marked individual have size-biased lifelengths. It is also analogous to the construction of size-biased Galton-Watson trees in Lyons et al. [53]. These trees arise by conditioning subcritical or critical GW-trees on non-extinction. See also [2, 27, 46]. In [18], T. Duquesne studied the so-called sin-trees that were introduced by D. Aldous in [2]. These trees are infinite trees with a unique infinite line of descent. He also considers the analogous problem for continuous trees and continuous state branching processes as made by other authors in [45, 46, 51].

We finally consider the case where X has paths with infinite variation. Its associated Laplace exponent is specified by the Lévy-Khintchine formula

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,\infty)} \Lambda(dr)(e^{-\lambda r} - 1 + \lambda r)$$

where $\alpha \geq 0$, $\int_{(0,\infty)} \Lambda(dr)(r \wedge r^2) < \infty$ and either $\beta > 0$ or $\int_{(0,1)} \Lambda(dr)r = \infty$. In order to compute the limit (4.1) in that case, we use the height process $(H_t, t \geq 0)$ defined in [19, 50] which is the analogue of the discrete-space height process in the finite variation case. We set $S_t := \sup_{[0,t]} X$. Then, since 0 is regular for itself for $S - X$, H is defined through local time. Indeed, for $t \geq 0$, H_t is the value at time t of the local time at level 0 of $S^{(t)} - X^{(t)}$ where $X^{(t)}$ is the time-reversed process of X at t

$$X_s^{(t)} := X_{t-s} - X_{(t-s)-} \quad s \in [0, t]$$

(with the convention $X_{0-} = X_0$) and $S_s^{(t)} := \sup_{0 \leq r \leq s} X_r^{(t)}$ is its past supremum.

Under the additional hypothesis

$$\int_{[1,+\infty)} \frac{d\lambda}{\psi(\lambda)} < \infty, \tag{4.2}$$

which implies that X has paths with infinite variation and that the height process H is locally bounded, we obtain a similar result to the finite variation case: the limit in (4.1) allows us to define a family of (honest) probabilities $(\mathbb{P}_x^\uparrow, x > 0)$ of Lévy processes conditioned to stay positive via a h -transform and the martingale

$$\int_0^{H_t} \rho_t(dz) e^{\alpha z} \mathbf{1}_{\{t \leq T_0\}}$$

where $\rho_t(\cdot)$ is a random positive measure on \mathbb{R}^+ which is a slight modification of the *exploration process* defined in [19, 50] and $\alpha = \psi'(0) \geq 0$ (since X does not drift to $+\infty$).

Again, in the recurrent case (i.e. if $\alpha = 0$), we observe that the previous quantity equals $X_{t \wedge T_0}$ and we recover the h -transform $h(x) = x$ of [15] in the spectrally positive case. Indeed, for general Lévy processes, the authors consider the law of the Lévy process conditioned to stay positive which is defined via the h -transform

$$h(x) = \mathbb{E} \left[\int_{[0, \infty)} \mathbf{1}_{\{I_t \geq -x\}} dL_t \right]$$

where I is the past infimum process and L is a local time at 0 for $X - I$. In the particular spectrally positive case, $L = -I$ and h is the identity.

Under \mathbb{P}_x^\uparrow , the height process $(H_t, t \geq 0)$ can be compared to the left height process \overleftarrow{H} studied in [18]. In that paper, T. Duquesne gives a genealogical interpretation of a continuous-state branching process with immigration by defining two continuous contour processes \overleftarrow{H} and \overrightarrow{H} that code the left and right parts of the infinite line of descent. We construct two similar processes for conditioned splitting trees in Section 4.3.

The paper is organized as follows. In Section 4.2, we treat the finite variation case and investigate the limiting process after stating some properties about splitting trees. Section 4.3 is devoted to studying the conditioned splitting tree and Section 4.4 to considering Lévy processes with infinite variation.

4.2 Finite variation case

4.2.1 Definitions and statement of result

Let Λ be a positive measure on $(0, \infty)$ such that $\Lambda \neq 0$ and

$$\int_{(0, \infty)} (x \wedge 1) \Lambda(dx) < \infty$$

and let $(X_t, t \geq 0)$ be a spectrally positive Lévy process with Lévy measure Λ and such that

$$\mathbb{E}_0 \left[e^{-\lambda X_t} \right] = e^{t\psi(\lambda)}, \quad \lambda > 0$$

where \mathbb{P}_x is the law of X conditioned to $X_0 = x$ and

$$\psi(\lambda) = \lambda - \int_{(0, \infty)} (1 - e^{-\lambda r}) \Lambda(dr).$$

We denote by $\mathcal{F}_t := \sigma(X_s, 0 \leq s \leq t)$ the natural filtration of X . We will suppose that $m := \int_{(0, \infty)} r \Lambda(dr) \leq 1$ that is, X is recurrent ($m = 1$) or drifts to $-\infty$ ($m < 1$). Then the hitting time $T_0 := \inf\{t \geq 0; X_t = 0\}$ is finite almost surely. Observe that since X is spectrally positive, the first hitting time of $(-\infty, 0]$ is T_0 .

Definition 4.1. The height process H associated with X is defined by

$$H_t := \# \left\{ 0 \leq s \leq t; X_{s-} < \inf_{s \leq r \leq t} X_r \right\}.$$

We set

$$\{s_t^1 < \dots < s_t^{H_t}\} := \{0 \leq s \leq t < T_0; X_{s-} < \inf_{s \leq r \leq t} X_r\},$$

$$I_s^t := \inf_{s \leq r \leq t} X_r, \quad 0 \leq s \leq t$$

and we denote the records of $(I_s^t, s \leq t)$ by $\rho_i^t := \inf_{s_i \leq r \leq t} X_r - X_{s_i-}$ for $1 \leq i \leq H_t$ and $\rho_0^t = \inf_{0 \leq r \leq t} X_r$ (see Figure 4.1).

The assumption $\int_{(0,\infty)} (x \wedge 1) \Lambda(dx) < \infty$ implies that the paths of X have finite variation and then for all positive t , H_t is finite a.s. (Lemma 3.1 in [50]).

Remark 4.2. The process X can be seen as a LIFO (last in-first out) queue [50, 63]. Indeed, a jump of X at time t corresponds to the entrance in the system of a new customer who requires a service $\Delta X_t := X_t - X_{t-}$. This customer is served in priority at rate 1 until a new customer enters the system. Then, the ρ_i^t 's are the remaining service times of the H_t customers present in the system at time t .

The sequence $(\rho_i^t, i \leq H_t)$ can be seen as a random positive measure on non-negative integers which puts weight ρ_i^t on $\{i\}$. Its total mass is X_t and its support is $\{0, \dots, H_t\}$. We denote by S the set of measures on \mathbb{N} with compact support. For ν in S , set $\nu_i := \nu(\{i\}), i \geq 0$ and

$$H(\nu) := \max\{i \geq 0; \nu_i \neq 0\}.$$

Then, according to [63, p.200], the process $(\rho^{t \wedge T_0}, t \geq 0)$ is a S -valued Markov process. Its infinitesimal generator \mathcal{A} is defined by

$$\begin{aligned} \mathcal{A}(f)(\nu) = & \left[\int_{(0,\infty)} \left(f(\nu_0, \nu_1, \dots, \nu_{H(\nu)}, r, 0, \dots) - f(\nu_0, \nu_1, \dots, \nu_{H(\nu)}, 0, \dots) \right) \Lambda(dr) \right. \\ & \left. - \frac{\partial f}{\partial x_{H(\nu)}}(\nu_0, \dots, \nu_{H(\nu)}, 0, \dots) \right] \mathbf{1}_{\{\nu \neq 0\}}. \end{aligned} \quad (4.3)$$

In the following proposition, we condition the Lévy process X to reach arbitrarily large heights before T_0 .

Theorem 4.3. (i) Assume that $b := \Lambda(\mathbb{R}^+)$ is finite and

$$\left(m = 1 \text{ and } \int_{(0,\infty)} z^2 \Lambda(dz) < \infty \right) \text{ or } \left(m < 1 \text{ and } \int_{[1,\infty)} z \log(z) \Lambda(dz) < \infty \right).$$

Then, for $t \geq 0$ and $\Theta \in \mathcal{F}_t$,

$$\lim_{a \rightarrow \infty} \mathbb{P}_x \left(\Theta, t < T_0 \mid \sup_{s \leq T_0} H_s \geq a \right) = \frac{1}{x} \mathbb{E}_x [M_{t \wedge T_0} \mathbf{1}_\Theta]$$

where

$$M_t = \sum_{i=0}^{H_t} \rho_i^t m^{-i}.$$

In particular, if $m = 1$, then $M_t = X_t$. Moreover, the process $(M_{t \wedge T_0}, t \geq 0)$ is a (\mathcal{F}_t) -martingale under \mathbb{P}_x .

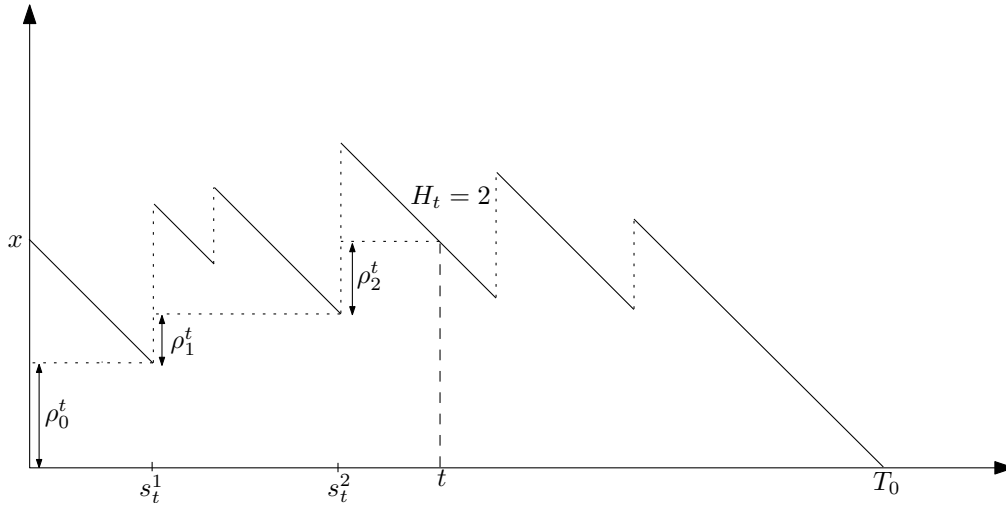


Figure 4.1: A trajectory of the process X started at x and killed when it reaches 0 and the remaining service times at time t ρ_i^t for $i \in \{0, \dots, H_t\}$.

(ii) If $b = \infty$, the conditioning with respect to $\{\sup_{s \in (0, T_0)} H_s \geq a\}$ is trivial in the sense that for all $a \geq 0$, $\mathbb{P}_x(\sup_{s \in (0, T_0)} H_s \geq a) = 1$.

Observe that if $b < \infty$, the process X is simply a compensated compound Poisson process whose jumps occur at rate b and are distributed as $\Lambda(\cdot)/b$.

The proof of this result will be made in Section 4.2.3. It uses the fact that X can be viewed as the contour process of a splitting tree visited at speed 1. The integrability hypotheses about Λ are made in order to use classical properties of (sub)critical BGW processes that appear in splitting trees.

Notice that the case where X is a Lévy process with Laplace exponent $\psi(\lambda) = \alpha\lambda - \int_{(0, \infty)} (1 - e^{-\lambda r}) \Lambda(dr)$ and $b < \infty$ can be treated in a same way if $m \leq \alpha$. Indeed, in that case, X is still the contour process of a splitting tree but it is visited at speed α . Theorem 4.3 is still valid but the martingale becomes

$$M_{t \wedge T_0} = \sum_{i=0}^{H_t} \rho_i^t \left(\frac{\alpha}{m}\right)^i \mathbf{1}_{\{t \leq T_0\}}.$$

Before the proof, we define the splitting trees and recall some of their properties.

4.2.2 Splitting Trees

Most of what follows is taken from [48]. We denote the set of finite sequences of positive integers by

$$\mathcal{U} = \bigcup_{n=0}^{\infty} (\mathbb{N}^*)^n$$

where $(\mathbb{N}^*)^0 = \{\emptyset\}$.

Definition 4.4. A discrete tree \mathcal{T} is a subset of \mathcal{U} such that

- (i) $\emptyset \in \mathcal{T}$ (root of the tree)
- (ii) if $uj \in \mathcal{T}$ for $j \in \mathbb{N}$, then $u \in \mathcal{T}$ (if an individual is in the tree, so is its mother)

(iii) $\forall u \in \mathcal{T}, \exists K_u \in \mathbb{N} \cup \{\infty\}, \forall j \in \{1, \dots, K_u\}, u_j \in \mathcal{T}$ (K_u is the offspring number of u).

If $u = (u_1, \dots, u_n) \in \mathcal{T}$, then its generation is $|u| := n$, its ancestor at generation i is denoted by $u|i$ and if $v = (v_1, \dots, v_m)$ we denote by uv the concatenation of u and v

$$uv := (u_1, \dots, u_n, v_1, \dots, v_m).$$

In chronological trees, each individual has a birth level α and a death level ω . Let p_1, p_2 be the two canonical projections of $\mathbb{U} := \mathcal{U} \times [0, +\infty)$ on \mathcal{U} and $[0, \infty)$. We will denote by \mathcal{T} the projection of $\mathbb{T} \subset \mathbb{U}$ on \mathcal{U}

$$\mathcal{T} := p_1(\mathbb{T}) = \{u : \exists \sigma \geq 0, (u, \sigma) \in \mathbb{T}\}.$$

Definition 4.5. A subset \mathbb{T} of \mathbb{U} is a chronological tree if

- (i) $\rho := (\emptyset, 0) \in \mathbb{T}$ (the root)
- (ii) \mathcal{T} is a discrete tree
- (iii) $\forall u \in \mathcal{T} \setminus \{\emptyset\}, \exists 0 \leq \alpha(u) < \omega(u) \leq \infty$ such that $(u, \sigma) \in \mathbb{T}$ if and only if $\alpha(u) < \sigma \leq \omega(u)$. $\alpha(u)$ (resp. $\omega(u)$) is the birth (resp. death) level of u
- (iv) if $ui \in \mathcal{T}$, then $\alpha(u) < \alpha(ui) < \omega(u)$ (an individual has only children during its life)
- (v) if $ui, uj \in \mathcal{T}$ then $i \neq j$ implies $\alpha(u_i) \neq \alpha(u_j)$ (no simultaneous births).

For $u \in \mathcal{T}$, we denote by $\zeta(u) := \omega(u) - \alpha(u)$ its *lifetime duration*. For two chronological trees \mathbb{T}, \mathbb{T}' and $x = (u, \sigma) \in \mathbb{T}$ such that $\sigma \neq \omega(u)$ (not a death point) and $\sigma \neq \alpha(ui)$ for any i (not a birth point), we denote by $G(\mathbb{T}', \mathbb{T}, x)$ the *graft* of \mathbb{T}' on \mathbb{T} at x

$$G(\mathbb{T}', \mathbb{T}, x) := \mathbb{T} \cup \{(uv, \sigma + \tau) : (v, \tau) \in \mathbb{T}'\}.$$

Recall that Λ is a σ -finite measure on $(0, \infty]$ such that $\int_{(0, \infty)} (r \wedge 1) \Lambda(dr) < \infty$. A *splitting tree* [26, 28] is a random chronological tree defined as follows. For $x \geq 0$, we denote by \mathbf{P}_x the law of a splitting tree starting from an ancestor individual \emptyset with lifetime $(0, x]$. We define recursively the family of probabilities $\mathbf{P} = (\mathbf{P}_x)_{x \geq 0}$. Let $(\alpha_i, \zeta_i)_{i \geq 1}$ be the atoms of a Poisson measure on $(0, x) \times (0, +\infty]$ with intensity measure $\text{Leb} \otimes \Lambda$ where Leb is the Lebesgue measure. Then \mathbf{P} is the unique family of probabilities on chronological trees \mathbb{T} such that

$$\mathbb{T} = \bigcup_{n \geq 1} G(\mathbb{T}_n, \emptyset \times (0, x), \alpha_n)$$

where, conditional on the Poisson measure, the (\mathbb{T}_n) are independent splitting trees and for $n \geq 1$, conditional on $\zeta_n = \zeta$, \mathbb{T}_n has law \mathbf{P}_ζ .

The measure Λ is called the *lifespan measure* of the splitting tree and when it has a finite mass b , there is an equivalent definition of a splitting tree:

- individuals behave independently from one another and have i.i.d. lifetime durations distributed as $\frac{\Lambda(\cdot)}{b}$,
- conditional on her birthdate α and her lifespan ζ , each individual reproduces according to a Poisson point process on $(\alpha, \alpha + \zeta)$ with intensity b ,
- births arrive singly.

We now display a branching process embedded in a splitting tree and which will be useful in the following. According to [48], when $b < \infty$, if for $n \in \mathbb{N}$, \mathcal{Z}_n is the number of alive individuals of generation n

$$\mathcal{Z}_n := \#\{v \in \mathcal{T} : |v| = n\}, \tag{4.4}$$

then under \mathbf{P} , $(\mathcal{Z}_n, n \geq 0)$ is a Bienaymé-Galton-Watson (BGW) process starting at 1 and with offspring distribution defined by

$$p_k := \int_{(0,\infty)} \frac{\Lambda(dz)}{b} \frac{(bz)^k}{k!} e^{-bz}, \quad k \geq 0. \quad (4.5)$$

Notice that under \mathbf{P}_x , $(\mathcal{Z}_n, n \geq 1)$ is still a BGW process with the same offspring distribution but starting at \mathcal{Z}_1 , distributed as a Poisson r.v. with parameter bx .

Because of the additional hypothesis $m < 1$ (resp. $m = 1$), the splitting trees that we consider are subcritical (resp. critical). Then, in both cases they have finite lengths a.s. and we can consider their associated JCCP (for jumping chronological contour process) $(Y_t, t \geq 0)$ as it is done in [48].

It is a càdlàg piecewise linear function with slope -1 which visits once each point of the tree. The visit of the tree begins at the death level of the ancestor. When the visit of an individual v of the tree begins, the value of the process is her death level $\omega(v)$. Then, it visits v backwards in time. If she has no child, her visit is interrupted after a time $\zeta(v)$; otherwise the visit stops when the birth level of her youngest child (call it w) is reached. Then, the contour process jumps from $\alpha(w)$ to $\omega(w)$ and starts the visit of w in the same way. When the visits of w and all her descendance will be completed, the visit of v can continue (at this point, the value of the JCCP is $\alpha(w)$) until another birth event occurs. When the visit of v is finished, the visit of her mother can resume (at level $\alpha(v)$). This procedure then goes on recursively until level 0 is encountered ($0 = \alpha(\emptyset) =$ birth level of the root) and after that the value of the process is 0 (see Figure 4.2). For a more formal definition of this process, read Section 3 in [48].

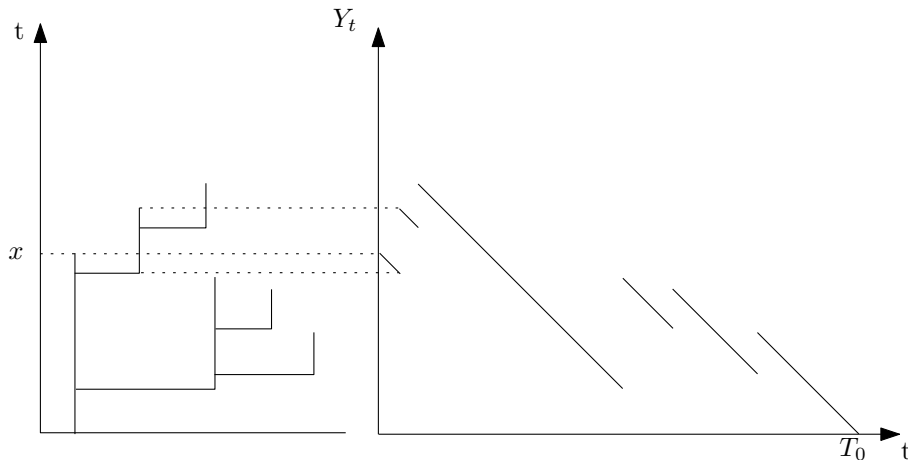


Figure 4.2: On the left panel, a splitting tree whose ancestor has lifespan duration x (vertical axis is time and horizontal axis shows filiation) and its associated Jumping Chronological Contour Process $(Y_t, t \geq 0)$ on the right panel.

Moreover, the splitting tree can be fully recovered from its JCCP and we will use this correspondence to prove Theorem 4.3. It enables us to link the genealogical height (or generation) in the chronological tree and the height process of the JCCP.

Proposition 4.6 (Lambert,[48]). *The process $(Y_t, t \geq 0)$ under \mathbf{P}_x has the law of the Lévy process $(X_t, 0 \leq t \leq T_0)$ under \mathbb{P}_x .*

Moreover, if, as in Definition 4.1, for $t \geq 0$, we consider

$$h(t) := \#\{0 \leq s \leq t; Y_{s-} < \inf_{s \leq r \leq t} Y_r\},$$

then $h(t)$ is exactly the genealogical height in \mathcal{T} of the individual visited at time t by the contour process.

4.2.3 Proof of Theorem 4.3

Thanks to Proposition 4.6, the process X is the JCCP of a splitting tree with lifespan measure Λ . For $a \geq 1$, let $\tau_a := \inf\{t \geq 0; H_t \geq a\}$. Then, $\{\sup_{s \leq T_0} H_s \geq a\} = \{\tau_a < T_0\}$. Furthermore, according to the second part of Proposition 4.6, the events $\{X$ reaches height a before $T_0\}$ and $\{\text{the splitting tree is alive at generation } a\}$ coincide.

We first prove the simpler point (ii) of Theorem 4.3 where $b = \infty$. According to [48], if for $n \geq 0$, Z_n denotes the sum of lifespans of individuals of generation n in the splitting tree

$$Z_n := \sum_{v \in \mathcal{T}, |v|=n} \zeta(v),$$

then under \mathbb{P}_x , the process $(Z_n, n \geq 0)$ is a Jirina process starting at x and with branching mechanism

$$F(\lambda) := \int_{(0, \infty)} (1 - e^{-\lambda r}) \Lambda(dr) = \lambda - \psi(\lambda),$$

that is, $Z := (Z_n, n \geq 0)$ is a time-homogeneous Markov chain with values in \mathbb{R}^+ , satisfying the branching property with respect to initial condition, and such that

$$\mathbb{E}_x \left[e^{-\lambda Z_n} \right] = e^{-x F_n(\lambda)}$$

where $F_n := F \circ \dots \circ F$ is the n -th iterate of F (for more details see the proof of Proposition 3.14). Hence, since H is the genealogical height in the splitting tree ,

$$\mathbb{P}_x(\tau_a < T_0) = \mathbb{P}_x(Z_a \neq 0) = 1 - \exp\left(-x \lim_{\lambda \rightarrow \infty} F_a(\lambda)\right) = 1$$

by monotone convergence and because the mass of Λ is infinite.

We now make the proof of Theorem 4.3(i) and suppose that b is finite.

$$\mathbb{P}_x \left(\Theta, t < T_0 \mid \sup_{s \leq T_0} H_s \geq a \right) = \frac{\mathbb{P}_x(\Theta, \tau_a \leq t < T_0) + \mathbb{P}_x(\Theta, t < \tau_a < T_0)}{\mathbb{P}_x(\tau_a < T_0)} \quad (4.6)$$

We will investigate the asymptotic behaviors of the three probabilities in the last display as $a \rightarrow \infty$.

As previously, H is the genealogical height in the splitting tree but in this case, we can use the process $(\mathcal{Z}_a, a \geq 1)$ defined by (4.4). As explained above, this process is a BGW process with offspring generating function defined by (4.5) and such that under \mathbb{P}_x , \mathcal{Z}_1 has a Poisson distribution with parameter bx . With an easy computation, one sees that the mean offspring number equals $m = \int_{(0, \infty)} r \Lambda(dr) \leq 1$. Hence the BGW-process is critical or subcritical. We have

$$\mathbb{P}_x(\tau_a < T_0) = \mathbb{P}_x(\mathcal{Z}_a \neq 0) = \sum_{k \geq 0} \mathbb{P}_x(\mathcal{Z}_a \neq 0 \mid \mathcal{Z}_1 = k) \mathbb{P}_x(\mathcal{Z}_1 = k)$$

and by the branching property,

$$\mathbb{P}_x(\mathcal{Z}_a \neq 0 \mid \mathcal{Z}_1 = k) = 1 - (1 - \mathbb{P}(\mathcal{Z}_{a-1} \neq 0))^k \underset{a \rightarrow \infty}{\sim} k \mathbb{P}(\mathcal{Z}_{a-1} \neq 0).$$

We first treat the subcritical case. According to Yaglom [71], if $(\mathcal{Z}_n, n \geq 0)$ is subcritical ($m < 1$) and if $\sum_{k \geq 1} p_k(k \log k) < \infty$, then there exists $c > 0$ such that

$$\lim_{n \rightarrow \infty} \frac{\mathbb{P}(\mathcal{Z}_n \neq 0)}{m^n} = c. \quad (4.7)$$

In the following lemma, we show that this log-condition holds with assumptions of Theorem 4.3.

Lemma 4.7. *If $\int_{[1, \infty)} z \log(z) \Lambda(dz) < \infty$, then $\sum_{k \geq 1} p_k(k \log k) < \infty$.*

Proof. According to (4.5),

$$\begin{aligned} \sum_{k \geq 2} p_k k \log k &= \sum_{k \geq 2} k \log k b^{-1} \int_{(0, \infty)} \Lambda(dz) \frac{(bz)^k}{k!} e^{-bz} \\ &= b^{-1} \int_{(0, \infty)} \Lambda(dz) e^{-bz} \sum_{k \geq 2} k \log k \frac{(bz)^k}{k!} \end{aligned}$$

by Fubini-Tonelli theorem. Since we have

$$\log k \leq \frac{k-z}{z} + \log z, \quad k \geq 2, z > 0,$$

$$\begin{aligned} \sum_{k \geq 2} k \log k \frac{(bz)^k}{k!} &\leq \sum_{k \geq 2} k \left(\frac{k}{z} - 1 + \log z \right) \frac{(bz)^k}{k!} \\ &\leq \sum_{k \geq 2} k \frac{z^{k-1} b^k}{(k-1)!} + z \log z \sum_{k \geq 2} \frac{z^{k-1} b^k}{(k-1)!} \\ &\leq b(z+1)e^{bz} + bz \log z e^{bz} \end{aligned}$$

and

$$\sum_{k \geq 2} p_k k \log k \leq \int_{(0, \infty)} \Lambda(dz) (z+1 + z \log z) < \infty.$$

□

Then if $\int_{[1, \infty)} r \log r \Lambda(dr) < \infty$, the log-condition of (4.7) is fulfilled and there exists a constant c such that

$$\lim_{a \rightarrow \infty} \mathbb{P}(\mathcal{Z}_{a-1} \neq 0 | \mathcal{Z}_1 = 1) / m^{a-1} = c.$$

Then,

$$\lim_{a \rightarrow \infty} \frac{\mathbb{P}(\mathcal{Z}_a \neq 0 | \mathcal{Z}_1 = k)}{m^{a-1}} = kc.$$

Moreover,

$$\frac{\mathbb{P}(\mathcal{Z}_a \neq 0 | \mathcal{Z}_1 = k)}{m^{a-1}} \leq \frac{k \mathbb{P}(\mathcal{Z}_a \neq 0 | \mathcal{Z}_1 = 1)}{m^{a-1}} \leq Ck$$

and

$$\sum_{k \geq 0} Ck \mathbb{P}_x(\mathcal{Z}_1 = k) = C \mathbb{E}_x[\mathcal{Z}_1] = Cbx < \infty$$

where C is some positive constant. Hence, using the dominated convergence theorem,

$$\lim_{a \rightarrow \infty} \frac{\mathbb{P}_x(\tau_a < T_0)}{m^{a-1}} = c \mathbb{E}_x[\mathcal{Z}_1] = cbx \quad (4.8)$$

Similarly, if $(\mathcal{Z}_n, n \geq 0)$ is critical ($m = 1$), since the variance of its reproduction law

$$\sigma^2 = \sum_{k \geq 1} k^2 p_k - m^2 = b \int_{(0, \infty)} z^2 \Lambda(dz) - m + m^2 = b \int_{(0, \infty)} z^2 \Lambda(dz)$$

is finite, one also knows [3] the asymptotic behavior of $\mathbb{P}(\mathcal{Z}_n \neq 0)$. Indeed, we have Kolmogorov's estimate

$$\lim_{n \rightarrow \infty} n \mathbb{P}(\mathcal{Z}_n \neq 0) = \frac{2}{\sigma^2}. \quad (4.9)$$

Then

$$\lim_{a \rightarrow \infty} \frac{\mathbb{P}_x(\tau_a < T_0)}{a-1} = \frac{2}{\sigma^2} b x. \quad (4.10)$$

We are now interested in the behavior of $\mathbb{P}_x(\Theta, \tau_a \leq t < T_0)$ as $a \rightarrow \infty$. In fact, we will show that it goes to 0 faster than m^{a-1} (resp. $1/a$) if $m < 1$ (resp. $m = 1$). Since $b = \Lambda(\mathbb{R}^+) < \infty$, the total number N_t of jumps of X before t has a Poisson distribution with parameter bt . Hence, since $\{\tau_a \leq t\} \subset \{N_t \geq a\}$,

$$\mathbb{P}_x(\Theta, \tau_a \leq t < T_0) \leq \mathbb{P}(N_t \geq a) = \sum_{i \geq a} e^{-bt} \frac{(bt)^i}{i!} \leq \frac{(bt)^a}{a!}.$$

Thus, using the last equation and equation (4.8) or (4.10), the first term of the r.h.s. of (4.6) vanishes as $a \rightarrow \infty$ for $m \leq 1$.

We finally study the term $\mathbb{P}_x(\Theta, t < \tau_a < T_0)$. For a word u , $i \in \mathbb{N}$ and $x \in \mathbb{R}$, we denote by $A(u, i, x)$ the event $\{u$ gives birth before age x to a daughter which has alive descendance at generation $|u| + i\}$, that is, if $|u| = j$,

$$A(u, i, x) := \{\exists v \in \mathcal{U}; |v| = i, uv \in \mathcal{T} \text{ and } \alpha(uv|j+1) - \alpha(u) \leq x\}.$$

Let v_t be the individual visited at time t . Hence, using the Markov property at time t and recalling that $\Theta \in \mathcal{F}_t$, we have

$$\begin{aligned} \mathbb{P}_x(\Theta, t < \tau_a < T_0) &= \mathbb{E}_x \left[\mathbf{1}_{\{t < T_0\}} \mathbf{1}_{\{t < \tau_a\}} \mathbf{1}_\Theta \mathbb{P}_x(\tau_a < T_0 | \mathcal{F}_t) \right] \\ &= \mathbb{E}_x \left[\mathbf{1}_{\{t < T_0\}} \mathbf{1}_{\{t < \tau_a\}} \mathbf{1}_\Theta \mathbb{P}_x \left(\bigcup_{i=0}^{H_t} A(v_t | i, a - i, \rho_i^t) \middle| \mathcal{F}_t \right) \right] \end{aligned}$$

and by the branching property,

$$\mathbb{P}_x \left(\bigcup_{i=0}^{H_t} A(v_t | i, a - i, \rho_i^t) \middle| \mathcal{F}_t \right) = 1 - \prod_{i=0}^{H_t} (1 - \mathbb{P}_{\rho_i^t}(\mathcal{Z}_{a-i} \neq 0)) \quad \text{a.s.}$$

As previously, since computations for subcritical and critical cases are equivalent, we only detail the first one. We have, with another use of (4.7),

$$\mathbb{P}_x \left(\bigcup_{i=0}^{H_t} A(v_t | i, a - i, \rho_i^t) \middle| \mathcal{F}_t \right) \underset{a \rightarrow \infty}{\sim} \sum_{i=0}^{H_t} m^{a-i-1} \rho_i^t c b \quad \text{a.s.}$$

We want to use the dominated convergence theorem to prove that

$$\lim_{a \rightarrow \infty} m^{1-a} \mathbb{P}_x(\Theta, t < \tau_a < T_0) = c b \mathbb{E}_x \left[\mathbf{1}_{\{t < T_0\}} \mathbf{1}_\Theta \sum_{i=0}^{H_t} \rho_i^t m^{-i} \right]$$

and then, using (4.8),

$$\lim_{a \rightarrow \infty} \mathbb{P}_x \left(\Theta, t < T_0 \mid \sup_{s \leq T_0} H_s \geq a \right) = \frac{1}{x} \mathbb{E}_x \left[\mathbf{1}_{\{t < T_0\}} \mathbf{1}_\Theta \sum_{i=0}^{H_t} \rho_i^t m^{-i} \right]$$

so that the proof of the subcritical case would be finished. We have almost surely

$$m^{-a} \mathbb{P}_x(\tau_a < T_0 | \mathcal{F}_t) \mathbf{1}_{\{t < T_0\}} \leq m^{-a} \sum_{i=0}^{H_t} \mathbb{P}_x(A(v_t | i, a - i, \rho_i^t)) \mathbf{1}_{\{t < T_0\}} \leq C' \sum_{i=0}^{H_t} \rho_i^t m^{-i} \mathbf{1}_{\{t < T_0\}}$$

where C' is a positive, deterministic constant. Hence, to obtain an integrable upper bound, since $\mathbb{E}_x[\rho_0^t \mathbf{1}_{\{t < T_0\}}] \leq x$, it is sufficient to prove that

$$\mathbb{E}_x \left[\sum_{i=1}^{H_t} \rho_i^t m^{-i} \right] < \infty \quad (4.11)$$

in order to use the dominated convergence theorem. Recall that $X^{(t)}$ denotes the time-reversal of X at time t

$$X_s^{(t)} := X_{t-s} - X_{(t-s)-} \quad s \in [0, t], \quad (X_{0-} = X_0)$$

and $S^{(t)}$ is its associated past supremum process

$$S_s^{(t)} := \sup\{X_r^{(t)} : 0 \leq r \leq s\}.$$

It is known that the process $X^{(t)}$ has the law of X under \mathbb{P}_0 [5, ch.II]. We also have

$$H_t = R_t := \#\{0 \leq s \leq t; X_s^{(t)} = S_s^{(t)}\}$$

which is the number of records of the process $S^{(t)}$ during $[0, t]$ and the ρ_i^t 's are the overshoots of the successive records. More precisely, if we denote by $\tilde{T}_1 < \tilde{T}_2 < \dots$ the record times of $X^{(t)}$, the overshoots are

$$\tilde{\rho}_i := X_{\tilde{T}_i}^{(t)} - \sup_{0 \leq s < \tilde{T}_i} X_s^{(t)}, \quad i \geq 1.$$

We come back to the proof of (4.11). We have

$$\sum_{i=1}^{H_t} \rho_i^t m^{-i} = \sum_{i=1}^{R_t} \tilde{\rho}_{R_t-i+1} m^{-i} = \sum_{i=1}^{R_t} \tilde{\rho}_i m^{i-R_t-1} = \sum_{i \geq 1} \tilde{\rho}_i m^{i-R_t-1} \mathbf{1}_{\{\tilde{T}_i \leq t\}}.$$

We denote by $(\tilde{\mathcal{F}}_s, s \geq 0)$ the natural filtration of $X^{(t)}$. Thus, using Fubini-Tonelli theorem and the strong Markov property for $X^{(t)}$ applied at time \tilde{T}_i

$$\mathbb{E}_x \left[\sum_{i=1}^{H_t} \rho_i^t m^{-i} \right] = \sum_{i \geq 1} \mathbb{E} \left[\tilde{\rho}_i m^{i-R_t-1} \mathbf{1}_{\{\tilde{T}_i \leq t\}} \right] = m^{-1} \sum_{i \geq 1} \mathbb{E} \left[\tilde{\rho}_i \mathbf{1}_{\{\tilde{T}_i \leq t\}} f(t - \tilde{T}_i) \right]$$

where

$$f(s) = \mathbb{E} \left[\frac{1}{m^{R_s}} \right], \quad s \geq 0.$$

However, as previously, we have almost surely $R_t \leq N_t$ where N_t is the number of jumps of X or $X^{(t)}$ before t and so has a Poisson distribution with parameter bt . Thus,

$$f(s) \leq \mathbb{E}[m^{-N_s}] = \exp\left(bs(m^{-1} - 1)\right) = e^{\kappa s}$$

where κ is some positive constant and

$$\mathbb{E}_x \left[\sum_{i=1}^{H_t} \rho_i^t m^{-i} \right] \leq m^{-1} e^{\kappa t} \sum_{i \geq 1} \mathbb{E} \left[\tilde{\rho}_i; \tilde{T}_i \leq t \right].$$

Moreover,

$$\begin{aligned} \mathbb{E} \left[\tilde{\rho}_i; \tilde{T}_i \leq t \right] &= \mathbb{E} \left[\tilde{\rho}_i \mathbf{1}_{\{\tilde{T}_i - \tilde{T}_{i-1} \leq t - \tilde{T}_{i-1}\}} \mathbf{1}_{\{\tilde{T}_{i-1} \leq t\}} \right] = \int_0^t \mathbb{P}(\tilde{T}_{i-1} \in ds) \mathbb{E} \left[\tilde{\rho}_1; \tilde{T}_1 \leq t - s \right] \\ &\leq \mathbb{E} \left[\tilde{\rho}_1; \tilde{T}_1 \leq t \right] \mathbb{P}(\tilde{T}_{i-1} \leq t) \end{aligned}$$

since $(\tilde{\rho}_i, \tilde{T}_i - \tilde{T}_{i-1}), i \geq 1$ are i.i.d. random variables. Then

$$\mathbb{E}_x \left[\sum_{i=1}^{H_t} \rho_i^t m^{-i} \right] \leq m^{-1} e^{\kappa t} \mathbb{E} \left[\tilde{\rho}_1; \tilde{T}_1 \leq t \right] \sum_{i \geq 0} \mathbb{P}(\tilde{T}_i \leq t) = m^{-1} e^{\kappa t} \mathbb{E} \left[\tilde{\rho}_1; \tilde{T}_1 \leq t \right] \sum_{i \geq 0} \mathbb{P}(R_t \geq i).$$

The sum in the r.h.s equals $\mathbb{E}[R_t]$ which is finite since $R_t \leq N_t$ a.s. According to Theorem VII.17 in [5], the joint law of $(X_{\tilde{T}_1}^{(t)}, \Delta X_{\tilde{T}_1}^{(t)})$ is given by

$$\mathbb{E} \left[F \left(X_{\tilde{T}_1}^{(t)}, \Delta X_{\tilde{T}_1}^{(t)} \right); \tilde{T}_1 < \infty \right] = \int_{(0, \infty)} \Lambda(dy) \int_0^y dx F(x, y). \quad (4.12)$$

Thus, since $\{\tilde{T}_1 \leq t\} \subset \{X_{\tilde{T}_1-}^{(t)} \geq -t\}$,

$$\begin{aligned} \mathbb{E} \left[\tilde{\rho}_1; \tilde{T}_1 \leq t \right] &\leq \mathbb{E} \left[\tilde{\rho}_1 \mathbf{1}_{\{\Delta X_{\tilde{T}_1}^{(t)} - X_{\tilde{T}_1}^{(t)} \leq t\}} \mathbf{1}_{\{\tilde{T}_1 < \infty\}} \right] = \int_{(0, \infty)} \Lambda(dy) \int_0^y x \mathbf{1}_{\{y-x \leq t\}} dx \\ &= \int_{(t, \infty)} \Lambda(dy) (yt - t^2/2) + \int_{(0, t]} \Lambda(dy) y^2/2 < \infty \end{aligned}$$

and the proof of (4.11) is completed.

Finally, in the critical case $m = 1$, computations are similar. Thanks to (4.9),

$$\mathbb{P}_x \left(\bigcup_{i=0}^{H_t} A(v_t | i, a - i, \rho_i^t) \middle| \mathcal{F}_t \right) \underset{a \rightarrow \infty}{\sim} \frac{1}{a} \sum_{i=0}^{H_t} \rho_i^t \frac{2}{\sigma^2} b = \frac{1}{a} \frac{2b}{\sigma^2} X_t \quad \text{a.s.}$$

by the definition of the ρ_i^t 's. Moreover, when $m = 1$, the process X is a (\mathcal{F}_t) -martingale and $\mathbb{E}_x[X_t] = \mathbb{E}_x[X_0] = x$. Then, by the dominated convergence theorem,

$$\lim_{a \rightarrow \infty} a \mathbb{P}_x(\Theta, t < \tau_a < T_0) = \frac{2b}{\sigma^2} \mathbb{E}_x[X_{t \wedge T_0} \mathbf{1}_\Theta]$$

and then, using (4.10),

$$\lim_{a \rightarrow \infty} \mathbb{P}_x \left(\Theta, t < T_0 \middle| \sup_{s \leq T_0} H_s \geq a \right) = \frac{1}{x} \mathbb{E}_x[X_{t \wedge T_0} \mathbf{1}_\Theta].$$

We conclude the proof by showing that $(M_{t \wedge T_0}, t \geq 0)$ is a (\mathcal{F}_t) -martingale. First, if $m = 1$, we have $M_t = X_t$ and as it was stated just before, in this case, the process X is a martingale and so is $X_{\cdot \wedge T_0}$. We now consider the case $m < 1$. Recall from (4.3) that the infinitesimal generator of $(\rho^{t \wedge T_0}, t \geq 0)$ is

$$\mathcal{A}(f)(\nu) = \left[\int_{(0, \infty)} \left(f(\nu_0, \nu_1, \dots, \nu_{H(\nu)}, r, 0, \dots) - f(\nu_0, \nu_1, \dots, \nu_{H(\nu)}, 0, \dots) \right) \Lambda(dr) - \frac{\partial f}{\partial x_{H(\nu)}}(\nu_0, \dots, \nu_{H(\nu)}, 0, \dots) \right] \mathbf{1}_{\{\nu \neq 0\}}, \quad \nu \in S.$$

Let g be the application from S to \mathbb{R} such that

$$g(\nu) := \sum_{i=0}^{H(\nu)} \nu_i m^{-i}, \quad \nu \in S. \quad (4.13)$$

Clearly, we have $g(\rho^{t \wedge T_0}) = M_{t \wedge T_0}$. According to [60, ch.VII], if $\mathcal{G}_t := \{\sigma(\rho^r), 0 \leq r \leq t\}$, to prove that M is a (\mathcal{G}_t) -martingale, it is sufficient to show that $\mathcal{A}(g) = 0$. For $\nu \in S$, we have

$$\begin{aligned} \mathcal{A}(g)(\nu) &= \left(\int_{(0, \infty)} (g(\nu) + r m^{-H(\nu)-1} - g(\nu)) \Lambda(dr) - m^{-H(\nu)} \right) \mathbf{1}_{\{\nu \neq 0\}} \\ &= \left(\int_{(0, \infty)} r \Lambda(dr) m^{-H(\nu)-1} - m^{-H(\nu)} \right) \mathbf{1}_{\{\nu \neq 0\}} = 0. \end{aligned}$$

Then, $M_{\cdot \wedge T_0}$ is a (\mathcal{G}_t) -martingale. Moreover, since $\langle \rho^t, \mathbf{1} \rangle = X_t$, $\mathcal{F}_t \subset \mathcal{G}_t$ and clearly $\mathcal{G}_t \subset \mathcal{F}_t$. Then, $\mathcal{F}_t = \mathcal{G}_t$ and $M_{\cdot \wedge T_0}$ is a (\mathcal{F}_t) -martingale. \square

4.2.4 A new probability

Definition 4.8. For $x > 0$, we define a new probability measure \mathbb{P}_x^\uparrow on (Ω, \mathcal{F}_t) by

$$\mathbb{P}_x^\uparrow(\Theta) = \frac{1}{x} \mathbb{E}_x [M_{t \wedge T_0} \mathbf{1}_\Theta], \quad \Theta \in \mathcal{F}_t.$$

First, notice that since $M_{\cdot \wedge T_0}$ is a martingale, this definition is consistent and that this new measure is honest, that is, it has mass 1. Second, $(X_t, t \geq 0)$ is no more Markovian under \mathbb{P}_x^\uparrow while $(\rho^t, t \geq 0)$ is because \mathbb{P}_x^\uparrow is obtained by a h -transform via the application g defined by (4.13).

Proposition 4.9. Under \mathbb{P}_x^\uparrow , in probability,

$$X_t \xrightarrow[t \rightarrow \infty]{} +\infty.$$

Proof. It is sufficient to prove that for all positive λ , $\mathbb{E}_x^\uparrow [e^{-\lambda X_t}] \rightarrow 0$ as t tends to ∞ . First, in the critical case $m = 1$,

$$\mathbb{E}_x^\uparrow [e^{-\lambda X_t}] = \frac{1}{x} \mathbb{E}_x [e^{-\lambda X_t} X_{t \wedge T_0}] \xrightarrow[t \rightarrow \infty]{} 0 \quad (4.14)$$

using dominated convergence theorem, because $(X_{t \wedge T_0}, t \geq 0)$ is a non-negative martingale converging a.s. to 0 and the mapping $x \mapsto e^{-\lambda x} x$ is bounded on \mathbb{R}^+ .

We now suppose that $m < 1$. Then, for all positive a , distinguishing between $H_t > a$ and $H_t \leq a$, we have

$$\begin{aligned} \mathbb{E}_x^\uparrow \left[e^{-\lambda X_t} \right] &= \frac{1}{x} \mathbb{E}_x \left[e^{-\lambda X_t} \sum_{i=0}^{H_t} m^{-i} \rho_i^t \mathbf{1}_{\{t \leq T_0\}} \right] \\ &\leq \frac{1}{x} \mathbb{E}_x \left[e^{-\lambda X_t} \sum_{i=1}^{H_t} m^{-i} \rho_i^t \mathbf{1}_{\{t \leq T_0\}} \mathbf{1}_{\{H_t > a\}} \right] + \mathbb{P}_x(t \leq T_0) + \frac{m^{-a}}{x} \mathbb{E}_x \left[e^{-\lambda X_t} X_{t \wedge T_0} \right] \end{aligned}$$

since $\sum_i \rho_i^t = X_t$. The second term of the r.h.s is an upper bound of the term $i = 0$ of the sum and it goes to 0 as t tends to ∞ since T_0 is finite a.s. So does the third term similarly to the critical case in (4.14). Then, to finish the proof, it is sufficient to show that the first term tends to 0 uniformly in t as a goes to ∞ . We have

$$A(t, a) := \mathbb{E}_x \left[e^{-\lambda X_t} \sum_{i=1}^{H_t} m^{-i} \rho_i^t \mathbf{1}_{\{t \leq T_0\}} \mathbf{1}_{\{H_t > a\}} \right] \leq \mathbb{E}_x \left[e^{-\lambda(X_t - I_t)} \sum_{i=1}^{H_t} m^{-i} \rho_i^t \mathbf{1}_{\{t \leq T_0\}} \mathbf{1}_{\{H_t > a\}} \right]$$

and as in proof of Theorem 4.3, by time reversing at time t , we have

$$A(t, a) \leq \sum_{i \geq 1} \mathbb{E} \left[e^{-\lambda S_t} m^{-(R_t - i + 1)} \tilde{\rho}_i \mathbf{1}_{\{\tilde{T}_i \leq t\}} \mathbf{1}_{\{R_t > a\}} \right] = \sum_{i \geq 1} m^{i-1} B_i(t, a)$$

where the \tilde{T}_i 's are the times of successive records of the supremum process S , the $\tilde{\rho}_i$'s are the associated overshoots and R_t is the number of records up to time t . In particular, $S_t = \sum_{j=1}^{R_t} \tilde{\rho}_j$.

On the one hand, let us treat the case when $i \leq a$. Since $\{R_t > a\} \subset \{\tilde{T}_a \leq t\}$ and by applying the strong Markov property at time \tilde{T}_a ,

$$\begin{aligned} B_i(t, a) &\leq \mathbb{E} \left[\prod_{j=1}^{R_t} e^{-\lambda \tilde{\rho}_j} m^{-R_t} \tilde{\rho}_i \mathbf{1}_{\{\tilde{T}_a \leq t\}} \right] \\ &= \mathbb{E} \left[\prod_{j=1}^a e^{-\lambda \tilde{\rho}_j} \tilde{\rho}_i \mathbf{1}_{\{\tilde{T}_a \leq t\}} \mathbb{E} \left[\prod_{j=a+1}^{R_t} e^{-\lambda \tilde{\rho}_j} m^{-R_t} \middle| \tilde{\mathcal{F}}_{\tilde{T}_a} \right] \right] \\ &= \mathbb{E} \left[\prod_{j=1}^a e^{-\lambda \tilde{\rho}_j} \tilde{\rho}_i \mathbf{1}_{\{\tilde{T}_a \leq t\}} m^{-a} f(t - \tilde{T}_a) \right] \end{aligned} \quad (4.15)$$

where for $s \geq 0$, $f(s) := \mathbb{E} \left[\prod_{j=1}^{R_s} e^{-\lambda \tilde{\rho}_j} m^{-R_s} \right]$. This function f is bounded. Indeed, for $s \geq 0$,

$$\begin{aligned} f(s) &= \sum_{i \geq 1} m^{-i} \mathbb{E} \left[\prod_{j=1}^i e^{-\lambda \tilde{\rho}_j}; \tilde{T}_i \leq s < \tilde{T}_{i+1} \right] \leq \sum_{i \geq 1} m^{-i} \mathbb{E} \left[\prod_{j=1}^i \left(e^{-\lambda \tilde{\rho}_j} \mathbf{1}_{\{\tilde{T}_j - \tilde{T}_{j-1} \leq s\}} \right) \right] \\ &\leq \sum_{i \geq 1} m^{-i} \mathbb{E} \left[e^{-\lambda \tilde{\rho}_1}; \tilde{T}_1 < \infty \right]^i \end{aligned}$$

since the r.v. $(\tilde{\rho}_j, \tilde{T}_j - \tilde{T}_{j-1})_{j \geq 1}$ are i.i.d. Applying equation (4.12) with the function $F(x, y) = e^{-\lambda x}$, we have

$$\mathbb{E} \left[e^{-\lambda \tilde{\rho}_1} \mathbf{1}_{\{\tilde{T}_1 < \infty\}} \right] = \int_{(0, \infty)} \Lambda(dy) \frac{1 - e^{-\lambda y}}{\lambda} = 1 - \frac{\psi(\lambda)}{\lambda}.$$

Then,

$$f(s) \leq \sum_{i \geq 1} m^{-i} \left(1 - \frac{\psi(\lambda)}{\lambda}\right)^i < \infty$$

since $1 - \psi(\lambda)/\lambda < m$ for all positive λ because $m = 1 - \psi'(0)$ and ψ is strictly convex.

We come back to (4.15). If we denote by C an upper bound for f , for $i \leq a$,

$$\begin{aligned} B_i(t, a) &\leq C m^{-a} \prod_{j=1, j \neq i}^a \mathbb{E} \left[e^{-\lambda \tilde{\rho}_j} \mathbf{1}_{\{\tilde{T}_j - \tilde{T}_{j-1} \leq t\}} \right] \mathbb{E} \left[e^{-\lambda \tilde{\rho}_i} \tilde{\rho}_i \mathbf{1}_{\{\tilde{T}_i - \tilde{T}_{i-1} \leq t\}} \right] \\ &\leq C m^{-a} \mathbb{E} \left[e^{-\lambda \tilde{\rho}_1} \mathbf{1}_{\{\tilde{T}_1 < \infty\}} \right]^{a-1} \mathbb{E} \left[e^{-\lambda \tilde{\rho}_1} \tilde{\rho}_1 \mathbf{1}_{\{\tilde{T}_1 < \infty\}} \right] \end{aligned}$$

Applying again (4.12) with $F(x, y) = x e^{-\lambda x}$, we obtain

$$\mathbb{E} \left[\tilde{\rho}_1 e^{-\lambda \tilde{\rho}_1} \mathbf{1}_{\{\tilde{T}_1 < \infty\}} \right] = \frac{\psi'(\lambda)}{\lambda} - \frac{\psi(\lambda)}{\lambda^2} < \infty. \quad (4.16)$$

Hence,

$$\begin{aligned} \sum_{i=1}^a m^{i-1} B_i(t, a) &\leq C(\lambda) m^{-a+1} \left(1 - \frac{\psi(\lambda)}{\lambda}\right)^{a-1} \sum_{i=1}^a m^i \\ &\leq C(\lambda) a m^{-a+1} \left(1 - \frac{\psi(\lambda)}{\lambda}\right)^{a-1}. \end{aligned} \quad (4.17)$$

On the other hand, for $i > a$, as in previous computations, we apply the strong Markov property at time \tilde{T}_i to obtain

$$\begin{aligned} \sum_{i \geq a+1} m^{i-1} B_i(t, a) &\leq \sum_{i \geq a+1} \mathbb{E} \left[\prod_{j=1}^i e^{-\lambda \tilde{\rho}_j} \tilde{\rho}_i \mathbf{1}_{\{\tilde{T}_i \leq t\}} m^{-1} f(t - \tilde{T}_i) \right] \\ &\leq C m^{-1} \sum_{i \geq a+1} \mathbb{E} \left[e^{-\lambda \tilde{\rho}_i} \tilde{\rho}_i \mathbf{1}_{\{\tilde{T}_i - \tilde{T}_{i-1} \leq t\}} \mathbf{1}_{\{\tilde{T}_{i-1} \leq t\}} \right] \\ &\leq C m^{-1} \left(\frac{\psi'(\lambda)}{\lambda} - \frac{\psi(\lambda)}{\lambda^2} \right) \sum_{i \geq a+1} \mathbb{P}(R_t \geq i-1) \end{aligned}$$

where we have first used the previous function f and its upper bound C and then that $(\tilde{\rho}_i, \tilde{T}_i - \tilde{T}_{i-1})$ is independent from \tilde{T}_{i-1} and equation (4.16).

Then

$$\sum_{i \geq a+1} m^{i-1} B_i(t, a) \leq \tilde{C}(\lambda) \sum_{i \geq a} \mathbb{P}(R_\infty \geq i) \quad (4.18)$$

where $\tilde{C}(\lambda)$ is a finite constant independent from t and a and $R_\infty := \lim_{t \rightarrow +\infty} R_t \in \mathbb{N}$ is the total number of records of S . Moreover, thanks to the strong Markov property, R_∞ follows a geometric distribution with success probability $\mathbb{P}(\tilde{T}_1 = \infty) = 1 - m$ according to (4.12) with $F \equiv 1$.

Finally, putting together (4.17) and (4.18), we obtain

$$A(t, a) \leq \sum_{i \geq 1} m^{i-1} B_i(t, a) \leq \bar{C}(\lambda) \left(a m^{-a+1} \left(1 - \frac{\psi(\lambda)}{\lambda}\right)^{a-1} + \sum_{i \geq a} \mathbb{P}(R_\infty \geq i) \right) \quad (4.19)$$

where $\bar{C}(\lambda)$ is a finite constant independent from t and a . The first term of the r.h.s. of (4.19) goes to 0 as a goes to ∞ since $1 - \psi(\lambda)/\lambda < m$. The second term also tends to 0 because $\mathbb{E}[R_\infty] = (1 - m)^{-1}$ is finite. Hence, we have proved that $A(t, a)$ tends to 0 as $a \rightarrow \infty$ uniformly in t and $\mathbb{E}_x^\uparrow[e^{-\lambda X_t}] \rightarrow 0$ which ends the proof. \square

4.3 Conditioned Subcritical and Critical Splitting Trees

In the previous section, we have defined the law of a conditioned Lévy process by using properties of (sub)critical splitting trees. Here, we study conditioned splitting trees by using the same properties about BGW-processes. More precisely, we are interested in the behavior as $n \rightarrow \infty$ of a (sub)critical splitting tree conditioned to be alive at generation n and we want to give a spine decomposition of the limiting tree. We also give the distribution of its contour process which visits the left part of the spine.

We will use notation of Section 4.4 of [47]. For $n \geq 0$ and $u \in \mathcal{T}$, we denote by $E_n(u)$ the event $\{u \text{ has an extant descendance at generation } |u| + n\}$. For simplicity of notation, we will denote by E_n the event $E_n(\emptyset)$. Let \mathbf{P}^n be the law of the splitting tree on this event

$$\mathbf{P}^n := \mathbf{P}(\cdot | E_n) = \mathbf{P}(\cdot | \mathcal{Z}_n > 0).$$

On the event E_n , we define a distinguished lineage $u_0^n u_1^n \cdots u_n^n$ as the first lineage of the tree that reaches generation n as explained below. First, $u_0^n = \emptyset$ and one defines recursively: for $i \geq 1$, $u_0^n u_1^n \cdots u_i^n$ is the youngest daughter of $u_0^n u_1^n \cdots u_{i-1}^n$ that has a descendance at generation n . We set $U^{(n)} := u_0^n u_1^n \cdots u_n^n$ and we denote by

$$\mathbb{B}^n := \{x \in \mathbb{T}; p_1(x) = U^{(n)} | k \text{ for } 0 \leq k \leq n\}$$

this lineage of \mathbb{T} defined on the event E_n . For $0 \leq k \leq n-1$, the age at which the individual $U^{(n)}|k$ gives birth to individual $U^{(n)}|k+1$ is

$$A_k^n := \alpha(U^{(n)}|k+1) - \alpha(U^{(n)}|k),$$

its residual lifetime is

$$R_k^n := \omega(U^{(n)}|k) - \alpha(U^{(n)}|k+1)$$

and its total lifelength is denoted by $T_k^n = A_k^n + R_k^n$. Observe that modulo labelling, the sequence $((A_k^n, R_k^n), 0 \leq k \leq n-1)$ and $A_n^n := \omega(U^{(n)}) - \alpha(U^{(n)})$ characterizes the spine \mathbb{B}^n .

In the following, when we say that a tree (marked with a special lineage) converges for finite truncations, we mean that there is convergence for events that are measurable with respect to the σ -field generated by the truncations $\{x \in \mathbb{T}; p_2(x) \leq \sigma\}$, $\sigma > 0$.

Theorem 4.10. *We suppose that $b < \infty$ and that either $m < 1$ and $\int_{[1,\infty)} z \log(z) \Lambda(dz) < \infty$ or $m = 1$ and $\int_{(0,\infty)} z^2 \Lambda(dz) < \infty$. Then, as $n \rightarrow \infty$, the law of \mathbb{B}^n under \mathbf{P}^n converges for finite truncations to an infinite spine \mathbb{B} whose distribution is characterized by an i.i.d sequence $(A_k, R_k)_{k \geq 0}$ such that*

$$(A_0, R_0) \stackrel{(d)}{=} (UD, (1-U)D)$$

where U is a uniform r.v. on $(0, 1)$ independent from the size-biased r.v. D

$$\mathbf{P}(D \in dz) = \frac{z \Lambda(dz)}{m} \quad z > 0.$$

Moreover, if $(x_i, i \in \mathbb{N})$ are the atoms of a Poisson measure on \mathbb{B} with intensity b , then as $n \rightarrow \infty$, the law of \mathbb{T} under \mathbf{P}^n converges for finite truncations to an infinite tree with a unique infinite branch \mathbb{B} on which are grafted at points (x_i) i.i.d. trees with common law \mathbf{P} .

Proof. First, for $t > 0$ we denote by $K_n(t)$ the number of individuals at time t that have alive descendance at generation n . Then, as $n \rightarrow \infty$, $\mathbf{P}(K_n(t) = 1 | \mathcal{Z}_n > 0) \rightarrow 1$. Hence, the limiting tree under \mathbf{P}^n has a unique infinite branch.

We now investigate the law of this limiting spine. Let p be a natural integer. We denote by \mathcal{F}^p the σ -field generated by the lifespans of all individuals until generation $p-1$ and the numbers and birthdates of their daughters.

Under \mathbf{P}^n , we denote by N_i the number of children of $U^{(n)}|i$ and by $\alpha_{ij}, 1 \leq j \leq N_i$ her age at their births. Then, for $n \geq p$, for $n_0, \dots, n_{p-1} \in \mathbb{N}^*$, for $z_0, \dots, z_{p-1} > 0$, for $0 < x_{ij} < z_i$ where $0 \leq i \leq p-1, 1 \leq j \leq n_i$ and for $u_1 u_2 \dots u_p$ any word of length p , we have

$$\begin{aligned} \mathbf{P}^n(\mathcal{E}^p) &:= \mathbf{P}^n(T_i^n \in dz_i, N_i = n_i, \alpha_{ij} \in dx_{ij}, 0 \leq i \leq p-1, 1 \leq j \leq n_i; U^{(n)}|p = u_1 \dots u_p) \\ &= \frac{1}{\mathbf{P}(E_n)} \mathbf{E} \left[\prod_{i=0}^{p-1} \left(\frac{\Lambda(dz_i)}{b} e^{-bz_i} \frac{(bz_i)^{n_i}}{n_i!} \prod_{j=1}^{n_i} \frac{dx_{ij}}{z_i} \prod_{l=1}^{u_{i+1}-1} \mathbf{1}_{(E_{n-i-1}(u_1 \dots u_{i+l}))^c} \right) \mathbf{1}_{E_{n-p}(u_1 \dots u_p)} \right] \end{aligned}$$

because by definition of \mathbb{B}^n , all the younger sisters of the marked individuals, that is, all individuals labeled by $u_1 \dots u_{i+l}$ for any $0 \leq i \leq p-1$ and any $1 \leq l < u_{i+1}$, have no alive descendance at generation n .

Conditioning with respect to the σ -field \mathcal{F}^p and thanks to the branching property, we have

$$\mathbf{P}^n(\mathcal{E}^p) = \prod_{i=0}^{p-1} \left(\frac{\Lambda(dz_i)}{b} e^{-bz_i} \frac{(bz_i)^{n_i}}{n_i!} \prod_{j=1}^{n_i} \frac{dx_{ij}}{z_i} \right) \prod_{i=1}^p \mathbf{P}(E_{n-i}^c)^{u_{i-1}} \frac{\mathbf{P}(E_{n-p})}{\mathbf{P}(E_n)}.$$

Since we consider subcritical or critical trees, $\mathbf{P}(E_{n-i}^c)$ goes to 1 as $n \rightarrow \infty$. Moreover, for $m < 1$, according to Lemma 4.7, if $\int_{[1, \infty)} z \log(z) \Lambda(dz) < \infty$, then the log-condition of (4.7) is fulfilled and there exists $c > 0$ such that

$$\frac{\mathbf{P}(E_n)}{m^n} = \frac{\mathbf{P}(\mathcal{Z}_n \neq 0)}{m^n} \xrightarrow{n \rightarrow \infty} c.$$

Furthermore, in critical case $m = 1$, if $\int_{(0, \infty)} z^2 \Lambda(dz)$ is finite, according to (4.9), there is $c' > 0$ such that

$$\lim_{n \rightarrow \infty} n \mathbf{P}(\mathcal{Z}_n \neq 0) = c'.$$

In both cases, we obtain by the dominated convergence theorem

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbf{P}^n(\mathcal{E}^p) &= \prod_{i=0}^{p-1} \left(\frac{\Lambda(dz_i)}{b} e^{-bz_i} \frac{(bz_i)^{n_i}}{n_i!} \prod_{j=1}^{n_i} \frac{dx_{ij}}{z_i} \right) m^{-p} \\ &= \prod_{i=0}^{p-1} \frac{z_i \Lambda(dz_i)}{m} \cdot \frac{e^{-bz_i} (bz_i)^{n_i-1}}{(n_i-1)!} \cdot \frac{1}{n_i} \cdot \prod_{j=1}^{n_i} \frac{dx_{ij}}{z_j}. \end{aligned}$$

Thus, we see that in the limit $n \rightarrow \infty$, the individuals of the spine have independent size-biased lifelengths (first term of the r.h.s.), give birth to one plus a Poissonian number of children (second term) whose birth dates are independent uniform during the life of their mother (fourth term) and one of them is marked uniformly among them (third term).

Hence, we obtain a limiting spine \mathbb{B} characterized by a sequence $(A_k + R_k, A_k)$ which are independent r.v. with the same joint distribution as (D, UD) . Moreover, conditional on $T_k = A_k + R_k$, the k -th individual of \mathbb{B} has a Poissonian number (with parameter $T_k b$) of non-marked daughters which are born independently and uniformly on $[0, T_k]$.

Intersecting the previous event \mathcal{E}^p with events involving trees descending from sisters of the marked individuals and applying the branching property leads by similar computations to the last statement of the theorem. \square

We can now define two contour processes of the conditioned splitting tree. We denote by $(X^\uparrow(t), t \geq 0)$ the contour of the limiting tree as defined previously for typical trees. Since the spine is infinite, only individuals on its left part are visited by this contour process. That is why we call it the *left contour process*. Notice that it does not characterize the conditioned tree because all information about the right part of the spine can not be recovered from X^\uparrow . That is why we need a second contour process $(X^\downarrow(t), t \geq 0)$.

For a non-conditioned (sub)critical tree, it is possible to define a process which visits it in the opposite direction by the time-reversal of the classical contour process X at T_0 :

$$\bar{X} := (X(T_0 - t), 0 \leq t \leq T_0).$$

Then, for the conditioned tree, this reversed contour process is trapped on the right part of the infinite spine: it starts at level 0 and visits the right part of the infinite spine in the opposite direction of the typical contour process. We denote it by X^\downarrow . Roughly speaking, it is the contour process of the right part that we reverse at $T_0 = +\infty$.

In [18], Duquesne obtains an equivalent result. He defines two processes \overleftarrow{H} and \overrightarrow{H} called left and right height processes and which describe the genealogy of a continuous state branching process with immigration. Notice that these height processes are linked to the height process $(H_t, t \geq 0)$ that we use in Section 4.4 for Lévy processes with infinite variation.

A consequence of the previous theorem deals with the joint law of the processes X^\uparrow and X^\downarrow .

Corollary 4.11. *Let $(U_k, D_k)_k$ be a sequence of i.i.d. pairs distributed as (U, D) in Theorem 4.10. We have*

$$(X^\uparrow, X^\downarrow) \stackrel{(d)}{=} \left(\sum_{k \geq 0} X_k(t - \varsigma_k) \mathbf{1}_{\{t \geq \varsigma_k\}}, \sum_{k \geq 0} \tilde{X}_k(t - \tilde{\varsigma}_k) \mathbf{1}_{\{t \geq \tilde{\varsigma}_k\}}, t \geq 0 \right)$$

where

- $(X_k, \tilde{X}_k)_k$ is a sequence of i.i.d. pairs of processes, where (X_k, \tilde{X}_k) is independent from (U_j, D_j) for $j \neq k$. Moreover, conditionally on U_k and D_k , they are independent and X_k is distributed as X started at level D_k and stopped at the hitting time η_k of level $U_k D_k$, and \tilde{X}_k is distributed as X started at level $U_k D_k$ and reversed at the hitting time $\tilde{\eta}_k$ of 0 by Y that is,

$$\tilde{X}_k \stackrel{(d)}{=} X((\tilde{\eta}_k - \cdot) \vee 0),$$

- $\varsigma_0 = \tilde{\varsigma}_0$, $\varsigma_k = \eta_1 + \dots + \eta_{k-1}$ and $\tilde{\varsigma}_k = \tilde{\eta}_1 + \dots + \tilde{\eta}_{k-1}$.

Remark 4.12. An open question is to prove that the left contour process X^\uparrow of the conditioned tree has the same distribution as the conditioned process X under \mathbb{P}^\uparrow defined in Section 4.2.

4.4 Infinite variation case

4.4.1 Definitions and main results

In this section, we consider the same setting as T. Duquesne and J.F. Le Gall in [19, Ch 1]. We assume that X is a Lévy process with no negative jumps, which does not drift to $+\infty$ (so that X hits 0 a.s.) and has paths with infinite variation. Its law is characterized by its Laplace transform

$$\mathbb{E}_0 \left[e^{-\lambda X_t} \right] = e^{t\psi(\lambda)}, \quad \lambda > 0$$

specified by the Lévy-Khintchine formula

$$\psi(\lambda) = \alpha_0 \lambda + \beta \lambda^2 + \int_{(0, \infty)} \Lambda(dr) \left(e^{-\lambda r} - 1 + \lambda r \mathbf{1}_{\{r < 1\}} \right)$$

where $\alpha_0 \in \mathbb{R}$, $\beta \geq 0$ and Λ is a positive measure on $(0, +\infty)$ such that $\int_0^\infty \Lambda(dr)(1 \wedge r^2)$ is finite. Actually, since X does not drift to $+\infty$, it has finite first moment and then $\int_{(0, \infty)} \Lambda(dr)(r \wedge r^2) < \infty$. We can rewrite ψ as

$$\psi(\lambda) = \alpha \lambda + \beta \lambda^2 + \int_{(0, \infty)} \Lambda(dr)(e^{-\lambda r} - 1 + \lambda r).$$

Then, X does not drift to $+\infty$ iff $\psi'(0) = \alpha \geq 0$. More precisely, if $\alpha = 0$, X is recurrent while if $\alpha < 0$, it drifts to $-\infty$.

According to Corollary VII.5 in [5], X has infinite variation iff $\lim_{\lambda \rightarrow \infty} \psi(\lambda)/\lambda = +\infty$, which is satisfied iff

$$\beta > 0 \text{ or } \int_{(0, 1)} r \Lambda(dr) = \infty. \quad (4.20)$$

In the following, we always suppose that (4.2) holds which implies (4.20).

We denote by I the past infimum process of X

$$I_t = \inf_{0 \leq s \leq t} X_s.$$

The process $X - I$ is a strong Markov process and because X has infinite variation, 0 is regular for itself for this process. Thus, we can consider its excursion measure N away from 0 and normalized so that $-I$ is the associated local time.

The point 0 is also regular for itself for the Markov process $X - S$. We denote by L a local time at 0 for this process and by N^* the associated excursion measure normalized so that for every Borel subset B of $(-\infty, 0)$,

$$N^* \left(\int_0^\sigma \mathbf{1}_B(X_s) ds \right) = m(B) \quad (4.21)$$

where m denotes the Lebesgue measure on \mathbb{R} (see Lemma 1.1.1 in [19] for more details). This also fixes the normalization of the local time L .

We now define the height process H associated with X which is the counterpart of the height process in the finite variation case. As in Section 4.2, we denote by $X^{(t)}$ the time-reversed process associated with X at time t and by $S^{(t)}$ its past supremum.

Definition 4.13. The height process $H = (H_t, t \geq 0)$ associated with the Lévy process X is the local time at level 0 at time t of the process $S^{(t)} - X^{(t)}$ with the normalization fixed in (4.21) and $H_0 := 0$.

According to Theorem 1.4.3 in [19], the condition (4.2) implies that H has a.s. continuous sample paths. There is an alternative definition of H_t : there exists a positive sequence $(\varepsilon_k, k \geq 0)$ such that a.s., for all $t \geq 0$,

$$H_t = \lim_{k \rightarrow \infty} \frac{1}{\varepsilon_k} \int_0^t \mathbf{1}_{\{X_t < I_s^t + \varepsilon_k\}} ds < \infty$$

where for $0 \leq s \leq t$, we set by

$$I_s^t := \inf_{s \leq r \leq t} X_r$$

the future infimum process of X at time t . Since this process is non-decreasing, one can define its right-continuous inverse I_t^{-1}

$$I_t^{-1}(u) := \inf\{s \geq 0; I_s^t > u\}, \quad 0 \leq u \leq X_t.$$

Observe that for $0 \leq u \leq I_t$, $I_t^{-1}(u) = 0$.

We now define the equivalent of the sequence $(\rho_i^t, i \leq H_t)$ defined in Section 4.2. Here, we consider the measure $\tilde{\rho}_t$ defined in [19, p.25] and called exploration process to which we add a weight I_t at 0.

Definition 4.14. For $t \geq 0$, the random positive measure ρ_t is defined by

$$\langle \rho_t, f \rangle = \langle \tilde{\rho}_t, f \rangle + I_t f(0) = \int_{[0, t]} d_s I_s^t f(H_s) + I_t f(0)$$

where $d_s I_s^t$ is the Stieltjes measure associated with the non-decreasing function $s \mapsto I_s^t$.

Notice that this random measure has support $[0, H_t]$, that $\rho_t(\{0\}) = I_t$ and that its total mass is $\langle \rho_t, \mathbf{1} \rangle = I_t + (X_t - I_t) = X_t$.

As in Section 4.2, we condition the Lévy process X to reach large heights before hitting 0 and the following theorem is the counterpart of Theorem 4.3.

Theorem 4.15. Recall that $\alpha = \psi'(0)$. Then, if (4.2) holds, for $t \geq 0$ and $\Theta \in \mathcal{F}_t$,

$$\lim_{a \rightarrow \infty} \mathbb{P}_x \left(\Theta, t < T_0 \mid \sup_{s \leq T_0} H_s \geq a \right) = \frac{1}{x} \mathbb{E}_x \left[\int_0^{H_t} \rho_t(dz) e^{\alpha z}; t \leq T_0, \Theta \right].$$

Moreover, if

$$M_t := \int_0^{H_t} \rho_t(dz) e^{\alpha z},$$

$(M_{t \wedge T_0}, t \geq 0)$ is a martingale under \mathbb{P}_x . In particular, similarly to the finite variation case, if $\alpha = 0$ (recurrent case), then $M_t = X_t$.

This theorem is proved in Section 4.4.2.

Remark 4.16. In the particular case $\Lambda \equiv 0$ (so we assume $\beta > 0$), we have $\psi(\lambda) = \alpha\lambda + \beta\lambda^2$ and X is a Brownian motion with drift $-\alpha$ and variance 2β . Then, as it is noticed in [52], the local time process at 0 for $S - X$ is S/β . Also, for $t \geq 0$, $H_t = (X_t - I_t)/\beta$ and $\tilde{\rho}_t/\beta$ is the Lebesgue measure on $[0, H_t]$. Finally, in that particular case, if $\alpha > 0$, the process

$$\left(I_{t \wedge T_0} + \frac{\beta}{\alpha} \left(e^{\frac{\alpha}{\beta}(X_t - I_t)} - 1 \right) \mathbf{1}_{\{t \leq T_0\}}, t \geq 0 \right)$$

is a martingale. In fact, this process belongs to a larger class of martingales called Kennedy martingales [43] and studied in [4].

As in the finite variation case, we define a new probability \mathbb{P}_x^\uparrow as a h -transform via the martingale $M_{\cdot \wedge T_0}$. The following proposition is the counterpart of Proposition 4.9 in the finite variation case.

Proposition 4.17. *Under \mathbb{P}_x^\uparrow , in probability*

$$X_t \xrightarrow[t \rightarrow \infty]{} +\infty.$$

This proposition, which will be proved in Section 4.4.3, justifies that \mathbb{P}_x^\uparrow can be considered as the law of a Lévy process started at $x > 0$ and conditioned to stay positive.

Remark 4.18. A natural improvement of the last property would be to prove that a.s. convergence also holds. In [5], Bertoin shows such a convergence result for spectrally negative Lévy processes and conditioned to stay positive under a different law \mathbb{P}_x^\uparrow defined with the help of the scale function of X . We cannot adapt his proof because in his case, X is still a Markov process under the new probability while in our case it is not.

Remark 4.19. Notice that the Definition 4.8 of \mathbb{P}_x^\uparrow does not make sense for $x = 0$. Then, a natural question is: can we define \mathbb{P}_0^\uparrow the law of a conditioned process beginning at 0 by finding a limit of the family of measures \mathbb{P}_x^\uparrow as $x \rightarrow 0$? This kind of problem has already been treated by Bertoin in [5, Prop VII.14] and by Chaumont and Doney in [15, Thm 2] for different hypotheses about X as mentioned previously.

4.4.2 Proof of Theorem 4.15

Before the proof, we give two technical lemmas concerning integrability of the exploration process and the height process.

Recall that $\tilde{\rho}$ is the exploration measure process defined in [19]

$$\langle \tilde{\rho}_t, f \rangle = \int_{[0,t]} d_s I_s^t f(H_s), \quad t \geq 0.$$

We denote by e_α the mapping

$$e_\alpha : x \mapsto e^{\alpha x}.$$

Lemma 4.20. *For $t > 0$,*

$$\mathbb{E} [\langle \tilde{\rho}_t, e_\alpha \rangle] = \mathbb{E} \left[\int_0^{H_t} \tilde{\rho}_t(dz) e^{\alpha z} \right] < \infty.$$

Proof. First, for $0 \leq s \leq t$, we have

$$H_s \leq H_t \quad d_s I_s^t \text{ a.e.}$$

because on $\{\Delta I_s^t > 0\} = \{\Delta S_s^{(t)} > 0\}$, $H_t = L_t^{(t)} = L_s^{(t)} + L_s^{(s)} \geq L_s^{(s)} = H_s$ where for $0 \leq u \leq v$, $L_u^{(v)}$ denotes the local time at level 0 at time u of the reverse process $S^{(v)} - X^{(v)}$. Then,

$$\begin{aligned} \mathbb{E} \left[\int_0^{H_t} \tilde{\rho}_t(dz) e^{\alpha z} \right] &= \mathbb{E} \left[\int_0^t d_s I_s^t e^{\alpha H_s} \right] \leq \mathbb{E} \left[\int_0^t d_s I_s^t e^{\alpha H_t} \right] \\ &\leq \mathbb{E} \left[(X_t - I_t) e^{\alpha H_t} \right] = \mathbb{E} [S_t^{(t)} e^{\alpha H_t}]. \end{aligned}$$

By time-reversion at t , we have $\mathbb{E}\left[S_t^{(t)} e^{\alpha H_t}\right] = \mathbb{E}[S_t e^{\alpha L_t}]$ where $(L_t, t \geq 0)$ is a local time at 0 for $S - X$. First, if $\alpha = 0$, $\mathbb{E}[S_t e^{\alpha L_t}] = \mathbb{E}[S_t]$ is finite according to [5, Thm VII.4]. Moreover, if $\alpha > 0$,

$$\mathbb{E}[S_t e^{\alpha L_t}] = \int_0^\infty da \mathbb{E}\left[S_t \mathbf{1}_{\{L_t \geq \ln a/\alpha\}}\right] = \mathbb{E}[S_t] + \int_1^\infty da \mathbb{E}\left[S_t \mathbf{1}_{\{L_t \geq \ln a/\alpha\}}\right]. \quad (4.22)$$

Since $\mathbb{E}[S_t]$ is finite, in order to prove the lemma, it is sufficient to show that

$$a \mapsto \mathbb{E}\left[S_t \mathbf{1}_{\{L_t \geq \ln a/\alpha\}}\right]$$

is integrable on $[1, +\infty)$. For $b > 0$, if $L^{-1}(\cdot) := \inf\{s \geq 0; L(s) > \cdot\}$ denotes the right-continuous inverse of L ,

$$\mathbb{E}\left[S_t \mathbf{1}_{\{L_t \geq b\}}\right] = \mathbb{E}\left[S_{L^{-1}(b)} \mathbf{1}_{\{L^{-1}(b) \leq t\}}\right] + \mathbb{E}\left[\left(S_t - S_{L^{-1}(b)}\right) \mathbf{1}_{\{L^{-1}(b) \leq t\}}\right] \quad (4.23)$$

On the one hand, if we denote by φ the bijective inverse mapping of ψ , the process $\left(\left(L^{-1}(b), S_{L^{-1}(b)}\right), b < L(\infty)\right)$ (called ascending ladder process) is a bivariate Lévy process killed at rate $\psi'(0) = \alpha$ and whose bivariate Laplace exponent κ is

$$\kappa(\mu, \lambda) := c \frac{\mu - \psi(\lambda)}{\varphi(\mu) - \lambda} \quad (4.24)$$

in the sense

$$\mathbb{E}\left[e^{-\mu L^{-1}(b) - \lambda S_{L^{-1}(b)}}\right] = e^{-b\kappa(\mu, \lambda)}.$$

Moreover, c is a constant depending on the normalization of the local time L (see Section VI.1 and Theorem VII.4 in [5]). In fact, as it is proved in the proof of Lemma 1.1.2 in [19], with the normalization defined by (4.21), we have $c = 1$. Then, for $\mu > 0$, using Markov inequality and (4.24), the first term of the r.h.s. of (4.23) satisfies

$$\begin{aligned} \mathbb{E}\left[S_{L^{-1}(b)} \mathbf{1}_{\{L^{-1}(b) \leq t\}}\right] &= \mathbb{E}\left[S_{L^{-1}(b)} \mathbf{1}_{\{e^{-\mu L^{-1}(b)} \geq e^{-\mu t}\}}\right] \leq e^{\mu t} \mathbb{E}\left[S_{L^{-1}(b)} e^{-\mu L^{-1}(b)}\right] \\ &\leq b e^{\mu t} \frac{\partial}{\partial \lambda} \kappa(\mu, 0) e^{-b\kappa(\mu, 0)} = b e^{\mu t} \frac{\mu - \alpha \varphi(\mu)}{\varphi(\mu)^2} e^{-b\mu/\varphi(\mu)}. \end{aligned} \quad (4.25)$$

On the other hand, by the strong Markov property applied at the stopping time $L^{-1}(b)$, the second term of r.h.s. of (4.23) is

$$\mathbb{E}\left[\left(S_t - S_{L^{-1}(b)}\right) \mathbf{1}_{\{L^{-1}(b) \leq t\}}\right] = \mathbb{E}\left[S'_{t-L^{-1}(b)} \mathbf{1}_{\{L^{-1}(b) \leq t\}}\right]$$

where S' has same distribution as process S and is independent from $L^{-1}(b)$. Then,

$$\mathbb{E}\left[\left(S_t - S_{L^{-1}(b)}\right) \mathbf{1}_{\{L^{-1}(b) \leq t\}}\right] \leq \mathbb{E}[S'_t] \cdot \mathbb{P}(L^{-1}(b) \leq t).$$

According to [5, Thm VII.4], $\mathbb{E}[S'_t]$ is finite and by similar computations as before, for $\mu > 0$,

$$\mathbb{P}(L^{-1}(b) \leq t) \leq e^{\mu t} e^{-b\mu/\varphi(\mu)}. \quad (4.26)$$

Then, using (4.25), (4.26) and choosing $b = \ln a/\alpha$, we have

$$\mathbb{E}\left[S_t \mathbf{1}_{\{L_t \geq \ln a/\alpha\}}\right] \leq C(\mu, t) \left(\frac{\ln a}{\alpha} + 1\right) a^{-\frac{\mu}{\varphi(\mu)\alpha}}$$

where $C(\mu, t)$ is a constant only depending on μ and t . Since $\varphi = \psi^{-1}$ is concave,

$$\frac{\varphi(\mu)}{\mu} \leq \varphi'(0) = \alpha^{-1}$$

and since by hypothesis $\psi(\lambda)/\lambda \rightarrow \infty$, for μ large enough, $\mu/(\varphi(\mu)\alpha) > 1$. Hence, the application $a \mapsto \mathbb{E} \left[S_t \mathbf{1}_{\{L_t \geq \ln a/\alpha\}} \right]$ is integrable on $[1, +\infty)$ and the proof is complete. \square

Lemma 4.21. *For any $\xi > 0$ and $t \geq 0$,*

$$\int_0^t \mathbb{E} \left[e^{\xi H_s} \right] ds < \infty.$$

Proof. By time reversibility at time s , $\mathbb{E} \left[e^{\xi H_s} \right] = \mathbb{E} \left[e^{\xi L_s} \right]$. Using (4.26) and similar techniques as in the previous proof, we have for all $\mu > 0$

$$\mathbb{E} \left[e^{\xi H_s} \right] \leq e^{\mu s} \int_1^\infty a^{-\mu/(\varphi(\mu)\xi)} da + 1.$$

Then, $\int_0^t \mathbb{E} \left[e^{\xi H_s} \right] ds$ is finite for μ large enough. \square

Proof of Theorem 4.15.

Step 1: We begin the proof with showing that $(M_{t \wedge T_0}, t \geq 0)$ is a martingale. In the case $\alpha = 0$, $M = X$ and it is known that $(X_t, t \geq 0)$ is a martingale.

We now suppose that $\alpha > 0$. For $t \geq 0$, we set

$$\nu_t := (I_t + x) \mathbf{1}_{\{I_t > -x\}} \delta_0 + \tilde{\rho}_t$$

and

$$\theta := \inf\{t \geq 0; \nu_t = 0\}.$$

Notice that ν is the exploration process starting at $x\delta_0$ defined in [19] by Duquesne and Le Gall. If we set $\tilde{\mathcal{G}}_t := \sigma(\tilde{\rho}_s, 0 \leq s \leq t)$ and $\mathcal{H}_t := \sigma(\nu_s, 0 \leq s \leq t)$, we have the inclusions $\tilde{\mathcal{G}}_t \subset \mathcal{H}_t \subset \mathcal{F}_t$. The second one is trivial and the first one holds because ν_t is the sum of the measure $\tilde{\rho}_t$ (which satisfies $\tilde{\rho}_t(\{0\}) = 0$) and a measure whose support is $\{0\}$. Moreover, $\tilde{\mathcal{G}}_t = \mathcal{F}_t$ because $\langle \tilde{\rho}_t, \mathbf{1} \rangle = X_t - I_t$ and because $-I_t$ is the local time at time t at level 0 of the process $\tilde{\rho}_t$. Hence, the three filtrations are equal and it is sufficient to prove that $M_{\cdot \wedge T_0}$ is a \mathcal{H} -martingale.

According to Corollary 2.5 and Remark 2.7 in [1], if f is a bounded non-negative function on $[0, \infty)$ of class C^1 such that its first derivative f' is bounded and such that f has a finite limit at $+\infty$, then, under \mathbb{P}_0 , the process $(M_t^f, t \geq 0)$ defined by

$$M_t^f := e^{-\langle \nu_t \wedge \theta, f \rangle} + \int_0^{t \wedge \theta} e^{-\langle \nu_s, f \rangle} (f'(H_s) - \psi(f(H_s))) ds \quad (4.27)$$

is a martingale with respect to the filtration \mathcal{H} . In fact, in [1], the authors only proved this result in the case $\beta = 0$ and $\int_{(0,1]} \Lambda(dr)r = \infty$. However, their proof can easily be adapted in the general setting using results of [19, Chap. 3].

Recall that $e_\alpha : z \mapsto e^{\alpha z}$ and let $(h_K)_{K \geq 1}$ be a sequence of functions of class C^1 such that for all $K \geq 1$, $0 \leq h_K \leq K \wedge e_\alpha$, $0 \leq h'_K \leq e'_\alpha$, h'_K is bounded and $(h_K)_K$ (resp. $(h'_K)_K$) is an increasing sequence which converges pointwise to e_α (resp. e'_α).

Then, for $a > 0, K \geq 1$, applying (4.27) with $f = ah_K$,

$$M_t^{a,K} := \frac{1 - e^{-a\langle \nu_{t \wedge \theta}, h_K \rangle}}{a} + \int_0^{t \wedge \theta} e^{-a\langle \nu_s, h_K \rangle} \left(\frac{\psi(ah_K(H_s))}{a} - h'_K(H_s) \right) ds$$

defines a martingale. We first have

$$M_t^{a,K} \xrightarrow{a \rightarrow 0} M_t^K := \langle \nu_{t \wedge \theta}, h_K \rangle + \int_0^{t \wedge \theta} (\psi'(0)h_K(H_s) - h'_K(H_s)) ds \quad \text{a.s.}$$

By Beppo Levi's theorem, since $(h_K)_K$ and $(h'_K)_K$ are non-negative and non-decreasing, we have with probability 1

$$\begin{aligned} M_t^\infty &:= \lim_{K \rightarrow \infty} M_t^K = \langle \nu_{t \wedge \theta}, e_\alpha \rangle + \int_0^{t \wedge \theta} (\alpha e^{\alpha H_s} - \alpha e^{\alpha H_s}) ds \\ &= \langle \nu_{t \wedge \theta}, e_\alpha \rangle. \end{aligned}$$

We want to show that $(M_t^\infty)_t$, that is the limit as $a \rightarrow 0$ and then as $K \rightarrow \infty$ of the martingales $M_t^{a,K}$, is still a martingale. To do this, we use the dominated convergence theorem and we have to find an upper bound for $M_t^{a,K}$ independent from a and K . We have

$$|M_t^{a,K}| \leq \langle \nu_t, e_\alpha \rangle + \int_0^t \alpha e^{\alpha H_s} ds + \int_0^t \frac{\psi(ah_K(H_s))}{ah_K(H_s)} h_K(H_s) ds.$$

Moreover, there exists $C_1, C_2 > 0$ such that $\frac{\psi(\lambda)}{\lambda} \leq C_1 + C_2 \lambda$. Hence,

$$\begin{aligned} |M_t^{a,K}| &\leq (I_t + x) \mathbf{1}_{\{I_t > -x\}} + \langle \tilde{\rho}_t, e_\alpha \rangle + \int_0^t \alpha e^{\alpha H_s} ds + \int_0^t \left(C_1 e^{\alpha H_s} + C_2 a (e^{\alpha H_s} - 1)^2 \right) ds \\ &\leq x + \langle \tilde{\rho}_t, e_\alpha \rangle + \int_0^t \left(AC_2 + (\alpha + C_1) e^{\alpha H_s} + AC_2 e^{2\alpha H_s} \right) ds \end{aligned} \quad (4.28)$$

if a belongs to the compact set $[0, A]$.

Thus, according to Lemmas 4.20 and 4.21, the r.h.s. of (4.28) is integrable. Then, applying the dominated convergence theorem, we have proved that $(\langle \nu_{t \wedge \theta}, e_\alpha \rangle, t \geq 0)$ is a martingale under \mathbb{P}_0 with respect to \mathcal{H} .

Furthermore,

$$\langle \nu_{t \wedge \theta}, e_\alpha \rangle = \left((I_t + x) \mathbf{1}_{\{I_t > -x\}} + \langle \tilde{\rho}_t, e_\alpha \rangle \right) \mathbf{1}_{\{t \leq \theta\}} = (x + I_t + \langle \tilde{\rho}_t, e_\alpha \rangle) \mathbf{1}_{\{t \leq T_{-x}\}}$$

where T_{-x} is the hitting time of $(-\infty, -x]$ by X under \mathbb{P}_0 .

Then, since $(x + I_t + \langle \tilde{\rho}_t, e_\alpha \rangle) \mathbf{1}_{\{t \leq T_{-x}\}}$ under \mathbb{P}_0 has the same law as

$$(I_t + \langle \tilde{\rho}_t, e_\alpha \rangle) \mathbf{1}_{\{t \leq T_0\}} = M_{t \wedge T_0}$$

under \mathbb{P}_x , $M_{\cdot \wedge T_0}$ is a martingale w.r.t. the filtration $\mathcal{H} = \mathcal{F}$.

Step 2: We are now able to prove the main part of Theorem 4.3. For $t > 0$ and $\Theta \in \mathcal{F}_t$, we have

$$\mathbb{P}_x \left(\Theta, t < T_0 \mid \sup_{s \leq T_0} H_s \geq a \right) = \frac{\mathbb{P}_x(\Theta, \tau_a \leq t < T_0) + \mathbb{P}_x(\Theta, t < \tau_a < T_0)}{\mathbb{P}_x(\sup_{s \leq T_0} H_s \geq a)} \quad (4.29)$$

and we will investigate the asymptotic behaviors of the three probabilities in the last equation. First, assumption (4.2) enables us to define the bijective mapping $\phi : (0, \infty) \rightarrow (0, \infty)$ such that

$$\phi(t) := \int_t^\infty \frac{d\lambda}{\psi(\lambda)}, \quad t > 0.$$

We denote by v its bijective inverse. Trivially, $v(a) \rightarrow 0$ as a goes to ∞ and thanks to Lemma 2.1 in [46], for $u \geq 0$, we have

$$\lim_{a \rightarrow \infty} \frac{v(a-u)}{v(a)} = e^{\alpha u}. \quad (4.30)$$

According to [19, Cor. 1.4.2], the mapping v can be linked to the excursion measure N :

$$v(a) = N(\sup H > a), \quad a > 0.$$

Then, by excursion theory and the exponential formula,

$$\mathbb{P}_x \left(\sup_{s \leq T_0} H_s \geq a \right) = 1 - \exp(-xN(\sup H > a)) = 1 - e^{-xv(a)}. \quad (4.31)$$

Hence,

$$\mathbb{P}_x \left(\sup_{s \leq T_0} H_s \geq a \right) \underset{a \rightarrow \infty}{\sim} xv(a). \quad (4.32)$$

The last display can be linked to continuous-state branching processes (CSBP) in the case $\beta = 0$. Indeed, according to [50, Thm 4.2], for any continuous and bounded function f on \mathbb{R}^+ ,

$$\int_0^{T_0} f(H_s) ds = \int_0^\infty f(u) Z_x(u) du, \quad \mathbb{P}_x - \text{a.s.}$$

where $(Z_x(u), u \geq 0)$ is a CSBP starting at x and with branching mechanism ψ . Then,

$$\mathbb{P}_x \left(\sup_{s \leq T_0} H_s \geq a \right) = 1 - \mathbb{P}_x(Z_x(a) \neq 0) = 1 - e^{-xv(a)}.$$

Then, we consider the probability $\mathbb{P}_x(\Theta, \tau_a \leq t < T_0) \leq \mathbb{P}_x(\tau_a \leq t)$ and we want to prove that it goes to 0 faster than $v(a)$ as $a \rightarrow \infty$. We define

$$\sigma_a := \sup\{t < \tau_a, X_t = I_t\}$$

the left-end point of the first excursion of $X - I$ which reaches height a and set $\varepsilon_a := \tau_a - \sigma_a$.

Let $(e(t), t \geq 0)$ be the excursion point process of $X - I$ at level 0, that is, for $t \geq 0$

$$e(t) = \begin{cases} \left((X - I)_{s+I^{-1}(t-)}, 0 \leq s \leq I^{-1}(t) - I^{-1}(t-) \right) & \text{if } I^{-1}(t-) < I^{-1}(t) \\ \partial & \text{otherwise} \end{cases}$$

where ∂ is a cemetery point.

For a generic excursion ε with duration $\sigma = \sigma(\varepsilon)$, we denote by

$$h^* := \sup_{[0, \sigma]} H(\varepsilon)$$

its maximum height. For $t \geq 0$, we denote by Δ_t the length of the excursion $e(t)$ of $X - I$ and we set $h_t^* := h^*(e(t))$. Then, since $(\Delta_t, h_t^*)_{t \geq 0}$ is the image by a measurable application of the Poisson point process $(e(t), t \geq 0)$, it is a Poisson point process on $(0, \infty) \times (0, \infty)$. Distinguishing its atoms (δ, h) between $h > a$ and $h < a$, we obtain that

$$Y_t^a := \sum_{s \leq t} \Delta_s \mathbf{1}_{\{h_s^* < a\}}$$

and

$$\tilde{Y}_t^a := \sum_{s \leq t} \Delta_s \mathbf{1}_{\{h_s^* > a\}}$$

are independent. Moreover, $(Y_t^a, t \geq 0)$ is a subordinator whose Lévy measure is

$$N(\Delta \in dr; \sup H < a).$$

If φ^a denotes its Laplace exponent, as a tends to ∞ , $\varphi^a \rightarrow \varphi$ where $\varphi = \psi^{-1}$ is the Laplace exponent of the subordinator $\sum_{s \leq t} \Delta_s$ (see Theorem VII.1 in [5]).

Furthermore, $\gamma_a := \inf\{s \geq 0; \tilde{Y}_s^a \neq 0\}$ is independent from Y^a and follows an exponential distribution with parameter $N(\sup H > a) = v(a)$. Hence,

$$\begin{aligned} \mathbb{E} \left[e^{-\lambda \sigma_a} \right] &= \mathbb{E} \left[e^{-\lambda Y^a(\gamma_a)} \right] = v(a) \int_0^\infty dt e^{-v(a)t} \mathbb{E} \left[e^{-\lambda Y_t^a} \right] = v(a) \int_0^\infty dt e^{-v(a)t} e^{-\varphi^a(\lambda)t} \\ &= \frac{v(a)}{v(a) + \varphi^a(\lambda)}. \end{aligned}$$

Hence, since σ_a and ε_a are independent,

$$\mathbb{P}_x(\tau_a \leq t) = \mathbb{P}_x(\sigma_a + \varepsilon_a \leq t) = \mathbb{E}_x \left[\mathbb{P}_x(\sigma_a \leq t - \varepsilon_a | \varepsilon_a) \mathbf{1}_{\{\varepsilon_a \leq t\}} \right]$$

and by Markov inequality, for all $\lambda > 0$,

$$\mathbb{P}_x(\tau_a \leq t) \leq \mathbb{E}_x \left[e^{\lambda(t-\varepsilon_a)} \frac{v(a)}{v(a) + \varphi^a(\lambda)} \mathbf{1}_{\{\varepsilon_a \leq t\}} \right] = \frac{v(a)e^{\lambda t}}{v(a) + \varphi^a(\lambda)} \mathbb{E}_x \left[e^{-\lambda \varepsilon_a} \mathbf{1}_{\{\varepsilon_a \leq t\}} \right].$$

Then,

$$\frac{\mathbb{P}_x(\tau_a \leq t)}{v(a)} \leq \frac{e^{\lambda t} \mathbb{P}_x(\varepsilon_a \leq t)}{v(a) + \varphi^a(\lambda)} \xrightarrow{a \rightarrow \infty} 0 \quad (4.33)$$

since $\varphi^a \rightarrow \varphi$ and the r.v. ε_a is the time to reach height a for an excursion of $X - I$ conditioned on $\sup H > a$, so that $\mathbb{P}_x(\varepsilon_a \leq t)$ vanishes in the limit $a \rightarrow \infty$.

We finally study the asymptotic behavior of the last term of (4.29). We have

$$\mathbb{P}_x(\Theta, t < \tau_a < T_0) = \mathbb{E}_x \left[\mathbf{1}_\Theta \mathbf{1}_{\{t \leq T_0 \wedge \tau_a\}} \mathbb{P}_x(\tau_a < T_0 | \mathcal{F}_t) \right]$$

and

$$1 - \mathbb{P}_x(\tau_a < T_0 | \mathcal{F}_t) = \mathbb{E}_x \left[\exp \left(- \sum_{0 \leq u \leq X_t} \chi(u, e(I_t^{-1}(u))) \right) \middle| \mathcal{F}_t \right]$$

where for $u > 0$ and for a generic excursion ε of $X - I$ of duration $\sigma = \sigma(\varepsilon)$,

$$\chi(u, \varepsilon) = \begin{cases} \infty & \text{if } \sup_{[0, \sigma]} H(\varepsilon) > a - H_{I_t^{-1}(u)} \\ 0 & \text{otherwise} \end{cases}$$

where we remind the reader that $I_t^{-1}(\cdot)$ is the right-continuous inverse of $(I_s^t, s \leq t)$. Then, by the exponential formula for Poisson point processes,

$$\begin{aligned} 1 - \mathbb{P}_x(\tau_a < T_0 | \mathcal{F}_t) &= \exp \left(- \int_0^{X_t} du \int N(d\varepsilon) \left(1 - e^{-\chi(u, \varepsilon)} \right) \right) \\ &= \exp \left(- \int_0^{X_t} du N(h^* > a - H_{I_t^{-1}(u)}) \right) \\ &= \exp \left(- \int_0^t (d_s I_s^t + I_t \delta_0(s)) N(h^* > a - H_s) \right) \\ &= \exp(-\langle \rho_t, v(a - \cdot) \rangle). \end{aligned}$$

Hence, since $\langle \rho_t, v(a - \cdot) \rangle \leq X_t v(a - H_t)$ goes to 0 as a goes to ∞ and according to (4.30),

$$\frac{\mathbb{P}_x(\tau_a < T_0 | \mathcal{F}_t)}{v(a)} = \frac{1 - \exp(-\langle \rho_t, v(a - \cdot) \rangle)}{v(a)} \xrightarrow{a \rightarrow \infty} \langle \rho_t, e_\alpha \rangle.$$

Then, using Fatou's Lemma,

$$\liminf_{a \rightarrow \infty} \frac{\mathbb{P}_x(\Theta, t < \tau_a < T_0)}{v(a)} \geq \mathbb{E}_x \left[\mathbf{1}_\Theta \mathbf{1}_{\{t \leq T_0\}} \langle \rho_t, e_\alpha \rangle \right] = \mathbb{E}_x [\mathbf{1}_\Theta M_{t \wedge T_0}].$$

Replacing Θ by Θ^c in the latter display, we have

$$\liminf_{a \rightarrow \infty} \frac{\mathbb{P}_x(t < \tau_a < T_0)}{v(a)} - \limsup_{a \rightarrow \infty} \frac{\mathbb{P}_x(\Theta, t < \tau_a < T_0)}{v(a)} \geq \mathbb{E}_x [M_{t \wedge T_0}] - \mathbb{E}_x [\mathbf{1}_\Theta M_{t \wedge T_0}].$$

Since $M_{\cdot \wedge T_0}$ is a martingale, $\mathbb{E}_x [M_{t \wedge T_0}] = x$. Moreover,

$$\frac{\mathbb{P}_x(t < \tau_a < T_0)}{v(a)} = \frac{\mathbb{P}_x(\tau_a < T_0)}{v(a)} - \frac{\mathbb{P}_x(\tau_a < T_0, \tau_a \leq t)}{v(a)}$$

and in the limit $a \rightarrow \infty$, the first term of the r.h.s. tends to x according to (4.32) and the second vanishes thanks to (4.33). Finally, we have

$$\limsup_{a \rightarrow \infty} \frac{\mathbb{P}_x(\Theta, t < \tau_a < T_0)}{v(a)} \leq \mathbb{E}_x [\mathbf{1}_\Theta M_{t \wedge T_0}] \leq \liminf_{a \rightarrow \infty} \frac{\mathbb{P}_x(\Theta, t < \tau_a < T_0)}{v(a)} \quad (4.34)$$

and putting together (4.32), (4.33) and (4.34), the proof is completed. \square

4.4.3 Proof of Proposition 4.17

We want to prove that under \mathbb{P}_x^\uparrow , X_t goes to infinity in probability as $t \rightarrow \infty$. We make a similar proof as that of Proposition 4.9, that is, we want to prove that for $\lambda > 0$, $\mathbb{E}_x^\uparrow [e^{-\lambda X_t}] \rightarrow 0$ as t tends to ∞ .

First, in the case $\alpha = 0$,

$$\mathbb{E}_x^\uparrow [e^{-\lambda X_t}] = \frac{1}{x} \mathbb{E}_x [e^{-\lambda X_t} X_{t \wedge T_0}] \xrightarrow{t \rightarrow \infty} 0$$

using dominated convergence theorem.

We now suppose that $\alpha > 0$. Then, for all positive a ,

$$\begin{aligned} \mathbb{E}_x^\uparrow [e^{-\lambda X_t}] &= \frac{1}{x} \mathbb{E}_x \left[e^{-\lambda X_t} \left(I_t + \int_0^t d_s I_s^t e^{\alpha H_s} \right); t \leq T_0 \right] \\ &\leq \mathbb{P}_x(t \leq T_0) + \frac{1}{x} \mathbb{E}_x \left[e^{-\lambda X_t} \int_0^t d_s I_s^t e^{\alpha H_s} \left(\mathbf{1}_{\{H_t \leq a\}} + \mathbf{1}_{\{H_t > a\}} \right) \mathbf{1}_{\{t \leq T_0\}} \right] \\ &\leq \mathbb{P}_x(t \leq T_0) + \frac{e^{a\alpha}}{x} \mathbb{E}_x \left[e^{-\lambda X_t} (X_t - I_t); t \leq T_0 \right] \\ &\quad + \frac{1}{x} \mathbb{E}_x \left[e^{-\lambda(X_t - I_t)} \int_0^t d_s I_s^t e^{\alpha H_s}; H_t > a \right]. \quad (4.35) \end{aligned}$$

The first and the second terms of the r.h.s. vanish as t goes to ∞ thanks to the dominated convergence theorem and because T_0 is finite a.s. We now want to show that the third term of (4.35) vanishes as a goes to infinity uniformly in t . By time-reversing at time t , we have

$$B(t, a) := \mathbb{E}_x \left[e^{-\lambda(X_t - I_t)} \int_0^t d_s I_s^t e^{\alpha H_s}; H_t > a \right] = \mathbb{E} \left[e^{-\lambda S_t} \int_0^t d_r S_r e^{\alpha(L_t - L_r)}; L_t > a \right].$$

Then, observing that $\{L_t > a\} = \{L^{-1}(a) < t\}$ and applying the strong Markov property at the stopping time $L^{-1}(a)$,

$$\begin{aligned} B(t, a) &= \mathbb{E} \left[e^{-\lambda S_{L^{-1}(a)}} e^{\alpha a} \mathbf{1}_{\{L_t > a\}} e^{-\lambda(S_t - S_{L^{-1}(a)})} e^{\alpha(L_t - a)} \right. \\ &\quad \left. \cdot \left(\int_0^{L^{-1}(a)} d_r S_r e^{-\alpha L_r} + \int_{L^{-1}(a)}^t d_r S_r e^{-\alpha L_r} \right) \right] \\ &= \mathbb{E} \left[e^{-\lambda S_{L^{-1}(a)}} e^{\alpha a} \mathbf{1}_{\{L_t > a\}} \left(f_1(t - L^{-1}(a)) \int_0^{L^{-1}(a)} d_r S_r e^{-\alpha L_r} \right. \right. \\ &\quad \left. \left. + e^{-\alpha a} f_2(t - L^{-1}(a)) \right) \right] \quad (4.36) \end{aligned}$$

where for $u \geq 0$,

$$f_1(u) = \mathbb{E} \left[e^{-\lambda S_u} e^{\alpha L_u} \right]$$

and

$$f_2(u) = \mathbb{E} \left[e^{-\lambda S_u} e^{\alpha L_u} \int_0^u d_r S_r e^{-\alpha L_r} \right] \leq \mathbb{E} \left[e^{-\lambda S_u} S_u e^{\alpha L_u} \right]$$

We first show that these two functions are bounded on $(0, \infty)$. As in (4.22), we have

$$f_1(u) = \mathbb{E} [e^{-\lambda S_u} e^{\alpha L_u}] = \int_1^\infty da \mathbb{E} [e^{-\lambda S_u} \mathbf{1}_{\{L_u > \ln a / \alpha\}}] + \mathbb{E} [e^{-\lambda S_u}].$$

For $b > 0$, using (4.24),

$$\mathbb{E} [e^{-\lambda S_u} \mathbf{1}_{\{L_u > b\}}] \leq \mathbb{E} [e^{-\lambda S_{L^{-1}(b)}} \mathbf{1}_{\{b < L(\infty)\}}] = \exp \left(-b \frac{\psi(\lambda)}{\lambda} \right).$$

Hence, choosing $b = \ln a / \alpha$,

$$f_1(s) \leq \int_1^\infty a^{-\psi(\lambda)/(\lambda\alpha)} da + 1 < \infty$$

because ψ is convex and $\psi(\lambda)/\lambda > \psi'(0) = \alpha$ as soon as $\lambda > 0$.

We do the same with f_2 . For $b > 0$, if M denotes an upper bound of the mapping $x \mapsto x e^{-\lambda x}$,

$$\begin{aligned} \mathbb{E} [e^{-\lambda S_s} S_s \mathbf{1}_{\{L_s > b\}}] &= \mathbb{E} [e^{-\lambda S_{L^{-1}(b)}} e^{-\lambda(S_s - S_{L^{-1}(b)})} (S_s - S_{L^{-1}(b)} + S_{L^{-1}(b)}) \mathbf{1}_{\{L_s > b\}}] \\ &\leq M \mathbb{E} [e^{-\lambda S_{L^{-1}(b)}} \mathbf{1}_{\{L_s > b\}}] + \mathbb{E} [e^{-\lambda S_{L^{-1}(b)}} S_{L^{-1}(b)} \mathbf{1}_{\{L_s > b\}}] \\ &\leq \exp \left(-b \frac{\psi(\lambda)}{\lambda} \right) \left(M + b \frac{\psi'(\lambda)\lambda - \psi(\lambda)}{\lambda^2} \right) \end{aligned}$$

using that $(S_{L^{-1}(b)}, b < L(\infty))$ is a subordinator with Laplace exponent $\psi(\lambda)/\lambda$ according to (4.24). Then, taking $b = \ln a / \alpha$,

$$f_2(s) \leq C(\lambda) \int_1^\infty (1 + \ln a) a^{-\alpha^{-1}\psi(\lambda)/\lambda} da + M < \infty.$$

We come back to (4.36). If C_1 (resp. C_2) denotes an upper bound of f_1 (resp. f_2) and observing that

$$\int_0^{L^{-1}(a)} d_r S_r e^{-\alpha L_r} \leq S_{L^{-1}(a)} \quad \text{a.s.,}$$

we have

$$\begin{aligned} B(t, a) &\leq C_1 e^{\alpha a} \mathbb{E} \left[e^{-\lambda S_{L^{-1}(a)}} S_{L^{-1}(a)} \mathbf{1}_{\{L_t > a\}} \right] + C_2 \mathbb{E} \left[e^{-\lambda S_{L^{-1}(a)}} \mathbf{1}_{\{L_t > a\}} \right] \\ &\leq C_1 \frac{\psi'(\lambda)\lambda - \psi(\lambda)}{\lambda^2} a e^{\alpha a} e^{-a \frac{\psi(\lambda)}{\lambda}} + C_2 e^{-a \frac{\psi(\lambda)}{\lambda}}. \end{aligned}$$

Thus, $B(t, a)$ goes to 0 uniformly in t as a tends to $+\infty$ and the proof is completed.

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